Combinatorics

Permutations and Combinations

Summary

LISTING 1 NUMBER TYPE

Permutations on n things taken r at a time with repetition	n^r
Permutations of n things taken r at a time without repetition	$\frac{n!}{(n-r)!}$

Permutations and Combinations

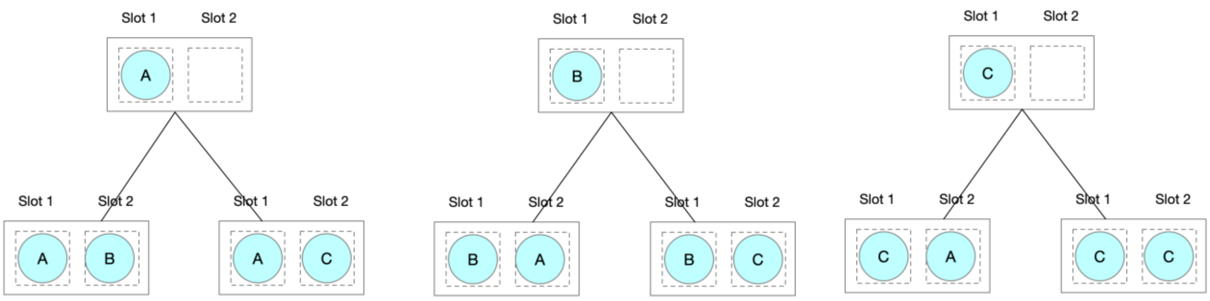
The fundamental difference between permutations and combinations is that with permutations order is significant whereas with combinations it is not. So with permutations A,B = B,A whereas with combinatios A,B = B,A.

Permutations

With permutations order is important. We consider two types of permutation; one which allows repetition and one which does not.

PERMUTATIONS WITHOUT REPETITION

If repetition is not allowed we have the following situation



We have three choices for the first slot. But for each of those choices we only have two choices for the second slot as once value from the set have been used up. So we have 3×2 ways of taking 3 objects 2 at a time without repetition. In full generality we have

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

Ways of taking n objects r at time withour repetition. R must be less than or equal to n. This can be expressed as $\frac{n!}{(n-r)!}$. We show why in the following section

Proof

$$n! = n \times (n-1) \times \ldots \times (n-r+1) \times \boldsymbol{(n-r)} \times \boldsymbol{(n-r-1)} \times\times \boldsymbol{2} \times \boldsymbol{1} \qquad \boldsymbol{(1)}$$

$$(n-r) \times (n-r-1) \times (n-r-2) \times \ldots \times 2 \times 1 = (n-r)! \qquad (2)$$

Substituting (2) into (1)

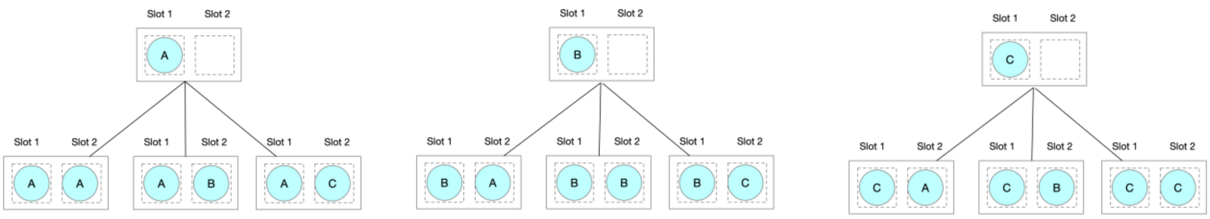
$$n! = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r)! \qquad (3)$$

And re-arranging

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

PERMUTATIONS WITH REPETITION

If repetition is allowed we have the following situation. We have 3 objects {A, B, C} taken two objects at a time. For each of the three possible values of the first slot we have 3 possible values for the second slot.



We have 3^2permutations. In full generality there are n^rways of arraning n objects taken r at a time with repetition

Combinations

COMBINATIONS WITHOUT REPETITION

Unlike permutations, where order is important, with combinations we are only concerned that we selected something. One way to visualize this is that the order of slots is unimportant so if we have three slots then

Slot 1	Slot 2	Slot 3	=	Slot 1	Slot 3	Slot 2	=
Slot 2	Slot 1	Slot 3	=	Slot 2	Slot 3	Slot 1	=
Slot 3	Slot 1	Slot 2	=	Slot 3	Slot 2	Slot 1	=

Because we consider all permutations of the same things equal, then the numbers of combinations equals the number of permutations divided by the number of permutations of slots. We can formulate the problem generally as

- The number of ways of selecting r items from a total of n where order in unimportant

