

# SIDIS cross section in TMD factorisation

Valerio Bertone

April 27, 2021

## Contents

<b>1</b>	<b>Structure of the observable</b>	<b>1</b>
<b>2</b>	<b>Integrating over the final-state kinematic variables</b>	<b>2</b>
2.1	Integrating over $x$ , $z$ , and $Q$ . . . . .	3

## 1 Structure of the observable

In this document we report the relevant formulas for the computation of semi-inclusive deep-inelastic scattering (SIDIS) multiplicities under the assumption that the (negative) virtuality of the  $Q^2$  of the exchanged vector boson is much smaller than the  $Z$  mass. This allows us to neglect weak contributions and write the cross section in TMD factorisation as:

$$\frac{d\sigma}{dx dQ dz dq_T} = \frac{4\pi\alpha^2 q_T}{zxQ^3} Y_+ H(Q, \mu) \sum_q e_q^2 \int_0^\infty db b J_0(bq_T) \bar{F}_q(x, b; \mu, \zeta_1) \bar{D}_q(z, b; \mu, \zeta_2), \quad (1)$$

with  $\zeta_1 \zeta_2 = Q^4$  and:

$$Y_+ = 1 + (1 - y)^2 = 1 + \left(1 - \frac{Q^2}{xs}\right)^2, \quad (2)$$

where  $s$  is the squared center of mass energy. The single TMDs are evolved and matched onto the respective collinear functions as usual:

$$\bar{F}_q(x, b; \mu, \zeta) = x F_q(x, b; \mu, \zeta) = R_q(\mu_0, \zeta_0 \rightarrow \mu, \zeta; b) \sum_j \int_x^1 dy \mathcal{C}_{qj}(y; \mu_0, \zeta_0) \left[ \frac{x}{y} f_j\left(\frac{x}{y}, \mu_0\right) \right], \quad (3)$$

and:

$$\bar{D}_i(z, b; \mu, \zeta) = z^3 D_q(z, b; \mu, \zeta) = R_q(\mu_0, \zeta_0 \rightarrow \mu, \zeta; b) \sum_j \int_z^1 dy [y^2 \mathcal{C}_{qj}(y; \mu_0, \zeta_0)] \left[ \frac{z}{y} d_j\left(\frac{z}{y}, \mu_0\right) \right]. \quad (4)$$

Notice that here we limit to the case  $Q \ll M_Z$  such that we can neglect the contribution of the  $Z$  boson and thus the electroweak couplings are given by the squared electric charges.

As usual, low- $q_T$  non-perturbative corrections are taken into account by introducing the monotonic function  $b_*(b)$  that behaves as:

$$\lim_{b \rightarrow 0} b_*(b) = b_{\min} \quad \text{and} \quad \lim_{b \rightarrow \infty} b_*(b) = b_{\max}. \quad (5)$$

This allows us to replace the TMDs in Eq. (1) with their “regularised” version:

$$\begin{aligned} \bar{F}_q(x, b; \mu, \zeta) &\rightarrow \bar{F}_q(x, b_*(b); \mu, \zeta) f_{\text{NP}}(x, b, \zeta), \\ \bar{D}_q(z, b; \mu, \zeta) &\rightarrow \bar{D}_q(z, b_*(b); \mu, \zeta) D_{\text{NP}}(z, b, \zeta), \end{aligned} \quad (6)$$

where we have introduced the non-perturbative functions  $f_{\text{NP}}$  and  $D_{\text{NP}}$ . It is important to stress that these functions further factorise as follows:

$$\begin{aligned} f_{\text{NP}}(x, b, \zeta) &= \tilde{f}_{\text{NP}}(x, b) \exp \left[ g_K(b) \ln \left( \frac{\zeta}{Q_0^2} \right) \right], \\ D_{\text{NP}}(z, b, \zeta) &= \tilde{D}_{\text{NP}}(x, b) \exp \left[ g_K(b) \ln \left( \frac{\zeta}{Q_0^2} \right) \right]. \end{aligned} \quad (7)$$

