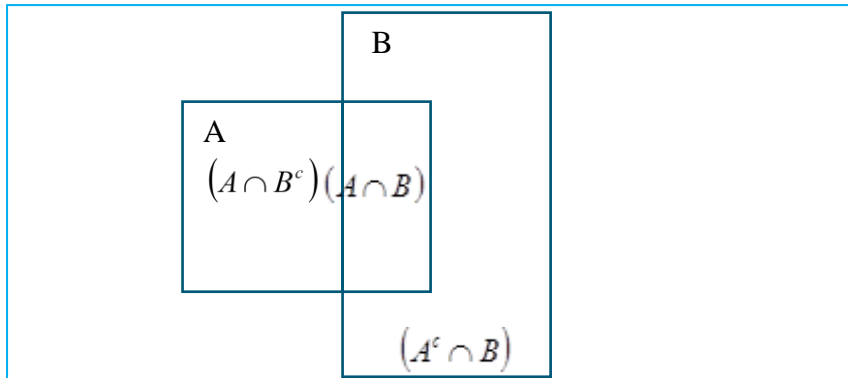


Foundations of Probability

Basic Rules

Probability that either of two events occurs $P(A \cup B)$

We can calculate the probability of either one of two events A or B occurring hence.

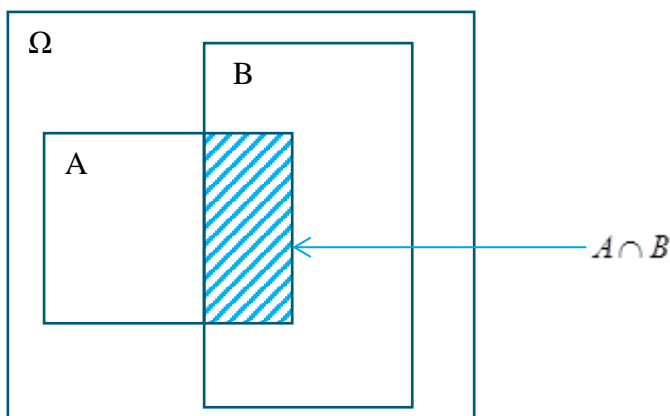


$$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$$

Because the three sets on the right hand side are disjoint. We can get a similar result by adding

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that both events occur $P(A \cap B)$



If we consider discrete probability then where every outcome is equally likely then the probability of $A \cap B$ is simply the number of outcomes in $A \cap B$ divided by the number of outcomes in the sample space Ω

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|}$$

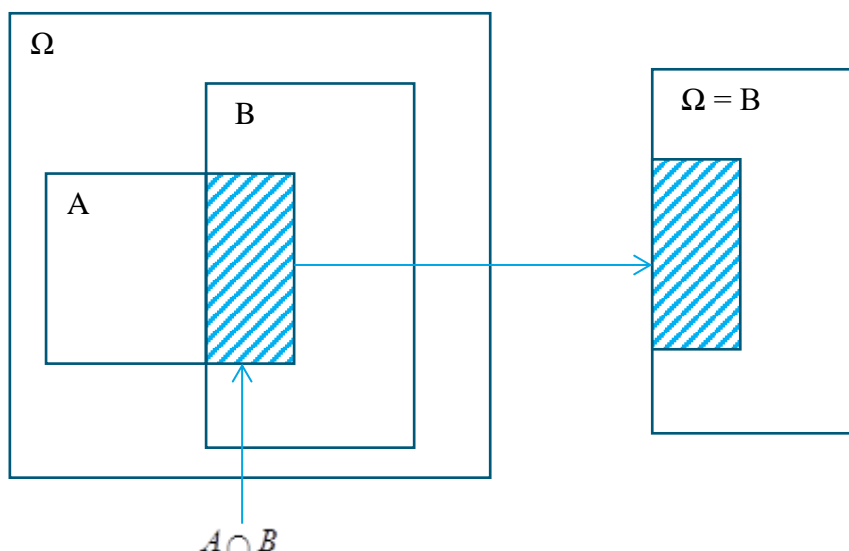
Conditional Probability

$P(A | B)$

The conditional probability $P(A|B)$ is the probability that the event A occurs given that the event B has occurred. Of course for A to occur given that B has occurred, the two events A and B must share outcomes. We know that

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|}$$

But if we know that B has occurred there is a higher probability than $P(A \cap B)$ that A occurs because the extra information that B has occurred allows us to reduce the sample space.



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Because.

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|B|} \div \frac{|\Omega|}{|\Omega|} = \frac{|A \cap B|}{|\Omega|} \div \frac{|B|}{|\Omega|} = \frac{P(A \cap B)}{P(B)}$$

Risk and Pricing Solutions

Multiplication Rule

Similarly if we are given $P(A_2|A_1)$ we can covert it back to $P(A_1 \cap A_2)$ by multiplying it through by $P(A_1)$

$$P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$$

We can extend this to three events

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_3|A_2 \cap A_1)P(A_2 \cap A_1) \\ &= P(A_3|A_2 \cap A_1)P(A_2|A_1)P(A_1) \end{aligned}$$

