Differentiation

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Table 1 Rules of Differentiation

1. Constant Function Rule $f'(c) = 0, if\ c\ is\ a\ constant$
2. Constant Multiple Rule $$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$
3. Sum rule $$\frac{d}{dx}f(x) + g(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
4. Product Rule $$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + \frac{d}{dx}f(x)g(x)$$
5. Quotient Rule $$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{d}{dx}f(x)g(x) - \frac{d}{dx}g(x)f(x)}{[g(x)]^2}$$
6. Power Rule $$\frac{d}{dx}[x^n] = nx^{n-1}$$
7. Chain Rule If $y = f(g(x))$ then $\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$
8. Derivative of e^x $$\frac{d}{dx}e^x = e^x$$
9. Inverse Function $$\frac{d}{dx}f^{-1}(y) = \frac{dx}{dy} \times \frac{dy}{dx} = 1$$
10. Natural logarithm $$\frac{dy}{dx}ln\,x = \frac{1}{x}$$

Definition of the derivative

The derivative measures the rate of change of one quantity with respect to another. Differentiation is then just the process of finding the derivative of a function. If we have a function of x then one of the many notations for specifying the derivative is as follows

$$\frac{d}{dx}f(x)$$

So if we took one or the simplest non-linear functions $f(x) = x^2$and differentiate it we see that

$$\frac{d}{dx}x^2 = 2x.$$

So in the simple case where x is equal to one

$$\frac{d}{dx}f(x) = 2$$

$$df(x) = dx \times 2$$

It is worth noting that differential calculus is concerned with finding the instantaneous rate of change of f with respect to x.

Why Bother?

Numerous problems in business, economics and finance are concerned with determining how one quantity is changing with respect to another. Differentiation also enables us to find where a function is highest and lowest both locally and across the entire domain. Also we often find where a rate of change is greatest or smallest and again differentiation provides us with this.

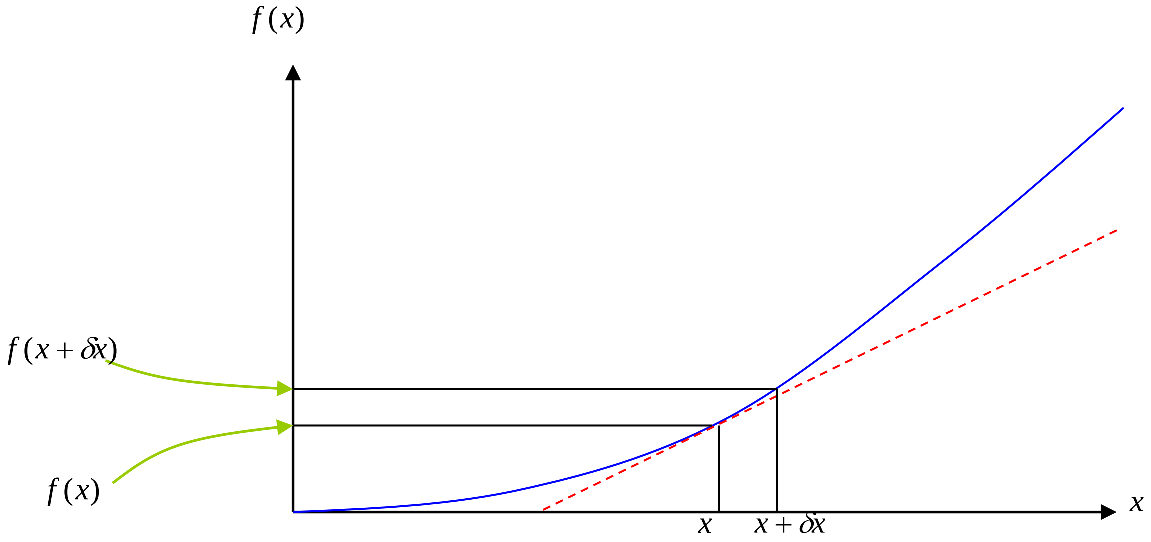
Where the derivative does not exist

There are three places where the derivative does not exist

- Discontinuity
- Cusp on a function
- Vertical inflection point

Calculation of the derivative

The approach



$$\frac{f(x+\delta x)-f(x)}{\delta x} \to \frac{df(x)}{dx}$$

Algebraic Derivation

If $y = f(x)$ then the derivative $\frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right)$is defined as $lim_{x\to\alpha}\frac{f(x+\delta x)-f(x)}{\delta x}$

