Random Processes

via coin tossing

Introduction

Imagine a random event that involves the tossing of a single coin. We have two outcomes, heads or tails in our *sample space* Ω

$$\Omega = \{H, T\}.$$

Furthermore, let us define a *random variable* x_1 that takes the value of plus one dollar if we obtain a head and minus one dollar if we obtain a tail.

$$x_1(H) = 1, x_1(T) = -1$$

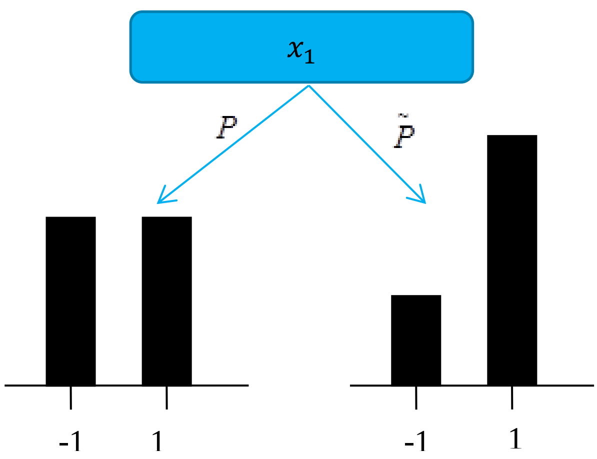
Notice that our random variable does not say anything about the probability of a head or tail. It just tells us what value we assign to the outcomes of the sample space. A probability measure is a real valued function that maps each outcome of the sample space to a probability. If our coin is fair we could have a measure P such that

$$P(H) = 0.5, P(T) = 0.5$$

We might however have a different measure for a loaded coin

$$\tilde{P}(H) = 0.75, \tilde{P}(T) = 0.25$$

Applying a probability measure to a random variable gives us a distribution. The distribution shows the probability of each value of the random variable. Different measures give different distributions.



Expectation or expected value

We can define the expectation or expected value of any random variable X under a probability measure P as.

$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

For our variable x_1 under the measure P we get

$$E(X) = x_1(H)P(H) + x_1(T)P(T) = 0.5 - 0.5 = 0$$

And under the measure $\tilde{P}$

$$\tilde{E}(X) = x_1(H)\tilde{P}(H) + x_1(T)\tilde{P}(T) = 0.75 - 0.25 = 0.5$$

Expectation of the variable squared

We are often interested in expectation of the square of the variable which we call the mean squared.

$$E({x_1}^2) = [x_1(H)]^2 P(H) + [x_1(T)]^2 P(T) = 0.5 + 0.5 = 1.0$$

$$\tilde{E}({x_1}^2) = [x_1(H)]^2 \tilde{P}(H) + [x_1(T)]^2 \tilde{P}(T) = 0.75 + 0.25 = 1.0$$

Variation from the expected value

For any actual value of a random variable X we can calculate the difference between that value and the expectation $X(\omega) - E(X)$ .We might ask the question "on average how much does a given value differ from the expected value?" We could calculate the average difference as $\sum_{\omega \in \Omega} [X(\omega) - E(X)] P(\omega)$ however where the distribution is symmetric around the mean this value will be zero. A more instructive measure is given by calculating the average of the difference squared

$$\sum_{\omega \in \Omega} [X(\omega) - E(X)]^2 P(\omega)$$

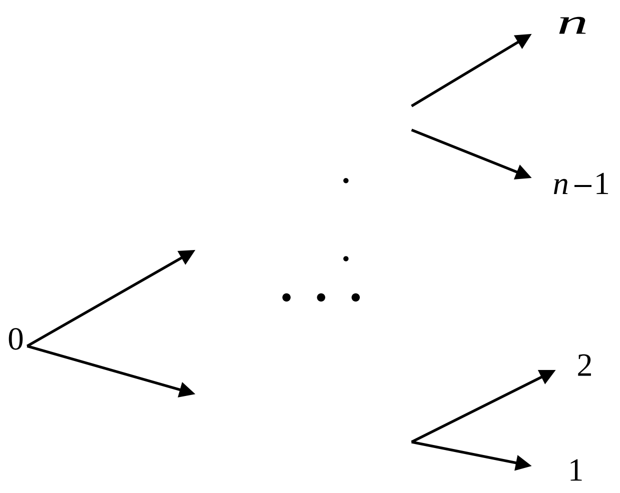
Under our two probability measures we get

$$[x_1(H) - E(x_1)]^2 P(H) + [x_1(T) - E(x_1)]^2 P(H) = 0.5 + 0.5 = 1.0$$

$$[x_1(H) - E(x_1)]^2 \tilde{P}(H) + [x_1(T) - E(x_1)]^2 \tilde{P}(H) = 0.5 + 0.5 = 1.5$$

