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Sri Lanka

EN4152 Assignment 3
Continuous and Discrete Wavelet Transforms

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Contents

1 Continuous Wavelet Transform

1.1 Introduction

The continuous wavelet transform (CWT) is defined by the following equation:

$$W(s,\tau) = \int x(t)\,\frac{1}{\sqrt{s}}\,\psi\left(\frac{t-\tau}{s}\right)dt$$

where $s =$ scaling factor, $\quad \tau =$ translation, $\quad$ and $\quad \psi =$ wavelet function.

There are many wavelet families such as *Haar*, *Shannon*, *Mexican Hat*, *Morlet*, and *Daubechies*, which are useful for different applications.

1.2 Wavelet Properties

1.2.1 Derivation of the Mexican hat function $m(t)$

Given

$$g(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2},$$

we compute the derivatives.

First derivative:

$$g'(t) = \frac{d}{dt}\left(\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\right) = -\frac{t}{\sqrt{2\pi}}\,e^{-t^2/2}.$$

Second derivative:

$$g''(t) = \frac{d}{dt}\left(-\frac{t}{\sqrt{2\pi}}e^{-t^2/2}\right) = -\frac{1}{\sqrt{2\pi}}e^{-t^2/2} + \frac{t^2}{\sqrt{2\pi}}e^{-t^2/2} = \frac{1}{\sqrt{2\pi}}(t^2-1)e^{-t^2/2}.$$

By definition the Mexican hat is (with the sign used in the assignment)

$$m(t) = -\,g''(t) = \frac{1}{\sqrt{2\pi}}\,(1-t^2)\,e^{-t^2/2}.$$

1.2.2 Normalization (unity energy)

Compute the energy

$$E = \int_{-\infty}^{\infty} m(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1-t^2)^2 e^{-t^2}\,dt.$$

Expand the integrand and use standard Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}, \qquad \int_{-\infty}^{\infty} t^4 e^{-t^2}\,dt = \frac{3\sqrt{\pi}}{4}.$$

Thus

$$\int_{-\infty}^{\infty}(1-2t^2+t^4)e^{-t^2}\,dt = \sqrt{\pi} - 2\cdot\frac{\sqrt{\pi}}{2} + \frac{3\sqrt{\pi}}{4} = \frac{3\sqrt{\pi}}{4}.$$

So

$$E = \frac{1}{2\pi}\cdot\frac{3\sqrt{\pi}}{4} = \frac{3}{8\sqrt{\pi}}.$$

Therefore the normalization factor is

$$\frac{1}{\sqrt{E}} = \sqrt{\frac{8\sqrt{\pi}}{3}} = \frac{2\sqrt{6}}{3}\,\pi^{1/4}$$

and the *normalized* Mexican-hat mother wavelet $\psi(t)$ is

$$\psi(t) = \frac{m(t)}{\sqrt{E}} = \frac{1}{\sqrt{E}} \cdot \frac{1}{\sqrt{2\pi}}(1-t^2)e^{-t^2/2}.$$

1.2.3 Daughter (scaled) wavelets

Combining the constants,The normalized Mexican hat mother wavelet is:

$$\boxed{\psi(t) \;=\; \frac{2\sqrt{3}}{3}\,\pi^{-1/4}\,(1-t^2)\,e^{-t^2/2}}$$

which satisfies $\int_{-\infty}^{\infty} \psi(t)^2\,dt = 1$.

The scaled (daughter) wavelet with scale $s > 0$ is

$$\psi_s(t,s) \;=\; \frac{1}{\sqrt{s}}\,\psi\!\left(\frac{t}{s}\right) \;=\; \frac{1}{\sqrt{s}} \cdot \frac{2\sqrt{3}}{3}\,\pi^{-1/4}\left(1-\frac{t^2}{s^2}\right)\,\exp\!\Big(-\frac{t^2}{2s^2}\Big).$$

1.2.4 Mexican hat daughter wavelet for scaling factors of 0.01 : 0.1 : 2

Figure 1 shows the Mexican-hat daughter wavelets generated for scales $s = 0.01 : 0.1 : 2$.

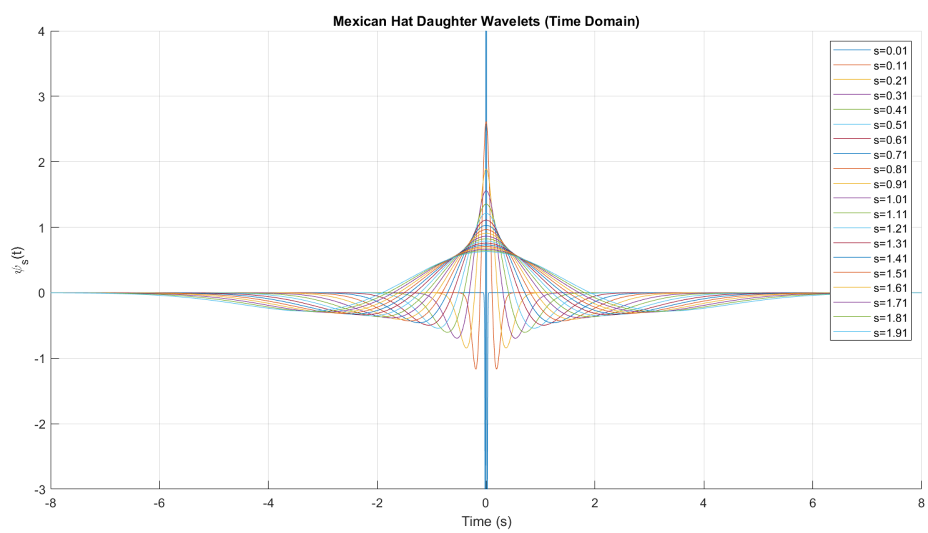

Figure 1: Mexican Hat Daughter Wavelets (Time Domain).

Verification (zero mean, unity energy, compact support). The script computed mean, energy and an observed support width for each scale. The printed verification table (MATLAB output) is included below.

Scale	Mean	Energy	SupportWidth
0.01	0	1	0.088
0.11	0	1	0.968
0.21	0	1	1.856
0.31	0	1	2.744
0.41	0	1	3.632
0.51	0	1	4.52
0.61	0	1	5.408
0.71	0	1	6.296
0.81	0	1	7.184
0.91	0	1	8.064
1.01	0	1	8.952
1.11	0	1	9.84
1.21	0	1	10.728
1.31	0	1	11.616
1.41	0	1	12.504
1.51	0	1	13.392
1.61	0	1	14.28
1.71	0	1	15.16
1.81	0	1	16.048
1.91	0	1	16.936

Observations:

- The **mean** for each daughter wavelet is effectively zero (rounded and printed as 0), verifying the zero-mean property.
- The **energy** for each daughter is unity (printed as 1), showing correct normalization.
- **Support width** increases with scale as expected — larger scales produce wider time-support wavelets (observational compact-support thresholding).

1.2.5 Spectra of daughter wavelets

Full and zoomed spectra. Figure 2 shows the full frequency spectra for the daughter wavelets; Figure 3 shows a zoomed-in view (0–6 Hz).

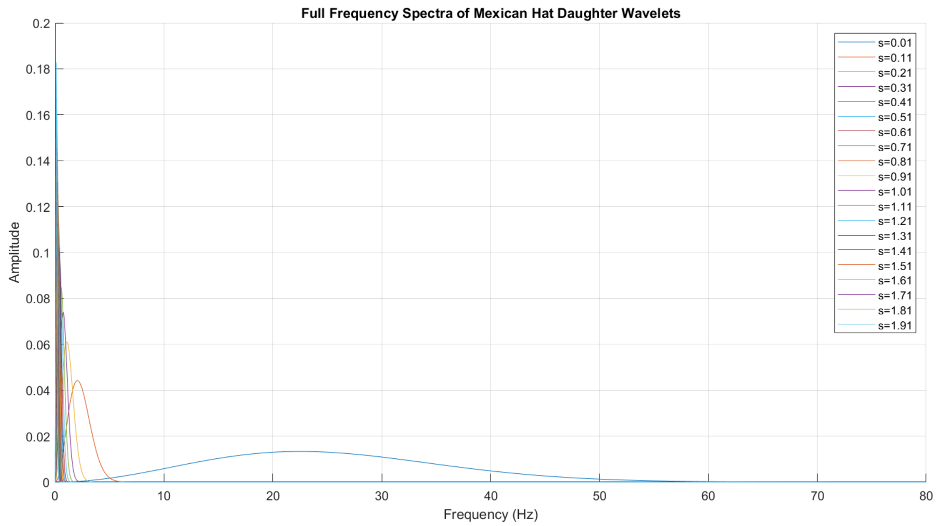

Figure 2: Full Frequency Spectra of Mexican Hat Daughter Wavelets.

