Department of Electronic & Telecommunication Engineering

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Branched Cylinders: Dendritic Tree Approximations

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27/04/2024

Question 01

General solution :

$$V_1(X) = A_1 e^{-X} + B_1 e^{X} \quad 0 \le X \le L_1 \quad — ①'$$

$$V_{21}(X) = A_{21} e^{-X} + B_{21} e^{X} \quad L_1 \le X \le L_{21} \quad — ②'$$

$$V_{22}(X) = A_{22} e^{-X} + B_{22} e^{X} \quad L_1 \le X \le L_{22} \quad — ③'$$

— ②

Boundary conditions :

$$\left.\frac{dV_1}{dX}\right|_{X=0} = -(r_i\lambda_c)_1 I_{app} \quad — ③$$

$$V_{21}(L_{21}) = V_{22}(L_{22}) = 0 \quad — ④$$

Nodal conditions :

$$V_1(L_1) = V_{21}(L_1) = V_{22}(L_1) \quad — ⑤$$

$$\frac{-1}{(r_i\lambda_c)_1}\left.\frac{dV_1}{dX}\right|_{X=L_1} = \frac{-1}{(r_i\lambda_c)_{21}}\left.\frac{dV_{21}}{dX}\right|_{X=L_1} + \frac{-1}{(r_i\lambda_c)_{22}}\left.\frac{dV_{22}}{dX}\right|_{X=L_1} \quad — ⑥$$

from ①'

$$\frac{dV_1}{dx} = -A_1 e^{-x} + B_1 e^{x}$$

$$\left.\frac{dV_1}{dx}\right|_{x=0} = -A_1 + B_1 \quad — ①''$$

①'' = ③

$$-A_1 + B_1 = -(r_i\lambda_c)_1 I_{app}$$

$$B_1 - A_1 = (r_i\lambda_c) I_{app}$$

from ②' and ④

$$V_{21}(L_{21}) = A_{21} e^{-L_{21}} + B_{21} e^{-L_{21}} = 0$$

Similarly,

$$V_{21}(L_{22}) = A_{22} e^{-L_{22}} + B_{22} e^{-L_{22}} = 0$$

from ⑤,

$$V_1(L_1) = V_{21}(L_1)$$

$$V_1(L_1) - V_{21}(L_1) = 0$$

$$A_1 e^{-L_1} + B_1 e^{-L_1} - A_{21} e^{-L_1} - B_{21} e^{-L_1} = 0$$

$$V_{21}(L_1) = V_{22}(L_1)$$

$$V_{21}(L_1) - V_{22}(L_1) = 0$$

$$A_{21} e^{-L_1} + B_{21} e^{-L_1} - A_{22} e^{-L_1} - B_{22} e^{-L_1} = 0$$

$$\frac{dV_1}{dx} = -A_1 e^{-x} + B_1 e^{-x}$$

$X = L_1$

$$\left.\frac{dV_1}{dx}\right|_{x=L_1} = -A_1 e^{-L_1} + B_1 e^{L_1}$$

$$\times \frac{-1}{(r_i d_c)_1} \Rightarrow \frac{-1}{(r_i d_c)_1} \left.\frac{dV_1}{dx}\right|_{x=L_1} = -\left(\frac{-A_1 e^{-L_1}}{(r_i d_c)_1} + \frac{B_1 e^{L_1}}{(r_i d)_1} \right) \quad — ⑧$$

Similarly

$$\frac{-1}{(r_i d_c)_{21}} \left.\frac{dV_{21}}{dx}\right|_{x=L_1} = -\left(\frac{-A_{21} e^{-L_1}}{(r_i d_c)_{21}} + \frac{B_{21} e^{L_1}}{(r_i d)_{21}} \right) \quad — ⑨$$

$$\frac{-1}{(r_i d_c)_{21}} \left.\frac{dV_{22}}{dx}\right|_{x=L_1} = -\left(\frac{-A_{22} e^{-L_1}}{(r_i d_c)_{22}} + \frac{B_{22} e^{L_1}}{(r_i d)_{22}} \right) \quad — ⑩$$

⑧ = ⑨ + ⑩

$$-\left(\frac{-A_1 e^{-L_1}}{(r_i d_c)_1} + \frac{B_1 e^{L_1}}{(r_i d)_1} \right) = -\left(\frac{-A_{21} e^{-L_1}}{(r_i d_c)_{21}} + \frac{B_{21} e^{L_1}}{(r_i d)_{21}} \right) + -\left(\frac{-A_{22} e^{-L_1}}{(r_i d_c)_{22}} + \frac{B_{22} e^{L_1}}{(r_i d)_{22}} \right)$$

$$-\frac{A_1 e^{-L_1}}{(r_i\lambda_c)_1} + \frac{B_1 e^{L_1}}{(r_i\lambda)_1} + \frac{A_{21} e^{-L_1}}{(r_i\lambda_c)_{21}} - \frac{B_{21} e^{L_1}}{(r_i\lambda)_{21}} + \frac{A_{22} e^{-L_1}}{(r_i\lambda_c)_{22}} - \frac{B_{22} e^{L_1}}{(r_i\lambda)_{22}} = 0$$

