Contents

1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

The evolution equation for GPDs reads:[1]

$$\mu^2\frac{d}{d\mu^2}f(x,\xi) = \int_{-1}^{1}\frac{dx'}{|2\xi|}\mathbb{V}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{2.1}$$

In general, the GPD f and the evolution kernel $\mathbb{V}$ should be respectively interpreted as a vector and a matrix in flavour space. However, for now, we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure.

The support of the evolution kernel $\mathbb{V}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 2.1. The Knowledge of the support region of

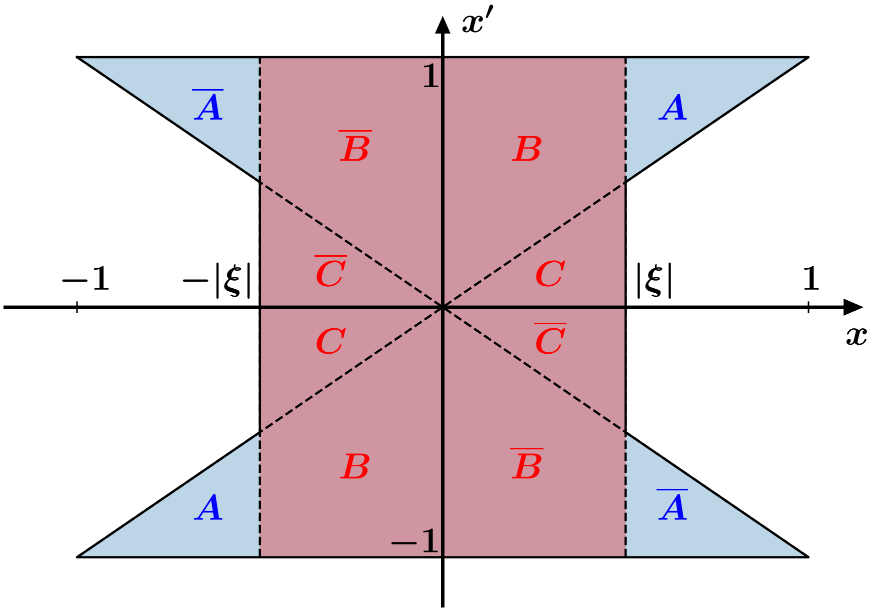

Fig. 2.1: Support domain of the evolution kernel $\mathbb{V}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

the evolution kernel allows us to rearrange Eq. (2.1) as follows:

$$\mu^2\frac{d}{d\mu^2}f(\pm x,\xi) = \int_{b(x)}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}\mathbb{V}\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}\mathbb{V}\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,, \tag{2.2}$$

with:

$$b(x) = |x|\theta\left(\left|\frac{x}{\xi}\right|-1\right)\,, \tag{2.3}$$

[1] It should be noticed that the integration bounds of the integration in Eq. (2.1) are dictated by the operator defintion of the distribution f on the light cone and not by the kernel $\mathbb{V}$.

and where we have used the symmetry property of the evolution kernels: $\mathbb{V}(y,y') = \mathbb{V}(-y,-y')$. In the unpolarised case, it is useful to define:[2]

$$\begin{aligned} f^{\pm}(x,\xi) &= f(x,\xi) \mp f(-x,\xi)\,, \\ \mathbb{V}^{\pm}(y,y') &= \mathbb{V}(y,y') \mp \mathbb{V}(-y,y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \frac{x'}{|2\xi|} \mathbb{V}^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,. \tag{2.5}$$

The $f^{\pm}$ distributions can be regarded as the GPD analogous of the $\pm$ forward distributions that can then be used to construct the usual singlet and non-singlet distributions in the QCD evolution basis. This uniquely determines the flavour structure of the evolution kernels $\mathbb{V}^{\pm}$.

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (DGLAP region, henceforth). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (ERBL region, henceforth). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ Eq. (2.5) needs to recover the DGLAP and ERBL equations, respectively.

2.1 End-point contributions

Some of the expressions for the anomalous dimensions discussed below contain +-prescribed terms. It is thus important to treat these terms properly. We are generally dealing with objects defined as:

$$\left[\mathbb{V}\left(x,x'\right)\right]_+ = \mathbb{V}\left(x,x'\right) - \delta(x-x') \int_{-1}^{1} dx\, \mathbb{V}\left(x,x'\right)\,. \tag{2.6}$$

where the function $\mathbb{V}$ has a pole at $x' = x$.

Let us take as an example the one-loop non-singlet anomalous dimension. For definiteness, we will refer for the precise expression to Eq. (101) of Ref. [1] and report it here for convenience (up to a factor $\alpha_s/4\pi$):

$$V_{\rm NS}^{(0)}(x,x') = 2C_F \left[\rho(x,x')\left\{\frac{1+x}{1+x'}\left(1+\frac{2}{x'-x}\right)\right\} + (x\to -x, x'\to -x')\right]_+\,, \tag{2.7}$$

with:[3]

$$\rho(x,x') = \theta(-x+x')\theta(1+x) - \theta(x-x')\theta(1-x) \tag{2.8}$$

In order for Eq. (2.7) to be consistent with the forward evolution, one should find:

$$\lim_{\xi\to 0} \frac{1}{|2\xi|} V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) \stackrel{?}{=} \frac{1}{x'} P_{\rm NS}\left(\frac{x}{x'}\right) = \frac{1}{x'} 2C_F \left[\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+\,, \tag{2.9}$$

such that Eq. (2.1) exactly reduces to the DGLAP equation. However, if one takes the explicit limit for $\xi \to 0$ of Eq. (2.7) one finds:[4]

$$\lim_{\xi\to 0} \frac{1}{|2\xi|} V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) = 2C_F \left[\frac{1}{x'}\theta\left(\frac{x}{x'}\right)\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+\,. \tag{2.10}$$

[2] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x) = -\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign. The opposite sign is absent in the longitudinally polarised case.