Summing multipleidentical Independent random variables

What if we toss the coin multiple times? If we toss the coin n times our sample space becomes. $\Omega = \{\omega_1\omega_2 \ldots \omega_1\}, \omega_i \in \{H, T\}$ We can define a random variable N_H that counts the number of heads. Such a random variable can take any value between 0 and n. What is the probability that in n tosses we will obtain k heads? Consider the following decision tree.



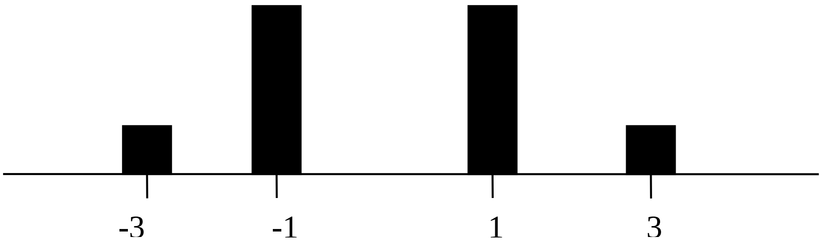
To calculate the probability of obtaining k heads in n tosses we need to take into account the probability of a head on a single toss which we call p and the number of paths through the tree that come to that number of heads. The paths are given by the binomial co-efficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

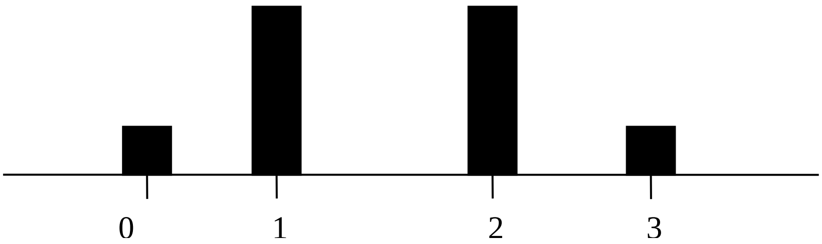
and the probability becomes

$$\binom{n}{k} p^k (1-p)^{n-k}$$

If we let N_T be the number of tails we have another random variable defined on the sample space. Clearly in all cases $n = N_H + N_T$ .We can define another random variable D that counts the number of heads minus the number of tails $D_n = N_H - N_T$ .Our distribution depends on both the random variable and the probability measure. In the case where n is equal to three, then under the measure P our distribution of D_n becomes.



Under the same measure the distribution of N_H becomes



We can intepret D as the distance from the origin if we move one unit in a positive direction whenever we obtain a head and one unit in a negative direction whenever we obtain a tail. This is the 'random walk' interpretation. What is the expectation of our random variables $D_n$and N_H ? If our coin is fairly weighted then $E(D_n) = 0$ and $E(N_H) = \frac{n}{2}$

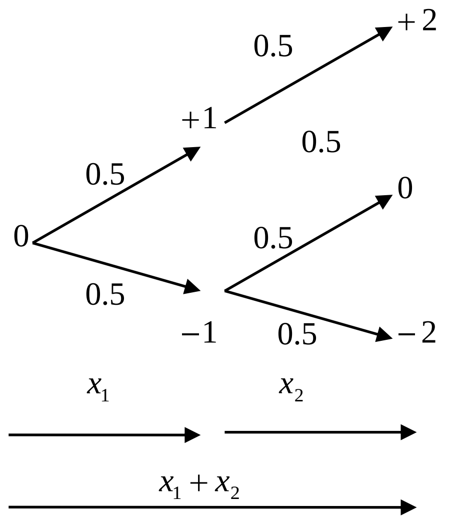
We can create a new game by playing the original games multiple times. If we play the original game twice then our new game effectively involves tossing the coin twice and our sample space becomes$\Omega = \{HH, HT, TH, TT\}$. We can define a new random variable X as the sum of two identical independent random variables

