Integration

Introduction

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- Introduction

Definite Integral

If f is a continuous function defined for $a \le x \le b$, we divide the sub-interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{(b-a)}{n}$. We let $x_0(=a), x_1, x_2, ..., x_n(=b)$ be the endpoints of these sub-intervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any sample points on these sub-intervals, so x_i^*, lies on the sub-interval $[x_{i-1}, x_i]$ then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \to \alpha} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$f(x)$ is the integrand, a and b are the limits of integration. The symbol dx has no official meaning by itself. The definite integral is a number that can be interpreted as the area under the curve from a to b

Properties of the Definite Integral

$$\int_a^b c\,dx = (b-a)c$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

Fundamental

FTOC

If $g(x)\int_a^x f(t)dt$ then $g'(x) = f(x)$

$\int_a^b f(x)dx = F(b) - F(a)$ where F is any anti-derivative of f, that is $F^{-1} = f$

Integrating complex numbers

One Show that

$$\int e^{ax}\cos bx dx = \frac{1}{a^2+b^2}e^{ax}\left(a\cos bx + b\sin bx\right)$$

And

$$\int e^{ax}\sin bx dx = \frac{1}{a^2+b^2}e^{ax}\left(-b\cos bx + a\sin bx\right)$$

Second express both more simply using a phase angle

Let $U = \int e^{ax}\cos bx dx$ and $V = \int e^{ax}\sin bx dx = \frac{1}{a^2+b^2}$

Consider $\int U + iV$

$$= \int e^{ax}\cos bx dx + i\int e^{ax}\sin bx dx$$

$$= \int e^{ax}\left(\cos bx + i\sin bx\right)dx$$

$$= \int e^{ax} e^{ibx}$$

$$= \int e^{ax} e^{x(a+ib)}$$

$$= \frac{1}{a+ib} e^{x(a+ib)}$$

$$= \frac{a-ib}{a^2+b^2} e^{ax} (\cos bx + i \sin bx)$$

$$= \frac{1}{a^2+b^2} e^{ax} (a \cos bx + b \sin bx) + i(-b \cos bx + a \sin bx)$$

Since real and imaginary parts must be equal

$$\int e^{ax} \cos bx dx = \frac{1}{a^2+b^2} e^{ax} (a \cos bx + b \sin bx)$$

$$\int e^{ax} \sin bx dx = \frac{1}{a^2+b^2} e^{ax} (-b \cos bx + a \sin bx)$$

Techniques

Integration is more challenging than differentiation and it is not always obvious which technique should be used. The following strategy may be useful.

1. Learn the basic integration formulas

Use the following table

$$\int cf(x)dx = c\int f(x)$$

$$\int kdx = kx + c\,]$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int \sin(x)dx = -\cos x + C$$

$$\int \cos(x)dx = \sin x + C$$

$$\int \sec^2(x)dx = \tan x + C$$

$$\int \csc^2(x)dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

2. Functions of linear functions of x

3. Partial Fractions

4. Integral of form $\int \frac{f'(x)}{f(x)} dx$

If we have a rational function where the numerator is the derivative of the denominator or the numerator is a multiple/sub-multiple of the denominator we can solve it using the following technique. We first show an example

$$\int \frac{2x+3}{x^2+3x-5} dx$$

We first let the denominator by z, that is $z = x^2 + 3x - 5$ so that

$$\frac{dz}{dx} = 2x+3 \therefore dz = dx(2x+3)$$

We can then re-write the original derivative in terms of z as follows

$$\int \frac{2x+3}{x^2+3x-5} dx = \int \frac{1}{z} dz = \ln z + C$$

5. Integral of form $\int f'(x)f(x)dx$

In very similar way to section four we can integrate such as function as

$$\int \sec^2 x \tan x dx$$

Let $z = \tan x \therefore \dfrac{dz}{dx} = \sec^2 x$

$$\int \sec^2 x \tan x dx = \int z dz = \frac{z^2}{2} = \frac{\tan^2 x}{2}$$

6. Integrating products of functions by parts

The product rule of differentiation states that if u and v are functions of x then

$$\frac{d}{dx}u.v = u\frac{dv}{dx} + v\frac{du}{dx}$$

If we integrate both sides of this expression we get

$$u.v = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

Re-arranging we get $\int u\dfrac{dv}{dx}dx = u.v - \int v\dfrac{du}{dx}dx$

We introduce shorthand to make it easier to remember

$$\int udv = u.v - \int v.du$$

We can then make use of this approach to integrate products of functions. We assume one of the functions is the derivative which we have to integrate. The following guidelines should be followed when choosing which term to use as u

