Transverse momentum resummation and matching to fixed order

Valerio Bertone

Contents

Part I. TMD evolution and its perturbative expansion

1 Evolution of the TMDs

In this section I will show how to solve the renormalisation-group equation (RGE) and the rapidity-evolution equation (often referred to as Collins-Soper (CS) equation) of a TMD distribution F. The distribution F can be either a PDF or a FF and can be associated to either to a quark (anti)flavour or to the gluon: the structure of the solution of the evolution equations is exactly the same. In fact, the solution of the equations only depends on whether we are evolving a quark or a gluon, while it does not distinguish between PDFs and FFs. In the impact-parameter space, F is a function of the transverse-momentum fraction x, of the bidimensional impact-parameter vector $\mathbf{b}_T$, of the renormalisation scale μ, and of the rapidity scale ζ, *i.e.* $F \equiv F(x, \mathbf{b}_T, \mu, \zeta)$. Since the evolution equations govern the behaviour of F w.r.t. the scale μ and ζ, in order to simplify the notation, in this section I will drop the dependence on x and $\mathbf{b}_T$, *i.e.* $F \equiv F(\mu, \zeta)$.[1]

The solution of the evolution equations allows one to express the distribution F at some final scales (μ, ζ) in terms of the same distribution at the initial scales (μ_0, ζ_0). It will turn out that this is accomplished by computing the evolution kernel $R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right]$, such that:

$$F(\mu, \zeta) = R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right] F(\mu_0, \zeta_0). \tag{1.1}$$

The evolution kernel R can be expressed in terms of perturbatively computable quantities. A collateral aspect that will be discussed below is the independence from the path that connects the initial and final scales (μ_0, ζ_0) and (μ, ζ). This in turn concerns the resummation of large logarithms that is required to ensure that the perturbative convergence is not spoiled.

The RGE and the CS equations read:

$$\begin{aligned}
\frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} &= K(\mu), \\
\frac{\partial \ln F}{\partial \ln \mu} &= \gamma(\mu, \zeta),
\end{aligned} \tag{1.2}$$

where γ and K are the anomalous dimensions of the evolution in μ and $\sqrt{\zeta}$, respectively. The equations above can be solved as follows. The first equation gives:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\right] F(\mu, \zeta_0). \tag{1.3}$$

The factor $F(\mu, \zeta_0)$ can then be evolved in μ using the second equation:

$$F(\mu, \zeta_0) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0), \tag{1.4}$$

such that:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0). \tag{1.5}$$

This equation has exactly the structure of Eq. (1.1). We now need to express the argument of the exponent in terms of perturbatively computable quantities.

In order to do so, we use the fact that the rapidity anomalous dimension K needs to be renormalised and thus it obeys its own RGE, that reads:

$$\frac{\partial K}{\partial \ln \mu} = -\gamma_K(\alpha_s(\mu)). \tag{1.6}$$

γ_K is said cusp anomalous dimension and obeys the perturbative expansion:

$$\gamma_K(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} \gamma_K^{(n)}, \tag{1.7}$$

---

[1] Notice that, despite the variables x and $\mathbf{b}_T$ will not appear explicitly, the symbol $\otimes$ indicates the Mellin convolution integral w.r.t. x while b_T indicates the length of the vector $\mathbf{b}_T$.

where $\gamma_K^{(n)}$ are numerical coefficients. The value of the coefficients up to $n = 3$ can be read from Eq. (59) of Ref. [1]. They coincide with those reported in Eq. (D.6) of Ref. [2] up to a factor two due to a different normalisation of the rapidity anomalous dimension K whose derivative w.r.t. ζ is exactly γ_K. In addition, the cusp anomalous dimensions for quarks and gluon are equal up to a factor C_F in the quark case and C_A in the gluon case.

Eq. (1.6) can be easily solved obtaining:

$$K(\mu) = K(\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) \,. \tag{1.8}$$

We anticipate that in the $\overline{\text{MS}}$ renormalisation scheme, there exists a particular scale, $\mu_b = 2e^{-\gamma_E}/b_T$, such that K computed at μ_b is free of logaritms and admits the following perturbative expansion:

$$K(\mu_b) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^{n+1} K^{(n,0)} \,, \tag{1.9}$$

where $K^{(n,0)}$ are numerical coefficients. Therefore, if $\mu_0 \simeq \mu_b$ the first term in the r.h.s. of Eq. (1.8) is free of large logs and thus its perturbative expansion, that reads:

$$K(\mu_0) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^{n+1} \sum_{m=0}^{n+1} K^{(n,m)} \ln^m \left(\frac{\mu_0}{\mu_b} \right) \,, \tag{1.10}$$

is reliable. The logarithmic terms in Eq. (1.10) can be computed by using Eq. (1.6) or Eq. (1.8) and expanding $\alpha_s(\mu_0)$ around $\alpha_s(\mu_b)$ using the appropriate RGE. Again, such an expansion is reliable only if μ_0 and μ_b are comparable such not to generate large logarithms. The second term in the r.h.s. of Eq. (1.8) instead takes care, through the evolution of α_s, of resumming large logarithms in the case in which $\mu \gg \mu_0$. The coefficients $K^{(n,m)}$ up to $n = 2$ are reported in Eq. (D.9) of Ref. [2] and up to $n = 1$ in Eq. (69) of Ref. [1]. They differ by a factor -2 due to a different definition of K. In addition, the logarithmic expansion is done in terms of $\ln(\mu_0/\mu_b)$ in Ref. [1] and in terms of $\ln(\mu_0^2/\mu_b^2)$ in Ref. [2]. Therefore, each coefficient differs by an additional factor 2^m, where m is the power of the logarithm that multiplies the coefficient itself.

For the sake of completeness, we compute the coefficients $K^{(n,m)}$ up to $n = 3$. To do so, we need to know the coefficients of the RGE of the coupling α_s, that we write:

$$\frac{d}{d\ln\mu} \left(\frac{\alpha_s(\mu)}{4\pi} \right) = -2 \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n \beta_n \,, \tag{1.11}$$

whose solution for the evolution between μ_b and μ' expanded to two loops reads:

$$\left(\frac{\alpha_s(\mu')}{4\pi} \right) = \left(\frac{\alpha_s(\mu_b)}{4\pi} \right) + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 \left(-2\beta_0 \ln \frac{\mu'}{\mu_b} \right) + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^3 \left(4\beta_0^2 \ln^2 \frac{\mu'}{\mu_b} - 2\beta_1 \ln \frac{\mu'}{\mu_b} \right) + \mathcal{O}(\alpha_s^4), \tag{1.12}$$

such that:

$$\left(\frac{\alpha_s(\mu')}{4\pi} \right)^2 = \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^3 \left(-4\beta_0 \ln \frac{\mu'}{\mu_b} \right) + \mathcal{O}(\alpha_s^4) \,, \tag{1.13}$$

and:

$$\left(\frac{\alpha_s(\mu')}{4\pi} \right)^3 = \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^3 + \mathcal{O}(\alpha_s^4) \,. \tag{1.14}$$

Therefore, using Eq. (1.7) up to three loops, one finds:

$$
\begin{aligned}
\gamma_K(\alpha_s(\mu')) &= \left(\frac{\alpha_s(\mu_b)}{4\pi} \right) \gamma_K^{(0)} + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 \left[-2\beta_0 \gamma_K^{(0)} \ln \frac{\mu'}{\mu_b} + \gamma_K^{(1)} \right] \\
&+ \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^3 \left[4\beta_0^2 \gamma_K^{(0)} \ln^2 \frac{\mu'}{\mu_b} + \left(-2\beta_1 \gamma_K^{(0)} - 4\beta_0 \gamma_K^{(1)} \right) \ln \frac{\mu'}{\mu_b} + \gamma_K^{(2)} \right] + \mathcal{O}(\alpha_s^4) \,.
\end{aligned}
\tag{1.15}
$$

Using Eq. (1.15) in the integral in the r.h.s. of Eq. (1.8) computed between μ_b and μ_0, one finds:

$$
\begin{aligned}
\int_{\mu_b}^{\mu_0} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) &= \int_0^{\ln \frac{\mu_0}{\mu_b}} d\ln \frac{\mu'}{\mu_b} \gamma_K(\alpha_s(\mu')) \\
&= \left(\frac{\alpha_s(\mu_b)}{4\pi} \right) \gamma_K^{(0)} \ln \frac{\mu_0}{\mu_b} + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 \left[-\beta_0 \gamma_K^{(0)} \ln^2 \frac{\mu_0}{\mu_b} + \gamma_K^{(1)} \ln \frac{\mu_0}{\mu_b} \right] \\
&+ \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^3 \left[\frac{4}{3} \beta_0^2 \gamma_K^{(0)} \ln^3 \frac{\mu_0}{\mu_b} + \left(-\beta_1 \gamma_K^{(0)} - 2\beta_0 \gamma_K^{(1)} \right) \ln^2 \frac{\mu_0}{\mu_b} + \gamma_K^{(2)} \ln \frac{\mu_0}{\mu_b} \right] + \mathcal{O}(\alpha_s^4) .
\end{aligned}
\tag{1.16}
$$

Replacing $\alpha_s(\mu_b)$ with $\alpha_s(\mu_0)$ through the expansion:

$$
\left(\frac{\alpha_s(\mu_b)}{4\pi} \right) = \left(\frac{\alpha_s(\mu_0)}{4\pi} \right) + 2\beta_0 \ln \frac{\mu_0}{\mu_b} \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 + \left(2\beta_1 \ln \frac{\mu_0}{\mu_b} + 4\beta_0^2 \ln^2 \frac{\mu_0}{\mu_b} \right) \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^3 + \mathcal{O}(\alpha_s^4) , \quad (1.17)
$$

the final result is:

$$
K(\mu_0) = \sum_{n=0}^{2} \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^{n+1} \sum_{m=0}^{n+1} K^{(n,m)} \ln^m \left(\frac{\mu_0}{\mu_b} \right) ,
\tag{1.18}
$$

where we read off:

$$
K^{(0,1)} = -\gamma_K^{(0)} ,
$$

$$
K^{(1,1)} = -\gamma_K^{(1)} + 2\beta_0 K^{(0,0)} , \quad K^{(1,2)} = -\beta_0 \gamma_K^{(0)}
$$

$$
K^{(2,1)} = -\gamma_K^{(2)} + 2\beta_1 K^{(0,0)} + 4\beta_0 K^{(1,0)} , \quad K^{(2,2)} = -\beta_1 \gamma_K^{(0)} - 2\beta_0 \gamma_K^{(1)} + 4\beta_0^2 K^{(0,0)} , \quad K^{(2,3)} = -\frac{4}{3} \beta_0^2 \gamma_K^{(0)}
\tag{1.19}
$$

A further important property of the anomalous dimensions can be derived by considering the fact that the crossed double derivarives of F must be equal:

$$
\frac{\partial}{\partial \ln \mu} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} = \frac{\partial}{\partial \ln \sqrt{\zeta}} \frac{\partial \ln F}{\partial \ln \mu} .
\tag{1.20}
$$

Using Eqs. (1.2) and (1.6) leads to the following additional differential equation:

$$
\frac{\partial \gamma}{\partial \ln \sqrt{\zeta}} = -\gamma_K(\alpha_s(\mu)) .
\tag{1.21}
$$

Using the point $\zeta = \mu^2$ as a boundary condition, the solution of the equation above is:

$$
\gamma(\mu, \zeta) = \gamma(\mu, \mu^2) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu} .
\tag{1.22}
$$

It turns out that $\gamma(\mu, \mu^2)$ has a purely perturbative expansion:

$$
\gamma(\mu, \mu^2) \equiv \gamma_F(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} \gamma_F^{(n)} ,
\tag{1.23}
$$

where $\gamma_F^{(n)}$ are again numerical coefficients that are given in Eq. (58) of Ref. [1] and Eq. (D.7) of Ref. [2] (Eq. (D.8) reports the coefficients for the gluon anomalous dimension). The two sets of coefficients differ by a minus sign due to the different definition of the constant (non-logarithmic) term of the RGE anomalous dimension. Therefore:

$$
\gamma(\mu, \zeta) = \gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu} .
\tag{1.24}
$$

Finally, plugging Eqs. (1.8) and (1.24) into Eq. (1.5), one finds:

$$
F(\mu, \zeta) = \exp \left\{ K(\mu_0) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'} \right] \right\} F(\mu_0, \zeta_0) .
\tag{1.25}
$$

Comparing Eq. (1.25) to Eq. (1.1) allows one to give an explicit expression to the evolution kernel:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]=\exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu'))-\gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}.\tag{1.26}$$

Eq. (1.25) has been obtained evolving the TMD distribution F first in the ζ direction (Eq. (1.3)) and then in the μ direction (Eq. (1.4)). However, it is easy to verify that exchanging the order of the evolutions leads to the exact same result, Eq. (1.25). In particular, the following relation holds:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta)\right]R\left[(\mu_0,\zeta)\leftarrow(\mu_0,\zeta_0)\right]=R\left[(\mu,\zeta)\leftarrow(\mu,\zeta_0)\right]R\left[(\mu,\zeta_0)\leftarrow(\mu_0,\zeta_0)\right]=R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right].\tag{1.27}$$

This is a direct consequence of the independence of evolution kernel R in Eq. (1.26) from the path $\mathcal{P}$ followed to connect the points (μ_0,ζ_0) to the point (μ,ζ):

$$R\left[(\mu,\zeta)\underset{\mathcal{P}}{\leftarrow}(\mu_0,\zeta_0)\right]\equiv R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right].\tag{1.28}$$

Another important piece of information comes from the fact that, for small values of b_T, the TMD F can be matched onto the respective collinear distribution f (a PDF or a FF) through the perturbative coefficients C[2]:

$$F(\mu,\zeta)=C(\mu,\zeta)\otimes f(\mu),\tag{1.29}$$

so that:

$$F(\mu,\zeta)=\exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu'))-\gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}C(\mu_0,\zeta_0)\otimes f(\mu_0).\tag{1.30}$$

Exactly as in the case of K, for $\mu_0=\sqrt{\zeta_0}=\mu_b$ the matching function admits the expansion:

$$C(\mu_b,\mu_b^2)=\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n C^{(n,0)},\tag{1.31}$$

where the coefficients $C^{(n,0)}$ are functions of x only. In order to be able to compute the function C for generic values of the scales μ and ζ, evolution equations can be derived. Deriving Eq. (1.29) with respect to μ and ζ gives:

$$\frac{\partial F}{\partial\ln\sqrt{\zeta}}=\frac{\partial C}{\partial\ln\sqrt{\zeta}}\otimes f(\mu),$$

$$\frac{\partial F}{\partial\ln\mu}=\frac{\partial C}{\partial\ln\mu}\otimes f(\mu)+C(\mu,\zeta)\otimes\frac{\partial f}{\partial\ln\mu}=\left[\frac{\partial C}{\partial\ln\mu}+C(\mu,\zeta)\otimes2P(\alpha_s(\mu))\right]\otimes f(\mu).\tag{1.32}$$

In the r.h.s. of the second equation I have used the DGLAP equation:

$$\frac{\partial f}{\partial\ln\mu}=2P(\alpha_s(\mu))\otimes f(\mu).\tag{1.33}$$

One can also take the derivative of Eq. (1.25) and, using Eq. (1.29), the result is:

$$\frac{\partial F}{\partial\ln\sqrt{\zeta}}=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]F(\mu,\zeta)=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]C(\mu,\zeta)\otimes f(\mu),$$

$$\frac{\partial F}{\partial\ln\mu}=\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]F(\mu,\zeta)=\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]C(\mu,\zeta)\otimes f(\mu).\tag{1.34}$$

Equating Eq. (1.32) and Eq. (1.34) and dropping the distribution f, the evolution equations for C are obtained:

$$\frac{\partial C}{\partial\ln\sqrt{\zeta}}=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]C(\mu,\zeta),$$

$$\frac{\partial C}{\partial\ln\mu}=\left\{\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]\Delta-2P(\alpha_s(\mu))\right\}\otimes C(\mu,\zeta),\tag{1.35}$$

---

[2] A sum over flavours is understood. As a matter of fact, the matching function C has to be regarded as a matrix in flavour space multiplying a vector of collinear PDFs/FFs.

where the Δ is a matrix in flavour space whose components are defined as:

$$\Delta_{ij}(x) = \delta_{ij}\delta(1-x)\,, \tag{1.36}$$

being i and j flavour indices. The equations above can be solved to determine the evolution of the matching function C. If one assumes $\mu_0 = \sqrt{\zeta_0} \simeq \mu_b$ in Eq. (1.25), the matching function C can be reliably expanded as:

$$C(\mu_0, \mu_0^2) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^n \sum_{m=0}^{2n} C^{(n,m)} \ln^m\left(\frac{\mu_0}{\mu_b}\right)\,. \tag{1.37}$$

The coefficient functions $C^{(n,m)}$ have been computed for both PDFs and FFs in Ref. [2]. The same functions have been computed also in Ref. [3] and reported in Ref. [1]. The authors of the latter paper have verified the equality of the two sets of functions.

For the sake of completeness, we rederive here the functions $C^{(n,m)}$, for $m \neq 0$, up to $n = 2$. This is done by solving the equations in Eq. (1.35) to evolve the coefficient C between (μ_b, μ_b^2) and (μ_0, ζ_0) and, under the assumption that $\mu_0 \simeq \sqrt{\zeta_0} \simeq \mu_b$, expand the solution to $\mathcal{O}(\alpha_s^2)$. Contrary to the usual assumption, we take $\mu_0 \neq \sqrt{\zeta_0}$ but in order to simplify the notation we introduce the following definitions:

$$a_b = \frac{\alpha_s(\mu_b)}{4\pi}\,, \quad a_0 = \frac{\alpha_s(\mu_0)}{4\pi}\,, \quad L_\mu = \ln\frac{\mu_0}{\mu_b}\,, \quad L_\zeta = \ln\frac{\sqrt{\zeta_0}}{\mu_b}\,, \tag{1.38}$$

and we set $C^{(0,0)} = \Delta \equiv 1$ keeping in mind that any scalar carries the flavour and x structure of Δ. Due to the fact the evolution kernel of the first equation in Eq. (1.35) is in dependent of ζ, the evolution of C in ζ reads:

$$
\begin{aligned}
C(\mu_b, \zeta_0) &= \exp\left[K(\mu_b)L_\zeta\right] C(\mu_b, \mu_b^2) \\
&= 1 + a_b\left[C^{(1,0)} + K^{(0,0)}L_\zeta\right] + a_b^2\left[C^{(2,0)} + \left(K^{(0,0)}C^{(1,0)} + K^{(1,0)}\right)L_\zeta + \frac{1}{2}\left(K^{(0,0)}\right)^2 L_\zeta^2\right] + \mathcal{O}(\alpha_s^3)
\end{aligned}
\tag{1.39}
$$

Now we need to solve also the second equation in Eq. (1.35) to evolve C from (μ_b, ζ_0) to (μ_0, ζ_0). The solution is:

$$C(\mu_0, \zeta_0) = \exp\left[\sum_{n=0}^{\infty}\int_{\mu_b}^{\mu_0}\frac{d\mu'}{\mu'}\left(\frac{\alpha_s(\mu')}{4\pi}\right)^{n+1}\left\{\left[\gamma_F^{(n)} - \gamma_K^{(n)}\ln\frac{\sqrt{\zeta_0}}{\mu'}\right]\Delta - 2P^{(n)}\right\}\right] \otimes C(\mu_b, \zeta_0) \tag{1.40}$$

Finally, we need to plug Eq. (1.39) into Eq. (1.40), replace $\alpha_s(\mu_b)$ with $\alpha_s(\mu_0)$ using Eq. (1.17), and truncate the result to $\mathcal{O}(\alpha_s^2)$. We refrain from writing the full expression because too lengthy that however can be written in a compact form as:

$$C(\mu_0, \zeta_0) = \sum_{n=0}^{2} a_0^n \sum_{m=0}^{2n}\sum_{k=0}^{m} \bar{C}^{(n,m-k,k)} L_\mu^{m-k} L_\zeta^k + \mathcal{O}(\alpha_s^3)\,, \tag{1.41}$$

with:

$$C^{(n,m)} = \sum_{k=0}^{m} \bar{C}^{(n,m-k,k)}\,. \tag{1.42}$$

In order to simplify the structure, one may take:

$$L_\mu = L_\zeta = L_0\,, \tag{1.43}$$

and accounting for the fact that $K^{(0,0)} = 0$, we have:

$$
\begin{aligned}
C(\mu_0, \mu_0^2) &= 1 + a_0\left[C^{(1,0)} + \left(\gamma_F^{(0)} - 2P^{(0)}\right)L_0 - \frac{1}{2}\gamma_K^{(0)}L_0^2\right] \\
&\quad + a_0^2\left[C^{(2,0)} + \left(\gamma_F^{(1)} + K^{(1,0)} - 2P^{(1)} + \left(\gamma_F^{(0)} + 2\beta_0\right)C^{(1,0)} - 2C^{(1,0)}\otimes P^{(0)}\right)L_0\right. \\
&\quad + \left(\beta_0\gamma_F^{(0)} + \frac{1}{2}\left(\gamma_F^{(0)}\right)^2 - 2\left(\beta_0 + \gamma_F^{(0)}\right)P^{(0)} + 2P^{(0)}\otimes P^{(0)} - \frac{1}{2}\gamma_K^{(1)} - \frac{1}{2}\gamma_K^{(0)}C^{(1,0)}\right)L_0^2 \\
&\quad + \left.\left(-\frac{1}{2}\gamma_K^{(0)}\gamma_F^{(0)} + \gamma_K^{(0)}P^{(0)} - \frac{2}{3}\beta_0\gamma_K^{(0)}\right)L_0^3 + \frac{1}{8}\left(\gamma_K^{(0)}\right)^2 L_0^4\right] + \mathcal{O}(\alpha_s^3)\,.
\end{aligned}
\tag{1.44}
$$

In order to use Eq. (1.30) in phenomenological applications, one needs to define the values of both the initial and final pairs of scales, (μ_0, ζ_0) and (μ, ζ). As discussed above, in the $\overline{\text{MS}}$ renormalisation scheme the natural value for the initial scales is $\mu_0 = \sqrt{\zeta_0} = \mu_b = 2e^{-\gamma_E}/b_T$. This particular choice is such that all logarithms of the initial scales nullify. However, in order to estimate higer-order corrections, one can displace the initial scales around the central value by a modest factors C_i^μ and C_i^ζ such that $(\mu_0, \zeta_0) = (C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2)$. To implement these variations, one needs to used Eq. (1.18) for the Collins-Soper evolution kernel K and Eq. (1.41) for the matching functions C.

Now we discuss the final scales μ and ζ. First it is important to stress that the choice of ζ in the computation of physical observables is in fact *immaterial*. The reason is that a physical observable that takes place at the typical hard scale Q is proportional to *two* TMD distributions:

$$\sigma(Q) \propto H(Q, \mu) \, F_1(\mu, \zeta_1) F_2(\mu, \zeta_2) \tag{1.45}$$

where H is the appropriate hard function and with the constraint $\zeta_1 \zeta_2 = Q^4$. Using the evolution kernel R, we find:

$$\sigma(Q) \propto H(Q, \mu) \, R\left[(\mu, \zeta_1) \leftarrow (\mu_0, \zeta_0)\right] R\left[(\mu, \zeta_2) \leftarrow (\mu_0, \zeta_0)\right] F_1(\mu_0, \zeta_0) F_2(\mu_0, \zeta_0). \tag{1.46}$$

But using Eq. (1.26):

$$R\left[(\mu, \zeta_1) \leftarrow (\mu_0, \zeta_0)\right] R\left[(\mu, \zeta_2) \leftarrow (\mu_0, \zeta_0)\right]$$

$$= \exp\left\{ K(\mu_0) \ln \frac{\sqrt{\zeta_1 \zeta_2}}{\zeta_0} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[2\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta_1 \zeta_2}}{\mu'^2} \right] \right\} \tag{1.47}$$

$$= \exp\left\{ 2K(\mu_0) \ln \frac{Q}{\sqrt{\zeta_0}} + 2 \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{Q}{\mu'} \right] \right\},$$

where in the second line we have used the constraint $\zeta_1 \zeta_2 = Q^4$. It thus turns out that the combination of the evolution kernels does not depend on the scales ζ_1 and ζ_2. Therefore, any choice for these scales that obeys the constraint above leads to *exactly* the same results. This being clear, the somewhat obvious choice is $\zeta_1 = \zeta_2 = Q^2$ and one does not need to consider any variations.

Finally, we consider the scale μ. The natural choice is $\mu = Q$, however, in order to estimate higer-order corrections, one may consider variations around this value by a modest factor C_f, $\mu = C_f Q$. This variations generate logarithms of μ/Q in the perturbative expansion of the hard factor H that can be computed by requiring that the physical observable σ is independent of μ. Specifically, one requires:

$$\frac{d\sigma}{d\ln \mu} = 0, \tag{1.48}$$

that, using Eq. (1.46), translates into the following evolution equation for H:

$$\frac{d\ln H}{d\ln \mu} = -2\gamma_F(\alpha_s(\mu)) - 2\gamma_K(\alpha_s(\mu)) \ln \frac{\mu}{Q}. \tag{1.49}$$

whose solution for the evolution between Q and μ is:

$$H(Q, \mu) = \exp\left\{ -2 \int_{Q}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) + \gamma_K(\alpha_s(\mu')) \ln \frac{\mu'}{Q} \right] \right\} H(Q, Q). \tag{1.50}$$

Knowing that:

$$H(Q, Q) \;=\; 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s(Q)}{4\pi} \right)^n H^{(n)}, \tag{1.51}$$

and assuming that Q and μ are not too far apart, Eq. (1.50) can be reliably expanded and truncated to $\mathcal{O}(\alpha_s^2)$. Defining:

$$a_\mu = \frac{\alpha_s(\mu)}{4\pi} \quad \text{and} \quad L_Q = \ln \frac{\mu}{Q}, \tag{1.52}$$

we find:

$$
\begin{aligned}
H(Q,\mu) \;=\; & 1 + a_\mu \left[H^{(1)} - 2\gamma_F^{(0)} L_Q - \gamma_K^{(0)} L_Q^2 \right] \\[2mm]
& + \; a_\mu^2 \bigg[H^{(2)} + \left(-2\gamma_F^{(1)} + 2\beta_0 H^{(1)} - 2\gamma_F^{(0)} H^{(1)} \right) L_Q \\[2mm]
& + \; \left(-\gamma_K^{(1)} - 2\beta_0 \gamma_F^{(0)} + 2\left(\gamma_F^{(0)}\right)^2 - \gamma_K^{(0)} H^{(1)} \right) L_Q^2 \\[2mm]
& + \; \left(-\frac{2}{3}\beta_0 \gamma_K^{(0)} + 2\gamma_F^{(0)} \gamma_K^{(0)} \right) L_Q^3 + \frac{1}{2}\left(\gamma_K^{(0)}\right)^2 L_Q^4 \bigg] + \mathcal{O}(\alpha_s^3) \,.
\end{aligned}
\tag{1.53}
$$

The formula above allows us to vary the final renormalisation scale μ around the natural value Q. To summarise, there are three possible variations that allow one to estimate higher-order corrections. The first two are associated to the variation of the initial scales μ_0 and $\sqrt{\zeta_0}$ around the natural scale μ_b and they are parametrised by the modest factors C_i^μ and C_i^ζ, such that $L_\mu = \ln C_i^\mu$ and $L_\zeta = \ln C_i^\zeta$. The third variation is associated to the final renormalisation scale μ around the hard scale Q and is parameterised by the factor C_f, such that $L_Q = \ln C_f$. In conclusion, the cross section formula with generic scales is proportional to:

$$
\sigma(Q) \propto H\left(Q, C_f Q\right) \left(R\left[(C_f Q, Q^2) \leftarrow (C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2) \right] \right)^2 F_1(C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2) F_2(C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2) \,.
\tag{1.54}
$$

However, notice that in Eq. (1.44) we have assumed $C_i^\mu = C_i^\zeta = C_i$. So effectively we are allowing for the variations of two scales parameterised by C_i and C_f.

