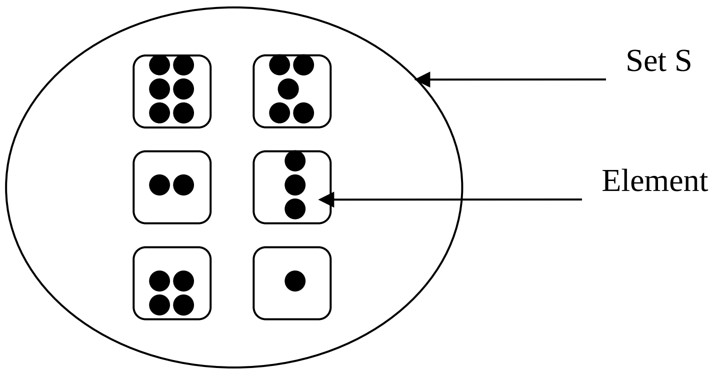
Foundations of Probability

Sets Theory

Set

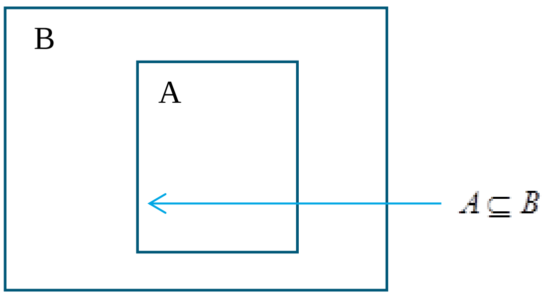
A set is a collection of things. We call the things elements of the set. If a set consists of the difference faces of a die we can write.

