# Convolution integrals

## Contents



This section discusses the approach implemented in APFEL++ to compute Mellin-convolution integrals. This kind of convolutions appears very often in the presence of collinearly factorised quantities as integrals over the longitudinal momentum fraction of partons. This is for example the case of all cross sections in collinear factorisation with hadrons in the initial and/or final state. As a consequence, Mellin-convolution integrals also appear in the DGLAP evolution equations. Also in transverse-momentum-dependent (TMD) factorisation they play an important role when matching TMD distributions onto the respective collinear ones. It is therefore of central importance to put in place an efficient and accurate strategy to numerically compute these convolutions. In doing so, it is also important to take into account as much as possible all possible flavours in which they may appear. For example, different variants apply in the presence of heavy-quark masses or when considering "non-forward" integrals in the framework of exclusive physics.

## 1 Structure of the integrals

The general structure of the integrals considered in APFEL++ has the form of a Mellin convolution between an *operator* $O$, this is a potentially complicated known function such as a splitting function or a partonic cross section,[1] and a *distribution* function $d$, as for example a parton distribution function (PDF) or a fragmentation function (FF). The explicit form of these integrals reads:

$$I(x) = x\int_0^1 dz \int_0^1 dy\, O(z) d(y) \delta(x - yz) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x\int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right) . \tag{1.1}$$

Typically in the presence of mass effects, the integration phase space may be modified and the convolution in Eq. (1.1) is generalised as:

$$I(x,\eta) = x\int_{x/\eta}^1 \frac{dz}{z} O(z,\eta) d\left(\frac{x}{\eta z}\right) , \tag{1.2}$$

where $\eta \leq 1$, with $\eta = 1$ reproducing Eq. (1.1). However, it is convenient to rewrite the integral in Eq. (1.2) in a form in which the lower bound of the integral is equal to $x$. This is easily done by performing the change of variable $y = \eta z$, so that:

$$I(x,\eta) = \int_x^\eta dy\, O\left(\frac{y}{\eta},\eta\right) \frac{x}{y} d\left(\frac{x}{y}\right) . \tag{1.3}$$

In order to enable the precomputation the expensive part of the integral in Eq. (1.3), we use the standard interpolation formula on the distribution $d$:

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right) , \tag{1.4}$$

[1] More precisely, $O$ is in general a distribution that may contain $\delta$ functions and +-distributions.

where $\alpha$ runs over the nodes of a grid in $x$ and the weights $w_\alpha$ are assumed to be Langrange polynomials of degree $k$. Now we plug Eq. (1.4) into Eq. (1.3) and specialise the computation of the integral $I$ to the girid point $x_\beta$. This gives:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^{\eta} dy\, O\left(\frac{y}{\eta},\eta\right)\, w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.5}$$

where we have defined $\overline{d}_\alpha = x_\alpha d(x_\alpha)$. As shown in the section devoted to the interpolation procedure, under some specific conditions the interpolating functions are such that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d\,, \tag{1.6}$$

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k})\,. \tag{1.7}$$

Therefore, Eq. (1.3) can be adjusted as:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{c}^{d} dy\, O\left(\frac{y}{\eta},\eta\right)\, w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.8}$$

Finally, we change the integration variable back to $z = y/\eta$ so that Eq. (1.8) becomes:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz\, O\,(z,\eta)\; w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)\right]}_{\Gamma_{\alpha\beta}} . \tag{1.9}$$

To summarise, Eq. (1.9) allows one to compute the integral $I$ on the nodes of a given grid as a weighted sum of the values of distribution $d$ on the nodes themselves. The weights are given by the appropriate integral of the operator $O$ and the interpolating functions $w_\alpha^{(k)}$. The value of $I$ for a generic value of $x$ can finally be obtained by interpolation using a formula similar to Eq. (1.4). Importantly, Eq. (1.9) provides a fast way of computing convolution integrals. Indeed, the weights $\Gamma_{\alpha\beta}$ can be precomputed and stored once and for all and the integral $I$ can be quickly computed with different distributions $d$ by simply taking the weighted average.

We now need an operational way to compute the weights $\Gamma_{\alpha\beta}$. To do so, it is necessary to know the general structure of the operator $O$. Quite generally, the operator $O$ splits as follows:

$$O(z,\eta) = R(z,\eta) + \sum_i \Big[P^{(i)}(z,\eta)\Big]_+ S^{(i)}(z,\eta) + L(\eta)\delta(1-z)\,, \tag{1.10}$$

where $R$ and $S^{(i)}$ are a regular functions at $z = 1$, that is:

$$R(1,\eta) = \lim_{z\to 1} R(z,\eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1,\eta) = \lim_{z\to 1} S^{(i)}(z,\eta) = J^{(i)}(\eta)\,, \tag{1.11}$$

being $K$, $J^{(i)}$, and $L$ a finite function of $\eta$. The functions $P^{(i)}$ are instead singular at $z = 1$ and non-integrable in the limit $\eta \to 1$ and thus are regularised through the +-prescription. Plugging Eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in Eq. (1.9) and making use of the definition of +-prescription gives:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \eta \int_{c/\eta}^{d/\eta} dz \left\{ R\,(z,\eta)\, w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) + \sum_i P^{(i)}(z,\eta) \left[ S^{(i)}\,(z,\eta)\, w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\
&+ \eta \left[ L(\eta) - \sum_i S^{(i)}(1,\eta) \int_0^{c/\eta} dz P^{(i)}(z,\eta)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) ,
\end{aligned} \tag{1.12}$$

