
BASIC BIT LEVEL

- ◆ [Bit Properties](#)
 - ◆ [Manipulation](#)
 - ◆ [Getting/Setting bits](#)
 - ◆ [Brain Teasers](#)
 - ◆ [Integers](#)
-

Risk and Pricing Solutions

Basic Bit

Properties

What is the result of $a \wedge 0s$?

a	00001101
0s	00000000
<hr/>	
$a \wedge 0s$	00001101

The result is a

What is the result of $a \wedge 1s$?

a	00001101
1s	11111111
<hr/>	
$a \wedge 1s$	11110010

The result is $\sim a$

What is the result of $a \wedge a$?

a	00001101
0s	00000000
<hr/>	
$a \wedge 0s$	00000000

The result is 0s

What is the result of $a \wedge 0s$?

a	00001101
0s	00000000
<hr/>	
$a \wedge 0s$	00000000

The result is 0s

What is the result of $a \wedge 1s$?

a	00001101
1s	11111111
<hr/>	
$a \wedge 1s$	00001101

The result is a

What is the result of $a \vee a$?

a	00001101
a	00001101
<hr/>	
$a \vee 1s$	00001101

The result is a

Risk and Pricing Solutions

What is the result of $a \wedge \sim a$?

a	00001101
$\sim a$	11110010
$a \wedge \sim a$	11111111

The results is 1s

Perform bitwise negation without using the \sim operator?

$$a \wedge 1s = a \wedge \sim 0$$

Risk and Pricing Solutions

Manipulation

Implement this function to return a mask of all 0s except a single 1 in bit location i

The following shows how this works with i=3

idx	7654 3 210
1	00000001
1 << 3	0000 1 000

```
public static sbyte MaskOne(int i) => (sbyte)(1 << i);
```

Implement this function to return a mask of all 1s except a single 0 in bit location i

The following shows how this works with i=3

idx	7654 3 210
1	00000001
1 << 3	0000 1 000
~(1 << 3)	1111 0 111

```
public static sbyte MaskTwo(int i) => (sbyte)(~(1 << i));
```

Implement this function to return a mask of all ones except for zeros in the i least significant bits from 0 to (i-1)

The following shows how this works with i=3

idx	76543 2 10
~0	11111111
~0 << 3	11111 0 00

```
public static sbyte MaskThree(int n) => (sbyte)(~0 << n);
```

Implement this function to return a mask of all zeroes except for ones in the n least significant bits

The following shows how this works with i=3

idx	76543 2 10
1 << 3	00001000
(1 << 3)-1	00000 1 11

```
public static sbyte MaskFour(int n) => (sbyte)((1 << n)-1);
```

Risk and Pricing Solutions

Implement this function to return a mask of all 0s except for digits i through j which

The following shows how this works with i=3, j=6

idx	7 6543 210
(1 << 6-3+1)	00010000
(1 << 6-3+1)-1	00001111
<hr/>	<hr/>
((1 << 6-3)-1)<< 3	0 1111 000

```
public static sbyte MaskFive(int i, int j) =>
    (sbyte)((1 << j-i+1) - 1) << i);
```

Implement this function to return a mask of all 1s except for digits i through j which contain 0s.

The following shows how this works with i=3, j=6

idx	7 6543 210
(1 << 6-3+1)	00010000
(1 << 6-3+1)-1	00001111
((1 << 6-3)-1)<< 3	01111000
<hr/>	<hr/>
~(((1 << 6-3)-1)<< 3)	1 0000 111

```
public static sbyte MaskSix(int i, int j)
    => (sbyte)~(((1 << j - i+1) - 1) << i);
```

Implement integer subtraction without using the – key.

We make use of the properties of twos complement

$$a - b = a + (2^n - b) = a + (b + \sim b + 1 - b) = a + \sim b + 1$$

```
public static sbyte Subtract(sbyte a, sbyte b) =>
    (sbyte)(a + ~b + 1);
```

Risk and Pricing Solutions

Getting/Setting bits

Write a function that returns true or false, reflecting whether or not the bit at index i is 1 or 0 respectively

Consider the specific case where $n = 5$ and $i=2$

idx	76543210
n	00000101
n >> 2	00000001
1	00000001
(n >> 2) & 1	00000001
((n >> 2) & 1) > 0	true

```
public bool GetBit(int n, int i) => ((n >> i) & 1) > 0;
```

Write a function set the bit at index i to 1

```
public int SetBit(int n, int i) => (1 << i) | n;
```

Write a function clear the bit at index i to 0

```
public int ClearBit(int n, int i) => ~(1 << i) & n;
```

Write a function that clears all bits from msb through to i inclusive

```
public int ClearFromMsbToI(int n, int i)
{
    return ((1 << i) - 1) & n;
}
```

Write a function that sets all bits from msb through to i inclusive

```
public int SetFromMsbToI(int n, int i)
{
    return (~0 << i) | n;
}
```

Write a function that clears all bits from 0 through to i inclusive

```
public int ClearFromLsbToI(int n, int i)
{
    return (~0 << i + 1) & n;
}
```

Write a function that sets all bits from 0 through to i inclusive

```
public int SetFromLsbToI(int n, int i)
{
    return ((1 << i + 1) - 1) | n;
}
```

Risk and Pricing Solutions

Bit Based Interview Questions

Right Shift Based

MINIMUM OF TWO INTEGERS NO BRANCHING

Write code to find the minimum of two signed integers. You may not use Math.min or branching constructs.