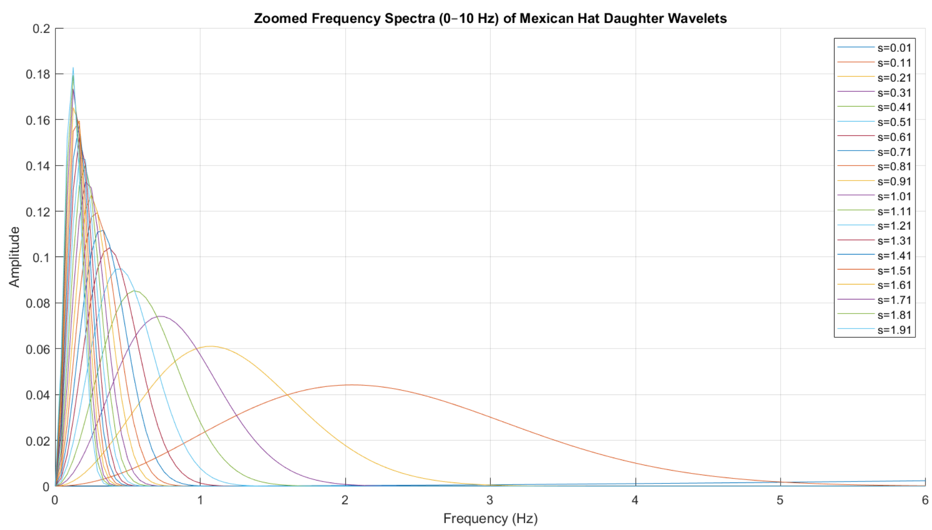

Figure 3: Zoomed Frequency Spectra (0–6 Hz) of Mexican Hat Daughter Wavelets.

Observations:

- As scale increases, the spectral content shifts toward lower frequencies and becomes more concentrated, consistent with the time-scale/frequency trade-off.
- The zoomed plot clarifies how small scales capture higher-frequency components while large scales capture low-frequency structure.
- Since all wavelets maintain unity power, it follows that the ratio of power to bandwidth increases with an increase in the scaling factor.

1.3 Continuous wavelet decomposition

1.3.1 Signal definition and time-domain plot.

A piecewise signal $x[n]$ (two different sinusoidal segments) was created with sampling frequency 250 Hz.

$$x[n] = \begin{cases} \sin(0.5\pi n), & 1 \leq n < \dfrac{3N}{2} \\ \sin(1.5\pi n), & \dfrac{3N}{2} \leq n < 3N \end{cases}$$

The time-domain waveform is shown in Figure 4.

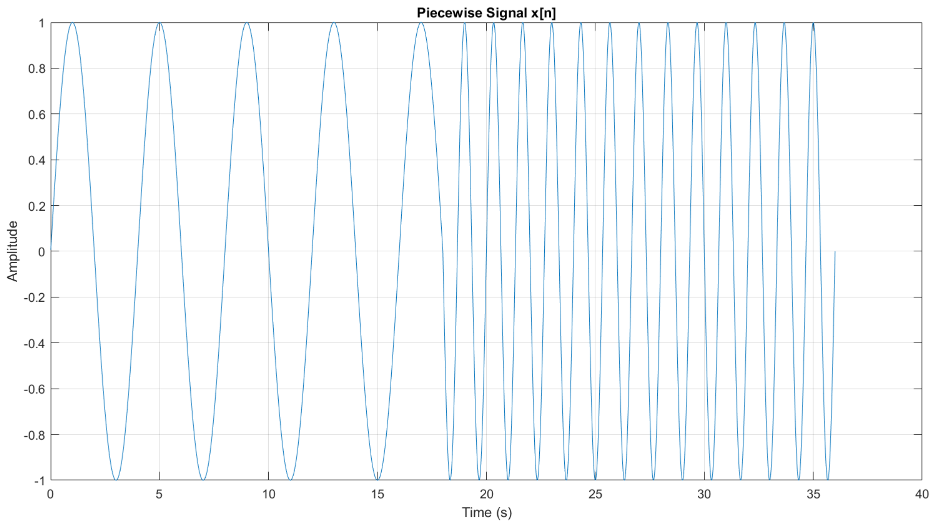

Figure 4: Piecewise signal $x[n]$.

1.3.2 Applying the scaled Mexican hat wavelets to $x[n]$

The continuous wavelet transform is then applied using the Mexican hat wavelet across scaling factors that vary from 0.01 to 2, incrementing in steps of 0.01.

1.3.3 CWT spectrogram

The CWT using the Mexican-hat wavelet (scales 0.01:0.01:2) and convolution-based translation produced a scale-time spectrogram shown in Figure 5.

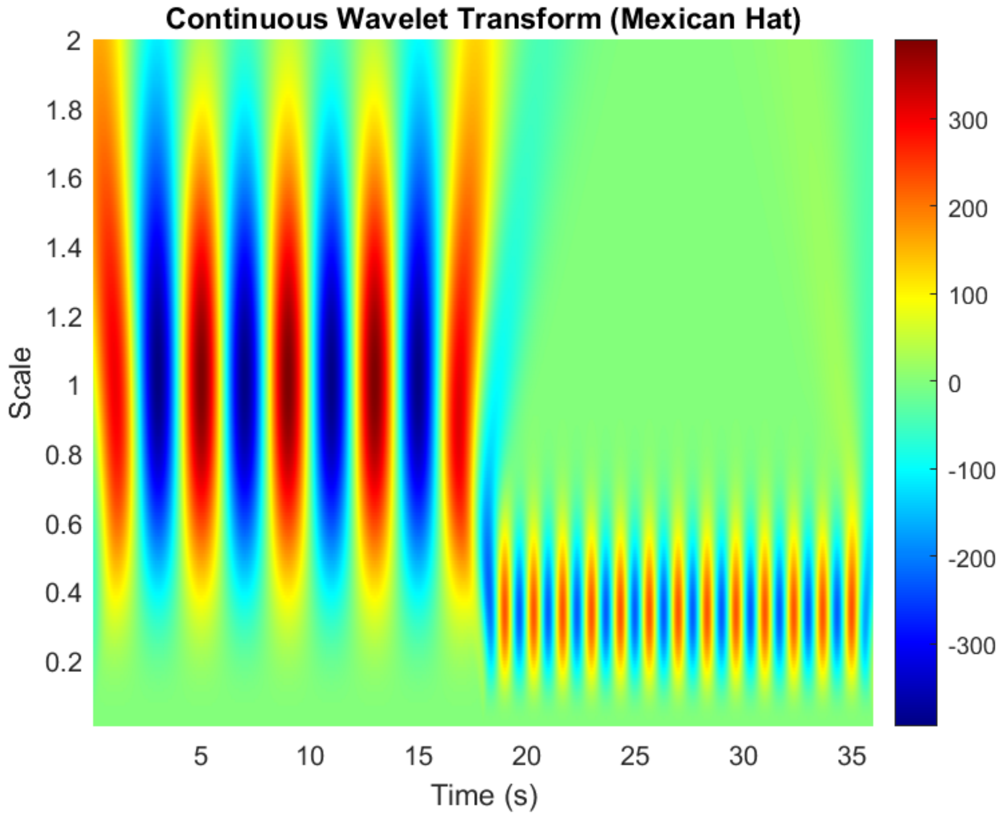

Figure 5: Continuous Wavelet Transform (Scale–Time spectrogram).

Observations:

- The spectrogram illustrates that wavelets with **smaller scaling factors** capture **higher frequency components**, whereas wavelets with **larger scaling factors** capture **lower frequency components**.
- increasing the scaling factor reduces the wavelet's frequency bandwidth.
- At a scaling factor near $s = 1.01$, the **median frequency** reaches approximately $0.5\pi\,\mathrm{rads}^{-1}$.
- As the scaling factor decreases towards $s = 0.31$, the median frequency increases to about $1.5\pi\,\mathrm{rads}^{-1}$.
- So, the spectrogram effectively demonstrates the **inverse relationship between the scaling factor and the wavelet's dominant frequency**.