that can be further manipulated changing the integration variable into $y = \eta z$:

$$\begin{aligned}
\Gamma_{\beta\alpha} &= \int_c^d dy \left\{ R\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) + \sum_i P^{(i)}\left(\frac{y}{\eta},\eta\right) \left[ S^{(i)}\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\
&+ \eta \left[ L(\eta) + \sum_i S^{(i)}(1,\eta) Q^{(i)}\left(\frac{c}{\eta},\eta\right)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right) .
\end{aligned} \tag{1.13}$$

where we have defined:

$$Q^{(i)}(a,\eta) \equiv -\int_0^a dz P^{(i)}(z,\eta)\,. \tag{1.14}$$

These integrals can, most of the times, be computed analytically.

Eqs. (1.12)-(1.13) expose the full potential complexity of the task of computing the weights $\Gamma_{\alpha\beta}$. However, in many practical applications some simplifications apply. For example, in the case of the perturbative coefficients of the ($\overline{\rm MS}$) splitting functions there are two simplications: the first is that $\eta = 1$, the second is that there is one single term in the sum over $i$ ($i = 0$) such that and the general form of the function $P$ is:

$$P^{(0)}(z,\eta) \to \frac{1}{1-z}\,, \tag{1.15}$$

so that:

$$Q^{(0)}(a,\eta) = \ln(1-a)\,. \tag{1.16}$$

Considering that:

$$w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}\,, \tag{1.17}$$

and that the expressions can always be manipulated in such a way that the coefficient of the +-prescripted term $S$ is a constant, one finds that:

$$\Gamma_{\beta\alpha} = \int_c^d dz \left\{ R(z)\, w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S}{1-z}\left[ w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} + \left[S\ln(1-c) + L\right]\delta_{\alpha\beta}\,. \tag{1.18}$$

The same kind of simplifications apply to the case of the perturbative Zero-Mass (ZM) coefficient functions with the only exception that the sum over $i$ extends to more terms depending on the perturbative order. For example at $\mathcal{O}(\alpha_s)$, *i.e.* at NLO, the general form of the coefficient functions reads:

$$\begin{aligned} \Gamma_{\beta\alpha} &= \int_c^d dz \left\{ R(z)\, w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S^{(0)} + S^{(1)}\ln(1-z)}{1-z}\left[ w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} \\ &+ \left[ S^{(0)}\ln(1-c) + \frac{1}{2}S^{(1)}\ln^2(1-c) + L\right]\delta_{\alpha\beta}\,. \end{aligned} \tag{1.19}$$

Thing get more complicated when considering massive coefficient functions. In this case, the role of $\eta$ is played by the ration $m/Q$, where $m$ is the mass of the quark and $Q$ the hard scale. In addition, the +-prescripted contributions can become more convoluted.

As an aside, it is interesting to notice that we have silently decided to use one of the two possible ways to compute the convolution integral $I(x)$ in Eq. (1.1). Specifically, we have chosen the rightmost equality. However, the equality:

$$I(x) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z)\,, \tag{1.20}$$

is equally valid. If one choses this particular version of the convolution integral and follows the interpolation procedure discussed above, the result for $I$ computed in $x_\beta$ would be identical to Eq. (1.9) (with $\eta = 1$) except for the fact that the weights $\Gamma_{\beta\alpha}$ are computed as:

$$\Gamma_{\beta\alpha} = \int_a^b dz\, O\left(\frac{x_\beta}{z}\right) w_\alpha^{(k)}(z) \tag{1.21}$$

with:

$$a \equiv \max(x_\beta, x_{\alpha-k}) \quad \text{and} \quad b \equiv \min(1, x_{\alpha+1})\,. \tag{1.22}$$

This immediately implies that the integral in Eq. (1.9) and that in Eq. (1.21) have to coincide. This is indeed the case and the only reason to choose Eq. (1.9) rather than Eq. (1.21) is convenience. Specifically, Eq. (1.21) implies computing the operator $O$ at $x_\beta/z$ rather than at $z$ and this makes the treatment of $\delta$-functions and +-distributions possibly present in $O$ a little more delicate. However, Eq. (1.21) remains a valid alternative that we just decided not to pursue.

## 1.1 Double convolutions

The technique discussed above to compute convolution integrals as a weighted sums of a given input distributions $d$ can be, under certain circumstances, generalised to convolution integrals involving *two* distributions. Relevant examples are the Drell-Yan (DY) and the semi-inclusive deep-inelastic scattering (SIDIS) cross sections integrated over the transverse momentum of the virtual boson. In these cases the cross section is proportional to a double convolutions between partonic cross sections and a pair of non-perturbative distributions. The general form of this double convolution is:

$$J(x_1,x_2) = x_1 x_2 \int_{x_1}^1 \frac{dy_1}{y_1} \int_{x_2}^1 \frac{dy_2}{y_2} O(y_1,y_2) d^{(1)}\left(\frac{x_1}{y_1}\right) d^{(2)}\left(\frac{x_2}{y_2}\right) . \tag{1.23}$$