Consider the case where we have two signed 8 bit integers a and b. If we take their difference (a-b) then the result can be classified as

- ◆ 0xxxxxxx If $a \geq b$ or 1xxxxxxx If $a < b$

If we perform a right arithmetic shift of 7 bits (size of the int -1) we get either

- ◆ 00000000 If $a \geq b$
- ◆ 11111111 If $a < b$

Now if we & the result of this shift with the original difference. $((a-b) \gg 7) \& a-b$

- ◆ 0 If $a \geq b$
- ◆ a-b If $a < b$

Now we add in b

- ◆ $0+b=b$ If $a \geq b$
- ◆ $a-b+b=a$ If $a < b$

So we have returned b if $a \geq b$ and a if $a < b$ which was the original aim

```
public sbyte Min(sbyte a, sbyte b)
{
    // Take the difference a-b. The result is one of two forms
    // a) 0xxxxxxx if a >= b
    // b) 1xxxxxxx if a < b
    sbyte difference = (sbyte) (a-b);

    // The result of the right shift is then one of two things
    // a) 00000000 if a >= b
    // b) 11111111 if a < b
    sbyte mask = (sbyte) (difference >> (sizeof(sbyte)*8-1));

    // Now if we & the mask and (a-b) we get one of two things
    // a) 00000000 if a >= b
    // b) a-b if a < b
    sbyte temp = (sbyte) (mask & difference);

    // If we add b to this temp variable we get one of two things which
    // is what we wanted
    // a) 0+b=b if a >= b
    // b) a-b+b=a if a < b
    return (sbyte) (temp + b);
}
```

Risk and Pricing Solutions

MAXIMUM OF TWO INTEGERS NO BRANCHING

Write code to find the maximum of two signed integers. You may not use Math.min or branching constructs.

This is the same as the previous code except for we take the complement of the shift.

```
public sbyte Max(sbyte a, sbyte b)
{
    // Take the difference a-b. The result is one of two forms
    // a) 0xxxxxxx if a >= b
    // b) 1xxxxxxx if a < b
    sbyte difference = (sbyte)(a-b);

    // The result of the complemented right shift is
    // then one of two things
    // a) 11111111 if a >= b
    // b) 00000000 if a < b
    sbyte mask = (sbyte)~(difference >> (sizeof(sbyte)*8-1));

    // Now if we & the mask and (a-b) we get one of two things
    // a) a-b      if a >= b
    // b) 0        if a < b
    sbyte temp = (sbyte)(mask & difference);

    // If we add b to this temp variable we get one of two things which
    // is what we wanted
    // a) a-b+b=a   if a >= b
    // b) 0+b=b     if a < b
    sbyte result = (sbyte)(temp + b);
    return result;
}
```


Risk and Pricing Solutions

INTEGER ABSOLUTE VALUE NO BRANCHING

Write code to find the absolute value of an integer without branching.

We first use our old shift right routine to form a mask. If x is positive the mask is 0s and if x is negative the mask is all 1s.

```
x=5          00000101
mask = x>>7   00000000

x=-5         11111011
mask = x>>7   11111111
```

Now if we xor the mask with x we get one of two things. If the mask is 0s then the result is just x . If the mask is negative the result is $\sim x$ because xor with 1s is the same as the complement operator.

```
x=5          00000101
mask = x>>7   00000000
mask ^ x      00000101

x=-5         11111011
mask = x>>7   11111111
mask ^ x      00000100
```

The final trick is to subtract the mask from the result of the xor. If the mask is zero then the subtraction has no effect and we return x . If the mask is 1s this represents -1 in 2s complement. In the negative case we have $x \wedge 1s$ -1 which is the same as positive x .

```
x=5          00000101
mask = x>>7   00000000
mask ^ x      00000101
(mask ^ x) - mask  00000101

x=-5         11111011
mask = x>>7   11111111
mask ^ x      00000100
(mask ^ x) - mask  00000101
```

The code is then

```
public sbyte AbsoluteValue(sbyte x)
{
    sbyte mask = (sbyte) (x >> 7);
    return (sbyte) ((mask ^ x) - mask);
}
```

Risk and Pricing Solutions

CALCULATE SIGN OF INTEGER

Write code to calculate the sign of an integer?

```
public sbyte GetSign(sbyte a)
    => (sbyte) (a >> ((sizeof(sbyte) * 8) - 1));

public sbyte GetSign2(sbyte a) =>
    (sbyte) (1 | (a >> ((sizeof(sbyte) * 8) - 1)));
```

CALCULATE $\lfloor \log_2 x \rfloor$

Write a function to calculate $\lfloor \log_2 x \rfloor$

The brute force algorithm simply shifts right one digit at a time until we reach zero. The number of times we can do this gives us the position of the most significant set bit and hence the number we are looking for

```
public byte IntLog(byte x)
{
    if (x <= 0) throw new ArgumentException();
    byte shiftCount = 0;

    while (x > 0)
    {
        x >>= 1;
        shiftCount++;
    }

    return (byte) (shiftCount - 1);
}
```

Risk and Pricing Solutions

More efficient $\log_2(n)$

```
public int LogOpt(int x)
{
    int e = 0;

    if ((x & (~((1<<16)-1))) != 0)
    {
        // We have set digits in location 16-31 so we don't
        // care about the digits in locations 0-15. Add 16
        // and shift right to home in on exact location
        x >>= 16; e += 16;
    }

    if ((x & (~((1<<8)-1))) != 0)
    {
        // We have set digits in location 8-15 so we don't
        // care about the digits in locations 0-7. Add 8
        // and shift right to home in on exact location
        x >>= 8; e += 8;
    }

    if ((x & (~((1<<4)-1))) != 0)
    {
        // We have set digits in location 4-7 so we don't
        // care about the digits in locations 0-3. Add 4
        // and shift right to home in on exact location
        x >>= 4; e += 4;
    }

    if ((x & (~((1<<2)-1))) != 0)
    {
        // We have set digits in location 2-3 so we don't
        // care about the digits in locations 0-1. Add 2
        // and shift right to home in on exact location
        x >>= 2; e += 2;
    }

    if ((x & (~((1<<1)-1))) != 0)
    {
        // Finally is the digit in slot index 0 or 1
        e += 1;
    }

    return e;
}
```

Risk and Pricing Solutions

Least Significant set bit based

UNSET LEAST SIGNIFICANT SET BIT

Write an expression to unset the least significant/rightmost set bit

We make use of the fact that subtracting one from a binary number has the effect of unsetting the least significant set bit and setting all bits to the right of that bit.

