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1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

The evolution equation for GPDs reads:[1]

$$\mu^2\frac{d}{d\mu^2}f(x,\xi) = \int_{-1}^{1}\frac{dx'}{|2\xi|}\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{2.1}$$

In general, the GPD f and the evolution kernel $\mathbb{P}$ should be respectively interpreted as a vector and a matrix in flavour space. However, for now, we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure.

The support of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 2.1. The Knowledge of the support region of

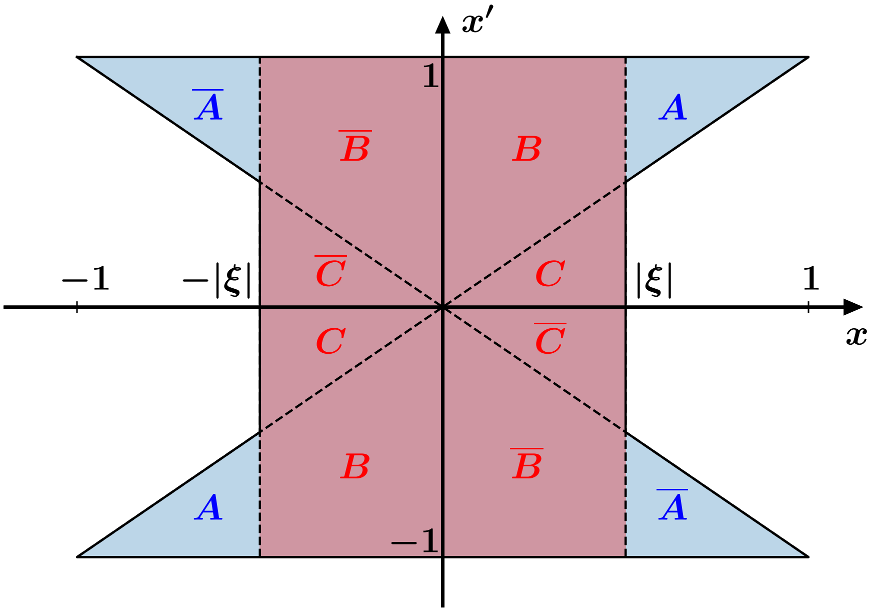

Fig. 2.1: Support domain of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

the evolution kernel allows us to rearrange Eq. (2.1) as follows:

$$\mu^2\frac{d}{d\mu^2}f(\pm x,\xi) = \int_{b(x)}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}\mathbb{P}\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}\mathbb{P}\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,, \tag{2.2}$$

[1] It should be noticed that the integration bounds of the integration in Eq. (2.1) are dictated by the operator defintion of the distribution f on the light cone and not by the kernel $\mathbb{P}$.

with:

$$b(x) = |x|\theta\left(\left|\frac{x}{\xi}\right| - 1\right)\,, \tag{2.3}$$

and where we have used the symmetry property of the evolution kernels: $\mathbb{P}(y, y') = \mathbb{P}(-y, -y')$. In the unpolarised case, it is useful to define:[2]

$$\begin{aligned} f^{\pm}(x, \xi) &= f(x, \xi) \mp f(-x, \xi)\,, \\ \mathbb{P}^{\pm}(y, y') &= \mathbb{P}(y, y') \mp \mathbb{P}(-y, y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x, \xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \frac{x'}{|2\xi|} \mathbb{P}^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x', \xi)\,. \tag{2.5}$$

The $f^{\pm}$ distributions can be regarded as the GPD analogous of the $\pm$ forward distributions that can then be used to construct the usual singlet and non-singlet distributions in the QCD evolution basis. This uniquely determines the flavour structure of the evolution kernels $\mathbb{P}^{\pm}$.

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (DGLAP region, henceforth). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (ERBL region, henceforth). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ Eq. (2.5) needs to recover the DGLAP and ERBL equations, respectively.

GPD anomalous dimensions are generally tricky to integrate numerically because of the intricate support. In order to simplify the integration procedure, we can decompose the anomalous dimensions using the labels given in Fig. 2.1 as a guide:

$$\begin{aligned} \mathbb{P}(y, y') &= \theta(y') \\ &\times \Big[\theta(y-1)\theta(y'-y)\mathbb{P}_A(y, y') + \theta(1-y)\theta(y'-y)\mathbb{P}_B(y, y') + \theta(1-y)\theta(y-y')\mathbb{P}_C(y, y') \\ &+ \theta(-y-1)\theta(y+y')\mathbb{P}_{\overline{A}}(y, y') + \theta(1+y)\theta(y+y')\mathbb{P}_{\overline{B}}(y, y') + \theta(1+y)\theta(-y'-y)\mathbb{P}_{\overline{C}}(y, y')\Big] \\ &+ \theta(-y') \\ &\times \Big[\theta(y-1)\theta(-y-y')\mathbb{P}_{\overline{A}}(y, y') + \theta(1-y)\theta(-y-y')\mathbb{P}_{\overline{B}}(y, y') + \theta(1-y)\theta(y'+y)\mathbb{P}_{\overline{C}}(y, y') \\ &+ \theta(-y-1)\theta(-y'+y)\mathbb{P}_A(y, y') + \theta(1+y)\theta(-y'+y)\mathbb{P}_B(y, y') + \theta(1+y)\theta(-y+y')\mathbb{P}_C(y, y')\Big]\,, \end{aligned} \tag{2.6}$$