Applying the same interpolation procedure as in the single-convolution case gives:

$$\begin{aligned} J(x_\delta,x_\gamma) &= \int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2) \left[\frac{x_\delta}{y_1} d^{(1)}\left(\frac{x_\delta}{y_1}\right)\right]\left[\frac{x_\gamma}{y_2} d^{(2)}\left(\frac{x_\gamma}{y_2}\right)\right] \\ &= \sum_{\alpha=0}^{N_x}\sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \overline{d}_\alpha^{(2)} \underbrace{\left[\int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2)\, w_\beta^{(k)}\left(\frac{x_\delta}{y_1}\right) w_\alpha^{(k)}\left(\frac{x_\gamma}{y_2}\right)\right]}_{\Theta^{\beta\delta,\alpha\gamma}} . \end{aligned} \tag{1.24}$$

In Eq. (1.24) we have assumes that there are no mass corrections and thus the convolutions take the simplest form. In the case of double convolutions, the operator $O$ is a function of two variables, $y_1$ and $y_2$, and, as in the case of the single convolutions, it receives three kinds of contribution in both these variables: local terms proportional to $\delta$-functions, singular terms proportional to +-prescripted functions, and regular terms. The complication here is that these contributions from the two variables $y_1$ and $y_2$ mix and thus, for example, terms local in $y_1$ and singular in $y_2$ may also appear. It is thus necessary to identify the general structure of the function $O$ to see whether it is possible to decompose the double operator $\Theta^{\beta\delta,\alpha\gamma}$ into products of single operators.

As we will explicitly show below, in the case of SIDIS up to $\mathcal{O}(\alpha_s)$ (NLO) the general structure of the function $O$ can be inferred looking at Eqs. (C.2)-(C.7) of Ref. [1]:

$$\begin{aligned} O(y_1,y_2) &= \mathrm{LL}\,\delta(1-y_1)\delta(1-y_2) + \mathrm{LS}\,\delta(1-y_1)\left[\frac{\ln(1-y_2)}{1-y_2}\right]_+ + \delta(1-y_1)\,\mathrm{LR}(y_2) \\ &+ \mathrm{SL}\left[\frac{\ln(1-y_1)}{1-y_1}\right]_+ \delta(1-y_2) + \mathrm{SS}\left[\frac{1}{1-y_1}\right]_+\left[\frac{1}{1-y_2}\right]_+ + \left[\frac{1}{1-y_1}\right]_+ \mathrm{SR}(y_2) \\ &+ \mathrm{RL}(y_1)\,\delta(1-y_2) + \mathrm{RS}(y_1)\left[\frac{1}{1-y_2}\right]_+ + \sum_i K_i \mathrm{R}_i^{(1)}(y_1)\mathrm{R}_i^{(2)}(y_2) . \end{aligned} \tag{1.25}$$

It is clear that in Eq. (1.25) all terms factorise into a part that only depends on $y_1$ and a part that only depends on $y_2$. This is the crucial feature that enables use the technology developed above for the single convolutions. Plugging Eq. (1.25) into Eq. (1.24), one finds that:

$$\begin{aligned} \Theta^{\beta\delta,\alpha\gamma} &= \mathrm{LL}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{LS}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{S1}}_{\alpha\gamma} + \Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{LR}}_{\alpha\gamma} \\ &+ \mathrm{SL}\,\Gamma^{\mathrm{S1}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{SS}\,\Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{SR}}_{\alpha\gamma} \\ &+ \Gamma^{\mathrm{RL}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \Gamma^{\mathrm{RS}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \sum_i K_i \Gamma^{R_i^{(1)}}_{\beta\delta}\Gamma^{R_i^{(2)}}_{\alpha\gamma} \end{aligned} \tag{1.26}$$

with:

$$\begin{aligned}
\Gamma^{\mathrm{L}}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, \delta(1-z) w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) = w_\beta^{(k)}(x_\alpha) = \delta_{\alpha\beta} \\
\Gamma^{\mathrm{S}n}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz \frac{\ln^n(1-z)}{1-z}\left[w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) - \delta_{\alpha\beta}\right] + \frac{1}{(n+1)!}\ln^{n+1}(1-c_{\alpha\beta})\,\delta_{\alpha\beta} \\
\Gamma^{f}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, f(z)\, w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right)
\end{aligned} \tag{1.27}$$

where $f$ is a regular function and the integration bounds are defined as:

$$c_{\alpha\beta} = \max(x_\alpha, x_\alpha/x_{\beta+1}) \quad \text{and} \quad d_{\alpha\beta} = \min(1, x_\alpha/x_{\beta-k})\,. \tag{1.28}$$

In general terms, we assume that it is always be possible to write an object of the kind of $\Theta^{\beta\delta,\alpha\gamma}$ as series of bilinear terms:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_j C_j \Gamma_j^{(1),\beta\delta}\Gamma_j^{(2),\alpha\gamma} \tag{1.29}$$

where $C_j$ are scalar coefficients, and the weights $\Gamma_j^{(1),\beta\delta}$ and $\Gamma_j^{(2),\alpha\gamma}$ can be computed using the technology discussed in the previous section. Plugging Eq. (1.29) into Eq. (1.24), one finds that:

$$J(x_\delta, x_\gamma) = \sum_j C_j f_j^{(1),\delta} f_j^{(2),\gamma}\,, \tag{1.30}$$

where we have defined:

$$f_j^{(1),\delta} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \Gamma_j^{(1),\beta\delta} \quad \text{and} \quad f_j^{(2),\gamma} \equiv \sum_{\alpha=0}^{N_x} \overline{d}_\alpha^{(2)} \Gamma_j^{(2),\alpha\gamma}\,. \tag{1.31}$$

