Stochastic Processes

Summary

TABLE 1 DEFINITIONS

Outcome	Each thing that can occur in an expiriment is called an outcome. In the example of tossing a coin we have two outcomes 'heads' or 'tails' which we can denote by the letters H and T respectively.
Sample Space	The set of all possible outcomes of an experiment is known as the sample space. By convention we label it Ω. In our simple coin tossing scenario we would have $\Omega = \{H, T\}$ If we toss two coins our sample space would become $\Omega = \{HH, HT, TH, TT\}$
Event	A subset of the probability space is ` an event. We define an event using the following notation. $A = \{\varpi \in \Omega; \varpi = H\}$ "This means the set of all outcomes ϖsuch that $\varpi$is a head".
Probability Measure	A probability measure P is a function that assigns to each element ϖ in Ω a probability such that $$\sum_{\omega \in \Omega} P(\varpi) = 1$$ Since an event A is a subset of Ω then the probability of an event is given by $$P(A) = \sum_{\omega \in A} P(\varpi)$$
Probability Space	A probability space (Ω, P) consists of a sample space and a probability measure. The sample space is the set of outcomes and the probability measure is a function that assigns to each element ϖin Ω a value in $[0,1]$ such that
Random Variable	A random variable x is a real valued function defined on Ω . Put another way a random variable maps each outcome from the sample space Ω to a real number.
Proabability Distribution	We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on Ω whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.

TABLE 2 PROPERTIES OF RANDOM VARIABLES

Expectation	$E(X) = \sum_{\omega\epsilon} X(\omega)P(\omega)$
Linearity of expectation	$E[aX + b] = aE[X] + b$
Expectation of a function of a random variable	$E[g(X)] = \sum_{\omega\epsilon\Omega} g(X(\omega))P(\omega)$
Expcetation of sum of random variables	$E[X + Y] = E[X] + E[Y]$
Variation from Expectation	$\sum_{\omega\in\Omega} [X(\omega) - E(X)]^2 P(\omega)$
Expectation of sum of n IIR variable	$n.E[X_n]$
Variance	$Var[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2$
Variance of constant	$Var[a] = 0$
Variance of a constant multiple	$Var[aX] = aVar[X]$
Variance of sum of two random variables	$Var[x + y] = Var[x] + Var[y]$
Variance of sum of n IIR variable	$n.Var[X_n]$

Worked example

A Single experiment

Imagine a random event that involves the tossing of a single coin. We have two *outcomes*, heads or tails giving us *a sample space* of

$$\Omega = \{H, T\}$$

Furthermore, let us define two *random variables*. The first x_1 takes the value of plus one if we obtain a head and minus one if we obtain a tail.

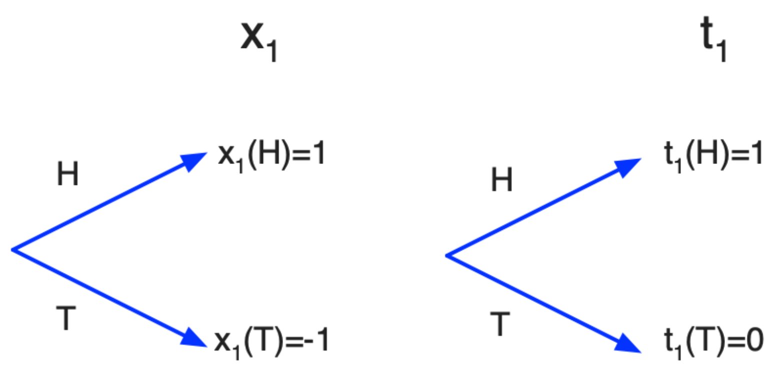
$$x_1(H) = 1, x_1(T) = -1$$

The second takes a value of plus one if we obtain a head and zero if we get a tail

$$t_1(H) = 1, t_1(T) = 0$$

Notice that our random variables do not say anything about the probability of a head or tail. They just tell us what value we assign to the outcomes of the sample space.