where the functions $\mathbb{P}_I$ and $\mathbb{P}_{\overline{I}}$, with $I = A, B, C$, are defined on the respective regions in Fig. 2.1.[3] Next, we take the combinations given in Eq. (2.8) relevant to implement the evolution equation in Eq. (2.5). By doing this, one obtains:

$$\mathbb{P}^{\pm}(y, y') = \theta(y-1)\mathbb{P}_A^{\pm}(y, y') + \theta(1-y)\left[\theta(y'-y)\mathbb{P}_B^{\pm}(y, y') + \theta(y-y')\mathbb{P}_C^{\pm}(y, y')\right]\,, \tag{2.7}$$

where we have defined:

$$\mathbb{P}_I^{\pm}(y, y') = \mathbb{P}_I(y, y') \mp \mathbb{P}_{\overline{I}}(-y, y')\,, \tag{2.8}$$

and omitted all the irrelevant/redundant terms and factors for the computation of the integral in the r.h.s. of Eq. (2.5). From Eq. (2.7), it should be clear that the anomalous dimension $\mathbb{P}_A^{\pm}$ is responsible for the evolution

[2] Notice the seemingly unusual fact that f^{+} is defined as difference and f^{-} as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x) = -\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign. The opposite sign is absent in the longitudinally polarised case.

[3] Note that $\mathbb{P}_I(y, y')$ and $\mathbb{P}_{\overline{I}}(y, y')$ are not required to be symmetric upon the transformation $(y \to -y, y' \to -y')$.

in the DGLAP region while $\mathbb{P}^{\pm}_B$ and $\mathbb{P}^{\pm}_C$ are responsible for the evolution in the ERBL region. The latter observation suggests that $\mathbb{P}^{\pm}_B$ and $\mathbb{P}^{\pm}_C$ are related. The relation can easily be established by observing that the general structure of the ERBL anomalous dimensions is:

$$V^{\mathrm{ERBL}}(y,y') = \theta(y'-y)F(y,y') + \theta(y-y')F(-y,-y')\,, \tag{2.9}$$

which immediately implies that:

$$\mathbb{P}^{\pm}_C(y,y') = \mathbb{P}^{\pm}_B(-y,-y')\,. \tag{2.10}$$

Finally, one finds that a convenient decomposition for the anomalous dimension in Eq. (2.5) is:

$$\mathbb{P}^{\pm}(y,y') = \theta(y-1)\mathbb{P}^{\pm}_A(y,y') + \theta(1-y)\left[\theta(y'-y)\mathbb{P}^{\pm}_B(y,y') + \theta(y-y')\mathbb{P}^{\pm}_B(-y,-y')\right]\,. \tag{2.11}$$

Eq. (2.5) can be further manipulated to make it resemble the structure of the DGLAP equation as much as possible. To this purpose, we define the parameter:

$$\kappa(x) = \frac{\xi}{x}\,, \tag{2.12}$$

so that:

$$\frac{x'}{|2\xi|}\mathbb{P}^{\pm}_I\left(\pm\frac{x}{\xi},\pm\frac{x'}{\xi}\right) = \mathrm{sign}(\xi)\frac{1}{2\kappa}\frac{x'}{x}\mathbb{P}^{\pm}_I\left(\pm\frac{1}{\kappa},\pm\frac{1}{\kappa}\frac{x'}{x}\right) \equiv \mathrm{sign}(\xi)\mathcal{P}^{\pm}_I\left(\pm\kappa,\frac{x}{x'}\right)\,, \tag{2.13}$$

where the last equality effectively defines the *DGLAP-like* splitting function:

$$\mathcal{P}^{\pm}_I(\pm\kappa,y) = \frac{1}{2\kappa y}\mathbb{P}^{\pm}_I\left(\pm\frac{1}{\kappa},\pm\frac{1}{\kappa y}\right)\,. \tag{2.14}$$

In the following we will assume $\xi > 0$ as, so far, this is the only experimentally accessible region. This allows us to get rid of $\mathrm{sign}(\xi)$ in Eq. (2.13). In addition, without loss of generality, we can also restrict ourselves to positive values of x because negative values can be easily accessed by symmetry using Eq. (2.8), *i.e.* $f^{\pm}(-x,\xi) = \mp f^{\pm}(x,\xi)$. Using the definition in Eq. (2.14) in the integral in the r.h.s. of Eq. (2.5) and finally performing a change of variable gives:

$$\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) = \int_{b(x)}^{1}\frac{dx'}{x'}\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)f^{\pm}(x',\xi) = \int_{x}^{x/b(x)}\frac{dy}{y}\mathcal{P}^{\pm}(\kappa,y)\,f^{\pm}\left(\frac{x}{y},\xi\right)\,, \tag{2.15}$$

with:

$$b(x) = x\,\theta(1-\kappa)\,, \tag{2.16}$$

and:

$$\mathcal{P}^{\pm}(\kappa,y) = \theta(1-\kappa)\mathcal{P}^{\pm}_A(\kappa,y) + \theta(\kappa-1)\left[\theta(1-y)\mathcal{P}^{\pm}_B(\kappa,y) + \theta(y-1)\mathcal{P}^{\pm}_B(-\kappa,y)\right]\,. \tag{2.17}$$

