Convolution integrals

Contents

1 Structure of the integrals

The general structure of the integrals considered in APFEL++ has the form of a Mellin convolution between an *operator* O, this is a potentially complicated known function such as a splitting function or a partonic cross section,[1] and a *distribution* function d, as for example a parton distribution function (PDF) or a fragmentation function (FF). The explicit form of these integrals reads:

$$I(x) = x\int_0^1 dz \int_0^1 dy\, O(z) d(y) \delta(x - yz) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x\int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right) . \tag{1.1}$$

Typically in the presence of mass effects, the integration phase space may be modified and the convolution in Eq. (1.1) is generalised as:

$$I(x,\eta) = x\int_{x/\eta}^1 \frac{dz}{z} O(z,\eta) d\left(\frac{x}{\eta z}\right) , \tag{1.2}$$

where $\eta \leq 1$, with $\eta = 1$ reproducing Eq. (1.1). However, it is convenient to rewrite the integral in Eq. (1.2) in a form in which the lower bound of the integral is equal to x. This is easily done by performing the change of variable $y = \eta z$, so that:

$$I(x,\eta) = \int_x^\eta dy\, O\left(\frac{y}{\eta},\eta\right) \frac{x}{y} d\left(\frac{x}{y}\right) . \tag{1.3}$$

In order to enable the precomputation the expensive part of the integral in Eq. (1.3), we use the standard interpolation formula on the distribution d:

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right) , \tag{1.4}$$

where α runs over the nodes of a grid in x and the weights w_α are assumed to be Langrange polynomials of degree k. Now we plug Eq. (1.4) into Eq. (1.3) and specialise the computation of the integral I to the girid point x_β. This gives:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^\eta dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.5}$$

where we have defined $\overline{d}_\alpha = x_\alpha d(x_\alpha)$. As shown in the section devoted to the interpolation procedure, under some specific conditions the interpolating functions are such that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d\,, \tag{1.6}$$

[1] More precisely, O is in general a distribution that may contain δ functions and +-distributions.

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k})\,. \tag{1.7}$$

Therefore, Eq. (1.3) can be adjusted as:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_c^d dy\, O\left(\frac{y}{\eta},\eta\right)\, w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right)\,. \tag{1.8}$$

Finally, we change the integration variable back to $z = y/\eta$ so that Eq. (1.8) becomes:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz\, O\left(z,\eta\right)\, w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)\right]}_{\Gamma_{\alpha\beta}}\,. \tag{1.9}$$

To summarise, Eq. (1.9) allows one to compute the integral I on the nodes of a given grid as a weighted sum of the values of distribution d on the nodes themselves. The weights are given by the appropriate integral of the operator O and the interpolating functions $w_\alpha^{(k)}$. The value of I for a generic value of x can finally be obtained by interpolation using a formula similar to Eq. (1.4). Importantly, Eq. (1.9) provides a fast way of computing convolution integrals. Indeed, the weights $\Gamma_{\alpha\beta}$ can be precomputed and stored once and for all and the integral I can be quickly computed with different distributions d by simply taking the weighted average.

We now need an operational way to compute the weights $\Gamma_{\alpha\beta}$. To do so, it is necessary to know the general structure of the operator O. Quite generally, the operator O splits as follows:

$$O(z,\eta) = R(z,\eta) + \sum_i \Big[P^{(i)}(z,\eta)\Big]_+ S^{(i)}(z,\eta) + L(\eta)\delta(1-z)\,, \tag{1.10}$$

where R and $S^{(i)}$ are a regular functions at $z = 1$, that is:

$$R(1,\eta) = \lim_{z\to 1} R(z,\eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1,\eta) = \lim_{z\to 1} S^{(i)}(z,\eta) = J^{(i)}(\eta)\,, \tag{1.11}$$

being K, $J^{(i)}$, and L a finite function of η. The functions $P^{(i)}$ are instead singular at $z = 1$ and non-integrable in the limit $\eta \to 1$ and thus are regularised through the +-prescription. Plugging Eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in Eq. (1.9) and making use of the definition of +-prescription gives:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \eta \int_{c/\eta}^{d/\eta} dz \left\{ R\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) + \sum_i P^{(i)}(z,\eta)\left[S^{(i)}\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\
&+ \eta\left[L(\eta) - \sum_i S^{(i)}(1,\eta)\int_0^{c/\eta} dz P^{(i)}(z,\eta)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,,
\end{aligned} \tag{1.12}$$

that can be further manipulated changing the integration variable into $y = \eta z$:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \int_c^d dy \left\{ R\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) + \sum_i P^{(i)}\left(\frac{y}{\eta},\eta\right)\left[S^{(i)}\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\
&+ \eta\left[L(\eta) + \sum_i S^{(i)}(1,\eta) Q^{(i)}\left(\frac{c}{\eta},\eta\right)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,.
\end{aligned} \tag{1.13}$$

where we have defined:

$$Q^{(i)}(a,\eta) \equiv -\int_0^a dz P^{(i)}\left(z,\eta\right)\,. \tag{1.14}$$

These integrals can, most of the times, be computed analytically.

Eqs. (1.12)-(1.13) expose the full potential complexity of the task of computing the weights $\Gamma_{\alpha\beta}$. However, in many practical applications some simplifications apply. For example, in the case of the perturbative coefficients

of the ($\overline{\text{MS}}$) splitting functions there are two simplications: the first is that $\eta = 1$, the second is that there is one single term in the sum over i ($i = 0$) such that and the general form of the function P is:

$$P^{(0)}(z,\eta) \to \frac{1}{1-z}\,, \tag{1.15}$$

so that:

$$Q^{(0)}(a,\eta) = \ln(1-a)\,. \tag{1.16}$$

Considering that:

$$w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}\,, \tag{1.17}$$

and that the expressions can always be manipulated in such a way that the coefficient of the +-prescripted term S is a constant, one finds that:

$$\Gamma_{\beta\alpha} = \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} + \left[S\ln\left(1-c\right) + L\right]\delta_{\alpha\beta}\,. \tag{1.18}$$

The same kind of simplifications apply to the case of the perturbative Zero-Mass (ZM) coefficient functions with the only exception that the sum over i extends to more terms depending on the perturbative order. For example at $\mathcal{O}(\alpha_s)$, *i.e.* at NLO, the general form of the coefficient functions reads:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S^{(0)} + S^{(1)}\ln(1-z)}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} \\ &+ \left[S^{(0)}\ln\left(1-c\right) + \frac{1}{2}S^{(1)}\ln^2\left(1-c\right) + L\right]\delta_{\alpha\beta}\,.\end{aligned} \tag{1.19}$$

Thing get more complicated when considering massive coefficient functions. In this case, the role of η is played by the ration m/Q, where m is the mass of the quark and Q the hard scale. In addition, the +-prescripted contributions can become more convoluted.

