Contents

1 Structure of the integrals

The general structure of the quantities to be computed in APFEL is the combination of terms I that have the form of Mellin convolutions between an operator O and distribution function d, that is:

$$I(x) = x\int_0^1 dz \int_0^1 dy\, O(z) d(y) \delta(x - yz) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x\int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right) . \tag{1.1}$$

However, very often, typically in the presence of mass effects, the integration phase space is modified and the convolution in eq. (1.1), limiting ourselves to the leftmost identity, is generalised as:

$$I(x,\eta) = x\int_{x/\eta}^1 \frac{dz}{z} O(z,\eta) d\left(\frac{x}{\eta z}\right) . \tag{1.2}$$

where $\eta \leq 1$, with $\eta = 1$ reproducing eq. (1.1). However, for purposes that will become clear later, we want to write the integral in eq. (1.2) in the form of eq. (1.1), that is in such a way that lower bound of the integral is not the rescaled variable x/η but the physical Bjorken x. To do so, one needs to perform the change of variable $y = \eta z$, so that:

$$I(x,\eta) = \int_x^\eta dy\, O\left(\frac{y}{\eta},\eta\right) \frac{x}{y} d\left(\frac{x}{y}\right) . \tag{1.3}$$

In order to precompute the expensive part of the integral in eq. (1.5), we use the standard interpolation formula to the distribution d:

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right) , \tag{1.4}$$

where α runs over the node of a give grid in x space and the weights w_α are typically polynomials of degree k (*i.e.* Lagrange interpolants). Now we use eq. (1.4) in eq. (1.5) and at the same time we limit the computation of the integral I to the point x_β of the grid used in eq. (1.4). This way we get:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^\eta dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.5}$$

where we have defined $\overline{d}_\alpha = x_\alpha d(x_\alpha)$. In the particular case of the Lagrange interpolants (in the APFEL procedure), one can show that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d\,, \tag{1.6}$$

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k})\,, \tag{1.7}$$

and thus eq. (1.5) can be adjusted as:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_c^d dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.8}$$

Finally, we change back the integration variable $z = y/\eta$ so that eq. (1.8) becomes:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz\, O(z,\eta)\; w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)\right]}_{\Gamma_{\alpha\beta}} . \tag{1.9}$$

The quantity in squared brackets is the interesting bit.

The expressions for the operator O that we have to deal with have this general form:

$$O(z,\eta) = R(z,\eta) + \sum_i \left[P^{(i)}(z,\eta)\right]_+ S^{(i)}(z,\eta) + L(\eta)\delta(1-z)\,, \tag{1.10}$$

where R and $S^{(i)}$ are a regular functions in $z=1$, that is:

$$R(1,\eta) = \lim_{z\to 1} R(z,\eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1,\eta) = \lim_{z\to 1} S^{(i)}(z,\eta) = J^{(i)}(\eta)\,, \tag{1.11}$$

being K, $J^{(i)}$, and L a finite function of η. The functions $P^{(i)}$ are instead plus-prescripted functions whose behaviour in $z=1$ is singular in the limit $\eta \to 1$. Notice that the sum over i in eq. (1.10) depends on the perturbative order of the expressions. Plugging eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in eq. (1.9) and making use of the definition of plus-prescription, we obtain:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \eta\int_{c/\eta}^{d/\eta} dz \left\{ R\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) + \sum_i P^{(i)}(z,\eta)\left[S^{(i)}\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\ &+ \eta\left[L(\eta) - \sum_i S^{(i)}(1,\eta)\int_0^{c/\eta} dz P^{(i)}(z,\eta)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,,\end{aligned} \tag{1.12}$$

that can be further manipulated changing the integration variable back, defining $\eta z = y$:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \int_c^d dy \left\{ R\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) + \sum_i P^{(i)}\left(\frac{y}{\eta},\eta\right)\left[S^{(i)}\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\ &+ \eta\left[L(\eta) + \sum_i S^{(i)}(1,\eta) Q^{(i)}\left(\frac{c}{\eta},\eta\right)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,.\end{aligned} \tag{1.13}$$

where we have defined:

$$Q^{(i)}(a,\eta) \equiv -\int_0^a dz P^{(i)}\left(z,\eta\right). \tag{1.14}$$

These integrals can, most of the times, be computed analytically.

Eqs. (1.12)-(1.13) express the full complexity of the task. However, there are a few remarks that apply is some particular cases and that reduce the complexity. In the case of the ($\overline{\text{MS}}$) splitting functions, there are two main simplications: the first is that $\eta = 1$, the second is that there is one single term in the sum over i ($i=0$) and the form of the function P is:

$$P^{(0)}(z,\eta) \to \frac{1}{1-z}\,, \tag{1.15}$$

so that:

$$Q^{(0)}(a,\eta) = \ln(1-a)\,. \tag{1.16}$$

Considering that:

$$w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}\,, \tag{1.17}$$

and that the expressions can be manipulated in such a way that the coefficient of the plus-prescripted term S is a constant, we have that:

$$\Gamma_{\beta\alpha} = \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} + \left[S\ln\left(1-c\right) + L\right]\delta_{\alpha\beta}\,. \tag{1.18}$$

The same kind of simplifications apply to the case of the Zero-Mass (ZM) coefficient functions with the only exception that the sum over i extends to more terms. In particular, since we will use exact expressions only up to $\mathcal{O}(\alpha_s)$, *i.e.* NLO, we have that:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S^{(0)} + S^{(1)}\ln(1-z)}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} \\ &+ \left[S^{(0)}\ln\left(1-c\right) + \frac{1}{2}S^{(1)}\ln^2\left(1-c\right) + L\right]\delta_{\alpha\beta}\,.\end{aligned} \tag{1.19}$$

As far as the massive coefficient functions up to $\mathcal{O}(\alpha_s^2)$ are concerned, things can be more complicated and we will discuss it later. In the next section we will consider the case of hadronic observables in the ZM scheme showing how to pre-compute the integral for the double-differential distributions in Drell-Yan production.