Simple Example

Let us consider a basic quadratic $y = f\left(x\right) = x^2$ then the derivative becomes

$lim_{x\to\alpha}\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{(x+\delta x)^2-x^2}{\delta x} = 2x$

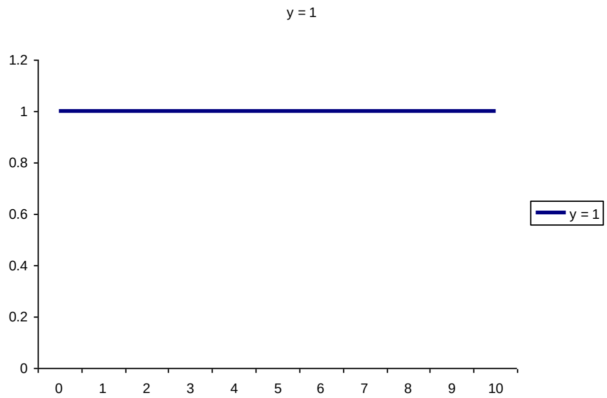
Proofs

Constant Function

The derivative of a constant function is zero

$$f'(c) = 0$$

$$f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = lim_{h\to 0}\frac{c-c}{h} = 0$$



Constant Multiple Rule

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

$$\frac{d}{dx}[cf(x)] = lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = lim_{h\to 0}\, c\frac{f(x+h)-cf(x)}{h} = c\frac{d}{dx}f(x)$$

Sum Rule

$$\frac{d}{dx}f(x)+g(x) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

$$\frac{d}{dx}f(x)+g(x) = \boldsymbol{lim}_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

Product Rule

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + \frac{d}{dx}f(x)g(x)$$

$$\frac{d}{dx}f(x)g(x) = lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add and subtract $f(x+h)g(x)$ to the numerator

$$\frac{d}{dx}f(x)g(x) = lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h}$$

Arrange the terms on the numerator

$$= lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Factorize

$$= lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$$= lim_{h\to 0} f\,(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}$$

$$= lim_{h\to 0} f\,(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Risk and Pricing Solutions

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{d}{dx}f(x)g(x) - \frac{d}{dx}g(x)f(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h}$$

Add and subtract $f(x)g(x)$ to the numerator

$$= \boldsymbol{lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h) + f(x)g(x) - f(x)g(x)}{g(x+h)g(x)h}}$$

Re-arrange

$$= lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{g(x+h)g(x)h}$$

Factorize

$$= lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)h}$$

$$= lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)h}$$

Because $\frac{\frac{a}{b}}{cd} = \frac{a}{bcd} = \frac{\frac{a}{bc}}{d}$ we re-arrange the above expression as

$$= lim_{h\to 0} \frac{\frac{g(x)[f(x+h)-f(x)]}{h} - \frac{f(x)[g(x+h)-g(x)]}{h}}{g(x+h)g(x)}$$

$$= \frac{lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h} - lim_{h\to 0} \frac{f(x)[g(x+h)-g(x)]}{h}}{lim_{h\to 0}\, g\,(x+h) \times lim_{h\to 0}\, g\,(x)}$$

$$= \frac{\frac{d}{dx} f(x)g(x) - \frac{d}{dx} g(x)f(x)}{[g(x)]^2}$$

Power Rule

$$\frac{d}{dx}f(x) = mx^{m-1}$$

By mathematical induction where m = 0

$f(x) = x^0 = 1\text{then}f'(x) = 0$ by the constant rule

So the rule works for m = 0 since $0 = 0x^{-1}$. Now we assume the rule works for m and prove it works for m+1

$$f(x) = x^{m+1} = x^m x$$

Using the power rule

$$f'(x) = \frac{d}{dx}x^m \times x + \frac{d}{dx}x \times x^m$$

$= (m-1)x^{m-1}x + 1x^m$Because we assume is correct for m and from the constant rule

$$= (m)x^m + x^m$$

$$= (m)x^m + x^m$$

$$= (m+1)x^m$$

Since the rule works for m = 0 and m = m+1 it works for all m

Chain Rule

The chain rule expresses a very simple notion in a slightly complex fashion. If $y = f(g(x))$