Eq. (1.30) shows that, under the hypothesis that the operator $O(y_1, y_2)$ can be expressed as a series of terms whose dependence on $y_1$ and $y_2$ factorizes,[2] the double convolution in Eq. (1.24) is given by a series of bilinear distributions ($f_j^{(1),\delta}$ and $f_j^{(2),\gamma}$) singularly obtained as convolutions of single operators with the input distributions $d^{(1)}$ and $d^{(2)}$. This is a particularly useful achievement that allows us to compute double convolutions without the need of extending the integration and the interpolation procedures to two dimensions with an obvious gain in accuracy and performance. As a matter of fact, the same argument can be extended to a multiple convolution of the function $O(\{y_i\})$, which again can be expressed as a series of $n$-linear terms, with $i = 1, \ldots, n$, with $n$ distributions:

$$J(\{x_{\alpha_i}\}) = \sum_j C_j \prod_{i=1}^{n} f_j^{(i),\alpha_i}\,, \tag{1.32}$$

with:

$$f_j^{(i),\alpha_i} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(i)} \Gamma_j^{(i),\beta\alpha_i}\,. \tag{1.33}$$

This technology could be useful for more complicated observables, like cross sections in $pp$ collisions with an identified hadron in the final state, that requires for example three convolutions.

The challenging part of the procedure just presented resides in the "pre-processing" of the function $O(y_1, y_2)$ that has to be analytically manipulated to disentangle the single terms. This step, however, has to be taken only once.

Before going into a concrete application, it is useful to connect Eq. (1.26) to Eq. (1.29) by identifying number and form of the coefficients and weights involved. Specifically, assuming that the series in the last term in the r.h.s. of Eq. (1.26) has $r$ terms, the series in Eq. (1.29) will have $8 + r$ terms, that is:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_{j=1}^{8+r} C_j \Gamma_j^{(1),\beta\delta}\Gamma_j^{(2),\alpha\gamma} \tag{1.34}$$

[2] This is the case for SIDIS and DY up to NLO. However, one may expect that this feature holds also beyond.

with:

$$\begin{array}{lcllll}
j=1 & : & C_1=\mathrm{LL}, & \Gamma_1^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, & \Gamma_1^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma}, \\
j=2 & : & C_2=\mathrm{LS}, & \Gamma_2^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, & \Gamma_2^{(2),\alpha\gamma}=\Gamma^{\mathrm{S1}}_{\alpha\gamma}, \\
j=3 & : & C_3=1, & \Gamma_3^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, & \Gamma_3^{(2),\alpha\gamma}=\Gamma^{\mathrm{LR}}_{\alpha\gamma}, \\
j=4 & : & C_4=\mathrm{SL}, & \Gamma_4^{(1),\beta\delta}=\Gamma^{\mathrm{S1}}_{\beta\delta}, & \Gamma_4^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma}, \\
j=5 & : & C_5=\mathrm{SS}, & \Gamma_5^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, & \Gamma_5^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma}, \\
j=6 & : & C_6=1, & \Gamma_6^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, & \Gamma_6^{(2),\alpha\gamma}=\Gamma^{\mathrm{SR}}_{\alpha\gamma}, \\
j=7 & : & C_7=1, & \Gamma_7^{(1),\beta\delta}=\Gamma^{\mathrm{RL}}_{\beta\delta}, & \Gamma_7^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma}, \\
j=8 & : & C_8=1, & \Gamma_8^{(1),\beta\delta}=\Gamma^{\mathrm{RS}}_{\beta\delta}, & \Gamma_8^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma}, \\
j=9 & : & C_9=K_1 & \Gamma_9^{(1),\beta\delta}=\Gamma^{R_1^{(1)}}_{\beta\delta}, & \Gamma_9^{(2),\alpha\gamma}=\Gamma^{R_1^{(2)}}_{\alpha\gamma}, \\
\vdots & & & & \\
j=8+r & : & C_{8+r}=K_r & \Gamma_{8+r}^{(1),\beta\delta}=\Gamma^{R_r^{(1)}}_{\beta\delta}, & \Gamma_{8+r}^{(2),\alpha\gamma}=\Gamma^{R_r^{(2)}}_{\alpha\gamma}
\end{array} \tag{1.35}$$

It should be noted that, despite the large number of terms in the series in Eq. (1.34), the number of weights to be precomputed is usually pretty limited. In addition, in many cases many of the terms of the series are zero so that the number of contributions is further reduced. We can now apply this procedure up to NLO in QCD to two specific cases: SIDIS first and DY second (incomplete).

### 1.1.1 Semi-inclusive deep inelastic scattering (SIDIS)

the structure of the ($p_T$-integrated) SIDIS cross section and the expressions of the respective hard coefficient functions can be found in Ref. [1]. Following this paper, the SIDIS differential cross section for the exchange of a virtual photon can be written as:

$$\frac{d^3\sigma}{dxdydz} = \frac{2\,\pi\alpha^2}{xyQ^2}\left[(1+(1-y)^2)2xF_1(x,z,Q^2) + 2(1-y)F_L(x,z,Q^2)\right] , \tag{1.36}$$

with $Q^2=-q^2$ the (negative) virtuality of the exchanged photon, $x$ and $z$ the momentum fractions that enter the PDFs and the FFs, and $y=Q^2/xs$ the inelasticity given in terms of $Q$, $x$, and the collision energy in the center of mass $s$. Notice that, as compared to Ref. [1], we have absorbed a factor $x$ into the definition of $F_L$ as customary for the longitudinal structure function in inclusive DIS.

