Stochastic Processes

Table 1 Definitions

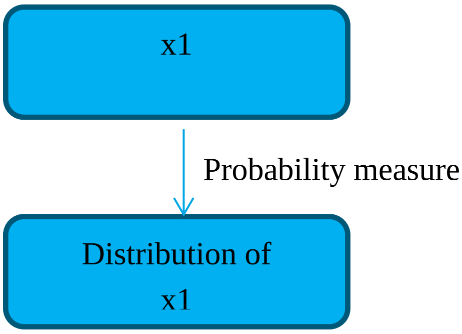
Outcome	Each thing that can occur in an expiriment is called an outcome. In the example of tossing a coin we have two outcomes 'heads' or 'tails' which we can denote by the letters H and T respectively.
Sample Space	The set of all possible outcomes of an experiment is known as the sample space. By convention we label it Ω. In our simple coin tossing scenario we would have $\Omega = \{H, T\}$ If we toss two coins our sample space would become $\Omega = \{HH, HT, TH, TT\}$
Event	A subset of the probability space is called an event. We define an event using the following notation. $A = \{\varpi \in \Omega; \varpi = H\}$ "This means the set of all outcomes ϖ such that ϖ is a head".
Probability Measure	A probability measure P is a function that assigns to each element ϖ in Ω a probability such that $$\sum_{\omega \in \Omega} P(\varpi) = 1$$ Since an event A is a subset of Ω then the probability of an event is given by $$P(A) = \sum_{\omega \in A} P(\varpi)$$
Probability Space	A probability space $P(\Omega, P)$ consists of a sample space and a probability measure. The sample space is the set of outcomes and the probability measure is a function that assigns to each element ϖ in Ω a value in $[0,1]$ such that
Random Variable	A random variable x is a real valued function defined on Ω . Put another way a random variable maps each outcome from the sample space Ω to a real number.
Proabability Distribution	We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on Ω whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.

TABLE 2 PROPERTIES OF RANDOM VARIABLES

Expectation	$E(X) = \sum_{\omega\epsilon} X(\omega)P(\omega)$
Linearity of expectation	$E[aX + b] = aE[X] + b$
Expectation of a function of a random variable	$E[g(X)] = \sum_{\omega\epsilon\Omega} g(X(\omega))P(\omega)$
Expcetation of sum of random variables	$E[X + Y] = E[X] + E[Y]$
Variation from Expectation	$\sum_{\omega\in\Omega} [X(\omega) - E(X)]^2 P(\omega)$
Expectation of sum of n IIR variable	$n.E[X_n]$
Variance	$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2$
Variance of constant	$Var[a] = 0$
Variance of a constant multiple	$Var[aX] = aVar[X]$

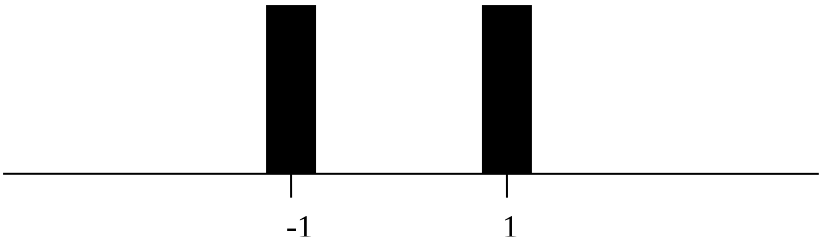
Probability Distribution

We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on Ω whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.



Under the probability measure P defined on Ω either a head or tail are equally likely so our distribution becomes

$$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$$



Risk and Pricing Solutions

Expectation

We can define the expectation or expected value of any random variable X under a probability measure P as.

$$E(X) = \sum_{\omega \epsilon} X(\omega)P(\omega)$$

- Weighted average of the values the random variable X can take
- Weighting by the probability of each value
- Measure of centrality

Expectation of Variable Squared

We are often interested in expectation of the square of the variable which we call the mean squared.

$$E({x_1}^2) = [x_1(H)]^2 P(H) + [x_1(T)]^2 P(T) = 0.5 + 0.5 = 1.0$$

$$\tilde{E}({x_1}^2) = [x_1(H)]^2 \tilde{P}(H) + [x_1(T)]^2 \tilde{P}(T) = 0.75 + 0.25 = 1.0$$