$$\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

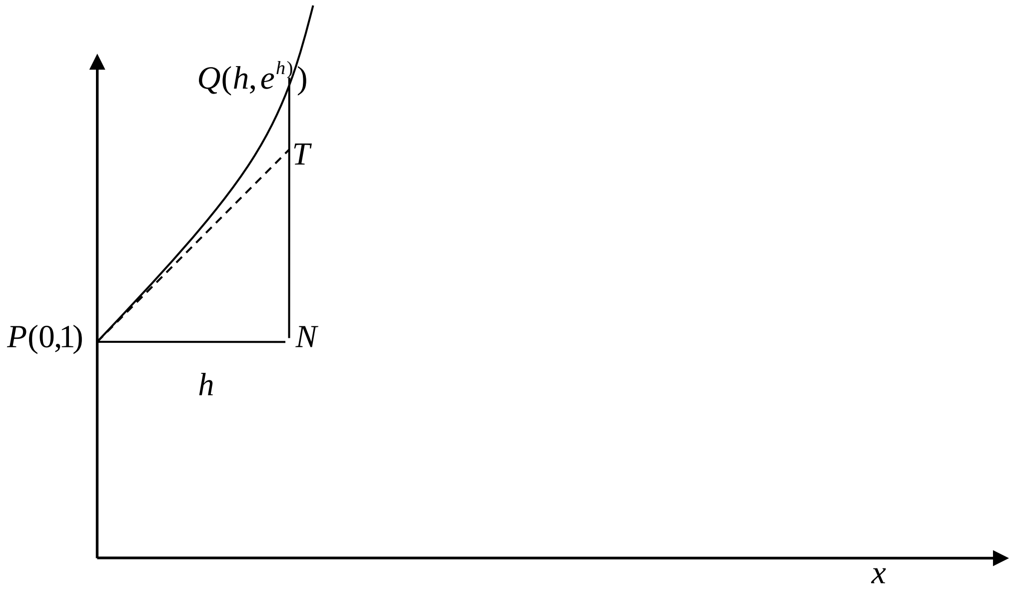
And by extension if $y = f(g(h(x)))$

$$\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$$

Etc.

The proof is fairly self-explanatory. If a unit change in x leads to a three unit change in g and a unit change in g leads to a four unit change in f then a unit change in x leads to a twelve unit change in f.

Derivative of e^x



The definition of the letter E is the number with the unique property that its gradient at the point it crosses the y-axis is one. From this it follows that

$$lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

So from the definition of the derivative we have

$$\frac{d}{dx} e^x = lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x (e^h - 1)}{h}$$

Plugging in the previous result that $lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$\frac{d}{dx} e^x = e^x$$

Derivative of inverse functions

The defining property of an inverse function is given by

$$y = f(x)$$

$$f^{-1}\big(f(x)\big) = x$$

$$f^{-1}(y) = x$$

So the inverse function is a function of a function. By the chain rule

$$\frac{d}{dx}f^{-1}(y) = \frac{d}{dy}f^{-1}(y) \times \frac{d}{dx}f(x)$$

And since $f^{-1}(y) = x$and $y = f(x)$ we get

$$\frac{d}{dx}f^{-1}(y) = \frac{dx}{dy} \times \frac{dy}{dx} = 1$$

Also we can not that an inverse function is a reflection of the original function about the line $y = x$. After such a reflection the new function will have a gradient equal to the reciprocal of the original function

Natural Logarithm

We can use the result of differentiating the exponent to differentiate the natural logarithm

$$y = ln\,x \therefore$$

$$x = e^y \therefore$$

$$\frac{dx}{dy} = e^y \therefore$$

$$dx = e^y dy \therefore$$

$$1 = e^y \frac{dy}{dx} \therefore$$

$$\frac{dy}{dx} = \frac{1}{e^y} \therefore$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Proof Natural logarithm 2

If $y = ln\,x$then $x = e^y$

We know that $\frac{dx}{dy} = e^y$and from the derivative of a reciprocal that $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