We now use the Callan-Gross relation:

$$F_2 = 2xF_1 + F_L \tag{1.37}$$

to replace $2xF_1$ with $F_2$ in Eq. (1.36):

$$\frac{d^3\sigma}{dxdydz} = \frac{2\,\pi\alpha^2}{xyQ^2}\left[Y_+F_2(x,z,Q^2) - y^2F_L(x,z,Q^2)\right] , \tag{1.38}$$

where we have defined:

$$Y_+ = 1+(1-y)^2 . \tag{1.39}$$

It is also useful to write Eq. (1.38) as differential in $x$, $Q^2$, and $z$:

$$\frac{d^3\sigma}{dx\,dQ^2\,dz} = \frac{2\,\pi\alpha^2}{xQ^4}\left[Y_+F_2(x,z,Q) - y^2F_L(x,z,Q)\right] . \tag{1.40}$$

The structure functions $F_2$ and $F_L$ are given at NLO by:

$$\begin{aligned}
F_{2,L}(x,z,Q) &= x\sum_{q,\bar{q}} e_q^2\Big[q(x,Q)\otimes C_{qq}^{2,L}(x,z)\otimes D_q(z,Q) \\
&+ q(x,Q)\otimes C_{qq}^{2,L}(x,z)\otimes D_g(z,Q) + g(x,Q)\otimes C_{qg}^{2,L}(x,z)\otimes D_q(z,Q)\Big] ,
\end{aligned} \tag{1.41}$$

where $\{q,g\}$ are the quark and gluon PDFs and $\{D_q,D_g\}$ are the quark and gluon FFs, $e_q$ is the electric charge of the quark $q$ and $\{C_{qq}^{2,L},C_{qg}^{2,L},C_{gq}^{2,L}\}$ are the relevant partonic cross sections. The partonic cross sections allow for a perturbative expansion in power of $\alpha_s$:

$$C = \sum_{n=0}\left(\frac{\alpha_s}{4\pi}\right)^n C^{(n)} \tag{1.42}$$

that we truncate to NLO, *i.e.* to $n = 1$. At LO ($n = 0$) we have the simple expression:

$$\begin{aligned} C_{qq}^{2,(0)}(x,z) &= \delta(1-x)\delta(1-z)\,, \\ C_{qg}^{2,(0)}(x,z) &= C_{gq}^{2,(0)}(x,z) = 0\,. \end{aligned} \tag{1.43}$$

At NLO ($n = 1$) we take the expressions from Appendix C of Ref. [1] being careful to take into account an additional factor two due to the difference in the expansion parameter ($\alpha_s/4\pi$ rather than $\alpha_s/2\pi$). We also need to combine the expressions for $F_1$ and $F_L$ using Eq. (1.37) to obtain the partonic cross sections for $F_2$. We start with the partonic cross sections for $F_L$ that read:

$$\begin{aligned} C_{qq}^{L,(1)} &= 8C_F xz\,, \\ C_{gq}^{L,(1)} &= 8C_F x(1-z)\,, \\ C_{qg}^{L,(1)} &= 8x(1-x)\,, \end{aligned} \tag{1.44}$$

while those for $F_2$ read:

$$\begin{aligned} \frac{C_{qq}^{2,(1)}}{2C_F} &= -8\delta(1-x)\delta(1-z) + 2\delta(1-x)\left(\frac{\ln(1-z)}{1-z}\right)_+ + \delta(1-x)\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\ &+ 2\left(\frac{\ln(1-x)}{1-x}\right)_+ \delta(1-z) + 2\left(\frac{1}{1-x}\right)_+\left(\frac{1}{1-z}\right)_+ - \left(\frac{1}{1-x}\right)_+(1+z) \\ &+ \left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\delta(1-z) - (1+x)\left(\frac{1}{1-z}\right)_+ + (2+6xz)\,, \\ \frac{C_{gq}^{2,(1)}}{2C_F} &= \delta(1-x)\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right] \\ &+ \left(\frac{1}{1-x}\right)_+ \frac{1+(1-z)^2}{z} \\ &+ 2(1+3x) - 6xz - (1+x)\frac{1}{z}\,, \\ C_{qg}^{2,(1)} &= \left[(x^2+(1-x)^2)\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\right]\delta(1-z) + (x^2+(1-x)^2)\left(\frac{1}{1-z}\right)_+ \\ &+ 2(-1+6x-6x^2) + (x^2+(1-x)^2)\frac{1}{z}\,. \end{aligned} \tag{1.45}$$

By inspection of Eqs. (1.44) and (1.45) we can deduce the various coefficients of Eq. (1.25). $F_L$ involves only regular functions so that all contributions are zero but the fully regular ones:

$$\begin{aligned} C_{qq}^{L,(1)}(x,z) &: \quad K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = z\,, \\ C_{gq}^{L,(1)}(x,z) &: \quad K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = 1-z\,, \\ C_{qg}^{L,(1)}(x,z) &: \quad K_1 = 8\,, \quad R_1^{(1)}(x) = x(1-x)\,, \quad R_1^{(2)}(z) = 1\,. \end{aligned} \tag{1.46}$$

