Contents

1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

The evolution equation for GPDs reads:[1]

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-1}^{1}\frac{dx'}{|2\xi|}\mathbb{V}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{2.1}$$

In general, the GPD f and the evolution kernel $\mathbb{V}$ should be respectively interpreted as a vector and a matrix in flavour space. However, for now, we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure.

The support of the evolution kernel $\mathbb{V}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 2.1. The knowledge of the support region of the evolution kernel allows us to rearrange Eq. (2.1) as follows:

$$\mu^2\frac{d}{d\mu^2}f(\pm x,\xi)=\int_{b(x)}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}\mathbb{V}\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}\mathbb{V}\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right], \tag{2.2}$$

with:

$$b(x)=|x|\theta\left(\left|\frac{x}{\xi}\right|-1\right), \tag{2.3}$$

and where we have used the symmetry property of the evolution kernels: $\mathbb{V}(y,y')=\mathbb{V}(-y,-y')$. In the unpolarised case, it is useful to define:[2]

$$\begin{aligned} f^{\pm}(x,\xi) &= f(x,\xi)\mp f(-x,\xi)\,,\\ \mathbb{V}^{\pm}(y,y') &= \mathbb{V}(y,y')\mp\mathbb{V}(-y,y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi)=\int_{b(x)}^{1}\frac{dx'}{x'}\frac{x'}{|2\xi|}\mathbb{V}^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)\,. \tag{2.5}$$

[1] It should be noticed that the integration bounds of the integration in Eq. (2.1) are dictated by the operator defintion of the distribution f on the light cone and not by the kernel $\mathbb{V}$.

[2] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x)=-\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign. The opposite sign is absent in the longitudinally polarised case.

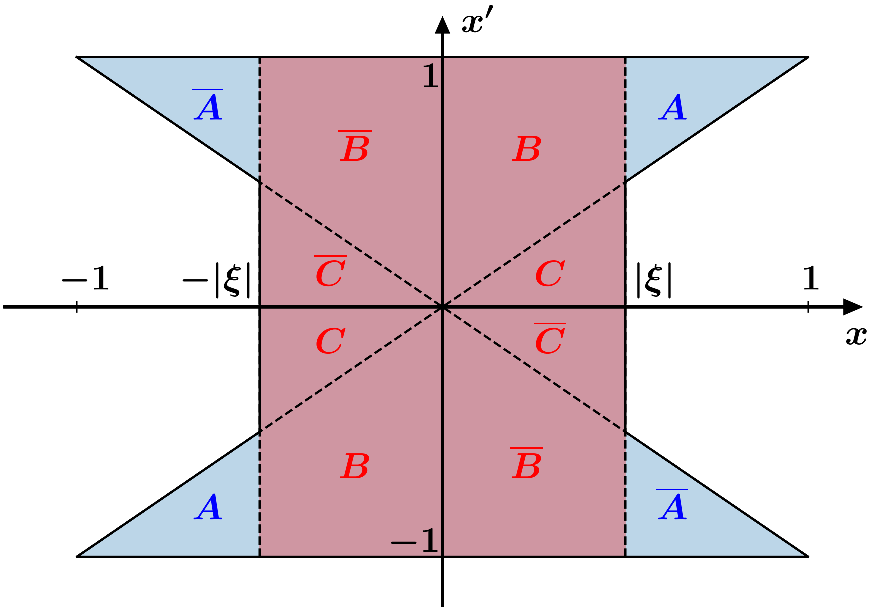

Fig. 2.1: Support domain of the evolution kernel.

The $f^{\pm}$ distributions can be regarded as the GPD analogous of the $\pm$ forward distributions that can then be used to construct the usual singlet and non-singlet distributions in the QCD evolution basis. This uniquely determines the flavour structure of the evolution kernels $\mathbb{V}^{\pm}$.

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (DGLAP region, henceforth). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (ERBL region, henceforth). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ Eq. (2.5) needs to recover the DGLAP and ERBL equations, respectively.

2.1 End-point contributions

Some of the expressions for the anomalous dimensions discussed below contain +-prescribed terms. It is thus important to treat these terms properly. We are generally dealing with objects defined as:

$$\left[\mathbb{V}(x,x')\right]_+ = \mathbb{V}(x,x') - \delta(x-x') \int_{-1}^{1} dx\, \mathbb{V}(x,x')\ . \tag{2.6}$$

where the function $\mathbb{V}$ has a pole at $x' = x$.

Let us take as an example the one-loop non-singlet anomalous dimension. For definiteness, we will refer for the precise expression to Eq. (101) of Ref. [1] and report it here for convenience (up to a factor $\alpha_s/4\pi$):

$$V_{\rm NS}^{(0)}(x,x') = 2C_F\left[\rho(x,x')\left\{\frac{1+x}{1+x'}\left(1+\frac{2}{x'-x}\right)\right\} + (x\to -x, x'\to -x')\right]_+ \ , \tag{2.7}$$

with:[3]

$$\rho(x,x') = \theta(-x+x')\theta(1+x) - \theta(x-x')\theta(1-x) \tag{2.8}$$

In order for Eq. (2.7) to be consistent with the forward evolution, one should find:

$$\lim_{\xi\to 0}\frac{1}{|2\xi|}V_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) \overset{?}{=} \frac{1}{x'}P_{\rm NS}\left(\frac{x}{x'}\right) = \frac{1}{x'}2C_F\left[\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+ \ , \tag{2.9}$$

[3] There is probably a typo in Eq. (102) of Ref. [1] as the second -1 should actually be a $+1$.

such that Eq. (2.1) exactly reduces to the DGLAP equation. However, if one takes the explicit limit for $\xi \to 0$ of Eq. (2.7) one finds:[4]

$$\lim_{\xi\to 0} \frac{1}{|2\xi|} V_{\rm NS}^{(0)}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) = 2C_F \left[\frac{1}{x'}\theta\left(\frac{x}{x'}\right)\left(1-\frac{x}{x'}\right)\frac{1+\left(\frac{x}{x'}\right)^2}{1-\left(\frac{x}{x'}\right)}\right]_+ . \tag{2.10}$$

Therefore, as compared to Eq. (2.9), the factor $1/x'$ in Eq. (2.10) appears *inside* the +-prescription sign rather than outside which makes the two expressions effectively different under integration. The difference amounts to a local term that can be quantified by knowing that:

$$[yg(y)]_+ = y\,[g(y)]_+ + \delta(1-y)\int_0^1 dz\,(1-z)g(z)\,. \tag{2.11}$$

Notice that, thanks to the factor $(1-z)$, the integral in the r.h.s. of the above equation converges despite the singularity of g. For example:

$$\left[\frac{y}{1-y}\right]_+ = y\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\,. \tag{2.12}$$

Finally, one finds that the forward limit of Eq. (2.7) gives:

$$\lim_{\xi\to 0} \frac{1}{|2\xi|} V_{\rm NS}^{(0)}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) = \frac{1}{x'}\left[P_{\rm NS}\left(\frac{x}{x'}\right) + \frac{4}{3}C_F\delta\left(1-\frac{x}{x'}\right)\right] , \tag{2.13}$$

which does *not* reproduce the DGLAP equation due to the presence of an additional local term.

2.2 On Vinnikov's code

The purpose of this section is to draw the attention on a possible incongruence of the GPD evolution code developed by Vinnikov and presented in Ref. [2]. For definiteness, we concentrate on the non-singlet $H_{\rm NS}$ GPD in the DGLAP region $x > \xi$, whose evolution equation is given in Eq. (29). For convenience, we report that equation here:

$$\begin{aligned}\frac{dH_{\rm NS}(x,\xi,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\Bigg[\int_x^1 dy \frac{x^2+y^2-2\xi^2}{(y-x)(y^2-\xi^2)}\left(H_{\rm NS}(y,\xi,Q^2)-H_{\rm NS}(x,\xi,Q^2)\right)\\ &+ H_{\rm NS}(x,\xi,Q^2)\Bigg(\frac{3}{2}+2\ln(1-x)+\frac{x-\xi}{2\xi}\ln((x-\xi)(1+\xi))\\ &- \frac{x+\xi}{2\xi}\ln((x+\xi)(1-\xi))\Bigg)\Bigg] ,\end{aligned} \tag{2.14}$$

and take the forward limit $\xi \to 0$, obtaining:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\Bigg[\int_x^1 dy \frac{x^2+y^2}{y^2(y-x)}\left(H_{\rm NS}(y,0,Q^2)-H_{\rm NS}(x,0,Q^2)\right)\\ &+ H_{\rm NS}(x,0,Q^2)\left(\frac{3}{2}+2\ln(1-x)\right)\Bigg] ,\end{aligned} \tag{2.15}$$

The limit for $\xi \to 0$ of the equation above should reproduce the usual DGLAP evolution equation:

$$\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{4\pi}\int_x^1 \frac{dy}{y}\left[\hat{P}_{\rm NS}\left(\frac{x}{y}\right)\right]_+ H_{\rm NS}\left(y,0,Q^2\right) , \tag{2.16}$$

where:

$$\hat{P}_{\rm NS}(z) = 2C_F\frac{1+z^2}{1-z} , \tag{2.17}$$

[4] The factor $\theta\left(\frac{x}{x'}\right)$ comes from the factor $\theta(-x+x')$ in Eq. (2.8) that can be rewritten as $\theta\left(\frac{x}{x'}\right)\theta\left(1-\frac{x}{x'}\right)$.

with $C_F = 4/3$. Written explicitly and accounting for the additional local term arising from the incompleteness of the convolution integral, one finds:

$$\begin{aligned}\frac{dH_{\rm NS}(x,0,Q^2)}{d\ln Q^2} &= \frac{2\alpha_s(Q^2)}{3\pi}\left[\int_x^1 dy\,\frac{x^2+y^2}{y^3(y-x)}\left(yH_{\rm NS}\left(y,0,Q^2\right)-xH_{\rm NS}(x,0,Q^2)\right)\right.\\ &+ \left. H_{\rm NS}(x,0,Q^2)\left(\frac{x(x+2)}{2}+2\ln(1-x)\right)\right],\end{aligned} \tag{2.18}$$

which evidently differs from Eq. (2.14). By inspection, one observes that the difference can be partially traced back to the issue discussed in Sect. (2.1). An interesting observation is that, for $x \to 1$, the two expressions tend to coincide. This means that the difference is larger at small values of x. This fact may have concurred to cause the oversight of this discrepancy in past numerical comparisons.

2.3 On Ji's evolution equation

In this section we discuss the evolution equations derived by Ji in Ref. [3]. This form of the evolution equation is dubbed "near-forward" in Ref. [4] because it closely resembles the DGLAP equation. However, in Ref. [3] two different equations apply to the regions $x < \xi$ and $x > \xi$. In this section, we will unify them showing that the resulting one-loop non-singlet off-forward anomalous dimension cannot be written as a fully +-prescribed distribution.