2 Discrete Wavelet Transform

2.1 Introduction

For the discrete wavelet transform (DWT), the equation is given by:

$$\psi_{m,n}(t) = \frac{1}{\sqrt{s_0^m}}\,\psi\left(\frac{t - n\tau_0 s_0^m}{s_0^m}\right) \tag{1}$$

where

$$s_0 = \text{scaling step size}, \qquad \tau_0 = \text{translation step size}.$$

Typically,

$$s_0 = 2, \qquad \tau_0 = 1$$

are chosen for efficient analysis. The parameters m and n represent integer multipliers.

2.2 Applying DWT with the Wavelet Toolbox in MATLAB

2.2.1 Creating the waveforms in MATLAB

Figures 6 and 7 show the original and noisy signals (AWGN with 10 dB SNR).

$$x_1[n] = \begin{cases} 2\sin(20\pi n) + \sin(80\pi n), & 0 \le n < 512 \\ 0.5\sin(40\pi n) + \sin(60\pi n), & 512 \le n < 1024 \end{cases}$$

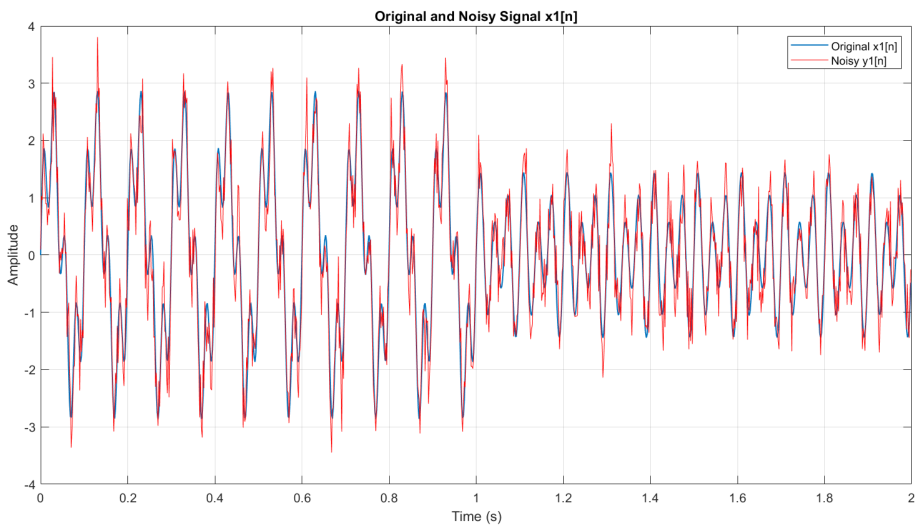

Figure 6: Original and Noisy Signal $x_1[n]$.

$$x_2[n] = \begin{cases} 1, & 0 \le n < 64 \\ 2, & 192 \le n < 256 \\ -1, & 128 \le n < 512 \\ 3, & 512 \le n < 704 \\ 1, & 704 \le n < 960 \\ 0, & \text{otherwise} \end{cases}$$

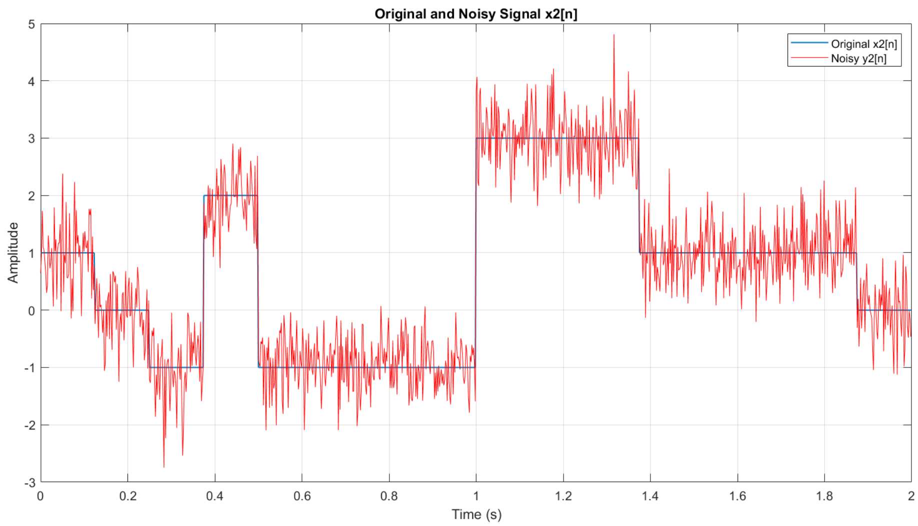

Figure 7: Original and Noisy Signal $x_2[n]$.

2.2.2 Wavelet morphology (Haar and db9)

Scaling and wavelet functions. Figures 8 and 9 display the scaling and wavelet functions for Haar and Daubechies-9, respectively.

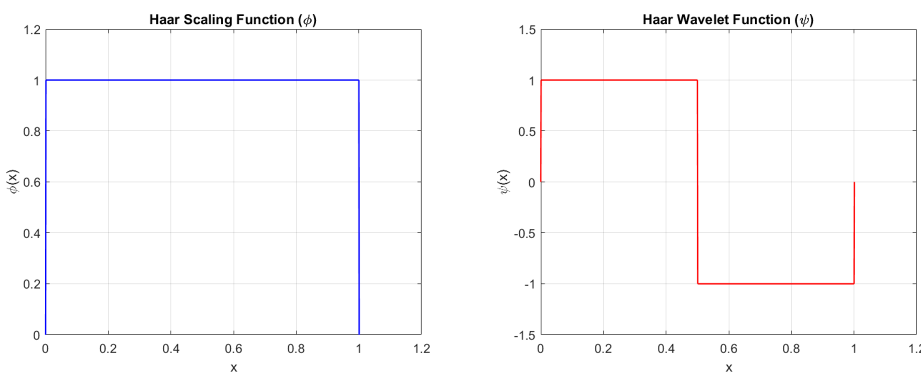

Figure 8: Haar scaling and wavelet functions.

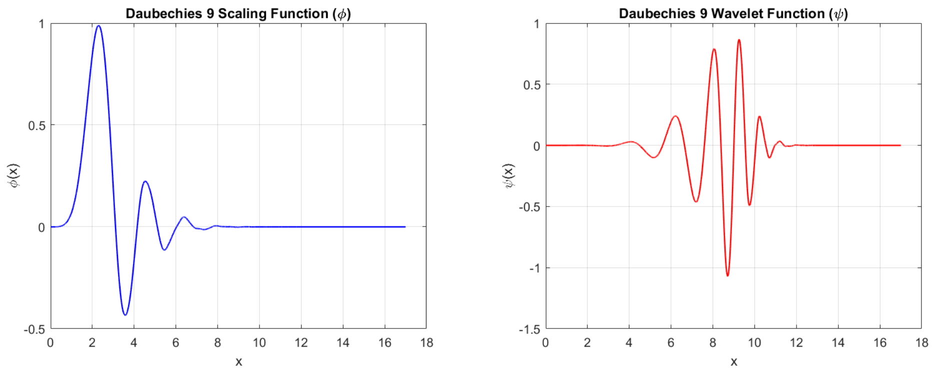

Figure 9: Daubechies-9 scaling and wavelet functions.

Observation:

- Haar is compact and discontinuous (good for abrupt changes); db9 is smoother and better for representing smooth structures.

2.2.3 10-level DWT decomposition

The 10-level decompositions for both signals/wavelets were plotted using a helper routine (detail and approximation coefficients). Example decomposition plot for y_1 with Haar/db9 are shown in Figures 10 and 11; similarly for y_2 in Figures 12 and 13.