Figure 1 Random variables are real valued functions on the sample space

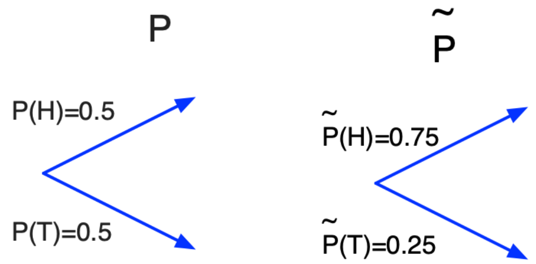


A probability measure is a real valued function that maps each outcome of the sample space to a probability. If our coin is fair we could have a measure P such that

$$P(H) = 0.5, P(T) = 0.5$$

We might however have a different measure for a loaded coin

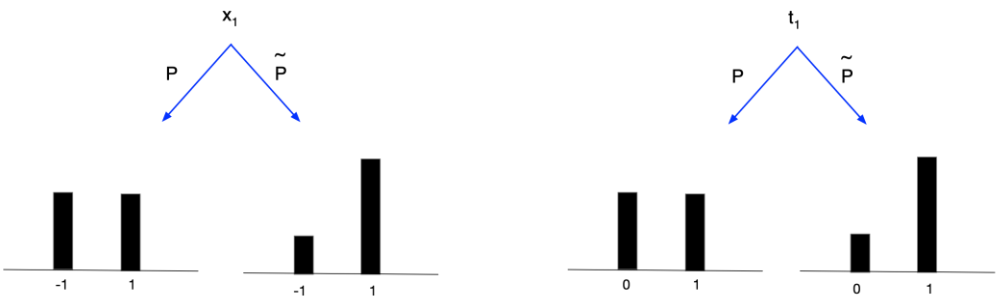
$$\tilde{P}(H) = 0.75, \tilde{P}(T) = 0.25$$



Distribution

Applying a probability measure to a random variable gives us a distribution. The distribution shows the probability of each value of the random variable. Since we have two random variable and two measures we have four probability distributions

Figure 2Applying measures to random variables give us distributions



Mean and variance

We can define the expectation or expected value of any random variable X under a probability measure P as

$$E(X) = \sum_{\omega \in \Omega} X(\omega)P(\omega)$$

For any actual value of a random variable X we can calculate the difference between that value and the expectation $X(\omega) - E(X)$ .We might ask the question "on average how much does a given value differ from the expected value?" We could calculate the average difference as $\sum_{\omega \in \Omega} [X(\omega) - E(X)]P(\omega)$ however where the distribution is symmetric around the mean this value will be zero. A more instructive measure is given by calculating the average of the difference squared

$$\sum_{\omega \in \Omega} [X(\omega) - E(X)]^2 P(\omega)$$

We now add the mean and variance values to the four distributions ontained by applying the two measures to our two random variables

Figure 3 Mean and variance

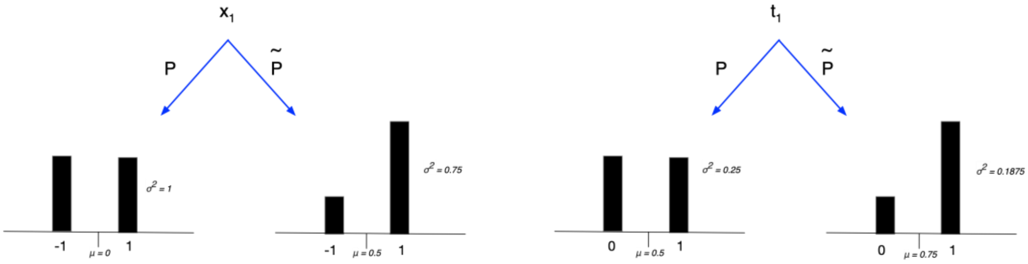


TABLE 3 SUMMARY

Outcome	H	Each outcome is a thing that can occur in an experiment
Sample Space	$\Omega = \{H, T\}$	The set of all possible outcomes that can occur in an experiment is called the sample space
Event	$A = \{\varpi \in \Omega; \varpi = H\}$	A subset of the sample space is called an event
Probability Measure	$P(H) = P(T) = 0.5$	A probability measure P is a function that assigns to each element ϖ in Ω a probability such that $\sum_{\omega \in \square} P(\varpi) = 1$
Probability Space	$(\{H, T\}, P(H) = P(T)) = 0.5)$	A probability space consists of a sample space and a probability measure
Random Variable	$x_1(H) = 1, x_1(T) = -1$	A random variable is a real valued function defined on the sample space.
Probability Distribution	$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$	Tabulation of the probabilities that the random variable takes its various values.
Expectation	$(X) = x_1(H)P(H) + x_1(T)P(T) = 0$	We define the expected value of our random variable under the probability measure P
Variation	$[x_1(H) - E(x_1)]^2 \tilde{P}(H) + [x_1(T) - E(x_1)]^2 \tilde{P}(H) = 1.5$	