1.1 Advantage of a logarithmic grid

Given the particular structure of the integral I in Eq. (1.1), it turns out to be very convenient to use a logarithmically distributed grid along with Lagrange interpolating functions. Let us specifically consider the (massless) integrals:

$$I_{\beta\alpha} = \int_{x_\beta}^1 dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right) . \tag{1.20}$$

If the grid is logarithmically spaced, *i.e.* $\ln x_{n+1} = \ln x_n + \delta x$ with δx a positive constant, and $\{w_\alpha(z)\}$ is a set of Lagrange interpolating functions of degree κ in $\ln z$, one has:

$$w_\alpha(z) = \widetilde{w}\left(\ln\frac{z}{x_\alpha}\right) \quad \Rightarrow \quad w_\alpha\left(\frac{x_\beta}{y}\right) = \widetilde{w}\left(\ln\frac{x_\beta}{x_\alpha} - \ln y\right) = \widetilde{w}\left((\beta-\alpha)\delta x - \ln y\right) . \tag{1.21}$$

Therefore, the integrand of the integral in Eq. (1.20) only depends on the difference $\beta - \alpha$ and not on β and α separately. Since the lower bound is x_β, this symmetry seems to broken at the level of the integral. However, the symmetry is preserved thanks to the support properties of the interpolating functions w_α and the fact that relevant functions (PDFs or FFs) are zero at $x = 1$. To see this, we consider the integration limits in Eq. (1.7) with $\eta = 1$. They can be written as:

$$c = \max(x_\beta, e^{(\beta-\alpha-1)\delta x}) \quad \text{and} \quad d = \min(1, e^{(\beta-\alpha+\kappa)\delta x}) . \tag{1.22}$$

While the limit d is manifestly only dependent on the difference $\beta - \alpha$, the limit c is not. However, c does not have this symmetry only when x_β is selected in place of $e^{(\beta-\alpha-1)\delta x}$ and this can only happen when:

$$x_\beta > e^{(\beta-\alpha-1)\delta x} . \tag{1.23}$$

Since the last point of the grid is $x_{N_x} = 1$, being N_x the number of grid nodes minus one, one can write:

$$x_\beta = \frac{x_\beta}{x_{N_x}} = e^{(\beta-N_x)\delta x} . \tag{1.24}$$

Finally, the inequality in Eq. (1.23) becomes:

$$\beta - N_x > \beta - \alpha - 1 \quad \Leftrightarrow \quad \alpha > N_x - 1 \quad \Leftrightarrow \quad \alpha = N_x . \tag{1.25}$$

Therefore, the integrals $I_{\beta N_x}$ do not respect the "$\beta - \alpha$" symmetry. However, as mentioned above, $I_{\beta N_x}$ will always multiply a function computed in $x_{N_x} = 1$. If this function is a PDF or a FF, it is identically zero at $x_{N_x} = 1$ and thus the symmetry is effectively preserved. In addition, c in Eq. (1.22) is such that if $\beta > \alpha$ one has $c \geq 1$. But being c the lower integration bound of and since in Eq. (1.20) the upper bound is 1, one immediately has that $I_{\beta\alpha} = 0$ for $\beta > \alpha$. The consequence of these observations is that computing the integrals $a_\alpha = I_{0\alpha}$ for $\alpha = 0, \ldots, N_x$ is enough to reconstruct the full set of $I_{\beta\alpha}$ because, in matricial representation, I will look like this:

$$I_{\beta\alpha} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N_x} \\ 0 & a_0 & a_1 & \cdots & a_{N_x-1} \\ 0 & 0 & a_0 & \cdots & a_{N_x-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} , \tag{1.26}$$

In conclusion, adopting a logarithmically-spaced grid allows one to compute $N_x + 1$ integrals rather than $(N_x + 1)(N_x + 2)/2$ integrals.

There is another detail that matters in terms of numerical efficiency of the computation of the integrals $I_{\beta\alpha}$. Given the support region of the functions w_α, the integral in Eq. (1.20) effectively reads:

$$I_{\beta\alpha} = \int_c^d dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right) , \tag{1.27}$$

with the integration limits given in Eq. (1.7). These limits can be rearranged as follows:

$$c = \frac{x_\beta}{x_{\min[N_x,\alpha+1]}} \quad \text{and} \quad d = \frac{x_\beta}{x_{\max[\beta,\alpha-\kappa]}}\,, \tag{1.28}$$

which makes it manifest the index range covered by the integration range. The basic observation is that the functions w_α are piecewise in correspondence of the grid nodes. This feature makes a numerical integration over the full range defined in Eq. (1.28) hard to converge due to the cusps at the grid nodes. However, the functions w_α are smooth between two consecutive nodes. Therefore, it turns out that it is convenient to compute the integrals $I_{\beta\alpha}$ by breaking the integration range as follows:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right)\,, \tag{1.29}$$

in such a way that the integrand of each single integrand is a smooth function and thus easier to integrate. Despite the number of intergrals to be computed increases, this procedure makes the computation faster and more accurate.