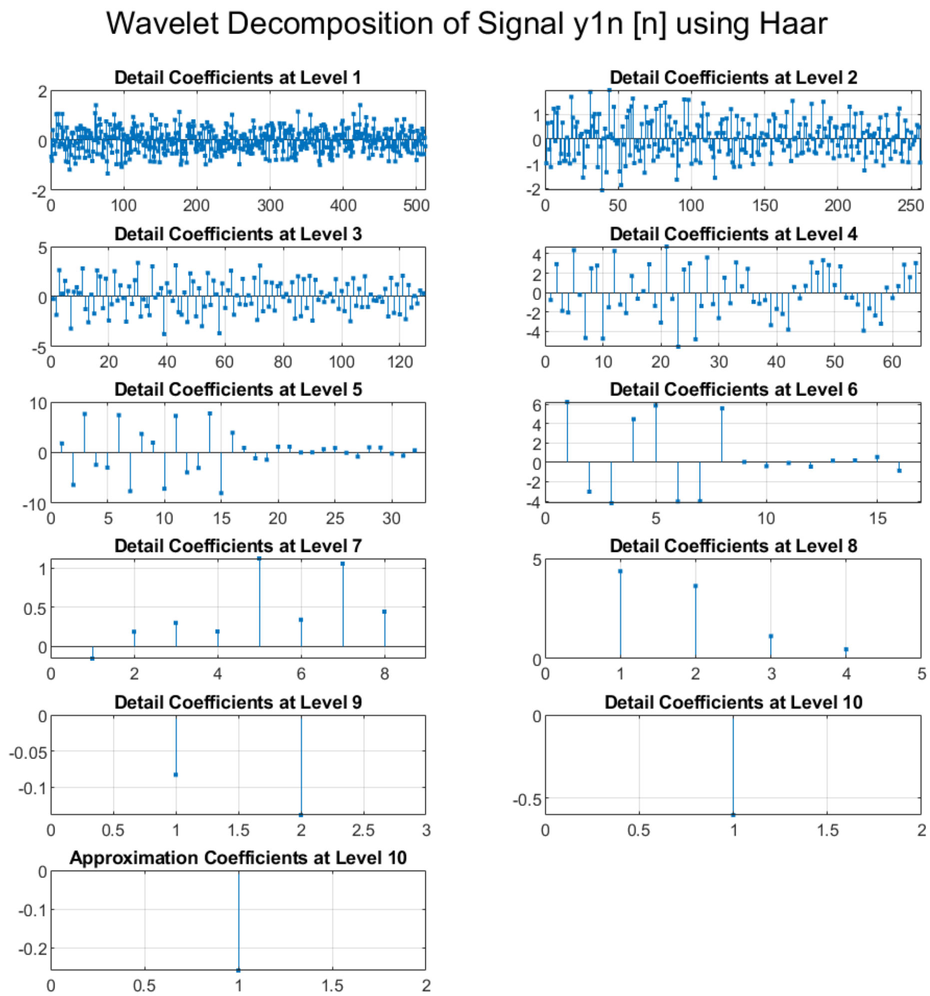

Figure 10: 10-level decomposition plots (example) – y1 with Haar (detail levels).

10-Level Decomposition of y_1 Using 'Haar'

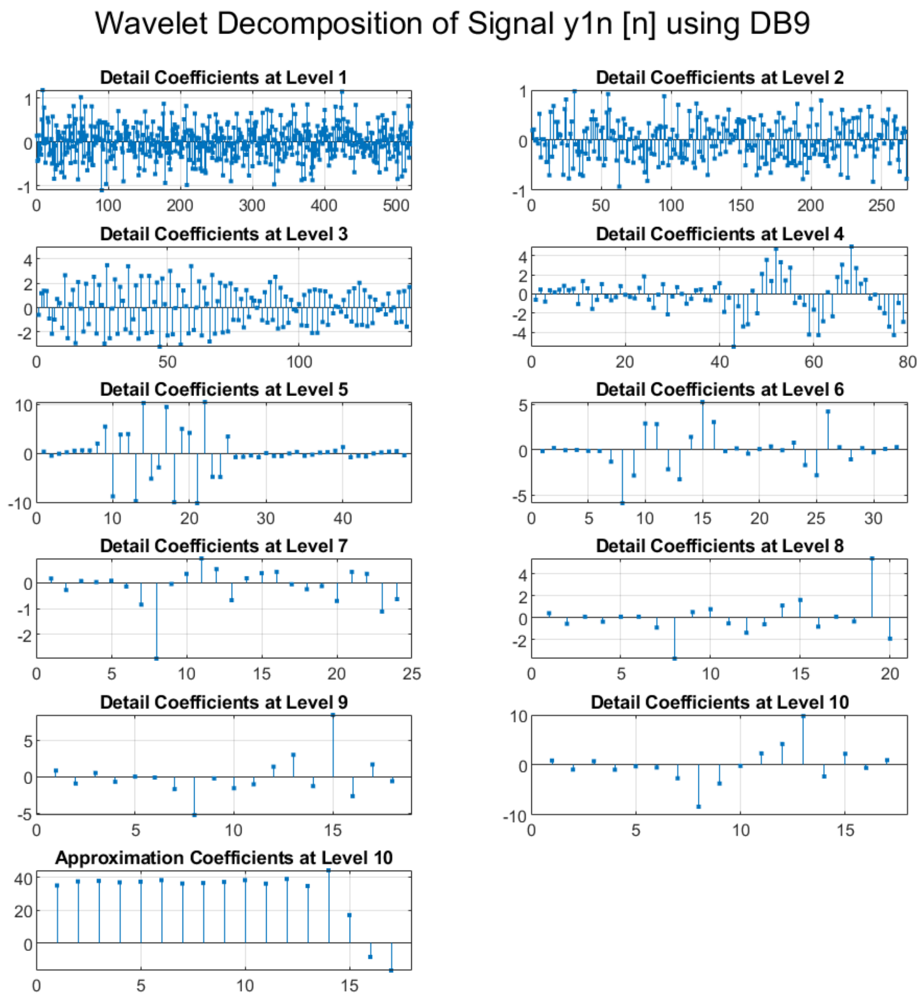

Figure 11: 10-level decomposition plots (example) – y1 with DB9 (detail levels).

10-Level Decomposition of y_1 Using 'DB9'

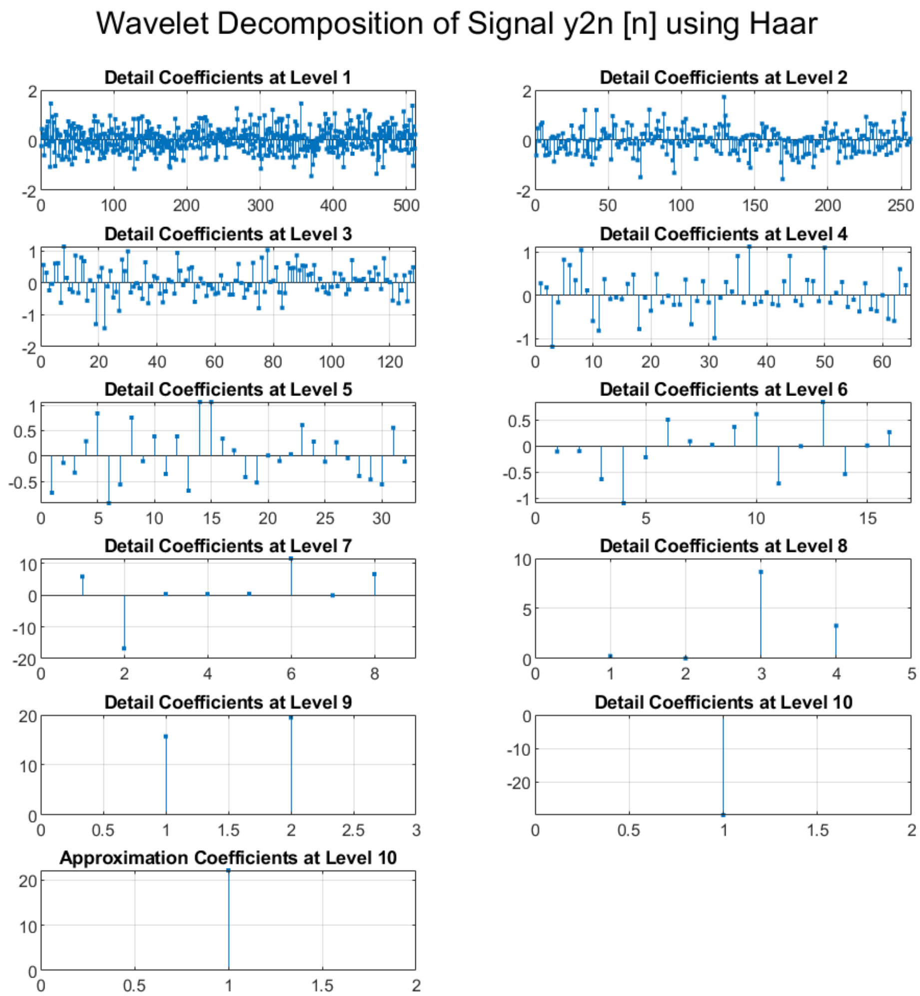

Figure 12: 10-level decomposition plots (example) – y2 with Haar (detail levels).

