

# Combining evolution and DIS operators

## 1 The structure of the observables

In all cases the inclusive DIS structure functions are conveniently expressed in terms of PDF combinations in the so-called physical basis  $\{q_i^\pm\}$ , with  $q_i^\mp = q_i \pm \bar{q}_i$ , where  $q_i$  and  $\bar{q}_i$  are the PDFs of the  $i$ -th quark-flavour, with  $i = u, d, s, c, b, t$ , and its antiparticle, respectively. Schematically, a DIS structure function can be written as:

$$F = C_g g + \sum_i (C_i^+ q_i^+ + C_i^- q_i^-) , \quad (1.1)$$

being  $C$  the appropriate coefficient functions. Conversely, the evolution of PDFs is usually computed in the so-called QCD evolution basis  $\{d_i^\pm\}$ , with  $d_1^+ = \Sigma$ ,  $d_2^+ = -T_3$ ,  $d_3^+ = T_8$ ,  $d_4^+ = T_{15}$ ,  $d_5^+ = T_{24}$ , and  $d_6^+ = T_{35}$  and  $d_1^- = V$ ,  $d_2^- = -V_3$ ,  $d_3^- = V_8$ ,  $d_4^- = V_{15}$ ,  $d_5^- = V_{24}$ , and  $d_6^- = V_{35}$ . The gluon remains unchanged.

$$g = \Gamma_{gg} g_0 + \Gamma_{gq} d_{1,0}^\pm \quad (1.2)$$

while:

$$\begin{aligned} d_i^\pm &= \theta_{i2} \theta(Q - m_i) \Gamma^\pm d_{i,0}^\pm + \theta(m_i - Q) \begin{cases} \Gamma_{qq} d_{1,0}^\pm + \Gamma_{qg} g_0 & \text{for } + \\ \Gamma^v d_{1,0}^\pm & \text{for } - \end{cases} \\ &= \begin{cases} \Gamma_{qq} d_{1,0}^\pm + \Gamma_{qg} g_0 & + \\ \Gamma^v d_{1,0}^\pm & - \end{cases} + \theta(Q - m_i) \left[ \theta_{i2} \Gamma^\pm d_{i,0}^\pm - \begin{cases} \Gamma_{qq} d_{1,0}^\pm + \Gamma_{qg} g_0 & + \\ \Gamma^v d_{1,0}^\pm & - \end{cases} \right] . \end{aligned} \quad (1.3)$$

so that:

$$q_i^- = \delta_{i1} \Gamma^v d_{1,0}^- + \Gamma^\pm \sum_{j=2}^6 \theta_{ji} \frac{1 - \delta_{ij} j}{j(j-1)} \theta(Q - m_j) d_{j,0}^\pm \quad (1.4)$$

Therefore, we need to relate these two bases. This is done through the linear transformation:

$$q_i^\pm = \sum_{j=1}^6 M_{ij} d_j^\pm , \quad (1.5)$$

where the transformation matrix  $M_{ij}$  can be written as:

$$\begin{aligned} M_{ij} &= \theta_{ji} \frac{1 - \delta_{ij} j}{j(j-1)} \quad j \geq 2 , \\ M_{i1} &= \frac{1}{6} , \end{aligned} \quad (1.6)$$

with  $\theta_{ji} = 1$  for  $j \geq i$  and zero otherwise.

Using eq. (1.5) we can make the following identifications:

$$D^\pm = q_{2j-1}^\pm \quad \text{and} \quad U^\pm = q_{2j}^\pm , \quad j = 1, 2, 3 , \quad (1.7)$$

so that we can write:

$$F^\pm = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 |V_{2i,(2j-1)}|^2 [C_\pm (q_{2j-1}^\pm \pm q_{2i}^\pm) + 4P^\pm C_g g] . \quad (1.8)$$

