Contents

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

1 Evolution equation

In general, the evolution equation for GPDs reads:

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-\infty}^{+\infty}\frac{dx'}{|2\xi|}P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{1.1}$$

The GPD f and the evolution kernel P may in general be a vector and a matrix in flavour space. For now we will just be concerned with the integral in the r.h.s. of Eq. (1.1) regardless of the flavour structure. The support of the evolution kernel $P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 1.1. Knowing the support of the evolution kernel, Eq. (1.1)

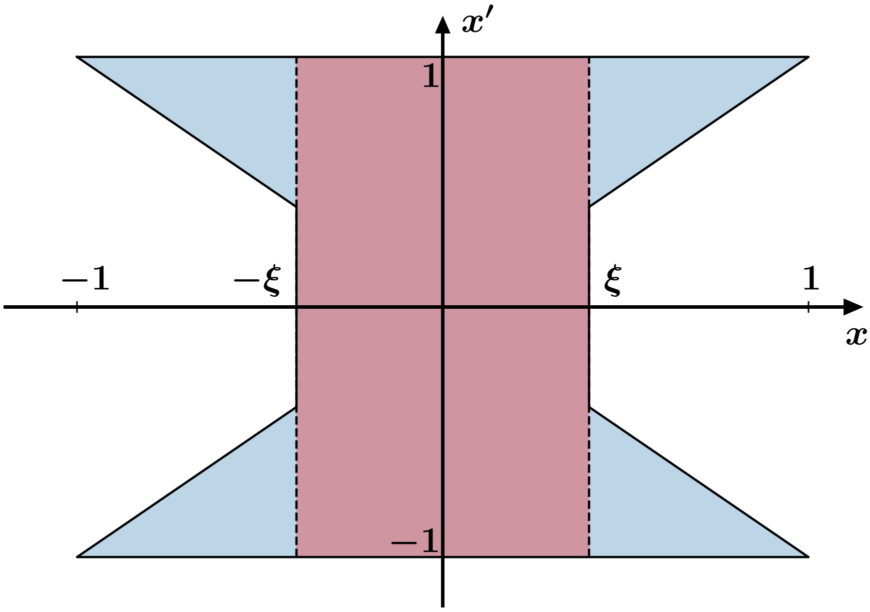

Fig. 1.1: Support domain of the evolution kernel $P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

can be split as follows:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}f(\pm x,\xi) &= \theta\left(\left|\frac{x}{\xi}\right|-1\right)\int_{|x|}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}P\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}P\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\\ &+ \theta\left(1-\left|\frac{x}{\xi}\right|\right)\int_{0}^{1}dx'\left[\frac{1}{|2\xi|}P\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{1}{|2\xi|}P\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right].\end{aligned} \tag{1.2}$$

where we have used the symmetry property of the evolution kernels $P(y,y')=P(-y,-y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned}f^{\pm}(x,\xi) &= f(x,\xi)\mp f(-x,\xi)\,,\\ P^{\pm}(y,y') &= P(y,y')\mp P(-y,y')\,,\end{aligned} \tag{1.3}$$

[1] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x)=-\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign.

so that the evolution equation for $f^{\pm}$ can be split as:

$$\begin{aligned}
\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) &= \theta\left(\left|\frac{x}{\xi}\right|-1\right)\int_{|x|}^{1}\frac{dx'}{x'}\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)\\
&+ \theta\left(1-\left|\frac{x}{\xi}\right|\right)\int_{0}^{1}\frac{dx'}{x'}\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)\\
&= I^{\pm,\mathrm{DGLAP}}(\xi,x)+I^{\pm,\mathrm{ERBL}}(\xi,x)\,.
\end{aligned} \tag{1.4}$$

The first term in the third line of the equation above, $I^{\pm,\mathrm{DGLAP}}$, corresponds to integrating over the blue regions in Fig. 1.1, while the second term, $I^{\pm,\mathrm{ERBL}}$, results from the integration over the red region. As indicated by the subscripts, $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ define the so-called DGLAP and ERBL regions in x relative ξ. Specifically, the presence of the θ-functions is such that for $x>\xi$ $I^{\pm,\mathrm{ERBL}}$ drops leaving only the DGLAP-like term $I^{\pm,\mathrm{DGLAP}}$. While for $x\leq\xi$ $I^{\pm,\mathrm{ERBL}}$ kicks in and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). Crucially, in the limits $\xi\to 0$ and $\xi\to\pm 1$ one should and does recover the DGLAP and ERBL equations, respectively.

In the DGLAP region, for convenience, we define the parameter:

$$\kappa(x)=\frac{\xi}{x}\,, \tag{1.5}$$

so that:

$$\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)=\frac{1}{2\kappa}\frac{x'}{x}P^{\pm}\left(\frac{1}{\kappa},\frac{1}{\kappa}\frac{x'}{x}\right)\equiv\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)\,, \tag{1.6}$$

where the last equality effectively defines the function:

$$\mathcal{P}^{\pm}(\kappa,y)=\frac{1}{2\kappa y}P^{\pm}\left(\frac{1}{\kappa},\frac{1}{\kappa y}\right)\,. \tag{1.7}$$

