Interest Rates

And discount factors

Interest rates are the bedrock of finance. Whether it is the man on the street taking out a mortgage to purchase a house or an investment bank trading complex derivatives, it is interest rates that specify the cost of capital and the return on investment of financial products.

In this article we discuss how interest rates are quoted. We show how to convert rates quoted in one quoting convention to an equivalent rate in a different convention. The important concept of a discount factor is also introduced at this stage. Building on this basic foundation we move on to the concept of a forward interest rate and show the relationship between spot starting rates and forward starting rates. Finally, we introduce the concept of a yield or discount curve or curve and show how different interpolation schemes can be used to extract rates for maturities that do not have actual data points on the curve.

Rate Definitions

There are three aspects that make interests rates more complicated than simple returns or growth factors

1. Quoted on an annualized basis
2. Compounding frequency
3. Day count conventions

Most people intuitively understand the first point. When we say the rate on our mortgage is 5% we know it means 5% per annum. The second aspect is more tricky. We consider that now

Rate Expressions and compounding

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. They are two different units of measurement. We can between a rate in one compounding frequency to the equivalent rate in another compounding frequency. Simple interest and has no compounding and its rate expression is given by

Simple Interest

$$\mathbf{(1 + r_s T)}$$

Moving on from simple interest we add compounding a given number of times per annum

M times per annum

$$\left(1 + \frac{r_m}{m}\right)^{mT}$$

Continuous

If we increase the compounding frequency without limit we get continuous compounding

$$\lim_{\boldsymbol{m} \to \infty} \left(\mathbf{1} + \frac{\boldsymbol{r_m}}{\boldsymbol{m}}\right)^{\boldsymbol{m}} = \boldsymbol{e^r}$$

Converting Between Rate Types

Given two rate expressions $\boldsymbol{x_1}, \boldsymbol{x_2}$ with respective rates $\boldsymbol{r_1}, \boldsymbol{r_2}$ we can create expressions for converting between rates by letting $\boldsymbol{x_1} = \boldsymbol{x_2}$and solving for $\boldsymbol{r_1}$in terms **of** $\boldsymbol{r_2}$ We use this rechnique to covert in the following situations.

M Times per annum to continuous $\mathbf{r_c = m \times \ln\left(1 + \frac{r_m}{m}\right)}$

Letting

$$e^{r_c T} = \left(1 + \frac{r_m}{m}\right)^{mT}$$

Taking the natural logarithm of each side

$$r_c T = \ln\left[\left(1 + \frac{r_m}{m}\right)^{mT}\right]$$

Dividing throught by

$$r_c = m \times \ln\left(1 + \frac{r_m}{m}\right)$$

Continuous to m times per annum

N time per annum to m times per annum $r_m = m\left[\left(1+\frac{r_n}{n}\right)^{n/m} - 1\right]$

$\left(1+\frac{r_m}{m}\right)^{mt} = \left(1+\frac{r_n}{n}\right)^{nt}$ Taking natural logarithms of each side

$log\left[\left(1+\frac{r_m}{m}\right)^{mt}\right] = log\left[\left(1+\frac{r_n}{n}\right)^{nt}\right]$ Note that $\log a^{mt} = mt \log a$

$mt\, log\left(1+\frac{r_m}{m}\right) = nt\, log\left(1+\frac{r_n}{n}\right)$ Divide each side by mt

$log\left(1+\frac{r_m}{m}\right) = \frac{n}{m} log\left(1+\frac{r_n}{n}\right)$ Note that $\frac{m}{n}\log a = \log a^{\frac{m}{n}}$

$log\left(1+\frac{r_m}{m}\right) = log\left(1+\frac{r_n}{n}\right)^{\frac{n}{m}}$ Taking exponents

$1+\frac{r_m}{m} = \left(1+\frac{r_n}{n}\right)^{\frac{n}{m}}$ Subtract one from each side

$\frac{r_m}{m} = \left(1+\frac{r_n}{n}\right)^{\frac{n}{m}} - 1$ Multiply both sides by m

$$r_m = m\left[\left(1+\frac{r_n}{n}\right)^{\frac{n}{m}} - 1\right]$$

Discount Factors

We are interested in how we can convert from a discount factor to each rate type and back again. In general we have an equation of the form

$$[RateExp] = \left[\frac{1}{df}\right]$$

So to convert from a given rate expression a discount factor we solve algebraically from the expression $df = \frac{\mathbf{1}}{\boldsymbol{RateExp}}$

- $df = \frac{1}{\left(1+\frac{r_m}{m}\right)^{mT}}$
- $df = \frac{1}{e^{r_c T}} = e^{-r_c T}$