We start by considering Eqs. (15)-(17) of Ref. [3]. The first step is to replace $\xi/2$ with ξ to match our notation. Then we consider the subtraction integrals in Eq. (16) keeping in mind that they apply to both regions $x < \xi$ and $x > \xi$:[5]

$$\int_{\pm\xi}^x \frac{dy}{y-x} = -\int_{\pm\kappa}^1 \frac{dz}{1-z} = -\int_0^1 \frac{dz}{1-z} + \int_{1\mp\kappa}^1 \frac{dt}{t} = -\int_0^1 \frac{dz}{1-z} - \ln(|1\mp\kappa|)\,, \tag{2.19}$$

with:

$$\kappa = \frac{\xi}{x}\,, \tag{2.20}$$

such that the full local term in Eq. (16) becomes:

$$\frac{3}{2} + \int_{\xi}^x \frac{dy}{y-x} + \int_{-\xi}^x \frac{dy}{y-x} = \frac{3}{2} - 2\int_0^1 \frac{dz}{1-z} - \ln\left(|1-\kappa^2|\right)\,, \tag{2.21}$$

Considering the symmetry for $\xi \leftrightarrow -\xi$ of the evolution kernel in Eq. (17) of Ref. [3], we can write Eq. (15) valid for $\kappa < 1$ in a more compact way as:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_x^1 \frac{dy}{y}\mathcal{P}_1^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)\,, \tag{2.22}$$

with:

$$\begin{aligned}\mathcal{P}_1^{-,(0)}(y,\kappa) &= 2P_{\rm NS}(y,2\kappa y) + \delta(1-y)2C_F\left(\frac{3}{2} - 2\int_0^1\frac{dz}{1-z} - \ln(|1-\kappa^2|)\right)\\ &= 2C_F\left\{\left(\frac{2}{1-y}\right)_+ - \frac{1+y}{1-\kappa^2y^2} + \delta(1-y)\left[\frac{3}{2} - \ln\left(|1-\kappa^2|\right)\right]\right\}\\ &= 2C_F\left\{\left[\frac{1+(1-2\kappa^2)y^2}{(1-y)(1-\kappa^2y^2)}\right]_+ + \delta(1-y)\left[\frac{3}{2} + \left(\frac{1}{2\kappa^2}-1\right)\ln\left(|1-\kappa^2|\right) + \frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\right\},\end{aligned} \tag{2.23}$$

[5] Note that all divergent integrals considered here are implicitly assumed to be principal-valued integrals such that:

$$\int_{-1}^1 \frac{dt}{t} = 0\,.$$

This allows us to omit the $\pm i\epsilon$ terms.

where $P_{\rm NS}$ is given in Eq. (17) of Ref. [3]. The splitting function $\mathcal{P}_1^{-,(0)}$ is such that:

$$\int_0^1 dy\, \mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{3}{2}+\left(\frac{1}{2k^2}-1\right)\ln\left(|1-\kappa^2|\right)+\frac{1}{2\kappa}\ln\left(\left|\frac{1-\kappa}{1+\kappa}\right|\right)\right]\,, \tag{2.24}$$

which means that it cannot be written as a fully +-prescribed distribution. However, the integral above correctly tends to zero as $\kappa\to 0$ allowing one to recover the usual DGLAP splitting function in the forward limit:

$$\lim_{\kappa\to 0}\mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+y^2}{1-y}\right]_+\,. \tag{2.25}$$

It should also be pointed out that also the limit for $\kappa\to 1$ of Eq. (2.23) is well-behaved:

$$\lim_{\kappa\to 1}\mathcal{P}_1^{-,(0)}(y,\kappa) = 2C_F\left\{\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\left[\frac{3}{2}-\ln(2)\right]\right\}\,. \tag{2.26}$$

which is necessary to have a smooth transition of the GPDs from the DGLAP ($x>\xi$) to the ERBL ($x<\xi$) region.

We now consider Eqs. (18) and (19) of Ref. [3] valid for $\kappa>1$. Interestingly, after some algebra, we find:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_x^1\frac{dy}{y}\mathcal{P}_1^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)+\int_x^\infty\frac{dy}{y}\mathcal{P}_2^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)\right]\,, \tag{2.27}$$

with $\mathcal{P}_1^{-,(0)}$ given by:

$$\mathcal{P}_1^{-,(0)}(y,\kappa) = 2P'_{\rm NS}(y,2\kappa y)+2P'_{\rm NS}(y,-2\kappa y)+\delta(1-x)2C_F\left(\frac{3}{2}-2\int_0^1\frac{dy}{1-y}-\ln(|1-\kappa^2|)\right)\,, \tag{2.28}$$

with $P'_{\rm NS}$ is given in Eq. (19) of Ref. [3] and remarkably equal to the expression in Eq. (2.23) signifying that:

$$P_{\rm NS}(y,2\kappa y) = P'_{\rm NS}(y,2\kappa y)+P'_{\rm NS}(y,-2\kappa y)\,. \tag{2.29}$$

While:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = -2P'_{\rm NS}(y,-2\kappa y)+2P'_{\rm NS}(-y,2\kappa y) = 2C_F(\kappa-1)\frac{y+(1+2\kappa)y^3}{(1-y^2)(1-\kappa^2y^2)}\,. \tag{2.30}$$

It is very interesting to notice that $\mathcal{P}_2^{-,(0)}$ is proportional to $(\kappa-1)$ that finally guarantees the continuity of GPDs at $\kappa=1$.

We observe that, within the integration interval, the splitting function $\mathcal{P}_2^{-,(0)}$ is singular at $y=1$.[6] However, as pointed out above, the second integral on the r.h.s. of Eq. (2.27) has to be regarded as principal-valued therefore it is well-defined. In order to treat this integral numerically we consider the specific case:

$$I = \int_x^\infty dy\,\frac{f(y)}{1-y}\,, \tag{2.31}$$

where f is a test function well-behaved over the full integration range. If one subtracts and adds back the divergence at $y=1$, *i.e.*:

$$f(1)\int_0^1\frac{dy}{1-y}\,, \tag{2.32}$$

one can rearrange the integral as follows:

$$I = \int_x^\infty\frac{dy}{1-y}\left[f(y)-f(1)\left(1+\theta(y-1)\frac{1-y}{y}\right)\right]+f(1)\ln(1-x)\equiv\int_x^\infty dy\left(\frac{1}{1-y}\right)_{++}f(y)\,, \tag{2.33}$$

which effectively defines the ++-distribution. It should be noticed that this definition is specific to the function $1/(1-y)$. In case of a different singular function the function that multiplies $\theta(y-1)$ would be different. The

[6] While the singularity at $y=-1$ is placed below the lower bound $y=x$ and thus does not cause any problem, there is an additional singularity at $y=\pm1/\kappa$ that needs to be considered. As discussed below, this singularity in $\mathcal{P}_2^{-,(0)}$ exactly cancels against an opposite singularity in $\mathcal{P}_1^{-,(0)}$.

advantage of this rearrangement is that the integrand is free of the divergence at $y=1$ and is thus amenable to numerical integration. Also, the ++-distribution reduces to the standard +-distribution when the upper integration bound is one rather than infinity. In this sense the ++-distribution generalises the +-distribution to ERBL-like integrals.

In view of the use of Eq. (2.33), it is convenient to rewrite Eq. (2.30) as follows:

$$\mathcal{P}_2^{-,(0)}(y,\kappa) = 2C_F\left[\frac{1+(1+\kappa)y+(1+\kappa-\kappa^2)y^2}{(1+y)(1-\kappa^2y^2)} - \left(\frac{1}{1-y}\right)_{++}\right], \tag{2.34}$$

where the first term in the squared bracket is regular at $y=1$.

Finally, Eqs. (2.22) and Eq. (2.27) can be combined as follows:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_x^1\frac{dy}{y}\mathcal{P}_1^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right) + \theta(\kappa-1)\int_x^\infty\frac{dy}{y}\mathcal{P}_2^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right)\right], \tag{2.35}$$

or even more compactly as:

$$\mu^2\frac{d}{d\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty\frac{dy}{y}\mathcal{P}^{-,(0)}(y,\kappa)f^-\left(\frac{x}{y},\xi\right), \tag{2.36}$$

with:

$$\mathcal{P}^{-,(0)}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{-,(0)}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{-,(0)}(y,\kappa), \tag{2.37}$$

to obtain a single DGLAP-like evolution equation valid for all values of κ. In fact, it should be pointed out that, when performing the integrals numerically, the form in Eq. (2.36) has to be adopted. The reason is that both functions $\mathcal{P}_1^{-,(0)}$ and $\mathcal{P}_2^{-,(0)}$, due to the factor $1-\kappa^2y^2$, are affected by a pole at $y=|\kappa|^{-1}$ that, for $|\kappa|>1$ or equivalently $|x|<|\xi|$ (*i.e.* in the ERBL region) have to cancel to give a finite result. Using the explicit expressions for $\mathcal{P}_1^{-,(0)}$ and $\mathcal{P}_2^{-,(0)}$, we find:

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}_1^{-,(0)}(y,\kappa) = -2C_F\frac{1+\kappa}{\kappa}, \tag{2.38}$$

and:

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}_2^{-,(0)}(y,\kappa) = 2C_F\frac{1+\kappa}{\kappa}. \tag{2.39}$$

Since the coefficient of the pole are equal in absolute value and opposite in sign they cancel in the integral. Below, we will explicitly verify this property also for the anomalous dimensions of the singlet sector.

Importantly, in the limit $\kappa\to 0$, the second integral in the r.h.s. of Eq. (2.35) drops and the splitting function $\mathcal{P}_1^{-,(0)}$ reduces to the one-loop non-singlet DGLAP splitting function (see Eq. (2.25)) so that, as expected, Eq. (2.35) becomes the DGLAP equation.

2.3.1 The ERBL equation

It is also interesting to verify that also the ERBL equation is recovered in the limit $\xi\to 1$. Given the definition of κ, Eq. (2.20), this limit is attained by taking $\kappa\to 1/x$. However, the limit procedure is more subtle than in the DGLAP case due to the presence of +-prescriptions and explicit local terms that need to cooperate to give the right result.

We make use of Eqs. (2.28) and (2.30) to write the evolution equation in terms of the function $P'_{\rm NS}$ in a form similar to that originally given in Ref. [3] but more compactly as:

$$\mu^2\frac{d}{d\mu^2}f^-(x,1) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_{-1}^1 dy\, V_{\rm NS}^{(0)}(x,y)f^-(y,1)\right]. \tag{2.40}$$

with:

$$\begin{aligned} V_{\rm NS}^{(0)}(x,y) &= \theta(y-x)\left[\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},\frac{2}{y}\right)\right] - 2C_F\delta(y-x)\int_{-1}^1 dz\,\frac{\theta(z-x)}{z-x} \\ &+ \theta(x-y)\left[-\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},-\frac{2}{y}\right)\right] + 2C_F\delta(x-y)\int_{-1}^1 dz\,\frac{\theta(x-z)}{z-x} \\ &+ 3C_F\delta(y-x), \end{aligned} \tag{2.41}$$

where:

$$\frac{2}{y}P'_{\rm NS}\left(\frac{x}{y},\frac{2}{y}\right) = 2C_F\frac{1+x}{1+y}\left(\frac{1}{2}+\frac{1}{y-x}\right)\,. \tag{2.42}$$

In order to make a step towards the ERBL equation, we change the variables x and y with:

$$\begin{aligned} t &= \frac{1}{2}(x+1)\,,\\ u &= \frac{1}{2}(y+1)\,, \end{aligned} \tag{2.43}$$

such that the evolution variable becomes:

$$\mu^2\frac{d}{d\mu^2}\Phi^-(t) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_0^1 du\,\overline{V}^{(0)}_{\rm NS}(t,u)\Phi^-(u)\right]\,. \tag{2.44}$$

with $\Phi^-(t) = f^-(x,1)$ and:

$$\begin{aligned} \overline{V}^{(0)}_{\rm NS}(t,u) &= C_F\left[\theta(u-t)\left(\frac{t-1}{u}+\frac{1}{u-t}-\delta(u-t)\int_0^1 du'\frac{\theta(u'-t)}{u'-t}\right)\right.\\ &- \left.\theta(t-u)\left(\frac{t}{1-u}+\frac{1}{u-t}-\delta(t-u)\int_0^1 du'\frac{\theta(t-u')}{u'-t}\right)+\frac{3}{2}\delta(u-t)\right]\,. \end{aligned} \tag{2.45}$$

Now we define:

$$[f(t,u)]_+ \equiv f(t,u) - \delta(u-t)\int_0^1 du'\,f(t,u')\,, \tag{2.46}$$

where f has a single pole at $u=t$, so that we can write Eq. (2.45) more compactly as:

$$\overline{V}^{(0)}_{\rm NS}(t,u) = C_F\left\{\left[\theta(u-t)\frac{t-1}{u}+\left(\frac{\theta(u-t)}{u-t}\right)_+\right]-\left[\theta(t-u)\frac{t}{1-u}+\left(\frac{\theta(t-u)}{u-t}\right)_+\right]+\frac{3}{2}\delta(u-t)\right\}\,. \tag{2.47}$$

This confirms the result of Ref. [4] modulo the fact that, for achieving a correct cancellation of the divergencies, the θ-function for the $+$-prescribed terms needs to be inside the $+$-prescription sign itself rather than outside. In fact, it is the very presence of the θ-function that generates the necessity of a $+$-prescription. Without the θ-function, the integral could be interpreted as principal-valued integral that requires no regularisation. The presence of the θ-function interdicts the cancellation between left and right sides of the pole resulting in a singular integral. The $+$-prescription reabsorbes this singularity producing a well-behaved integral.