Plugging Eq. (2.17) into Eq. (2.15), one obtains:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) &= \theta(1-\kappa)\int_x^1\frac{dy}{y}\mathcal{P}^{\pm}_A(\kappa,y)\,f^{\pm}\left(\frac{x}{y},\xi\right)\\ &+ \theta(\kappa-1)\int_x^{\infty}\frac{dy}{y}\left[\theta(1-y)\mathcal{P}^{\pm}_B(\kappa,y) + \theta(y-1)\mathcal{P}^{\pm}_B(-\kappa,y)\right]f^{\pm}\left(\frac{x}{y},\xi\right)\,.\end{aligned} \tag{2.18}$$

Eq. (2.18) has almost the form of a "standard" DGLAP equation except for the upper bound of the integral in the second line that extends up to infinity. However, this kind of integrals can be handled within APFEL with minor modifications of the integration strategy and up to a numerical approximation to be assessed.

2.1 On continuity of GPDs

It is well known that GPDs are required to be continuous at $x=\xi$ for factorisation to be valid [3]. It is thus interesting to consider the consequence of this constraint. To this end, let us consider the limits of Eq. (2.18) for $x\to\xi^{\pm}$, which corresponds to $\kappa\to 1^{\pm}$:

$$\lim_{x\to\xi^+}\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) = \int_x^1\frac{dy}{y}\mathcal{P}^{\pm}_B(1,y)\,f^{\pm}\left(\frac{x}{y},\xi\right) + \int_1^{\infty}\frac{dy}{y}\mathcal{P}^{\pm}_B(-1,y)\,f^{\pm}\left(\frac{x}{y},\xi\right)\,, \tag{2.19}$$

and:

$$\lim_{x\to\xi^-}\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi)=\mu^2\frac{d}{d\mu^2}f^{\pm}(\xi,\xi)=\int_x^1\frac{dy}{y}\mathcal{P}_A^{\pm}(1,y)\,f^{\pm}\left(\frac{x}{y},\xi\right)\,. \tag{2.20}$$

Taking the difference between Eqs. (2.19) and (2.20), using the continuity of f at $x=\xi$, and considering that:[4]

$$\mathcal{P}_A^{\pm}(1,y)=\mathcal{P}_B^{\pm}(1,y)\;, \tag{2.21}$$

one finds:

$$\int_1^\infty\frac{dy}{y}\mathcal{P}_B^{\pm}(-1,y)\,f^{\pm}\left(\frac{x}{y},\xi\right)=0\,, \tag{2.22}$$

which has to be valid at any scale and for any $f^{\pm}$. This immediately implies that:

$$\mathcal{P}_B^{\pm}(-1,y)=0\,, \tag{2.23}$$

for all values of y and order-by-order in perturbation theory. We will explicitly verify this constraint when we will discuss the explicit expressions.

2.2 End-point contributions

Some of the expressions for the anomalous dimensions discussed below contain +-prescribed terms. It is thus important to treat these terms properly. We are generally dealing with objects defined as (see Eq. (2.1)):

$$\frac{1}{|2\xi|}\left[\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\right]_+=\frac{1}{|2\xi|}\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)-\frac{1}{|2\xi|}\delta(x-x')\int_{-\infty}^{\infty}dx'\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\,. \tag{2.24}$$

where the function $\mathbb{P}$ has a pole at $x'=x$. Notice that the integral in the r.h.s. of the above definition runs over the entire real axis. It is then the support of the function $\mathbb{P}$ to possibly redefine the integration bounds. When implementing the definition in Eq. (2.14), mostly due to the presence of the factor $1/y$, one needs to be careful in deriving the appropriate subtraction term.

Let us take as an example the one-loop non-singlet anomalous dimension. For definiteness, we will refer for the precise expression to Eq. (101) of Ref. [1] and report it here for convenience (up to a factor $\alpha_s/2\pi$):

$$V_{\rm NS}^{(0)}(x,x')=2C_F\left[\rho(x,x')\left\{\frac{1+x}{1+x'}\left(1+\frac{2}{x'-x}\right)\right\}+(x\to -x,x'\to -x')\right]_+\,, \tag{2.25}$$

with:[5]

$$\rho(x,x')=\theta(-x+x')\theta(1+x)-\theta(x-x')\theta(1-x) \tag{2.26}$$

In order for Eq. (2.25) to be consistent with the forward evolution, one should find:

$$\lim_{\xi\to 0}\frac{1}{|2\xi|}V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\overset{?}{=}\frac{1}{x'}P_{\rm NS}\left(\frac{x}{x'}\right)=\frac{1}{x'}2C_F\left[\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+\,, \tag{2.27}$$

such that Eq. (2.1) exactly reduces to the collinear DGLAP equation. However, if one takes the explicit limit for $\xi\to 0$ of Eq. (2.25) one finds:[6]

$$\lim_{\xi\to 0}\frac{1}{|2\xi|}V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)=2C_F\left[\frac{1}{x'}\theta\left(\frac{x}{x'}\right)\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+\,. \tag{2.28}$$

