A Common basis for the coupled QED×QCD evolution

Abstract

In this document I will present a suitable flavour basis for the coupled QCD×QED DGLAP evolution of PDFs.

Contents

1 The structure of the DGLAP equation

The DGLAP equation that governs PDF evolution has a general structure that in QCD holds at any perturbative order. Suppose one wants to study the coupled evolution of the gluon distribution function $g(x,\mu)$, the i-th quark distribution function $q_i(x,\mu)$ and the j-th anti-quark distribution function $\overline{q}_j(x,\mu)$. In this case evolution equation would look like this:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} = \sum_{k,l}\begin{pmatrix} P_{q_iq_k} & P_{q_ig} & P_{q_i\overline{q}_l} \\ P_{gq_k} & P_{gg} & P_{g\overline{q}_l} \\ P_{\overline{q}_jq_k} & P_{\overline{q}_jg} & P_{\overline{q}_j\overline{q}_l} \end{pmatrix}\begin{pmatrix} q_k \\ g \\ \overline{q}_l \end{pmatrix}, \tag{1.1}$$

where we are understanding the convolution and where the sum over k and l runs over all n_f the active flavours. Because of charge conjugation invariance and $SU(n_f)$ flavour symmetry, one can show that:

$$\begin{aligned} &P_{q_iq_j} = P_{\overline{q}_i\overline{q}_j} = \delta_{ij}P^V_{qq} + P^S_{qq} \\ &P_{\overline{q}_iq_j} = P_{q_i\overline{q}_j} = \delta_{ij}P^V_{\overline{q}q} + P^S_{\overline{q}q} \\ &P_{q_ig} = P_{\overline{q}_ig} = P_{qg} \\ &P_{gq_i} = P_{g\overline{q}_i} = P_{gq}\,. \end{aligned} \tag{1.2}$$

Plugging Eq. (1.2) into Eq. (1.1), one finds:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} = \begin{pmatrix} P^V_{qq} & P_{qg} & P^V_{q\overline{q}} \\ P_{gq} & P_{gg} & P_{gq} \\ P^V_{q\overline{q}} & P_{qg} & P^V_{qq} \end{pmatrix}\begin{pmatrix} q_i \\ g \\ \overline{q}_j \end{pmatrix} + \begin{pmatrix} P^S_{qq} & 0 & P^S_{q\overline{q}} \\ 0 & 0 & 0 \\ P^S_{q\overline{q}} & 0 & P^S_{qq} \end{pmatrix}\begin{pmatrix} \sum_k q_k \\ g \\ \sum_l \overline{q}_l \end{pmatrix}. \tag{1.3}$$

Setting $i=j$ and summing and subtracting the first and the third row/column, we find:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} q_i^+ \\ g \\ q_i^- \end{pmatrix} = \begin{pmatrix} (P^V_{qq}+P^V_{q\overline{q}}) & 2P_{qg} & 0 \\ P_{gq} & P_{gg} & 0 \\ 0 & 0 & (P^V_{qq}-P^V_{q\overline{q}}) \end{pmatrix}\begin{pmatrix} q_i^+ \\ g \\ q_i^- \end{pmatrix} + \begin{pmatrix} (P^S_{qq}+P^S_{q\overline{q}}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (P^S_{qq}-P^S_{q\overline{q}}) \end{pmatrix}\begin{pmatrix} \sum_k q_k^+ \\ g \\ \sum_k q_k^- \end{pmatrix}, \tag{1.4}$$

where we have defined:

$$q_i^{\pm} \equiv q_i \pm \overline{q}_i \,. \tag{1.5}$$

It is evident that in this way we have semi-diagonalised the initial system because now the third equation is decoupled from the rest of the system. Using the following definitions:

$$\begin{aligned} &\Sigma \equiv \sum_k q_k^+ \\ &V \equiv \sum_k q_k^- \\ &P^{\pm} \equiv P_{qq}^V \pm P_{q\overline{q}}^V \\ &P_{qq} \equiv P^+ + n_f(P_{qq}^S + P_{q\overline{q}}^S) \\ &P^V \equiv P^- + n_f(P_{qq}^S - P_{q\overline{q}}^S) \end{aligned} \quad , \tag{1.6}$$

we have:

$$\begin{cases} \mu^2 \dfrac{\partial}{\partial \mu^2} g = P_{gg} g + P_{gq} \Sigma \\ \mu^2 \dfrac{\partial}{\partial \mu^2} q_i^+ = P^+ q_i^+ + \dfrac{1}{n_f}(P_{qq} - P^+)\Sigma + 2P_{qg} g \\ \mu^2 \dfrac{\partial}{\partial \mu^2} q_i^- = P^- q_i^- + \dfrac{1}{n_f}(P^V - P^-)V \end{cases} . \tag{1.7}$$

At this point we want to generalise this discussion including QED corrections. There are two main differences. The first is obviously the fact that we need to introduce in the DGLAP equation the parton distribution associated to the photon $\gamma(x,\mu)$. The second difference is the fact that the all-order splitting functions no longer undergo the stringent simplications of Eq. (1.2). In fact, the QED corrections introduce an asymmetry between down-like quarks (d, s and b) and up-like quarks (u, c, t), due essentially to the different electric charge, that breaks flavour symmetry for the quark splitting functions. We then must consider the following extended evolution system:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_k \\ \overline{u}_h \end{pmatrix} = \sum_{e,l,m,n} \begin{pmatrix} \mathcal{P}_{u_j u_e} & \mathcal{P}_{u_j d_l} & \mathcal{P}_{u_j g} & \mathcal{P}_{u_j \gamma} & \mathcal{P}_{u_j \overline{d}_m} & \mathcal{P}_{u_j \overline{u}_n} \\ \mathcal{P}_{d_i u_e} & \mathcal{P}_{d_i d_l} & \mathcal{P}_{d_i g} & \mathcal{P}_{d_i \gamma} & \mathcal{P}_{d_i \overline{d}_m} & \mathcal{P}_{d_i \overline{u}_n} \\ \mathcal{P}_{g u_e} & \mathcal{P}_{g d_l} & \mathcal{P}_{gg} & \mathcal{P}_{g\gamma} & \mathcal{P}_{g \overline{d}_m} & \mathcal{P}_{g \overline{u}_n} \\ \mathcal{P}_{\gamma u_e} & \mathcal{P}_{\gamma d_l} & \mathcal{P}_{\gamma g} & \mathcal{P}_{\gamma\gamma} & \mathcal{P}_{\gamma \overline{d}_m} & \mathcal{P}_{\gamma \overline{u}_n} \\ \mathcal{P}_{\overline{d}_k u_e} & \mathcal{P}_{\overline{d}_k d_l} & \mathcal{P}_{\overline{d}_k g} & \mathcal{P}_{\overline{d}_k \gamma} & \mathcal{P}_{\overline{d}_k \overline{d}_m} & \mathcal{P}_{\overline{d}_k \overline{u}_n} \\ \mathcal{P}_{\overline{u}_h u_e} & \mathcal{P}_{\overline{u}_h d_l} & \mathcal{P}_{\overline{u}_h g} & \mathcal{P}_{\overline{u}_h \gamma} & \mathcal{P}_{\overline{u}_h \overline{d}_m} & \mathcal{P}_{\overline{u}_h \overline{u}_n} \end{pmatrix} \begin{pmatrix} u_e \\ d_l \\ g \\ \gamma \\ \overline{d}_m \\ \overline{u}_n \end{pmatrix} \tag{1.8}$$

where:

$$u_i = \{u,c,t\}\,, \qquad d_i = \{d,s,b\}\,, \qquad \overline{u}_i = \{\overline{u},\overline{c},\overline{t}\}\,, \qquad \overline{d}_i = \{\overline{d},\overline{s},\overline{b}\}\,. \tag{1.9}$$

Each splitting function in Eq. (1.8) can be split into two pieces:

$$\mathcal{P}_{ab} = P_{ab} + \widetilde{P}_{ab}\,, \tag{1.10}$$

where P_{ab} is the usual QCD splitting function which does not contain any powers of the fine structure constant α and therefore undergoes to the same simplifications discussed above. As a further consequence if a or b is equal to γ, P_{ab} must vanish. $\widetilde{P}_{ab}$ instead contains at least one power of α. In this way we can rearrange Eq. (1.8) as follows:

$$
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_k \\ \overline{u}_h \end{pmatrix} = \sum_{e,l,m,n} \left[\begin{pmatrix} \delta_{je} P_{qq}^V + P_{qq}^S & P_{qq}^S & P_{qg} & 0 & P_{q\overline{q}}^S & \delta_{jn} P_{q\overline{q}}^V + P_{q\overline{q}}^S \\ P_{qq}^S & \delta_{il} P_{qq}^V + P_{qq}^S & P_{qg} & 0 & \delta_{im} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S \\ P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ P_{q\overline{q}}^S & \delta_{kl} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{qg} & 0 & \delta_{km} P_{qq}^V + P_{qq}^S & P_{qq}^S \\ \delta_{he} P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S & P_{qg} & 0 & P_{qq}^S & \delta_{hn} P_{qq}^V + P_{qq}^S \end{pmatrix} \right.
$$

$$
\left. + \begin{pmatrix} \widetilde{P}_{u_j u_e} & \widetilde{P}_{u_j d_l} & \widetilde{P}_{u_j g} & \widetilde{P}_{u_j \gamma} & \widetilde{P}_{u_j \overline{d}_m} & \widetilde{P}_{u_j \overline{u}_n} \\ \widetilde{P}_{d_i u_e} & \widetilde{P}_{d_i d_l} & \widetilde{P}_{d_i g} & \widetilde{P}_{d_i \gamma} & \widetilde{P}_{d_i \overline{d}_m} & \widetilde{P}_{d_i \overline{u}_n} \\ \widetilde{P}_{g u_e} & \widetilde{P}_{g d_l} & \widetilde{P}_{gg} & \widetilde{P}_{g\gamma} & \widetilde{P}_{g \overline{d}_m} & \widetilde{P}_{g \overline{u}_n} \\ \widetilde{P}_{\gamma u_e} & \widetilde{P}_{\gamma d_l} & \widetilde{P}_{\gamma g} & \widetilde{P}_{\gamma\gamma} & \widetilde{P}_{\gamma \overline{d}_m} & \widetilde{P}_{\gamma \overline{u}_n} \\ \widetilde{P}_{\overline{d}_k u_e} & \widetilde{P}_{\overline{d}_k d_l} & \widetilde{P}_{\overline{d}_k g} & \widetilde{P}_{\overline{d}_k \gamma} & \widetilde{P}_{\overline{d}_k \overline{d}_m} & \widetilde{P}_{\overline{d}_k \overline{u}_n} \\ \widetilde{P}_{\overline{u}_h u_e} & \widetilde{P}_{\overline{u}_h d_l} & \widetilde{P}_{\overline{u}_h g} & \widetilde{P}_{\overline{u}_h \gamma} & \widetilde{P}_{\overline{u}_h \overline{d}_m} & \widetilde{P}_{\overline{u}_h \overline{u}_n} \end{pmatrix} \right] \begin{pmatrix} u_e \\ d_l \\ g \\ \gamma \\ \overline{d}_m \\ \overline{u}_n \end{pmatrix} \tag{1.11}
$$

Since α comes always with an electric charge associated, every $\widetilde{P}_{ab}$ can factor out at least an electric charge $e_u^2 = 4/9$ or $e_d^2 = 1/9$. In order to see how this factorisation takes place, we should analyse one by one the splitting functions $\widetilde{P}_{ab}$. Defining:

$$
e_\Sigma^2 = N_c(e_u^2 n_u + e_d^2 n_d)\,, \tag{1.12}
$$

where n_u and n_d are respectively the number of up- and down-type active quarks such that $n_u + n_d = n_f$ and $N_c = 3$ is the number of colours, we have that:

$$
\begin{aligned} &\widetilde{P}_{gg} \to e_\Sigma^2 \widetilde{P}_{gg}\,, \quad \widetilde{P}_{g\gamma} \to e_\Sigma^2 \widetilde{P}_{g\gamma}\,, \\ &\widetilde{P}_{\gamma g} \to e_\Sigma^2 \widetilde{P}_{\gamma g}\,, \quad \widetilde{P}_{\gamma\gamma} \to e_\Sigma^2 \widetilde{P}_{\gamma\gamma}\,. \end{aligned} \tag{1.13}
$$

This is the consequence of the fact that, having only bosons as external particles, the presence of any fermion in the splitting must be summed over all flavours. This is (should be) true at any perturbative order.

Now we consider the splitting functions involving one boson and one quark. Here the situation is more involved because at higher orders it may happen that the incoming/outgoing quark never couples to a photon and thus, given that there is at least one power of α, apart from a term proportional to the charge of the incoming/outgoing quark, there must also be a term proportional to the charge e_Σ^2. However, such contributions only appear at three loops (NNLO) and since here we are only intersted in the two-loop splitting functions, we have:

$$
\begin{aligned} &\widetilde{P}_{gu_i} = \widetilde{P}_{g\overline{u}_i} = e_u^2 \widetilde{P}_{gq}\,, \quad \widetilde{P}_{gd_i} = \widetilde{P}_{g\overline{d}_i} = e_d^2 \widetilde{P}_{gq}\,, \\ &\widetilde{P}_{u_i g} = \widetilde{P}_{\overline{u}_i g} = e_u^2 \widetilde{P}_{qg}\,, \quad \widetilde{P}_{d_i g} = \widetilde{P}_{\overline{d}_i g} = e_d^2 \widetilde{P}_{qg}\,, \\ &\widetilde{P}_{\gamma u_i} = \widetilde{P}_{\gamma\overline{u}_i} = e_u^2 \widetilde{P}_{\gamma q}\,, \quad \widetilde{P}_{\gamma d_i} = \widetilde{P}_{\gamma\overline{d}_i} = e_d^2 \widetilde{P}_{\gamma q}\,, \\ &\widetilde{P}_{u_i \gamma} = \widetilde{P}_{\overline{u}_i \gamma} = e_u^2 \widetilde{P}_{q\gamma}\,, \quad \widetilde{P}_{d_i \gamma} = \widetilde{P}_{\overline{d}_i \gamma} = e_d^2 \widetilde{P}_{q\gamma}\,. \end{aligned} \tag{1.14}
$$

Finally, we consider the splitting functions involving quarks or anti-quarks in the final and initial states. Again we will limit ourselves to two loops and under this restriction we have:

$$
\begin{array}{l} \widetilde{P}_{u_i u_j} = \widetilde{P}_{\overline{u}_i \overline{u}_j} = e_u^2 \delta_{ij} \widetilde{P}_{qq}^V + e_u^4 \widetilde{P}_{qq}^S \\ \widetilde{P}_{d_i d_j} = \widetilde{P}_{\overline{d}_i \overline{d}_j} = e_d^2 \delta_{ij} \widetilde{P}_{qq}^V + e_d^4 \widetilde{P}_{qq}^S \\ \widetilde{P}_{\overline{u}_i u_j} = \widetilde{P}_{u_i \overline{u}_j} = e_u^4 \widetilde{P}_{qq}^S \\ \widetilde{P}_{\overline{d}_i d_j} = \widetilde{P}_{d_i \overline{d}_j} = e_d^4 \widetilde{P}_{qq}^S \\ \widetilde{P}_{u_i d_j} = \widetilde{P}_{d_i u_j} = \widetilde{P}_{\overline{u}_i d_j} = \widetilde{P}_{d_i \overline{u}_j} = \widetilde{P}_{\overline{d}_i u_j} = \widetilde{P}_{u_i \overline{d}_j} = \widetilde{P}_{\overline{u}_i \overline{d}_j} = \widetilde{P}_{\overline{d}_i \overline{u}_j} = e_u^2 e_d^2 \widetilde{P}_{qq}^S \end{array} . \tag{1.15}
$$

