SIDIS cross section in TMD factorisation

Valerio Bertone

November 23, 2020

Contents

1 Structure of the observable

In this document we report the relevant formulas for the computation of semi-inclusive deep-inelastic scattering (SIDIS) multiplicities under the assumption that the (negative) virtuality of the Q^2 of the exchanged vector boson is much smaller than the Z mass. This allows us to neglect weak contributions and write the cross section in TMD factorisation as:

$$\frac{d\sigma}{dxdQdzdq_T} = \frac{4\pi\alpha^2 q_T}{zxQ^3} Y_+ H(Q,\mu) \sum_q e_q^2 \int_0^\infty db\, b J_0\left(bq_T\right) \overline{F}_q(x,b;\mu,\zeta_1)\overline{D}_q(z,b;\mu,\zeta_2)\,, \quad (1)$$

with $\zeta_1\zeta_2 = Q^4$ and:

$$Y_+ = 1 + (1-y)^2 = 1 + \left(1 - \frac{Q^2}{xs}\right)^2\,, \quad (2)$$

where s is the squared center of mass energy. The single TMDs are evolved and matched onto the respective collinear functions as usual:

$$\overline{F}_i(x,b;\mu,\zeta) = xF_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_j \int_x^1 dy\, \mathcal{C}_{ij}(y;\mu_0,\zeta_0) \left[\frac{x}{y} f_j\left(\frac{x}{y},\mu_0\right)\right]\,, \quad (3)$$

and:

$$\overline{D}_i(z,b;\mu,\zeta) = z^3 D_i(z,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_j \int_z^1 dy\, \left[y^2 \mathbb{C}_{ij}(y;\mu_0,\zeta_0)\right] \left[\frac{z}{y} d_j\left(\frac{z}{y},\mu_0\right)\right]\,. \quad (4)$$

More often, the SIDIS cross section is presented as differential w.r.t. the transverse momentum of the outgoing hadron p_{Th} that is connected to the transverse momentum of the vector boson as:

$$p_{Th} = zq_T\,. \quad (5)$$

The resulting cross section reads

$$\frac{d\sigma}{dxdQdzdp_{Th}} = \frac{4\pi\alpha^2 p_{Th}}{xz^3Q^3} Y_+ H(Q,\mu) \sum_q e_q^2 \int_0^\infty db\, b J_0\left(b\frac{p_{Th}}{z}\right) \overline{F}_q(x,b;\mu,\zeta_1)\overline{D}_q(z,b;\mu,\zeta_2)\,, \quad (6)$$

Notice that here we limit to the case $Q \ll M_Z$ such that we can neglect the contribution of the Z boson and thus the electroweak couplings are given by the squared electric charges.

As usual, low-q_T non-perturbative corrections are taken into account by introducing the monotonic function $b_*(b)$ that behaves as:

$$\lim_{b\to 0} b_*(b) = b_{\rm min} \quad \text{and} \quad \lim_{b\to\infty} b_*(b) = b_{\rm max}\,. \tag{7}$$

This allows us to replace the TMDs in Eq. (6) with their "regularised" version:

$$\begin{aligned} \overline{F}_i(x,b;\mu,\zeta) &\rightarrow \overline{F}_i(x,b_*(b);\mu,\zeta) f_{\rm NP}(x,b,\zeta)\,, \\ \overline{D}_i(z,b;\mu,\zeta) &\rightarrow \overline{D}_i(z,b_*(b);\mu,\zeta) D_{\rm NP}(z,b,\zeta)\,, \end{aligned} \tag{8}$$

where we have introduced the non-perturbative functions $f_{\rm NP}$ and $D_{\rm NP}$. It is important to stress that these functions further factorise as follows:

$$\begin{aligned} f_{\rm NP}(x,b,\zeta) &= \widetilde{f}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,, \\ D_{\rm NP}(z,b,\zeta) &= \widetilde{D}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,. \end{aligned} \tag{9}$$

The common exponential function represents the non-perturbative corrections to TMD evolution and the specific functional form is driven by the solution of the Collins-Soper equation where Q_0 is some initial scale. Finally the set of non-perturbative functions to be determined from fits to data are $\widetilde{f}_{\rm NP}$, $\widetilde{D}_{\rm NP}$, and $g_K(b)$. It is worth noticing that by definition

$$f_{\rm NP}(x,b,\zeta) = \frac{\overline{F}_i(x,b;\mu,\zeta)}{\overline{F}_i(x,b_*(b);\mu,\zeta)}\,, \tag{10}$$

and similarly for $D_{\rm NP}$. Therefore, one has a partial handle on the b-dependence of these functions from the region in which b is small enough to make both numerator and denominator perturbatively computable. Making use of Eq. (9) and setting $\zeta_1 = \zeta_2 = Q^2$ allows us to rewrite Eq. (6) as:

$$\begin{aligned} \frac{d\sigma}{dxdQdzdp_{Th}} &= \frac{4\pi\alpha^2 p_{Th}}{xz^3Q^3} Y_+ H(Q,\mu)\sum_q e_q^2 \\ &\times \int_0^\infty db\, J_0\left(b\frac{p_{Th}}{z}\right) b\overline{F}_i(x,b_*(b);\mu,Q^2)\overline{D}_i(z,b_*(b);\mu,Q^2) f_{\rm NP}(x,b,Q^2) D_{\rm NP}(z,b,Q^2)\,. \end{aligned} \tag{11}$$