<u>Alternative</u>

$$y = ln\,x \therefore$$

$$x = e^y \therefore$$

$$\frac{dx}{dy} = e^y \therefore$$

$$dx = e^y dy \therefore$$

$$1 = e^y \frac{dy}{dx} \therefore$$

$$\frac{dy}{dx} = \frac{1}{e^y} \therefore$$

$\frac{dy}{dx} = \frac{1}{x}$

SINE

We first note that that $lim_{h\to 0}\frac{sin(x)}{x} = 1$which we will need later

$$f''(sin\,x) = lim_{h\to 0}\frac{sin(x+h) - sin(x)}{h} =$$

$$lim_{h\to 0}\frac{sin(\,x)\,cos(\,h) + sin(\,h)\,cos(\,x) - sin\,x}{h}$$

Now we can re-arrange the terms to give us

$$lim_{h\to 0}\frac{sin(\,x)[cos(\,h) - 1}{h} + \frac{sin(\,h)\,cos(\,x)}{h}$$

Now the first term will tend to zero and the second term will tend to $\cos(x)\bullet 1 = \cos(x)$

Derivative of x^n

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof (a)

We can easily proof this result using the chain rule. First we note that

$$x^n = (e^{\ln x})^n = e^{n \ln x}$$

Then by the chain rule and let $u = n \ln x$

$$\frac{d}{dx}[x^n] = \frac{d}{dx} e^{n \ln x} = \frac{d}{du} e^u \frac{d}{dx} n \ln x$$

$$= e^u n \frac{1}{x}$$

$$= x^n n \frac{1}{x}$$

$$= nx^{n-1}$$

Proof (b)

$$\frac{d}{dx}f(x) = mx^{m-1}$$

By mathematical induction where m = 0

$f(x) = x^0 = 1$ then $f'(x) = 0$ by the constant rule

So the rule works for m = 0 since $0 = 0x^{-1}$. Now we assume the rule works for m and prove it works for m+1

$$f(x) = x^{m+1} = x^m x$$

Using the power rule

$$f'(x) = \frac{d}{dx}x^m \times x + \frac{d}{dx}x \times x^m$$

$= (m-1)x^{m-1}x + 1x^m$ Because we assume is correct for m and from the constant rule

$$= (m)x^m + x^m$$

$$= (m)x^m + x^m$$

$$= (m+1)x^m$$

Since the rule works for m = 0 and m = m+1 it works for all m

Derivative of sin x

We first note that that $\lim_{h\to 0}\frac{\sin(x)}{x} = 1$ which we will need later

$$f'(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} =$$

$$\lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin x}{h}$$

Now we can re-arrange the terms to give us

$$\lim_{h\to 0}\frac{\sin(x)[\cos(h)-1}{h}+\frac{\sin(h)\cos(x)}{h}$$

Now the first term will tend to zero and the second term will tend to $\cos(x)\bullet 1 = \cos(x)$

Worked Examples

Worked Example 1Differentiate $f(x) = \sqrt{x}$

$$f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

If we multiply the top and bottom by $\sqrt{x}+h+\sqrt{x}$ we get

$$f'(x) = lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

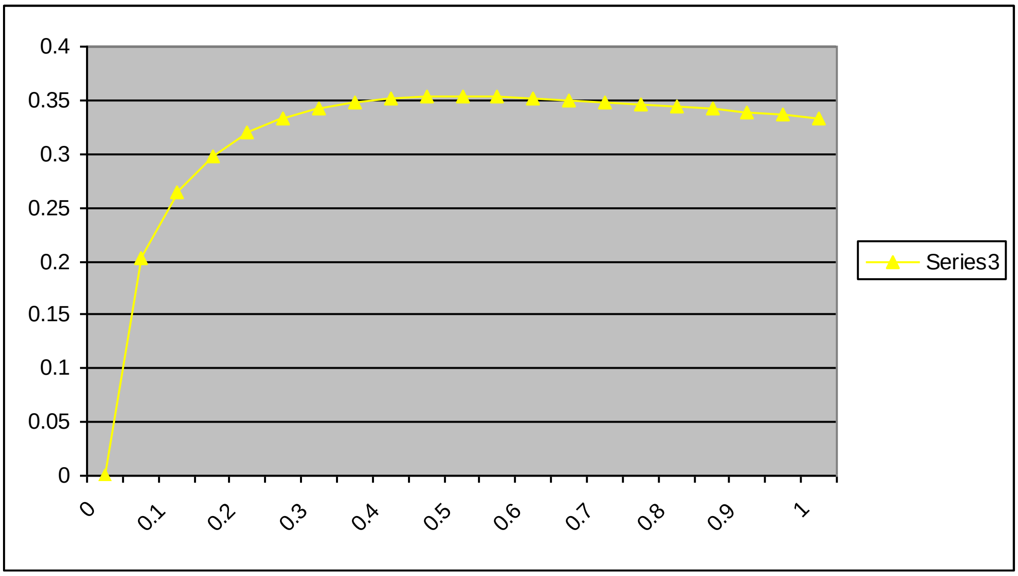
$$= lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$



