Contents

1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

The evolution equation for GPDs reads:

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-\infty}^{+\infty}\frac{dx'}{|2\xi|}\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{2.1}$$

In general, the GPD f and the evolution kernel $\mathbb{P}$ should be respectively interpreted as a vector and a matrix in flavour space. However, for now, we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure.

The support of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 2.1. The Knowledge of the support region of

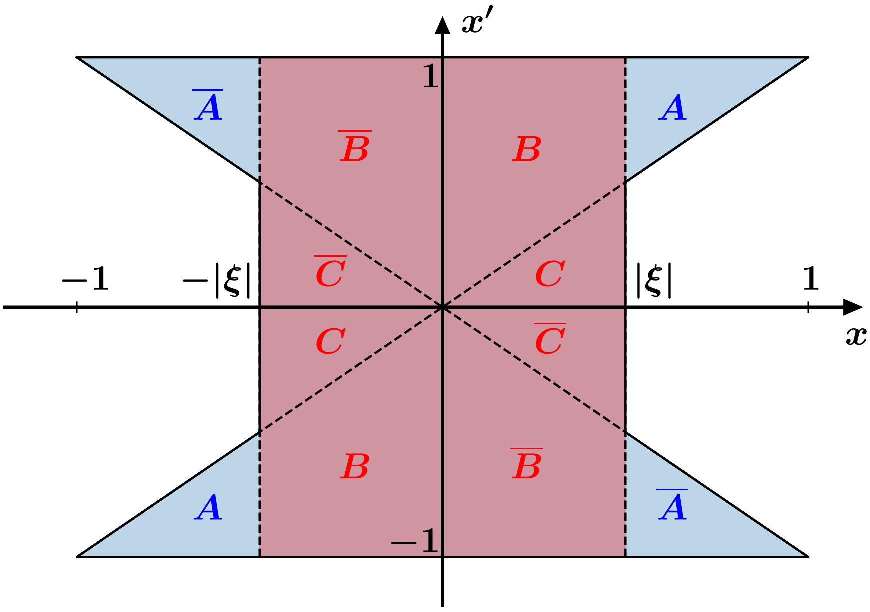

Fig. 2.1: Support domain of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

the evolution kernel allows us to rearrange Eq. (2.1) as follows:

$$\mu^2\frac{d}{d\mu^2}f(\pm x,\xi)=\int_{b(x)}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}\mathbb{P}\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}\mathbb{P}\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,, \tag{2.2}$$

with:

$$b(x)=|x|\theta\left(\left|\frac{x}{\xi}\right|-1\right)\,, \tag{2.3}$$

and where we have used the symmetry property of the evolution kernels: $\mathbb{P}(y,y') = \mathbb{P}(-y,-y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned} f^{\pm}(x,\xi) &= f(x,\xi) \mp f(-x,\xi)\,, \\ \mathbb{P}^{\pm}(y,y') &= \mathbb{P}(y,y') \mp \mathbb{P}(-y,y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \frac{x'}{|2\xi|} \mathbb{P}^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,. \tag{2.5}$$

The $f^{\pm}$ distributions can be regarded as the GPD analogous of the $\pm$ forward distributions that can then be used to construct the usual singlet and non-singlet distributions in the QCD evolution basis. This uniquely determines the flavour structure of the evolution kernels $\mathbb{P}^{\pm}$.

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (DGLAP region, henceforth). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (ERBL region, henceforth). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ Eq. (2.5) needs to recover the DGLAP and ERBL equations, respectively.

GPD anomalous dimensions are generally tricky to integrate numerically because of the intricate support. In order to simplify the integration procedure, we can decompose the anomalous dimensions using the labels given in Fig. 2.1 as a guide:

$$\begin{aligned} \mathbb{P}(y,y') &= \theta(y') \\ &\times \Big[\theta(y-1)\theta(y'-y)\mathbb{P}_A(y,y') + \theta(1-y)\theta(y'-y)\mathbb{P}_B(y,y') + \theta(1-y)\theta(y-y')\mathbb{P}_C(y,y') \\ &+ \theta(-y-1)\theta(y+y')\mathbb{P}_{\overline{A}}(y,y') + \theta(1+y)\theta(y+y')\mathbb{P}_{\overline{B}}(y,y') + \theta(1+y)\theta(-y'-y)\mathbb{P}_{\overline{C}}(y,y')\Big] \\ &+ \theta(-y') \\ &\times \Big[\theta(y-1)\theta(-y-y')\mathbb{P}_{\overline{A}}(y,y') + \theta(1-y)\theta(-y-y')\mathbb{P}_{\overline{B}}(y,y') + \theta(1-y)\theta(y'+y)\mathbb{P}_{\overline{C}}(y,y') \\ &+ \theta(-y-1)\theta(-y'+y)\mathbb{P}_A(y,y') + \theta(1+y)\theta(-y'+y)\mathbb{P}_B(y,y') + \theta(1+y)\theta(-y+y')\mathbb{P}_C(y,y')\Big]\,, \end{aligned} \tag{2.6}$$

where the functions $\mathbb{P}_I$ and $\mathbb{P}_{\overline{I}}$, with $I = A, B, C$, are defined on the respective regions in Fig. 2.1.[2] Next, we take the combinations given in Eq. (2.4) relevant to implement the evolution equation in Eq. (2.5). By doing this, one obtains:

$$\mathbb{P}^{\pm}(y,y') = \theta(y-1)\mathbb{P}_A^{\pm}(y,y') + \theta(1-y)\left[\theta(y'-y)\mathbb{P}_B^{\pm}(y,y') + \theta(y-y')\mathbb{P}_C^{\pm}(y,y')\right]\,, \tag{2.7}$$

where we have defined:

$$\mathbb{P}_I^{\pm}(y,y') = \mathbb{P}_I(y,y') \mp \mathbb{P}_{\overline{I}}(-y,y')\,, \tag{2.8}$$

and omitted all the irrelevant/redundant terms and factors for the computation of the integral in the r.h.s. of Eq. (2.5). From Eq. (2.7), it should be clear that the anomalous dimension $\mathbb{P}_A^{\pm}$ is responsible for the evolution in the DGLAP region while $\mathbb{P}_B^{\pm}$ and $\mathbb{P}_C^{\pm}$ are responsible for the evolution in the ERBL region. The latter observation suggests that $\mathbb{P}_B^{\pm}$ and $\mathbb{P}_C^{\pm}$ are related. The relation can easily be established by observing that the general structure of the ERBL anomalous dimensions is:

$$V^{\mathrm{ERBL}}(y,y') = \theta(y'-y)F(y,y') + \theta(y-y')F(-y,-y')\,, \tag{2.9}$$

[1] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x) = -\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign. The opposite sign is absent in the longitudinally polarised case.

[2] Note that $\mathbb{P}_I(y,y')$ and $\mathbb{P}_{\overline{I}}(y,y')$ are not required to be symmetric upon the transformation $(y \to -y, y' \to -y')$.

which immediately implies that:

$$\mathbb{P}_C^{\pm}(y,y') = \mathbb{P}_B^{\pm}(-y,-y')\,. \tag{2.10}$$

Finally, one finds that a convenient decomposition for the anomalous dimension in Eq. (2.5) is:

$$\mathbb{P}^{\pm}(y,y') = \theta(y-1)\mathbb{P}_A^{\pm}(y,y') + \theta(1-y)\left[\theta(y'-y)\mathbb{P}_B^{\pm}(y,y') + \theta(y-y')\mathbb{P}_B^{\pm}(-y,-y')\right]\,. \tag{2.11}$$

Eq. (2.5) can be further manipulated to make it resemble the structure of the DGLAP equation as much as possible. To this purpose, we define the parameter:

$$\kappa(x) = \frac{\xi}{x}\,, \tag{2.12}$$

so that:

$$\frac{x'}{|2\xi|}\mathbb{P}_I^{\pm}\left(\pm\frac{x}{\xi},\pm\frac{x'}{\xi}\right) = \mathrm{sign}(\xi)\frac{1}{2\kappa}\frac{x'}{x}\mathbb{P}_I^{\pm}\left(\pm\frac{1}{\kappa},\pm\frac{1}{\kappa}\frac{x'}{x}\right) \equiv \mathrm{sign}(\xi)\mathcal{P}_I^{\pm}\left(\pm\kappa,\frac{x}{x'}\right)\,, \tag{2.13}$$

where the last equality effectively defines the *DGLAP-like* splitting function:

$$\mathcal{P}_I^{\pm}(\pm\kappa,y) = \frac{1}{2\kappa y}\mathbb{P}_I^{\pm}\left(\pm\frac{1}{\kappa},\pm\frac{1}{\kappa y}\right)\,. \tag{2.14}$$