One can check that integrating $\overline{V}^{(0)}_{\rm NS}$ over t gives zero:[7]

$$\int_0^1 dt\,\overline{V}^{(0)}_{\rm NS}(t,u) = 0\,. \tag{2.48}$$

This finally confirms that $\overline{V}^{(0)}_{\rm NS}$ as derived from Ref. [3] admits a fully $+$-prescribed form, that is:

$$\overline{V}^{(0)}_{\rm NS}(t,u) = C_F\left\{\theta(u-t)\left[\frac{t-1}{u}+\frac{1}{u-t}\right]-\theta(t-u)\left[\frac{t}{1-u}+\frac{1}{u-t}\right]\right\}_+\,. \tag{2.49}$$

This was also explicitly derived in Ref. [5] and argued that this property must hold for symmetry reasons.

It is now interesting to derive an ERBL-like evolution equation for GPDs at one loop. This equation can be written as:

$$\frac{d}{d\ln\mu^2}f^-(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_{-1}^1\frac{dy}{\xi}\mathbb{V}^{(0)}_{\rm NS}\left(\frac{x}{\xi},\frac{y}{\xi}\right)f^-(y,\xi)\,, \tag{2.50}$$

with:

$$\begin{aligned} \frac{1}{\xi}\mathbb{V}^{(0)}_{\rm NS}\left(\frac{x}{\xi},\frac{y}{\xi}\right) &= \frac{2}{y}\left\{\theta(x-\xi)\theta(y-x)P_{\rm NS}\left(\frac{x}{y},\frac{2\xi}{y}\right)-\theta(-x-\xi)\theta(x-y)P_{\rm NS}\left(\frac{x}{y},\frac{2\xi}{y}\right)\right.\\ &+ \theta(\xi-x)\theta(x+\xi)\left[\theta(y-x)P'_{\rm NS}\left(\frac{x}{y},\frac{2\xi}{y}\right)-\theta(x-y)P'_{\rm NS}\left(\frac{x}{y},-\frac{2\xi}{y}\right)\right]\\ &+ \left.\delta\left(1-\frac{x}{y}\right)C_F\left[\frac{3}{2}+\int_\xi^x\frac{dz}{z-x}+\int_{-\xi}^x\frac{dz}{z-x}\right]\right\}\,. \end{aligned} \tag{2.51}$$

[7] Note that the two $+$-prescribed terms when integrated over t do not individually give zero but their combination does.

The integration domain of Eq. (2.51) is displayed in Fig. 2.2. As we will see below, in order to compute conformal

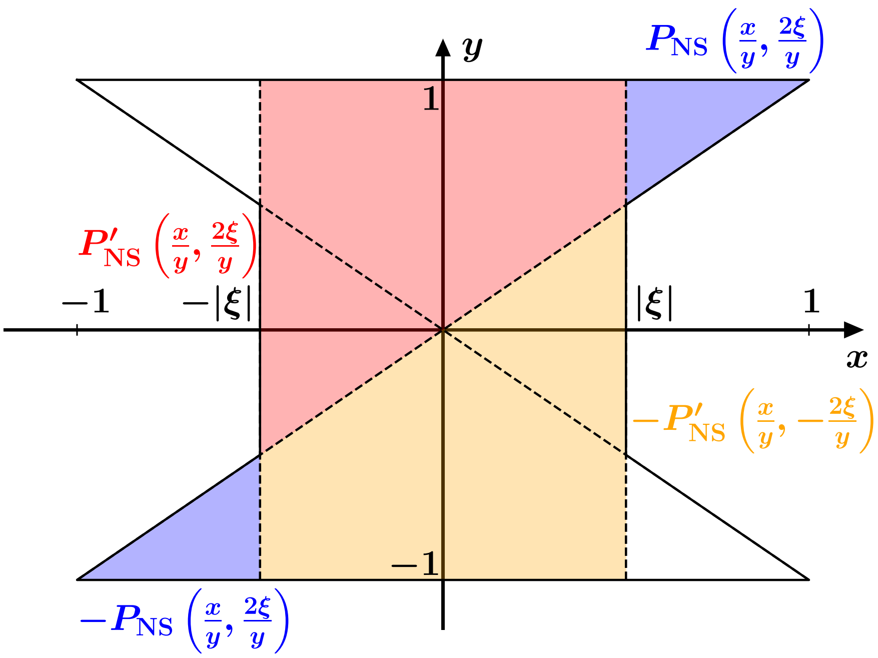

Fig. 2.2: Support domain of the evolution kernel in Eq. (2.51).

moments one needs to convolute the evolution kernel with some function integrating over the x variable. To do so, we use Eq. (2.29) to write $P_{\rm NS}$ in terms of $P'_{\rm NS}$ then, referring to Fig. 2.2, we integrate over the blue, read, and orange regions separately reducing the convolution to integrals between ξ and y and $-\xi$ and y that can the be gathered. Given a well-behaved test function f, the final result reads:

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\xi}\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right)f(x) &= 2C_F\left\{\frac{3}{2}f(y)+\int_{\xi}^{y}dx\left[\frac{1}{C_Fy}P'_{\rm NS}\left(\frac{x}{y},-\frac{2\xi}{y}\right)f(x)-\frac{f(y)}{y-x}\right]\right.\\ &+\left.\int_{-\xi}^{y}dx\left[\frac{1}{C_Fy}P'_{\rm NS}\left(\frac{x}{y},\frac{2\xi}{y}\right)f(x)-\frac{f(y)}{y-x}\right]\right\}.\end{aligned}\tag{2.52}$$

Considering that:

$$\frac{1}{C_Fy}P'_{\rm NS}\left(\frac{x}{y},\frac{2\xi}{y}\right)=\frac{x-\xi}{2\xi(y+\xi)}+\frac{1}{y-x},\tag{2.53}$$

and thus:

$$\frac{1}{C_Fy}P'_{\rm NS}\left(\frac{x}{y},-\frac{2\xi}{y}\right)=-\frac{x+\xi}{2\xi(y-\xi)}+\frac{1}{y-x},\tag{2.54}$$

one finally has:

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\xi}\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right)f(x) &= 2C_F\left\{\frac{3}{2}f(y)-\frac{1}{2}\int_{\xi}^{y}dx\left[\frac{x+\xi}{\xi(y-\xi)}f(x)-2\frac{f(x)-f(y)}{y-x}\right]\right.\\ &+\left.\frac{1}{2}\int_{-\xi}^{y}dx\left[\frac{x-\xi}{\xi(y+\xi)}f(x)+2\frac{f(x)-f(x)}{y-x}\right]\right\}.\end{aligned}\tag{2.55}$$

2.3.2 Conformal moments

A relevant question is whether Eq. (2.51) is such that the so-called conformal moments of non-singlet GPDs (see Eq. (111) of Ref. [1]):

$$\mathcal{C}_n^-(\xi)=\xi^n\int_{-1}^{1}dx\,C_n^{3/2}\left(\frac{x}{\xi}\right)f^-(x,\xi)\,,\tag{2.56}$$

where $C_n^{3/2}$ are Gegenbauer polynomials, "diagonalise" the leading-order evolution equation in Eq. (2.136). To state it differently, we would like to check whether conformal moments evolve multiplicatively. Multiplying Eq. (2.136) by $\xi^n C_n^{3/2}(x/\xi)$ and integrating over x between -1 and 1, yields:

$$\frac{d\mathcal{C}_n^-(\xi)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\xi^n \int_{-1}^1 dy f^-(y,\xi) \int_{-1}^1 \frac{dx}{\xi} C_n^{3/2}\left(\frac{x}{\xi}\right) \mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) . \tag{2.57}$$

Therefore, the aim is to check whether the following equality holds:

$$\int_{-1}^1 \frac{dx}{\xi} C_n^{3/2}\left(\frac{x}{\xi}\right) \mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) = \mathcal{V}_n^{(0)}(\xi) C_n^{3/2}\left(\frac{y}{\xi}\right) , \tag{2.58}$$

where $\mathcal{V}_n^{(0)}$ is a number that may generally depend on ξ. If Eq. (2.58) held, Eq. (2.57) would become:

$$\frac{d\mathcal{C}_n^-(\xi)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\mathcal{V}_n^{(0)}(\xi)\mathcal{C}_n^-(\xi) , \tag{2.59}$$

that is to say that conformal moments evolve multiplicative exactly like Mellin moments for forward distributions. As a matter of fact, up to numerical factor, conformal moments do coincide with Mellin moments in the limit $\xi \to 0$. This can be seen by observing that the Gegenbauer polynomials admit the following expansion:

$$\xi^n C_n^{3/2}\left(\frac{x}{\xi}\right) = \frac{n+1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k}\binom{n+2}{k+1}(x+\xi)^k(x-\xi)^{n-k} , \tag{2.60}$$

that is such that:

$$\lim_{\xi\to 0} \xi^n C_n^{3/2}\left(\frac{x}{\xi}\right) = \frac{(2n+1)!}{2^n(n!)^2} x^n . \tag{2.61}$$

Therefore, the conformal moments of the non-singlet distribution in the forward limit become:

$$\lim_{\xi\to 0} \mathcal{C}_n^-(\xi) = \frac{(2n+1)!}{2^n(n!)^2}[1+(-1)^n] f^-(n+1) , \tag{2.62}$$

where, with abuse of notation, we have defined the Mellin moments of the forward distribution as:

$$f^-(n) = \lim_{\xi\to 0}\int_0^1 dx\, x^{n-1} f^-(x,\xi) , \tag{2.63}$$

that are known to diagonalise the DGLAP equation to all orders. Using Eq. (2.61) and the fact that:

$$\lim_{\xi\to 0} \frac{1}{\xi}\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) = [\theta(x)\theta(y-x) - \theta(-x)\theta(x-y)]\,\frac{1}{y} P_{\rm NS}^{(0)}\left(\frac{x}{y}\right) , \tag{2.64}$$

where $P_{\rm NS}^{(0)}$ is the leading-order non-singlet DGLAP splitting function, one finds:

$$\lim_{\xi\to 0}\int_{-1}^1 \frac{dx}{\xi} C_n^{3/2}\left(\frac{x}{\xi}\right) \mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) = \frac{(2n+1)!}{2^n(n!)^2} y^n P_{\rm NS}^{(0)}(n+1) = \left[\lim_{\xi\to 0}\xi^n C_n^{3/2}\left(\frac{y}{\xi}\right)\right] P_{\rm NS}^{(0)}(n+1) , \tag{2.65}$$

where, again with abuse of notation, the Mellin moments of the splitting functions are given by (see, *e.g.*, Eq. (4.129) of Ref. [6]):

$$P_{\rm NS}^{(0)}(n) = \int_0^1 dx\, x^{n-1} P_{\rm NS}^{(0)}(x) = 2C_F\left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2\sum_{k=2}^n \frac{1}{k}\right] . \tag{2.66}$$

Finally, Eq. (2.59) is fulfilled and the evolution kernel can be directly read off:

$$\lim_{\xi\to 0}\mathcal{V}_n^{(0)}(\xi) = P_{\rm NS}^{(0)}(n+1) = 2C_F\left[-\frac{1}{2} + \frac{1}{(n+1)(n+2)} - 2\sum_{k=2}^{n+1} \frac{1}{k}\right] . \tag{2.67}$$

On the other hand, it is well known that the one-loop ERBL kernel $V_{\rm NS}^{(0)}$ obeys the relation:

$$\int_{-1}^{1} dx\, C_n^{3/2}(x)\, V_{\rm NS}^{(0)}(x,y) = V_n^{(0)} C_n^{3/2}(y)\,. \tag{2.68}$$

Evidently, $V_n^{(0)}$ is the evolution kernel in the $\xi \to 1$ limit and its expression can be read off from Eq. (2.19) of Ref. [7] where, to match our notation, we have to flip the sign and removed the β-function:

$$\lim_{\xi\to 1} \mathcal{V}_n(\xi) = V_n^{(0)} = 2C_F\left[-\frac{1}{2} + \frac{1}{(n+1)(n+2)} - 2\sum_{k=2}^{n+1}\frac{1}{k}\right] = 2C_F\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} - 2\sum_{k=1}^{n}\frac{1}{k}\right]. \tag{2.69}$$

Remarkably, this is identical to Eq. (2.67). This finally means that Eq. (2.59) is fulfilled in both the $\xi \to 0$ and $\xi \to 1$ limits and that in these limits the evolution kernels are equal. In other words, conformal moments of forward distributions and distribution amplitudes at leading order evolve multiplicatively with the same anomalous dimension. This suggests that the kernels $\mathcal{V}_n$ may be constant in ξ and equal to Eq. (2.67).

