Pricing Derivatives Using Black-Scholes-Merton Model

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Introduction

In this blog, we will explore how to price simple equity derivatives using the Black-Scholes-Merton (BSM) model. We will derive the mathematical formula and then provide Python code to implement it.

Background and Preliminaries

Before proceeding to the deep of the discussion, we need to know some definition and terminology

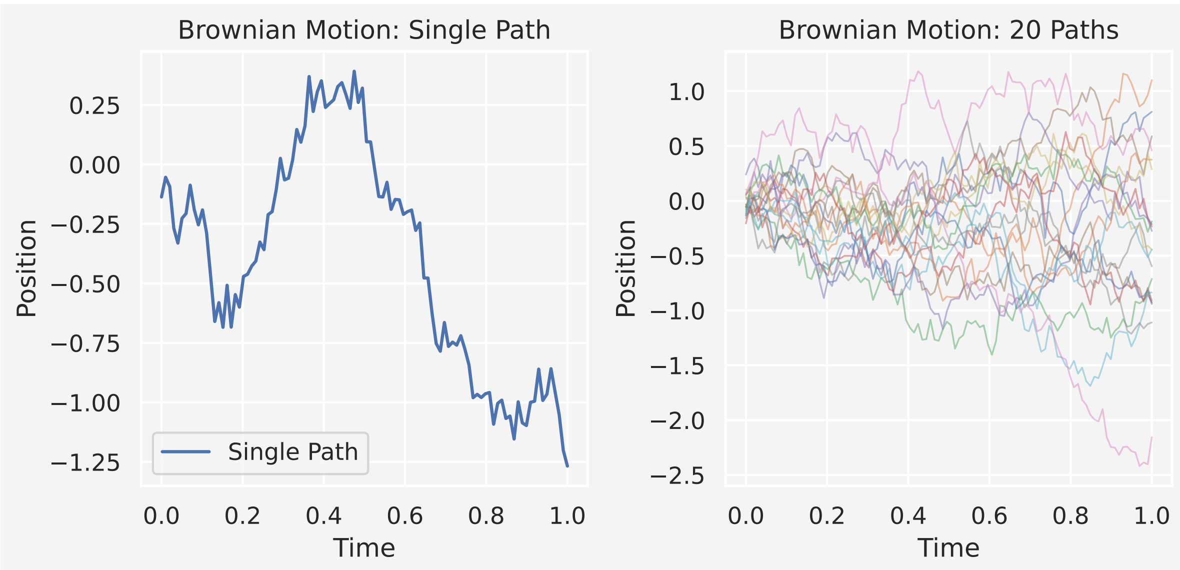
Brownian Motion: Brownian motion is a concept with definitions and applications across various disciplines, named after the botanist Robert Brown, is the random, erratic movement of particles suspended in a fluid (liquid or gas) due to their collisions with the fast-moving molecules of the fluid.

Brownian motion is a stochastic process $(B_t)_{t\geq 0}$ *defined as a continuous-time process with the following properties:*

- $B_0 = 0$ almost surely.
- B_t has independent increments.
- For $t > s$, $B_t - B_s \sim N(0, t-s)$ (normally distributed with mean 0 and variance $t-s$).
- B_t has continuous paths almost surely.

```
from mywebstyle import plot_style
plot_style('#f4f4f4')
import numpy as np
import matplotlib.pyplot as plt

# Parameters
n_steps = 100  # Number of steps
n_paths = 20   # Number of paths
time_horizon = 1  # Total time
dt = time_horizon / n_steps  # Time step
t = np.linspace(0, time_horizon, n_steps)  # Time array

# Generate Brownian motion
def generate_brownian_paths(n_paths, n_steps, dt):
    # Standard normal increments scaled by sqrt(dt)
    increments = np.random.normal(0, np.sqrt(dt), (n_paths, n_steps))
    # Cumulative sum to generate paths
    return np.cumsum(increments, axis=1)

# Generate one path and multiple paths
single_path = generate_brownian_paths(1, n_steps, dt)[0]
multiple_paths = generate_brownian_paths(n_paths, n_steps, dt)

# Plotting
fig, axes = plt.subplots(1, 2, figsize=(7.9, 3.9))

# Single path
axes[0].plot(t, single_path, label="Single Path")
axes[0].set_title("Brownian Motion: Single Path")
axes[0].set_xlabel("Time")
```

```
axes[0].set_ylabel("Position")
axes[0].legend()

# Multiple paths
for path in multiple_paths:
    axes[1].plot(t, path, alpha=0.5, linewidth=0.8)
axes[1].set_title(f"Brownian Motion: {n_paths} Paths")
axes[1].set_xlabel("Time")
axes[1].set_ylabel("Position")

plt.tight_layout()
plt.show()
```



Geometric Brownian Motion (GBM)