Finally, if the grid is logarithmically distributed, and one defines:

$$s = \exp\left[\delta x\right]\,, \tag{1.30}$$

Eq. (1.29) can also be written as:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right)\,. \tag{1.31}$$

1.1.1 GPD-related integrals

When considering, for example, GPD-related computations such as the evolution of GPDs, another kind of integral structure comes into play, that is:

$$J(x) = x\int_0^1 \frac{dz}{z} O\left(\frac{x}{z}, x\right) d(z)\,, \tag{1.32}$$

which differes from that in Eq. (1.1) in two respects: by the lower integration bound that is zero rather than x and by the fact that the operator O may also depend explicitly on the external variable x. Under the assumption that this integral is convergent, we need to adapt the strategy for the computation of numerical integrals discussed above to Eq. (1.32). As a first step, we make the change of integration variable $z \to x/y$ and use the interpolation formula in Eq. (1.4), to obtain:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^{\infty} dy\, O(y, x_\beta)\, w_\alpha\left(\frac{x_\beta}{y}\right) = \sum_{\alpha=0}^{N_x} J_{\beta\alpha}\overline{d}_\alpha\,. \tag{1.33}$$

The ∞ in the upper bound is a new feature. However, the support region of the functions w_α is the same so that:

$$J_{\beta\alpha} = \int_c^{\overline{d}} dy\, O(y, x_\beta)\, w_\alpha\left(\frac{x_\beta}{y}\right)\,, \tag{1.34}$$

with integration bounds:

$$c = \frac{x_\beta}{x_{\min[N_x,\alpha+1]}} \quad \text{and} \quad \overline{d} = \frac{x_\beta}{x_{\max[0,\alpha-\kappa]}}\,. \tag{1.35}$$

Following the procedure discussed in the previous section, the actual computation is achieved by:

$$\begin{aligned} J_{\beta\alpha} &= \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha]} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y, x_\beta) w_\alpha\left(\frac{x_\beta}{y}\right) \\ &= \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y, x_\beta) w_\alpha\left(\frac{x_\beta}{y}\right)\,. \end{aligned} \tag{1.36}$$

Therefore, at the level of the single integral, the only difference with respect to the "standard" case is a change of the upper summation bound. It should be clear, however, that this procedure necessarily implies an approximation. This is due to the fact that the interpolation grid, understood to be logarithmically distributed, does not reach zero. This approximation manifests itself in the fact that the upper bound of the intergrals in Eq. (1.36) never reaches infinity but it gets as large as s^κ. The size of s^κ can range widely depending on the lower bound of the grid x_0, on the number of nodes N_x, and on the interpolation degree κ, according to:

$$s^\kappa = \exp\left[-\frac{\kappa}{N_x}\ln x_0\right] = x_0^{-\frac{\kappa}{N_x}}\,. \tag{1.37}$$

In particular, s^κ becomes larger as: x_0 gets smaller, N_x gets smaller, and κ gets larger. We will need to assess the impact of this approximation numerically.

If the operator O in Eq. (1.32) did not explicitly depend on x, the integral $J_{\beta\alpha}$ would only depend on the difference $\beta-\alpha$. However, under this assumption that we will release below, at variance with the standard case, $J_{\beta\alpha}\neq 0$ also for $\beta>\alpha$. The reason is that the lower bound c is now allowed to exceed 1. Using Eq. (1.22), this implies that $\beta-\alpha$ can also be larger than zero. This means in turn that computing the integrals $b_\alpha = J_{0\alpha}$ and $b_{-\alpha} = J_{\alpha 0}$ for $\alpha = 0,\ldots,N_x$ is enough to reconstruct the full set of $J_{\beta\alpha}$. In matricial representation, J takes the form:

$$J_{\beta\alpha} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N_x} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N_x-1} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N_x-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{-N_x} & b_{-N_x+1} & b_{-N_x+2} & \cdots & b_0 \end{pmatrix}\,, \tag{1.38}$$

Therefore, adopting a logarithmically-spaced grid allows one to compute $2N_x+1$ integrals rather than N_x^2 integrals.

1.2 Joining grids

As discussed above, logarithmically distributed grids are particularly advantageous because the allow one to substantially reduce the number of integrals to be compute. However, logarithmic grids have two main drawbacks. First, they cannot get down to $x=0$ and this is a potential problem for integrals like those in Eq. (1.32). Second, logarithmic grids that start from a relatively low value of x tend to be relatively sparse at large values of x. This is also a problem because all integrals that we are considering are such that the function being interpolated is integrated up to $x=1$ and the interpolation there can thus potentially degrade in accuracy.

A possible solution to this problem is to increase the density in a stepwise fashion as x gets closer to one. This produces locally logarithmically distributed grids that allow to exploit the symmetry discussed above while making the grid denser and thus more accurate at large x. The implementation of this procedure in APFEL is achieved by means of *locked* subgrids. In practice, one starts with a logarithmic grid with a given lower bound, *e.g.* $x_{\rm min}^{(0)} = 10^{-5}$. Starting from a given node, $x_{\rm min}^{(1)}$, the density of the grid is increased by some integer factor. This procedure can be repeated an arbitrary number of times as one moves towards large x, effectively defining logarithmic subgrids that are increasingly denser and thus guarantee a better interpolation accuracy.