To prove this we need to compute Eq. (2.58) for a generic value of ξ. To do so, we use Eq. (2.55) for $\mathbb{V}_{\rm NS}^{(0)}$ replacing $f(x)$ with $C_n^{3/2}(x/\xi)$:

$$\begin{aligned}
\int_{-1}^{1}\frac{dx}{\xi}\, C_n^{3/2}\left(\frac{x}{\xi}\right)\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) &= 2C_F\Bigg\{\frac{3}{2}C_n^{3/2}\left(\frac{y}{\xi}\right)\\
&- \frac{1}{2}\int_{\xi}^{y} dx\left[\frac{x+\xi}{\xi(y-\xi)}C_n^{3/2}\left(\frac{x}{\xi}\right) - 2\frac{C_n^{3/2}(x/\xi) - C_n^{3/2}(y/\xi)}{y-x}\right]\\
&+ \frac{1}{2}\int_{-\xi}^{y} dx\left[\frac{x-\xi}{\xi(y+\xi)}C_n^{3/2}\left(\frac{x}{\xi}\right) + 2\frac{C_n^{3/2}(x/\xi) - C_n^{3/2}(y/\xi)}{y-x}\right]\Bigg\}.
\end{aligned} \tag{2.70}$$

Of course, in the limits $\xi \to 0$ and $\xi \to 1$ this expression produces the Gegenbauer polynomials in the respective limits times V_n^0. The calculation for *all* n and for a generic value of ξ is not straightforward. To start with, we compute the integral above for $n = 0$ where we expected to obtain zero because $V^{(0)} = 0$. Since $C_0^{\alpha}(z) = 1$, the zero-th conformal moment evaluates to:

$$\begin{aligned}
\int_{-1}^{1}\frac{dx}{\xi}\, C_0^{3/2}\left(\frac{x}{\xi}\right)\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) &= C_F\left[3 - \int_{\xi}^{y} dx\frac{x+\xi}{\xi(y-\xi)} + \int_{-\xi}^{y} dx\frac{x-\xi}{\xi(y+\xi)}\right]\\
&= C_F\left[3 - \frac{y+\xi}{2\xi} - 1 + \frac{y-\xi}{2\xi} - 1\right] = 0\,,
\end{aligned} \tag{2.71}$$

providing further evidence that indeed Gegenbauer polynomials do diagonalise GPD leading-order evolution with evolution kernels $\mathcal{V}_n^{(0)}(\xi) = V_n^{(0)}$ independent of ξ.

Now let us consider the case $n = 1$. In the case the Gegenbauer polynomial is:

$$C_1^{3/2}\left(\frac{x}{\xi}\right) = \frac{3x}{\xi}\,. \tag{2.72}$$

Since:

$$V_1^{(0)} = -\frac{8C_F}{3}\,, \tag{2.73}$$

we expect:

$$\int_{-1}^{1}\frac{dx}{\xi}\, C_1^{3/2}\left(\frac{x}{\xi}\right)\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) \overset{?}{=} -\frac{8C_F y}{\xi}\,. \tag{2.74}$$

Using Eq. (2.70) with $n = 1$, we find:

$$\begin{aligned}
\int_{-1}^{1}\frac{dx}{\xi}\, C_1^{3/2}\left(\frac{x}{\xi}\right)\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) &= \frac{3C_F}{\xi}\left[3y - \int_{\xi}^{y} dx\left[\frac{x(x+\xi)}{\xi(y-\xi)} + 2\right] + \int_{-\xi}^{y} dx\left[\frac{x(x-\xi)}{\xi(y+\xi)} - 2\right]\right]\\
&= \frac{3C_F}{\xi}\left[3y - 4y - \frac{5}{3}y\right] = -\frac{8C_F y}{\xi}\,,
\end{aligned} \tag{2.75}$$

as expected. We can now attempt a general proof. To do so, we define $z = y/\xi$ and, using a the change of variable $v = x/\xi$ in the integrals, rewrite the r.h.s. of Eq. (2.70) without the factor $2C_F$ as follows:

$$\begin{aligned} I &= \frac{3}{2}C_n^{3/2}(z) - \frac{1}{2}\int_1^z dv \left[\frac{v+1}{z-1}C_n^{3/2}(v) - 2\frac{C_n^{3/2}(v) - C_n^{3/2}(z)}{z-v}\right] \\ &+ \frac{1}{2}\int_{-1}^z dv \left[\frac{v-1}{z+1}C_n^{3/2}(v) + 2\frac{C_n^{3/2}(v) - C_n^{3/2}(z)}{z-v}\right] . \end{aligned} \tag{2.76}$$

Now we use the fact that $C_n^{3/2}$ is indeed a polynomial of degree n whose expansion reads:

$$C_n^{3/2}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_k^{(n)} x^\ell \quad \text{with} \quad a_k^{(n)} = (-1)^k 2^\ell \frac{\Gamma(\ell+k+3/2)}{\Gamma(3/2)k!\ell!} , \tag{2.77}$$

with $\ell = n - 2k$, that allows us to write:

$$I = \frac{3}{2}C_n^{3/2}(z) - \frac{1}{2}\sum_{k=0}^{\lfloor n/2 \rfloor} a_k^{(n)} \left[\int_1^z dv \frac{v^{\ell+1}+v^\ell}{z-1} - \int_{-1}^z dv \frac{v^{\ell+1}-v^\ell}{z+1} + 2\int_1^z dv \frac{z^\ell - v^\ell}{z-v} + 2\int_{-1}^z dv \frac{z^\ell - v^\ell}{z-v}\right] . \tag{2.78}$$

Now we can solve the integrals. The first integral gives:

$$\int_1^z dv \frac{v^{\ell+1}+v^\ell}{z-1} = \frac{1}{\ell+2}\frac{1-z^{\ell+2}}{1-z} + \frac{1}{\ell+1}\frac{1-z^{\ell+1}}{1-z} = \frac{1}{\ell+2}\sum_{j=0}^{\ell+1} z^j + \frac{1}{\ell+1}\sum_{j=0}^{\ell} z^j , \tag{2.79}$$

and similarly the second:

$$\int_{-1}^z dv \frac{v^{\ell+1}-v^\ell}{z+1} = -\frac{(-1)^{\ell+2}}{\ell+2}\sum_{j=0}^{\ell+1}(-z)^j + \frac{(-1)^{\ell+1}}{\ell+1}\sum_{j=0}^{\ell}(-z)^j , \tag{2.80}$$

where we have used the geometric series:

$$\sum_{j=0}^{n} v^j = \frac{1-v^{n+1}}{1-v} . \tag{2.81}$$

Their combination evaluates to:

$$\int_1^z dv \frac{v^{\ell+1}+v^\ell}{z-1} - \int_{-1}^z dv \frac{v^{\ell+1}-v^\ell}{z+1} = \left[\frac{1}{\ell+2} + \frac{1}{\ell+1}\right]\sum_{j=0}^{\ell}\left[1+(-1)^{\ell-j}\right] z^j . \tag{2.82}$$

Notice that the $z^{\ell+1}$ term vanishes because the projector is null for $j = \ell+1$. Now we turn to the third and fourth integrals in Eq. (2.76):

$$\int_1^z dv \frac{z^\ell - v^\ell}{z-v} = z^{\ell-1}\int_1^z dv \frac{1-(v/z)^\ell}{1-v/z} = \sum_{j=0}^{\ell-1} z^{\ell-j-1}\int_1^z dv\, v^j = \sum_{j=0}^{\ell-1}\frac{z^\ell - z^{\ell-j-1}}{j+1} = -\sum_{j=0}^{\ell-1}\frac{z^j}{\ell-j} + z^\ell\sum_{j=1}^{\ell}\frac{1}{j} , \tag{2.83}$$

and:

$$\int_{-1}^z dv \frac{z^\ell - v^\ell}{z-v} = -\sum_{j=0}^{\ell-1}\frac{(-1)^{\ell-j}z^j}{\ell-j} + z^\ell\sum_{j=1}^{\ell}\frac{1}{j} , \tag{2.84}$$

so that their combination gives:

$$2\int_1^z dv \frac{z^\ell - v^\ell}{z-v} + 2\int_{-1}^z dv \frac{z^\ell - v^\ell}{z-v} = -2\sum_{j=0}^{\ell-1}\frac{1+(-1)^{\ell-j}}{\ell-j} z^j + z^\ell\sum_{j=1}^{\ell}\frac{4}{j} . \tag{2.85}$$

Putting everything together one finds:

$$
\begin{aligned}
I &= \frac{3}{2}C_n^{3/2}(z) \\
&- \sum_{k=0}^{\lfloor n/2\rfloor} a_k^{(n)} \left[\sum_{j=0}^{\ell-1}\left(\frac{1}{\ell+2}+\frac{1}{\ell+1}-\frac{2}{\ell-j}\right)\frac{1+(-1)^{\ell-j}}{2}z^j + \left(\frac{1}{\ell+2}+\frac{1}{\ell+1}+\sum_{j=0}^{\ell-1}\frac{2}{j+1}\right)z^\ell\right].
\end{aligned}
\tag{2.86}
$$

Now we exchange the sums over k and the first sum over j in the second line of the equation above by using the following equality:

$$
\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\ell-1}\cdots = \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{n-2k-1}\cdots = \sum_{j=0}^{n-1}\sum_{k=0}^{\lfloor\frac{n-j-1}{2}\rfloor}\cdots, \tag{2.87}
$$

finding:

$$
\begin{aligned}
I &= \frac{3}{2}C_n^{3/2}(z) \\
&- \sum_{j=0}^{n-1} z^j \frac{1+(-1)^{n-j}}{2}\sum_{k=0}^{\lfloor\frac{n-j-1}{2}\rfloor} a_k^{(n)}\left(\frac{1}{n-2k+2}+\frac{1}{n-2k+1}-\frac{2}{n-2k-j}\right) \\
&- \sum_{k=0}^{\lfloor n/2\rfloor} z^\ell a_k^{(n)}\left(\frac{1}{\ell+2}+\frac{1}{\ell+1}+\sum_{j=0}^{\ell-1}\frac{2}{j+1}\right).
\end{aligned}
\tag{2.88}
$$

where in the first line we have made explicit $\ell = n-2k$ and used the equality:

$$
\frac{1+(-1)^{\ell-j}}{2} = \frac{1+(-1)^{n-2k-j}}{2} = \frac{1+(-1)^{n-j}}{2}. \tag{2.89}
$$

This projector nullifies all the terms in the first series over j for which $n-j$ is odd selecting only the even ones. Therefore we can identify the combination $n-j$ with an even index, *i.e.* $n-j=2h$, and remove the projector. Replacing the summation index k with h and making explicit the index ℓ also in the second line gives:

$$
\begin{aligned}
I &= \frac{3}{2}C_n^{3/2}(z) \\
&- \sum_{h=0}^{\lfloor n/2\rfloor} a_h^{(n)} z^{n-2h}\left[\frac{1}{n-2h+2}+\frac{1}{n-2h+1}+2\sum_{j=1}^{n-2h}\frac{1}{j}+\sum_{j=1}^{h}\frac{a_{h-j}^{(n)}}{a_h^{(n)}}\left(\frac{1}{n-2h+2j+2}+\frac{1}{n-2h+2j+1}-\frac{1}{j}\right)\right].
\end{aligned}
\tag{2.90}
$$

It turns out that the term in the square brackets is independent of the summation index h (this statement can be easily verified numerically but an analytic proof looks very hard to achieve). Therefore, without loss of generality, we can set $h=0$ and pull it out from the summation symbol, obtaining:

$$
\begin{aligned}
I &= \frac{3}{2}C_n^{3/2}(z) - \left[\frac{1}{n+2}+\frac{1}{n+1}+2\sum_{j=1}^{n}\frac{1}{j}\right]\sum_{h=0}^{\lfloor n/2\rfloor} a_h^{(n)} z^{n-2h} \\
&= \left[\frac{3}{2}-\frac{1}{n+2}-\frac{1}{n+1}-2\sum_{j=1}^{n}\frac{1}{j}\right]C_n^{3/2}(z)
\end{aligned}
\tag{2.91}
$$

which finally proves the identity:

$$
\int_{-1}^{1}\frac{dx}{\xi}\,C_n^{3/2}\left(\frac{x}{\xi}\right)\mathbb{V}_{\rm NS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) = V_n^{(0)}C_n^{3/2}\left(\frac{y}{\xi}\right), \tag{2.92}
$$

with $V_n^{(0)}$ given in Eq. (2.69).

