Differential Equations

## Introduction

This document covers

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A first order ODE is an equation involving only a function y, its first derivative and any given functions of the independent variable y

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A function is called a solution of a given ODE on some open interval if h(x) is defined and differentiable throughout the interval and is such that the ODE becomes an identity when y is replaced by h and y’ is replaced by h’

# Solvable by integration (I)

Where the first order ODE is off the form

Or the equivalent implicit form

Then we can simply solve by integration

Consider the example

Then integrating gives us

# Solvable by integration (II)

We can also first order ODE’s of the following form where a and b are constants

Or the equivalent implicit form

We re-arrange to get

Integrating both sides with respect to x

# Integrating Factors

We will use the first order ODE

To highlight a method that can be used to solve equations which do not easily lend themselves to integration (Note this simple example could of course be solved using the technique described in the previous section)

The general idea is we multiply the ordinary differential equation by a function such that the left hand side of the equation is equal to the derivative of the product If we find such a function we can easily integrate the lhs to get rid of the y’ term , obtain and solve

We want to find a function r(x)

Note now that the left hand side looks like the derivative of the product ry. Remembering from the product rule of differentiation that

To ensure that the left hand side of (i) is indeed the derivative of the product ry we need to equate the left hand side of (i) with the right hand side of (ii)

Solving for r(x)

Integrating with respect to x

Let’s make use of our new integrating factor by inserting it into the original ODE

We can simplify by dividing through by c

And now of course the left hand side is equal to the derivative of the product so we can integrate

You can if you wish verify this solution by using the method in the previous section

# General Solution of Linear First Order Differential Equations

We can use the same technique of integrating factors to find the general form of linear first order ordinary differential equations

As before we want to find an integrating factor r(x) such that the left hand side of the linear ODE becomes the derivative of the product r(x)y given by

Essentially we need

Inserting back into the original equation we obtain

+c

# Integrating Factors involving a function

We can use the same technique of integrating factors where we have a function of the form

Or implicitly

We find a function r(x) such that

Equating the right left side of the ode with the right side of the derivative of the product we get

’ Hence

So we can use the integrating factor remembering that the c drops out. Inserting the factor into the original equation we obtain

Integrating with respect to x we obtain

+c

Therefore

Of course at this stage a solution depends on being able to calculate the integral on the r.h.s

# A short cut

The integrating factor for an ODE of the form is always given by

# Separable ODE’s

ODE’s of the form

Can be directly solved by integration by noting that

Let’s look at some examples

## Example 1 Radiocarbon dating

Radioactive decay is governed by the ODE or the rate of change is proportional to the amount of the substance already present. We can solve for the general solution of this ODE using separation of variables as follows

Re-arranging

Integrating with respect to x

Noting that we get

Taking exponents of both sides

If we are told the half life of a particular substance is 5715 years we can find an exact solution by noting

Dividing through by A and taking logarithms