Question 02

$$\mathrm{x} = \begin{pmatrix} A_1 \\ B_1 \\ A_{21} \\ B_{21} \\ A_{22} \\ B_{22} \end{pmatrix} \qquad \mathrm{b} = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

and

$$\mathrm{A} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-L_{21}} & e^{L_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-L_{22}} & e^{L_{22}} \\ e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} & 0 & 0 \\ 0 & 0 & e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} \\ -e^{-L_1}/(r_i\lambda_c)_1 & e^{L_1}/(r_i\lambda_c)_1 & e^{-L_1}/(r_i\lambda_c)_{21} & -e^{L_1}/(r_i\lambda_c)_{21} & e^{-L_1}/(r_i\lambda_c)_{22} & -e^{L_1}/(r_i\lambda_c)_{22} \end{pmatrix}$$

$$Ax = b$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-L_{21}} & e^{L_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-L_{22}} & e^{L_{22}} \\ e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} & 0 & 0 \\ 0 & 0 & e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} \\ -e^{-L_1}/(r_i\lambda_c)_1 & e^{L_1}/(r_i\lambda_c)_1 & e^{-L_1}/(r_i\lambda_c)_{21} & -e^{L_1}/(r_i\lambda_c)_{21} & e^{-L_1}/(r_i\lambda_c)_{22} & -e^{L_1}/(r_i\lambda_c)_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_{21} \\ B_{21} \\ A_{22} \\ B_{22} \end{pmatrix} = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A_1 - b \\ A_{21} e^{-L_{21}} + B_{21} e^{L_{21}} \\ A_{22} e^{-L_{22}} + B_{22} e^{L_{22}} \\ A_1 e^{-L_1} + B_1 e^{L_1} - A_{21} e^{L_1} - B_{21} e^{L_1} \\ A_{21} e^{-L_1} + B_{21} e^{L_1} - A_{22} e^{L_1} - B_{22} e^{L_1} \\ -\frac{A_1 e^{-L_1}}{(r_i\lambda_c)_1} + \frac{B_1 e^{L_1}}{(r_i\lambda)_1} + \frac{A_{21} e^{-L_1}}{(r_i\lambda_c)_{21}} - \frac{B_{21} e^{L_1}}{(r_i\lambda)_{21}} + \frac{A_{22} e^{-L_1}}{(r_i\lambda_c)_{22}} - \frac{B_{22} e^{L_1}}{(r_i\lambda)_{22}} \end{pmatrix} = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix multiplication gives equations (7).

Solving For the constants

Parameters and the corresponding MATLAB variables to be used for a single branched cable having the electrical properties of the squid giant axon.

parameter	MATLAB variable	value	units
d_1	d1	75	10^{-4} cm
d_{21}	d21	30	10^{-4} cm
d_{22}	d22	15	10^{-4} cm
R_m	Rm	6	$10^3\,\Omega\,\mathrm{cm}^2$
R_c	Rc	90	$\Omega\,\mathrm{cm}$
L_1	l1	1.5	–
L_{21}, L_{22}	l21, l22	3.0	–
I_{app}	iapp	1	10^{-9} A
$(r_i\lambda_c)_1$, $(r_i\lambda_c)_{21}$, $(r_i\lambda_c)_{22}$	rl1, rl21, rl22	–	Ω
π	pi	3.14159...	–
R_s	Rs	1	$10^6\,\Omega$

Defining matrix A and boundray conditions

In [1]:
```
% electrical constants and derived quantities for typical
% mammalian dendrite

% Dimensions of compartments

d1 = 75e-4;                     % cm
d21 = 30e-4;                    % cm
d22 = 15e-4;                    % cm
%d21 = 47.2470e-4;       % E9 cm
%d22 = d21;               % E9 cm

l1 = 1.5;                       % dimensionless
l21 = 3.0;                      % dimensionless
l22 = 3.0;                      % dimensionless

% Electrical properties of compartments

Rm = 6e3;                       % Ohms cm^2
Rc = 90;                        % Ohms cm
Rs = 1e6;                       % Ohms
```

```
c1 = 2*(Rc*Rm)^(1/2)/pi;

rl1 = c1*d1^(-3/2);              % Ohms
rl21 = c1*d21^(-3/2);            % Ohms
rl22 = c1*d22^(-3/2);            % Ohms


% Applied current

iapp = 1e-9;     % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21 rl1*exp(-l1)/rl22 -rl1*e

b = [rl1*iapp 0 0 0 0 0]';
```