When dealing with integrals such as that in Eq. (1.20), the simplest, but possibly not optimal, way to exploit the subgrid structure is to switch to a denser grid at the level of the *integral* essentially using one grid when x_β is below the transition node and the other when it is above. The advantage of this approach is that the integration procedure discussed above applies verbatim with the only detail that according to the position of x_β one grid is used rather than another. The disadvantage of this procedure is that integrals with low values of x_β do not take advantage of the denser grids at large x. When dealing with functions like PDFs that vanish rapidly as x tends to one, this is typically fine because the bulk of the integrals is typically due to the region at $x\gtrsim x_\beta$. This usually makes possible interpolation inaccuracies at large x negligible.

The procedure discussed above when applied to integrals of the kind of Eq. (1.32) may lead to severe inaccuracies a large values of x_β. This is due to the fact that the integral extends down to zero and the naive procedure does not make use of the low-x grids effectively truncating the integral to increasingly larger values of x.

In the following we discuss a more convoluted but more effective way to exploit the subgrids that solves the problem mentioned above, *i.e.* the lack of exploitation of the large-x grids when x_β is small and the lack of exploitation of the low-x grids when x_β is large. In this approach the switch from one grid to the other does not happen, as above, at the level of the integral but at the level of the *integrand*.

In this respect, the basic observation is that, thanks to the locking of the subgrids, the integral for a DGLAP-like convolution as in Eq. (1.1) computed in the point x_β of the joint grid can be decomposed as follows:

$$I(x_\beta) = \sum_{g=0}^{N_g=1} \int_{x_\beta/\max[x_\beta, x_{\gamma^{(g)}}]}^{x_\beta/\max[x_\beta, x_{\gamma^{(g+1)}}]} dy\, O(y) f\left(\frac{x_\beta}{y}\right) = \sum_{g=1}^{N_g} I^{(g)}(x_\beta)\,, \tag{1.39}$$

while for and ERBL-like convolution as that in Eq. (1.32) the decomposition reads:

$$J(x_\beta) = \sum_{g=0}^{N_g-1} \int_{x_\beta/x_{\gamma^{(g+1)}}}^{x_\beta/x_{\gamma^{(g)}}} dy\, O(y) f\left(\frac{x_\beta}{y}\right) = \sum_{g=1}^{N_g} J^{(g)}(x_\beta)\,, \tag{1.40}$$

where $\gamma^{(g)}$ is the index corresponding to the lower bound of the g-th grid on the joint grid with the conditions $\gamma^{(0)} = 0$ and $\gamma^{(N_g)} = N_x$. It is interesting to observe that if $x_{\gamma^{(g+1)}} \leq x_\beta$, the integration bounds of the integral in the g-th term in Eq. (1.39) coincide giving a vanishing contribution. Therefore, only the subgrids having $x_{\gamma^{(g+1)}} > x_\beta$ will eventually contribute to the full integral. Conversely, for Eq. (1.40) all grids contribute to the full convolution integral no matter the position of x_β w.r.t. the subgrid bounds.

Let us first consider the g-th contribution to the integral in Eq. (1.40), $J^{(g)}$, and apply the usual interpolation procedure on the g-th subgrid. To do so, we need to take into account the fact that the g-th subgrid starts from the index $\gamma^{(g)}$ of the joint grid and that in the range $\gamma^{(g)} \leq \alpha \leq \gamma^{(g+1)}$ joint grid and g-th subgrid coincide up to an index offset given by $\gamma^{(g)}$ (*i.e.* $x_\alpha = x^{(g)}_{\alpha-\gamma^{(g)}}$). This gives:

$$J^{(g)}(x_\beta) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} \overline{d}_\alpha \int_{\overline{c}}^{\overline{d}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} J^{(g)}_{\beta\alpha}\, \overline{d}_\alpha\,, \tag{1.41}$$

where we have exploited the fact that, if $\alpha \geq \gamma^{(g+1)} + \kappa^{(g)}$, the lower bounds of the sum above exceeds to upper one giving no contribution to set the upper limit of the sum. In addition, the integration bounds are defined as:

$$\begin{aligned} \ln \overline{c} &= (\beta - \alpha - 1 + \max[0, \alpha + 1 - \gamma^{(g+1)}])\delta x^{(g)}\,, \\ \ln \overline{d} &= (\beta - \alpha + \min[\kappa^{(g)}, \alpha - \gamma^{(g)}])\delta x^{(g)}\,. \end{aligned} \tag{1.42}$$

This allows us to write the factors $J^{(g)}_{\beta\alpha}$ in Eq. (1.41) as follows:

$$J^{(g)}_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-\gamma^{(g+1)}]}^{\min[\kappa^{(g)}, \alpha-\gamma^{(g)}]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right)\,, \tag{1.43}$$

with $s = \exp(\delta x^{(g)})$. Therefore, Eq. (1.40) can finally be written as:

$$J(x_\beta) = \sum_{g=0}^{N_g-1} \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} J^{(g)}_{\beta\alpha}\, \overline{d}_\alpha\,. \tag{1.44}$$

We now introduce the discreet θ-function $\theta_{\rho,\sigma}$ defined such that:

$$\theta_{\rho,\sigma} = \begin{cases} 1 & \rho \geq \sigma \\ 0 & \rho < \sigma \end{cases}\,, \tag{1.45}$$

which allows to write:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x+\kappa^{(N_g-1)}-1} \overline{d}_\alpha \left(\sum_{g=0}^{N_g-1} \theta_{\alpha,\gamma^{(g)}} \theta_{\gamma^{(g+1)}+\kappa^{(g)}-1,\alpha} J^{(g)}_{\beta\alpha} \right)\,. \tag{1.46}$$