Using the information above, we can now write the QED correction matrix to the splitting functions up to two loops as follows:

$$
\begin{pmatrix}
e_u^2\delta_{je}\widetilde{P}_{qq}^V + e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^4\widetilde{P}_{qq}^S \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^2\delta_{il}\widetilde{P}_{qq}^V + e_d^4\widetilde{P}_{qq}^S & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
e_u^2\widetilde{P}_{gq} & e_d^2\widetilde{P}_{gq} & e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & e_d^2\widetilde{P}_{gq} & e_u^2\widetilde{P}_{gq} \\
e_u^2\widetilde{P}_{\gamma q} & e_d^2\widetilde{P}_{\gamma q} & e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & e_d^2\widetilde{P}_{\gamma q} & e_u^2\widetilde{P}_{\gamma q} \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^4\widetilde{P}_{qq}^S & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & e_d^2\delta_{km}\widetilde{P}_{qq}^V + e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^2\delta_{hn}\widetilde{P}_{qq}^V + e_u^4\widetilde{P}_{qq}^S
\end{pmatrix} =
$$

$$
\begin{pmatrix}
e_u^2\delta_{je}\widetilde{P}_{qq} & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & 0 \\
0 & e_d^2\delta_{il}\widetilde{P}_{qq}^V & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & 0 & 0 \\
e_u^2\widetilde{P}_{gq} & e_d^2\widetilde{P}_{gq} & e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & e_d^2\widetilde{P}_{gq} & e_u^2\widetilde{P}_{gq} \\
e_u^2\widetilde{P}_{\gamma q} & e_d^2\widetilde{P}_{\gamma q} & e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & e_d^2\widetilde{P}_{\gamma q} & e_u^2\widetilde{P}_{\gamma q} \\
0 & 0 & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & e_d^2\delta_{km}\widetilde{P}_{qq}^V & 0 \\
0 & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & e_u^2\delta_{hn}\widetilde{P}_{qq}^V
\end{pmatrix}
+
\begin{pmatrix}
e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & 0 & 0 & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^4\widetilde{P}_{qq}^S \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^4\widetilde{P}_{qq}^S & 0 & 0 & e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^4\widetilde{P}_{qq}^S & 0 & 0 & e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & 0 & 0 & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^4\widetilde{P}_{qq}^S
\end{pmatrix}
\tag{1.16}
$$

We now apply the same decomposition to the purely QCD matrix, obtaining:

$$
\begin{pmatrix}
\delta_{je}P_{qq}^V + P_{qq}^S & P_{qq}^S & P_{qg} & 0 & P_{q\overline{q}}^S & \delta_{jn}P_{q\overline{q}}^V + P_{q\overline{q}}^S \\
P_{qq}^S & \delta_{il}P_{qq}^V + P_{qq}^S & P_{qg} & 0 & \delta_{im}P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\
0 & 0 & 0 & 0 & 0 & 0 \\
P_{q\overline{q}}^S & \delta_{kl}P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{qg} & 0 & \delta_{km}P_{qq}^V + P_{qq}^S & P_{qq}^S \\
\delta_{he}P_{q\overline{q}}^V + P_{q\overline{q}}^S & P_{q\overline{q}}^S & P_{qg} & 0 & P_{qq}^S & \delta_{hn}P_{qq}^V + P_{qq}^S
\end{pmatrix} =
$$

$$
\begin{pmatrix}
\delta_{je}P_{qq}^V & 0 & P_{qg} & 0 & 0 & \delta_{jm}P_{q\overline{q}}^V \\
0 & \delta_{il}P_{qq}^V & P_{qg} & 0 & \delta_{in}P_{q\overline{q}}^V & 0 \\
P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_{kl}P_{q\overline{q}}^V & P_{qg} & 0 & \delta_{km}P_{qq}^V & 0 \\
\delta_{he}P_{q\overline{q}}^V & 0 & P_{qg} & 0 & 0 & \delta_{hn}P_{qq}^V
\end{pmatrix}
+
\begin{pmatrix}
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S
\end{pmatrix}
\tag{1.17}
$$

Finally, plugging Eqs. (1.16) and (1.17) into Eq. (1.11), performing the sum over e, l, m and n and identifying $k = i$ and $h = j$, we obtain:

$$
\begin{aligned}
\mu^2\frac{\partial}{\partial\mu^2}
\begin{pmatrix} u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_i \\ \overline{u}_j \end{pmatrix}
&= \left[
\begin{pmatrix}
P_{qq}^V & 0 & P_{qg} & 0 & 0 & P_{q\overline{q}}^V \\
0 & P_{qq}^V & P_{qg} & 0 & P_{q\overline{q}}^V & 0 \\
P_{gq} & P_{gq} & P_{gg} & 0 & P_{gq} & P_{gq} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_{q\overline{q}}^V & P_{qg} & 0 & P_{qq}^V & 0 \\
P_{q\overline{q}}^V & 0 & P_{qg} & 0 & 0 & P_{qq}^V
\end{pmatrix}
+
\begin{pmatrix}
e_u^2\widetilde{P}_{qq} & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & 0 \\
0 & e_d^2\widetilde{P}_{qq} & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & 0 & 0 \\
e_u^2\widetilde{P}_{gq} & e_d^2\widetilde{P}_{gq} & e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & e_d^2\widetilde{P}_{gq} & e_u^2\widetilde{P}_{gq} \\
e_u^2\widetilde{P}_{\gamma q} & e_d^2\widetilde{P}_{\gamma q} & e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & e_d^2\widetilde{P}_{\gamma q} & e_u^2\widetilde{P}_{\gamma q} \\
0 & 0 & e_d^2\widetilde{P}_{qg} & e_d^2\widetilde{P}_{q\gamma} & e_d^2\widetilde{P}_{qq}^V & 0 \\
0 & 0 & e_u^2\widetilde{P}_{qg} & e_u^2\widetilde{P}_{q\gamma} & 0 & e_u^2\widetilde{P}_{qq}^V
\end{pmatrix}
\right]
\begin{pmatrix} u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_i \\ \overline{u}_j \end{pmatrix} \\
&+ \left[
\begin{pmatrix}
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
P_{qq}^S & P_{qq}^S & 0 & 0 & P_{q\overline{q}}^S & P_{q\overline{q}}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S \\
P_{q\overline{q}}^S & P_{q\overline{q}}^S & 0 & 0 & P_{qq}^S & P_{qq}^S
\end{pmatrix}
+
\begin{pmatrix}
e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & 0 & 0 & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^4\widetilde{P}_{qq}^S \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^4\widetilde{P}_{qq}^S & 0 & 0 & e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
e_u^2e_d^2\widetilde{P}_{qq}^S & e_d^4\widetilde{P}_{qq}^S & 0 & 0 & e_d^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S \\
e_u^4\widetilde{P}_{qq}^S & e_u^2e_d^2\widetilde{P}_{qq}^S & 0 & 0 & e_u^2e_d^2\widetilde{P}_{qq}^S & e_u^4\widetilde{P}_{qq}^S
\end{pmatrix}
\right]
\begin{pmatrix} \sum_e u_e \\ \sum_l d_l \\ g \\ \gamma \\ \sum_m \overline{d}_m \\ \sum_n \overline{u}_n \end{pmatrix}
\end{aligned}
\tag{1.18}
$$

In order to have the same evolution system in terms of plus- and minus-distributions, we apply to Eq. (1.18) the following transformation:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{T}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.19}$$

so that we get:

$$\begin{aligned}
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} u_j^+ \\ d_i^+ \\ g \\ \gamma \\ d_i^- \\ u_j^- \end{pmatrix} \quad = \quad & \left[\begin{pmatrix} P^+ & 0 & 2P_{qg} & 0 & 0 & 0 \\ 0 & P^+ & 2P_{qg} & 0 & 0 & 0 \\ P_{gq} & P_{gq} & P_{gg} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P^- & 0 \\ 0 & 0 & 0 & 0 & 0 & P^- \end{pmatrix} + \begin{pmatrix} e_u^2 \widetilde{P}^+ & 0 & 2e_u^2 \widetilde{P}_{qg} & 2e_u^2 \widetilde{P}_{q\gamma} & 0 & 0 \\ 0 & e_d^2 \widetilde{P}^+ & 2e_d^2 \widetilde{P}_{qg} & 2e_d^2 \widetilde{P}_{q\gamma} & 0 & 0 \\ e_u^2 \widetilde{P}_{gq} & e_d^2 \widetilde{P}_{gq} & e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & 0 & 0 \\ e_u^2 \widetilde{P}_{\gamma q} & e_d^2 \widetilde{P}_{\gamma q} & e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_d^2 \widetilde{P}^- & 0 \\ 0 & 0 & 0 & 0 & 0 & e_u^2 \widetilde{P}^- \end{pmatrix} \right] \begin{pmatrix} u_j^+ \\ d_i^+ \\ g \\ \gamma \\ d_i^- \\ u_j^- \end{pmatrix} \\
+ \; & \left[\begin{pmatrix} P_{qq} - P^+ & P_{qq} - P^+ & 0 & 0 & 0 & 0 \\ P_{qq} - P^+ & P_{qq} - P^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P^V - P^- & P^V - P^- \\ 0 & 0 & 0 & 0 & P^V - P^- & P^V - P^- \end{pmatrix} \right. \\
+ \; & \left. \begin{pmatrix} e_u^4 (\widetilde{P}_{qq} - \widetilde{P}^+) & e_u^2 e_d^2 (\widetilde{P}_{qq} - \widetilde{P}^+) & 0 & 0 & 0 & 0 \\ e_u^2 e_d^2 (\widetilde{P}_{qq} - \widetilde{P}^+) & e_d^4 (\widetilde{P}_{qq} - \widetilde{P}^+) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] \frac{1}{n_f} \begin{pmatrix} \sum_e u_e^+ \\ \sum_l d_l^+ \\ g \\ \gamma \\ \sum_m d_m^- \\ \sum_n u_n^- \end{pmatrix}
\end{aligned} \tag{1.20}$$

Using the following definitions:

$$\begin{aligned} \Sigma_u &= \sum_{k=i}^{n_u} u_k^+ \qquad & \Sigma_d &= \sum_{k=i}^{n_d} d_k^+ \\ V_u &= \sum_{k=i}^{n_u} u_k^- \qquad & V_d &= \sum_{k=i}^{n_d} d_k^- \,, \end{aligned} \tag{1.21}$$

which are such that:

$$\Sigma = \Sigma_u + \Sigma_d \quad \text{and} \quad V = V_u + V_d \,, \tag{1.22}$$

we can further manipulate Eq. (5.9) obtaining the coupled system:

$$\left\{\begin{array}{lcl}
\mu^2\dfrac{\partial g}{\partial\mu^2} &=& (P_{gq}+e_u^2\widetilde{P}_{gq})\Sigma_u+(P_{gq}+e_d^2\widetilde{P}_{gq})\Sigma_d+(P_{gg}+e_\Sigma^2\widetilde{P}_{gg})g+e_\Sigma^2\widetilde{P}_{g\gamma}\gamma \\
\mu^2\dfrac{\partial \gamma}{\partial\mu^2} &=& e_u^2\widetilde{P}_{\gamma q}\Sigma_u+e_d^2\widetilde{P}_{\gamma q}\Sigma_d+e_\Sigma^2\widetilde{P}_{\gamma g}g+e_\Sigma^2\widetilde{P}_{\gamma\gamma}\gamma \\
\mu^2\dfrac{\partial d_i^+}{\partial\mu^2} &=& (P^++e_d^2\widetilde{P}^+)d_i^++2(P_{qg}+e_d^2\widetilde{P}_{qg})g+2e_d^2\widetilde{P}_{q\gamma}\gamma \\
&+& \dfrac{1}{n_f}[(P_{qq}-P^+)+e_u^2e_d^2(\widetilde{P}_{qq}-\widetilde{P}^+)]\Sigma_u+\dfrac{1}{n_f}[(P_{qq}-P^+)+e_d^4(\widetilde{P}_{qq}-\widetilde{P}^+)]\Sigma_d \\
\mu^2\dfrac{\partial u_j^+}{\partial\mu^2} &=& (P^++e_u^2\widetilde{P}^+)u_j^++2(P_{qg}+e_u^2\widetilde{P}_{qg})g+2e_u^2\widetilde{P}_{q\gamma}\gamma \\
&+& \dfrac{1}{n_f}[(P_{qq}-P^+)+e_u^4(\widetilde{P}_{qq}-\widetilde{P}^+)]\Sigma_u+\dfrac{1}{n_f}[(P_{qq}-P^+)+e_u^2e_d^2(\widetilde{P}_{qq}-\widetilde{P}^+)]\Sigma_d \\
\mu^2\dfrac{\partial d_i^-}{\partial\mu^2} &=& (P^-+e_d^2\widetilde{P}^-)d_i^-+\dfrac{1}{n_f}(P^V-P^-)V_u+\dfrac{1}{n_f}(P^V-P^-)V_d \\
\mu^2\dfrac{\partial u_j^-}{\partial\mu^2} &=& (P^-+e_u^2\widetilde{P}^-)u_j^-+\dfrac{1}{n_f}(P^V-P^-)V_u+\dfrac{1}{n_f}(P^V-P^-)V_d
\end{array}\right. . \tag{1.23}$$

2 Evolution basis

In order to diagonalise as much as possible the evolution matrix in the presence of QED corrections avoiding unnecessary couplings between parton distributions, we propose the following evolution basis:

$$\begin{array}{ll}
1)\ g & \\
2)\ \gamma & \\
3)\ \Sigma=\Sigma_u+\Sigma_d & 9)\ V=V_u+V_d \\
4)\ \Delta_\Sigma=\Sigma_u-\Sigma_d & 10)\ \Delta_V=V_u-V_d \\
5)\ T_1^u=u^+-c^+ & 11)\ V_1^u=u^--c^- \\
6)\ T_2^u=u^++c^+-2t^+ & 12)\ V_2^u=u^-+c^--2t^- \\
7)\ T_1^d=d^+-s^+ & 13)\ V_1^d=d^--s^- \\
8)\ T_2^d=d^++s^+-2b^+ & 14)\ V_2^d=d^-+s^--2b^-
\end{array} \tag{2.1}$$

In this basis the evolution system becomes:

$$
\left\{
\begin{array}{rcl}
\mu^2 \dfrac{\partial g}{\partial \mu^2} &=& (P_{gq} + \eta^+ \widetilde{P}_{gq})\Sigma + \eta^- \widetilde{P}_{gq}\Delta_\Sigma + (P_{gg} + e_\Sigma^2 \widetilde{P}_{gg})g + e_\Sigma^2 \widetilde{P}_{g\gamma}\gamma \\
\mu^2 \dfrac{\partial \gamma}{\partial \mu^2} &=& \eta^+ \widetilde{P}_{\gamma q}\Sigma + \eta^- \widetilde{P}_{\gamma q}\Delta_\Sigma + e_\Sigma^2 \widetilde{P}_{\gamma g} g + e_\Sigma^2 \widetilde{P}_{\gamma\gamma}\gamma \\
\mu^2 \dfrac{\partial \Sigma}{\partial \mu^2} &=& \left[P_{qq} + \eta^+ \widetilde{P}^+ + \dfrac{\eta^+ e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+)\right]\Sigma + \left[\eta^- \widetilde{P}^+ + \dfrac{\eta^- e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+)\right]\Delta_\Sigma \\
&+& 2(n_f P_{qg} + e_\Sigma^2 \widetilde{P}_{qg})g + 2e_\Sigma^2 \widetilde{P}_{q\gamma}\gamma \\
\mu^2 \dfrac{\partial \Delta_\Sigma}{\partial \mu^2} &=& \left[\eta^- \widetilde{P}^+ + \dfrac{n_u - n_d}{n_f}(P_{qq} - P^+) + \dfrac{\eta^+ \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+)\right]\Sigma + \left[P^+ + \eta^+ \widetilde{P}^+ + \dfrac{\eta^- \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+)\right]\Delta_\Sigma \\
&+& 2[(n_u - n_d)P_{qg} + \delta_e^2 \widetilde{P}_{qg}]g + 2\delta_e^2 \widetilde{P}_{q\gamma}\gamma
\end{array}
\right.
$$

$$
\begin{array}{rcl}
\mu^2 \dfrac{\partial T_{1,2}^u}{\partial \mu^2} &=& (P^+ + e_u^2 \widetilde{P}^+)T_{1,2}^u \\
\mu^2 \dfrac{\partial T_{1,2}^d}{\partial \mu^2} &=& (P^+ + e_d^2 \widetilde{P}^+)T_{1,2}^d
\end{array}
$$

$$
\left\{
\begin{array}{rcl}
\mu^2 \dfrac{\partial V}{\partial \mu^2} &=& (P^V + \eta^+ \widetilde{P}^-)V + \eta^- \widetilde{P}^- \Delta_V \\
\mu^2 \dfrac{\partial \Delta_V}{\partial \mu^2} &=& \left[\dfrac{n_u - n_d}{n_f}(P^V - P^-) + \eta^- \widetilde{P}^-\right]V + \left[P^- + \eta^+ \widetilde{P}^-\right]\Delta_V
\end{array}
\right.
$$

$$
\begin{array}{rcl}
\mu^2 \dfrac{\partial V_{1,2}^u}{\partial \mu^2} &=& (P^- + e_u^2 \widetilde{P}^-)V_{1,2}^u \\
\mu^2 \dfrac{\partial V_{1,2}^d}{\partial \mu^2} &=& (P^- + e_d^2 \widetilde{P}^-)V_{1,2}^d
\end{array}
\tag{2.2}
$$

with the definitions:

$$
\begin{aligned}
e_\Sigma^2 &= N_c(n_u e_u^2 + n_d e_d^2) \\
\delta_e^2 &= N_c(n_u e_u^2 - n_d e_d^2) \\
\eta^\pm &= \frac{1}{2}(e_u^2 \pm e_d^2)
\end{aligned}
\tag{2.3}
$$

and where we have used the curly bracket to denote the coupled equations. The main thing to notice is that there are two coupled sub-systems. This is in contrast with what we had in pure QCD where there was only one coupled system.