A stochastic process S_t is said to follow a geometric Brownian motion if it satisfies the following equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Which can be written as

$$S_t - S_0 = \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u$$

To solve the GBM, we apply Ito's formula to the function $Z_t = f(t, S_t) = \ln(S_t)$ and then by Taylor's expansion, we have

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial s}dtdS_t$$

By definition we have

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \\ (dS_t)^2 &= \mu^2 (dt)^2 + 2\mu\sigma dt dB_t + \sigma^2 (dB_t)^2 \end{aligned}$$

The term $(dt)^2$ is negligible compared to the term dt and it is also assume that the product $dtdB_t$ is negligible. Furthermore, the quadratic variation of B_t i.e., $(dB_t)^2 = dt$. With these values, we obtain

$$\begin{aligned} dZ_t &= \frac{1}{S_t}dS_t + \frac{1}{2}\left\{-\frac{1}{S_t^2}\right\}[dS_t]^2 \\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\left\{-\frac{1}{S_t^2}\right\}\sigma^2 S_t^2 dt \\ \implies dZ_t &= (\mu dt + \sigma dB_t) - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t \end{aligned}$$

with $Z_0 = \ln S_0$. Now we have the following

$$\begin{aligned} \int_0^t dZ_s &= \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dB_s \\ \implies Z_t - Z_0 &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t \\ \implies \ln S_t - \ln S_0 &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t \\ \implies \ln\left(\frac{S_t}{S_0}\right) &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t \\ \implies S_t &= S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} \end{aligned}$$

Black-Scholes-Merton Formula

Now we are ready to derive the BSM PDE. The payoff of an *option* $V(S, T)$ at maturity is is known. To find the value at an earlier stage, we need to know how V behaves as a function of

S and t. By Ito's lemma we have

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dB.$$

Now let's consider a portfolio consisting of a short one option and long $\frac{\partial V}{\partial S}$ shares at time t. The value of this portfolio is

$$\Pi = -V + \frac{\partial V}{\partial S} S$$

over the time $[t, t + \Delta t]$, the total profit or loss from the changes in the values of the portfolio is

$$\Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S$$

Now by the discretization we have,

$$\begin{aligned}
\Delta S &= \mu S \Delta t + \sigma S \Delta B \\
\Delta V &= \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta B \\
\implies \Delta \Pi &= \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) \Delta t
\end{aligned}$$

At this point, if r is the risk-free interest rate then we will have following relationship

$$r \Pi \Delta t = \Delta \Pi$$

The rationale of this relation is that no-aribtrage assumption. Thus, we have

$$\begin{aligned}
\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) \Delta t &= r \left(-V + \frac{\partial V}{\partial S} S\right) \Delta t \\
\implies \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - rV = 0
\end{aligned}$$

This is the famous Black-Scholes-Merton PDF, formally written with the boundary conditions as follows

$$
\begin{aligned}
\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 c^2 \frac{\partial^2 c}{\partial S^2} + rc\frac{\partial c}{\partial S} - rc = 0 \\
c(0,t) = 0 \\
c(S_{+\infty}, t) = S - Ke^{-r(T-t)} \\
c(S,T) = max\{S-K, 0\}
\end{aligned}
$$

This Black-Scholes-Merton PDE can be reduced to the heat equation using the substitutions $S = Ke^x$, $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, and $c(S,t) = Kv(x,\tau)$. Let's derive the solution step by step in full mathematical detail and show how this leads to the normal CDF.

Step 1: Substitutions

We aim to reduce the BSM PDE:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0$$

to the heat equation. Using the substitutions:

- $S = Ke^x$, where $x = \ln(S/K)$, and $S \in (0, \infty)$ maps $x \in (-\infty, \infty)$,
- $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, so $\tau = \frac{1}{2}\sigma^2(T-t)$,
- $c(S,t) = Kv(x,\tau)$, where $v(x,\tau)$ is the transformed function.

Step 2: Derivative Transformations

For $c(S,t) = Kv(x,\tau)$, we compute derivatives.

1. The first derivative of c with respect to S:
$$\frac{\partial c}{\partial S} = \frac{\partial}{\partial S}\big(Kv(x,\tau)\big) = K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S},$$
where $x = \ln(S/K)$ implies $\frac{\partial x}{\partial S} = \frac{1}{S}$. Thus:
$$\frac{\partial c}{\partial S} = K\frac{\partial v}{\partial x}\frac{1}{S}.$$

2. The second derivative of c with respect to S:
$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x}\frac{1}{S}\right).$$
Using the product rule:
$$\frac{\partial^2 c}{\partial S^2} = K\frac{\partial^2 v}{\partial x^2}\frac{1}{S^2} - K\frac{\partial v}{\partial x}\frac{1}{S^2}.$$

3. The time derivative:

$$\frac{\partial c}{\partial t} = K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}, \quad \text{and } \frac{\partial \tau}{\partial t} = -\frac{1}{\frac{1}{2}\sigma^2}.$$

Step 3: Transforming the PDE

Substituting the above derivatives into the BSM PDE, we rewrite each term.

