Putting it together: scaling and shifting

We now know that if we have a process x with mean zero and variance one we can scale and shift it to a process with mean μ and variance σ^2 by adding μ and multiplying by $\sqrt{\sigma^2} = \sigma$. Our new random variable $(x+\mu)\sigma$ is now distributed with mean μ and variance σ^2

Even more useful is the fact that if we know that a random variable x is distributed with mean μ and variance σ^2 then we also know that the random variable $\frac{(x-\mu)}{\sigma}$ is distributed with mean zero and variance one

Increasing the number of steps

From our previous sections we can see that if we sum n identical independent random variables with mean zero and unit variance we obtain a new random variable

$$X_n = \sum_{i=1}^{n} x_n$$

With mean zero and variance n. But what happens as we increase the number of steps? In the limit as $n \to \infty$ X_n is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Using the standard normal distribution

The standard normal distribution with mean zero and standard deviation has been studied extensively and its properties are well known. So if we have a random variable x which we know is normally distributed with mean μ and variance σ^2 then we can represent it via a scale and shift of the standard normal distribution $\phi(0,1)$ as follows

$$x = \mu + \sigma\phi(0,1)$$

Stochastic time series

Overview

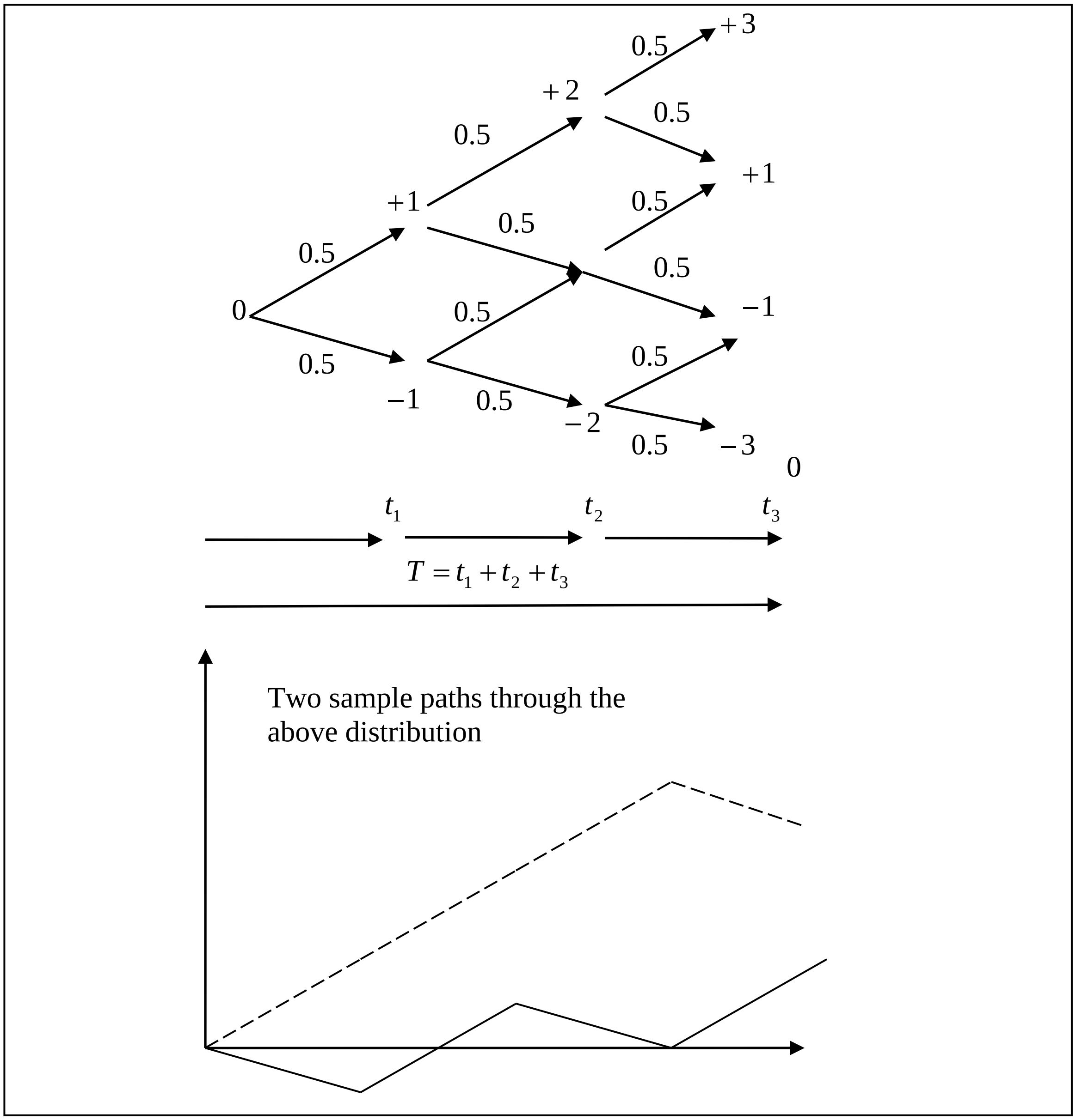
We know from the previous section that the sum of n identical, independently distributed random variables with mean μ and finite non-zero variances σ^2 is a random variable with mean $n\mu$ and variance $n\sigma^2$.