[3] There is probably a typo in Eq. (102) of Ref. [1] as the second -1 should actually be a $+1$.

[4] The factor $\theta\left(\frac{x}{x'}\right)$ comes from the factor $\theta(-x+x')$ in Eq. (2.8) that can be rewritten as $\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)$.

Therefore, as compared to Eq. (2.9), the factor $1/x'$ in Eq. (2.10) appears *inside* the +-prescription sign rather than outside which makes the two expressions effectively different under integration. The difference amounts to a local term that can be quantified by knowing that:

$$[yg(y)]_+ = y\,[g(y)]_+ + \delta(1-y)\int_0^1 dz\,(1-z)g(z)\,. \tag{2.11}$$

Notice that, thanks to the factor $(1-z)$, the integral in the r.h.s. of the above equation converges despite the singularity of g. For example:

$$\left[\frac{y}{1-y}\right]_+ = y\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\,. \tag{2.12}$$

Finally, one finds that the forward limit of Eq. (2.7) gives:

$$\lim_{\xi\to 0}\frac{1}{|2\xi|}V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) = \frac{1}{x'}\left[P_{\rm NS}\left(\frac{x}{x'}\right) + \frac{4}{3}C_F\delta\left(1-\frac{x}{x'}\right)\right]\,, \tag{2.13}$$

which does *not* reproduce the DGLAP equation due to the presence of an additional local term.

2.2 On Vinnikov's code

The purpose of this section is to draw the attention on a possible incongruence of the GPD evolution code developed by Vinnikov and presented in Ref. [5]. For definiteness, we concentrate on the non-singlet $H_{\rm NS}$ GPD in the DGLAP region $x > \xi$, whose evolution equation is given in Eq. (29). For convenience, we report that equation here:

$$\begin{aligned}\frac{dH_{\rm NS}(x,\xi,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\frac{x^2+y^2-2\xi^2}{(y-x)(y^2-\xi^2)}\left(H_{\rm NS}(y,\xi,Q^2)-H_{\rm NS}(x,\xi,Q^2)\right)\right.\\ &+ H_{\rm NS}(x,\xi,Q^2)\left(\frac{3}{2}+2\ln(1-x)+\frac{x-\xi}{2\xi}\ln((x-\xi)(1+\xi))\right.\\ &- \left.\left.\frac{x+\xi}{2\xi}\ln((x+\xi)(1-\xi))\right)\right]\,,\end{aligned} \tag{2.14}$$

and take the forward limit $\xi \to 0$, obtaining:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\frac{x^2+y^2}{y^2(y-x)}\left(H_{\rm NS}(y,0,Q^2)-H_{\rm NS}(x,0,Q^2)\right)\right.\\ &+ \left.H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)\right)\right]\,,\end{aligned} \tag{2.15}$$

The limit for $\xi \to 0$ of the equation above should reproduce the usual DGLAP evolution equation:

$$\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{4\pi}\int_x^1\frac{dy}{y}\left[\hat{P}_{\rm NS}\left(\frac{x}{y}\right)\right]_+ H_{\rm NS}\left(y,0,Q^2\right)\,, \tag{2.16}$$

where:

$$\hat{P}_{\rm NS}(z) = 2C_F\frac{1+z^2}{1-z}\,, \tag{2.17}$$

with $C_F = 4/3$. Written explicitly and accounting for the additional local term arising from the incompleteness of the convolution integral, one finds:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\,\frac{x^2+y^2}{y^3(y-x)}\left(yH_{\rm NS}\left(y,0,Q^2\right)-xH_{\rm NS}(x,0,Q^2)\right)\right.\\ &+ \left.H_{\rm NS}(x,0,Q^2)\left(\frac{x(x+2)}{2}+2\ln(1-x)\right)\right]\,,\end{aligned} \tag{2.18}$$

which evidently differs from Eq. (2.15). By inspection, one observes that the difference can be partially traced back to the issue discussed in Sect. (2.1). An interesting observation is that, for $x \to 1$, the two expressions tend to coincide. This means that the difference is larger at small values of x. This fact may have concurred to cause the oversight of this discrepancy in past numerical comparisons.

2.3 On Ji's evolution equation

In this section we discuss the evolution equations derived by Ji in Ref. [4]. This form of the evolution equation is dubbed "near-forward" in Ref. [2] because it closely resembles the DGLAP equation. However, in Ref. [4] two different equations apply to the regions $x < \xi$ and $x > \xi$. In this section, we will unify them showing that the resulting one-loop non-singlet off-forward anomalous dimension cannot be written as a fully +-prescribed distribution.

