Supplementary Material: "On Exponential Utility and Conditional Value-at-Risk as Risk-Averse Performance Criteria"

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February 2023

This document provides some technical details to accompany the main paper entitled above. The reference numbers in the current document pertain to the works listed at the end of the document.

1 Notes Regarding Exponential Utility

1.1 Note Regarding Limit

Recall the statement from the main paper: it can be shown under certain conditions that $\lim_{\theta \to 0} \rho^{\pi}_{\theta,x}(Z) = \underline{b} + E^{\pi}_x(Z')$. We explain this statement by providing conditions under which $\lim_{\theta \to 0} \rho^{\pi}_{\theta,x}(Z') = E^{\pi}_x(Z')$ holds.

Assume that $Z' \in L^2 := L^2(\Omega, \mathcal{B}_\Omega, P^\pi_x)$ and there are real numbers a and b such that $a < 0 < b$ and $\exp(\frac{-\theta}{2} Z') \in L^2$ for all $\theta \in [a, b]$. Under these conditions, one can use [1, Thm. 2.27, p. 56] and Hölder's Inequality to find that

$$\tfrac{\mathrm{d}}{\mathrm{d}\theta} E^\pi_x(\exp(\tfrac{-\theta}{2} Z')) = \tfrac{-1}{2} E^\pi_x(Z' \exp(\tfrac{-\theta}{2} Z')) \tag{1}$$

$$\lim_{\theta \to 0} E^\pi_x(Z' \exp(\tfrac{-\theta}{2} Z')) = E^\pi_x(Z') \tag{2}$$

$$\lim_{\theta \to 0} E^\pi_x(\exp(\tfrac{-\theta}{2} Z')) = 1. \tag{3}$$

For (1), define $\tilde{f} : \Omega \times [a, b] \to \mathbb{R}$ such that $\tilde{f}(\omega, \theta) := \exp(\frac{-\theta}{2} Z'(\omega))$. It holds that $\tilde{f}(\cdot, \theta) \in L^2 \subseteq L^1 := L^1(\Omega, \mathcal{B}_\Omega, P^\pi_x)$ for all $\theta \in [a, b]$. The partial derivative of $\tilde{f}$ with respect to θ is given by

$$\tfrac{\partial}{\partial\theta} \tilde{f}(\omega, \theta) = \tfrac{-1}{2} Z'(\omega) \exp(\tfrac{-\theta}{2} Z'(\omega)), \tag{4}$$

and

$$|\tfrac{\partial}{\partial\theta} \tilde{f}(\omega, \theta)| \le \tfrac{1}{2} Z'(\omega) \exp(\tfrac{|a|}{2} Z'(\omega)) \quad \forall \omega \in \Omega \;\; \forall \theta \in [a, b]. \tag{5}$$

We denote the function on the right of (5) by $\tilde{g} := \frac{1}{2} Z' \exp(\frac{|a|}{2} Z')$. To derive (5), note that $a < 0$ and

$$-b \le -\theta \le -a = |a| \qquad \forall \theta \in [a, b]. \tag{6}$$

By Hölder's Inequality [2, p. 82], we know that $\tilde{g} \in L^1$ because $Z' \in L^2$ and $\exp(\frac{-\theta}{2} Z') \in L^2$ for all $\theta \in [a, b]$, and in particular, for $\theta = a$. Then, we use [1, Thm. 2.27b], which allows us to interchange the derivative and the integral, to conclude (1).

To show (2), define $\bar{f} : \Omega \times [a, b] \to \mathbb{R}$ such that $\bar{f}(\omega, \theta) := Z'(\omega) \exp(\frac{-\theta}{2} Z'(\omega))$. Note that $\bar{f}(\cdot, \theta) \in L^1$ for all $\theta \in [a, b]$ as a consequence of Hölder's Inequality. It holds that

$$|\bar{f}(\omega, \theta)| \leq \bar{g}(\omega) := Z'(\omega) \exp(\tfrac{|a|}{2} Z'(\omega)) \quad \forall \omega \in \Omega \ \ \forall \theta \in [a, b], \tag{7}$$

where $\bar{g} \in L^1$ by Hölder's Inequality. By continuity of the exponential function, we have

$$\lim_{\theta \to 0} \bar{f}(\omega, \theta) = \lim_{\theta \to 0} Z'(\omega) \exp(\tfrac{-\theta}{2} Z'(\omega)) = Z'(\omega) \quad \forall \omega \in \Omega. \tag{8}$$

As we have verified the conditions that are required for [1, Thm. 2.27a], we apply this result to interchange the limit and the integral and thereby conclude (2). The derivation of (3) uses a similar argument.