$X_1 = x_1$	Mean μ and standard deviation σ^2
$X_2 = x_1 + x_2$	Mean 2μ and standard deviation $2\sigma^2$
$X_3 = x_1 + x_2 + x_3$	Mean 3μ and standard deviation $3\sigma^2$
$X_4 = x_1 + x_2 + x_3 + x_4$	Mean 4μ and standard deviation $4\sigma^2$
$X_n = x_1 + x_2 + x_3 + ... + x_n$	Mean $n\mu$ and standard deviation $n\sigma^2$

What if we were to replace the subscript n with the subscript t as follows?

$X_1 = x_1$	Mean μ and standard deviation σ^2
$X_2 = x_1 + x_2$	Mean 2μ and standard deviation $2\sigma^2$
$X_3 = x_1 + x_2 + x_3$	Mean 3μ and standard deviation $3\sigma^2$
$X_4 = x_1 + x_2 + x_3 + x_4$	Mean 4μ and standard deviation $4\sigma^2$
$X_t = x_1 + x_2 + x_3 + ... + x_t$	Mean $t\mu$ and standard deviation $t\sigma^2$

We can view X_t as a random variable which describes the displacement of a particle that starts at the origin and is displaced the distance X_t at time t.

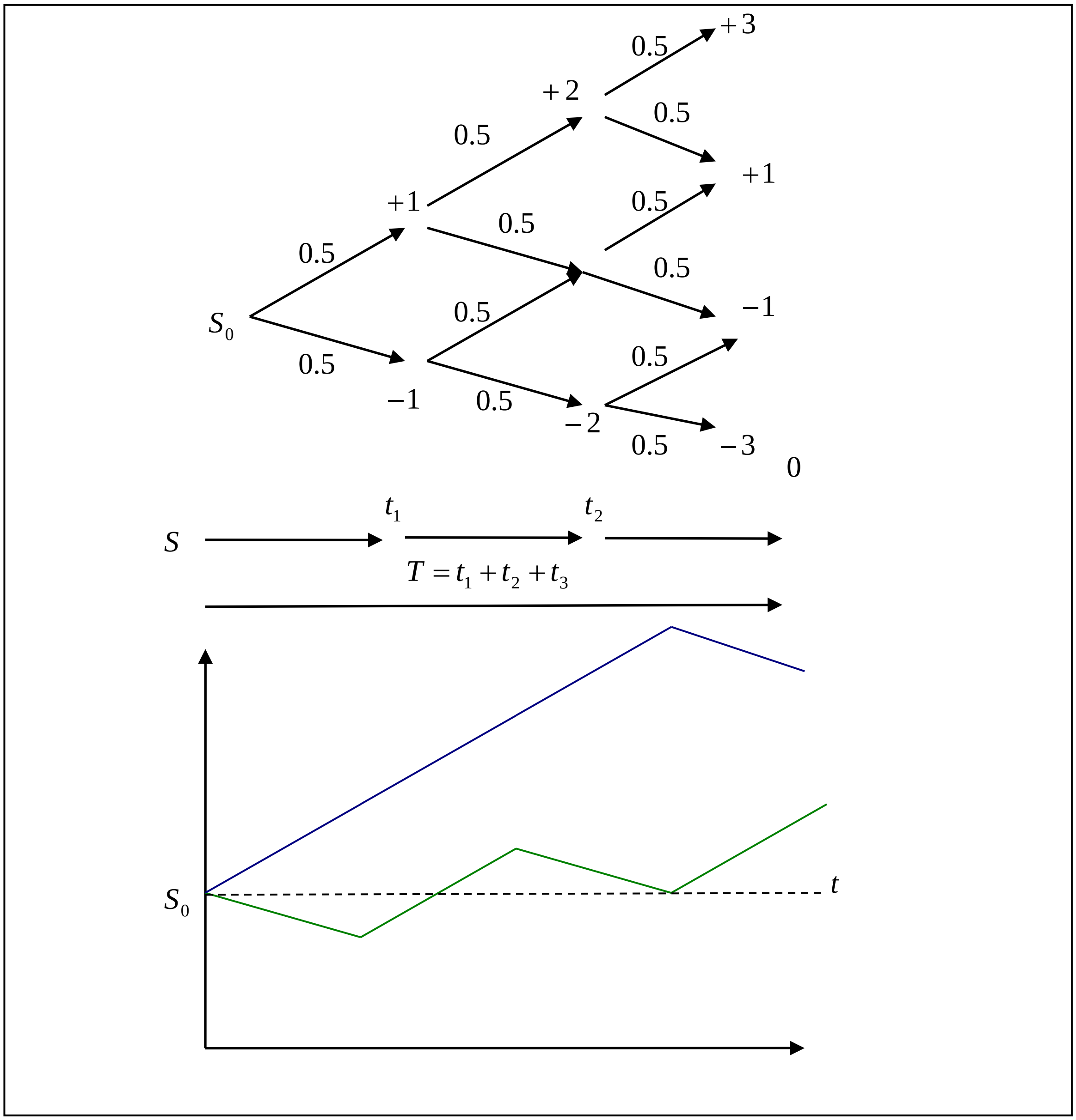


Once again note that it is the variance that scales linearly. The total variance in this example is three so each individual part has variance one.

A stock price process

Zero interest rates

Consider a stock price that starts at some level S_0.On each tick one of two equally probable outcomes can occur; either the stock moves up by one dollar or it moves down by one dollar. Over three ticks we could model our stock using the process in the previous section with one small modification; we adjust the mean to be the current stock level.



We have created a simple stochastic process to model the change in value of our stock over time. We know from the introductory sections that if over one unit of time our stochastic process has variance σ^2 and mean zero then over a period of time T it will have variance $\sigma^2 T$

We also know that if we break down a stochastic process with variance $\sigma^2 T$ into n equal steps represented by the random variables x_i then at each step we will need the variance of x_i to equal $\frac{\sigma^2 T}{n}$. The change in our asset price can then be modelled as $X = \sum_{i=1}^{n} x_i$

In the limit as n tends to infinity the steps in our stochastic process become finer and finer but the total variance remains at $\sigma^2 T$ Also our process converges to a Gaussian variable of mean zero and variance $\sigma^2 T$ and hence $\frac{X}{\sigma\sqrt{T}}$ is normally distributed with mean zero and variance one.