1. For $\frac{\partial c}{\partial t}$:

$$\frac{\partial c}{\partial t} = -\frac{1}{\frac{1}{2}\sigma^2}K\frac{\partial v}{\partial \tau}.$$

2. For $\frac{\partial c}{\partial S}$:

$$S\frac{\partial c}{\partial S} = S \cdot \left(K\frac{\partial v}{\partial x}\frac{1}{S}\right) = K\frac{\partial v}{\partial x}.$$

3. For $\frac{\partial^2 c}{\partial S^2}$:

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} = \frac{1}{2}\sigma^2 S^2\left(K\frac{\partial^2 v}{\partial x^2}\frac{1}{S^2} - K\frac{\partial v}{\partial x}\frac{1}{S^2}\right) = \frac{1}{2}\sigma^2 K\frac{\partial^2 v}{\partial x^2}.$$

Substituting all these into the BSM PDE:

$$-\frac{1}{\frac{1}{2}\sigma^2}K\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 K\frac{\partial^2 v}{\partial x^2} + rK\frac{\partial v}{\partial x} - rKv = 0.$$

Divide through by K:

$$-\frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial x^2} + \frac{2r}{\sigma^2}\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v = 0.$$

To simplify, let $v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$, where α and β are constants. Substituting and choosing $\alpha = -\frac{r}{\sigma^2}$ and $\beta = -\frac{r^2}{2\sigma^2}$, the equation reduces to:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Step 4: Solving the Heat Equation

The heat equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ has a well-known solution using Fourier methods:

$$u(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\tau}} f(y)\, dy,$$

where $f(y)$ is the initial condition.

For the BSM problem, the initial condition is the payoff:

$$f(y) = \max(e^y - 1, 0).$$

Performing the integration leads to the final solution involving the cumulative normal distribution function:

$$v(x, \tau) = N(d_1) - e^{-x} N(d_2),$$

where:

$$d_1 = \frac{x + \frac{1}{2}\tau}{\sqrt{\tau}}, \quad d_2 = \frac{x - \frac{1}{2}\tau}{\sqrt{\tau}}.$$

Transforming back to the original variables gives the Black-Scholes formula:

$$C(S, t) = Se^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where:

$$d_1 = \frac{\ln(S/K) + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}.$$

Similarly, we can derive the price of a European put option:

$$P = Ke^{-rT} N(-d_2) - Se^{-qT} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Asymptotic Behavior of the BSM formula for call and put options

What if $K \to 0$? In that case,

1. $\ln(S_0/K) \to \infty$, causing $d_1 \to \infty$ and $d_2 \to \infty$
2. The cdf $N(d_1) \to 1$ and $N(d_2) \to 1$
3. The second term $Ke^{-rT} N(d_2) \to 0$ as $K \to 0$

In this case, the price of a call option $C \to S_0$ and the price of a put option $P \to 0$

Greeks: Delta and Gamma

Delta (Δ) is the sensitivity of the option price to changes in the underlying asset price:

$$\Delta = \frac{\partial C}{\partial S} \approx \frac{C(S_0 + h) - C(S_0 - h)}{2h}$$

This is the **central difference approximation**, which provides a more accurate estimate of delta compared to the forward or backward difference methods.

- $C(S_0 + h)$: Calculate the option price with the spot price increased by h.
- $C(S_0 - h)$: Calculate the option price with the spot price decreased by h.

Gamma (Γ) measures the rate of change of delta with respect to the underlying asset price:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{\Delta(S_0 + h) - \Delta(S_0 - h)}{2h} \approx \frac{C(S_0 + h) - 2C(S_0) + C(S_0 - h)}{h^2}$$

Gamma (Γ) measures the rate of change of delta (Δ) with respect to the underlying spot price (S_0).

- $C(S_0 + h)$: Option price with the spot price increased by h.
- $C(S_0)$: Option price at the current spot price.
- $C(S_0 - h)$: Option price with the spot price decreased by h.

Relationship Between Delta and Gamma:

- Gamma represents how much delta changes for a small change in S_0.
- If gamma is high, delta is more sensitive to changes in S_0, which is important for hedging strategies.