The situation is more complicated for $F_2$ but we can still identify the different contributions:

$$\begin{aligned} C_{qq}^{2,(1)} \;:\; & \mathrm{LL} = -16C_F\,, \quad \mathrm{LS} = 4C_F\,, \quad \mathrm{LR}(z) = 2C_F\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\ & \mathrm{SL} = 4C_F\,, \quad \mathrm{SS} = 4C_F\,, \quad \mathrm{SR}(z) = -2C_F(1+z) \\ & \mathrm{RL}(x) = 2C_F\left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\,, \quad \mathrm{RS}(x) = -2C_F(1+x)\,, \\ & \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1,\, R_1^{(2)}(z) = 1\right\}\,, \\ & \left\{K_2 = 12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \end{aligned} \tag{1.47}$$

$$\begin{aligned} C_{gq}^{2,(1)} \;:\; & \mathrm{LR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right]\,, \\ & \mathrm{SR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\right]\,, \\ & \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1+3x,\, R_1^{(2)}(z) = 1\right\}\,, \\ & \left\{K_2 = -12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \\ & \left\{K_3 = -2C_F,\, R_3^{(1)}(x) = 1+x,\, R_3^{(2)}(z) = \frac{1}{z}\right\}\,, \end{aligned} \tag{1.48}$$

$$\begin{aligned} C_{qg}^{2,(1)} \;:\; & \mathrm{RL}(x) = \left[x^2 + (1-x)^2\right]\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\,, \quad \mathrm{RS}(x) = x^2 + (1-x)^2\,, \\ & \left\{K_1 = 2,\, R_1^{(1)}(x) = -1 + 6x - 6x^2,\, R_1^{(2)}(z) = 1\right\}\,, \\ & \left\{K_2 = 1,\, R_2^{(1)}(x) = x^2 + (1-x)^2,\, R_2^{(2)}(z) = \frac{1}{z}\right\}\,. \end{aligned} \tag{1.49}$$

Analogously, for the only LO partonic cross sections we find that:

$$C_{qq}^{2,(0)} \;:\; \mathrm{LL} = 1\,. \tag{1.50}$$

All the coefficients that are not mentioned are equal to zero. We can now explicitly implement Eq. (1.26). The one thing that is left to sort out is the structure of $F_2$ and $F_L$ in terms of the appropriate PDF and FF combinations. Looking Eq. (1.41), we observe that none of the coefficient functions depends on the particular quark flavour (this is a feature of the ZM scheme). Therefore, simplifying the notation, we can rewrite Eq. (1.41) as:

$$F \;=\; C_{qq}\sum_q e_q^2\left[qD_q + \overline{q}D_{\overline{q}}\right] + C_{gq}\sum_q e_q^2\left[q + \overline{q}\right]\,D_g + C_{qg}g\sum_q e_q^2\left[D_q + D_{\overline{q}}\right]\,, \tag{1.51}$$

where now the sums run only over the quark flavours and not over the antiflavours.

### 1.1.2 Drell Yan (DY)

In this section we apply to the Drell-Yan (DY) process the same procedure followed above for SIDIS. As a matter of fact, SIDIS and DY are strictly connected in that DY can be regarded as the time-like counterpart of SIDIS. As a consequence, the structure of the relevant observables as well as the form of the expressions involved are very similar. Therefore, the application of the method described above is straightforward.

## 1.2 Advantage of a logarithmic grid

Given the particular structure of the integral $I$ in Eq. (1.1), it turns out to be very convenient to use a logarithmically distributed grid along with Lagrange interpolating functions. Let us specifically consider the (massless) integrals:

$$I_{\beta\alpha} = \int_{x_\beta}^{1} dy\, O(y) w_\alpha\left(\frac{x_\beta}{y}\right) . \tag{1.52}$$

A logarithmically-spaced grid is defined such that $\ln x_{n+1} = \ln x_n + \delta x$ with $\delta x$ a positive constant. In addition, we consider a set of Lagrange interpolating functions of degree $\kappa$ polynomial in $\ln z$, $\{w_\alpha(z)\}$, that thus have the following form (see section on the interpolation):

$$w_\alpha^{(k)}(z) = \sum_{\substack{j=0 \\ j\leq\alpha}}^{k} \theta(z - x_{\alpha-j})\theta(x_{\alpha-j+1} - z) \prod_{\substack{\delta=0 \\ \delta\neq j}}^{k} \left[\frac{\ln(z) - \ln(x_{\alpha-j+\delta})}{\ln(x_\alpha) - \ln(x_{\alpha-j+\delta})}\right] . \tag{1.53}$$

Due to the fact that the grid is logarithmically distributed with step $\delta x$, the function above can be rearranged as follows:

$$w_\alpha^{(k)}(z) = \sum_{\substack{j=0 \\ j\leq\alpha}}^{k} \theta(z - x_{\alpha-j})\theta(x_{\alpha-j+1} - z) \prod_{\substack{\delta=0 \\ \delta\neq j}}^{k} \left[\frac{1}{\delta x} \ln\left(\frac{z}{x_\alpha}\right) \frac{1}{j-\delta} + 1\right] . \tag{1.54}$$

