Contents

1 PDF matching conditions

If the (Zero Mass) Variable Flavour Number Scheme (ZM−VFNS) at NNLO is considered, matching conditions for the PDFs and the coupling constant at the heavy quarks thresholds (m_c^2, m_b^2 and m_t^2) must be implemented. This is due to the fact that we are working with an "effective theory" where before a certain threshold, say m_h^2, the heavy quark flavour h is treated as infinitely massive, while after the crossing of the same threshold the same flavour is treated as massless. This results in discontinuities of PDFs and coupling constant in correspondence of the thresholds. Such discontinuities can be evaluated in perturbation theory, and in particular one can see that they start at NNLO, so that PDFs and a_s are continuous at LO and NLO [1].

The discontinuity of the PDF l (in the Mellin space) of a light quark([1]) just beyond the threshold $m_h^2(= m_c^2, m_b^2, m_t^2)$, where the effective flavour number passes from n_f to n_f+1, is given as a function of the same PDF just before the threshold by the following relation [2]:

$$l^{(n_f+1)}(N, m_h^2) = [1 + a_s^2(m_h^2) A_{qq,h}^{NS,(2)}(N)] l^{(n_f)}(N, m_h^2) \,. \tag{1}$$

with $l = u, \overline{u}, d, \overline{d}, \dots$, while the gluon distribution function is given by:

$$g^{(n_f+1)}(N, m_h^2) = [1 + a_s^2(m_h^2) A_{gg,h}^{S,(2)}(N)] g^{(n_f)}(N, m_h^2) + a_s^2(m_h^2) A_{gq,h}^{S,(2)}(N) \Sigma^{(n_f)}(N, m_h^2) \tag{2}$$

and, in the end, the sum of heavy quark h and its anti-quark $\overline{h}$, which are going to be produced after the threshold m_h^2, is:

$$(h^{(n_f+1)} + \overline{h}^{(n_f+1)})(N, m_h^2) = a_s^2(m_h^2)[\tilde{A}_{hq}^{S,(2)}(N) \Sigma^{(n_f)}(N, m_h^2) + \tilde{A}_{hg}^{S,(2)}(N) g^{(n_f)}(N, m_h^2)] \,. \tag{3}$$

Of course, we have $h = \overline{h}$.

Now, since:

$$\Sigma^{(n_f+1)} = \sum_{l=1}^{n_f} (l^{(n_f+1)} + \overline{l}^{(n_f+1)}) + (h^{(n_f+1)} + \overline{h}^{(n_f+1)}) \tag{4}$$

we find that the matching condition for the singlet is:

$$\begin{aligned} \Sigma^{(n_f+1)}(N, m_h^2) = [1 + a_s^2(m_h^2) A_{qq,h}^{NS,(2)}(N)] \Sigma^{(n_f)}(N, m_h^2) + \\ a_s^2(m_h^2)[\tilde{A}_{hq}^{S,(2)}(N) \Sigma^{(n_f)}(N, m_h^2) + \tilde{A}_{hg}^{S,(2)}(N) g^{(n_f)}(N, m_h^2)] \end{aligned} \tag{5}$$

so that, from eqs. (2) and (5):

$$\begin{aligned} \begin{pmatrix} \Sigma^{(n_f+1)} \\ g^{(n_f+1)} \end{pmatrix} &= \begin{pmatrix} 1 + a_s^2[A_{qq,h}^{NS,(2)} + \tilde{A}_{hq}^{S,(2)}] & a_s^2 \tilde{A}_{hg}^{S,(2)} \\ a_s^2 A_{gq,h}^{S,(2)} & 1 + a_s^2 A_{gg,h}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(n_f)} \\ g^{(n_f)} \end{pmatrix} \\ &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2 A_{qq,h}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2 \begin{pmatrix} \tilde{A}_{hq}^{S,(2)} & \tilde{A}_{hg}^{S,(2)} \\ A_{gq,h}^{S,(2)} & A_{gg,h}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(n_f)} \\ g^{(n_f)} \end{pmatrix} \end{aligned} \tag{6}$$

where we have omitted all the dependencies. So we have obtained the matching conditions for the singlet and the gluon distribution functions.

Now we consider the other distributions. The valence distribution V for $n_f + 1$ active (light) flavours just beyond the threshold m_h^2 is defined as:

$$V^{(n_f+1)} = \sum_{l=1}^{n_f} (l^{(n_f+1)} - \overline{l}^{(n_f+1)}) + (h^{(n_f+1)} - \overline{h}^{(n_f+1)}) \tag{7}$$

[1]Note that the light quarks run from 1 to n_f.

but, since $h = \overline{h}$, the last term vanish and we are left with:

$$
\begin{gathered}
V^{(n_f+1)} = \sum_{l=1}^{n_f} (l^{(n_f+1)} - \overline{l}^{(n_f+1)}) = \\
[1 + a_s^2 A_{qq,h}^{NS,(2)}] \sum_{l=1}^{n_f} (l^{(n_f)} - \overline{l}^{(n_f)}) = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V^{(n_f)} \,.
\end{gathered} \tag{8}
$$

So, this is the matching condition for the valence distribution V.

Now we consider the valence distribution V_3 and V_8 which are both composed only by light quarks, namely:

$$
V_3 = (u - \overline{u}) - (d - \overline{d}) \quad \text{and} \quad V_8 = (u - \overline{u}) + (d - \overline{d}) - 2(s - \overline{s}) \tag{9}
$$

so that, in these cases, the matching conditions work as in the case of V, i.e.:

$$
V_{3,8}^{(n_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V_{3,8}^{(n_f)} \,. \tag{10}
$$

The same holds for T_3 and T_8, which are defined as:

$$
T_3 = (u + \overline{u}) - (d + \overline{d}) \quad \text{and} \quad V_8 = (u + \overline{u}) + (d + \overline{d}) - 2(s + \overline{s}) \tag{11}
$$

so:

$$
T_{3,8}^{(n_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] T_{3,8}^{(n_f)} \,. \tag{12}
$$

The remaining valence distribution V_{15}, V_{24} and V_{35} are defined as:

$$
\begin{aligned}
V_{15} &= (u - \overline{u}) + (d - \overline{d}) + (s - \overline{s}) - 3(c - \overline{c}) \\
V_{24} &= (u - \overline{u}) + (d - \overline{d}) + (s - \overline{s}) + (c - \overline{c}) - 4(b - \overline{b}) \\
V_{35} &= (u - \overline{u}) + (d - \overline{d}) + (s - \overline{s}) + (c - \overline{c}) + (b - \overline{b}) - 5(t - \overline{t})
\end{aligned} \tag{13}
$$

and since in each one of them the heavy quarks appear always as difference between quark and anti-quark, they cancel exactly. For example, at the m_b^2 threshold, V_{15} is entirely composed by light quarks so there is no problem, while V_{24} and V_{35} have also a b-quark contribution, given by $-4(b - \overline{b})$ and $(b - \overline{b})$ respectively (of course, the $t(\overline{t})$ distribution is zero). Anyway, this terms give no matching condition since the b contribution is exactly equal to the $\overline{b}$ contribution, so that they cancel. So:

$$
V_{15,24,35}^{(n_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V_{15,24,35}^{(n_f)} \,. \tag{14}
$$

In the end, to deal with T_{15}, T_{24} and T_{35}, we have to specify the threshold. Indeed, in these cases the heavy quark contribution does not cancel. Their definition is:

$$
\begin{aligned}
T_{15} &= (u + \overline{u}) + (d + \overline{d}) + (s + \overline{s}) - 3(c + \overline{c}) \\
T_{24} &= (u + \overline{u}) + (d + \overline{d}) + (s + \overline{s}) + (c + \overline{c}) - 4(b + \overline{b}) \\
T_{35} &= (u + \overline{u}) + (d + \overline{d}) + (s + \overline{s}) + (c + \overline{c}) + (b + \overline{b}) - 5(t + \overline{t}) \,.
\end{aligned} \tag{15}
$$

Just before the threshold m_c^2 we have only 3 active light flavours (u, d and s), while just beyond m_c^2 we have 4 active flavours and among them the flavour c is considered to be heavy. Of course, we have no b and t contribution (so T_{24} and T_{35} are equal). So the matching conditions are:

$$
\begin{gathered}
T_{15}^{(4)} = [1 + a_s^2 A_{qq,c}^{NS,(2)}] \underbrace{\sum_{l=u,d,s} (l^{(3)} + \overline{l}^{(3)})}_{\Sigma^{(3)}} - 3a_s^2 [\tilde{A}_{cq}^{S,(2)} \Sigma^{(3)} + \tilde{A}_{cg}^{S,(2)} g^{(3)}] = \\
\begin{pmatrix} 1 + a_s^2 [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2 \tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}
\end{gathered} \tag{16}
$$

while:

$$T_{24,35}^{(4)} = \begin{pmatrix} 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2 \tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix} . \tag{17}$$

We can put the above relation in a matricial form:

$$\begin{pmatrix} T_{15}^{(4)} \\ T_{24}^{(4)} \\ T_{35}^{(4)} \end{pmatrix} = \begin{pmatrix} 1 + a_s^2[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2 \tilde{A}_{cg}^{S,(2)} \\ 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2 \tilde{A}_{cg}^{S,(2)} \\ 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2 \tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix} . \tag{18}$$