10-Level Decomposition of y_2 Using 'Haar'

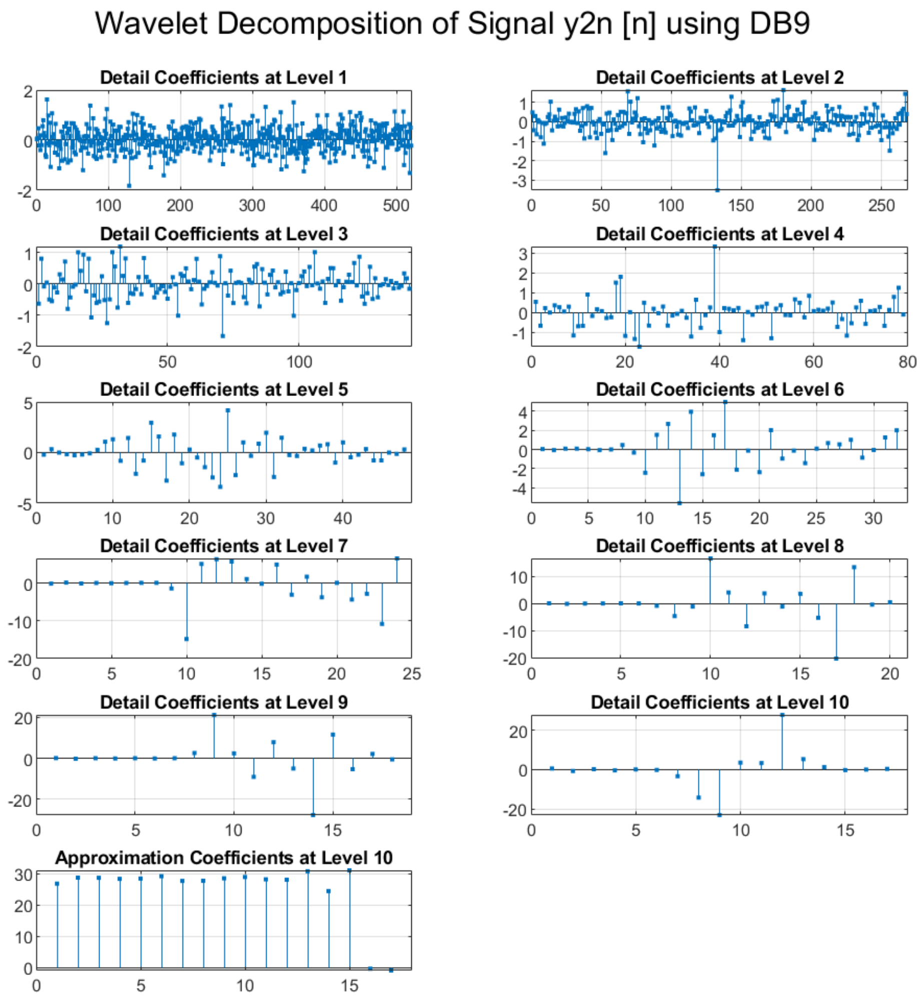

Figure 13: 10-level decomposition plots (example) – y2 with DB9 (detail levels).

10-Level Decomposition of y_2 Using 'DB9'

2.2.4 Reconstruction and energy verification

Reconstruction via inverse DWT (A10 + D10 + ... + D1). Reconstructed waveforms (sum of approximation and detail components) are shown for y_1 and y_2 with both Haar and DB9 (Figures 14–17). Energy checks and printed console outputs confirming reconstruction accuracy are included below.

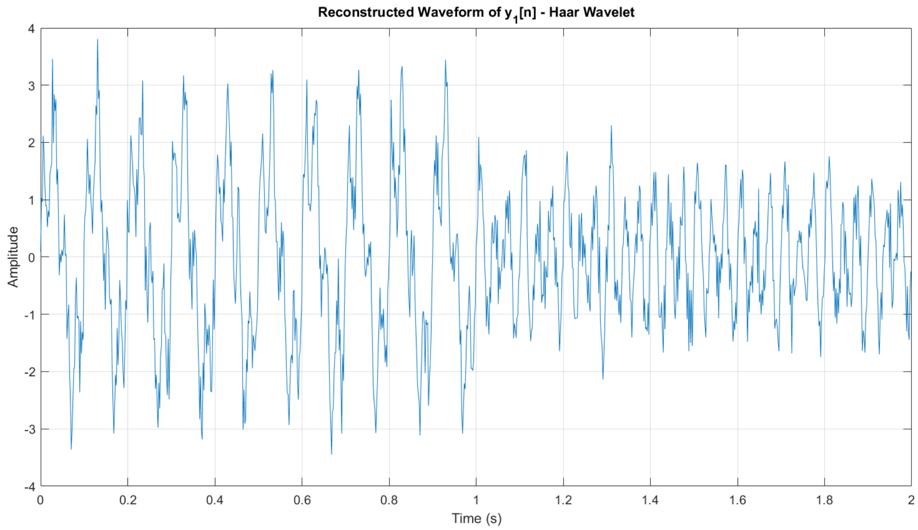

Figure 14: Reconstructed waveform of $y_1[n]$ using Haar.

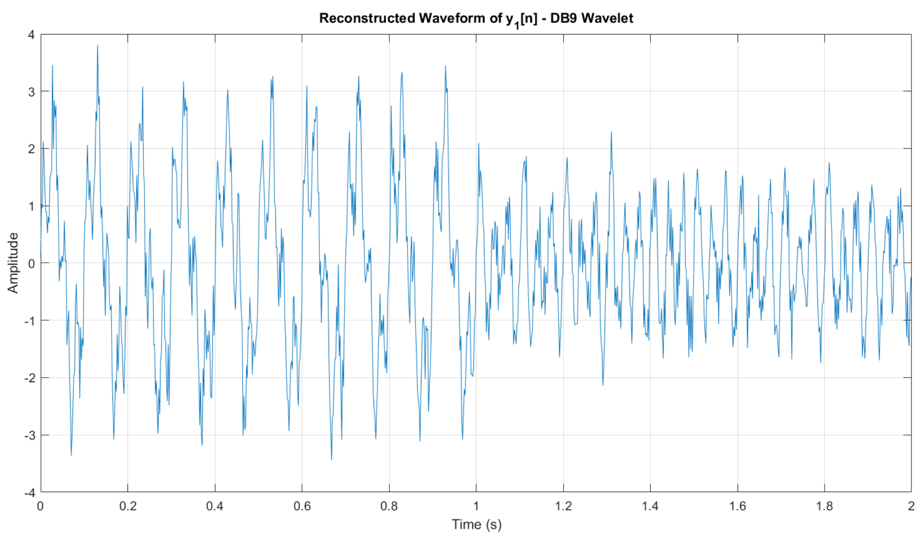

Figure 15: Reconstructed waveform of $y_1[n]$ using DB9.

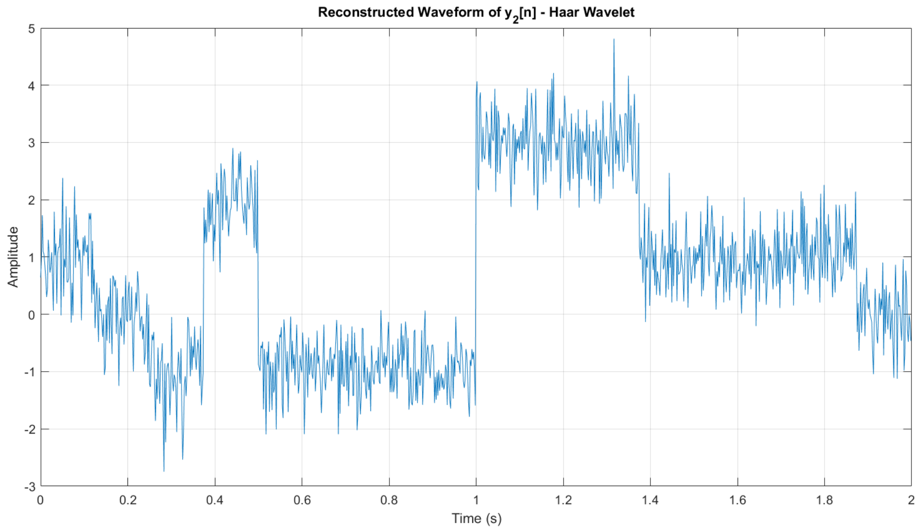

Figure 16: Reconstructed waveform of $y_2[n]$ using Haar.