Question 03

Calculating Values of coefficients

In [2]:
```
x=A\b;
display(x);
```

Out[2]: x = 6x1 double

0.0007

0.0000

0.0011

-0.0000

0.0011

-0.0000

Values of coefficients

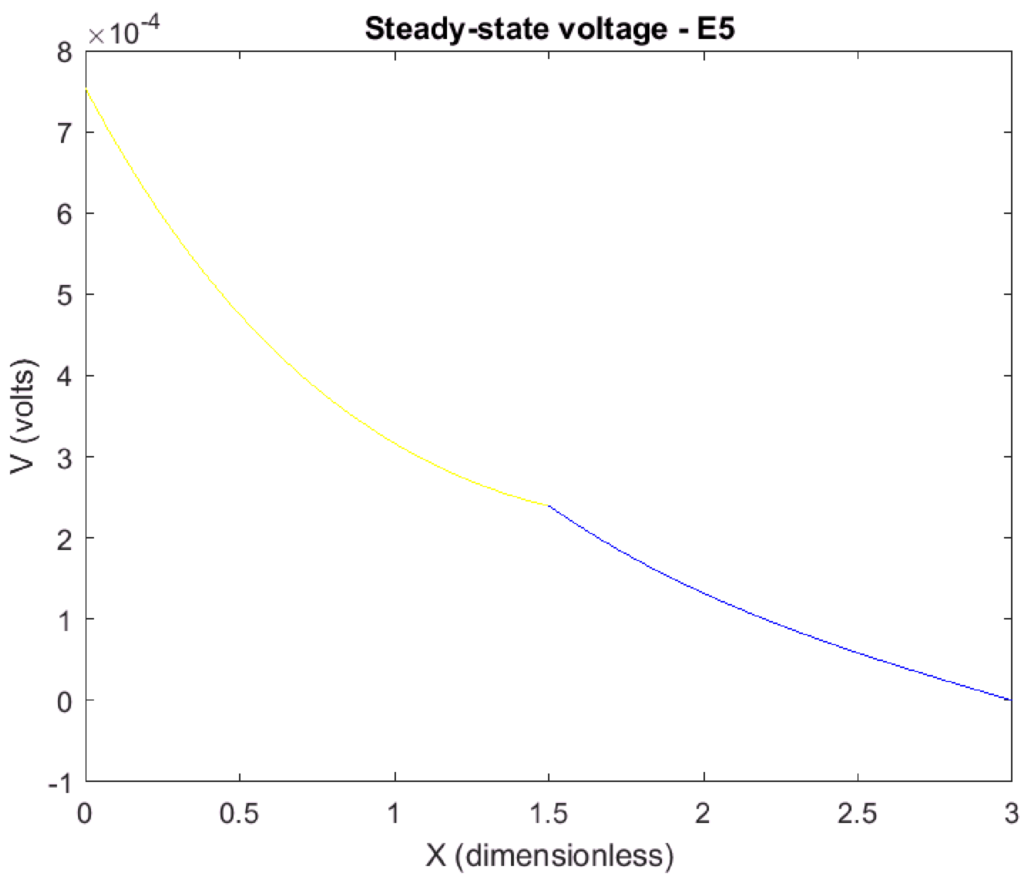
- A1= 0.0007
- B1= 0.0000
- A21= 0.0011
- B21= -0.0000
- A22= 0.0011
- B22= -0.0000

Plotting the steady state voltage profiles

Question 04

In [3]:
```
y1 = linspace(0, l1, 20);
y21 = linspace(l1, l21, 20);
y22 = linspace(l1, l22, 20);
v1 = x(1) * exp(-y1) + x(2) * exp(y1);
v21 = x(3) * exp(-y21) + x(4) * exp(y21);
v22 = x(5) * exp(-y22) + x(6) * exp(y22);
plot(y1, v1, 'y-', y21, v21, 'r-', y22, v22, 'b-');
xlabel('X (dimensionless)');
ylabel('V (volts)');
title('Steady-state voltage - E5');
```

Out[3]:



Observations

In this analysis, it's evident that the red line is not visible in the diagram, indicating either its equality or extreme proximity to the blue line. Since the yellow line, representing the parent branch, does not affect the daughter branches, it's reasonable to deduce that the red and blue lines are equivalent. Consequently, the steady-state voltage profiles of both daughter branches are identical.

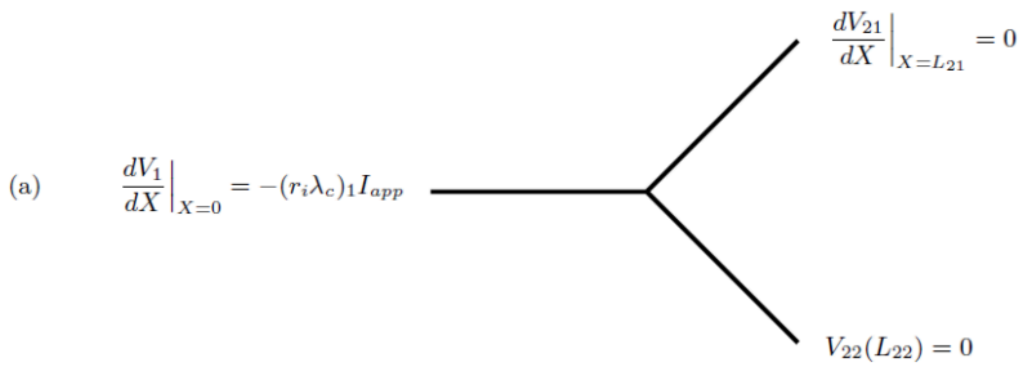
Moreover, the code confirms this deduction. By plotting the red and blue lines for the daughter branches, with the red line being imperceptible, it's apparent that the blue line has overwritten it. This alignment implies equality between the two lines and thus between the steady-state voltage profiles of the daughter branches.

