Bits, Bytes and Numbers

THIS DOCUMENT COVERS

Bit Operators

<<

```
a          11010101
a << 2     01010100
```

>> (Signed Integers)

```
a          11010101
a >> 2     11110101

a          01010101
a >> 2     00010101
```

~

```
a          00001101
~a         11110010
```

&

```
a          00001101
b          11101011
----------------------
a&b        00001001
```

|

```
a          00001101
b          11101011
----------------------
a|b        11101111
```

^

```
a          00001101
b          11101011
----------------------
a^b        11100110
```

Bit Properties

a ^ 0s = a

```
a           00001101
0s          00000000
a ^ 0s      00001101
```

a ^ 1s = ~a

```
a           00001101
1s          11111111
a ^ 1s      11110010
```

a ^ a = 0

```
a           00001101
0s          00000000
a & 0s      00000000
```

a & 0s = 0

```
a           00001101
0s          00000000
a & 0s      00000000
```

a & 1s = a

```
a           00001101
1s          11111111
a & 1s      00001101
```

a & a = a

```
a           00001101
a           00001101
a & a       00001101
```

a | 1s = 1s

```
a           00001101
1s          11111111
a | 1s      11111111
```

a | a = a

```
a           00001101
a           00001101
a | 1s      00001101
```

a ^ ~a = 1s

```
a      00001101
~a     11110010
a^~a   11111111
```

Bit Manipulation

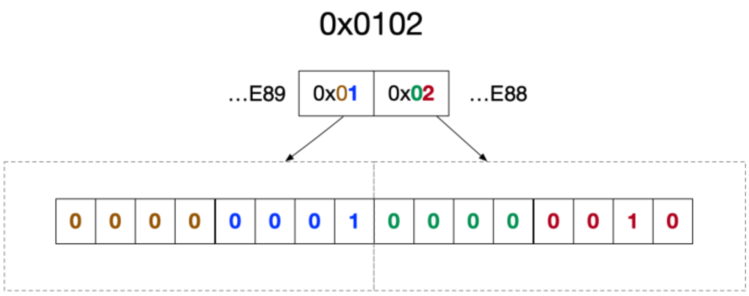
1 << i	`1 << 3`	0001000	Create a mask with all zeros except a single 1 at bit location i
~(1 << i)	`~(1 << 3)`	11110111	Create a mask with all ones except a single 0 at bit location i
~0 << n	`~0 << 3`	11111000	Create a mask of all 1s except for 0s in the n least significant digits
(1 << i)-1	`(1 << 3)-1`	00000111	Create a mask of all 0s except for 1s in the n least significant digits
(1 << j-i+1)-1)<<i	`(1 << 4-2+1)-1)<<2`	00011100	Create a mask of all 0s except for digits i through j which contain 1s
~(((1 << i - j + 1) - 1) << i))	`~(((1 << i - j + 1) - 1) << i))`	111000111	Create a mask of all 1s except for digits i through j which contain 0s
a+(~b+1)	`5 + (~3+1)`	2	Perform subtraction without using the - key

Binary representations

Most numeric types consist of multiple bytes. The order in which the bytes are arranged in memory is known as endianness. On a little endian system, a numeric object's least to most signifant bytes are arranged in order from lower memory addresses to higher memory addresses. Consider a .NET unsigned short which occupies 2 bytes or 16 bits

```
ushort a = 0x0102;
```

Figure 1 Endianess



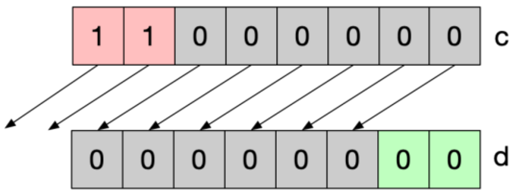
Bit Operators

<< Left Shift

A left shift moves everything n place to the left. The left most n bits are dropped and the rightmost n bits are filled with zeros.

```
byte c = byte.MaxValue;
byte d = (byte)(a >> 2);
```