These calculations confirm that conformal moments evolve multiplicatively as:

$$\frac{d\mathcal{C}_n^-(\xi)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}V_n^{(0)}\mathcal{C}_n^-(\xi)\,. \tag{2.93}$$

The solution to this equation is simply:

$$\mathcal{C}_n^-(\xi,\mu) = \exp\left[\frac{V_n^{(0)}}{4\pi}\int_{\mu_0}^{\mu} d\ln\mu^2\alpha_s(\mu)\right]\mathcal{C}_n^-(\xi,\mu_0)\,, \tag{2.94}$$

in which remarkably the evolution kernel does not depend on ξ. This in turn means that the ξ dependence of conformal moments is entirely encoded in the initial scale moments and that evolution leave it unchanged.

The question that remains to be answered is whether the knowledge of the conformal moments of a GPD is enough to reconstruct its x dependence.

2.3.3 On the expansion of GPDs in terms of Gegenbauer polynomials

In the previous section, we have shown that the conformal moments of a non-singlet GPD, defined as in Eq. (2.56), evolve multiplicatively at LO according to Eq. (2.94). The question that remains to be answered is whether it is possible to reconstruct any non-singlet GPD by knowing all its conformal moments. In order to do so, one should be able to expand any given GPD f^- in terms of conformal moments:

$$f^-(x,\xi,\mu) = \sum_{m=0}^{\infty}\mathcal{C}_m^-(\xi,\mu)f_m(x,\xi)\,, \tag{2.95}$$

where f_n is some set of functions. In this way, the evolution of f^- would simply amount to evolving the single conformal moments $\mathcal{C}_m^-$ using Eq. (2.94). However, in order for Eq. (2.95) to be true the set of functions f_n must obey the relation:

$$\xi^n\int_{-1}^{1}dx\,C_n^{3/2}\left(\frac{x}{\xi}\right)f_m(x,\xi) = \xi^{n-1}\int_{-1/\xi}^{1/\xi}dz\,C_n^{3/2}(z)\,f_m(\xi z,\xi) = \delta_{mn}\,. \tag{2.96}$$

For $\xi = 1$, as is the case for the ERBL evolution equation of DAs, one could exploit the orthogonality of Gegenbauer polynomials in the interval $[-1:1]$ and would have that:

$$f_m(x,1) = \frac{2m!(m+3/2)[\Gamma(3/2)]^2}{\pi\Gamma(m+3)}(1-x^2)C_m^{3/2}(x)\,, \tag{2.97}$$

that exactly verifies Eq. (2.96) for $\xi = 1$. This allows one to expand DAs in terms of Gegenbauer polynomials and compute the evolution by evolving the single conformal moments. However, for $\xi < 1$ Eq. (2.96) requires integrating the Gegenbauer polynomials outside the region $[-1:1]$ thus preventing the use of orthogonality. To overcome this problem, one could define:

$$f_m(x,\xi) = \xi^{1-n}\theta(x+\xi)\theta(\xi-x)\frac{2m!(m+3/2)[\Gamma(3/2)]^2}{\pi\Gamma(m+3)}\left(1-\frac{x^2}{\xi^2}\right)C_m^{3/2}\left(\frac{x}{\xi}\right)\,, \tag{2.98}$$

that indeed satisfies Eq. (2.96) but at the price of introducing θ-functions that require f^- to be zero outside the ERBL region, *i.e.* for $x > \xi$ and $x < -\xi$. In other words, in order for the expansion in Eq. (2.95) to be possible, one needs to require that:

$$f^-(x,\xi,\mu) = 0 \quad \text{for} \quad x > \xi \quad \text{and} \quad x < -\xi\,, \tag{2.99}$$

that is clearly not acceptable for a GPD. However, in Ref. [8] one reads: "[...] the restricted support property of each individual term does not imply that the GPD vanishes in the outer region [$x > \xi$ or $x < -\xi$]. Rather one should understand this expansion [Eq. (2.95)] as an ill-convergent sum of distributions (in the mathematical sense) that yields a result which is non-zero in the outer region." Indeed, the authors of Ref. [8] managed to find a way to make Eq. (2.95) different from zero for $x > \xi$ or $x < -\xi$. This needs to be understood.

2.3.4 Sum rules

A question arises: does the fact that the GPD anomalous dimension (*cfr.* Eq. (2.24)) does not admit a fully +-prescribed form violate any conservation law? To answer this question, we notice that the fact that the non-singlet DGLAP anomalous dimension integrates to zero (see Eq. (2.25)), and thus admits a +-prescribed form, derives from the conservation of the total number of quarks minus anti-quarks (valence sum rule):

$$\int_0^1 dx\, f^-(x,0) = \text{constant}\,. \tag{2.100}$$

Taking the derivative of this equation w.r.t. $\ln\mu^2$ and using the DGLAP equation gives:

$$\begin{aligned} 0 &= \int_0^1 dx \int_x^1 \frac{dy}{y}\mathcal{P}_1^-(y,0) f^-\left(\frac{x}{y},0\right) = \int_0^1 dy\,\mathcal{P}_1^-(y,0)\int_0^y \frac{dx}{y}\, f^-\left(\frac{x}{y},0\right) \\ &= \int_0^1 dy\,\mathcal{P}_1^-(y,0)\int_0^1 dz\, f^-(z,0) = \text{constant}\times\int_0^1 dy\,\mathcal{P}_1^-(y,0) \quad\Leftrightarrow\quad \int_0^1 dy\,\mathcal{P}_1^-(y,0) = 0\,. \end{aligned} \tag{2.101}$$

This clearly justifies the requirement for $\mathcal{P}_1^-(y,0)$ to be fully +-prescribed.

One may try to apply the same argument to GPDs. In this case the valence sum rule generalises in:

$$\int_0^1 dx\, f^-(x,\xi) = F\,, \tag{2.102}$$

where F is independent of μ and ξ but may (and does) depend on the momentum transfer t. F is usually referred to as form factor. One should now take the derivative w.r.t. $\ln\mu^2$ and use Eq. (2.35) but in doing this one needs to take into account that $\kappa = \xi/x$:

$$\begin{aligned} 0 &= \int_0^1 dx \int_0^1 \frac{dy}{y}\left[\theta(y-x)\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] f^-(y,\xi) \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^y \frac{dx}{y}\,\mathcal{P}_1^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right) + \int_0^\xi \frac{dx}{y}\,\mathcal{P}_2^{-,(0)}\left(\frac{x}{y},\frac{\xi}{x}\right)\right] \\ &= \int_0^1 dy\, f^-(y,\xi)\left[\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right)\right]\,, \end{aligned} \tag{2.103}$$

In order for this relation to be identically true, it is necessary that:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 0\,. \tag{2.104}$$

Notice that for $\xi\to 0$, the equality above reduces to Eq. (2.101). It is interesting to verify Eq. (2.104) plugging in the explicit expressions for $\mathcal{P}_1^{-,(0)}$, Eq. (2.23), and $\mathcal{P}_2^{-,(0)}$, Eq. (2.30). One finds:

$$\int_0^1 dz\,\mathcal{P}_1^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[-\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} - \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.105}$$

that correctly tends to zero as $\xi\to 0$, and:

$$\int_0^{\xi/y} dz\,\mathcal{P}_2^{-,(0)}\left(z,\frac{\xi}{yz}\right) = 2C_F\left[\frac{3}{2}\frac{\xi^2}{\xi^2-y^2} + \ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.106}$$

such that Eq. (2.104) is fulfilled. Despite Eq. (2.104) has been explicitly proved at one-loop, the same relation must hold order by order in perturbation theory.

It is important to notice that the constraint on the non-singlet GPD anomalous dimensions deriving from the valence sum rule, and resulting in Eq. (2.104), does not take the form of a +-prescription, Eq. (2.6). A further proof can be given by considering the non-singlet GPD evolution equation given in Eq. (2.1) (see also Eq. (99) of Ref. [1]). The independence of the form factor from μ immediately leads to:

$$\int_{-1}^1 dx' f(x',\xi)\int_{-1}^1 dx\left[\hat{V}_{\rm NS}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\right]_+ = 0\,, \tag{2.107}$$

where $\hat{V}_{\rm NS}$ is nothing but $V_{\rm NS}$ stripped of the supposedly global +-prescription. Assuming ξ positive and different from zero ($\xi > 0$), a simple change of variables gives:

$$\int_{-1/\xi}^{1/\xi} dy' f(\xi y', \xi) \int_{-1/\xi}^{1/\xi} dy \left[\hat{V}_{\rm NS}(y, y')\right]_+ = 0\,. \tag{2.108}$$

Given the fact that the bounds of the inner integral are not -1 and 1, the effect of the +-prescription as given in Eq. (2.6) cannot give zero. This prevents the above equation to be identically fulfilled violating polynomiality of GPDs. We can thus conclude that $V_{\rm NS}$ *cannot* be written as a fully +-prescribed function.

2.4 The singlet sector

Having ascertained that the evolution equation from Ref. [3] is well-behaved for the non-singlet distribution f^-, we move on to consider the singlet $f_{\rm S}$ and gluon $f_{\rm G}$ distributions. As in the standard DGLAP evolution equation, singlet and gluon GPDs couple under evolution. Defining f^+ as the column vector of singlet and gluon GPDs, the corresponding anomalous dimension $\mathcal{P}^+$ is a matrix in flavour space:

$$\mathcal{P}^+ = \begin{pmatrix} \mathcal{P}_{\rm SS} & \mathcal{P}_{\rm SG} \\ \mathcal{P}_{\rm GS} & \mathcal{P}_{\rm GG} \end{pmatrix}\,. \tag{2.109}$$

Following the same procedure discussed above for the non-singlet distribution f^-, the one-loop evolution equation for f^+ reads:

$$\mu^2 \frac{d}{d\mu^2} f^+(x,\xi) = \frac{\alpha_s(\mu)}{4\pi} \int_x^\infty \frac{dy}{y} \mathcal{P}^{+,(0)}(y,\kappa) f^+\left(\frac{x}{y}, \xi\right)\,, \tag{2.110}$$

with:

$$\mathcal{P}^{+,(0)}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{+,(0)}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{+,(0)}(y,\kappa)\,. \tag{2.111}$$

The single splitting function matrices $\mathcal{P}_1$ and $\mathcal{P}_2$ are derived from the expression Ref. [3] as:

$$\begin{aligned} \mathcal{P}_{1,\rm IJ}(y,\kappa) &= 2P_{\rm IJ}(y, 2\kappa y) = 2P'_{\rm IJ}(y, 2\kappa y) + 2P'_{\rm IJ}(y, -2\kappa y)\,, \\ \mathcal{P}_{2,\rm IJ}(y,\kappa) &= -2P'_{\rm IJ}(y, -2\kappa y) - 2P'_{\rm IJ}(-y, 2\kappa y)\,, \end{aligned} \tag{2.112}$$

with $\rm I, J = S, G$. This leads to:

$$\left\{ \begin{aligned} \mathcal{P}_{1,\rm SS}^{(0)}(y,\kappa) &= \mathcal{P}_1^{-,(0)}(y,\kappa)\,, \\ \mathcal{P}_{2,\rm SS}^{(0)}(y,\kappa) &= 2C_F\left[\frac{1+y+\kappa y+\kappa^3 y^2}{\kappa(1+y)(1-\kappa^2 y^2)} - \left(\frac{1}{1-y}\right)_{++}\right]\,, \end{aligned} \right. \tag{2.113}$$

$$\left\{ \begin{aligned} \mathcal{P}_{1,\rm SG}^{(0)}(y,\kappa) &= 4n_f T_R\left[\frac{y^2+(1-y)^2-\kappa^2 y^2}{(1-\kappa^2 y^2)^2}\right]\,, \\ \mathcal{P}_{2,\rm SG}^{(0)}(y,\kappa) &= 4n_f T_R(1-\kappa)\left[\frac{1-\kappa(\kappa+2)y^2}{\kappa(1-\kappa^2 y^2)^2}\right]\,, \end{aligned} \right. \tag{2.114}$$

$$\left\{ \begin{aligned} \mathcal{P}_{1,\rm GS}^{(0)}(y,\kappa) &= 2C_F\left[\frac{1+(1-y)^2-\kappa^2 y^2}{y(1-\kappa^2 y^2)}\right]\,, \\ \mathcal{P}_{2,\rm GS}^{(0)}(y,\kappa) &= -2C_F\frac{(1-\kappa)^2}{\kappa(1-\kappa^2 y^2)}\,, \end{aligned} \right. \tag{2.115}$$

$$\left\{ \begin{aligned} \mathcal{P}_{1,\rm GG}^{(0)}(y,\kappa) &= 4C_A\left[\left(\frac{1}{1-y}\right)_+ - \frac{1+\kappa^2 y}{1-\kappa^2 y^2} + \frac{1}{(1-\kappa^2 y^2)^2}\left(\frac{1-y}{y} + y(1-y)\right)\right] \\ &+ \delta(1-y)\left[\left(\frac{11C_A - 4n_f T_R}{3}\right) - 2C_A\ln(|1-\kappa^2|)\right]\,, \\ \mathcal{P}_{2,\rm GG}^{(0)}(y,\kappa) &= 2C_A\left[\frac{2(1-\kappa)(1+y^2)}{(1-\kappa^2 y^2)^2} + \frac{\kappa^2(1+y)}{1-\kappa^2 y^2} + \frac{1-\kappa^2}{1-\kappa^2 y^2}\left(2 - \frac{1}{\kappa} - \frac{1}{1+y}\right) - \left(\frac{1}{1-y}\right)_{++}\right]\,. \end{aligned} \right. \tag{2.116}$$