In the following we will assume $\xi > 0$ as, so far, this is the only experimentally accessible region. This allows us to get rid of $\mathrm{sign}(\xi)$ in Eq. (2.13). In addition, without loss of generality, we can also restrict ourselves to positive values of x because negative values can be easily accessed by symmetry using Eq. (2.4), *i.e.* $f^{\pm}(-x,\xi) = \mp f^{\pm}(x,\xi)$. Using the definition in Eq. (2.14) in the integral in the r.h.s. of Eq. (2.5) and finally performing a change of variable gives:

$$\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) = \int_{b(x)}^1 \frac{dx'}{x'}\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)f^{\pm}\left(x',\xi\right) = \int_x^{x/b(x)}\frac{dy}{y}\mathcal{P}^{\pm}\left(\kappa,y\right)f^{\pm}\left(\frac{x}{y},\xi\right)\,, \tag{2.15}$$

with:

$$b(x) = x\,\theta(1-\kappa)\,, \tag{2.16}$$

and:

$$\mathcal{P}^{\pm}\left(\kappa,y\right) = \theta(1-\kappa)\mathcal{P}_A^{\pm}(\kappa,y) + \theta(\kappa-1)\left[\theta(1-y)\mathcal{P}_B^{\pm}(\kappa,y) + \theta(y-1)\mathcal{P}_B^{\pm}(-\kappa,y)\right]\,. \tag{2.17}$$

Plugging Eq. (2.17) into Eq. (2.15), one obtains:

$$\begin{aligned}
\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) &= \int_x^1\frac{dy}{y}\left[\theta(1-\kappa)\mathcal{P}_A^{\pm}\left(\kappa,y\right) + \theta(\kappa-1)\mathcal{P}_B^{\pm}\left(\kappa,y\right)\right]f^{\pm}\left(\frac{x}{y},\xi\right)\\
&+ \theta(\kappa-1)\int_1^{\infty}\frac{dy}{y}\mathcal{P}_B^{\pm}\left(-\kappa,y\right)f^{\pm}\left(\frac{x}{y},\xi\right)\\
&= \int_x^1\frac{dy}{y}\left[\theta(1-\kappa)\mathcal{P}_A^{\pm}\left(\kappa,y\right) + \theta(\kappa-1)\left(\mathcal{P}_B^{\pm}\left(\kappa,y\right) - \mathcal{P}_B^{\pm}\left(-\kappa,y\right)\right)\right]f^{\pm}\left(\frac{x}{y},\xi\right)\\
&+ \theta(\kappa-1)\int_x^{\infty}\frac{dy}{y}\mathcal{P}_B^{\pm}\left(-\kappa,y\right)f^{\pm}\left(\frac{x}{y},\xi\right)\,.
\end{aligned} \tag{2.18}$$

Eq. (2.18) has almost the form of a "standard" DGLAP equation except for the term in the last line whose integration range extends between x and ∞. This kind of integral can be handled within APFEL with minor modifications of the integration strategy and up to a numerical approximation to be assessed. In addition, one can prove that at one-loop accuracy, the function $\theta(1-\kappa)\mathcal{P}_A^{\pm}\left(\kappa,y\right) + \theta(\kappa-1)\mathcal{P}_B^{\pm}\left(\kappa,y\right)$ in the integral in the first line is continuos at the point $\kappa = 1$ (but not smooth) because:

$$\mathcal{P}_A^{\pm}\left(1,y\right) = \mathcal{P}_B^{\pm}\left(1,y\right)\,. \tag{2.19}$$