10	000010 10
10 - 1	000010 01

If we then & the result of this operation with the original number the effect is to unset the least significant(rightmost) set bit

10	000010 10
10 - 1	00001001
10 & (10 - 1)	000010 00

SET ALL BITS TO RIGHT OF RIGHTMOST SET BIT

Write an expression to set all bits to the right of the least significant set bit

We make use of the fact that subtracting one from a binary number has the effect of unsetting the least significant set bit and setting all bits to the right of that bit.

10	000010 10
10 - 1	000010 01

If we then | the result of this operation with the original number the effect is to set all bits to the right of the least significant set bit to 1.

10	000011 00
10 - 1	00001011
10 (10 - 1)	000011 11

Risk and Pricing Solutions

CHECK IS POWER OF 2

Write a function to check if a given unsigned integer is a power of 2

We make use of the fact the binary representation of any power of 2 is a single 1 followed by all zeros

$2^0 = 1$	00000001
$2^1 = 2$	00000010
$2^2 = 4$	00000100

Secondly we note that subtractive 1 from such a representation flips the single 1 to zero and changes all zeros following it to 1s

$2^0 - 1$	00000000
$2^1 - 1 = 1$	00000001
$2^2 - 1 = 3$	00000011

Finally we use the fact that anding the two forms gives a result of zero.

2^0	00000001
$2^0 - 1$	00000000
$2^0 \wedge (2^0 - 1)$	00000000

2^1	00000010
$2^1 - 1$	00000001
$2^1 \wedge (2^1 - 1)$	00000000

2^2	00000100
$2^2 - 1$	00000011
$2^2 \wedge (2^2 - 1)$	00000000

The code is given as follows. Note the special case for zero which is not a power of 2

```
public bool IsPowerOfTwo(uint a)
{
    return (a != 0) && (a & (a-1)) == 0;
}
```

Risk and Pricing Solutions

CALCULATE NUMBER OF SET BITS

Write a function to calculate the number of 1s in an integers binary representation

*We can use a simple linear traversal of the integers bits counting the 1s as we go. Such as algorithm is constant time and always takes $O(\text{sizeof}(\text{int}) * 8)$.*

```
public int BitCount(int a)
{
    int numBits = sizeof(int) * 8;
    int bitCount = 0;

    for (int i=0; i < numBits;i++)
    {
        if ((a >> i) & 1) > 0)
            bitCount++;
    }

    return bitCount;
}
```

But actually we can do better. The following describes a clever algorithm invented by Brian Kernigan. Its key idea is that if we subtract 1 from any integer then the result is that every bit from the lsb up to and including the least significant 1 is flipped. If we then perform an & operation we are effectively removing the least significant 1.

5	00000101
5 - 1	<u>00000100</u>
$5 \wedge (5 - 1)$	00000100

2^1	00011100
$2^1 - 1$	<u>00000001</u>
$2^1 \wedge (2^1 - 1)$	00000000

2^2	00000100
$2^2 - 1$	<u>00000011</u>
$2^2 \wedge (2^2 - 1)$	00000000

Risk and Pricing Solutions

COUNT NUMBER OF DIFFERING BITS

Given two integers a and b find the number of bits you would need to change to modify x to be y?

A simple Xor is enough to give us the bit that differ between a and b. We can then use kernighans algorithm to count the number of bits

```
public int CountDifference(byte a, byte b)
{
    byte diff = (byte) (a ^ b);
    int count = 0;

    while (diff != 0)
    {
        diff = (byte) (diff & (diff-1));
        count++;
    }
    return count;
}
```

Risk and Pricing Solutions

Most Significant set bit based

SET ALL BITS TO RIGHT OF MOST SIGNIFICANT BIT

Write an expression to set all bits to the right of the most significant set bit

x	01000000
$x = x \gg 1$	01100000
$x = x \gg 2$	01111000
$x = x \gg 4$	01111111

LARGEST POWER OF 2 $\leq x$

Write statements to calculate the largest power of 2 less than or equal to x

Let e be the power we are looking for. Applying the result of the previous question we obtain a number $y = (2 \times e) - 1$ The power we are looking for then becomes $\frac{(y+1)}{2}$

SMALLEST POWER OF 2 $\geq x$

Write statements to calculate the smallest power of 2 greater than or equal to x

Let e be the power we are looking for. Applying the result of the previous question we obtain a number $y = e - 1$ The power we are looking for then becomes $y+1$

Risk and Pricing Solutions

MSB > N

Write code to calculate if msb if is in location > n

```
idx          76543210
x0           00010000
( (1 << 5) - 1) 00011111
~( (1 << 5) - 1) 11100000
x & ~( (1 << 5) - 1) != 0  true
```

Risk and Pricing Solutions

Miscellaneous

PRINT DECIMALS BINARY REPRESENTATION

Given a decimal fraction such as 0.46 return a string representating its binary. If the number cannot be reprinted exactly in binary in n bits throw an exception

```
private string ConvertIntegralPart(double b, int maxDigits)
{
    StringBuilder result = new StringBuilder("0.");
    if (b >= 1.0) throw new ArgumentException("Input must be a fraction");

    double frac = 0.5;

    while (b >= 0 && maxDigits-- > 0)
    {
        if (b >= frac)
        {
            result.Append("1");
            b -= frac;
        }
        else
        {
            result.Append("0");
        }

        frac /= 2;
    }

    return result.ToString();
}
```

Risk and Pricing Solutions

NEXT INT SAME 1 COUNT

Given an integer find the next largest integer that has the same number of 1s in its binary representation.

```
public byte NextLargestSame1Count(byte x)
{
    int onesCount = 0;

    for (int i = 0; i < sizeof(byte) * 8; i++)
    {
        // We have found the first non-trailing zero
        if ((x >> i) & 1) == 0)
        {
            if (onesCount > 0)
            {
                // Flip first non-trailing zero to 1
                x |= (byte) (1 << i);

                // Zero locations right of flipped digit
                x &= (byte) (~1 << (i-1));

                // add back in onesCount-1 1s in lsb locations
                x |= (byte) ((1 << onesCount-1)-1);

                break;
            }
        }
        else
        {
            onesCount++;
        }
    }
    return x;
}
```

Risk and Pricing Solutions

PREVIOUS INT SAME 1 COUNT

Given an integer find the next smallest integer that has the same number of 1s in its binary representation.