Assuming that $\overline{d}_\alpha = 0$ for $\alpha \geq N_x$, as is often the case, and defining:

$$J_{\beta\alpha} = \sum_{g=0}^{N_g-1} \theta_{\alpha,\gamma^{(g)}} \theta_{\gamma^{(g+1)}+\kappa^{(g)}-1,\alpha} J^{(g)}_{\beta\alpha}\,, \tag{1.47}$$

one finally obtains:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x-1} J_{\beta\alpha}\overline{d}_\alpha \,. \tag{1.48}$$

In is useful to introduce the helper function h that returns the grid index given as an argument the index on the joint grid and that can be represented as:

$$h(\alpha) = \sum_{g'=0}^{N_g-1} \theta_{\alpha,\gamma^{(g')}} - 1 \,. \tag{1.49}$$

It is easy to see that:

$$h(\gamma^{(g)}) = g \,. \tag{1.50}$$

Making use on the monotonicity of h, Eq. (1.47) can be reduced to:

$$J_{\beta\alpha} = \sum_{g=\max[0,h(\alpha)-1]}^{h(\alpha)} J^{(g)}_{\beta\alpha} \,. \tag{1.51}$$

We can now apply the same procedure to the integral I in Eq. (1.39). As above, one finds:

$$I^{(g)}(x_\beta) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} \overline{d}_\alpha \int_c^d dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} I^{(g)}_{\beta\alpha}\,\overline{d}_\alpha \,, \tag{1.52}$$

where the logarithm of the integration bounds this time read:

$$\begin{aligned} \ln c &= (\beta - \alpha - 1 + \max[0, \alpha + 1 - \max[\beta, \gamma^{(g+1)}]])\delta x^{(g)} \,, \\ \ln d &= (\beta - \alpha + \min[\kappa^{(g)}, \alpha - \max[\beta, \gamma^{(g)}]])\delta x^{(g)} \,. \end{aligned} \tag{1.53}$$

which gives:

$$I^{(g)}_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-\max[\beta,\gamma^{(g+1)}]]}^{\min[\kappa^{(g)},\alpha-\max[\beta,\gamma^{(g)}]]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right) \,, \tag{1.54}$$

allowing one to define:

$$I_{\beta\alpha} = \sum_{g=\max[0,h(\alpha)-1]}^{h(\alpha)} I^{(g)}_{\beta\alpha} \,, \tag{1.55}$$

such that:

$$I(x_\beta) = \sum_{\alpha=0}^{N_x-1} I_{\beta\alpha}\overline{d}_\alpha \,. \tag{1.56}$$

2 Double convolution

In the case of Drell-Yan or Semi-Inclusive DIS (SIDIS), cross sections are the result of double convolutions between partonic cross sections and a pair of non-perturbative distributions. In this case the kind of expressions we are dealing with has the following form:

$$\begin{aligned} J(x_\delta, x_\gamma) &= x_\delta x_\gamma \int_{x_\delta}^1 \frac{dy_1}{y_1} \int_{x_\gamma}^1 \frac{dy_2}{y_2} O(y_1, y_2) d^{(1)}\left(\frac{x_\delta}{y_1}\right) d^{(2)}\left(\frac{x_\gamma}{y_2}\right) \\ &= \int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1, y_2) \left[\frac{x_\delta}{y_1} d^{(1)}\left(\frac{x_\delta}{y_1}\right)\right] \left[\frac{x_\gamma}{y_2} d^{(2)}\left(\frac{x_\gamma}{y_2}\right)\right] \\ &= \sum_{\alpha=0}^{N_x} \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \overline{d}_\alpha^{(2)} \underbrace{\left[\int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1, y_2)\, w_\beta^{(k)}\left(\frac{x_\delta}{y_1}\right) w_\alpha^{(k)}\left(\frac{x_\gamma}{y_2}\right)\right]}_{\Theta^{\beta\delta,\alpha\gamma}} . \end{aligned} \tag{2.1}$$

In Eq. (2.1) we assume that there are no mass corrections and thus the convolutions take the simplest form. In the case of double convolutions, the partonic cross sections O are functions of two variables, y_1 and y_2, and, as in the case of the single convolutions, they get three kinds of contributions in both these variables: local terms proportional to δ-functions, singular terms proportional to plus-prescripted functions, and regular terms. The complication here is that these contributions from the two variables y_1 and y_2 mix and thus, for example, terms local in y_1 and singular in y_2 appear. It is thus necessary to identify the general structure of the function O to see whether it is possible to decompose the double operator $\Theta^{\beta\delta,\alpha\gamma}$ into products of single operators of the kind $\Gamma^{\beta\delta}$ and $\Gamma^{\alpha\gamma}$.

In the case of the NLO corrections to SIDIS, the general structure of the function O can be inferred looking at Eqs. (C.2)-(C.7) of Ref. [1]:

$$\begin{aligned} O(y_1, y_2) &= \mathrm{LL}\,\delta(1-y_1)\delta(1-y_2) + \mathrm{LS}\,\delta(1-y_1)\left[\frac{\ln(1-y_2)}{1-y_2}\right]_+ + \delta(1-y_1)\,\mathrm{LR}(y_2) \\ &+ \mathrm{SL}\left[\frac{\ln(1-y_1)}{1-y_1}\right]_+ \delta(1-y_2) + \mathrm{SS}\left[\frac{1}{1-y_1}\right]_+ \left[\frac{1}{1-y_2}\right]_+ + \left[\frac{1}{1-y_1}\right]_+ \mathrm{SR}(y_2) \\ &+ \mathrm{RL}(y_1)\,\delta(1-y_2) + \mathrm{RS}(y_1)\left[\frac{1}{1-y_2}\right]_+ + \sum_i K_i \mathrm{R}_i^{(1)}(y_1)\mathrm{R}_i^{(2)}(y_2) . \end{aligned} \tag{2.2}$$