In all cases, the limit for $\kappa \to 0$ of $\mathcal{P}_1$ reproduces the one-loop DGLAP splitting functions. In addition, we also notice that all $\mathcal{P}_2$'s are proportional to $\kappa - 1$. Along with the fact that all $\mathcal{P}_1$ are well-behaved at $\kappa = 1$:

$$\begin{aligned}
\mathcal{P}^{(0)}_{1,\mathrm{SS}}(y,1) &= 2C_F\left\{\left[\frac{1}{1-y}\right]_+ + \delta(1-y)\left[\frac{3}{2}-\ln(2)\right]\right\}\,, \\
\mathcal{P}^{(0)}_{1,\mathrm{SG}}(y,1) &= \frac{4n_f T_R}{(1+y)^2}\,, \\
\mathcal{P}^{(0)}_{1,\mathrm{GS}}(y,1) &= \frac{4C_F}{y(1+y)}\,, \\
\mathcal{P}^{(0)}_{1,\mathrm{GG}}(y,1) &= 4C_A\left[\left(\frac{1}{1-y^2}\right)_+ + \frac{1}{y(1+y)^2}\right] + \delta(1-y)\left(\frac{11C_A - 4n_f T_R}{3}\right)\,,
\end{aligned} \tag{2.117}$$

this guarantees the continuity of GPDs across the point $x = \xi$.

Now, we explicitly verify that the pole at $y = |\kappa|^{-1}$ that affects all splitting functions above cancels between $\mathcal{P}^{(0)}_{1,\mathrm{IJ}}$ and $\mathcal{P}^{(0)}_{2,\mathrm{IJ}}$. We find:

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}^{(0)}_{1,\mathrm{SS}}(y,\kappa) = -\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}^{(0)}_{2,\mathrm{SS}}(y,\kappa) = -2C_F\frac{1+\kappa}{\kappa}\,, \tag{2.118}$$

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)^2\mathcal{P}^{(0)}_{1,\mathrm{SG}}(y,\kappa) = -\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)^2\mathcal{P}^{(0)}_{2,\mathrm{SG}}(y,\kappa) = \frac{8n_fT_R(1-\kappa)}{\kappa}\,, \tag{2.119}$$

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}^{(0)}_{1,\mathrm{GS}}(y,\kappa) = -\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)\mathcal{P}^{(0)}_{2,\mathrm{GS}}(y,\kappa) = 2C_F\frac{(1-\kappa)^2}{\kappa}\,, \tag{2.120}$$

$$\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)^2\mathcal{P}^{(0)}_{1,\mathrm{GG}}(y,\kappa) = -\lim_{y\to\kappa^{-1}}(1-\kappa^2y^2)^2\mathcal{P}^{(0)}_{2,\mathrm{GG}}(y,\kappa) = 4C_A\frac{(\kappa-1)(\kappa^2+1)}{\kappa^2}\,. \tag{2.121}$$

These results confirm the cancellation of the pole at $y = |\kappa|^{-1}$ in the integral in the r.h.s. of Eq. (2.110).

We now compute the ERBL limit by taking $\kappa \to 1/x$. To do so, we use the ERBL-compliant form of the evolution equation:

$$\mu^2\frac{d}{d\mu^2}\Phi^+(t) = \frac{\alpha_s(\mu)}{4\pi}\left[\int_0^1 du\,\overline{V}^{(0)}_{\mathrm{S}}(t,u)\Phi^+(u)\right]\,, \tag{2.122}$$

with $\Phi^+(t) = f^+(x,1)$ and where u and t are defined in Eq. (2.43). $\overline{V}^{(0)}_{\mathrm{S}}$ is a matrix in flavour space with the same structure of $\mathcal{P}^+$ in Eq. (2.109), whose components can be written in terms of the $P'_{\rm IJ}$ functions as follows:

$$\begin{aligned}
V_{\rm SS}^{(0)}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm SS}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right] - C_F\delta\left(u-t\right)\int_0^1 du'\,\frac{\theta(u'-t)}{u'-t}\\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm SS}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right] + C_F\delta\left(t-u\right)\int_0^1 du'\,\frac{\theta(t-u')}{u'-t}\\
&+ \frac{3}{2}C_F\delta\left(u-t\right)\,,\\
V_{\rm SG,GS}^{(0)}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm SG,GS}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right]\\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm SG,GS}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right]\,,\\
V_{\rm GG}^{(0)}(t,u) &= \theta(u-t)\left[\frac{2}{2u-1}P'_{\rm GG}\left(\frac{2t-1}{2u-1},\frac{2}{2u-1}\right)\right] - C_A\delta\left(u-t\right)\int_0^1 du'\,\frac{\theta(u'-t)}{u'-t}\\
&+ \theta(t-u)\left[-\frac{2}{2u-1}P'_{\rm GG}\left(\frac{2t-1}{2u-1},-\frac{2}{2u-1}\right)\right] + C_A\delta\left(t-u\right)\int_0^1 du'\,\frac{\theta(t-u')}{u'-t}\\
&+ \left(\frac{11C_A-4n_fT_R}{6}\right)\delta\left(u-t\right)\,.
\end{aligned} \tag{2.123}$$

The explicit expressions read:

$$\begin{aligned}
V_{\rm SS}^{(0)}(t,u) &= V_{\rm NS}^{(0)}(t,u)\,,\\
V_{\rm SG}^{(0)}(t,u) &= 2n_fT_R\left(\frac{2u-1}{2}\right)\left[\theta(u-t)\frac{t}{u}\left(\frac{2t-1}{u}-2\frac{1-t}{1-u}\right)-\theta(t-u)\left(\frac{1-t}{1-u}\right)\left(\frac{1-2t}{1-u}-2\frac{t}{u}\right)\right]\,,\\
V_{\rm GS}^{(0)}(t,u) &= C_F\left(\frac{2}{2t-1}\right)\left[\theta(u-t)\left(2t-\frac{t^2}{u}\right)-\theta(t-u)\left(2(1-t)-\frac{(1-t)^2}{1-u}\right)\right]\,,\\
V_{\rm GG}^{(0)}(t,u) &= \text{This is a little convoluted... leave it for when I feel like doing the calculation}\,.
\end{aligned} \tag{2.124}$$

2.4.1 Sum rules

As done above for the non-singlet splitting function, here we exploit polynomiality and the GPDs and the momentum sum rule (MSR) of PDFs to check the correctness of the singlet splitting functions. Before considering GPDs, let us derive the constrain that the MSR has on the forward splitting functions. The MSR states that the integral over the longitudinal momentum fractions x of the sum of all PDFs weighted by x represents the momentum fraction of all partons in the hadron and thus is to be equal to one at all scales. The respective formula reads:

$$\int_0^1 dx\,x\left[f_{\rm S}(x,0)+f_{\rm G}(x,0)\right]=1\,. \tag{2.125}$$

We can now that the derivative w.r.t. $\ln\mu^2$ of both side of the the equation above and, using the evolution equation we find:

$$\int_0^1 dy\,y\left[\mathcal{P}_{1,\rm SS}(y,0)+\mathcal{P}_{1,\rm GS}(y,0)\right]\left[\int_0^1 dz\,zf_{\rm S}(z,0)\right]+\int_0^1 dy\,y\left[\mathcal{P}_{1,\rm SG}(y,0)+\mathcal{P}_{1,\rm GG}(y,0)\right]\left[\int_0^1 dz\,zf_{\rm G}(z,0)\right]=0\,. \tag{2.126}$$

In order for this equation to be identically fulfilled for any pair of distributions $f_{\rm S}$ and $f_{\rm G}$, one needs to have:

$$\begin{aligned} \int_0^1 dy\, y\left[\mathcal{P}_{1,\rm SS}(y,0)+\mathcal{P}_{1,\rm GS}(y,0)\right] &= 0\,, \\ \int_0^1 dy\, y\left[\mathcal{P}_{1,\rm SG}(y,0)+\mathcal{P}_{1,\rm GG}(y,0)\right] &= 0\,. \end{aligned} \tag{2.127}$$

Effectively, these equations are fulfilled by the DGLAP splitting functions. We now wish to extend this argument to GPDs to find similar relations for the splitting functions that apply to $\xi \neq 0$. To do so, we use polynomiality of GPDs that for both singlet and gluon reads:

$$\int_0^1 dx\, x f_{\rm S(G)}(x,\xi) = A_{q(g)} + \xi^2 D_{q(g)}\,, \tag{2.128}$$

where the coefficients A and D generally depend on the scale μ. However, it is well known that helicity-conserving (H) and helicity-flip (E) GPDs have the same D-term but with opposite sign such that, if we define $f_{\rm S(G)}$ as the sum of H and E the ξ^2 term cancels and one is left with:

$$\int_0^1 dx\, x f_{\rm S(G)}(x,\xi) = A_{q(g)}\,. \tag{2.129}$$

Therefore the sum of $f_{\rm S}$ and $f_{\rm G}$ gives:

$$\int_0^1 dx\, x\left[f_{\rm S}(x,\xi)+f_{\rm G}(x,\xi)\right] = A\,, \tag{2.130}$$

with $A = A_q + A_g$. Importantly A is an observable and thus independent of the scale μ. In addition, H and E obey the same evolution equations. This allows us to take the derivative with respect to $\ln\mu^2$ of both sides of the equation above and use the evolution equations. Following the same steps as in Sect. 2.3.4 leads to the following equalities:

$$\begin{aligned} \int_0^1 dz\, z\left[\mathcal{P}_{1,\rm SS}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}_{1,\rm GS}\left(z,\frac{\xi}{yz}\right)\right]+\int_0^{\xi/y} dz\, z\left[\mathcal{P}_{2,\rm SS}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}_{2,\rm GS}\left(z,\frac{\xi}{yz}\right)\right] &= 0\,, \\ \int_0^1 dz\, z\left[\mathcal{P}_{1,\rm SG}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}_{1,\rm GG}\left(z,\frac{\xi}{yz}\right)\right]+\int_0^{\xi/y} dz\, z\left[\mathcal{P}_{2,\rm SG}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}_{2,\rm GG}\left(z,\frac{\xi}{yz}\right)\right] &= 0\,, \end{aligned} \tag{2.131}$$

that hold at each single order in α_s and automatically reduce to Eq. (2.127) for $\xi \to 0$. It is now important to verify that the one-loop splitting functions written above obey these relations. We find:

$$\int_0^1 dz\, z\left[\mathcal{P}^{(0)}_{1,\rm SS}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}^{(0)}_{1,\rm GS}\left(z,\frac{\xi}{yz}\right)\right] = 2C_F\left[-\frac{1}{2}\frac{\xi^2}{y^2-\xi^2}-\ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.132}$$

and:

$$\int_0^{\xi/y} dz\, z\left[\mathcal{P}^{(0)}_{2,\rm SS}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}^{(0)}_{2,\rm GS}\left(z,\frac{\xi}{yz}\right)\right] = 2C_F\left[\frac{1}{2}\frac{\xi^2}{y^2-\xi^2}+\ln\left(\left|1-\frac{\xi^2}{y^2}\right|\right)\right]\,, \tag{2.133}$$

that evidently cancel so that the first of Eq. (2.131) is satisfied. In addition, they both tend to zero as $\xi \to 0$ as required by the first of Eq. (2.127). Then we find:

$$\begin{aligned} \int_0^1 dz\, z\left[\mathcal{P}^{(0)}_{1,\rm SG}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}^{(0)}_{1,\rm GG}\left(z,\frac{\xi}{yz}\right)\right] &= \frac{y^2\xi^2}{3(y^2-\xi^2)^2}\left[C_A\left(\frac{11\xi^2}{y^2}-4\right)+2n_fT_R\left(1-\frac{2\xi^2}{y^2}\right)\right] \\ &- 2C_A\ln\left(\left|\frac{y^2-\xi^2}{y^2}\right|\right)\,, \end{aligned} \tag{2.134}$$

and:

$$\begin{aligned} \int_0^{\xi/y} dz\, z\left[\mathcal{P}^{(0)}_{2,\rm SG}\left(z,\frac{\xi}{yz}\right)+\mathcal{P}^{(0)}_{2,\rm GG}\left(z,\frac{\xi}{yz}\right)\right] &= -\frac{y^2\xi^2}{3(y^2-\xi^2)^2}\left[C_A\left(\frac{11\xi^2}{y^2}-4\right)+2n_fT_R\left(1-\frac{2\xi^2}{y^2}\right)\right] \\ &+ 2C_A\ln\left(\left|\frac{y^2-\xi^2}{y^2}\right|\right)\,, \end{aligned} \tag{2.135}$$

so that also the second of Eq. (2.131) is satisfied. Again, their limit for $\xi \to 0$ tends to zero that is consistent with the second of Eq. (2.127).

