Derivation of BS PDF

Assume our stock price follows a Wiener process given by $dS = \mu S dt + \sigma S \varepsilon \sqrt{dt}$ then changes in a function of S and t will be given by an Ito process

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

We could consider this the process for a call option struck on S at time t. Furthermore we can set up a portfolio of a short call and long $\frac{\partial f}{\partial S}$ stocks. The change in this portfolio will then be given by

$$d\Pi = -\left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} \left[\mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} \right]$$

$$d\Pi = -\left(\frac{\partial f}{\partial S} \mu S - \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} \sigma S dz$$

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$

We have now eliminated all randomness from our process. It must hence grow at the risk-free rate

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt = \left(-f + \frac{\partial f}{\partial S} S \right) r dt$$

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + rS \frac{\partial f}{\partial S} = rf$$

Risk-Neutral proof of call price

We have shown above that a portfolio of a short call and long delta stocks is riskless over infinitesimal intervals. This means that the portfolio must grow at the risk-free rate.

$$\Pi_t = \Delta_t S_0 - C_t$$

And that its expected value at time T is given as

$$e^{-rT} \left(\Delta_t S_0 - C_t \right)$$

But we can also use the boundary values of Δ and C to calculate the expected value of Π. If the option expires in the money the delta will be one and the portfolio will be worth $S-(S-K)=K$ On the other hand if the stock expired out of the money the delta will be zero and the portfolio will be worth zero so the portfolio expected value at expiry is then pK where p is the probability of the option expiring in the money. If we equate this value with the above we obtain.

$$e^{-rT}\left(\Delta_t S_0 - C_t\right) = pK$$

But we can calculate the probability of expiring in the money from the properties of the Wiener browning motion model of the stock price evolution – see appendix A.

$$e^{-rT}\left(\Delta_t S_0 - C_t\right) = N(d_2)K$$

Re-arranging we get

$$C_t = \Delta_t S_0 - e^{-rT} N(d_2)K$$

But we now need to derive Δ_t

First we note that it can be shown that

$$N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$$

So we take our original expression

$$C_t = \Delta_t S_0 - e^{-rT} N(d_2)K$$

And differentiate with respect to S

$$\frac{\partial C_t}{\partial S_t} = \frac{\partial}{\partial S_t}\left(\Delta_t S_0\right) - e^{-rT} K \frac{\partial}{\partial S_t}\left(N(d_2)\right)$$

By definition

$$\frac{\partial C_t}{\partial S_t} = \Delta_t$$

So we get

$$\Delta_t = \frac{\partial}{\partial S_t}(\Delta_t S_0) - e^{-rT} K \frac{\partial}{\partial S_t}(N(d_2))$$

Expanding the first term on the rhs using the product rule

$$\frac{\partial}{\partial S_t}(\Delta_t S_0) = \left(\frac{\partial}{\partial S_t}\Delta_t \times S_0\right) + \left(\frac{\partial S_t}{\partial S_t} \times \Delta_t\right)$$

Note that

$$\frac{\partial}{\partial S_t}\Delta_t = \frac{\partial}{\partial S_t}\left(\frac{\partial C_t}{\partial S_t}\right) = \frac{\partial^2 C_t}{\partial S_t^{\ 2}} = \Gamma_t$$

So now we have

$$\Delta_t = \Gamma_t S_t + \Delta_t - e^{-rT} K \frac{\partial}{\partial S_t}(N(d_2))$$

Now we expand the third term using the chain rule

$$\frac{\partial}{\partial S_t}(N(d_2)) = \frac{\partial N(d_2)}{\partial d_2} \times \frac{\partial d_2}{\partial S_t} = N'(d_2) \times \frac{\partial d_2}{\partial S_t}$$

But we know that

$$N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$$

So we rewrite the third term

$$-e^{-rT} K \frac{S_T}{K} e^{r(T)} N'(d_1) \times \frac{\partial d_2}{\partial S_t}$$

So we need to calculate

$$\frac{\partial d_2}{\partial S_t} = \frac{1}{S_t \sigma \sqrt{T}}$$

Substituting in

$$-e^{-rT} K \frac{S_T}{K} e^{r(T)} N'(d_1) \times \frac{1}{S_t \sigma \sqrt{T}}$$