We start by considering Eqs. (15)-(17) of Ref. [4]. The first step is to replace $\xi/2$ with ξ to match our notation. Then we consider the subtraction integrals in Eq. (16) keeping in mind that they apply to both regions $x < \xi$ and $x > \xi$:[5]

$$\int_{\pm\xi}^{x} \frac{dy}{y-x} = -\int_{\pm\kappa}^{1} \frac{dz}{1-z} = -\int_{0}^{1} \frac{dz}{1-z} + \int_{1\mp\kappa}^{1} \frac{dt}{t} = -\int_{0}^{1} \frac{dz}{1-z} - \ln(|1\mp\kappa|)\,, \tag{2.19}$$

with:

$$\kappa = \frac{\xi}{x}\,, \tag{2.20}$$

such that the full local term in Eq. (16) becomes:

$$\frac{3}{2} + \int_{\xi}^{x} \frac{dy}{y-x} + \int_{-\xi}^{x} \frac{dy}{y-x} = \frac{3}{2} - 2\int_{0}^{1} \frac{dz}{1-z} - \ln\left(|1-\kappa^2|\right)\,, \tag{2.21}$$

Considering the symmetry for $\xi \leftrightarrow -\xi$ of the evolution kernel in Eq. (17) of Ref. [4], we can write Eq. (15) valid for $\kappa < 1$ in a more compact way as:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \int_x^1 \frac{dy}{y} \mathcal{P}_1^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right)\,, \tag{2.22}$$

with:

$$\begin{aligned}
\mathcal{P}_1^{-,(0)}(y,\kappa) &= 2P_{\rm NS}(y,2\kappa y) + \delta(1-x)2C_F\left(\frac{3}{2} - 2\int_0^1 \frac{dy}{1-y} - \ln(|1-\kappa^2|)\right)\\
&= 2C_F\left\{\left(\frac{2}{1-y}\right)_+ - \frac{1+y}{1-\kappa^2 y^2} + \delta(1-y)\left[\frac{3}{2} - \ln\left(|1-\kappa^2|\right)\right]\right\}\\
&= 2C_F\left\{\left[\frac{1+(1-2\kappa^2)y^2}{(1-y)(1-\kappa^2y^2)}\right]_+ + \delta(1-y)\left[\frac{3}{2} + \left(\frac{1}{2\kappa^2}-1\right)\ln\left(|1-\kappa^2|\right) + \frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\right\}\,,
\end{aligned} \tag{2.23}$$

where $P_{\rm NS}$ is given in Eq. (17) of Ref. [4]. The splitting function $\mathcal{P}_1^{-,(0)}$ is such that:

$$\int_0^1 dy\,\mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{3}{2} + \left(\frac{1}{2k^2}-1\right)\ln\left(|1-\kappa^2|\right) + \frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\,, \tag{2.24}$$

which means that it cannot be written as a fully +-prescribed distribution. However, the integral above correctly tends to zero as $\kappa \to 0$ allowing one to recover the usual DGLAP splitting function in the forward limit:

$$\lim_{\kappa\to 0} \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+y^2}{1-y}\right]_+\,. \tag{2.25}$$

[5] Note that all divergent integrals considered here are implicitly assumed to be principal-valued integrals such that:

$$\int_{-1}^{1} \frac{dt}{t} = 0\,.$$

This allows us to omit the $\pm i\epsilon$ terms.

It should also be pointed out that also the limit for $\kappa \to 1$ of Eq. (2.23) is well-behaved:

$$\lim_{\kappa\to 1} \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F \left\{ \left[\frac{1}{1-y}\right]_+ + \delta(1-y)\left[\frac{3}{2} - \ln(2)\right]\right\} . \tag{2.26}$$

which is necessary to have a smooth transition of the GPDs from the DGLAP ($x > \xi$) to the ERBL ($x < \xi$) region.

We now consider Eqs. (18) and (19) of Ref. [4] valid for $\kappa > 1$. Interestingly, after some algebra, we find:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_x^1 \frac{dy}{y}\mathcal{P}_1^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) + \int_x^\infty \frac{dy}{y}\mathcal{P}_2^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right)\right] , \tag{2.27}$$

with $\mathcal{P}_1^{-,(0)}$ given by:

$$\mathcal{P}_1^{-,(0)}(y,\kappa) = 2P'_{\rm NS}(y,2\kappa y) + 2P'_{\rm NS}(y,-2\kappa y) + \delta(1-x)2C_F\left(\frac{3}{2} - 2\int_0^1 \frac{dy}{1-y} - \ln(|1-\kappa^2|)\right) , \tag{2.28}$$

with $P'_{\rm NS}$ is given in Eq. (19) of Ref. [4] and remarkably equal to the expression in Eq. (2.23) signifying that:

$$P_{\rm NS}(y,2\kappa y) = P'_{\rm NS}(y,2\kappa y) + P'_{\rm NS}(y,-2\kappa y) . \tag{2.29}$$

While:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = -2P'_{\rm NS}(y,-2\kappa y) + 2P'_{\rm NS}(-y,2\kappa y) = 2C_F(\kappa-1)\frac{y+(1+2\kappa)y^3}{(1-y^2)(1-\kappa^2 y^2)} . \tag{2.30}$$

It is very interesting to notice that $\mathcal{P}_2^{-,(0)}$ is proportional to $(\kappa - 1)$ that finally guarantees the continuity of GPDs at $k = 1$.

