Differential Equations

Introduction

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A first order ODE is an equation involving only a function y, its first derivative and any given functions of the independent variable y

$F(x, y, y')$=0

$y' = f(x, y)$

A function $h(x)$is called a solution of a given ODE on some open interval $a < x < b$ if h(x) is defined and differentiable throughout the interval and is such that the ODE becomes an identity when y is replaced by h and y' is replaced by h'

Solvable by integration (I)

Where the first order ODE is off the form

$y' = g(x)$

Or the equivalent implicit form

$y' - g(x) = 0$

Then we can simply solve by integration

$$\int y' dx = \int g(x)\, dx$$

$$y = \int g(x)\, dx$$

Risk and Pricing Solutions

Consider the example

$y' - 3t^2 - 2t = 0$

Then integrating gives us

$y = t^3 - t^2$

Solvable by integration (II)

We can also first order ODE's of the following form where a and b are constants

$$y' = ay + b$$

Or the equivalent implicit form

$$y' + ay + b = 0$$

We re-arrange to get

$$y' = a\left(y + \frac{b}{a}\right)$$

$$\frac{1}{y + \frac{b}{a}} y' = a$$

Integrating both sides with respect to x

$$\int \frac{1}{y + \frac{b}{a}} y' dx = \int a dx$$

$$\int \frac{1}{y + \frac{b}{a}} dy = ax + c$$

$$ln\left|y + \frac{b}{a}\right| = ax + c$$

$$y = -\frac{b}{a} + e^{ax}e^{c}$$

Integrating Factors

We will use the first order ODE

$y' + 2y = 4$

To highlight a method that can be used to solve equations which do not easily lend themselves to integration (Note this simple example could of course be solved using the technique described in the previous section)

The general idea is we multiply the ordinary differential equation by a function $r(x)$such that the left hand side of the equation is equal to the derivative of the product $r(x)y$ If we find such a function we can easily integrate the lhs to get rid of the y' term , obtain $r(x)y$and solve

We want to find a function r(x)

(i) $r(x)y' + r(x)2y = 4r(x)$

Note now that the left hand side looks like the derivative of the product ry. Remembering from the product rule of differentiation that

(ii) $(ry)' = ry' + r'y$

To ensure that the left hand side of (i) is indeed the derivative of the product ry we need to equate the left hand side of (i) with the right hand side of (ii)

$r(x)y' + r(x)2y = ry' + r'y$

Solving for r(x)

$r(x)2y = r'y$

$\frac{1}{r}r' = 2$

Integrating with respect to x

$\int \frac{1}{r}r'\,dx = \int 2dx$

$\int \frac{1}{r}dr = 2x + c$

$r = e^{2x+c} = ce^{2x}$

Let's make use of our new integrating factor by inserting it into the original ODE

$$y'ce^{2x} + 2yce^{2x} = 4ce^{2x}$$

We can simplify by dividing through by c

$$e^{2x}y' + 2e^{2x}y = 4e^{2x}$$

And now of course the left hand side is equal to the derivative of the product $e^{2t}y$ so we can integrate

$$\int e^{2x}y' + 2e^{2x}y\,dx = \int 4e^{2x}dx$$

$$e^{2x}y = 2e^{2x} + c$$

$$y = \frac{2e^{2x} + c}{e^{2x}}$$

$$y = 2 + ce^{-2x}$$

You can if you wish verify this solution by using the method in the previous section

General Solution of Linear First Order Differential Equations

We can use the same technique of integrating factors to find the general form of linear first order ordinary differential equations

$$y' + p(x)y = g(x)$$

As before we want to find an integrating factor r(x) such that the left hand side of the linear ODE becomes the derivative of the product r(x)y given by $(r(x)y)' = y'r(x) + yr'(x)$

Essentially we need

$$y'r + yr' = y'r + pyr$$

$$yr' = pyr$$

$$r' = pr$$

$$\frac{1}{r}r' = p$$

$$log|r| = \int p\,dx$$

$$r = e^{\int pdx}$$

Inserting back into the original equation we obtain

$$y'e^{\int pdx} + p(x)ye^{\int pdx} = g(x)e^{\int pdx}$$

$\frac{d}{dx}[ye^{\int pdx}] = g(x)e^{\int pdx}dx$+c

$$ye^{\int pdx} = \int g(x)e^{\int pdx}\,dx + x$$

$$y = e^{-\int pdx}\left[\int g(x)e^{\int pdx} + c\right]$$

Integrating Factors involving a function

We can use the same technique of integrating factors where we have a function of the form

$y' + ay = g(x)$

Or implicitly

$y' + ay - g(x) = 0$

We find a function r(x) such that $\left(yr(x)\right)' = y'r + yr'$

$y'r(x) + r(x)ay = g(x)r(x)$

Equating the right left side of the ode with the right side of the derivative of the product we get

$r(x)ay = \ yr$' Hence $\frac{1}{r(x)}r' = a$

$\mathrm{r} = \mathrm{ce}^{\mathrm{ax}}$

So we can use the integrating factor e^{ax} remembering that the c drops out. Inserting the factor into the original equation we obtain

$y'\mathrm{e}^{\mathrm{ax}} + ya\mathrm{e}^{\mathrm{ax}} = g(x)\mathrm{e}^{\mathrm{ax}}$

Integrating with respect to x we obtain

$\mathrm{ye}^{\mathrm{ax}} = \int \mathrm{e}^{\mathrm{ax}}\mathrm{g(x)}\,\mathrm{dx}$+c

Therefore

$$\mathrm{y} = \mathrm{e}^{-\mathrm{ax}} \int \mathrm{e}^{\mathrm{ax}}\mathrm{g(x)}\,\mathrm{dx} + c\mathrm{e}^{-\mathrm{ax}}$$

Of course at this stage a solution depends on being able to calculate the integral on the r.h.s

A short cut

The integrating factor for an ODE of the form $y' + ay = g(x)$is always given by $r(x) = e^{\int adx}$

Separable ODE's

ODE's of the form

$$g(y)y' = f(x)$$

Can be directly solved by integration by noting that

$$\int g(y)\, y' dx = \int f(x) dx$$

$$\int g(y)\, dy = \int f(x) dx$$

Let's look at some examples

Risk and Pricing Solutions

Example 1 Radiocarbon dating

Radioactive decay is governed by the ODE $y' = ky$or the rate of change is proportional to the amount of the substance already present. We can solve for the general solution of this ODE using separation of variables as follows

$$y' = ky$$

Re-arranging

$$\frac{1}{y}y' = k$$

Integrating with respect to x

$$\int \frac{1}{y}y' dx = \int k dx$$

Noting that $y' dx = \frac{dy}{dx}dx = dy$ we get

$$log|y| = kx + C$$

Taking exponents of both sides

$$y = e^{kx+C} = e^{kx}e^{c} = Ae^{kx}$$

If we are told the half life of a particular substance is 5715 years we can find an exact solution by noting

$$Ae^{5715k} = 0.5A$$

Dividing through by A and taking logarithms

$$5715k = log0.5, k = \frac{log0.5}{5715}$$

$$y = Ae^{-00001213x}$$