The integral in the r.h.s. can be numerically computed using the Ogata quadrature of zero-th degree (because J_0 enters the integral):

$$\begin{aligned} \frac{d\sigma}{dxdQdzdp_{Th}} &\simeq \frac{4\pi\alpha^2}{xz^2Q^3} Y_+ H(Q,\mu)\sum_q e_q^2 \\ &\times \sum_{n=1}^N w_n^{(0)} \frac{\xi_n^{(0)} z}{p_{Th}} \overline{F}_i\left(x, b_*\left(\frac{\xi_n^{(0)} z}{p_{Th}}\right);\mu,Q^2\right) \overline{D}_i\left(z, b_*\left(\frac{\xi_n^{(0)} z}{p_{Th}}\right);\mu,Q^2\right) \\ &\times f_{\rm NP}\left(x,\frac{\xi_n^{(0)} z}{p_{Th}},Q^2\right) D_{\rm NP}\left(z,\frac{\xi_n^{(0)} z}{p_{Th}},Q^2\right)\,, \end{aligned} \tag{12}$$

where $w_n^{(0)}$ and $\xi_n^{(0)}$ are the Ogata weights and coordinates, respectively, and the sum over n is truncated to the N-th term that should be chosen in such a way to guarantee a given target accuracy. The equation above can be conveniently recasted as follows:

$$\frac{d\sigma}{dxdQdzdp_{Th}} \simeq \sum_{n=1}^N w_n^{(0)} \frac{\xi_n^{(0)} z}{p_{Th}} S\left(x,z,\frac{\xi_n^{(0)} z}{p_{Th}};\mu,Q^2\right) f_{\rm NP}\left(x,\frac{\xi_n^{(0)} z}{p_{Th}},Q^2\right) D_{\rm NP}\left(z,\frac{\xi_n^{(0)} z}{p_{Th}},Q^2\right)\,, \tag{13}$$

where:

$$S\left(x,z,b;\mu,Q^2\right) \simeq \frac{4\pi\alpha^2}{xz^2Q^3} Y_+ H(Q,\mu) \sum_q e_q^2 \left[\overline{F}_i\left(x,b_*(b);\mu,Q^2\right)\right]\left[\overline{D}_i\left(z,b_*(b);\mu,Q^2\right)\right]\,. \tag{14}$$

Since it is often the case that data is presented as cross sections within finite p_{Th} bins, one can exploit a property of the Bessel functions to compute the indefinite integral in p_{Th} of the cross section in Eq. (11). Specifically, we now compute:

$$K(x,z,Q,p_{Th}) = \int dp_{Th}\left[\frac{d\sigma}{dxdQdzdp_{Th}}\right]. \tag{15}$$

This is easily done by using the following property of the Bessel functions:

$$\int dx\, x J_0(x) = x J_1(x)\,, \tag{16}$$

that is equivalent to:

$$\int dp_{Th}\, p_{Th} J_0\left(b\frac{p_{Th}}{z}\right) = \frac{z}{b} p_{Th} J_1\left(b\frac{p_{Th}}{z}\right)\,. \tag{17}$$

Therefore:

$$\begin{aligned} K(x,z,Q,p_{Th}) \quad &= \quad \frac{4\pi\alpha^2 p_{Th}}{xz^2Q^3} Y_+ H(Q,\mu)\sum_q e_q^2 \\ &\times \quad \int_0^\infty db\, J_1\left(b\frac{p_{Th}}{z}\right) \overline{F}_i(x,b_*(b);\mu,Q^2)\overline{D}_i(z,b_*(b);\mu,Q^2) f_{\rm NP}(x,b,Q^2) D_{\rm NP}(z,b,Q^2)\,. \end{aligned} \tag{18}$$

The integral can again be computed using the Ogata quadrature as:

$$K(x,z,Q,p_{Th}) \simeq z\sum_{n=1}^N w_n^{(1)}\, S\left(x,z,\frac{\xi_n^{(1)}z}{p_{Th}};\mu,Q^2\right) f_{\rm NP}\left(x,\frac{\xi_n^{(1)}z}{p_{Th}},Q^2\right) D_{\rm NP}\left(z,\frac{\xi_n^{(1)}z}{p_{Th}},Q^2\right)\,, \tag{19}$$

with S given in Eq. (14). Once K is known, the integral of the cross section over the bin $p_{Th} \in [p_{Th,\min} : p_{Th,\max}]$ is given by:

$$\int_{p_{Th,\min}}^{p_{Th,\max}} dp_{Th}\left[\frac{d\sigma}{dxdQdzdp_{Th}}\right] = K(x,z,Q,p_{Th,\max}) - K(x,z,Q,p_{Th,\min})\,. \tag{20}$$

This allows one to compute analytically one of the integrals that are often required to compare predictions to data.