We observe that, within the integration interval, the splitting function $\mathcal{P}_2^{-,(0)}$ is singular at $y = 1$.[6] However, as pointed out above, the second integral on the r.h.s. of Eq. (2.27) has to be regarded as principal-valued therefore it is well-defined. In order to treat this integral numerically we consider the specific case:

$$I = \int_x^\infty dy\, \frac{f(y)}{1-y} , \tag{2.31}$$

where f is a test function well-behaved over the full integration range. If one subtracts and adds back the divergence at $y = 1$, *i.e.*:

$$f(1)\int_0^1 \frac{dy}{1-y} , \tag{2.32}$$

one can rearrange the integral as follows:

$$I = \int_x^\infty \frac{dy}{1-y}\left[f(y) - f(1)\left(1 + \theta(y-1)\frac{1-y}{y}\right)\right] + f(1)\ln(1-x) \equiv \int_x^\infty dy \left(\frac{1}{1-y}\right)_{++} f(y) , \tag{2.33}$$

which effectively defines the ++-distribution. It should be noticed that this definition is specific to the function $1/(1-y)$. In case of a different singular function the function that multiplies $\theta(y-1)$ would be different. The advantage of this rearrangement is that the integrand is free of the divergence at $y = 1$ and is thus amenable to numerical integration. Also, the ++-distribution reduces to the standard +-distribution when the upper integration bound is one rather than infinity. In this sense the ++-distribution generalises the +-distribution to ERBL-like integrals.

In view of the use of Eq. (2.33), it is convenient to rewrite Eq. (2.30) as follows:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+(1+\kappa)y+(1+\kappa-\kappa^2)y^2}{(1+y)(1-\kappa^2 y^2)} - \left(\frac{1}{1-y}\right)_{++}\right] , \tag{2.34}$$

where the first term in the squared bracket is regular at $y = 1$.

[6] The singularities at $y = -1$ and $y = \pm 1/\kappa$ are all placed below $y = x$ that is the lower integration bound and thus do not cause any problem.

Finally, Eqs. (2.22) and Eq. (2.27) can be combined as follows:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi)=\frac{\alpha_s(\mu)}{4\pi}\left[\int_x^1\frac{dy}{y}\mathcal{P}_1^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)+\theta(\kappa-1)\int_x^\infty\frac{dy}{y}\mathcal{P}_2^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)\right]\,, \quad (2.35)$$

or even more compactly as:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi)=\frac{\alpha_s(\mu)}{4\pi}\int_x^\infty\frac{dy}{y}\mathcal{P}^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)\,, \quad (2.36)$$

with:

$$\mathcal{P}^{-,(0)}(y,\kappa)=\theta(1-y)\mathcal{P}_1^{-,(0)}(y,\kappa)+\theta(\kappa-1)\mathcal{P}_2^{-,(0)}(y,\kappa)\,, \quad (2.37)$$

to obtain a single DGLAP-like evolution equation valid for all values of κ. In fact, it should be pointed out that, when performing the integrals numerically, the form in Eq. (2.36) has to be adopted. The reason is that both functions $\mathcal{P}_1^{-,(0)}$ and $\mathcal{P}_2^{-,(0)}$, due to the factor $1-\kappa^2y^2$, are affected by a pole at $y=|\kappa|^{-1}$ that, for $|\kappa|>1$ or equivalently $|x|<|\xi|$ (*i.e.* in the ERBL region) have to cancel to give a finite result. Using the explicit expressions for $\mathcal{P}_1^{-,(0)}$ and $\mathcal{P}_2^{-,(0)}$, we find:

$$\lim_{y\to\kappa}(1-\kappa^2y^2)\mathcal{P}_1^{-,(0)}(y,\kappa)=-2C_F(1+\kappa)\,, \quad (2.38)$$

and:

$$\lim_{y\to\kappa}(1-\kappa^2y^2)\mathcal{P}_2^{-,(0)}(y,\kappa)=2C_F(1+\kappa)\,. \quad (2.39)$$

Since the coefficient of the pole are equal in absolute value and opposite in sign they cancel in the integral. Below, we will explicitly verify this property also for the anomalous dimensions of the singlet sector.

Importantly, in the limit $\kappa\to 0$, the second integral in the r.h.s. of Eq. (2.35) drops and the splitting function $\mathcal{P}_1^{-,(0)}$ reduces to the one-loop non-singlet DGLAP splitting function (see Eq. (2.25)) so that, as expected, Eq. (2.35) becomes the DGLAP equation.

It is also interesting to verify that also the ERBL equation is recovered in the limit $\xi\to 1$. Given the definition of κ, Eq. (2.20), this limit is attained by taking $\kappa\to 1/x$. However, the limit procedure is more subtle than in the DGLAP case due to the presence of +-prescriptions and explicit local terms that need to cooperate to give the right result.