Now let us write the Eq. (2.2) in a matricial form, separating the pure QCD splitting functions (those without tilde) from the QED contributions:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} = \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0\\ 0 & 0 & 0 & 0\\ 2n_f P_{qg} & 0 & P_{qq} & 0\\ \frac{n_u-n_d}{n_f}2n_f P_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq}-P^+) & P^+\end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} e_\Sigma^2\widetilde{P}_{gg} & e_\Sigma^2\widetilde{P}_{g\gamma} & \eta^+\widetilde{P}_{gq} & \eta^-\widetilde{P}_{gq}\\ e_\Sigma^2\widetilde{P}_{\gamma g} & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & \eta^+\widetilde{P}_{\gamma q} & \eta^-\widetilde{P}_{\gamma q}\\ 2e_\Sigma^2\widetilde{P}_{qg} & 2e_\Sigma^2\widetilde{P}_{q\gamma} & \eta^+\widetilde{P}^+ + \frac{\eta^+ e_\Sigma^2}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+) & \eta^-\widetilde{P}^+ + \frac{\eta^- e_\Sigma^2}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\\ 2\delta_e^2\widetilde{P}_{qg} & 2\delta_e^2\widetilde{P}_{q\gamma} & \eta^-\widetilde{P}^+ + \frac{\eta^+\delta_e^2}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+) & \eta^+\widetilde{P}^+ + \frac{\eta^-\delta_e^2}{n_f}(\widetilde{P}_{qq}-\widetilde{P}^+)\end{pmatrix}\right]\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} \tag{2.4}$$

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} V\\ \Delta_V\end{pmatrix} = \left[\begin{pmatrix} P^V & 0\\ \frac{n_u-n_d}{n_f}(P^V-P^-) & P^-\end{pmatrix} + \begin{pmatrix}\eta^+\widetilde{P}^- & \eta^-\widetilde{P}^-\\ \eta^-\widetilde{P}^- & \eta^+\widetilde{P}^-\end{pmatrix}\right]\begin{pmatrix} V\\ \Delta_V\end{pmatrix} \tag{2.5}$$

It should finally be noticed that, every time one of the quark flavours is not active, the associated non-singlet distributions $T_{1,2}^{u,d}$ and $V_{1,2}^{u,d}$ involving that quark flavour, start evolving as a singlet distribution according to the following equations:

$$\begin{aligned} T_{1,2}^u &= \frac{\Sigma+\Delta_\Sigma}{2}\,,\\ T_{1,2}^d &= \frac{\Sigma-\Delta_\Sigma}{2}\,,\\ V_{1,2}^u &= \frac{V+\Delta_V}{2}\,,\\ V_{1,2}^d &= \frac{V-\Delta_V}{2}\,. \end{aligned} \tag{2.6}$$

3 QED corrections at LO

If we consider only LO QED corrections to the PDF evolution equations, there are a few simplifications that make the evolution sistem simpler. In particular we have that:

$$\widetilde{P}_{gg} = \widetilde{P}_{g\gamma} = \widetilde{P}_{\gamma g} = \widetilde{P}_{gq} = \widetilde{P}_{qg} = 0\,. \tag{3.1}$$

In addition:

$$\widetilde{P}^+ = \widetilde{P}^- = \widetilde{P}_{qq}\,. \tag{3.2}$$

With these simplifications we can rewrite the above evolution systems as follows:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} = \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0\\ 0 & 0 & 0 & 0\\ 2n_f P_{qg} & 0 & P_{qq} & 0\\ \frac{n_u-n_d}{n_f}2n_f P_{qg} & 0 & \frac{n_u-n_d}{n_f}(P_{qq}-P^+) & P^+\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & e_\Sigma^2\widetilde{P}_{\gamma\gamma} & \eta^+\widetilde{P}_{\gamma q} & \eta^-\widetilde{P}_{\gamma q}\\ 0 & 2e_\Sigma^2\widetilde{P}_{q\gamma} & \eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq}\\ 0 & 2\delta_e^2\widetilde{P}_{q\gamma} & \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq}\end{pmatrix}\right]\begin{pmatrix} g\\ \gamma\\ \Sigma\\ \Delta_\Sigma\end{pmatrix} \tag{3.3}$$

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} V\\ \Delta_V\end{pmatrix} = \left[\begin{pmatrix} P^V & 0\\ \frac{n_u-n_d}{n_f}(P^V-P^-) & P^-\end{pmatrix} + \begin{pmatrix}\eta^+\widetilde{P}_{qq} & \eta^-\widetilde{P}_{qq}\\ \eta^-\widetilde{P}_{qq} & \eta^+\widetilde{P}_{qq}\end{pmatrix}\right]\begin{pmatrix} V\\ \Delta_V\end{pmatrix} \tag{3.4}$$

$$\begin{aligned}
\mu^2\frac{\partial T^u_{1,2}}{\partial\mu^2} &= (P^+ + e_u^2\widetilde{P}_{qq})T^u_{1,2}\\
\mu^2\frac{\partial T^d_{1,2}}{\partial\mu^2} &= (P^+ + e_d^2\widetilde{P}_{qq})T^d_{1,2}\\
\mu^2\frac{\partial V^u_{1,2}}{\partial\mu^2} &= (P^- + e_u^2\widetilde{P}_{qq})V^u_{1,2}\\
\mu^2\frac{\partial V^d_{1,2}}{\partial\mu^2} &= (P^- + e_d^2\widetilde{P}_{qq})V^d_{1,2}
\end{aligned} \tag{3.5}$$

Notice that Eq. (3.3), recognising that $2e_\Sigma^2 = \theta^-$ and $2\delta_e^2 = \theta^+$, is consistent with Eq. (9) of the APFEL paper [1].

4 Matching conditions in the QED evolution basis

In order to couple QCD and QED evolutions in the VFNS, it is necessary to work out the structure of the QCD matching conditions in the evolution basis defined in Sect. 2. The matching conditions up to $\mathcal{O}(\alpha_s^2)$ have been computed in Ref. [2] and for the transition between the n_f- and the (n_f+1)-scheme for the gluon g, the light quarks l and the heavy quarks h and $\overline{h}$ read:

$$g^{(n_f+1)} = \left[1+\left(\frac{\alpha_s}{4\pi}\right)A^{S,(1)}_{gg,h} + \left(\frac{\alpha_s}{4\pi}\right)^2 A^{S,(2)}_{gg,h}\right]g^{(n_f)} + \left(\frac{\alpha_s}{4\pi}\right)^2 A^{S,(2)}_{gq,h}\Sigma^{(n_f)} \tag{4.1}$$

$$l^{(n_f+1)} = \left[1+\left(\frac{\alpha_s}{4\pi}\right)^2 A^{NS,(2)}_{qq,h}\right]l^{(n_f)}\,. \tag{4.2}$$

$$h^{(n_f+1)} = \overline{h}^{(n_f+1)} = \frac{1}{2}\left(\frac{\alpha_s}{4\pi}\right)^2\tilde{A}^{S,(2)}_{hq}\Sigma^{(n_f)} + \frac{1}{2}\left[\left(\frac{\alpha_s}{4\pi}\right)\tilde{A}^{S,(1)}_{hq} + \left(\frac{\alpha_s}{4\pi}\right)^2\tilde{A}^{S,(2)}_{hg}\right]g^{(n_f)}\,. \tag{4.3}$$

The first observation is that, given that the dynamical production of heavy quarks at threshold always happens in pairs, it is clear that all distributions that involve only differences between flavours and anti-flavours match multiplicatively at the thresholds as the light flavours do. In particular:

$$\begin{pmatrix} V^{(n_f+1)} \\ \Delta_V^{(n_f+1)}\end{pmatrix} = \left[1+\left(\frac{\alpha_s}{4\pi}\right)^2 A^{NS,(2)}_{qq,h}\right]\begin{pmatrix} V^{(n_f)} \\ \Delta_V^{(n_f)}\end{pmatrix} = \tag{4.4}$$

Now we turn to consider Σ and Δ_Σ. The first case is easy and we have that:

$$\begin{aligned}
\Sigma^{(n_f+1)} &= \sum_{l=1}^{n_f}(l^{(n_f+1)}+\overline{l}^{(n_f+1)}) + (h^{(n_f+1)}+\overline{h}^{(n_f+1)})\\
&= \left[1+\left(\frac{\alpha_s}{4\pi}\right)^2\left(\tilde{A}^{S,(2)}_{hq} + A^{NS,(2)}_{qq,h}\right)\right]\Sigma^{(n_f)} + \left[\left(\frac{\alpha_s}{4\pi}\right)\tilde{A}^{S,(1)}_{hg} + \left(\frac{\alpha_s}{4\pi}\right)^2\tilde{A}^{S,(2)}_{hg}\right]g^{(n_f)}\,.
\end{aligned} \tag{4.5}$$

The second case is instead a bit trickier because:

$$\Delta_\Sigma^{(n_f+1)} = \sum_{q=n_{\rm up}}(q^{(n_f+1)}+\overline{q}^{(n_f+1)}) - \sum_{q=n_{\rm down}}(q^{(n_f+1)}+\overline{q}^{(n_f+1)}) = \Sigma_{\rm up}^{(n_f+1)} - \Sigma_{\rm down}^{(n_f+1)} \tag{4.6}$$

and thus the way how matching conditions have to be applied depends on whether the (n_f+1)-th quark is of type up or down. In particular, every time that a threshold is crossed, only one of the components $\Sigma_{\rm up}$ or $\Sigma_{\rm down}$ will get the contribution from the heavy quark h:

$$\begin{aligned}
\Delta_\Sigma^{(n_f+1)} &= \left[1+\left(\frac{\alpha_s}{4\pi}\right)^2 A^{NS,(2)}_{qq,h}\right]\Delta_\Sigma^{(n_f)}\\
&\pm \left(\frac{\alpha_s}{4\pi}\right)^2\tilde{A}^{S,(2)}_{hq}\Sigma^{(n_f)} \pm \left[\left(\frac{\alpha_s}{4\pi}\right)\tilde{A}^{S,(1)}_{hg} + \left(\frac{\alpha_s}{4\pi}\right)^2\tilde{A}^{S,(2)}_{hg}\right]g^{(n_f)}
\end{aligned} \tag{4.7}$$

with $+$ if h is an up-type quark and $-$ if it is a down-type quark. The matching conditions for g, Σ, and Δ_Σ can be written in a matricial form as:

$$
\begin{pmatrix} g^{(n_f+1)} \\ \Sigma^{(n_f+1)} \\ \Delta_\Sigma^{(n_f+1)} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(\frac{\alpha_s}{4\pi}\right) \begin{pmatrix} A_{gg,h}^{S,(1)} & 0 & 0 \\ \tilde{A}_{hg}^{S,(1)} & 0 & 0 \\ \pm\tilde{A}_{hg}^{S,(1)} & 0 & 0 \end{pmatrix} + \left(\frac{\alpha_s}{4\pi}\right)^2 \begin{pmatrix} A_{gg,h}^{S,(2)} & A_{gq,h}^{S,(2)} & 0 \\ \tilde{A}_{hg}^{S,(2)} & \tilde{A}_{hq}^{S,(2)} + A_{qq,h}^{NS,(2)} & 0 \\ \pm\tilde{A}_{hg}^{S,(2)} & \pm\tilde{A}_{hq}^{S,(2)} & A_{qq,h}^{NS,(2)} \end{pmatrix} \right] \begin{pmatrix} g^{(n_f)} \\ \Sigma^{(n_f)} \\ \Delta_\Sigma^{(n_f)} \end{pmatrix} \tag{4.8}
$$

If we define:

$$
\begin{aligned}
A^{NS} &= 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 A_{qq,h}^{NS,(2)} \\
A_{gg} &= 1 + \left(\frac{\alpha_s}{4\pi}\right) A_{gg,h}^{S,(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 A_{gg,h}^{S,(2)} \\
A_{gq} &= \left(\frac{\alpha_s}{4\pi}\right)^2 A_{gq,h}^{S,(2)} \\
A_{qg} &= \left(\frac{\alpha_s}{4\pi}\right) \tilde{A}_{hg}^{S,(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \tilde{A}_{hg}^{S,(2)} \\
A_{qq} &= 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\tilde{A}_{hq}^{S,(2)} + A_{qq,h}^{NS,(2)}\right)
\end{aligned} \tag{4.9}
$$

we can shrink the system of equations above into:

$$
\begin{pmatrix} g^{(n_f+1)} \\ \gamma^{(n_f+1)} \\ \Sigma^{(n_f+1)} \\ \Delta_\Sigma^{(n_f+1)} \end{pmatrix} = \begin{pmatrix} A_{gg} & 0 & A_{gq} & 0 \\ 0 & 1 & 0 & 0 \\ A_{qg} & 0 & A_{qq} & 0 \\ \pm A_{qg} & 0 & \pm(A_{qq} - A^{NS}) & A^{NS} \end{pmatrix} \begin{pmatrix} g^{(n_f)} \\ \gamma^{(n_f)} \\ \Sigma^{(n_f)} \\ \Delta_\Sigma^{(n_f)} \end{pmatrix} \tag{4.10}
$$

and in addition:

$$
\begin{pmatrix} V^{(n_f+1)} \\ \Delta_V^{(n_f+1)} \end{pmatrix} = A^{NS} \begin{pmatrix} V^{(n_f)} \\ \Delta_V^{(n_f)} \end{pmatrix} \tag{4.11}
$$

and:

$$
h^{(n_f+1)} + \overline{h}^{(n_f+1)} = (A_{qq} - A^{NS})\Sigma^{(n_f)} + A_{qg} g^{(n_f)} \,. \tag{4.12}
$$

We finally turn to consider the matching of T_1^u, T_2^u, T_1^d, and T_2^d. Due to the way these distributions are constructed, they might behave in different ways below and above a given threshold. Therefore we need to consider what happens at each of them.