1.1 Renormalisation and factorisation scale variations

In order to match the q_T-resummation approach, in this section we address the question of varying renormalisation of factorisation scales. Before going into the details, we remind that q_T-resummation also introduces the so-called resummation scale that is identified with the upper integration bound of the Sudakov form factor. However, we have already come across this scale in the previous section. More precisely, variation of the resummation scale are achieved by varying the parameter C_f.

1.1.1 The renormalisation scale

Introducing renormalisation scale variations in the formalism discussed above amounts to replacing any instance of the strong coupling α_s that is computed in, say, μ with the coupling itself computed in $\kappa_R \mu$, where κ_R is a factor not too far from one, and accounting for the evolution between $\kappa_R \mu$ and μ at a given fixed order. This is a perhaps complicated way of using the following equality:

$$
\begin{aligned}
a_s(\mu) \;=\; & a_s(\kappa_R \mu) \bigg[1 + a_s(\kappa_R \mu)\beta_0 \ln \kappa_R^2 + a_s^2(\kappa_R \mu)\left(\beta_0^2 \ln^2 \kappa_R^2 + \beta_1 \ln \kappa_R^2 \right) \\[2mm]
& + \; a_s^3(\kappa_R \mu)\left(\beta_0^3 \ln^3 \kappa_R^2 + \frac{5}{2}\beta_0 \beta_1 \ln^2 \kappa_R^2 + \beta_2 \ln \kappa_R^2 \right) \bigg] + \mathcal{O}(\alpha_s^5) \,,
\end{aligned}
\tag{1.55}
$$

that can be derived extending Eq. (1.12). At $N^k LL$ the square bracket has to include corrections up to $\mathcal{O}\left(\alpha_s^k\right)$.

1.1.2 The factorisation scale

Variation of the factorisation scale arise from a similar procedure as that discussed above for the strong coupling applied to the evolution of the collinear distributions. However, in this case the collinear distributions, that are originally computed at the scale $\mu_0 = C_i \mu_b$, are first evolved from μ_0 to $\mu_F = \kappa_F Q$, with κ_F close to one, but the interval of evolution between μ_F and Q is expanded to fixed order. More specifically, suppressing the flavour indices and the dependence on the momentum fraction x, and denoting with Γ the evolution operator

(see Sect. 2.2), one has:

$$
\begin{aligned}
f(\mu_0) &= \Gamma(\mu_0, \mu_F) \otimes f(\mu_F) = \Gamma(\mu_0, Q) \otimes \Gamma(Q, \mu_F) \otimes f(\mu_F) \\[2ex]
&= \Gamma(\mu_0, Q) \otimes \left[\Delta - a_s(\kappa_R Q) \ln \kappa_F^2 P^{(0)} \right. \\[2ex]
&\quad + \left. a_s^2(\kappa_R Q) \left(-\ln \kappa_F^2 P^{(1)} + \left(\frac{1}{2} \ln \kappa_F^2 - \ln \kappa_R^2 \right) \ln \kappa_F^2 \beta_0 P^{(0)} + \frac{1}{2} \ln^2 \kappa_F^2 P^{(0)} \otimes P^{(0)} \right) \right] \otimes f(\kappa_F Q) \\[2ex]
&= \Gamma(\mu_0, Q) \otimes \widetilde{f}(Q, \kappa_f) \,,
\end{aligned}
$$
(1.56)

where we have used Eqs. (2.22) and (1.55) to replace $a_s(Q)$ with $a_s(\kappa_R Q)$. In addition, we have defined $\widetilde{f}(Q, \kappa_f)$ as the collinear distribution at Q obtained by evolving the distribution from $\mu_F = \kappa_F Q$ to Q by means of an evolution operator expanded to the appropriate fixed-order. At LL and NLL the square bracket does not need to include any α_s corrections while N^{k}LL requires including corrections up to $\mathcal{O}(\alpha_s^{k-1})$. As a simpler alternative that may however give rise to larger variations, one may just evolve the collinear distributions from μ_0 to $\kappa_F \mu_0$ without first going to the hard scale μ_F. The result is:

$$
\begin{aligned}
f(\mu_0) &= \Gamma(\mu_0, \kappa_F \mu_0) \otimes f(\kappa_F \mu_0) \\[2ex]
&= \left[\Delta - a_s(\kappa_R \mu_0) \ln \kappa_F^2 P^{(0)} \right. \\[2ex]
&\quad + \left. a_s^2(\kappa_R \mu_0) \left(-\ln \kappa_F^2 P^{(1)} + \left(\frac{1}{2} \ln \kappa_F^2 - \ln \kappa_R^2 \right) \ln \kappa_F^2 \beta_0 P^{(0)} + \frac{1}{2} \ln^2 \kappa_F^2 P^{(0)} \otimes P^{(0)} \right) \right] \otimes f(\kappa_F \mu_0) \,.
\end{aligned}
$$
(1.57)

This procedure may potentially give larger variations because the expansion of the evolution operator is done around the scale $\mu_0 \simeq \mu_b$ that can become small. Therefore, at this scale the coupling can become large and so do the neglected higher-order corrections.

1.2   Non-perturbative component

In the previous section, I have considered the evolution of TMDs and thus I concentrated on their dependence on the renormalisation and rapidity scales μ and ζ, leaving aside the dependence on b_T. The computation of the rapidity evolution kernel in the $\overline{\text{MS}}$ scheme has led to the introduction of the scale:

$$
\mu_b = \frac{2e^{-\gamma_E}}{b_T} \,,
$$
(1.58)

This scale, within a modest factor, provides a natural choice for the evolution initial scales μ_0 and $\sqrt{\zeta_0}$ that prevents the appearance of large logarithms. Crucially, the strong coupling α_s has to be computed in the vicinity of the scale μ_b. Therefore, if the impact parameters b_T becomes large, $\alpha_s(\mu_b)$ may potentially become very large invalidating any perturbative calculation. Since the computation of q_T dependent observables requires Fourier transforming TMDs, they need to be accessed also at large values of b_T where the perturbative computation of the previous section breaks down. To overcome this limitation, it is customary to introduce an *arbitrary* scale $b_{\max}$ that denotes the maximum value of b_T at which one trusts perturbation theory. The value has to be such that:

$$
\alpha_s \left(\frac{2e^{-\gamma_E}}{b_{\max}} \right) \ll 1 \,.
$$
(1.59)

Then one introduces a monotonic function b_* of b_T with the following behaviour:

$$
\begin{aligned}
b_*(b_T) &\simeq b_T & \text{for} \quad b_T \to 0 \,, \\
b_*(b_T) &\to b_{\max} & \text{for} \quad b_T \to \infty \,.
\end{aligned}
$$
(1.60)

A common choice is:

$$
b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}} \,,
$$
(1.61)

but more complicated functions can be employed to have better control on the low-b_T region. Now, one writes:

$$F(x, b_T, \mu, \zeta) = \left[\frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)} \right] F(x, b_*(b_T), \mu, \zeta) \equiv f_{\mathrm{NP}}(x, b_T, \zeta) F(x, b_*(b_T), \mu, \zeta) \,. \tag{1.62}$$

This separation, often referred to as CSS prescription after Collins, Soper, and Sterman, is advantageous because, due to the behaviour of $b_*(b_T)$, $F(x, b_*(b_T), \mu, \zeta)$ can be computed in perturbation theory for any value of b_T, while $f_{\mathrm{NP}}(x, b_T, \zeta)$ embodies the non-perturbative dependence. It is important to stress that this separation is arbitrary and depends of the particular choice of b_* and b_{max}. Therefore, for any particular choice, only the combination in Eq. (1.62) is meaningful and it is misleading to refer to f_{NP} as to the non-perturbative part of TMDs in a universal sense. The reason why the function f_{NP} in Eq. (1.62) does not depend on the renormalisation scale μ is that this dependence cancels out in the ratio. To be more specific, if one chooses $\mu_0 = \mu_b = 2e^{-\gamma_E}/b_T$ and uses Eq. (1.25), one finds:

$$f_{\mathrm{NP}}(x, b_T, \zeta) = \frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)} =$$

$$\exp\left\{ K(\mu_b) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_b}} - K(\mu_{b_*}) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_{b_*}}} + \int_{\mu_b}^{\mu_{b_*}} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'} \right] \right\} \frac{F(\mu_b, \zeta_b)}{F(\mu_{b_*}, \zeta_{b_*})} \,, \tag{1.63}$$

with $\mu_{b_*} = \sqrt{\zeta_{b_*}} = 2e^{-\gamma_E}/b_*(b_T)$. It is thus apparent that the dependence on μ cancels. Due to the required behaviour of b_*, if b_T becomes small, b_T and b_* get closer and so do μ_{b_*} and μ_b as well as ζ_{b_*} and ζ_b. Therefore, from the equation above, we see that $f_{\mathrm{NP}} \to 1$ for $b_T \to 0$. Conversely, if b_T becomes large b_* saturates to b_{max}. In this limit, we can write:

$$f_{\mathrm{NP}}(x, b_T, \zeta) \xrightarrow[b_T \to \infty]{}$$

$$\exp\left\{ K(\mu_b) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_b}} - K(\mu_{\mathrm{min}}) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_{\mathrm{min}}}} + \int_{\mu_b}^{\mu_{\mathrm{min}}} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'} \right] \right\} \frac{F(\mu_b, \zeta_b)}{F(\mu_{\mathrm{min}}, \zeta_{\mathrm{min}})} \,, \tag{1.64}$$

where I have defined $\mu_{\mathrm{min}} = \sqrt{\zeta_{\mathrm{min}}} = 2e^{-\gamma_E}/b_{\mathrm{max}}$. As b_T becomes large, μ_b and ζ_b become increasingly small and the exponential takes the form of an evolution kernel between two sets of scales far apart from each other. As well known, the evolution kernel suppresses the distribution for large final scales. Extrapolating, one would then expect f_{NP} to be exponentially suppressed as b_T becomes large.

The definition of f_{NP} in Eq. (1.64) allows for a more systematic study of this quantity that may help adopting an optimal parameterisation when fitting it to data. The idea is to plot the ratio in the r.h.s. of the first line of Eq. (1.64) as a function of b_T for fixed values of x and ζ. We know that for b_T larger than some value, the numerator of this ratio will become unreliable. To identify this value, one can change the definition of μ_b (as well as that for μ_{b_*}) by introducing a factor C_2:

$$\mu_b = C_2 \frac{2e^{-\gamma_E}}{b_T} \,. \tag{1.65}$$

and vary C_2 around one by, say, a factor two up and down. When the effect of varying C_2 on the ratio becomes large, one can say that non-perturbative effects are large. On the other hand, in the region of b_T where variations are small perturbation theory is still reliable. Therefore this region can be used to put some constraint on f_{NP}.

1.3 Alternative to the CSS prescription

As discussed above, the CSS prescription has the purpose to avoid integrating the Landau pole both in the strong coupling α_s and in the collinear distributions. This is implemented by *globally* replacing b_T with b_* in Eq. (1.61) in the TMDs that in turn naturally leads to the introduction of the non-perturbative function f_{NP} through Eq. (1.62). Despite very simple and natural, this approach implies using b_* even where not strictly required. Specifically in the logarithms appearing explicitly in the exponent of the Sudakov form factor that will now depend on b_* rather than on b_T even though they are not problematic. Of course, the presence of f_{NP} will reabsorb these effects as clear from Eq. (1.62).

However, one can decide take a less invasive perspective to leave the explicit logarithms in the Sudakov unchanged while only regularising only α_s and the collinear distributions. This can be done by computing them

at some modified scale $\mu_*(mu)$ su that:

$$
\begin{aligned}
\mu_*(\mu) &\simeq \mu && \text{for} && \mu \to \infty \,, \\
\mu_*(\mu) &\to \mu_{\min} && \text{for} && \mu \to 0 \,.
\end{aligned}
\tag{1.66}
$$

This can be easily connected to the b_* prescription as:

$$
\mu_*(\mu) = \frac{b_0}{b_* \left(\frac{b_0}{\mu} \right)} \,,
\tag{1.67}
$$

with $b_0 = 2e^{-\gamma_E}$. Adopting the particular b_* in Eq. (1.61) and choosing $b_{\max} = b_0$ gives the surprisingly simple relation:

$$
\mu_*(\mu) = \sqrt{\mu^2 + 1} \,.
\tag{1.68}
$$

Therefore, the alternative recipe to the CSS procedure amounts to computing α_s and collinear distributions in $\sqrt{\mu^2 + 1}$ rather than in μ. This suffices to avoid integrating over the Landau pole, regularising only what actually needs to be regularised and leaving the rest untouched. The one unpleasant drawback of this approach is the fact that it does not make as transparent as in CSS the introduction of the function $f_{\rm NP}$ that in any case has to be present to encapsulate non-perturbative effects.

The difference between the standard CSS prescription and its alternative discussed here is displayed in Fig. 1.1 for the q_T distribution of the Drell-Yan cross section at $\sqrt{s} = 13$ TeV, $Q = M_Z$, and $y = 0$.

Fig. 1.1: .

2   Extraction of the singular behaviour for $q_T \to 0$

In this section we derive the expansion of the different ingredients of TMDs in order to finally extract the singular behaviour of a cross section involving TMDs for $q_T \to 0$. This will eventually serve to match the low-q_T resummed calculation to the fixed-order one valid at $q_T \simeq Q$.

2.1   Expansion of the evolution kernel

In this subsection, we work out the perturbative expansion of the evolution kernel R up to $\mathcal{O}(\alpha_s^2)$. This expansion is useful to extract the (singular) asymptotic behaviour of a cross section computed at fixed order in pQCD as q_T tends to zero. Since TMDs always appear in pairs in the computation of a cross section and that the evolution kernel is universal, it is convenient to compute the expansion of R^2. To do so, we start from Eq. (1.26) where we set $\mu_0 = \sqrt{\zeta_0} = \mu_b = 2e^{-\gamma_E}/b_T$ and $\mu = \sqrt{\zeta} = Q$, possible scale variations can be reinstated at a later stage:

$$
R^2 = \exp\left\{ K(\mu_b) \ln \frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \left[\gamma_F(\alpha_s(\mu')) - \frac{1}{2} \gamma_K(\alpha_s(\mu')) \ln \frac{Q^2}{\mu'^2} \right] \right\} \,.
\tag{2.1}
$$

Now we use the expansions in Eqs. (1.7), (1.9) and (1.23):

$$R^2 = \exp\left\{\sum_{n=0}^{\infty} a_s^{n+1}(\mu_b) K^{(n,0)} \ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} a_s^{n+1}(\mu')\left(\gamma_F^{(n)} - \frac{1}{2}\gamma_K^{(n)}\ln\frac{Q^2}{\mu'^2}\right)\right\}, \tag{2.2}$$

where we have defined:

$$a_s = \frac{\alpha_s}{4\pi}. \tag{2.3}$$

In order to carry out the perturbative expansion we need to write the argument of the exponential in terms of a common value of α_s computed at some hard scale: the natural choice in view of the matching is $\alpha_s(Q)$. This can be achieved by using the RGE for a_s at leading order:

$$\mu^2 \frac{da_s}{d\mu^2} = -\beta_0 a_s^2(\mu), \tag{2.4}$$

whose solution is:

$$a_s(\mu) = \frac{a_s(Q)}{1 + a_s(Q)\beta_0\ln(\mu^2/Q^2)} \simeq a_s(Q)\left[1 + a_s(Q)\beta_0\ln(Q^2/\mu^2) + \mathcal{O}(a_s^2)\right]. \tag{2.5}$$

Plugging the expansion in the r.h.s. of Eq. (2.5) into Eq. (2.2) and retaining only terms up to $\mathcal{O}(a_s^2)$, one finds:

$$\begin{aligned}
R^2 &= \exp\left\{a_s(Q)\left[K^{(0,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)} - \frac{1}{2}\gamma_K^{(0)}\ln\frac{Q^2}{\mu'^2}\right)\right]\right.\\
&\quad + a_s^2(Q)\beta_0\left[K^{(0,0)}\ln^2\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)}\ln\frac{Q^2}{\mu'^2} - \frac{1}{2}\gamma_K^{(0)}\ln^2\frac{Q^2}{\mu'^2}\right)\right]\\
&\quad \left. + a_s^2(Q)\left[K^{(1,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(1)} - \frac{1}{2}\gamma_K^{(1)}\ln\frac{Q^2}{\mu'^2}\right)\right] + \mathcal{O}(a_s^3)\right\}.
\end{aligned} \tag{2.6}$$

The final step before carrying out the expansion is computing the integrals:

$$\int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_{\ln(\mu_b^2)}^{\ln Q^2} d\ln\mu'^2\ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_0^{\ln(Q^2/\mu_b^2)} dx\, x^k = \frac{1}{k+1}\ln^{k+1}\left(\frac{Q^2}{\mu_b^2}\right). \tag{2.7}$$

so that:

$$\begin{aligned}
R^2 &\simeq \exp\left\{a_s(Q)\left[\left(K^{(0,0)} + \gamma_F^{(0)}\right)L - \frac{1}{4}\gamma_K^{(0)}L^2\right]\right.\\
&\quad \left. + a_s^2(Q)\left[\left(K^{(1,0)} + \gamma_F^{(1)}\right)L + \left(\beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)}\right)L^2 - \frac{1}{6}\beta_0\gamma_K^{(0)}L^3\right]\right\},
\end{aligned} \tag{2.8}$$

where we have defined:

$$L \equiv \ln\frac{Q^2}{\mu_b^2}, \tag{2.9}$$

Extending the computation to $\mathcal{O}(\alpha_s^3)$, Eq. (2.6) can be conveniently written as:

$$R^2 = \exp\left\{\sum_{n=1}^{3} a_s^n(Q)\sum_{k=1}^{n+1} S^{(n,k)}L^k\right\}, \tag{2.10}$$

with:

$$S^{(1,1)} = K^{(0,0)} + \gamma_F^{(0)}, \quad S^{(1,2)} = -\frac{1}{4}\gamma_K^{(0)},$$

$$S^{(2,1)} = K^{(1,0)} + \gamma_F^{(1)}, \quad S^{(2,2)} = \beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)}, \quad S^{(2,3)} = -\frac{1}{6}\beta_0\gamma_K^{(0)},$$

$$S^{(3,1)} = K^{(2,0)} + \gamma_F^{(2)}, \quad S^{(3,2)} = \frac{1}{2}\beta_1\gamma_F^{(0)} + \beta_1 K^{(0,0)} + \beta_0\gamma_F^{(1)} + 2\beta_0 K^{(1,0)} - \frac{1}{4}\gamma_K^{(2)},$$

$$S^{(3,3)} = \frac{1}{3}\beta_0^2\gamma_F^{(0)} - \frac{1}{6}\beta_1\gamma_K^{(0)} + \beta_0^2 K^{(0,0)} - \frac{1}{3}\beta_0\gamma_K^{(1)}, \quad S^{(3,4)} = -\frac{1}{8}\beta_0^2\gamma_K^{(0)}.$$

$$\tag{2.11}$$

Finally, Eq. (2.10) can be expanded up to order a_s^3 as:

$$R^2 = 1 + \sum_{n=1}^{3} a_s^n(Q) \sum_{k=1}^{2n} \widetilde{S}^{(n,k)} L^k \,, \tag{2.12}$$

with:

$$\widetilde{S}^{(1,1)} = S^{(1,1)} \,, \qquad\qquad\qquad\qquad \widetilde{S}^{(1,2)} = S^{(1,2)} \,,$$

$$\widetilde{S}^{(2,1)} = S^{(2,1)} \,, \qquad\qquad\qquad\qquad \widetilde{S}^{(2,2)} = S^{(2,2)} + \frac{1}{2}\left[S^{(1,1)}\right]^2 \,,$$

$$\widetilde{S}^{(2,3)} = S^{(2,3)} + S^{(1,1)}S^{(1,2)} \,, \qquad\qquad \widetilde{S}^{(2,4)} = \frac{1}{2}\left[S^{(1,2)}\right]^2$$

$$\widetilde{S}^{(3,1)} = S^{(3,1)} \,, \qquad\qquad\qquad\qquad \widetilde{S}^{(3,2)} = S^{(1,1)}S^{(2,1)} + S^{(3,2)} \,,$$

$$\widetilde{S}^{(3,3)} = S^{(1,1)}S^{(2,2)} + \frac{1}{6}\left[S^{(1,1)}\right]^3 + S^{(1,2)}S^{(2,1)} + S^{(3,3)} \,, \quad \widetilde{S}^{(3,4)} = \frac{1}{2}\left[S^{(1,1)}\right]^2 S^{(1,2)} + S^{(1,1)}S^{(2,3)} + S^{(1,2)}S^{(2,2)} + S^{(3,4)} \,,$$

$$\widetilde{S}^{(3,5)} = \frac{1}{2}S^{(1,1)}\left[S^{(1,2)}\right]^2 + S^{(1,2)}S^{(2,3)} \,, \qquad\qquad \widetilde{S}^{(3,6)} = \frac{1}{6}\left[S^{(1,2)}\right]^3 \,.$$

$$\tag{2.13}$$

2.2 Expansion of the DGLAP evolution

A further step towards the extraction of the singular behaviour of the resummed cross section in the limit $q_T \to 0$ requires expanding the solution of the DGLAP equation that also resums logarithms as that in Eq. (2.9). The solution of the DGLAP equation in Eq. (1.33) that evolves PDFs from the scale $\mu_0 = \mu_b$ to the scale Q can be written as([3]):

$$f(Q) = \Gamma(Q, \mu_b) \otimes f(\mu_b) \,, \tag{2.14}$$

where the evolution operator Γ obeys the equation:

$$\frac{\partial \Gamma(Q, \mu_b)}{\partial \ln Q^2} = P(Q) \otimes \Gamma(Q, \mu_b) \,, \tag{2.15}$$

with the splitting functions P having the perturbative expansion:

$$P(Q) = \sum_{n=0}^{\infty} a_s^{n+1}(Q) P^{(n)} \,. \tag{2.16}$$

The coefficients $P^{(n)}$ are functions of x (or z) only and thus (in $\overline{\text{MS}}$) P depends on Q only through the coupling a_s. We now need to expand Γ in powers of a_s up to $\mathcal{O}(a_s^2)$. To this end, we assume that Γ obeys the expansion:

$$\Gamma(Q, \mu_b) = \Delta + \sum_{n=1}^{\infty} a_s^n(Q) \sum_{k=1}^{n} L^k \Gamma^{(n,k)} \,, \tag{2.17}$$

where the coefficients $\Gamma^{(n,k)}$ are functions of x only that we need to determine up to $n = 2$. Eq. (2.15) can be solved iteratively. At $\mathcal{O}(a_s)$, considering that:

$$\frac{da_s}{d \ln Q^2} = \mathcal{O}(a_s^2) \,, \tag{2.18}$$

this gives:

$$\Gamma^{(1,1)} = P^{(0)} \,. \tag{2.19}$$

At $\mathcal{O}(a_s^2)$, this gives:

$$-L\beta_0 P^{(0)} + \Gamma^{(2,1)} + 2L\Gamma^{(2,2)} = P^{(1)} + LP^{(0)} \otimes P^{(0)} \,, \tag{2.20}$$

---

[3] A summation over the flavour indices is understood

so that:

$$\Gamma^{(2,1)} = P^{(1)},$$

$$\Gamma^{(2,2)} = \frac{1}{2}\beta_0 P^{(0)} + \frac{1}{2} P^{(0)} \otimes P^{(0)}. \tag{2.21}$$

Putting all pieces together and extending the expansion to $\mathcal{O}(\alpha_s^3)$, we find:

$$
\begin{aligned}
\Gamma(Q, \mu_b) &= \Delta + a_s(Q) L P^{(0)} + a_s^2(Q) \left[L P^{(1)} + L^2 \left(\frac{\beta_0 P^{(0)}}{2} + \frac{\left(P^{(0)}\right)^2}{2} \right) \right] \\
&+ a_s^3(Q) \left[L P^{(2)} + L^2 \left(\frac{P^{(0)} P^{(1)} + P^{(1)} P^{(0)}}{2} + \frac{\beta_1 P^{(0)}}{2} + \beta_0 P^{(1)} \right) \right. \\
&+ \left. L^3 \left(\frac{1}{2}\beta_0 \left(P^{(0)}\right)^2 + \frac{1}{3}\beta_0^2 P^{(0)} + \frac{\left(P^{(0)}\right)^3}{6} \right) \right] + \mathcal{O}(a_s^4).
\end{aligned}
\tag{2.22}
$$

However, the quantity we are interested in is the inverse of the operator Γ, that evolves the PDFs from the scale Q to μ_b:

$$
\begin{aligned}
\Gamma(\mu_b, Q) &= \Gamma^{-1}(Q, \mu_b) \\
&= \Delta - a_s(Q) L P^{(0)} - a_s^2(Q) \left[L P^{(1)} + L^2 \left(\frac{\beta_0 P^{(0)}}{2} - \frac{\left(P^{(0)}\right)^2}{2} \right) \right] \\
&- a_s^3(Q) \left[L P^{(2)} + L^2 \left(-\frac{P^{(0)} P^{(1)} + P^{(1)} P^{(2)}}{2} + \frac{1}{2}\beta_1 P^{(0)} + \beta_0 P^{(1)} \right) \right. \\
&+ \left. L^3 \left(-\frac{1}{2}\beta_0 \left(P^{(0)}\right)^2 + \frac{1}{3}\beta_0^2 P^{(0)} + \frac{1}{6}\left(P^{(0)}\right)^3 \right) \right] + \mathcal{O}(a_s^4).
\end{aligned}
\tag{2.23}
$$

2.3   Expansion of the matching coefficients at the hard scale

The final step is the expansion of the matching coefficients C defined in Eq. (1.29) for $\mu_0 = \mu_b$ around the scale Q up to order a_s^3. To do this, we use the expansion in Eq. (1.31), in which a_s is computed at μ_b, and combine it with Eq. (2.5). This yields:

$$
\begin{aligned}
C(\mu_b, \mu_b^2) &= C^{(0)} + a_s(Q) C^{(1)} + a_s^2(Q) \left[C^{(2)} + L\beta_0 C^{(1)} \right] \\
&+ a_s^3(Q) \left[C^{(3)} + L \left(\beta_1 C^{(1)} + 2\beta_0 C^{(2)} \right) + L^2 \beta_0^2 C^{(1)} \right] + \mathcal{O}(a_s^4)
\end{aligned}
\tag{2.24}
$$

2.4   Expansion of the single low-scale TMD

Before going into the more convoluted (but more relevant) case of a cross section, it is useful to compute the expansion of a single low-scale TMD, *i.e.* the convolution between matching coefficients and collinear

distributions at the scale $\mu_0 = \mu_b$. This can be achieved combining Eqs. (2.23) and (2.24):

$$
\begin{aligned}
C(\mu_b, \mu_b^2) \otimes f(\mu_b) &= C(\mu_b, \mu_b^2) \otimes \Gamma(\mu_b, Q) \otimes f(Q) \\[2mm]
&= \Bigg\{ C^{(0)} + a_s(Q) \left[C^{(1)} - L P^{(0)} \right] \\[2mm]
&\quad + a_s^2(Q) \left[C^{(2)} + L \left(-C^{(1)} P^{(0)} + \beta_0 C^{(1)} - P^{(1)} \right) + L^2 \left(\frac{\left(P^{(0)}\right)^2}{2} - \frac{\beta_0 P^{(0)}}{2} \right) \right] \\[2mm]
&\quad + a_s^3(Q) \Bigg[C^{(3)} + L \left(-C^{(2)} P^{(0)} - C^{(1)} P^{(1)} + \beta_1 C^{(1)} + 2\beta_0 C^{(2)} - P^{(2)} \right) \\[2mm]
&\quad + L^2 \left(-\frac{3}{2} \beta_0 C^{(1)} P^{(0)} + \frac{1}{2} C^{(1)} \left(P^{(0)} \right)^2 + \frac{P^{(0)} P^{(1)} + P^{(1)} P^{(0)}}{2} - \frac{\beta_1 P^{(0)}}{2} + \beta_0^2 C^{(1)} - \beta_0 P^{(1)} \right) \\[2mm]
&\quad + L^3 \left(\frac{1}{2} \beta_0 \left(P^{(0)} \right)^2 - \frac{1}{3} \beta_0^2 P^{(0)} - \frac{1}{6} \left(P^{(0)} \right)^3 \right) \Bigg] \Bigg\} \otimes f(Q) + \mathcal{O}(a_s^4) \,.
\end{aligned}
$$
(2.25)

2.5 Expansion of the cross section

We are finally in the position to gather all pieces and write down the perturbative expansion of a cross section involving TMDs, relevant in the limit $q_T \to 0$. The only additional information required is the hard factor H that also admits a perturbative expansion. It is now opportune to reinstate the flavour indices. In general, the hard factor H has a flavour structure, meaning that it carries a pair of flavour indices. Choosing $\mu = Q$, the expansion of the hard factor reads:

$$
H_{ij}(Q) = \sum_{n=0}^{\infty} a_s^n(Q) H_{ij}^{(n)} = H_{ij}^{(0)} + a_s(Q) H_{ij}^{(1)} + a_s^2(Q) H_{ij}^{(2)} + a_s^3(Q) H_{ij}^{(3)} + \mathcal{O}(a_s^4) \,,
\tag{2.26}
$$

where $H_{ij}^{(n)}$ are numerical factors. In relevant processes, such as Drell-Yan and SIDIS, it turns out that up to $\mathcal{O}(\alpha_s^2)$ the coefficients of the expansion of hard factor H are diagonal in flavour. Specifically, for scales sufficiently below the Z-mass scale, we have:

$$
H_{ij}^{(n)} \propto e_i^2 \delta_{ij} \,, \quad \text{for} \quad n = 0, 1, 2 \,,
\tag{2.27}
$$

where e_i is the electric charge of the i-th quark flavour. For higher scales, one just needs to replace the electric charges with the effective electro-weak charges. This simplification does not hold beyond $\mathcal{O}(\alpha_s^2)$.