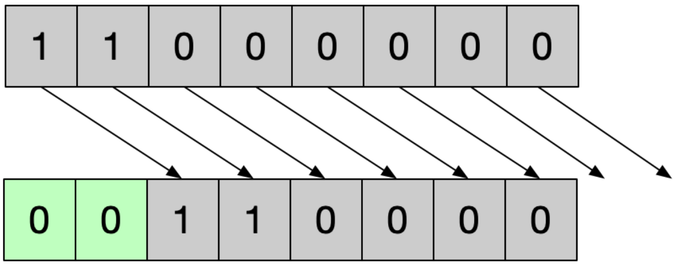
Figure 2 Left Shift



>> Right Shift

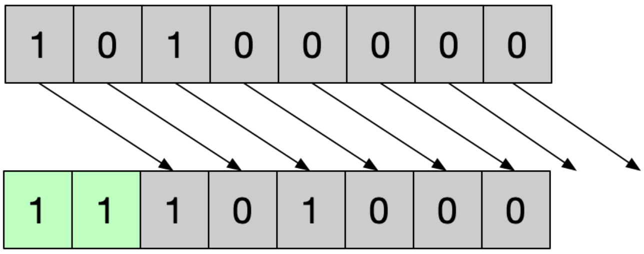
The right shift operator shifts all bits n places to the right. The rightmost n digits are dropeed and the leftmost n digits are filled as follows. If the operand is unsigned the left n bits are filled with zeros

```
byte c = 128+64;
byte d = (byte)(c >> 2);
```



If the operand is signed the sign of the bits filled on the left matches the sign bit in the most significant position.

```
sbyte a = -96;
sbyte b = (sbyte)(a >> 2);
```



This extra complexity ensures that shifting right 1 place is equivalent to divinding by two when the operand is negative.

~ bitwise complement

Inverts all the bits

```
a      00001101
~a     11110010
```

& bitwise and

Copies a 1 into the result if the corresponding bits in each operand are 1

```
a      00001101
b      11101011
a&b    00001001
```

| bitwise or

```
a      00001101
b      11101011
a|b    11101111
```

^ exclusive or

```
a      00001101
b      11101011
a^b    11100110
```

Numeric Bit Representations

We now move on to show how number systems of the following form are represented in .NET

$$\pm(d_{\infty}\beta^{\infty}+\ldots d_1\beta^1 + d_0\beta^0 + b_{-1}\beta^{-1} + b_{-1}\beta^{-2}+\ldots b_{-\alpha}\beta^{-\alpha}) = \pm\left(\sum_{k=-\infty}^{\infty} d_k\beta^k\right)$$

Unsigned Integers

If we only need to represent positive whole numbers, that is to say unsigned integers, we can use a n-bit binary representation. We don't need any bits to represent fractions.

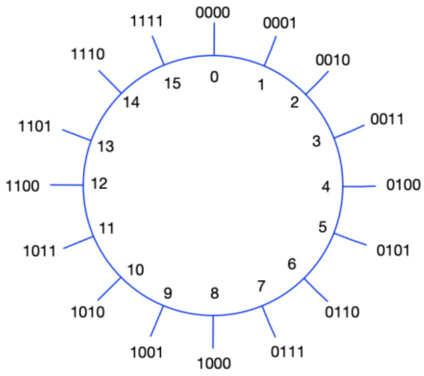
$$(d_{n-1}\ldots d_1 d_0)_2 = (d_{n-1}2^{n-1}+\ldots d_1 2^1 + d_0 2^0) = \left(\sum_{k=0}^{n-1} d_k n^k\right)$$

Such a representation can distinguish between 2^n different values which we can use to represent positive integers in the range $[0, 2^n - 1]$ To highlight the approach consider the specific case of $n = 4$

```
00    0000
01    0001
02    0010
03    0011
04    0100
05    0101
06    0110
07    0111
08    1000
09    1001
10    1010
11    1011
12    1100
13    1101
14    1110
15    1111
```