But now it is easy to generalize. At m_b^2, T_{15} does not contain, so:

$$T_{15}^{(5)} = [1 + a_s^2 A_{qq,b}^{NS,(2)}] T_{15}^{(4)} \tag{19}$$

while:

$$\begin{pmatrix} T_{24}^{(5)} \\ T_{35}^{(5)} \end{pmatrix} = \begin{pmatrix} 1 + a_s^2[A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] & -4a_s^2 \tilde{A}_{bg}^{S,(2)} \\ 1 + a_s^2[A_{qq,b}^{NS,(2)} + \tilde{A}_{bq}^{S,(2)}] & a_s^2 \tilde{A}_{bg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix} . \tag{20}$$

Finally, at m_t^2 we have:

$$\begin{aligned} T_{15}^{(6)} &= [1 + a_s^2 A_{qq,t}^{NS,(2)}] T_{15}^{(5)} \\ T_{24}^{(6)} &= [1 + a_s^2 A_{qq,t}^{NS,(2)}] T_{25}^{(5)} \end{aligned} \tag{21}$$

and:

$$T_{35}^{(6)} = \begin{pmatrix} 1 + a_s^2[A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] & -5a_s^2 \tilde{A}_{tg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(5)} \\ g^{(5)} \end{pmatrix} . \tag{22}$$

An explicit calculation for the coefficients $A^{(2)}$ in the x-space can be found in [hep-ph/9612398]. Anyhow, that calculation is performed more generally in the case $m_h^2 \neq \mu_F^2$. This results in extra-terms proportional to $\ln(m_h^2/\mu_F^2)$, which vanish if, as we do, one takes the factorisation scale μ^2 coinciding with the scale of the process Q^2. Moreover, in that case also NLO ($\propto a_s$) appear in the matching conditions.

To summarise the PDF matching conditions at the threshold m_h^2, we have that:

- singlet and gluon couple as follows:

$$\begin{pmatrix} \Sigma^{(n_f+1)} \\ g^{(n_f+1)} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2 A_{qq,h}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2 \begin{pmatrix} \tilde{A}_{hq}^{S,(2)} & \tilde{A}_{hg}^{S,(2)} \\ A_{gq,h}^{S,(2)} & A_{gg,h}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(n_f)} \\ g^{(n_f)} \end{pmatrix} \tag{23}$$

- from eqs. (8), (10), (14) and (12), one can see that V, $V_{3,8,\dots,35}$ and $T_{3,8}$ behave in the same way, i.e.:

$$P^{(n_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] P^{(n_f)} \quad \text{with} \quad P = V, V_3, \dots, V_{35}, T_3, T_8 \tag{24}$$

- T_{15}, T_{24} and T_{35} have different matching conditions depending on the threshold. In particular: for $m_h^2 = m_c^2$ they are given by eq. (18), for $m_h^2 = m_b^2$ they are given by eqs. (19) and (20) and for $m_h^2 = m_t^2$ they are given by eqs. (21) and (22)

In the following Sections we will discuss how to write the evolution kernels in the presence of the matching conditions. We will explicitly consider only the forward evolution, i.e. the final scale Q^2 greater than the initial one Q_0^2. Anyway the backward evolution ($Q_0^2 > Q^2$) can be easily obtained from the forward one. In fact, given the evolution kernel Γ, the following relation holds:

$$\Gamma(Q^2, Q_0^2)\Gamma(Q_0^2, Q^2) = 1 \quad \Longrightarrow \quad \Gamma(Q^2, Q_0^2) = \Gamma^{-1}(Q_0^2, Q^2). \tag{25}$$

so, if $Q^2 > Q_0^2$ the code computes directly $\Gamma(Q^2, Q_0^2)$, else if $Q_0^2 > Q^2$ the code evaluates first the forward evolution $\Gamma(Q_0^2, Q^2)$ and then, to get $\Gamma(Q^2, Q_0^2)$, it calculates $\Gamma^{-1}(Q_0^2, Q^2)$.

1.1 Matching Conditions on the Evolution Kernels: 0 Thresholds Crossing

Before to discuss the crossing of the thresholds, it would be useful to write down the evolution kernels in the "trivial" situation of no threshold crossing. There are 4 particular cases: 1) $Q_0^2 < Q^2 < m_c^2$ with $n_f = 3$ active flavours, 2) $m_c^2 < Q_0^2 < Q^2 < m_b^2$ with $n_f = 4$ active flavours, 3) $m_b^2 < Q_0^2 < Q^2 < m_t^2$ with $n_f = 5$ active flavours and 4) $m_t^2 < Q_0^2 < Q^2$ with $n_f = 6$ active flavours, which do not need the introduction of the matching conditions. So, in the following Subsections we will write down the evolution of the whole PDF set the these cases.

1.1.1 $Q_0^2 < Q^2 < m_c^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(3)}(Q^2) \\ g^{(3)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{26}$$

- **V**:

$$V^{(3)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(3)}(Q_0^2) \tag{27}$$

- $\mathbf{V_3}$ and $\mathbf{V_8}$:

$$V_{3,8}^{(3)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8}^{(3)}(Q_0^2) \tag{28}$$

- $\mathbf{V_{15}}$, $\mathbf{V_{24}}$ and $\mathbf{V_{35}}$:

$$V_{15,24,35}^{(3)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(3)}(Q_0^2) \tag{29}$$

- $\mathbf{T_3}$ and $\mathbf{T_8}$:

$$T_{3,8}^{(3)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8}^{(3)}(Q_0^2) \tag{30}$$

- $\mathbf{T_{15}}$, $\mathbf{T_{24}}$ and $\mathbf{T_{35}}$:

$$T_{15,24,35}^{(3)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{31}$$

1.1.2 $m_c^2 < Q_0^2 < Q^2 < m_b^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(4)}(Q^2) \\ g^{(4)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \tag{32}$$

- **V**:

$$V^{(4)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(4)}(Q_0^2) \tag{33}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$ and $\mathbf{V_{15}}$:

$$V_{3,8,15}^{(4)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8,15}^{(4)}(Q_0^2) \tag{34}$$

- $\mathbf{V_{24}}$ and $\mathbf{V_{35}}$:

$$V_{24,35}^{(4)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(4)}(Q_0^2) \tag{35}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$ and $\mathbf{T_{15}}$:

$$T_{3,8,15}^{(4)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8,15}^{(4)}(Q_0^2) \tag{36}$$

- $\mathbf{T_{24}}$ and $\mathbf{T_{35}}$:

$$T_{24,35}^{(4)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \tag{37}$$

1.1.3 $m_b^2 < Q_0^2 < Q^2 < m_t^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \tag{38}$$

- **V**:

$$V^{(5)}(Q^2) = \Gamma^v(Q^2,Q_0^2)V^{(5)}(Q_0^2) \tag{39}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$, $\mathbf{V_{15}}$ and $\mathbf{V_{24}}$:

$$V_{3,8,15,24}^{(5)}(Q^2) = \Gamma^-(Q^2,Q_0^2)V_{3,8,15,24}^{(5)}(Q_0^2) \tag{40}$$

- $\mathbf{V_{35}}$:

$$V_{35}^{(5)}(Q^2) = \Gamma^v(Q^2,Q_0^2)V^{(5)}(Q_0^2) \tag{41}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$, $\mathbf{T_{15}}$ and $\mathbf{T_{24}}$:

$$T_{3,8,15,24}^{(5)}(Q^2) = \Gamma^+(Q^2,Q_0^2)T_{3,8,15,24}^{(5)}(Q_0^2) \tag{42}$$

- $\mathbf{T_{35}}$:

$$T_{35}^{(5)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \tag{43}$$

1.1.4 $m_t^2 < Q_0^2 < Q^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2,Q_0^2)} \begin{pmatrix} \Sigma^{(6)}(Q_0^2) \\ g^{(6)}(Q_0^2) \end{pmatrix} \tag{44}$$

- **V**:

$$V^{(6)}(Q^2) = \Gamma^v(Q^2,Q_0^2)V^{(6)}(Q_0^2) \tag{45}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$, $\mathbf{V_{15}}$, $\mathbf{V_{24}}$ and $\mathbf{V_{35}}$:

$$V_{3,8,15,24,35}^{(6)}(Q^2) = \Gamma^-(Q^2,Q_0^2)V_{3,8,15,24,35}^{(6)}(Q_0^2) \tag{46}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$, $\mathbf{T_{15}}$, $\mathbf{T_{24}}$ and $\mathbf{T_{35}}$:

$$T_{3,8,15,24,35}^{(6)}(Q^2) = \Gamma^+(Q^2,Q_0^2)T_{3,8,15,24,35}^{(6)}(Q_0^2) \tag{47}$$

1.2 Matching Conditions on the Evolution Kernels: 1 Threshold Crossing

In order to implement the matching conditions in our code, we will show how to transfer them from the PDFs to the evolution kernels.

In this Section we suppose that the evolution crosses only one threshold. We will show how the matching conditions on the PDFs modify the form of the evolution kernels in the cases: 1) $Q_0^2 < m_c^2 \leq Q^2$, 2) $Q_0^2 < m_b^2 \leq Q^2$ and $Q_0^2 < m_c^2 \leq Q^2$.

1.2.1 $Q_0^2 < m_c^2 \leq Q^2$

In order to evolve PDFs from the scale Q_0^2 to Q^2 passing through the threshold m_c^2, we have to: first evolve them from Q_0^2 to m_c^2, where there are 3 active flavours, then increase the number of active flavour from 3 to 4 by imposing the matching conditions, and in the end evolve the PDFs, now having 4 active flavours, from m_c^2 to the scale Q^2.

In what follows we will work only in the Mellin space, so we will drop any dependence on N.