- charm threshold ($n_f = 3$):

$$
\begin{pmatrix} g^{(4)} \\ \gamma^{(4)} \\ \Sigma^{(4)} \\ \Delta_\Sigma^{(4)} \\ T_1^{u,(4)} \\ T_2^{u,(4)} \\ T_1^{d,(4)} \\ T_2^{d,(4)} \\ V^{(4)} \\ \Delta_V^{(4)} \\ V_1^{u,(4)} \\ V_2^{u,(4)} \\ V_1^{d,(4)} \\ V_2^{d,(4)} \end{pmatrix} = \left(\begin{array}{c|cccc|cccc|cc|cccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
0 & A_{gg} & 0 & A_{gq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & A_{qg} & 0 & A_{qq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & A_{qg} & 0 & (A_{qq} - A^{NS}) & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & A_{qg} & 0 & -A_{qq} + \frac{3}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & A_{qg} & 0 & A_{qq} - \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & \frac{1}{2}A^{NS} & -\frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 \\
\hline
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & -\frac{1}{2}A^{NS} & 0 & 0 & 0 & 0
\end{array} \right) \begin{pmatrix} g^{(3)} \\ \gamma^{(3)} \\ \Sigma^{(3)} \\ \Delta_\Sigma^{(3)} \\ T_1^{u,(3)} \\ T_2^{u,(3)} \\ T_1^{d,(3)} \\ T_2^{d,(3)} \\ V^{(3)} \\ \Delta_V^{(3)} \\ V_1^{u,(3)} \\ V_2^{u,(3)} \\ V_1^{d,(3)} \\ V_2^{d,(3)} \end{pmatrix} \tag{4.13}
$$

- bottom threshold ($n_f = 4$):

$$
\begin{pmatrix} g^{(5)} \\ \gamma^{(5)} \\ \Sigma^{(5)} \\ \Delta_{\Sigma}^{(5)} \\ T_1^{u,(5)} \\ T_2^{u,(5)} \\ T_1^{d,(5)} \\ T_2^{d,(5)} \\ V^{(5)} \\ \Delta_V^{(5)} \\ V_1^{u,(5)} \\ V_2^{u,(5)} \\ V_1^{d,(5)} \\ V_2^{d,(5)} \end{pmatrix}
=
\left(\begin{array}{c|cccc|cccc|cc|cccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
0 & A_{gg} & 0 & A_{gq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & A_{qg} & 0 & A_{qq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -A_{qg} & 0 & -(A_{qq} - A^{NS}) & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & -2A_{qg} & 0 & -2A_{qq} + \frac{5}{2}A^{NS} & -\frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 \\
\hline
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & -\frac{1}{2}A^{NS} & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} g^{(4)} \\ \gamma^{(4)} \\ \Sigma^{(4)} \\ \Delta_{\Sigma}^{(4)} \\ T_1^{u,(4)} \\ T_2^{u,(4)} \\ T_1^{d,(4)} \\ T_2^{d,(4)} \\ V^{(4)} \\ \Delta_V^{(4)} \\ V_1^{u,(4)} \\ V_2^{u,(4)} \\ V_1^{d,(4)} \\ V_2^{d,(4)} \end{pmatrix}
\tag{4.14}
$$

- top threshold ($n_f = 5$):

$$
\begin{pmatrix} g^{(6)} \\ \gamma^{(6)} \\ \Sigma^{(6)} \\ \Delta_{\Sigma}^{(6)} \\ T_1^{u,(6)} \\ T_2^{u,(6)} \\ T_1^{d,(6)} \\ T_2^{d,(6)} \\ V^{(6)} \\ \Delta_V^{(6)} \\ V_1^{u,(6)} \\ V_2^{u,(6)} \\ V_1^{d,(6)} \\ V_2^{d,(6)} \end{pmatrix}
=
\left(\begin{array}{c|cccc|cccc|cc|cccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
0 & A_{gg} & 0 & A_{gq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & A_{qg} & 0 & A_{qq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & A_{qg} & 0 & (A_{qq} - A^{NS}) & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -2A_{qg} & 0 & -2A_{qq} + \frac{5}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 & 0 \\
\hline
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}A^{NS} & \frac{1}{2}A^{NS} & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS} & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{NS}
\end{array}\right)
\begin{pmatrix} g^{(5)} \\ \gamma^{(5)} \\ \Sigma^{(5)} \\ \Delta_{\Sigma}^{(5)} \\ T_1^{u,(5)} \\ T_2^{u,(5)} \\ T_1^{d,(5)} \\ T_2^{d,(5)} \\ V^{(5)} \\ \Delta_V^{(5)} \\ V_1^{u,(5)} \\ V_2^{u,(5)} \\ V_1^{d,(5)} \\ V_2^{d,(5)} \end{pmatrix}
\tag{4.15}
$$

5 Including the lepton PDFs

In order to include the lepton PDFs in the coupled QCD×QED DGLAP evolution, we first need to make some preliminary considerations. The first thing to notice is that, considering only QED corrections and not electroweak corrections, we do not need to introduce neutrino PDFs as neutrinos do not couple neither to the gluon nor to the photon. Therefore the only PDFs that need to be introduced are those of the charged leptons $e^{\pm}$, $\mu^{\pm}$ and $\tau^{\pm}$. The second point to consider is that the absolute value of the charge of all leptons is always equal to one and, since charges enter the DGLAP evolution as squares, this allows us to maintain, at least in the leptonic sector, the isospin symmetry $l^+ \leftrightarrow l^-$. As a final remark, we notice that, while the muon and the electron mass, $\simeq 0.5$ MeV and $\simeq 105$ MeV respectively, are below the $\Lambda_{\rm QCD}$ and thus they do not introduce any threshold in the DGLAP evolution, the tauon mass, whose mass is $m_\tau = 1.777$ GeV, is well above $\Lambda_{\rm QCD}$ and above the initial scale at which PDFs are usually parameterised ($Q_0 = 1 - 1.4$ GeV). As a consequence, the presence of tauons in the evolution implies the introduction of a new threshold between m_c and m_b at which the τ PDFs are dynamically generated from the photon. On the contrary, e and μ PDFs cannot be dynamically generated by evolution and need to be parameterised at the initial scale. We will see later how the functional form of the e and μ PDFs at the initial scale can be guessed by assuming a dynamical generation by photon splitting at their respective mass thresholds.

In order to write the full DGLAP equations in the presence of quarks, leptons, gluon and photon, we start considering Eq. (1.8) where we add the leptons ℓ_α and $\overline{\ell}_\beta$:

$$
\mu^2 \frac{\partial}{\partial \mu^2}
\begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_k \\ \overline{u}_h \\ \overline{\ell}_\beta \end{pmatrix}
= \sum_{e,l,m,n,\gamma,\delta}
\begin{pmatrix}
\mathcal{P}_{\ell_\alpha \ell_\gamma} & \mathcal{P}_{\ell_\alpha u_e} & \mathcal{P}_{\ell_\alpha d_l} & \mathcal{P}_{\ell_\alpha g} & \mathcal{P}_{\ell_\alpha \gamma} & \mathcal{P}_{\ell_\alpha \overline{d}_m} & \mathcal{P}_{\ell_\alpha \overline{u}_n} & \mathcal{P}_{\ell_\alpha \overline{\ell}_\delta} \\
\mathcal{P}_{u_j \ell_\gamma} & \mathcal{P}_{u_j u_e} & \mathcal{P}_{u_j d_l} & \mathcal{P}_{u_j g} & \mathcal{P}_{u_j \gamma} & \mathcal{P}_{u_j \overline{d}_m} & \mathcal{P}_{u_j \overline{u}_n} & \mathcal{P}_{u_j \overline{\ell}_\delta} \\
\mathcal{P}_{d_i \ell_\gamma} & \mathcal{P}_{d_i u_e} & \mathcal{P}_{d_i d_l} & \mathcal{P}_{d_i g} & \mathcal{P}_{d_i \gamma} & \mathcal{P}_{d_i \overline{d}_m} & \mathcal{P}_{d_i \overline{u}_n} & \mathcal{P}_{d_i \overline{\ell}_\delta} \\
\mathcal{P}_{g \ell_\gamma} & \mathcal{P}_{g u_e} & \mathcal{P}_{g d_l} & \mathcal{P}_{gg} & \mathcal{P}_{g \gamma} & \mathcal{P}_{g \overline{d}_m} & \mathcal{P}_{g \overline{u}_n} & \mathcal{P}_{g \overline{\ell}_\delta} \\
\mathcal{P}_{\gamma \ell_\gamma} & \mathcal{P}_{\gamma u_e} & \mathcal{P}_{\gamma d_l} & \mathcal{P}_{\gamma g} & \mathcal{P}_{\gamma \gamma} & \mathcal{P}_{\gamma \overline{d}_m} & \mathcal{P}_{\gamma \overline{u}_n} & \mathcal{P}_{\gamma \overline{\ell}_\delta} \\
\mathcal{P}_{\overline{d}_k \ell_\gamma} & \mathcal{P}_{\overline{d}_k u_e} & \mathcal{P}_{\overline{d}_k d_l} & \mathcal{P}_{\overline{d}_k g} & \mathcal{P}_{\overline{d}_k \gamma} & \mathcal{P}_{\overline{d}_k \overline{d}_m} & \mathcal{P}_{\overline{d}_k \overline{u}_n} & \mathcal{P}_{\overline{d}_k \overline{\ell}_\delta} \\
\mathcal{P}_{\overline{u}_h \ell_\gamma} & \mathcal{P}_{\overline{u}_h u_e} & \mathcal{P}_{\overline{u}_h d_l} & \mathcal{P}_{\overline{u}_h g} & \mathcal{P}_{\overline{u}_h \gamma} & \mathcal{P}_{\overline{u}_h \overline{d}_m} & \mathcal{P}_{\overline{u}_h \overline{u}_n} & \mathcal{P}_{\overline{u}_h \overline{\ell}_\delta} \\
\mathcal{P}_{\overline{\ell}_\beta \ell_\gamma} & \mathcal{P}_{\overline{\ell}_\beta u_e} & \mathcal{P}_{\overline{\ell}_\beta d_l} & \mathcal{P}_{\overline{\ell}_\beta g} & \mathcal{P}_{\overline{\ell}_\beta \gamma} & \mathcal{P}_{\overline{\ell}_\beta \overline{d}_m} & \mathcal{P}_{\overline{\ell}_\beta \overline{u}_n} & \mathcal{P}_{\overline{\ell}_\beta \overline{\ell}_\delta}
\end{pmatrix}
\begin{pmatrix} \ell_\gamma \\ u_e \\ d_l \\ g \\ \gamma \\ \overline{d}_m \\ \overline{u}_n \\ \overline{\ell}_\delta \end{pmatrix} ,
\tag{5.1}
$$

where $\ell_\alpha, \ell_\gamma \in \{e^-, \mu^-, \tau^-\}$ and $\overline{\ell}_\beta, \overline{\ell}_\delta \in \{e^+, \mu^+, \tau^+\}$. It should be clear that the introduction of leptons automatically carries one or more powers of α. As a consequence Eq. (1.11) is untouched by the inclusion of the leptons in the evolution and only the QED-correction matrix gets contributions. In other words only the $\widetilde{P}_{ab}$ component of the splitting-function matrix (see Eq. (1.10)) is modified when considering also leptons.

Now we can start analising the splitting functions involving leptons. First of all we observe that up to two loops leptons and gluon do not couple so that one has:

$$\mathcal{P}_{\ell_\alpha g} = \mathcal{P}_{\overline{\ell}_\beta g} = \mathcal{P}_{g \ell_\gamma} = \mathcal{P}_{g \overline{\ell}_\delta} = 0 \,. \tag{5.2}$$

For the splitting functions involving leptons and photon, we can use charge conjugation invariance and flavour symmetry, up to two loops, to obtain:

$$\begin{aligned}
&\mathcal{P}_{\ell_\alpha \ell_\beta} = \mathcal{P}_{\overline{\ell}_\alpha \overline{\ell}_\beta} = \delta_{ij} \widetilde{P}^V_{\ell\ell} + \widetilde{P}^S_{\ell\ell} \\
&\mathcal{P}_{\overline{\ell}_\alpha \ell_\beta} = \mathcal{P}_{\ell_\alpha \overline{\ell}_\beta} = \delta_{ij} \widetilde{P}^V_{\overline{\ell}\ell} + \widetilde{P}^S_{\ell\ell} \\
&\mathcal{P}_{\ell_\alpha \gamma} = \mathcal{P}_{\overline{\ell}_\alpha \gamma} = \widetilde{P}_{\ell\gamma} \\
&\mathcal{P}_{\gamma \ell_\alpha} = \mathcal{P}_{\gamma \overline{\ell}_\alpha} = \widetilde{P}_{\gamma\ell} \,.
\end{aligned} \tag{5.3}$$

In addition, since to this perturbative order they are pure QED splitting functions, they can be derived starting from the pure QCD splitting functions just by adjusting the colour factors. Therefore, apart from replacing the strong coupling α_s with the fine-structure running constant α and assuming that the lepton charge is one, we can simply write:

$$\begin{aligned}
\widetilde{P}^{V,S}_{\ell\ell} &= P^{V,S}_{qq}(T_R = 1, C_F = 1, C_A = 0) \,, \\
\widetilde{P}^{V}_{\overline{\ell}\ell} &= P^{V}_{\overline{q}q}(T_R = 1, C_F = 1, C_A = 0) \,, \\
\widetilde{P}_{\ell\gamma} &= P_{qg}(T_R = 1, C_F = 1, C_A = 0) \,, \\
\widetilde{P}_{\gamma\ell} &= P_{gq}(T_R = 1, C_F = 1, C_A = 0) \,.
\end{aligned} \tag{5.4}$$

The last category of splitting functions that we need to consider is that connecting quarks and leptons of the kind $\mathcal{P}_{\ell_\alpha q_i}$ or $\mathcal{P}_{q_i \ell_\alpha}$. They start at two loops, *i.e.* $\mathcal{O}(\alpha^2)$, and at this order they can be written as:

$$\mathcal{P}_{\ell_\alpha q_i} = \mathcal{P}_{q_i \ell_\alpha} = \mathcal{P}_{\overline{\ell}_\alpha q_i} = \mathcal{P}_{q_i \overline{\ell}_\alpha} = \mathcal{P}_{\ell_\alpha \overline{q}_i} = \mathcal{P}_{\overline{q}_i \ell_\alpha} = \mathcal{P}_{\overline{\ell}_\alpha \overline{q}_i} = \mathcal{P}_{\overline{q}_i \overline{\ell}_\alpha} = e^2_{q_i} \widetilde{P}^S_{q\ell} \,. \tag{5.5}$$