Therefore, as compared to Eq. (2.31), the factor $1/x'$ in Eq. (2.28) appears *inside* the +-prescription sign rather than outside which makes the two expressions different. The difference amounts to a local term that can be quantified by knowing that:

$$\left[\theta(y)\theta(1-y)yg(y)\right]_+=y\left[\theta(y)\theta(1-y)g(y)\right]_++\delta(1-y)\int_0^1dz\,(1-z)g(z)\,. \tag{2.29}$$

[4] We will prove this equality case by case.
[5] There is probably a typo in Eq. (102) of Ref. [1] and the second -1 should actually be a $+1$.
[6] The factor $\theta\left(\frac{x}{x'}\right)$ comes from the factor $\theta(-x+x')$ in Eq. (2.26) that can be rewritten as $\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)$.

Notice that, thanks to the factor $(1-z)$, the integral in the r.h.s. of the above equation converges. For example:

$$\left[y\theta(y)\theta(1-y)\frac{y}{1-y}\right]_+ = y\left[\theta(y)\theta(1-y)\frac{1}{1-y}\right]_+ + \delta(1-y)\,. \tag{2.30}$$

Finally, one finds that the forward limit of Eq. (2.25) gives:

$$\lim_{\xi\to 0}\frac{1}{|2\xi|}V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) = \frac{1}{x'}\left[P_{\rm NS}\left(\frac{x}{x'}\right) + \frac{4}{3}C_F\delta\left(1-\frac{x}{x'}\right)\right], \tag{2.31}$$

which does *not* reproduce the DGLAP equation due to the presence of an additional local term. One could reverse the argument by saying that, in order to reproduce the DGLAP equation, the well-known expression for the one-loop non-singlet GPD anomalous dimension misses a local term whose forward limit is $-4C_F/3x$.

2.3 On Vinnikov's code

The purpose of this Appendix is to draw the attention on a possible incongruence of the GPD evolution code developed by Vinnikov and presented in Ref. [5]. For definiteness, we concentrate on the non-singlet $H_{\rm NS}$ GPD in the DGLAP region $x>\xi$, whose evolution equation is given in Eq. (29). For completeness, I report that equation here:

$$\begin{aligned}\frac{dH_{\rm NS}(x,\xi,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\frac{x^2+y^2-2\xi^2}{(y-x)(y^2-\xi^2)}\left(H_{\rm NS}(y,\xi,Q^2)-H_{\rm NS}(x,\xi,Q^2)\right)\right.\\ &+ H_{\rm NS}(x,\xi,Q^2)\left(\frac{3}{2}+2\ln(1-x)+\frac{x-\xi}{2\xi}\ln((x-\xi)(1+\xi))\right.\\ &- \left.\left.\frac{x+\xi}{2\xi}\ln((x+\xi)(1-\xi))\right)\right],\end{aligned} \tag{2.32}$$

and take the forward limit $\xi\to 0$:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\frac{x^2+y^2}{y^2(y-x)}\left(H_{\rm NS}(y,0,Q^2)-H_{\rm NS}(x,0,Q^2)\right)\right.\\ &+ \left. H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)\right)\right],\end{aligned} \tag{2.33}$$

The limit for $\xi\to 0$ of the equation above should reproduce the usual DGLAP evolution equation:

$$\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{4\pi}\int_x^1\frac{dy}{y}\left[\hat{P}_{\rm NS}\left(\frac{x}{y}\right)\right]_+ H_{\rm NS}\left(y,0,Q^2\right)\,, \tag{2.34}$$

where:

$$\hat{P}_{\rm NS}(z) = 2C_F\frac{1+z^2}{1-z}\,, \tag{2.35}$$

with $C_F=4/3$. Written explicitly and accounting for the additional local term arising from the incompleteness of the convolution integral, one finds:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\,\frac{x^2+y^2}{y^3(y-x)}\left(yH_{\rm NS}\left(y,0,Q^2\right)-xH_{\rm NS}(x,0,Q^2)\right)\right.\\ &+ \left. H_{\rm NS}(x,0,Q^2)\left(\frac{x(x+2)}{2}+2\ln(1-x)\right)\right],\end{aligned} \tag{2.36}$$

which evidently differs from Eq. (2.33). By inspection, one can argue that the difference can be partially traced back to the issue discussed in Sect. (2.2). An interesting observation is that, for $x\to 1$, the two expressions tend to coincide. This may have concurred to cause the oversight of this discrepancy in past numerical comparisons.

2.4 On Ji's evolution equation

In this section we discuss the evolution equations derived by Ji in Ref. [4]. This form of the evolution equation is dubbed "near-forward" in Ref. [2] because it closely resembles the DGLAP equation. However, in Ref. [4] two different equations apply to the regions $x < \xi$ and $x > \xi$. In this section, we will unify them showing that the resulting one-loop non-singlet off-forward anomalous dimension cannot be written as a fully +-prescribed distribution.