We make use of Eqs. (2.28) and (2.30) to write the evolution equation in terms of the function $P'_{\rm NS}$ in a form similar to that originally given in Ref. [4] but more compactly as:

$$\mu^2\frac{d}{d\mu^2}f^-(x,1)=\frac{\alpha_s(\mu)}{4\pi}\left[\int_{-1}^1 dy\,V_{\rm NS}^{(0)}(x,y)f^-(y,1)\right]\,. \quad (2.40)$$

with:

$$\begin{aligned}V_{\rm NS}^{(0)}(x,y) &= \theta(y-x)\left[\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},\frac{2}{y}\right)\right]-2C_F\delta(y-x)\int_{-1}^1 dz\,\frac{\theta(z-x)}{z-x}\\ &+ \theta(x-y)\left[-\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},-\frac{2}{y}\right)\right]+2C_F\delta(x-y)\int_{-1}^1 dz\,\frac{\theta(x-z)}{z-x}\\ &+ 3C_F\delta(y-x)\,,\end{aligned} \quad (2.41)$$

where:

$$\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},\frac{2}{y}\right)=2C_F\frac{1+x}{1+y}\left(\frac{1}{2}+\frac{1}{y-x}\right)\,. \quad (2.42)$$

In order to make a step towards the ERBL equation, we change the variables x and y with:

$$\begin{aligned}t&=\frac{1}{2}(x+1)\,,\\ u&=\frac{1}{2}(y+1)\,,\end{aligned} \quad (2.43)$$

such that the evolution variable becomes:

$$\mu^2 \frac{d}{d\mu^2}\Phi^-(t) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_0^1 du\,\overline{V}_{\rm NS}^{(0)}(t,u)\Phi^-(u)\right]. \tag{2.44}$$

with $\Phi^-(t) = f^-(x,1)$ and:

$$\begin{aligned}\overline{V}_{\rm NS}^{(0)}(t,u) &= C_F\Bigg[\theta(u-t)\left(\frac{t-1}{u}+\frac{1}{u-t}-\delta(u-t)\int_0^1 du'\frac{\theta(u'-t)}{u'-t}\right)\\ &- \theta(t-u)\left(\frac{t}{1-u}+\frac{1}{u-t}-\delta(t-u)\int_0^1 du'\frac{\theta(t-u')}{u'-t}\right)+\frac{3}{2}\delta(u-t)\Bigg].\end{aligned} \tag{2.45}$$

Now we define:

$$[f(t,u)]_+ \equiv f(t,u) - \delta(u-t)\int_0^1 du'\, f(t,u'), \tag{2.46}$$

where f has a single pole at $u=t$, so that we can write Eq. (2.45) more compactly as:

$$\overline{V}_{\rm NS}^{(0)}(t,u) = C_F\left\{\left[\theta(u-t)\frac{t-1}{u}+\left(\frac{\theta(u-t)}{u-t}\right)_+\right]-\left[\theta(t-u)\frac{t}{1-u}+\left(\frac{\theta(t-u)}{u-t}\right)_+\right]+\frac{3}{2}\delta(u-t)\right\}. \tag{2.47}$$

This confirms the result of Ref. [2] modulo the fact that, for achieving a correct cancellation of the divergencies, the θ-function for the +-prescribed terms needs to be inside the +-prescription sign itself rather than outside.

One can check that integrating $\overline{V}_{\rm NS}^{(0)}$ over t gives zero:[7]

$$\int_0^1 dt\,\overline{V}_{\rm NS}^{(0)}(t,u) = 0. \tag{2.48}$$

This finally confirms that $\overline{V}_{\rm NS}^{(0)}$ as derived from Ref. [4] admits a fully +-prescribed form, that is:

$$\overline{V}_{\rm NS}^{(0)}(t,u) = C_F\left\{\theta(u-t)\left[\frac{t-1}{u}+\frac{1}{u-t}\right]-\theta(t-u)\left[\frac{t}{1-u}+\frac{1}{u-t}\right]\right\}_+. \tag{2.49}$$

This was also explicitly derived in Ref. [7] and argued that this property must hold for symmetry reasons.

It is now interesting to derive an ERBL-like evolution equation for GPDs at one loop. Assuming $\xi>0$, this equation can be written as:

$$\frac{d}{d\ln\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_{-1}^1\frac{dy}{\xi}\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right)f^-(y,\xi). \tag{2.50}$$

A simple change of variables allows one to write this equation as:

$$\frac{d}{d\ln\mu^2}f^-(\xi x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_{-1/\xi}^{1/\xi} dy\,\mathbb{V}_{\rm NS}^{(0)}(x,y)\,f^-(\xi y,\xi), \tag{2.51}$$

where the evolution kernel is given by:

$$\begin{aligned}\mathbb{V}_{\rm NS}^{(0)}(x,y) &= \theta(1-|x|)V_{\rm NS}^{(0)}(x,y)\\ &+ \theta(|x|-1)2C_F\Bigg\{\left[\theta(y-|x|)\left(\frac{2}{y-|x|}+\frac{|x|+y}{1-y^2}\right)\right]_+\\ &+ \delta(|x|-y)\Bigg[\frac{3}{2}-\ln\left(\frac{|x|-1}{|x|+1}\right)\\ &+ 2\ln\left(\frac{1-\xi|x|}{\xi(|x|-1)}\right)\left[\theta\left(\frac{1+\xi}{2\xi}-|x|\right)-\theta\left(|x|-\frac{1+\xi}{2\xi}\right)\right]\\ &+ \frac{1}{2}(|x|+1)\ln\left(\frac{1-\xi}{\xi(|x|-1)}\right)-\frac{1}{2}(|x|-1)\ln\left(\frac{1+\xi}{\xi(|x|+1)}\right)\Bigg]\Bigg\},\end{aligned} \tag{2.52}$$