The proof of

$$\lim_{\theta \to 0^+} \tfrac{-2}{\theta} \log E_x^\pi(\exp(\tfrac{-\theta}{2} Z')) = \lim_{\theta \to 0^-} \tfrac{-2}{\theta} \log E_x^\pi(\exp(\tfrac{-\theta}{2} Z')) = E_x^\pi(Z') \tag{9}$$

follows from $E_x^\pi(\exp(\frac{-\theta}{2} Z'))$ being positive and finite for all $\theta \in [a, b]$, (1)–(3), and L'Hôpital's Rule.

Remark: If c and c_N are bounded, then Z' is an element of L^2, in particular. L^p spaces are formally presented by [1, Chap. 6], for example.

1.2 Note Regarding Mean-Variance Approximation

Here, we provide details regarding Eq. (6) from the main paper. Let Y be a non-negative random variable on a probability space $(\Omega, \mathcal{F}, \mu)$. Let $E(g(Y)) := \int_\Omega g(Y) \mathrm{d}\mu$ denote the expectation of $g(Y)$, where $g : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function. We paraphrase the statement of interest from the main paper: if the magnitude of θ is sufficiently small and if there is an $M < +\infty$ such that $E(Y^n) \leq M$ for all $n \in \mathbb{N}$, then the EU of Y approximates a weighted sum of the expectation $E(Y)$ and variance $\text{var}(Y)$,

$$\rho_\theta(Y) := \tfrac{-2}{\theta} \log E(\exp(\tfrac{-\theta}{2} Y)\big) \approx E(Y) - \tfrac{\theta}{4} \text{var}(Y).$$

By the definition of the exponential function, e.g., see [3, Eq. 1, p. 1], it holds that

$$\exp(\tfrac{-\theta}{2} y) = \sum_{n=0}^{\infty} \frac{(\frac{-\theta}{2} y)^n}{n!} \tag{10}$$

for all $y \in \mathbb{R}$. Recall that we consider $\theta \in \Theta \subseteq (-\infty, 0)$, and thus,

$$h_n := \frac{(\frac{-\theta}{2} Y)^n}{n!} \tag{11}$$

is a non-negative Borel-measurable function for each $n \in \mathbb{N}$. Since any series of non-negative Borel-measurable functions can be integrated term by term [2, Corollary 1.6.4 (a), p. 46], it holds that

$$E(\exp(\tfrac{-\theta}{2} Y)) = \sum_{n=0}^{\infty} \frac{(\frac{-\theta}{2})^n}{n!} E(Y^n) = 1 + \underbrace{\tfrac{-\theta}{2} E(Y) + \frac{(\frac{-\theta}{2})^2}{2} E(Y^2) + \sum_{n=3}^{\infty} \frac{(\frac{-\theta}{2})^n}{n!} E(Y^n)}_{\phi_\theta}, \tag{12}$$

where each integral is guaranteed to exist (i.e., not be of the form $+\infty - \infty$) because each function inside each integral is non-negative and Borel measurable.