It is clear that in Eq. (2.2) all terms factorise into a part that only depends on y_1 and a part that only depends on y_2. This is crucial to use the same technology developed above for the single convolutions. Plugging Eq. (2.2) into Eq. (2.1), one finds that:

$$\begin{aligned} \Theta^{\beta\delta,\alpha\gamma} &= \mathrm{LL}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{LS}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{S1}}_{\alpha\gamma} + \Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{LR}}_{\alpha\gamma} \\ &+ \mathrm{SL}\,\Gamma^{\mathrm{S1}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{SS}\,\Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{SR}}_{\alpha\gamma} \\ &+ \Gamma^{\mathrm{RL}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \Gamma^{\mathrm{RS}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \sum_i K_i \Gamma^{R_i^{(1)}}_{\beta\delta}\Gamma^{R_i^{(2)}}_{\alpha\gamma} \end{aligned} \tag{2.3}$$

with:

$$\begin{aligned} \Gamma^{\mathrm{L}}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\,\delta(1-z) w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) = w_\beta^{(k)}(x_\alpha) = \delta_{\alpha\beta} \\ \Gamma^{\mathrm{S}n}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz \frac{\ln^n(1-z)}{1-z}\left[w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) - \delta_{\alpha\beta}\right] + \frac{1}{(n+1)!}\ln^{n+1}(1-c_{\alpha\beta})\,\delta_{\alpha\beta} \\ \Gamma^{f}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, f(z)\, w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) \end{aligned} \tag{2.4}$$

where f is a regular function and the integration bounds are defined as:

$$c_{\alpha\beta} = \max(x_\alpha, x_\alpha/x_{\beta+1}) \quad \text{and} \quad d_{\alpha\beta} = \min(1, x_\alpha/x_{\beta-k})\,. \tag{2.5}$$

In general terms, it should always be possible to write an object of the kind of $\Theta^{\beta\delta,\alpha\gamma}$ as series of bilinear terms:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_j C_j \Gamma_j^{(1),\beta\delta} \Gamma_j^{(2),\alpha\gamma} \tag{2.6}$$

where C_j is a scalar, and $\Gamma_j^{(1),\beta\delta}$ and $\Gamma_j^{(2),\alpha\gamma}$ are single operators that can be computed using the technology discussed in the previous section. Plugging Eq. (2.6) into Eq. (2.1), one finds that:

$$J(x_\delta, x_\gamma) = \sum_j C_j f_j^{(1),\delta} f_j^{(2),\gamma}\,, \tag{2.7}$$

where we have defined:

$$f_j^{(1),\delta} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \Gamma_j^{(1),\beta\delta} \quad \text{and} \quad f_j^{(2),\gamma} \equiv \sum_{\alpha=0}^{N_x} \overline{d}_\alpha^{(2)} \Gamma_j^{(2),\alpha\gamma}\,. \tag{2.8}$$

Eq. (2.7) shows that, in the hypothesis that the double function $O(y_1, y_2)$ can be expressed in terms of a series of terms where the dependence on y_1 and y_2 factorizes[1], the double convolution in Eq. (2.1) can be expressed as a series of bilinear terms of distributions ($f_j^{(1),\delta}$ and $f_j^{(2),\gamma}$) singularly obtained as convolutions of a limited number of single operators with distributions. This is a particularly useful achievement that allows us to compute double convolutions without the need of extending the integration and the interpolation procedures with an obvious gain in accuracy and performance. As a matter of fact, the same argument can be extended to a multiple convolution of the function $O(\{y_i\})$, which again can be expressed as a series of n-linear terms, with $i = 1, \ldots, n$, with n distributions:

$$J(\{x_{\alpha_i}\}) = \sum_j C_j \prod_{i=1}^{n} f_j^{(i),\alpha_i}\,, \tag{2.9}$$

with:

$$f_j^{(i),\alpha_i} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(i)} \Gamma_j^{(i),\beta\alpha_i}\,. \tag{2.10}$$

This technology could be useful for more complicated observables, like cross sections in pp collisions with an identified hadron in the final state, that requires for example three convolutions.

The challenging part of the technology just presented resides in the "pre-processing" of the function $O(y_1, y_2)$, that is the analytical work required to disentangle the different terms. This step however has to be taken only once.

Before employing this procedure for any concrete application, it is appropriate to connect Eq. (2.3) to Eq. (2.6) by identifying number and form of the coefficients and operators involved. Specifically, assuming that the series in the last term in the r.h.s. of Eq. (2.3) has r terms, the series in Eq. (2.6) will have $8 + r$ terms, that is:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_{j=1}^{8+r} C_j \Gamma_j^{(1),\beta\delta} \Gamma_j^{(2),\alpha\gamma} \tag{2.11}$$

[1] This is the case for SIDIS and DY up to NLO but we expect this feature to hold also beyond, despite complications due to the more involved plus-prescripted functions are also expected.