2.1 On continuity of GPDs

It is well known that GPDs are required to be continuous at $x = \xi$ for factorisation to be valid [3]. It is thus interesting to consider the consequence of this constraint. To this end, let us consider the limits of Eq. (2.18) for $x \to \xi^\pm$, which corresponds to $\kappa \to 1^\pm$:

$$\lim_{x\to\xi^+} \mu^2 \frac{d}{d\mu^2} f^\pm(x,\xi) = \int_x^1 \frac{dy}{y} \mathcal{P}_B^\pm(1,y)\, f^\pm\left(\frac{x}{y},\xi\right) + \int_1^\infty \frac{dy}{y} \mathcal{P}_B^\pm(-1,y)\, f^\pm\left(\frac{x}{y},\xi\right), \tag{2.20}$$

and:

$$\lim_{x\to\xi^-} \mu^2 \frac{d}{d\mu^2} f^\pm(x,\xi) = \mu^2 \frac{d}{d\mu^2} f^\pm(\xi,\xi) = \int_x^1 \frac{dy}{y} \mathcal{P}_A^\pm(1,y)\, f^\pm\left(\frac{x}{y},\xi\right). \tag{2.21}$$

Taking the difference between Eqs. (2.20) and (2.21) and using Eq. (2.19) as well as the continuity of f at $x = \xi$, one finds that:

$$\int_1^\infty \frac{dy}{y} \mathcal{P}_B^\pm(-1,y)\, f^\pm\left(\frac{x}{y},\xi\right) = 0\,, \tag{2.22}$$

which has to be valid at any scale and for any $f^\pm$. This immediately implies that:

$$\mathcal{P}_B^\pm(-1,y) = 0\,, \tag{2.23}$$

for all values of y and order-by-order in perturbation theory. We will explicitly verify this constraint when we will discuss the explicit expressions. Also notice that this constraint, along with Eq. (2.19), is such that the function $\theta(1-\kappa)\mathcal{P}_A^\pm(\kappa,y) + \theta(\kappa-1)\left(\mathcal{P}_B^\pm(\kappa,y) - \mathcal{P}_B^\pm(-\kappa,y)\right)$ appearing in the integral in the third line of Eq. (2.18) is continuous at $\kappa = 1$.

2.2 End-point contributions

Some of the expressions for the anomalous dimensions discussed above contain a +-prescribed terms. It is important to treat these terms properly accounting for additional local terms stemming from the "incompleteness" of the convolution integrals. More specifically, the defintion of the +-prescription for the function g (singular in $y = 1$) convoluted with a smooth test function f is:

$$\int_0^1 dy\, [g(y)]_+\, f(y) = \int_0^1 dy\, g(y)\, [f(y) - f(1)]\,. \tag{2.24}$$

Now, we need to work out the action of the +-prescription on integrals of the following kind (*cfr.* Eq. (2.18)):

$$I = \int_x^c dy\, [g(y)]_+\, f(y)\,, \quad x < 1 \quad \text{and} \quad c \geq 1\,. \tag{2.25}$$

In order to apply the definition of +-prescription, we manipulate the integral above as follows:

$$\begin{aligned} I &= \int_0^1 dy\, [g(y)]_+\, f(y) - \int_0^x dy\, g(y) f(y) + \int_1^c dy\, [g(y)]_+\, f(y) \\ &= \int_0^1 dy\, g(y)\, [f(y) - f(1)] - \int_0^x dy\, g(y) f(y) + \int_1^c dy\, g(y) f(y) \\ &= \int_x^c dy\, g(y)\, [f(y) - f(1)] + f(1)\, [L_1(x) + L_2(c)] \end{aligned} \tag{2.26}$$

where for shortness we have defined:

$$L_1 = -\int_0^x dy\, g(y) \quad \text{and} \quad L_2 = \int_1^c dy\, g(y)\,. \tag{2.27}$$

The term L_1 is the usual term arising from the incompleteness of the convolution integral and is finite for $x < 1$. The term L_2 is new. In the DGLAP region ($\kappa < 1$) the upper integration bound c is equal to one so that $L_2 = 0$ and we recover the usual DGLAP structure. In the ERBL region ($\kappa > 1$), instead, one can have $c = +\infty$ so that:

$$L_2 = \int_1^\infty dy\, g(y)\,. \tag{2.28}$$

One such integral emerges in the computation of the last line of Eq. (2.18) and one needs to compute:

$$L_2(\kappa) = \int_1^\infty dy\, \mathcal{P}_B^{\pm}(-\kappa, y)\ . \tag{2.29}$$

Given Eq. (2.23), one immediately finds that $L_2(1) = 0$.