As an aside, it is interesting to notice that we have silently decided to use one of the two possible ways to compute the convolution integral $I(x)$ in Eq. (1.1). Specifically, we have chosen the rightmost equality. However, the equality:

$$I(x) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z)\,, \tag{1.20}$$

is equally valid. If one choses this particular version of the convolution integral and follows the interpolation procedure discussed above, the result for I computed in x_β would be identical to Eq. (1.9) (with $\eta = 1$) except for the fact that the weights $\Gamma_{\beta\alpha}$ are computed as:

$$\Gamma_{\beta\alpha} = \int_a^b dz\, O\left(\frac{x_\beta}{z}\right) w_\alpha^{(k)}(z) \tag{1.21}$$

with:

$$a \equiv \max(x_\beta, x_{\alpha-k}) \quad \text{and} \quad b \equiv \min(1, x_{\alpha+1})\,. \tag{1.22}$$

This immediately implies that the integral in Eq. (1.9) and that in Eq. (1.21) have to coincide. This is indeed the case and the only reason to choose Eq. (1.9) rather than Eq. (1.21) is convenience. Specifically, Eq. (1.21) implies computing the operator O at x_β/z rather than at z and this makes the treatment of δ-functions and +-distributions possibly present in O a little more delicate. However, Eq. (1.21) remains a valid alternative that we just decided not to pursue.

1.1 Double convolutions

The technique discussed above to compute convolution integrals as a weighted sums of a given input distributions d can be, under certain circumstances, generalised to convolution integrals involving *two* distributions. Relevant examples are the Drell-Yan (DY) and the semi-inclusive deep-inelastic scattering (SIDIS) cross sections integrated over the transverse momentum of the virtual boson. In these cases the cross section is proportional to

a double convolutions between partonic cross sections and a pair of non-perturbative distributions. The general form of this double convolution is:

$$J(x_1,x_2) = x_1 x_2 \int_{x_1}^1 \frac{dy_1}{y_1} \int_{x_2}^1 \frac{dy_2}{y_2} O(y_1,y_2) d^{(1)}\left(\frac{x_1}{y_1}\right) d^{(2)}\left(\frac{x_2}{y_2}\right) . \tag{1.23}$$

Applying the same interpolation procedure as in the single-convolution case gives:

$$\begin{aligned} J(x_\delta,x_\gamma) &= \int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2) \left[\frac{x_\delta}{y_1} d^{(1)}\left(\frac{x_\delta}{y_1}\right)\right]\left[\frac{x_\gamma}{y_2} d^{(2)}\left(\frac{x_\gamma}{y_2}\right)\right] \\ &= \sum_{\alpha=0}^{N_x}\sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \overline{d}_\alpha^{(2)} \underbrace{\left[\int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2)\, w_\beta^{(k)}\left(\frac{x_\delta}{y_1}\right) w_\alpha^{(k)}\left(\frac{x_\gamma}{y_2}\right)\right]}_{\Theta^{\beta\delta,\alpha\gamma}} . \end{aligned} \tag{1.24}$$

In Eq. (1.24) we have assumes that there are no mass corrections and thus the convolutions take the simplest form. In the case of double convolutions, the operator O is a function of two variables, y_1 and y_2, and, as in the case of the single convolutions, it receives three kinds of contribution in both these variables: local terms proportional to δ-functions, singular terms proportional to +-prescripted functions, and regular terms. The complication here is that these contributions from the two variables y_1 and y_2 mix and thus, for example, terms local in y_1 and singular in y_2 may also appear. It is thus necessary to identify the general structure of the function O to see whether it is possible to decompose the double operator $\Theta^{\beta\delta,\alpha\gamma}$ into products of single operators.

As we will explicitly show below, in the case of SIDIS up to $\mathcal{O}(\alpha_s)$ (NLO) the general structure of the function O can be inferred looking at Eqs. (C.2)-(C.7) of Ref. [1]:

$$\begin{aligned} O(y_1,y_2) &= \mathrm{LL}\,\delta(1-y_1)\delta(1-y_2) + \mathrm{LS}\,\delta(1-y_1)\left[\frac{\ln(1-y_2)}{1-y_2}\right]_+ + \delta(1-y_1)\,\mathrm{LR}(y_2) \\ &+ \mathrm{SL}\left[\frac{\ln(1-y_1)}{1-y_1}\right]_+ \delta(1-y_2) + \mathrm{SS}\left[\frac{1}{1-y_1}\right]_+\left[\frac{1}{1-y_2}\right]_+ + \left[\frac{1}{1-y_1}\right]_+ \mathrm{SR}(y_2) \\ &+ \mathrm{RL}(y_1)\,\delta(1-y_2) + \mathrm{RS}(y_1)\left[\frac{1}{1-y_2}\right]_+ + \sum_i K_i \mathrm{R}_i^{(1)}(y_1)\mathrm{R}_i^{(2)}(y_2) . \end{aligned} \tag{1.25}$$