[7] Note that the two +-prescribed terms when integrated over t do not individually give zero but their combination does.

with $V_{\rm NS}^{(0)}$ given in Eq. (2.45) and where we have exploited the fact that $f^-(x,\xi) = f^-(-x,\xi)$. In addition, the +-prescription in the second line of the above equation has the following *ad hoc* definition:

$$[f(x,y)]_+ = f(x,y) - \delta(x-y)\int_1^{1/\xi} dy\, f(x,y)\,. \tag{2.53}$$

A question arises: does the fact that the GPD anomalous dimension (*cfr.* Eq. (2.24)) does not admit a fully +-prescribed form violate any conservation law? To answer this question, we notice that the fact that the non-singlet DGLAP anomalous dimension integrates to zero (see Eq. (2.25)), and thus admits a +-prescribed form, derives from the conservation of the total number of quarks minus anti-quarks (valence sum rule):

$$\int_0^1 dx\, f^-(x,0) = \text{constant}\,. \tag{2.54}$$

Taking the derivative of this equation w.r.t. $\ln\mu^2$ and using the DGLAP equation gives:

$$\begin{aligned} 0 &= \int_0^1 dx \int_x^1 \frac{dy}{y}\mathcal{P}_1^-(y,0) f^-\left(\frac{x}{y},0\right) = \int_0^1 dy\,\mathcal{P}_1^-(y,0)\int_0^y \frac{dx}{y}\, f^-\left(\frac{x}{y},0\right) \\ &= \int_0^1 dy\,\mathcal{P}_1^-(y,0)\int_0^1 dz\, f^-(z,0) = \text{constant}\times\int_0^1 dy\,\mathcal{P}_1^-(y,0) \quad\Leftrightarrow\quad \int_0^1 dy\,\mathcal{P}_1^-(y,0) = 0\,. \end{aligned} \tag{2.55}$$

This clearly justifies the requirement for $\mathcal{P}_1^-(y,0)$ to be fully +-prescribed.

One may try to apply the same argument to GPDs. In this case the valence sum rule generalises in:

$$\int_0^1 dx\, f^-(x,\xi) = F\,, \tag{2.56}$$

where F is independent of μ and ξ but may (and does) depend on the momentum transfer t. F is usually referred to as form factor. One should now take the derivative w.r.t. $\ln\mu^2$ and use Eq. (2.35) but in doing this one needs to take into account that $\kappa = \xi/x$:

$$\begin{aligned} 0 &= \int_0^1 dx\int_0^1 \frac{dy}{y}\left[\theta(y-x)\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] f^-(y,\xi) \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^y \frac{dx}{y}\,\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \int_0^\xi \frac{dx}{y}\,\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right)\right]\,, \end{aligned} \tag{2.57}$$

In order for this relation to be identically true, it is necessary that:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 0\,. \tag{2.58}$$

Notice that for $\xi\to 0$, the equality above reduces to Eq. (2.55). It is interesting to verify Eq. (2.58) plugging in the explicit expressions for $\mathcal{P}_1^{-,(0)}$, Eq. (2.23), and $\mathcal{P}_2^{-,(0)}$, Eq. (2.30). One finds:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[-\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} - \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.59}$$

that correctly tends to zero as $\xi\to 0$, and:

$$\int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} + \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.60}$$

such that Eq. (2.58) is fulfilled. Despite Eq. (2.58) has been explicitly proved at one-loop, the same relation must hold order by order in perturbation theory.

It is important to notice that the constraint on the non-singlet GPD anomalous dimensions deriving from the valence sum rule, and resulting in Eq. (2.58), does not take the form of a +-prescription, Eq. (2.6). A further proof can be given by considering the non-singlet GPD evolution equation given in Eq. (2.1) (see also Eq. (99) of Ref. [1]). The independence of the form factor from μ immediately leads to:

$$\int_{-1}^{1} dx' f(x',\xi) \int_{-1}^{1} dx \left[\hat{V}_{\rm NS}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\right]_+ = 0\,, \tag{2.61}$$

where $\hat{V}_{\rm NS}$ is nothing but $V_{\rm NS}$ stripped of the supposedly global +-prescription. A simple change of variables (assuming ξ positive) gives:

$$\int_{-1/\xi}^{1/\xi} dy' f(\xi y',\xi) \int_{-1/\xi}^{1/\xi} dy \left[\hat{V}_{\rm NS}\left(y,y'\right)\right]_+ = 0\,. \tag{2.62}$$

Given the fact that the bounds of the inner integral are not -1 and 1, the effect of the +-prescription as given in Eq. (2.6) cannot give zero. This prevents the above equation to be identically fulfilled violating polynomiality of GPDs. We can thus conclude that $V_{\rm NS}$ *cannot* be written as a fully +-prescribed function.