We can hence represent the change in our stock price over the time T as the stochastic process

$$\sigma\sqrt{T}\,N(0,1)$$

And we can represent the terminal value of our stock price over the time T as the stochastic process

$$S_0 + \sigma\sqrt{T}\,N(0,1)$$

A more realistic stock price process

In the previous section we created a very simplistic stock price model where at each point in time a stock would move up or down by some equal amount and mean value of the stock price process at the end of the period was the same as its value at the beginning. We now consider a more realistic stock price model

The Markov property

The value of X_t given X_s is that determined by $X_t - X_s$ and so the behaviour of X_t is totally unaffected by the values of X_r for r less than s. This is known as the Markov property.

Normal distribution notation

$\phi(0,1)$ Normal distribution with mean zero and variance of one

$\phi(0,2)$ Normal distribution with mean zero and variance of two

$\phi(0,2) = \sqrt{2}\phi(0,1)$

$\phi(0,t) = \sqrt{t}\phi(0,1)$

If a process has a variance of σ^2 per unit of time then we get

$\phi(0,\sigma^2 t) = \sigma^2 \sqrt{t}\phi(0,1)$

Brownian Motion

We say that a stochastic process W_t is a Brownian motion if it has the following properties.

- $W_0 = 0$
- For every s less than t, $W_t - W_s$ is normally distributed with variance $(t-s)$.

From these properties it follows that

$$W_t - W_s = \phi(0, t-s).$$

In the limit as $(t-s)$ tends to zero we get

$$dW_t = (W_t - W_s) = \phi(0, t-s) = \sqrt{(t-s)}\phi(0,1).$$

We can then represent a process with non unit variance as

$$\sigma dW_t = \sigma(W_t - W_s) = \sigma\phi(0, t-s) = \sigma\sqrt{(t-s)}\phi(0,1)$$

Stochastic differential equation

We extend this notion to define a family X of random variables X_t that satisfy the stochastic differential equation.

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Dropping the parameters of the drift and volatility we get a general stochastic differential equation for a variable with a constant growth or drift rate plus some noise.

$$dX_t = \mu dt + \sigma dW_t$$

Ito Calculus

Taylor series expansion of function of two variables

Given a normal function of two variables $G(x,y)$ the Taylor series expansion gives us

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \tfrac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \frac{\partial^2 G}{\partial y \partial x}\Delta y \Delta x$$

In the limit as $\Delta t, \Delta x$ tend to zero we can drop the second order terms and obtain

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$$

Extending Taylor series expansion to Ito

Now consider the situation where a variable x follows the Ito process

$$\Delta x = \mu \Delta t + b\varepsilon\sqrt{\Delta t}$$

Furthermore if G is a function of x and t then we can write

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \frac{\partial^2 G}{\partial t \partial x}\Delta t \Delta x$$

In this case we cannot exclude the $\frac{\partial^2 G}{\partial x^2}\Delta x^2$ term for the following reason

$$\frac{\partial^2 G}{\partial x^2}\Delta x^2 =$$

$$\frac{\partial^2 G}{\partial x^2}\left[\mu \Delta t + b\varepsilon\sqrt{\Delta t}\right]^2 =$$

$$\frac{\partial^2 G}{\partial x^2}\left[\mu^2 \Delta t^2 + 2\mu \Delta t b\varepsilon\sqrt{\Delta t} + b^2\varepsilon^2\Delta t\right]$$

Inside the square brackets, we obtain a first order term so we need to include this to get

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{\partial^2 G}{\partial t^2}b^2\varepsilon^2\Delta t$$

However, it can be shown that in the limit as change in t tends to zero then $b^2\varepsilon^2\Delta t$ tends to $b^2\Delta t$. We now obtain

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{\partial^2 G}{\partial x^2}b^2\Delta t$$

