Bit Questions

BASIC BIT LEVEL

Bits

Properties

What is the result of a ^ 0s?

```
a              00001101
0s             00000000
-----------------------
a ^ 0s         00001101
```

The result is a

What is the result of a ^ 1s?

```
a              00001101
1s             11111111
-----------------------
a ^ 1s         11110010
```

The result is ~a

What is the result of a^a?

```
a              00001101
0s             00000000
-----------------------
a & 0s         00000000
```

The result is 0s

What is the result of a&0s?

```
a              00001101
0s             00000000
-----------------------
a & 0s         00000000
```

The result is 0s

What is the result of a&1s?

```
a              00001101
1s             11111111
-----------------------
a & 1s         00001101
```

The result is a

What is the result of a | a?

```
a              00001101
a              00001101
-----------------------
a | 1s         00001101
```

The result is a

What is the result of a ^ a ?

```
a       00001101
~a      11110010
a^~a    11111111
```

The resuls is 1s

Perform bitwise negation without using the ~ operator?

a ^ 1s = a ^ ~0

Manipulation

Implement this function to return a mask of all 0s except a single 1 in bit location i

The following shows how this works with i=3

```
idx           76543210
1             00000001
1 << 3        00001000
```

```
public static sbyte MaskOne(int i) => (sbyte)(1 << i);
```

Implement this function to return a mask of all 1s except a single 0 in bit location i

The following shows how this works with i=3

```
idx           76543210
1             00000001
1 << 3        00001000
~(1 << 3)     11110111
```

```
public static sbyte MaskTwo(int i) => (sbyte)(~(1 << i));
```

Implement this function to return a mask of all ones except for zeros in the i least significant bits from 0 to (i-1)

The following shows how this works with i=3

```
idx           76543210
~0            11111111
~0 << 3       11111000
```

```
public static sbyte MaskThree(int n) => (sbyte)(~0 << n);
```

Implement this function to return a mask of all zeroes except for ones in the n least significant bits

The following shows how this works with i=3

```
idx           76543210
1 << 3        00001000
(1 << 3)-1    00000111
```

```
public static sbyte MaskFour(int n) => (sbyte)((1 << n)-1);
```

Implement this function to return a mask of all 0s except for digits i through j which

The following shows how this works with i=3, j=6

```
idx                       76543210
(1 << 6-3+1)              00010000
(1 << 6-3+1)-1            00001111
((1 << 6-3)-1)<< 3        01111000
```

```
public static sbyte MaskFive(int i, int j) =>
  (sbyte)(((1 << j-i+1) - 1) << i);
```

Implement this function to return a mask of all 1s except for digits i through j which contain 0s.

The following shows how this works with i=3, j=6

```
idx                       76543210
(1 << 6-3+1)              00010000
(1 << 6-3+1)-1            00001111
((1 << 6-3)-1)<< 3        01111000
~(((1 << 6-3)-1)<< 3)     10000111
```

```
public static sbyte MaskSix(int i, int j)
      => (sbyte)~(((1 << j - i+1) - 1) << i);
```

Implement integer subtraction without using the – key.

We make use of the properties of twos complement

$$a - b = a + (2^n - b) = a + (b + {\sim}b + 1 - b) = a + {\sim}b + 1$$

```
public static sbyte Subtract(sbyte a, sbyte b) =>
(sbyte)(a + ~b +1);
```

Getting/Setting bits

Write a function that returns true or false, reflecting whether or not the bit at index i is 1 or 0 respectively

Consider the specific case where n = 5 and i=2

```
idx                  76543210
n                    00000101
n >> 2               00000001
1                    00000001
(n >> 2)&1           00000001
((n >> 2)&1)>0       true
```

```
public bool GetBit(int n, int i) => ((n >> i) & 1) > 0;
```

Write a function set the bit at index i to 1

```
public int SetBit(int n, int i) => (1 << i) | n;
```

Write a function set the bit at index i to 0

```
public int ClearBit(int n, int i) => ~(1 << i ) & n;
```

Write a function that clears all bits from msb through to i inclusive

```
public int ClearFromMsbToI(int n, int i)
{
  return ((1 << i-1 )-1) & n;
}
```

Write a function that sets all bits from msb through to i inclusive

```
public int SetFromMsbToI(int n, int i)
{
  return (~0 << i) | n;
}
```

Write a function that clears all bits from 0 through to i inclusive

```
public int ClearFromLsbToI(int n, int i)
{
  return (~0 << i+1) & n;
}
```

Write a function that sets all bits from 0 through to i inclusive

```
public int SetFromLsbToI(int n, int i)
{
  return ((1 << i+1)-1) | n;
}
```

Brain Teasers

Write code to find the minimum of two signed integers. You may not use Math.min or branching constructs.