3 Anomalous dimensions

A crucial ingredient for an efficient implementation of GPD evolution is the availability of the DGLAP-like splitting functions $\mathcal{P}^{\pm}$, Eq. (2.14), in a closed form amenable to be easily integrated as in Eq. (2.15).

The explicit form of the one-loop anomalous dimensions can be found, for example, in Ref. [2]. In order to address this question, we need to make the flavour structure of the evolution kernels explicit. Working in the QCD evolution basis, we have:

$$\mathcal{P}^{+} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}, \tag{3.1}$$

and:

$$\mathcal{P}^{-} = P^{\rm NS}, \tag{3.2}$$

where $P^{\rm NS}$ is the appropriate evolution kernel for the particular non-singlet distribution. At one loop it turns out that all the non-singlet splitting functions are equal amongst themselves and to $P_{qq}^{(0)}$, *i.e.* $P^{{\rm NS},(0)} = P_{qq}^{(0)}$.[3]

In order to define the basic steps to reduce the anomalous dimension to a suitable form, let us consider the non-singlet unpolarised anomalous dimension at one loop. Using Eq. (2.14), one finds:

$$P^{{\rm NS},(0)}(\kappa,y) = \begin{cases} 2C_F\left[\dfrac{1+(1-2\kappa^2)y^2}{(1-y)(1-\kappa^2y^2)}\right]_+, & 0\leq\kappa\leq 1\,, \\[2ex] 2C_F\left[\dfrac{1}{1-y}+\dfrac{1-\kappa}{2\kappa}\dfrac{1}{1+\kappa y}\right]_+, & 1\leq\kappa\leq\dfrac{1}{x}\,. \end{cases} \tag{3.3}$$

It is interesting to take the DGLAP limit at $\kappa\to 0$:

$$P^{{\rm NS},(0)}(0,y) = 2C_F\left[\frac{1+y^2}{1-y}\right]_+, \tag{3.4}$$

that coincides with the usual DGLAP splitting function. While the crossover point at $\kappa = 1$ gives:

$$P^{{\rm NS},(0)}(1,y) = 2C_F\left[\frac{1}{1-y}\right]_+, \tag{3.5}$$

regardless of whether the limit is taken from the DGLAP region ($\kappa\to 1^-$) of from the ERBL region ($\kappa\to 1^+$). This tells us that $\mathcal{P}_{qq}^{+(0)}(\kappa,y)$ is a continuos function of κ. Fig. 3.1 displays the behaviour of the regular part of the anomalous dimension $\mathcal{P}_{qq}^{+(0)}$ defined in Eq. (3.3) as a function of the parameter κ for different values of the variable y. This plot shows the continuity of $\mathcal{P}_{qq}^{+(0)}$ at $\kappa = 1$. Notice that, for $y > 1$, $\mathcal{P}_{qq}^{+(0)}$ develops a pole at $\kappa = 1/y < 1$ below which, being it in the DGLAP region, no integration over y is required. Since $P_{qq}^{(0)} = P^{{\rm NS},(0)}$, what we discussed above applies verbatim to $P_{qq}^{(0)}$.

Let us now turn to the remaining splitting functions $P_{qg}^{(0)}$, $P_{gq}^{(0)}$, and $P_{gg}^{(0)}$. Their explicit expressions read[4]:

$$P_{qg}^{(0)}(\kappa,y) = \begin{cases} 4n_fT_R\dfrac{1-2y+(2-\kappa^2)y^2}{(1-\kappa^2y^2)^2}, & 0\leq\kappa\leq 1\,, \\[2ex] 2n_fT_R\left[\dfrac{1+\kappa}{\kappa^2(1+\kappa y)}\right]\left[\dfrac{1-\kappa}{\kappa y}+\dfrac{1}{1+\kappa y}\right], & 1\leq\kappa\leq\dfrac{1}{x}\,, \end{cases} \tag{3.6}$$

[3] When going beyond one loop, three different non-singlet structures emerge. In the QCD evolution basis, they are associated to the total-valence and ±-like distributions.