We can now generalise Eq. (1.18) to include the presence of leptons in the DGLAP evolution into the system:

$$\begin{aligned}
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_i \\ \overline{u}_j \\ \overline{\ell}_\alpha \end{pmatrix}
&= \left[\mathrm{QCD_{NS}} + \mathrm{QED_{NS}} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^2 \widetilde{P}_{qq} & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & 0 & 0 \\
0 & 0 & e_d^2 \widetilde{P}_{qq} & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & 0 & 0 & 0 \\
0 & e_u^2 \widetilde{P}_{gq} & e_d^2 \widetilde{P}_{gq} & e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & e_d^2 \widetilde{P}_{gq} & e_u^2 \widetilde{P}_{gq} & 0 \\
0 & e_u^2 \widetilde{P}_{\gamma q} & e_d^2 \widetilde{P}_{\gamma q} & e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & e_d^2 \widetilde{P}_{\gamma q} & e_u^2 \widetilde{P}_{\gamma q} & 0 \\
0 & 0 & 0 & e_d^2 \widetilde{P}_{qg} & e_d^2 \widetilde{P}_{q\gamma} & e_d^2 \widetilde{P}^V_{qq} & 0 & 0 \\
0 & 0 & 0 & e_u^2 \widetilde{P}_{qg} & e_u^2 \widetilde{P}_{q\gamma} & 0 & e_u^2 \widetilde{P}^V_{qq} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \right] \begin{pmatrix} \ell_\alpha \\ u_j \\ d_i \\ g \\ \gamma \\ \overline{d}_i \\ \overline{u}_j \\ \overline{\ell}_\alpha \end{pmatrix} \\
&+ \left[\mathrm{QCD_{SG}} + \mathrm{QED_{SG}} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^4 \widetilde{P}^S_{qq} & e_u^2 e_d^2 \widetilde{P}^S_{qq} & 0 & 0 & e_u^2 e_d^2 \widetilde{P}^S_{qq} & e_u^4 \widetilde{P}^S_{qq} & 0 \\
0 & e_u^2 e_d^2 \widetilde{P}^S_{qq} & e_d^4 \widetilde{P}^S_{qq} & 0 & 0 & e_d^4 \widetilde{P}^S_{qq} & e_u^2 e_d^2 \widetilde{P}^S_{qq} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_u^2 e_d^2 \widetilde{P}^S_{qq} & e_d^4 \widetilde{P}^S_{qq} & 0 & 0 & e_d^4 \widetilde{P}^S_{qq} & e_u^2 e_d^2 \widetilde{P}^S_{qq} & 0 \\
0 & e_u^4 \widetilde{P}^S_{qq} & e_u^2 e_d^2 \widetilde{P}^S_{qq} & 0 & 0 & e_u^2 e_d^2 \widetilde{P}^S_{qq} & e_u^4 \widetilde{P}^S_{qq} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \right] \begin{pmatrix} \sum_\gamma \ell_\gamma \\ \sum_e u_e \\ \sum_l d_l \\ g \\ \gamma \\ \sum_m \overline{d}_m \\ \sum_n \overline{u}_n \\ \sum_\delta \overline{\ell}_\delta \end{pmatrix}
\end{aligned} \tag{5.6}$$

where:

$$\mathrm{QED}_{\mathrm{NS}} = \begin{pmatrix} \widetilde{P}^V_{\ell\ell} & 0 & 0 & 0 & \widetilde{P}_{\ell\gamma} & 0 & 0 & \widetilde{P}^V_{\bar{\ell}\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{P}_{\gamma\ell} & 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{P}_{\gamma\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{P}^V_{\bar{\ell}\ell} & 0 & 0 & 0 & \widetilde{P}_{\ell\gamma} & 0 & 0 & \widetilde{P}^V_{\ell\ell} \end{pmatrix} \tag{5.7}$$

and:

$$\mathrm{QED}_{\mathrm{SG}} = \begin{pmatrix} \widetilde{P}^S_{\ell\ell} & e_u^2\widetilde{P}^S_{q\ell} & e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & e_d^2\widetilde{P}^S_{q\ell} & e_u^2\widetilde{P}^S_{q\ell} & \widetilde{P}^S_{\ell\ell} \\ e_u^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & e_u^2\widetilde{P}^S_{q\ell} \\ e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & e_d^2\widetilde{P}^S_{q\ell} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & e_d^2\widetilde{P}^S_{q\ell} \\ e_u^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & e_u^2\widetilde{P}^S_{q\ell} \\ \widetilde{P}^S_{\ell\ell} & e_u^2\widetilde{P}^S_{q\ell} & e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & e_d^2\widetilde{P}^S_{q\ell} & e_u^2\widetilde{P}^S_{q\ell} & \widetilde{P}^S_{\ell\ell} \end{pmatrix} \tag{5.8}$$

Now we use the tranformation:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.9}$$

For the leptonic parts, we get again the system in Eq. (1.20), defining:

$$\begin{aligned} &\widetilde{P}^{\pm}_{\ell} \equiv \widetilde{P}^V_{\ell\ell} \pm \mathcal{P}^V_{\ell\bar{\ell}} \\ &\widetilde{P}_{\ell\ell} \equiv \widetilde{P}^{+} + 2n_\ell \widetilde{P}^S_{\ell\ell} \quad , \\ &\widetilde{P}^V \equiv \widetilde{P}^{-} \end{aligned} \tag{5.10}$$

being n_ℓ the number of active leptons, we have that:

$$\mathbf{T}\mathrm{QED}_{\mathrm{NS}}\mathbf{T}^{-1} = \begin{pmatrix} \widetilde{P}^{+} & 0 & 0 & 0 & 2\widetilde{P}_{\ell\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{P}_{\gamma\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{P}^{-} \end{pmatrix} \tag{5.11}$$

and:

$$\mathbf{T}\mathrm{QED}_{\mathrm{SG}}\mathbf{T}^{-}1 = \frac{1}{n_\ell}\begin{pmatrix} (\widetilde{P}_{\ell\ell} - \widetilde{P}^{+}) & 2n_\ell e_u^2\widetilde{P}^S_{q\ell} & 2n_\ell e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 \\ 2n_\ell e_u^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2n_\ell e_d^2\widetilde{P}^S_{q\ell} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

In conclusion, the single equations are:

$$
\begin{aligned}
\mu^2 \frac{\partial \ell^+_\alpha}{\partial \mu^2} &= \widetilde{P}^+ \ell^+_\alpha + \frac{1}{n_\ell}(\widetilde{P}_{\ell\ell} - \widetilde{P}^+)\Sigma_\ell + 2\widetilde{P}^S_{q\ell}(\eta^+\Sigma + \eta^-\Delta_\Sigma) + 2\widetilde{P}_{\ell\gamma}\gamma \\
\mu^2 \frac{\partial \ell^-_\alpha}{\partial \mu^2} &= \widetilde{P}^- \ell^-_\alpha \\
\mu^2 \frac{\partial u^+_j}{\partial \mu^2} &= \cdots + 2e_u^2 \widetilde{P}^S_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial d^+_i}{\partial \mu^2} &= \cdots + 2e_d^2 \widetilde{P}^S_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial \gamma}{\partial \mu^2} &= \cdots + \widetilde{P}_{\gamma\ell}\Sigma_\ell
\end{aligned}
\tag{5.13}
$$

where, as usual, we have defined:

$$
\ell^\pm_\alpha = \ell_\alpha \pm \overline{\ell}_\alpha \,. \tag{5.14}
$$

Now, let us consider the following combinations:

$$
\begin{aligned}
\Sigma_\ell &= \sum_{\alpha=e,\mu,\tau} \ell^+_\alpha \\
V_\ell &= \sum_{\alpha=e,\mu,\tau} \ell^-_\alpha \\
T^\ell_1 &= \ell^+_e - \ell^+_\mu \\
T^\ell_2 &= \ell^+_e + \ell^+_\mu - 2\ell^+_\tau \\
V^\ell_1 &= \ell^-_e - \ell^-_\mu \\
V^\ell_2 &= \ell^-_e + \ell^-_\mu - 2\ell^-_\tau
\end{aligned}
\tag{5.15}
$$

It is easy to see that above the τ mass threshold, *i.e.* where $n_\ell = 3$, they evolve according to the following equations:

$$
\begin{aligned}
\mu^2 \frac{\partial \Sigma_\ell}{\partial \mu^2} &= 2n_\ell \widetilde{P}_{\ell\gamma}\gamma + 2n_\ell \widetilde{P}^S_{q\ell}(\eta^+\Sigma + \eta^-\Delta_\Sigma) + \widetilde{P}_{\ell\ell}\Sigma_\ell \\
\mu^2 \frac{\partial V_\ell}{\partial \mu^2} &= \widetilde{P}^- V_\ell = \widetilde{P}^V V_\ell \\
\mu^2 \frac{\partial T^\ell_{1,2}}{\partial \mu^2} &= \widetilde{P}^+ T^\ell_{1,2} \\
\mu^2 \frac{\partial V^\ell_{1,2}}{\partial \mu^2} &= \widetilde{P}^- V^\ell_{1,2}
\end{aligned}
\tag{5.16}
$$

In addition, the photon and the QCD singlet distributions Σ and Δ_Σ acquire the following terms:

$$
\begin{aligned}
\mu^2 \frac{\partial \Sigma}{\partial \mu^2} &= \cdots + 2e_\Sigma^2 \widetilde{P}^S_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial \Delta_\Sigma}{\partial \mu^2} &= \cdots + 2\delta_e^2 \widetilde{P}^S_{q\ell}\Sigma_\ell \\
\mu^2 \frac{\partial \gamma}{\partial \mu^2} &= \cdots + \widetilde{P}_{\gamma\ell}\Sigma_\ell
\end{aligned}
\tag{5.17}
$$

In conclusion, the full system of equations in the evolution basis including leptons is the following:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix} = \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2n_f P_{qg} & 0 & P_{qq} & 0 & 0 \\ \frac{n_u - n_d}{n_f} 2n_f P_{qg} & 0 & \frac{n_u - n_d}{n_f}(P_{qq} - P^+) & P^+ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right.$$
$$+ \begin{pmatrix} e_\Sigma^2 \widetilde{P}_{gg} & e_\Sigma^2 \widetilde{P}_{g\gamma} & \eta^+ \widetilde{P}_{gq} & \eta^- \widetilde{P}_{gq} & 0 \\ e_\Sigma^2 \widetilde{P}_{\gamma g} & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & \eta^+ \widetilde{P}_{\gamma q} & \eta^- \widetilde{P}_{\gamma q} & 0 \\ 2e_\Sigma^2 \widetilde{P}_{qg} & 2e_\Sigma^2 \widetilde{P}_{q\gamma} & \eta^+ \widetilde{P}^+ + \frac{\eta^+ e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & \eta^- \widetilde{P}^+ + \frac{\eta^- e_\Sigma^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & 0 \\ 2\delta_e^2 \widetilde{P}_{qg} & 2\delta_e^2 \widetilde{P}_{q\gamma} & \eta^- \widetilde{P}^+ + \frac{\eta^+ \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & \eta^+ \widetilde{P}^+ + \frac{\eta^- \delta_e^2}{n_f}(\widetilde{P}_{qq} - \widetilde{P}^+) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.18}$$
$$\left. + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{P}_{\gamma\ell} \\ 0 & 0 & 0 & 0 & 2e_\Sigma^2 \widetilde{P}_{q\ell}^S \\ 0 & 0 & 0 & 0 & 2\delta_e^2 \widetilde{P}_{q\ell}^S \Sigma_\ell \\ 0 & 2n_\ell \widetilde{P}_{\ell\gamma} & 2n_\ell \eta^+ \widetilde{P}_{q\ell}^S & 2n_\ell \eta^- \widetilde{P}_{q\ell}^S & \widetilde{P}_{\ell\ell} \end{pmatrix} \right] \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix}$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} V \\ \Delta_V \end{pmatrix} = \left[\begin{pmatrix} P^V & 0 \\ \frac{n_u - n_d}{n_f}(P^V - P^-) & P^- \end{pmatrix} + \begin{pmatrix} \eta^+ \widetilde{P}^- & \eta^- \widetilde{P}^- \\ \eta^- \widetilde{P}^- & \eta^+ \widetilde{P}^- \end{pmatrix} \right] \begin{pmatrix} V \\ \Delta_V \end{pmatrix} \tag{5.19}$$

$$\mu^2 \frac{\partial V_\ell}{\partial \mu^2} = \widetilde{P}^- V_\ell = \widetilde{P}^V V_\ell \tag{5.20}$$

$$\begin{aligned} \mu^2 \frac{\partial T_{1,2}^u}{\partial \mu^2} &= (P^+ + e_u^2 \widetilde{P}^+) T_{1,2}^u \\ \mu^2 \frac{\partial T_{1,2}^d}{\partial \mu^2} &= (P^+ + e_d^2 \widetilde{P}^+) T_{1,2}^d \end{aligned} \tag{5.21}$$

$$\begin{aligned} \mu^2 \frac{\partial V_{1,2}^u}{\partial \mu^2} &= (P^- + e_u^2 \widetilde{P}^-) V_{1,2}^u \\ \mu^2 \frac{\partial V_{1,2}^d}{\partial \mu^2} &= (P^- + e_d^2 \widetilde{P}^-) V_{1,2}^d \end{aligned} \tag{5.22}$$

$$\begin{aligned} \mu^2 \frac{\partial T_{1,2}^\ell}{\partial \mu^2} &= \widetilde{P}^+ T_{1,2}^\ell \\ \mu^2 \frac{\partial V_{1,2}^\ell}{\partial \mu^2} &= \widetilde{P}^- V_{1,2}^\ell \end{aligned} \tag{5.23}$$

5.1 Evolution equations at LO

At LO in QED, considering that $\widetilde{P}_{q\ell}^S = 0$ and that $\mathcal{P}_{ij} = \widetilde{P}_{ij}$, the evolution equations above reduce to:

$$
\begin{aligned}
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix} &= \left[\begin{pmatrix} P_{gg} & 0 & P_{gq} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2n_f P_{qg} & 0 & P_{qq} & 0 & 0 \\ \frac{n_u - n_d}{n_f} 2n_f P_{qg} & 0 & \frac{n_u - n_d}{n_f}(P_{qq} - P^+) & P^+ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & e_\Sigma^2 \widetilde{P}_{\gamma\gamma} & \eta^+ \widetilde{P}_{\gamma q} & \eta^- \widetilde{P}_{\gamma q} & \widetilde{P}_{\gamma q} \\ 0 & 2e_\Sigma^2 \widetilde{P}_{q\gamma} & \eta^+ \widetilde{P}_{qq} & \eta^- \widetilde{P}_{qq} & 0 \\ 0 & 2\delta_e^2 \widetilde{P}_{q\gamma} & \eta^- \widetilde{P}_{qq} & \eta^+ \widetilde{P}_{qq} & 0 \\ 0 & 2n_\ell \widetilde{P}_{q\gamma} & 0 & 0 & \widetilde{P}_{qq} \end{pmatrix} \right] \begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \\ \Sigma_\ell \end{pmatrix}
\end{aligned}
\tag{5.24}
$$

$$
\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} V \\ \Delta_V \end{pmatrix} = \left[\begin{pmatrix} P^V & 0 \\ \frac{n_u - n_d}{n_f}(P^V - P^-) & P^- \end{pmatrix} + \begin{pmatrix} \eta^+ \widetilde{P}_{qq} & \eta^- \widetilde{P}_{qq} \\ \eta^- \widetilde{P}_{qq} & \eta^+ \widetilde{P}_{qq} \end{pmatrix} \right] \begin{pmatrix} V \\ \Delta_V \end{pmatrix} \tag{5.25}
$$

$$
\mu^2 \frac{\partial V_\ell}{\partial \mu^2} = \widetilde{P}_{qq} V_\ell \tag{5.26}
$$

$$
\begin{aligned}
\mu^2 \frac{\partial T_{1,2}^u}{\partial \mu^2} &= (P^+ + e_u^2 \widetilde{P}_{qq}) T_{1,2}^u \\
\mu^2 \frac{\partial T_{1,2}^d}{\partial \mu^2} &= (P^+ + e_d^2 \widetilde{P}_{qq}) T_{1,2}^d \\
\mu^2 \frac{\partial V_{1,2}^u}{\partial \mu^2} &= (P^- + e_u^2 \widetilde{P}_{qq}) V_{1,2}^u \\
\mu^2 \frac{\partial V_{1,2}^d}{\partial \mu^2} &= (P^- + e_d^2 \widetilde{P}_{qq}) V_{1,2}^d
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
\mu^2 \frac{\partial T_{1,2}^\ell}{\partial \mu^2} &= \widetilde{P}_{qq} T_{1,2}^\ell \\
\mu^2 \frac{\partial V_{1,2}^\ell}{\partial \mu^2} &= \widetilde{P}_{qq} V_{1,2}^\ell
\end{aligned}
\tag{5.28}
$$

There is one last detail to be discussed. Contrary to electrons and muons, whose masses are well below $\Lambda_{\rm QCD}$, the τ has a mass equal to $m_\tau = 1.777$ GeV which is well above $\Lambda_{\rm QCD}$ and even above the typical initial scale $Q_0 \simeq 1$ GeV from which PDFs are usually evolved. As a consequence, we need to account for the possibility to cross the τ mass threshold. To do so, we just need to realise that below the τ threshold, where $n_\ell = 2$, the T_8^ℓ and V_8^ℓ reduce to the lepton singlet Σ_ℓ and total valence V_ℓ distributions and thus evolve as such.

From the implementation point of view, the main problem is the fact that we need to introduce a new threshold between the charm and the bottom thresholds and this will complicate the structure of the code.