Cancelling

$$-\frac{N'(d_1)}{\sigma\sqrt{T}}$$

Putting the whole thing together

$$\Delta_t = \Gamma_t S_t + \Delta_t - \frac{N'(d_1)}{\sigma\sqrt{T}}$$

Re arranging

$$\Gamma_t = \frac{N'(d_1)}{S_t \sigma\sqrt{T}}$$

We can integrate this with respect to S noting first that

$$\frac{\partial N(d_1)}{\partial S_t} = \frac{N'(d_1)}{S_t \sigma\sqrt{T}}$$

So by definition

$$\int \frac{N'(d_1)}{S_t \sigma\sqrt{T}} dS = N(d_1)$$

And of course

$$\int \Gamma_t dS = \Delta_t$$

So we get

$$\Delta_t = N(d_1) + C$$

And to meet the boundary conditions C must of course be zero

So we have now proved in a non-rigorous manner the BS formula for European call on non-dividend paying stock

Show that $d_2 = d_1 - \sigma\sqrt{T-t}$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

We start by separating out the terms on the numerator

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

To get

$$= \frac{\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}} + \frac{rT}{\sigma\sqrt{T}} - \frac{\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

Now note that

$$-\frac{\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} - \frac{\sigma^2 T}{\sigma\sqrt{T}}$$

To get

$$= \frac{\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}} + \frac{rT}{\sigma\sqrt{T}} + \frac{\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} - \frac{\sigma^2 T}{\sigma\sqrt{T}}$$

And finally

$$\frac{\sigma^2 T}{\sigma\sqrt{T}} = \sigma\sqrt{T}$$

Show that $N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$

Show that if the equation $C_{t_0} = \Delta S_{t_0} - B_{t_0}$ then a hedging portfolio of ΔS_{t_0} stocks and $-B_{t_0}$ bond is self-financing

First we note that if $C_{t_0} = \Delta S_{t_0} - B_{t_0}$ is the correct value of a European call we can replicate is by entering into a hedging portfolio of ΔS_{t_0} stocks and $-B_{t_0}$ bond and then continually rebalance it so that at all times we hold ΔS_t stocks and $-B_t$ bonds.

The cost of this hedging strategy is equal to the initial setup costs $\Delta S_{t_0} - B_{t_0} = C_{t_0}$ plus any rebalancing costs. We can say $RC = (\Delta S_{t_0} - B_{t_0}) + RB$

But if the formula is correct the value C_{t_0} must be equal to the NPV of all future cash-flows then there cannot be any other cash flows not accounted for by $\Delta S_{t_0} - B_{t_0}$

Since there are no costs other that the initial setup costs we can see that the hedging portfolio must be self-financing.

From the BS formula for European Call along with Put-Call parity to derive the price of a European put

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

And put-call parity is of course

$$C_t - P_t = S_t - K e^{-rT}$$

Substituting for C we get

$$S_t N(d_1) - e^{-rT} N(d_2) K - P_t = S_t - K e^{-rT}$$

Re-arranging to get P on the LHS

$$P_t = S_t N(d_1) - e^{-rT} N(d_2) K - S_t + K e^{-rT}$$

Bringing terms in S and K together and factorising

$$P_t = S_t (N(d_1) - 1) - K e^{-rT} (N(d_2) - 1)$$

Therefore

$$P_t = -S_t (1 - N(d_1)) + K e^{-rT} (1 - N(d_2))$$

Now we note that

$$1 - N(x) = N(-x)$$

To write the put price as

$$P_t = -S_t N(-d_1) + K e^{-rT} N(-d_2)$$

$$P_t = K e^{-rT} N(-d_2) - S_t N(-d_1)$$

From the BS formula for European Call derive the call delta and then put delta

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Therefore

$$\frac{\partial C_t}{\partial S_t} = \frac{\partial}{\partial S_t}\left(S_t N(d_1)\right) - e^{-rT} K \frac{\partial}{\partial S_t} N(d_2)$$

First we know that

$$\frac{\partial}{\partial S_t} d_1 = \frac{\partial}{\partial S_t} d_2 = \frac{1}{s\sigma\sqrt{T}}$$

And by the chain rule we know that

$$\frac{\partial}{\partial S_t} N(d_1) = \frac{\partial}{\partial d_1} N(d_1) \times \frac{\partial}{\partial S_t} d_2 \text{ and } \frac{\partial}{\partial S_t} N(d_2) = \frac{\partial}{\partial d_2} N(d_2) \times \frac{\partial}{\partial S_t} d_2$$