Variation from expected value

For any actual value of a random variable X we can calculate the difference between that value and the expectation $X(\omega) - E(X)$ .We might ask the question "on average how much does a given value differ from the expected value?" We could calculate the average difference as $\sum_{\omega \in \square} [X(\omega) - E(X)]P(\omega)$ however where the distribution is symmetric around the mean this value will be zero. A more instructive measure is given by calculating the average of the difference squared

$$\sum_{\omega \in \Omega} [X(\omega) - E(X)]^2 P(\omega)$$

Under our two probability measures we get

$$[x_1(H) - E(x_1)]^2 P(H) + [x_1(T) - E(x_1)]^2 P(H) = 0.5 + 0.5 = 1.0$$

$$[x_1(H) - E(x_1)]^2 \tilde{P}(H) + [x_1(T) - E(x_1)]^2 \tilde{P}(H) = 0.5 + 0.5 = 1.5$$

Expectation of a constant multiple of a random variable

$$E[aX + b] = aE[X] + b$$

$E[aX + b] = \sum_{\omega\in\Omega}(aX(\omega) + b)P(\omega)$	From definition 1
$= \sum_{\omega\in\Omega}\big(aX(\omega)\big)P(\omega) + \sum_{\omega\in\Omega} bP(\omega)$	By multiplying out the brackets
$= a\sum_{\omega\in\Omega}\big(X(\omega)\big)P(\omega) + \sum_{\omega\in\Omega} bP(\omega)$	From the properties of summation
$= aE[X] + b\sum_{\omega\in\Omega} P(\omega)$	From definition 1
$= aE[X] + b.1$	From axioms of probability
$= aE[X] + b$	

Risk and Pricing Solutions

Expectation of a function of random variable

$$E[g(X)] = \sum_{\omega \epsilon \Omega} g(X(\omega))P(\omega)$$

- The expectation of a function of a random variable is **<u>not equal</u>** to the function of the expectation $E[g(X)] \neq g[E(X)]$

Variance

$$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

Let $\mu = E[X]$

$E[(X-\mu)^2] = \sum_{\omega\epsilon\Omega}(X(\omega)-\mu)^2\ P(\omega)$ From definition

$=\sum_{\omega\epsilon\Omega}\left(\left(X(\omega)\right)^2 - 2\mu X(\omega) + \mu^2\right)P(\omega)$ Multiplying out

$=\sum_{\omega\epsilon\Omega}\left(X(\omega)\right)^2 P(\omega) + \sum_{\omega\epsilon\Omega} -2\mu X(\omega)P(\omega) + \sum_{\omega\epsilon\Omega}\mu^2 P(\omega)$

$= E[X^2] + \sum_{\omega\epsilon\Omega} -2\mu X(\omega)P(\omega) + \sum_{\omega\epsilon\Omega}\mu^2 P(\omega)$ From definition 3

$= E[X^2] + -2\mu\sum_{\omega\epsilon\Omega} X(\omega)P(\omega) + \mu^2\sum_{\omega\epsilon\Omega} P(\omega)$ Properties of summations

$$= E[X^2] - 2\mu\mu + \mu^2 \sum_{\omega\epsilon\Omega} P(\omega)$$

$= E[X^2] - 2\mu\mu + \mu^2$ Axioms of probability

$= E[X^2] - \mu^2$

$= E[X^2] - (E[X])^2$

Risk and Pricing Solutions

Variance of a constant property

$$Var[a] = 0$$

Variance of a constant multiple

$Var[aX] = aVar[X]$

$Var[aX] = E[(aX - E[aX])^2]$

$= E[(aX - aE[X])^2]$ From definition 2

$= E[(aX - a\mu)^2]$ Letting $\mu = E[X]$

$= \sum_{\omega\in\Omega}(aX(\omega) - a\mu)^2 \ P(\omega)$ From definition

$= \sum_{\omega\in\Omega} a^2(X(\omega) - \mu)^2 \ P(\omega)$

$$= a^2 \sum_{\omega\in\Omega}(X(\omega) - \mu)^2 \ P(\omega)$$

$= a^2 Var[X]$ From definition 4

Adding Random variables

Expectation of the sum of two finite countable variables

If X is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$and Y is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$ then the sample space of the joint distribution will be given by a set of pairs

