

Basic Terminology

OUTCOME

Each thing that can occur in an experiment is called an outcome. In the example of tossing a coin we have two outcomes ‘heads’ or ‘tails’ which we can denote by the letters H and T respectively.

SAMPLE SPACE

The set of all possible outcomes of an experiment is known as the sample space. By convention we label it Ω . In our simple coin tossing scenario we would have $\Omega = \{H, T\}$ If we toss two coins our sample space would become $\Omega = \{HH, HT, TH, TT\}$

EVENT

A subset of the probability space is called an event. We define an event using the following notation.

$A = \{\omega \in \Omega; \omega = H\}$ “This means the set of all outcomes ω such that ω is a head”.

PROBABILITY MEASURE

A probability measure P is a function that assigns to each element ω in Ω a probability such that

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Since an event A is a subset of Ω then the probability of an event is given by

$$P(A) = \sum_{\omega \in A} P(\omega)$$

PROBABILITY SPACE

A probability space $P(\Omega, P)$ consists of a sample space and a probability measure. The sample space is the set of outcomes and the probability measure is a function that assigns to each element ω in Ω a value in $[0,1]$ such that

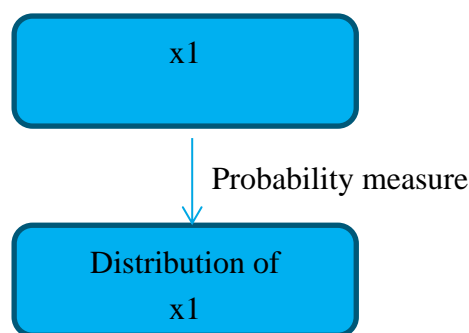
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RANDOM VARIABLE

A random variable x is a real valued function defined on Ω . Put another way a random variable maps each outcome from the sample space Ω to a real number.

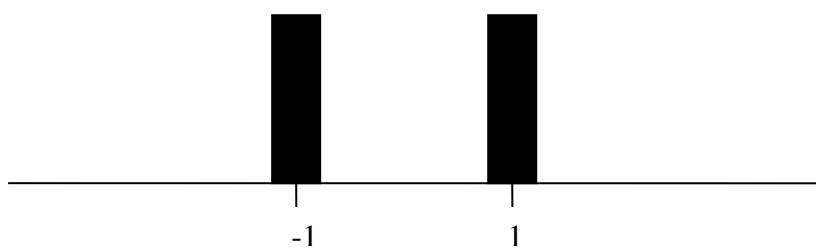
PROBABILITY DISTRIBUTION

We now introduce one more important concept, that of a probability distribution. A random variable is a function defined on Ω whereas its distribution is a tabulation of the probabilities that the random variable takes its various values. A random variable is **not** a distribution.



Under the probability measure P defined on Ω either a head or tail are equally likely so our distribution becomes

$$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$$



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EXPECTATION

We can define the expectation or expected value of any random variable X under a probability measure P as.

$$E(X) = \sum_{\omega \in \Omega} X(\omega)P(\omega)$$

- ◆ Weighted average of the values the random variable X can take
- ◆ Weighting by the probability of each value
- ◆ Measure of centrality

EXPECTATION OF VARIABLE SQUARED

We are often interested in expectation of the square of the variable which we call the mean squared.

$$E(x_1^2) = [x_1(H)]^2P(H) + [x_1(T)]^2P(T) = 0.5 + 0.5 = 1.0$$

$$\tilde{E}(x_1^2) = [x_1(H)]^2\tilde{P}(H) + [x_1(T)]^2\tilde{P}(T) = 0.75 + 0.25 = 1.0$$

VARIATION FROM EXPECTED VALUE

For any actual value of a random variable X we can calculate the difference between that value and the expectation $X(\omega) - E(X)$. We might ask the question “on average how much does a given value differ from the expected value?” We could calculate the average difference as $\sum_{\omega \in \Omega} [X(\omega) - E(X)]P(\omega)$ however where the distribution is symmetric around the mean this value will be zero. A more instructive measure is given by calculating the average of the difference squared

$$\sum_{\omega \in \Omega} [X(\omega) - E(X)]^2P(\omega)$$

Under our two probability measures we get

$$[x_1(H) - E(x_1)]^2P(H) + [x_1(T) - E(x_1)]^2P(H) = 0.5 + 0.5 = 1.0$$

$$[x_1(H) - E(x_1)]^2\tilde{P}(H) + [x_1(T) - E(x_1)]^2\tilde{P}(H) = 0.5 + 0.5 = 1.5$$