Now, recall the following relation for the natural logarithm,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \quad \text{for } -1 < z \le 1, \tag{13}$$

e.g., see [4, Example 2, pp. 212–213]. Since $\frac{-\theta}{2}Y$ is non-negative and the exponential is increasing, it holds that $\exp(\frac{-\theta}{2}Y) \ge \exp(0) = 1$ everywhere, and thus, $E(\exp(\frac{-\theta}{2}Y)) \ge 1$. In addition, we use (12) and the assumed existence of an $M < +\infty$ such that $E(Y^n) \le M$ for all $n \in \mathbb{N}$ to find that

$$\begin{aligned} 0 \le E(\exp(\tfrac{-\theta}{2}Y)) - 1 = \phi_\theta &= \tfrac{-\theta}{2}E(Y) + \frac{(\frac{-\theta}{2})^2}{2}E(Y^2) + \sum_{n=3}^{\infty}\frac{(\frac{-\theta}{2})^n}{n!}E(Y^n) \\ &\le \tfrac{-\theta}{2}M + \frac{(\frac{-\theta}{2})^2}{2}M + \sum_{n=3}^{\infty}\frac{(\frac{-\theta}{2})^n}{n!}M \\ &= M\sum_{n=1}^{\infty}\frac{(\frac{-\theta}{2})^n}{n!}. \end{aligned} \tag{14}$$

By the definition of the exponential, e.g., use (10) with $y = 1$, it holds that

$$\exp(\tfrac{-\theta}{2}) = \sum_{n=0}^{\infty}\frac{(\frac{-\theta}{2})^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{(\frac{-\theta}{2})^n}{n!}, \tag{15}$$

and by (14) and (15),

$$0 \le \phi_\theta \le M(\exp(\tfrac{-\theta}{2}) - 1). \tag{16}$$

Note that there is a $\theta < 0$ whose magnitude is sufficiently small so that $0 \le \phi_\theta \le 1$ holds. Using such a θ, we apply $E(\exp(\frac{-\theta}{2}Y)) = 1 + \phi_\theta$ (12) and (13) with $z = \phi_\theta$ to write

$$\log E(\exp(\tfrac{-\theta}{2}Y)) \overset{(12)}{=} \log(1+\phi_\theta) \overset{(13)}{=} \phi_\theta - \frac{\phi_\theta^2}{2} + \frac{\phi_\theta^3}{3} - \frac{\phi_\theta^4}{4} + \dots. \tag{17}$$

By discarding the terms of order three or greater, we have the following approximation,

$$\log E(\exp(\tfrac{-\theta}{2}Y)) \approx \phi_\theta - \frac{\phi_\theta^2}{2}, \tag{18}$$

whose accuracy improves when we have chosen θ so that ϕ_θ is closer to zero. By substituting the expression for ϕ_θ, see (12), and discarding terms of order three or greater, we have

$$\begin{aligned} \log E(\exp(\tfrac{-\theta}{2}Y)) &\approx \tfrac{-\theta}{2}E(Y) + \frac{(\frac{-\theta}{2})^2}{2}E(Y^2) - \frac{(\frac{-\theta}{2}E(Y))^2}{2} \\ &= \tfrac{-\theta}{2}E(Y) + \frac{(\frac{-\theta}{2})^2}{2}\operatorname{var}(Y). \end{aligned} \tag{19}$$

Finally, by multiplying by $\frac{-2}{\theta}$, we obtain the desired approximation,

$$\tfrac{-2}{\theta}\log E(\exp(\tfrac{-\theta}{2}Y)) \approx E(Y) - \tfrac{\theta}{4}\operatorname{var}(Y). \tag{20}$$

2 Some Details about CVaR Optimal Control

For convenience, we first repeat some information from the main paper. The function $J^* : S \times \mathbb{R} \to \mathbb{R}$ is defined by

$$J^*(x,s) := \inf_{\pi \in \Pi} E_x^\pi(\max\{Z' - s, 0\}),$$

where Z' is a non-negative, everywhere-bounded cumulative random cost incurred by a control system over time. In particular, each realization of Z' is an element of $[0, \bar{a}]$, where $\bar{a} \in \mathbb{R}_+$. Details regarding the precise meaning of $E_x^\pi(\cdot)$ in the definition of J^* will be provided below.

We define $\mathcal{Z} := [-\bar{a}, \bar{a}] \subseteq \mathbb{R}$.