The key idea is that we need to swap a set bit with an unset bit. If the bit we unset is to the left (more significant) than the bit we set we have decreased the number. Consider the specific input of 62. The first step is to find the 1 that we will flip to a zero. In order to be valid the 1 bit must have a 0 bit to the right of it (less significant) We are hence looking for the first non-trailing 1. We walk from least significant bits to most significant counting zeroes on the way and stopping when we reach the first non trailing 1.

idx	76543210
x ₀	01001111
i	6

The index of the non-trailing 1 is $i = 6$, the number of zeroes is $iz = 2$ and the number of ones is $io + 1 - iz = 5$ In order to unset the first non-trailing 1 which is at index 6 we create a mask $mask1 = \sim(1 \ll i)$

idx	76543210
x ₀	01001111
mask1	10111111
x ₁ = x ₀ & mask1	00001111

Rather than look for a single bit to set to the right of i, we instead clear all bits to the right of i and insert $io - 1$ ones immediately to the right of i. First we create a mask for the zeroing $mask2 = \sim 0 \ll i$

idx	76543210
x ₁	00001111
mask2	11000000
x ₂ = x ₁ & mask2	00000000

Now we put the $io - 1$ ones immediately to the right of i. Our mask is

$$mask3 = ((1 \ll (i - 1)) - 1) \ll (i - io)$$

idx	76543210
x ₂	00000000
mask3	00111110
x ₃ = x ₂ mask3	00111110

The source code is then

Risk and Pricing Solutions

```
public byte NextSmallestSame1Count(byte x)
{
    Console.WriteLine($"{x}      {Convert.ToString(x, 2).PadLeft(8, '0')}");

    int zeroCount = 0;

    for (int i = 0; i < sizeof(byte) * 8; i++)
    {
        if ((x >> i) & 1) != 0)
        {
            // If this condition is true the bit at the
            // current index is set and there exists
            // unset bits to the right of it.
            // We can do a switch
            if (zeroCount > 0)
            {
                // 1. Unset the bit at the current index.
                // To do this we form a mask of all 1s
                // except at index i where it has a zero.
                // The mask is anded with x to unset bit i
                byte mask1 = (byte)~(1 << i);
                x &= mask1;

                // 2. The index of the unset bit is i. We want to
                // clear all bits to the right of i. That is
                // we want to clear bits 0 through i-1 or
                // the leftmost i bits. We define a mask that
                // consists of i 0s in positions 0 through i-1
                // and the rest 1s. We and the mask with x to clear
                byte mask2 = (byte)(~0 << i);
                x &= mask2;

                // 4. We originally had (i+1-zeroCount) 1 digits.
                // We need to these back in location i-1
                // i-1-(i+1-zeroCount)
                int oneCount = i + 1 - zeroCount;
                byte mask3 = (byte)((1 << oneCount) - 1);

                // 5. Shift the mask into position
                mask3 = (byte)(mask3 << (i-oneCount));
                // 6. Apply the mask
                x |= mask3;
                break;
            }
        }
        else
        {
            zeroCount++;
        }
    }
    return x;
}
```

Risk and Pricing Solutions

An array holds all the values from 0 to n inclusive with the exception of one number in the range which is missing. Write code to find which one

The simplest solution makes use of the fact that we know that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. We can then just walk the array summing as we go and then subtract the result from the known value. This technique is $O(n)$

```
public int FindMissing(int[] a)
{
    int n = a.Length;
    int expected = (n*(n+1))/2 ;
    int actual = a.Aggregate ((x, y) => x+y);

    return expected - actual;
}
```

If the array is sorted we can use the relationship between each array element and its index to perform a binary search for $O(\log n)$

	lo	mid	hi	0	1	2	3	4	5	6	7
0	0	3	7	0	1	2	3	4	5	6	8
1	3	5	7	0	1	2	3	4	5	6	8
2	5	6	7	0	1	2	3	4	5	6	8
3	6	6	7	0	1	2	3	4	5	6	8

```
public int SearchIterative(int[] ar)
{
    int lo = 0;
    int hi = ar.Length-1;
    int mi = 0;

    // Special cases for off the front and back off the
    // sequence
    if (ar[lo] != 0) return 0;
    if (ar[hi] == hi) return hi + 1;

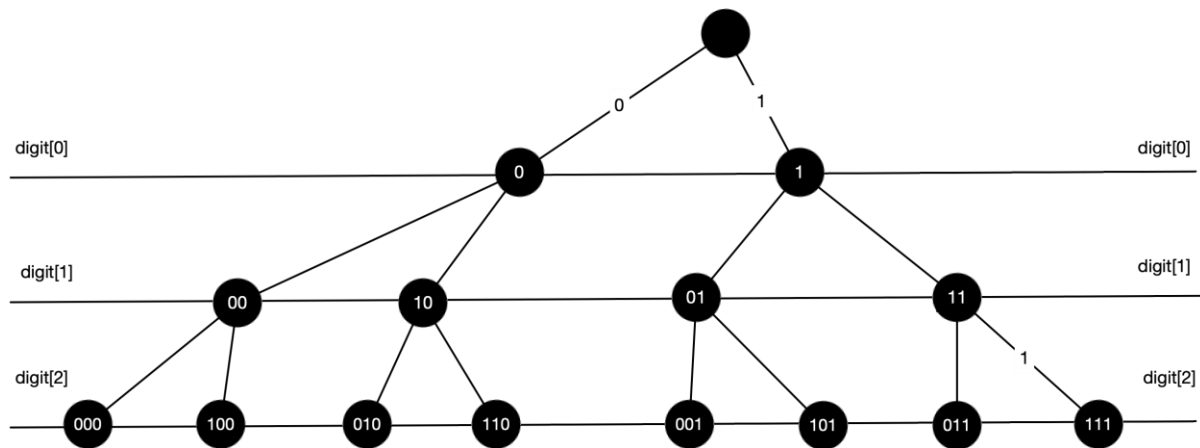
    while ((hi-lo)>1)
    {
        mi = (lo+hi)/2;

        if ((ar[mi]-mi) != 0)
            hi = mi;
        else
            lo = mi;
    }

    return lo+1;
}
```