The common exponential function represents the non-perturbative corrections to TMD evolution and the specific functional form is driven by the solution of the Collins-Soper equation where  $Q_0$  is some initial scale. Finally the set of non-perturbative functions to be determined from fits to data are  $\tilde{f}_{\text{NP}}$ ,  $\tilde{D}_{\text{NP}}$ , and  $g_K(b)$ . It is worth noticing that by definition

$$f_{\text{NP}}(x, b, \zeta) = \frac{\bar{F}_q(x, b; \mu, \zeta)}{\bar{F}_q(x, b_*(b); \mu, \zeta)}, \quad (8)$$

and similarly for  $D_{\text{NP}}$ . Therefore, one has a partial handle on the  $b$ -dependence of these functions from the region in which  $b$  is small enough to make both numerator and denominator perturbatively computable. Making use of Eq. (7) and setting  $\zeta_1 = \zeta_2 = Q^2$  allows us to rewrite Eq. (1) as:

$$\begin{aligned} \frac{d\sigma}{dx dQ dz dq_T} &= \frac{4\pi\alpha^2 q_T}{xzQ^3} Y_+ H(Q, \mu) \sum_q e_q^2 \\ &\times \int_0^\infty db J_0(bq_T) \bar{F}_q(x, b_*(b); \mu, Q^2) \bar{D}_q(z, b_*(b); \mu, Q^2) f_{\text{NP}}(x, b, Q^2) D_{\text{NP}}(z, b, Q^2). \end{aligned} \quad (9)$$

The integral in the r.h.s. can be numerically computed using the Ogata quadrature of zero-th degree (because  $J_0$  enters the integral):

$$\begin{aligned} \frac{d\sigma}{dx dQ dz dq_T} &\simeq \frac{4\pi\alpha^2}{xzQ^3} Y_+ H(Q, \mu) \sum_q e_q^2 \\ &\times \sum_{n=1}^N w_n^{(0)} \frac{\xi_n^{(0)}}{q_T} \bar{F}_q \left( x, b_* \left( \frac{\xi_n^{(0)}}{q_T} \right); \mu, Q^2 \right) \bar{D}_q \left( z, b_* \left( \frac{\xi_n^{(0)}}{q_T} \right); \mu, Q^2 \right) \\ &\times f_{\text{NP}} \left( x, \frac{\xi_n^{(0)}}{q_T}, Q^2 \right) D_{\text{NP}} \left( z, \frac{\xi_n^{(0)}}{q_T}, Q^2 \right), \end{aligned} \quad (10)$$

where  $w_n^{(0)}$  and  $\xi_n^{(0)}$  are the Ogata weights and coordinates, respectively, and the sum over  $n$  is truncated to the  $N$ -th term that should be chosen in such a way to guarantee a given target accuracy. The equation above can be conveniently recasted as follows:

$$\frac{d\sigma}{dx dQ dz dq_T} \simeq \sum_{n=1}^N w_n^{(0)} \frac{\xi_n^{(0)}}{q_T} S \left( x, z, \frac{\xi_n^{(0)}}{q_T}; \mu, Q^2 \right) f_{\text{NP}} \left( x, \frac{\xi_n^{(0)}}{q_T}, Q^2 \right) D_{\text{NP}} \left( z, \frac{\xi_n^{(0)}}{q_T}, Q^2 \right), \quad (11)$$

where:

$$S(x, z, b; \mu, Q^2) = \frac{4\pi\alpha^2}{xzQ^3} Y_+ H(Q, \mu) \sum_q e_q^2 [\bar{F}_q(x, b_*(b); \mu, Q^2)] [\bar{D}_q(z, b_*(b); \mu, Q^2)]. \quad (12)$$

## 2 Integrating over the final-state kinematic variables

Experimental measurements of differential distributions for SIDIS production are often delivered as integrated over finite regions of the final-state kinematic phase space.

More specifically, the cross section is not integrated of the transverse momentum of the vector boson,  $q_T$ , but over the transverse momentum of the outgoing hadron,  $p_{Th}$ , that is connected to the former through:

$$p_{Th} = zq_T. \quad (13)$$

The integrated cross section then reads:

$$\tilde{\sigma} = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{x_{\min}}^{x_{\max}} dx \int_{z_{\min}}^{z_{\max}} dz \int_{p_{Th,\min}/z}^{p_{Th,\max}/z} dq_T \left[ \frac{d\sigma}{dx dQ dz dq_T} \right]. \quad (14)$$

One can exploit a property of the Bessel functions to compute the indefinite integral in  $q_T$  of the cross section. Specifically, we now compute:

$$K(x, z, Q, q_T) = \int dq_T \left[ \frac{d\sigma}{dx dQ dz dq_T} \right]. \quad (15)$$

This is easily done by using the following property of the Bessel functions:

$$\int dx x J_0(x) = x J_1(x), \quad (16)$$

that is equivalent to:

$$\int dq_T q_T J_0(bq_T) = \frac{q_T}{b} J_1(bq_T). \quad (17)$$

Therefore:

$$\begin{aligned} K(x, z, Q, q_T) &= \frac{4\pi\alpha^2 q_T}{xzQ^3} Y_+ H(Q, \mu) \sum_q e_q^2 \\ &\times \int_0^\infty db J_1(bq_T) \bar{F}_q(x, b_*(b); \mu, Q^2) \bar{D}_q(z, b_*(b); \mu, Q^2) f_{\text{NP}}(x, b, Q^2) D_{\text{NP}}(z, b, Q^2). \end{aligned} \quad (18)$$

