Integration

## Introduction

This document covers

Introduction

# Definite Integral

If is a continuous function defined for , we divide the sub-interval into n subintervals of equal width . We let be the endpoints of these sub-intervals and we let be any sample points on these sub-intervals, so lies on the sub-interval then the definite integral of from a to b is



is the integrand, a and b are the limits of integration. The symbol dx has no official meaning by itself. The definite integral is a number that can be interpreted as the area under the curve from a to b

## Properties of the Definite Integral

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# Fundamental

## FTOC

If then 

where F is any anti-derivative of f, that is 

# Integrating complex numbers

## One Show that



And

## 

Second express both more simply using a phase angle

Let and 

Consider 





# 

# 

# 

# 

# 

Since real and imaginary parts must be equal





# Techniques

Integration is more challenging than differentiation and it is not always obvious which technique should be used. The following strategy may be useful.

## Learn the basic integration formulas

Use the following table



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## Functions of linear functions of x

## Partial Fractions

## Integral of form

If we have a rational function where the numerator is the derivative of the denominator or the numerator is a multiple/sub-multiple of the denominator we can solve it using the following technique. We first show an example



We first let the denominator by z, that is so that



We can then re-write the original derivative in terms of z as follows



## Integral of form

In very similar way to section four we can integrate such as function as



Let 



## Integrating products of functions by parts

The product rule of differentiation states that if u and v are functions of x then



If we integrate both sides of this expression we get



Re-arranging we get 

We introduce shorthand to make it easier to remember



We can then make use of this approach to integrate products of functions. We assume one of the functions is the derivative which we have to integrate. The following guidelines should be followed when choosing which term to use as u

* If one of factors is a log function that function must be chosen as u
* If there is no log term present the power of x is taken as u
* This only holds good for positive whole number powers of x
* If there is neither a log function or a power of x, the exponential function is taken as u

### Example 1



Let 

# Integrating Trig Functions

## 1. Integrate

First we make use of the trigonometric identity to express the integral as a function of a linear function of x.



So now we rewrite it



To integrate  let 





## 2. Integrate

We first note that



Now we make use of the trigonometric identity



This enables us to write the integral in the form



The first tem in the integral is of the form and the second term is a standard integral.

For the first term we let 



We now have the whole thing



## 3. Integrate

First we note that



Now we use the following trigonometric identity



To rewrite our integral as





Next we make sure of the trigonometric identity



Lets substitute this into our integral





# 

## 2. Integrate

 From the trigonometric identities, therefore



To integrate  let 





## 3. Integrate

## 



4. Integrate

Integrate

# 

# 

# Integrating Powers of Sine



 // Let 



// Let 

# Integrating Powers of Cosecant



**Proof**



Let 



But note also that



## Integrating powers of cosecant higher than two



 Now we integrate by parts

















# Integrating Powers of Cosine

## 







# Integrating Powers of Secant



// Basic anti derivative

For powers of secant greater than two we can use the following reduction formula



We can prove this result using integration by parts as follows

Let and hence 











Re-arranging we get





# Integrating Powers of Tangent









# Approaches to solving

Integration is more challenging than differentiation and it is not always obvious which technique should be used. The following strategy may be useful.

## 1. Learn the basic integration formulas

Use the following table



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## 2. Simply the Integrand if possible

* Algebraic manipulation
* Trign

## 2. Functions of Form

Use the substitution rule. This is the integration rule that corresponds to the chain rule.

## 2. Integration by parts

Integration by parts is the integration rule that corresponds to the product rule from differentiation.









If we let u = f(x) and v=g(x) we can re-write the integration by parts rule as

## 

## 2. Functions of Form

Use the substitution rule. This is the integration rule that corresponds to the chain rule.

## Worked Examples

Integrate

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# Net Change

The fundamental theorem tells us that if a functionis a continuous onthen



Where is any anti-derivative of. Saying that is any anti-derivative ofis the same as saying that enables us to re-write the fundamental theorem as follows.



In other words the definite integral of a rate of change is the net change in the original function.

## Examples

Ifis the volume of water in a reservoir at time t thenis the rate at which water flows into the reservoir at time t and



Which is the change in the amount of water in the reservoir between time a and time b.

# Substitution Rule

# Average Value

The average value of a functionis on the intervalis given by 

# Inverse functions and differentiability

# E

## Rate of Change of an exponential function

### Rate of change proportional to the function itself

We can show that the rate of change of any exponential function is proportional to the function itself by noting that







Since doesn’t depend on h we can take it outside the limit to get



But we also know that 

So



### Definition of the letter E

The letter e defined as the number such that . We can then easily see that the derivative of an exponential function with base e will be given as



It then very easily follows that the indefinite integral of is given by



## L Functions

Overview

If a is greater than zero and less than one then the exponential function is increasing or decreasing and is hence one to one. As such it has an inverse function which we call the logarithmic function.



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### Natural

Of all the choices of base the most common is the base e and write 

Furthermore we can use the properties of logarithms to convert from other bases to base e.



### Derivative of the natural logarithm

We can obtain the derivative of a natural logarithm very easily from the derivate of the natural exponent.





### Derivative of logarithms of base other than e

We can combine the result for the derivative of the natural logarithm with the relationship between logarithms of different bases to obtain the derivative of logarithms of all bases.







Since is a constant we can take it outside the differentiation



### Derivative of the exponential function with base other than e

We can now use the fact that we know the derivative of a logarithmic function of any base to calculate the derivative of an exponential function of any base



