# Stochatics Processes

## Basic Probability

### Terminology

**Outcome**

Each thing that can occur in the expiriment is called an outcome. In the example of tossing a coin we have two outcomes 'heads' or 'tails' which we can denote by the letters H and T respectively.

**Sample Space**

The set of all possible outcomes of an expiriment is called the sample space. By convention we label it $\Omega$. In our simple coin tossing scenario we would have $\Omega = \{H, T\}$ If we toss two coins our sample space would become $\Omega = \{HH, HT, TH, TT\}$

**Event**

A subset of the probability space is called an event. We define an event using the following notation.

$A = \{\varpi \in \Omega; \varpi = H\}$ "This means the set of all outcomes $\varpi$ such that $\varpi$ is a head".

**Probability Measure**

A probability measure P is a function that assigns to each element $\varpi$ in $\Omega$ a probability such that

$$\sum_{\omega \in \Omega} P(\varpi) = 1$$

Since an event A is a subset of $\Omega$ then the probability of an event is given by

$$P(A) = \sum_{\omega \in A} P(\varpi)$$

**Probability Space**

A probability space $P(\Omega, P)$ consists of a sample space and a probability measure. The sample space is the set of outcomes and the probability measure is a function that assigns to each element $\varpi$ in $\Omega$ a value in $[0,1]$ such that

$$\sum_{\omega \in \Omega} P(\varpi) = 1$$

# Risk and Pricing Solutions

**Random Variable**

A random variable x is a real valued function defined on $\Omega$ . Put another way a random variable maps each outcome from the sample space $\Omega$ to a real number.

## Axioms

**Conditional Probability**

Given two events E and F

$$P(E \mid F) = \frac{P(EF)}{P(F)}$$

**Independent Events**

If $P(E \mid F) = P(E)P(F)$ then the events E and F are said to be independent. The occurent of E does not change the probability of F occuring.

## An example

Consider a game where we toss a fair coin and if it lands heads we receive one dollar and if it lands tails we loose one dollar. We have the following.

$\Omega = \{H, T\}$ (Sample Space)

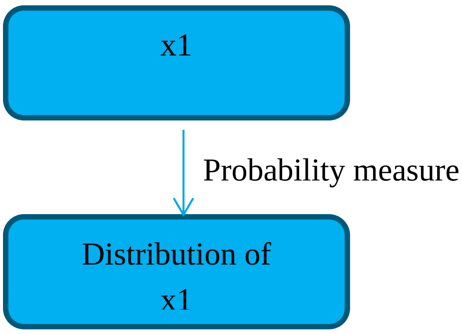
And our probability measure is then given by

$P(H) = P(T) = 0.5$ (Probability Measure)

We define a random variable, $x_1$ on the outcomes that takes the value of plus one dollar if we obtain a head and minus one dollar if we obtain a tail.

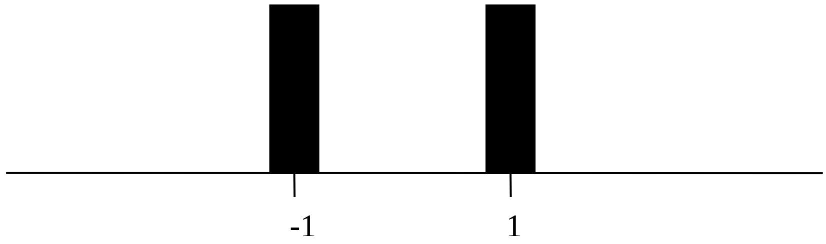
$x_1(H) = 1, x_1(T) = -1$ (Random Variable)

We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on $\Omega$ whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.

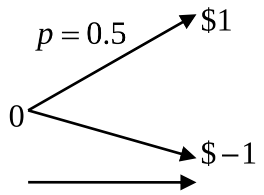



Under the probability measure P defined on $\Omega$ either a head or tail are equally likely so our distribution becomes

$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$




We can then calculate the mean $\mu$ and the variance, $\sigma^2$ of our distribution.




- $\mu = 0$

- $\sigma^2 = 0.5(1-0)^2 + 0.5(-1-0)^2 = 1.0$

## Summing two Identical Independent random variables

We can create a new game by playing the original games twice. We define $x_1$ as the profit/loss from the first coin toss and $x_1$ as the profit/loss from the second toss. We can then define the random variable $X = x_1 + x_2$ giving the total profit or loss at the end of the two tosses.