We start by considering Eqs. (15)-(17) of Ref. [4]. The first step is to replace $\xi/2$ with ξ to match our notation. Then we consider the subtraction integrals in Eq. (16) keeping in mind that they apply to both regions $x < \xi$ and $x > \xi$, *i.e.* over the full range in κ:[7]

$$\int_{\pm\xi}^{x} \frac{dy}{y-x} = -\int_{\pm\kappa}^{1} \frac{dz}{1-z} = -\int_{0}^{1} \frac{dz}{1-z} + \int_{1\mp\kappa}^{1} \frac{dt}{t} = -\int_{0}^{1} \frac{dz}{1-z} - \ln(|1\mp\kappa|)\,, \tag{2.37}$$

such that the full local term in Eq. (16) becomes:

$$\frac{3}{2} + \int_{\xi}^{x} \frac{dy}{y-x} + \int_{-\xi}^{x} \frac{dy}{y-x} = \frac{3}{2} - 2\int_{0}^{1} \frac{dz}{1-z} - \ln\left(|1-\kappa^2|\right)\,, \tag{2.38}$$

Considering the symmetry for $\xi \leftrightarrow -\xi$ of the evolution kernel in Eq. (17) of Ref. [4], we can write Eq. (15) valid for $\kappa < 1$ in a more compact way as:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \int_{x}^{1} \frac{dy}{y} \mathcal{P}_1^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right)\,, \tag{2.39}$$

with:

$$\begin{aligned}
\mathcal{P}_1^{-,(0)}(y,\kappa) &= 2P_{\rm NS}(y,\kappa y) + \delta(1-x)2C_F\left(\frac{3}{2} - 2\int_0^1 \frac{dy}{1-y} - \ln(|1-\kappa^2|)\right) \\
&= 2C_F\left\{\left(\frac{2}{1-y}\right)_+ - \frac{1+y}{1-\kappa^2 y^2} + \delta(1-y)\left[\frac{3}{2} - \ln\left(|1-\kappa^2|\right)\right]\right\} \\
&= 2C_F\left\{\left[\frac{1+(1-2\kappa^2)y^2}{(1-y)(1-\kappa^2y^2)}\right]_+ + \delta(1-y)\left[\frac{3}{2} + \left(\frac{1}{2\kappa^2}-1\right)\ln\left(|1-\kappa^2|\right) + \frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\right\}\,,
\end{aligned} \tag{2.40}$$

where $P_{\rm NS}$ is given in Eq. (17) of Ref. [4]. The splitting function $\mathcal{P}_1^{-,(0)}$ is such that:

$$\int_0^1 dy\, \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{3}{2} + \left(\frac{1}{2k^2}-1\right)\ln\left(|1-\kappa^2|\right) + \frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\,, \tag{2.41}$$

which means that it cannot be written as a fully +-prescribed distribution. However, the integral above correctly tends to zero as $\kappa \to 0$ allowing one to recover the usual DGLAP splitting function in the forward limit:

$$\lim_{\kappa\to 0} \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+y^2}{1-y}\right]_+\,. \tag{2.42}$$

It should also be pointed out that also the limit for $\kappa \to 1$ of Eq. (2.40) is well-behaved:

$$\lim_{\kappa\to 1} \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left\{\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\left[\frac{3}{2} - \ln(2)\right]\right\}\,. \tag{2.43}$$

which is required to have a smooth transition from the DGLAP to the ERBL region.

[7] Note that all divergent integrals considered here are implicitly assumed to be principal-valued integrals such that:

$$\int_{-1}^{1} \frac{dt}{t} = 0\,.$$

This allows us to omit the $\pm i\epsilon$ terms.

We now consider Eqs. (18) and (19) of Ref. [4] valid for $\kappa > 1$. Interestingly, after some algebra, we find:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \left[\int_x^1 \frac{dy}{y} \mathcal{P}_1^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) + \int_x^\infty \frac{dy}{y} \mathcal{P}_2^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) \right], \tag{2.44}$$

with $\mathcal{P}_1^{-,(0)}$ given by:

$$\mathcal{P}_1^{-,(0)}(y,\kappa) = 2P'_{\rm NS}(y,\kappa y) + 2P'_{\rm NS}(y,-\kappa y) + \delta(1-x) 2C_F \left(\frac{3}{2} - 2\int_0^1 \frac{dy}{1-y} - \ln(|1-\kappa^2|) \right), \tag{2.45}$$

with $P'_{\rm NS}$ is given in Eq. (19) of Ref. [4] and remarkably equal to the expression in Eq. (2.40) signifying that:

$$P_{\rm NS}(y,\kappa y) = P'_{\rm NS}(y,\kappa y) + P'_{\rm NS}(y,-\kappa y)\,. \tag{2.46}$$

While:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = -2P'_{\rm NS}(y,-\kappa y) + 2P'_{\rm NS}(-y,\kappa y) = 2C_F(\kappa-1)\frac{y+(1+2\kappa)y^3}{(1-y^2)(1-\kappa^2 y^2)}\,. \tag{2.47}$$

It is very interesting to notice that $\mathcal{P}_2^{-,(0)}$ is proportional to $(\kappa-1)$ that guarantees the continuity of GPDs at $k=1$.