In summary, both the diagram and the code strongly support the conclusion that the steady-state voltage profiles of the two daughbranches arecare equual.

Solutions for different boundary conditions

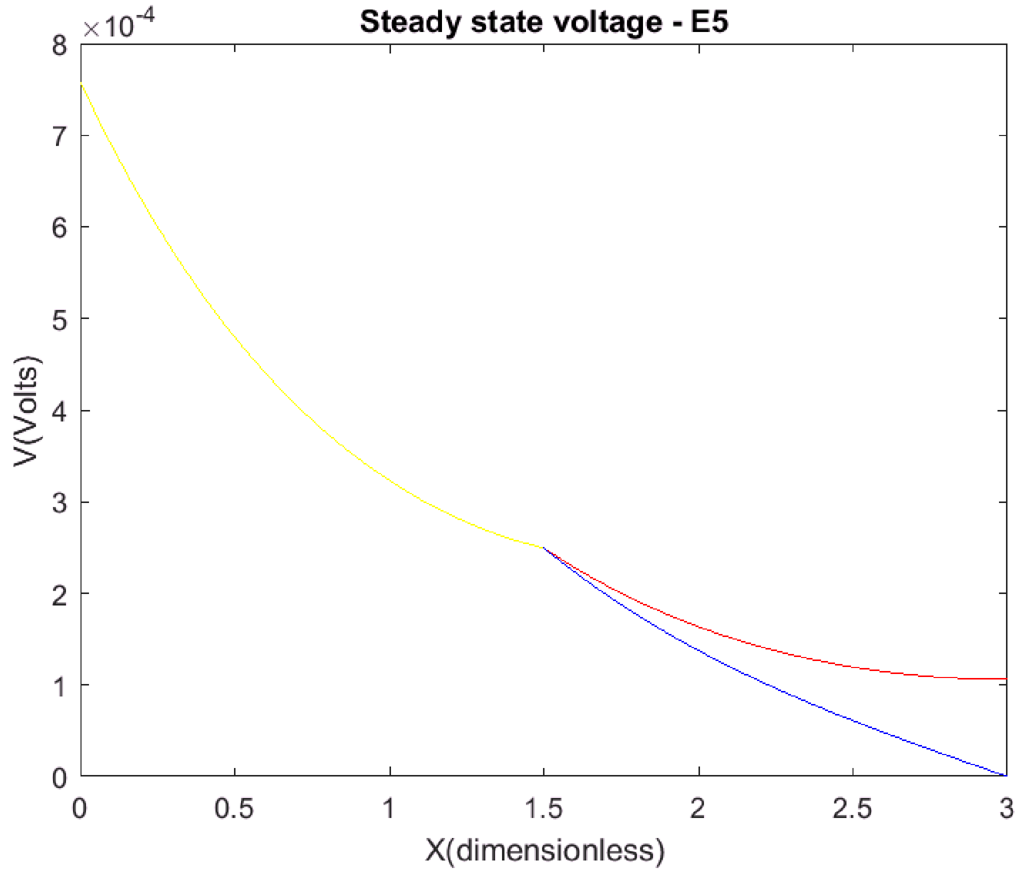
Plotting steady state voltage profiles for different boundary contions

Boundary condition 1-2(a)