A physical cross section differential in q_T can finally be computed as:

$$
\frac{d\sigma}{dq_T^2} \propto \sum_{ij} H_{ij}(Q) \int d^2 \mathbf{b}_T \, e^{i\mathbf{b}_T \cdot \mathbf{q}_T} F_i(x, \mathbf{b}_T, Q, Q^2) D_j(z, \mathbf{b}_T, Q, Q^2)
\tag{2.28}
$$

where F_i and D_j are two different sets of TMD distributions (*e.g.* a set of PDFs and a set of FFs) indexed by a flavour index. Assuming that the flavour indices i and j run *either* on quarks *or* on the gluon and writing explicitly all perturbative factors, one has:

$$
\begin{aligned}
\frac{d\sigma}{dq_T^2} &\propto \sum_{ij} H_{ij}(Q) \int \frac{d^2 \mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T \cdot \mathbf{q}_T} R^2 \left[(\mu_b, \mu_b^2) \to (Q, Q^2) \right] \\[2mm]
&\quad \times \sum_k \left[\mathcal{C}_{ik}(x; \mu_b, \mu_b^2) \otimes f_k(x, \mu_b) \right] \sum_l \left[\mathbb{C}_{jl}(z; \mu_b, \mu_b^2) \otimes d_l(z, \mu_b) \right] \\[2mm]
&= \sum_{n=0}^{\infty} a_s^n(Q) \sum_{p=0}^{2n} I_p(q_T, Q) \sum_{ij,kl} B_{ij,kl}^{(n,p)}(x, z) \underset{x}{\otimes} f_k(x, Q) \underset{z}{\otimes} d_l(z, Q) \,.
\end{aligned}
\tag{2.29}
$$

In the equation above we used different symbols to distinguish the perturbative components of the distributions F and D because in general they are different. In addition, we have defined:

$$I_p(q_T, Q) \equiv \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T \cdot \mathbf{q}_T} \ln^p \left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}} \right) = \frac{1}{2} \int_0^\infty db_T \, b_T J_0(b_T q_T) \ln^p \left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}} \right). \qquad (2.30)$$

Results for I_p have been computed up to $p = 4$ in Eq. (136) of Appendix B of Ref. [4] and can be extended to $p = 6$ using Eqs. (133) and (135). Specifically, and including the trivial tranform with $p = 0$, they read:

$$I_0(q_T, Q) = \delta(q_T^2),$$

$$I_1(q_T, Q) = -\frac{1}{q_T^2},$$

$$I_2(q_T, Q) = -\frac{2}{q_T^2} \ln \left(\frac{Q^2}{q_T^2} \right),$$

$$I_3(q_T, Q) = -\frac{3}{q_T^2} \ln^2 \left(\frac{Q^2}{q_T^2} \right),$$

$$I_4(q_T, Q) = -\frac{4}{q_T^2} \left[\ln^3 \left(\frac{Q^2}{q_T^2} \right) - 4\zeta_3 \right] \qquad (2.31)$$

$$I_5(q_T, Q) = -\frac{5}{q_T^2} \left[\ln^4 \left(\frac{Q^2}{q_T^2} \right) - 16\zeta_3 \ln \left(\frac{Q^2}{q_T^2} \right) \right],$$

$$I_6(q_T, Q) = -\frac{6}{q_T^2} \left[\ln^5 \left(\frac{Q^2}{q_T^2} \right) - 40\zeta_3 \ln^2 \left(\frac{Q^2}{q_T^2} \right) - 48\zeta_5 \right].$$

The functions above are compute under the assumption that there is no non-perturbative component. Upon this assumption, that we will relax below, all terms with $p = 0$ will be proportional to $\delta(q_T^2)$. Analogous terms are *not* included in the fixed-order calculation. Therefore, when matching the resummed calculation to the fixed-order one, one should remove from Eq. (2.29) all terms with $p = 0$. Importantly, this means removing the full $\mathcal{O}(1)$ terms such that the leading-order term for $q_T > 0$ is $\mathcal{O}(a_s)$.

The coefficients $B^{(n,k)}$ in Eq. (2.29) can be determined by using the expansions worked out above in Eqs. (2.12), (2.25), and (2.26). The leading order is obtained by just retaining the $\mathcal{O}(a_s^0)$ terms in all the expansions. This gives:

$$B_{ij,kl}^{(0,0)}(x, z) = H_{ij}^{(0)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z). \qquad (2.32)$$

The $\mathcal{O}(a_s)$ terms are obtained by combining the leading-order ones of all expansions but one. Finally organising the terms in powers of L, this yields:

$$B_{ij,kl}^{(1,0)}(x, z) = H_{ij}^{(1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) + H_{ij}^{(0)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) + H_{ij}^{(0)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z),$$

$$B_{ij,kl}^{(1,1)}(x, z) = H_{ij}^{(0)} \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) - H_{ij}^{(0)} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - H_{ij}^{(0)} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z),$$

$$B_{ij,kl}^{(1,2)}(x, z) = H_{ij}^{(0)} \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z).$$

$$(2.33)$$

The $\mathcal{O}(a_s^2)$ terms are more convoluted and read:

$$
\begin{aligned}
B_{ij,kl}^{(2,0)}(x,z) &= H_{ij}^{(2)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) + H_{ij}^{(1)}\left[\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z)\right] \\[6pt]
&+ H_{ij}^{(0)}\left[\mathcal{C}_{ik}^{(2)}(x)\mathbb{C}_{jl}^{(0)}(z) + \mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(1)}(z) + \mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(2)}(z)\right], \\[6pt]
B_{ij,kl}^{(2,1)}(x,z) &= H_{ij}^{(1)}\left[\tilde{S}^{(1,1)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z)\right] \\[6pt]
&+ H_{ij}^{(0)}\left[\tilde{S}^{(2,1)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,1)}\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,1)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(x)\right. \\[6pt]
&+ \beta_0\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \beta_0\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z) \\[6pt]
&- \left[\mathcal{C}^{(1)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \mathcal{C}_{ik}^{(1)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) \\[6pt]
&- \mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(1)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(1)}(z) \\[6pt]
&\left.- \left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(1)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(1)}\right]_{jl}(z)\right], \\[6pt]
B_{ij,kl}^{(2,2)}(x,z) &= H_{ij}^{(1)}\tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) \\[6pt]
&+ H_{ij}^{(0)}\left[\tilde{S}^{(2,2)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z)\right. \\[6pt]
&- \tilde{S}^{(1,1)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(1,1)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[6pt]
&- \frac{1}{2}\beta_0\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \frac{1}{2}\beta_0\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[6pt]
&+ \frac{1}{2}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) + \frac{1}{2}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[6pt]
&\left.+ \left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z)\right], \\[6pt]
B_{ij,kl}^{(2,3)}(x,z) &= \tilde{S}^{(2,3)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(y) - \tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z), \\[6pt]
B_{ij,kl}^{(2,4)}(x,z) &= \tilde{S}^{(2,4)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z).
\end{aligned}
\tag{2.34}
$$

Finally, the $\mathcal{O}(a_s^3)$ terms are even more convoluted and read:

$$
\begin{aligned}
B_{ij,kl}^{(3,0)}(x,z) &= H_{ij}^{(3)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) \\[6pt]
&+ H_{ij}^{(2)}\left[\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z)\right] \\[6pt]
&+ H_{ij}^{(1)}\left[\mathcal{C}_{ik}^{(2)}(x)\mathbb{C}_{jl}^{(0)}(z) + \mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(1)}(z) + \mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(2)}(z)\right] \\[6pt]
&+ H_{ij}^{(0)}\left[\mathcal{C}_{ik}^{(3)}(x)\mathbb{C}_{jl}^{(0)}(z) + \mathcal{C}_{ik}^{(2)}(x)\mathbb{C}_{jl}^{(1)}(z) + \mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(2)}(z) + \mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(3)}(z)\right],
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
B_{ij,kl}^{(3,1)}(x,z) &= H_{ij}^{(2)} \left[\tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) \right] \\
&+ H_{ij}^{(1)} \left[\tilde{S}^{(2,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \right. \\
&+ \beta_0 \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \beta_0 \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\
&- \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) \\
&\left. - \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) \right] \\
&+ H_{ij}^{(0)} \left[\tilde{S}^{(3,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,1)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \right. \\
&+ \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(2)}(x) \mathbb{C}_{jl}^{(0)} + \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(1)}(z) + \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(2)}(z) \\
&+ \beta_1 \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \beta_1 \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\
&+ 2\beta_0 \mathcal{C}_{ik}^{(2)}(x) \mathbb{C}_{jl}^{(0)}(z) + 2\beta_0 \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(1)}(z) + 2\beta_0 \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(2)}(z) \\
&- \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) \\
&- \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) \\
&- \mathcal{C}_{ik}^{(2)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(2)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(2)} \right]_{jl}(z) \\
&\left. - \left[\mathcal{C}^{(2)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(2)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(2)}(z) \right],
\end{aligned}
\tag{2.36}
$$

$$
\begin{aligned}
B_{ij,kl}^{(3,2)}(x,z) \;=\;& H_{ij}^{(2)} \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&+ H_{ij}^{(1)} \bigg[\tilde{S}^{(2,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \tilde{S}^{(1,1)} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&- \frac{1}{2} \beta_0 \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \frac{1}{2} \beta_0 \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&+ \frac{1}{2} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) + \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) + \frac{1}{2} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \bigg] \\[4pt]
&+ H_{ij}^{(0)} \bigg[\tilde{S}^{(3,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,2)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \tilde{S}^{(2,1)} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(2,1)} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&+ \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(2)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(1)}(z) + \tilde{S}^{(1,2)} \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(2)}(z) \\[4pt]
&- \tilde{S}^{(1,1)} \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,1)} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,1)} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \tilde{S}^{(1,1)} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) \\[4pt]
&+ \beta_0^2 \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \beta_0^2 \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \frac{3}{2} \beta_0 \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) - \frac{3}{2} \beta_0 \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \beta_0 \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) - \beta_0 \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&- \frac{3}{2} \beta_0 \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) - \frac{3}{2} \beta_0 \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&+ \tilde{S}^{(1,1)} \beta_0 \mathcal{C}_{ik}^{(1)}(x) \mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,1)} \beta_0 \mathcal{C}_{ik}^{(0)}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&+ \frac{1}{2} \mathcal{C}_{ik}^{(1)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) + \frac{1}{2} \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&+ \frac{1}{2} \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) + \frac{1}{2} \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&- \frac{1}{2} \beta_1 \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) + \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&- \frac{1}{2} \beta_1 \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) + \mathcal{C}_{ik}^{(0)}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) \\[4pt]
&+ \left[\mathcal{C}^{(1)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) + \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \left[\mathbb{C}^{(1)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) \\[4pt]
&+ \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(1)} \right]_{ik}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(0)} \right]_{jl}(z) + \left[\mathcal{C}^{(0)} \otimes \mathcal{P}^{(0)} \right]_{ik}(x) \left[\mathbb{C}^{(0)} \otimes \mathbb{P}^{(1)} \right]_{jl}(z) \bigg],
\end{aligned}
$$

$$
(2.37)
$$

$$
\begin{aligned}
B_{ij,kl}^{(3,3)}(x,z) =\ & H_{ij}^{(1)}\left[\tilde{S}^{(2,3)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z)\right] \\[4pt]
&+ H_{ij}^{(0)}\left[\tilde{S}^{(3,3)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,3)}\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,3)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z)\right. \\[4pt]
&- \tilde{S}^{(2,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(2,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&- \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(1)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(1)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(1)}\right]_{jl}(z) \\[4pt]
&- \tilde{S}^{(1,2)}\left[\mathcal{C}^{(1)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(1)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(1)}(z) \\[4pt]
&+ \frac{1}{2}\tilde{S}^{(1,1)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) + \frac{1}{2}\tilde{S}^{(1,1)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&+ \tilde{S}^{(1,2)}\beta_0\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(1,2)}\beta_0\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(z) + \beta_0\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) \\[4pt]
&- \frac{1}{2}\tilde{S}^{(1,1)}\beta_0\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) + \tilde{S}^{(1,1)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \frac{1}{2}\tilde{S}^{(1,1)}\beta_0\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&- \frac{1}{3}\beta_0^2\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) + \frac{1}{2}\beta_0\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \frac{1}{6}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) \\[4pt]
&- \frac{1}{3}\beta_0^2\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) + \frac{1}{2}\beta_0\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) - \frac{1}{6}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&\left.- \frac{1}{2}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \frac{1}{2}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z)\right],
\end{aligned}
$$
$$(2.38)$$

$$
\begin{aligned}
B_{ij,kl}^{(3,4)}(x,z) =\ & H_{ij}^{(1)}\tilde{S}^{(2,4)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&+ H_{ij}^{(0)}\left[\tilde{S}^{(3,4)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,4)}\mathcal{C}_{ik}^{(1)}(x)\mathbb{C}_{jl}^{(0)}(z) + \tilde{S}^{(2,4)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1)}(x)\right. \\[4pt]
&- \tilde{S}^{(2,3)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(2,3)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&- \frac{1}{2}\tilde{S}^{(1,2)}\beta_0\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \frac{1}{2}\tilde{S}^{(1,2)}\beta_0\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z) \\[4pt]
&+ \frac{1}{2}\tilde{S}^{(1,2)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) \\[4pt]
&+ \tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) \\[4pt]
&\left.+ \frac{1}{2}\tilde{S}^{(1,2)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z)\right],
\end{aligned}
$$
$$(2.39)$$

$$
B_{ij,kl}^{(3,5)}(x,z) = H_{ij}^{(0)}\left[\tilde{S}^{(3,5)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z) - \tilde{S}^{(2,4)}\mathcal{C}_{ik}^{(0)}(x)\left[\mathbb{C}^{(0)}\otimes\mathbb{P}^{(0)}\right]_{jl}(z) - \tilde{S}^{(2,4)}\left[\mathcal{C}^{(0)}\otimes\mathcal{P}^{(0)}\right]_{ik}(x)\mathbb{C}_{jl}^{(0)}(z)\right], \quad (2.40)
$$

$$
B_{ij,kl}^{(3,6)}(x,z) = H_{ij}^{(0)}\tilde{S}^{(3,6)}\mathcal{C}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(0)}(z). \quad (2.41)
$$

2.6 Scale variations

In this section, we work out the expressions for the renormalisation- and factorisation-scale variations of the expanded cross section up to $\mathcal{O}(\alpha_s^2)$. This is done by displacing the coupling from Q to μ_{R} and the PDFs/FFs from Q to μ_{F}. Under the assumption that $\mu_{\mathrm{R}} \simeq \mu_{\mathrm{R}} \simeq Q$, this displacements can be obtained by expanding the solutions of the evolution equation between Q and μ_{R} or μ_{F} up to $\mathcal{O}(\alpha_s^2)$. For the coupling, this expansion can be read off from Eq. (1.12):

$$
a_s(Q) = a_s(\mu_{\mathrm{R}})\left[1 + a_s(\mu_{\mathrm{R}})\beta_0\ln\frac{\mu_{\mathrm{R}}^2}{Q^2}\right] + \mathcal{O}(\alpha_s^3). \quad (2.42)
$$

For the PDFs/FFs, the expansion of the evolution operator can be read off from Eq. (2.22):

$$
\Gamma(Q,\mu_{\mathrm{F}}) = \Delta - a_s(Q)\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}P^{(0)} - a_s^2(Q)\left[\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}P^{(1)} - \frac{1}{2}\ln^2\frac{\mu_{\mathrm{F}}^2}{Q^2}\left(\beta_0 P^{(0)} + P^{(0)}\otimes P^{(0)}\right)\right] + \mathcal{O}(\alpha_s^3), \quad (2.43)
$$

with $\Delta_{ij}(x) = \delta_{ij}\delta(1-x)$. We now use Eq. (2.42) to replace $a_s(Q)$ with $a_s(\mu_{\mathrm{R}})$ in the expression above and get:

$$
\begin{aligned}
\Gamma(Q,\mu_{\mathrm{F}}) \;=\; & \Delta - a_s(\mu_{\mathrm{R}})\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}P^{(0)} \\[6pt]
& - a_s^2(\mu_{\mathrm{R}})\left[\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}P^{(1)} + \left(\ln\frac{\mu_{\mathrm{R}}^2}{Q^2} - \frac{1}{2}\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}\right)\ln\frac{\mu_{\mathrm{F}}^2}{Q^2}\beta_0 P^{(0)} - \frac{1}{2}\ln^2\frac{\mu_{\mathrm{F}}^2}{Q^2}P^{(0)}\otimes P^{(0)}\right] + \mathcal{O}(\alpha_s^3).
\end{aligned}
\tag{2.44}
$$

Since we have two such operators, one for the PDFs and one for the FFs, it is convenient to take the product beforehand and truncate it to $\mathcal{O}(\alpha_s^2)$ so that:

$$
\begin{aligned}
f_k(x,Q)d_l(z,Q) \;=\; & \Bigg[\delta_{km}\delta(1-x)\delta_{ln}\delta(1-z) \\[6pt]
& - a_s(\mu_{\mathrm{R}})L_{\mathrm{F}}\left(\delta_{km}\delta(1-x)\mathbb{P}^{(0)}_{ln}(z) + \mathcal{P}^{(0)}_{km}(x)\delta_{ln}\delta(1-z)\right) \\[6pt]
& - a_s^2(\mu_{\mathrm{R}})\Bigg[L_{\mathrm{F}}\left(\delta_{km}\delta(1-x)\mathbb{P}^{(1)}_{ln}(z) + \mathcal{P}^{(1)}_{km}(x)\delta_{ln}\delta(1-z)\right) \\[6pt]
& + \left(L_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{F}}\right)L_{\mathrm{F}}\beta_0\left(\delta_{km}\delta(1-x)\mathbb{P}^{(0)}_{ln}(z) + \mathcal{P}^{(0)}_{km}(x)\delta_{ln}\delta(1-z)\right) \\[6pt]
& - \frac{1}{2}L_{\mathrm{F}}^2\left(\delta_{km}\delta(1-x)\mathbb{P}^{(0)}_{l\beta}(z)\otimes\mathbb{P}^{(0)}_{\beta n}(z) + 2\mathcal{P}^{(0)}_{km}(x)\mathbb{P}^{(0)}_{ln}(z) + \mathcal{P}^{(0)}_{k\beta}(x)\otimes\mathcal{P}^{(0)}_{\beta m}(x)\delta_{ln}\delta(1-z)\right)\Bigg] \\[6pt]
& \Bigg] \underset{x}{\otimes} f_m(x,\mu_{\mathrm{F}})\underset{z}{\otimes} d_n(z,\mu_{\mathrm{F}}),
\end{aligned}
\tag{2.45}
$$

where the greek indices are undestood to be summed over and where we have defined:

$$
L_{\mathrm{R}} = \ln\frac{\mu_{\mathrm{R}}^2}{Q^2} \quad \text{and} \quad L_{\mathrm{F}} = \ln\frac{\mu_{\mathrm{F}}^2}{Q^2}.
\tag{2.46}
$$

Finally, using Eqs. (2.42) and (2.45) the expansion of the cross section in Eq. (2.29) can be generalised to:

$$
\frac{d\sigma}{dq_T^2} \;\propto\; \sum_{s=0}^{\infty} a_s^s(\mu_{\mathrm{R}})\sum_{p=0}^{2s} I_p(q_T,Q)\sum_{ij,kl}\overline{B}^{(s,p)}_{ij,mn}(x,z)\underset{x}{\otimes} f_m(x,\mu_{\mathrm{F}})\underset{z}{\otimes} d_n(z,\mu_{\mathrm{F}}).
\tag{2.47}
$$

where:

$$\overline{B}^{(0,0)}_{ij,mn}(x,z) = B^{(0,0)}_{ij,mn}(x,z),$$

$$\overline{B}^{(1,0)}_{ij,mn}(x,z) = B^{(1,0)}_{ij,mn}(x,z) - L_{\mathrm{F}}H^{(0)}_{ij}\left(\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) + \mathcal{P}^{(0)}_{im}(x)\delta_{jn}\delta(1-z)\right),$$

$$\overline{B}^{(1,1)}_{ij,mn}(x,z) = B^{(1,1)}_{ij,mn}(x,z),$$

$$\overline{B}^{(1,2)}_{ij,mn}(x,z) = B^{(1,2)}_{ij,mn}(x,z),$$

$$\overline{B}^{(2,0)}_{ij,mn}(x,z) = B^{(2,0)}_{ij,mn}(x,z) + \beta_0 L_{\mathrm{R}}H^{(0)}_{ij}\left(\delta_{im}\delta(1-x)\mathbb{C}^{(1,0)}_{jn}(z) + \mathcal{C}^{(1,0)}_{im}(x)\delta_{jn}\delta(1-z)\right)$$

$$- L_{\mathrm{F}}H^{(0)}_{ij}\bigg(\delta_{im}\delta(1-x)\mathbb{C}^{(1,0)}_{j\beta}(z)\underset{z}{\otimes}\mathbb{P}^{(0)}_{\beta n}(z) + \mathcal{C}^{(1,0)}_{im}(x)\mathbb{P}^{(0)}_{jn}(z)$$

$$+ \mathcal{P}^{(0)}_{im}(x)\mathbb{C}^{(1,0)}_{jn}(z) + \mathcal{C}^{(1,0)}_{i\beta}(x)\underset{x}{\otimes}\mathcal{P}^{(0)}_{\beta m}(x)\delta_{jn}\delta(1-z)\bigg)$$

$$+ H^{(1)}_{ij}\left(\beta_0 L_{\mathrm{R}}\delta_{im}\delta(1-x)\delta_{jn}\delta(1-z) - L_{\mathrm{F}}\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) - L_{\mathrm{F}}\mathcal{P}^{(0)}_{im}(x)\delta_{jn}\delta(1-z)\right)$$

$$- H^{(0)}_{ij}\bigg[L_{\mathrm{F}}\left(\delta_{im}\delta(1-x)\mathbb{P}^{(1)}_{jn}(z) + \mathcal{P}^{(1)}_{im}(x)\delta_{jn}\delta(1-z)\right)$$

$$+ \left(L_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{F}}\right)L_{\mathrm{F}}\beta_0\left(\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) + \mathcal{P}^{(0)}_{im}(x)\delta_{ln}\delta(1-z)\right)$$

$$- \frac{1}{2}L_{\mathrm{F}}^2\left(\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{j\beta}(z)\otimes\mathbb{P}^{(0)}_{\beta n}(z) + 2\mathcal{P}^{(0)}_{im}(x)\mathbb{P}^{(0)}_{jn}(z) + \mathcal{P}^{(0)}_{i\beta}(x)\otimes\mathcal{P}^{(0)}_{\beta m}(x)\delta_{jn}\delta(1-z)\right)\bigg],$$

$$\overline{B}^{(2,1)}_{ij,mn}(x,z) = B^{(2,1)}_{ij,mn}(x,z) - \beta_0 L_{\mathrm{R}}H^{(0)}_{ij}\left(\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) + \mathcal{P}^{(0)}_{im}(x)\delta_{jn}\delta(1-z)\right)$$

$$+ L_{\mathrm{F}}H^{(0)}_{ij}\left(\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{j\beta}(z)\underset{z}{\otimes}\mathbb{P}^{(0)}_{\beta n}(z) + 2\mathcal{P}^{(0)}_{im}(x)\mathbb{P}^{(0)}_{jn}(z) + \mathcal{P}^{(0)}_{i\beta}(x)\underset{x}{\otimes}\mathcal{P}^{(0)}_{\beta m}(x)\delta_{jn}\delta(1-z)\right)$$

$$+ H^{(0)}_{ij}S^{(1,1)}\left(\beta_0 L_{\mathrm{R}}\delta_{im}\delta(1-x)\delta_{jn}\delta(1-z) - L_{\mathrm{F}}\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) - L_{\mathrm{F}}\mathcal{P}^{(0)}_{im}(x)\delta_{jn}\delta(1-z)\right),$$

$$\overline{B}^{(2,2)}_{ij,mn}(x,z) = B^{(2,2)}_{ij,mn}(x,z)$$

$$+ H^{(0)}_{ij}S^{(1,2)}\left(\beta_0 L_{\mathrm{R}}\delta_{im}\delta(1-x)\delta_{jn}\delta(1-z) - L_{\mathrm{F}}\delta_{im}\delta(1-x)\mathbb{P}^{(0)}_{jn}(z) - L_{\mathrm{F}}\mathcal{P}^{(0)}_{im}(x)\delta_{jn}\delta(1-z)\right),$$

$$\overline{B}^{(2,3)}_{ij,mn}(x,z) = B^{(2,3)}_{ij,mn}(x,z),$$

$$\overline{B}^{(2,4)}_{ij,mn}(x,z) = B^{(2,4)}_{ij,mn}(x,z),$$

$$(2.48)$$

with the coefficients $B^{(s,p)}_{ij,mn}$ given in Eqs. (2.32), (2.33), and (2.34). Of course, if $\mu_{\mathrm{R}} = \mu_{\mathrm{F}} = Q$, all logs vanish and the $\overline{B}^{(s,p)}_{ij,mn}$ coefficients automatically reduce to $B^{(s,p)}_{ij,mn}$.