6 QED corrections at NLO

In this section we discuss the details of the implementation of the NLO QED corrections. While the inclusion of the LO corrections presents many simplifications, *e.g.* QED and QCD corrections do not mix and thus the DGLAP equations as well as the α_s and α evolution equations are decoupled, when including NLO corrections QED and QCD corrections mix both in the DGLAP and in the coupling evolution equations. In addition, as far as the DIS structure functions are concerned, such corrections induce photon initiated diagrams that have to be included in order to have the full set of corrections. In the following, we will first discuss how to generalise the coupling evolution equations, we will the consider the DIS structure functions, and finally we will turn to the DGLAP.

6.1 Evolution equations for the couplings

As already mentioned, NLO QED corrections induce a mixing with QCD. At the level of the couplings, this essentially means that the QCD β-function will get corrections proportional to α and vice-versa, the QED β-function will get corrections proportional to α_s, that is:

$$\begin{aligned} \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} &= \beta^{\rm QCD}(\alpha_s, \alpha)\,. \\ \mu^2 \frac{\partial \alpha}{\partial \mu^2} &= \beta^{\rm QED}(\alpha_s, \alpha)\,. \end{aligned} \tag{6.1}$$

As a consequence, these evolution equations form a set of coupled differential equations.

Before discussing the numerical solution of these equations, we first need to know explicitly the new contributions to the β-functions. In particular, up to NLO one has:

$$\beta^{\rm QCD}(\alpha_s, \alpha) = -\alpha_s \left[\beta_0^{(\alpha_s)}\left(\frac{\alpha_s}{4\pi}\right) + \beta_1^{(\alpha_s\alpha)}\left(\frac{\alpha_s}{4\pi}\right)\left(\frac{\alpha}{4\pi}\right) + \beta_1^{(\alpha_s^2)}\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right], \tag{6.2}$$

and:

$$\beta^{\rm QED}(\alpha_s, \alpha) = -\alpha \left[\beta_0^{(\alpha)}\left(\frac{\alpha}{4\pi}\right) + \beta_1^{(\alpha\alpha_s)}\left(\frac{\alpha}{4\pi}\right)\left(\frac{\alpha_s}{4\pi}\right) + \beta_1^{(\alpha^2)}\left(\frac{\alpha}{4\pi}\right)^2 + \dots\right], \tag{6.3}$$

where the new terms $\beta_1^{(\alpha_s\alpha)}$ and $\beta_1^{(\alpha\alpha_s)}$ can be read from Ref. [3]. Taking into account the additional factor four in the definition of the expansion parameter and writing explicitly the colour factors, one finds:

$$\beta_1^{(\alpha_s\alpha)} = -2\sum_{i=1}^{n_f} q_i^2 \quad \text{and} \quad \beta_1^{(\alpha\alpha_s)} = -\frac{16}{3}N_c\sum_{i=1}^{n_f} q_i^2\,, \tag{6.4}$$

where n_f is the number of active quark flavours and $N_c = 3$ is the number of colours. We also need the term $\beta_1^{(\alpha^2)}$ which can again be taken from the same paper:

$$\beta_1^{(\alpha^2)} = -4\left(n_l + N_c\sum_{i=1}^{n_f} q_i^2\right), \tag{6.5}$$

where n_l is the number of active lepton flavours.

Eq. (6.1) can be written in the vectorial form:

$$\frac{\partial \boldsymbol{\alpha}}{\partial t} = \boldsymbol{\beta}\left(\boldsymbol{\alpha}(t)\right), \tag{6.6}$$

with $t = \ln\mu^2$ and:

$$\boldsymbol{\alpha} = \begin{pmatrix}\alpha_s \\ \alpha\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\beta} = \begin{pmatrix}\beta^{\rm QCD} \\ \beta^{\rm QED}\end{pmatrix}. \tag{6.7}$$

Eq. (6.6) is an ordinary differential equation that can be numerically solved using the Runge-Kutta method.

The vector form of the fourth order Runge-Kutta algorithm for a step h is:

$$\begin{aligned} \boldsymbol{k}_1 &= h\boldsymbol{\beta}\left(\boldsymbol{\alpha}(t)\right) \\ \boldsymbol{k}_2 &= h\boldsymbol{\beta}\left(\boldsymbol{\alpha}(t) + \frac{\boldsymbol{k}_1}{2}\right) \\ \boldsymbol{k}_3 &= h\boldsymbol{\beta}\left(\boldsymbol{\alpha}(t) + \frac{\boldsymbol{k}_2}{2}\right) \\ \boldsymbol{k}_4 &= h\boldsymbol{\beta}\left(\boldsymbol{\alpha}(t) + \boldsymbol{k}_3\right) \\ \boldsymbol{\alpha}(t+h) &= \boldsymbol{\alpha}(t) + \frac{\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4}{6} + \mathcal{O}(h^5). \end{aligned} \tag{6.8}$$

This formulation assumes that α_s and α evolve from the same scale t to $t + h$. In practice, this means that the reference scale at which the two coupling are given must be the same. This is usually not the case because one might want to define α_s, say, at the Z mass scale M_Z and α, say, at the τ mass scale m_τ. This is no longer possible when introducing mixed $\mathcal{O}(\alpha_s\alpha)$ corrections because the subtraction of the ultraviolet divergences must happen at the same scale μ.

6.2 NLO QED corrections to DIS structure functions

The first photon-initiated corrections to the DIS structure functions represent the NLO QED corrections. Such corrections provide a direct handle on the photon PDF from DIS data. In fact, at LO in QED the photon PDF does not contribute directly to structure functions and it is only indirectly constrained from data through its coupling to the singlet PDF in the DGLAP evolution.

When considering NLO QED corrections to DIS structure functions, one has to include into the hard cross sections all the $\mathcal{O}(\alpha)$ diagrams where one single photon is either in the initial state or emitted from an incoming quark (or possibly an incoming lepton). Such diagrams are purely of QED origin and no QCD contributions are present. As a consequence, the corresponding coefficient functions can be easily derived from the QCD expressions just by properly adjusting the colour factors. In addition, this correspondence holds regardless of whether mass effects are included or not and thus, starting from the pure NLO QCD expressions, one can obtain the NLO QED corrections to be added to any general-mass scheme.

The main complication arises from the flavour structure. In fact, due to the fact that the coupling of the photon is proportional to the squared charge of the parton it couples to (a quark or a lepton), in the case of the quarks the isospin symmetry is broken.

Let us start with the ZM coefficient functions. We concentrate on the $\mathcal{O}(\alpha)$ contribution to the generic NC structure function F. This correction can easily be derived from the structure of the $\mathcal{O}(\alpha_s)$ correction. The algorithm is very simple, for the coefficient functions one has:

$$\begin{aligned} C_+^{(\alpha)} &= \frac{C_+^{(\alpha_s)}}{C_F} \\ C_g^{(\alpha)} &= \frac{C_g^{(\alpha_s)}}{T_R} \end{aligned} \tag{6.9}$$

as there is no pure-singlet contribution at this order where $C_F = 4/3$ and $T_R = 1/2$ are the usual QCD colour factors. For constructing the structure functions, given the simple structure of the diagrams involved, one just has to do the following replacements in the pure QCD contributions:

$$\begin{aligned} B_q(Q) &\rightarrow B_q(Q)e_q^2 \quad \text{for} \quad F_2, F_L \\ D_q(Q) &\rightarrow D_q(Q)e_q^2 \quad \text{for} \quad F_3 \end{aligned} \tag{6.10}$$

where $B_q(Q)$ and $D_q(Q)$ are the NC couplings. With this algorithm at hand one can write the $\mathcal{O}(\alpha)$ contributions to the light and heavy quark structure functions as:

$$\begin{aligned} F^{(\alpha),l} &= \langle B_l e_l^2 \rangle \left[\frac{C_g^{(\alpha_s)}}{T_R}\gamma + \frac{1}{n_f}\frac{C_+^{(\alpha_s)}}{C_F}\Sigma \right] + \frac{1}{2}(B_u e_u^2 - B_d e_d^2)\frac{C_+^{(\alpha_s)}}{C_F}T_3 \\ &+ \frac{1}{6}(B_u e_u^2 + B_d e_d^2 - 2B_s e_s^2)\frac{C_+^{(\alpha_s)}}{C_F}T_8 + \langle B_l e_l^2 \rangle \frac{C_+^{(\alpha_s)}}{C_F}\sum_{j=4}^{n_f}\frac{1}{j(j-1)}f_j \,. \end{aligned} \tag{6.11}$$

The heavy-quark components instead are:

$$\begin{aligned} F^{(\alpha),c} &= \theta(Q^2 - m_c^2)B_c e_c^2 \left\{ \left[\frac{C_g^{(\alpha_s)}}{T_R}\gamma + \frac{1}{n_f}\frac{C_+^{(\alpha_s)}}{C_F}\Sigma \right] - \frac{1}{4}\frac{C_+^{(\alpha_s)}}{C_F}T_{15} + \frac{C_+^{(\alpha_s)}}{C_F}\sum_{j=5}^{n_f}\frac{1}{j(j-1)}f_j \right\}, \\ F^{(\alpha),b} &= \theta(Q^2 - m_b^2)B_b e_b^2 \left\{ \left[\frac{C_g^{(\alpha_s)}}{T_R}\gamma + \frac{1}{n_f}\frac{C_+^{(\alpha_s)}}{C_F}\Sigma \right] - \frac{1}{5}\frac{C_+^{(\alpha_s)}}{C_F}T_{24} + \frac{C_+^{(\alpha_s)}}{C_F}\sum_{j=6}^{n_f}\frac{1}{j(j-1)}f_j \right\}, \\ F^{(\alpha),t} &= \theta(Q^2 - m_t^2)B_t e_t^2 \left\{ \left[\frac{C_g^{(\alpha_s)}}{T_R}\gamma + \frac{1}{n_f}\frac{C_+^{(\alpha_s)}}{C_F}\Sigma \right] - \frac{1}{6}\frac{C_+^{(\alpha_s)}}{C_F}T_{35} \right\}. \end{aligned} \tag{6.12}$$

The CC structure functions are more complicated to treat because their flavour structure is more complex. As a first step, we write the $\mathcal{O}(\alpha_s)$ contribution to $F = F_2, F_L$ (we will consider F_3 later) in a convenient way as:

$$F^{\nu,(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[C_\pm^{(\alpha_s)}\left(D+\overline{U}\right)+2C_g^{(\alpha_s)}g\right] \tag{6.13}$$

and:

$$F^{\overline{\nu},(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[C_\pm^{(\alpha_s)}\left(\overline{D}+U\right)+2C_g^{(\alpha_s)}g\right] \tag{6.14}$$

where we have omitted the convolution symbol and overall factor $2x$. At this order we do not have to worry about whether C_+ or C_- has to be used because they coincide. However, in the following it will appear naturally which one has be used where so we keep them distinguished. One can combine the expressions above conveniently as:

$$F^{\nu,(\alpha_s)} + F^{\overline{\nu},(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[C_+^{(\alpha_s)}\left(D^+ + U^+\right)+4C_g^{(\alpha_s)}g\right] \tag{6.15}$$

and:

$$F^{\nu,(\alpha_s)} - F^{\overline{\nu},(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[C_-^{(\alpha_s)}\left(D^- - U^-\right)\right] \tag{6.16}$$

Using the usual algorithm one can write down the $\mathcal{O}(\alpha)$ contribution to the CC structure functions as:

$$F^{\nu,(\alpha)} + F^{\overline{\nu},(\alpha)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[\frac{C_+^{(\alpha_s)}}{C_F}\left(e_D^2 D^+ + e_U^2 U^+\right)+2(e_D^2+e_U^2)\frac{C_g^{(\alpha_s)}}{T_R}\gamma\right] \tag{6.17}$$

and:

$$F^{\nu,(\alpha)} - F^{\overline{\nu},(\alpha)} = \sum_{U=u,c,t}\sum_{D=d,s,b} |V_{UD}|^2 \left[\frac{C_-^{(\alpha_s)}}{C_F}\left(e_D^2 D^- - e_U^2 U^-\right)\right] \tag{6.18}$$

Now the question is expressing these combinations in terms of PDFs in the evolution basis. The starting point are the relations:

$$q_i^\pm = \sum_{j=1}^{6} M_{ij} d_j^\pm\,, \tag{6.19}$$

where $d_j^\pm$ belong to the QCD evolution basis, that is: $d_1^+ = \Sigma$, $d_2^+ = -T_3$, $d_3^+ = T_8$, $d_4^+ = T_{15}$, $d_5^+ = T_{24}$, and $d_6^+ = T_{35}$ and $d_1^- = V$, $d_2^- = -V_3$, $d_3^- = V_8$, $d_4^- = V_{15}$, $d_5^- = V_{24}$, and $d_6^- = V_{35}$. Note that here we are using the more "natural" ordering for the distributions $q_i = \{d,u,s,c,b,t\}$ rather than that where u comes before d; this is the reason of the minus sign in front of T_3 and V_3. The trasformation matrix M_{ij} can be written as:

$$\begin{aligned} M_{ij} &= \theta_{ji}\frac{1-\delta_{ij}j}{j(j-1)} \quad j\geq 2\,, \\ M_{i1} &= \frac{1}{6}\,, \end{aligned} \tag{6.20}$$

with $\theta_{ji} = 1$ for $j \geq i$ and zero otherwise. In addition, one can showM_{ij} is such that:

$$\sum_{j=1}^{6} M_{ij} = 0\,, \quad \text{and} \quad \sum_{i=1}^{6} M_{ij} = \delta_{1j}\,. \tag{6.21}$$

Using Eq. (6.19) we can make the following identifications:

$$D^\pm = q_{2j-1}^\pm \quad \text{and} \quad U^\pm = q_{2j}^\pm\,, \quad j=1,2,3\,. \tag{6.22}$$

In addition e_U^2 and e_D^2 do not depend on the particular "value" of U and D. So that we can write:

$$F^{\nu,(\alpha)} + F^{\overline{\nu},(\alpha)} = \sum_{i=1}^{3}\sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 \left[\frac{C_+^{(\alpha_s)}}{C_F}\left(e_D^2 q_{2j-1}^+ + e_U^2 q_{2i}^+\right)+2(e_D^2+e_U^2)\frac{C_g^{(\alpha_s)}}{T_R}\gamma\right] \tag{6.23}$$

and:

$$F^{\nu,(\alpha)} - F^{\overline{\nu},(\alpha)} = \sum_{i=1}^{3}\sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 \left[\frac{C_-^{(\alpha_s)}}{C_F} \left(e_D^2 q_{2j-1}^- - e_U^2 q_{2i}^- \right) \right] \tag{6.24}$$

Now, let us concentrate on the combinations:

$$e_D^2 q_{2j-1}^\pm \pm e_U^2 q_{2i}^\pm = e_D^2 \sum_{k=1}^{6} M_{(2j-1),k} d_k^\pm \pm e_U^2 \sum_{k=1}^{6} M_{2i,k} d_k^\pm = \sum_{k=1}^{6} \left(e_D^2 M_{(2j-1),k} \pm e_U^2 M_{2i,k} \right) d_k^\pm \tag{6.25}$$

Now we use the explicit for of M_{ij} given in Eq. (6.20) to make some simplifications:

$$e_D^2 q_{2j-1}^\pm \pm e_U^2 q_{2i}^\pm \quad = \quad \frac{1}{6}(e_D^2 \pm e_U^2) d_1^\pm + \sum_{k=2}^{6} \left(e_D^2 \theta_{k,(2j-1)} \frac{1-\delta_{(2j-1),k} k}{k(k-1)} \pm e_U^2 \theta_{k,2i} \frac{1-\delta_{2i,k} k}{k(k-1)} \right) d_k^\pm \tag{6.26}$$