Π is a class of policies that are history-dependent through the augmented state (X_t, S_t). Any $\pi \in \Pi$ takes the form $\pi = (\pi_0, \pi_1, \dots, \pi_{N-1})$, where $\pi_t(\cdot|\cdot,\cdot)$ is a Borel-measurable stochastic kernel on A given $S \times \mathcal{Z}$ for each t.

In the main paper, we have defined $\Omega := (S \times A)^N \times S$, and we have stated that P_x^π is a probability measure on $(\Omega, \mathcal{B}_\Omega)$ that is parametrized by an initial condition $x \in S$ and a policy π. We have said that the notation $E_x^\pi(\cdot)$ denotes the expectation with respect to P_x^π. Now, in the case of CVaR, we use different definitions for Ω and P_x^π to accommodate an extended state space. In particular, we use $\Omega := (S \times \mathcal{Z} \times A)^N \times S \times \mathcal{Z}$. Let δ_y denote the Dirac measure on $(\mathcal{M}, \mathcal{B}_\mathcal{M})$ concentrated at $y \in \mathcal{M}$, where $\mathcal{M}$ is a metrizable space. Let $Q(\cdot|\cdot,\cdot)$ be the *transition law*, which is a Borel-measurable stochastic kernel on S given $S \times A$. That is, if $(x_t, u_t) \in S \times A$ is the realization of (X_t, U_t), then the probability that X_{t+1} is in $B \in \mathcal{B}_S$ is given by

$$Q(B|x_t, u_t) := p(\{w_t \in D : f(x_t, u_t, w_t) \in B\}|x_t, u_t). \tag{21}$$

Let $(x, s) \in S \times \mathcal{Z}$ and $\pi \in \Pi$ be given. P_x^π takes the following form on measurable rectangles in Ω,

$$\begin{aligned}
&P_x^\pi(\underline{S}_0 \times \underline{\mathcal{Z}}_0 \times \underline{A}_0 \times \underline{S}_1 \times \underline{\mathcal{Z}}_1 \times \underline{A}_1 \times \cdots \times \underline{S}_{N-1} \times \underline{\mathcal{Z}}_{N-1} \times \underline{A}_{N-1} \times \underline{S}_N \times \underline{\mathcal{Z}}_N) = \\
&\int_{\underline{S}_0} \int_{\underline{\mathcal{Z}}_0} \int_{\underline{A}_0} \int_{\underline{S}_1} \int_{\underline{\mathcal{Z}}_1} \int_{\underline{A}_1} \cdots \int_{\underline{S}_{N-1}} \int_{\underline{\mathcal{Z}}_{N-1}} \int_{\underline{A}_{N-1}} \int_{\underline{S}_N} \int_{\underline{\mathcal{Z}}_N} \delta_{(s_{N-1} - c'(x_{N-1}, u_{N-1}))}(\mathrm{d}s_N)\, Q(\mathrm{d}x_N | x_{N-1}, u_{N-1}) \\
&\quad \pi_{N-1}(\mathrm{d}u_{N-1}|x_{N-1}, s_{N-1})\, \delta_{(s_{N-2} - c'(x_{N-2}, u_{N-2}))}(\mathrm{d}s_{N-1})\, Q(\mathrm{d}x_{N-1}|x_{N-2}, u_{N-2}) \cdots \\
&\quad \pi_1(\mathrm{d}u_1|x_1, s_1)\, \delta_{(s_0 - c'(x_0, u_0))}(\mathrm{d}s_1)\, Q(\mathrm{d}x_1|x_0, u_0)\, \pi_0(\mathrm{d}u_0|x_0, s_0)\, \delta_s(\mathrm{d}s_0)\, \delta_x(\mathrm{d}x_0),
\end{aligned} \tag{22}$$

where $\underline{S}_t \in \mathcal{B}_S$, $\underline{\mathcal{Z}}_t \in \mathcal{B}_\mathcal{Z}$, and $\underline{A}_t \in \mathcal{B}_A$ for each t [5, Prop. 7.28, pp. 140–141]. Note that P_x^π depends on s, which we do not write explicitly to follow the convention in the literature, e.g., see [6].

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