$$S = \{x: 1 \leq x \leq 9, x \in Z\}$$



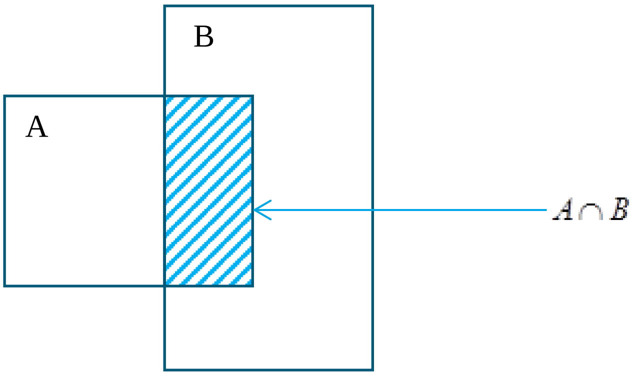
Subsets

Every element of A is also an element of B



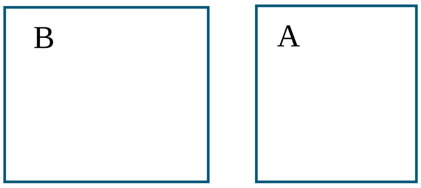
Intersection

The set containing the elements in A and B



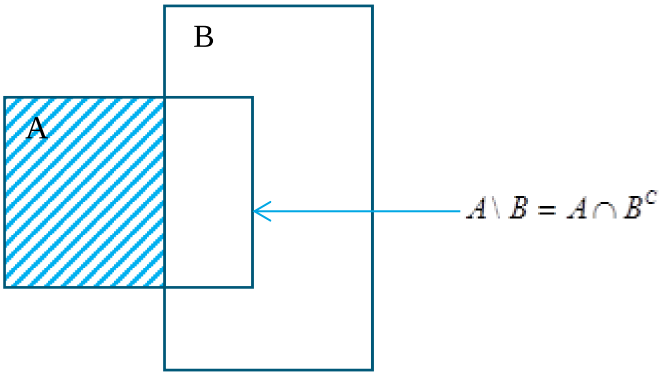
Disjoint sets

$A \cap B = 0$



Difference

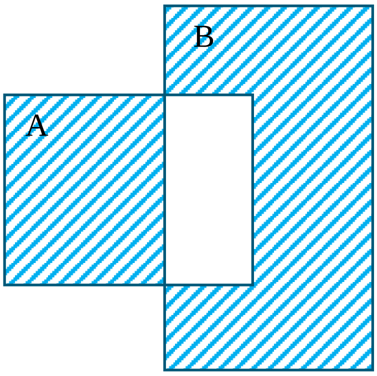
The elements in A but not in B



Symmetric Difference

The elements in A or B but not in both

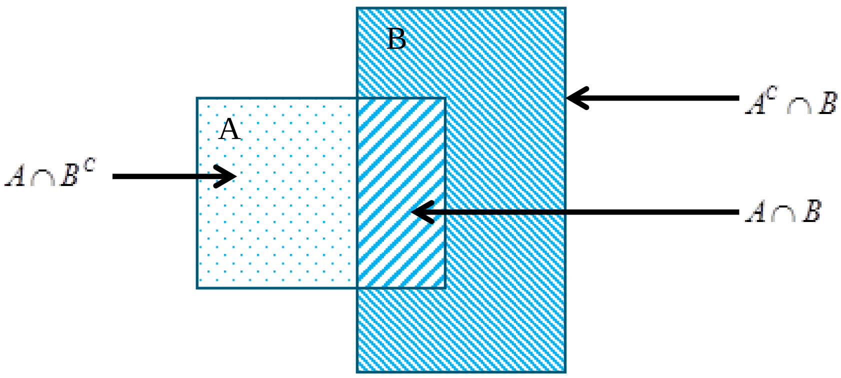
$$A \Delta B = (A \backslash B) \cup (B \backslash A)$$



$$A \Delta B = (A \backslash B) \cup (B \backslash A)$$

Union

Given two sets A and B we can define the set of all elements either in A or B or both is denoted AUB.



$$A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$$
$$= A \backslash B \cup B \backslash A \cup (A \cap B)$$

Product of two sets

If A and B are sets we can form the product C as

$$C = \{(a, b): a \in A, b \in B\}$$

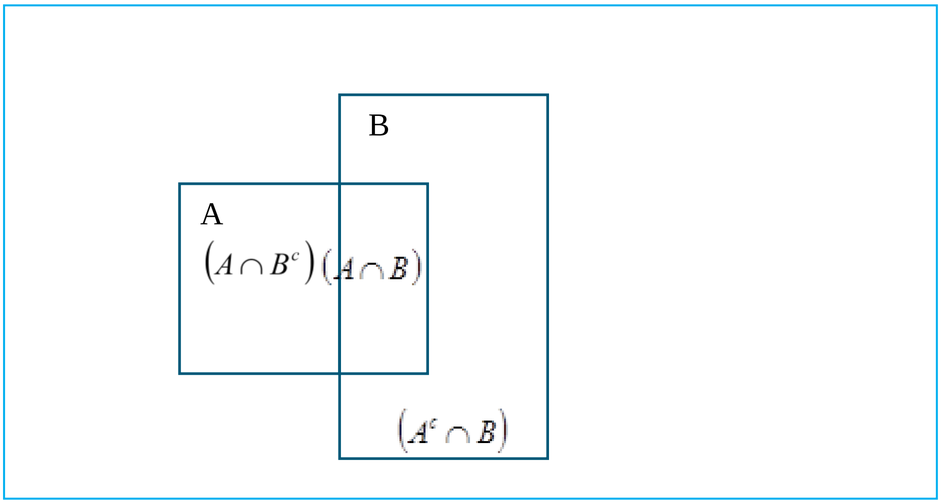
And we write

$$C = A \times B$$

Basic Rules

Probability that either of two events occurs P(AB)

We can calculate the probability of either one of two events A or B occuring hence.

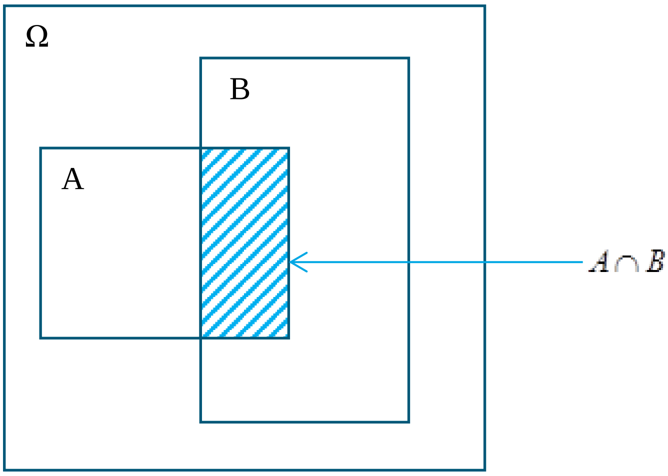


$$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$$

Because the three sets on the right hand side are disjoint. We can get a similar result by adding