$$X_2 = x_1 + x_2.$$

$$X_2(HH) = 2, X_2(HT) = 0, X_2(TH) = 0, X_2(TT) = -2$$

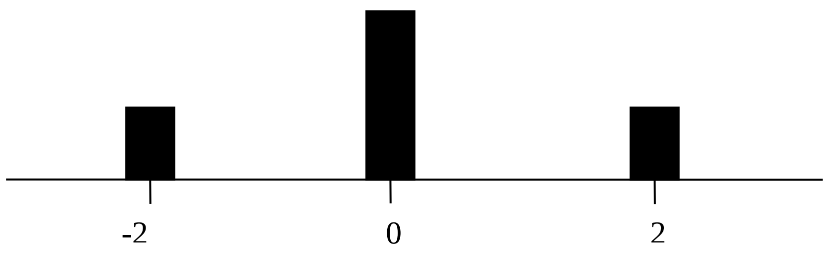
If we use the original measure P we get

$$P(HH) = 0.5^2, P(HT) = 0.5^2, P(TH) = 0.5^2, P(TT) = 0.5^2$$



And our distribution becomes

$$P(X = 2) = 0.25, P(X = 0) = 0.5, P(X = -2) = 0.25$$



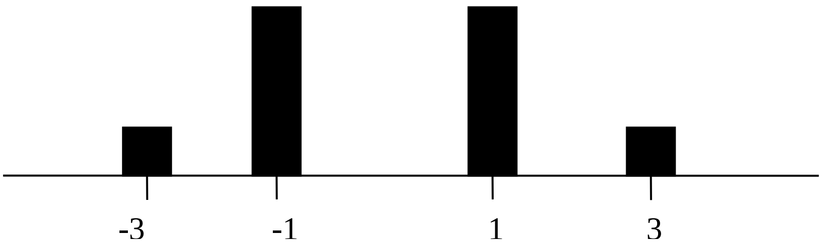
By setting up our random variable in this way X_2 is actually measuring the number of heads less the number of tails. The reason for the jump of 2 between the possible values is that if we go from 1 head to 2 heads then the number of tails decreases from one tail to zero tails and the value $N_H - N_T$ increases by two.

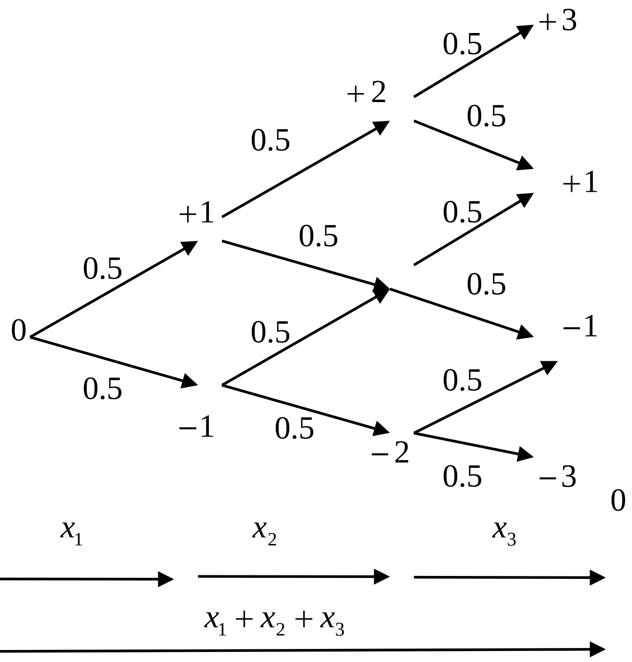
If we then perform n identical tosses of the coin and define n identical random variables $x_1, x_2, x_3, \ldots, x_n$ each will also have mean zero, and variance, σ^2 of one. hen the distribution of our profit and loss is as follows. The mean of the distribution is zero and the variance is two.