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EXPECTATION OF A CONSTANT MULTIPLE OF A RANDOM VARIABLE

$$E[aX + b] = aE[X] + b$$

$E[aX + b] = \sum_{\omega \in \Omega} (aX(\omega) + b)P(\omega)$	From definition 1
$= \sum_{\omega \in \Omega} (aX(\omega))P(\omega) + \sum_{\omega \in \Omega} bP(\omega)$	By multiplying out the brackets
$= a \sum_{\omega \in \Omega} (X(\omega))P(\omega) + \sum_{\omega \in \Omega} bP(\omega)$	From the properties of summation
$= aE[X] + b \sum_{\omega \in \Omega} P(\omega)$	From definition 1
$= aE[X] + b \cdot 1$	From axioms of probability
$= aE[X] + b$	

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EXPECTATION OF A FUNCTION OF RANDOM VARIABLE

$$E[g(X)] = \sum_{\omega \in \Omega} g(X(\omega))P(\omega)$$

- ♦ The expectation of a function of a random variable is **not equal** to the function of the expectation $E[g(X)] \neq g[E(X)]$

VARIANCE

$$\text{Var}[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

Let $\mu = E[X]$

$$E[(X - \mu)^2] = \sum_{\omega \in \Omega} (X(\omega) - \mu)^2 P(\omega) \quad \text{From definition}$$

$$= \sum_{\omega \in \Omega} \left((X(\omega))^2 - 2\mu X(\omega) + \mu^2 \right) P(\omega) \quad \text{Multiplying out}$$

$$= \sum_{\omega \in \Omega} (X(\omega))^2 P(\omega) + \sum_{\omega \in \Omega} -2\mu X(\omega) P(\omega) + \sum_{\omega \in \Omega} \mu^2 P(\omega)$$

$$= E[X^2] + \sum_{\omega \in \Omega} -2\mu X(\omega) P(\omega) + \sum_{\omega \in \Omega} \mu^2 P(\omega) \quad \text{From definition 3}$$

$$= E[X^2] + -2\mu \sum_{\omega \in \Omega} X(\omega) P(\omega) + \mu^2 \sum_{\omega \in \Omega} P(\omega) \quad \text{Properties of summations}$$

$$= E[X^2] - 2\mu\mu + \mu^2 \sum_{\omega \in \Omega} P(\omega)$$

$$= E[X^2] - 2\mu\mu + \mu^2 \quad \text{Axioms of probability}$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

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VARIANCE OF A CONSTANT PROPERTY

$$\text{Var}[a] = 0$$

VARIANCE OF A CONSTANT MULTIPLE

$$\text{Var}[aX] = a\text{Var}[X]$$

$$\text{Var}[aX] = E[(aX - E[aX])^2]$$

$$= E[(aX - aE[X])^2]$$

From definition 2

$$= E[(aX - a\mu)^2]$$

Letting $\mu = E[X]$

$$= \sum_{\omega \in \Omega} (aX(\omega) - a\mu)^2 P(\omega)$$

From definition

$$= \sum_{\omega \in \Omega} a^2 (X(\omega) - \mu)^2 P(\omega)$$

$$= a^2 \sum_{\omega \in \Omega} (X(\omega) - \mu)^2 P(\omega)$$

$$= a^2 \text{Var}[X]$$

From definition 4

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Adding Random variables

Expectation of the sum of two finite countable variables

If X is a random variable with sample space $\{x_1, x_2, \dots, x_m\}$ and Y is an independent random variable with sample space $\{y_1, y_2, \dots, y_n\}$ then the sample space of the joint distribution will be given by a set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}, \dots, \{x_1, y_n\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}, \dots, \{x_2, y_n\}$$

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$$\{x_m, y_1\}, \{x_m, y_2\}, \dots, \{x_m, y_n\}$$

The expectation of the sum of the two variables is then given by

$$\sum_{i=1}^m \sum_{j=1}^n (x_i + y_j) p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^m \sum_{j=1}^n x_i p(x_i, y_j) + \sum_{i=1}^m \sum_{j=1}^n y_j p(x_i, y_j)$$

Noting that $\sum_{j=1}^n p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^m p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^m x_i p(x_i) + \sum_{j=1}^n y_j p(y_j)$$