It is clear that in Eq. (1.25) all terms factorise into a part that only depends on y_1 and a part that only depends on y_2. This is the crucial feature that enables use the technology developed above for the single convolutions. Plugging Eq. (1.25) into Eq. (1.24), one finds that:

$$\begin{aligned} \Theta^{\beta\delta,\alpha\gamma} &= \mathrm{LL}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{LS}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{S1}}_{\alpha\gamma} + \Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{LR}}_{\alpha\gamma} \\ &+ \mathrm{SL}\,\Gamma^{\mathrm{S1}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{SS}\,\Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{SR}}_{\alpha\gamma} \\ &+ \Gamma^{\mathrm{RL}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \Gamma^{\mathrm{RS}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \sum_i K_i \Gamma^{R_i^{(1)}}_{\beta\delta}\Gamma^{R_i^{(2)}}_{\alpha\gamma} \end{aligned} \tag{1.26}$$

with:

$$\begin{aligned} \Gamma^{\mathrm{L}}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\,\delta(1-z) w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) = w_\beta^{(k)}(x_\alpha) = \delta_{\alpha\beta} \\ \Gamma^{\mathrm{S}n}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz \frac{\ln^n(1-z)}{1-z}\left[w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) - \delta_{\alpha\beta}\right] + \frac{1}{(n+1)!}\ln^{n+1}(1-c_{\alpha\beta})\,\delta_{\alpha\beta} \\ \Gamma^{f}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, f(z)\, w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) \end{aligned} \tag{1.27}$$

where f is a regular function and the integration bounds are defined as:

$$c_{\alpha\beta} = \max(x_\alpha, x_\alpha/x_{\beta+1}) \quad \text{and} \quad d_{\alpha\beta} = \min(1, x_\alpha/x_{\beta-k}) . \tag{1.28}$$

In general terms, we assume that it is always be possible to write an object of the kind of $\Theta^{\beta\delta,\alpha\gamma}$ as series of bilinear terms:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_j C_j \Gamma_j^{(1),\beta\delta} \Gamma_j^{(2),\alpha\gamma} \tag{1.29}$$

where C_j are scalar coefficients, and the weights $\Gamma_j^{(1),\beta\delta}$ and $\Gamma_j^{(2),\alpha\gamma}$ can be computed using the technology discussed in the previous section. Plugging Eq. (1.29) into Eq. (1.24), one finds that:

$$J(x_\delta, x_\gamma) = \sum_j C_j f_j^{(1),\delta} f_j^{(2),\gamma}\,, \tag{1.30}$$

where we have defined:

$$f_j^{(1),\delta} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \Gamma_j^{(1),\beta\delta} \quad \text{and} \quad f_j^{(2),\gamma} \equiv \sum_{\alpha=0}^{N_x} \overline{d}_\alpha^{(2)} \Gamma_j^{(2),\alpha\gamma}\,. \tag{1.31}$$

Eq. (1.30) shows that, under the hypothesis that the operator $O(y_1, y_2)$ can be expressed as a series of terms whose dependence on y_1 and y_2 factorizes,[2] the double convolution in Eq. (1.24) is given by a series of bilinear distributions ($f_j^{(1),\delta}$ and $f_j^{(2),\gamma}$) singularly obtained as convolutions of single operators with the input distributions $d^{(1)}$ and $d^{(2)}$. This is a particularly useful achievement that allows us to compute double convolutions without the need of extending the integration and the interpolation procedures to two dimensions with an obvious gain in accuracy and performance. As a matter of fact, the same argument can be extended to a multiple convolution of the function $O(\{y_i\})$, which again can be expressed as a series of n-linear terms, with $i = 1, \ldots, n$, with n distributions:

$$J(\{x_{\alpha_i}\}) = \sum_j C_j \prod_{i=1}^{n} f_j^{(i),\alpha_i}\,, \tag{1.32}$$

with:

$$f_j^{(i),\alpha_i} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(i)} \Gamma_j^{(i),\beta\alpha_i}\,. \tag{1.33}$$

This technology could be useful for more complicated observables, like cross sections in pp collisions with an identified hadron in the final state, that requires for example three convolutions.

The challenging part of the procedure just presented resides in the "pre-processing" of the function $O(y_1, y_2)$ that has to be analytically manipulated to disentangle the single terms. This step, however, has to be taken only once.

Before going into a concrete application, it is useful to connect Eq. (1.26) to Eq. (1.29) by identifying number and form of the coefficients and weights involved. Specifically, assuming that the series in the last term in the r.h.s. of Eq. (1.26) has r terms, the series in Eq. (1.29) will have $8 + r$ terms, that is:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_{j=1}^{8+r} C_j \Gamma_j^{(1),\beta\delta} \Gamma_j^{(2),\alpha\gamma} \tag{1.34}$$

with:

$$\begin{array}{lcllll}
j=1 & : & C_1 = \mathrm{LL}, & \Gamma_1^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{L}}, & \Gamma_1^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{L}}, \\
j=2 & : & C_2 = \mathrm{LS}, & \Gamma_2^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{L}}, & \Gamma_2^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{S1}}, \\
j=3 & : & C_3 = 1, & \Gamma_3^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{L}}, & \Gamma_3^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{LR}}, \\
j=4 & : & C_4 = \mathrm{SL}, & \Gamma_4^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{S1}}, & \Gamma_4^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{L}}, \\
j=5 & : & C_5 = \mathrm{SS}, & \Gamma_5^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{S0}}, & \Gamma_5^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{S0}}, \\
j=6 & : & C_6 = 1, & \Gamma_6^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{S0}}, & \Gamma_6^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{SR}}, \\
j=7 & : & C_7 = 1, & \Gamma_7^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{RL}}, & \Gamma_7^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{L}}, \\
j=8 & : & C_8 = 1, & \Gamma_8^{(1),\beta\delta} = \Gamma_{\beta\delta}^{\mathrm{RS}}, & \Gamma_8^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{\mathrm{S0}}, \\
j=9 & : & C_9 = K_1 & \Gamma_9^{(1),\beta\delta} = \Gamma_{\beta\delta}^{R_1^{(1)}}, & \Gamma_9^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{R_1^{(2)}}, \\
\vdots & & & & \\
j=8+r & : & C_{8+r} = K_r & \Gamma_{8+r}^{(1),\beta\delta} = \Gamma_{\beta\delta}^{R_r^{(1)}}, & \Gamma_{8+r}^{(2),\alpha\gamma} = \Gamma_{\alpha\gamma}^{R_r^{(2)}}
\end{array} \tag{1.35}$$

[2] This is the case for SIDIS and DY up to NLO. However, one may expect that this feature holds also beyond.