- $\mu = 0$
- $\sigma^2 = 0.25(2-0)^2 + 0.25(-2-0)^2 = 2.0$

Summing three Identical Independent random variables

Let us go one-step further and look at the event obtained by summing three of the original events. $X_3 = x_1 + x_2 + x_3$ We get the following distribution, whose mean is zero and whose variance is three.



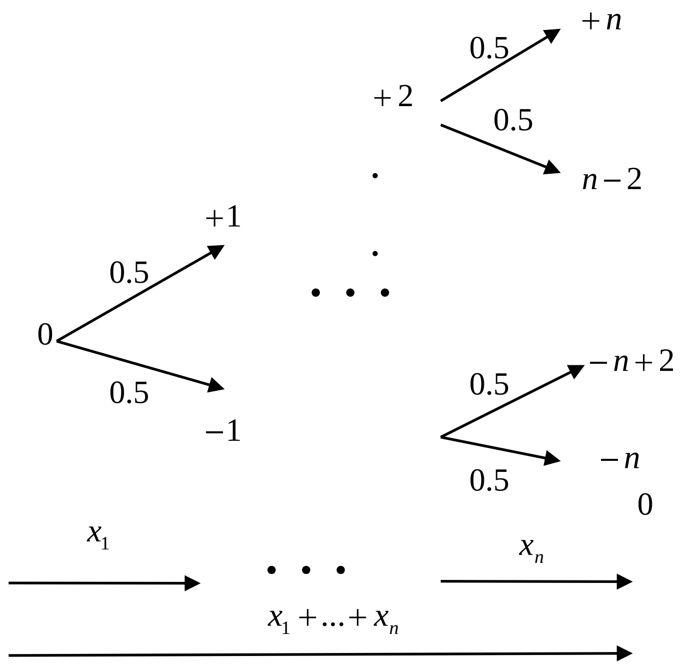


- $\mu = 0$

- $\sigma^2 = 0.125(3-0)^2 + 0.125(-3-0)^2 + 0.375(1-0)^2 + 0.375(-1-0)^2 = 3.0$

Sum of n identical Independent random variables

Taking this process to its logical conclusion by summing n of our independent, identically distributed random variables we obtain the random variable $X_n = x_1 + \ldots + x_n$ which is distributed with mean zero and variance n.

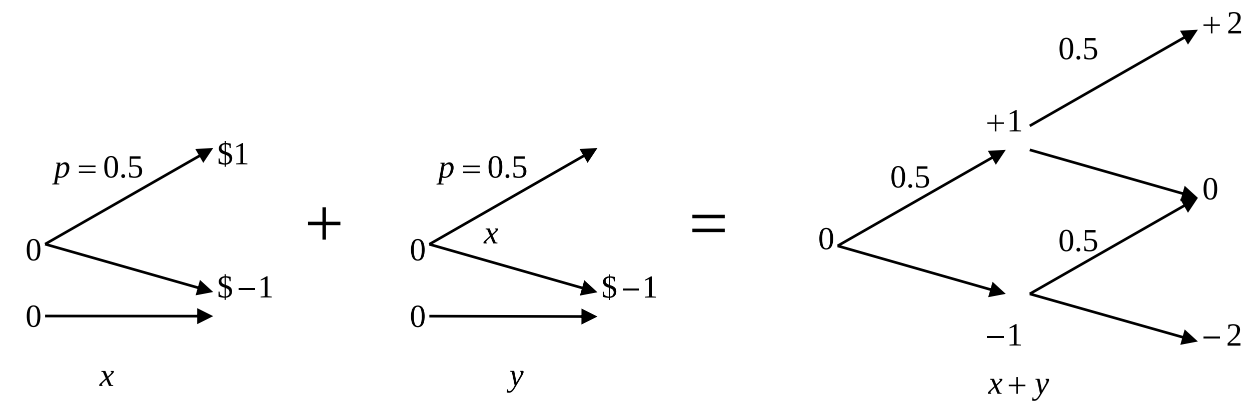


From a proof of why the sum of n independent, identically distributed random variables with mean μ and variance σ^2 is a random variable with mean $n\mu$ and variance $n\sigma^2$ see below

Expectation of sum of I.I.D random variables

$$E[X+X+X+X+........+X]=NE[X]$$

Highlighting the principle



We have random variable x with sample space $\{x_1, x_2\} = \{1,0\}$ and another identically distributed random variable y with sample space $\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x+y$ is given by the set of pairs