Let's start with singlet and gluon. We have:

$$\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2)\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(m_c^2)\,. \tag{48}$$

From eq. (23):

$$\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(m_c^2) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_c^2)\begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix}\right]\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2) \tag{49}$$

now, substituting the above relation into the eq. (48), we get:

$$\begin{aligned}\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2)&\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \right.\\ &\left. a_s^2(m_c^2)\begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix}\right]\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2)\,.\end{aligned} \tag{50}$$

But:

$$\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2) = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(m_c^2, Q_0^2)\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(Q_0^2)\,. \tag{51}$$

So, in the end:

$$\begin{aligned}\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) = \left\{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2)\right.&\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \right. \\ a_s^2(m_c^2)\begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix}&\left.\left.\right]\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(m_c^2, Q_0^2)\right\}\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(Q_0^2)\end{aligned}\,. \tag{52}$$

Now we consider the distributions V, $V_{3,8}$ and $T_{3,8}$ which evolve respectively through Γ^v, Γ^- and Γ^+, but which obey the same matching conditions. So:

$$P^{(4)}(Q^2) = \Gamma^{(P)}(Q^2, m_c^2)P^{(4)}(m_c^2) \tag{53}$$

so that:

$$P = \begin{cases} V & \to \Gamma^{(P)} = \Gamma^v \\ V_{3,8} & \to \Gamma^{(P)} = \Gamma^- \\ T_{3,8} & \to \Gamma^{(P)} = \Gamma^+ \end{cases} \tag{54}$$

but:

$$P^{(4)}(m_c^2) = [1 + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]P^{(3)}(m_c^2) \tag{55}$$

and:

$$P^{(3)}(m_c^2) = \Gamma^{(P)}(m_c^2, Q_0^2)P^{(3)}(Q_0^2) \tag{56}$$

so that:

$$P^{(4)}(Q^2) = \left\{\Gamma^{(P)}(Q^2, m_c^2)[1 + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^{(P)}(m_c^2, Q_0^2)\right\}P^{(3)}(Q_0^2) \tag{57}$$

Now we consider V_{15}, which before m_c^2 evolves as:

$$V_{15}^{(3)}(m_c^2) = \Gamma^v(m_c^2, Q_0^2) V^{(3)}(Q_0^2) \tag{58}$$

while after m_c^2 it evolves as:

$$V_{15}^{(4)}(Q^2) = \Gamma^-(Q_0^2, m_c^2) V_{15}^{(4)}(m_c^2)\,. \tag{59}$$

From eq. (24), we find that the matching condition at m_c^2 is:

$$V_{15}^{(4)}(m_c^2) = [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] V_{15}^{(3)}(m_c^2) \tag{60}$$

so:

$$V_{15}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \tag{61}$$

Instead, for $Q_0^2 < m_c^2 \leq Q^2$, both V_{24} and V_{35} evolve as:

$$V_{24,35}^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \tag{62}$$

Now, we consider T_{15}. After m_c^2, it evolves as:

$$T_{15}^{(4)}(Q^2) = \Gamma^+(Q^2, m_c^2) T_{15}^{(4)}(m_c^2)\,. \tag{63}$$

This time the matching condition is given by the first line of eq. (18):

$$T_{15}^{(4)}(m_c^2) = \begin{pmatrix} 1 + a_s^2(m_c^2)[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2)\tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)}(m_c^2) \\ g^{(3)}(m_c^2) \end{pmatrix}. \tag{64}$$

But:

$$\begin{pmatrix} \Sigma^{(3)}(m_c^2) \\ g^{(3)}(m_c^2) \end{pmatrix} = \begin{pmatrix} \Gamma_{qq}(m_c^2, Q_0^2) & \Gamma_{qg}(m_c^2, Q_0^2) \\ \Gamma_{gq}(m_c^2, Q_0^2) & \Gamma_{gg}(m_c^2, Q_0^2) \end{pmatrix} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{65}$$

In the end, one finds that:

$$\begin{aligned} T_{15}^{(4)}(Q^2) = & \Bigg\{ \Gamma^+(Q^2, m_c^2) \begin{pmatrix} 1 + a_s^2(m_c^2)[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2)\tilde{A}_{cg}^{S,(2)} \end{pmatrix} \times \\ & \begin{pmatrix} \Gamma_{qq}(m_c^2, Q_0^2) & \Gamma_{qg}(m_c^2, Q_0^2) \\ \Gamma_{gq}(m_c^2, Q_0^2) & \Gamma_{gg}(m_c^2, Q_0^2) \end{pmatrix} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{66}$$

Now we are left only with T_{24} and T_{35}, which evolve as the single before and after the threshold, so they evolve exactly as the first line of eq. (52), i.e:

$$\begin{aligned} T_{24,35}^{(4)}(Q^2) = & \Bigg\{ \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix} (Q^2, m_c^2) \Bigg[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \\ & a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \Bigg] \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix} (m_c^2, Q_0^2) \Bigg\} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix} (Q_0^2) \end{aligned}\,. \tag{67}$$

Now, let us summarise what happens to the evolution kernels, by introducing the matching conditions, if one crosses the m_c^2 threshold. We remind that the matching conditions appear only from the NNLO.

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(4)}(Q^2) \\ g^{(4)}(Q^2) \end{pmatrix} = \Bigg\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{68}$$

where:

$$\begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \tag{69}$$

- **V**:

$$V^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \tag{70}$$

- $\mathbf{V_3}$ and $\mathbf{V_8}$:

$$V_{3,8}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^-(m_c^2, Q_0^2) \right\} V_{3,8}^{(3)}(Q_0^2) \tag{71}$$

- $\mathbf{V_{15}}$:

$$V_{15}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \tag{72}$$

- $\mathbf{V_{24}}$ and $\mathbf{V_{35}}$:

$$V_{24,35}^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \tag{73}$$

- $\mathbf{T_3}$ and $\mathbf{T_8}$:

$$T_{3,8}^{(4)}(Q^2) = \left\{ \Gamma^+(Q^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^+(m_c^2, Q_0^2) \right\} T_{3,8}^{(3)}(Q_0^2) \tag{74}$$

- $\mathbf{T_{15}}$:

$$\begin{aligned} T_{15}^{(4)}(Q^2) \quad = \quad & \Big\{ \Gamma^+(Q^2, m_c^2) \Big(1 + a_s^2(m_c^2)[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] \quad -3a_s^2(m_c^2)\tilde{A}_{cg}^{S,(2)} \Big) \times \\ & \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Big\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{75}$$

- $\mathbf{T_{24}}$ and $\mathbf{T_{35}}$:

$$T_{24,35}^{(4)}(Q^2) = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2, m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{76}$$

Now, it is very easy to rewrite the above summary for the crossing of the remaining thresholds m_b^2 ($Q_0^2 < m_b^2 \le Q^2$) and m_t^2 ($Q_0^2 < m_t^2 \le Q^2$).

1.2.2 $Q_0^2 < m_b^2 \le Q^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \tag{77}$$

where:

$$\begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_b^2) A_{qq,b}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_b^2) \begin{pmatrix} \tilde{A}_{bq}^{S,(2)} & \tilde{A}_{bg}^{S,(2)} \\ A_{gq,b}^{S,(2)} & A_{gg,b}^{S,(2)} \end{pmatrix} \tag{78}$$

- $\mathbf{V}$:

$$V^{(5)}(Q^2) = \left\{\Gamma^v(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2, Q_0^2)\right\} V^{(4)}(Q_0^2) \tag{79}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$ and $\mathbf{V_{15}}$:

$$V_{3,8,15}^{(5)}(Q^2) = \left\{\Gamma^-(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^-(m_b^2, Q_0^2)\right\} V_{3,8,15}^{(4)}(Q_0^2) \tag{80}$$

- $\mathbf{V_{24}}$:

$$V_{24}^{(5)}(Q^2) = \left\{\Gamma^-(Q_0^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2, Q_0^2)\right\} V^{(4)}(Q_0^2) \tag{81}$$

- $\mathbf{V_{35}}$:

$$V_{35}^{(5)}(Q^2) = \left\{\Gamma^v(Q_0^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2, Q_0^2)\right\} V^{(4)}(Q_0^2) \tag{82}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$ and $\mathbf{T_{15}}$:

$$T_{3,8,15}^{(5)}(Q^2) = \left\{\Gamma^+(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^+(m_b^2, Q_0^2)\right\} T_{3,8,15}^{(4)}(Q_0^2) \tag{83}$$

- $\mathbf{T_{24}}$:

$$\begin{aligned} T_{24}^{(5)}(Q^2) \quad = \quad & \Bigg\{\Gamma^+(Q^2, m_b^2)\left(1 + a_s^2(m_b^2)[A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] \quad -4a_s^2(m_b^2)\tilde{A}_{bg}^{S,(2)}\right) \times \\ & \underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2, Q_0^2)}\Bigg\} \begin{pmatrix}\Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2)\end{pmatrix} \end{aligned} \tag{84}$$

- $\mathbf{T_{35}}$:

$$T_{35}^{(5)}(Q^2) = \left\{\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix}M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b\end{pmatrix} \underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2, Q_0^2)}\right\} \begin{pmatrix}\Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2)\end{pmatrix} \tag{85}$$