- If one of factors is a log function that function must be chosen as u
- If there is no log term present the power of x is taken as u
- This only holds good for positive whole number powers of x
- If there is neither a log function or a power of x, the exponential function is taken as u

Example 1

$$\int e^{5x} \sin 3x dx$$

Let $u = e^{5x}, du = 5e^{5x}, dv = \sin 3x, v = \frac{-\cos 3x}{3}$

Integrating Trig Functions

1. Integrate $\int \sin^2 x dx$

First we make use of the trigonometric identity to express the integral as a function of a linear function of x.

$$\cos 2x = 1 - 2\sin^2 x \therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

So now we rewrite it

$$\int \sin^2 x dx = \frac{1}{2}\int 1 - \cos 2x = \frac{1}{2}\left(x - \int \cos 2x dx\right)$$

To integrate $\int \cos 2x dx$ let $u = 2x \therefore \frac{du}{dx} = 2 \therefore dx = \frac{du}{2}$

$$\int \cos 2x dx = \frac{1}{2}\int \cos u du = \frac{\sin u}{2} = \frac{\sin 2x}{2}$$

$$\int \sin^2 x dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]$$

2. Integrate $\int \sin^3 x dx$

We first note that

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

Now we make use of the trigonometric identity

$$\sin^2 x = 1 - \cos_2 x$$

This enables us to write the integral in the form $\int F(f(x))f'(x)dx$

$$\int \sin^3 x dx = \int \left(\cos^2 x - 1\right)\sin x dx = \int \cos^2 x \sin x - \sin x dx$$

The first tem in the integral is of the form $\int F(f(x))f'(x)dx$ and the second term is a standard integral.

For the first term we let $u = \cos x \therefore \frac{du}{dx} = -\sin x \therefore du = -\sin x dx$

$$\int \cos^2 x \sin x dx = -\int u^2 du = -\frac{u^3}{3} = \frac{\cos^3 x}{3}$$

We now have the whole thing

$$\int \sin^3 x dx = -\cos x + \frac{\cos^3 x}{3} + C$$

3. Integrate $\int \sin^4 x dx$

First we note that

$$\int \sin^4 x dx = \int \left(\sin^2 x\right)^2 dx$$

Now we use the following trigonometric identity

$$\cos 2x = 1 - \sin^2 x \therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

To rewrite our integral as

$$\int \left(\sin^2 x\right)^2 dx = \int \frac{1}{2}\left([1 - \cos 2x]\right)^2 dx$$

$$= \frac{1}{4}\int (1 - \cos 2x)^2 dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

Next we make sure of the trigonometric identity

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \therefore$$

$$\cos^2 2x = \frac{1}{2}(1 + \cos 2(2x)) = \frac{1}{2}(1 + \cos 4x)$$

Lets substitute this into our integral

$$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right)dx$$

$$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{4}\right)dx$$

$$= \frac{1}{4}\left(x - \sin 2x + \frac{x}{2} + \frac{\sin 4x}{8}\right)$$

2. Integrate $\int \cos^2 x dx$

$\cos 2x = 2\cos^2 x + 1 \therefore \cos^2 x = \frac{1}{2}(1 + \cos 2x)$ From the trigonometric identities, therefore

$$\int \cos^2 x dx = \frac{1}{2}\int 1 + \cos 2x = \frac{1}{2}\left(x + \int \cos 2x dx\right)$$