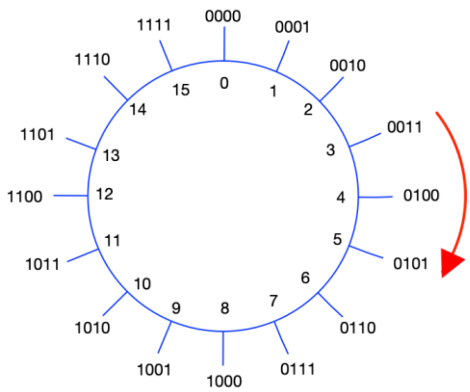
It is useful to visualize such a system as a circle

Figure 3 Unsigned Integers



Unsigned Addition

Addition $(a + b)$ is achieved by starting at a and moving b places clockwise around the wheel. Consider the specific case of $(3 + 2)_{10} = (0011 + 0010)_2$ We visualize this as follows



This is very simple binary addition

```
      0011
+     0010
----------
      0101
```

The following C# code shows how we might achieve such addition

Figure 4 Adding Unsigned Integers

```
public uint Add(uint a, uint b)
{
	uint carry = 0;
	uint result = 0;

	for (int bitIdx = 0; bitIdx < SizeInBits; bitIdx++)
	{
		// We deal in one binary digit at multiplicand time. By right
		// shifting multiplicand and bitIdx times we set the bit we want
		// into the least significant bit.
		uint aShifted = (a >> bitIdx);
		uint bShifted = (b >> bitIdx);

		// Now we make use of the fact that the number 1 has
		// in our unsigned representation consists of SizeInBits
		// zeros followed by multiplicand solitary one in the least significant
		// position. We can hence take our shifted valued and logically
		// and them with 1 to ensure we only have the digit values in the least
		// significant locations remaining.
		uint aDigit = aShifted & 1;
		uint bDigit = bShifted & 1;

		// We have three binary digits that feed into the current digit
		// {the multiplicand digit, the multiplier digit and the carry}
		//If one or all three
		// of these are one then the digit will be one, otherwise it will be
		// zero.
		uint sumBit = (aDigit ^ bDigit) ^ carry;

		// We now shift the bit into its correct location and add it into the
		// result
		result = result | (sumBit << bitIdx);

		// Finally calculate the carry for the next digit
		carry = (aDigit & bDigit) | (aDigit & carry) | (bDigit & carry);
	}

	return result;
}
```

Notice in our add method we do not deal with the overflow from the most significant bit. When we add one to the largest representable binary digit which consists of all ones the result is the smallest binary digit consisting of all zeros. In a four bit unsigned integer we would have as follows. Note the bold red overflow is discarded.

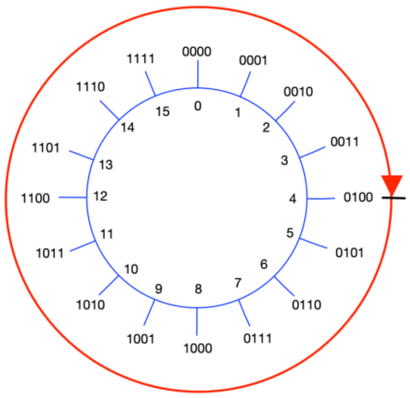
```
     1111
+    0001
    10000
```

By implementing add in this way we have created a modulo number system. If there are n bits in our unsigned integer then addition is mod_{2^n}. For any unsigned integers a and m we have