We also know that by the product rule

$$\frac{\partial}{\partial S_t}\left(S_t N(d_1)\right) = \left(\frac{\partial}{\partial S_t} S_t\right) N(d_1) + \left(\frac{\partial}{\partial S_t} N(d_1)\right) S_t$$

Putting this all together

$$\frac{\partial C_t}{\partial S_t} = \left(\frac{\partial}{\partial S_t} S_t\right) N(d_1) + \left(\frac{\partial}{\partial d_1} N(d_1) \times \frac{\partial}{\partial S_t} d_2\right) S_t - e^{-rT} K \frac{\partial}{\partial d_2} N(d_2) \times \frac{\partial}{\partial S_t} d_2$$

$$= N(d_1) + \left(N'(d_1) \times \frac{1}{s\sigma\sqrt{T}} \right) S_t - e^{-rT} K N'(d_2) \times \frac{1}{s\sigma\sqrt{T}}$$

Finally we use the result that

$$N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$$

To get

$$\frac{\partial C_t}{\partial S_t} = N(d_1) + \left(N'(d_1) \times \frac{1}{s\sigma\sqrt{T}} \right) S_t - e^{-rT} K \frac{S_T}{K} e^{r(T)} N'(d_1) \times \frac{1}{s\sigma\sqrt{T}}$$

Cancelling

$$\frac{\partial C_t}{\partial S_t} = N(d_1) + N'(d_1) \frac{1}{\sigma\sqrt{T}} - N'(d_1) \times \frac{1}{\sigma\sqrt{T}}$$

$$\frac{\partial C_t}{\partial S_t} = N(d_1)$$

And of course we can use put-call parity to show that

$$\frac{\partial P_t}{\partial S_t} = N(d_1) - 1$$

Now use the put/call delta to derive the gamma

$$\frac{\partial C_t}{\partial S_t} = N(d_1)$$

Hence

$$\frac{\partial^2 C_t}{\partial S_t^{\ 2}} = \frac{\partial}{\partial S_t} N(d_1)$$

So by the product rule

$$= \frac{\partial}{\partial d_1} N(d_1) \frac{\partial}{\partial S_t} d_1$$

So we have

$$\frac{\partial^2 C_t}{\partial S_t^{\ 2}} = \frac{N'(d_1)}{S_t \sigma \sqrt{T}}$$

And of course since

$$\frac{\partial P_t}{\partial S_t} = N(d_1) - 1$$

Then of course put gamma is just the same

$$\frac{\partial^2 P_t}{\partial S_t^{\ 2}} = \frac{N'(d_1)}{S_t \sigma \sqrt{T}}$$

From the BS formula for European Call derive the call rho

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

First we note that

$$\frac{\partial}{\partial r} d_1 = \frac{\partial}{\partial r} d_2 = \frac{T}{\sigma\sqrt{T}}$$

$$\frac{\partial C_t}{\partial r} = \frac{\partial}{\partial r} S_t N(d_1) - K \frac{\partial}{\partial r} e^{-rT} N(d_2)$$

Consider the first term

$$\frac{\partial}{\partial r} S_t N(d_1) = \frac{SN'(d_1)T}{\sigma\sqrt{T}}$$

Now the second term

$$-K\left(-Te^{-rT} N(d_2) + \frac{e^{-rT} N'(d_2)T}{\sigma\sqrt{T}}\right)$$

And note that

$$N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$$

So we get

$$-K\left(-Te^{-rT} N(d_2) + \frac{e^{-rT} N'(d_2)T}{\sigma\sqrt{T}}\right) = KTe^{-rT} N(d_2) - \frac{Ke^{-rT}T}{\sigma\sqrt{T}} \frac{S_T}{K} e^{r(T)} N'(d_1)$$

Cancelling terms and adding back in the original first term

$$\frac{SN'(d_1)T}{\sigma\sqrt{T}} + KTe^{-rT}N(d_2) - \frac{TN'(d_1)S}{\sigma\sqrt{T}}$$

Hence

$$\frac{\partial C_t}{\partial r} = KTe^{-rT}N(d_2)$$

From the BS formula for European Call derive the call vega

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

So we get

$$\frac{\partial C_t}{\partial \sigma} = S_t \frac{\partial}{\partial \sigma} N(d_1) - Ke^{-rT} \frac{\partial}{\partial \sigma} N(d_2)$$

Applying the chain rule to each term

$$\frac{\partial C_t}{\partial \sigma} = S_t \frac{\partial}{\partial d_1} N(d_1) \frac{\partial}{\partial \sigma} d_1 - Ke^{-rT} \frac{\partial}{\partial d_2} N(d_2) \frac{\partial}{\partial \sigma} d_2$$

