Series

THIS DOCUMENT COVERS

Definitions

Sequence

A list of numbers in a definitive order order $a_1, a_2, \ldots., a_n = \{a_1, a_2, \ldots., a_n, \ldots\} = \{a_n\}$.

Series

Obtained by adding the terms of a sequence $s = \sum_{k=1}^{n} a_k$

geometric Series

The ratio of each term to its predecessor is constant. $S = \sum_{k=0}^{n} = 1 + a + ar + ar^2 + \ldots.. + ar^n$

Taylor Series

We can approximate a function $f(x)$ for small values around some anchor point a, as a power series $f(x) = P(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Properties of sums

Law	Name	Notes
Constant multiple	$\sum_{r=1}^{n} k.f(r) = k.\sum_{r=1}^{n} f(r)$	
Adding	$\sum_{r=1}^{n} \big(f(r) + g(r)\big) =.\sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$	
Linearity	$\sum_{r=1}^{n} \big(c.f(r) + g(r)\big) = c.\sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$	
First n integers	$\sum_{r=1}^{n} r = \frac{n(n-1)}{2}$	
First n integers squared	$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$	
First n integers cubed	$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$	
Value of geometric series	$S = \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$	
Value of ininite geometric series where absolute value of x is less than 1	$lim_{N\to\infty}\frac{1}{1-x} - \frac{x^N}{1-x} = \frac{1}{1-x}$	

Introduction

Even elementary functions can be difficult to work with. Even simple functions like $\sin(x)$ can be difficult to evaluate because every integer input results in an irrational. A small subset, however, known as the polynomials are much easier to work with because they are

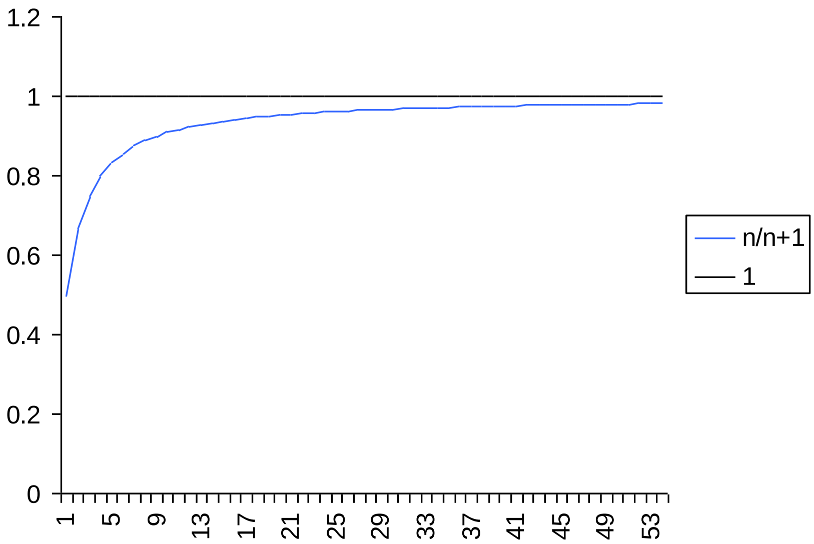
- Easy to integrate
- Easy to evaluate for any value of x
- Infinitely differentiable

The tactic of expressing complicated functions as infinite series motivates much of the study of infinite series

Sequences

Definition

- List of numbers in a definitive order $a_1, a_2, \ldots, a_n, \ldots$
- If $\{a_n\}$ has a limit L we can make $\{a_n\}$ as close to L as we like by increasing n
- $\{a_n\}$ is bounded above if there is some M such that $a_n < M$
- $\{a_n\}$ is bounded below if there is some M such that $a_n > M$
- Every bounded monotonic sequence is convergent



Notation

The following notations are equivalent

- $\{a_1, a_2, \ldots, a_n, \ldots\}$

- $\{a_n\}$
- $\{a_n\}_{n=1}^{\infty}$

Some sequences can be defined by a formula for the nth term.

- $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$
- $a_n = \frac{n}{n+1}$
- $\left\{\frac{1}{2}, \frac{2}{3}, \frac{4}{4}, \ldots\ldots, \frac{n}{n+1}, \ldots.\right\}_{n=1}^{\infty}$

Series

Definition

- Obtained by adding the terms of an infinite sequence
- $\sum_{n=1}^{\infty} a_n = s$
- Given a series $\sum_{n=1}^{\infty} a_n = s$we can generate a sequence of its partial sums
- $s_n = \sum_{i=1}^{n} a_i$
- If the resulting sequence $\{s_n\}$is convergent we say the series is convergent
- With any series $\sum_{n=1}^{\infty} a_n$we associate two sequences
- $\{a_n\}$ the terms
- $\{s_n\}$the partial sums
- If $\sum_{n=1}^{\infty} a_n$is convergent $lim_{n\to\infty} a_n = 0$
- If $lim_{n\to\infty} a_n$ does not exist $\sum_{n=1}^{\infty} a_n$is divergent
- $lim_{n\to\infty} a_n \neq 0$then$\sum_{n=1}^{\infty} a_n$is divergent

Geometric Series

A geometric series is a series where the ratio of each term to its predecessor is constant.