Worked Example 2 Differentiate $f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

$$lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\left(\sqrt{x+h}\right)\sqrt{x}}$$

If we multiply the top and bottom by $\sqrt{x}+h+\sqrt{x}$ we get

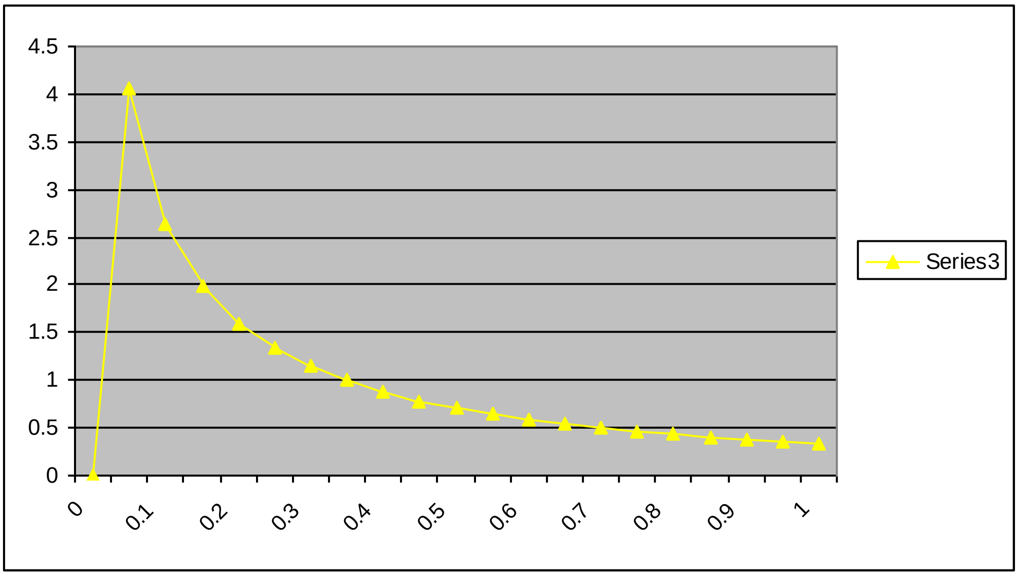
$$lim_{h\to 0}\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{x+h}\right)^2}{h\left(\sqrt{x+h}\right)\sqrt{x}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$lim_{h\to 0}\frac{x-x-h}{h\left(\sqrt{x+h}\right)\sqrt{x}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$lim_{h\to 0}\frac{-1}{\left(\sqrt{x+h}\sqrt{x+h}\sqrt{x}\right)+\left(\sqrt{x+h}\sqrt{x}\sqrt{x}\right)}$$

$$lim_{h\to 0}\frac{-1}{\left(x+h\sqrt{x}\right)+\left(x\sqrt{x+h}\right)} = \frac{-1}{x\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{-1}{x\sqrt{x}+x\sqrt{x}}$$

$$= \frac{-1}{2x\sqrt{x}} = \frac{-1}{2}x^{\frac{-3}{2}}$$



Worked Example 3 Differentiate $f(x) = \frac{1}{\sqrt{x+2}}$

$$f'(x) = lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h+2}} - \frac{1}{\sqrt{x+2}}}{h}$$

$$lim_{h \to 0} \frac{\sqrt{x+2} - \sqrt{x+h+2}}{h\left(\sqrt{x+h+2}\right)\sqrt{x+2}}$$