We observe that, within the integration interval, the splitting function $\mathcal{P}_2^{-,(0)}$ is singular at $y=1$.[8] However, as pointed out above, the second integral on the r.h.s. of Eq. (2.44) has to be regarded as principal-valued therefore it is well-defined. In order to treat this integral numerically we consider the specific integral:

$$I = \int_x^\infty dy\, \frac{f(y)}{1-y}\,, \tag{2.48}$$

where f is a test function well-behaved over the full integration range. If one subtracts and adds back the divergence at $y=1$, *i.e.*:

$$f(1)\int_0^1 dy\, g(y)\,, \tag{2.49}$$

one can rearrange the integral as follows:

$$I = \int_x^\infty \frac{dy}{1-y}\left[f(y) - f(1)\left(1+\theta(y-1)\frac{1-y}{y}\right)\right] + f(1)\ln(1-x) \equiv \int_x^\infty dy \left(\frac{1}{1-y}\right)_{++} f(y)\,, \tag{2.50}$$

which effectively defines the ++-distribution. However, it should be noticed that this definition is specific to the function $1/(1-y)$. In case of a different singular function the function that multiplies $\theta(y-1)$ would be different. The advantage of this rearrangement is that the integrand is free of the divergence at $y=1$ and is thus amenable to numerical integration. Also, the ++-distribution reduces to the standard +-distribution when the upper integration bound is one rather than infinity. In this sense the ++-distribution generalises the +-distribution to ERBL-like integrals. However, it is important to notice that, contrary to the +-distribution, the ++-distribution does not modify the function it applies to, specifically:

$$\left(\frac{1}{1-y}\right)_{++} = \frac{1}{1-y} \tag{2.51}$$

for all values of y, including $y=1$.

In view of the use of Eq. (2.50), it is convenient to rewrite Eq. (2.47) as follows:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+(1+\kappa)y+(1+\kappa-\kappa^2)y^2}{(1+y)(1-\kappa^2y^2)} - \left(\frac{1}{1-y}\right)_{++}\right], \tag{2.52}$$

where the first term in the squared bracket is regular at $y=1$.

Finally, Eqs. (2.39) and Eq. (2.44) can be combined as follows:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \left[\int_x^1 \frac{dy}{y} \mathcal{P}_1^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) + \theta(\kappa-1)\int_x^\infty \frac{dy}{y} \mathcal{P}_2^{-,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) \right], \tag{2.53}$$

[8] The singularities at $y=-1$ and $y=\pm 1/\kappa$ are all placed below $y=x$ that is the lower integration bound and thus do not cause any problem.

to obtain a singe DGLAP-like evolution equation valid for all values of κ.

In the limit $\kappa \to 0$, the second integral in the r.h.s. of Eq. (2.53) drops and the splitting function $\mathcal{P}_1^{-,(0)}$ reduces to the one-loop non-singlet DGLAP splitting function (see Eq. (2.42)) so that, as expected, Eq. (2.53) becomes the DGLAP equation.

It is also interesting to verify that also the ERBL equation is recovered in the limit $\xi \to 1$. Given the definition of κ, Eq. (2.12), this limit is attained by taking $\kappa \to 1/x$. However, the limit procedure is more subtle than in the DGLAP case due to the presence of +-prescriptions and explicit local terms that need to cooperate to give the right result.

We make use of Eqs. (2.45) and (2.47) to write the evolution equation in terms of the function $P'_{\rm NS}$ in a form similar to that originally given in Ref. [4] but more compactly as:

$$\mu^2 \frac{d}{d\mu^2} f^-(x,1) = \frac{\alpha_s(\mu)}{4\pi} \left[\int_{-1}^{1} dy\, V_{\rm NS}^{(0)}(x,y) f^-(y,1) \right] . \tag{2.54}$$

with:

$$\begin{aligned} V_{\rm NS}^{(0)}(x,y) &= \theta(y-x)\left[\frac{1}{y} P'_{\rm NS}\left(\frac{x}{y},\frac{1}{x}\right)\right] - 2C_F \delta(y-x) \int_{-1}^{1} dz\, \frac{\theta(z-x)}{z-x} \\ &+ \theta(x-y)\left[-\frac{1}{y} P'_{\rm NS}\left(\frac{x}{y},-\frac{1}{x}\right)\right] + 2C_F \delta(x-y) \int_{-1}^{1} dz\, \frac{\theta(x-z)}{z-x} \\ &+ 3C_F \delta(y-x) , \end{aligned} \tag{2.55}$$

where:

$$\frac{1}{y} P'_{\rm NS}\left(\frac{x}{y},\frac{1}{x}\right) = 2C_F \frac{1+x}{1+y}\left(\frac{1}{2} + \frac{1}{y-x}\right) . \tag{2.56}$$