$$S = \sum_{k=0}^{n} = ax^k\ a + ar + ar^2 + \ldots.. + ar^n$$

The value of a geometric series is given by

$$S = \sum_{k=0}^{n} ar^k = \frac{a(1 - r^{n+1})}{1 - r}$$

Infinite Geometric Series

If the first N elements in a geometric series are given by $\frac{1-x^{N+1}}{1-x}$ then the first N-1 elements will be given by $\frac{1-x^N}{1-x} = \frac{1}{1-x} - \frac{x^N}{1-x}$ and in the case where x is greater than -1 and less than one then in the limit as the number of elements tends to infinity we get

$$lim_{N\to\infty} \frac{1}{1-x} - \frac{x^N}{1-x} = \frac{1}{1-x}$$

Since in the case where the absolute value of x is less than zero x^N will tend to zero as N tends to infinity the second term also tends to zero and we have an expression for an infinite geometric series.

Power Series

- $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots.$
- For each x the series is a series of constants which we can test for convergence
- The sum of the series is a function $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$
- The domain of this function is the set of x for which the series converges

Taylor Series

We can approximate a function $f(x)$ for small values around some anchor point a, as a power series.

$$f(x) = P(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For a given $f(x)$ the Taylor series will only be valid for a given subset of the domain, known as the radius of convergence.

Fibonacci

Questions Series

Fibonacci

Fibonacci Iterative

Write Fibonacci iterative

```
public static int FibonacciIterative(int n)
{
    // f0 f1 f2 f3 f4
    //  0  1  1  2  3

    // fn  = Fibonacci(n)
```

```
            // fn1 = Fibonacci(n+1)
            // fn2 = Fibonacci(n+2)
            int fn = 0, fn1 = 1;

            for (int i = 0; i < n; i++)
            {
                int fn2 = fn + fn1;
                fn = fn1;
                fn1 = fn2;
            }
            return fn;
        }
```

Analyse the runtime?

O(N)

FIBONACCI RECURSIVE

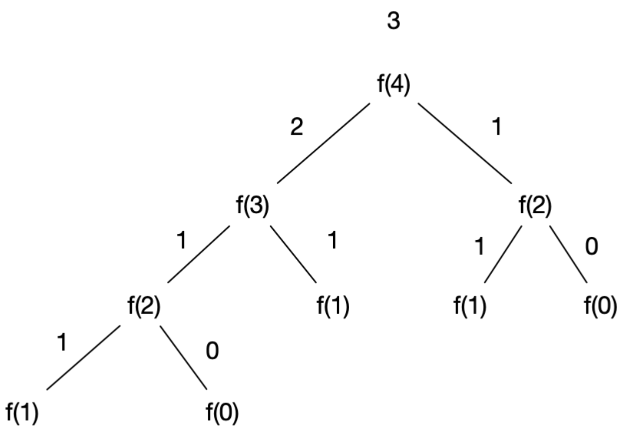
Write Fibonacci recursive

```
        public static int FibonacciRecursive(int n)
        {
            if (n == 0)
                return 0;
            if (n == 1)
                return 1;

            return FibonacciRecursive(n - 1) + FibonacciRecursive(n - 2);
        }
```

Analyse the runtime?

Consider the following call graph of f(4)



The runtime is upper bounded by $O(2^n)$*. There is a slighter tighter bound*

Improve the performance of the recursive algorithm

This is O(2n) =O(n)

```
public static int FibonacciRecursiveMemo(int i)
{
    var cache = new int[i + 1];

    int F(int x)
    {
        if (x == 0 || x == 1) return x;

        if (cache[x] == 0) cache[x] = F(x - 1) + F(x - 2);

        return cache[x];
    }

    return F(i);
}
```

Proofs

Prove that $\sum_{r=1}^{n} r = \frac{n(n-1)}{2}$

$\sum_{r=1}^{n} r = 1 + 2 + 3 + 4 + , \ldots , + n$ and

$\sum_{r=1}^{n} r = n + (n-1) + (n-2) + , \ldots\ldots\ldots + 1$ therefore

$$\sum_{r=1}^{n} r + \sum_{r=1}^{n} r = 2\sum_{r=1}^{n} r = (n+1) + (n+1) + , \ldots\ldots\ldots + (n+1) = n(n+1) \therefore \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Prove that $S = \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$

We show a now a proof of the value of a geometric series

$$S = \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$

$$S = a + ar + ar^2 + \ldots\ldots + ar^n$$

$$Sr = ar + ar^2 + \ldots\ldots + ar^n + ar^{n+1}$$

Subtracting the second expression from the first we get

$$S - Sr = a - ar^{n+1}$$

Factoring both sides

$$S(1-r) = a(1-r^{n+1})$$

Dividing both sides by $(1-r)$

$$S = \frac{a(1-r^{n+1})}{(1-r)}$$

Prove that $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ $if\ abs(r) < 1$

We prove this useful result by looking at the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} ar^k = \frac{a\left(1-r^{n+1}\right)}{1-r}$$

And noting that the numerator $a\left(1-r^{n+1}\right)$will tend to a as n tends to infinity

$$\sum_{k=0}^{\infty} kr^{k-1} = \frac{1}{1-r}$$