2 Integrating over the final-state kinematic variables

Experimental measurements of differential distributions are often delivered as integrated over finite regions of the final-state kinematic phase space. In other words, experiments measure quantities like:

$$\widetilde{\sigma} = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{z_{\min}}^{z_{\max}} dz \int_{x_{\min}}^{x_{\max}} dx \int_{p_{Th,\min}}^{p_{Th,\max}} dp_{Th}\left[\frac{d\sigma}{dxdQdzdp_{Th}}\right]. \tag{21}$$

We have already seen abobe how the integral over p_{Th} can be computed analytically. We now move to considering the integral of the cross section over x, z, and Q. Since integrals usually comes over x, z, and Q together with an integration in p_{Th}, we consider the primitive function K in Eq. (19) rather than the cross section itself.

2.1 Integrating over x, z, and Q

The amount of numerical computation required to carry out the integration of a single bin is very large. Indicatively, it amounts to computing a three-dimensional integral for each of the terms of the Ogata quadrature that usually range from a few tens to hundreds. Therefore, in order to be able to do the integrations in a reasonable amount of time and yet obtain accurate results, it is necessary to put in place an efficient integration strategy. This goal can be achieved by exploiting the numerical integration based on interpolation techniques as implemented in APFEL++. To this purpose, we first define one grid in x, $\{x_\alpha\}$ with $\alpha = 0, \dots, N_x$, one grid in z, $\{z_\beta\}$ with $\beta = 0, \dots, N_z$, and one grid in Q, $\{Q_\tau\}$ with $\tau = 0, \dots, N_Q$, each of which with a set of interpolating functions $\mathcal{I}$ associated. The grids should be such to cover the kinematics of given data set. Then the value of K in Eq. (19) for any kinematics can be obtained through interpolation as:

$$
\begin{aligned}
K(x, z, Q, p_{Th}) \quad &\simeq \quad \sum_{n=1}^{N}\sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} w_n^{(1)} z_\beta S\left(x_\alpha, z_\beta, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}; \mu, Q_\tau^2\right) \mathcal{I}_\alpha(x)\mathcal{I}_\beta(z)\mathcal{I}_\tau(Q) \\
&\times \quad f_{\mathrm{NP}}\left(x_\alpha, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}, Q_\tau^2\right) D_{\mathrm{NP}}\left(z_\beta, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}, Q_\tau^2\right) .
\end{aligned}
\tag{22}
$$

Once we have K in this form, integrating over x, z, and Q amounts to integrating the interpolating functions $\mathcal{I}$ over the relevant intervals that can be done analytically within APFEL. Of course, this also requires tabulating S but this can be done once and for all. The final result is:

$$
\begin{aligned}
\widetilde{K}(p_{Th}) \quad &= \quad \int_{x_{\min}}^{x_{\max}} dx \int_{z_{\min}}^{z_{\max}} dz \int_{Q_{\min}}^{Q_{\max}} dQ\, K(x, z, Q, p_{Th}) \\
&\simeq \quad \sum_{n=1}^{N}\sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} W_{n\alpha\beta\tau}(p_{Th}) f_{\mathrm{NP}}\left(x_\alpha, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}, Q_\tau^2\right) D_{\mathrm{NP}}\left(z_\beta, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}, Q_\tau^2\right) ,
\end{aligned}
\tag{23}
$$

with:

$$
W_{n\alpha\beta\tau}(p_{Th}) = w_n^{(1)} z_\beta S\left(x_\alpha, z_\beta, \frac{\xi_n^{(1)} z_\beta}{p_{Th}}; \mu, Q_\tau^2\right) \int_{x_{\min}}^{x_{\max}} dx\, \mathcal{I}_\alpha(x) \int_{z_{\min}}^{z_{\max}} dz\, \mathcal{I}_\beta(z) \int_{Q_{\min}}^{Q_{\max}} dQ\, \mathcal{I}_\tau(Q) . \tag{24}
$$

Finally, the integrated cross section in Eq. (21) is computed as:

$$
\widetilde{\sigma} = \widetilde{K}(p_{Th,\max}) - \widetilde{K}(p_{Th,\min}) . \tag{25}
$$

Since the aim is to fit the functions f_{NP} and D_{NP} to data, one can precompute and store the coefficients W defined in Eq. (24) and compute the cross sections in a fast way making use of Eq. (23).

References