$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$

$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$

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$\{x_m, y_1\}, \{x_m, y_2\}, \ldots\ldots, \{x_m, y_n\}$

The expectation of the sum of the two variables in then given by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \, p(x_i, y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j \, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i\,, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i\,, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i p(x_i) + \sum_{j=1}^{n} x_j p(y_j)$$

Therefore we can note that

$E[X + Y] = E[X] + E[Y]$

Expectation of the sum of n identically distributed random variables

We can calculate the expectation of the sum of n identically distributed random variables denoted by $X_1, X_2, ..., X_n$ as $E[X_1] + E[X_2] + ... + E[X_n]$ which is equal to

$$n.E[X_n]$$

Worked example

Imagine a random event that involves the tossing of a single coin. We have two outcomes, heads or tails in our *sample space* Ω

$$\Omega = \{H, T\}.$$

Furthermore, let us define a *random variable* x_1 that takes the value of plus one dollar if we obtain a head and minus one dollar if we obtain a tail.

$$x_1(H) = 1, x_1(T) = -1$$

Notice that our random variable does not say anything about the probability of a head or tail. It just tells us what value we assign to the outcomes of the sample space. A probability measure is a real valued function that maps each outcome of the sample space to a probability. If our coin is fair we could have a measure P such that

$$P(H) = 0.5, P(T) = 0.5$$

We might however have a different measure for a loaded coin

$$\tilde{P}(H) = 0.75, \tilde{P}(T) = 0.25$$

Applying a probability measure to a random variable gives us a distribution. The distribution shows the probability of each value of the random variable. Different measures give different distributions.

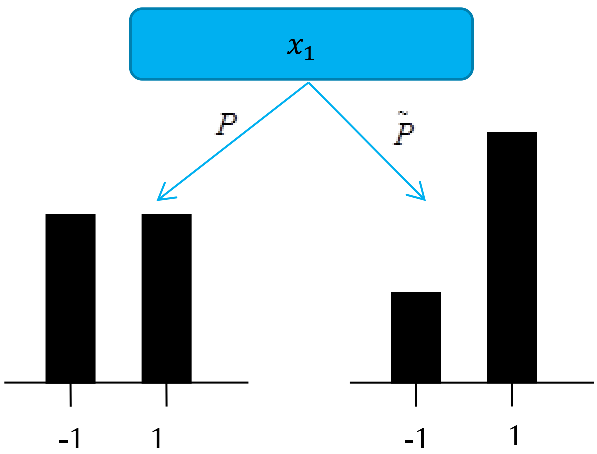


TABLE 3 SUMMARY

Outcome	H	Each outcome is a thing that can occur in an experiment
Sample Space	$\Omega = \{H, T\}$	The set of all possible outcomes that can occur in an experiment is called the sample space
Event	$A = \{\varpi \in \Omega; \varpi = H\}$	A subset of the sample space is called an event
Probability Measure	$P(H) = P(T) = 0.5$	A probability measure P is a function that assigns to each element ϖ in Ω a probability such that $\sum_{\omega \in \square} P(\varpi) = 1$
Probability Space	$(\{H, T\}, P(H) = P(T)) = 0.5)$	A probability space consists of a sample space and a probability measure
Random Variable	$x_1(H) = 1, x_1(T) = -1$	A random variable is a real valued function defined on the sample space.
Probability Distribution	$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$	Tabulation of the probabilities that the random variable takes its various values.
Expectation	$(X) = x_1(H)P(H) + x_1(T)P(T) = 0$	We define the expected value of our random variable under the probability measure P
Variation	$[x_1(H) - E(x_1)]^2 \tilde{P}(H) + [x_1(T) - E(x_1)]^2 \tilde{P}(H) = 1.5$	