We obtain the following sample space

$$\Omega = \{\omega_1\omega_2\} = \{HH, HT, TH, TT\}$$

With the following measure

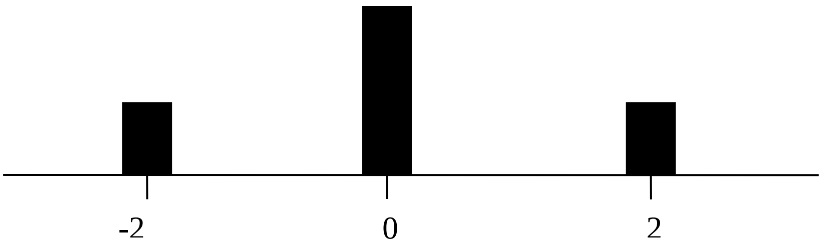
$$P(HH) = P(HT) = P(TH) = P(TT) = 0.25$$

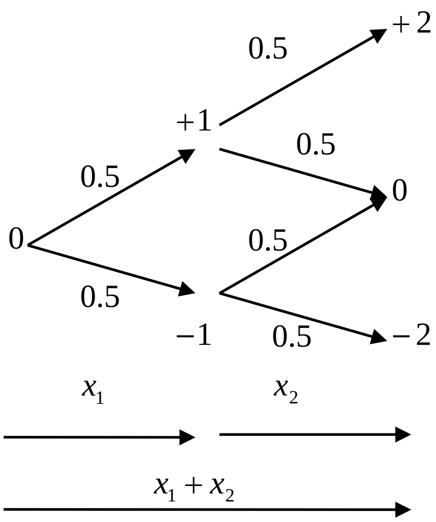
Giving us the random variable

$$X(HH) = 2, X(HT) = (TH) = 0, X(TT) - 2$$

Applying our measure to the random variable we get our probability distribution

$$P(X = 2) = 0.25, P(X = 0) = 0.5, P(X = -2) = 0.25$$



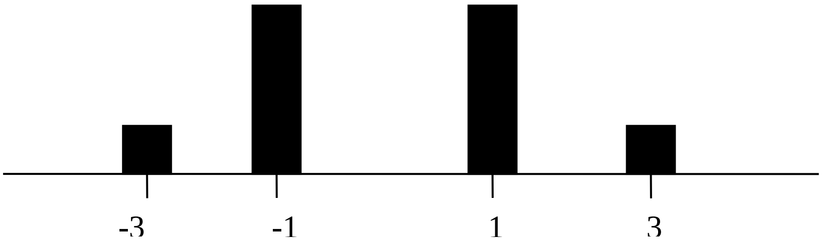



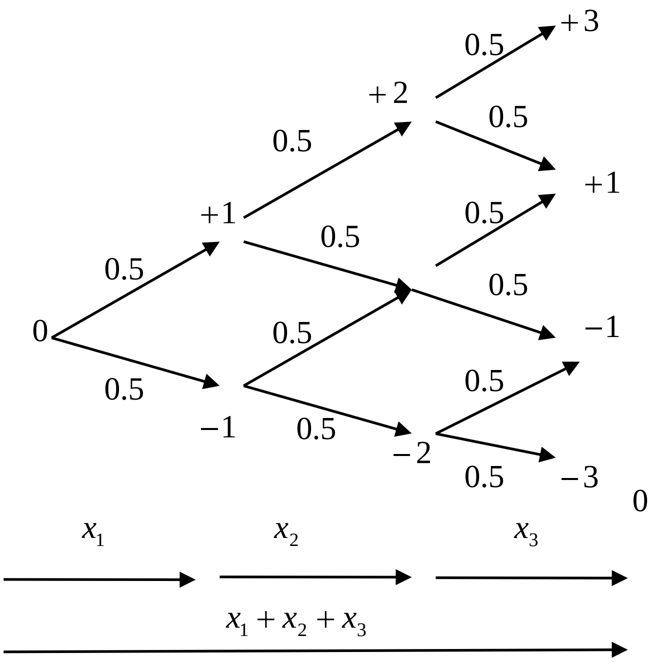

- $\mu = 0$

- $\sigma^2 = 0.25(2-0)^2 + 0.25(-2-0)^2 = 2.0$

## Summing three Identical Independent random variables

Let us go one-step further and look at the event obtained by summing three of the original events. $X_3 = x_1 + x_2 + x_3$ We get the following distribution, whose mean is zero and whose variance is three.