with:

$$\begin{array}{lcll}
j=1 & : & C_1=\mathrm{LL}, & \Gamma_1^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_1^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=2 & : & C_2=\mathrm{LS}, & \Gamma_2^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_2^{(2),\alpha\gamma}=\Gamma^{\mathrm{S1}}_{\alpha\gamma},\\
j=3 & : & C_3=1, & \Gamma_3^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_3^{(2),\alpha\gamma}=\Gamma^{\mathrm{LR}}_{\alpha\gamma},\\
j=4 & : & C_4=\mathrm{SL}, & \Gamma_4^{(1),\beta\delta}=\Gamma^{\mathrm{S1}}_{\beta\delta}, \quad \Gamma_4^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=5 & : & C_5=\mathrm{SS}, & \Gamma_5^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, \quad \Gamma_5^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma},\\
j=6 & : & C_6=1, & \Gamma_6^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, \quad \Gamma_6^{(2),\alpha\gamma}=\Gamma^{\mathrm{SR}}_{\alpha\gamma},\\
j=7 & : & C_7=1, & \Gamma_7^{(1),\beta\delta}=\Gamma^{\mathrm{RL}}_{\beta\delta}, \quad \Gamma_7^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=8 & : & C_8=1, & \Gamma_8^{(1),\beta\delta}=\Gamma^{\mathrm{RS}}_{\beta\delta}, \quad \Gamma_8^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma},\\
j=9 & : & C_9=K_1 & \Gamma_9^{(1),\beta\delta}=\Gamma^{R_1^{(1)}}_{\beta\delta}, \quad \Gamma_9^{(2),\alpha\gamma}=\Gamma^{R_1^{(2)}}_{\alpha\gamma},\\
\vdots & & & \\
j=8+r & : & C_{8+r}=K_r & \Gamma_{8+r}^{(1),\beta\delta}=\Gamma^{R_r^{(1)}}_{\beta\delta}, \quad \Gamma_{8+r}^{(2),\alpha\gamma}=\Gamma^{R_r^{(2)}}_{\alpha\gamma}
\end{array} \tag{2.12}$$

It should be noted that, despite the large number of terms in the series in Eq. (2.11), the number of operators to be precomputed it pretty limited. In addition, in many cases many of the terms of the series are zero so the number of contributions is greatly reduced.

We can now apply this procedure up to NLO in QCD to two specific cases: SIDIS first and DY second.

2.1 Semi-inclusive deep inelastic scattering (SIDIS)

the structure of the SIDIS observables and the expressions for the respective hard cross sections can be found in Ref. [1]. Following this paper, the SIDIS differential cross section for the exchange of a virtual photon can be written as:

$$\frac{d^3\sigma}{dx\,dy\,dz}=\frac{2\,\pi\alpha^2}{xyQ^2}\left[(1+(1-y)^2)2xF_1(x,z,Q^2)+2(1-y)F_L(x,z,Q^2)\right]\,, \tag{2.13}$$

with $Q^2=-q^2$ the (negative) virtuality of the exchanged photon, x and z the momentum fractions of PDFs and FFs, and $Q^2=xys$ the definition of the inelasticity y in terms of the squared collision energy in the center of mass s. Notice that, as compared to Ref. [1], we have absorbed into the definition of F_L a factor x as customary in the definition of the longitudinal structure function in inclusive DIS.

We now use the Callan-Gross relation:

$$F_2=2xF_1+F_L \tag{2.14}$$

to replace $2xF_1$ with F_2 in Eq. (2.13):

$$\frac{d^3\sigma}{dx\,dy\,dz}=\frac{2\,\pi\alpha^2}{xyQ^2}\left[Y_+F_2(x,z,Q^2)-y^2F_L(x,z,Q^2)\right]\,, \tag{2.15}$$

where we have defined:

$$Y_+=1+(1-y)^2\,. \tag{2.16}$$

It is also useful to write Eq. (2.15) as differential in x, Q^2, and z:

$$\frac{d^3\sigma}{dx\,dQ^2\,dz}=\frac{2\,\pi\alpha^2}{xQ^4}\left[Y_+F_2(x,z,Q)-y^2F_L(x,z,Q)\right]\,. \tag{2.17}$$

The structure functions F_2 and F_L are given at NLO by:

$$\begin{aligned}
F_{2,L}(x,z,Q) &= x\sum_{q,\overline{q}}e_q^2\Big[q(x,Q)\otimes C_{qq}^{2,L}(x,z)\otimes D_q(z,Q)\\
&+ q(x,Q)\otimes C_{gq}^{2,L}(x,z)\otimes D_g(z,Q)+g(x,Q)\otimes C_{qg}^{2,L}(x,z)\otimes D_q(z,Q)\Big]\,,
\end{aligned} \tag{2.18}$$

where $\{q,g\}$ are the quark and gluon PDFs and $\{D_q,D_g\}$ are the quark and gluon FFs, e_q is the electric charge of the quark q and $\{C_{qq}^{2,L},C_{qg}^{2,L},C_{gq}^{2,L}\}$ are the relevant partonic cross sections. The partonic cross sections allow for a perturbative expansion in power of α_s:

$$C=\sum_{n=0}\left(\frac{\alpha_s}{4\pi}\right)^n C^{(n)} \tag{2.19}$$

that we truncate to NLO, *i.e.* to $n = 1$. At LO ($n = 0$) we have the simple expression:

$$C_{qq}^{2,(0)}(x,z) = \delta(1-x)\delta(1-z) \quad \text{and all others zero.} \tag{2.20}$$