Having ascertained that the evolution equation from Ref. [4] is well-behaved for the non-singlet distribution f^-, we move on to consider the singlet $f_{\rm S}$ and gluon $f_{\rm G}$ distributions. As in the standard DGLAP evolution equation, singlet and gluon GPDs couple under evolution. Defining f^+ as the column vector of singlet and gluon GPDs, the corresponding anomalous dimension $\mathcal{P}^+$ is a matrix in flavour space:

$$\mathcal{P}^+ = \begin{pmatrix} \mathcal{P}_{\rm SS} & \mathcal{P}_{\rm SG} \\ \mathcal{P}_{\rm GS} & \mathcal{P}_{\rm GG} \end{pmatrix}\,. \tag{2.63}$$

Following the same procedure discussed above for the non-singlet distribution f^-, the one-loop evolution equation for f^+ reads:

$$\mu^2\frac{d}{d\mu^2} f^+(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty \frac{dy}{y}\mathcal{P}^{+,(0)}(y,\kappa) f^+\left(\frac{x}{y},\xi\right)\,, \tag{2.64}$$

with:

$$\mathcal{P}^{+,(0)}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{+,(0)}(y,\kappa)\theta(\kappa-1)\mathcal{P}_2^{+,(0)}(y,\kappa)\,. \tag{2.65}$$

The single splitting function matrices $\mathcal{P}_1$ and $\mathcal{P}_2$ are derived from the expression Ref. [4] as:

$$\begin{aligned} \mathcal{P}_{1,\rm IJ}(y,\kappa) &= 2P_{\rm IJ}(y,2\kappa y) = 2P'_{\rm IJ}(y,2\kappa y) + 2P'_{\rm IJ}(y,-2\kappa y)\,,\\ \mathcal{P}_{2,\rm IJ}(y,\kappa) &= -2P'_{\rm IJ}(y,-2\kappa y) - 2P'_{\rm IJ}(-y,2\kappa y)\,, \end{aligned} \tag{2.66}$$

with $\rm I, J = S, G$. This leads to:

$$\begin{cases} \mathcal{P}^{(0)}_{1,\rm SS}(y,\kappa) = \mathcal{P}_1^{-,(0)}(y,\kappa)\,,\\ \mathcal{P}^{(0)}_{2,\rm SS}(y,\kappa) = 2C_F\left[\dfrac{(1+\kappa)(1+y)+\kappa^3y^2}{(1+y)(1-\kappa^2y^2)} - \left(\dfrac{1}{1-y}\right)_{++}\right]\,, \end{cases} \tag{2.67}$$

$$\begin{cases} \mathcal{P}^{(0)}_{1,\rm SG}(y,\kappa) = 4n_fT_R\left[\dfrac{y^2+(1-y)^2-\kappa^2y^2}{(1-\kappa^2y^2)^2}\right]\,,\\ \mathcal{P}^{(0)}_{2,\rm SG}(y,\kappa) = 4n_fT_R(1-\kappa)\left[\dfrac{1-\kappa(\kappa+2)y^2}{\kappa(1-\kappa^2y^2)^2}\right]\,, \end{cases} \tag{2.68}$$

$$\begin{cases} \mathcal{P}^{(0)}_{1,\rm GS}(y,\kappa) = 2C_F\left[\dfrac{1+(1-y)^2-\kappa^2y^2}{y(1-\kappa^2y^2)}\right]\,,\\ \mathcal{P}^{(0)}_{2,\rm GS}(y,\kappa) = -2C_F\dfrac{(1-\kappa)^2}{\kappa(1-\kappa^2y^2)}\,, \end{cases} \tag{2.69}$$

$$\begin{cases} \mathcal{P}_{1,\mathrm{GG}}^{(0)}(y,\kappa) &=& 4C_A\Bigg[\left(\dfrac{1}{1-y}-\dfrac{\kappa^2 y}{1-\kappa^2y^2}\right)_+ + \dfrac{1}{(1-\kappa^2y^2)^2}\left(\dfrac{1-y}{y}+y(1-y)-1+\kappa^2y^2\right) \\ & + & \delta(1-y)\left(\dfrac{11C_A-4T_Rn_f}{3}\right), \\ \mathcal{P}_{2,\mathrm{GG}}^{(0)}(y,\kappa) &=& 2C_A\dfrac{1-\kappa^2}{1-\kappa^2y^2}\left[\dfrac{2(1+y^2)}{(1+\kappa)(1-\kappa^2y^2)}-\dfrac{1}{\kappa}+2-\dfrac{1}{1+y}-\left(\dfrac{1}{1-y}\right)_{++}\right]. \end{cases} \tag{2.70}$$

Notice that for $\mathcal{P}_{1,\mathrm{GG}}^{(0)}$ we have used the integral:

$$\int_0^1 dz \frac{\kappa^2 z}{1-\kappa^2 z^2} = -\frac{1}{2}\ln(|1-\kappa^2|)\,, \tag{2.71}$$

to make the limit $\kappa \to 1$ explicitly convergent.

In all cases, the limit for $\kappa \to 0$ of $\mathcal{P}_1$ reproduces the one-loop DGLAP splitting functions. In addition, we also notice that all $\mathcal{P}_2$'s are proportional to $\kappa - 1$. Along with the fact that all $\mathcal{P}_1$ are well-behaved at $\kappa = 1$:

$$\begin{aligned} \mathcal{P}_{1,\mathrm{SS}}^{(0)}(y,1) &= 2C_F\left\{\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\left[\frac{3}{2}-\ln(2)\right]\right\}, \\ \mathcal{P}_{1,\mathrm{SG}}^{(0)}(y,1) &= \frac{4n_fT_R}{(1+y)^2}, \\ \mathcal{P}_{1,\mathrm{GS}}^{(0)}(y,1) &= \frac{4C_F}{y(1+y)}, \\ \mathcal{P}_{1,\mathrm{GG}}^{(0)}(y,1) &= 4C_A\left[\left(\frac{1}{1-y^2}\right)_+ + \frac{1}{y(1+y)^2}\right] + \delta(1-y)\left(\frac{11C_A-4T_Rn_f}{3}\right), \end{aligned} \tag{2.72}$$

this guarantees the continuity of GPDs across the point $x = \xi$.

Now, we explicitly verify that the pole at $y = |\kappa|^{-1}$ that affects all splitting functions above cancels between $\mathcal{P}_{1,\mathrm{IJ}}^{(0)}$ and $\mathcal{P}_{2,\mathrm{IJ}}^{(0)}$. Since $\mathcal{P}_{1(2),\mathrm{SS}}^{(0)} = \mathcal{P}_{1(2)}^{-,(0)}$, we do not need to do it again for the SS splitting functions. For the others we find:

$$\lim_{y\to\kappa}(1-\kappa^2y^2)^2\mathcal{P}_{1,\mathrm{SG}}^{(0)}(y,\kappa) = -\lim_{y\to\kappa}(1-\kappa^2y^2)^2\mathcal{P}_{2,\mathrm{SG}}^{(0)}(y,\kappa) = \frac{8n_fT_R(1-\kappa)}{\kappa}\,, \tag{2.73}$$

$$\lim_{y\to\kappa}(1-\kappa^2y^2)\mathcal{P}_{1,\mathrm{GS}}^{(0)}(y,\kappa) = -\lim_{y\to\kappa}(1-\kappa^2y^2)\mathcal{P}_{2,\mathrm{GS}}^{(0)}(y,\kappa) = 2C_F\frac{(1-\kappa)^2}{\kappa}\,, \tag{2.74}$$

$$\lim_{y\to\kappa}(1-\kappa^2y^2)^2\mathcal{P}_{1,\mathrm{GG}}^{(0)}(y,\kappa) = -\lim_{y\to\kappa}(1-\kappa^2y^2)^2\mathcal{P}_{2,\mathrm{GG}}^{(0)}(y,\kappa) = 4C_A\frac{(\kappa-1)(\kappa^2+1)}{\kappa^2}\,. \tag{2.75}$$

These results confirm the cancellation of the pole at $y = |\kappa|^{-1}$ in the integral in the r.h.s. of Eq. (2.64).

We now compute the ERBL limit by taking $\kappa \to 1/x$. To do so, we use the ERBL-compliant form of the evolution equation:

$$\mu^2\frac{d}{d\mu^2}\Phi^+(t) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_0^1 du\,\overline{V}_{\mathrm{S}}^{(0)}(t,u)\Phi^+(u)\right], \tag{2.76}$$

with $\Phi^+(t) = f^+(x,1)$ and where u and t are defined in Eq. (2.43). $\overline{V}_{\mathrm{S}}^{(0)}$ is a matrix in flavour space with the same structure of $\mathcal{P}^+$ in Eq. (2.63), whose components can be written in terms of the $P'_{\rm IJ}$ functions as follows:

$$
\begin{aligned}
V^{(0)}_{\rm SS}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm SS}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right] - C_F\delta\left(u-t\right)\int_0^1 du'\,\frac{\theta(u'-t)}{u'-t} \\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm SS}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right] + C_F\delta\left(t-u\right)\int_0^1 du'\,\frac{\theta(t-u')}{u'-t} \\
&+ \frac{3}{2}C_F\delta\left(u-t\right)\,, \\
V^{(0)}_{\rm SG,GS}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm SG,GS}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right] \\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm SG,GS}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right]\,, \\
V^{(0)}_{\rm GG}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm GG}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right] - C_A\delta\left(u-t\right)\int_0^1 du'\,\frac{\theta(u'-t)}{u'-t} \\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm GG}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right] + C_A\delta\left(t-u\right)\int_0^1 du'\,\frac{\theta(t-u')}{u'-t} \\
&+ \left(\frac{11C_A-4T_Rn_f}{6}\right)\delta\left(u-t\right)\,.
\end{aligned}
\tag{2.77}
$$

The explicit expressions read:

$$
\begin{aligned}
V^{(0)}_{\rm SS}(t,u) &= V^{(0)}_{\rm NS}(t,u)\,, \\
V^{(0)}_{\rm SG}(t,u) &= 2n_fT_R\left(\frac{2u-1}{2}\right)\left[\theta(u-t)\frac{t}{u}\left(\frac{2t-1}{u}-2\frac{1-t}{1-u}\right)-\theta(t-u)\left(\frac{1-t}{1-u}\right)\left(\frac{1-2t}{1-u}-2\frac{t}{u}\right)\right]\,, \\
V^{(0)}_{\rm GS}(t,u) &= C_F\left(\frac{2}{2t-1}\right)\left[\theta(u-t)\left(2t-\frac{t^2}{u}\right)-\theta(t-u)\left(2(1-t)-\frac{(1-t)^2}{1-u}\right)\right]\,, \\
V^{(0)}_{\rm GG}(t,u) &= \text{This is a little convoluted\ldots leave it for when I feel like doing the calculation}\,.
\end{aligned}
\tag{2.78}
$$

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