Computation of the charged current structure functions

1 The structure of the observables

The structure in terms of PDFs of the charged current (CC) structure functions is complicated by the mixing between down- and up-type quarks provided by the CKM matrix. As a first step, we write the $\mathcal{O}(\alpha_s)$ contribution to $F = F_2, F_L$ (we will consider F_3 later) in a convenient way as:

$$F^{\nu} = \sum_{U=u,c,t} \sum_{D=d,s,b} |V_{UD}|^2 \left[C_{\pm}\left(D+\overline{U}\right) + 2C_g g\right] \tag{1.1}$$

and:

$$F^{\overline{\nu}} = \sum_{U=u,c,t} \sum_{D=d,s,b} |V_{UD}|^2 \left[C_{\pm}\left(\overline{D}+U\right) + 2C_g g\right] \tag{1.2}$$

where we have omitted the convolution symbol and an overall factor $2x$. At this order we don't have to worry about whether C_+ or C_- has to be used because they coincide. However, in the following it will appear naturally which one has be used and where. One can combine the expressions above conveniently as:

$$F^{\pm} \equiv \frac{F^{\nu} \pm F^{\overline{\nu}}}{2} = \frac{1}{2} \sum_{U=u,c,t} \sum_{D=d,s,b} |V_{UD}|^2 \left[C_{\pm}\left(D^{\pm} \pm U^{\pm}\right) + P^{\pm} 4C_g g\right] \tag{1.3}$$

where we have used the usual definition $q^{\pm} = q \pm \overline{q}$ and defined the projector:

$$P^{\pm} = \frac{1 \pm 1}{2} \,. \tag{1.4}$$

It should be noted that the subscript $\pm$ to the quark coefficient function $C_{\pm}$ because is now associated to each of $F^{\pm}$.

Now we need to express these observables in terms of PDFs in the evolution basis. The starting point is the relation:

$$q_i^{\pm} = \sum_{j=1}^{6} M_{ij} d_j^{\pm} \,, \tag{1.5}$$

where $d_j^{\pm}$ belong to the QCD evolution basis, that is: $d_1^+ = \Sigma$, $d_2^+ = -T_3$, $d_3^+ = T_8$, $d_4^+ = T_{15}$, $d_5^+ = T_{24}$, and $d_6^+ = T_{35}$ and $d_1^- = V$, $d_2^- = -V_3$, $d_3^- = V_8$, $d_4^- = V_{15}$, $d_5^- = V_{24}$, and $d_6^- = V_{35}$. Note that here we are using the more "natural" ordering for the distibutions $q_i = \{d, u, s, c, b, t\}$ rather than that where u comes before d; this is the reason of the minus sign in front of T_3 and V_3. The trasformation matrix M_{ij} can be written as:

$$\begin{aligned} M_{ij} &= \theta_{ji} \frac{1-\delta_{ij} j}{j(j-1)} \quad j \geq 2 \,, \\ M_{i1} &= \frac{1}{6} \,, \end{aligned} \tag{1.6}$$

with $\theta_{ji} = 1$ for $j \geq i$ and zero otherwise. In addition, one can show that M_{ij} is such that:

$$\sum_{j=1}^{6} M_{ij} = 0 \,, \quad \text{and} \quad \sum_{i=1}^{6} M_{ij} = \delta_{1j} \,. \tag{1.7}$$

Using eq. (1.5) we can make the following identifications:

$$D^{\pm} = q_{2j-1}^{\pm} \quad \text{and} \quad U^{\pm} = q_{2j}^{\pm} \,, \quad j = 1, 2, 3 \,, \tag{1.8}$$

so that we can write:

$$F^{\pm} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|V_{2i,(2j-1)}|^2\left[C_{\pm}\left(q_{2j-1}^{\pm} \pm q_{2i}^{\pm}\right) + 4P^{\pm}C_g g\right]. \tag{1.9}$$