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1) \frac{\partial}{\partial \sigma} d_1 - Ke^{-rT} N'(d_2) \frac{\partial}{\partial \sigma} d_2$$

But we know that

$$N'(d_2) = \frac{S_T}{K} e^{r(T)} N'(d_1)$$

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1) \frac{\partial}{\partial \sigma} d_1 - Ke^{-rT} \frac{S_T}{K} e^{r(T)} N'(d_1) \frac{\partial}{\partial \sigma} d_2$$

Cancelling

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1) \frac{\partial}{\partial \sigma} d_1 - S_t N'(d_1) \frac{\partial}{\partial \sigma} d_2$$

Now we note that

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)$$

So by the product rule

$$\frac{\partial}{\partial\sigma}d_1 = \left(\frac{\partial}{\partial r}\frac{1}{\sigma\sqrt{T}}\right)\left[\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right]+\left(\frac{\partial}{\sigma r}\left(\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)\right)\frac{1}{\sigma\sqrt{T}}$$

Taking each term in turn

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma\sqrt{T}}\right)=\frac{1}{\sigma^2\sqrt{T}}$$

$$\frac{\partial}{\partial\sigma}\left(\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)=\sigma T$$

Putting it together

$$\frac{\partial}{\partial\sigma}d_1 = \frac{1}{\sigma^2\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right]+\sigma T\frac{1}{\sigma\sqrt{T}}$$

$$\frac{\partial}{\partial\sigma}d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma^2\sqrt{T}}+\frac{T}{\sqrt{T}}$$

And similarly

$$\frac{\partial}{\partial\sigma}d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma^2\sqrt{T}}+\frac{T}{\sqrt{T}}$$

Now we put these back into our original

$$\frac{\partial C_t}{\partial\sigma}=S_t N'(d_1)\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma^2\sqrt{T}}+\frac{T}{\sqrt{T}}-S_t N'(d_1)\frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma^2\sqrt{T}}+\frac{T}{\sqrt{T}}$$

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1)\left(\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma^2 \sqrt{T}}\right) + \frac{T}{\sqrt{T}} - \frac{T}{\sqrt{T}} - \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma^2 \sqrt{T}}\right)\right)$$

Cancelling all the terms we get

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1)\sqrt{T}$$

The Vega of a European put is the same as the Vega of a European call.

Why is the vega of a European put equal to Vega of a European call?

From the BS formula for European Call derive the call Theta

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

So we start the differentiation with respect to time

$$C_t = S_t N(d_1) - e^{-rT} N(d_2) K$$

$$\frac{\partial C_t}{\partial T} = S_t \frac{\partial}{\partial T} N(d_1) - K \frac{\partial}{\partial T} e^{-rT} N(d_2)$$

Applying the chain rule and the product rule

$$\frac{\partial C_t}{\partial T} = SN'(d_1)\frac{\partial}{\partial T} d_1 - Ke^{-rT} N'(d_2)\frac{\partial}{\partial T} d_2 + KTe^{-rT} N'(d_2)$$

Extend the Black-Scholes equation for a put and a call to deal with the situation where the underlying stock pays Lumpy Dividends

If we define D as the NPV of all dividends from value time until the option expiry given by

$$D = \sum_{i=1}^{n} d_i e^{-r_{t_i} t_i}$$

Then we calculate

$$S^* = S - D$$

And

$$\sigma^* = \frac{S}{S^*}\sigma$$

The value of the call/put is then just the value of the call and put using the original Black-Scholes formula using S^* as the spot level and σ^* as the volatility

Extend the Black-Scholes equation for a put and a call to deal with the situation where the underlying stock pays a Continuous dividend yield

The value of the call and put is just the value of the call and put using the original Black-Scholes formula and substituting Se^{-qT} for every occurrence of S. where q is the continuous dividend yield from value time until expiry of the option.