Risk and Pricing Solutions

Finally if the array is not sorted we can use the relationship between the number of 1s and 0s at each bit position to work out which number is missing. This is by far the most complex technique but it does yield $O(\log n)$ on an unsorted array.



```
public int SearchIterative(int[] ar)
{
    int numBits = (int)Math.Ceiling(Math.Log(ar.Count() + 2) /
Math.Log(2));
    int result = 0;

    List<int> searchArray = new List<int>(ar);
    for (int i = 0; i < numBits; i++)
    {
        List<int> ones = new List<int>();
        List<int> zeroes = new List<int>();

        for (int j = 0; j < searchArray.Count; j++)
        {
            if ((searchArray[j] >> i & 1) == 0)
                zeroes.Add(searchArray[j]);
            else
                ones.Add(searchArray[j]);
        }

        if (ones.Count >= zeroes.Count)
        {
            searchArray = zeroes;
        }
        else
        {
            searchArray = ones;
            result |= 1 << i;
        }
    }

    return result;
}
```

Risk and Pricing Solutions

Given an array of integers called ints and an array of elements called elements that returns the maximum xor value of each value in elements against the elements in integers.

This is a tricky one and we need to use a data structure called a trie to solve it

Given an integer find the longest sequence of 1s you can form if you are allowed to flip one zero to a 1.

Risk and Pricing Solutions

SWAP EVEN AND ODD BITS

Write code to swap the even and odd bits of a given integer

The first stage is to separate out the even and odd digits. We use masks . Consider the specific example

<code>idx</code>	76543210
<code>x</code>	10111101
<code>mask odd (0xaa)</code>	10101010
<code>mask even (0x55)</code>	01010101

We apply the masks

<code>idx</code>	76543210
<code>x</code>	10111101
<code>mask_{odd}</code>	10101010
<code>x_{odd} = x & mask_{odd}</code>	10101000
<hr/>	
<code>idx</code>	76543210
<code>x</code>	10111101
<code>mask_{even}</code>	01010101
<code>x_{even} = x & mask_{even}</code>	00010101

We shift the odd bits into even positions and even bits into odd positions

<code>idx</code>	76543210
<code>x_{even} = x_{even} <<< 1</code>	00101010
<code>x_{odd} = x_{odd} >>> 1</code>	01010100
<code>result = x_{even} x_{odd}</code>	01111110

The code is then

Risk and Pricing Solutions

```
sbyte SwapEvenAndOdd(sbyte x)
{
    // 1. Define the masks
    sbyte oddMask = unchecked((sbyte)0xaa);
    sbyte evenMask = unchecked((sbyte)0b01010101);

    // 2. Separate out the even and odd bits
    sbyte xEven = (sbyte)(x & evenMask);
    sbyte xOdd = (sbyte)(x & oddMask);

    // 3. Move odd bits into even positions and
    //     even bits into odd bit. Notice the cast to int
    //     to compensate fro C# having only arithmetic shift
    //     operators.
    xEven = (sbyte)(xEven << 1);
    xOdd = (sbyte)((((byte)xOdd) >> 1);
    return (sbyte)(xEven | xOdd);
}
```

Risk and Pricing Solutions

Integers

Write a function to add two signed bytes. Do not use the + operator?

```
public uint Add(uint a, uint b)
{
    uint carry = 0;
    uint result = 0;

    for (int bitIdx = 0; bitIdx < SizeInBits; bitIdx++)
    {
        // We deal in one binary digit at multiplicand time. By right
        // shifting multiplicand and bitIdx times we set the bit we want
        // into the least significant bit.
        uint aShifted = (a >> bitIdx);
        uint bShifted = (b >> bitIdx);

        // Now we make use of the fact that the number 1 has
        // in our unsigned representation consists of SizeInBits
        // zeros followed by multiplicand solitary one in the
        // least significant
        // position. We can hence take our shifted valued and logically
        // and them with 1 to ensure we only have the digit values in the
        // least significant locations remaining.
        uint aDigit = aShifted & 1;
        uint bDigit = bShifted & 1;

        // We have three binary digits that feed into the current digit
        // {the multiplicand digit, the multiplier digit and the carry}
        // If one or all three
        // of these are one then the digit will be one, otherwise it will be
        // zero.
        uint sumBit = (aDigit ^ bDigit) ^ carry;

        // We now shift the bit into its correct location and add it into
        // the result
        result = result | (sumBit << bitIdx);

        // Finally calculate the carry for the next digit
        carry = (aDigit & bDigit) | (aDigit & carry) | (bDigit & carry);
    }

    return result;
}
```

Risk and Pricing Solutions

User your function to write unsigned subtraction?

We make use of the fact that $a - b = a + \sim b + 1$

```
public int Subtract(int a, int b) => a + ~b + 1;
```

Write code to do simple binary multiplication?

In binary multiplication is simply shiting the multiplicand left by a number of digits equal to the multiplier.

```
      00001101
*     00000011
-----
      01101000
```

```
public int UnsignedMultiply(int multiplicand, int multiplier)
{
    int result = 0;

    int numBits = sizeof(int)*8;

    for (int i = 0; i < numBits; i++)
    {
        if ( ((multiplier >> i) & 1) > 0)
            result |= (multiplicand << i);
    }

    return result;
}
```

Risk and Pricing Solutions

Write code to do very simple division?

Write a function `Log(double x, double b)` that takes a double value and a double base and returns $\text{Log}_b x$. It should work for any valid real base.