Therefore we can note that

$$E[X + Y] = E[X] + E[Y]$$

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Expectation of the sum of n identically distributed random variables

We can calculate the expectation of the sum of n identically distributed random variables denoted by X_1, X_2, \dots, X_n as $E[X_1] + E[X_2] + \dots + E[X_n]$ which is equal to

$$n \cdot E[X_n]$$

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Worked example

Imagine a random event that involves the tossing of a single coin. We have two outcomes, heads or tails in our *sample space* Ω

$$\Omega = \{H, T\}.$$

Furthermore, let us define a *random variable* x_1 that takes the value of plus one dollar if we obtain a head and minus one dollar if we obtain a tail.

$$x_1(H) = 1, x_1(T) = -1$$

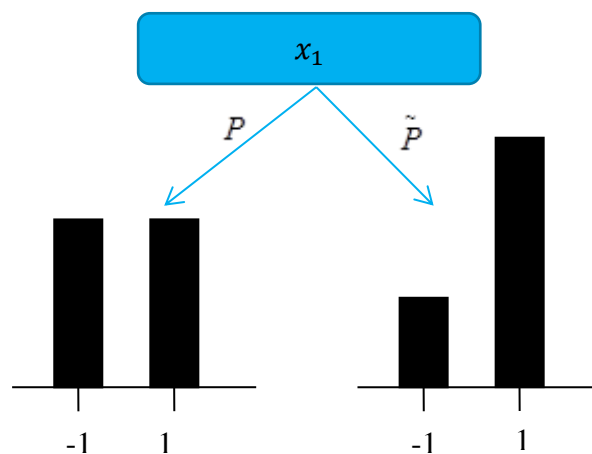
Notice that our random variable does not say anything about the probability of a head or tail. It just tells us what value we assign to the outcomes of the sample space. A probability measure is a real valued function that maps each outcome of the sample space to a probability. If our coin is fair we could have a measure P such that

$$P(H) = 0.5, P(T) = 0.5$$

We might however have a different measure for a loaded coin

$$\tilde{P}(H) = 0.75, \tilde{P}(T) = 0.25$$

Applying a probability measure to a random variable gives us a distribution. The distribution shows the probability of each value of the random variable. Different measures give different distributions.



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TABLE 1 SUMMARY

Outcome	H	Each outcome is a thing that can occur in an experiment
Sample Space	$\Omega = \{H, T\}$	The set of all possible outcomes that can occur in an experiment is called the sample space
Event	$A = \{\omega \in \Omega; \omega = H\}$	A subset of the sample space is called an event
Probability Measure	$P(H) = P(T) = 0.5$	A probability measure P is a function that assigns to each element ω in Ω a probability such that $\sum_{\omega \in \Omega} P(\omega) = 1$
Probability Space	$(\{H, T\}, P(H) = P(T) = 0.5)$	A probability space consists of a sample space and a probability measure
Random Variable	$x_1(H) = 1, x_1(T) = -1$	A random variable is a real valued function defined on the sample space.
Probability Distribution	$P(x_1 = 1) = 0.5, P(x_1 = -1) = 0.5$	Tabulation of the probabilities that the random variable takes its various values.
Expectation	$E(X) = x_1(H)P(H) + x_1(T)P(T) = 0$	We define the expected value of our random variable under the probability measure P
Variation	$[x_1(H) - E(x_1)]^2 \tilde{P}(H) + [x_1(T) - E(x_1)]^2 \tilde{P}(H) = 1.5$	

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Summing Identical Independent random variables

We can create a new game by playing the original games twice. We define x_1 as the profit/loss from the first coin toss and x_2 as the profit/loss from the second toss. We can then define the random variable $X = x_1 + x_2$ as the total profit or loss at the end of the two tosses.