2.4.2 Conformal moments

In this section, we discuss the diagonalisation of the GPD evolution equation for the single distributions in conformal space. More specifically, we first write the evolution kernels in an ERBL-like form and then compute the conformal moments using the appropriate Gegenbauer polynomials showing that they are eigenfunctions of the kernels.

Similarly to the non-singlet sector, the GPD evolution equation for the vector of singlet distributions reads:

$$\frac{d}{d\ln\mu^2} f^+(x,\xi) = \frac{\alpha_s(\mu)}{4\pi}\int_{-1}^{1}\frac{dy}{\xi}\mathbf{V}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) f^+(y,\xi)\,, \tag{2.136}$$

where $\mathbf{V}^{(0)}$ is a two by two matrix with components:

$$\begin{aligned}\frac{1}{\xi}\mathbf{V}_{\rm IJ}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right) &= \frac{2}{y}\Bigg\{\left[\theta(x-\xi)\theta(y-x)+\theta(\xi-x)\theta(x+\xi)\theta(y-x)-\theta(-x-\xi)\theta(x-y)\right]P'_{\rm IJ}\left(\frac{x}{y},\frac{2\xi}{y}\right)\\ &+ \left[\theta(x-\xi)\theta(y-x)-\theta(\xi-x)\theta(x+\xi)\theta(x-y)-\theta(-x-\xi)\theta(x-y)\right]P'_{\rm IJ}\left(\frac{x}{y},-\frac{2\xi}{y}\right)\\ &+ \delta_{\rm IJ}\delta\left(1-\frac{x}{y}\right)C_{\rm I}\left[K_{\rm I}+\int_{\xi}^{x}\frac{dz}{z-x}+\int_{-\xi}^{x}\frac{dz}{z-x}\right]\Bigg\}\,,\quad {\rm I,J}={\rm S,G}\,,\end{aligned} \tag{2.137}$$

with $C_{\rm S}=C_F$ and $C_{\rm G}=C_A$ and:

$$K_{\rm S}=\frac{3}{2}\quad\text{and}\quad K_{\rm G}=\frac{11C_A-4n_fT_R}{6C_A}\,. \tag{2.138}$$

The conformal moments of the singlet kernels can be found by generalising Eq. (2.52):

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\xi}\mathbf{V}_{\rm IJ}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right)f(x) &= 2C_{\rm I}\Bigg\{\delta_{\rm IJ}K_{\rm I}f(y)+\int_{\xi}^{y}dx\left[\frac{1}{C_{\rm I}y}P'_{\rm IJ}\left(\frac{x}{y},-\frac{2\xi}{y}\right)f(x)-\delta_{\rm IJ}\frac{f(y)}{y-x}\right]\\ &+ \int_{-\xi}^{y}dx\left[\frac{1}{C_{\rm I}y}P'_{\rm IJ}\left(\frac{x}{y},\frac{2\xi}{y}\right)f(x)-\delta_{\rm IJ}\frac{f(y)}{y-x}\right]\Bigg\}\,,\end{aligned} \tag{2.139}$$

and replacing $f(x)$ with the appropriate Gegenbauer polynomial computed in x/ξ. Since $\mathbf{V}_{\rm SS}^{(0)}=\mathbb{V}_{\rm NS}^{(0)}$, the SS component has $C_n^{3/2}(x/\xi)$ as eigenfunctions with eigenvalues $V_{{\rm SS},n}^{(0)}=V_n^{(0)}$ given in Eq. (2.69). The eigenvalues of the remaining kernels are expected to be:

$$\begin{aligned}V_{{\rm SS},n}^{(0)} &= V_n^{(0)}\,,\\ V_{{\rm SG},n}^{(0)} &= P_{\rm SG}^{(0)}(n+1)=2T_R\frac{4+3n+n^2}{(n+1)(n+2)(n+3)}\,,\\ V_{{\rm GS},n}^{(0)} &= P_{\rm GS}^{(0)}(n+1)=2C_F\frac{4+3n+n^2}{n(n+1)(n+2)}\,,\\ V_{{\rm GG},n}^{(0)} &= P_{\rm GG}^{(0)}(n+1)=2C_A\left[K_{\rm G}+\frac{1}{n(n+1)}+\frac{1}{(n+2)(n+3)}-\sum_{k=1}^{n+1}\frac{1}{k}\right]\,,\end{aligned} \tag{2.140}$$

However, we still need to determine the degree of the Gegenbauer polynomials that provide eigenfunction to the kernles. To do so, we replace $f(x)=C_1^\alpha(x)=2\alpha x$ in Eq. (2.139) and determine the value of α by requiring that the integral equals $V_{{\rm SS},1}^{(0)}C_1^\alpha(x)$. We start with the GS kernel:

$$\int_{-1}^{1}\frac{dx}{\xi}\mathbf{V}_{\rm GS}^{(0)}\left(\frac{x}{\xi},\frac{y}{\xi}\right)C_n^\alpha\left(\frac{x}{\xi}\right) = C_F\left[\int_{-1}^{z}\frac{dv}{v}\frac{(v+1)(2z-v+1)}{z+1}C_n^\alpha\left(\frac{x}{\xi}\right)-\int_{1}^{z}\frac{dv}{v}\frac{(v-1)(2z-v-1)}{z-1}C_n^\alpha\left(\frac{x}{\xi}\right)\right]\,, \tag{2.141}$$

with $z = y/\xi$. For $n = 1$ we get:

$$\int_{-1}^{1} \frac{dx}{\xi} \mathbf{V}_{\mathrm{GS}}^{(0)} \left(\frac{x}{\xi}, \frac{y}{\xi}\right) C_1^{\alpha} \left(\frac{x}{\xi}\right) = C_F \frac{16}{3} \alpha z = V_{\mathrm{GS},1}^{(0)} C_1^{\alpha}(z) \quad \forall \alpha \,, \tag{2.142}$$

so we get no information on α from the first moment. For the third moment we find:

$$\int_{-1}^{1} \frac{dx}{\xi} \mathbf{V}_{\mathrm{GS}}^{(0)} \left(\frac{x}{\xi}, \frac{y}{\xi}\right) C_3^{\alpha} \left(\frac{x}{\xi}\right) \quad = \quad \frac{11}{15} C_F \left[-2\alpha(\alpha+1) z \frac{116 - 2\alpha}{33} + \frac{4}{3} \alpha(\alpha+1)(\alpha+2) z^3 \right] \tag{2.143}$$

For the SG component we find:

$$\int_{-1}^{1} \frac{dx}{\xi} \mathbf{V}_{\mathrm{SG}}^{(0)} \left(\frac{x}{\xi}, \frac{y}{\xi}\right) C_n^{3/2} \left(\frac{x}{\xi}\right) = \frac{T_R \xi z^2}{z^2 - 1} \left[\int_{-1}^{z} dv (v - z) \left(1 - \frac{2v}{z+1}\right) C_n^{3/2}(v) - \int_{1}^{z} dv (v - z) \left(1 - \frac{2v}{z-1}\right) C_n^{3/2}(v) \right] , \tag{2.144}$$

with $x = y/\xi$

3 Compton form factor

In this section, we discuss the numerical implementation of the Compton form factors (CFFs) for double deeply-virtual Compton scattering (DDVCS). The CFFs can be regarded as the GPD counterpart of the deep-inelastic scattering (DIS) structure function. The relation is easily established by observing that the diagrams necessary to compute the DIS structure functions at a given order are exactly the same diagrams of DDVCS in the forward limit with a cut on the final state. It is well-known that cut and uncut digrams are related through dispersion relation. Specifically, ($2\pi i$ times) a given cut diagram with an external on-shell momentum p is equal to the discontinuity of the uncut diagram in the forward limit along the branch cut on the real axis determined by $p^2 \geq m^2$, where m is the rest mass of the external particle. For the CCF $\mathcal{H}$, this translates into the following relation with the DIS structure function F_1:([8])

$$\lim_{\xi\to 0} \left[\mathcal{H}(x+i\varepsilon, \xi, t, Q) - \mathcal{H}(x-i\varepsilon, \xi, t, Q)\right] = 4\pi i F_1(x, Q)\,. \tag{3.1}$$

Analyticity of the amplitude implies that $\mathcal{H}(x-i\varepsilon, \xi, t, Q) = \mathcal{H}^*(x+i\varepsilon, \xi, t, Q)$ which finally gives:

$$\frac{1}{2\pi} \lim_{\xi\to 0} \mathrm{Im}\left[\mathcal{H}(x, \xi, t, Q)\right] = F_1(x, Q)\,. \tag{3.2}$$

An analogous relation exists between the CFF $\widetilde{\mathcal{H}}$ and the polarised structure function g_5 as well as between the longitudinal projections $\mathcal{H}_L$ and $\widetilde{\mathcal{H}}_L$, and F_L and g_L, respectively. These relation can be used to check that coefficient functions for the CCFs reproduce the well-know DIS coefficient functions in the forward limit.

Factorisation for DDVCS allows one to write CFFs as convolution between short-distance partonic cross sections (or coefficient functions) and nucleon GPDs. For $\mathcal{H}$ factorisation reads:

$$\mathcal{H}(x, \xi, t, Q) = \sum_q e_q^2 \int_{-1}^{1} \frac{dy}{y} \left[C_q\left(\frac{x}{y}, \frac{\xi}{x}, Q\right) F_q(y, \xi, t, Q) + C_g\left(\frac{x}{y}, \frac{\xi}{x}, Q\right) F_g(y, \xi, t, Q)\right] \tag{3.3}$$

where e_q^2 is the electric charge of the flavour q([9]) with the sum running over the *active* flavours (and not over the anti-flavours). F_q and F_g are the quark q and gluon GPDs, respectively, and C_q and C_g the respective coefficient functions that admit a perturbative expansion in powers of α_s:

$$C_q\left(y, \kappa, Q\right) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n C_q^{(n)}\left(y, \kappa\right) \quad \text{and} \quad C_g\left(y, \kappa, Q\right) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n C_g^{(n)}\left(y, \kappa\right)\,. \tag{3.4}$$

Currently, corrections up to $n = 2$ are known [9, 10, 11, 12]. For implementation purposes, it is convenient to rewrite Eq. (3.3) as follows:

$$\begin{aligned} x\mathcal{H}(x, \xi, t, Q) \;&=\; \sum_q e_q^2 \int_x^{\infty} dy \Bigg[C_q\left(y, \kappa, Q\right) \frac{x}{y} F_q\left(\frac{x}{y}, \xi, t, Q\right) + C_q\left(y, -\kappa, Q\right) \frac{x}{y} F_{\overline{q}}\left(\frac{x}{y}, \xi, t, Q\right) \\ &+\; \left[C_g\left(y, \kappa, Q\right) + C_g\left(y, -\kappa, Q\right)\right] \frac{x}{y} F_g\left(\frac{x}{y}, \xi, t, Q\right)\Bigg]\,, \end{aligned} \tag{3.5}$$

with $\kappa = \xi/x$ and where we have exploited the symmetries $C_{q,g}\left(y, \kappa, Q\right) = C_{q,g}\left(-y, -\kappa, Q\right)$ and $F_g(y, \xi, t, Q) = -F_g(-y, \xi, t, Q)$, and defined $F_{\overline{q}}(y, \xi, t, Q) = -F_q(-y, \xi, t, Q)$.