Substituting in $\Delta x = \mu\Delta t + b\varepsilon\sqrt{\Delta t}$ we obtain

$$\Delta G = \frac{\partial G}{\partial x}\mu\Delta t + \frac{\partial G}{\partial x}b\varepsilon\sqrt{\Delta t} + \frac{\partial G}{\partial t}\Delta t + \frac{\partial^2 G}{\partial x^2}b^2\Delta t$$

Re-arranging we get

$$\Delta G = \left(\frac{\partial G}{\partial x}\mu + \frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial x^2}b^2\right)\Delta t + \frac{\partial G}{\partial x}b\varepsilon\sqrt{\Delta t}$$

In the limit as the change in t tends to zero we obtain

$$dG = \left(\frac{\partial G}{\partial x}\mu + \frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

Deriving the lognormal process for stock prices from Ito

If we believe that the change in stock prices over a small interval of time can be modelled by the following stochastic differential equation

$$dS = \mu S dt + \sigma S dW_t$$

Then we can use Ito to show that the distribution of prices at some later time T is lognormal. We consider the function $G = \ln(S)$. Ito tells us that

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 G}{\partial S^2} S^2\sigma^2 \right)dt + \frac{\partial G}{\partial S}\sigma S dW_t$$

We know from ordinary calculus that if $G = \ln(S)$ then $\frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S} = -\frac{1}{S^2}, \frac{\partial G}{\partial t} = 0$

Plugging these into our stochastic differential equation we obtain

$$dG = \left(\mu + \frac{\sigma^2}{2} \right)dt + \sigma dW_t$$

So we have shown that the change in the logarithm of the stock prices is a Wiener process with mean $\mu + \frac{\sigma^2}{2}$ and standard deviation σ. This is the same as saying the continuously compounded return is a Wiener process with mean $\mu + \frac{\sigma^2}{2}$ and standard deviation σ.

$$\ln(S_T) - \ln(S_0) = \left(\mu + \frac{\sigma^2}{2} \right)t + \sigma\sqrt{T}\phi(0,1)$$

From this we can see that the logarithm of the terminal stock price is normally distributed with mean $\ln(S_0) + \left(\mu + \frac{\sigma^2}{2} \right)t$ and variance $\sigma^2 t$

$$\ln(S_T) = \ln(S_0) + \left(\mu + \frac{\sigma^2}{2} \right)t + \sigma\sqrt{t}\phi(0,1)$$

One well known result from statistics is that if the logarithm of variable is normally distributed then the variable itself will be lognormal.

$$S_T = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)t + \sigma\sqrt{T}\phi(0,1)}$$

Appendix A – A short introduction to differential equations

If we have the differential equation $\frac{dx}{dt} = 3x, dx = 3xdt$ how do we back out a solution. A little intuition can show that $x = e^{3t}$ is a solution since

$$\frac{d}{dt}e^{3t} = 3e^{3t} = 3x$$

Is it the only solution though? It is not because any equation of the form

$$x = Ae^{3t}$$

Where A is any constant is also a solution. We now solve the same equation but change the variable names.

$$\frac{dB_t}{dt} = 3B_t, dB_t = 3B_t dt$$

Once again we have the same solution

$$B_t = e^{3t}$$

because

$$\frac{d}{dt}e^{3t} = 3e^{3t} = 3B_t$$

We can also very easily change the original differential equation to

$$\frac{dB_t}{dt} = rB_t, dB_t = rB_t dt$$

the solution becomes

$$B_t = e^{rt} \text{ since } \frac{d}{dt}e^{rt} = re^{rt} = rB_t$$

Since any equation of the form $B_t = Ae^{rt}$ is a solution we choose the initial condition to be the value of B at the start which we denote B_0 and the solution to our differential equation becomes