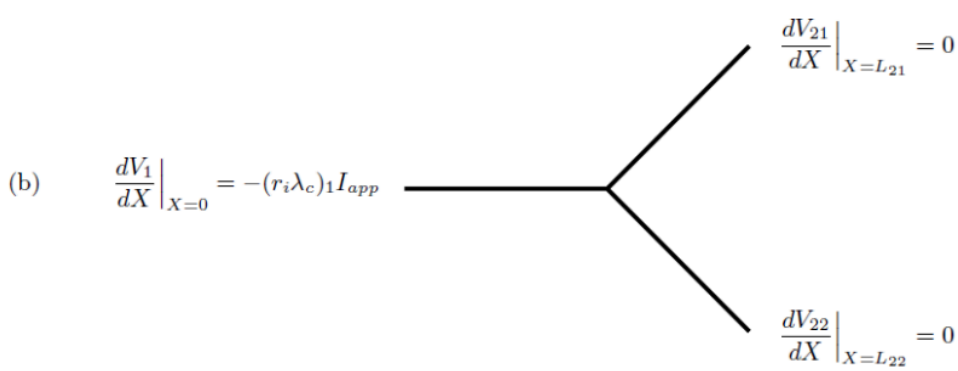


In [4]:
```
A2_a=A;
A2_a(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
x=A2_a\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

Out[4]:



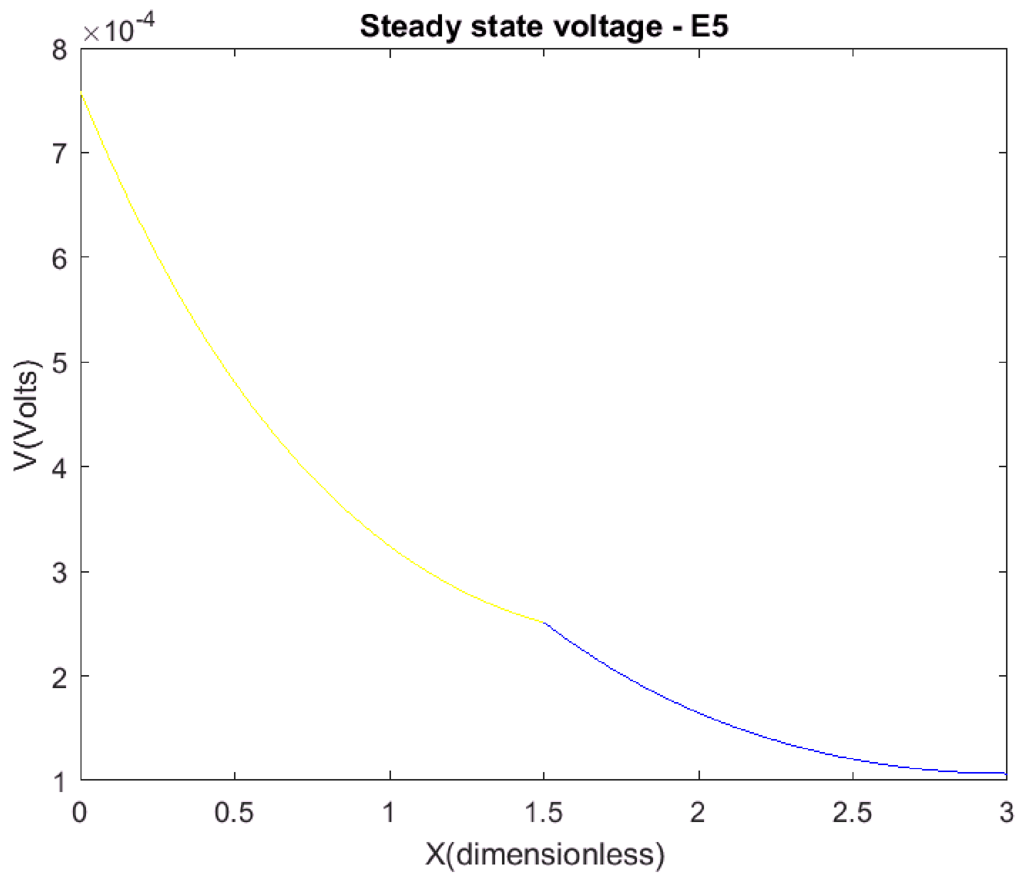
Boundary condition 2-2(b)



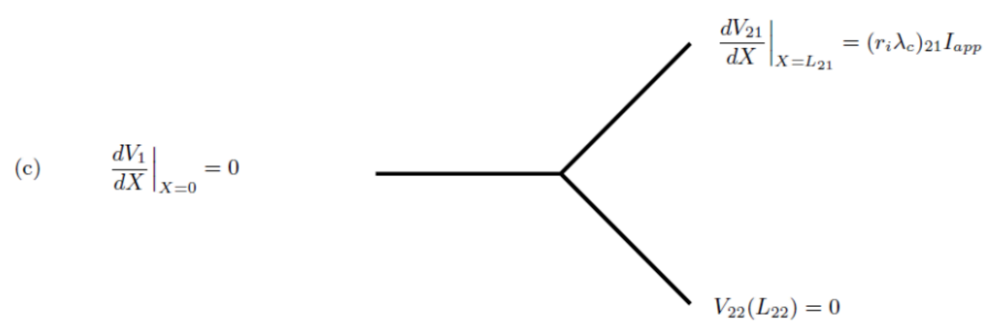
In [5]:

```
A2_b=A;
A2_b(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
A2_b(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x=A2_b\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

Out[5]:



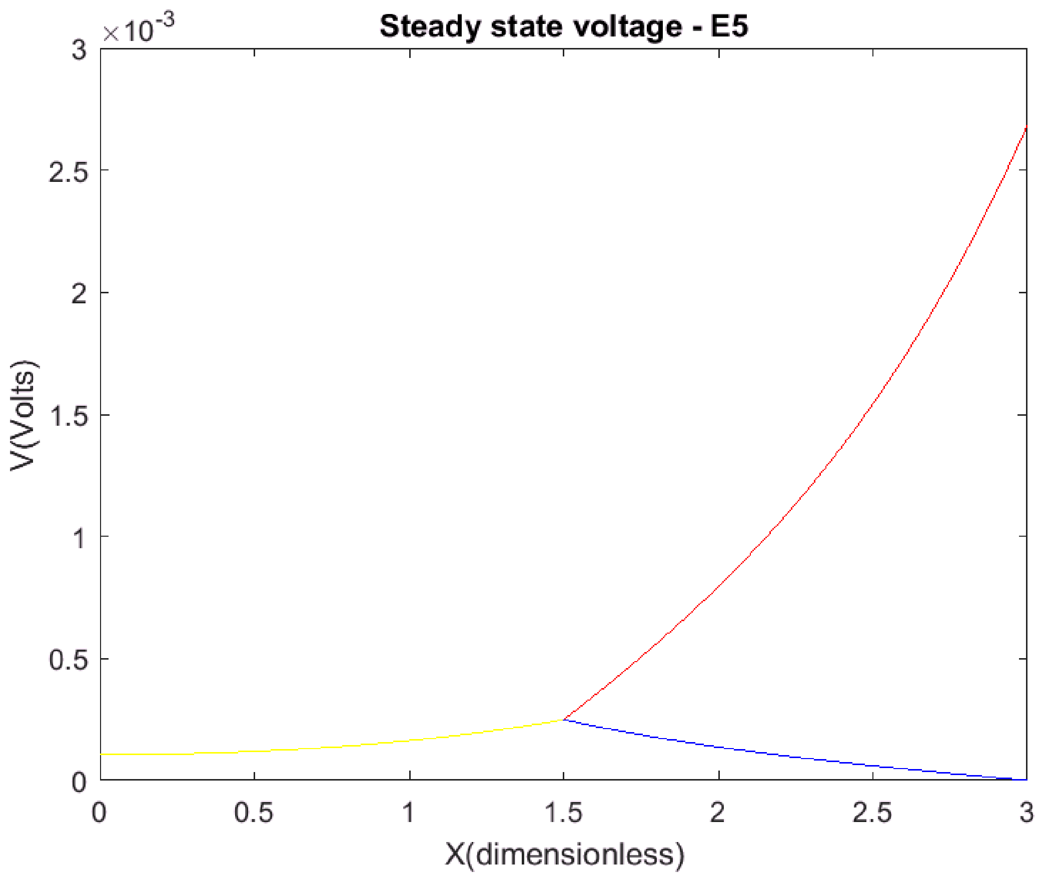
Boundary condition 3-2(c)



In [6]:

```
A2_c=A2_a;
b(1) = 0;
b(2) = rl21*iapp;
x=A2_c\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

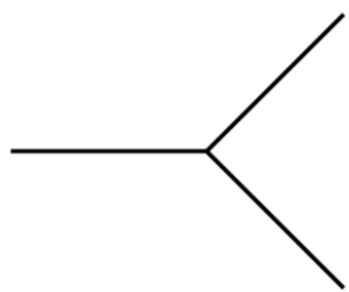
Out[6]:



Boundary condition 4-2(b)

$$\left.\frac{dV_{21}}{dX}\right|_{X=L_{21}} = (r_i\lambda_c)_{21} I_{app}$$

(d) $\left.\frac{dV_1}{dX}\right|_{X=0} = 0$

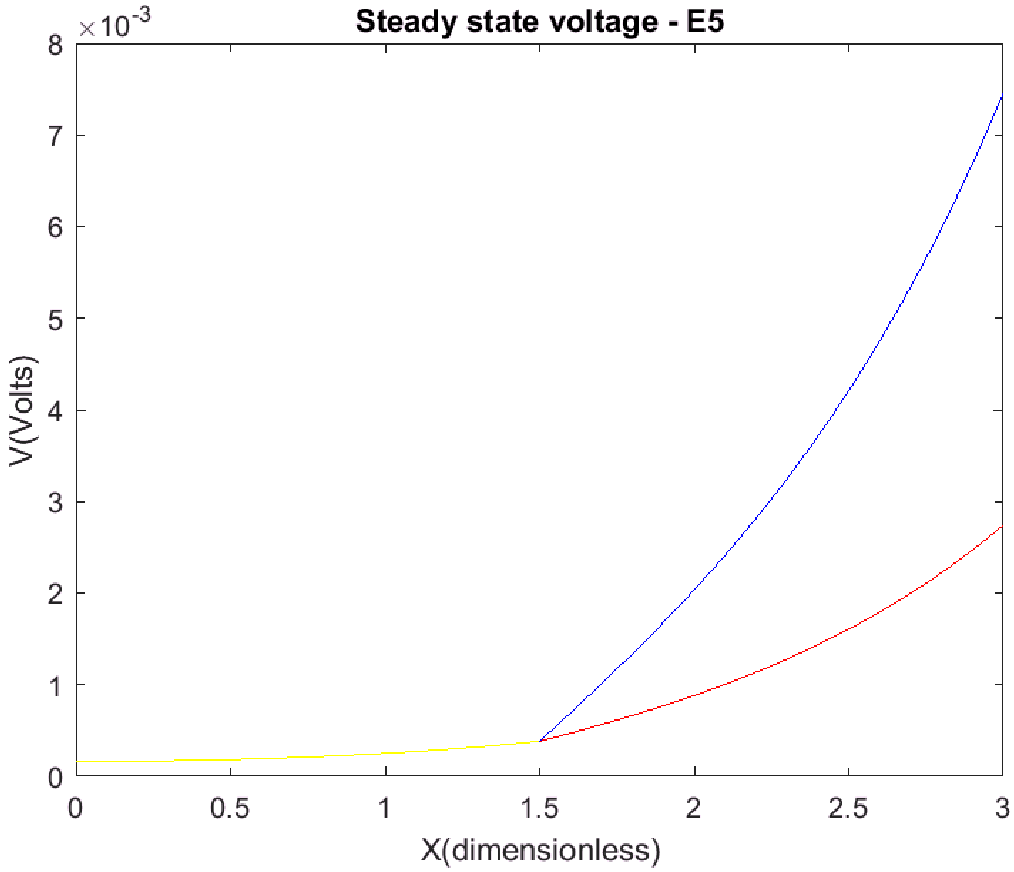


$$\left.\frac{dV_{22}}{dX}\right|_{X=L_{22}} = (r_i\lambda_c)_{22} I_{app}$$

In [7]:

```
A2_d=A2_b;
b(3) = rl22*iapp;
x=A2_d\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

Out[7]:



Question 05- Meaning of the positive right hand sides

The positive right-hand side signifies a positive rate of change of voltage with respect to electrotonic distance X at the end of the second daughter branch L22

These positive rates of change suggest that the voltage increases as one moves away from the terminals of the daughter branches. This suggests an increase in membrane potential along the length of the daughter branch. A positive rate of change of voltage at the end of a daughter branch implies that the incoming signals are being summated or combined, potentially leading to an enhanced response in downstream neuronal elements.

Overall, these positive derivatives indicate an upward trend in voltage towards the ends of the daughter branches in response to the specified boundary conditions.

Steady-state voltage profile for for d21 = d22 = 47.2470 x 10^−4cm

Question 06

Redefining matrix A and boundray conditions

In [8]:

```
% electrical constants and derived quantities for typical
% mammalian dendrite
```

```
% Dimensions of compartments

d1 = 75e-4;                      % cm
%d21 = 30e-4;                    % cm
%d22 = 15e-4;                    % cm
d21 = 47.2470e-4;        % E9 cm
d22 = d21;                % E9 cm

l1 = 1.5;                        % dimensionless
l21 = 3.0;                       % dimensionless
l22 = 3.0;                       % dimensionless

% Electrical properties of compartments

Rm = 6e3;                        % Ohms cm^2
Rc = 90;                         % Ohms cm
Rs = 1e6;                        % Ohms

c1 = 2*(Rc*Rm)^(1/2)/pi;

rl1 = c1*d1^(-3/2);              % Ohms
rl21 = c1*d21^(-3/2);            % Ohms
rl22 = c1*d22^(-3/2);            % Ohms


% Applied current

iapp = 1e-9;     % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21 rl1*exp(-l1)/rl22 -rl1*e

b = [rl1*iapp 0 0 0 0 0]';
```

Recalculating coefficients of the equation

In [9]:
```
x=A\b;
display(x);
```

Out[9]: x = 6x1 double

1.0e+-3 *

0.7189

-0.0014

0.7275

-0.0018

0.7275

-0.0018

Values of coefficients

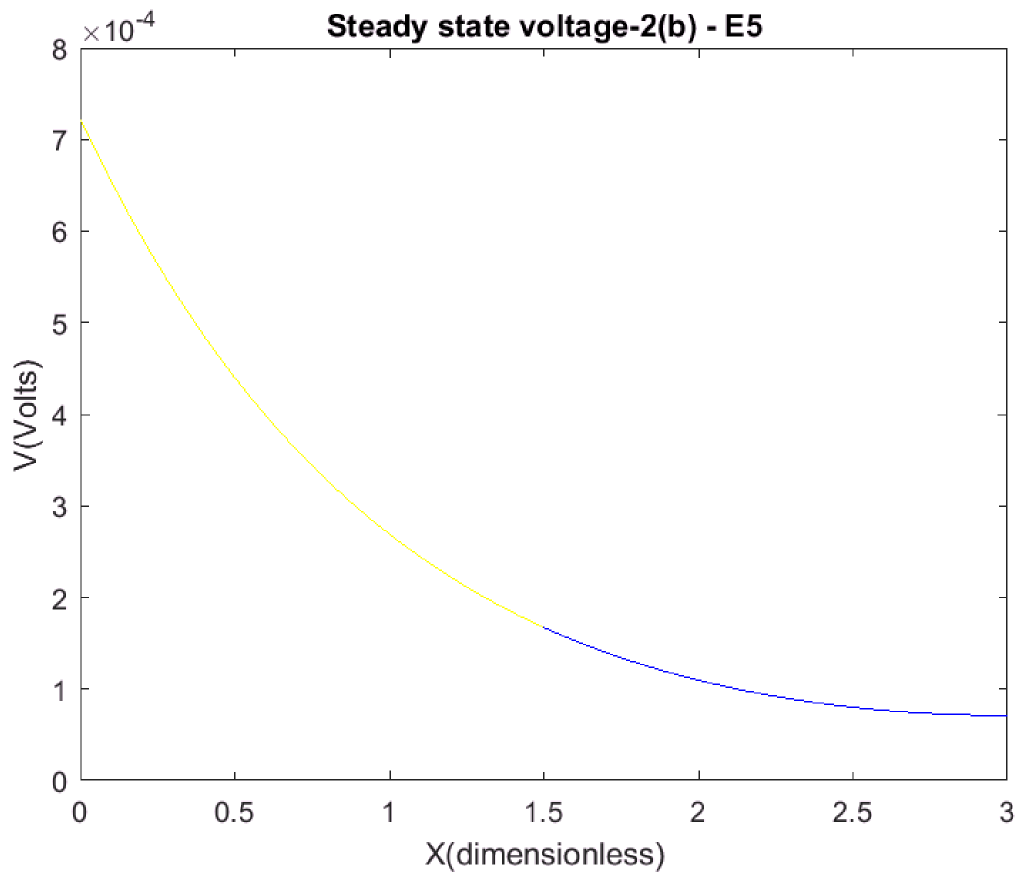
- A1= 0.7189
- B1= -0.0014
- A21= 0.7275
- B21= -0.0018
- A22= 0.7275
- B22= -0.0018