Consider the case where we have two signed 8 bit integers a and b. If we take their difference (a-b) then the result can be classified as

- *0xxxxxxxx* *If a >= b*
- *1xxxxxxxx* *If a < b*

If we perform a right arithmetic shift of 7 bits (sizeo the int -1) we get either

- *00000000* *If a >= b*
- *11111111* *If a < b*

Now if we & the result of this shift with the original difference. ((a-b) >> 7) & a-b

- *0* *If a >= b*
- *a-b* *If a < b*

Now we add in b

- *0+b=b* *If a >= b*
- *a-b+b =a* *If a < b*

So we have returned b if a>=b and a if a < b which was the original aim

```
public sbyte Min(sbyte a, sbyte b)
{
    // Take the differnce a-b. The result is one of two forms
    // a) 0xxxxxxx if a >= b
    // b) 1xxxxxxx if a < b
    sbyte difference = (sbyte)(a-b);

    // The result of the right shift is then one of two things
    // a) 00000000 if a >= b
    // b) 11111111 if a < b
    sbyte mask = (sbyte)(difference >> (sizeof(sbyte)*8-1));

    // Now if we & the mask and (a-b) we get one of of two things
    // a) 00000000 if a >= b
    // b) a-b      if a < b
    sbyte temp = (sbyte)(mask & difference);

    // If we add b to this temp variable we get one of two things which
    // is what we wanted
    // a) 0+b=b     if a >= b
    // b) a-b+b=a   if a < b
    return (sbyte)(temp + b);
}
```

Write code to find the maximum of two signed integers. You may not use Math.min or branching constructs.

This is the same as the previous code except for we take the completment of the shift.

```
public sbyte Max(sbyte a, sbyte b)
{
    // Take the differnce a-b. The result is one of two forms
    // a) 0xxxxxxx if a >= b
    // b) 1xxxxxxx if a < b
    sbyte difference = (sbyte)(a-b);

    // The result of the complemented right shift is
    //  then one of two things
    // a) 11111111 if a >= b
    // b) 00000000 if a < b
    sbyte mask = (sbyte)~(difference >> (sizeof(sbyte)*8-1));

    // Now if we & the mask and (a-b) we get one of of two things
    // a) a-b      if a >= b
    // b) 0        if a < b
    sbyte temp = (sbyte)(mask & difference);

    // If we add b to this temp variable we get one of two things which
    // is what we wanted
    // a) a-b+b=a    if a >= b
    // b) 0+b=b      if a < b
    sbyte result = (sbyte)(temp + b);
    return result;
}
```

Write a function to check if a given unsigned integer is a powe of 2

We make use of the fact the binary representation of any power of 2 is a single 1 followed by all zeros

$2^0 = 1$	00000001
$2^1 = 2$	00000010
$2^2 = 4$	00000100

Secondly we note that subtractive 1 from such a representation flips the single 1 to zero and changes all zeros following it to 1s

$2^0 - 1$	00000000
$2^1 - 1 = 1$	00000001
$2^2 - 1 = 3$	00000011

Finally we use the fact that anding the two forms gives a result of zero.]

2^0	00000001
$2^0 - 1$	00000000
$2^0 \wedge (2^0 - 1)$	00000000
2^1	
$2^1 - 1$	00000010
$2^1 \wedge (2^1 - 1)$	00000001
	00000000
2^2	00000100
$2^2 - 1$	00000011
$2^2 \wedge (2^2 - 1)$	00000000

The code is given as follows. Note the special case for zero which is not a power of 2

```
public bool IsPowerOfTwo(uint a)
{
  return (a != 0) && (a & (a-1)) == 0;
}
```

Write a function to calculate the number of 1s in an integers binary representation

*We can use a simple linear traversal of the integers bits counting the 1s as we go. Such as algorith is constant time and always takes O(sizeof(int)*8).*

```
public int BitCount(int a)
{
    int numBits = sizeof(int) * 8;
    int bitCount = 0;

    for (int i=0; i < numBits;i++)
    {
        if (((a >> i) & 1) > 0)
            bitCount++;
    }

    return bitCount;
}
```

But actually we can do better. The following describes a clever aglorithm invented by Brian Kernigan. It key idea is that if we siubtract 1 from any integer then the result is that ever bit from the lsb up to a and including the least significant 1 is flipped. If we then perform an & operation we are effectively removing the least signigicant 1.

