Properies of Discrete Random variables

1. Expectation/Mean/Expected Value

$$E(X) = \sum_{i=1}^{n} x_i p(x_i)$$

- Weighted average of the values the random variable X can take
- Weighting by the probability of each value
- Measure of centrality

2. Expectation of a constant multiple of a random variable

$$E[aX + b] = aE[X] + b$$

$E[aX + b] = \sum_{i=1}^{n}(ax_i + b)p(x_i)$ From definition 1

$= \sum_{i=1}^{n}(ax_i)p(x_i) + \sum_{i=1}^{n} bp(x_i)$ By multiplying out the brackets

$= a\sum_{i=1}^{n} x_i p(x_i) + b\sum_{i=1}^{n} p(x_i)$ From the properties of summation

$= aE[X] + b\sum_{i=1}^{n} p(x_i)$ From definition 1

$= aE[X] + b.1$ From axioms of probability

$= aE[X] + b$

3. Expectation of a function of random variable

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

- The expectation of a function of a random variable is **not equal** to the function of the expectation $E[g(X)] \neq g[E(X)]$

4. Variance

$$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

Let $\mu = E[X]$

$E[(X-\mu)^2] = \sum_{i=1}^{n}(x_i-\mu)^2 p(x_i)$ From definition

$= \sum_{i=1}^{n}(x_i{}^2 - 2\mu x_i + \mu^2)p(x_i)$ Multiplying out

$= \sum_{i=1}^{n} x_i{}^2 p(x_i) + \sum_{i=1}^{n} -2\mu x_i p(x_i) + \sum_{i=1}^{n} \mu^2 p(x_i)$

$= E[X^2] + \sum_{i=1}^{n} -2\mu x_i p(x_i) + \sum_{i=1}^{n} \mu^2 p(x_i)$ From definition 3

$= E[X^2] - 2\mu \sum_{i=1}^{n} x_i p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i)$ Properties of summations

$$= E[X^2] - 2\mu\mu + \mu^2 \sum_{i=1}^{n} p(x_i)$$

$= E[X^2] - 2\mu\mu + \mu^2$ Axioms of probability

$= E[X^2] - \mu^2$

$= E[X^2] - (E[X])^2$

5. Variance of a constant property

$$Var[a] = 0$$

6. Variance of a constant multiple

$Var[aX] = aVar[X]$

$Var[aX] = E[(aX - E[aX])^2]$

$= E[(aX - aE[X])^2]$ From definition 2

$= E[(aX - a\mu)^2]$ Letting $\mu = E[X]$

$= \sum_{i=1}^{n}\left(ax_i \quad - a\mu\right)^2 p(x_i)$ From definition

$$= \sum_{i=1}^{n} a^2\left(x_i \quad - \mu\right)^2 p(x_i)$$

$$= a^2 \sum_{i=1}^{n}\left(x_i \quad - \mu\right)^2 p(x_i)$$

$= a^2 Var[X]$ From definition 4

7. Expectation of the sum of two finite countable variables

If X is a random variable with sample space $\{x_1, x_2, \ldots\ldots, x_m\}$and Y is an independent random variable with sample space $\{y_1, y_2, \ldots\ldots, y_n\}$then the sample space of the joint distribution will be given by a set of pairs

$\{x_1, y_1\}, \{x_1, y_2\}, \ldots\ldots, \{x_1, y_n\}$

$\{x_2, y_1\}, \{x_2, y_2\}, \ldots\ldots, \{x_2, y_n\}$

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$\{x_m, y_1\}, \{x_m, y_2\}, \ldots\ldots, \{x_m, y_n\}$

The expectation of the sum of the two variables in then given by

$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) p(x_i, y_j)$

Multiplying out we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \, p(x_i, y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j \, p(x_i, y_j)$$

Noting that $\sum_{j=1}^{n} p(x_i \,, y_j) = p(x_i)$ and $\sum_{i=1}^{m} p(x_i \,, y_j) = p(y_j)$

$$\sum_{i=1}^{m} x_i p(x_i) + \sum_{j=1}^{n} x_j p(y_j)$$

Therefore we can note that

$$E[X + Y] = E[X] + E[Y]$$

8. Expectation of the sum of n identically distributed random variables

We can calculate the expectation of the sum of n identically distributed random variables denoted by $X_1, X_2, ..., X_n$ as $E[X_1] + E[X_2] + ... + E[X_n]$ which is equal to

\ $n. E[X_n]$