At first order ($n = 0$), only the quark channel contributes and the respective coefficient function is:

$$C_q^{(0)}\left(y, \kappa\right) = -\frac{1}{1-y+i\varepsilon} - \frac{1}{1+y-i\varepsilon}\,. \tag{3.6}$$

In order to get rid of the $i\varepsilon$ that complicates the implementation, we use the following identity:

$$\frac{1}{X \mp i\varepsilon} = \mathrm{PV}\frac{1}{X} \pm \pi i \delta(X) \tag{3.7}$$

where PV stands for principal value, so that:

$$C_q^{(0)}\left(y, \kappa\right) = \mathrm{PV}\left[-\frac{1}{1-y} - \frac{1}{1+y}\right] + \pi i \left[\delta(1-y) - \delta(1+y)\right]\,. \tag{3.8}$$

[8] An additional factor 2 is included by convention in the definition of F_1.
[9] We are assuming that $Q \ll M_Z$ so that weak corrections to the electroweak coupling can be neglected.

Plugging this into Eq. (3.5), defining $\overline{F}_q^+(y,\xi,t,Q) = y(F_q(y,\xi,t,Q) + F_{\overline{q}}(y,\xi,t,Q))$, and understanding the principal value gives:

$$x\mathcal{H}^{\rm LO}(x,\xi,t,Q) = \sum_q e_q^2 \int_x^\infty dy \left[-\left(\frac{1}{1-y}\right)_{++} - \frac{1}{1+y} + \pi i \delta(1-y) \right] \overline{F}_q^+\left(\frac{x}{y},\xi,t,Q\right)\,, \tag{3.9}$$

where we have also introduced the ++-prescription defined in Eq. (2.33) that makes the integrand finite over the integration range making it numerically computable. It is intersting to observe that applying Eq. (3.2) to the leading-order CFF gives the expected result:

$$\frac{1}{2\pi} \lim_{\xi\to 0} \mathrm{Im}\left[x\mathcal{H}^{\rm LO}(x,\xi,t,Q)\right] = \frac{1}{2}\sum_q e_q^2 x f_q^+(x,Q) = xF_1^{\rm LO}(x,Q)\,, \tag{3.10}$$

where f_q^+ are the forward PDFs. The particular structure of Eq. (3.5) allows one to relate real and imaginary part of the CFF by observing that:

$$\begin{aligned} \mathrm{Im}\left[x\mathcal{H}^{\rm LO}(x,\xi,t,Q)\right] &= \pi \sum_q e_q^2 \overline{F}_q^+(x,\xi,t,Q)\,, \\ \mathrm{Re}\left[x\mathcal{H}^{\rm LO}(x,\xi,t,Q)\right] &= \int_x^\infty dy \left[-\left(\frac{1}{1-y}\right)_{++} - \frac{1}{1+y} \right] \sum_q e_q^2 \overline{F}_q^+\left(\frac{x}{y},\xi,t,Q\right). \end{aligned} \tag{3.11}$$

Therefore, one can plug the first into the second obtaining:

$$\mathrm{Re}\left[x\mathcal{H}^{\rm LO}(x,\xi,t,Q)\right] = \frac{1}{\pi}\int_x^\infty dy \left[-\left(\frac{1}{1-y}\right)_{++} - \frac{1}{1+y} \right] \mathrm{Im}\left[x\mathcal{H}^{\rm LO}\left(\frac{x}{y},\xi,t,Q\right)\right]. \tag{3.12}$$

This relation, often referred to as *dispersion* relation, is strictly valid only at leading order and is a useful tool when extracting GPDs (or CFFs) from experimental data.

We can now move to consider the $\mathcal{O}(\alpha_s)$ corrections to the CCF $\mathcal{H}$. At this order, both the quark and the gluon channels contribute. We take the analytic expressions from Ref. [11]. For the quark channel we use the finite part of Eq. (46) and for the gluon the finite part of Eq. (47). Since we are assuming $x_{\rm B} > 0$ ($= x$ in our notation), we need to take $Q^2 > 0$. In addition, our definition of coefficient functions in Eq. (3.4) is such that the expressions of Ref. [11] have to be identified with $(1/x)C_{q,g}^{(1)}(x_{\rm B}/x, \xi/x_{\rm B})$. This gives:

$$\begin{aligned} C_q^{(1)}(y,\kappa) &= -\frac{C_F}{1-y+i\varepsilon}\Bigg\{ -9 + 3\frac{1-\kappa^2y^2-2y}{1-\kappa^2y^2}\ln\left(-\frac{1-y}{y}-i\varepsilon\right) + \frac{1+y^2-2\kappa^2y^2}{1-\kappa^2y^2}\ln^2\left(-\frac{1-y}{y}-i\varepsilon\right) \\ &+ 6\frac{\kappa(1-\kappa)y^2}{1-\kappa^2y^2}\ln(1-\kappa-i\varepsilon) + \frac{(1-\kappa)(1-y-2\kappa^2y^2)}{\kappa(1-\kappa^2y^2)}\ln^2(1-\kappa-i\varepsilon)\Bigg\} \\ &+ (y\to -y, \kappa\to-\kappa)\,, \\ C_g^{(1)}(y,\kappa) &= T_R\left(-\frac{1}{1-y+i\varepsilon} - \frac{1}{1+y-i\varepsilon}\right) \\ &\times \Bigg\{ 2\ln\left(-\frac{1-y}{y}-i\varepsilon\right) + \frac{1-2y+2y^2-\kappa^2y^2}{1-\kappa^2y^2}\ln^2\left(-\frac{1-y}{y}-i\varepsilon\right) \\ &+ \frac{2(1-\kappa)}{\kappa}\ln(1-\kappa-i\varepsilon) + \frac{(1-\kappa)(1-2\kappa y^2-\kappa^2y^2)}{\kappa(1-\kappa^2y^2)}\ln^2(1-\kappa-i\varepsilon)\Bigg\} \\ &+ (y\to -y, \kappa\to-\kappa)\,. \end{aligned} \tag{3.13}$$

Now, we need to split the expressions above into real and imaginary parts. To do so, we use Eq. (3.7) and:

$$\ln\left(-\frac{1-y}{y}-i\varepsilon\right) = \ln\left(\left|\frac{1-y}{y}\right|\right) - i\pi\theta(1-y)\,. \tag{3.14}$$

The presence of the θ-function in front of $i\pi$ is a consequence of the fact that that term only arises if the argument of the logarithm is negative that only happens if $y < 1$. It follows that:

$$\ln^2\left(-\frac{1-y}{y} - i\varepsilon\right) = \ln^2\left(\left|\frac{1-y}{y}\right|\right) - \pi^2\theta(1-y) - 2\pi i\theta(1-y)\ln\left(\frac{1-y}{y}\right), \tag{3.15}$$

Similarly:

$$\ln(1-\kappa-i\varepsilon) = \ln(|1-\kappa|) - i\pi\theta(\kappa-1)\,. \tag{3.16}$$

Before computing the full general expressions for real and imaginary parts, it is instructive to compute the imaginary part in the forward limit to check that Eq. (3.2) is fulfilled also at $\mathcal{O}(\alpha_s)$. Let us start with the quark coefficient function. The forward limit is obtained by taking $\kappa \to 0$. By doing so, we observe that the terms $(y \to -y, \kappa \to -\kappa)$ in Eq. (3.13) do not contribute to the imaginary part. The reason is that the integral in Eq. (3.5) only covers positive values of y where those terms are regular, so that the $i\varepsilon$ prescription has no effect and no imaginary terms are generated. The limit produces:([10])

$$\begin{aligned}\frac{1}{2\pi}\lim_{\kappa\to 0}\mathrm{Im}\left[C_q^{(1)}(y,\kappa)\right] &= C_F\theta(1-y)\left\{2\frac{\ln(1-y)}{1-y} - \frac{3}{2}\frac{1}{1-y} - (1+y)\ln(1-y) - \frac{1+y^2}{1-y}\ln(y) + 3\right\}\\ &- C_F\delta(1-y)\left\{\frac{9}{2} + \pi^2 + \frac{3}{2}\ln(1-y) - \ln^2(1-y)\right\}.\end{aligned} \tag{3.18}$$

It is important to notice the θ-function in the first term of the r.h.s.. Despite its presence exposes the singularity of $1/(1-y)$, this is exactly what is needed to cancel the singularities generated by $\delta(1-y)\ln(1-y)$ and $\delta(1-y)\ln^2(1-y)$. To see this, we write the logarithms in the second line as:

$$\delta(1-y)\ln(1-y) = -\delta(1-y)\int_0^1\frac{dz}{1-z} \quad \text{and} \quad \delta(1-y)\ln^2(1-y) = -2\delta(1-y)\int_0^1 dz\frac{\ln(1-z)}{1-z}\,, \tag{3.19}$$

that combined with the $1/(1-y)$ and $\ln(1-y)/(1-y)$ terms in the first line gives rise to the +-prescription so that the final result is:

$$\begin{aligned}\frac{1}{2\pi}\lim_{\kappa\to 0}\mathrm{Im}\left[C_q^{(1)}(y,\kappa)\right] &= C_F\theta(1-y)\left[2\left(\frac{\ln(1-y)}{1-y}\right)_+ - \frac{3}{2}\left(\frac{1}{1-y}\right)_+ - (1+y)\ln(1-y)\right.\\ &- \left.\frac{1+y^2}{1-y}\ln(y) + 3 - \delta(1-y)\left(\pi^2 + \frac{9}{2}\right)\right].\end{aligned} \tag{3.20}$$

This result *almost* perfectly agrees with the expected result. The expression for the $\mathcal{O}(\alpha_s)$ correction to the quark channel of F_1 can be read off from Eq. (4.85) of Ref. [6]. First we notice that the expansion parameter in that reference is $\alpha_s/(2\pi)$ while we use $\alpha_s/(4\pi)$. Therefore, Eq. (4.85) has to be multiplied by a factor 2 in the first place to match our notation. In addition, Eq. (4.85) corresponds to F_2. But since $2xF_1 = F_2 - F_L$ and the corresponding coefficient function for F_L is $C_{L,q}^{(1)}(x) = 4C_F x$ (see Eq. (4.86) of Ref. [6]), it is sufficient to subtract this term and finally divide by 2 to obtain the coefficient function for F_1. The net result should be exactly Eq. (4.85) of Ref. [6] with the term $2z$ removed. Our Eq. (3.20) matches up to the π^2 term in the coefficient of $\delta(1-y)$ that multiplies a different factor. This discrepancy needs to be investigated further.

We now consider the gluon coefficient function. In this case, due to the overall factor that is singular in

[10] Despite not contributing to the imaginary part, notice that:

$$\lim_{\kappa\to 0}\frac{\ln(1-k)}{\kappa} = -1 \quad \text{and} \quad \lim_{\kappa\to 0}\frac{\ln^2(1-k)}{\kappa} = 0\,. \tag{3.17}$$

both $y = 1$ and $y = -1$, we also need to consider the $(y \to -y, \kappa \to -\kappa)$ term. This limit gives:

$$
\begin{aligned}
\lim_{\kappa\to 0} \mathrm{Im}\left[C_g^{(1)}(y,\kappa)\right] &= 2\pi T_R \left(\frac{1}{1-y} + \frac{1}{1+y}\right)\left\{1 + (1-2y+2y^2)\ln\left(\frac{1-y}{y}\right)\right\} \\
&+ (y \to -y) \\
&+ \pi T_R \left(\delta(1-y) - \delta(1+y)\right) \\
&\times \left\{2\ln\left(\frac{1-y}{y}\right) + (1-2y+2y^2)\left[\ln^2\left(\frac{1-y}{y}\right) - \pi^2\right]\right\} \\
&+ (y \to -y)\,.
\end{aligned} \tag{3.21}
$$

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