Plugging this definition into the first integral in the r.h.s. of Eq. (1.4) gives:

$$\begin{aligned}
I^{\pm,\mathrm{DGLAP}}(\xi,x) &= \theta\left(\frac{1}{|\kappa|}-1\right)\int_{|x|}^{1}\frac{dx'}{x'}\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)\\
&= \theta\left(\frac{1}{|\kappa|}-1\right)\int_{|x|}^{1}\frac{dx'}{x'}\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)f^{\pm}(x',\xi)\\
&\equiv \theta\left(\frac{1}{|\kappa|}-1\right)\mathcal{P}^{\pm}\left(\kappa,x\right)\otimes f^{\pm}(x,\xi)\,.
\end{aligned} \tag{1.8}$$

Therefore, $I^{\pm,\mathrm{DGLAP}}$ has the form of a “standard” Mellin convolution that, up to minor modifications due to the fact that κ depends on the physical x, is easily handled by APFEL. Assuming a grid in x indexed by α or β, we have:

$$x_\beta I^{\pm,\mathrm{DGLAP}}(\xi,x_\beta)=\theta\left(\frac{1}{|\kappa_\beta|}-1\right)\sum_{\alpha}\mathcal{P}_{\beta\alpha}^{\pm}(\xi)f_{\alpha}^{\pm}(\xi)\,, \tag{1.9}$$

with:

$$f_{\alpha}^{\pm}(\xi)=x_\alpha f^{\pm}(x_\alpha,\xi)\,, \tag{1.10}$$

and:

$$\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{DGLAP}}(\xi)=\theta\left(\frac{1}{|\kappa_\beta|}-1\right)\int_{c}^{d}dx'\,\mathcal{P}^{\pm}\left(\kappa_\beta,x'\right)w_{\alpha}^{(k)}\left(\frac{x_\beta}{x'}\right)\,, \tag{1.11}$$

where $\kappa_\beta=\kappa(x_\beta)=\xi/x_\beta$ and $\{w_{\alpha}^{(k)}\}$ is a set of Lagrange interpolating functions of degree k and the integration bounds are:

$$c=\max(x_\beta,x_\beta/x_{\alpha+1})\quad\text{and}\quad c=\min(1,x_\beta/x_{\alpha-k})\,. \tag{1.12}$$

Now we need to treat $I^{\pm,\mathrm{ERBL}}$ in Eq. (1.4), that we write:

$$x_\beta I^{\pm,\mathrm{ERBL}}(\xi,x_\beta)=\sum_{\alpha}\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi)f_{\alpha}^{\pm}(\xi)\,, \tag{1.13}$$

with:

$$\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi) = \theta\left(1-\frac{1}{|\kappa_\beta|}\right)\frac{x_\beta}{x_\alpha}\int_0^1 dx'\,\mathcal{P}^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\frac{w_\alpha^{(k)}(x')}{x'}\,. \tag{1.14}$$

It is necessary to give a proper treatment of the integral in Eq. (1.14). The kernel $\mathcal{P}^{\pm}$ is generally a distribution having the following structure:

$$\mathcal{P}^{\pm}(\kappa,y) = R^{\pm}(\kappa,y) + \left[S^{\pm}(\kappa,y)\right]_+ + L^{\pm}(\kappa)\delta(1-y)\,. \tag{1.15}$$

Plugging Eq. (1.15) into Eq. (1.14), one finds:

$$\begin{aligned}\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi) &= \theta\left(1-\frac{1}{|\kappa_\beta|}\right)\frac{x_\beta}{x_\alpha}\left\{\int_0^1 dx'\left[R^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\frac{w_\alpha^{(k)}(x')}{x'}\right.\right.\\ &+ \left.\left. S^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\left(\frac{w_\alpha^{(k)}(x')}{x'}-\frac{\delta_{\alpha\beta}}{x_\beta}\right)\right] + L^{\pm}(\kappa_\beta)\delta_{\alpha\beta}\right\}.\end{aligned} \tag{1.16}$$

where we have used the following property of the interpolating functions: $w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}$. In addition, since the interpolating functions are such that:

$$w_\alpha^{(k)}(x) \neq 0 \quad \text{for} \quad x_{\alpha-k} < x < x_{\alpha+1}\,, \tag{1.17}$$

the integral can be computed more efficiently as:

$$\begin{aligned}\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi) &= \theta\left(1-\frac{1}{|\kappa_\beta|}\right)\frac{x_\beta}{x_\alpha}\left\{\int_a^b dx'\left[R^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\frac{w_\alpha^{(k)}(x')}{x'}\right.\right.\\ &+ \left.\left. S^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\left(\frac{w_\alpha^{(k)}(x')}{x'}-\frac{\delta_{\alpha\beta}}{x_\beta}\right)\right] + L^{\pm}(\kappa_\beta)\delta_{\alpha\beta}\right\},\end{aligned} \tag{1.18}$$

with:

$$a = \max(0, x_{\alpha-k}) \quad \text{and} \quad b = \min(1, x_{\alpha+1})\,. \tag{1.19}$$

Finally, summing $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ and multiplying by a factor x_β, the evolution equation in Eq. (1.1) can be approximated on an grid in x as:

$$\mu^2\frac{d}{d\mu^2}f_\beta^{\pm}(\xi) = \sum_\alpha\left[\mathcal{P}_{\beta\alpha}^{\pm,\mathrm{DGLAP}}(\xi) + \mathcal{P}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi)\right]f_\alpha^{\pm}(\xi) \tag{1.20}$$

This is a system of coupled differential equation that can be solved numerically using the fourth-order Runge-Kutta algorithm as implemented in APFEL.

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