The sample space becomes

$$\Omega = \{\omega_1 \omega_2\} = \{HH, HT, TH, TT\}$$

The measure becomes

$$P(HH) = P(HT) = P(TH) = P(TT) = 0.25$$

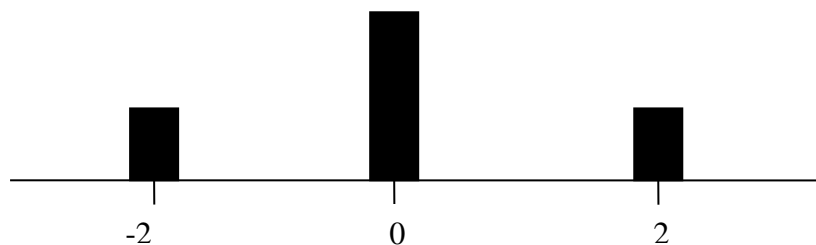
Giving us the random variable

$$X(HH) = 2, X(HT) = (TH) = 0, X(TT) = -2$$

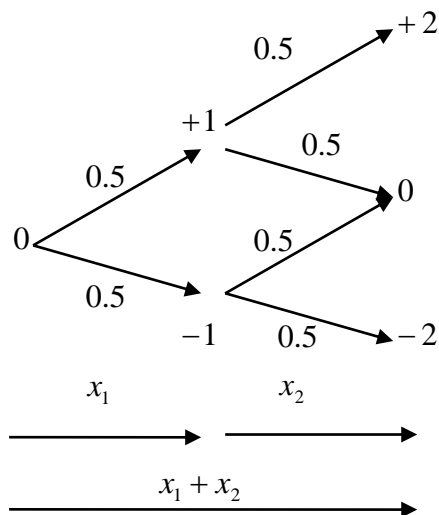
Applying our measure to the random variable we get our probability distribution

$$P(X = 2) = 0.25, P(X = 0) = 0.5, P(X = -2) = 0.25$$

Which is visualized as follows

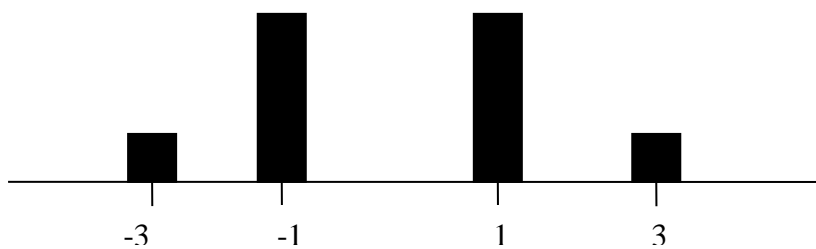


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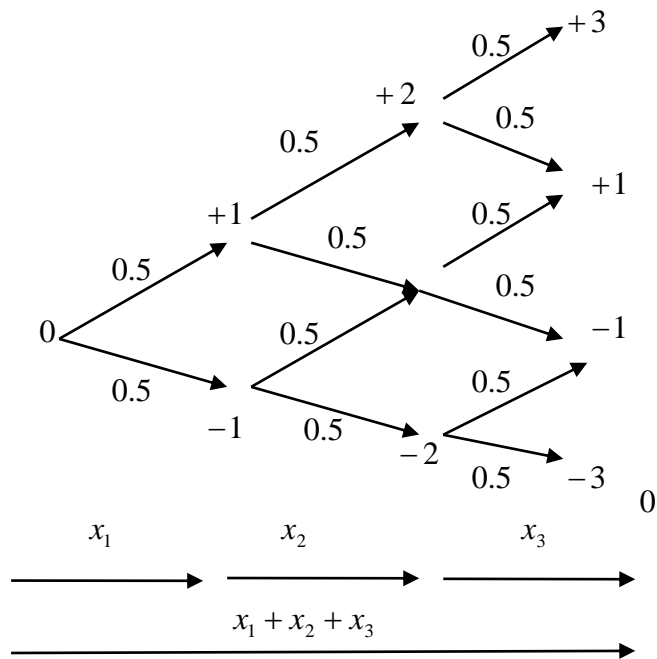


- ◆ $\mu = 0$
- ◆ $\sigma^2 = 0.25(2 - 0)^2 + 0.25(-2 - 0)^2 = 2.0$
- ◆

Let us go one-step further and look at summing three coin tosses. $X_3 = x_1 + x_2 + x_3$ We get the following distribution, whose mean is zero and whose variance is three.