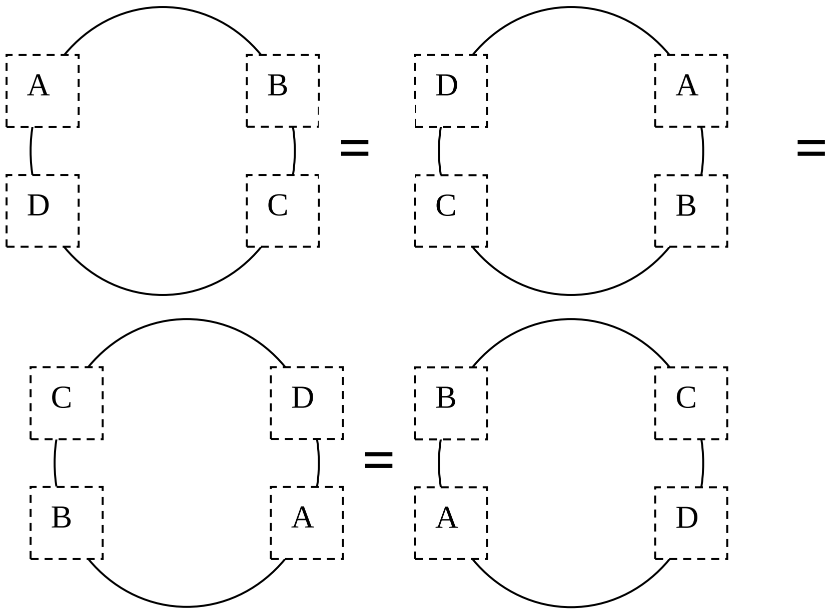
$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \equiv {}^{n}C_{r} \equiv {}_{n}C_{r} \equiv C(n,r)$$

Combinations without repetition

$$\left(\binom{n}{r}\right)\left(\binom{n+r-1}{r}\right) =$$

Circular Permutations

With a circular permutation we consider two different permutations where each entry has the same element to its right/left as identical



From this we can see that the number of circular permutations of n items taken r at a time is given by

$$\frac{n!}{r(n-r)!}$$

Binomial Theorum

Overview – A recurrence relation for the co-efficients

Expressions of the form$(1 + x)^n$, where n is a positive integer are known as binomial expressions. Multiplying out a binomial expansion gives us a binomial expansion.

$(1 + x)^1 = 1 + x$

$(1 + x)^2 = 1 + 2x + x^2$

$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$

$(1 + x)^4 = 1 + 4x + 6x^2 + x4^3 + x^4$

Notice the emerging pattern! The co-efficient of x^2 in the expansion of $(1 + x)^4$is equal to the sum of the co-efficient of x^2 and the co-efficient of x in the expansion of $(1 + x)^3$. In general the co-efficient of x^min the expansion of $(1 + x)^n$is equal to the sum of the co-efficient of $x^m$and x^{m-1}in the expansion of $(1 + x)^{n-1}$. We can easily see why this is by looking at the following example

$(1 + x)^5 = (1 + x)(1 + x)^4$

$1 + 4x + 6x^2 + 4x^3 + x^4$

$1 + x$

$1 + 4x + 6x^2 + 4x^3 + x^4$

$x + 4x^2 + 6x^3 + 4x^4 + x^5$

$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

A closd form solution for Binomial Co-efficients

The coefficients of the binomial expansion $(1+x)^n$ are given by

$$1+{}_1^nCx^1+{}_2^nCx^2+\ldots+{}_{n-1}^{\;\;n}Cx^{n-1}+{}_n^nCx^n$$

It is worth spending a little time looking at why this might be. Consider the expansion of

(1.1). $(1+x)^3=(1_1+x_1)(1_2+x_2)(1_3+x_3)=$

(1.2) $1_1.1_2.1_2+1_1.1_2.x_3+1_1..x_2.1_1+1_1.x_2x_3+x_1.1_2.1_3+x_1.1_2.x_3+x_1x_2.1_3+x_1x_2x_3$

(1.3) $1+x_3+x_2+x_2x_3+x_1+x_1.x_3+x_1x_2+x_1x_2x_3=$

(1.4) $1+(x_3+x_2+x_1)+(x_1.x_3+x_1x_2+x_2x_3)+(x_1x_2x_3)=$

(1.5) $1+3x+3x^2+x^3$

Look at line (1.4) and notice that the number of ways of obtaining a unit power of x is the number of ways of selecting one item from the set (x_3,x_2,x_1) The ways of obtaining a square power of x is the number of ways of selecting two items from (x_3,x_2,x_1) The ways of obtaining a cube power of x is the number of ways of selecting three items from (x_3,x_2,x_1) And not forgetting the unit term, the number of ways of obtaining 1 is the number of ways of selecting zero x's from three things

Since the 1's have no effect the co-efficient of the x^r term is the number of combinations of n x's taken r at a time $+{}_r^nCx^r$

Extending the solution to (a+b)

Now we know how to obtain the coefficients of $(1+x)^n$ we can extend the methodology to the expansion of $(a+b)^n$ by noting that.