Since things are getting quite complicated, we make the approximation of diagonal CKM matrix for the quark contributions. The error one does in taking this approximation is proportional to α times the off diagonal CMK matrix elements which is clearly quite small. In practice, this means setting:

$$|V_{2i,(2j-1)}|^2 = \delta_{ij} \tag{6.27}$$

so that:

$$F_\pm^{(\alpha)} \equiv F^{\nu,(\alpha)} \pm F^{\overline{\nu},(\alpha)} = \sum_{i=1}^{3} \left[\frac{C_\pm^{(\alpha_s)}}{C_F} \left(e_D^2 q_{2i-1}^\pm \pm e_U^2 q_{2i}^\pm \right) + P_\pm 2 (e_D^2 \pm e_U^2) \frac{C_g^{(\alpha_s)}}{T_R} \gamma \right] =$$

$$\sum_{i=1}^{3} \left\{ \frac{C_\pm^{(\alpha_s)}}{C_F} \left[\frac{1}{6}(e_D^2 \pm e_U^2) d_1^\pm + \sum_{k=2}^{6} \left(e_D^2 \theta_{k,(2i-1)} \frac{1-\delta_{(2i-1),k} k}{k(k-1)} \pm e_U^2 \theta_{k,2i} \frac{1-\delta_{2i,k} k}{k(k-1)} \right) d_k^\pm \right] + P_\pm 2 (e_D^2 \pm e_U^2) \frac{C_g^{(\alpha_s)}}{T_R} \gamma \right\} \tag{6.28}$$

with:

$$P_\pm = \frac{1 \pm 1}{2} \tag{6.29}$$

According to our definition of light, charm, bottom, and top components we have that $i = 1$ belongs to the light structure function, $i = 2$ to the charm structure function, and $i = 2$ to the top structure function. The bottom structure function does not get any contribution because, according to our definition, it is proportional to off-diagonal elements of the CKM matrix, so that:

$$F_\pm^{(\alpha)} = F_\pm^{(\alpha),l} + F_\pm^{(\alpha),c} + F_\pm^{(\alpha),t} \tag{6.30}$$

with:

$$F_\pm^{(\alpha),l} = \frac{C_\pm^{(\alpha_s)}}{C_F} \left[\frac{1}{6}(e_D^2 \pm e_U^2) d_1^\pm + \frac{e_D^2 \mp e_U^2}{2} d_2^\pm + (e_D^2 \pm e_U^2) \sum_{k=3}^{6} \frac{1}{k(k-1)} d_k^\pm \right] + P_\pm 2 (e_D^2 \pm e_U^2) \frac{C_g^{(\alpha_s)}}{T_R} \gamma \tag{6.31}$$

$$F_\pm^{(\alpha),c} = \frac{C_\pm^{(\alpha_s)}}{C_F} \left[\frac{1}{6}(e_D^2 \pm e_U^2) d_1^\pm - \frac{e_D^2}{3} d_3^\pm + \frac{e_D^2 \mp 3 e_U^2}{12} d_4^\pm + (e_D^2 \pm e_U^2) \sum_{k=5}^{6} \frac{1}{k(k-1)} d_k^\pm \right] + P_\pm 2 (e_D^2 \pm e_U^2) \frac{C_g^{(\alpha_s)}}{T_R} \gamma \tag{6.32}$$

$$F_\pm^{(\alpha),t} = \frac{C_\pm^{(\alpha_s)}}{C_F} \left[\frac{1}{6}(e_D^2 \pm e_U^2) d_1^\pm - \frac{e_D^2}{5} d_5^\pm + \frac{e_D^2 \mp 5 e_U^2}{30} d_6^\pm \right] + P_\pm 2 (e_D^2 \pm e_U^2) \frac{C_g^{(\alpha_s)}}{T_R} \gamma \tag{6.33}$$

Let us define:

$$\epsilon_\pm = \frac{e_D^2 \pm e_U^2}{2} \tag{6.34}$$

and recall that:

$$F^{\nu,(\alpha)} = \frac{F_+^{(\alpha)} + F_-^{(\alpha),t}}{2} \tag{6.35}$$

and:

$$F^{\overline{\nu},(\alpha)} = \frac{F_+^{(\alpha)} - F_-^{(\alpha),t}}{2} \tag{6.36}$$

and use the fact that:

$$C_+^{(\alpha_s)} = C_-^{(\alpha_s)} = C_q^{(\alpha_s)} \tag{6.37}$$

so that:

$$F^{\nu,(\alpha),l} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ + \frac{\epsilon_-}{6}d_1^- + \frac{\epsilon_-}{2}d_2^+ + \frac{\epsilon_+}{2}d_2^- + \sum_{k=3}^{6}\frac{\epsilon_+ d_k^+ + \epsilon_- d_k^-}{k(k-1)}\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.38}$$

$$F^{\overline{\nu},(\alpha),l} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ - \frac{\epsilon_-}{6}d_1^- + \frac{\epsilon_-}{2}d_2^+ - \frac{\epsilon_+}{2}d_2^- + \sum_{k=3}^{6}\frac{\epsilon_+ d_k^+ - \epsilon_- d_k^-}{k(k-1)}\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.39}$$

$$F^{\nu(\alpha),c} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ + \frac{\epsilon_+}{6}d_1^- - \frac{e_D^2}{6}(d_3^+ + d_3^-) + \frac{e_D^2+3e_U^2}{24}d_4^+ + \frac{e_D^2-3e_U^2}{24}d_4^- + \sum_{k=5}^{6}\frac{\epsilon_+ d_k^+ + \epsilon_- d_k^-}{k(k-1)}\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.40}$$

$$F^{\overline{\nu}(\alpha),c} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ - \frac{\epsilon_+}{6}d_1^- - \frac{e_D^2}{6}(d_3^+ - d_3^-) + \frac{e_D^2+3e_U^2}{24}d_4^+ - \frac{e_D^2-3e_U^2}{24}d_4^- + \sum_{k=5}^{6}\frac{\epsilon_+ d_k^+ - \epsilon_- d_k^-}{k(k-1)}\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.41}$$

$$F^{\nu,(\alpha),t} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ + \frac{\epsilon_-}{6}d_1^- - \frac{e_D^2}{10}(d_5^+ + d_5^-) + \frac{e_D^2+5e_U^2}{60}d_6^+ + \frac{e_D^2-5e_U^2}{60}d_6^-\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.42}$$

$$F^{\overline{\nu},(\alpha),t} = \frac{C_q^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^+ - \frac{\epsilon_-}{6}d_1^- - \frac{e_D^2}{10}(d_5^+ - d_5^-) + \frac{e_D^2+5e_U^2}{60}d_6^+ - \frac{e_D^2-5e_U^2}{60}d_6^-\right] + 2\epsilon_+\frac{C_g^{(\alpha_s)}}{T_R}\gamma \tag{6.43}$$

The relations above hold for F_2 and F_L. For F_3 the analogous of Eqs. (6.44) and (6.45) are:

$$F_3^{\nu,(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b}|V_{UD}|^2\left[C_{3,q}^{(\alpha_s)}\left(D-\overline{U}\right) + 2C_{3,g}^{(\alpha_s)}g\right] \tag{6.44}$$

and:

$$F_3^{\overline{\nu},(\alpha_s)} = \sum_{U=u,c,t}\sum_{D=d,s,b}|V_{UD}|^2\left[C_{3,q}^{(\alpha_s)}\left(-\overline{D}+U\right) + 2C_{3,g}^{(\alpha_s)}g\right] \tag{6.45}$$

In practice, as compared to F_2 and F_L, the structure of the observables is the same with the only difference that the anti-quark get a minus sign. In terms of the distributions in the evolution basis $d_k^\pm$ this is equivalent to exchange the plus with the minus distributions, *i.e.* $d_k^+ \leftrightarrow d_k^-$. Starting from the relations above we can directly write the results for F_3:

$$F_3^{\nu,(\alpha),l} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- + \frac{\epsilon_-}{6}d_1^+ + \frac{\epsilon_-}{2}d_2^- + \frac{\epsilon_+}{2}d_2^+ + \sum_{k=3}^{6}\frac{\epsilon_+ d_k^- + \epsilon_- d_k^+}{k(k-1)}\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.46}$$

$$F_3^{\overline{\nu},(\alpha),l} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- - \frac{\epsilon_-}{6}d_1^+ + \frac{\epsilon_-}{2}d_2^- - \frac{\epsilon_+}{2}d_2^+ + \sum_{k=3}^{6}\frac{\epsilon_+ d_k^- - \epsilon_- d_k^+}{k(k-1)}\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.47}$$

$$F_3^{\nu(\alpha),c} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- + \frac{\epsilon_+}{6}d_1^+ - \frac{e_D^2}{6}(d_3^- + d_3^+) + \frac{e_D^2+3e_U^2}{24}d_4^- + \frac{e_D^2-3e_U^2}{24}d_4^+ + \sum_{k=5}^{6}\frac{\epsilon_+ d_k^- + \epsilon_- d_k^+}{k(k-1)}\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.48}$$

$$F_3^{\overline{\nu}(\alpha),c} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- - \frac{\epsilon_+}{6}d_1^+ - \frac{e_D^2}{6}(d_3^- - d_3^+) + \frac{e_D^2+3e_U^2}{24}d_4^- - \frac{e_D^2-3e_U^2}{24}d_4^+ + \sum_{k=5}^{6}\frac{\epsilon_+ d_k^- - \epsilon_- d_k^+}{k(k-1)}\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.49}$$

$$F_3^{\nu,(\alpha),t} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- + \frac{\epsilon_-}{6}d_1^+ - \frac{e_D^2}{10}(d_5^- + d_5^+) + \frac{e_D^2+5e_U^2}{60}d_6^- + \frac{e_D^2-5e_U^2}{60}d_6^+\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.50}$$

$$F_3^{\overline{\nu},(\alpha),t} = \frac{C_{3,q}^{(\alpha_s)}}{C_F}\left[\frac{\epsilon_+}{6}d_1^- - \frac{\epsilon_-}{6}d_1^+ - \frac{e_D^2}{10}(d_5^- - d_5^+) + \frac{e_D^2+5e_U^2}{60}d_6^- - \frac{e_D^2-5e_U^2}{60}d_6^+\right] + 2\epsilon_+\frac{C_{3,g}^{(\alpha_s)}}{T_R}\gamma \tag{6.51}$$

6.3 The DGLAP equations

In this section we address the question of implementing the full NLO QCD⊗QED corrections to the DGLAP equation. For the moment we limit ourselves to considering only the presence of the photon PDF and only later we will include also the lepton PDFs. The starting point is Eqs. (2.4)-(2.5) and what we have to do is identifying the form of the $\widetilde{P}$ splitting functions. By definition they are all proportional to at least one power of α. In particular, when including only LO QED corrections:

$$\widetilde{P} = \alpha\mathcal{P}^{(0,1)} + \dots \tag{6.52}$$

where we are using the notation of Refs. [4, 5] to indicate the power of α_s and α that a given splitting function multiplies. The inclusion of the full NLO QCD⊗QED corrections implies including more terms to $\widetilde{P}$. In particular, we have to include all terms whose sum of the power of α_s and α is equal to two excluding the terms which are proportional to α_s only. As a consequence, we have that:

$$\widetilde{P} = \alpha\mathcal{P}^{(0,1)} + \alpha_s\alpha\mathcal{P}^{(1,1)} + \alpha^2\mathcal{P}^{(0,2)} \dots \tag{6.53}$$

and the corrections $\mathcal{P}^{(1,1)}$ and $\mathcal{P}^{(0,2)}$ have been recently computed and published in Refs. [4, 5].

It is now necessary to analyse the structure of the two additional terms $\mathcal{P}^{(1,1)}$ and $\mathcal{P}^{(0,2)}$ in order to construct the corresponding splitting matrices to be included in the DGLAP equations. Let us start with the $\mathcal{O}(\alpha_s\alpha)$ correction. We first identify the vanishing terms and from Eq. (32) of Ref. [4] we read that:

$$\mathcal{P}_{qq}^{S(1,1)} = \mathcal{P}_{q\overline{q}}^{S(1,1)} = 0 \tag{6.54}$$

while from Eq. (1.6) we deduce that:

$$\begin{aligned}\mathcal{P}_{qq}^{(1,1)} &= \mathcal{P}^{+(1,1)}\\ \mathcal{P}^{V(1,1)} &= \mathcal{P}^{-(1,1)}\end{aligned} \tag{6.55}$$

In the following we list the remaining terms taking them from Ref. [4] but stripping them of the electric charges which are already accounted for in our formulation. An additional factor 4 is also included due to the difference in the definition of the expansion parameters:

$$
\begin{aligned}
\mathcal{P}_{q\gamma}^{(1,1)} &= 2C_F\left\{4-9x-(1-4x)\ln x-(1-2x)\ln^2 x+4\ln(1-x)\right.\\
&+ \left. p_{qg}(x)\left[2\ln^2\left(\frac{1-x}{x}\right)-4\ln\left(\frac{1-x}{x}\right)-\frac{2\pi^2}{3}+10\right]\right\}\,,\\
\mathcal{P}_{g\gamma}^{(1,1)} &= 4C_F\left\{-16+8x+\frac{20}{3}x^2+\frac{4}{3x}-(6+10x)\ln x-2(1+x)\ln^2 x\right\}\,,\\
\mathcal{P}_{\gamma\gamma}^{(1,1)} &= -4C_F\delta(1-x)\,,\\
\mathcal{P}_{qg}^{(1,1)} &= \frac{T_R}{C_F}\mathcal{P}_{q\gamma}^{(1,1)}\,,\\
\mathcal{P}_{\gamma g}^{(1,1)} &= \frac{T_R}{C_F}\mathcal{P}_{g\gamma}^{(1,1)}\,,\\
\mathcal{P}_{gg}^{(1,1)} &= \frac{T_R}{C_F}\mathcal{P}_{\gamma\gamma}^{(1,1)}\,,\\
\mathcal{P}_{qq}^{V(1,1)} &= 8C_F\left[-\left(2\ln x\ln(1-x)+\frac{3}{2}\ln x\right)p_{qq}(x)-\frac{3+7x}{2}\ln x-\frac{1+x}{2}\ln^2 x\right.\\
&- \left.5(1-x)-\left(\frac{\pi^2}{2}-\frac{3}{8}-6\zeta_3\right)\delta(1-x)\right]\,,\\
\mathcal{P}_{q\bar{q}}^{V(1,1)} &= 8C_F\left[4(1-x)+2(1+x)\ln x+2p_{qq}(-x)S_2(x)\right]\,,\\
\mathcal{P}^{+(1,1)}=\mathcal{P}_{qq}^{V(1,1)}+\mathcal{P}_{q\bar{q}}^{V(1,1)} &= 8C_F\left[-\left(2\ln x\ln(1-x)+\frac{3}{2}\ln x\right)p_{qq}(x)+\frac{1-3x}{2}\ln x-\frac{1+x}{2}\ln^2 x\right.\\
&- \left.(1-x)+2p_{qq}(-x)S_2(x)-\left(\frac{\pi^2}{2}-\frac{3}{8}-6\zeta_3\right)\delta(1-x)\right]\,,\\
\mathcal{P}^{-(1,1)}=\mathcal{P}_{qq}^{V(1,1)}-\mathcal{P}_{q\bar{q}}^{V(1,1)} &= 8C_F\left[-\left(2\ln x\ln(1-x)+\frac{3}{2}\ln x\right)p_{qq}(x)-\frac{7+11x}{2}\ln x-\frac{1+x}{2}\ln^2 x\right.\\
&- \left.9(1-x)-2p_{qq}(-x)S_2(x)-\left(\frac{\pi^2}{2}-\frac{3}{8}-6\zeta_3\right)\delta(1-x)\right]\,,\\
\mathcal{P}_{\gamma q}^{(1,1)} &= 4C_F\left[-(3\ln(1-x)+\ln^2(1-x))p_{gq}(x)+\left(2+\frac{7}{2}x\right)\ln x\right.\\
&- \left.\left(1-\frac{x}{2}\right)\ln^2 x-2x\ln(1-x)-\frac{7}{2}x-\frac{5}{2}\right]\,,\\
\mathcal{P}_{gq}^{(1,1)} &= \mathcal{P}_{\gamma q}^{(1,1)}\,,
\end{aligned}
\tag{6.56}
$$

The function $S_2(x)$ is given by:

$$
S_2(x)=\int_{\frac{x}{1+x}}^{\frac{1}{1+x}}\frac{dz}{z}\ln\left(\frac{1-z}{z}\right)=\mathrm{Li}_2\left(-\frac{1}{x}\right)-\mathrm{Li}_2(-x)+\ln^2\left(\frac{x}{1+x}\right)-\ln^2\left(\frac{1}{1+x}\right)\,. \tag{6.57}
$$

and:

$$\begin{aligned} p_{qq}(x) &= \frac{1+x^2}{(1-x)_+} \\ p_{qg}(x) &= x^2+(1-x)^2 \\ p_{gq}(x) &= \frac{1+(1-x)^2}{x} \end{aligned} \tag{6.58}$$

It should be noticed that, excluding the colour factors and some additional pre-factors, all functions appearing above appear also in the $\mathcal{O}(\alpha_s^2)$ expressions of the splitting functions. As a consequence, there is no additional work to be done to reduce these expressions to the form needed by APFEL, that is to separate regular, singular, and local contributions.