5	00000101
$5-1$	00000100
$5 \wedge (5-1)$	00000100
2^1	00011100
2^1-1	00000001
$2^1 \wedge (2^1-1)$	00000000
2^2	00000100
2^2-1	00000011
$2^2 \wedge (2^2-1)$	00000000

but we can go better and use Brian Kernigans algortithm. This makes use of the fact that when we subtract 1 from an integer the rightmost 1 bit and all bits following the rightmost bits are flipped. If we hence subtract 1 from a number and and the result we effectively remove the rightmost bit. If we keep doing this until the number becomes zero we count the number of 1s

Given an integer find the next largest integer that has the same number of 1s in its binary representation.

Given an integer find the next smallest integer that has the same number of 1s in its binary representation.

```
a                     0000000000000011
b                     0000000000000101
a-b                   1111111111111110
a-b >> 15             1111111111111111
(a-b >> 15) & a       1111111111111110
((a-b >> 15) & a)+b   0000000000000011

a                     0000000000000101
b                     0000000000000011
a-b                   0000000000000010
a-b >> 15             0000000000000000
(a-b >> 15) & a       0000000000000000
((a-b >> 15) & a)+b   0000000000000011
```

Integers

Write a function to add two signed bytes. Do not use the + operator?

```
public uint Add(uint a, uint b)
{
	uint carry = 0;
	uint result = 0;

	for (int bitIdx = 0; bitIdx < SizeInBits; bitIdx++)
	{
		// We deal in one binary digit at multiplicand time. By right
		// shifting multiplicand and bitIdx times we set the bit we want
		// into the least significant bit.
		uint aShifted = (a >> bitIdx);
		uint bShifted = (b >> bitIdx);

		// Now we make use of the fact that the number 1 has
		// in our unsigned representation consists of SizeInBits
		// zeros followed by multiplicand solitary one in the
		//least significant
		// position. We can hence take our shifted valued and logically
		// and them with 1 to ensure we only have the digit values in the
		// least significant locations remaining.
		uint aDigit = aShifted & 1;
		uint bDigit = bShifted & 1;

		// We have three binary digits that feed into the current digit
		// {the multiplicand digit, the multiplier digit and the carry}
		//If one or all three
		// of these are one then the digit will be one, otherwise it will be
		// zero.
		uint sumBit = (aDigit ^ bDigit) ^ carry;

		// We now shift the bit into its correct location and add it into
		// the result
		result = result | (sumBit << bitIdx);

		// Finally calculate the carry for the next digit
		carry = (aDigit & bDigit) | (aDigit & carry) | (bDigit & carry);
	}

	return result;
}
```

User your function to write unsigned subtraction?

Write a function Log(double x, double b) that takes a double value and a double base and returns $Log_b x$ It should work for any valid real base.

We only have natural logarithm and logarithm base 10 in the mathematics package but we can make use of the following to calculate any base from the natural logarithm or the base 10 logarithm

$$log_a x = \frac{log_b x}{log_b a}$$

Proof

$$Let\ x = a^y \text{ and hence } log_a x = y$$

$$log_b x = log_b a^y$$

$$log_b x = y \times log_b a$$

$$log_b x = log_a x \times log_b a$$

$$log_a x = \frac{log_b x}{log_b a}$$

The C# source code is then given by

Logarithm any base

```
public double Log(double x, double b)
{
    return Math.Log(x) / Math.Log(b);
}
```

Given an integer value calculate the number of digits d required to represent that integer in base b nunber system

A number n represented in a base b number system will consist of k digits if and only if $b^{k-1} \leq n < b^k$. *In other words* b^{k-1}*is the smallest number that requires k digits. Based on these facts we can derive expressions that calculate the number of digits k required to represent n in base b.*

Expression using the floor function

Taking logarithms our inequality becomes.

$k - 1 \leq log_b\, n < k.$

From the properties of the floor function we know that $\lfloor x \rfloor = m \;\leftrightarrow m \;\leq x < m + 1$ *and hence in our case*

$\lfloor log_b\, n \rfloor = k - 1$

Expression using the ceiling function

We can achive a similar result that uses the ceiling function by adding one to the inequality $b^{k-1} \leq n < b^k$. *so we get*

$b^{k-1} < n + 1 \leq b^k$ *and taking logarithms we get*

$k - 1 < log_b(n + 1) \leq k$

From the properties of the ceiling function we know that $\lceil x \rceil = m \leftrightarrow m - 1 < x \leq m$ *and hence that*

$\lceil log_b(n + 1) \rceil = k$

Number of digits code

The following code uses the ceiling function approach. It requires a function that gives the logarithm of any base.

```
public double DigitsRequired(int x, int b)
    => Math.Ceiling(Log(x+1,b));

public double Log(double x, double b)
    => Math.Log(x) / Math.Log(b);
```