Summing Identical Independent random variables

We can create a new game by playing the original games twice. We define x_1 as the profit/loss from the first coin toss and x_2 as the profit/loss from the second toss. We can then define the random variable $X = x_1 + x_2$ as the total profit or loss at the end of the two tosses. The sample space becomes

$$\Omega = \{\omega_1\omega_2\} = \{HH, HT, TH, TT\}$$

The measure becomes

$$P(HH) = P(HT) = P(TH) = P(TT) = 0.25$$

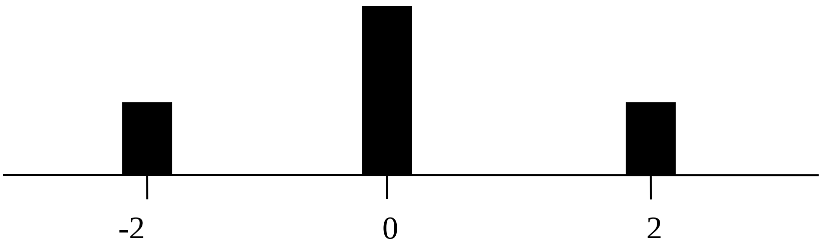
Giving us the random variable

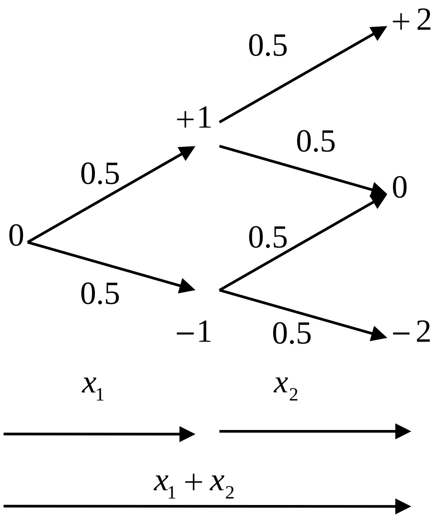
$$X(HH) = 2, X(HT) = (TH) = 0, X(TT) - 2$$