At NLO ($n = 1$) we take the expressions from Appendix C of Ref. [1] being careful to take into account an additional factor two due to the difference in the expansion parameter ($\alpha_s/4\pi$ rather than $\alpha_s/2\pi$) and to combine the expressions for F_1 and F_L using Eq. (2.14) to obtain the partonic cross sections for F_2. We start with the partonic cross sections for F_L that read:

$$\begin{array}{lcl} C_{qq}^{L,(1)} &=& 8C_F xz\,, \\ C_{gq}^{L,(1)} &=& 8C_F x(1-z)\,, \\ C_{qg}^{L,(1)} &=& 8x(1-x)\,, \end{array} \tag{2.21}$$

while those for F_2 read:

$$\begin{aligned} \frac{C_{qq}^{2,(1)}}{2C_F} &= -8\delta(1-x)\delta(1-z) + 2\delta(1-x)\left(\frac{\ln(1-z)}{1-z}\right)_+ + \delta(1-x)\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\ &+ 2\left(\frac{\ln(1-x)}{1-x}\right)_+ \delta(1-z) + 2\left(\frac{1}{1-x}\right)_+\left(\frac{1}{1-z}\right)_+ - \left(\frac{1}{1-x}\right)_+(1+z) \\ &+ \left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\delta(1-z) - (1+x)\left(\frac{1}{1-z}\right)_+ + (2+6xz) \\ \frac{C_{gq}^{2,(1)}}{2C_F} &= \delta(1-x)\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right] \\ &+ \left(\frac{1}{1-x}\right)_+ \frac{1+(1-z)^2}{z} \\ &+ 2(1+3x) - 6xz - (1+x)\frac{1}{z} \\ C_{qg}^{2,(1)} &= \left[(x^2+(1-x)^2)\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\right]\delta(1-z) + (x^2+(1-x)^2)\left(\frac{1}{1-z}\right)_+ \\ &+ 2(-1+6x-6x^2) + (x^2+(1-x)^2)\frac{1}{z} \end{aligned} \tag{2.22}$$

By inspection of Eqs. (2.21) and (2.22) we can deduce the various coefficients of Eq. (2.2). In particular, as usual, F_L involves only regular functions so that all contributions are zero but the fully regular ones:

$$\begin{array}{lcl} C_{qq}^{L,(1)}(x,z) &:& K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = z\,, \\ C_{gq}^{L,(1)}(x,z) &:& K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = 1-z\,, \\ C_{qg}^{L,(1)}(x,z) &:& K_1 = 8\,, \quad R_1^{(1)}(x) = x(1-x)\,, \quad R_1^{(2)}(z) = 1\,. \end{array} \tag{2.23}$$

For F_2 the situation is more complicated but we can still identify the different contributions:

$$
\begin{aligned}
C_{qq}^{2,(1)} \;:\;& \mathrm{LL} = -16C_F\,, \quad \mathrm{LS} = 4C_F\,, \quad \mathrm{LR}(z) = 2C_F\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\
& \mathrm{SL} = 4C_F\,, \quad \mathrm{SS} = 4C_F\,, \quad \mathrm{SR}(z) = -2C_F(1+z) \\
& \mathrm{RL}(x) = 2C_F\left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\,, \quad \mathrm{RS}(x) = -2C_F(1+x)\,, \\
& \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = 12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \\
C_{gq}^{2,(1)} \;:\;& \mathrm{LR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right]\,, \\
& \mathrm{SR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\right]\,, \\
& \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1+3x,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = -12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \\
& \left\{K_3 = -2C_F,\, R_3^{(1)}(x) = 1+x,\, R_3^{(2)}(z) = \frac{1}{z}\right\}\,, \\
C_{qg}^{2,(1)} \;:\;& \mathrm{RL}(x) = \left[x^2 + (1-x)^2\right]\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\,, \quad \mathrm{RS}(x) = x^2 + (1-x)^2\,, \\
& \left\{K_1 = 2,\, R_1^{(1)}(x) = -1 + 6x - 6x^2,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = 1,\, R_2^{(1)}(x) = x^2 + (1-x)^2,\, R_2^{(2)}(z) = \frac{1}{z}\right\}\,.
\end{aligned}
\tag{2.24}
$$

Analogously, for the only LO partonic cross sections we find that:

$$
C_{qq}^{2,(0)} \;:\; \mathrm{LL} = 1\,. \tag{2.25}
$$

All the coefficients that are not mentioned are equal to zero. We can now implement explicitly Eq. (2.3). The one thing that is left to sort out is the structure of the structure functions in terms of the appropriate PDF and FF combinations. Considering Eq. (2.18), we observe that none of the coefficient functions depends on the particular quark flavour (this is a feature of the ZM scheme). Therefore, simplifying the notation, we can rewrite Eq. (2.18) as:

$$
F = C_{qq}\sum_q e_q^2\left[qD_q + \overline{q}D_{\overline{q}}\right] + C_{gq}\sum_q e_q^2\left[q + \overline{q}\right]\, D_g + C_{qg}g\sum_q e_q^2\left[D_q + D_{\overline{q}}\right]\,, \tag{2.26}
$$

where now the sums run only over the quark flavours and not the antiflavours.

2.2 Drell Yan (DY)

In this section we apply to the Drell-Yan (DY) process the same procedure followed above for SIDIS. As a matter of fact, SIDIS and DY are strictly connected in that DY can be regarded as the time-like counterpart of SIDIS. As a consequence, the structure of the relevant observables as well as the form of the expressions involved are very similar. Therefore, the application of the method described above is straightforward.

References

[1] D. de Florian, M. Stratmann and W. Vogelsang, Phys. Rev. D **57** (1998) 5811 doi:10.1103/PhysRevD.57.5811 [hep-ph/9711387].