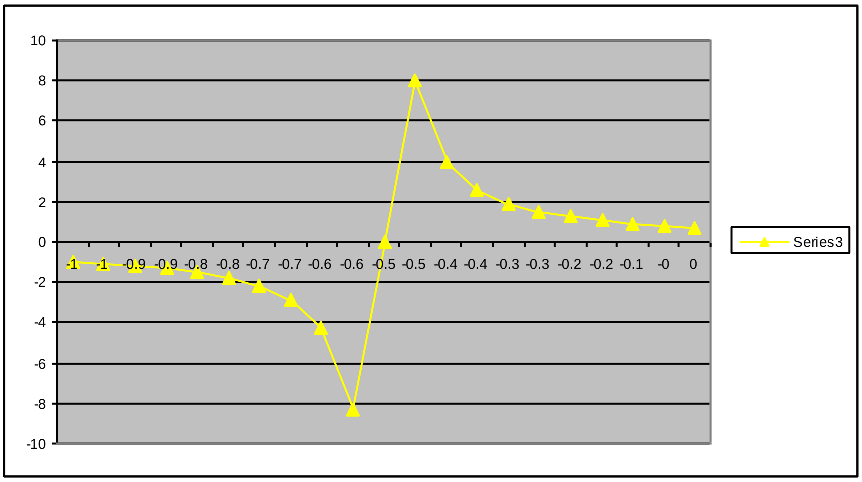
If we multiply the top and bottom by $\sqrt{x+h+2} + \sqrt{x+2}$ we get

$$lim_{h \to 0} \frac{\left(\sqrt{x+2}\right)^2 - \left(\sqrt{x+h+2}\right)^2}{h\sqrt{x+h+2}\sqrt{x+2}\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$lim_{h \to 0} \frac{-1}{\sqrt{x+h+2}\sqrt{x+h+2}\sqrt{x+2} + \sqrt{x+2}\sqrt{x+2}\sqrt{x+h+2}}$$

$$lim_{h \to 0} \frac{-1}{(x+h+2)\sqrt{x+2} + (x+2)\sqrt{x+h+2}}$$

$$\frac{-1}{(x+2)\sqrt{x+2}+(x+2)\sqrt{x+2}}=\frac{-1}{2(x+2)\sqrt{x+2}}=\frac{-1}{2(x+2)^{\frac{3}{2}}}$$



Worked Example 4 If $F(x) = x^3 - 5x + 1$, find $F'(1)$ and use it to find the tangent line to the curve $y = x^3 - 5x + 1$ at the point $(0,`1)$.

$$F'(x) = \frac{F(x+h) - F(x)}{h}$$

$$F'(x) = \frac{(x+h)^3 - 5(x+h) + 1 - [x^3 - 5x + 1]}{h}$$

<u>Binomial Expansion</u>

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$(x+h)^3 = 1.x^3h^0 + 3x^2h^1 + 3x^1h^2 + 1.^0h^3$$

$$(x+h)^3 - 5(x+h) + 1 - [x^3 - 5x + 1] = h(3x^2 + 3xh + h^2 - 5)$$

$$F'(x) = lim_{h-0} \frac{h(3x^2 + 3xh + h^2 - 5)}{h}$$

$$F'(x) = 3x^2 - 5$$

$$\boldsymbol{F'(1) = -2}$$

We know that the slope of the tangent and that is passes through the point $(0,1)$

$$y = y^1 + m(x - x^1) = 1 + -2(x - 0)$$

$y = -2x + 1$

Worked Example 5

If $G(x) = \frac{x}{1+2x}$find $G'(x)$and use it to find the equation of the tangent to the curve at the point $G(-{}^{1}/_{4}, {}^{-1}/_{2})$

$$G'(x) = lim_{h\to 0} \frac{G(x+h) - G(x)}{h}$$

$$= lim_{h\to 0} \left(\frac{x+h}{1+2(x+h)} - \frac{x}{1+2x}\right)\frac{1}{h}$$

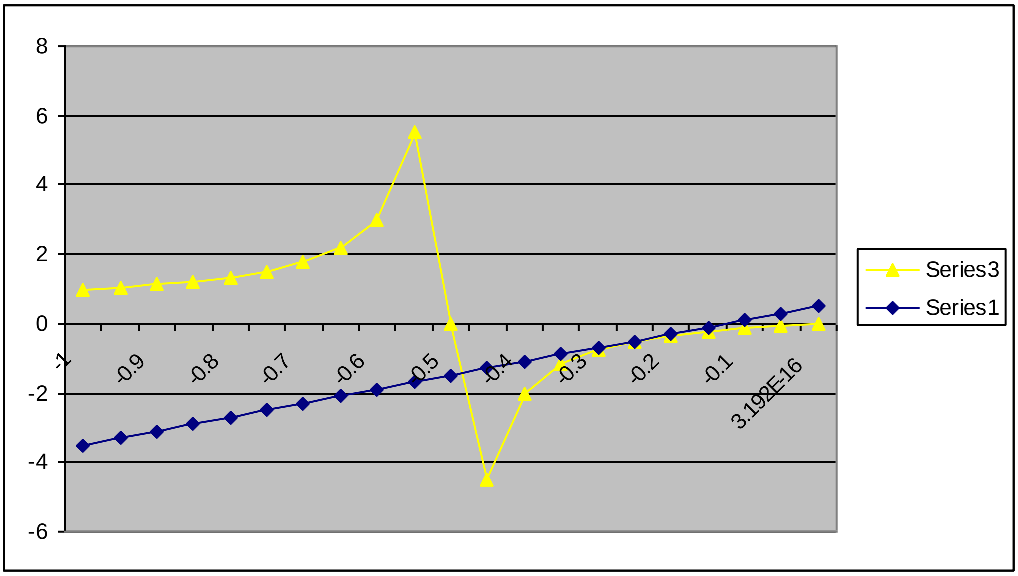
$$= lim_{h\to 0} \left(\frac{(x+h)(1+2x) - x(1+2x+2h)}{1+2(x+h)(1+2x)}\right)\frac{1}{h}$$