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that both events



If we consider discrete probability then where every outcome is equally likely then the probability of A∩B is simply the number of outcomes is A∩B divided by the number of outcomes in the sample space Ω

$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} A \cap B^C$

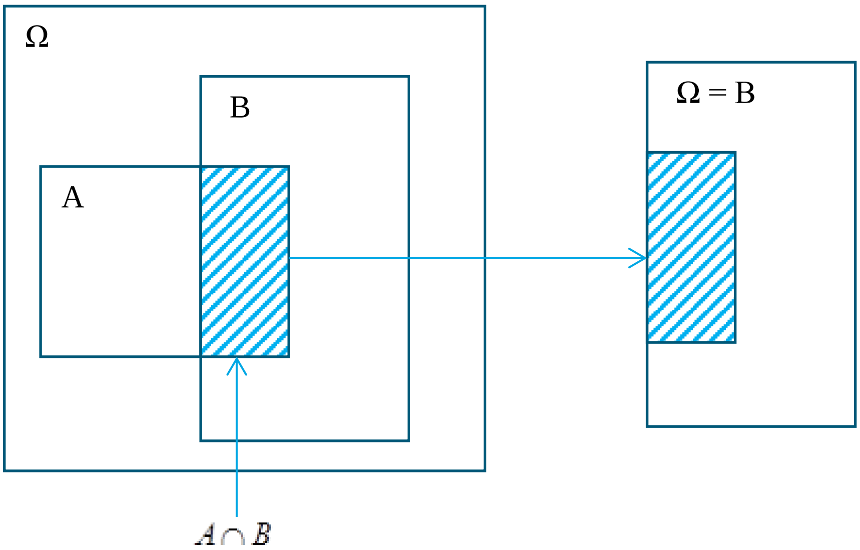
Conditional Probability

P(A | B)

The conditional probability P(A|B) is the probability that the event A occurs given that the event B has occurred. Of course for A to occur given that B has occurred, the two events A and B must share outcomes. We know that

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|}$$

But if we know that B has occurred there is a higher probability than P(A∩B) that A occurs because the extra information that B has occurred allows us to reduce the sample space.



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Because.

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|B|} \div \frac{|\Omega|}{|\Omega|} = \frac{|A \cap B|}{|\Omega|} \div \frac{|B|}{|\Omega|} = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule

Similarly if we are given $P(A_2|A_1)$ we can covert it back to $P(A_1 \cap A_2)$ by multiplying it through by $P(A_1)$

$$P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$$

We can extend this to three events

$$P(A_1 \cap A_2 \cap A_3) = P(A_3|A_2 \cap A)P(A_2 \cap A_1)$$
$$= P(A_3|A_2 \cap A)P(A_2|A_1)P(A_1)$$

And then n events

$$P(A_1 \cap A_2 \ldots \cap \ldots A_n) = P(A_n|A_{n-1} \cap \ldots \cap A_1)P(A_{n-1} \cap \ldots \cap A_1)$$
$$\times P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1)P(A_{n-2} \cap \ldots \cap A_1)$$

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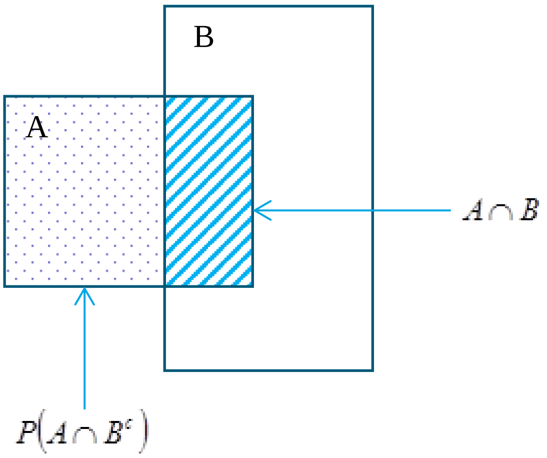
Partitition Rule

Any event A can be partitioned into those outsomes it shares with a second event B and those outcomes it doesn't share with B.