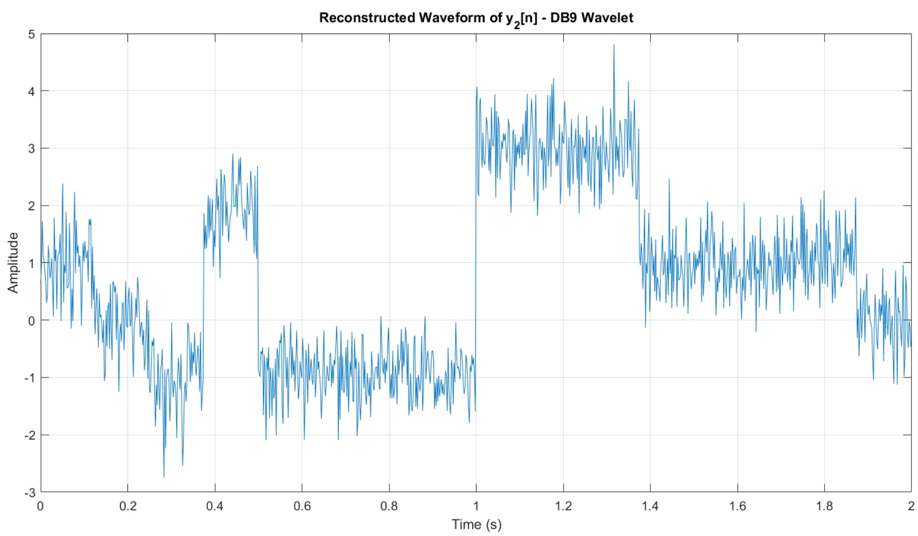

Figure 17: Reconstructed waveform of $y_2[n]$ using DB9.

Energy verification (MATLAB console outputs):

```
y1[n] - HAAR:
  Original Energy      = 1756.0850
  Reconstructed Energy = 1756.0850
  Difference           = 1.818989e-12

y1[n] - DB9:
  Original Energy      = 1756.0850
  Reconstructed Energy = 1756.0850
  Difference           = 3.033572e-07
```

```
y2[n] - HAAR:
  Original Energy      = 2823.8624
  Reconstructed Energy = 2823.8624
  Difference           = 7.730705e-12

y2[n] - DB9:
  Original Energy      = 2823.8624
  Reconstructed Energy = 2823.8624
  Difference           = 4.600629e-07
```

Observations:

- The very small energy differences (on the order of 10^{-7}–10^{-12}) confirm that the inverse DWT reconstruction equals the original (within numerical precision).
- Both Haar and DB9 decompositions reconstruct the signal; differences are numerical and acceptable.

2.3 Signal denoising with DWT

2.3.1 Plotting the magnitude of wavelet coefficients

Sorted absolute coefficient plots were produced for each case (Figures 18–21).

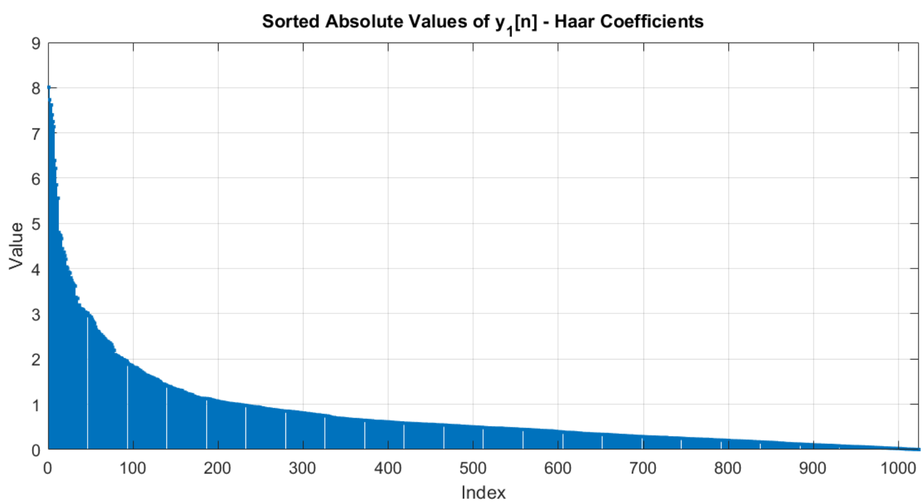

Figure 18: Sorted absolute values of y1 (Haar coefficients).

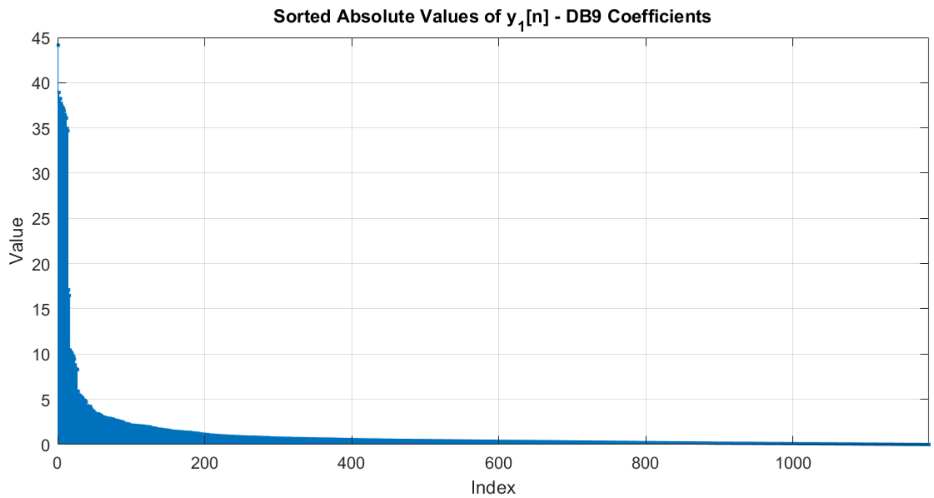

Figure 19: Sorted absolute values of y1 (DB9 coefficients).

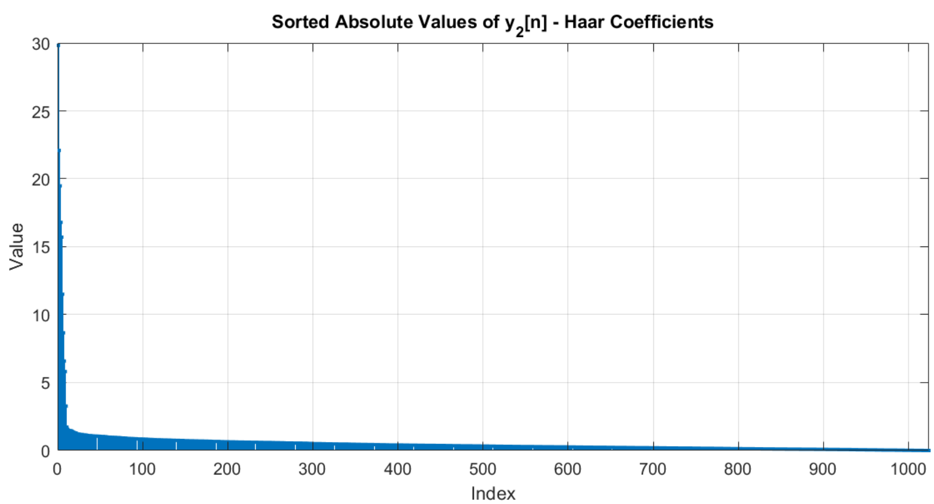

Figure 20: Sorted absolute values of y2 (Haar coefficients).