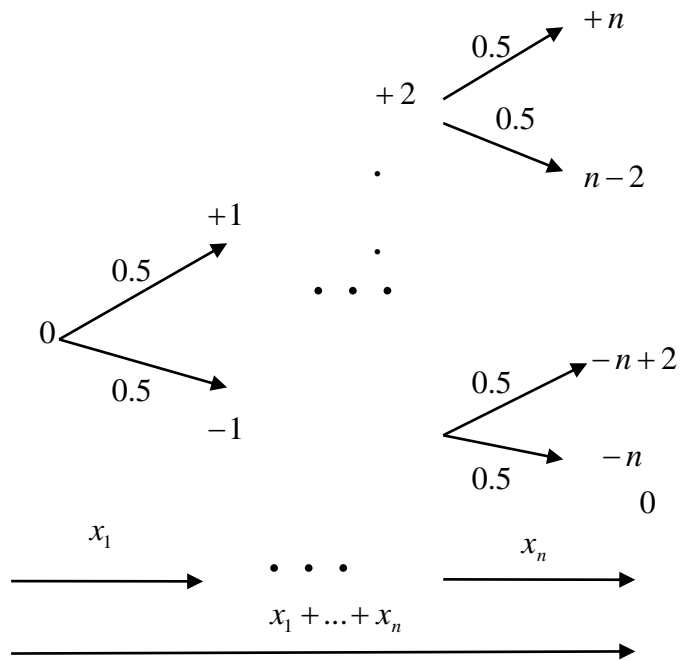
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- ♦ $\mu = 0$
- ♦ $\sigma^2 = 0.125(3 - 0)^2 + 0.125(-3 - 0)^2 + 0.375(1 - 0)^2 + 0.375(-1 - 0)^2 = 3.0$

Taking this process to its logical conclusion by summing n of our independent, identically distributed random variables we obtain the random variable $X_n = x_1 + \dots + x_n$ which is distributed with mean zero and variance n .

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From a proof of why the sum of n independent, identically distributed random variables with mean μ and variance σ^2 is a random variable with mean $n\mu$ and variance $n\sigma^2$ see below

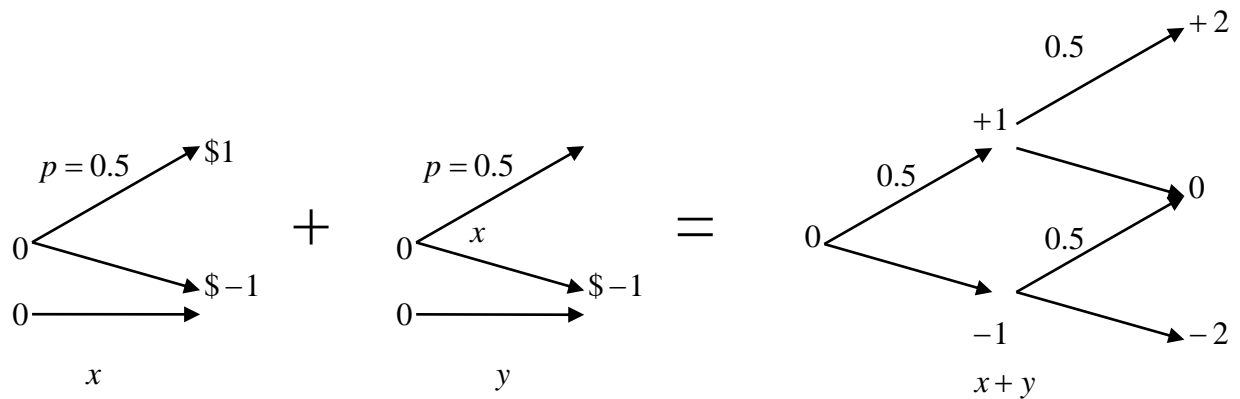
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Proofs

Expectation of sum of N I.I.D random variables

$$E[X + X + X + X + \dots + X] = NE[X]$$

The following figure shows the general approach



We have random variable x with sample space $\{x_1, x_2\} = \{1, 0\}$ and a second identically distributed random variable y with sample space $\{y_1, y_2\} = \{1, 0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

The expectation of the sum of the variables is then given by

$$\sum_{i=1}^2 \sum_{j=1}^2 (x_i + y_j) p(x_i, y_j)$$

Multiplying out

$$\sum_{i=1}^2 \sum_{j=1}^2 x_i p(x_i, y_j) + \sum_{i=1}^2 \sum_{j=1}^2 y_j p(x_i, y_j)$$

Noting that $\sum_{j=1}^n p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^m p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^m x_i p(x_i) + \sum_{j=1}^n y_j p(y_j)$$

