Differentiation

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Definition of the derivative

### Table 1 Rules of Differentiation

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## Definition of the derivative

The derivative measures the rate of change of one quantity with respect to another. Differentiation is then just the process of finding the derivative of a function. If we have a function of x then one of the many notations for specifying the derivative is as follows

So if we took one or the simplest non-linear functions and differentiate it we see that

.

So in the simple case where x is equal to one

It is worth noting that differential calculus is concerned with finding the instantaneous rate of change of f with respect to x.

Why Bother?

Numerous problems in business, economics and finance are concerned with determining how one quantity is changing with respect to another. Differentiation also enables us to find where a function is highest and lowest both locally and across the entire domain. Also we often find where a rate of change is greatest or smallest and again differentiation provides us with this.

Where the derivative does not exist

There are three places where the derivative does not exist

* Discontinuity
* Cusp on a function
* Vertical inflection point

## Calculation of the derivative

### The approach



Algebraic Derivation

If then the derivative is defined as

Simple Example

Let us consider a basic quadratic then the derivative becomes

### Proofs

Constant Function

The derivative of a constant function is zero



Constant Multiple Rule

Sum Rule

Product Rule

Add and subtract to the numerator

Arrange the terms on the numerator

Factorize

Quotient Rule

Add and subtract to the numerator

Re-arrange

Factorize

Because we re-arrange the above expression as

Power Rule

By mathematical induction where m = 0

then by the constant rule

So the rule works for m = 0 since . Now we assume the rule works for m and prove it works for m+1

Using the power rule

Because we assume is correct for m and from the constant rule

Since the rule works for m = 0 and m = m+1 it works for all mChain Rule

The chain rule expresses a very simple notion in a slightly complex fashion. If

And by extension if

Etc.

The proof is fairly self-explanatory. If a unit change in x leads to a three unit change in g and a unit change in g leads to a four unit change in f then a unit change in x leads to a twelve unit change in f.

Derivative of



The definition of the letter E is the number with the unique property that its gradient at the point it crosses the y-axis is one. From this it follows that

So from the definition of the derivative we have

Plugging in the previous result that

Derivative of inverse functions

The defining property of an inverse function is given by

So the inverse function is a function of a function. By the chain rule

And since and we get

Also we can not that an inverse function is a reflection of the original function about the line . After such a reflection the new function will have a gradient equal to the reciprocal of the original function

Natural Logarithm

We can use the result of differentiating the exponent to differentiate the natural logarithm

Proof Natural logarithm 2

If then

We know that and from the derivative of a reciprocal that

Alternative

Sine

We first note that that which we will need later

Now we can re-arrange the terms to give us

Now the first term will tend to zero and the second term will tend to 

Derivative of

Proof (a)

We can easily proof this result using the chain rule. First we note that

Then by the chain rule and let

Proof (b)



By mathematical induction where m = 0

then by the constant rule

So the rule works for m = 0 since . Now we assume the rule works for m and prove it works for m+1



Using the power rule



Because we assume is correct for m and from the constant rule







Since the rule works for m = 0 and m = m+1 it works for all m

Derivative of sin x

We first note that that which we will need later





Now we can re-arrange the terms to give us



Now the first term will tend to zero and the second term will tend to 

## *Worked Examples*

##### Worked Example 1Differentiate

If we multiply the top and bottom by we get

## Worked Example 2 Differentiate

If we multiply the top and bottom by we get



##### Worked Example 3 Differentiate

If we multiply the top and bottom by we get



##### Worked Example 4 If, find and use it to find the tangent line to the curve at the point.

Binomial Expansion



where

We know that the slope of the tangent and that is passes through the point

Worked Example 5

If find and use it to find the equation of the tangent to the curve at the point

The gradient of the tangent to the curve at point is given by and the equation of the tangent is



##### Worked Example 5

Differentiate 

First re-write the function as 

Now differentiate each term in turn using the chain rule



Simplifying



Express using positive powers only

Note that to rewrite

Note that to rewrite as

Simplify

Note that 

Finally expressing the whole derivative in surd notation



##### Worked Example 6











##### Worked Example 7

Differentiate 

By the quotient rule



Multiple by 



##### Worked Example 8





Notice that the derivative is zero when the function has a horizontal tangent and that as in the given interval the function is always decreasing so the derivative is always negative



##### Worked Example 9

If and find when 



, 



If  then 

 Multiply by 