Implementation

Notation

- S: Spot price of the stock.
- K: Strike price of the option.
- T: Time to maturity (in years).
- r: Risk-free rate (continuously compounded).
- q: Dividend yield (continuously compounded).
- σ: Volatility of the stock.
- $N(\cdot)$: Cumulative distribution function of the standard normal distribution.

```python
from dataclasses import dataclass
import numpy as np
from scipy.stats import norm

@dataclass
class Equity:
    spot: float
    dividend_yield: float
    volatility: float

@dataclass
class EquityOption:
    strike: float
    time_to_maturity: float
    put_call: str

@dataclass
class EquityForward:
    strike: float
    time_to_maturity: float

def bsm(underlying: Equity, option: EquityOption, rate: float) -> float:
    S = underlying.spot
    K = option.strike
    T = option.time_to_maturity
    r = rate
    q = underlying.dividend_yield
    sigma = underlying.volatility

    # Handle edge case where strike is effectively zero
    if K < 1e-8:
        if option.put_call.lower() == "call":
            return S
        else:
            return 0.0

    d1 = (np.log(S / K) + (r - q + 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    if option.put_call.lower() == "call":
        price = S * np.exp(-q * T) * norm.cdf(d1) \
                - K * np.exp(-r * T) * norm.cdf(d2)
```

```python
    elif option.put_call.lower() == "put":
        price = K * np.exp(-r * T) * norm.cdf(-d2) \
                - S * np.exp(-q * T) * norm.cdf(-d1)
    else:
        raise ValueError("Invalid option type. Must be 'call' or 'put'.")

    return price

def delta(underlying: Equity, option: EquityOption, rate: float) -> float:
    bump = 0.01 * underlying.spot
    bumped_up = Equity(spot=underlying.spot + bump,
                       dividend_yield=underlying.dividend_yield,
                       volatility=underlying.volatility)
    bumped_down = Equity(spot=underlying.spot - bump,
                         dividend_yield=underlying.dividend_yield,
                         volatility=underlying.volatility)
    price_up = bsm(bumped_up, option, rate)
    price_down = bsm(bumped_down, option, rate)
    return (price_up - price_down) / (2 * bump)

def gamma(underlying: Equity, option: EquityOption, rate: float) -> float:
    bump = 0.01 * underlying.spot
    bumped_up = Equity(spot=underlying.spot + bump,
                       dividend_yield=underlying.dividend_yield,
                       volatility=underlying.volatility)
    bumped_down = Equity(spot=underlying.spot - bump,
                         dividend_yield=underlying.dividend_yield,
                         volatility=underlying.volatility)
    original_price = bsm(underlying, option, rate)
    price_up = bsm(bumped_up, option, rate)
    price_down = bsm(bumped_down, option, rate)
    return (price_up - 2 * original_price + price_down) / (bump**2)

def fwd(underlying: Equity, forward: EquityForward, rate: float) -> float:
    S = underlying.spot
    K = forward.strike
    T = forward.time_to_maturity
    r = rate
    q = underlying.dividend_yield
    forward_price = S * np.exp((r - q) * T) - K

    return forward_price
```

```python
def check_put_call_parity(
    underlying: Equity,
    call_option: EquityOption,
    put_option: EquityOption,
    rate: float
    ) -> bool:

    call_price = bsm(underlying, call_option, rate)
    put_price = bsm(underlying, put_option, rate)
    S = underlying.spot
    K = call_option.strike
    T = call_option.time_to_maturity
    r = rate
    q = underlying.dividend_yield

    parity_lhs = call_price - put_price
    parity_rhs = S * np.exp(-q * T) - K * np.exp(-r * T)

    return np.isclose(parity_lhs, parity_rhs, atol=1e-4)
```

Example Usage

Say, we want to price a call option on an equity with spot price $S_0 = 450$ with dividend yield $q = 1.4\%$, and volatility 14%. The strike price of the call is $K = 470$, with time to maturity in years $T = 0.23$ and the risk free rate $r = 0.05$. Next, we want to see the asymptotic behavior of the call option if the strike price $K \to 0$ with interest rate 0. Next, we want to price a put option on the same equity but strike price $K = 500$, time to maturity in years $T = 0.26$ and interest rate is 0. Finally, we want to check if the put-call parity relationship is hold. In each case, we consider $h = 0.01$ a bump or small change in the stock price.

```python
if __name__ == "__main__":
    eq = Equity(450, 0.014, 0.14)
    option_call = EquityOption(470, 0.23, "call")
    option_put = EquityOption(500, 0.26, "put")

    print(bsm(eq, option_call, 0.05))
    print(bsm(eq, EquityOption(1e-15, 0.26, "call"), 0.0))
    print(bsm(Equity(450, 0.0, 1e-9), option_put, 0.0))

    # Check put-call parity
    eq = Equity(450, 0.015, 0.15)
```

```
    option_call = EquityOption(470, 0.26, "call")
    option_put = EquityOption(470, 0.26, "put")
    print(check_put_call_parity(eq, option_call, option_put, 0.05))
```

```
5.834035584709966
450
50.0
True
```

References

- Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus.*
- Options, Futures, and Other Derivatives by John C. Hull
- Arbitrage Theory in Continuous Time Book by Tomas Björk