2.7   A note on the logarithmic ordering

The logarithmic accuracy is driven by the evolution kernel R defined in Eq. (1.1). As it should be clear from Eq. (2.12), the evolution kernel squared R^2, also referred to as Sudakov form factor, admits the following

expansion:

$$R^2 = \sum_{n=0}^{\infty} a_s^n \sum_{k=1}^{2n} \widetilde{S}^{(n,k)} L^k \,. \tag{2.49}$$

This expansion naturally allows us to define a logarithmic ordering such that:

$$R^2 = \sum_{m=0}^{\infty} R^2_{\mathrm{N^m LL}} \,, \tag{2.50}$$

with:

$$R^2_{\mathrm{N^m LL}} = \sum_{n=[m/2]}^{\infty} \widetilde{S}^{(n,2n-m)} a_s^n L^{2n-m} \,, \tag{2.51}$$

where $[m/2]$ stands for "integer part of $m/2$". Now, I multiply $R^2_{\mathrm{N^m LL}}$ by a_s to a generic power p and do some manipulation defining $n + p = j$. This way, I get:

$$a_s^p R^2_{\mathrm{N^m LL}} = \sum_{j=[(m+2p)/2]}^{\infty} \widetilde{S}^{(j-p,2j-(m+2p))} a_s^j L^{2j-(m+2p)} \sim R^2_{\mathrm{N^{m+2p} LL}} \,, \tag{2.52}$$

where the symbol $\sim$ means that the terms at two sides have the same logarithmic accuracy. This finding is relevant in that, in order to compute a cross section, the Sudakov from factor has to be multiplied by the hard factor H and a pair of matching functions C. Both these components admit an expansion in powers of a_s without enhancing logaritms L (see Eqs. (1.31) and (2.26)). Therefore R^2 will eventually be multiplied by some power of a_s.

Suppose we compute the Sudakov form factor to leading-logarithmic (LL) accuracy ($m = 0$): Eq. (2.52) tells us that, in terms of logarithmic accuracy, including $\mathcal{O}(a_s)$ corrections in H and C implies introducing next-to-next-to-leading-logarithmic (NNLL) corrections. In practice, this means that to LL and NLL the functions H and C can be taken at leading order, at NNLL we only need to introduce $\mathcal{O}(a_s)$ corrections in H and C, while at N³LL we have to introduce $\mathcal{O}(a_s^2)$ corrections in H and C.

As a final remark, we notice that, as opposed to the Sudakov form factor, the evolution of collinear distributions and α_s provide resummation of single logarithms. Therefore, the inclusion of perturbative corrections to the β-functions and splitting functions should be accordingly. Specifically, the perturbative accuracy at which they are computed should be the same as that of the non-cusp anomalous dimension γ_F.

3   Non-perturbative effect on the asymptotic cross section

As discussed in Sect. 1.2, TMDs have a non-perturbative component that can be parameterised as in Eq. (1.62). This implies two main ingredients: the introduction of a function $b_*(b_T)$ that prevents the perturbative component to enter the non-perturbative regime and a function f_{NP} that parametrises the actual non-perturbative component. These changes leave almost unchanged the derivation of the asymptotic behavior of a cross section for $q_T \to 0$. The only change concerns the functions I_p defined in Eq. (2.30), that have to be replaced with:

$$\widetilde{I}_p(x,z,q_T,Q) \equiv \frac{1}{2} \int_0^{\infty} db_T \, b_T J_0(b_T q_T) f_{\mathrm{NP}}^{(1)}(x,b_T,Q^2) f_{\mathrm{NP}}^{(2)}(z,b_T,Q^2) \ln^p \left(\frac{b_*^2(b_T) Q^2}{4 e^{-2\gamma_E}} \right) \,, \tag{3.1}$$

where $f_{\mathrm{NP}}^{(1)}$ and $f_{\mathrm{NP}}^{(2)}$ are the non-perturbative functions associated to the two TMD distributions F and D involved in the cross sections. Here we are assuming $\mu = \sqrt{\zeta} = Q$ and $\mu_0 = \sqrt{\zeta_0} = \mu_b$. In general, the integral in Eq. (3.1) cannot be evaluated analytically but one can use the Ogata quadrature method to compute it numerically.[4] In the case $k = 0$ Eq. (3.1) has an intersting consequence. Specifically, contrary to the case in which no non-perturbative contribution is introduced, $\widetilde{I}_0$ is different from zero also for $q_T > 0$. Therefore, also the terms in the expansion in Eq. (2.29) that are independent of L contribute to the cross section differential in q_T. Clearly, any non-perturbative contribution is expected to give a substantial contribution only in the vicinity of $q_T = 0$.

---

[4] The convergence of the Ogata quadrature method crucially depends on the function f_{NP} being exponentially suppressed for large values of b_T, such that full integrand is dumped in this region.

In this respect, one should check that the introduction of the non-perturbative components does not affect the large-q_T region. This is amounts to showing that:

$$\widetilde{I}_p(x, z, q_T, Q) \underset{q_T \gtrsim Q}{\sim} I_p(q_T, Q) \,. \tag{3.2}$$

Since for small values of b_T we (must) have that $b_*(b_T) \simeq b_T$ and that $f_{\mathrm{NP}}(x, b_T, \zeta) \simeq 1$, this implies that for large values of q_T (that is the conjugate variable of b_T) $\widetilde{I}_p$ has to tend to I_p. This can be shown numerically on a case-by-case base. In particular, we can compute the integral in Eq. (3.1) using the Ogata quadrature method and compare the results to those in Eq. (2.31). For the comparison we use Eq. (1.61) with $b_{\max} = 2e^{-\gamma_E}$ and choose as a non-perturbative functions f_{NP} the following x-independent form:

$$f_{\mathrm{NP}}^{(1)}(b_T, \zeta) = f_{\mathrm{NP}}^{(2)}(b_T, \zeta) = \exp\left[\left(-g_1 - g_2 \ln\left(\frac{\sqrt{\zeta}}{2Q_0}\right)\right) \frac{b_T^2}{2}\right] \,, \tag{3.3}$$

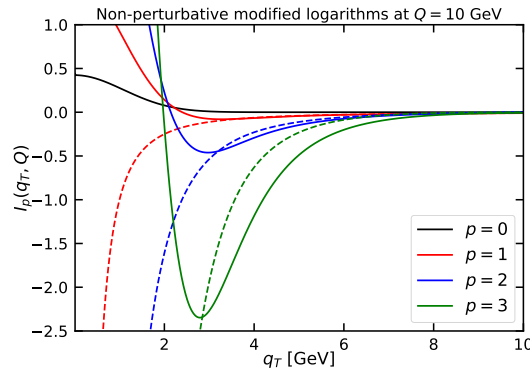with $g_1 = 0.02$, $g_1 = 0.5$, and $Q_0 = 1.6$ GeV.

Fig. 3.1: Comparison of the integral in Eq. (3.1) for $p = 0, 1, 2, 3$ (solid lines) to the expressions in Eq. (2.31) (dashed lines) as functions of q_T at $Q = 10$ GeV. The black dashed line is not present because it corresponds to $\delta(q_T^2)$.

In Fig. 3.1, I compare the integral in Eq. (3.1) for $p = 0, 1, 2, 3$ (solid lines) to the expressions in Eq. (2.31) (dashed lines) at $Q = 10$ GeV as functions of q_T. Notice that the black dashed line is not present because it corresponds to $\delta(q_T^2)$. It is thus true that $\widetilde{I}_p$ tends to I_p as q_T increases. However, this is not enough to satisfy Eq. (3.2). As a matter of fact, if one zooms in around $q_T \lesssim Q$, one finds a substantial difference between $\widetilde{I}_p$ and I_p. In Fig 3.2 the zoom is showed for $p = 1$. Even if q_T is abundantly in the fixed-order region, the difference
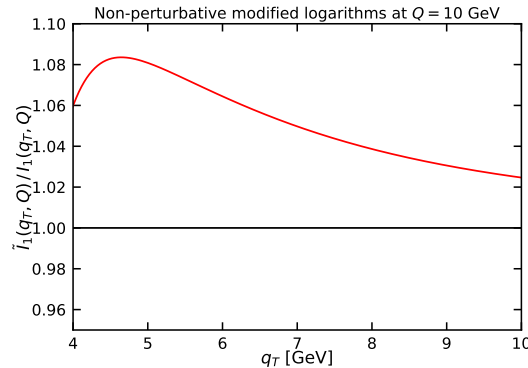

Fig. 3.2: Ratio between $\widetilde{I}_1$ and I_1 in the region of q_T close to Q.

between $\widetilde{I}_1$ and I_1 is still large. Therefore, it appears that the relation in Eq. (3.2) is not plainly fulfilled. Moreover, the discrepancy tends to become larger as p increases. Therefore, when performing the matching between resummed and fixed-order cross sections, it is very important to include non-perturbative effects in the

expanded calculation in order to ensure a proper cancellation with the resummed calculation in the large-q_T region.

On the other hand, by design, the inclusion of the non-perturbative effects modifies significantly also the low-q_T region. The modification is such to prevent the cancellation at small q_T between the expansion of the resummed calculation and the fixed-order one. A possible solution is to use yet another definition of the functions I_p in the expanded calculation. These functions have to be such to tend to those in Eq. (2.31) for small values of q_T but converge to the definition in Eq. (3.1) more rapidly as q_T increases. To do this, one may exploit the fact that power-suppressed contributions do not affect the logarithmic expansion of the resummed calculation to combine the two definitions as follows:

$$\hat{I}_p(x, z, q_T, Q) = \left[1 - \left(\frac{q_T}{Q} \right)^S \right] I_p(q_T, Q) + \left(\frac{q_T}{Q} \right)^S \widetilde{I}_p(x, z, q_T, Q) \,, \tag{3.4}$$

where the exponent S can be adjusted to make the transition from one regime to the other more or less strong. Clearly, this definition should not be pushed to values of q_T much above Q. As an example, Fig. 3.3 shows the shape of $\hat{I}_1$ defined in Eq. (3.4) for $S = 0.2$ along with its components given in Eqs. (2.31) and (3.1).
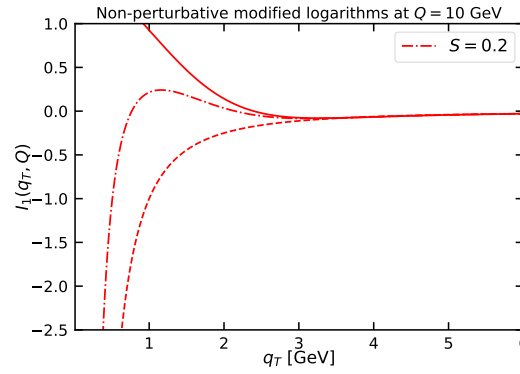

Fig. 3.3: Behaviour of $\hat{I}_1$ defined in Eq. (3.4) for $S = 0.2$ compared to the single components defined in Comparison of the integral in Eqs. (2.31) and (3.1).

4   Resummation scheme

The resummed cross section in Eqs. (2.28)-(2.29) is written in terms of process-independent TMDs, F and D, and a hard factor H that is instead process dependent. However, in the original q_T-resummation formalism applied to Drell-Yan production the same cross section was written in a different but equivalent way as:

$$\frac{d\sigma}{dq_T^2} \quad \propto \quad \sum_{ij} \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T \cdot \mathbf{q}_T} \widetilde{R}^2 \left[(\mu_b, \mu_b^2) \to (Q, Q^2) \right]$$

$$\times \quad \sum_k \left[\widetilde{\mathcal{C}}_{ik}(x; \mu_b, \mu_b^2) \otimes f_k(x, \mu_b) \right] \sum_l \left[\widetilde{\mathbb{C}}_{jl}(z; \mu_b, \mu_b^2) \otimes d_l(z, \mu_b) \right] \,, \tag{4.1}$$

where there is no hard function H and the Sudakov form factor $\widetilde{R}^2$ as well as the matching functions $\widetilde{\mathcal{C}}$ and $\widetilde{\mathbb{C}}$ are now process dependent. The correspondence between the various ingredients of Eqs. (2.29) and (4.1) requires that:

$$H(Q)R^2(Q, \mu_b)\mathcal{C}(\mu_b)\mathbb{C}(\mu_b) = \widetilde{R}^2(Q, \mu_b)\widetilde{\mathcal{C}}(\mu_b)\widetilde{\mathbb{C}}(\mu_b) \,, \tag{4.2}$$

where we only indicated the relevant scales, and can be established by considering the RGE of the hard factor H. Dropping the flavour indices and considering that $H(\mu) = H(a_s(\mu))$ with $a_s = \alpha_s/4\pi$, the chain rule gives the following differential equation:

$$\frac{d \ln H}{d \ln \mu} = \frac{d \ln a_s}{d \ln \mu} \frac{d \ln H}{d \ln a_s} = -2\beta(a_s)H^{-1}\frac{dH}{da_s} \,, \tag{4.3}$$

whose solution is:

$$H(Q) = H(\mu_0) \exp\left[-2\int_{\mu_0}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) H^{-1}(\mu) \frac{dH(\mu)}{da_s}\right].$$

(4.4)

Setting $\mu_0 = \mu_b = 2e^{-\gamma_E}/b_T$ and replacing $H(Q)$ in the l.h.s. of Eq. (4.2) with the r.h.s. of Eq. (4.4) allows us to write:

$$H(Q) R^2(Q, \mu_b) \mathcal{C}(\mu_b) \mathbb{C}(\mu_b) = R^2(Q, \mu_b) \exp\left[-2\int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) H^{-1}(\mu) \frac{dH(\mu)}{da_s}\right] [H(\mu_b)]^{1/2} \mathcal{C}(\mu_b) [H(\mu_b)]^{1/2} \mathbb{C}(\mu_b),$$

(4.5)

so that we can identify:

$$
\begin{aligned}
\widetilde{R}(Q, \mu_b) &= R(Q, \mu_b) \exp\left[-\int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) H^{-1}(\mu) \frac{dH(\mu)}{da_s}\right], \\[2mm]
\widetilde{\mathcal{C}}(\mu_b) &= [H(\mu_b)]^{1/2} \mathcal{C}(\mu_b), \\[2mm]
\widetilde{\mathbb{C}}(\mu_b) &= [H(\mu_b)]^{1/2} \mathbb{C}(\mu_b).
\end{aligned}
$$

(4.6)

Given Eq. (1.26), the identification of the Sudakov form factor is equivalent to the following change of the non-cusp anomalous dimension:

$$\widetilde{\gamma}_F(a_s(\mu)) = \gamma_F(a_s(\mu)) - \beta(a_s(\mu)) H^{-1}(\mu) \frac{dH(\mu)}{da_s}.$$

(4.7)

The relations above define a particular *resummation scheme* trasformation that leaves the cross section unchanged. In fact, there is an infinite number of possible resummation scheme trasformations. Given any perturbative function $h(a_s(\mu))$, the cross section in Eq. (2.29) is invariant under the following tranformations:

$$
\begin{aligned}
H(Q) &\rightarrow h^{-1}(Q) H(Q), \\[2mm]
\gamma_F(a_s(\mu)) &\rightarrow \gamma_F(a_s(\mu)) - \beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s}, \\[2mm]
\mathcal{C}(\mu_b) &\rightarrow [h(\mu_b)]^{1/2} \mathcal{C}(\mu_b), \\[2mm]
\mathbb{C}(\mu_b) &\rightarrow [h(\mu_b)]^{1/2} \mathbb{C}(\mu_b).
\end{aligned}
$$

(4.8)

The particular choice $h = H$ gives the transformations in Eqs. (4.6) and (4.7). These tranformations can be expanded up to the nominal order such that to N^3LL the relevant shifts of the perturbative coefficients are:

$$
\begin{aligned}
\widetilde{\mathcal{C}}_{ij}^{(0)}(x) &= \mathcal{C}_{ij}^{(0)}(x) = \delta_{ij}\delta(1-x), \\[2mm]
\widetilde{\mathcal{C}}_{ij}^{(1)}(x) &= \mathcal{C}_{ij}^{(1)}(x) + \frac{1}{2} H^{(1)} \delta_{ij}\delta(1-x), \\[2mm]
\widetilde{\mathcal{C}}^{(2)}(x) &= \mathcal{C}_{ij}^{(2)}(x) + \frac{1}{2} H^{(1)} \mathcal{C}_{ij}^{(1)}(x) + \frac{1}{2}\left[H^{(2)} - \frac{1}{4}(H^{(1)})^2\right]\delta_{ij}\delta(1-x),
\end{aligned}
$$

(4.9)

and similarly for $\widetilde{\mathbb{C}}$, while:

$$
\begin{aligned}
\widetilde{\gamma}_F^{(0)} &= \gamma_F^{(0)}, \\[2mm]
\widetilde{\gamma}_F^{(1)} &= \gamma_F^{(1)} - \beta_0 H^{(1)}, \\[2mm]
\widetilde{\gamma}_F^{(2)} &= \gamma_F^{(2)} - \beta_1 H^{(1)} - 2\beta_0\left[H^{(2)} - \frac{1}{2}(H^{(1)})^2\right].
\end{aligned}
$$

(4.10)

4.1 Non-equivalence of resummation schemes beyond accuracy

In this section, we explicitly show that a change of resummation scheme leads to differences that are beyond the nominal accuracy. To this purpose, we restrict ourselves to considering NNLL accuracy and prove that the expansions required to implement a change of scheme lead to order $\mathcal{O}(\alpha_s^2)$ mismatch terms. Such corrections are of the kind $\alpha_s^k L^{2k-4}$ and are clearly beyond NNLL accuracy.[5] The proof is simply achieved by using the equivalent of the equality in Eq. (4.4) for a generic scheme-change function h that we write as follows:

$$h(Q) = \left[\sqrt{h(\mu_b)} \right]^2 \exp \left[-2 \int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s} \right] , \tag{4.11}$$

as a consequence of the fact that each of the two $\sqrt{h(\mu_b)}$ is absorbed into one of the matching functions C. At NNLL it is sufficient to limit to $\mathcal{O}(\alpha_s)$ corrections to h:

$$h(\mu) = 1 + a_s(\mu) h^{(1)} , \tag{4.12}$$

such that:

$$\sqrt{h(\mu_b)} = 1 + \frac{1}{2} a_s(\mu_b) h^{(1)} + \mathcal{O}(\alpha_s^2) , \tag{4.13}$$

and:

$$\beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s} = a_s^2(\mu) h^{(1)} \beta_0 + \mathcal{O}(\alpha_s^3) . \tag{4.14}$$

We also need to expand the running of α_s:

$$a_s(\mu) = a_s(\mu_0) - 2 a_s^2(\mu_0) \beta_0 \ln \frac{\mu}{\mu_0} + \mathcal{O}(\alpha_s^3) , \tag{4.15}$$

that allows us to write:

$$\exp \left[-2 \int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s} \right] = 1 - 2 a_s^2(\mu_b) h^{(1)} \beta_0 \ln \frac{Q}{\mu_b} + \mathcal{O}(\alpha_s^3) . \tag{4.16}$$

so that the r.h.s. of Eq. (4.11) can be expanded as:

$$\left[\sqrt{h(\mu_b)} \right]^2 \exp \left[-2 \int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s} \right] = 1 + a_s(\mu_b) h^{(1)} + a_s^2(\mu_b) h^{(1)} \left(\frac{1}{4} h^{(1)} - 2\beta_0 \ln \frac{Q}{\mu_b} \right) + \mathcal{O}(\alpha_s^3) . \tag{4.17}$$

Now we use Eq. (4.15) to expand the l.h.s. of Eq. (4.11) as follows:

$$h(Q) = 1 + a_s(\mu_b) h^{(1)} - 2 a_s^2(\mu_0) h^{(1)} \beta_0 \ln \frac{\mu}{\mu_0} + \mathcal{O}(\alpha_s^3) . \tag{4.18}$$

Finally, taking the difference between l.h.s. and r.h.s. of Eq. (4.11) one immediately finds:

$$h(Q) - \left[\sqrt{h(\mu_b)} \right]^2 \exp \left[-2 \int_{\mu_b}^{Q} \frac{d\mu}{\mu} \beta(a_s(\mu)) h^{-1}(\mu) \frac{dh(\mu)}{da_s} \right] = - \left[\frac{1}{2} a_s(\mu_b) h^{(1)} \right]^2 + \mathcal{O}(\alpha_s^3) = \mathcal{O}(\alpha_s^2) . \tag{4.19}$$

This confirms that a resummation-scheme change at NNLL leads to non-vanishing differences that are however beyond accuracy. This is a particular case of a result that is extensible to all fixed logarithmic orders.

---

[5] Note that it may look like that this correction is a N^{4}LL one. However, for exponentiating observables the counting is best done at the level of the cumulant cross section which translates this term into a N^{3}LL correction.

Part II. Semi-inclusive deep-inelastic-scattering

5 Fixed order and asymptotic limit

In order to validate the results above, it is opportune to compare the $\mathcal{O}(a_s)$ expressions to those present in the literature. To this end, we write explicitly the expression for the SIDIS cross section differential in q_T for $q_T > 0$, *i.e.* without the $\delta(q_T^2)$ terms, and with no non-perturbative effect. Considering that, $S^{(1,1)} = 6C_F$, $S^{(1,2)} = -2C_F$, and $H_{ij}^{(0)} = e_i^2 \delta_{ij}$, this yields:

$$
\begin{aligned}
\frac{d\sigma}{dxdydzdq_T^2} \quad \propto \quad & a_s(Q) \sum_{ij,kl} \left[-B_{ij,kl}^{(1,1)}(x,z) \frac{1}{q_T^2} - B_{ij,kl}^{(1,2)}(x,z) \frac{2}{q_T^2} \ln\left(\frac{Q^2}{q_T^2}\right) \right] \underset{x}{\otimes} f_k(x,Q) \underset{z}{\otimes} d_l(z,Q) \\
= \quad & \frac{a_s(Q)}{q_T^2} \sum_{ij,kl} e_i^2 \delta_{ij} \left[4C_F \left(\ln\left(\frac{Q^2}{q_T^2}\right) - \frac{3}{2} \right) \delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \right. \\
+ \quad & \left. \mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(0)}(z) \right] \underset{x}{\otimes} f_k(x,Q) \underset{z}{\otimes} d_l(z,Q) \\
= \quad & \frac{a_s(Q)}{q_T^2} \sum_i e_i^2 \left[4C_F \left(\ln\left(\frac{Q^2}{q_T^2}\right) - \frac{3}{2} \right) f_i(x,Q)d_i(z,Q) \right. \\
+ \quad & \left. \left(\sum_k \mathcal{P}_{ik}^{(0)}(x) \underset{x}{\otimes} f_k(x,Q) \right) d_i(z,Q) + f_i(x,Q) \left(\sum_l \mathbb{P}_{il}^{(0)}(z) \underset{z}{\otimes} d_l(z,Q) \right) \right] + \mathcal{O}(a_s^2).
\end{aligned}
\tag{5.1}
$$

This result, up to pre-factors that will be made explicit below, nicely agrees with that of, *e.g.*, Refs. [5, 6, 7].

In order to check that the matching is actually removing the double counting terms, it is instructive to derive Eq. (5.1) extracting the asymptote from the fixed-order computation at $\mathcal{O}(a_s)$. We take the expressions for the coefficient functions from Eqs. (106)-(109) of Appendix B of Ref. [7] or from Eqs. (4.6)-(4.20) of Ref. [8]. Referring to the second reference, some simplifications apply. In particular, we consider cross sections with unpolarised projectiles ($\lambda_e = 0$) on unpolarised targets ($S_\perp^\mu = 0$) and integrated over the azimuthal angles ϕ_H and ϕ_S. By doing so and after a simple manipulation, the cross section simplifies greatly and can be written in terms of structure functions as:

$$
\frac{d\sigma}{dxdydzdq_T^2} = \frac{2\pi\alpha^2}{xyQ^2} \left[Y_+ F_{UU,T} + 2(1-y)F_{UU,L} \right] = \frac{2\pi\alpha^2}{xyQ^2} Y_+ \left[F_{UU,2} - \frac{y^2}{Y_+} F_{UU,L} \right],
\tag{5.2}
$$

with:

$$
Y_+ \equiv 1 + (1-y)^2,
\tag{5.3}
$$

and where we have defined the structure function:

$$
F_{UU,2} \equiv F_{UU,T} + F_{UU,L}.
\tag{5.4}
$$

Notice that, as compared to Ref. [8], we have factored out from the structure functions a factor $1/(\pi z^2)$([6]) so that they factorize as:

$$
\begin{aligned}
F_{UU,S} \quad = \quad & a_s \frac{x}{Q^2} \sum_i e_i^2 \int_x^1 \frac{d\bar{x}}{\bar{x}} \int_z^1 \frac{d\bar{z}}{\bar{z}} \delta\left(\frac{q_T^2}{Q^2} - \frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}} \right) \left[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_i\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) \right. \\
+ \quad & \left. \hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_g\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_i\left(\frac{x}{\bar{x}}\right) d_g\left(\frac{z}{\bar{z}}\right) \right] + \mathcal{O}(a_s^2).
\end{aligned}
\tag{5.5}
$$

---

[6] The factor z^2 is the consequence of the fact that we are writing the cross section differential in q_T^2 that is the transverse momentum of the exchanged photon while in Ref. [8] the cross section is differentila in p_T^2 that is the transverse momentum of the of the outgoing hadrons. Since $p_T = zq_T$, the factor z^2 cancels.

with $S = 2, L$ and where the sum over i runs over the active quark and antiquark flavours. The explicit expressions for the coefficient functions are:

$$\hat{B}_{qq}^{2,\text{FO}}(x, z, q_T) = 2C_F \left[(1-x)(1-z) + 4xz + \frac{1+x^2z^2}{xz}\frac{Q^2}{q_T^2}\right],$$

$$\hat{B}_{qq}^{L,\text{FO}}(x, z, q_T) = 8C_F xz,$$

$$\hat{B}_{qg}^{2,\text{FO}}(x, z, q_T) = 2T_R \left[[x^2 + (1-x)^2][z^2 + (1-z)^2]\frac{1-x}{xz^2}\frac{Q^2}{q_T^2} + 8x(1-x)\right],$$

$$\hat{B}_{qg}^{L,\text{FO}}(x, z, q_T) = 16T_R x(1-x).$$

$$\hat{B}_{gq}^{2,\text{FO}}(x, z, q_T) = 2C_F \left[(1-x)z + 4x(1-z) + \frac{1+x^2(1-z)^2}{xz}\frac{1-z}{z}\frac{Q^2}{q_T^2}\right],$$

$$\hat{B}_{gq}^{L,\text{FO}}(x, z, q_T) = 8C_F x(1-z),$$

These expressions are enough to compute the SIDIS cross section at $\mathcal{O}(a_s)$ in the region $q_T \lesssim Q$. In order to match Eq. (5.1), one has to take the limit $q_T/Q \to 0$ and retain in the coefficient functions only the terms enhanced as $\ln(Q^2/q_T^2)$. This automatically means that $F_{UU,L}$ does not contribute in this limit because it contains no logarithmic enhancements such that:

$$F_{UU,L} \underset{q_T/Q \to 0}{\ll} F_{UU,2}.$$ (5.7)

Another crucial observation is that the δ-function in Eq. (5.5) can be expanded as follows([7]):

$$\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-x)(1-z)}{xz}\right) \underset{q_T^2/Q^2 \to 0}{\longrightarrow} \ln\left(\frac{Q^2}{q_T^2}\right)\delta(1-x)\delta(1-z) + \frac{x\delta(1-z)}{(1-x)_+} + \frac{z\delta(1-x)}{(1-z)_+},$$ (5.8)

so that:

$$F_{UU,2} \underset{q_T/Q \to 0}{\longrightarrow} a_s \frac{x}{q_T^2}\sum_i e_i^2 \int_x^1 \frac{d\bar{x}}{\bar{x}}\int_z^1 \frac{d\bar{z}}{\bar{z}}\left[\hat{B}_{qq}^{2,\text{asy}}(\bar{x}, \bar{z}, q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\right.$$

$$+ \left.\hat{B}_{qg}^{2,\text{asy}}(\bar{x}, \bar{z}, q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}_{gq}^{2,\text{asy}}(\bar{x}, \bar{z}, q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\right] + \mathcal{O}(a_s^2).$$ (5.9)

with:

$$\hat{B}_{qq}^{2,\text{asy}}(x, z, q_T) = 2C_F \left[2\ln\left(\frac{Q^2}{q_T^2}\right) + \frac{1+x^2}{(1-x)_+}\delta(1-z) + \delta(1-x)\frac{1+z^2}{(1-z)_+}\right]$$

$$= 2C_F \left[2\ln\left(\frac{Q^2}{q_T^2}\right) - 3\right]\delta(1-x)\delta(1-z) + \mathcal{P}_{qq}^{(0)}(x)\delta(1-z) + \delta(1-x)\mathbb{P}_{qq}^{(0)}(z),$$ (5.10)

$$\hat{B}_{qg}^{2,\text{asy}}(x, z, q_T) = 2T_R \left[x^2 + (1-x)^2\right]\delta(1-z) = \mathcal{P}_{qg}^{(0)}(x)\delta(1-z),$$

$$\hat{B}_{gq}^{2,\text{asy}}(x, z, q_T) = \delta(1-x)2C_F \left[\frac{1+(1-z)^2}{z}\right] = \delta(1-x)\mathbb{P}_{qg}^{(0)}(z).$$

It is thus easy to see that we can rewrite Eq. (5.9) as:

$$F_{UU,2} \underset{q_T/Q \to 0}{\longrightarrow} a_s \frac{x}{q_T^2}\sum_i e_i^2 \left[4C_F \left(\ln\left(\frac{Q^2}{q_T^2}\right) - \frac{3}{2}\right)f_i(x)d_i(z)\right.$$

$$+ \left.\left(\sum_{k=q,g}\mathcal{P}_{ik}^{(0)}(x)\otimes f_k(x)\right)d_i(z) + f_i(x)\left(\sum_{k=q,g}\mathbb{P}_{ik}^{(0)}(z)\otimes d_k(z)\right)\right] + \mathcal{O}(a_s^2).$$ (5.11)

---

[7] The proof of this relation is given in Appendix A.