Replotting the steady state voltage profile for 2(b)

In [10]:

```
A2_b2=A;
A2_b2(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
A2_b2(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x=A2_b2\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage-2(b) - E5');
```

Out[10]:



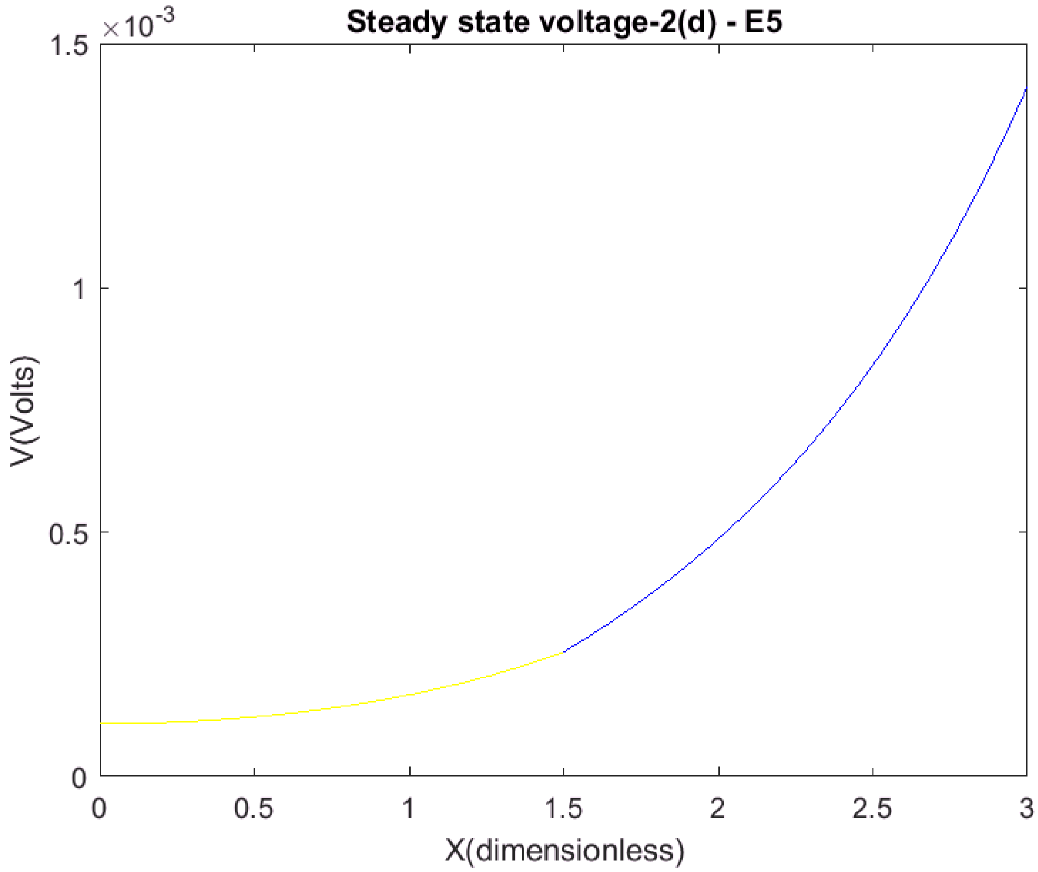
Replotting the steady state voltage profile for 2(d)

In [11]:

```
A2_d2=A2_b2;

b(3) = rl22*iapp;
b(1) = 0;
b(2) = rl21*iapp;
x=A2_d2\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage-2(d) - E5');
```

Out[11]:



In both sets of graphs, as the boundary conditions for daughter branches are equal, the steady-state voltage profiles of the two daughter branches are equal. In comparing the current graphs with the previous ones the notable differences are:

- These graphs appear smoother, showing consistent voltage transitions along the daughter branches. This allows electrical impulses to be transmitted smoothly to the daughter cells.
- These two graphs are continuously differentiabl while the previous two are not. t

In [ ]: