SIDIS cross section in TMD factorisation

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1 Structure of the observable

In this document we report the relevant formulas for the computation of semi-inclusive deep-inelastic scattering (SIDIS) multiplicities under the assumption that the (negative) virtuality of the Q^2 of the exchanged vector boson is much smaller than the Z mass. This allows us to neglect weak contributions and write the cross section in TMD factorisation as:

$$\frac{d\sigma}{dxdQdzdq_T} = \frac{4\pi\alpha^2 q_T}{zxQ^3} Y_+ H(Q,\mu) \sum_q e_q^2 \int_0^\infty db\, b J_0(bq_T)\, \overline{F}_q(x,b;\mu,\zeta_1)\overline{D}_q(z,b;\mu,\zeta_2)\,, \tag{1}$$

with $\zeta_1\zeta_2 = Q^4$ and:

$$Y_+ = 1 + (1-y)^2 = 1 + \left(1 - \frac{Q^2}{xs}\right)^2\,, \tag{2}$$

where s is the squared center of mass energy. The single TMDs are evolved and matched onto the respective collinear functions as usual:

$$\overline{F}_q(x,b;\mu,\zeta) = xF_q(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_j \int_x^1 dy\, \mathcal{C}_{qj}(y;\mu_0,\zeta_0) \left[\frac{x}{y} f_j\left(\frac{x}{y},\mu_0\right)\right]\,, \tag{3}$$

and:

$$\overline{D}_i(z,b;\mu,\zeta) = z^3 D_q(z,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_j \int_z^1 dy\, \left[y^2 \mathbb{C}_{qj}(y;\mu_0,\zeta_0)\right] \left[\frac{z}{y} d_j\left(\frac{z}{y},\mu_0\right)\right]\,. \tag{4}$$

Notice that here we limit to the case $Q \ll M_Z$ such that we can neglect the contribution of the Z boson and thus the electroweak couplings are given by the squared electric charges.

As usual, low-q_T non-perturbative corrections are taken into account by introducing the monotonic function $b_*(b)$ that behaves as:

$$\lim_{b\to 0} b_*(b) = b_{\rm min} \quad \text{and} \quad \lim_{b\to\infty} b_*(b) = b_{\rm max}\,. \tag{5}$$

This allows us to replace the TMDs in Eq. (1) with their "regularised" version:

$$\begin{aligned} \overline{F}_q(x,b;\mu,\zeta) &\rightarrow \overline{F}_q(x,b_*(b);\mu,\zeta) f_{\rm NP}(x,b,\zeta)\,, \\ \overline{D}_q(z,b;\mu,\zeta) &\rightarrow \overline{D}_q(z,b_*(b);\mu,\zeta) D_{\rm NP}(z,b,\zeta)\,, \end{aligned} \tag{6}$$

where we have introduced the non-perturbative functions $f_{\rm NP}$ and $D_{\rm NP}$. It is important to stress that these functions further factorise as follows:

$$\begin{aligned} f_{\rm NP}(x,b,\zeta) &= \widetilde{f}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,, \\ D_{\rm NP}(z,b,\zeta) &= \widetilde{D}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,. \end{aligned} \tag{7}$$

The common exponential function represents the non-perturbative corrections to TMD evolution and the specific functional form is driven by the solution of the Collins-Soper equation where Q_0 is some initial scale. Finally the set of non-perturbative functions to be determined from fits to data are $\widetilde{f}_{\rm NP}$, $\widetilde{D}_{\rm NP}$, and $g_K(b)$. It is worth noticing that by definition

$$f_{\rm NP}(x,b,\zeta) = \frac{\overline{F}_q(x,b;\mu,\zeta)}{\overline{F}_q(x,b_*(b);\mu,\zeta)}\,, \tag{8}$$

and similarly for $D_{\rm NP}$. Therefore, one has a partial handle on the b-dependence of these functions from the region in which b is small enough to make both numerator and denominator perturbatively computable. Making use of Eq. (7) and setting $\zeta_1=\zeta_2=Q^2$ allows us to rewrite Eq. (1) as:

$$\begin{aligned} \frac{d\sigma}{dxdQdzdq_T} &= \frac{4\pi\alpha^2 q_T}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2 \\ &\times \int_0^\infty db\, J_0\left(bq_T\right)\, b\overline{F}_q(x,b_*(b);\mu,Q^2)\overline{D}_q(z,b_*(b);\mu,Q^2)f_{\rm NP}(x,b,Q^2)D_{\rm NP}(z,b,Q^2)\,. \end{aligned} \tag{9}$$

The integral in the r.h.s. can be numerically computed using the Ogata quadrature of zero-th degree (because J_0 enters the integral):

$$\begin{aligned} \frac{d\sigma}{dxdQdzdq_T} &\simeq \frac{4\pi\alpha^2}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2 \\ &\times \sum_{n=1}^N w_n^{(0)}\frac{\xi_n^{(0)}}{q_T}\overline{F}_q\left(x,b_*\left(\frac{\xi_n^{(0)}}{q_T}\right);\mu,Q^2\right)\overline{D}_q\left(z,b_*\left(\frac{\xi_n^{(0)}}{q_T}\right);\mu,Q^2\right) \\ &\times f_{\rm NP}\left(x,\frac{\xi_n^{(0)}}{q_T},Q^2\right)D_{\rm NP}\left(z,\frac{\xi_n^{(0)}}{q_T},Q^2\right)\,, \end{aligned} \tag{10}$$

where $w_n^{(0)}$ and $\xi_n^{(0)}$ are the Ogata weights and coordinates, respectively, and the sum over n is truncated to the N-th term that should be chosen in such a way to guarantee a given target accuracy. The equation above can be conveniently recasted as follows:

$$\frac{d\sigma}{dxdQdzdq_T} \simeq \sum_{n=1}^N w_n^{(0)}\frac{\xi_n^{(0)}}{q_T}S\left(x,z,\frac{\xi_n^{(0)}}{q_T};\mu,Q^2\right)f_{\rm NP}\left(x,\frac{\xi_n^{(0)}}{q_T},Q^2\right)D_{\rm NP}\left(z,\frac{\xi_n^{(0)}}{q_T},Q^2\right)\,, \tag{11}$$

where:

$$S\left(x,z,b;\mu,Q^2\right) = \frac{4\pi\alpha^2}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2\left[\overline{F}_q\left(x,b_*(b);\mu,Q^2\right)\right]\left[\overline{D}_q\left(z,b_*(b);\mu,Q^2\right)\right]\,. \tag{12}$$

2 Integrating over the final-state kinematic variables

Experimental measurements of differential distributions for SIDIS production are often delivered as integrated over finite regions of the final-state kinematic phase space.

More specifically, the cross section is not integrated of the transverse momentum of the vector boson, q_T, but over the transverse momentum of the outgoing hadron, p_{Th}, that is connected to the former through:

$$p_{Th} = zq_T\,. \tag{13}$$

The integrated cross section then reads:

$$\widetilde{\sigma} = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{x_{\rm min}}^{x_{\rm max}} dx \int_{z_{\rm min}}^{z_{\rm max}} dz \int_{p_{Th,{\rm min}}/z}^{p_{Th,{\rm max}}/z} dq_T \left[\frac{d\sigma}{dxdQdzdq_T}\right]\,. \tag{14}$$

One can exploit a property of the Bessel functions to compute the indefinite integral in q_T of the cross section. Specifically, we now compute:

$$K(x,z,Q,q_T) = \int dq_T \left[\frac{d\sigma}{dxdQdzdq_T}\right]\,. \tag{15}$$

This is easily done by using the following property of the Bessel functions:

$$\int dx\, x J_0(x) = x J_1(x)\,, \tag{16}$$

that is equivalent to:

$$\int dq_T\, q_T J_0\left(bq_T\right) = \frac{q_T}{b} J_1\left(bq_T\right)\,. \tag{17}$$

Therefore:

$$\begin{aligned} K(x,z,Q,q_T) &= \frac{4\pi\alpha^2 q_T}{xzQ^3} Y_+ H(Q,\mu) \sum_q e_q^2 \\ &\times \int_0^\infty db\, J_1\left(bq_T\right) \overline{F}_q(x,b_*(b);\mu,Q^2)\overline{D}_q(z,b_*(b);\mu,Q^2) f_{\rm NP}(x,b,Q^2) D_{\rm NP}(z,b,Q^2)\,. \end{aligned} \tag{18}$$

The integral can again be computed using the Ogata quadrature as:

$$K(x,z,Q,q_T) \simeq \sum_{n=1}^{N} w_n^{(1)}\, S\left(x,z,\frac{\xi_n^{(1)}}{q_T};\mu,Q^2\right) f_{\rm NP}\left(x,\frac{\xi_n^{(1)}}{q_T},Q^2\right) D_{\rm NP}\left(z,\frac{\xi_n^{(1)}}{q_T},Q^2\right)\,, \tag{19}$$

with S given in Eq. (12). Once K is known, the integral of the cross section over the bin $q_T \in [p_{Th,{\rm min}}/z : p_{Th,{\rm max}}/z]$ is computed as:

$$\int_{p_{Th,{\rm min}}/z}^{p_{Th,{\rm max}}/z} dq_T \left[\frac{d\sigma}{dxdQdzdq_T}\right] = K(x,z,Q,p_{Th,{\rm max}}/z) - K(x,z,Q,p_{Th,{\rm min}}/z)\,. \tag{20}$$

This allows one to compute analytically one of the integrals that are often required to compare predictions to data.

2.1 Integrating over x, z, and Q

We now move to considering the integral of the cross section over x, z, and Q. Since these integrals usually come together with an integration in q_T, in the following we will consider the primitive function K in Eq. (19) rather than the cross section itself, that is:

$$\widetilde{K}(p_{Th}) = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{z_{\rm min}}^{z_{\rm max}} dz \int_{x_{\rm min}}^{x_{\rm max}} dx\, K(x,z,Q,p_{Th}/z)\,, \tag{21}$$

so that:

$$\widetilde{\sigma} = \widetilde{K}(p_{Th,{\rm max}}) - \widetilde{K}(p_{Th,{\rm min}})\,. \tag{22}$$

The amount of numerical computation required to carry out the integration of a single bin is very large. Indicatively, it amounts to computing a three-dimensional integral for each of the terms of the Ogata quadrature that usually range from a few tens to hundreds. Therefore, in order to be able to do the integrations in a reasonable amount of time and yet obtain accurate results, it is necessary to put in place an efficient integration strategy. This goal can be achieved by exploiting a numerical integration based on interpolation techniques to precompute the relevant quantities. To this purpose, we first define one grid in x, $\{x_\alpha\}$ with $\alpha = 0, \ldots, N_x$, one grid in z, $\{z_\beta\}$ with $\beta = 0, \ldots, N_z$, and one grid in Q, $\{Q_\tau\}$ with $\tau = 0, \ldots, N_Q$, each of which with a set of interpolating functions $\mathcal{I}$ associated. The grids should be such to span the full kinematic range covered by given data set. Then the value of K in Eq. (19) for any kinematics can be obtained through interpolation as:

$$\begin{aligned} K(x,z,Q,q_T) \;&\simeq\; \sum_{n=1}^{N} w_n^{(1)} S\left(x,z,\frac{\xi_n^{(1)}}{q_T};\mu,Q^2\right) \sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} \mathcal{I}_\alpha(x)\mathcal{I}_\beta(z)\mathcal{I}_\tau(Q) \\ &\times\; f_{\rm NP}\left(x_\alpha,\frac{\xi_n^{(1)}}{q_T},Q_\tau^2\right) D_{\rm NP}\left(z_\beta,\frac{\xi_n^{(1)}}{q_T},Q_\tau^2\right) . \end{aligned} \tag{23}$$

Once we have K in this form, the integration over x, z, and Q in Eq. (21) does not involve the non-perturbative functions $f_{\rm NP}$ and $D_{\rm NP}$ and can be written as:

$$\widetilde{K}(p_{Th}) = \sum_{n=1}^{N}\sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} W_{n\alpha\beta\tau}(p_{Th}) f_{\rm NP}\left(x_\alpha,\frac{z_\beta\xi_n^{(1)}}{p_{Th}},Q_\tau^2\right) D_{\rm NP}\left(z_\beta,\frac{z_\beta\xi_n^{(1)}}{p_{Th}},Q_\tau^2\right) , \tag{24}$$

with:

$$W_{n\alpha\beta\tau}(p_{Th}) = w_n^{(1)} \int_{Q_{\rm min}}^{Q_{\rm max}} dQ\,\mathcal{I}_\tau(Q) \int_{z_{\rm min}}^{z_{\rm max}} dz\,\mathcal{I}_\beta(z) \int_{x_{\rm min}}^{x_{\rm max}} dx\,\mathcal{I}_\alpha(x) S\left(x,z,\frac{z\xi_n^{(1)}}{p_{Th}};\mu,Q^2\right) . \tag{25}$$

Since the aim is to fit the functions $f_{\rm NP}$ and $D_{\rm NP}$ to data, one can precompute and store the coefficients W defined in Eq. (25) and compute the cross sections in a fast way making use of Eq. (24).

It is often the case that the integrated cross section, Eq. (14), is given within a certain acceptance region which is typically defined as:

$$W = \sqrt{\frac{(1-x)Q^2}{x}} \geq W_{\rm min}\,, \quad y_{\rm min} \leq y\left(=\frac{Q^2}{sx}\right) \leq y_{\rm max}\,. \tag{26}$$

These constraints can be expressed as constraints on the variable x for a fixed value of Q:

$$x \leq \frac{Q^2}{W_{\rm min}^2 + Q^2}\,, \quad x \geq \frac{Q^2}{sy_{\rm max}}\,, \quad x \leq \frac{Q^2}{sy_{\rm min}}\,. \tag{27}$$

Therefore, in order to implement the acceptance cuts in the computation of the integrated cross sections, it is enough to replace the integration bounds of the integral in x in Eq. (14) as follows:

$$x_{\rm min} \to \overline{x}_{\rm min}(Q) = \max\left[x_{\rm min},\frac{Q^2}{sy_{\rm max}}\right]\,, \quad x_{\rm max} \to \overline{x}_{\rm max}(Q) = \min\left[x_{\rm max},\frac{Q^2}{sy_{\rm min}},\frac{Q^2}{W_{\rm min}^2+Q^2}\right]\,. \tag{28}$$

Given the number and complexity of integrals as that in Eq. (25) to be computed, a numerical implementation requires devising a strategy that maximises the efficiency. In order to do so, we need to "unpac" the function S and perform the integrations in a way that unnecessary computations are avoided as much as possible. Taking $\mu = Q$ and $b = z\xi_n^{(1)}/p_{Th}$, using the fact that the Sudakov evolution factor for quarks R_q is universal, and dropping the unnecessary dependencies, Eq. (12) can be conveniently reorganised as:

$$
\begin{aligned}
S\left(x, z, \frac{z\xi_n^{(1)}}{p_{Th}}, Q\right) &= 4\pi \frac{\alpha^2(Q)H(Q)}{Q^3} R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right), Q\right) \\
&\times \frac{1}{z}\sum_q e_q^2 \overline{D}_q\left(z, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right)\frac{Y_+(x,Q)}{x}\overline{F}_q\left(x, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right).
\end{aligned}
\tag{29}
$$

with:

$$
\overline{F}_q(x,b) = \sum_j \int_x^1 dy\, \mathcal{C}_{qj}(y)\left[\frac{x}{y} f_j\left(\frac{x}{y}, \frac{b_0}{b}\right)\right], \tag{30}
$$

and:

$$
\overline{D}_i(z,b) = \sum_j \int_z^1 dy\, \left[y^2 \mathbb{C}_{qj}(y)\right]\left[\frac{z}{y} d_j\left(\frac{z}{y}, \frac{b_0}{b}\right)\right], \tag{31}
$$

being the initial-scale TMD distributions before Sudakov evolution. The weights in Eq. (25) can then be computed as:

$$
\begin{aligned}
W_{n\alpha\beta\tau}(p_{Th}) &= w_n^{(1)} 4\pi \int_{Q_{\min}}^{Q_{\max}} dQ\, \mathcal{I}_\tau(Q)\frac{\alpha^2(Q)H(Q)}{Q^3} \\
&\times \int_{z_{\min}}^{z_{\max}} dz\, \mathcal{I}_\beta(z) R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right), Q\right)\frac{1}{z}\sum_q e_q^2 \overline{D}_q\left(z, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right)\right) \\
&\times \int_{\overline{x}_{\min}(Q)}^{\overline{x}_{\max}(Q)} dx\, \mathcal{I}_\alpha(x)\frac{Y_+(x,Q)}{x}\overline{F}_q\left(x, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right)\right).
\end{aligned}
\tag{32}
$$

This nesting should optimise the computation of the integrals.

3 Normalising the SIDIS cross section

It is by now well known that the computation of q_T-dependent SIDIS cross sections at moderate and low energies largely undershoots the data. The origin of this normalisation mismatch is as yet unclear. Somewhat surprisingly though, the computation of q_T-integrated cross sections nicely describes the data. This points to a mismatch between q_T-dependent and q_T-integrated calculations. In the low-q_T region this mismatch can be explicitly derived up to $\mathcal{O}(\alpha_s)$ showing that indeed the q_T-dependent resummed calculation does not include terms that are instead present in the q_T-integrated calculation, amongst which threshold logarithms. Despite a careful investigation on the origin of this mismatch would be of great interest, for phenomenological studies a more empirical approach suffices to obtain a better description of the data. Specifically, one can exploit the fact that the q_T-integrated calculation describes well the data to normalise the q_T-dependent cross section. The prescription is as simple as this:

$$
\left[\frac{d\sigma}{dxdQdzdq_T}\right]_{\text{TMD norm.}} = \frac{\left[\frac{d\sigma}{dxdQdz}\right]_{\text{F.O.}}}{\int_0^\infty dq_T\left[\frac{d\sigma}{dxdQdzdq_T}\right]_{\text{TMD}}}\left[\frac{d\sigma}{dxdQdzdq_T}\right]_{\text{TMD}}, \tag{33}
$$

where the subscript F.O. stands for fixed order. This prescription clearly ensures that:

$$
\int_0^\infty dq_T\left[\frac{d\sigma}{dxdQdzdq_T}\right]_{\text{TMD norm.}} = \left[\frac{d\sigma}{dxdQdz}\right]_{\text{F.O.}}, \tag{34}
$$

guaranteeing a correct normalisation. The F.O. integrated cross section is currently fully known up to $\mathcal{O}(\alpha_s)$. The integral in the denominator of Eq. (33) can be explicitly calculated and, using a

b_*-prescription such that $b_*(b_T) \sim b_T$ for $b_T \to 0$, it turns out to vanish invalidating the argument. Fortunately, using a modified version of the b_*-prescription that behaves as $b_*(b_T) \to b_0/Q$ for $b_T \to 0$, the integral becomes finite and computable. In particular, setting for simplicity $\mu = Q$, one finds:

$$
\begin{aligned}
\int_0^\infty dq_T \left[\frac{d\sigma}{dxdQdzdq_T}\right]_{\rm TMD} &= \frac{2\pi\alpha^2}{xzQ^3} Y_+ H(Q,Q) \sum_q e_q^2 \\
&\times \left\{\sum_j \int_x^1 dy\, \mathcal{C}_{qj}(y;Q,Q^2) \left[\frac{x}{y} f_j\left(\frac{x}{y},Q\right)\right]\right\} \\
&\times \left\{\sum_j \int_z^1 dy\, \left[y^2 \mathbb{C}_{qj}(y;Q,Q^2)\right] \left[\frac{z}{y} d_j\left(\frac{z}{y},Q\right)\right]\right\}.
\end{aligned}
\tag{35}
$$

This allows us to easily compute the normalisation factor on the r.h.s. of Eq. (33). It is interesting to observe that at leading order the normalisation factor is identically equal to one. This means that in a LL or NLL analysis of TMDs no normalisation is needed. In principle, when considering logarithmic accuracies higher than NNLL, one would need to included $\mathcal{O}(\alpha_s^2)$ corrections in the normalisation factor. Since the fixed-order integrated cross section in the numerator is currently know up to $\mathcal{O}(\alpha_s)$, also the denominator should be computed to the same order, *i.e.* including up to $\mathcal{O}(\alpha_s)$ in H, $\mathcal{C}_{qj}$, and $\mathbb{C}_{qj}$, also beyond NNLL.