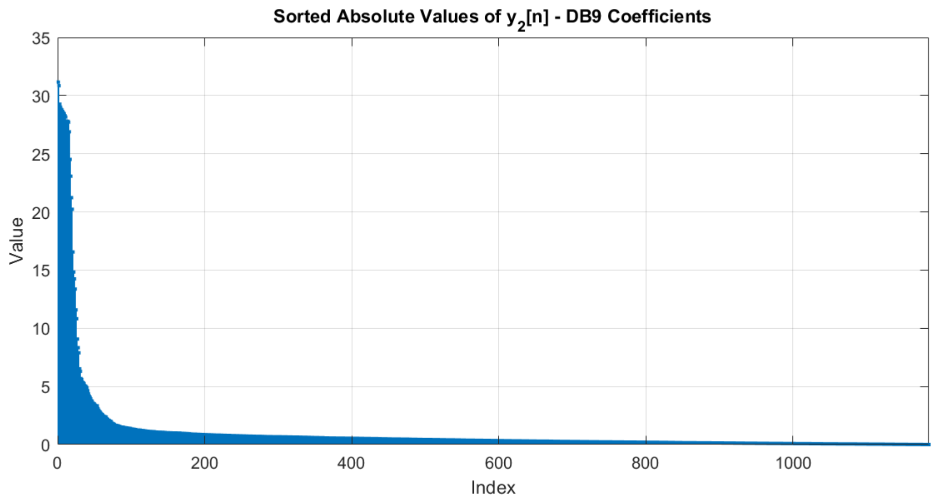

Figure 21: Sorted absolute values of y2 (DB9 coefficients).

2.3.2 Selected Thresholds

The thresholds selected for denoising each signal and wavelet combination are as follows:

- $y_1[n]$ with Haar wavelet – 0.75
- $y_1[n]$ with DB9 wavelet – 0.1
- $y_2[n]$ with Haar wavelet – 2.5
- $y_2[n]$ with DB9 wavelet – 1.25

The denoised signals for each method are shown in Figures 22–25.

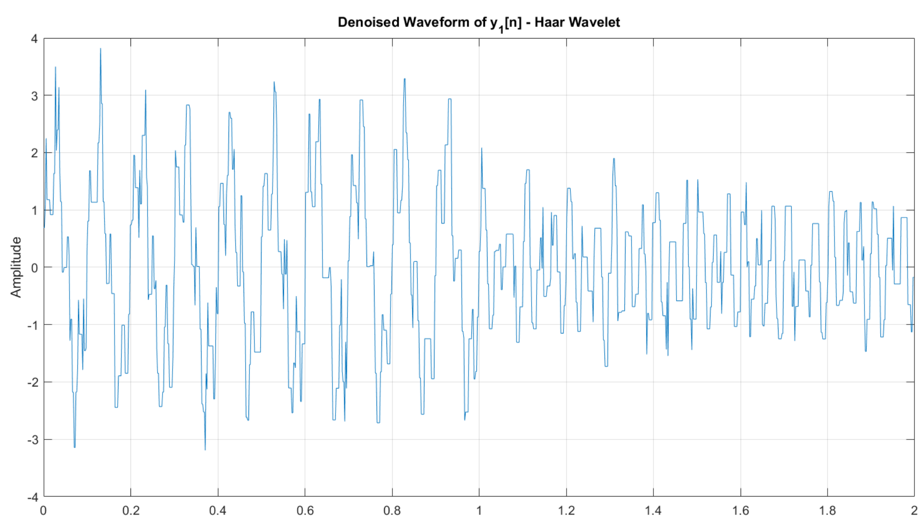

Figure 22: Denoised waveform of y1 using Haar.

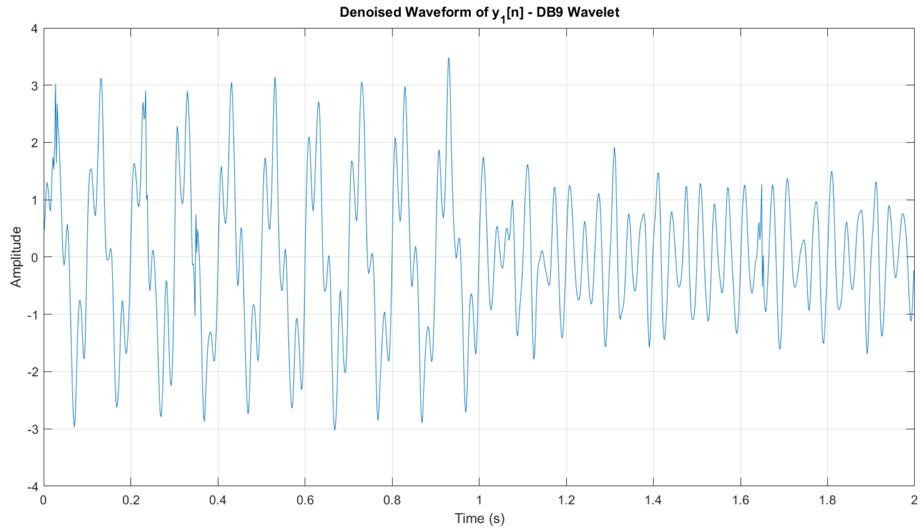

Figure 23: Denoised waveform of y1 using DB9.

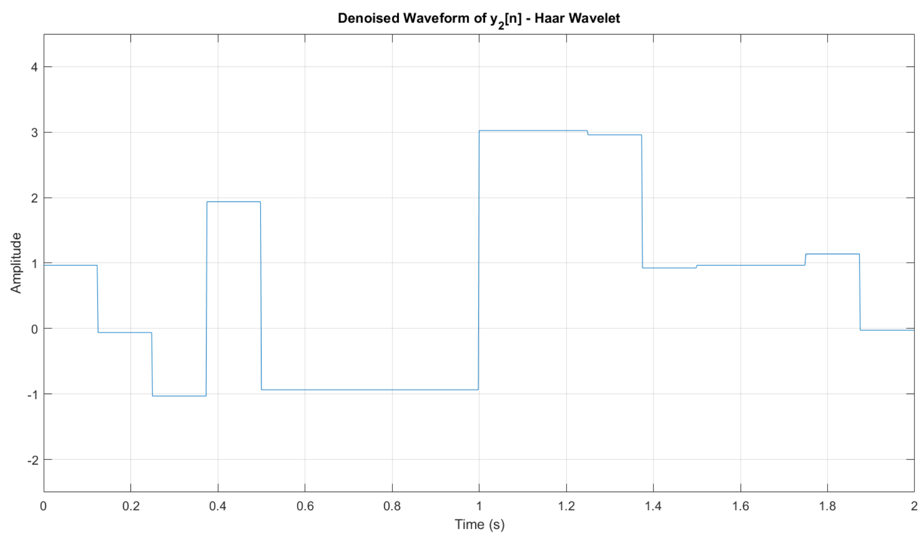

Figure 24: Denoised waveform of y2 using Haar.

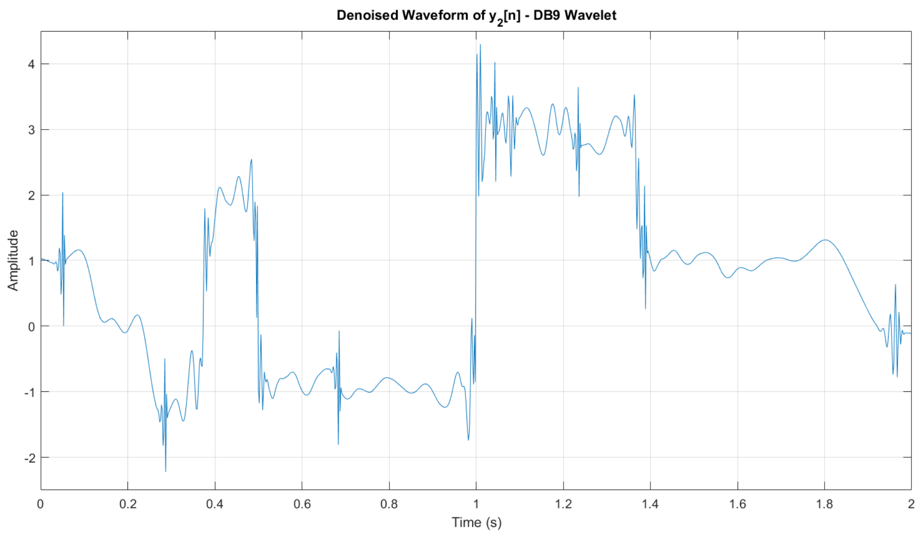

Figure 25: Denoised waveform of y2 using DB9.