1.2.3 $Q_0^2 < m_t^2 \leq Q^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix}\Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2)\end{pmatrix} = \left\{\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(Q^2, m_t^2)} \begin{pmatrix}M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t\end{pmatrix} \underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_t^2, Q_0^2)}\right\} \begin{pmatrix}\Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2)\end{pmatrix} \tag{86}$$

where:

$$\begin{pmatrix}M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + a_s^2(m_t^2)A_{qq,t}^{NS,(2)} \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} + a_s^2(m_t^2) \begin{pmatrix}\tilde{A}_{tq}^{S,(2)} & \tilde{A}_{tg}^{S,(2)} \\ A_{gq,t}^{S,(2)} & A_{gg,t}^{S,(2)}\end{pmatrix} \tag{87}$$

- $\mathbf{V}$:

$$V^{(6)}(Q^2) = \left\{\Gamma^v(Q^2, m_t^2)[1 + a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^v(m_t^2, Q_0^2)\right\} V^{(5)}(Q_0^2) \tag{88}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$, $\mathbf{V_{15}}$ and $\mathbf{V_{24}}$:

$$V_{3,8,15,24}^{(6)}(Q^2) = \left\{\Gamma^-(Q^2, m_t^2)[1 + a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^-(m_t^2, Q_0^2)\right\} V_{3,8,15,24}^{(5)}(Q_0^2) \tag{89}$$

- $\mathbf{V_{35}}$:

$$V_{35}^{(6)}(Q^2) = \left\{\Gamma^-(Q_0^2, m_t^2)[1 + a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^v(m_t^2, Q_0^2)\right\} V^{(5)}(Q_0^2) \tag{90}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$, $\mathbf{T_{15}}$ and $\mathbf{T_{24}}$:

$$T_{3,8,15,24}^{(6)}(Q^2) = \left\{\Gamma^+(Q^2, m_t^2)[1 + a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^+(m_t^2, Q_0^2)\right\} T_{3,8,15,24}^{(5)}(Q_0^2) \tag{91}$$

- $\mathbf{T_{35}}$:

$$\begin{aligned} T_{35}^{(6)}(Q^2) &= \Bigg\{\Gamma^+(Q^2, m_t^2)\left(1 + a_s^2(m_t^2)[A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] \quad -5a_s^2(m_t^2)\tilde{A}_{tg}^{S,(2)}\right) \times \\ &\quad \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, Q_0^2)}\Bigg\}\begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{92}$$

1.3 Matching Conditions on the Evolution Kernels: 2 Thresholds Crossing

In this Section we will discuss the case in which the evolution crosses two thresholds. Therefore there are only two situations: 1) $Q_0^2 < m_c^2 < m_b^2 \leq Q^2$ and 2) $Q_0^2 < m_b^2 < m_t^2 \leq Q^2$. Anyway, there is nothing new, indeed to obtain such evolution kernels we have just to "merge" together what we have already done in the previous Section.

1.3.1 $Q_0^2 < m_c^2 < m_b^2 \leq Q^2$

- **Singlet** and **gluon**:

$$\begin{aligned} \begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} &= \Bigg\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \times \\ &\quad \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{93}$$

- $\mathbf{V}$:

$$\begin{aligned} V^{(5)}(Q^2) &= \Big\{\Gamma^v(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\times \\ &\quad \Gamma^v(m_b^2, m_c^2)[1 + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2, Q_0^2)\Big\}V^{(3)}(Q_0^2) \end{aligned} \tag{94}$$

- $\mathbf{V_3}$ and $\mathbf{V_8}$:

$$\begin{aligned} V_{3,8}^{(5)}(Q^2) &= \Big\{\Gamma^-(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\times \\ &\quad \Gamma^-(m_b^2, m_c^2)[1 + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^-(m_c^2, Q_0^2)\Big\}V_{3,8}^{(3)}(Q_0^2) \end{aligned} \tag{95}$$

- $\mathbf{V_{15}}$:

$$\begin{aligned} V_{15}^{(5)}(Q^2) &= \Big\{\Gamma^-(Q^2, m_b^2)[1 + a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\times \\ &\quad \Gamma^-(m_b^2, m_c^2)[1 + a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2, Q_0^2)\Big\}V^{(3)}(Q_0^2) \end{aligned} \tag{96}$$

- $\mathbf{V_{24}}$:

$$V_{24}^{(5)}(Q^2) = \Big\{\Gamma^{-}(Q^2, m_b^2)[1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}]\times \Gamma^{v}(m_b^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}]\Gamma^{v}(m_c^2, Q_0^2)\Big\} V^{(3)}(Q_0^2) \quad (97)$$

- $\mathbf{V_{35}}$:

$$V_{35}^{(5)}(Q^2) = \Big\{\Gamma^{v}(Q^2, m_b^2)[1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}]\times \Gamma^{v}(m_b^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}]\Gamma^{v}(m_c^2, Q_0^2)\Big\} V^{(3)}(Q_0^2) \quad (98)$$

- $\mathbf{T_3}$ and $\mathbf{T_8}$:

$$T_{3,8}^{(5)}(Q^2) = \Big\{\Gamma^{+}(Q^2, m_b^2)[1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}]\times \Gamma^{+}(m_b^2, m_c^2)[1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}]\Gamma^{+}(m_c^2, Q_0^2)\Big\} T_{3,8}^{(3)}(Q_0^2) \quad (99)$$

- $\mathbf{T_{15}}$:

$$T_{15}^{(5)}(Q^2) = \Bigg\{\Gamma^{+}(Q^2, m_b^2)[1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}]\Gamma^{+}(m_b^2, m_c^2)\times \left(1 + a_s^2(m_c^2)[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] \quad -3a_s^2(m_c^2)\tilde{A}_{cg}^{S,(2)}\right) \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (100)$$

- $\mathbf{T_{24}}$:

$$T_{24}^{(5)}(Q^2) = \Bigg\{\Gamma^{+}(Q^2, m_b^2)\left(1 + a_s^2(m_b^2)[A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] \quad -4a_s^2(m_b^2)\tilde{A}_{bg}^{S,(2)}\right)\times \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (101)$$

- $\mathbf{T_{35}}$:

$$T_{35}^{(5)}(Q^2) = \Bigg\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \times \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \Bigg\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (102)$$

1.3.2 $Q_0^2 < m_b^2 < m_t^2 \leq Q^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix}\Sigma^{(6)}(Q^2)\\ g^{(6)}(Q^2)\end{pmatrix} = \Bigg\{\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(Q^2,m_t^2)}\begin{pmatrix}M_{11}^t & M_{12}^t\\ M_{21}^t & M_{22}^t\end{pmatrix}\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_t^2,m_b^2)}\times$$
$$\begin{pmatrix}M_{11}^b & M_{12}^b\\ M_{21}^b & M_{22}^b\end{pmatrix}\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2,Q_0^2)}\Bigg\}\begin{pmatrix}\Sigma^{(4)}(Q_0^2)\\ g^{(4)}(Q_0^2)\end{pmatrix} \tag{103}$$

- **V**:

$$V^{(6)}(Q^2) = \Big\{\Gamma^v(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^v(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,Q_0^2)\Big\}V^{(4)}(Q_0^2) \tag{104}$$

- $\mathbf{V_3}$, $\mathbf{V_8}$ and $\mathbf{V_{15}}$:

$$V_{3,8,15}^{(6)}(Q^2) = \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^-(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^-(m_b^2,Q_0^2)\Big\}V_{3,8,15}^{(4)}(Q_0^2) \tag{105}$$

- $\mathbf{V_{24}}$:

$$V_{24}^{(6)}(Q^2) = \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^-(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,Q_0^2)\Big\}V^{(4)}(Q_0^2) \tag{106}$$

- $\mathbf{V_{35}}$:

$$V_{35}^{(6)}(Q^2) = \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^v(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,Q_0^2)\Big\}V^{(4)}(Q_0^2) \tag{107}$$

- $\mathbf{T_3}$, $\mathbf{T_8}$ and $\mathbf{T_{15}}$:

$$T_{3,8,15}^{(6)}(Q^2) = \Big\{\Gamma^+(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^+(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^+(m_b^2,Q_0^2)\Big\}T_{3,8,15}^{(4)}(Q_0^2) \tag{108}$$

- $\mathbf{T_{24}}$:

$$T_{24}^{(6)}(Q^2) = \Bigg\{\Gamma^+(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^+(m_t^2,m_b^2)\times$$
$$\Big(1+a_s^2(m_b^2)[A_{qq,b}^{NS,(2)}-4\tilde{A}_{bq}^{S,(2)}] \quad -4a_s^2(m_b^2)\tilde{A}_{bg}^{S,(2)}\Big)\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2,Q_0^2)}\Bigg\}\begin{pmatrix}\Sigma^{(4)}(Q_0^2)\\ g^{(4)}(Q_0^2)\end{pmatrix} \tag{109}$$

- $\mathbf{T_{35}}$:

$$T_{35}^{(6)}(Q^2) = \Big\{\Gamma^+(Q^2,m_t^2)\Big(1+a_s^2(m_t^2)[A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] \quad -5a_s^2(m_t^2)\tilde{A}_{tg}^{S,(2)}\Big)\times$$
$$\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_t^2,m_b^2)}\begin{pmatrix}M_{11}^b & M_{12}^b\\ M_{21}^b & M_{22}^b\end{pmatrix}\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2,Q_0^2)}\Big\}\begin{pmatrix}\Sigma^{(4)}(Q_0^2)\\ g^{(4)}(Q_0^2)\end{pmatrix} \tag{110}$$

1.4 Matching Conditions on the Evolution Kernels: 3 Thresholds Crossing

In this Section we will discuss the case in which the evolution crosses three thresholds. Therefore there ais only one situation: $Q_0^2 < m_c^2 < m_b^2, m_t^2 \leq Q^2$