It should be noted that, despite the large number of terms in the series in Eq. (1.34), the number of weights to be precomputed is usually pretty limited. In addition, in many cases many of the terms of the series are zero so that the number of contributions is further reduced. We can now apply this procedure up to NLO in QCD to two specific cases: SIDIS first and DY second (incomplete).

1.1.1 Semi-inclusive deep inelastic scattering (SIDIS)

the structure of the (p_T-integrated) SIDIS cross section and the expressions of the respective hard coefficient functions can be found in Ref. [1]. Following this paper, the SIDIS differential cross section for the exchange of a virtual photon can be written as:

$$\frac{d^3\sigma}{dxdydz} = \frac{2\,\pi\alpha^2}{xyQ^2}\left[(1+(1-y)^2)2xF_1(x,z,Q^2) + 2(1-y)F_L(x,z,Q^2)\right]\,, \tag{1.36}$$

with $Q^2 = -q^2$ the (negative) virtuality of the exchanged photon, x and z the momentum fractions that enter the PDFs and the FFs, and $y = Q^2/xs$ the inelasticity given in terms of Q, x, and the collision energy in the center of mass s. Notice that, as compared to Ref. [1], we have absorbed a factor x into the definition of F_L as customary for the longitudinal structure function in inclusive DIS.

We now use the Callan-Gross relation:

$$F_2 = 2xF_1 + F_L \tag{1.37}$$

to replace $2xF_1$ with F_2 in Eq. (1.36):

$$\frac{d^3\sigma}{dxdydz} = \frac{2\,\pi\alpha^2}{xyQ^2}\left[Y_+F_2(x,z,Q^2) - y^2F_L(x,z,Q^2)\right]\,, \tag{1.38}$$

where we have defined:

$$Y_+ = 1+(1-y)^2\,. \tag{1.39}$$

It is also useful to write Eq. (1.38) as differential in x, Q^2, and z:

$$\frac{d^3\sigma}{dx\,dQ^2\,dz} = \frac{2\,\pi\alpha^2}{xQ^4}\left[Y_+F_2(x,z,Q) - y^2F_L(x,z,Q)\right]\,. \tag{1.40}$$

The structure functions F_2 and F_L are given at NLO by:

$$\begin{aligned} F_{2,L}(x,z,Q) &= x\sum_{q,\overline{q}} e_q^2\Big[q(x,Q)\otimes C_{qq}^{2,L}(x,z)\otimes D_q(z,Q) \\ &+ q(x,Q)\otimes C_{gq}^{2,L}(x,z)\otimes D_g(z,Q) + g(x,Q)\otimes C_{qg}^{2,L}(x,z)\otimes D_q(z,Q)\Big]\,, \end{aligned} \tag{1.41}$$

where $\{q,g\}$ are the quark and gluon PDFs and $\{D_q, D_g\}$ are the quark and gluon FFs, e_q is the electric charge of the quark q and $\{C_{qq}^{2,L}, C_{qg}^{2,L}, C_{gq}^{2,L}\}$ are the relevant partonic cross sections. The partonic cross sections allow for a perturbative expansion in power of α_s:

$$C = \sum_{n=0}\left(\frac{\alpha_s}{4\pi}\right)^n C^{(n)} \tag{1.42}$$

that we truncate to NLO, *i.e.* to $n=1$. At LO ($n=0$) we have the simple expression:

$$\begin{aligned} C_{qq}^{2,(0)}(x,z) &= \delta(1-x)\delta(1-z)\,, \\ C_{qg}^{2,(0)}(x,z) &= C_{gq}^{2,(0)}(x,z) = 0\,. \end{aligned} \tag{1.43}$$

At NLO ($n=1$) we take the expressions from Appendix C of Ref. [1] being careful to take into account an additional factor two due to the difference in the expansion parameter ($\alpha_s/4\pi$ rather than $\alpha_s/2\pi$). We also need to combine the expressions for F_1 and F_L using Eq. (1.37) to obtain the partonic cross sections for F_2. We start with the partonic cross sections for F_L that read:

$$\begin{aligned} C_{qq}^{L,(1)} &= 8C_F xz\,, \\ C_{gq}^{L,(1)} &= 8C_F x(1-z)\,, \\ C_{qg}^{L,(1)} &= 8x(1-x)\,, \end{aligned} \tag{1.44}$$

while those for F_2 read:

$$\begin{aligned}
\frac{C_{qq}^{2,(1)}}{2C_F} &= -8\delta(1-x)\delta(1-z) + 2\delta(1-x)\left(\frac{\ln(1-z)}{1-z}\right)_+ + \delta(1-x)\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\
&+ 2\left(\frac{\ln(1-x)}{1-x}\right)_+ \delta(1-z) + 2\left(\frac{1}{1-x}\right)_+ \left(\frac{1}{1-z}\right)_+ - \left(\frac{1}{1-x}\right)_+ (1+z) \\
&+ \left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\delta(1-z) - (1+x)\left(\frac{1}{1-z}\right)_+ + (2+6xz)\,, \\
\frac{C_{gq}^{2,(1)}}{2C_F} &= \delta(1-x)\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right] \\
&+ \left(\frac{1}{1-x}\right)_+ \frac{1+(1-z)^2}{z} \\
&+ 2(1+3x) - 6xz - (1+x)\frac{1}{z}\,, \\
C_{qg}^{2,(1)} &= \left[(x^2+(1-x)^2)\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\right]\delta(1-z) + (x^2+(1-x)^2)\left(\frac{1}{1-z}\right)_+ \\
&+ 2(-1+6x-6x^2) + (x^2+(1-x)^2)\frac{1}{z}\,.
\end{aligned} \tag{1.45}$$