[4] The expression for $P_{gq}^{(0)}$ derived from Ref. [2] seems to be wrong. I have derived the seemingly correct expression from Ref. [4] by performing the replacement $\xi\to 2\kappa y$ in Eq. (24).

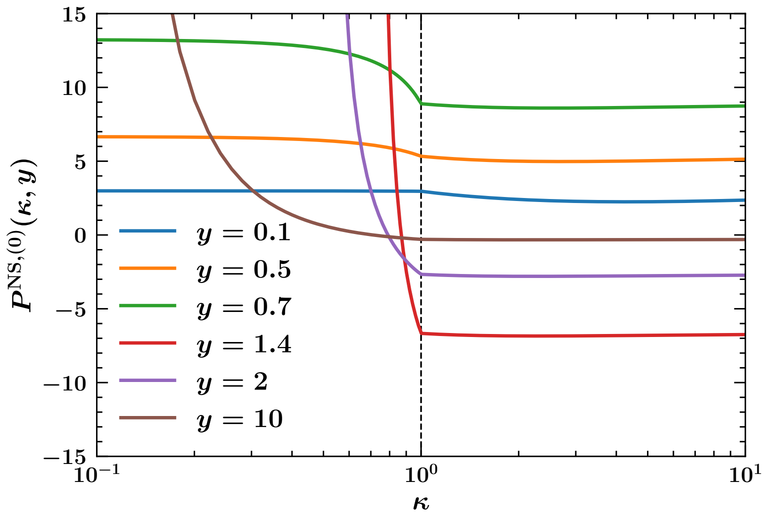

Fig. 3.1: Behaviour of the anomalous dimension $P^{\mathrm{NS},(0)}$ as a function of κ for different values of y.

$$P_{gq}^{(0)}(\kappa, y) = \begin{cases} 2C_F \left[\dfrac{2 - 2y + (1-\kappa^2)y^2}{y(1-\kappa^2 y^2)}\right], & 0 \le \kappa \le 1\,, \\ C_F \left[\dfrac{2(1+\kappa) - (1-\kappa^2)y}{\kappa y(1+\kappa y)}\right], & 1 \le \kappa \le \dfrac{1}{x}\,, \end{cases} \tag{3.7}$$

$$P_{gg}^{(0)}(\kappa, y) = \begin{cases} 4C_A \left[\dfrac{-2 + y(1 - y + \kappa^2(1+y))}{(1-\kappa^2 y^2)^2} + \dfrac{1}{y} + \dfrac{1}{(1-y)_+}\right] + \delta(1-x)\dfrac{11C_A - 4n_f T_R}{3}, & 0 \le \kappa \le 1\,, \\ C_A \left[\dfrac{-1 + 3\kappa^2 - \kappa(2 + (1-\kappa)^2\kappa)y}{\kappa^3 y(1+\kappa y)^2} + \dfrac{2}{y} + \dfrac{2}{(1-y)_+}\right] + \delta(1-x)\dfrac{11C_A - 4n_f T_R}{3}, & 1 \le \kappa \le \dfrac{1}{x}\,. \end{cases} \tag{3.8}$$

Their forward limit ($\kappa = 0$) is:

$$\begin{aligned} P_{qg}^{(0)}(0, y) &= 4n_f T_R \left[y^2 + (1-y)^2\right] \\ P_{gq}^{(0)}(0, y) &= 2C_F \left[\frac{1 + (1-y)^2}{y}\right], \\ P_{gg}^{(0)}(0, y) &= 4C_A \left[-2 + y(1-y) + \frac{1}{y} + \frac{1}{(1-y)_+}\right] + \delta(1-x)\frac{11C_A - 4n_f T_R}{3}, \end{aligned} \tag{3.9}$$

which coincides with the usual one-loop DGLAP splitting functions. At the crossover point $\kappa = 1$ the evolution kernels reduce to:

$$P_{qg}^{(0)}(1, y) = \frac{4n_f T_R}{(1+y)^2}, \tag{3.10}$$

$$P_{gq}^{(0)}(1, y) = \frac{4C_F}{y(1+y)}, \tag{3.11}$$

$$P_{gg}^{(0)}(1, y) = 4C_A \left[\frac{1 + y^2}{y(1+y)^2\,(1-y)_+}\right] + \delta(1-y)\frac{11C_A - 4n_f T_R}{3}\,. \tag{3.12}$$

Interestingly, like $P^{\mathrm{NS},(0)}$ and $P_{qq}^{(0)}$, also $P_{qg}^{(0)}$, $P_{gq}^{(0)}$, and $P_{gg}^{(0)}$ are continuos in $\kappa = 1$.