2.3.3 RMSE results

```
The RMSE value for y_1[n] signal when denoised using "Haar" wavelet is 0.398471.
The RMSE value for y_1[n] signal when denoised using "DB9" wavelet is 0.249889.
The RMSE value for y_2[n] signal when denoised using "Haar" wavelet is 0.059772.
The RMSE value for y_2[n] signal when denoised using "DB9" wavelet is 0.280877.
```

2.3.4 Comparison and interpretation:

- For y_1: DB9 produced a lower RMSE (0.2499) compared to Haar (0.3985), indicating DB9 yielded better denoising fidelity for this smooth sinusoidal mixture.
- For y_2: Haar gave a much lower RMSE (0.0598) than DB9 (0.2809). This reflects Haar's suitability for piecewise-constant / abrupt signals.
- Overall: Choose the wavelet that matches the signal characteristics — Haar for abrupt changes, smoother wavelets (db9) for smoothly varying signals.

Combined plots. The original vs denoised comparisons in one figure are shown in Figure 26.

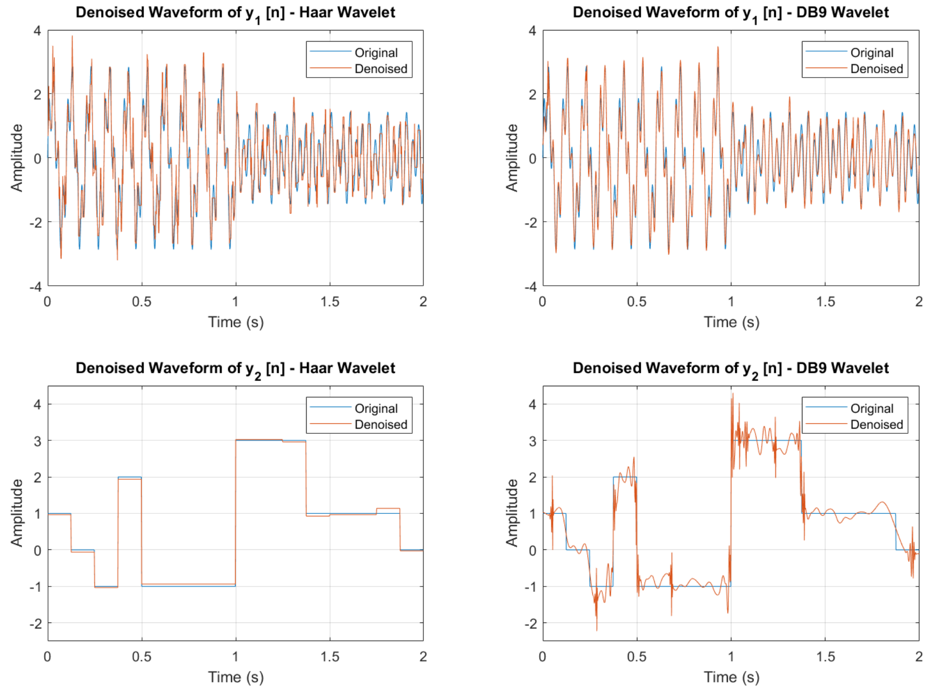

Figure 26: Original and denoised signals (combined subplot).

2.4 Signal compression with DWT

2.4.1 ECG waveform.

The provided aVR ECG waveform (sampled at 257 Hz) is plotted in Figure 27.

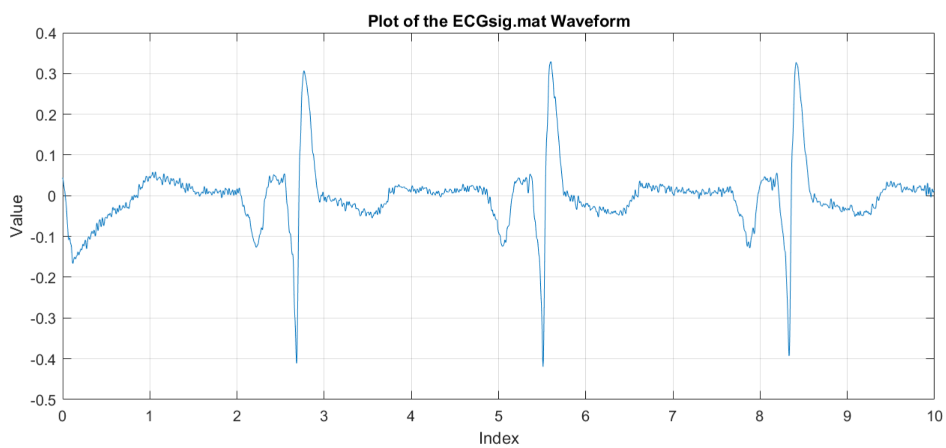

Figure 27: Plot of the ECG aVR waveform.

2.4.2 Coefficient sorting and energy accumulation.

Sorted absolute coefficient plots are shown in Figure 28.

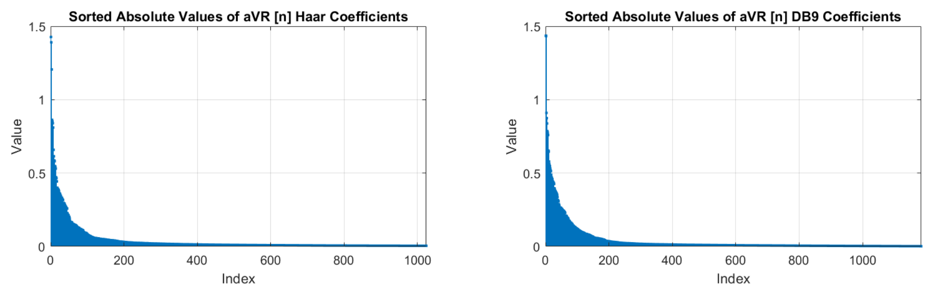

Figure 28: Sorted absolute values of aVR coefficients (Haar and DB9).

2.4.3 Number of coefficients for 99% energy and thresholds (MATLAB outputs):

```
Number of Haar coefficients required for 99% energy: 177
Number of DB9 coefficients required for 99% energy: 170
```

2.4.4 Compression and reconstructed signals.

Reconstructed signals after thresholding/compression (Haar and DB9) are shown in Figure 29.

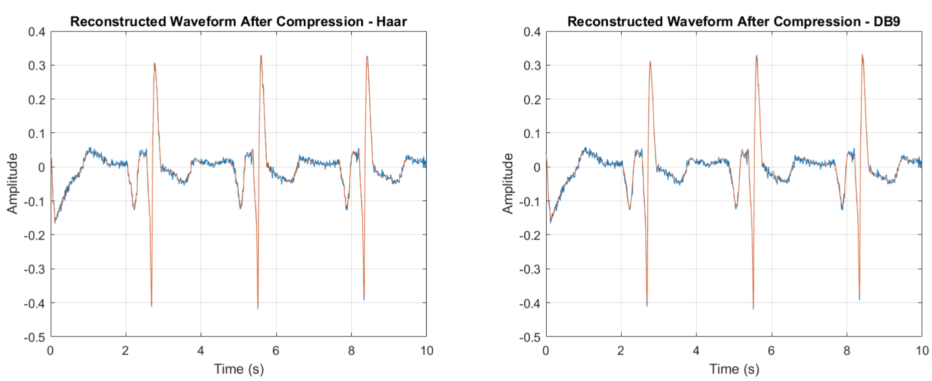

Figure 29: Reconstructed waveform after compression (Haar left, DB9 right).

Compression ratios and RMSE (MATLAB outputs):

```
Compression ratio for Haar wavelet: 14.56
Compression ratio for DB9 wavelet: 16.08

The RMSE value for aVR signal when compressed using "Haar" wavelet is 0.007820.
The RMSE value for aVR signal when compressed using "DB9" wavelet is 0.007129.
```

Observations:

- DB9 achieved a slightly higher compression ratio (16.08 vs 14.56) and a slightly lower RMSE (0.007129 vs 0.007820) — indicating marginally better performance for this ECG in preserving morphology at the 99% energy level.
- Visual inspection of Figure 29 shows that both reconstructions retain the ECG morphology very well.