By inspection of Eqs. (1.44) and (1.45) we can deduce the various coefficients of Eq. (1.25). F_L involves only regular functions so that all contributions are zero but the fully regular ones:

$$\begin{array}{lcl}
C_{qq}^{L,(1)}(x,z) & : & K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = z\,, \\
C_{gq}^{L,(1)}(x,z) & : & K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = 1-z\,, \\
C_{qg}^{L,(1)}(x,z) & : & K_1 = 8\,, \quad R_1^{(1)}(x) = x(1-x)\,, \quad R_1^{(2)}(z) = 1\,.
\end{array} \tag{1.46}$$

The situation is more complicated for F_2 but we can still identify the different contributions:

$$\begin{aligned}
C_{qq}^{2,(1)} \;:\; & \mathrm{LL} = -16C_F\,, \quad \mathrm{LS} = 4C_F\,, \quad \mathrm{LR}(z) = 2C_F\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\
& \mathrm{SL} = 4C_F\,, \quad \mathrm{SS} = 4C_F\,, \quad \mathrm{SR}(z) = -2C_F(1+z) \\
& \mathrm{RL}(x) = 2C_F\left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\,, \quad \mathrm{RS}(x) = -2C_F(1+x)\,, \\
& \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = 12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,,
\end{aligned} \tag{1.47}$$

$$
\begin{aligned}
C_{gq}^{2,(1)} \quad : \quad & \mathrm{LR}(z) = 2C_F \left[\frac{1+(1-z)^2}{z} \ln\left[z(1-z)\right] + z \right] , \\
& \mathrm{SR}(z) = 2C_F \left[\frac{1+(1-z)^2}{z} \right] , \\
& \left\{ K_1 = 4C_F,\, R_1^{(1)}(x) = 1+3x,\, R_1^{(2)}(z) = 1 \right\} , \\
& \left\{ K_2 = -12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z \right\} , \\
& \left\{ K_3 = -2C_F,\, R_3^{(1)}(x) = 1+x,\, R_3^{(2)}(z) = \frac{1}{z} \right\} ,
\end{aligned}
\tag{1.48}
$$

$$
\begin{aligned}
C_{qg}^{2,(1)} \quad : \quad & \mathrm{RL}(x) = \left[x^2 + (1-x)^2\right] \ln\left(\frac{1-x}{x}\right) + 2x(1-x) , \quad \mathrm{RS}(x) = x^2 + (1-x)^2 , \\
& \left\{ K_1 = 2,\, R_1^{(1)}(x) = -1 + 6x - 6x^2,\, R_1^{(2)}(z) = 1 \right\} , \\
& \left\{ K_2 = 1,\, R_2^{(1)}(x) = x^2 + (1-x)^2,\, R_2^{(2)}(z) = \frac{1}{z} \right\} .
\end{aligned}
\tag{1.49}
$$

Analogously, for the only LO partonic cross sections we find that:

$$
C_{qq}^{2,(0)} \quad : \quad \mathrm{LL} = 1 . \tag{1.50}
$$

All the coefficients that are not mentioned are equal to zero. We can now explicitly implement Eq. (1.26). The one thing that is left to sort out is the structure of F_2 and F_L in terms of the appropriate PDF and FF combinations. Looking Eq. (1.41), we observe that none of the coefficient functions depends on the particular quark flavour (this is a feature of the ZM scheme). Therefore, simplifying the notation, we can rewrite Eq. (1.41) as:

$$
F \quad = \quad C_{qq} \sum_q e_q^2 \left[q D_q + \overline{q} D_{\overline{q}} \right] + C_{gq} \sum_q e_q^2 \left[q + \overline{q} \right] D_g + C_{qg} g \sum_q e_q^2 \left[D_q + D_{\overline{q}} \right] , \tag{1.51}
$$

where now the sums run only over the quark flavours and not over the antiflavours.

1.1.2 Drell Yan (DY)

In this section we apply to the Drell-Yan (DY) process the same procedure followed above for SIDIS. As a matter of fact, SIDIS and DY are strictly connected in that DY can be regarded as the time-like counterpart of SIDIS. As a consequence, the structure of the relevant observables as well as the form of the expressions involved are very similar. Therefore, the application of the method described above is straightforward.

1.2 Advantage of a logarithmic grid

Given the particular structure of the integral I in Eq. (1.1), it turns out to be very convenient to use a logarithmically distributed grid along with Lagrange interpolating functions. Let us specifically consider the (massless) integrals:

$$
I_{\beta\alpha} = \int_{x_\beta}^1 dy\, O(y) w_\alpha \left(\frac{x_\beta}{y} \right) . \tag{1.52}
$$

A logarithmically-spaced is defined such that $\ln x_{n+1} = \ln x_n + \delta x$ with δx a positive constant. In addition, we consider a set of Lagrange interpolating functions of degree κ polynomial in $\ln z$, $\{w_\alpha(z)\}$, that thus have the following form (see section on the interpolation):

$$
w_\alpha^{(k)}(z) = \sum_{\substack{j=0 \\ j \le \alpha}}^{k} \theta(z - x_{\alpha-j}) \theta(x_{\alpha-j+1} - z) \prod_{\delta=0, \delta\neq j}^{k} \left[\frac{\ln(z) - \ln(x_{\alpha-j+\delta})}{\ln(x_\alpha) - \ln(x_{\alpha-j+\delta})} \right] . \tag{1.53}
$$

Due to the fact that the grid is logarithmically distributed with step δx, the function above can be rearranged as follows:

$$w_\alpha^{(k)}(z) = \sum_{j=0, j\leq\alpha}^{k} \theta(z - x_{\alpha-j})\theta(x_{\alpha-j+1} - z) \prod_{\delta=0,\delta\neq j}^{k} \left[\frac{1}{\delta x}\ln\left(\frac{z}{x_\alpha}\right)\frac{1}{j-\delta} + 1\right]. \tag{1.54}$$