Applying our measure to the random variable we get our probability distribution

$$P(X = 2) = 0.25, P(X = 0) = 0.5, P(X = -2) = 0.25$$

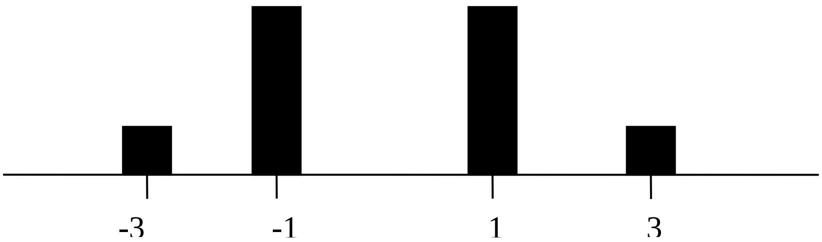
Which is visualizesd as flollows

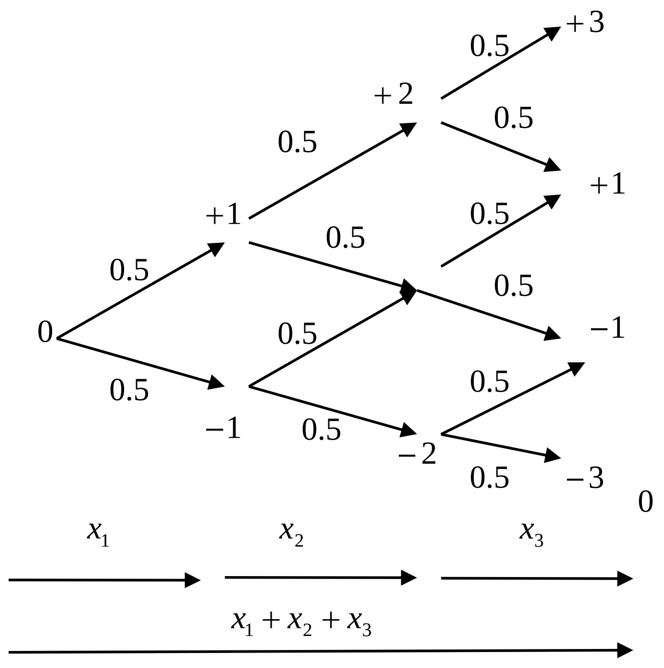




- $\mu = 0$
- $\sigma^2 = 0.25(2-0)^2 + 0.25(-2-0)^2 = 2.0$
-

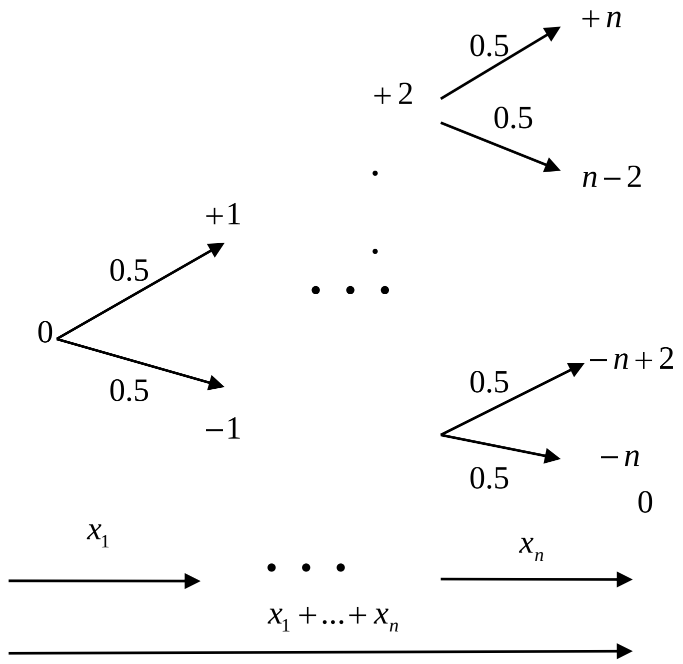
Let us go one-step further and look at summing three toin tosses. $X_3 = x_1 + x_2 + x_3$We get the following distribution, whose mean is zero and whose variance is three.





- $\mu = 0$
- $\sigma^2 = 0.125(3-0)^2 + 0.125(-3-0)^2 + 0.375(1-0)^2 + 0.375(-1-0)^2 = 3.0$

Taking this process to its logical conclusion by summing n of our independent, identically distributed random variables we obtain the random variable $X_n = x_1 + \ldots + x_n$ which is distributed with mean zero and variance n.



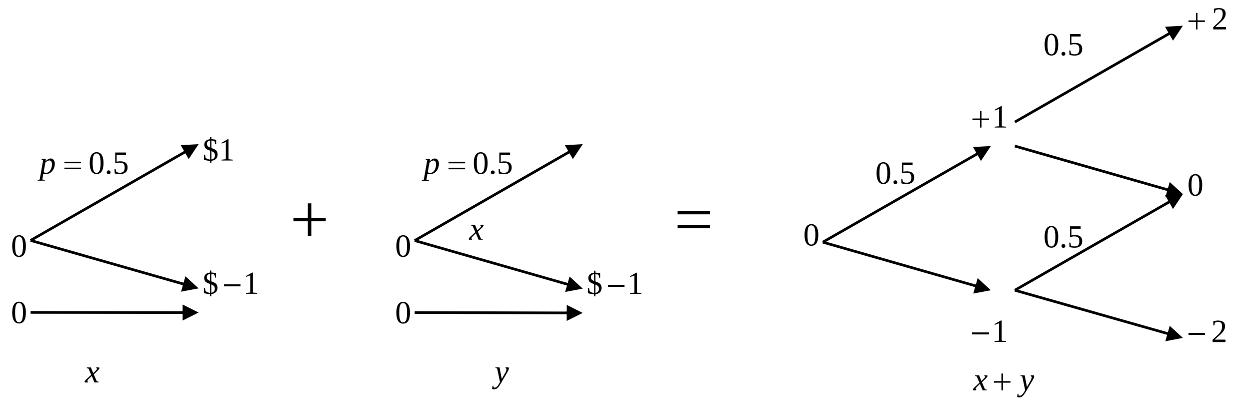
From a proof of why the sum of n independent, identically distributed random variables with mean μ and variance $\sigma^2$is a random variable with mean $n\mu$ and variance $n\sigma^2$see below

Proofs

Expectation of sum of N I.I.D random variables

$$E[X + X + X + X + \ldots\ldots + X] = NE[X]$$

The following figure shows the general approach



We have random variable x with sample space $\{x_1, x_2\} = \{1,0\}$ and a second identically distributed random variable $y$with sample space$\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

The expectation of the sum of the variables is then given by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(x_i + y_j)p(x_i, y_j)$$