We only have natural logarithm and logarithm base 10 in the mathematics package but we can make use of the following to calculate any base from the natural logarithm or the base 10 logarithm

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

Let $x = a^y$ and hence $\log_a x = y$

$$\log_b x = \log_b a^y$$

$$\log_b x = y \times \log_b a$$

$$\log_b x = \log_a x \times \log_b a$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The C# source code is then given by

Logarithm any base

```
public double Log(double x, double b)
{
    return Math.Log(x) / Math.Log(b);
}
```

Risk and Pricing Solutions

BITS REQUIRED TO REPRESENT INTEGER IN BASE B

Given an integer value calculate the number of digits d required to represent that integer in base b number system

A number n represented in a base b number system will consist of k digits if and only if $b^{k-1} \leq n < b^k$. In other words b^{k-1} is the smallest number that requires k digits. Based on these facts we can derive expressions that calculate the number of digits k required to represent n in base b .

Expression using the floor function

Taking logarithms our inequality becomes.

$$k - 1 \leq \log_b n < k.$$

From the properties of the floor function we know that $\lfloor x \rfloor = m \Leftrightarrow m \leq x < m + 1$ and hence in our case

$$\lfloor \log_b n \rfloor = k - 1$$

Expression using the ceiling function

We can achieve a similar result that uses the ceiling function by adding one to the inequality $b^{k-1} \leq n < b^k$. so we get

$$b^{k-1} < n + 1 \leq b^k \text{ and taking logarithms we get}$$

$$k - 1 < \log_b(n + 1) \leq k$$

From the properties of the ceiling function we know that $\lceil x \rceil = m \Leftrightarrow m - 1 < x \leq m$ and hence that

$$\lceil \log_b(n + 1) \rceil = k$$

Number of digits code

The following code uses the ceiling function approach. It requires a function that gives the logarithm of any base.

```
public double DigitsRequired(int x, int b)
    => Math.Ceiling(Log(x+1,b));

public double Log(double x, double b)
    => Math.Log(x) / Math.Log(b);
```

Risk and Pricing Solutions

CONVERT INTEGER BASE

Given a string representation of an integer N in base λ convert it to a string representation of an integer in base β . For example given the input “10” with $\lambda = 10$ and $\beta = 2$ it would return “1010”

Let $N = \pm(a_n\lambda^n + \dots + a_2\lambda^2 + a_1\lambda^1 + a_0\lambda^0)_\lambda$ then we want to find the coefficients c_i such that

$$N = \pm(c_n\beta^n + \dots + c_2\beta^2 + c_1\beta^1 + c_0\beta^0)_\beta$$

We have a number N

$$N = (a_na_{n-1}\dots a_2a_1a_0)_\lambda$$

That we want to convert to base β such that

$$N = (c_nc_{n-1}\dots c_2c_1c_0)_\beta$$

We can rewrite this as

$$N = c_0 + \beta(c_1 + \beta(c_2 + \dots + \beta(c_{n-1})))_\beta$$

If we divide it by β then the remainder is clearly c_0 and the quotient is

$$c_1 + \beta(c_2 + \beta(c_3 + \dots + \beta(c_{n-1})))_\beta$$

If we repeat this until the quotient is zero we can read off the value of c_0 to c_n giving us the required number in the new base $(c_nc_{n-1}\dots c_2c_1c_0)_\beta$

Risk and Pricing Solutions

Integer change of base

```
private string ConvertIntegralPart(string input, int l, int b)
{
    var result = new StringBuilder();

    // Calculate the decimal equivalent
    var idx = 0;
    var d = input[idx].ToIntDigit();
    while(++idx < input.Length) d = (d * l) + input[idx].ToIntDigit();

    var quotient = d;
    do
    {
        var r = quotient % b ;
        quotient = quotient / b;
        result.Append(r.ToChar());
    }
    while (quotient > 0 );

    var chars = result.ToString().ToCharArray().Reverse().ToArray();
    return new string(chars);
}
```

Risk and Pricing Solutions

CONVERT FRACTIONAL BASE

Given a string representation of an fraction N in base convert it to a string representation of a fraction in base β . For example given the input “0.75” with $\lambda = 10$ and $\beta = 2$ it would return “0.11”

Consider the situation where we have a fraction part $0 < x < 1$ in some base λ and we want to find the digits d_k in the representation

$$x = \sum_{k=1}^{\infty} d_k \beta^{-k} = (0.d_1 d_2 d_3 \dots)_{\beta}$$

We first note that

$$\beta x = (d_1.d_2 d_3 \dots)_{\beta}$$

So if we take our fractional part and multiply it by β then the resulting integral component is the d_1 we can similarly repeat the process to find the digits $d_2 \dots d_m$

Fractional change of base

```
private string ConvertFractionalPart(string input, int l, int b,
    int maxDigits=16)
{
    var fractionString = input.Split('.')[1];
    var result = new StringBuilder("0.");

    // Calculate the decimal Fraction
    double decimalFraction = 0.0;
    for (int i = 0; i < fractionString.Length; i++)
    {
        decimalFraction +=
            fractionString[i].ToIntDigit() * Math.Pow(1, -(i+1));
    }

    int digitIdx=0;
    while (decimalFraction > 0.0 && digitIdx++ < maxDigits)
    {
        decimalFraction = (decimalFraction * b);
        int digit = (int)decimalFraction;
        result.Append(digit.ToChar());

        decimalFraction -= digit;
    }

    return result.ToString();
}
```

Risk and Pricing Solutions

SIMPLE UNSIGNED DIVISIONS

Write the simplest possible code to perform unsigned integer division. What is the runtime?