$(a + 2^n)_{mod_{2^n}} = a_{mod_{2^n}}$

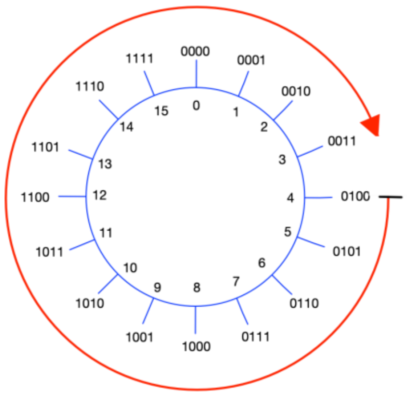
$(a + m.2^n)_{mod_{2^n}} = a_{mod_{2^n}}$ for

In our case adding $2^4 = 16$ to any value gets back to the same value. We show $4 + 2^4 = 4$



Unsigned subtraction

In our 4 bit integer notice what happens if we add $2^4 - 1 = 15$ to 4. We only rotate to 3. So adding $2^4 - 1$ is the same as adding -1.



Similarly, adding $2^4 - 2$ is the same as subtracting 2 and adding $2^4 - b$ is the same as subtracting b. We noted in the previous section that $(a + 2^n)_{mod_{2^n}} = a_{mod_{2^n}}$ and so it is self evident that

$$(a + [2^n - b])_{mod_{2^n}} = (a - b)_{mod_{2^n}}$$

This is a very useful result if we combine it with the following observation. Adding any binary number to its complement gives a number consisting solely of 1s.

$$b + \sim b = 11 \ldots 1$$

And in our representation we have that 11 ... 1 $= 2^n - 1$ hence it follows that

$$b + {\sim}b = 2^n - 1$$

If we substitute this into the expression

$$(a + [2^n - b])_{mod_{2^n}} = (a - b)_{mod_{2^n}}$$

We get

$$(a - b)_{mod_{2^n}} (a + {\sim}b + 1)_{mod_{2^n}}$$

This means we can use our method for addition of unsigned integers to perform subtraction of unsigned integers. The following shows the simple C# code

```
public uint Subtract(uint a, uint b) => Add(a, Add(~b, 1));
```

b (Decimal)	**B (Binary)**	**~b (Binary)**	**Adding (Clockwise)**	**Subtraction (Anticlockwise)**
0	0000	1111	15	$15 - 2^4 = -1$
1	0001	1110	14	$14 - 2^4 = -2$
2	0010	1101	13	$13 - 2^4 = -3$
3	0011	1100	12	$12 - 2^4 = -4$

Proof that $(a + {\sim}b + 1)_{mod_{2^n}} = (a - b)_{mod_{2^n}}$ ∴

From properties of modulo numbers we know that

$(a + 2^n)_{mod_{2^n}} = a_{mod_{2^n}}$ and hence

$(a - b + 2^n)_{mod_{2^n}} = (a - b)_{mod_{2^n}}$ rearranging

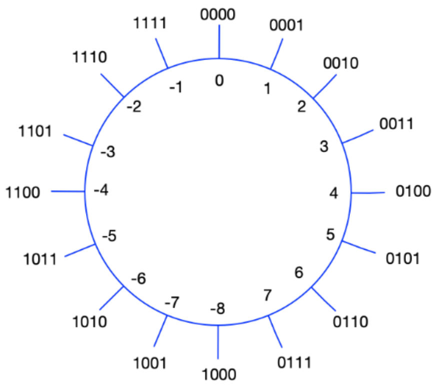
$(a + [2^n - b])_{mod_{2^n}} = (a - b)_{mod_{2^n}}$ adding and subtracting 1

$(a + [2^n - 1 - b] + 1)_{mod_{2^n}} = (a - b)_{mod_{2^n}}$

2S COMPLEMENT SIGNED INTEGERS

A n bit 2s complement representation supports the values from $-2^{n-1}\ldots(2^{n-1}-1)$

Figure 5 2s Complement Signed Integers




```
+00   0000
+01   0001
+02   0010
+03   0011
+04   0100
+05   0101
+06   0110
+07   0111
-08   1000
-07   1001
-06   1010
-05   1011
-04   1100
-03   1101
-02   1110
-01   1111
```

The table shows that to negate a number we complement it and add one. $-a = \sim a + 1$ We say we are taking the twos complement. When adding a pair of twos complement numbers where one of them is negative we simply move around the wheel the number of places in the positive direction of the twos complement binary representation.