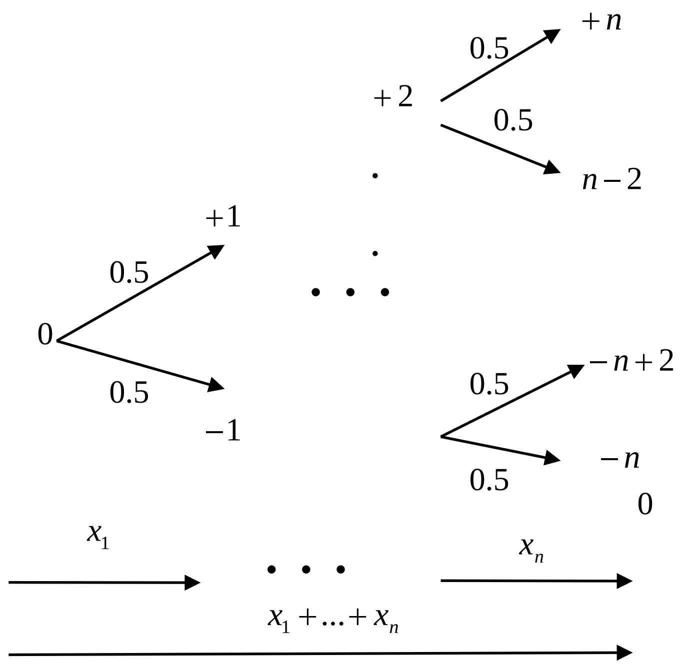




- $\mu = 0$

- $\sigma^2 = 0.125(3-0)^2 + 0.125(-3-0)^2 + 0.375(1-0)^2 + 0.375(-1-0)^2 = 3.0$

## Sum of n identical Independent random variables

Taking this process to its logical conclusion by summing n of our independent, identically distributed random variables we obtain the random variable $X_n = x_1 + ... + x_n$ which is distributed with mean zero and variance n.

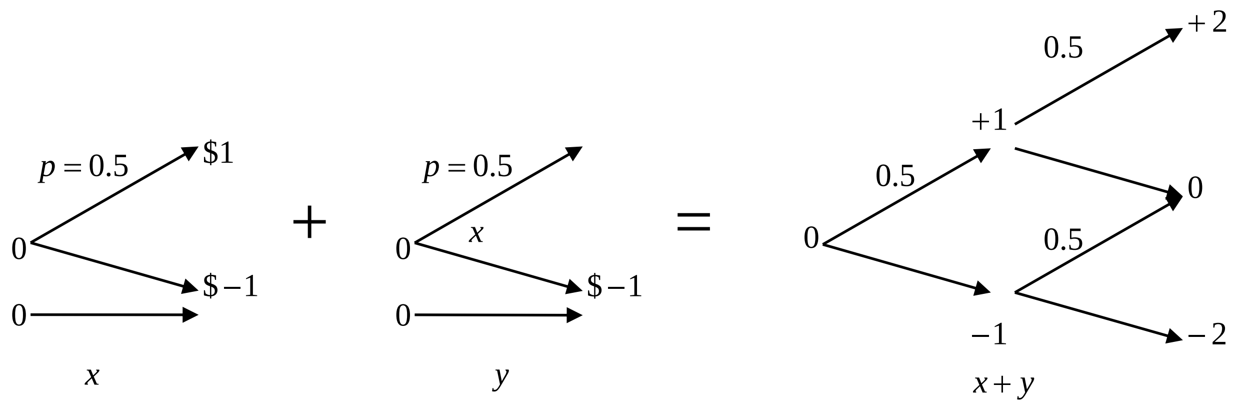



From a proof of why the sum of n independent, identically distributed random variables with mean $\mu$ and variance $\sigma^2$ is a random variable with mean $n\mu$ and variance $n\sigma^2$ see below

## Expectation of sum of I.I.D random variables

$$E[X+X+X+X+........+X]=NE[X]$$

### Highlighting the principle




We have random variable $x$ with sample space $\{x_1, x_2\} = \{1,0\}$ and another identically distributed random variable $y$ with sample space $\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x+y$ is given by the set of pairs