It is finally easy to see that Eq. (1.53) is such that:

$$w_\alpha(z) = \widetilde{w}\left(\ln\frac{z}{x_\alpha}\right) \quad \Rightarrow \quad w_\alpha\left(\frac{x_\beta}{y}\right) = \widetilde{w}\left(\ln\frac{x_\beta}{x_\alpha} - \ln y\right) = \widetilde{w}\left((\beta-\alpha)\delta x - \ln y\right). \tag{1.55}$$

Therefore, the integrand of the integral in Eq. (1.52) only depends on the difference $\beta - \alpha$ and not on β and α separately. Since the lower bound is x_β, this symmetry seems to broken at the level of the integral. However, the symmetry is preserved thanks to the support properties of the interpolating functions w_α and the fact that relevant functions (PDFs or FFs) are zero at $x = 1$. To see this, we consider the integration limits in Eq. (1.7) with $\eta = 1$. They can be written as:

$$c = \max(x_\beta, e^{(\beta-\alpha-1)\delta x}) \quad \text{and} \quad d = \min(1, e^{(\beta-\alpha+\kappa)\delta x}). \tag{1.56}$$

While the limit d is manifestly only dependent on the difference $\beta - \alpha$, the limit c is not. However, c does not have this symmetry only when x_β is selected in place of $e^{(\beta-\alpha-1)\delta x}$ and this can only happen when:

$$x_\beta > e^{(\beta-\alpha-1)\delta x}. \tag{1.57}$$

Since the last point of the grid is $x_{N_x} = 1$, being N_x the number of grid nodes minus one, one can write:

$$x_\beta = \frac{x_\beta}{x_{N_x}} = e^{(\beta-N_x)\delta x}. \tag{1.58}$$

Finally, the inequality in Eq. (1.57) becomes:

$$\beta - N_x > \beta - \alpha - 1 \quad \Leftrightarrow \quad \alpha > N_x - 1 \quad \Leftrightarrow \quad \alpha = N_x. \tag{1.59}$$

Therefore, the integrals $I_{\beta N_x}$ do not respect the "$\beta - \alpha$" symmetry. However, as mentioned above, $I_{\beta N_x}$ will always multiply a function computed in $x_{N_x} = 1$. If this function is a PDF or a FF, it is identically zero at $x_{N_x} = 1$ and thus the symmetry is effectively preserved. In addition, c in Eq. (1.56) is such that if $\beta > \alpha$ one has $c \geq 1$. But being c the lower integration bound of and since in Eq. (1.52) the upper bound is 1, one immediately has that $I_{\beta\alpha} = 0$ for $\beta > \alpha$. The consequence of these observations is that computing the integrals $a_\alpha = I_{0\alpha}$ for $\alpha = 0, \ldots, N_x$ is enough to reconstruct the full set of $I_{\beta\alpha}$ because, in matricial representation, I will look like this:

$$I_{\beta\alpha} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N_x} \\ 0 & a_0 & a_1 & \cdots & a_{N_x-1} \\ 0 & 0 & a_0 & \cdots & a_{N_x-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}, \tag{1.60}$$

In conclusion, adopting a logarithmically-spaced grid allows one to compute $N_x + 1$ integrals rather than $(N_x + 1)(N_x + 2)/2$ integrals.

There is another detail that matters in terms of numerical efficiency of the computation of the integrals $I_{\beta\alpha}$. Given the support region of the functions w_α, the integral in Eq. (1.52) effectively reads:

$$I_{\beta\alpha} = \int_c^d dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right), \tag{1.61}$$

with the integration limits given in Eq. (1.7). These limits can be rearranged as follows:

$$c = \frac{x_\beta}{x_{\min[N_x,\alpha+1]}} \quad \text{and} \quad d = \frac{x_\beta}{x_{\max[\beta,\alpha-\kappa]}}, \tag{1.62}$$

which makes it manifest the index range covered by the integration range. The basic observation is that the functions w_α are piecewise in correspondence of the grid nodes. This feature makes a numerical integration over the full range defined in Eq. (1.62) hard to converge due to the cusps at the grid nodes. However, the functions

w_α are smooth between two consecutive nodes. Therefore, it turns out that it is convenient to compute the integrals $I_{\beta\alpha}$ by breaking the integration range as follows:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y) w_\alpha \left(\frac{x_\beta}{y}\right) , \tag{1.63}$$

in such a way that the integrand of each single integrand is a smooth function and thus easier to integrate. Despite the number of intergrals to be computed increases, this procedure makes the computation faster and more accurate.

Finally, if the grid is logarithmically distributed, and one defines:

$$s = \exp\left[\delta x\right] , \tag{1.64}$$

Eq. (1.63) can also be written as:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_\alpha \left(\frac{x_\beta}{y}\right) . \tag{1.65}$$

1.2.1 GPD-related integrals

When considering, for example, GPD-related computations such as the evolution of GPDs, another kind of integral structure comes into play, that is:

$$J(x) = x \int_0^1 \frac{dz}{z} O\left(\frac{x}{z}, x\right) d(z) , \tag{1.66}$$

which differes from that in Eq. (1.1) in two respects: by the lower integration bound that is zero rather than x and by the fact that the operator O may also depend explicitly on the external variable x. Under the assumption that this integral is convergent, we need to adapt the strategy for the computation of numerical integrals discussed above to Eq. (1.66). As a first step, we make the change of integration variable $z \to x/y$ and use the interpolation formula in Eq. (1.4), to obtain:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^{\infty} dy\, O(y, x_\beta)\, w_\alpha \left(\frac{x_\beta}{y}\right) = \sum_{\alpha=0}^{N_x} J_{\beta\alpha} \overline{d}_\alpha . \tag{1.67}$$

The ∞ in the upper bound is a new feature. However, the support region of the functions w_α is the same so that:

$$J_{\beta\alpha} = \int_c^{\overline{d}} dy\, O(y, x_\beta)\, w_\alpha \left(\frac{x_\beta}{y}\right) , \tag{1.68}$$

with integration bounds:

$$c = \frac{x_\beta}{x_{\min[N_x,\alpha+1]}} \quad \text{and} \quad \overline{d} = \frac{x_\beta}{x_{\max[0,\alpha-\kappa]}} . \tag{1.69}$$