Manipulating Binary

Operations

In this section we assume an n bit integer's bit are labeled from bit 0 to bit n-1 where bit 0 is the least significant and bit n-1 is the most significant bit

bool IsBitSet(int intVal, int bitIdx)

The following code shows how to check if the bit at a particular index in an integer's binary representation is set. We illustrate the idea with the specific example of getting the bit at index 2 on the number 27 which is zero.

```
27                  00011011
1 << 2              00000100
27 & (1 << 2)       00000000
```

```
  public bool IsBitSet(int intVal, int bitIdx)
  {
      var mask = 1 << bitIdx;
      return (mask & intVal) != 0;
  }
```

int SetBitValueToOne(int intVal, int bitIdx)

The following code shows how to set a bit at a particular index in an integers binary representation. We illustrate the idea with the specific example of setting the bit at index 2 on the number 27 to give the number 31

```
27                  00011011
1 << 2              00000100
27 | (1 << 2)       00011111
```

```
  public int SetBitValueToOne(int a, int bitIdx)
  {
      var mask = 1 << bitIdx;
      return mask | a;
  }
```

int SetBitValueToZero(int intVal, int bitIdx)

The following code shows how to clear a bit at a particular index in an integers binary representation. We illustrate the idea with the specific example of clearing the bit at index 3 on the number 27 to give the number 19

```
27                  00011011
1 << 2              00001000
~(1 << 2)           11110101
27 & ~(1 << 2)      00010011
```

```
  public int SetBitValueToZero(int a, int bitIdx)
  {
      var mask = ~(1 << bitIdx);
      return mask & a;
  }
```

All ones except the first j binary digits

This code shows how to get a binary representation with all ones except the first j least significant bits which are all zero. That is to say given an n bit integer and and an index j we have

$\forall i \mid j > i \geq 0\,,\ b_i = 0$

$\forall i \mid n > i \geq j\,,\ b_i = 1$

The code is very simple given by the expression

```
  ~0 << j
```

Listing 1 Example

```
0                   00000000
~0                  11111111
~0 << 3             11111000
```

All zeroes except the first j binary digits

This code shows how to get a binary representation with all zeroes except the first j least significant bits which are all one. That is to say given an n bit integer and and an index j we have

$$\forall i \mid j > i \geq 0 \, , \; b_i = 1$$

$$\forall i \mid n > i \geq j \, , \; b_i = 0$$

The code is slightly more complex than the previous example. We make use of the fact that if we start with a binary representation of all zeroes except for a single one at location j then the value of that representation as an integer is 2^j If we substract one from this value we get $2^j - 1 = 2^0 + \cdots + 2^{j-1}$which in binary is represented as the value one in bit locations $[0, j-1]$which gives us the first j locations as one and all other locations as zero. The code is given by

```
(1 << n)-1
```

Listing 2Example

```
1 << 3              00001000
(1 << 3)-1          00000111
```

Mask with ones in location

ClearLsBits

Clear the least significant bits from bit zero through to bit i inclusive

```
public int ClearLsBits(int a, int i) => (-1 << i) & a;
```

ClearMsBits

Clear the most significant bits from MSB though to bit i inclusive

```
public int ClearMsBits(int a, int i) => ~(-1 << i) & a;
```

Tricks

Adding the same number

Performing integer addition where both operands are the same equal to multiplying by two which is equal to shifting left one place.

```
         00001101
+        00001101
         00011010
```

Multiplication

In binary multiplication is simply shiting the multiplicand left by a number of digits equal to the multiplier.

```
         00001101
*        00000011
         01101000
```

Clear rightmost bits a & (~0 << n)

```
a                  00011011
~0 << 2            11111100
a & (~0 << 2)      00011000
```

Problems and Interview Questions