1.4.1 $Q_0^2 < m_c^2 < m_b^2 < m_t^2 \leq Q^2$

- **Singlet** and **gluon**:

$$\begin{pmatrix}\Sigma^{(6)}(Q^2)\\ g^{(6)}(Q^2)\end{pmatrix} = \Big\{\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(Q^2,m_t^2)}\begin{pmatrix}M_{11}^t & M_{12}^t\\ M_{21}^t & M_{22}^t\end{pmatrix}\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_t^2,m_b^2)}\times$$
$$\begin{pmatrix}M_{11}^b & M_{12}^b\\ M_{21}^b & M_{22}^b\end{pmatrix}\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_b^2,m_c^2)}\begin{pmatrix}M_{11}^c & M_{12}^c\\ M_{21}^c & M_{22}^c\end{pmatrix} \tag{111}$$
$$\underbrace{\begin{pmatrix}\Gamma_{qq} & \Gamma_{qg}\\ \Gamma_{gq} & \Gamma_{gg}\end{pmatrix}}_{(m_c^2,Q_0^2)}\Big\}\begin{pmatrix}\Sigma^{(3)}(Q_0^2)\\ g^{(3)}(Q_0^2)\end{pmatrix}$$

- $\mathbf{V}$:

$$V^{(6)}(Q^2) = \Big\{\Gamma^v(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^v(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,m_c^2) \tag{112}$$
$$[1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2,Q_0^2)\Big\}V^{(3)}(Q_0^2)$$

- $\mathbf{V_3}$ and $\mathbf{V_8}$:

$$V_{3,8}^{(6)}(Q^2) = \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^-(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^-(m_b^2,m_c^2) \tag{113}$$
$$[1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^-(m_c^2,Q_0^2)\Big\}V_{3,8}^{(3)}(Q_0^2)$$

- $\mathbf{V_{15}}$:

$$V_{15}^{(6)}(Q^2) = \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times$$
$$\Gamma^-(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^-(m_b^2,m_c^2) \tag{114}$$
$$[1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2,Q_0^2)\Big\}V^{(3)}(Q_0^2)$$

- $\mathbf{V_{24}}$:

$$\begin{aligned} V_{15}^{(6)}(Q^2) &= \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times \\ & \Gamma^-(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,m_c^2) \\ & [1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2,Q_0^2)\Big\}V^{(3)}(Q_0^2) \end{aligned} \tag{115}$$

- $\mathbf{V_{35}}$:

$$\begin{aligned} V_{15}^{(6)}(Q^2) &= \Big\{\Gamma^-(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times \\ & \Gamma^v(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^v(m_b^2,m_c^2) \\ & [1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^v(m_c^2,Q_0^2)\Big\}V^{(3)}(Q_0^2) \end{aligned} \tag{116}$$

- $\mathbf{T_3}$ and $\mathbf{T_8}$:

$$\begin{aligned} T_{3,8}^{(6)}(Q^2) &= \Big\{\Gamma^+(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times \\ & \Gamma^+(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^+(m_b^2,m_c^2) \\ & [1+a_s^2(m_c^2)A_{qq,c}^{NS,(2)}]\Gamma^+(m_c^2,Q_0^2)\Big\}T_{3,8}^{(3)}(Q_0^2) \end{aligned} \tag{117}$$

- $\mathbf{T_{15}}$:

$$\begin{aligned} T_{15}^{(6)}(Q^2) &= \Big\{\Gamma^+(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\times \\ & \Gamma^+(m_t^2,m_b^2)[1+a_s^2(m_b^2)A_{qq,b}^{NS,(2)}]\Gamma^+(m_b^2,m_c^2) \\ & \begin{pmatrix} 1+a_s^2(m_c^2)[A_{qq,c}^{NS,(2)}-3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2)\tilde{A}_{cg}^{S,(2)} \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2,Q_0^2)}\Bigg\}\begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{118}$$

- $\mathbf{T_{24}}$:

$$\begin{aligned} T_{24}^{(6)}(Q^2) &= \Big\{\Gamma^+(Q^2,m_t^2)[1+a_s^2(m_t^2)A_{qq,t}^{NS,(2)}]\Gamma^+(m_t^2,m_b^2)\times \\ & \begin{pmatrix} 1+a_s^2(m_b^2)[A_{qq,b}^{NS,(2)}-4\tilde{A}_{bq}^{S,(2)}] & -4a_s^2(m_b^2)\tilde{A}_{bg}^{S,(2)} \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2,m_c^2)}\times \\ & \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2,Q_0^2)}\Bigg\}\begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \tag{119}$$

- $\mathbf{T_{35}}$:

$$T_{35}^{(6)}(Q^2) \;=\; \Big\{\Gamma^+(Q^2,m_t^2)\Big(1+a_s^2(m_b^2)[A_{qq,t}^{NS,(2)}-5\tilde{A}_{tq}^{S,(2)}] \quad -5a_s^2(m_t^2)\tilde{A}_{tg}^{S,(2)}\Big)\times \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2,m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2,Q_0^2)}\Big\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \tag{120}$$

2 Matching conditions with intrinsic distributions

The (quite outdated) discussion presented above concerning the PDF matching conditions is restricted to forward evolution (*i.e.* $Q > Q_0$) and under the assumption that possible intrinsic heavy-quark contributions are absent.[2] This corresponds to assuming that heavy-quark PDFs are entirely dynamically generated at the respective threshold and are identically vanishing below threshold. While this turns out to be a good approximation for PDFs, this is not the case for fragmentation functions (FFs). Indeed, intrinsic heavy-quark FFs are sizeable. In addition, assuming that there are no heavy-quark PDFs makes backward evolution problematic. The basic problem is that the matching-function matrix is non-squared which forbids its inversion thus complicating backward evolution.

The purpose of this section is the relaxation of the no-intrinsic-PDF assumption. This will imply the introduction of new matching functions due to the presence match heavy-quark PDFs below threshold. Importantly, the perturbative expansion of these additional functions is different from zero at $\mathcal{O}(\alpha_s)$ even choosing $\mu_h = m_h$, with μ_h the matching scale, producing a discontinuity already at NLO. However, in what follows we will assume that matching conditions are not computed beyond $\mathcal{O}(\alpha_s^2)$, *i.e.* NNLO, and that intrinsic heavy-quark and heavy-antiquark contributions are equal below threshold. These assumptions lead to:

$$h^{(n_f)}(\mu_h) = \overline{h}^{(n_f)}(\mu_h) \quad \text{and} \quad h^{(n_f+1)}(\mu_h) = \overline{h}^{(n_f+1)}(\mu_h)\,, \tag{121}$$

where we dropped the non-scale dependence. It immediately follows that the valence distributions $\{V, V_3, V_8, V_{15}, V_{24}, V_{35}\}$ match multiplicatively as:[3]

$$V_i^{(n_f+1)}(\mu_h) = [1+K_{ll}](\mu_h)V^{(n_f)}(\mu_h) = [1+K_{ll}(\mu_h)]V_i^{(n_f)}(\mu_h)\,, \quad i = 3,8,15,24,35\,, \tag{122}$$

with K_{ll} some given function that starts at $\mathcal{O}(\alpha_s^2)$. This equivalently amounts to say that no non-perturbative quark-antiquark asymmetry is originally present nor it is generated by the matching procedure.[4]

This allows us to concentrate the discussion on the singlet sector. We start by generalising Eqs. (1),

[2] In fact, the discussion also assumes that the PDF matching takes place at the heavy quark-mass value. This is also a non-necessary assumption that we will however not discuss here.

[3] Note that "1" in Eq. (122) is to be understood as $\delta(1-x)$ in x space.

[4] The assumption of absence of non-perturbative quark-antiquark asymmetry is badly violated for light-valence distributions, such as up and down in the proton. However, we will never realistically need to match PDFs or FFs at the light-quark scales. On the other hand, this assumption helps us keep the discussion general.

(2), and (3) (changing and simplifying a bit the notation) as:

$$
\begin{aligned}
g^{(n_f+1)} &= [1+K_{gg}]g^{(n_f)} + K_{gl}\sum_l l^{+(n_f+1)} + K_{gh}h^{+(n_f)}\,, \\
l^{+(n_f+1)} &= [1+K_{ll}]l^{+(n_f)}\,, \\
h^{+(n_f+1)} &= K_{hg}g^{(n_f)} + K_{hl}\sum_l l^{+(n_f+1)} + [1+K_{hh}]h^{+(n_f)}\,, \\
H^{+(n_f+1)} &= H^{+(n_f)}\,,
\end{aligned}
\tag{123}
$$

where $q^+ = q+\overline{q}$ and l runs between 1 and n_f. We remind that h is the (n_f+1)-th heavier flavour, *i.e.* the one that becomes active at the threshold under consideration. We have also introduced the "super" heavy quark flavour H, heavier than h, that is not affected by the matching at the n_f-th threshold. Note that there are two new functions coming into play, namely K_{gh} and K_{hh}, both multiplying the intrinsic heavy-quark contribution $h^{+(n_f)}$. Moreover, we have written Eq. (123) in such a way that the coefficients K start at least at $\mathcal{O}(\alpha_s)$. In fact, we have already seen that $K_{ll} = \mathcal{O}(\alpha_s^2)$; the same applies to K_{hl} and K_{gl}. Defining the column vector P with components $(g, d^+, u^+, s^+, c^+, b^+, t^+)$, we can represent the matching in a matricial form as:

$$
P^{(n_f+1)} = \left[\mathbb{I} + \mathbb{K}^{(n_f)}\right] P^{(n_f)}\,,
\tag{124}
$$

where $\mathbb{I}$ is the 7×7 unity matrix and $\mathbb{K}^{(n_f)}$ are given, depending on the value of n_f, by:

$$
\mathbb{K}^{(0)} = \begin{pmatrix}
K_{gg} & K_{gh} & 0 & 0 & 0 & 0 & 0 \\
K_{hg} & K_{hh} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\mathbb{K}^{(1)} = \begin{pmatrix}
K_{gg} & K_{gl} & K_{gh} & 0 & 0 & 0 & 0 \\
0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\
K_{hg} & K_{hl} & K_{hh} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\mathbb{K}^{(2)} = \begin{pmatrix}
K_{gg} & K_{gl} & K_{gl} & K_{gh} & 0 & 0 & 0 \\
0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\
K_{hg} & K_{hl} & K_{hl} & K_{hh} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\mathbb{K}^{(3)} = \begin{pmatrix}
K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gh} & 0 & 0 \\
0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\
K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hh} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\mathbb{K}^{(4)} = \begin{pmatrix}
K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gh} & 0 \\
0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{ll} & 0 & 0 \\
K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hh} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\mathbb{K}^{(5)} = \begin{pmatrix}
K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gh} \\
0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{ll} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_{ll} & 0 \\
K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hh}
\end{pmatrix}
\tag{125}
$$