$\{x_1, y_1\}, \{x_1, y_2\}$

$\{x_2, y_1\}, \{x_2, y_2\}$

The expectation of the sum of the variables is then given by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(x_i + y_j)p(x_i, y_j)$$

Multiplying out

$$\sum_{i=1}^{2}\sum_{j=1}^{2} x_i\, p(x_i, y_j) + \sum_{i=1}^{2}\sum_{j=1}^{2} y_j\, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i\, p(x_i) + \sum_{j=1}^{n} x_j\, p(y_j)$$

Therefore we can note that

$$E[X+Y] = E[X] + E[Y]$$

## Generalising

If $x$ is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$ and $y$ is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$ then the sample space of the joint distribution will be given by a set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$$

.

.

.

$$\{x_m, y_1\}, \{x_m, y_2\}, \ldots\ldots, \{x_m, y_n\}$$

The expectation of the sum of the two variables in then given by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i p(x_i, y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i p(x_i) + \sum_{j=1}^{n} x_j p(y_j)$$

Therefore we can note that

$$E[X+Y] = E[X] + E[Y]$$

Furthermore

$$E[X+X] = E[X] + E[X] = 2E[X]$$

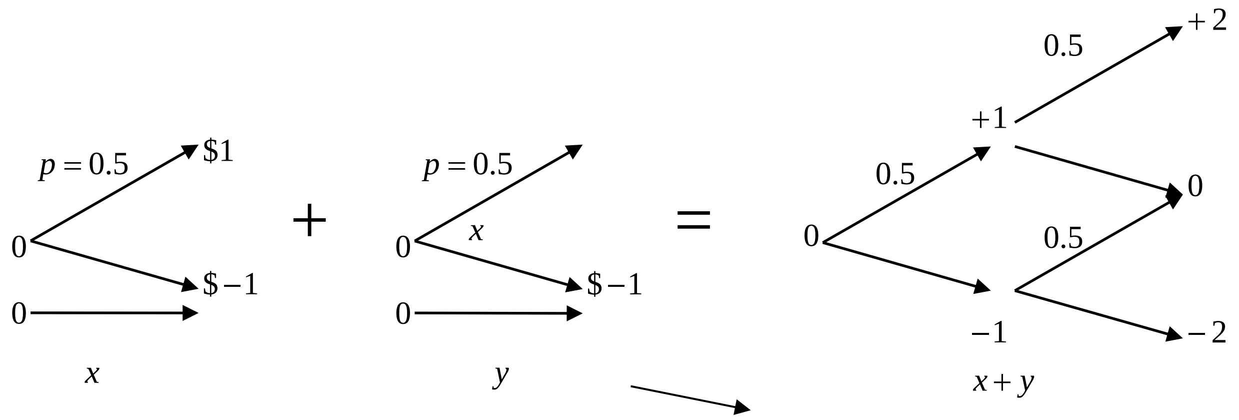
And

$$E[X + X + X + X + \dots\dots + X] = NE[X]$$

## Variance of sum of I.I.D random variables

$Var[x+y] = Var[x] + Var[y]$

### Highlighting the principle




We have random variable $x$ with sample space $\{x_1, x_2\} = \{1,0\}$ and another identically distributed random variable $y$ with sample space $\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

# Risk and Pricing Solutions

$\{x_1, y_1\}, \{x_1, y_2\}$

$\{x_2, y_1\}, \{x_2, y_2\}$

## Proof

$$Var[x+y] = E\left[(x+y)^2\right] - \{E[x+y]\}^2$$

$$Var[x+y] = E\left[\left(x^2 + 2xy + y^2\right)\right] - \{E[x] + E[Y]\}^2$$

$$Var[x+y] = E\left[x^2\right] + E\left[y^2\right] + E[2xy] - \{E[x] + E[Y]\}^2$$

$$Var[x+y] = E\left[x^2\right] + E\left[y^2\right] + E[2xy] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x+y] = E\left[x^2\right] + E\left[y^2\right] + 2E[x][y] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x+y] = E\left[x^2\right] - E[x]^2 + E\left[y^2\right] - E[y]^2$$

$$Var[x+y] = Var[x] + E[y]$$