Performing twice

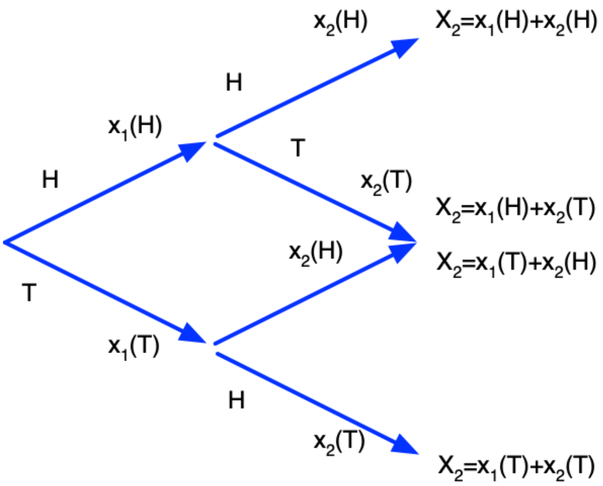
We can create a new game by playing the original games multiple times. If we play the original game twice then our new game effectively involves tossing the coin two times and our sample space becomes $\Omega = \{HH, HT, TH, TT\}$. We can define two new random variables by summing the original variables

$X_2 = x_1 + x_2.$

$T_2 = t_1 + t.$

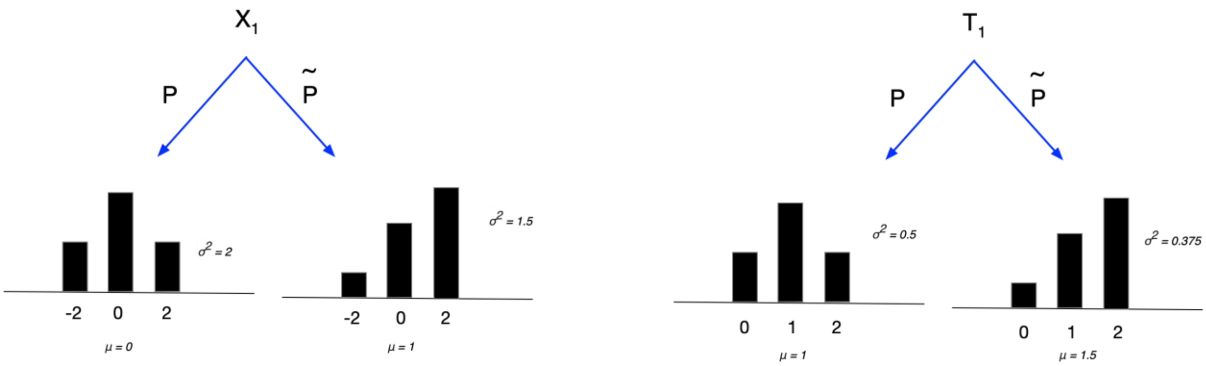
The following diagram shows how summing two independent random variables works for $X_2 = x_1 + x_2$. Notice how there are two ways of achieving the outcome that a head and a tail occur $\{HT, TH\}$

Figure 4 Summing two instances of the same random variable



Under our two measures our distributions of the two variables are as follows

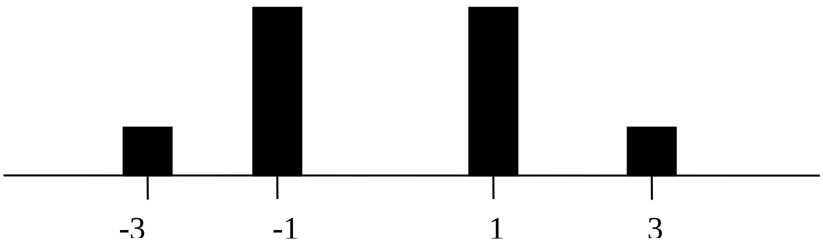
Figure 5 Mean and variance



Performing three times

Let us go one-step further and look at the event obtained by summing three of the original events. $X_3 = x_1 + x_2 + x_3$ Under the probability measure P we get the following distribution, whose mean is zero and whose variance is three.