It is finally easy to see that Eq. (1.53) is such that:

$$w_\alpha(z) = \widetilde{w}\left(\ln\frac{z}{x_\alpha}\right) \quad \Rightarrow \quad w_\alpha\left(\frac{x_\beta}{y}\right) = \widetilde{w}\left(\ln\frac{x_\beta}{x_\alpha} - \ln y\right) = \widetilde{w}\left((\beta-\alpha)\delta x - \ln y\right) . \tag{1.55}$$

Therefore, the integrand of the integral in Eq. (1.52) only depends on the difference $\beta - \alpha$ and not on $\beta$ and $\alpha$ separately. Since the lower bound is $x_\beta$, this symmetry seems to broken at the level of the integral. However, the symmetry is preserved thanks to the support properties of the interpolating functions $w_\alpha$ and the fact that the relevant functions (PDFs or FFs) are zero at $x = 1$. To see this, we consider the integration limits in Eq. (1.7) with $\eta = 1$. They can be written as:

$$c = \max(x_\beta, e^{(\beta-\alpha-1)\delta x}) \quad \text{and} \quad d = \min(1, e^{(\beta-\alpha+\kappa)\delta x}) . \tag{1.56}$$

While the limit $d$ is manifestly only dependent on the difference $\beta - \alpha$, the limit $c$ is not. However, $c$ does not have this symmetry only when $x_\beta$ is selected in place of $e^{(\beta-\alpha-1)\delta x}$ and this can only happen when:

$$x_\beta > e^{(\beta-\alpha-1)\delta x} . \tag{1.57}$$

Since the last point of the grid is $x_{N_x} = 1$, being $N_x$ the number of grid intervals, one can write:

$$x_\beta = \frac{x_\beta}{x_{N_x}} = e^{(\beta-N_x)\delta x} . \tag{1.58}$$

Finally, relying on the monotonicity of the exponential function, the inequality in Eq. (1.57) becomes:

$$\beta - N_x > \beta - \alpha - 1 \quad \Leftrightarrow \quad \alpha > N_x - 1 \quad \Leftrightarrow \quad \alpha = N_x . \tag{1.59}$$

Therefore, the integrals $I_{\beta N_x}$ do not respect the "$\beta - \alpha$" symmetry. However, as mentioned above, $I_{\beta N_x}$ will always multiply a function computed in $x_{N_x} = 1$. In all cases of interest this function is identically zero at $x_{N_x} = 1$ and thus the symmetry is effectively preserved. In addition, $c$ in Eq. (1.56) is such that if $\beta > \alpha$ one has $c \geq 1$. But being $c$ the lower integration bound of and since in Eq. (1.52) the upper bound is 1, one immediately has that $I_{\beta\alpha} = 0$ for $\beta > \alpha$. The consequence of these observations is that computing the integrals $a_\alpha = I_{0\alpha}$ for $\alpha = 0, \ldots, N_x$ is enough to reconstruct the full set of integrals $I_{\beta\alpha}$ because, in matricial representation, $I$ will look like this:

$$I_{\beta\alpha} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N_x} \\ 0 & a_0 & a_1 & \cdots & a_{N_x-1} \\ 0 & 0 & a_0 & \cdots & a_{N_x-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} . \tag{1.60}$$

In conclusion, adopting a logarithmically-spaced grid allows one to compute $N_x + 1$ integrals rather than $(N_x + 1)(N_x + 2)/2$ integrals.

There is another aspect that matters in terms of numerical efficiency of the computation of the integrals $I_{\beta\alpha}$. Given the support region of the interpolating functions $w_\alpha$, the integral in Eq. (1.52) effectively reads:

$$I_{\beta\alpha} = \int_c^d dy\, O(y) w_\alpha \left(\frac{x_\beta}{y}\right) , \tag{1.61}$$

with the integration limits given in Eq. (1.7). These limits can be rearranged as follows:

$$c = \frac{x_\beta}{x_{\min[N_x,\alpha+1]}} \quad \text{and} \quad d = \frac{x_\beta}{x_{\max[\beta,\alpha-\kappa]}} , \tag{1.62}$$

which makes it manifest the index range covered by the integration range. The basic observation is that the functions $w_\alpha$ are piecewise in correspondence of the grid nodes. This feature makes a numerical integration over the full range defined in Eq. (1.62) hard to converge due to the cusps at the grid nodes. However, the functions $w_\alpha$ are smooth between two consecutive nodes. Therefore, it turns out to be convenient to compute the integrals $I_{\beta\alpha}$ by breaking the integration range as follows:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y) w_\alpha \left(\frac{x_\beta}{y}\right) , \tag{1.63}$$

in such a way that the integrand of each single integral is a smooth function and thus easier to integrate. Despite the number of intergrals to be computed increases, this procedure makes the computation faster and more accurate. Finally, if the grid is logarithmically distributed, and one defines:

$$s = \exp\left[\delta x\right] , \tag{1.64}$$

Eq. (1.63) can also be written as:

$$I_{\beta\alpha} = \sum_{j=\max[0,\alpha+1-N_x]}^{\min[\kappa,\alpha-\beta]} \int_{s^{\beta-\alpha+j-1}}^{s^{\beta-\alpha+j}} dy\, O(y) w_\alpha \left(\frac{x_\beta}{y}\right) . \tag{1.65}$$

These are the basic objects computed by APFEL++ to define a DGLAP-like operator.