Division is just repeated subtraction so we have

```
private (int q, int r) UnsignedDivide(int dividend, int divisor)
{
    int quotient = 0;
    int remainder = dividend;

    while (remainder >= divisor)
    {
        remainder -= divisor;
        quotient++;
    }

    return (quotient, remainder);
}
```

This algorithm is very slow. Its runtime is $O(q)$ where q is the quotient

SIMPLE SIGNED DIVISIONS

Use the code from the previous question to perform signed division

```
public (int q, int r) Divide(int dividend, int divisor)
{
    if (divisor == 0) throw new DivideByZeroException();

    if ( dividend < 0 && divisor < 0 )
        return UnsignedDivide(-dividend,-divisor);

    if (dividend < 0)
    {
        (int q, int r) =UnsignedDivide(-dividend, divisor);
        return (-q,r);
    }

    if (divisor < 0)
    {
        (int q, int r) = UnsignedDivide(dividend, -divisor);
        return (-q, r);
    }

    return UnsignedDivide(dividend,divisor);
}
```

Risk and Pricing Solutions

UNSIGNED BINARY LONG DIVISION

Write code to perform binary long division unsigned

```
public (int quotient, int remainder) UnsignedDivide(int dividend,
int divisor)
{
    int numBits = sizeof(byte) * 8;
    int quotient = 0;

    int remainder = 0;

    for (int i = numBits-1; i >= 0; i--)
    {
        // Get the value of the dividend's bit index i
        byte d_i = (byte)((dividend >> i) & 1);

        // Shift the remainder left by 1 bit and add in the
        // bit i from the dividend
        remainder = ((remainder << 1) | d_i);

        // The value of the quotient at index i can only be 1 or
0.
        // It is 1 if the divisor is greater than or equal to
        // remainder, otherwise it is zero
        int q_i = (((remainder >= divisor) ? 1 : 0) << i);

        // copy q_i into the quotient
        quotient |= q_i;

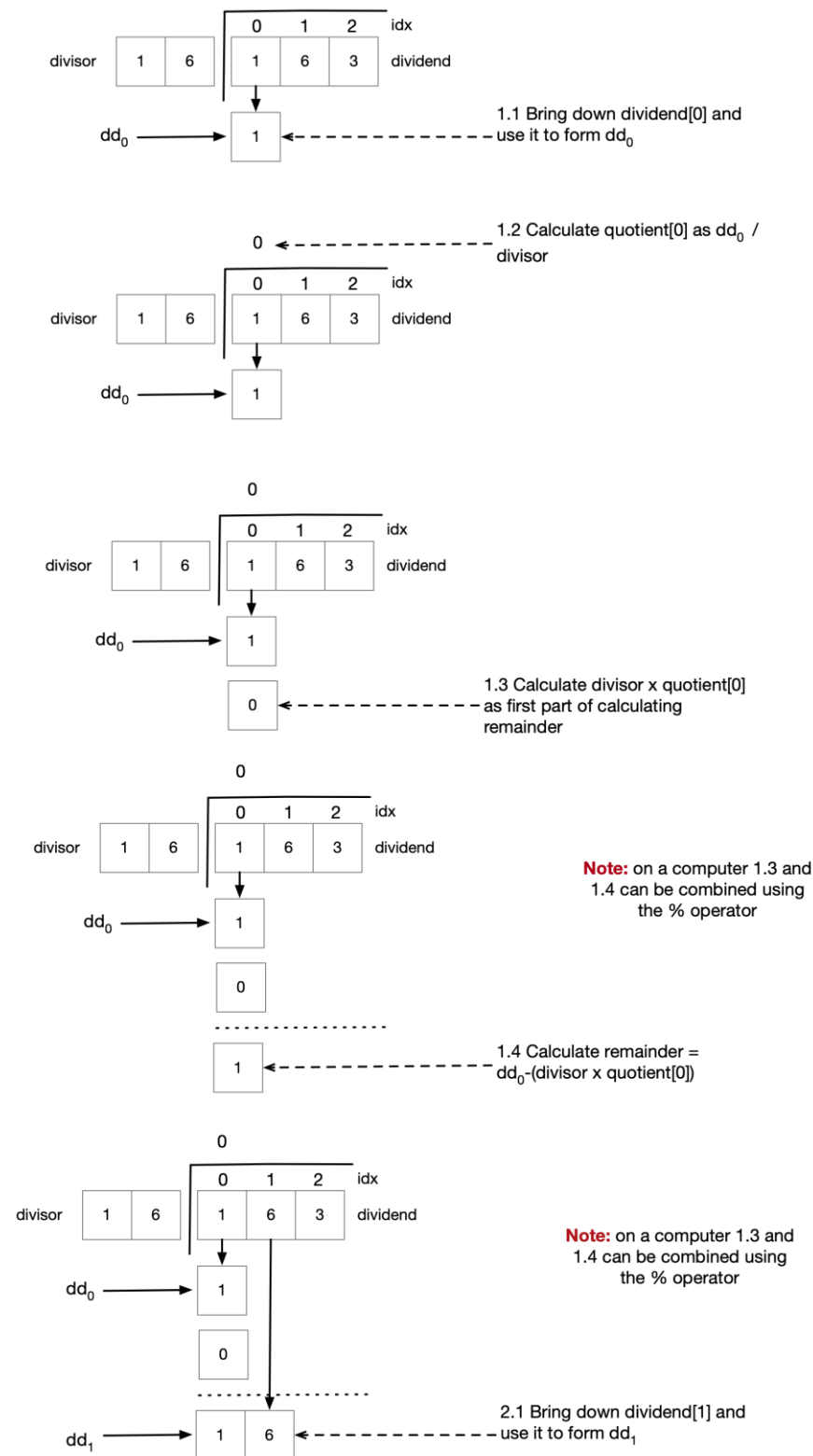
        // If the quotient digit q_i is non zero we subtract the
        // divisor from the dividendTemp
        if ( q_i > 0 )
            remainder -= divisor;
    }

    return (quotient,remainder);
}
```

Risk and Pricing Solutions

LONG DIVISION ANY BASE

Write code to perform integer division using a long division algorithm. The dividend is specified using a string. The base of the dividend and the divisor are given as simple ints?