To integrate $\int \cos 2x dx$ let $u = 2x \therefore \frac{du}{dx} = 2 \therefore dx = \frac{du}{2}$

$$\int \cos 2x dx = \frac{1}{2}\int \cos u du = \frac{\sin u}{2} = \frac{\sin 2x}{2}$$

$$\int \cos^2 x dx = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]$$

3. Integrate

$$\int \sin^3 x dx$$

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

4. Integrate

Integrate $\int \tan^2 x dx$

Risk and Pricing Solutions

$$\int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} = \int \frac{1-\cos^2 x}{\cos^2 x} = \int \frac{1}{\cos^2 x} - 1 = \int \sec^2 x - 1$$

$$= \tan x - x + C$$

Integrating Powers of Sine

$$\int \sin^2 x dx = \frac{1}{2}\int 1 - \cos 2x$$

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int \left(1-\cos^2 x\right)\sin x dx \quad \text{// Let } u = \cos x$$

$$\int \sin^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$\int \sin^5 x dx = \int \left(\cos^2 x - 1\right)^2 \sin x dx \text{ // Let } u = \cos x$$

Integrating Powers of Cosecant

$$\int \sin^{-1} x dx = \int \frac{1}{\sin x} dx = \int \csc x dx = \ln\left|\csc x - \cot x\right| = \ln\left|\tan\frac{x}{2}\right|$$

Proof

$$\int \sin^{-1} x dx = \int \csc x dx = \int \csc x \left(\frac{\csc x - \cot x}{\csc x - \cot x}\right) dx = \int \left(\frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\right) dx$$

Let $u = \csc x - \cot x \therefore du = -\csc x \cot x + \csc^2 x dx$

$$\int \frac{1}{u} du = \ln\left|\csc x - \cot x\right| + C$$

But note also that

$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x} = \tan\frac{x}{2}$$

Integrating powers of cosecant higher than two

$$\int \sin^{-n} x dx = \int \frac{1}{\sin^n x} dx = \int \csc^n x dx$$

$\int \csc^n x dx = \int \csc^2 x \csc^{n-2} x dx$ Now we integrate by parts

$$= -\cot x \csc^{n-2} x - \int (-\cot x)(n-2)(\csc^{n-3} x)(-\csc x \cot x) dx$$

$$= -\cot x \csc^{n-2} x - (n-2)\int (\cot^2 x)(\csc^{n-2} x) dx$$

$$= -\cot x \csc^{n-2} x - (n-2)\int (\csc^2 x - 1)(\csc^{n-2} x) dx$$

$$= -\cot x \csc^{n-2} x - (n-2)\int \csc^n x - \csc^{n-2} x dx$$

$$= -\cot x \csc^{n-2} x - (n-2)\int \csc^n x dx + (n-2)\int \csc^{n-2} x dx$$

$$\int \csc^n x dx + (n-2)\int \csc^n x dx = -\cot x \csc^{n-2} x + (n-2)\int \csc^{n-2} x dx$$

$$(n-1)\int \csc^n x dx = -\cot x \csc^{n-2} x + (n-2)\int \csc^{n-2} x dx$$

$$\int \csc^n x dx = \frac{-\cot x \csc^{n-2} x}{(n-1)} + \frac{(n-2)}{(n-1)}\int \csc^{n-2} x dx$$

Integrating Powers of Cosine

$$\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx$$

$$\int \cos^3 x dx = \int \cos^2 x \cos x dx = \int (1-\sin^2 x)\cos x dx$$

$$\int \cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\int \cos^5 x dx = \int (\sin^2 x - 1)^2 \cos x dx$$

Integrating Powers of Secant

$$\int \cos^{-1} x dx = \int \frac{1}{\cos x} dx = \int \sec x dx = \ln\left|\sec x + \tan x\right|$$

$$\int \cos^{-2} x dx = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx = \cot x + C \text{ // Basic anti derivative}$$

For powers of secant greater than two we can use the following reduction formula

$$\int \cos^{-3} x dx = \int \frac{1}{\cos^3 x} dx = \int \sec^3 x dx = \frac{\sec^{n-2} x \tan x}{n-1} - \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