The integral can again be computed using the Ogata quadrature as:

$$K(x, z, Q, q_T) \simeq \sum_{n=1}^N w_n^{(1)} S \left( x, z, \frac{\xi_n^{(1)}}{q_T}; \mu, Q^2 \right) f_{\text{NP}} \left( x, \frac{\xi_n^{(1)}}{q_T}, Q^2 \right) D_{\text{NP}} \left( z, \frac{\xi_n^{(1)}}{q_T}, Q^2 \right), \quad (19)$$

with  $S$  given in Eq. (12). Once  $K$  is known, the integral of the cross section over the bin  $q_T \in [p_{Th,\min}/z : p_{Th,\max}/z]$  is computed as:

$$\int_{p_{Th,\min}/z}^{p_{Th,\max}/z} dq_T \left[ \frac{d\sigma}{dx dQ dz dq_T} \right] = K(x, z, Q, p_{Th,\max}/z) - K(x, z, Q, p_{Th,\min}/z). \quad (20)$$

This allows one to compute analytically one of the integrals that are often required to compare predictions to data.

## 2.1 Integrating over $x$ , $z$ , and $Q$

We now move to considering the integral of the cross section over  $x$ ,  $z$ , and  $Q$ . Since these integrals usually come together with an integration in  $q_T$ , in the following we will consider the primitive function  $K$  in Eq. (19) rather than the cross section itself, that is:

$$\tilde{K}(p_{Th}) = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{z_{\min}}^{z_{\max}} dz \int_{x_{\min}}^{x_{\max}} dx K(x, z, Q, p_{Th}/z), \quad (21)$$

so that:

$$\tilde{\sigma} = \tilde{K}(p_{Th,\max}) - \tilde{K}(p_{Th,\min}). \quad (22)$$

The amount of numerical computation required to carry out the integration of a single bin is very large. Indicatively, it amounts to computing a three-dimensional integral for each of the terms of the Ogata quadrature that usually range from a few tens to hundreds. Therefore, in order to be able to do the integrations in a reasonable amount of time and yet obtain accurate results, it is necessary to put in place an efficient integration strategy. This goal can be achieved by exploiting a numerical integration based on interpolation techniques to precompute the relevant quantities. To this purpose, we first define one grid in  $x$ ,  $\{x_\alpha\}$  with  $\alpha = 0, \dots, N_x$ , one grid in  $z$ ,  $\{z_\beta\}$  with  $\beta = 0, \dots, N_z$ , and one grid in  $Q$ ,  $\{Q_\tau\}$  with  $\tau = 0, \dots, N_Q$ , each of which with a set of interpolating functions  $\mathcal{I}$  associated. The grids should be such to span the full kinematic range covered by given data set. Then the value of  $K$  in Eq. (19) for any kinematics can be obtained through interpolation as:

$$K(x, z, Q, q_T) \simeq \sum_{n=1}^N w_n^{(1)} S\left(x, z, \frac{\xi_n^{(1)}}{q_T}; \mu, Q^2\right) \sum_{\alpha=1}^{N_x} \sum_{\beta=1}^{N_z} \sum_{\tau=1}^{N_Q} \mathcal{I}_\alpha(x) \mathcal{I}_\beta(z) \mathcal{I}_\tau(Q) \times f_{\text{NP}}\left(x_\alpha, \frac{\xi_n^{(1)}}{q_T}, Q_\tau^2\right) D_{\text{NP}}\left(z_\beta, \frac{\xi_n^{(1)}}{q_T}, Q_\tau^2\right). \quad (23)$$

Once we have  $K$  in this form, the integration over  $x$ ,  $z$ , and  $Q$  in Eq. (21) does not involve the non-perturbative functions  $f_{\text{NP}}$  and  $D_{\text{NP}}$  and can be written as:

$$\tilde{K}(p_{Th}) = \sum_{n=1}^N \sum_{\alpha=1}^{N_x} \sum_{\beta=1}^{N_z} \sum_{\tau=1}^{N_Q} W_{n\alpha\beta\tau}(p_{Th}) f_{\text{NP}}\left(x_\alpha, \frac{z_\beta \xi_n^{(1)}}{p_{Th}}, Q_\tau^2\right) D_{\text{NP}}\left(z_\beta, \frac{z_\beta \xi_n^{(1)}}{p_{Th}}, Q_\tau^2\right), \quad (24)$$

with:

$$W_{n\alpha\beta\tau}(p_{Th}) = w_n^{(1)} \int_{Q_{\min}}^{Q_{\max}} dQ \mathcal{I}_\tau(Q) \int_{z_{\min}}^{z_{\max}} dz \mathcal{I}_\beta(z) \int_{x_{\min}}^{x_{\max}} dx \mathcal{I}_\alpha(x) S\left(x, z, \frac{\xi_n^{(1)}}{p_{Th}}; \mu, Q^2\right). \quad (25)$$