A On Vinnikov's code

The purpose of this Appendix is to draw the attention on a possible incongruence of the GPD evolution code developed by Vinnikov and presented in Ref. [5]. For definiteness, we concentrate on the non-singlet $H_{\rm NS}$ GPD in the DGLAP region $x > \xi$, whose evolution equation is given in Eq. (29). For completeness, I report that equation here:

$$\begin{aligned} \frac{dH_{\rm NS}(x,\xi,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy \frac{x^2+y^2-2\xi^2}{(y-x)(y^2-\xi^2)}\left(H_{\rm NS}(y,\xi,Q^2)-H_{\rm NS}(x,\xi,Q^2)\right)\right. \\ &+ H_{\rm NS}(x,\xi,Q^2)\left(\frac{3}{2}+2\ln(1-x)+\frac{x-\xi}{2\xi}\ln((x-\xi)(1+\xi))\right. \\ &- \left.\left.\frac{x+\xi}{2\xi}\ln((x+\xi)(1-\xi))\right)\right]. \end{aligned} \tag{A.1}$$

The limit for $\xi \to 0$ of the equation above should reproduce the usual DGLAP evolution equation:

$$\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{4\pi}\int_x^1 \frac{dz}{z}\left[\hat{P}_{\rm NS}(z)\right]_+ H_{\rm NS}\left(\frac{x}{z},0,Q^2\right), \tag{A.2}$$

where:

$$\hat{P}_{\rm NS}(z) = 2C_F\frac{1+z^2}{1-z} = 2C_F\left[\frac{2}{1-z}-(1+z)\right], \tag{A.3}$$

with $C_F = 4/3$. Written explicitly:

$$\begin{aligned} \frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{\alpha_s(Q^2)}{4\pi}2C_F\left[\int_x^1 dz\,\frac{2}{1-z}\left(\frac{1}{z}H_{\rm NS}\left(\frac{x}{z},0,Q^2\right)-H_{\rm NS}(x,0,Q^2)\right)\right. \\ &- \int_x^1\frac{dz}{z}(1+z)H_{\rm NS}\left(\frac{x}{z},0,Q^2\right) \\ &+ \left.H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)\right)\right]. \end{aligned} \tag{A.4}$$

Now I explicitly take the limit for $\xi \to 0$ of Eq. (A.1). The result is:

$$\begin{aligned} \frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\frac{x^2+y^2}{y^2(y-x)}\left(H_{\rm NS}(y,0,Q^2)-H_{\rm NS}(x,0,Q^2)\right)\right. \\ &+ \left.H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)\right)\right]. \end{aligned} \tag{A.5}$$

Some further simple algebraic manipulation finally gives:

$$\begin{aligned} \frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{\alpha_s(Q^2)}{4\pi}2C_F\left[\int_x^1 dz\,\frac{2}{1-z}\left(\frac{1}{z}H_{\rm NS}\left(\frac{x}{z},0,Q^2\right)-H_{\rm NS}(x,0,Q^2)\right)\right. \\ &- \int_x^1\frac{dz}{z}(1+z)H_{\rm NS}\left(\frac{x}{z},0,Q^2\right) \\ &+ \left.H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)+\ln(x)+(1-x)\right)\right], \end{aligned} \tag{A.6}$$

that is close to the correct results, Eq. (A.4), except for the two additional terms in the third line. Since the $\xi \to 0$ limit does not seem to produce the correct result, this suggests that the evolution code presented in Ref. [5] may not be entirely correct.[5]

[5] It is, in fact, possible to correct Eq. (A.1) in such a way that its $\xi \to 0$ limit gives the correct DGLAP evolution equation.

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