Following the procedure discussed in the previous section, the actual computation is achieved by:

$$\begin{aligned} J_{\beta\alpha} &= \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha]} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y, x_\beta) w_\alpha \left(\frac{x_\beta}{y}\right) \\ &= \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y, x_\beta) w_\alpha \left(\frac{x_\beta}{y}\right) . \end{aligned} \tag{1.70}$$

Therefore, at the level of the single integral, the only difference with respect to the "standard" case is a change of the upper summation bound. It should be clear, however, that this procedure necessarily implies an approximation. This is due to the fact that the interpolation grid, understood to be logarithmically distributed, does not reach zero. This approximation manifests itself in the fact that the upper bound of the intergrals in

Eq. (1.70) never reaches infinity but it gets as large as s^κ. The size of s^κ can range widely depending on the lower bound of the grid x_0, on the number of nodes N_x, and on the interpolation degree κ, according to:

$$s^\kappa = \exp\left[-\frac{\kappa}{N_x}\ln x_0\right] = x_0^{-\frac{\kappa}{N_x}}\,. \tag{1.71}$$

In particular, s^κ becomes larger as: x_0 gets smaller, N_x gets smaller, and κ gets larger. We will need to assess the impact of this approximation numerically.

If the operator O in Eq. (1.66) did not explicitly depend on x, the integral $J_{\beta\alpha}$ would only depend on the difference $\beta-\alpha$. However, under this assumption that we will release below, at variance with the standard case, $J_{\beta\alpha}\neq 0$ also for $\beta>\alpha$. The reason is that the lower bound c is now allowed to exceed 1. Using Eq. (1.56), this implies that $\beta-\alpha$ can also be larger than zero. This means in turn that computing the integrals $b_\alpha = J_{0\alpha}$ and $b_{-\alpha} = J_{\alpha 0}$ for $\alpha = 0,\ldots,N_x$ is enough to reconstruct the full set of $J_{\beta\alpha}$. In matricial representation, J takes the form:

$$J_{\beta\alpha} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N_x} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N_x-1} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N_x-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{-N_x} & b_{-N_x+1} & b_{-N_x+2} & \cdots & b_0 \end{pmatrix}, \tag{1.72}$$

Therefore, adopting a logarithmically-spaced grid allows one to compute $2N_x+1$ integrals rather than N_x^2 integrals.

1.3 Joining grids

As discussed above, logarithmically distributed grids are particularly advantageous because the allow one to substantially reduce the number of integrals to be compute. However, logarithmic grids have two main drawbacks. First, they cannot get down to $x=0$ and this is a potential problem for integrals like those in Eq. (1.66). Second, logarithmic grids that start from a relatively low value of x tend to be relatively sparse at large values of x. This is also a problem because all integrals that we are considering are such that the function being interpolated is integrated up to $x=1$ and the interpolation there can thus potentially degrade in accuracy.

A possible solution to this problem is to increase the density in a stepwise fashion as x gets closer to one. This produces locally logarithmically distributed grids that allow to exploit the symmetry discussed above while making the grid denser and thus more accurate at large x. The implementation of this procedure in APFEL is achieved by means of *locked* subgrids. In practice, one starts with a logarithmic grid with a given lower bound, *e.g.* $x_{\rm min}^{(0)} = 10^{-5}$. Starting from a given node, $x_{\rm min}^{(1)}$, the density of the grid is increased by some integer factor. This procedure can be repeated an arbitrary number of times as one moves towards large x, effectively defining logarithmic subgrids that are increasingly denser and thus guarantee a better interpolation accuracy.

When dealing with integrals such as that in Eq. (1.52), the simplest, but possibly not optimal, way to exploit the subgrid structure is to switch to a denser grid at the level of the *integral* essentially using one grid when x_β is below the transition node and the other when it is above. The advantage of this approach is that the integration procedure discussed above applies verbatim with the only detail that according to the position of x_β one grid is used rather than another. The disadvantage of this procedure is that integrals with low values of x_β do not take advantage of the denser grids at large x. When dealing with functions like PDFs that vanish rapidly as x tends to one, this is typically fine because the bulk of the integrals is typically due to the region at $x\gtrsim x_\beta$. This usually makes possible interpolation inaccuracies at large x negligible.

The procedure discussed above when applied to integrals of the kind of Eq. (1.66) may lead to severe inaccuracies a large values of x_β. This is due to the fact that the integral extends down to zero and the naive procedure does not make use of the low-x grids effectively truncating the integral to increasingly larger values of x.

In the following we discuss a more convoluted but more effective way to exploit the subgrids that solves the problem mentioned above, *i.e.* the lack of exploitation of the large-x grids when x_β is small and the lack of exploitation of the low-x grids when x_β is large. In this approach the switch from one grid to the other does not happen, as above, at the level of the integral but at the level of the *integrand*.

In this respect, the basic observation is that, thanks to the locking of the subgrids, the integral for a DGLAP-like convolution as in Eq. (1.1) computed in the point x_β of the joint grid can be decomposed as follows:

$$I(x_\beta) = \sum_{g=0}^{N_g=1}\int_{x_\beta/\max[x_\beta,x_{\gamma(g+1)}]}^{x_\beta/\max[x_\beta,x_{\gamma(g)}]} dy\, O(y) f\left(\frac{x_\beta}{y}\right) = \sum_{g=1}^{N_g} I^{(g)}(x_\beta)\,, \tag{1.73}$$

while for and ERBL-like convolution as that in Eq. (1.66) the decomposition reads:

$$J(x_\beta) = \sum_{g=0}^{N_g-1} \int_{x_\beta/x_{\gamma^{(g)}}}^{x_\beta/x_{\gamma^{(g)}}} dy\, O(y) f\left(\frac{x_\beta}{y}\right) = \sum_{g=1}^{N_g} J^{(g)}(x_\beta)\,, \tag{1.74}$$

where $\gamma^{(g)}$ is the index corresponding to the lower bound of the g-th grid on the joint grid with the conditions $\gamma^{(0)} = 0$ and $\gamma^{(N_g)} = N_x$. It is interesting to observe that if $x_{\gamma^{(g+1)}} \leq x_\beta$, the integration bounds of the integral in the g-th term in Eq. (1.73) coincide giving a vanishing contribution. Therefore, only the subgrids having $x_{\gamma^{(g+1)}} > x_\beta$ will eventually contribute to the full integral. Conversely, for Eq. (1.74) all grids contribute to the full convolution integral no matter the position of x_β w.r.t. the subgrid bounds.