$$(a+b) = a\left(\frac{a+b}{a}\right).$$

Also

$$\left(\frac{a+b}{a}\right)^n = \frac{(a+b)^n}{a^n}$$

So

$$(a+b)^n = a^n\left[1+\frac{b}{a}\right]^n$$

Binomial expansion of E

Binomial expansion of E is given by

$$\left(1+\frac{1}{n}\right)^n = 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^2+,\ldots\ldots,+\frac{1}{n^n}$$

$$=1+\left(\frac{n}{n}\right)+\frac{n(n-1)}{2!n^2}+\frac{n(n-1)(n-2)}{3!n^4}+,\ldots\ldots,+\frac{1}{n^n}$$

$$=1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+,\ldots\ldots,+\frac{1}{n^n}$$

In the limit $\underset{n\to\infty}{Lim}\left(\frac{1}{n}\right)=0$ then the above expression tends to

$$=1+1+\frac{1-0}{2!}+\frac{(1-0)(1-0)}{3!}+,\ldots\ldots=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+,\ldots\ldots=\sum_{r=0}^{\infty}\frac{1}{r!}$$

Properties of Combinatorial Coefficients

1. ${}^nC_n={}^nC_0$ Because ${}^nC_n=\frac{n!}{(n-n)!}=\frac{n!}{0!n!}=1$ and ${}^nC_0=\frac{n!}{n!0!}=1$

2. ${}^nC_{n-r}={}^nC_r$ Because ${}^nC_{n-r}=\frac{n!}{[n-(n-r)]!(n-r)!}=\frac{n!}{(n-r)!r!}={}^nC_r$

3. ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$ Because **Insert Proof**

Permutation Example

Iteration 1

Visit Permutation

1	2	3	4	Initial permutation

Find J

We want to find the smallest index j such that we have visited every permutation starting with $a_0 \dots a_j$ IWe achieve this by setting $j = n - 1$ and decrementing j until $a_j < a_{j+1}$ Once this condition is met we know we have visited every permutation beginning with $a_0 \dots a_j$ In this specific case we have $j = 3$ and we have visited every permutation beginning with $\{1,2,3\}$ namely the single permutation$\{1,2,3\}\{4\}$

		j	
1	2	3	4

Increase a_j

We know from the previous step that we have visisted ever permutation beginning with $a_0 \dots a_j$. We want to find the smallest element greater than a_j that can legitemately follow $a_0 \dots a_{j-1}$in a permutation. We achieve this by setting $l = n$ and then decreasing l until $a_j < a_l$ Because the tail is sorted in decreasing order we know $a_{j+1} \geq \dots \geq a_n$ so the first element walking back from $a_n$that is greater than a_j is also the lowest possible value that is greater than a_j

		j	l
1	2	3	4

Swap $a_j \leftrightarrow a_l$

		j	l
1	2	4	3

Reverse

We know that everthing after a_j is in decreasing order. But to be lexigographic we need it to be in increasing order so we reverse $a_{j+1} .. a_n$. In this case we have a single element so no reversing is needed.

Iteration 2

Visit Permutation

1	2	4	3

Find J

We want to find the smallest index j such that we have visited every permutation starting with $a_0 \dots a_j$ IWe achieve this by setting $j = n - 1$ and decrementing j until $a_j < a_{j+1}$

	j		
1	2	4	3

Once this condition is met we know we have visited every permutation beginning with $a_0 \dots a_j$ In this specific case we have $j = 2$ and we have visited every permutation beginning with $\{1,2\}$ namely the $\{1,2\}\{3,4\}$ and $\{1,2\}\{4,3\}$

Increase a_j

We know from the previous step we have visited all permutation beginning with $\{1,2\}$ so the key now it to increase $a_2$by the smallest amount possible. We know that $a_{j+1} \geq \dots \geq a_n$ in our case $\{4,3\}$ so the first element moving from highest to lowest index greater than $a_2$is also the smallest element greater than a_2

	j		l
1	2	4	3

Swap $a_j \leftrightarrow a_l$

	j		l
1	3	4	2

Reverse

No we know everything after $\{1,3\}$is in decreasing order. As we have increased $a_2$we want to reverse everying after it so we end up with the next lexicographical element.

Iteration 3

Visit Permutation

1	3	2	4

Find J

We want to find the smallest index j such that we have visited every permutation starting with $a_0 \ldots a_j$ IWe achieve this by setting $j = n - 1$ and decrementing j until $a_j < a_{j+1}$ We have visited the single permutation starting {1,3,2} namely {1,3.2}{4}

		j	
1	3	2	4

Increase a_j

We want to find the smallest value greater than a_j

		j	l
1	3	2	4

Swap $a_j \leftrightarrow a_l$

1	3	4	2

Reverse

There is only a single element in position a_{j+1}so reversing does nothing

Iteration 4

Visit Permutation

1	3	4	2

Find j

	j		
1	3	4	2

Increase a_j