In order to make a step towards the ERBL equation, we change the variables with:

$$\begin{aligned} t &= \frac{1}{2}(x+1) , \\ u &= \frac{1}{2}(y+1) , \end{aligned} \tag{2.57}$$

such that the evolution variable becomes:

$$\mu^2 \frac{d}{d\mu^2} \Phi(t) = \frac{\alpha_s(\mu)}{4\pi} \left[\int_0^1 du\, \overline{V}_{\rm NS}^{(0)}(t,u) \Phi(u) \right] . \tag{2.58}$$

with $\Phi(t) = f^-(x,1)$ and:

$$\begin{aligned} \overline{V}_{\rm NS}^{(0)}(t,u) &= 2C_F \Bigg[\theta(u-t)\left(\frac{t-1}{u} + \frac{1}{u-t} - \delta(u-t)\int_0^1 du' \frac{\theta(u'-t)}{u'-t}\right) \\ &- \theta(t-u)\left(\frac{t}{1-u} + \frac{1}{u-t} - \delta(t-u)\int_0^1 du' \frac{\theta(t-u')}{u'-t}\right) + \frac{3}{4}\delta(u-t) \Bigg] . \end{aligned} \tag{2.59}$$

Now, with a slight abuse of notation, we define:[9]

$$\theta\left[\pm(u-t)\right] f(t,u) - \delta(u-t) \int_0^1 du'\, \theta\left[\pm(u'-t)\right] f(t,u') \equiv \left[f(t,u)\right]_+ , \tag{2.60}$$

where f has a single pole at $u=t$, so that we can write Eq. (2.59) more compactly as:

$$\overline{V}_{\rm NS}^{(0)}(t,u) = 2C_F \left\{ \theta(u-t)\left[\frac{t-1}{u} + \left(\frac{1}{u-t}\right)_+\right] - \theta(t-u)\left[\frac{t}{1-u} + \left(\frac{1}{u-t}\right)_+\right] + \frac{3}{2}\delta(u-t) \right\} . \tag{2.61}$$

[9] The abuse of notation stems from the fact that this definition of the +-prescription is not identical to the usual one. The presence of the θ-functions is crucial to avoid a principal value and guarantee the cancellation of the divergence proper to the +-prescription.

This confirms the result of Ref. [2].

One can check that integrating $\overline{V}^{(0)}_{\rm NS}$ over t gives zero:[10]

$$\int_0^1 dt\, \overline{V}^{(0)}_{\rm NS}(t,u) = 0\,. \tag{2.62}$$

This finally confirms that $\overline{V}^{(0)}_{\rm NS}$ as derived from Ref. ([4]) admits a fully +-prescribed form. This was also explicitly derived in Ref. [7] and argued that this property must for symmetry reasons.

A question arises: does the fact that the GPD anomalous dimension (*cfr.* Eq. (2.41)) does not admit a fully +-prescribed form violate any conservation law? To answer this question, we notice that the fact that non-singlet DGLAP anomalous dimension integrates to zero (see Eq. (2.42)), and thus admits a +-prescribed form, derives from the conservation of the total number of quarks minus anti-quarks (valence sum rule):

$$\int_0^1 dx\, f^-(x,0) = \text{constant}\,. \tag{2.63}$$

Taking the derivative of this equation w.r.t. $\ln\mu^2$ and using the DGLAP equation gives:

$$\begin{aligned} 0 &= \int_0^1 dx \int_x^1 \frac{dy}{y}\mathcal{P}_1^-(y,0) f^-\left(\frac{x}{y},0\right) = \int_0^1 dy\, \mathcal{P}_1^-(y,0)\int_0^y \frac{dx}{y}\, f^-\left(\frac{x}{y},0\right) \\ &= \int_0^1 dy\, \mathcal{P}_1^-(y,0)\int_0^1 dz\, f^-(z,0) = \text{constant}\times\int_0^1 dy\, \mathcal{P}_1^-(y,0) \quad \Leftrightarrow \quad \int_0^1 dy\, \mathcal{P}_1^-(y,0) = 0\,. \end{aligned} \tag{2.64}$$

This clearly justifies the requirement for $\mathcal{P}_1^-(y,0)$ to be fully +-prescribed.

One may try to apply the same argument to GPDs. In this case the valence sum rule generalises in:

$$\int_0^1 dx\, f^-(x,\xi) = F\,, \tag{2.65}$$

where F is independent of μ and ξ but may (and does) depend on the momentum transfer t. F is usually referred to as form factor. One should now take the derivative w.r.t. $\ln\mu^2$ and use Eq. (2.53). In doing this one needs to take into account that $\kappa = \xi/x$:

$$\begin{aligned} 0 &= \int_0^1 dx \int_0^1 \frac{dy}{y}\left[\theta(y-x)\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] f^-(y,\xi) \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^y \frac{dx}{y}\,\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \int_0^\xi \frac{dx}{y}\,\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right)\right]\,, \end{aligned} \tag{2.66}$$

In order for this relation to be identically true, it is necessary that:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 0\,. \tag{2.67}$$

Notice that for $\xi\to 0$, the equality above reduces to Eq. (2.64). It is interesting to verify Eq. (2.67) plugging in the explicit expressions for $\mathcal{P}_1^{-,(0)}$, Eq. (2.40), and $\mathcal{P}_2^{-,(0)}$, Eq. (2.47). One finds:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[-\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} - \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.68}$$

that correctly tends to zero as $\xi\to 0$, and:

$$\int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} + \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.69}$$

[10] Note that the two +-prescribed terms when integrated over t do not individually give zero but their combination does.

such that (to my big relief) Eq. (2.67) is fulfilled. Despite Eq. (2.67) has been explicitly proved at one-loop, the same relation must hold order by order in perturbation theory.