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Therefore we can note that

$$E[X + Y] = E[X] + E[Y]$$

GENERALISING

If x is a random variable with sample space $\{x_1, x_2, \dots, x_m\}$ and y is an independent random variable with sample space $\{y_1, y_2, \dots, y_n\}$ then the sample space of the joint distribution will be given by a set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}, \dots, \{x_1, y_n\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}, \dots, \{x_2, y_n\}$$

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$$\{x_m, y_1\}, \{x_m, y_2\}, \dots, \{x_m, y_n\}$$

The expectation of the sum of the two variables is then given by

$$\sum_{i=1}^m \sum_{j=1}^n (x_i + y_j) p(x_i, y_j)$$

Multiplying out we get

$$\sum_{i=1}^m \sum_{j=1}^n x_i p(x_i, y_j) + \sum_{i=1}^m \sum_{j=1}^n y_j p(x_i, y_j)$$

Noting that $\sum_{j=1}^n p(x_i, y_j) = p(x_i)$ and $\sum_{i=1}^m p(x_i, y_j) = p(y_j)$

$$\sum_{i=1}^m x_i p(x_i) + \sum_{j=1}^n y_j p(y_j)$$

Therefore we can note that

$$E[X + Y] = E[X] + E[Y]$$

Furthermore

$$E[X + X] = E[X] + E[X] = 2E[X]$$

And

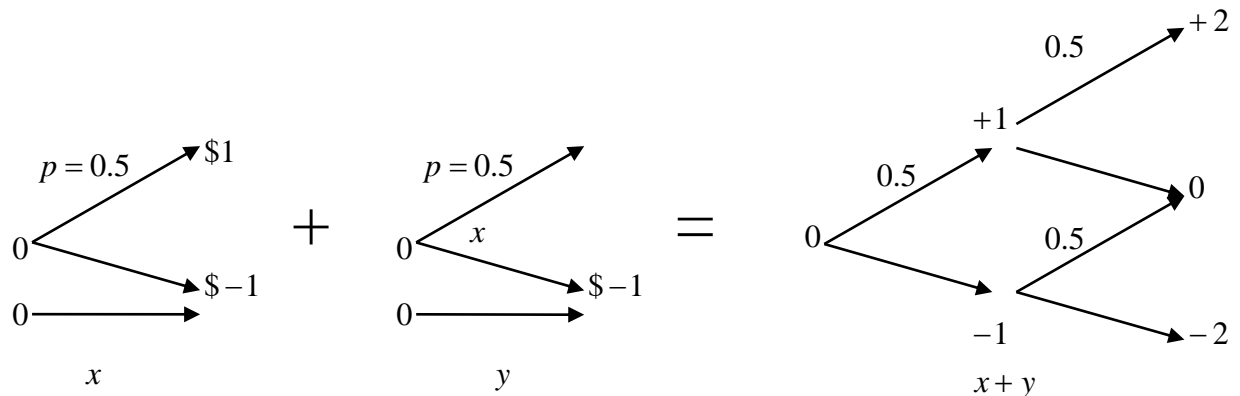
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$$E[X + X + X + X + \dots + X] = NE[X]$$

Variance of sum of I.I.D random variables

$$\text{Var}[x + y] = \text{Var}[x] + \text{Var}[y]$$

The following diagram shows the general approach.



We have random variable x with sample space $\{x_1, x_2\} = \{1, 0\}$ and another identically distributed random variable y with sample space $\{y_1, y_2\} = \{1, 0\}$. The sample space of the joint distribution $x + y$ is given by the set of pairs

$$\{x_1, y_1\}, \{x_1, y_2\}$$

$$\{x_2, y_1\}, \{x_2, y_2\}$$

PROOF

$$\text{Var}[x + y] = E[(x + y)^2] - \{E[x + y]\}^2$$

$$\text{Var}[x + y] = E[(x^2 + 2xy + y^2)] - \{E[x] + E[Y]\}^2$$

$$\text{Var}[x + y] = E[x^2] + E[y^2] + E[2xy] - \{E[x] + E[Y]\}^2$$

$$\text{Var}[x + y] = E[x^2] + E[y^2] + E[2xy] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$\text{Var}[x + y] = E[x^2] + E[y^2] + 2E[x][y] - E[x]^2 - E[y]^2 - 2E[x][y]$$

$$\text{Var}[x + y] = E[x^2] - E[x]^2 + E[y^2] - E[y]^2$$

$$\text{Var}[x + y] = \text{Var}[x] + \text{Var}[y]$$