The algorithm to determine the matrix $\mathbb{K}^{(n_f)}$, indexing the gluon distribution with 0 and the quark distributions from 1 to 6, reads:

$$\mathbb{K}_{ij}^{(n_f)} = \begin{cases} K_{gg} & i=j=0 \\ K_{gl} & i=0\,,\ 1\leq j\leq n_f \\ K_{gh} & i=0\,,\ j=n_f+1 \\ K_{ll} & i=j\,,\ 1\leq i,j\leq n_f \\ K_{hg} & i=n_f+1\,,\ j=0 \\ K_{hl} & i=n_f+1\,,\ 1\leq j\leq n_f \\ K_{hh} & i=j=n_f+1 \\ 0 & \text{elsewhere} \end{cases} . \tag{126}$$

Now, in order to apply the matching condition to the distributions in the evolution basis $E = (g, \Sigma, -T_3, T_8, T_{15}, T_{24}, T_{35})$, it is necessary to rotate the matching matrices from the physical basis to the evolution basis through $T\mathbb{K}^{(n_f)}T^{-1}$, where T transforms the vector P into the evolution-basis vector E:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -4 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & -5 \end{pmatrix} . \tag{127}$$

The resulting matching matrices are much more complicated and generally less sparse than those in Eq. (125). However, we can still compute them algorithmically depending on the number of light flavours n_f. To do so, we define the following linearly independent combinations:

$$\begin{array}{rcl} M_0 & = & 1\,, \\ M_1 & = & K_{gg}\,, \\ M_2 & = & K_{gh} + n_f K_{gl}\,, \\ M_3 & = & K_{gh} - K_{gl}\,, \\ M_4 & = & K_{hg}\,, \\ M_5 & = & K_{hh} + n_f(K_{hl} + K_{ll})\,, \\ M_6 & = & K_{hh} - (K_{hl} + K_{ll})\,, \\ M_7 & = & K_{ll}\,, \end{array} \tag{128}$$

such that:

$$\left\{T\left[\mathbb{I} + \mathbb{K}^{(n_f)}\right]T^{-1}\right\}_{ij} = \begin{cases} M_0 + M_1 & i=0 & j=0 \\ \frac{1}{6}M_2 & i=0 & j=1 \\ 0 & i=0 & 2\leq j\leq n_f \\ -\frac{1}{n_f+1}M_3 & i=0 & j=n_f+1 \\ \frac{1}{j(j-1)}M_2 & i=0 & n_f+2\leq j\leq 6 \\ & & \\ M_4 & i=1 & j=0 \\ M_0 + \frac{1}{6}M_5 & i=1 & j=1 \\ 0 & i=1 & 2\leq j\leq n_f \\ -\frac{1}{n_f+1}M_6 & i=1 & j=n_f+1 \\ \frac{1}{j(j-1)}M_5 & i=1 & n_f+2\leq j\leq 6 \\ & & \\ \delta_{ij}\left(M_0 + M_7\right) & 2\leq i\leq n_f & 0\leq j\leq 6 \\ & & \\ -n_f M_4 & i=n_f+1 & j=0 \\ \frac{n_f}{6}\left(-M_5 + (n_f+1)M_7\right) & i=n_f+1 & j=1 \\ 0 & i=n_f+1 & 2\leq j\leq n_f \\ M_0 + \frac{1}{n_f+1}\left(n_f M_6 + (n_f+1)M_7\right) & i=n_f+1 & j=n_f+1 \\ \frac{n_f}{j(j-1)}\left(-M_5 + (n_f+1)M_7\right) & i=n_f+1 & n_f+2\leq j\leq 6 \\ & & \\ M_4 & n_f+2\leq i\leq 6 & j=0 \\ \frac{1}{6}M_5 & n_f+2\leq i\leq 6 & j=1 \\ 0 & n_f+2\leq i\leq 6 & 2\leq j\leq n_f \\ -\frac{1}{n_f+1}M_6 & n_f+2\leq i\leq 6 & j=n_f+1 \\ \delta_{ij}M_0 + \frac{1}{j(j-1)}M_5 & n_f+2\leq i\leq 6 & n_f+2\leq j\leq 6 \end{cases} \tag{129}$$

In order to be able to perform the backward evolution, it is necessary to invert the matching matrix $\mathbb{I}+\mathbb{K}^{(n_f)}$. This is conveniently done perturbatively by expanding the matrix $\left[\mathbb{I}+\mathbb{K}^{(n_f)}\right]^{-1}$ in powers of α_s and truncating the expansion at the appropriate order. Given the assumption discussed above, we are not interested in going beyond $\mathcal{O}(\alpha_s^2)$. In addition, we know that:

$$\mathbb{K}^{(n_f)} = a_s \mathbb{K}^{(1)(n_f)} + a_s^2 \mathbb{K}^{(2)(n_f)} + \mathcal{O}(\alpha_s^3)\,, \tag{130}$$

so that:

$$\left[\mathbb{I}+\mathbb{K}^{(n_f)}\right]^{-1} = 1 - a_s \mathbb{K}^{(1)(n_f)} - a_s^2\left[\mathbb{K}^{(2)(n_f)} - \left(\mathbb{K}^{(1)(n_f)}\right)^2\right] + \mathcal{O}(\alpha_s^3)\,. \tag{131}$$

But the matrix $\mathbb{K}^{(1)(n_f)}$ is particularly simple because at $\mathcal{O}(\alpha_s)$ the matching functions K_{ll}, K_{hl}, and K_{gl} vanish. Therefore, using Eq. (126) we find:

$$\mathbb{K}_{ij}^{(1)(n_f)} = \delta_{i,0}\left(\delta_{j,0}K_{gg}^{(1)} + \delta_{j,n_f+1}K_{gh}^{(1)}\right) + \delta_{i,n_f+1}\left(\delta_{j,0}K_{hg}^{(1)} + \delta_{j,n_f+1}K_{hh}^{(1)}\right)\,, \tag{132}$$

such that its square has the same structure and can be written as:

$$\begin{aligned}\left(\mathbb{K}_{ij}^{(1)(n_f)}\right)^2 &= \delta_{i,0}\left(\delta_{j,0}\sum_{k=g,h}K_{gk}^{(1)}K_{kg}^{(1)} + \delta_{j,n_f+1}\sum_{k=g,h}K_{gk}^{(1)}K_{kh}^{(1)}\right)\\ &+ \delta_{i,n_f+1}\left(\delta_{j,0}\sum_{k=g,h}K_{hk}^{(1)}K_{kg}^{(1)} + \delta_{j,n_f+1}\sum_{k=g,h}K_{hk}^{(1)}K_{kh}^{(1)}\right)\,,\end{aligned} \tag{133}$$

where, in x space, the product of two K functions is to be understood as a convolution. We report below the expression in x space of the functions $K_{gh}^{(1)}$ and $K_{gh}^{(1)}$ for the matching of PDFs:

$$\begin{aligned}K_{gh}^{(1)}(x) &= 2C_F\frac{1+(1-x)^2}{x}\left(\ln\frac{\mu_h^2}{m_h^2} - 1 - 2\ln x\right)\,,\\ K_{hh}^{(1)}(x) &= 2C_F\left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_h^2}{m_h^2} - 1 - 2\ln(1-x)\right)\right]_+\,.\end{aligned} \tag{134}$$

We notice that the $\mathcal{O}(\alpha_s^2)$ correction to these functions is currently unknown, therefore we only use the $\mathcal{O}(\alpha_s)$ expressions. For the sake of completeness, we also report here the remaining $\mathcal{O}(\alpha_s)$ matching functions:

$$\begin{aligned}K_{gg}^{(1)}(x) &= -\frac{4}{3}T_R\delta(1-x)\ln\frac{\mu_h^2}{m_h^2}\,,\\ K_{hg}^{(1)}(x) &= 4T_R\left[x^2+(1-x)^2\right]\ln\frac{\mu_h^2}{m_h^2}\,,\end{aligned} \tag{135}$$

Of course, the matrix $\left[\mathbb{I}+\mathbb{K}^{(n_f)}\right]^{-1}$ in Eq. (131) has to be rotated into the evolution basis by means of the transformation matrix T in Eq. (127)

In order to implement the possibility to compute the evolution operator evolution in the VFNS, we need to know how the evolution operator for a given number of active flavours n_f matches at the following threshold. The first step is to generalise the structure of the DGLAP evolution equations in the QCD evolution basis allowing for the presence of (scale-independent) intrinsic contributions. It is important to realise that DGLAP equations govern the evolution in μ of the collinear distributions but do not give us any information on the non-dynamic part of these distributions, *i.e.* its intrinsic contribution. The starting point is the DGLAP equation for the quark distribution q_k^+[5] in its basic form:

$$\frac{dq_k^+}{d\ln\mu^2} = \frac{dq_k^+}{dt} = 2P_{qg}\otimes g + (P_{qq}^V + P_{q\bar{q}}^V)\otimes q_k^+ + (P_{qq}^S + P_{q\bar{q}}^S)\otimes\sum_{i=1}^{n_f}q_i^+\,. \tag{136}$$

[5]We remain under the assumption that intrinsic contributions are symmetric upon charge conjugation. This allow us to forget about the valence sector that will always evolve multiplicatively.