Figure 6 Distribution of X_3 under P



- $\mu = 0$
- $\sigma^2 = 0.125(3-0)^2 + 0.125(-3-0)^2 + 0.375(1-0)^2 + 0.375(-1-0)^2 = 3.0$

Figure 7Distribution of T2 under P

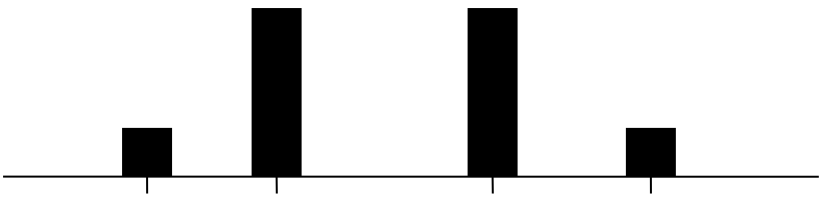
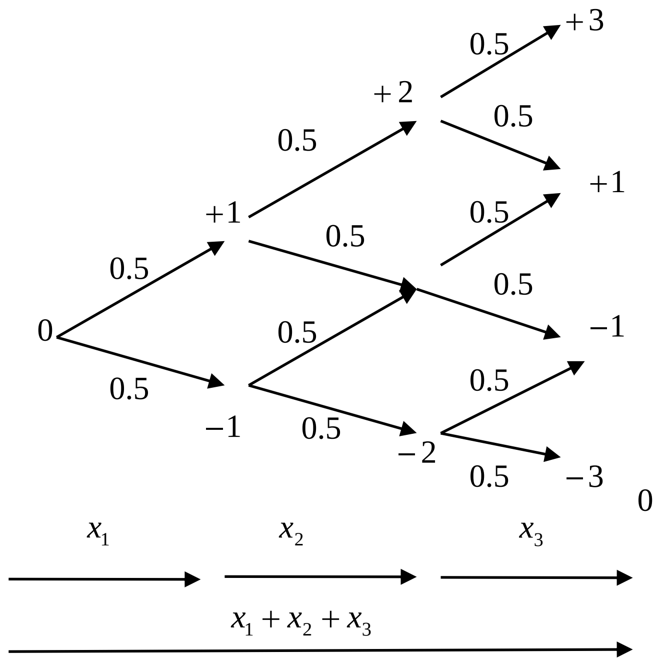


Figure 8 Tree for X_3



Generalizing

If we then perform n identical tosses of the coin and define n identical random variables $x_1, x_2, x_3, \ldots, x_n$ we can define a new random variable as the sum of the individual random variables

$$X_n = x_1 + \cdots + x_n.$$

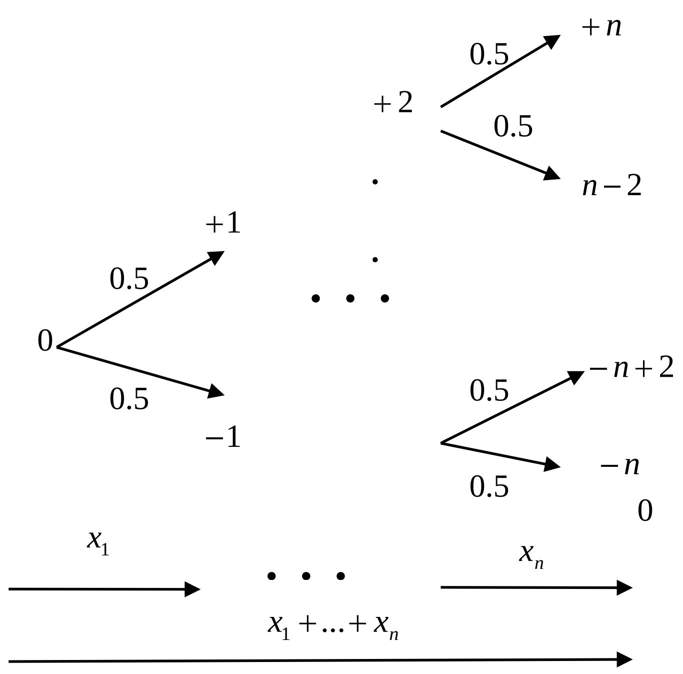
And similarly for $t_1, t_2, t_3, \ldots, t_n$ we can define a new random variable

$$T_n = t_1 + \cdots + t_n$$

The sample space of T_n and X_n is then $\Omega = \{\omega_1\omega_2 \ldots \omega_1\}, \omega_i \in \{H, T\}$