Using the definition of  $M_{ij}$  in eq. (1.6), we can rewrite  $F^\pm$  in terms of PDFs in the evolution basis as:

$$F^\pm = \sum_{i=1}^3 \sum_{j=1}^3 |V_{2i,(2j-1)}|^2 F_{ij}^\pm , \quad (1.9)$$

with:

$$F_{ij}^{\pm} = C_g 2P^{\pm} g + C_{\pm}^S P^{\pm} \frac{1}{6} d_1^{\pm} + C_{\pm} \sum_{k=2}^6 \frac{\theta_{k,2j-1}(1 - \delta_{2j-1,k}k) \pm \theta_{k,2i}(1 - \delta_{2i,k}k)}{2k(k-1)} d_k^{\pm}. \quad (1.10)$$

Eq. (1.10) is valid only for  $F_2$  and  $F_3$ . In order to obtain a similar equation also for  $F_3$ , one needs to change sign to the antiquark distributions, *i.e.*  $\bar{q}_i \rightarrow -\bar{q}_i$ . In the QCD evolution basis, this has the consequence of exchanging the  $T$ -like distributions with the  $V$ -like ones, that is to say  $d_k^+ \leftrightarrow d_k^-$ . It is the easy to see that:

$$F_3^{\pm} = \sum_{i=1}^3 \sum_{j=1}^3 |V_{2i,(2j-1)}|^2 F_{3,ij}^{\pm}, \quad (1.11)$$

with:

$$F_{3,ij}^{\pm} = C_g 2P^{\pm} g + C_{\pm}^S P^{\mp} \frac{1}{6} d_1^{\pm} + C_{\pm} \sum_{k=2}^6 \frac{\theta_{k,2j-1}(1 - \delta_{2j-1,k}k) \mp \theta_{k,2i}(1 - \delta_{2i,k}k)}{2k(k-1)} d_k^{\pm}. \quad (1.12)$$

It is now useful to consider the inclusive structure functions and exploit the unitarity of the CKM matrix elements  $V_{UD}$ :

$$\sum_{i=1}^3 |V_{2i,(2j-1)}|^2 = \sum_{j=1}^3 |V_{2i,(2j-1)}|^2 = 1 \quad \Rightarrow \quad \sum_{i=1}^3 \sum_{j=1}^3 |V_{2i,(2j-1)}|^2 = 3. \quad (1.13)$$

Summing over  $i$  and  $j$  in eq. (1.9) and using eq. (1.10), one obtains:

$$F^{\pm} = C_g 6P^{\pm} g + C_{\pm}^S P^{\pm} \frac{1}{2} d_1^{\pm} + \frac{1}{2} C_{\pm} \sum_{k=2}^6 d_k^{\pm} \sum_{l=1}^6 (\pm 1)^{l+1} M_{lk}. \quad (1.14)$$

Considering separately  $F^+$  and  $F^-$  and using eq. (??), one finds:

$$F^+ = C_g 6g + C_+^S \frac{1}{2} d_1^+ \quad (1.15)$$

and:

$$F^- = \frac{1}{2} C_- \sum_{k=2}^6 \left[ \frac{P_k^+}{k-1} - \frac{P_k^-}{k} \right] d_k^-, \quad (1.16)$$

with the even/odd projectors defined as:

$$P_k^{\pm} = \frac{1 \pm (-1)^k}{2}. \quad (1.17)$$

It should be pointed out that such simple expressions (independent of the CMK matrix elements) is achievable only if it is possible to factorize the non-singlet coefficient functions as implicitly done in eqs. (1.10) and (1.12). In fact, this is possible only in the ZM case in which the coefficient functions of each PDF combination is the same.

For  $F_3$  we find:

$$F_3^+ = C_g 6g + C_-^S \frac{1}{2} d_1^- \quad (1.18)$$

and:

$$F_3^- = \frac{1}{2} C_+ \sum_{k=2}^6 \left[ \frac{P_k^+}{k-1} - \frac{P_k^-}{k} \right] d_k^+. \quad (1.19)$$