We can prove this result using integration by parts as follows

Let $u = \sec^{n-2} x, dv = \sec^2 x$ and hence $du = (n-2)\sec^{n-2} x \tan x, v = \tan x$

$$\int \sec^n x dx = \sec^{n-2} x \tan x - \int \tan x(n-2)\sec^{n-2} x \tan x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \tan^2 x \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \left(\sec^2 x - 1\right)\sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x - \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx$$

Re-arranging we get

$$\int \sec^n xdx + (n-2)\int \sec^n xdx = \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} xdx$$

$$\int \sec^n xdx+ = \frac{\sec^{n-2} x\tan x}{n-1} - \frac{(n-2)}{n-1}\int \sec^{n-2} xdx$$

Integrating Powers of Tangent

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x|$$

$$\int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} = \int \frac{1-\cos^2 x}{\cos^2 x} = \int \frac{1}{\cos^2 x} - 1 = \int \sec^2 x - 1 = \tan x - x + C$$

$$\int \tan^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} = \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^3 x} = -\int \frac{1-u^2}{u^3} du$$

$$\int \tan^4 x dx = \int \left(\sec^2 x - 1\right)\tan^2 x dx = \int \sec^2 x \tan^2 x dx - \int \tan^2 x dx$$

Approaches to solving

Integration is more challenging than differentiation and it is not always obvious which technique should be used. The following strategy may be useful.

1. Learn the basic integration formulas

Use the following table

$$\int cf(x)dx = c\int f(x)$$

$$\int kdx = kx + c\,]$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int \sin(x)dx = -\cos x + C$$

$$\int \cos(x)dx = \sin x + C$$

$$\int \sec^2(x)dx = \tan x + C$$

$$\int \csc^2(x)dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

2. Simply the Integrand if possible

- Algebraic manipulation
- Trign

2. Functions of Form $\int f(g(x))g'(x)dx = \int f(u)du$

Use the substitution rule. This is the integration rule that corresponds to the chain rule.

2. Integration by parts

Integration by parts is the integration rule that corresponds to the product rule from differentiation.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \therefore$$

$$\int [f(x)g'(x) + g(x)f'(x)]dx = f(x)g(x) \therefore$$

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x) \therefore$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \therefore$$

If we let u = f(x) and v=g(x) we can re-write the integration by parts rule as

$$\int u dv = uv - \int v du \therefore$$

2. Functions of Form $\int f(g(x))g'(x)dx = \int f(u)du$

Use the substitution rule. This is the integration rule that corresponds to the chain rule.

Worked Examples

Integrate $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} = \int \frac{1-\cos^2 x}{\cos^2 x} = \int \frac{1}{\cos^2 x} - 1 = \int \sec^2 x - 1$$

$$= \tan x - x + C$$

Net Change

The fundamental theorem tells us that if a function f is a continuous on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is any anti-derivative of f. Saying that F is any anti-derivative of f is the same as saying that $F' = f$ enables us to re-write the fundamental theorem as follows.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

In other words the definite integral of a rate of change is the net change in the original function.

Examples

If $V(t)$ is the volume of water in a reservoir at time t then $V'(t)$ is the rate at which water flows into the reservoir at time t and

$$\int_a^b V'(t)dt = V(b) - V(a)$$

Which is the change in the amount of water in the reservoir between time a and time b.

Substitution Rule

Average Value

The average value of a function f is on the interval $[a,b]$ is given by $f_{average} = \frac{1a}{b-a}\int_a^b f(x)dx$

Inverse functions and differentiability

E

Rate of Change of an exponential function

Rate of change proportional to the function itself

We can show that the rate of change of any exponential function is proportional to the function itself by noting that

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}$$

Since a^x doesn't depend on h we can take it outside the limit to get

$$= a^x \lim_{h\to 0}\frac{\left(a^h-1\right)}{h}$$

But we also know that $\frac{d}{dx}a^0 = \lim_{h\to 0}\frac{a^h - a^0}{h}$

So

$$\frac{d}{dx}a^x = a^x f'(0)$$

Definition of the letter E

The letter e defined as the number such that $\lim_{h\to 0}\frac{e^h - e^0}{h} = 1$. We can then easily see that the derivative of an exponential function with base e will be given as