In the general case to calculate the probability of obtaining k heads in n tosses we need to take into account the probability of a head on a single toss which we call p and the number of paths through the decision tree that come to that number of heads. The paths are given by the binomial co-efficients $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and the probability becomes $\binom{n}{k} p^k (1-p)^{n-k}$ The

Figure 9 Paths through the tree for X_n under measure P



Our distribution depends on both the random variable and the probability measure. Under our measure P representing a fairly weighted coin the expectations of our two random variables are given by $E(X_n) = 0$ and $E(T_n) = \frac{n}{2}$

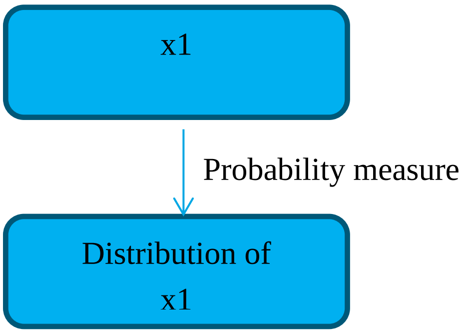
We can intepret X as the distance from the origin if we move one unit in a positive direction whenever we obtain a head and one unit in a negative direction whenever we obtain a tail. This is the ‘random walk’ interpretation.

In general the sum of n independent, identically distributed random variables with mean μ and variance σ^2 is a random variable with mean $n\mu$ and variance $n\sigma^2$ For a proof of why this is the case see the proofs section below.

Details

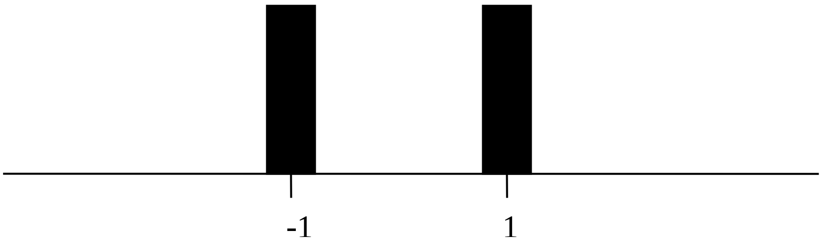
Probability Distribution

We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on Ω whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.



Under the probability measure P defined on Ω either a head or tail are equally likely so our distribution becomes

$$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$$



Expectation

We can define the expectation or expected value of any random variable X under a probability measure P as.

$$E(X) = \sum_{\omega \epsilon} X(\omega)P(\omega)$$

- Weighted average of the values the random variable X can take
- Weighting by the probability of each value
- Measure of centrality

Expectation of Variable Squared

We are often interested in expectation of the square of the variable which we call the mean squared.

$$E({x_1}^2) = [x_1(H)]^2P(H) + [x_1(T)]^2P(T) = 0.5 + 0.5 = 1.0$$

$$\tilde{E}({x_1}^2) = [x_1(H)]^2\tilde{P}(H) + [x_1(T)]^2\tilde{P}(T) = 0.75 + 0.25 = 1.0$$

Variation from expected value

For any actual value of a random variable X we can calculate the difference between that value and the expectation $X(\omega) - E(X)$ .We might ask the question "on average how much does a given value differ from the expected value?" We could calculate the average difference as $\sum_{\omega \in \square}[X(\omega) - E(X)]P(\omega)$ however where the distribution is symmetric around the mean this value will be zero. A more instructive measure is given by calculating the average of the difference squared

$$\sum_{\omega \in \Omega}[X(\omega) - E(X)]^2P(\omega)$$

Under our two probability measures we get

$$[x_1(H) - E(x_1)]^2P(H) + [x_1(T) - E(x_1)]^2P(H) = 0.5 + 0.5 = 1.0$$

$$[x_1(H) - E(x_1)]^2\tilde{P}(H) + [x_1(T) - E(x_1)]^2\tilde{P}(H) = 0.5 + 0.5 = 1.5$$

Expectation of a function of random variable

$$E[g(X)] = \sum_{\omega \epsilon \Omega} g(X(\omega))P(\omega)$$

- The expectation of a function of a random variable is **<u>not equal</u>** to the function of the expectation $E[g(X)] \neq g[E(X)]$

Proofs

Show that $\boldsymbol{E[X+Y]=E[X]+E[Y]}$

If X is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$and Y is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$ then the sample space of X+Y is

$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$

$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$

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$\{x_m, y_1\}, \{x_m, y_2\}, \ldots\ldots, \{x_m, y_n\}$