The resulting evolution equations at $\mathcal{O}(\alpha_s\alpha)$ are:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \end{pmatrix}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} = \begin{pmatrix} e_\Sigma^2\mathcal{P}_{gg}^{(1,1)} & e_\Sigma^2\mathcal{P}_{g\gamma}^{(1,1)} & \eta^+\mathcal{P}_{gq}^{(1,1)} & \eta^-\mathcal{P}_{gq}^{(1,1)} \\ e_\Sigma^2\mathcal{P}_{\gamma g}^{(1,1)} & e_\Sigma^2\mathcal{P}_{\gamma\gamma}^{(1,1)} & \eta^+\mathcal{P}_{\gamma q}^{(1,1)} & \eta^-\mathcal{P}_{\gamma q}^{(1,1)} \\ 2e_\Sigma^2\mathcal{P}_{qg}^{(1,1)} & 2e_\Sigma^2\mathcal{P}_{q\gamma}^{(1,1)} & \eta^+\mathcal{P}^{+(1,1)} & \eta^-\mathcal{P}^{+(1,1)} \\ 2\delta_e^2\mathcal{P}_{qg}^{(1,1)} & 2\delta_e^2\mathcal{P}_{q\gamma}^{(1,1)} & \eta^-\mathcal{P}^{+(1,1)} & \eta^+\mathcal{P}^{+(1,1)} \end{pmatrix}\begin{pmatrix} g \\ \gamma \\ \Sigma \\ \Delta_\Sigma \end{pmatrix} \tag{6.59}$$

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} V \\ \Delta_V \end{pmatrix}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} = \begin{pmatrix} \eta^+\mathcal{P}^{-(1,1)} & \eta^-\mathcal{P}^{-(1,1)} \\ \eta^-\mathcal{P}^{-(1,1)} & \eta^+\mathcal{P}^{-(1,1)} \end{pmatrix}\begin{pmatrix} V \\ \Delta_V \end{pmatrix} \tag{6.60}$$

$$\begin{aligned} \mu^2\frac{\partial T_{1,2}^u}{\partial\mu^2}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} &= e_u^2\mathcal{P}^{+(1,1)}T_{1,2}^u \\ \mu^2\frac{\partial T_{1,2}^d}{\partial\mu^2}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} &= e_d^2\mathcal{P}^{+(1,1)}T_{1,2}^d \\ \mu^2\frac{\partial V_{1,2}^u}{\partial\mu^2}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} &= e_u^2\mathcal{P}^{-(1,1)}V_{1,2}^u \\ \mu^2\frac{\partial V_{1,2}^d}{\partial\mu^2}\Bigg|_{\mathcal{O}(\alpha_s\alpha)} &= e_d^2\mathcal{P}^{-(1,1)}V_{1,2}^d \end{aligned} \tag{6.61}$$

Now we turn to $\mathcal{P}^{(0,2)}$ and, referring to Eq. (58) of Ref. [5], we first observe that:

$$\mathcal{P}_{qq}^{S(0,2)} = \mathcal{P}_{q\overline{q}}^{S(0,2)} \neq 0 \tag{6.62}$$

so that:

$$\begin{aligned} &\mathcal{P}_{qq}^{(0,2)} \neq \mathcal{P}^{+(0,2)} \\ &\mathcal{P}^{V(0,2)} = \mathcal{P}^{-(0,2)} \end{aligned} \tag{6.63}$$

The fact that $\mathcal{P}^V = \mathcal{P}^-$ was actually one of our assumptions which is verified. Of course if we were to include QED corrections beyond NLO this assumption would no longer hold. Also, from Eq. (53) of the same reference we read:

$$\begin{aligned} &\mathcal{P}_{qg}^{(0,2)} = \mathcal{P}_{gq}^{(0,2)} = 0 \\ &\mathcal{P}_{gg}^{(0,2)} = \mathcal{P}_{g\gamma}^{(0,2)} = \mathcal{P}_{\gamma g}^{(0,2)} = 0 \end{aligned} \tag{6.64}$$

which provide a big simplification. The non-vanishing $\mathcal{O}(\alpha^2)$ splitting functions can mostly be written in terms of the $\mathcal{O}(\alpha_s\alpha)$ ones and are:

$$
\begin{aligned}
\mathcal{P}^{(0,2)}_{\gamma\gamma} &= \frac{1}{C_F}\mathcal{P}^{(1,1)}_{g\gamma} - 4\delta(1-x) \\
\mathcal{P}^{(0,2)}_{q\gamma} &= \frac{1}{C_F}\mathcal{P}^{(1,1)}_{q\gamma} \\
\mathcal{P}^{(0,2)}_{\gamma q} &= \frac{1}{C_F}\mathcal{P}^{(1,1)}_{\gamma q} + \left(\frac{e^2_\Sigma}{e^2_q}\right) 4\left[-\frac{4}{3}x - p_{gq}(x)\left(\frac{20}{9}+\frac{4}{3}\ln(1-x)\right)\right] , \\
\mathcal{P}^{\pm(0,2)}_{qq} &= \frac{1}{2C_F}\mathcal{P}^{\pm(1,1)} + \left(\frac{e^2_\Sigma}{e^2_q}\right) 4\left[-\frac{4}{3}(1-x) - p_{qq}(x)\left(\frac{2}{3}\ln x+\frac{10}{9}\right) - \left(\frac{2\pi^2}{9}+\frac{1}{6}\right)\delta(1-x)\right] , \\
\mathcal{P}^{S(0,2)}_{qq} &= 4\left[\frac{20}{9x} - 2 + 6x - \frac{56}{9}x^2 + \left(1+5x+\frac{8}{3}x^2\right)\ln x - (1+x)\ln^2 x\right] ,
\end{aligned}
\tag{6.65}
$$

It is interesting to notice that the expressions above for the $\mathcal{O}(\alpha^2)$ splitting functions coincide with the $\mathcal{O}(\alpha_s^2)$ where C_F and T_R are set to one and C_A set to zero.

Contrary to all cases we have treated so far in which the electric charges appeared to the second power at most, here they appear to the fourth power and thus we need to adjust the couplings accordingly. Although, we could adjust the couplings relying on basic considerations, we adopt the brute force approach and re-derive the DGLAP equations in the evolution basis defined above but concentrating only on the $\mathcal{O}(\alpha^2)$ contributions. This will provide a more solid result. In order to simplify the notation, we will get rid of all unneeded indices under the assumption that we are dealing only with the $\mathcal{O}(\alpha^2)$ contributions to the splitting functions. We now define:

$$
e^4_\Sigma = N_c(e^4_u n_u + e^4_d n_d) , \tag{6.66}
$$

so that:

$$
P_{\gamma\gamma} \to e^4_\Sigma \mathcal{P}_{\gamma\gamma} . \tag{6.67}
$$

In addition, for the splitting functions involving one photon and one quark we have:

$$
\begin{aligned}
&P_{\gamma u_i} = P_{\gamma \overline{u}_i} \to e^4_u \mathcal{P}_{\gamma u} , \quad P_{\gamma d_i} = P_{\gamma \overline{d}_i} \to e^4_d \mathcal{P}_{\gamma d} , \\
&P_{u_i\gamma} = P_{\overline{u}_i\gamma} \to e^4_u P_{q\gamma} , \quad P_{d_i\gamma} = P_{\overline{d}_i\gamma} \to e^4_d P_{q\gamma} .
\end{aligned}
\tag{6.68}
$$

Finally, we consider the splitting functions involving quarks or anti-quarks in the final and initial states:

$$
\begin{array}{l}
P_{u_iu_j} = P_{\overline{u}_i\overline{u}_j} \to e^4_u\delta_{ij}\mathcal{P}^V_{uu} + e^4_u\mathcal{P}^S_{qq} \\
P_{d_id_j} = P_{\overline{d}_i\overline{d}_j} \to e^4_d\delta_{ij}\mathcal{P}^V_{dd} + e^4_d\mathcal{P}^S_{qq} \\
P_{\overline{u}_iu_j} = P_{u_i\overline{u}_j} \to e^4_u\delta_{ij}\mathcal{P}^V_{q\overline{q}} + e^4_u\mathcal{P}^S_{qq} \\
P_{\overline{d}_id_j} = P_{d_i\overline{d}_j} \to e^4_d\delta_{ij}\mathcal{P}^V_{q\overline{q}} + e^4_d\mathcal{P}^S_{qq} \\
P_{u_id_j} = P_{d_iu_j} = P_{\overline{u}_id_j} = P_{d_i\overline{u}_j} = P_{\overline{d}_iu_j} = P_{u_i\overline{d}_j} = P_{\overline{u}_i\overline{d}_j} = P_{\overline{d}_i\overline{u}_j} \to e^2_ue^2_d\mathcal{P}^S_{qq}
\end{array} .
\tag{6.69}
$$

so that we finally get:

$$
\begin{aligned}
\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} u_j^+ \\ d_i^+ \\ g \\ \gamma \\ d_i^- \\ u_j^- \end{pmatrix}\Bigg|_{\mathcal{O}(\alpha^2)} &= \begin{pmatrix}
e^4_u\mathcal{P}^+_{uu} & 0 & 0 & 2e^4_u\mathcal{P}_{q\gamma} & 0 & 0 \\
0 & e^4_d\mathcal{P}^+_{dd} & 0 & 2e^4_d\mathcal{P}_{q\gamma} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
e^4_u\mathcal{P}_{\gamma u} & e^4_d\mathcal{P}_{\gamma d} & 0 & e^4_\Sigma\mathcal{P}_{\gamma\gamma} & 0 & 0 \\
0 & 0 & 0 & 0 & e^4_d\mathcal{P}^-_{dd} & 0 \\
0 & 0 & 0 & 0 & 0 & e^4_u\mathcal{P}^-_{uu}
\end{pmatrix}\begin{pmatrix} u_j^+ \\ d_i^+ \\ g \\ \gamma \\ d_i^- \\ u_j^- \end{pmatrix} \\
&+ \begin{pmatrix}
e^4_u\mathcal{P}^S_{qq} & e^2_ue^2_d\mathcal{P}^S_{qq} & 0 & 0 & 0 & 0 \\
e^2_ue^2_d\mathcal{P}^S_{qq} & e^4_d\mathcal{P}^S_{qq} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix} \Sigma_u \\ \Sigma_d \\ g \\ \gamma \\ V_d \\ V_u \end{pmatrix}
\end{aligned}
\tag{6.70}
$$

that can also be written as:

$$\begin{cases} \mu^2 \dfrac{\partial g}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & 0 \\ \mu^2 \dfrac{\partial \gamma}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & e_u^4 \mathcal{P}_{\gamma u}\Sigma_u + e_d^4 \mathcal{P}_{\gamma d}\Sigma_d + e_\Sigma^4 \mathcal{P}_{\gamma\gamma}\gamma \\ \mu^2 \dfrac{\partial \Sigma_d}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & e_d^4 \left(\mathcal{P}_{dd}^+ + n_d \mathcal{P}_{qq}^S\right)\Sigma_d + n_d e_d^2 e_u^2 \mathcal{P}_{qq}^S \Sigma_u + 2n_d e_d^4 \mathcal{P}_{q\gamma}\gamma \\ \mu^2 \dfrac{\partial \Sigma_u}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & e_u^4 \left(\mathcal{P}_{uu}^+ + n_u \mathcal{P}_{qq}^S\right)\Sigma_u + n_u e_u^2 e_d^2 \mathcal{P}_{qq}^S \Sigma_d + 2n_u e_u^4 \mathcal{P}_{q\gamma}\gamma \\ \mu^2 \dfrac{\partial V_d}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & e_d^4 \mathcal{P}_{dd}^- V_d \\ \mu^2 \dfrac{\partial V_u}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & e_u^4 \mathcal{P}_{uu}^- V_u \end{cases} . \tag{6.71}$$

Now, using the fact that:

$$\begin{array}{l} \Sigma = \Sigma_u + \Sigma_d \\ \Delta_\Sigma = \Sigma_u - \Sigma_d \\ V = V_u + V_d \\ \Delta_V = V_u - V_d \end{array} \tag{6.72}$$

we can write:

$$\begin{cases} \mu^2 \dfrac{\partial g}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & 0 \\ \mu^2 \dfrac{\partial \gamma}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{\gamma u} + e_d^4 \mathcal{P}_{\gamma d}\right)\Sigma + \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{\gamma u} - e_d^4 \mathcal{P}_{\gamma d}\right)\Delta_\Sigma + e_\Sigma^4 \mathcal{P}_{\gamma\gamma}\gamma \\ \mu^2 \dfrac{\partial \Sigma}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^+ + e_d^4 \mathcal{P}_{dd}^+ + 2\eta^+ e_\Sigma^2 \mathcal{P}_{qq}^S\right)\Sigma + \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^+ - e_d^4 \mathcal{P}_{dd}^+ + 2\eta^- e_\Sigma^2 \mathcal{P}_{qq}^S\right)\Delta_\Sigma + 2e_\Sigma^4 \mathcal{P}_{q\gamma}\gamma \\ \mu^2 \dfrac{\partial \Delta_\Sigma}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^+ - e_d^4 \mathcal{P}_{dd}^+ + 2\eta^- \delta_e^2 \mathcal{P}_{qq}^S\right)\Sigma + \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^+ + e_d^4 \mathcal{P}_{dd}^+ + 2\eta^+ \delta_e^2 \mathcal{P}_{qq}^S\right)\Delta_\Sigma + 2\delta_e^4 \mathcal{P}_{q\gamma}\gamma \\ \mu^2 \dfrac{\partial V}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^- + e_d^4 \mathcal{P}_{dd}^-\right)V + \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^- - e_d^4 \mathcal{P}_{dd}^-\right)\Delta_V \\ \mu^2 \dfrac{\partial \Delta_V}{\partial \mu^2}\Big|_{\mathcal{O}(\alpha^2)} & = & \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^- - e_d^4 \mathcal{P}_{dd}^-\right)V + \dfrac{1}{2}\left(e_u^4 \mathcal{P}_{uu}^- + e_d^4 \mathcal{P}_{dd}^-\right)\Delta_V \end{cases} . \tag{6.73}$$

And the remaining distributions evolving multiplicatively as:

$$
\begin{aligned}
\mu^2 \frac{\partial T^u_{1,2}}{\partial \mu^2}\bigg|_{\mathcal{O}(\alpha^2)} &= e_u^4 \mathcal{P}^+_{uu} T^u_{1,2} \\
\mu^2 \frac{\partial T^d_{1,2}}{\partial \mu^2}\bigg|_{\mathcal{O}(\alpha^2)} &= e_d^4 \mathcal{P}^+_{dd} T^d_{1,2} \\
\mu^2 \frac{\partial V^u_{1,2}}{\partial \mu^2}\bigg|_{\mathcal{O}(\alpha^2)} &= e_u^4 \mathcal{P}^-_{uu} V^u_{1,2} \\
\mu^2 \frac{\partial V^d_{1,2}}{\partial \mu^2}\bigg|_{\mathcal{O}(\alpha^2)} &= e_d^4 \mathcal{P}^-_{dd} V^d_{1,2}
\end{aligned}
\tag{6.74}
$$

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