Appendix A – Probability of a stock exceeding a given level

Initial derivation How would we go about calculating the probability that a given stock with price S_t at time t will exceed some level K at time T if the instantaneous rate of return is r and the annual volatility is σ? In other words, we need to know the probability that $S_T \geq K$. In order for the terminal stock price to be greater than K then we need

$$\ln\left(\frac{S_T}{S_t}\right) \geq \ln\left(\frac{K}{S_t}\right)$$

The first point of note is that if the stock price process follows a Brownian motion then the returns over some time interval $(T-t)$ will be normally distributed with mean $(r-\sigma^2/2)(T-t)$ and standard deviation $\sigma\sqrt{T-t}$. Furthermore, from the properties of the normal distribution we know that the random variable

$$\frac{\ln\left(\frac{S_T}{S_t}\right) - \left[\left(r - \frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

will be normally distributed having mean zero and standard deviation one. We re-write our original inequality in terms of the new distribution.

$$\ln\left(\frac{S_T}{S_t}\right) \geq \ln\left(\frac{K}{S_t}\right)$$

$$\frac{\ln\left(\frac{S_T}{S_t}\right) - \left[\left(r - \frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}} \geq \frac{\ln\left(\frac{K}{S_t}\right) - \left[\left(r - \frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

The left hand side of this inequality is a normally distributed random variable having mean zero and standard deviation of one. In order to use the standard cumulative normal distribution we require an inequality with a less than sign. We multiply both sides by negative one.

$$\frac{\ln\left(\frac{S_t}{S_T}\right) + \left[\left(r - \frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}} \leq \frac{\ln\left(\frac{S_t}{K}\right) + \left[\left(r - \frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

Re-arrangement

We can re-arrange the right hand side of our inequality as follows.

$$\frac{\ln\left(e^{r(T-t)}\frac{S_t}{K}\right)-(T-t)\left(\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}$$

We simply note that

$$(T-t)r = \log\left(e^{(T-t)r}\right)$$

And

$$(T-t)\left(r-\frac{\sigma^2}{2}\right) = (T-t)r-(T-t)\frac{\sigma^2}{2}$$

$$= \log\left(e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}$$

So re-writing the original write hand side, we get

$$\frac{\ln\left(\frac{S_t}{K}\right)+\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{S_t}{K}\right)+\ln\left(e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{S_t}{K}e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}$$

Interpretation

$$C_t = N(d_1)S_t - e^{-rT}KN(d_2)$$
$$P_t = (N(d_1)-1)S_t + e^{-rT}K(1-N(d_2))$$

Call

$$C = e^{-rT}\left[S_0 N(d_1)e^{rT} - KN(d_2)\right]$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The two terms inside the square brackets can be interpreted as the

- Expected value of the stock price if it exceeds the strike
- The probability the strike will be paid multiplied by the strike

We consider each in turn. Consider the probability that the strike is paid. We know the continuously compounded return is a normally distributed random variable having mean $\left(r - \sigma^2/2\right)T$ and standard deviation $\sigma\sqrt{T}$. In order for the strike to be paid

$$\ln\left(\frac{S_T}{S_0}\right) > \ln\left(\frac{K}{S_0}\right)$$

We need to covert our normally distributed random variable to a random variable with a mean of zero and a standard deviation of one. We note that if a random variable with having mean $\left(r - \sigma^2/2\right)T$ and standard deviation $\sigma\sqrt{T}$ then the random variable

$\dfrac{X - \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}$ is normally distributed with mean zero and standard deviation one. We convert our original inequality so we get

$$\ln\left(\frac{S_T}{S_0}\right) > \ln\left(\frac{K}{S_0}\right)$$

$$\frac{\ln\left(\frac{S_T}{S_0}\right)-\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}} > \frac{\ln\left(\frac{K}{S_0}\right)-\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}$$

The left hand side is now a normally distributed random variably with mean zero and standard deviation of one. In order to use the cumulative normal function however we need a less than or equal to inequality. We multiply the right hand side by minus one.

$$\frac{\ln\left(\frac{S_T}{S_0}\right)-\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}} \leq \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}$$

We now have now derived d_2.

Put

$$P_t = \left(N\left(d_1\right) - 1\right)S_t + e^{-rT} K\left(1 - N\left(d_2\right)\right)$$

We know that the cumulative normal distribution has the property $1 - N(x) = N(-x)$ and that $\left(N\left(d_1\right) - 1\right)S = -S\left(1 - N\left(d_1\right)\right)$

$$P_t = e^{-rT}\left[-N\left(-d_1\right)S_t e^{rT} + e^{-rT} K\left(N\left(-d_2\right)\right)\right]$$

In the same way as the call example, we can interpret the two terms inside the brackets as follows. The first term is the expected value of the stock price if the value is below the strike multiplied by negative one to indicate this is what we pay. The second term is the strike price multiplied by the probability of the terminal stock level finishing below the strike strike.