$$B_t = B_0 e^{rt}$$

Appendix B – Probability of a stock exceeding a given level

Initial derivation

How would we go about calculating the probability that a given stock with price S_t at time t will exceed some level K at time T if the instantaneous rate of return is r and the annual volatility is σ? In other words, we need to know the probability that $S_T \geq K$. In order for the terminal stock price to be greater than K then we need

$$\ln\left(\frac{S_T}{S_t}\right) \geq \ln\left(\frac{K}{S_t}\right)$$

The first point of note is that if the stock price process follows a Brownian motion then the returns over some time interval $(T-t)$ will be normally distributed with mean $\left(r-\sigma^2/2\right)(T-t)$ and standard deviation $\sigma\sqrt{T-t}$. Furthermore, from the properties of the normal distribution we know that the random variable

$$\frac{\ln\left(\frac{S_T}{S_t}\right) - \left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

will be normally distributed having mean zero and standard deviation one. We re-write our original inequality in terms of the new distribution.

$$\ln\left(\frac{S_T}{S_t}\right) \geq \ln\left(\frac{K}{S_t}\right)$$

$$\frac{\ln\left(\frac{S_T}{S_t}\right) - \left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}} \geq \frac{\ln\left(\frac{K}{S_t}\right) - \left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

The left hand side of this inequality is a normally distributed random variable having mean zero and standard deviation of one. In order to use the standard cumulative normal distribution we require an inequality with a less than sign. We multiply both sides by negative one.

$$\frac{\ln\left(\frac{S_t}{S_T}\right) + \left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}} \leq \frac{\ln\left(\frac{S_t}{K}\right) + \left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

Re-arrangement

We can re-arrange the right hand side of our inequality as follows.

$$\frac{\ln\left(e^{r(T-t)}\frac{S_t}{K}\right)-(T-t)\left(\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}$$

We simply note that

$$(T-t)r=\log\left(e^{(T-t)r}\right)$$

And

$$(T-t)\left(r-\frac{\sigma^2}{2}\right)=(T-t)r-(T-t)\frac{\sigma^2}{2}$$

$$=\log\left(e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}$$

So re-writing the original write hand side, we get

$$\frac{\ln\left(\frac{S_t}{K}\right)+\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{\sigma\sqrt{T-t}}=\frac{\ln\left(\frac{S_t}{K}\right)+\ln\left(e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}$$

$$=\frac{\ln\left(\frac{S_t}{K}e^{(T-t)r}\right)-(T-t)\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}$$

Deriving the Black-Scholes differential equation

Assume our stock price follows a Wiener process given by $dS = \mu S dt + \sigma S \varepsilon \sqrt{dt}$ then changes in a function of S and t will be given by an Ito process

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

We could consider this the process for a call option struck on S at time t. Furthermore we can set up a portfolio of a short call and long $\frac{\partial f}{\partial S}$ stocks. The change in this portfolio will then be given by

$$d\Pi = -\left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} \left[\mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} \right]$$

$$d\Pi = -\left(\frac{\partial f}{\partial S} \mu S - \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} \sigma S dz$$

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$

We have now eliminated all randomness from our process. It must hence grow at the risk-free rate

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt = \left(-f + \frac{\partial f}{\partial S} S \right) r dt$$

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + rS \frac{\partial f}{\partial S} = rf$$

Might need stuff

Transforming to standard normal distribution

One standard property of the mean of a distribution is that $E[X+b]=E[X]+b$. If a random variable X has mean

$$E[nX]=n\mu=\sum_{i=1}^{n} nx_i\, p_i$$

Then $X-n\mu$ is a random variable with mean zero

$$E[nX-n\mu]=\sum_{i=1}^{n}\left(nx_i-n\mu\right)p_i=\sum_{i=1}^{n} nx_i\, p_i-\sum_{i=1}^{n} n\mu p_i=n\mu-n\mu=0$$

Finally we can scale our variable to have a standard deviation of one by performing

$$\frac{X_n-n\mu}{\sigma\sqrt{n}}$$

Again for a proof of this scaling find a proper mathematician