Risk and Pricing Solutions

```
public (string quotient, string remainder) IntegerLongDivision(string
dividend, int divisor, int b = 10)
{
    StringBuilder quotient = new StringBuilder();
    int remainder = 0;
    int dd = 0;

    for (int idx = 0; idx < dividend.Length; idx++)
    {
        // idx.1 copy in next digit into temporary dividend dd
        dd = (dd * b) + dividend[idx].ToIntDigit();

        // idx.2 calculate partial quotient and set into quotient[idx]
        int partialQuotient = dd / divisor;
        quotient.Append(partialQuotient.ToChar());

        // idx.3 calculate this temporary as part of
        //calculating the remainder
        int temp = partialQuotient * divisor;

        // idx.4 Calculate the remainder
        remainder = dd % divisor;

        // the remainder will form the basis of dd[idx+1]
        dd = remainder;
    }

    return (quotient.ToString(), remainder.ToChar().ToString());
}

public static class Extensions
{
    public static int ToIntDigit(this char c)
    {
        if (char.IsNumber(c)) return (int)char.GetNumericValue(c);

        return char.ToLower(c) - 'a' + 10;
    }

    public static char ToChar(this int i)
    {
        if (i >= 0 && i < 10)
            return (char)(i + '0');

        return (char)(i + 'a' - 10);
    }
}
```

Risk and Pricing Solutions

LOG BASE 2 FLOOR (INTEGER)

Calculate Log base 2 floor integer using basic algorithm what is the runtime?

```
public byte LogBrute(byte x)
{
    if (x<=0) throw new ArgumentException();
    byte shiftCount =0;

    while (x >0)
    {
        x >>=1;
        shiftCount++;
    }

    return (byte)(shiftCount -1);
}
```

Risk and Pricing Solutions

Optimise your algorithm from the previous section

```
// Question: optimised algorithm to calculate
// -----floor(log(x)) to base 2
public int LogOpt(int x)
{
    int e = 0;

    if ((x & (~((1<<16)-1))) != 0)
    {
        // We have set digits in location 16-31 so we don't
        // care about the digits in locations 0-15. Add 16
        // and shift right to home in on exact location
        x >>= 16; e += 16;
    }

    if ((x & (~((1<<8)-1))) != 0)
    {
        // We have set digits in location 8-15 so we don't
        // care about the digits in locations 0-7. Add 8
        // and shift right to home in on exact location
        x >>= 8; e += 8;
    }

    if ((x & (~((1<<4)-1))) != 0)
    {
        // We have set digits in location 4-7 so we don't
        // care about the digits in locations 0-3. Add 4
        // and shift right to home in on exact location
        x >>= 4; e += 4;
    }

    if ((x & (~((1<<2)-1))) != 0)
    {
        // We have set digits in location 2-3 so we don't
        // care about the digits in locations 0-1. Add 2
        // and shift right to home in on exact location
        x >>= 2; e += 2;
    }

    if ((x & (~((1<<1)-1))) != 0)
    {
        // Finally is the digit in slot index 0 or 1
        e += 1;
    }

    return e;
}
```


Risk and Pricing Solutions

DIVISION TO FLOATING POINT

Modify your answer from the previous section to return a floating point result rather than quotient and remainder?

Risk and Pricing Solutions

Number Theory

IS PRIME NAÏVE

Calculate is prime using simple brute force. What is the runtime?

The following naïve implementation is $O(n)$

```
public static bool IsPrimeNaive(int n)
{
    if (n <= 1) return false;

    for (int i = 2; i < n; i++)
    {
        if (n % i == 0)
            return false;
    }
    return true;
}
```

The runtime is $O(N)$

IS PRIME SIMPLE OPTIMISATION

Optimise it. What is the runtime?

```
public bool IsPrimeUsingSquareRoot(int n)
{
    if (n < 2)
        return false;

    if (n == 2)
        return true;

    // The definition of a prime is an integer x
    // which is not exactly divisible by any
    // number other than itself and one. If a
    // number x is not prime it can be written as
    // the product of two factors a x b. If both
    // a and b were greater than the square root of
    // x then a x b would also be greater than x and hence
    // a x b is not x. SO testing all factors up to floor(root(x))
    // is sufficient as if one factor is floor(root(x)) the other factor must
    // be less than that

    // hence test the n-2 integers from
    // 2,..., Floor(Root(N))
    return Enumerable.Range(2, (int)Math.Floor(Math.Sqrt(n)))
        .All(i => n % i > 0);
}
```

Risk and Pricing Solutions

HIGHEST COMMON FACTOR

Implement Euclids algorithm for HCF. What is the runtime?

```
/// <summary>
/// Implementation of Euclids algorithm
/// </summary>
/// <param name="a"></param>
/// <param name="b"></param>
/// <returns></returns>
public static int HighestCommonFactor(int a, int b)
{
    if (a < b)
    {
        return HighestCommonFactor(b, a);
    }
    else
    {
        int remainder = a % b;

        if (remainder == 0)
        {
            return b;
        }
        else
        {
            return HighestCommonFactor(b, remainder);
        }
    }
}
```

LOWEST COMMON MULTIPLE

Calculate LCM using the algorithm from the previos section?

$$lcm(x, y) = \frac{x \times y}{hcf(x, y)}$$

Risk and Pricing Solutions

Floating Point

Precision and Range

PRECISION OF FLOAT

What is the precision of a single precision floating point number and why?

Six significant figures

The binary machine number $\varepsilon = 2^{-23}$ is the machine epsilon and is hence the smallest positive value such that $1 + \varepsilon \neq 1$. Because $2^{-23} \approx 1.2 \times 10^{-7}$ which if we write it out we see

0.00000012 If we see this value what it really means is that the value is

$$0.00000018 > x > 0.00000006$$

So only the sixth significant figure is accurate.

RANGE OF FLOAT

What is the range of a single precision floating point and why?

From $\approx 2^{128}$ to $\approx -(2^{128})$ which is approximately from 3.4×10^{38} to $-(3.4 \times 10^{38})$

The reason being that the largest absolute value representable in single precision is given by $(2 - 2^{-23})2^{127} \approx 2^{128} \approx 3.4 \times 10^{38}$ as the mantissa has 23 bits and the exponent has 8 bits.

PRECISION OF DOUBLE

What is the precision of a double precision floating point number and why?

The binary machine number $\varepsilon = 2^{-53}$ is the machine epsilon and is hence the smallest positive value such that $1 + \varepsilon \neq 1$. Because $2^{-53} \approx 1.1 \times 10^{-16}$ so only to the 15 significant figure is correct.