Using the definition of M_{ij} in eq. (1.6), we can rewrite $F^{\pm}$ in terms of PDFs in the evolution basis as:

$$F^{\pm} = \sum_{i=1}^{3}\sum_{j=1}^{3}|V_{2i,(2j-1)}|^2 F_{ij}^{\pm}, \tag{1.10}$$

with:

$$F_{ij}^{\pm} = C_g 2P^{\pm}g + C_{\pm}^{\mathrm{S}}P^{\pm}\frac{1}{6}d_1^{\pm} + C_{\pm}\sum_{k=2}^{6}\frac{\theta_{k,2j-1}(1-\delta_{2j-1,k}k) \pm \theta_{k,2i}(1-\delta_{2i,k}k)}{2k(k-1)}d_k^{\pm}. \tag{1.11}$$

Eq. (1.11) is valid only for F_2 and F_3. In order to obtain a similar equation also for F_3, one needs to change sign to the antiquark distributions, *i.e.* $\overline{q}_i \to -\overline{q}_i$. In the QCD evolution basis, this has the consequence of exchanging the T-like distributions with the V-like ones, that is to say $d_k^{+} \leftrightarrow d_k^{-}$. It is the easy to see that:

$$F_3^{\pm} = \sum_{i=1}^{3}\sum_{j=1}^{3}|V_{2i,(2j-1)}|^2 F_{3,ij}^{\pm}, \tag{1.12}$$

with:

$$F_{3,ij}^{\pm} = C_g 2P^{\pm}g + C_{\pm}^{\mathrm{S}}P^{\mp}\frac{1}{6}d_1^{\pm} + C_{\pm}\sum_{k=2}^{6}\frac{\theta_{k,2j-1}(1-\delta_{2j-1,k}k) \mp \theta_{k,2i}(1-\delta_{2i,k}k)}{2k(k-1)}d_k^{\pm}. \tag{1.13}$$

It is now useful to consider the inclusive structure functions and exploit the unitarity of the CKM matrix elements V_{UD}:

$$\sum_{i=1}^{3}|V_{2i,(2j-1)}|^2 = \sum_{j=1}^{3}|V_{2i,(2j-1)}|^2 = 1 \quad \Rightarrow \quad \sum_{i=1}^{3}\sum_{j=1}^{3}|V_{2i,(2j-1)}|^2 = 3. \tag{1.14}$$

Summing over i and j in eq. (1.10) and using eq. (1.11), one obtains:

$$F^{\pm} = C_g 6P^{\pm}g + C_{\pm}^{\mathrm{S}}P^{\pm}\frac{1}{2}d_1^{\pm} + \frac{1}{2}C_{\pm}\sum_{k=2}^{6}d_k^{\pm}\sum_{l=1}^{6}(\pm 1)^{l+1}M_{lk}. \tag{1.15}$$

Considering separately F^{+} and F^{-} and using eq. (1.7), one finds:

$$F^{+} = C_g 6g + C_{+}^{\mathrm{S}}\frac{1}{2}d_1^{+} \tag{1.16}$$

and:

$$F^{-} = \frac{1}{2}C_{-}\sum_{k=2}^{6}\left[\frac{P^{+}}{k-1} - \frac{P^{-}}{k}\right]d_k^{-}, \tag{1.17}$$

with the even/odd projectors defined as:

$$P_k^{\pm} = \frac{1 \pm (-1)^k}{2}. \tag{1.18}$$

It should be pointed out that such simple expressions (independent of the CMK matrix elements) is achievable only if it is possible to factorize the non-singlet coefficient functions as implicitly done in eqs. (1.11) and (1.13). In fact, this is possible only in the ZM case in which the coefficient functions of each PDF combination is the same.

For F_3 we find:

$$F_3^{+} = C_g 6g + C_{-}^{\mathrm{S}}\frac{1}{2}d_1^{-} \tag{1.19}$$

and:

$$F_3^{-} = \frac{1}{2}C_{+}\sum_{k=2}^{6}\left[\frac{P_k^{+}}{k-1} - \frac{P_k^{-}}{k}\right]d_k^{+}. \tag{1.20}$$