Therefore, up to factors, Eq. (5.11) agrees with Eq. (5.1). This confirms that the expansion of the resummed calculation, as well as the asymptotic limit of the fixed order, removes the double-counting terms when doing the matching.

In order to provide a version of Eq. (5.5) that can be readily implemented, we need to perform one of the integrals making use of the δ-function. We integrate over $\bar{x}$ so that we write:

$$\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) = \frac{\bar{z}\bar{x}_0^2}{1-\bar{z}}\delta(\bar{x} - \bar{x}_0)\,, \tag{5.12}$$

with:

$$\bar{x}_0 = \frac{1-\bar{z}}{1-\bar{z}\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.13}$$

This allows us to write:

$$
\begin{aligned}
F_{UU,S} &= a_s \frac{x}{Q^2} \sum_i e_i^2 \int_z^{z_{\max}} \frac{d\bar{z}}{1-\bar{z}} \bar{x}_0 \left[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T) f_i\left(\frac{x}{\bar{x}_0}\right) d_i\left(\frac{z}{\bar{z}}\right) \right. \\
&+ \left. \hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T) f_g\left(\frac{x}{\bar{x}_0}\right) d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T) f_i\left(\frac{x}{\bar{x}_0}\right) d_g\left(\frac{z}{\bar{z}}\right) \right] + \mathcal{O}(a_s^2)\,,
\end{aligned}
\tag{5.14}
$$

with:

$$z_{\max} = \frac{1-x}{1-x\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.15}$$

Now we can rewrite the cross section above in such a way that it matches that at $\mathcal{O}(a_s)$ of Ref. [9]. That would allow us to confidently use the $\mathcal{O}(a_s^2)$ calculation presented in that reference for the matching to the resummed calculation. This is made tricky by the different notation used in Ref. [9] and from the fact that in that paper the cross section is differential in a different set of variables. Specifically, we would like it to be differential in x, y, z, and q_T^2 while in Ref. [9] it is differential in x, Q^2, η, and p_T^2, where the last two are the rapidity and the transverse momentum of the outgoing hadron, respectively. Eq. (13) of Ref. [9], can be translated into our notation by noticing that:

$$\frac{d\sigma}{dx dQ^2 dp_T^2 d\eta} = \frac{x}{zQ^2} \sum_{i,j} \int_z^{z_{\max}} \frac{d\bar{z}}{1-\bar{z}} f_i\left(\frac{x}{\bar{x}_0}\right) d_j\left(\frac{z}{\bar{z}}\right) \frac{d\sigma_{ij}^{(1)}}{dx dQ^2 d\bar{z} d\eta} + \mathcal{O}(a_s^2)\,, \tag{5.16}$$

where we have exploited the δ-functions in Eqs. (18)-(20) to get rid of the integral over z.([8])

The $\mathcal{O}(a_s)$ partonic cross sections in Eqs. (18)-(20) of Ref. [9], setting $\varepsilon = 0$, can be written as:

$$\frac{d\sigma_{ij}^{(1)}}{dx dQ^2 d\bar{z} d\eta} = \frac{2\pi\alpha^2 a_s e_q^2 \bar{x}_0}{xQ^4} Y_+ \left[\underbrace{\left(F_{UU,M}^{ij}(\bar{x}_0, \bar{z}) + \frac{3}{2}F_{UU,L}^{ij}(\bar{x}_0, \bar{z})\right)}_{F_{UU,2}^{ij}} - \frac{y^2}{Y_+} F_{UU,L}^{ij}(\bar{x}_0, \bar{z}) \right]\,. \tag{5.17}$$

One can verify that $F_{UU,2}^{qq}(\bar{x}_0, \bar{z})$, $F_{UU,L}^{qq}(\bar{x}_0, \bar{z})$, $F_{UU,M}^{qg}(\bar{x}_0, \bar{z})$, $F_{UU,L}^{qg}(\bar{x}_0, \bar{z})$, $F_{UU,M}^{gq}(\bar{x}_0, \bar{z})$, and $F_{UU,L}^{gq}(\bar{x}_0, \bar{z})$ correspond exactly to $\hat{B}_{qq}^{2,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$, $\hat{B}_{qq}^{L,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$, $\hat{B}_{qg}^{M,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$, $\hat{B}_{qg}^{L,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$, $\hat{B}_{gq}^{M,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$, and $\hat{B}_{gq}^{L,\mathrm{FO}}(\bar{x}_0, \bar{z}, q_T)$ in Eq. (5.6) of Ref. [8]. It is crucial to notice that the correspondence holds only if $\bar{x}_0$ defined in Eq. (5.13) as a function of $\bar{z}$ is used. As an example, taking into account the different factors (a factor 2 for $F_{UU,M}$ and a factor 4 for $F_{UU,L}$) and using our notation for the integration variables ($y \to \bar{z}$, $\rho(z=0) \to \bar{x}_0$), we read off from Eq. (18) of Ref. [9]:

$$
\begin{aligned}
F_{UU,M}^{qq}(\bar{x}_0, \bar{z}) &= 2C_F \left[\frac{(\bar{x}_0 + \bar{z})^2 + 2(1 - \bar{x}_0 - \bar{z})}{(1 - \bar{x}_0)(1 - \bar{z})} \right]\,, \\
F_{UU,L}^{qq}(\bar{x}_0, \bar{z}) &= 8C_F \bar{x}_0 \bar{z}\,,
\end{aligned}
\tag{5.18}
$$

---

[8] Notice that the z variable of Ref. [9] does not coincide with our definition.

so that:

$$
\begin{aligned}
F_{UU,2}^{qq}(\bar{x}_0, \bar{z}) &= F_{UU,M}^{qq}(\bar{x}_0, \bar{z}) + \frac{3}{2} F_{UU,L}^{qq}(\bar{x}_0, \bar{z}) \\[2mm]
&= 2C_F \left[\frac{(\bar{x}_0 + \bar{z})^2 + 2(1 - \bar{x}_0 - \bar{z})}{(1 - \bar{x}_0)(1 - \bar{z})} + 3\bar{x}_0 \bar{z} \right] \\[2mm]
&= 2C_F \left[(1 - \bar{x}_0)(1 - \bar{z}) + 4\bar{x}_0 \bar{z} + \frac{1 + \bar{x}_0^2 \bar{z}^2}{\bar{x}_0 \bar{z}} \left(\frac{\bar{x}_0 \bar{z}}{(1 - \bar{x}_0)(1 - \bar{z})} \right) \right] .
\end{aligned}
\tag{5.19}
$$

Using Eq. (5.13), it is easy to see that the factor in round brackets in the last line of the equation above is equal to Q^2/q_T^2. Therefore, it reduces exactly to the first relation in Eq. (5.6) of Ref. [8]. The same holds also for the two remaining partonic channels.

Putting all pieces together, Eq. (5.16) can be recast as:

$$
\frac{d\sigma}{dx dQ^2 dp_T^2 d\eta} = \frac{2\pi\alpha^2}{zxQ^4} Y_+ \left[F_{UU,2} - \frac{y^2}{Y_+} F_{UU,L} \right] ,
\tag{5.20}
$$

with:

$$
\begin{aligned}
F_{UU,S} &= a_s \frac{x}{Q^2} \sum_i e_i^2 \int_z^{z_{\max}} \frac{d\bar{z}}{1 - \bar{z}} \bar{x}_0 \left[F_{UU,L}^{qq}(\bar{x}_0, \bar{z}) f_i \left(\frac{x}{\bar{x}_0} \right) d_i \left(\frac{z}{\bar{z}} \right) \right. \\[2mm]
&\left. + F_{UU,L}^{qg}(\bar{x}_0, \bar{z}) f_g \left(\frac{x}{\bar{x}_0} \right) d_i \left(\frac{z}{\bar{z}} \right) + F_{UU,L}^{gq}(\bar{x}_0, \bar{z}) f_i \left(\frac{x}{\bar{x}_0} \right) d_g \left(\frac{z}{\bar{z}} \right) \right] + \mathcal{O}(a_s^2) .
\end{aligned}
\tag{5.21}
$$

Therefore, the structure of the observables is exactly the same. What is left to work out is the Jacobian to express the cross section differential in the same variables as in Eq. (5.2). What we need is to know is how the variables Q^2, p_T, and η are related to y, z, and q_T. The relevant relations are:

$$
\begin{cases}
Q^2 &= xyS_H \\[2mm]
p_T^2 &= z^2 q_T^2 \\[2mm]
\eta &= \frac{1}{2} \ln \left(\frac{y(1-x)S_H}{q_T^2} \right)
\end{cases}
\implies \quad dQ^2 dp_T^2 d\eta = \frac{zQ^2}{y} dy dz dq_T^2 .
\tag{5.22}
$$

where S_H is the squared center-of-mass energy of the collision, so that:

$$
\frac{d\sigma}{dx dy dz dq_T^2} = \frac{zQ^2}{y} \frac{d\sigma}{dx dQ^2 dp_T^2 d\eta} = \frac{2\pi\alpha^2}{xyQ^2} Y_+ \left[F_{UU,2} - \frac{y^2}{Y_+} F_{UU,L} \right] ,
\tag{5.23}
$$

exactly like in Eq. (5.2). This confirms that the computation of Ref. [9] can be matched to the resummed calculation provided that the correct Jacobian is taken into account.

In Fig. 5.1, a comparison between the $\mathcal{O}(\alpha_s)$ computation of Ref. [9] for the differential cross section in Eq. (5.16), implemented in the computer code TIMBA, and the implementation in the APFEL++ framework [10] of the expressions in Eq. (5.14) is presented. The cross section as a function of the hadron transverse momentum p_T obtained with APFEL++ is displayed as ratios to TIMBA for representative values of the kinematic variables: the agreement between the two codes is excellent. Since we will be using TIMBA for the NLO (i.e. $\mathcal{O}(a_s^2)$) cross section calculation to be matched to the NNLL resummed computation, this provides a solid ground to start from.

In order to be able to compare the theoretic predictions to the data, it is useful to work out the for of the cross section differential in the variables x, Q^2, z, and p_T^2. This is easily done starting from Eq. (5.23) and considering that:

$$
\eta = \ln \left(\frac{zW}{p_T} \right) , \quad \text{with} \quad W = \sqrt{\frac{Q^2(1-x)}{x}} , \quad \Rightarrow \quad d\eta = \frac{dz}{z} .
\tag{5.24}
$$

It follows that:

$$
\frac{d\sigma}{dx dQ^2 dz dp_T^2} = \frac{2\pi\alpha^2}{xQ^4} Y_+ \left[F_{UU,2} - \frac{y^2}{Y_+} F_{UU,L} \right] ,
\tag{5.25}
$$

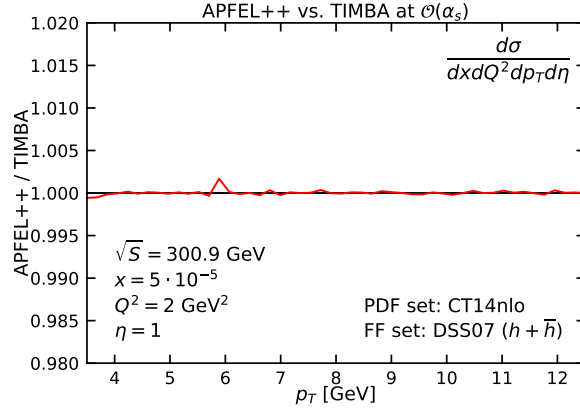with $y = Q^2/xS_H$ (see Eq. (5.22)).

Fig. 5.1: Comparison at $\mathcal{O}(\alpha_s)$ between the TIMBA code, implementation of the results of Ref. [9], and the implementation of the expressions in Eq. (5.14) in the APFEL++ code [10].

6 Integrating over q_T

It is interesting to carry out the integration over q_T of Eq. (5.5) analytically. The result can eventually be compared to those presented in Appendix C of Ref. [11]. Exploiting the δ-function yields:

$$
\begin{aligned}
\int_0^\infty dq_T^2\, F_{UU,S} \;=\; & a_s x \sum_i e_i^2 \int_x^1 \frac{d\bar{x}}{\bar{x}} \int_z^1 \frac{d\bar{z}}{\bar{z}} \left[\hat{B}_{qq}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_i\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) \right. \\[2mm]
& +\; \hat{B}_{qg}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_g\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) \\[2mm]
& +\; \left. \hat{B}_{gq}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_i\left(\frac{x}{\bar{x}}\right) d_g\left(\frac{z}{\bar{z}}\right) \right] + \mathcal{O}(a_s^2).
\end{aligned}
\tag{6.1}
$$

replacing q_T^2/Q^2 in Eq. (5.6) has no effect on the coefficient functions for $F_{UU,L}$. The reason is that $F_{UU,L}$ is independent of q_T and therefore integrating over q_T has essentially the effect of removing the δ-function. On the other hand, integrating $F_{UU,2}$ gives:

$$
\hat{B}_{qq}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) = 2C_F \left[(1-x)(1-z) + 4xz + \frac{1+x^2 z^2}{(1-x)(1-z)}\right],
$$

$$
\hat{B}_{qg}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) = 2T_R \left[[x^2 + (1-x)^2]\left(\frac{1}{1-z} + \frac{1}{z} - 2\right) + 8x(1-x)\right],
\tag{6.2}
$$

$$
\hat{B}_{gq}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) = 2C_F \left[(1-x)z + 4x(1-z) + \frac{1+x^2(1-z)^2}{z(1-x)}\right],
$$

This expressions only account for real emissions and as a consequence they are affected by soft divergences for $x \to 1$ and $z \to 1$. These divergences must eventually cancel against the virtual contributions. The inclusion of these virtual contributions has as a net effect the appearance of the plus prescription in all terms that behave as $(1-x)^{-1}$ and $(1-z)^{-1}$ and of terms proportional to $\delta(1-x)$ and/or $\delta(1-z)$ (local terms). As a matter of fact, one can see that up to the local terms and the plus prescription the expressions of Appendix C of Ref. [11] are reproduced([9]). The local terms correspond to virtual corrections that thus give contributions at $q_T = 0$. As such, they should coincide with the terms proportional to $\delta(q_T^2)$ generated in the expansion of the resummed calculation. Specifically, they should be equal to the coefficients $B_{ij,kl}^{(n,0)}$ in the expansion in Eq. (2.29). This is evidently the case for $B_{ij,kl}^{(0,0)}$ that gives the leading-order contribution to the integrated cross section. Unfortunately, this is not true at next-to-leading order because the expressions for the local terms in

---

[9] The expressions reported in Ref. [11] are those computed long time ago in Ref. [12].

Appendix C of Ref. [11] (Eqs. (C.2)-(C.4)) do not match $B_{ij,kl}^{(1,0)}$. To be more precise by first writing down the explicit expression for $B_{ij,kl}^{(1,0)}$ summed over the indices of the hard function:

$$
\begin{aligned}
\overline{C}_{kl}(x,z) &= e_q^{-2} \sum_{ij} B_{ij,kl}^{(1,0)}(x,z) = H^{(1)} \delta_{kl} \delta(1-x)\delta(1-z) + \delta(1-x)\mathbb{C}_{kl}^{(1,0)}(z) + \mathcal{C}_{lk}^{(1,0)}(x)\delta(1-z)\,, \\
&= 2C_F \left(-8 + \frac{\pi^2}{6}\right) \delta_{kl}\delta(1-x)\delta(1-z) + \delta(1-x)\mathbb{C}_{kl}^{(1,0)}(z) + \mathcal{C}_{lk}^{(1,0)}(x)\delta(1-z)\,.
\end{aligned}
\tag{6.3}
$$

The single channels read:

$$
\begin{aligned}
\overline{C}_{qq}(x,z) &= 2C_F \left[-8\delta(1-x)\delta(1-z) + \delta(1-x)\left(1 - z + 2\frac{1+z^2}{1-z}\ln z\right) + (1-x)\,\delta(1-z) \right]\,, \\
\overline{C}_{gq}(x,z) &= 2C_F \left[\delta(1-x)\left(z + 2\frac{1+(1-z)^2}{z}\ln z\right) + x\delta(1-z) \right]\,, \\
\overline{C}_{qg}(x,z) &= 4T_R \left[\delta(1-x)\left(z(1-z) + (z^2 + (1-z)^2)\ln z\right) + x(1-x)\delta(1-z) \right]\,.
\end{aligned}
\tag{6.4}
$$

7   Integrating over z

In order to understand which of the two is the correct set of expressions, whether those computed in Ref. [12] or those derived here, we may try to integrate over z summing over all possible hadrons produced in the final state. By doing so, we should obtain the expressions of the hard cross sections for inclusive deep-inelastic scattering. This last set of expressions has also been computed in Ref. [12] and is very likely to be correct. The presence of terms proportional to $\ln(1-x)$ in the inclusive DIS hard cross sections suggests that the expressions reported in Ref. [11] and computed in Ref. [12] are the correct ones. To integrate the expressions in Ref. [11] we need to use the fact that the SIDIS cross section integrate in q_T for the production of a hadron h has the following structure:

$$
\frac{d\sigma^h}{dxdydz} = \sum_{ij} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} C_{ij}\left(\frac{x}{\xi}, \frac{z}{\zeta}\right) d_{i/h}(\zeta) f_j(\xi)\,,
\tag{7.1}
$$

where $d_{i/h}$ is the fragmentation function for the parton i fragmenting into the hadron species h. What we need to compute is:

$$
\frac{d\sigma}{dxdy} = \sum_h \int_0^1 dz\, z\, \frac{d\sigma^h}{dxdydz}\,,
\tag{7.2}
$$

where the sum over h extends over all possible hadronic species. In order to compute the integral above we need to use the following property of the Mellin convolution:

$$
C(z) \otimes d(z) = \int_z^1 \frac{d\zeta}{\zeta} C\left(\frac{z}{\zeta}\right) d(\zeta) = \int_0^1 d\zeta \int_0^1 d\eta\, C(\zeta) f(\eta) \delta(z - \zeta\eta)\,.
\tag{7.3}
$$

By using this property we have:

$$
\begin{aligned}
\frac{d\sigma}{dxdy} &= \sum_{ij} \int_x^1 \frac{d\xi}{\xi} \int_0^1 d\zeta\, C_{ij}\left(\frac{x}{\xi}, \zeta\right) f_j(\xi) \sum_h \int_0^1 d\eta\, d_{i/h}(\eta) \int_0^1 dz\, z\delta(z - \zeta\eta) \\
&= \sum_j \int_x^1 \frac{d\xi}{\xi} \left[\int_0^1 d\zeta\, \zeta \sum_i C_{ij}\left(\frac{x}{\xi}, \zeta\right) \right] f_j(\xi) = \sum_j \int_x^1 \frac{d\xi}{\xi} \widetilde{C}_j\left(\frac{x}{\xi}\right) f_j(\xi)\,.
\end{aligned}
\tag{7.4}
$$

where we have defined the inclusive DIS coefficient functions as:

$$
\widetilde{C}_j(\xi) \equiv \int_0^1 d\zeta\, \zeta \sum_i C_{ij}(\xi, \zeta)\,,
\tag{7.5}
$$

and we have used the momentum sum rule:

$$
\sum_h \int_0^1 d\eta\, \eta d_{i/h}(\eta) = 1\,,
\tag{7.6}
$$

that essentially encodes the fact that the total probability for a parton to fragment into any hadron in one. The correctness of Eq. (7.5) can be verified explicitly by integrating over z the expressions reported in Appendix C of Ref. [11] for both structure functions F_2 and F_L. The leading order is trivial:

$$\widetilde{C}_{2,q}^{(0)}(\xi) \equiv \int_0^1 d\zeta\, \zeta \delta(1-\xi)\delta(1-\zeta) = \delta(1-\xi)\,, \tag{7.7}$$

as well know, while the gluon as well as the longitudinal coefficient functions are zero. The next-to-leading order coefficient functions are computed as:

$$\widetilde{C}_q^{(1)}(\xi) \equiv \int_0^1 d\zeta\, \zeta \left[C_{qq}^{(1)}(\xi,\zeta) + C_{gq}^{(1)}(\xi,\zeta) \right]\,,$$

$$\widetilde{C}_g^{(1)}(\xi) \equiv \int_0^1 d\zeta\, \zeta C_{qg}^{(1)}(\xi,\zeta)\,. \tag{7.8}$$

For F_L they read:

$$\widetilde{C}_{L,q}^{(1)}(\xi) = 8C_F\xi\,,$$

$$\widetilde{C}_{L,g}^{(1)}(\xi) = 8T_R\xi(1-\xi)\,, \tag{7.9}$$

while for F_2:

$$\begin{aligned}
\widetilde{C}_{2,q}^{(1)}(\xi) &= 2C_F\left[2\left(\frac{\ln(1-\xi)}{1-\xi}\right)_+ - \frac{3}{2}\left(\frac{1}{1-\xi}\right)_+ - (1+\xi)\ln(1-\xi) \right. \\
&\quad \left. - \frac{1+\xi^2}{1-\xi}\ln\xi + 3 + 2\xi - \left(\frac{9}{2} + 2\zeta_2\right)\delta(1-\xi) \right]\,, \\
\widetilde{C}_{2,g}^{(1)}(\xi) &= 2T_R\left[(\xi^2 + (1-\xi)^2)\ln\left(\frac{1-\xi}{\xi}\right) + 8\xi(1-\xi) - 1 \right]\,.
\end{aligned} \tag{7.10}$$

These expressions coincide with those reported, *e.g.*, in Chapter 4 of Ref. [13].

Part III. Drell-Yan production

Part IV. Appendices

A   Expansion of the kinematic δ-function

In order to prove the equality in Eq. (5.8), we consider the following integral:

$$I(\varepsilon) = \int_0^1 dx \int_0^1 dy \, \delta(xy - \varepsilon) f(x, y) \,, \tag{A.1}$$

where $f(x, y)$ is a test function "well-behaved" over the integration region. Now I split the integral as follows:

$$I(\varepsilon) = \left(\int_0^1 dx \int_x^1 dy + \int_0^1 dy \int_y^1 dx \right) \delta(xy - \varepsilon) f(x, y) \,, \tag{A.2}$$

where the first term in the r.h.s. corresponds to the integral over the grey region *above* the blue line while the second over the grey region *below* the blue line in Fig. A.1.
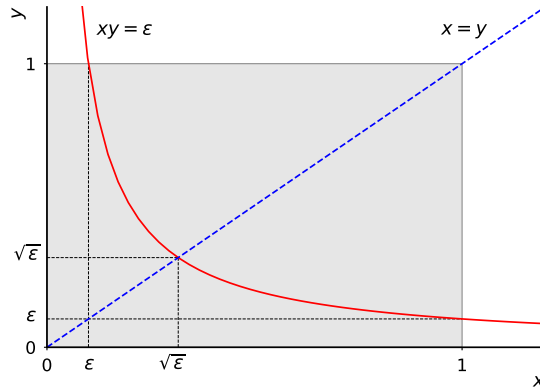

Fig. A.1: Integration region of the integral in Eq. (A.1). The integral is along the red curve defined by the δ-function.