And then n events

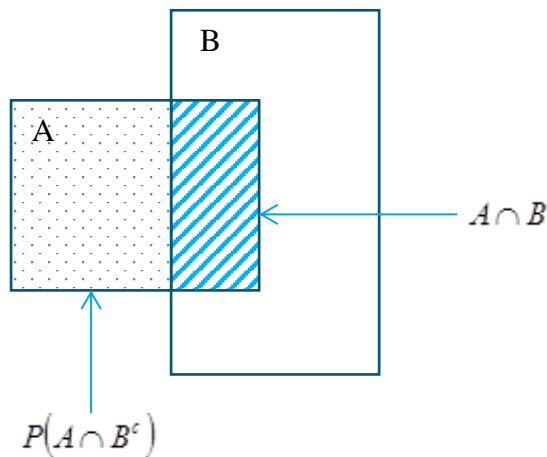
$$\begin{aligned} P(A_1 \cap A_2 \dots \cap A_n) &= P(A_n|A_{n-1} \cap \dots \cap A_1)P(A_{n-1} \cap \dots \cap A_1) \\ &\times P(A_{n-1}|A_{n-2} \cap \dots \cap A_1)P(A_{n-2} \cap \dots \cap A_1) \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

Risk and Pricing Solutions

Partition Rule

Any event A can be partitioned into those outcomes it shares with a second event B and those outcomes it doesn't share with B.

$$P(A) = P(A \cap B) + P(A \cap B^c)$$



We can express this using conditional probabilities as.

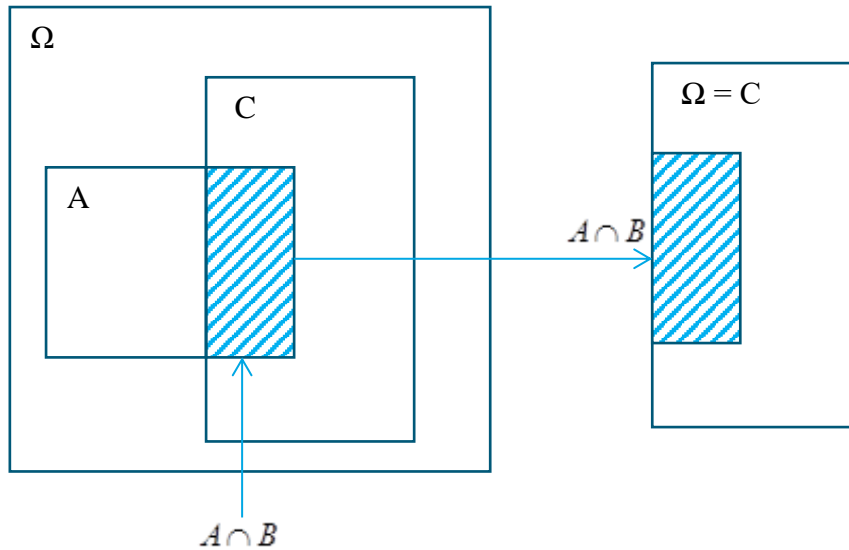
$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Risk and Pricing Solutions

Conditional Partition Rule

$$P(A|C) = P(A|B \cap C)P(B|C) + P(A|B^c \cap C)P(B^c|C)$$

For a proof of this consider the following

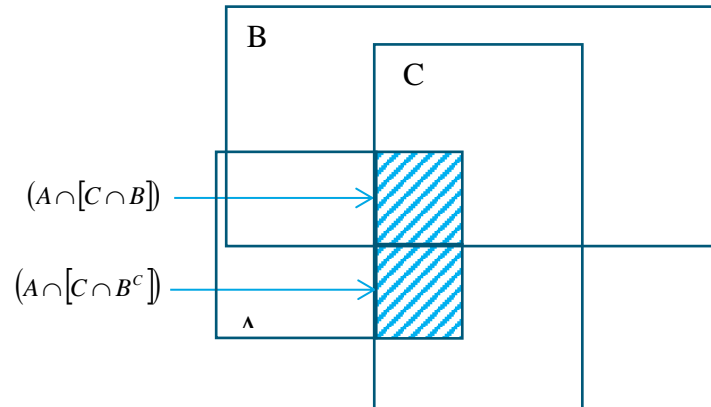


$$P(A|C) = \frac{P(A \cap C)}{P(C)} \quad (1)$$

But the set $A \cap C$ can be broken up into the part that intersects with a third set B and the part that doesn't intersect with the third set B

$$A \cap C = (A \cap [C \cap B]) \cup (A \cap [C \cap B^c]) \quad (2)$$

Risk and Pricing Solutions



So we can insert 2 into 1

$$P(A|C) = \frac{P(A \cap [(C \cap B) \cup (C \cap B^c)])}{P(C)} \quad (3)$$

Now we need to remember that $P(A \cap C \cap B) = P(A|C \cap B)P(C \cap B)$ we update the numerator on the RHS of 3

$$P(A|C) = \frac{P(A|C \cap B)P(C \cap B) + P(A|C \cap B^c)P(C \cap B^c)}{P(C)} \quad (4)$$

Finally we note that $P(C \cap B) = P(B|C)P(C)$ and use this to update the numerator on the RHS

$$P(A|C) = \frac{P(A|C \cap B)P(B|C)P(C) + P(A|C \cap B^c)P(B^c|C)P(C)}{P(C)} \quad (5)$$

Finally we cancel the $P(C)$'s

$$P(A|C) = P(A|C \cap B)P(B|C) + P(A|C \cap B^c)P(B^c|C) \quad (6)$$