Given a string representation of an integer N in base convert it to a string representation of an integer in base $\boldsymbol{\beta}$. For example given the input “10” with $\boldsymbol{\lambda} = 10$ and $\boldsymbol{\beta} = 2$ it would return “1010”

Let $N = \pm(a_n\lambda^{\infty} + \ldots + a_2\lambda^1 + a_1\lambda^0)_\lambda$ then we want to find the coefficients c_i such that

$$N = \pm(c_n\beta^{\infty} + \ldots + c_2\beta^1 + c_1\beta^0)_\beta$$

We have a number N

$$\boldsymbol{N = (a_n a_{n-1} \ldots a_2 a_1)_\lambda}$$

That we want to convert to base β such that

$$N = (c_m c_{m-1} \ldots c_1 c_1)_\beta$$

We can rewrite this as

$$N = c_1 + \beta\big(c_2 + \beta\big(c_3 + \ldots + \beta(c_m)\big)\ldots\big)_\beta$$

If we divide it by β then the remainder is clearly c_1 and the quotient is

$$c_2 + \beta\big(c_3 + \beta\big(c_4 + \ldots + \beta(c_m)\big)\ldots\big)_\beta$$

If we repeat this until the quotient is zero we can read off the value of c_1 to c_n giving us the required number in the new base $(c_m c_{m-1} \ldots c_1 c_1)_\beta$

Integer change of base

```
private string ConvertIntegralPart(string input, int l, int b)
{
    var result = new StringBuilder();

    // Calculate the decimal equivalent
    var idx =0;
    var d = input[idx].ToIntDigit();
    while(++idx < input.Length) d = (d * l) + input[idx].ToIntDigit();

    var quotient = d;
    do
    {
          var r = quotient % b ;
          quotient = quotient / b;
          result.Append(r.ToChar());
    }
    while (quotient > 0 );

    var chars = result.ToString().ToCharArray().Reverse().ToArray();
    return new string(chars);
}
```

Given a string representation of an fraction N in base convert it to a string representation of a fraction in base $\boldsymbol{\beta}$. For example given the input “0.75” with $\boldsymbol{\lambda} = 10$ and $\boldsymbol{\beta} = 2$ it would return “0.11”

Consider the situation where we have a fraction part $0 < x < 1$ in some base λ and we want to find the digits $d_k$in the representation

$$x = \sum_{k=1}^{\infty} d_h \beta^{-k} = (0.d_1 d_2 d_3 \ldots)_\beta$$

We first note that

$$\beta x = (d_1.d_2 d_3 \ldots)_\beta$$

So if we take our fractional part and multiply it by β then the resulting integral component is the $d_1$we can similarly repeate the process to find the digits $d_2 .. d_m$

Fractional change of base

```
private string ConvertFractionalPart(string input, int l, int b,
     int maxDigits=16)
{
     var fractionString = input.Split('.')[1];
     var result = new StringBuilder("0.");

     // Calculate the decimal Fraction
     double decimalFraction = 0.0;
     for (int i = 0; i < fractionString.Length; i++)
     {
            decimalFraction +=
                  fractionString[i].ToIntDigit() * Math.Pow(l,-(i+1));

     }

     int digitIdx=0;
     while (decimalFraction > 0.0 && digitIdx++ < maxDigits)
     {
            decimalFraction = (decimalFraction * b);
            int digit = (int)decimalFraction;
            result.Append(digit.ToChar());

            decimalFraction -= digit;
     }

     return result.ToString();
}
```

Write the simplest possible code to perform unsigned integer division?