Now I use the following equalities:

$$\delta(xy - \varepsilon) = \begin{cases} \dfrac{1}{x} \delta \left(y - \dfrac{\varepsilon}{x} \right) \theta(y - \sqrt{\varepsilon}) & \text{integral over } y, \\[3mm] \dfrac{1}{y} \delta \left(x - \dfrac{\varepsilon}{y} \right) \theta(x - \sqrt{\varepsilon}) & \text{integral over } x. \end{cases} \tag{A.3}$$

The θ-functions arise from the fact that the first integral has to be done along the upper branch on the red curve while the second along the lower branch. The two branches are joint at the point $x = y = \sqrt{\varepsilon}$ and thus the integration ranges are bounded from below by this point. Therefore, I find:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} f \left(x, \frac{\varepsilon}{x} \right) + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} f \left(\frac{\varepsilon}{y}, y \right) \,. \tag{A.4}$$

It is crucial to realise that in the first and the second integral the following conditions hold: $\varepsilon < \sqrt{\varepsilon} \leq x$ and $\varepsilon < \sqrt{\varepsilon} \leq y$, respectively. Therefore, in the limit $\varepsilon \to 0$, the arguments ε/x and ε/y of the function f will tend to zero. I now add and subtract a term proportional to $f(0, 0)$ to both integrals, so that:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} \left[f \left(x, \frac{\varepsilon}{x} \right) - f(0, 0) \right] + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} \left[f \left(\frac{\varepsilon}{y}, y \right) - f(0, 0) \right] + 2f(0, 0) \underbrace{\int_{\sqrt{\varepsilon}}^1 \frac{d\xi}{\xi}}_{-\ln \sqrt{\varepsilon}} \,. \tag{A.5}$$

Finally, I take the limit for $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} I(\varepsilon) = \int_0^1 \frac{dx}{x} \left[f(x, 0) - f(0, 0) \right] + \int_0^1 \frac{dy}{y} \left[f(0, y) - f(0, 0) \right] - \ln(\varepsilon) f(0, 0) \,, \tag{A.6}$$

that I rewrite as:

$$\lim_{\varepsilon \to 0} I(\varepsilon) = \int_0^1 dx \int_0^1 dy \left\{ \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y) \right\} f(x,y) \,. \tag{A.7}$$

Comparing the equation above with Eq. (A.1), one deduces that:

$$\delta(xy - \varepsilon) \xrightarrow[\varepsilon \to 0]{} \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y) \,. \tag{A.8}$$

Finally, substituting:

$$x \to \frac{1-x}{x}\,, \quad y \to \frac{1-z}{z}\,, \quad \text{and} \quad \varepsilon \to \frac{q_T^2}{Q^2}\,, \tag{A.9}$$

it is easy to recover Eq. (5.8). In particular, one needs to use the fact that:

$$\delta\left(\frac{1-x}{x}\right) = \delta(1-x) \,. \tag{A.10}$$

A.1   Including the next corrections in ε

The limit in Eq. (A.8) can be written as:

$$\delta(xy - \varepsilon) = \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y) + \mathcal{O}(\varepsilon \ln \varepsilon) \,. \tag{A.11}$$

The aim of this section is to calculate the first contributions to the $\mathcal{O}(\varepsilon \ln \varepsilon)$ term: specifically we are after terms proportional to $\varepsilon \ln \varepsilon$ and ε. We start by considering the integral in Eq. (A.1) and write:

$$
\begin{aligned}
J(\varepsilon) = I(\varepsilon) - \lim_{\varepsilon \to 0} I(\varepsilon) \;=\; & \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} \left[f\left(x, \frac{\varepsilon}{x}\right) - f(x,0) \right] + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} \left[f\left(\frac{\varepsilon}{y}, y\right) - f(0,y) \right] \\
& - \int_0^{\sqrt{\varepsilon}} \frac{dx}{x} \left[f(x,0) - f(0,0) \right] - \int_0^{\sqrt{\varepsilon}} \frac{dy}{y} \left[f(0,y) - f(0,0) \right] \,.
\end{aligned}
\tag{A.12}
$$

We now use the following Taylor expansions([10]):

$$
\begin{aligned}
f(x,0) &= f(0,0) + \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} x + \frac{1}{2} \left.\frac{\partial^2 f(x',0)}{\partial x'^2}\right|_{x'=0} x^2 + \mathcal{O}(x^3)\,, \\
f(0,y) &= f(0,0) + \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} y + \frac{1}{2} \left.\frac{\partial^2 f(0,y')}{\partial y'^2}\right|_{y'=0} y^2 + \mathcal{O}(y^3)\,, \\
f\left(x,\frac{\varepsilon}{x}\right) &= f(x,0) + \left.\frac{\partial f(x,y')}{\partial y'}\right|_{y'=0} \frac{\varepsilon}{x} + \frac{1}{2} \left.\frac{\partial^2 f(x,y')}{\partial y'^2}\right|_{y'=0} \frac{\varepsilon^2}{x^2} + \mathcal{O}\left(\frac{\varepsilon^3}{x^3}\right)\,, \\
f\left(\frac{\varepsilon}{y},y\right) &= f(0,y) + \left.\frac{\partial f(x',y)}{\partial x'}\right|_{x'=0} \frac{\varepsilon}{y} + \frac{1}{2} \left.\frac{\partial^2 f(x',y)}{\partial x'^2}\right|_{x'=0} \frac{\varepsilon^2}{y^2} + \mathcal{O}\left(\frac{\varepsilon^3}{y^3}\right)\,,
\end{aligned}
\tag{A.13}
$$

---

[10] Notice that the expansions in the third and fourth lines in Eq. (A.13) are allowed because the variables x and y are bounded as $1 \leq x, y \leq \sqrt{\varepsilon}$, therefore the ratios ε/x and ε/y tend to zero as ε tends to zero over the full integration range.

so that, neglecting anything more suppressed than ε, we find:

$$
\begin{aligned}
J(\varepsilon) \;=\;& \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^2} \left.\frac{\partial f(x,y')}{\partial y'}\right|_{y'=0} + \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^3} \left.\frac{\partial^2 f(x,y')}{\partial y'^2}\right|_{y'=0} + \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^2} \left.\frac{\partial f(x',y)}{\partial x'}\right|_{x'=0} + \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^3} \left.\frac{\partial^2 f(x',y)}{\partial x'^2}\right|_{x'=0} \\[4pt]
& -\; \sqrt{\varepsilon}\, \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} - \frac{\varepsilon}{4} \left.\frac{\partial^2 f(x',0)}{\partial x'^2}\right|_{x'=0} - \sqrt{\varepsilon}\, \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} - \frac{\varepsilon}{4} \left.\frac{\partial^2 f(0,y')}{\partial y'^2}\right|_{y'=0} \\[8pt]
\;=\;& \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^2} \left[\left.\frac{\partial f(x,y')}{\partial y'}\right|_{y'=0} - \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} \right] + \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^2} \left[\left.\frac{\partial f(x',y)}{\partial x'}\right|_{x'=0} - \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} \right] \\[4pt]
& +\; \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^3} \left[\left.\frac{\partial^2 f(x,y')}{\partial y'^2}\right|_{y'=0} - \left.\frac{\partial^2 f(0,y')}{\partial y'^2}\right|_{y'=0} \right] + \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^3} \left[\left.\frac{\partial^2 f(x',y)}{\partial x'^2}\right|_{x'=0} - \left.\frac{\partial^2 f(x',0)}{\partial x'^2}\right|_{x'=0} \right] \\[4pt]
& +\; \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^2} \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} + \varepsilon \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^2} \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} - \sqrt{\varepsilon}\, \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} - \sqrt{\varepsilon}\, \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} \\[4pt]
& +\; \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^3} \left.\frac{\partial^2 f(0,y')}{\partial y'^2}\right|_{y'=0} + \frac{\varepsilon^2}{2} \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^3} \left.\frac{\partial^2 f(x',0)}{\partial x'^2}\right|_{x'=0} - \frac{\varepsilon}{4} \left.\frac{\partial^2 f(x',0)}{\partial x'^2}\right|_{x'=0} - \frac{\varepsilon}{4} \left.\frac{\partial^2 f(0,y')}{\partial y'^2}\right|_{y'=0} \\[8pt]
\;=\;& \varepsilon \frac{\partial}{\partial y'} \int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^2} \left[f(x,y') - f(0,y') \right]\bigg|_{y'=0} + \varepsilon \frac{\partial}{\partial x'} \int_{\sqrt{\varepsilon}}^{1} \frac{dy}{y^2} \left[f(x',y) - f(x',0) \right]\bigg|_{x'=0} \\[4pt]
& -\; \varepsilon \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} - \varepsilon \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} \,,
\end{aligned}
$$
$$\tag{A.14}$$

where, remarkably, the terms proportional to $\sqrt{\varepsilon}$ cancel out as well as those proportional to ε but involving a second derivative. The integrals in the next-to-last line are potentially logarithmically divergent as $\varepsilon \to 0$ and we wish to isolate this divergence as well as the next-to-leading term that is a constant in ε. Of course, the logarithmic divergence is tamed by the overall factor ε therefore the full J remains power suppressed. To isolate these terms we simply expand the integrand around zero of the integration variable so that for example:

$$
\int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x^2} \left[f(x,y') - f(0,y') \right] = -\frac{1}{2} \left.\frac{\partial f(x',y')}{\partial x'}\right|_{x'=0} \ln \varepsilon + \frac{1}{2} \left.\frac{\partial^2 f(x',y')}{\partial x'^2}\right|_{x'=0} \,,
\tag{A.15}
$$

where we neglected the $\mathcal{O}(\sqrt{\varepsilon})$ terms. We finally have:

$$
\begin{aligned}
J(\varepsilon) \;=\;& -\varepsilon \ln \varepsilon \left.\frac{\partial^2 f(x',y')}{\partial x'\partial y'}\right|_{x'=y'=0} + \frac{\varepsilon}{2} \left.\frac{\partial^3 f(x',y')}{\partial x'^2\partial y'}\right|_{x'=y'=0} + \frac{\varepsilon}{2} \left.\frac{\partial^3 f(x',y')}{\partial x'\partial y'^2}\right|_{x'=y'=0} \\[4pt]
& -\; \varepsilon \left.\frac{\partial f(0,y')}{\partial y'}\right|_{y'=0} - \varepsilon \left.\frac{\partial f(x',0)}{\partial x'}\right|_{x'=0} \,,
\end{aligned}
\tag{A.16}
$$

which isolates the terms proportional to $\varepsilon \ln \varepsilon$ and ε as required. Now, for the sake of simplicity, we focus on the leading term $\varepsilon \ln \varepsilon$[11] and use the following identity:

$$
\int_{0}^{1} dy \left[\frac{\delta(y)}{y} \right]_{+} g(y) = \left.\frac{dg(y')}{dy'}\right|_{y'=0} \,,
\tag{A.17}
$$

where one should interpret:

$$
\left[\frac{\delta(y)}{y} \right]_{+} \equiv \lim_{\eta \to 0^+} \left[\frac{\delta(y-\eta)}{y} \right]_{+} \,.
\tag{A.18}
$$

This finally gives:

$$
J(\varepsilon) \;=\; -\varepsilon \ln \varepsilon \int_{0}^{1} dx \int_{0}^{1} dy \left[\frac{\delta(x)}{x} \right]_{+} \left[\frac{\delta(y)}{y} \right]_{+} f(x,y) + \mathcal{O}(\varepsilon) \,,
\tag{A.19}
$$

---

[11] In fact, also considering $\mathcal{O}(\varepsilon)$ terms would imply also considering the contribution to F_L to the asymptotic cross section.

which allows us to write:

$$\delta(xy - \varepsilon) = -\ln \varepsilon \, \delta(x)\delta(y) + \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \varepsilon \ln \varepsilon \left[\frac{\delta(x)}{x}\right]_+ \left[\frac{\delta(y)}{y}\right]_+ + \mathcal{O}(\varepsilon) \, . \tag{A.20}$$

We have thus isolated in this way the next leading term to the expansion in q_T of the kinematic δ-function in Eq. (5.8). More precisely, using the replacements in Eq. (A.9), one finds:

$$
\begin{aligned}
\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-x)(1-z)}{xz}\right) &= \ln\left(\frac{Q^2}{q_T^2}\right)\delta(1-x)\delta(1-z) + \frac{x\delta(1-z)}{(1-x)_+} + \frac{z\delta(1-x)}{(1-z)_+} \\
&+ \frac{q_T^2}{Q^2}\ln\left(\frac{Q^2}{q_T^2}\right) xz \left[\frac{\delta(1-x)}{1-x}\right]_+ \left[\frac{\delta(1-z)}{1-z}\right]_+ + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right) \, .
\end{aligned}
\tag{A.21}
$$

B Solution of an RGE in perturbation theory

In this section we address the question of how a renormalisation group equation (RGE) can be solved within the framework of perturbation theory. The relevance of this issue concernes the estimation of theoretical uncertainties stemming from higher-order corrections. More specifically, we will show that the so-called *resummation scale* originates from a perturbative solution of the TMD evolution equations.

In order to define the problem, we consider a generic RGE in QCD:

$$\frac{d\ln R}{d\ln \mu} = \gamma(\alpha_s(\mu))\,, \tag{B.1}$$

where R is a generic renormalisation-scheme dependent (and thus unobservable) quantity, μ is the scale resulting from the renormalisation procedure, and γ is the log-free anomalous dimension that is computable in perturbation theory as a power series in the strong coupling α_s truncated at order k:

$$\gamma(\alpha_s(\mu)) = \frac{\alpha_s(\mu)}{4\pi} \sum_{n=0}^{k} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \gamma^{(n)}\,. \tag{B.2}$$

Notice that Eq. (B.1) applies to different relevant quantities in QCD such as the strong coupling itself and to collinear and TMD distributions. Of course, Eq. (B.1) is an ordinary differential equation that defines a family of solutions. In order to identify a particular solution, we need a boundary condition given by the knowledge of R at some particular scale μ_0: $R(\mu_0) = R_0$.

B.1 The strong coupling

The question that we want to address here is how to solve Eq. (B.1), along with its boundary condition $R(\mu_0) = R_0$, within the framework of perturbation theory. To rephrase it: how do we compute $R(\mu)$ to some perturbative accuracy for a generic scale μ given Eq. (B.1) and its boundary condition $R(\mu_0) = R_0$? The first case to be considered, because it underlies all others, is the strong coupling, *i.e.* we take $R = a_s = \alpha_s/4\pi$ that obeys the RGE[12]:

$$\frac{d\ln a_s}{d\ln \mu} = \overline{\beta}(a_s(\mu)) = a_s(\mu) \sum_{n=0}^{k} a_s^n(\mu)\beta^{(n)}\,. \tag{B.3}$$

The possibly more natural perturbative approach to this problem is that to assume that $\alpha_s(\mu)$ admits a perturbative expansion around $a_s(\mu_0) = 0$ truncated to the same order k of the anomalous dimension:

$$a_s(\mu) = a_s(\mu_0) \sum_{n=0}^{k} c_n(\mu)a_s^n(\mu_0)\,. \tag{B.4}$$

Plugging Eq. (B.4) into both left- and right-hand side of Eq. (B.3), one finds the following recursive equations for the first few coefficients c_n:

$$\begin{aligned}
\frac{dc_0}{d\ln \mu} &= 0 \\[2mm]
\frac{dc_1}{d\ln \mu} &= \beta^{(0)}c_0^2 \\[2mm]
\frac{dc_2}{d\ln \mu} &= 2\beta^{(0)}c_0 c_1 + \beta^{(1)}c_0^3 \\[2mm]
\frac{dc_3}{d\ln \mu} &= \beta^{(0)}(c_1^2 + 2c_0 c_2) + 3\beta^{(1)}c_0^2 c_1 + \beta^{(2)}c_0^4 \\[2mm]
&\ \vdots
\end{aligned} \tag{B.5}$$

---

[12] Notice that $\overline{\beta}$ in Eq. (B.3) is related to the usual QCD β-function by $\overline{\beta} = -\beta/a_s$.

Integrating these differential equations between μ_0 and μ and using the boundary condition gives:

$$
\begin{aligned}
c_0 &= 1 \\[2ex]
c_1 &= \beta^{(0)} \ln\left(\frac{\mu}{\mu_0}\right) \\[2ex]
c_2 &= \left(\beta^{(0)}\right)^2 \ln^2\left(\frac{\mu}{\mu_0}\right) + \beta^{(1)} \ln\left(\frac{\mu}{\mu_0}\right) \\[2ex]
c_3 &= \left(\beta^{(0)}\right)^3 \ln^3\left(\frac{\mu}{\mu_0}\right) + \frac{5}{2}\beta^{(0)}\beta^{(1)} \ln^2\left(\frac{\mu}{\mu_0}\right) + \beta^{(2)} \ln\left(\frac{\mu}{\mu_0}\right)
\end{aligned}
\tag{B.6}
$$

$$\vdots$$

Computing these coefficients up to order k allows one to obtain the perturbative expansion of $a_s(\mu)$ in terms of $a_s(\mu_0)$ up the fixed-order perturbative accuracy N^{k}LO. However, it should be clear from Eq. (B.6) that, for $\mu \gg \mu_0$ or $\mu \ll \mu_0$, $c_n \sim \ln^n(\mu/\mu_0) \gg 1$. Therefore, despite the smallness of $\alpha_s(\mu_0)$, the presence of such large logarithms is such that the truncated series in Eq. (B.4), *i.e.* a fixed-order computation, may not provide an accurate approximation of $\alpha_s(\mu)$.

However, nothing prevents us from extending the series in Eq. (B.4) from $k+1$ terms to and infinite number of terms even though the anomalous dimension in Eq. (B.3) is truncated to order k:

$$
a_s(\mu) = a_s(\mu_0) \sum_{n=0}^{\infty} c_n(\mu) a_s^n(\mu_0) .
\tag{B.7}
$$

The coefficients c_n can be computed iteratively exactly as shown above for any value of n. The calculation above can be used to infer the structure of the coefficients c_n:

$$
c_n(\mu) = \sum_{i=n-k}^{n} c_{n,i} \ln^i\left(\frac{\mu}{\mu_0}\right) ,
\tag{B.8}
$$

where $c_{n,i}$ are numerical coefficients given by combinations of the coefficients β_j, $j = 0, \ldots, k$. Using this equation and rearranging the series finally gives:

$$
a_s(\mu) = a_s(\mu_0) \sum_{l=0}^{k} a_s^l(\mu_0) \sum_{n=0}^{\infty} c_{n+l,n} \left[a_s(\mu_0) \ln\left(\frac{\mu}{\mu_0}\right)\right]^n .
\tag{B.9}
$$

This rearrangment exposes the resummation procedure: for each power of α_s at which the β-function is known, all powers of the kind $(a_s \ln(\mu/\mu_0))^n$ are included. In jargon, one usually says that the knowledge of the anomalous dimension to order k allows N^{k}LL resummation. The question that remains to be answered is how do we calculate all the $c_{n+l,n}$ coefficients to effectively enable resummation? This is usually done by observing that Eq. (B.3) with $k = 0$ can be easily solved in a closed form and gives:

$$
a_s^{\mathrm{LL}}(\mu) = \frac{a_s(\mu_0)}{1 - \beta^{(0)} a_s(\mu_0) \ln\left(\frac{\mu}{\mu_0}\right)} ,
\tag{B.10}
$$

which provides the LL resummation of the running coupling. Comparing this result with Eq. (B.9) with $k = 0$ tells us that:

$$
c_{n,n} = \left(\beta^{(0)}\right)^n ,
\tag{B.11}
$$

which is also consistent with Eq. (B.6). In order to go further, one needs to include the term $k = 1$ in Eq. (B.3) and integrate the differential equation between μ_0 and μ. The result is a transcendental equation in a_s:

$$
\frac{1}{\beta^{(0)}}\left(-\frac{1}{a_s(\mu)} + \frac{1}{a_s(\mu_0)}\right) + \frac{b_1}{\beta^{(0)}} \ln\left(\frac{a_s(\mu_0)(1 + b_1 a_s(\mu))}{a_s(\mu)(1 + b_1 a_s(\mu_0))}\right) = \ln\left(\frac{\mu}{\mu_0}\right)
\tag{B.12}
$$

with $b_n = \beta^{(n)}/\beta^{(0)}$. Of course, the solution of this transcendental equation solves exactly the differential equation in Eq. (B.3). However, in order to obtain an analytic equation for $a_s(\mu)$ we need to resort on perturbation theory. This is done by first rearranging the above equation as:

$$\frac{a_s(\mu_0)}{a_s(\mu)} = 1 - a_s(\mu_0)\beta^{(0)} \ln\left(\frac{\mu}{\mu_0}\right) + a_s(\mu_0)b_1 \ln\left(\frac{a_s(\mu_0)(1 + b_1 a_s(\mu))}{a_s(\mu)(1 + b_1 a_s(\mu_0))}\right). \tag{B.13}$$

This equation tells us that the second logarithm appearing in the r.h.s., being proportional to α_s can be expanded and truncated to leading order. Notice that this operation is allowed because no potentially large logarithms of the scales are present. Therefore, we can use Eq. (B.10) inside this term, obtaining:

$$\ln\left(\frac{a_s(\mu_0)(1 + b_1 a_s(\mu))}{a_s(\mu)(1 + b_1 a_s(\mu_0))}\right) = \ln\left(\frac{a_s(\mu_0)}{a_s^{\mathrm{LL}}(\mu)}\right) + \mathcal{O}(a_s), \tag{B.14}$$

where we have also used the fact that:

$$a_s(\mu) = a_s^{\mathrm{LL}}(\mu)\left[1 + \mathcal{O}(a_s)\right], \tag{B.15}$$

that descends directly from Eq. (B.9). This leads us to:

$$a_s(\mu) = \frac{a_s^{\mathrm{LL}}(\mu)}{1 + a_s^{\mathrm{LL}}(\mu)b_1 \ln\left(\frac{a_s(\mu_0)}{a_s^{\mathrm{LL}}(\mu)}\right)} + \mathcal{O}(a_s^3) \tag{B.16}$$

that can be further expanded, finally obtaining:

$$a_s^{\mathrm{NLL}}(\mu) = a_s^{\mathrm{LL}}(\mu)\left[1 - b_1 a_s^{\mathrm{LL}}(\mu) \ln\left(\frac{a_s(\mu_0)}{a_s^{\mathrm{LL}}(\mu)}\right)\right], \tag{B.17}$$

which gives the NLL solution for the evolution of the strong coupling (see also Ref. [14]). The same kind of procedure can be applied iteratively to obtain perturbatively more accurate solutions. In the following we will derive the NNLL-accurate solution for the running of the strong coupling that we will need below. This can be achieved by including $k = 2$ term in Eq. (B.3) and integrating the resulting differential equation between μ_0 and μ. This results in the following transcendental equation:

$$\begin{aligned}\beta^{(0)} \ln\left(\frac{\mu}{\mu_0}\right) &=& \left. -\frac{1}{a_s} - b_1 \ln(a_s) + \mathrm{const.} + a_s(b_1^2 - b_2) + \mathcal{O}(a_s^2)\right|_{a_s(\mu_0)}^{a_s(\mu)} \\ &=& -\frac{1}{a_s(\mu)} + \frac{1}{a_s(\mu_0)} + b_1 \ln\left(\frac{a_s(\mu_0)}{a_s^{\mathrm{NLL}}(\mu)}\right) + (b_1^2 - b_2)(a_s^{\mathrm{LL}}(\mu) - a_s(\mu_0)) + \mathcal{O}(a_s^2).\end{aligned} \tag{B.18}$$

where we have replaced $a_s(\mu)$ with a_s^{LL} and a_s^{NLL} where appropriate using Eq. (B.15) and:

$$a_s(\mu) = a_s^{\mathrm{NLL}}(\mu)\left[1 + \mathcal{O}(a_s^2)\right]. \tag{B.19}$$

Eq. (B.18) can finally be solved for $a_s(\mu)$ obtaining:

$$\begin{aligned}a_s^{\mathrm{NNLL}}(\mu) &=& a_s^{\mathrm{LL}}(\mu)\left[1 + b_1 a_s^{\mathrm{LL}}(\mu) \ln\left(\frac{a_s^{\mathrm{LL}}(\mu)}{a_s(\mu_0)}\right) + (b_2 - b_1^2)a_s^{\mathrm{LL}}(\mu)\left[a_s^{\mathrm{LL}}(\mu) - a_s(\mu_0)\right]\right. \\ && \left. + b_1^2 \left(a_s^{\mathrm{LL}}(\mu)\right)^2 \ln\left(\frac{a_s^{\mathrm{LL}}(\mu)}{a_s(\mu_0)}\right)\left(1 + \ln\left(\frac{a_s^{\mathrm{LL}}(\mu)}{a_s(\mu_0)}\right)\right)\right] \\ &=& a_s^{\mathrm{LL}}(\mu)\left[1 + b_1 \left[a_s^{\mathrm{NLL}}(\mu) + b_1 \left(a_s^{\mathrm{LL}}(\mu)\right)^2\right] \ln\left(\frac{a_s^{\mathrm{LL}}(\mu)}{a_s(\mu_0)}\right) + (b_2 - b_1^2)a_s^{\mathrm{LL}}(\mu)\left[a_s^{\mathrm{LL}}(\mu) - a_s(\mu_0)\right]\right],\end{aligned} \tag{B.20}$$

i.e. the NNLL-accurate running of the strong coupling.

It is now important to stress an important aspect that is central to this section, *i.e.* the difference between, *e.g.* Eqs. (B.12) and (B.17). Eq. (B.17) has been obtained from Eq. (B.12) by neglecting genuinely subleading corrections. This makes the two solutions equivalent from the point of view of accuracy. However, while

Eq. (B.12) satisfies the evolution equation Eq. (B.3) with $k = 1$ *exactly*, Eq. (B.17) violates it by subleading terms. In other words, one finds:

$$\frac{d \ln a_s^{\text{NLL}}(\mu)}{d \ln \mu} = \beta^{(0)} a_s^{\text{NLL}}(\mu) + \beta^{(1)} (a_s^{\text{NLL}}(\mu))^2 + \mathcal{O}(a_s^3) . \tag{B.21}$$

Therefore, the neglect of subleading terms required to go from Eq. (B.12) to Eq. (B.17) has the effect of introducing subleading terms in the evolution equation that is no longer exactly satisfied. We point out that these spurious subleading terms originates from insisting on obtaining a closed-form analytic expression for the evolution of the coupling.

Eq. (B.17) has the additional (perhaps undesired) feature that it makes the evolution of the coupling *non-conservative*. In practice, this means that evolving α_s from μ_0 to μ using Eq. (B.17) and then back from μ to μ_0 one does not recover the initial value. This is clearly not the case if one uses the exact solution deriving from Eq. (B.12). To show this, Fig. B.1 displays the behaviour of the NLL analytical solution, Eq. (B.21), with $n_f = 5$ active flavours evolved from $\alpha_s(M_Z) = 0.118$ down to 1 GeV and then up back to M_Z. As clear, the backward and forward evolution curves do not coincide leading to a mismatch of the value of $\alpha_s(M_Z)$.
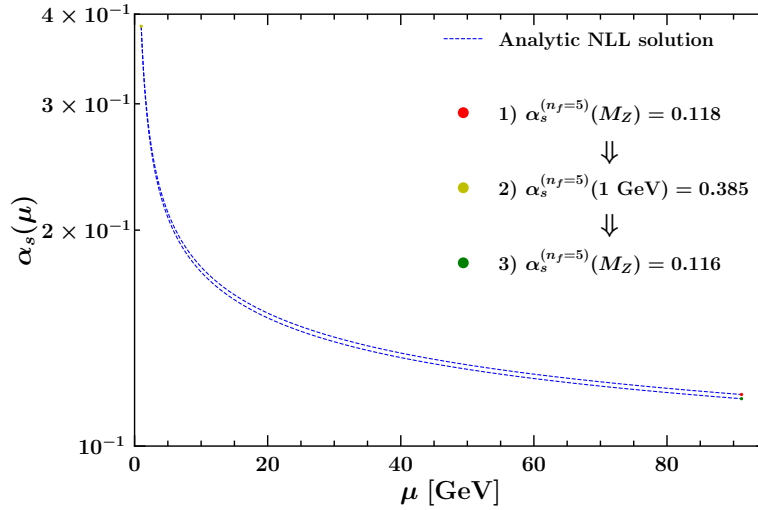

Fig. B.1: Non-conservativity of the NLL analytic evolution of the strong coupling.