## 2 GPD-related integrals

When considering computations involving GPDs, another kind of integral structure comes into play, that is:

$$J(x) = x \int_0^1 \frac{dz}{z} O\left(\frac{x}{z}, x\right) d(z) . \tag{2.1}$$

This integral differs from that in Eq. (1.1) in two respects: the lower integration bound is zero rather than $x$ and the operator $O$ may also depend explicitly on the external variable $x$. These differences make the strategy devised for the numerical computation of convolution integrals as in Eq. (1.1) partially invalid. Let us discuss it in detail to motivate the particular strategy adopted to compute Eq. (2.1) on a grid.

As discussed above, logarithmically distributed grids are particularly advantageous for integrals like those in Eq. (1.1) because they allow for a substantial reduction of the number of integrals to be computed. However, logarithmic grids have two main drawbacks. First, logarithmic grids that start from a relatively low value of $x$ tend to be relatively sparse at large values of $x$. This is a problem because all integrals that we are considering are such that the function being interpolated is integrated up to $x = 1$ and the interpolation there can thus potentially degrade in accuracy. A possible solution to this problem is to increase the density in a stepwise fashion as $x$ gets closer to one. This produces locally logarithmically distributed grids that allow one to exploit the symmetry discussed above while making the grid denser, and thus more accurate, at large $x$. The implementation of this procedure in APFEL++ is achieved by means of *locked* subgrids. In practice, one starts with a logarithmic grid with a given lower bound, *e.g.* $x_{\min}^{(0)} = 10^{-5}$. Starting from a given node, $x_{\min}^{(1)}$, the density of the grid is increased by some integer factor. This procedure can be repeated an arbitrary number of times as one moves towards large $x$, effectively defining logarithmic subgrids that are increasingly denser

and thus guarantee a better interpolation accuracy. When dealing with integrals such as that in Eq. (1.52), the simplest way to exploit the subgrid structure is to switch to a denser grid at the level of the *integral*, essentially using one grid when $x_\beta$ is below the transition node and the other when it is above. The advantage of this approach is that the integration procedure discussed above applies verbatim with the only difference that, according to the position of $x_\beta$, one grid is used rather than another. The disadvantage of this procedure is that integrals with low values of $x_\beta$ do not take advantage of the denser grids at large $x$. When dealing with functions like PDFs that vanish rapidly as $x$ tends to one, this is typically fine because the bulk of the integrals is typically due to the region at $x \gtrsim x_\beta$. This usually makes possible interpolation inaccuracies at large $x$ negligible. Unfortunately, when applied to integrals of the kind of Eq. (2.1), this strategy may lead to severe inaccuracies at large values of $x_\beta$. This is due to the fact that the integral extends down to zero and this procedure does not make use of the low-$x$ grids, effectively truncating the integral to increasingly larger values of $x$ as $x_\beta$ approaches one.

The second problem is that logarithmic grids get down to $x = 0$. This is not an issue for the computation in Eq. (1.1) because the lower integration bound is $x$ and thus, as long as the integral in not computed below the lower bound of the grid, this does not introduce any inaccuracy. Conversely, this is a potential problem for integrals like those in Eq. (2.1) that extend down to zero for any value of $x$. This may suggest that logarithmic grids are not suitable in the first place for integrals as in Eq. (2.1). However, we observe that GPD-related integrands are usually well-behaved around $y = 0$ implying that a logarithmic grid with lower bound $x_0$ close enough to zero (*e.g.* $x_0 = 10^{-5}$) is expected to be enough to make the contribution to the integral due to the region $y \in [0, x_0]$ negligibly small. Therefore, also for the computation of Eq. (2.1), we stick to logarithmically distributed grids which allows us to use the same class for constructing and managing grids developed to compute Eq. (1.1).

In addition, GPD-related integrals, due to the explicit dependence on the external variable of the operator (see Eq. (2.1)), do not enjoy $\beta - \alpha$ symmetry discussed above. This is a major limitation in that it does not allow us to reduce the number of integrals to be computed.

These observation, along with the discussion on the interpolation grid, leads us to take an approach in which the operator is computed for all possible pairs $(\alpha, \beta)$ over a grid that extends as low as possible in $x$ and is dense at large $x$: *i.e.* the joint grid.

Using the usual interpolation formula gives us that the integral in Eq. (2.1) is computed on the grid as follows:

$$J(x_\beta) = \sum_{\alpha=}^{N_x-1} J_{\beta\alpha}\overline{d}_\alpha \,. \tag{2.2}$$

with $\overline{d}_\alpha = x_\alpha d(x_\alpha)$ and the assumption $d(1) = 0$, and with:

$$J_{\beta\alpha} = \sum_{j=0}^{\kappa} \int_{x_\beta/x_{\alpha-j+1}}^{x_\beta/x_{\alpha-j}} dy\, O(y, x_\beta) w_\alpha^{(\kappa)}\left(\frac{x_\beta}{y}\right) , \tag{2.3}$$

where the indices $\alpha$ and $\beta$ are intended to run over the joint grid. We point out again that in this case it is necessary to compute all the $N_x^2$ integrals making up $J_{\beta\alpha}$.

## References

[1] D. de Florian, M. Stratmann, and W. Vogelsang, "QCD analysis of unpolarized and polarized Lambda baryon production in leading and next-to-leading order," *Phys. Rev. D*, vol. 57, pp. 5811–5824, 1998.