$$= lim_{h\to 0} \frac{x + 2x^2 + h + 2xh - x - 2x^2 - 2xh}{(1 + 2x + 2x + 4x^2 + 2h + 2xh)h}$$

$$= lim_{h\to 0} \frac{1}{(4x + 4x^2 + 1 + 2h + 2xh)}$$

$$= \frac{1}{(4x + 4x^2 + 1)}$$

The gradient of the tangent to the curve at point $G(-{}^{1}/_{4}, {}^{-1}/_{2})$is given by $G'(-1/4) =$ 4and the equation of the tangent is $y = 4x + .5$



Worked Example 5

Differentiate $y = \sqrt[3]{3x^2} - \dfrac{1}{\sqrt{5x}}$

First re-write the function as $y = \left(3x^2\right)^{1/3} - (5x)^{1/2}$

Now differentiate each term in turn using the chain rule

$$\frac{dy}{dx} = \frac{1}{3}\left(3x^2\right)^{-2/3} 6x + \frac{1}{2}(5x)^{-1/2} \times 5$$

Simplifying

$$\frac{dy}{dx} = 2x\left(3x^2\right)^{-2/3} + \frac{5}{2}(5x)^{-1/2}$$

Express using positive powers only

$$\frac{dy}{dx}=\frac{2x}{\left(3x^2\right)^{2/3}}+\frac{5}{2(5x)^{1/2}}$$ Note that $\left(3x^2\right)^{2/3}=\left(9x^4\right)^{1/3}$ to rewrite

$$\frac{dy}{dx}=\frac{2x}{\left(9x^4\right)^{1/3}}+\frac{5}{2(5x)^{1/2}}$$ Note that $2(5x)^{1/2}=2(5x)(5x)^{1/2}$ to rewrite as

$$\frac{dy}{dx}=\frac{2x}{\left(9x^4\right)^{1/3}}+\frac{5}{2\times 5\times x\times (5x)^{1/2}}$$ Simplify

$$\frac{dy}{dx}=\frac{2x}{\left(9x^4\right)^{1/3}}+\frac{1}{2x(5x)^{1/2}}$$ Note that $\frac{2x}{\left(9x^4\right)^{1/3}}=\frac{2x}{9^{1/3}x^{4/3}}=\frac{2}{9^{1/3}x^{1/3}}=\frac{2}{\sqrt[3]{9x}}$

Finally expressing the whole derivative in surd notation

$$\frac{dy}{dx}=\frac{2}{\sqrt[3]{9x}}+\frac{1}{2x\sqrt{5x}}$$

Worked Example 6

$$y = \sqrt[3]{t}\left(t^2 + t + t^{-1}\right) = t^{1/3}\left(t^2 + t + t^{-1}\right) = t^{7/3} + t^{4/3} + t^{-2/3}$$

$$\frac{dy}{dx} = \frac{7}{3}t^{4/3} + \frac{4}{3}t^{1/3} - \frac{2}{3}t^{-5/3}$$