Of course, one can argue that this effect is subleading and thus allowed. On the other hand, one may wonder whether a violation of the evolution equation introduced with the sole purpose of obtaining an analytic expression for the evolution is justified. In fact, it appears to be a better practice to employ an evolution that obeys the evolution equation exactly order by order in perturbation theory. The reason is simply a better control on the theoretical uncertainty related to missing higher-order corrections. To be more specific, the subleading terms introduced to obtain Eq. (B.17) would affect the computation of an observable at fixed-order in perturbation theory in the exact same way as a renormalisation scale variation would do. Therefore, it appears more transparent to use an exact solution to the evolution equation and leave the estimate of theoretical uncertainties to renormalisation-scale variations.

B.2 DGLAP evolution

As a second application of Eq. (B.1), we consider the DGLAP evolution in which the quantity R is identified with a collinear distribution f. In order to simplify the discussion, we consider the Mellin moments of a non-singlet distribution whose evolution equation can be written as exactly as in Eq. (B.1) with $R = f$ and γ the appropriate anomalous dimension. Our aim is to find a solution to this equation in perturbation theory by computing f at some final scale μ when knowing f at some initial scale μ_0. Of course, the solution to the

evolution equation is:

$$
\begin{aligned}
f(\mu) &= \exp\left[\int_{\mu_0}^{\mu} d\ln\mu'\, \gamma(a_s(\mu'))\right] f(\mu_0) \\[2mm]
&= \exp\left[\sum_{n=0}^{k} \gamma^{(n)} \int_{\mu_0}^{\mu} d\ln\mu'\, a_s^{n+1}(\mu')\right] f(\mu_0) \\[2mm]
&= f(\mu_0) \prod_{n=0}^{k} \exp\left[\gamma^{(n)} I_n\right] f(\mu_0)\,,
\end{aligned}
\tag{B.22}
$$

with:

$$
I_n = \int_{a_s(\mu_0)}^{a_s(\mu)} da_s \left(\frac{a_s^n}{\overline{\beta}(a_s)}\right)\,.
\tag{B.23}
$$

In the second line of Eq. (B.22) we have used the perturbative expansion of the anomalous dimension γ and in the third line made a change of variable in the integral using Eq. (B.3). It is now opportune to analyse the integrals I_n. The first question that we ask is whether the perturbative expansion of γ and $\overline{\beta}$ are related. More specifically, given a $k+1$ terms in the expansion of $\overline{\beta}$, is there any bound on the number of terms in the expansion of γ in terms of logarithmic accuracy? In order to answer this question, it is necessary to first understand what kind of logarithms Eq. (B.22) is actually resumming. Indeed, one may wonder whether the evolution of α_s alone is enough to rusum all the large logarithms. If so, we would be enabled to expand and truncate the exponential to the desired perturbative accuracy because higher orders would be suppressed by powers of α_s. The answer to this question boils down to determining whether I_n is logarithmically enhanced for some value of n. We assume that $\overline{\beta}$ is truncated at some power $k'+1$, with k' generally different from k in Eq. (B.22). To give a quantitative argument let us take $k' = 2$, such that

$$
\overline{\beta}(a_s) = \beta^{(0)} a_s \left[1 + b_1 a_s\right]\,.
\tag{B.24}
$$

We then compute I_0, I_1, and I_2:

$$
\begin{aligned}
I_0 &= \frac{1}{\beta^{(0)}} \int_{a_s(\mu_0)}^{a_s(\mu)} da_s \left(\frac{1}{a_s + b_1 a_s^2}\right) = \frac{1}{\beta^{(0)}} \left[\ln a_s - \ln(1 + b_1 a_s)\right]\Big|_{a_s(\mu_0)}^{a_s(\mu)} = \frac{1}{\beta^{(0)}} \ln\left(\frac{a_s(\mu)}{a_s(\mu_0)}\right) + \mathcal{O}(a_s)\,, \\[3mm]
I_1 &= \frac{1}{\beta^{(0)}} \int_{a_s(\mu_0)}^{a_s(\mu)} da_s \left(\frac{1}{1 + b_1 a_s}\right) = \frac{1}{\beta^{(0)}} \left[\frac{\ln(1 + b_1 a_s)}{b_1}\right]\Big|_{a_s(\mu_0)}^{a_s(\mu)} = \mathcal{O}(a_s)\,, \\[3mm]
I_2 &= \frac{1}{\beta^{(0)}} \int_{a_s(\mu_0)}^{a_s(\mu)} da_s \left(\frac{a_s}{1 + b_1 a_s}\right) = \frac{1}{\beta^{(0)}} \left[\frac{a_s}{b_1} - \frac{\ln(1 + b_1 a_s)}{b_1^2}\right]\Big|_{a_s(\mu_0)}^{a_s(\mu)} = \mathcal{O}(a_s^2)\,.
\end{aligned}
\tag{B.25}
$$

In general one finds that, for $n \geq 1$, $I_n = \mathcal{O}(a_s^n)$. Therefore, the exponential in Eq. (B.22) can safely be expanded:

$$
\exp\left[\gamma^{(n)} I_n\right] = 1 + \gamma^{((n))} \times \mathcal{O}(a_s^n)\,, \quad n \geq 1\,.
\tag{B.26}
$$

Somewhat surprisingly, this reveals that the inclusion of perturbative corrections beyond leading order to the anomalous dimension γ does not directly improve the logarithmic accuracy of the resummation provided by the evolution. The exception to this pattern is I_0 that cannot be (fully) expanded due to the presence of a logarithm of $a_s(\mu)/a_s(\mu_0)$ that, as can be see from Eq. (B.10), is potentially large:

$$
\exp\left[\gamma^{(0)} I_0\right] = \exp\left[\frac{\gamma^{(0)}}{\beta^{(0)}} \ln\left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)\right] + \mathcal{O}(a_s) = \left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} + \mathcal{O}(a_s)\,.
\tag{B.27}
$$

This factor alone provides the resummation up to some logarithmic accuracy determined uniquely by the evolution of $a_s(\mu)$. Finally one has:

$$
f(\mu) = \left[1 + \mathcal{O}(a_s)\right] \left(\frac{a_s(\mu)}{a_s(\mu_0)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} f(\mu_0)\,.
\tag{B.28}
$$

However, the inclusion of correction terms in the square brackets in the r.h.s. of Eq. (B.28) is still required to attain some given logarithmic accuracy. Since the exponential in Eq. (B.27) resums single logarithms of the kind $a_s^m \ln^m(\mu/\mu_0)$, a term proportional to a_s^n (without any logarithm) belongs to the tower $a_s^{m+n} \ln^m(\mu/\mu_0)$ and thus contributes to $N^n LL$. To make an example, suppose we want to achieve NLL resummation. To do so, we need to include the $\mathcal{O}(a_s)$ in the squared bracket of Eq. (B.28) because that contributes to NLL. That term can be easily computed by retaining the $\mathcal{O}(a_s)$ terms in Eq. (B.25). Moreover, $a_s(\mu)$ in the last factor has to be computed using the NLL evolution given in Eq. (B.21). This finally gives:

$$f^{\text{NLL}}(\mu) = \left[1 + \frac{1}{\beta^{(0)}}\left(\gamma^{(1)} - b_1\gamma^{(0)}\right)\left(a_s^{\text{LL}}(\mu) - a_s(\mu_0)\right)\right]\left(\frac{a_s^{\text{NLL}}(\mu)}{a_s(\mu_0)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} f(\mu_0).$$
(B.29)

Also note that in the squared bracket we are entitled to replaced a_s with a_s^{LL} becase the difference is $\mathcal{O}(a_s^2)$ and thus NNLL.

In order to make a step towards the notation typically used in q_T resummation, we introduce the quantity:

$$\lambda \equiv \lambda(\mu) = a_s(\mu_0)\beta^{(0)}\ln\left(\frac{\mu}{\mu_0}\right),$$
(B.30)

that allows us to write the LL and NLL RGE solution for a_s as:

$$a_s^{\text{LL}}(\mu) = a_s(\mu_0)\frac{1}{1-\lambda},$$
(B.31)

and:

$$a_s^{\text{NLL}}(\mu) = a_s(\mu_0)\frac{1}{1-\lambda}\left[1 - a_s(\mu_0)\frac{b_1\ln(1-\lambda)}{1-\lambda}\right],$$
(B.32)

such that:

$$\begin{aligned}
f^{\text{NLL}}(\mu) &= \left[1 + a_s(\mu_0)\frac{1}{\beta^{(0)}}\left(\gamma^{(1)} - b_1\gamma^{(0)}\right)\frac{\lambda}{1-\lambda}\right] \\
&\times \exp\left[-\frac{\gamma^{(0)}}{\beta^{(0)}}\ln(1-\lambda) - a_s(\mu_0)\frac{\gamma^{(0)}b_1}{\beta^{(0)}}\frac{\ln(1-\lambda)}{1-\lambda}\right]f(\mu_0).
\end{aligned}$$
(B.33)

We now introduce the functions:

$$\begin{aligned}
g^{(0)}(\lambda) &= 1 + a_s(\mu_0)\frac{1}{\beta^{(0)}}\left(\gamma^{(1)} - b_1\gamma^{(0)}\right)\frac{\lambda}{1-\lambda}, \\[2mm]
g^{(1)}(\lambda) &= -\frac{\gamma^{(0)}}{\beta^{(0)}}\ln(1-\lambda), \\[2mm]
g^{(2)}(\lambda) &= -\frac{\gamma^{(0)}b_1}{\beta^{(0)}}\frac{\ln(1-\lambda)}{1-\lambda},
\end{aligned}$$
(B.34)

such that:

$$f^{\text{NLL}}(\mu) = g^{(0)}(\lambda)\exp\left[g^{(1)}(\lambda) + a_s(\mu_0)g^{(2)}(\lambda)\right]f(\mu_0).$$
(B.35)

This procedure can be extended to $N^k LL$ accuracy just by introducing the appropriate $g^{(n)}$ functions, with $n \leq k+1$, in the exponential and including corrections up to $\mathcal{O}(a_s^k)$ in $g^{(0)}$. The notation is purposely reminiscent of soft-gluon and q_T resummation.

We can use the parallel with soft-gluon resummation to introduce the analogous of a "resummation" scale. This is done by introducing an *arbitrary* scale $\mu_R \sim \mu_0$ along with the trivial equality:

$$\ln\left(\frac{\mu}{\mu_0}\right) = \ln\left(\frac{\mu}{\mu_R}\right) + \ln\left(\frac{\mu_R}{\mu_0}\right),$$
(B.36)

such that $\ln(\mu_R/\mu_0) \sim 1$. Using Eq. (B.30), one has:

$$\lambda = a_s(\mu_0)\beta^{(0)}\ln\left(\frac{\mu}{\mu_R}\right) + a_s(\mu_0)\beta^{(0)}\ln\left(\frac{\mu_R}{\mu_0}\right) = \bar{\lambda} + a_s(\mu_0)\beta^{(0)}\ln\left(\frac{\mu_R}{\mu_0}\right),$$
(B.37)

where the second term in the r.h.s. is not enhanced by a large logarithm and can thus be treated perturbatively. Writing the functions $g^{(n)}$ in terms of $\overline{\lambda}$ and neglecting subleading terms gives the new functions:

$$
\begin{aligned}
\overline{g}^{(0)}(\overline{\lambda}) &= 1 + a_s(\mu_0)\frac{1}{\beta^{(0)}}\left(\gamma^{(1)} - b_1\gamma^{(0)}\right)\frac{\overline{\lambda}}{1-\overline{\lambda}}\,, \\[2mm]
\overline{g}^{(1)}(\overline{\lambda}) &= -\frac{\gamma^{(0)}}{\beta^{(0)}}\ln\left(1-\overline{\lambda}\right)\,, \\[2mm]
\overline{g}^{(2)}(\overline{\lambda}) &= -\frac{\gamma^{(0)}}{1-\overline{\lambda}}\left[\frac{b_1}{\beta^{(0)}}\ln\left(1-\overline{\lambda}\right) - \ln\left(\frac{\mu_{\mathrm{R}}}{\mu_0}\right)\right]\,,
\end{aligned}
\tag{B.38}
$$

so that:

$$
f^{\mathrm{NLL}}(\mu) = \overline{g}^{(0)}(\overline{\lambda})\exp\left[\overline{g}^{(1)}(\overline{\lambda}) + a_s(\mu_0)\overline{g}^{(2)}(\overline{\lambda})\right]f(\mu_0)\,.
\tag{B.39}
$$

Eqs. (B.39) and (B.35) are both NLL accurate and only differ by subleading terms. In fact, Eq. (B.39) generalises Eq. (B.35) because the latter reduces to the former for $\mu_{\mathrm{R}} = \mu_0$. Therefore, one should be able to use modest variations of μ_{R} around μ_0 to estimate the possible impact of higher-order corrections to the anomalous dimension.

Eq. (B.39) provides a closed-form, fully analytical solution to the RGE, Eq. (B.1), for collinear distributions. However, exactly like in the case of the analytical solution for a_s, Eq. (B.28) will satisfy Eq. (B.1) only up to subleading terms with a consequent non-conservative pattern of the evolution. Conversely, a numerical evaluation of the solution in Eq. (B.22) will obey exactly Eq. (B.1) and the evolution will be conservative.

B.3 The Sudakov form factor

Now, we turn to address the question of solving the TMD evolution equations in Eq. (1.2) in perturbation theory. We have already derived the solution, Eq. (1.25), in terms of the relevant anomalous dimensions. However, this solution contains an integral. A possibility is that of computing that integral numerically. This ensures that the results obeys *exactly* the evolution equations in Eq. (1.2) and the evolution is conservative. Another possibility is that of attempting to solve the integral analytically. This turns out to be possible provided that one uses the appropriate analytic solution for the running of a_s. As discussed above, this comes at the price that the result violates the equations in Eq. (1.2) by subleading terms. As we will see below, this very violation gives origin to the resummation scale.

In order to derive a fully analytic expression for the expression in Eq. (1.25) we need to consider the perturbative expansion of the relevant anomalous dimensions. This is done in Eq. (2.2) for the square of the evolution operator that is relevant to Drell-Yan production. In order to match the notation of this section, we identify $\mu_b = \mu$ and $Q = \mu_0$ in Eq. (2.2). In addition, we limit ourselves to consider the NNLL accuracy that means truncating the expansion at $n = 2$ for γ_K and at $n = 1$ for K and γ_F. This finally gives:

$$
\begin{aligned}
R^{\mathrm{NNLL}} &= \exp\Bigg\{-\sum_{n=0}^{1}K^{(n,0)}\left[a_s^{\mathrm{N}^{1-n}\mathrm{LL}}(\mu)\right]^{n+1}\ln\frac{\mu}{\mu_0} - \sum_{n=0}^{1}\gamma_F^{(n)}\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[a_s^{\mathrm{N}^{1-n}\mathrm{LL}}(\mu')\right]^{n+1} \\[2mm]
&\quad - \sum_{n=0}^{2}\gamma_K^{(n)}\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[a_s^{\mathrm{N}^{2-n}\mathrm{LL}}(\mu')\right]^{n+1}\ln\frac{\mu'}{\mu_0}\Bigg\}\,,
\end{aligned}
\tag{B.40}
$$

where we have also used the fact that:

$$
a_s = a_s^{\mathrm{N}^k\mathrm{LL}}\left[1 + \mathcal{O}(a_s^k)\right]\,.
\tag{B.41}
$$

It is convenient to express the above equation in terms of the variable λ defined in Eq. (B.30). This leads to:

$$
\begin{aligned}
R^{\mathrm{NNLL}} &= \exp\Bigg\{-\frac{1}{\beta^{(0)}a_s(\mu_0)}\sum_{n=0}^{1}\left(K^{(n,0)}\lambda\left[a_s^{\mathrm{N}^{1-n}\mathrm{LL}}(\lambda)\right]^{n+1} + \gamma_F^{(n)}\int_0^{\lambda}d\lambda'\left[a_s^{\mathrm{N}^{1-n}\mathrm{LL}}(\lambda')\right]^{n+1}\right) \\[2mm]
&\quad - \left(\frac{1}{\beta^{(0)}a_s(\mu_0)}\right)^2\sum_{n=0}^{2}\gamma_K^{(n)}\int_0^{\lambda}d\lambda'\,\lambda'\left[a_s^{\mathrm{N}^{2-n}\mathrm{LL}}(\lambda')\right]^{n+1}\Bigg\}\,,
\end{aligned}
\tag{B.42}
$$

where $\lambda' = \lambda(\mu')$ and $a_s(\lambda) \equiv a_s(\mu)$. We write it more explicitly as:

$$
\begin{aligned}
R^{\mathrm{NNLL}} \;=\;\; & \exp\Bigg\{ -\frac{1}{\beta^{(0)}a_s(\mu_0)} \left(K^{(0,0)}\lambda a_s^{\mathrm{NLL}}(\lambda) + \gamma_F^{(0)}\int_0^\lambda d\lambda'\, a_s^{\mathrm{NLL}}(\lambda') + K^{(1,0)}\lambda \left[a_s^{\mathrm{LL}}(\lambda)\right]^2 + \gamma_F^{(1)}\int_0^\lambda d\lambda' \left[a_s^{\mathrm{LL}}(\lambda')\right]^2 \right) \\[2mm]
& - \left(\frac{1}{\beta^{(0)}a_s(\mu_0)}\right)^2 \left(\gamma_K^{(0)}\int_0^\lambda d\lambda'\, \lambda' a_s^{\mathrm{NNLL}}(\lambda') + \gamma_K^{(1)}\int_0^\lambda d\lambda'\, \lambda' \left[a_s^{\mathrm{NLL}}(\lambda')\right]^2 + \gamma_K^{(2)}\int_0^\lambda d\lambda'\, \lambda' \left[a_s^{\mathrm{LL}}(\lambda')\right]^3 \right) \Bigg\}\,.
\end{aligned}
$$
(B.43)

We now need to compute the integrals involving the strong coupling a_s. This can be done using the analytic expressions derived in Sect. B.1.The task is simplified by the fact that $a_s^{\mathrm{N}^k\mathrm{LL}}$, for $k > 0$, can be written in terms of a_s^{LL} (see Eqs. (B.17) and (B.20)) so that we can exploit the relations:

$$
\int_0^\lambda d\lambda'\, f\left(a_s^{\mathrm{LL}}(\lambda')\right) = a_s(\mu_0) \int_{a_s(\mu_0)}^{a_s^{\mathrm{LL}}(\lambda)} \frac{da}{a^2}\, f(a)\,,
$$
(B.44)

$$
\int_0^\lambda d\lambda'\, \lambda'\, f\left(a_s^{\mathrm{LL}}(\lambda')\right) = \int_0^\lambda d\lambda'\, f\left(a_s^{\mathrm{LL}}(\lambda')\right) - a_s^2(\mu_0) \int_{a_s(\mu_0)}^{a_s^{\mathrm{LL}}(\lambda)} \frac{da}{a^3}\, f(a)\,.
$$
(B.45)

C   Iterative solution of the strong coupling running

In this section, I will derive a general formula to compute iteratively the running of the strong coupling. In order to simplify the notation, I will refer to the strong coupling computed at some generic scale as to a, while I will call a_0 the value of coupling at the reference scale. The corresponding RGE will then be:

$$
\frac{d\ln a}{d\ln \mu} = \beta(a)\,,
$$
(C.1)

with:

$$
\beta(a) = a\sum_{n=0}^{\infty} \beta_n a^n\,.
$$
(C.2)

If one truncates the β function at the first order, the resulting evolution equation reads:

$$
\frac{d\ln a_{\mathrm{LL}}}{d\ln \mu} = a_{\mathrm{LL}}\beta_0\,,
$$
(C.3)

that admits a simple closed-form exact solution:

$$
a_{\mathrm{LL}}(\mu) = \frac{a_0}{1 - a_0\beta_0 \ln\left(\frac{\mu}{\mu_0}\right)}\,.
$$
(C.4)

Since we know how to write a_{LL} in a closed form, we can combine Eqs. (C.1) and (C.4) as follows:

$$
\frac{d\ln a}{d\ln a_{\mathrm{LL}}} = \frac{a}{a_{\mathrm{LL}}}\sum_{n=0}^{\infty} b_n a^n\,, \quad \text{with} \quad b_n = \frac{\beta_n}{\beta_0}\,.
$$
(C.5)

Now we can attempt to solve this equation whose general solution is:

$$
\int_{a_0}^{a_{\mathrm{LL}}} \frac{dy}{y^2} = \int_{a_0}^{a} \frac{dx}{x^2}\left(\sum_{n=0}^{\infty} b_n x^n\right)^{-1}\,.
$$
(C.6)

Assuming that $x \ll 1$ (as it should be in the perturbative regime), one can write:

$$
\left(\sum_{n=0}^{\infty} b_n x^n\right)^{-1} = \sum_{n=0}^{\infty} \bar{b}_n x^n\,,
$$
(C.7)

where the coefficients $\bar{b}_n$ can be iteratively determined in terms of the coefficients b_n. Considering that $b_0 = 1$, the terms up to x^4 read:

$$
\begin{aligned}
\bar{b}_0 &= 1\,, \\
\bar{b}_1 &= -b_1\,, \\
\bar{b}_2 &= -b_2 + b_1^2\,, \\
\bar{b}_3 &= -b_3 + 2b_2 b_1 - b_1^3 \\
\bar{b}_4 &= -b_4 + 2b_1 b_3 - 3b_1^2 b_2 + b_2^2 + b_1^4\,.
\end{aligned}
\tag{C.8}
$$

Plugging this expansion into Eq. (C.6) and solving the integrals gives:

$$
\frac{1}{a_{\mathrm{LL}}} = \frac{1}{a} - \bar{b}_1 \ln\left(\frac{a}{a_0}\right) - \sum_{k=1}^{\infty} \frac{\bar{b}_{k+1}}{k}\left(a^k - a_0^k\right)\,.
\tag{C.9}
$$

This is a transcendental equation in a that can be solved iteratively assuming that $a, a_0 \ll 1$. Upon this assumption, the terms in the r.h.s. of the equation above are increasingly smaller. Therefore, at the first order one has:

$$
\frac{1}{a_{\mathrm{LL}}} = \frac{1}{a} \quad \implies \quad a = a_{\mathrm{LL}}\,,
\tag{C.10}
$$

as expected. At second order instead:

$$
\frac{1}{a_{\mathrm{LL}}} = \frac{1}{a} - \bar{b}_1 \ln\left(\frac{a}{a_0}\right)\,.
\tag{C.11}
$$

This is a transcendental equation that can be solved by observing that the second term in the r.h.s. is subleading w.r.t. the first and thus one can plug into that term the first order solution, that is:

$$
\frac{1}{a_{\mathrm{LL}}} = \frac{1}{a} - \bar{b}_1 \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)\,.
\tag{C.12}
$$

This equation can now be resolved for a obtaining:

$$
a = a_{\mathrm{NLL}} = \frac{a_{\mathrm{LL}}}{1 + a_{\mathrm{LL}}\bar{b}_1 \ln\left(\dfrac{a_{\mathrm{LL}}}{a_0}\right)}\,.
\tag{C.13}
$$

One can keep iterating in the same way finally obtaining the general recursive formula for the analytic $\mathrm{N}^n\mathrm{LL}$ solution:

$$
a_{\mathrm{N}^n\mathrm{LL}} = a_{\mathrm{LL}}\left\{1 + a_{\mathrm{LL}}\left[\bar{b}_1 \ln\left(\frac{a_{\mathrm{N}^{n-1}\mathrm{LL}}}{a_0}\right) + \sum_{k=1}^{n-1}\frac{\bar{b}_{k+1}}{k}\left(a_{\mathrm{N}^{n-k-1}\mathrm{LL}}^k - a_0^k\right)\right]\right\}^{-1} + \mathcal{O}(a^{n+2})\,, \quad n \geq 1\,.
\tag{C.14}
$$

Note also that the curly bracket, that appears in the denominator, can be further expanded and truncated to the relevant order because the single terms appearing there are increasingly subleading.

At this stage it is important to highlight a feature of the analytic solution of running of the strong coupling. Due to its iterative nature, the $\mathrm{N}^n\mathrm{LL}$ solution will only depend on a_{LL}. This allows us to write the solution to all other RGEs, such as the DGLAP evolution equations and the Collins-Soper equation, in terms of a_{LL} only.