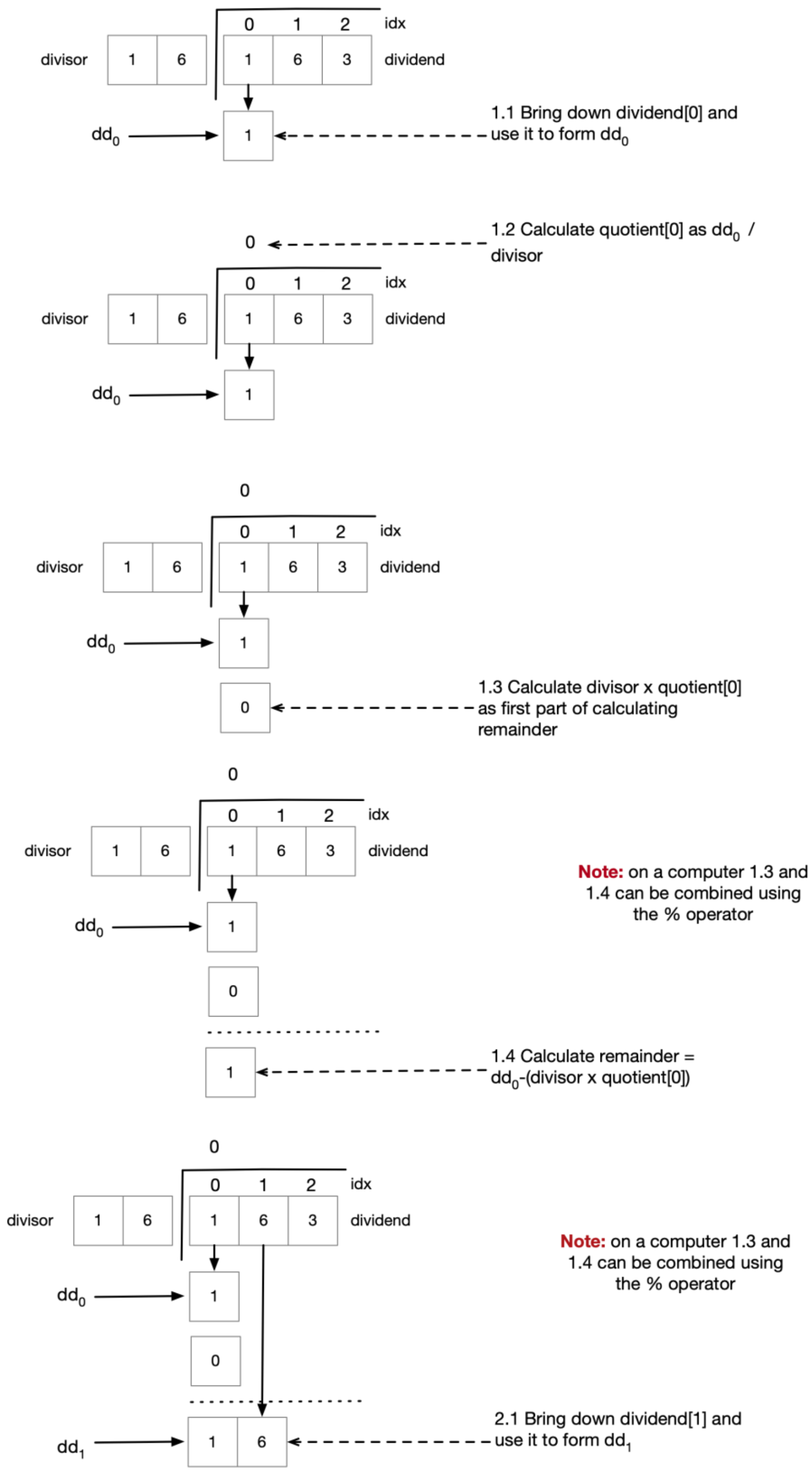
Division is just repeated subtraction so we have

```
public (uint q, uint r) Divide(uint dividend, uint divisor)
{
    uint quotient = 0;
    uint remainder = dividend;

    while (divisor <= remainder)
    {
        remainder = remainder - divisor;
        quotient++;
    }

    return (quotient,remainder);
}
```

Write code to perform integer division using a long division algorithm. The dividend is specified using a string. The base of the dividend and the divisor are given as simple ints?



```
public (string quotient, string remainder) IntegerLongDivision(string
dividend, int divisor, int b = 10)
{
    StringBuilder quotient = new StringBuilder();
    int remainder = 0;
    int dd = 0;

    for (int idx = 0; idx < dividend.Length; idx++)
    {
          // idx.1 copy in next digit into temporary dividend dd
          dd = (dd * b) + dividend[idx].ToIntDigit();

          // idx.2 calculate partial quotient and set into quotient[idx]
          int partialQuotient = dd / divisor;
          quotient.Append(partialQuotient.ToChar());

          // idx.3 calculate this temporary as part of
          //calculating the remainder
          int temp = partialQuotient * divisor;

          // idx.4 Calculate the remainder
          remainder = dd % divisor;

          // the remainder will form the basis of dd[idx+1]
          dd = remainder;
    }

    return (quotient.ToString(), remainder.ToChar().ToString());
}

public static class Extensions
{
    public static int ToIntDigit(this char c)
    {
          if (char.IsNumber(c)) return (int)char.GetNumericValue(c);

          return char.ToLower(c) - 'a' + 10;
    }

    public static char ToChar(this int i)
    {
          if (i >= 0 && i < 10)
                return (char)(i + '0');

          return (char)(i + 'a' - 10);
    }
}
```

Modify your answer from the previous section to return a floating point result rather than quotient and remainder?