It is crucial to notice that the sum on the r.h.s. of the equation above runs over the n_f active flavours only. Now, we sum the index k up to n_f and we get:

$$\frac{d}{dt}\sum_{k=1}^{n_f} q_k^+ = 2n_f P_{qg} \otimes g + P_{qq} \otimes \sum_{k=1}^{n_f} q_k^+ \,, \tag{137}$$

where we have defined:

$$P_{qq} = (P_{qq}^V + P_{q\bar{q}}^V) + n_f (P_{qq}^S + P_{q\bar{q}}^S). \tag{138}$$

In the presence of intrinsic heavy-quark distributions and after some algebra, one finds that:

$$\frac{d\Sigma}{dt} = 2n_f P_{qg} \otimes g + n_f P_{qq} \otimes \left(\frac{\Sigma}{6} + \sum_{j=n_f+1}^{6} \frac{E_j}{j(j-1)} \right) \,, \tag{139}$$

with $E_j \in \{\Sigma, -T_3, T_8, T_{15}, T_{24}, T_{35}\}$. Therefore, the singlet distribution, on top of the gluon, couples also to the other quark distributions. Notice that:

$$\sum_{j=n_f+1}^{6} \frac{1}{j(j-1)} = \frac{1}{n_f} - \frac{1}{6} \,, \tag{140}$$

such that, for $E_j = \Sigma$, Eq. (139) reproduces the usual form of the DGLAP equation for the singlet. We now need to determine how the distributions E_j, for $j > 1$, evolve. We find:

$$\frac{dE_j}{dt} = (1 - \theta_{n_f j}) \left[2n_f P_{qg} \otimes g + n_f P_{qq} \otimes \left(\frac{\Sigma}{6} + \sum_{i=n_f+1}^{6} \frac{E_i}{i(i-1)} \right) \right] + \theta_{n_f j} P^+ \otimes E_j \,, \tag{141}$$

with:

$$\theta_{n_f j} = \begin{cases} 1 & n_f \geq j \\ 0 & n_f < j \end{cases} \,. \tag{142}$$

Therefore, as expected, for a given n_f (and thus a given energy) the distribution E_j evolves multiplicatively through P^+ if $j \leq n_f$ and evolves exactly like the singlet if $j > n_f$. As an example, for $n_f = 4$, that is to say for energies between the charm and the bottom thresholds, $E_2 = T_3$, $E_3 = T_8$, and $E_4 = T_{15}$ evolve multiplicatively, while $E_5 = T_{24}$ and $E_6 = T_{35}$ evolve like the singlet.

Finally, the gluon distribution evolves as:

$$\frac{dg}{dt} = P_{gg} \otimes g + P_{gq} \otimes \sum_{i=1}^{n_f} q_i^+ \,, \tag{143}$$

that translates into:

$$\frac{dg}{dt} = P_{gg} \otimes g + n_f P_{gq} \otimes \left(\frac{\Sigma}{6} + \sum_{j=n_f+1}^{6} \frac{E_j}{j(j-1)} \right) \,. \tag{144}$$

Eqs. (139), (141), and (144) fully define the splitting kernel matrix in the general case of possible presence of intrinsic heavy-quark contributions. Noticeably, these equations correctly reduce to the more familiar ones if these contributions are set to zero.

Therefore, the evolution of the vector of distributions E with n_f active flavours and possible intrinsic heavy-quark contributions takes the matricial form:

$$\frac{dE}{dt} = \mathbb{P}^{(n_f)} \otimes E \,, \tag{145}$$

where the entries of the splitting-function matrix $\mathbb{P}^{(n_f)}$ can be defined algorithmically as:

$$\mathbb{P}_{ij}^{(n_f)} = \begin{cases} P_{gg} & i=0 & j=0 \\ \frac{n_f}{6}P_{gq} & i=0 & j=1 \\ 0 & i=0 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\frac{n_f}{6}P_{gq} & i=0 & n_f+1\leq j\leq 6 \\ & & \\ 2n_fP_{qg} & i=1 & j=0 \\ \frac{n_f}{6}P_{qq} & i=1 & j=1 \\ 0 & i=1 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\frac{n_f}{6}P_{qq} & i=1 & n_f+1\leq j\leq 6 \\ & & \\ \delta_{ij}P^+ & 2\leq i\leq n_f & 0\leq j\leq 6 \\ & & \\ 2n_fP_{qg} & n_f+1\leq i\leq 6 & j=0 \\ \frac{n_f}{6}P_{qq} & n_f+1\leq i\leq 6 & j=1 \\ 0 & n_f+1\leq i\leq 6 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\frac{n_f}{6}P_{qq} & n_f+1\leq i\leq 6 & n_f+1\leq j\leq 6 \end{cases} . \tag{146}$$

2.1 Evolution operator

Assuming that the set of distributions E evolves through the operator $\mathbf{\Gamma}$ as:

$$E(t) = \mathbf{\Gamma}(t,t_0)\otimes E(t_0)\,, \tag{147}$$

Given the possible presence of heavy-quark thresholds in the evolution interval $[t_0,t]$, the operator $\mathbf{\Gamma}$ is in fact a product of operators. More specifically. it can be written as:

$$\mathbf{\Gamma}(t,t_0) = \mathbf{\Gamma}^{(N)}(t,t_N)\otimes\prod_{n_f=N-1}^{0}\mathcal{M}^{(n_f)}\otimes\mathbf{\Gamma}^{(n_f)}(t_{n_f+1}-\epsilon,t_{n_f}) \tag{148}$$

with $\mathcal{M}^{(n_f)} = T\left[\mathbb{I}+\mathbb{K}^{(n_f)}\right]T^{-1}$ given in Eq. (129) and $t>t_N>\dots>t_1>t_0$, being $t_1,\dots,t_N$ the heavy-quark thresholds enclosed in the evolution interval. Using Eq. (145), each evolution operator $\mathbf{\Gamma}^{(n_f)}$ obeys the evolution equation:

$$\frac{d}{dt}\mathbf{\Gamma}^{(n_f)}(t,t_{n_f}) = \mathbb{P}^{(n_f)}(t)\otimes\mathbf{\Gamma}^{(n_f)}(t,t_{n_f})\,, \tag{149}$$

with boundary condition $\mathbf{\Gamma}^{(n_f)}(t_{n_f},t_{n_f}) = \mathbb{I}$. The solution of the equation above can be written as:

$$\mathbf{\Gamma}^{(n_f)}(t,t_{n_f}) = \mathcal{P}\exp\left[\int_{t_{n_f}}^{t}dt'\,\mathbb{P}^{(n_f)}(t')\right]\,, \tag{150}$$

where $\mathcal{P}$ symbolises the path ordering. In order to achieve an efficient computation of the evolution operator, we need to identify the general structure of $\mathbf{\Gamma}^{(n_f)}$ Given the structure of the splitting-function matrix in Eq. (146), one can infer the structure of $\mathbf{\Gamma}^{(n_f)}$. It turns out that the structure of

$\mathbf{\Gamma}^{(n_f)}$ is almost the same as that of $\mathbb{P}^{(n_f)}$. Specifically:

$$\left[\mathbf{\Gamma}^{(n_f)} - \mathbb{I}\right]_{ij} = \begin{cases} \Gamma_{gg} & i=0 & j=0 \\ \Gamma_{gq} & i=0 & j=1 \\ 0 & i=0 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\Gamma_{gq} & i=0 & n_f+1\leq j\leq 6 \\ & & \\ \Gamma_{qg} & i=1 & j=0 \\ \Gamma_{qq} & i=1 & j=1 \\ 0 & i=1 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\Gamma_{qq} & i=1 & n_f+1\leq j\leq 6 \\ & & \\ \delta_{ij}\Gamma^{+} & 2\leq i\leq n_f & 0\leq j\leq 6 \\ & & \\ \Gamma_{qg} & n_f+1\leq i\leq 6 & j=0 \\ \Gamma_{qq} & n_f+1\leq i\leq 6 & j=1 \\ 0 & n_f+1\leq i\leq 6 & 2\leq j\leq n_f \\ \frac{6}{j(j-1)}\Gamma_{qq} & n_f+1\leq i\leq 6 & n_f+1\leq j\leq 6 \end{cases} . \tag{151}$$