The expectation of the sum of the two variables in then given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(X(x_i) + Y(y_j)).\,p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n}X(x_i)\,.\,p(x_i, y_j) + \sum_{i=1}^{m}\sum_{j=1}^{n}Y(y_j)\,p(x_i, y_j)$$

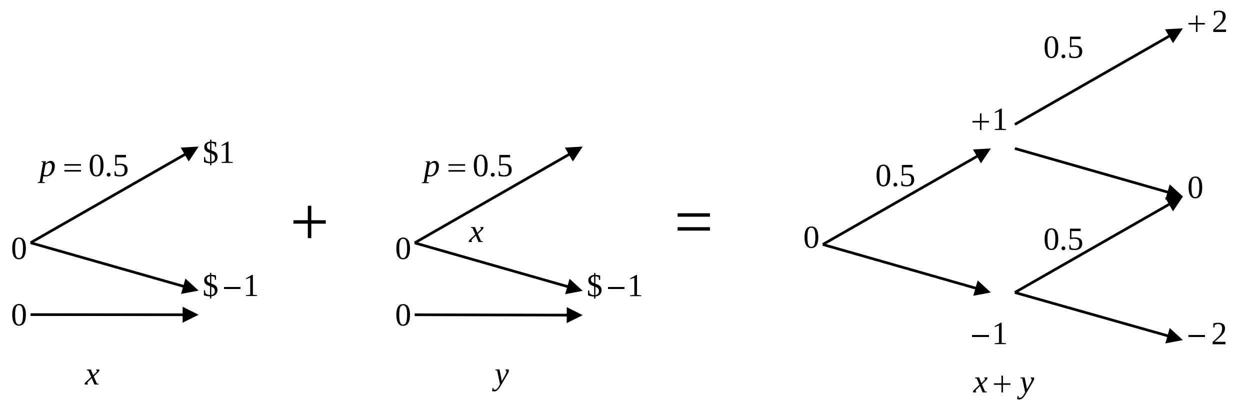
Noting that $\sum_{j=1}^{n}p(x_i\,, y_j) = p(x_i)$ and $\sum_{i=1}^{m}p(x_i\,, y_j) = p(y_j)$

$$\sum_{i=1}^{m}x_i p(x_i) + \sum_{j=1}^{n}x_j p(y_j)$$

Therefore we can note that

$E[X+Y] = E[X] + E[Y]$

The following figure shows the a specific example approach



We have random variable x with sample space $\{x_1, x_2\} = \{1,0\}$ and a second identically distributed random variable $y$with sample space$\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

The expectation of the sum of the variables is then given by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(x_i + y_j)p(x_i, y_j)$$

Multiplying out

$$\sum_{i=1}^{2}\sum_{j=1}^{2}x_i\, p(x_i, y_j) + \sum_{i=1}^{2}\sum_{j=1}^{2}y_j\, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i\,, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i\,, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i p(x_i) + \sum_{j=1}^{n} x_j p(y_j)$$

Therefore we can note that

$$\boldsymbol{E}[\boldsymbol{X} + \boldsymbol{Y}] = \boldsymbol{E}[\boldsymbol{X}] + \boldsymbol{E}[\boldsymbol{Y}]$$

Risk and Pricing Solutions

***Show that the* expectation of the *sum of n iid random variables is* $n.E[X_n]$**

We can calculate the expectation of the sum of n identically distributed random variables denoted by $X_1, X_2, ..., X_n$ as $E[X_1] + E[X_2] + ... + E[X_n]$ which is equal to

$$n.E[X_n]$$

***Show that the* $E[aX + b] = aE[X] + b$**

$E[aX + b] = \sum_{\omega\in\Omega}(aX(\omega) + b)P(\omega)$ From definition 1

$= \sum_{\omega\in\Omega}\big(aX(\omega)\big)P(\omega) + \sum_{\omega\in\Omega} bP(\omega)$ By multiplying out the brackets

$= a\sum_{\omega\in\Omega}\big(X(\omega)\big)P(\omega) + \sum_{\omega\in\Omega} bP(\omega)$ From the properties of summation

$= aE[X] + b\sum_{\omega\in\Omega} P(\omega)$ From definition 1

$= aE[X] + b.1$ From axioms of probability

$$= aE[X] + b$$

***Show that* $Var[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2$**

Let $\mu = E[X]$

$E[(X - \mu)^2] = \sum_{\omega\epsilon\Omega}(X(\omega) - \mu)^2\ P(\omega)$ From definition