Having ascertained that the evolution equation from Ref. [4] is well-behaved for the non-singlet distribution f^-, we move on to consider the singlet f^+ and the gluon f_g distributions. As in the standard DGLAP evolution equation, singlet and gluon GPDs couple under evolution. Defining f^+ as the column vector of singlet and gluon GPDs, the corresponding anomalous dimension $\mathcal{P}^+$ is a matrix in flavour space:

$$\mathcal{P}^+ = \begin{pmatrix} \mathcal{P}_{\rm SS} & \mathcal{P}_{\rm SG} \\ \mathcal{P}_{\rm GS} & \mathcal{P}_{\rm GG} \end{pmatrix} . \tag{2.70}$$

Following the same procedure discussed above for the non-singlet distribution f^-, the one-loop evolution equation for f^+ reads:

$$\mu^2 \frac{d}{d\mu^2} f^+(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \left[\int_x^1 \frac{dy}{y} \mathcal{P}_1^{+,(0)}(y,\kappa) f^-\left(\frac{x}{y},\xi\right) + \theta(\kappa-1) \int_x^\infty \frac{dy}{y} \mathcal{P}_2^{+,(0)}(y,\kappa) f^+\left(\frac{x}{y},\xi\right)\right] . \tag{2.71}$$

The single splitting functions $\mathcal{P}_1$ and $\mathcal{P}_2$ are derived from the expression Ref. [4] as:

$$\begin{aligned} \mathcal{P}_{I,1}(y,\kappa) &= 2P_I(y,\kappa y) = 2P'_I(y,\kappa y) + 2P'_I(y,-\kappa y) , \\ \mathcal{P}_{I,2}(y,\kappa) &= -2P'_I(y,-\kappa y) - 2P'_I(-y,\kappa y) , \end{aligned} \tag{2.72}$$

with $I = \mathrm{SS}, \mathrm{SG}, \mathrm{GS}, \mathrm{SS}$. This leads to:

$$\begin{cases} \mathcal{P}_{\rm SS,1}^{(0)}(y,\kappa) = \mathcal{P}_1^{-,(0)}(y,\kappa) , \\ \mathcal{P}_{\rm SS,2}^{(0)}(y,\kappa) = 2C_F(1-\kappa)\left[\dfrac{1-(1+2\kappa+2\kappa^2)y^2}{\kappa(1-y^2)(1-\kappa^2y^2)}\right] = 2C_F\left[\dfrac{(1+\kappa)(1+y)+\kappa^3y^2}{(1+y)(1-\kappa^2y^2)} - \left(\dfrac{1}{1-y}\right)_{++}\right] , \end{cases} \tag{2.73}$$

$$\begin{cases} \mathcal{P}_{\rm SG,1}^{(0)}(y,\kappa) = 4n_fT_R\left[\dfrac{y^2+(1-y)^2-\kappa^2y^2}{(1-\kappa^2y^2)^2}\right] , \\ \mathcal{P}_{\rm SG,2}^{(0)}(y,\kappa) = 4n_fT_R(1-\kappa)\left[\dfrac{1-\kappa(\kappa+2)y^2}{\kappa(1-\kappa^2y^2)^2}\right] , \end{cases} \tag{2.74}$$

$$\begin{cases} \mathcal{P}_{\rm GS,1}^{(0)}(y,\kappa) = 2C_F\left[\dfrac{1+(1-y)^2-\kappa^2y^2}{y(1-\kappa^2y^2)}\right] , \\ \mathcal{P}_{\rm GS,2}^{(0)}(y,\kappa) = 2C_F(1-\kappa)\left[\dfrac{2-\kappa(\kappa+1)y^2}{\kappa y(1-\kappa^2y^2)}\right] , \end{cases} \tag{2.75}$$

$$\begin{cases} \mathcal{P}_{\rm GG,1}^{(0)}(y,\kappa) = \dfrac{4C_A}{(1-\kappa^2y^2)^2}\left[(1-\kappa^2)(1-\kappa^2y^2)\dfrac{y}{(1-y)_+} + \dfrac{1-y}{y} + y(1-y)\right] + \delta(1-y)\left(\dfrac{11C_A-4T_Rn_f}{3}\right) , \\ \mathcal{P}_{\rm GG,2}^{(0)}(y,\kappa) = 2C_A\dfrac{1-\kappa^2}{1-\kappa^2y^2}\left[\dfrac{2(1+y^2)}{(1+\kappa)(1-\kappa^2y^2)} - \dfrac{1}{\kappa} + 2 - \dfrac{1}{1+y} - \left(\dfrac{1}{1-y}\right)_{++}\right] . \end{cases} \tag{2.76}$$

In all cases, the limit for $\kappa \to 0$ of $\mathcal{P}_1$ reproduces the one-loop DGLAP splitting functions.

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