$$= \frac{1}{3}t^{-5/3}\left(7t^{9/3} + 4t^{6/3} - 2\right)$$

$$= \frac{1}{3}t^{-5/3}\left(7t^2 + 4t^2 - 2\right)$$

$$= \frac{\left(7t^2 + 4t^2 - 2\right)}{3t^{5/3}}$$

Worked Example 7

Differentiate $f(x) = \dfrac{x}{x + \frac{c}{x}}$

$$f'(x) = \frac{x + \frac{c}{x} - \left[\frac{d}{dx}\left(x + \frac{c}{x}\right) \times x\right]}{\left(x + \frac{c}{x}\right)^2}$$ By the quotient rule

$$f = \frac{x + \frac{c}{x} - x + \frac{c}{x}}{\left(\frac{x^2 + c}{x}\right)^2}$$

$$f = \frac{\frac{2c}{x}}{\frac{(x^2+c)^2}{(x)^2}} \text{ Multiple by } \frac{x^2}{x^2}$$

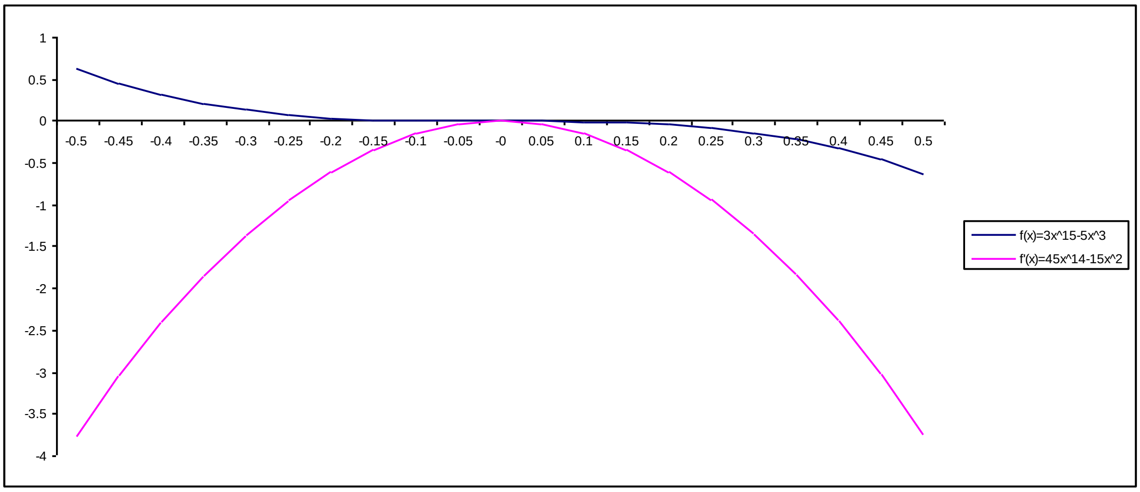
$$f = \frac{2cx}{(x^2+c)^2}$$

Worked Example 8

$$f(x) = 3x^{15} - 5x^3$$

$$f'(x) = 45x^{14} - 15x^2$$

Notice that the derivative is zero when the function has a horizontal tangent and that as in the given interval the function is always decreasing so the derivative is always negative



Risk and Pricing Solutions

Worked Example 9

If $y = x^2 - 4x$ and $x = \sqrt{2t^2 + 1}$ find $\frac{dy}{dt}$ when $t = \sqrt{2}$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dx} = 2x - 4, \quad \frac{dx}{dt} = \frac{1}{2}\left(2t^2 + 1\right)^{\frac{-1}{2}} 4t = \frac{2t}{\left(2t^2 + 1\right)^{\frac{1}{2}}}$$

$$\frac{dy}{dt} = \frac{2t(2x - 4)}{\left(2t^2 + 1\right)^{\frac{1}{2}}} = \frac{4t(x - 2)}{\left(2t^2 + 1\right)^{\frac{1}{2}}}$$

If $t = \sqrt{2}$ then $x = \sqrt{\left(\sqrt{2}\right)^2 + 1} = \sqrt{5}$

$$\frac{dy}{dt} = \frac{4\sqrt{2}\left(2\sqrt{5} - 4\right)}{\left(2\left(\sqrt{2}\right)^2 + 1\right)^{\frac{1}{2}}} = \frac{4\sqrt{2}\left(\sqrt{5} - 2\right)}{\sqrt{5}} \quad \text{Multiply by } \frac{\sqrt{5}}{\sqrt{5}}$$

$$\frac{4\sqrt{2}\left(\sqrt{5} - 2\right)\sqrt{5}}{5} = \frac{4\sqrt{2}}{5}\left(5 - 2\sqrt{5}\right)$$