$\{x_1, y_1\}, \{x_1, y_2\}$

$\{x_2, y_1\}, \{x_2, y_2\}$

The expectation of the sum of the variables is then given by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(x_i + y_j)p(x_i, y_j)$$

Multiplying out

$$\sum_{i=1}^{2}\sum_{j=1}^{2} x_i\, p(x_i, y_j) + \sum_{i=1}^{2}\sum_{j=1}^{2} y_j\, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i\, p(x_i) + \sum_{j=1}^{n} x_j\, p(y_j)$$

Therefore we can note that

$$E[X + Y] = E[X] + E[Y]$$

Generalising

If x is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$ and y is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$ then the sample space of the joint distribution will be given by a set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$$

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$$\{x_m, y_1\},\{x_m, y_2\},.....,\{x_m, y_n\}$$

The expectation of the sum of the two variables in then given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + y_j)p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n}x_i p(x_i, y_j) + \sum_{i=1}^{m}\sum_{j=1}^{n}y_j p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^{m}x_i p(x_i) + \sum_{j=1}^{n}x_j p(y_j)$$

Therefore we can note that

$$E[X+Y] = E[X] + E[Y]$$

Furthermore

$$E[X+X] = E[X] + E[X] = 2E\left[X\right]$$

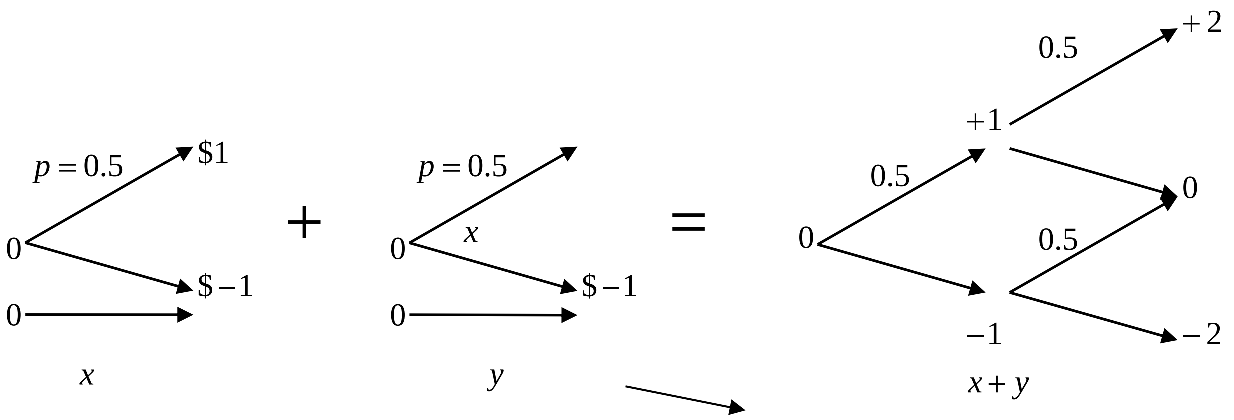
And

$$E[X + X + X + X + \dots\dots + X] = NE[X]$$

Variance of sum of I.I.D random variables

$Var[x+y] = Var[x] + Var[y]$

Highlighting the principle



We have random variable x with sample space $\{x_1, x_2\} = \{1,0\}$ and another identically distributed random variable y with sample space $\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

Risk and Pricing Solutions

$\{x_1, y_1\}, \{x_1, y_2\}$

$\{x_2, y_1\}, \{x_2, y_2\}$

Proof

$$Var[x+y] = E[(x+y)^2] - \{E[x+y]\}^2$$

$$Var[x+y] = E[(x^2+2xy+y^2)] - \{E[x]+E[Y]\}^2$$

$$Var[x+y] = E[x^2]+E[y^2]+E[2xy] - \{E[x]+E[Y]\}^2$$

$$Var[x+y] = E[x^2]+E[y^2]+E[2xy] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x+y] = E[x^2]+E[y^2]+2E[x][y] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$Var[x+y] = E[x^2] - E[x]^2 + E[y^2] - E[y]^2$$

$$Var[x+y] = Var[x] + E[y]$$