$=\sum_{\omega\epsilon\Omega}\left(\big(X(\omega)\big)^2 - 2\mu X(\omega) + \mu^2\right)P(\omega)$ Multiplying out

$=\sum_{\omega\epsilon\Omega}\big(X(\omega)\big)^2P(\omega) + \sum_{\omega\epsilon\Omega} -2\mu X(\omega)P(\omega) + \sum_{\omega\epsilon\Omega}\mu^2P(\omega)$

$= E[X^2] + \sum_{\omega\epsilon\Omega} -2\mu X(\omega)P(\omega) + \sum_{\omega\epsilon\Omega}\mu^2P(\omega)$ From definition 3

$= E[X^2] + -2\mu\sum_{\omega\epsilon\Omega} X(\omega)P(\omega) + \mu^2\sum_{\omega\epsilon\Omega} P(\omega)$ Properties of summations

$= E[X^2] - 2\mu\mu + \mu^2 \sum_{\omega\epsilon\Omega} P(\omega)$

$= E[X^2] - 2\mu\mu + \mu^2$ Axioms of probability

$= E[X^2] - \mu^2$

$= E[X^2] - (E[X])^2$

Show that $Var[aX] = aVar[X]$

$Var[aX] = E[(aX - E[aX])^2]$

$= E[(aX - aE[X])^2]$ From definition 2

$= E[(aX - a\mu)^2]$ Letting $\mu = E[X]$

$= \sum_{\omega\in\Omega}(aX(\omega) - a\mu)^2 \; P(\omega)$ From definition

$= \sum_{\omega\in\Omega} a^2(X(\omega) - \mu)^2 \; P(\omega)$

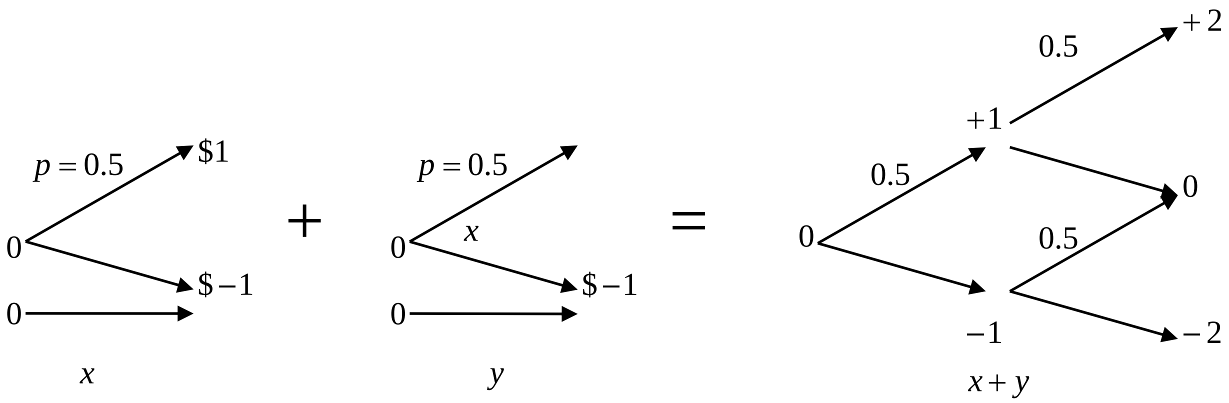
$$= a^2 \sum_{\omega\in\Omega}(X(\omega) - \mu)^2 \; P(\omega)$$

$= a^2 Var[X]$ From definition 4

Risk and Pricing Solutions

Show that $\boldsymbol{Var[x + y] = Var[x] + Var[y]}$

The following diagram shows the general approach.



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$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

Therefore

$$Var[x + y] = E[(x + y)^2] - \{E[x + y]\}^2$$

$$Var[x + y] = E[(x^2 + 2xy + y^2)] - \{E[x] + E[Y]\}^2$$

$$Var[x + y] = E[x^2] + E[y^2] + E[2xy] - \{E[x] + E[Y]\}^2$$

$$Var[x + y] = E[x^2] + E[y^2] + E[2xy] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x + y] = E[x^2] + E[y^2] + 2E[x][y] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x + y] = E[x^2] - E[x]^2 + E[y^2] - E[y]^2$$

$$Var[x + y] = Var[x] + E[y]$$