Since the aim is to fit the functions  $f_{\text{NP}}$  and  $D_{\text{NP}}$  to data, one can precompute and store the coefficients  $W$  defined in Eq. (25) and compute the cross sections in a fast way making use of Eq. (24).

It is often the case that the integrated cross section, Eq. (14), is given within a certain acceptance region which is typically defined as:

$$W = \sqrt{\frac{(1-x)Q^2}{x}} \geq W_{\min}, \quad y_{\min} \leq y \left(= \frac{Q^2}{sx}\right) \leq y_{\max}. \quad (26)$$

These constraints can be expressed as constraints on the variable  $x$  for a fixed value of  $Q$ :

$$x \leq \frac{Q^2}{W_{\min}^2 + Q^2}, \quad x \geq \frac{Q^2}{sy_{\max}}, \quad x \leq \frac{Q^2}{sy_{\min}}. \quad (27)$$

Therefore, in order to implement the acceptance cuts in the computation of the integrated cross sections, it is enough to replace the integration bounds of the integral in  $x$  in Eq. (14) as follows:

$$x_{\min} \rightarrow \bar{x}_{\min}(Q) = \max\left[x_{\min}, \frac{Q^2}{sy_{\max}}\right], \quad x_{\max} \rightarrow \bar{x}_{\max}(Q) = \min\left[x_{\max}, \frac{Q^2}{sy_{\min}}, \frac{Q^2}{W_{\min}^2 + Q^2}\right]. \quad (28)$$

Given the number and complexity of integrals as that in Eq. (25) to be computed, a numerical implementation requires devising a strategy that maximises the efficiency. In order to do so, we need to “unpac” the function  $S$  and perform the integrations in a way that unnecessary computations are avoided as much as possible. Taking  $\mu = Q$  and  $b = z\xi_n^{(1)}/p_{Th}$ , using the fact that the Sudakov evolution factor for quarks  $R_q$  is universal, and dropping the unnecessary dependencies, Eq. (12) can

be conveniently reorganised as:

$$\begin{aligned}
S\left(x, z, \frac{z\xi_n^{(1)}}{p_{Th}}, Q\right) &= 4\pi \frac{\alpha^2(Q)H(Q)}{Q^3} R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right), Q\right) \\
&\times \frac{1}{z} \sum_q e_q^2 \bar{D}_q\left(z, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right) \frac{Y_+(x, Q)}{x} \bar{F}_q\left(x, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right).
\end{aligned} \tag{29}$$

with:

$$\bar{F}_q(x, b) = \sum_j \int_x^1 dy \mathcal{C}_{qj}(y) \left[ \frac{x}{y} f_j\left(\frac{x}{y}, \frac{b_0}{b}\right) \right], \tag{30}$$

and:

$$\bar{D}_i(z, b) = \sum_j \int_z^1 dy [y^2 \mathcal{C}_{qj}(y)] \left[ \frac{z}{y} d_j\left(\frac{z}{y}, \frac{b_0}{b}\right) \right], \tag{31}$$

being the initial-scale TMD distributions before Sudakov evolution. The weights in Eq. (25) can then be computed as:

$$\begin{aligned}
W_{n\alpha\beta\tau}(p_{Th}) &= w_n^{(1)} 4\pi \int_{Q_{\min}}^{Q_{\max}} dQ \mathcal{I}_\tau(Q) \frac{\alpha^2(Q)H(Q)}{Q^3} \\
&\times \int_{z_{\min}}^{z_{\max}} dz \mathcal{I}_\beta(z) R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right), Q\right) \frac{1}{z} \sum_q e_q^2 \bar{D}_q\left(z, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right)\right) \\
&\times \int_{\bar{x}_{\min}(Q)}^{\bar{x}_{\max}(Q)} dx \mathcal{I}_\alpha(x) \frac{Y_+(x, Q)}{x} \bar{F}_q\left(x, b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}, Q\right)\right).
\end{aligned} \tag{32}$$

This nesting should optimise the computation of the integrals.