$$\frac{d}{dx}e^x = e^x \cdot 1 = e^x$$

It then very easily follows that the indefinite integral of e^x is given by

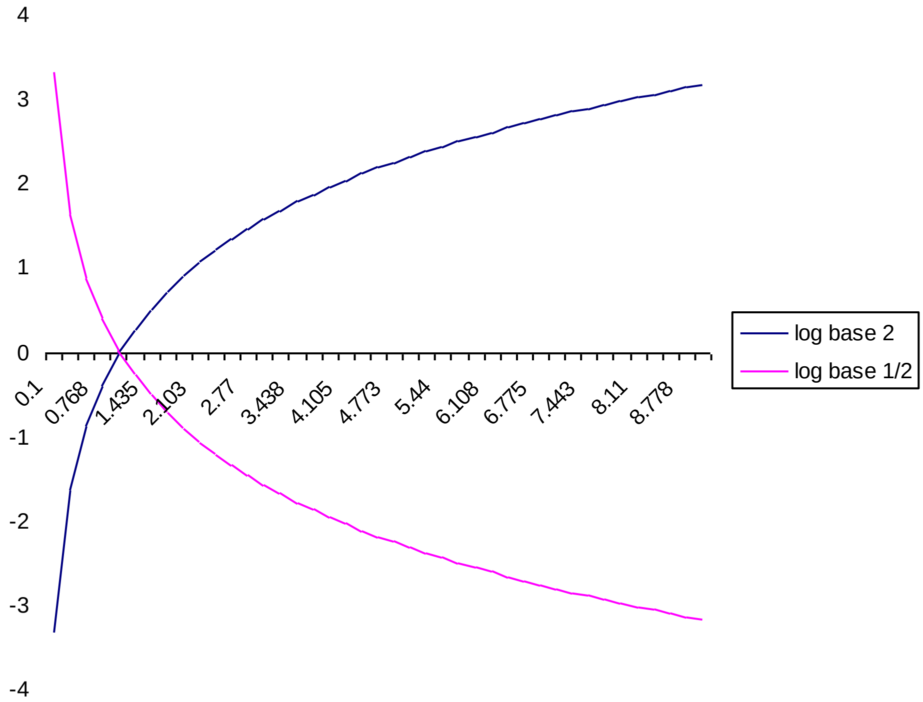
$$\int e^x = e^x + C$$

L Functions

Overview

If a is greater than zero and less than one then the exponential function a^x is increasing or decreasing and is hence one to one. As such it has an inverse function which we call the logarithmic function.

$$y = a^x \therefore \log_a y = x$$



Natural

Of all the choices of base the most common is the base e and write $y = e^x \therefore \log_e y = x$

Furthermore we can use the properties of logarithms to convert from other bases to base e.

$$\log_a x = \log_b x \cdot \log_a b \therefore \log_b x = \frac{\log_a x}{\log_a b}$$

Derivative of the natural logarithm

We can obtain the derivative of a natural logarithm very easily from the derivate of the natural exponent.

$$y = e^x \therefore \frac{dy}{dx} e^x = e^x$$

$$x = \log_e{}^y \therefore \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{e^x} = \frac{1}{y}$$

Derivative of logarithms of base other than e

We can combine the result for the derivative of the natural logarithm with the relationship between logarithms of different bases to obtain the derivative of logarithms of all bases.

$$\log_a x = \log_b x \cdot \log_a b \therefore \log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_a x = \frac{\log_e x}{\log_e a}$$

$$\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)$$

Since $\log_e a$ is a constant we can take it outside the differentiation

$$\frac{1}{\log_e a} \bullet \frac{d}{dx}(\log_e x) = \frac{1}{\log_e a} \bullet \frac{1}{x} = \frac{1}{x \log_e a}$$

Derivative of the exponential function with base other than e

We can now use the fact that we know the derivative of a logarithmic function of any base to calculate the derivative of an exponential function of any base

$y = a^x \therefore x = \ln y$

$$\frac{dx}{dy} = \frac{1}{y \log_e a} \therefore$$

$$\frac{dy}{dx} = \frac{1}{\frac{1}{y \log_e a}} = y \log_e a = a^x \log_e a$$