Now, using Eqs. (146) and (151), we need to work out a compact algorithm to compute the product $\mathbb{P}^{(n_f)} \otimes \mathbf{\Gamma}^{(n_f)}$ required for the computation of the evolution operator. This reads:

$$\left[\mathbb{P}^{(n_f)}\mathbf{\Gamma}^{(n_f)}\right]_{ik} = \mathbb{P}^{(n_f)}_{ik} + \begin{cases} P_{gg}\Gamma_{gg} + P_{gq}\Gamma_{qg} & i=0 & k=0 \\ P_{gg}\Gamma_{gq} + P_{gq}\Gamma_{qq} & i=0 & k=1 \\ 0 & i=0 & 2\leq k\leq n_f \\ \frac{6}{k(k-1)}\left[P_{gg}\Gamma_{gq} + P_{gq}\Gamma_{qq}\right] & i=0 & n_f+1\leq k\leq 6 \\ & & \\ 2n_f P_{qg}\Gamma_{gg} + P_{qq}\Gamma_{qg} & i=1 & k=0 \\ 2n_f P_{qg}\Gamma_{gq} + P_{qq}\Gamma_{qq} & i=1 & k=1 \\ 0 & i=1 & 2\leq k\leq n_f \\ \frac{6}{k(k-1)}\left[2n_f P_{qg}\Gamma_{gq} + P_{qq}\Gamma_{qq}\right] & i=1 & n_f+1\leq k\leq 6 \\ & & \\ \delta_{ik}P^{+}\Gamma^{+} & 2\leq i\leq n_f & 0\leq k\leq 6 \\ & & \\ \frac{6}{k(k-1)}\left[2n_f P_{qg}\Gamma_{gq} + P_{qq}\Gamma_{qq}\right] & n_f+1\leq i\leq 6 & n_f+1\leq k\leq 6 \end{cases} \tag{152}$$

The number of independent evolution kernels equals the number of independent splitting functions, that is *five* for the singlet sector and *two* for the non-singlet sector. It is thus convenient to define the following vector of operators:

$$\boldsymbol{\gamma} = \begin{pmatrix} \Gamma^{V} \\ \Gamma^{-} \\ \Gamma^{+} \\ \Gamma_{qq} \\ \Gamma_{qg} \\ \Gamma_{gq} \\ \Gamma_{gg} \end{pmatrix} , \tag{153}$$

whose evolution equation reads:

$$\frac{d\boldsymbol{\gamma}}{dt} = \mathcal{P}(\boldsymbol{\gamma})\,. \tag{154}$$

Based on Eq. (152), the action of the function $\mathcal{P}$ on the vector $\boldsymbol{\gamma}$ can be written as follows:

$$\mathcal{P}\left(\boldsymbol{\gamma}\right)=\begin{pmatrix} P^{\rm V}\Gamma_0+P^{\rm V}\Gamma^{\rm V} \\ P^{-}\Gamma_0+P^{-}\Gamma^{-} \\ P^{+}\Gamma_0+P^{+}\Gamma^{+} \\ P_{qq}\Gamma_0+P_{qq}\Gamma_{qq}+2n_fP_{qg}\Gamma_{gq} \\ P_{qq}\Gamma_{qg}+2n_fP_{qg}\Gamma_{gg} \\ P_{gq}\Gamma_{qq}+P_{gg}\Gamma_{gq} \\ P_{gg}\Gamma_0+P_{gq}\Gamma_{qg}+P_{gg}\Gamma_{gg} \end{pmatrix}. \tag{155}$$

where Γ_0 is the identity operator. This is enough to compute the evolution of the vector $\boldsymbol{\gamma}$ for all the required numbers of active flavours n_f. Once this is done, the operators $\boldsymbol{\Gamma}^{(n_f)}$ can be reconstructed and combined to the matching conditions according to Eq. (148). To do so, we need to compute the product:

$$T\left[\mathbb{I}+\mathbb{K}^{(n_f)}\right]T^{-1}\boldsymbol{\Gamma}^{(n_f)}=\mathbb{I}+T\mathbb{K}^{(n_f)}T^{-1}+\boldsymbol{\Gamma}^{(n_f)}+\mathbb{S}^{(n_f)}\,, \tag{156}$$

where we have defined $\mathbb{S}^{(n_f)}=T\mathbb{K}^{(n_f)}T^{-1}\boldsymbol{\Gamma}^{(n_f)}$ that is given by:

$$\mathbb{S}_{ik}^{(n_f)}=\begin{cases} K_{gg}\Gamma_{gg}+K_{gl}\Gamma_{qg} & i=0 & k=0 \\ K_{gg}\Gamma_{gq}+K_{gl}\Gamma_{qq} & i=0 & k=1 \\ 0 & i=0 & 2\leq k\leq n_f \\ \frac{1}{k(k-1)}\left[K_{gg}\Gamma_{gq}+K_{gl}\Gamma_{qq}\right] & i=0 & n_f+1\leq k\leq 6 \\ \\ K_{hg}\Gamma_{gg}+K_{hl}\Gamma_{qg}+K_{ll}\Gamma_{qg} & i=1 & k=0 \\ K_{hg}\Gamma_{gq}+K_{hl}\Gamma_{qq}+K_{ll}\Gamma_{qq} & i=1 & k=1 \\ 0 & i=1 & 2\leq k\leq n_f \\ \frac{1}{k(k-1)}\left[K_{hg}\Gamma_{gq}+K_{hl}\Gamma_{qq}+K_{ll}\Gamma_{qq}\right] & i=1 & n_f+1\leq k\leq 6 \\ \\ \delta_{ik}K_{ll}\Gamma^{+} & 2\leq i\leq n_f & 0\leq k\leq 6 \\ \\ -n_f\left(K_{hg}\Gamma_{gg}+K_{hl}\Gamma_{qg}\right)+K_{ll}\Gamma_{qg} & i=n_f+1 & k=0 \\ -n_f\left(K_{hg}\Gamma_{gq}+K_{hl}\Gamma_{qq}\right)+K_{ll}\Gamma_{qq} & i=n_f+1 & k=1 \\ 0 & i=n_f+1 & 2\leq k\leq n_f \\ \frac{6}{k(k-1)}\left[-n_f\left(K_{hg}\Gamma_{gq}+K_{hl}\Gamma_{qq}\right)+K_{ll}\Gamma_{qq}\right] & i=n_f+1 & n_f+1\leq k\leq 6 \\ \\ \frac{6}{k(k-1)}\left[K_{hg}\Gamma_{gq}+K_{hl}\Gamma_{qq}+K_{ll}\Gamma_{qq}\right] & n_f+2\leq i\leq 6 & n_f+2\leq k\leq 6 \end{cases} \tag{157}$$

$$\mathbb{S}_{ik}^{(n_f)}=\begin{cases} M_1\Gamma_{gg}+\frac{1}{1+n_f}\left(M_2-M_3\right)\Gamma_{qg} & i=0 & k=0 \\ M_1\Gamma_{gq}+\frac{1}{1+n_f}\left(M_2-M_3\right)\Gamma_{qq} & i=0 & k=1 \\ 0 & i=0 & 2\leq k\leq n_f \\ \frac{1}{k(k-1)}\left[M_1\Gamma_{gq}+\frac{1}{1+n_f}\left(M_2-M_3\right)\Gamma_{qq}\right] & i=0 & n_f+1\leq k\leq 6 \\ \\ M_4\Gamma_{gg}+\frac{1}{n_f+1}\left(M_5-M_6\right)\Gamma_{qg} & i=1 & k=0 \\ M_4\Gamma_{gq}+\frac{1}{n_f+1}\left(M_5-M_6\right)\Gamma_{qq} & i=1 & k=1 \\ 0 & i=1 & 2\leq k\leq n_f \\ \frac{1}{k(k-1)}\left[M_4\Gamma_{gq}+\frac{1}{n_f+1}(M_5-M_6)\Gamma_{qq}\right] & i=1 & n_f+1\leq k\leq 6 \\ \\ \delta_{ik}M_7\Gamma^{+} & 2\leq i\leq n_f & 0\leq k\leq 6 \\ \\ -n_f\left(M_4\Gamma_{gg}+\frac{1}{n_f+1}(M_5-M_6)\Gamma_{qg}\right)+(n_f+1)M_7\Gamma_{qg} & i=n_f+1 & k=0 \\ -n_f\left(M_4\Gamma_{gq}+\frac{1}{n_f+1}(M_5-M_6)\Gamma_{qq}\right)+(n_f+1)M_7\Gamma_{qq} & i=n_f+1 & k=1 \\ 0 & i=n_f+1 & 2\leq k\leq n_f \\ \frac{6}{k(k-1)}\left[-n_f\left(M_4\Gamma_{gq}+\frac{1}{n_f+1}(M_5-M_6)\Gamma_{qq}\right)+(n_f+1)M_7\Gamma_{qq}\right] & i=n_f+1 & n_f+1\leq k\leq 6 \\ \\ \frac{6}{k(k-1)}\left[M_4\Gamma_{gq}+\frac{1}{n_f+1}(M_5-M_6)\Gamma_{qq}\right] & n_f+2\leq i\leq 6 & n_f+2\leq k\leq 6 \end{cases} \tag{158}$$

We finally comment on FFs. So far, to the best of my knowledge, none of the $\mathcal{O}(\alpha_s^2)$ corrections to the FF matching conditions has been computed. Therefore, we must limit to $\mathbb{K}^{(1)(n_f)}$ that is simply given by the transpose of that for PDFs.

References

[1] M. Buza, Y. Matiounine, J. Smith and W. L. van Neerven, Eur. Phys. J. C **1**, 301 (1998) [arXiv:hep-ph/9612398].

[2] A. Vogt, Comput. Phys. Commun. **170** (2005) 65 [arXiv:hep-ph/0408244].