Additionally, it is interesting to observe that there are *two* resummation structures. The first is contained in a_{LL} and given by the geometric summation in Eq. (C.4). The second is $\ln\left(\frac{a_{\mathrm{N}^{n-1}\mathrm{LL}}}{a_0}\right)$. This structure first appears at NLL as:

$$
a_{\mathrm{NLL}} = a_{\mathrm{LL}}\left[1 + a_{\mathrm{LL}}\bar{b}_1 \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)\right]^{-1} + \mathcal{O}(a^3) = a_{\mathrm{LL}}\left[1 - a_{\mathrm{LL}}\bar{b}_1 \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)\right] + \mathcal{O}(a^3)\,.
\tag{C.15}
$$

When going further, the content of the logarithm will be the coupling computed at higher logarithmic orders. However, beyond LL, the logarithm can be expanded around $a_{\mathrm{LL}} = 0$ finally leading to $\ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)$. As an example, we consider a_{NNLL} that, using Eq. (C.14), reads:

$$
\begin{aligned}
a_{\mathrm{NNLL}} &= a_{\mathrm{LL}}\left\{1 + a_{\mathrm{LL}}\left[\bar{b}_1 \ln\left(\frac{a_{\mathrm{NLL}}}{a_0}\right) + \bar{b}_2\left(a_{\mathrm{LL}} - a_0\right)\right]\right\}^{-1} + \mathcal{O}(a^4) \\
&= a_{\mathrm{LL}}\left\{1 - a_{\mathrm{LL}}\left[\bar{b}_1 \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right) + \bar{b}_2\left(a_{\mathrm{LL}} - a_0\right)\right] + a_{\mathrm{LL}}^2 \bar{b}_1^2 \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)\left(1 + \ln\left(\frac{a_{\mathrm{LL}}}{a_0}\right)\right)\right\} + \mathcal{O}(a^4)\,.
\end{aligned}
\tag{C.16}
$$

We can also go further and compute the N^3LL running:

$$a_{N^3LL} = a_{LL} \left\{ 1 + a_{LL} \left[\bar{b}_1 \ln\left(\frac{a_{N^2LL}}{a_0}\right) + \bar{b}_2 (a_{NLL} - a_0) + \frac{\bar{b}_3}{2} (a_{LL}^2 - a_0^2) \right] \right\}^{-1} + \mathcal{O}(a^5), \quad (C.17)$$

D   Analytic calculation of the Collins-Soper kernel

In this section, we compute analytically to different logarithmic accuracies the solution of the Collins-Soper evolution equation Eq. (1.6). This amounts to computing Eq. (1.8) plugging in the appropriate perturbative corrections. To do so, we choose as an initial scale $\mu_0 = \mu_b = 2e^{-\gamma_E}/b_T$ that is such that the expansion of K at this scale admits a log-free perturbative expansion (see Eq. (1.9)). Therefore, the integral to be computed is:

$$K(\mu) = K(\mu_b) - \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')), \quad (D.1)$$

that upon expansion of the K and γ_K anomalous dimensions becomes:

$$K(\mu) = \sum_{n=0}^{\infty} a_s^{n+1}(\mu_b) K^{(n,0)} - \sum_{n=0}^{\infty} \gamma_K^{(n)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^{n+1}(\mu'). \quad (D.2)$$

Following the usual counting, to N^kLL accuracy the truncation of the perturbative series is:

$$K(\mu) = \sum_{n=0}^{k-1} a_s^{n+1}(\mu_b) K^{(n,0)} - \sum_{n=0}^{k} \gamma_K^{(n)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^{n+1}(\mu'), \quad (D.3)$$

such that at LL one has:

$$K_{LL}(\mu) = -\gamma_K^{(0)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s(\mu'), \quad (D.4)$$

at NLL:

$$K_{NLL}(\mu) = a_s(\mu_b) K^{(0,0)} - \gamma_K^{(0)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s(\mu') - \gamma_K^{(1)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^2(\mu'), \quad (D.5)$$

at NNLL:

$$K_{NNLL}(\mu) = a_s(\mu_b) K^{(0,0)} + a_s^2(\mu_b) K^{(1,0)} - \gamma_K^{(0)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s(\mu') - \gamma_K^{(1)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^2(\mu') - \gamma_K^{(2)} \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^3(\mu'), \quad (D.6)$$

and so on. The question now boils down to computing analytically the integrals of different powers of a_s. To do so, we need to use the analytic solution of the running of the strong coupling. This can be done by using the g-function approach developed in Ref. [15] that is such that, using $a_s(\mu_b)$ as boundary condition, the N^kLL evolution reads:

$$a_s^{N^kLL}(\mu') = a_s(\mu_b) \sum_{l=0}^{k} a_s^l(\mu_b) g_{l+1}^{(\beta)}(\lambda), \quad (D.7)$$

with the g-functions relevant up to NNLL given by:

$$g_1^{(\beta)}(\lambda) = \frac{1}{1-\lambda},$$

$$g_2^{(\beta)}(\lambda) = -\frac{b_1 \ln(1-\lambda)}{(1-\lambda)^2} - \frac{\beta_0 \ln \kappa}{(1-\lambda)^2},$$

$$g_3^{(\beta)}(\lambda) = \frac{b_2 \lambda - b_1^2 \left(\lambda + \ln(1-\lambda) - \ln^2(1-\lambda)\right)}{(1-\lambda)^3} + \frac{b_1(2\ln(1-\lambda) - 1)\beta_0 \ln \kappa + \beta_0^2 \ln^2 \kappa}{(1-\lambda)^3}$$

$$g_4^{(\beta)}(\lambda) = \frac{(b_3 - b_2 b_1)\lambda - \frac{1}{2}(b_1^3 - 2b_2 b_1 + b_3)\lambda^2 + (2b_1(b_1^2 - b_2)\lambda - b_2 b_1)\ln(1-\lambda) + b_1^3(\frac{5}{2} - \ln(1-\lambda))\ln^2(1-\lambda)}{(1-\lambda)^4}$$

$$+ \frac{(2(b_1^2 - b_2)\lambda + b_1^2(5 - 3\ln(1-\lambda))\ln(1-\lambda) - b_2)\beta_0 \ln \kappa + b_1(-3\ln(1-\lambda) + \frac{5}{2})\beta_0^2 \ln^2 \kappa - \beta_0^3 \ln^3 \kappa}{(1-\lambda)^4}, \quad (D.8)$$

where:

$$\lambda = a_s(\mu_b)\beta^{(0)} \ln\left(\frac{\mu'}{\mu_b}\right) , \tag{D.9}$$

and $b_n = \beta^{(n)}/\beta^{(0)}$. Since the integration variable only appears through λ, it is convenient to make a change of variable and, with a little abuse of notation, rewrite the integrals as follows:

$$\int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} a_s^p(\mu') = \frac{1}{a_s(\mu_b)\beta^{(0)}} \int_0^{\lambda_\mu} d\lambda \, a_s^p(\lambda) , \tag{D.10}$$

with:

$$\lambda_\mu = a_s(\mu_b)\beta^{(0)} \ln\left(\frac{\mu}{\mu_b}\right) . \tag{D.11}$$

At LL, we can plug the LL solution of the strong coupling into Eq. (D.4) producing:

$$K_{\mathrm{LL}}(\mu) = \frac{\gamma_K^{(0)}}{\beta^{(0)}} \ln(1 - \lambda_\mu) . \tag{D.12}$$

At NLL, we plug the NLL solution of the strong coupling into the first integral of Eq. (D.5) and the LL solution into the second integral finding:

$$K_{\mathrm{NLL}}(\mu) = K_{\mathrm{LL}}(\mu) + a_s(\mu_b)\left[K^{(0,0)} + \frac{b_1\gamma_K^{(0)}}{\beta^{(0)}}\frac{\lambda_\mu + \ln(1-\lambda_\mu)}{1-\lambda_\mu} - \frac{\gamma_K^{(1)}}{\beta^{(0)}}\frac{\lambda_\mu}{1-\lambda_\mu}\right] . \tag{D.13}$$

At NNLL, following the same pattern, we find:

$$\begin{aligned}
K_{\mathrm{NNLL}}(\mu) &= K_{\mathrm{NLL}}(\mu) \\[2mm]
&+ a_s^2(\mu_b)\left[K^{(1,0)} - \frac{b_2\gamma_K^{(0)}}{\beta^{(0)}}\frac{\lambda_\mu^2}{2(1-\lambda_\mu)^2} + \frac{b_1^2\gamma_K^{(0)}}{\beta^{(0)}}\frac{\lambda_\mu^2 - \ln^2(1-\lambda_\mu)}{2(1-\lambda_\mu)^2}\right. \\[2mm]
&+ \left.\frac{b_1\gamma_K^{(1)}}{\beta^{(0)}}\frac{2\ln(1-\lambda_\mu) - \lambda_\mu(\lambda_\mu-2)}{2(1-\lambda_\mu)^2} + \frac{\gamma_K^{(2)}}{\beta^{(0)}}\frac{\lambda_\mu(\lambda_\mu-2)}{2(1-\lambda_\mu)^2}\right] .
\end{aligned} \tag{D.14}$$

It is clear that to $\mathrm{N}^k\mathrm{LL}$ this generalises to:

$$K_{\mathrm{N}^k\mathrm{LL}}(\mu) = \sum_{n=0}^{k} a_s^n(\mu_b) g_n^{(K)}(\lambda_\mu) , \tag{D.15}$$

where the appropriate g-functions up to NNLL can be read off from the calculations above.

E   Analytic perturbation theory

The fact highlighted above that the solution to all other RGEs relevant to resum large logarithms of q_T can be written in terms of a_{LL} only is particularly relevant. As a matter of fact, this will eventually allow us to eliminate the Landau pole simply by requiring that a_{LL}, or equivalently the one-loop β-function, has the correct analytic structure in the complex μ plane. This approach is often referred to analytic perturbation theory (APT) and was originally suggested by Bogolyubov, Logiunov, and Shirkov back in 1959 [16] to obtain the correct causality properties of the QED Green's functions also when working at fixed-order in perturbation theory. At one-loop order, this boils down to requiring that the strong coupling a_{LL} admits a spectral representation allowing to restore the analytic properties of the two-point Green's function required for causality to be fulfilled. The direct consequence of the procedure is the natural appearance of power suppressed terms of the form Λ/μ that automatically eliminate the Landau pole making the strong coupling a_{LL} finite for all values of μ.

However, a_{LL} does not only appears as integer powers but also within logs. It is therefore crucial to be able to treat these terms correctly preserving the correct analytical structure on the complex μ. Here fractional analytical perturbation theory (FAPT) comes to rescue [17].

F   Derivation of analytic expression for g-functions in q_T-resummation

F.1   The Sudakov integral

The Sudakov exponential has the following explicit form:

$$S = -\int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[A\left(\alpha_s\left(q^2\right)\right) \ln \frac{M^2}{q^2} + \tilde{B}\left(\alpha_s\left(q^2\right)\right) \right], \tag{F.1}$$

where M is the invariant mass of the final state, Q is the resummation scale, and $A, \tilde{B}$ are power series in $\alpha_s\left(Q^2\right)$.

The A function is completely determined by the cusp anomalous dimensions.

The $\tilde{B}$ function contains the usual B coefficients and two additional terms coming from PDF evolution and coefficient functions.

From the PDF evolution equations in Mellin space

$$f_{a,N}\left(\mu_F^2\right) = \sum_b U_{ab,N}\left(\frac{b_0^2}{b^2}, \mu_F^2\right) f_{b,N}\mu_F^2 \tag{F.2}$$

$$\frac{dU_{ab,N}\left(\mu^2, \mu_F^2\right)}{d\ln\mu^2} = \sum_c \gamma_{ac,N}\left(\alpha_s\left(\mu^2\right)\right) U_{cb,N}\left(\mu^2, \mu_F^2\right) \tag{F.3}$$

one obtains (for a single species of partons) the solution

$$U_N\left(b_0^2/b^2, \mu_F^2\right) = \exp\left\{ -\int_{b_0^2/b^2}^{\mu_F^2} \frac{dq^2}{q^2} \gamma_N\left(\alpha_s\left(q^2\right)\right) \right\} \tag{F.4}$$

Thus we have the first additional contribution to B coefficients: $B \longrightarrow B + 2\,\gamma_N\,\alpha_s$.

As for the coefficient functions (again, in Mellin space), we can use the following renormalisation-group identity:

$$C_N\left(\alpha_s\left(b_0^2/b^2\right)\right) = C_N\left(\alpha_s\left(M^2\right)\right) \exp\left\{ -\int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2} \beta\left(\alpha_s\left(q^2\right)\right) \frac{d\ln C_N\left(\alpha_s\left(q^2\right)\right)}{d\ln\alpha_s\left(q^2\right)} \right\} \tag{F.5}$$

to derive the second additional contribution to B coefficients: $B \longrightarrow B + 2\,\beta\left(\alpha_s\right) \frac{d\ln C_N(\alpha_s)}{d\ln\alpha_s}$.

The $\tilde{B}$ function is thus given by

$$\tilde{B} = B + 2\,\gamma_N\,\alpha_s + 2\,\beta\left(\alpha_s\right) \frac{d\ln C_N\left(\alpha_s\right)}{d\ln\alpha_s}. \tag{F.6}$$

With simple math and a change of the integration variable, we have

$$S = -\int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[A\left(\alpha_s\left(q^2\right)\right) \left(\ln \frac{M^2}{Q^2} + \ln \frac{Q^2}{q^2} \right) + \tilde{B}\left(\alpha_s\left(q^2\right)\right) \right] \tag{F.7}$$

$$= -\int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[A\left(\alpha_s\left(q^2\right)\right) \ln \frac{Q^2}{q^2} + \bar{B}\left(\alpha_s\left(q^2\right)\right) \right] \tag{F.8}$$

$$= \int_\lambda^0 \frac{dt}{\beta_0 \alpha_s} \left[A\left(\alpha_s\left(t\right)\right) \frac{t}{\beta_0 \alpha_s} + \bar{B}\left(\alpha_s\left(t\right)\right) \right], \tag{F.9}$$

where $\lambda = \beta_0 \alpha_s \ln\left(Q^2 b^2/b_0^2\right) = \beta_0 \alpha_s L$, and the $A, \bar{B}$ functions are given by

$$A = \sum_{n=1}^\infty A_n \alpha_s^n, \tag{F.10}$$

$$\bar{B} = \sum_{n=1}^\infty \left(\tilde{B}_n + A_n \ln \frac{M^2}{Q^2} \right) \alpha_s^n \tag{F.11}$$

i.e., the $\ln\left(M^2/Q^2\right)$ dependence is entirely embodied in the shift $\tilde{B} \longrightarrow \bar{B}$.

We will now explicitly compute the g_n functions, including the μ_R-dependent terms coming from the replacement $Q \longrightarrow \mu_R$.

F.2 α_s **evolution**

The evolution of α_s from Q^2 to q^2 up to NNLL order is given by:

$$\alpha_s\left(q^2\right) = \frac{\alpha_s}{1-t} - \left(\frac{\alpha_s}{1-t}\right)^2 \frac{\beta_1}{\beta_0} \ln\left(1-t\right) + \left(\frac{\alpha_s}{1-t}\right)^3 \left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2\left(1-t\right) - \ln\left(1-t\right) - t\right) + \frac{\beta_2}{\beta_0}t\right) + \mathcal{O}\left(\alpha_s^4\right) \quad \text{(F.12)}$$

with $\alpha_s = \alpha_s\left(Q^2\right)$ and $t = \beta_0 \alpha_s \ln\frac{Q^2}{q^2}$.

From now on, I will call the first three terms of this expansion LLα_s, NLLα_s and NNLLα_s, respectively.

For future reference, using the expansions $\frac{1}{1-t} = 1 + t + t^2 + \ldots$ and $\ln\left(1-t\right) = -t - \frac{1}{2}t^2 + \ldots$, we get the following expansion for α_s:

$$\alpha_s\left(q^2\right) = \alpha_s - \alpha_s^2\beta_0 \ln\frac{q^2}{Q^2} - \alpha_s^3\left(\beta_1 \ln\frac{q^2}{Q^2} - \beta_0^2 \ln^2\frac{q^2}{Q^2}\right) \quad \text{(F.13)}$$

F.3 g_1

The LL function is easily obtained by integrating the A_1 term:

$$\int_\lambda^0 \frac{t\,\mathrm{d}t}{\beta_0^2\alpha_s^2} \left[A_1\alpha_s^{\mathrm{LL}}\left(t\right)\right] \quad \text{(F.14)}$$

The only possibility here is to use the LL expression for α_s, since any other choice would be subleading. The integrand is of $\mathcal{O}(\alpha_s^{-1})$.

The result (multiplied by $\beta_0\alpha_s/\lambda$ in order to write the integral as Lg_1) is:

$$g_1(\lambda) = \frac{A_1(\lambda + \ln(1-\lambda))}{\beta_0\lambda} \quad \text{(F.15)}$$

F.4 g_2

The NLL function has three contributions:

$$\int_\lambda^0 \frac{t\,\mathrm{d}t}{\beta_0^2\alpha_s^2} \left[A_1\alpha_s^{\mathrm{NLL}}\left(t\right) + A_2\left(\alpha_s^{\mathrm{LL}}\left(t\right)\right)^2\right] + \int_\lambda^0 \frac{\mathrm{d}t}{\beta_0\alpha_s} \left[\bar{B}_1\alpha_s^{\mathrm{LL}}\left(t\right)\right] \quad \text{(F.16)}$$

The combinations are chosen in such a way that all the integrands are $\mathcal{O}(\alpha_s^0)$.

We obtain

$$g_2(\lambda) = \frac{\bar{B}_1}{\beta_0}\ln(1-\lambda) + \frac{A_1\beta_1}{\beta_0^3}\left(\frac{\lambda}{1-\lambda} + \frac{1}{2}\ln^2(1-\lambda) + \frac{\ln(1-\lambda)}{1-\lambda}\right) - \frac{A_2}{\beta_0^2}\left(\frac{\lambda}{1-\lambda} + \ln(1-\lambda)\right). \quad \text{(F.17)}$$

F.5 g_3

The NNLL function has five contributions:

$$\int_\lambda^0 \frac{t\,\mathrm{d}t}{\beta_0^2\alpha_s^2} \left[A_1\alpha_s^{\mathrm{NNLL}}\left(t\right) + A_2\left(2\alpha_s^{\mathrm{LL}}\left(t\right)\alpha_s^{\mathrm{NLL}}\left(t\right)\right) + A_3\left(\alpha_s^{\mathrm{LL}}\left(t\right)\right)^3\right] + \int_\lambda^0 \frac{\mathrm{d}t}{\beta_0\alpha_s} \left[\bar{B}_1\alpha_s^{\mathrm{NLL}}\left(t\right) + \bar{B}_2\left(\alpha_s^{\mathrm{LL}}\left(t\right)\right)^2\right]$$
$$\text{(F.18)}$$

The combinations are chosen in such a way that all the integrands are $\mathcal{O}(\alpha_s)$.

Summing up the five terms, we obtain

$$
\begin{aligned}
g_3(\lambda) \;=\; & -\frac{\bar{B}_2}{\beta_0}\frac{\lambda}{(1-\lambda)} + \frac{\bar{B}_1\beta_1}{\beta_0^2}\frac{\lambda+\ln(1-\lambda)}{1-\lambda} - \frac{A_3}{2\beta_0^2}\frac{\lambda^2}{(1-\lambda)^2} \\
& + \; A_1\left(\frac{\lambda\left(\beta_1^2\lambda+\beta_0\beta_2(2-3\lambda)\right)}{2\beta_0^4(1-\lambda)^2} + \frac{\beta_1^2(1-2\lambda)\ln^2(1-\lambda)}{(2\beta_0^4)(1-\lambda)^2} + \left(\frac{\beta_1^2}{\beta_0^4(1-\lambda)} + \frac{\beta_0\beta_2-\beta_1^2}{\beta_0^4}\right)\ln(1-\lambda)\right) \\
& + \; \frac{A_2\beta_1}{\beta_0^3}\left(\frac{\lambda(3\lambda-2)}{2(1-\lambda)^2} - \frac{(1-2\lambda)\ln(1-\lambda)}{(1-\lambda)^2}\right).
\end{aligned}
\tag{F.19}
$$

F.6   Introduction of μ_R-dependence

We now allow the resummation scale Q to be different from the renormalisation scale μ_R, and consequently compute the variation of g_1 with respect to this change of scale.

Considering the expansion of α_s up to $\mathcal{O}(\alpha_s^2)$ in Eq.F.13, we derive the following formulae for g_1 variation:

$$
\lambda \;\longrightarrow\; \lambda - \beta_0^2\alpha_s^2(\mu_R^2)L\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.20}
$$

$$
Lg_1 \;\longrightarrow\; Lg_1 + Lg_1'(\lambda)\Delta\lambda = Lg_1 - \lambda^2 g_1'(\lambda)\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.21}
$$

The result is a next-to-leading order contribution that has to be added to g_2:

$$
\Delta g_1^{LL}(\lambda) = \frac{A_1}{\beta_0}\left(\frac{\lambda}{1-\lambda} + \ln(1-\lambda)\right)\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.22}
$$

Considering the expansion of α_s up to $\mathcal{O}(\alpha_s^3)$ in Eq.F.13, we get two shifts:

$$
\lambda \;\longrightarrow\; \lambda - \beta_0\beta_1\alpha_s^3(\mu_R^2)L\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.23}
$$

$$
Lg_1 \;\longrightarrow\; Lg_1 + Lg_1'(\lambda)\Delta\lambda = Lg_1 - \alpha_s\frac{\beta_1}{\beta_0}\lambda^2 g_1'(\lambda)\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.24}
$$

and

$$
\lambda \;\longrightarrow\; \lambda + \beta_0^3\alpha_s^3(\mu_R^2)L\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.25}
$$

$$
Lg_1 \;\longrightarrow\; Lg_1 + Lg_1'(\lambda)\Delta\lambda = Lg_1 + \beta_0\alpha_s\lambda^2 g_1'(\lambda)\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.26}
$$

These are both next-to-next-to-leading order contributions and will be added to g_3:

$$
\Delta g_1^{NLL1}(\lambda) = \alpha_s\frac{A_1}{\beta_0^2}\left(\frac{\lambda}{1-\lambda} + \ln(1-\lambda)\right)\left(\beta_1 - \beta_0^2\ln\left(\frac{Q^2}{\mu_R^2}\right)\right)\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.27}
$$

A further term generating a contribution of the same order is the third term of the Taylor expansion (computed with first order squared variation of α_s:

$$
\lambda \;\longrightarrow\; \lambda - \beta_0^2\alpha_s^2(\mu_R^2)L\ln\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.28}
$$

$$
Lg_1 \;\longrightarrow\; Lg_1 + \frac{1}{2}g_1''(\lambda)(\Delta\lambda)^2 = Lg_1 + \frac{1}{2}\beta_0\alpha_s\lambda^3 g_1''(\lambda)\ln^2(\frac{Q^2}{\mu_R^2})
\tag{F.29}
$$

which gives

$$
\Delta g_1^{NLL2}(\lambda) = \alpha_s A_1\left(\frac{\lambda(2-3\lambda)}{2(\lambda-1)^2} + \ln(1-\lambda)\right)\ln^2\left(\frac{Q^2}{\mu_R^2}\right)
\tag{F.30}
$$

Applying the same variation at g_2 we get

$$\lambda \quad \longrightarrow \quad \lambda - \beta_0^2 \alpha_s^2(\mu_R^2) L \ln\left(\frac{Q^2}{\mu_R^2}\right) \tag{F.31}$$

$$g_2 \quad \longrightarrow \quad g_2 + g_2^{'}(\lambda)\Delta\lambda = g_2 - \beta_0 \alpha_s(\mu_R^2)\lambda g_2^{'}(\lambda) \ln\left(\frac{Q^2}{\mu_R^2}\right) \tag{F.32}$$

The resulting next-to-next-to-leading contribution to add to g_3 is:

$$\Delta g_2^{LL}(\lambda) = \alpha_s\left(\bar{B}_1\frac{\lambda}{1-\lambda} + \frac{A_2}{\beta_0}\frac{\lambda^2}{(1-\lambda)^2} - \frac{A_1\beta_1}{\beta_0^2}\frac{\lambda^2}{(1-\lambda)^2}\ln(1-\lambda)\right)\ln\left(\frac{Q^2}{\mu_R^2}\right). \tag{F.33}$$

F.7 Final expressions

The final expression for g-functions including renormalisation-scale dependence is:

$$g_1(\lambda) = \frac{A_1(\lambda + \ln(1-\lambda))}{\beta_0\lambda} \tag{F.34}$$

$$\begin{aligned}g_2(\lambda) =& \frac{\bar{B}_1}{\beta_0}\ln(1-\lambda) + \frac{A_1\beta_1}{\beta_0^3}\left(\frac{\lambda}{1-\lambda} + \frac{1}{2}\ln^2(1-\lambda) + \frac{\ln(1-\lambda)}{1-\lambda}\right) - \frac{A_2}{\beta_0^2}\left(\frac{\lambda}{1-\lambda} + \ln(1-\lambda)\right) \\ &+ \frac{A_1}{\beta_0}\left(\frac{\lambda}{1-\lambda} + \ln(1-\lambda)\right)\ln\left(\frac{Q^2}{\mu_R^2}\right)\end{aligned} \tag{F.35}$$

$$\begin{aligned}g_3(\lambda) =& -\frac{\bar{B}_2}{\beta_0}\frac{\lambda}{(1-\lambda)} + \frac{\bar{B}_1\beta_1}{\beta_0^2}\frac{\lambda + \ln(1-\lambda)}{1-\lambda} - \frac{A_3}{2\beta_0^2}\frac{\lambda^2}{(1-\lambda)^2} \\ &+ A_1\left(\frac{\lambda\left(\beta_1^2\lambda + \beta_0\beta_2(2-3\lambda)\right)}{2\beta_0^4(1-\lambda)^2} + \frac{\beta_1^2(1-2\lambda)\ln^2(1-\lambda)}{(2\beta_0^4)(1-\lambda)^2} + \left(\frac{\beta_1^2}{\beta_0^4(1-\lambda)} + \frac{\beta_0\beta_2 - \beta_1^2}{\beta_0^4}\right)\ln(1-\lambda)\right) \\ &+ \frac{A_2\beta_1}{\beta_0^3}\left(\frac{\lambda(3\lambda-2)}{2(1-\lambda)^2} - \frac{(1-2\lambda)\ln(1-\lambda)}{(1-\lambda)^2}\right) - \frac{A_1}{2}\frac{\lambda^2}{(1-\lambda)^2}\ln^2\left(\frac{Q^2}{\mu_R^2}\right) \\ &+ \left(\bar{B}_1\frac{\lambda}{1-\lambda} + \frac{A_2}{\beta_0}\frac{\lambda^2}{(1-\lambda)^2} + \frac{A_1\beta_1}{\beta_0^2}\left(\frac{\lambda}{1-\lambda} + \frac{(1-2\lambda)\ln(1-\lambda)}{(1-\lambda)^2}\right)\right)\ln\left(\frac{Q^2}{\mu_R^2}\right)\end{aligned} \tag{F.36}$$

References

[1] J. Collins and T. C. Rogers, "Connecting Different TMD Factorization Formalisms in QCD," *Phys. Rev. D*, vol. 96, no. 5, p. 054011, 2017.

[2] M. G. Echevarria, I. Scimemi, and A. Vladimirov, "Unpolarized Transverse Momentum Dependent Parton Distribution and Fragmentation Functions at next-to-next-to-leading order," *JHEP*, vol. 09, p. 004, 2016.

[3] S. Catani, L. Cieri, D. de Florian, G. Ferrera, and M. Grazzini, "Vector boson production at hadron colliders: hard-collinear coefficients at the NNLO," *Eur. Phys. J. C*, vol. 72, p. 2195, 2012.

[4] G. Bozzi, S. Catani, D. de Florian, and M. Grazzini, "Transverse-momentum resummation and the spectrum of the Higgs boson at the LHC," *Nucl. Phys. B*, vol. 737, pp. 73–120, 2006.

[5] R. Meng, F. I. Olness, and D. E. Soper, "Semiinclusive deeply inelastic scattering at small q(T)," *Phys. Rev. D*, vol. 54, pp. 1919–1935, 1996.

[6] J. Collins, L. Gamberg, A. Prokudin, T. Rogers, N. Sato, and B. Wang, "Relating Transverse Momentum Dependent and Collinear Factorization Theorems in a Generalized Formalism," *Phys. Rev. D*, vol. 94, no. 3, p. 034014, 2016.

[7] P. M. Nadolsky, D. Stump, and C. Yuan, "Semiinclusive hadron production at HERA: The Effect of QCD gluon resummation," *Phys. Rev. D*, vol. 61, p. 014003, 2000. [Erratum: Phys.Rev.D 64, 059903 (2001)].

[8] A. Bacchetta, D. Boer, M. Diehl, and P. J. Mulders, "Matches and mismatches in the descriptions of semi-inclusive processes at low and high transverse momentum," *JHEP*, vol. 08, p. 023, 2008.

[9] A. Daleo, D. de Florian, and R. Sassot, "O(alpha**2(s)) QCD corrections to the electroproduction of hadrons with high transverse momentum," *Phys. Rev. D*, vol. 71, p. 034013, 2005.

[10] V. Bertone, "APFEL++: A new PDF evolution library in C++," *PoS*, vol. DIS2017, p. 201, 2018.

[11] D. de Florian, M. Stratmann, and W. Vogelsang, "QCD analysis of unpolarized and polarized Lambda baryon production in leading and next-to-leading order," *Phys. Rev. D*, vol. 57, pp. 5811–5824, 1998.

[12] W. Furmanski and R. Petronzio, "Lepton - Hadron Processes Beyond Leading Order in Quantum Chromodynamics," *Z. Phys. C*, vol. 11, p. 293, 1982.

[13] R. Ellis, W. Stirling, and B. Webber, *QCD and collider physics*, vol. 8. Cambridge University Press, 2 2011.

[14] L. Del Debbio, S. Forte, J. I. Latorre, A. Piccione, and J. Rojo, "Neural network determination of parton distributions: The Nonsinglet case," *JHEP*, vol. 03, p. 039, 2007.

[15] V. Bertone, G. Bozzi, and F. Hautmann, "Perturbative hysteresis and emergent resummation scales," 2 2022.

[16] N. Bogolyubov, A. Logunov, and D. Shirkov, "The method of dispersion relations and perturbation theory," *Sov. Phys. JETP*, vol. 10, no. 3, pp. 574–581, 1960.

[17] A. Bakulev, S. Mikhailov, and N. Stefanis, "QCD analytic perturbation theory: From integer powers to any power of the running coupling," *Phys. Rev. D*, vol. 72, p. 074014, 2005.