Multiplying out

$$\sum_{i=1}^{2}\sum_{j=1}^{2} x_i\, p(x_i, y_j) + \sum_{i=1}^{2}\sum_{j=1}^{2} y_j\, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i\, , y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i\, , y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i p(x_i) + \sum_{j=1}^{n} x_j p(y_j)$$

Therefore we can note that

$$\boldsymbol{E}[\boldsymbol{X}+\boldsymbol{Y}]=\boldsymbol{E}[\boldsymbol{X}]+\boldsymbol{E}[\boldsymbol{Y}]$$

Generalising

If x is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$ and y is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$then the sample space of the joint distribution will be given by a set of pairs

$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$

$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$

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$\{x_m, y_1\}, \{x_m, y_2\}, \ldots\ldots, \{x_m, y_n\}$

The expectation of the sum of the two variables in then given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i+y_j)p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n}x_i\,p(x_i, y_j)+\sum_{i=1}^{m}\sum_{j=1}^{n}y_j\,p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n}p(x_i\,, y_j)=p(x_i)$ and $\sum_{i=1}^{m}p(x_i, y_j)=p(y_j)$

$$\sum_{i=1}^{m}x_i p(x_i)+\sum_{j=1}^{n}x_j p(y_j)$$

Therefore we can note that

$$E[X+Y]=E[X]+E[Y]$$

Furthermore

$$E[X+X]=E[X]+E[X]=2E[X]$$

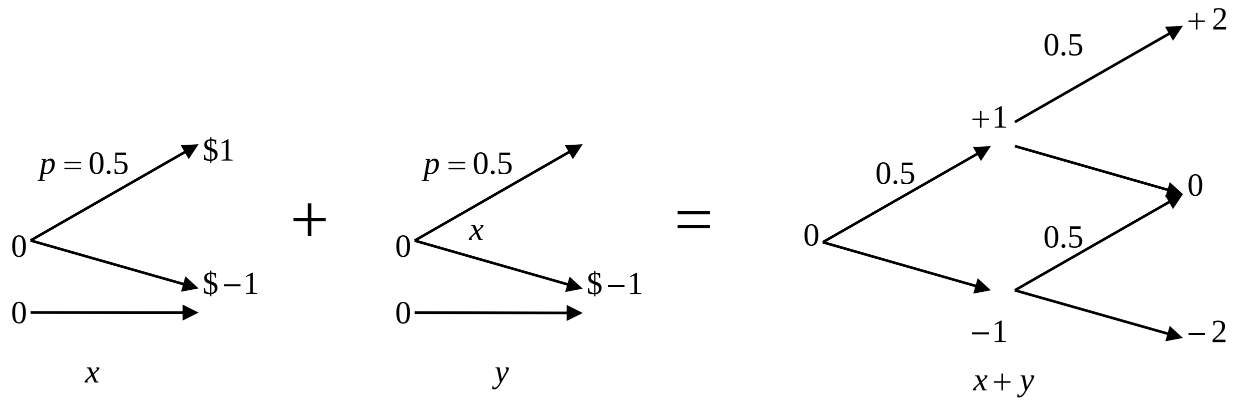
And

$$E[X + X + X + X+........+X] = NE[X]$$

Variance of sum of I.I.D random variables

$$Var[x + y] = Var[x] + Var[y]$$

The following diagram shows the general approach.



We have random variable x with sample space $\{x_1, x_2\} = \{1,0\}$and another identically distributed random variable y with sample space$\{y_1, y_2\} = \{1,0\}$. The sample space of the joint distribution $x + y$is given by the set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

Proof

$Var[x + y] = E[(x + y)^2] - \{E[x + y]\}^2$

$Var[x + y] = E[(x^2 + 2xy + y^2)] - \{E[x] + E[Y]\}^2$

$Var[x + y] = E[x^2] + E[y^2] + E[2xy] - \{E[x] + E[Y]\}^2$

$Var[x + y] = E[x^2] + E[y^2] + E[2xy] - E[x]^2 - E[y]^2 - 2E[x][y]$

$Var[x + y] = E[x^2] + E[y^2] + 2E[x][y] - E[x]^2 - E[y]^2 - 2E[x][y]$

$Var[x + y] = E[x^2] - E[x]^2 + E[y^2] - E[y]^2$

$Var[x + y] = Var[x] + E[y]$