$P(A) = P(A \cap B) + P(A \cap B^c)$



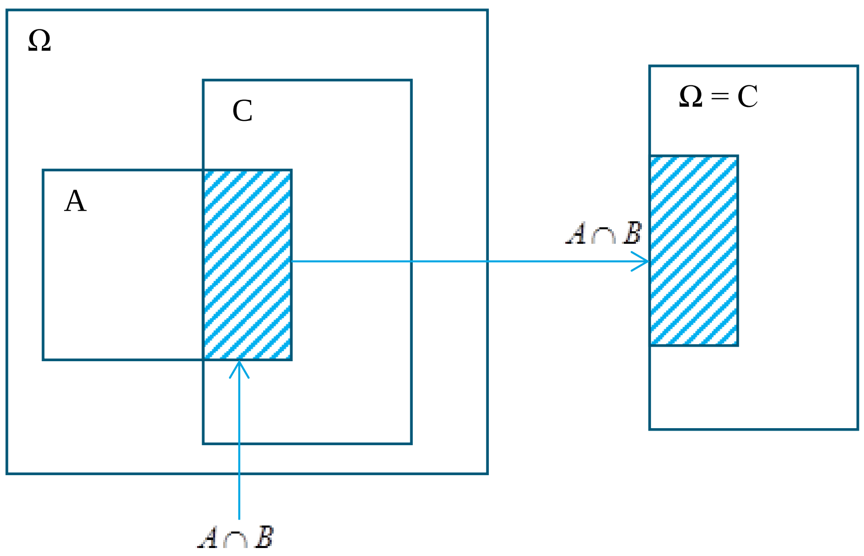
We can express this using conditional probabilities as.

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Conditional Partitition Rule

$$P(A|C) = P(A|B \cap C)P(B|C) + P(A|B^c \cap C)P(B^c|C)$$

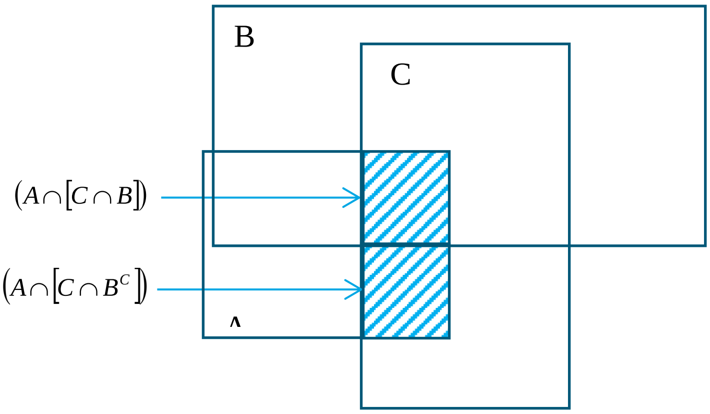
For a proof of this consider the following



$$P(A|C) = \frac{P(A \cap C)}{P(C)} \tag{1}$$

But the set $A \cap C$ can be broken up into the part that interects with a third set B and the part that doesn't intersect with the third set B

$$A \cap C = (A \cap [C \cap B]) \cup (A \cap [C \cap B^C]) \tag{2}$$



So we can insert 2 into 1

$$P(A|C) = \frac{P(A \cap [(C \cap B) \cup (C \cap B^C)])}{P(C)} \qquad (3)$$

Now we need to remember that $P(A \cap C \cap B) = P(A|C \cap B)P(C \cap B)$we update the numerator on the RHS of 3

$$P(A|C) = \frac{P(A|C \cap B)P(C \cap B) + P(A|C \cap B)P(C \cap B^C)}{P(C)} \qquad (4)$$

Finally we note that $P(C \cap B) = P(B|C)P(B)$and use this to update the numerator on the RHS

$$P(A|C) = \frac{P(A|C \cap B)P(B|C)P(C) + P(A|C \cap B)P(B^C|C)P(C)}{P(C)} \qquad (5)$$

Finally we cancel the P(C)'s

$$P(A|C) = P(A|C \cap B)P(B|C) + P(A|C \cap B)P(B^C|C) \qquad (6)$$