Let us first consider the g-th contribution to the integral in Eq. (1.74), $J^{(g)}$, and apply the usual interpolation procedure on the g-th subgrid. To do so, we need to take into account the fact that the g-th subgrid starts from the index $\gamma^{(g)}$ of the joint grid and that in the range $\gamma^{(g)} \leq \alpha \leq \gamma^{(g+1)}$ joint grid and g-th subgrid coincide up to an index offset given by $\gamma^{(g)}$ (*i.e.* $x_\alpha = x^{(g)}_{\alpha-\gamma^{(g)}}$). This gives:

$$J^{(g)}(x_\beta) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} \overline{d}_\alpha \int_{\overline{c}}^{\overline{d}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} J^{(g)}_{\beta\alpha}\, \overline{d}_\alpha\,, \tag{1.75}$$

where we have exploited the fact that, if $\alpha \geq \gamma^{(g+1)} + \kappa^{(g)}$, the lower bounds of the sum above exceeds to upper one giving no contribution to set the upper limit of the sum. In addition, the integration bounds are defined as:

$$\begin{aligned} \ln \overline{c} &= (\beta - \alpha - 1 + \max[0, \alpha + 1 - \gamma^{(g+1)}])\delta x^{(g)}\,, \\ \ln \overline{d} &= (\beta - \alpha + \min[\kappa^{(g)}, \alpha - \gamma^{(g)}])\delta x^{(g)}\,. \end{aligned} \tag{1.76}$$

This allows us to write the factors $J^{(g)}_{\beta\alpha}$ in Eq. (1.75) as follows:

$$J^{(g)}_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-\gamma^{(g+1)}]}^{\min[\kappa^{(g)},\alpha-\gamma^{(g)}]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right)\,, \tag{1.77}$$

with $s = \exp(\delta x^{(g)})$. Therefore, Eq. (1.74) can finally be written as:

$$J(x_\beta) = \sum_{g=0}^{N_g-1} \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} J^{(g)}_{\beta\alpha}\, \overline{d}_\alpha\,. \tag{1.78}$$

We now introduce the discreet θ-function $\theta_{\rho,\sigma}$ defined such that:

$$\theta_{\rho,\sigma} = \begin{cases} 1 & \rho \geq \sigma \\ 0 & \rho < \sigma \end{cases}\,, \tag{1.79}$$

which allows to write:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x+\kappa^{(N_g-1)}-1} \overline{d}_\alpha \left(\sum_{g=0}^{N_g-1} \theta_{\alpha,\gamma^{(g)}} \theta_{\gamma^{(g+1)}+\kappa^{(g)}-1,\alpha} J^{(g)}_{\beta\alpha} \right)\,. \tag{1.80}$$

Assuming that $\overline{d}_\alpha = 0$ for $\alpha \geq N_x$, as is often the case, and defining:

$$J_{\beta\alpha} = \sum_{g=0}^{N_g-1} \theta_{\alpha,\gamma^{(g)}} \theta_{\gamma^{(g+1)}+\kappa^{(g)}-1,\alpha} J^{(g)}_{\beta\alpha}\,, \tag{1.81}$$

one finally obtains:

$$J(x_\beta) = \sum_{\alpha=0}^{N_x-1} J_{\beta\alpha} \overline{d}_\alpha\,. \tag{1.82}$$

In is useful to introduce the helper function h that returns the grid index given as an argument the index on the joint grid and that can be represented as:

$$h(\alpha) = \sum_{g'=0}^{N_g-1} \theta_{\alpha,\gamma^{(g')}} - 1\,. \tag{1.83}$$

It is easy to see that:

$$h(\gamma^{(g)}) = g\,. \tag{1.84}$$

Making use on the monotonicity of h, Eq. (1.81) can be reduced to:

$$J_{\beta\alpha} = \sum_{g=\max[0,h(\alpha)-1]}^{h(\alpha)} J^{(g)}_{\beta\alpha}\,. \tag{1.85}$$

We can now apply the same procedure to the integral I in Eq. (1.73). As above, one finds:

$$I^{(g)}(x_\beta) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} \overline{d}_\alpha \int_c^d dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right) = \sum_{\alpha=\gamma^{(g)}}^{\gamma^{(g+1)}+\kappa^{(g)}-1} I^{(g)}_{\beta\alpha}\,\overline{d}_\alpha\,, \tag{1.86}$$

where the logarithm of the integration bounds this time read:

$$\begin{aligned} \ln c &= (\beta - \alpha - 1 + \max[0, \alpha + 1 - \max[\beta, \gamma^{(g+1)}]])\delta x^{(g)}\,, \\ \ln d &= (\beta - \alpha + \min[\kappa^{(g)}, \alpha - \max[\beta, \gamma^{(g)}]])\delta x^{(g)}\,. \end{aligned} \tag{1.87}$$

which gives:

$$I^{(g)}_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-\max[\beta,\gamma^{(g+1)}]]}^{\min[\kappa^{(g)},\alpha-\max[\beta,\gamma^{(g)}]]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_{\alpha-\gamma^{(g)}}\left(\frac{x_\beta}{y}\right)\,, \tag{1.88}$$

allowing one to define:

$$I_{\beta\alpha} = \sum_{g=\max[0,h(\alpha)-1]}^{h(\alpha)} I^{(g)}_{\beta\alpha}\,, \tag{1.89}$$

such that:

$$I(x_\beta) = \sum_{\alpha=0}^{N_x-1} I_{\beta\alpha}\overline{d}_\alpha\,. \tag{1.90}$$

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