Transverse momentum resummation and matching to fixed order

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Part I. TMD evolution and its perturbative expansion

1 Evolution of the TMDs

In this section I will show how to solve the renormalisation-group equation (RGE) and the rapidity-evolution equation (often referred to as Collins-Soper (CS) equation) of a TMD distribution F. The distribution F can be either a PDF or a FF and can be associated to either to a quark (anti)flavour or to the gluon: the structure of the solution of the evolution equations is exactly the same. In fact, the solution of the equations only depends on whether we are evolving a quark or a gluon, while it does not distinguish between PDFs and FFs. In the impact-parameter space, F is a function of the transverse-momentum fraction x, of the bidimensional impact-parameter vector $\mathbf{b}_T$, of the renormalisation scale μ, and of the rapidity scale ζ, *i.e.* $F \equiv F(x, \mathbf{b}_T, \mu, \zeta)$. Since the evolution equations govern the behaviour of F w.r.t. the scale μ and ζ, in order to simplify the notation, in this section I will drop the dependence on x and $\mathbf{b}_T$, *i.e.* $F \equiv F(\mu, \zeta)$.([1])

The solution of the evolution equations allows one to express the distribution F at some final scales (μ, ζ) in terms of the same distribution at the initial scales (μ_0, ζ_0). It will turn out that this is accomplished by computing the evolution kernel $R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right]$, such that:

$$F(\mu, \zeta) = R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{1.1}$$

The evolution kernel R can be expressed in terms of perturbatively computable quantities. A collateral aspect that will be discussed below is the independence from the path that connects the initial and final scales (μ_0, ζ_0) and (μ, ζ). This in turn concerns the resummation of large logarithms that is required to ensure that the perturbative convergence is not spoiled.

The RGE and the CS equations read:

$$\begin{aligned} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} &= K(\mu)\,, \\ \frac{\partial \ln F}{\partial \ln \mu} &= \gamma(\mu, \zeta)\,, \end{aligned} \tag{1.2}$$

where γ and K are the anomalous dimensions of the evolution in μ and $\sqrt{\zeta}$, respectively. The equations above can be solved as follows. The first equation gives:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\right] F(\mu, \zeta_0)\,. \tag{1.3}$$

The factor $F(\mu, \zeta_0)$ can then be evolved in μ using the second equation:

$$F(\mu, \zeta_0) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,, \tag{1.4}$$

such that:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{1.5}$$

This equation has exactly the structure of Eq. (1.1). We now need to express the argument of the exponent in terms of perturbatively computable quantities.

In order to do so, we use the fact that the rapidity anomalous dimension K needs to be renormalised and thus it obeys its own RGE, that reads:

$$\frac{\partial K}{\partial \ln \mu} = -\gamma_K(\alpha_s(\mu))\,. \tag{1.6}$$

γ_K is said cusp anomalous dimension and obeys the perturbative expansion:

$$\gamma_K(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} \gamma_K^{(n)}\,, \tag{1.7}$$

[1] Notice that, despite the variables x and $\mathbf{b}_T$ will not appear explicitly, the symbol $\otimes$ indicates the Mellin convolution integral w.r.t. x while b_T indicates the length of the vector $\mathbf{b}_T$.

where $\gamma_K^{(n)}$ are numerical coefficients. The value of the coefficients up to $n = 3$ can be read from Eq. (59) of Ref. [3]. They coincide with those reported in Eq. (D.6) of Ref. [2] up to a factor two due to a different normalisation of the rapidity anomalous dimension K whose derivative w.r.t. ζ is exactly γ_K. In addition, the cusp anomalous dimensions for quarks and gluon are equal up to a factor C_F in the quark case and C_A in the gluon case.

Eq. (1.6) can be easily solved obtaining:

$$K(\mu) = K(\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) \,. \tag{1.8}$$

We anticipate that in the $\overline{\rm MS}$ renormalisation scheme, there exists a particular scale, $\mu_b = 2e^{-\gamma_E}/b_T$, such that K computed at μ_b is free of logaritms and admits the following perturbative expansion:

$$K(\mu_b) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^{n+1} K^{(n,0)} \,, \tag{1.9}$$

where $K^{(n,0)}$ are numerical coefficients. Therefore, if $\mu_0 \simeq \mu_b$ the first term in the r.h.s. of Eq. (1.8) is free of large logs and thus its perturbative expansion, that reads:

$$K(\mu_0) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{n+1} \sum_{m=0}^{n+1} K^{(n,m)} \ln^m\left(\frac{\mu_0}{\mu_b}\right) \,, \tag{1.10}$$

is reliable. The logarithmic terms in Eq. (1.10) can be computed by using Eq. (1.6) or Eq. (1.8) and expanding $\alpha_s(\mu_0)$ around $\alpha_s(\mu_b)$ using the appropriate RGE. Again, such an expansion is reliable only if μ_0 and μ_b are comparable such not to generate large logarithms. The second term in the r.h.s. of Eq. (1.8) instead takes care, through the evolution of α_s, of resumming large logarithms in the case in which $\mu \gg \mu_0$. The coefficients $K^{(n,m)}$ up to $n = 2$ are reported in Eq. (D.9) of Ref. [2] and up to $n = 1$ in Eq. (69) of Ref. [3]. They differ by a factor -2 due to a different definition of K. In addition, the logarithmic expansion is done in terms of $\ln(\mu_0/\mu_b)$ in Ref. [3] and in terms of $\ln(\mu_0^2/\mu_b^2)$ in Ref. [2]. Therefore, each coefficient differs by an additional factor 2^m, where m is the power of the logarithm that multiplies the coefficient itself.

For the sake of completeness, we compute the coefficients $K^{(n,m)}$ up to $n = 3$. To do so, we need to know the coefficients of the RGE of the coupling α_s, that we write:

$$\frac{d}{d\ln\mu}\left(\frac{\alpha_s(\mu)}{4\pi}\right) = -2\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \beta_n \,, \tag{1.11}$$

whose solution for the evolution between μ_b and μ' expanded to two loops reads:

$$\left(\frac{\alpha_s(\mu')}{4\pi}\right) = \left(\frac{\alpha_s(\mu_b)}{4\pi}\right) + \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2 \left(-2\beta_0 \ln\frac{\mu'}{\mu_b}\right) + \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^3 \left(4\beta_0^2 \ln^2\frac{\mu'}{\mu_b} - 2\beta_1 \ln\frac{\mu'}{\mu_b}\right) + \mathcal{O}(\alpha_s^4) \,, \tag{1.12}$$

such that:

$$\left(\frac{\alpha_s(\mu')}{4\pi}\right)^2 = \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2 + \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^3 \left(-4\beta_0 \ln\frac{\mu'}{\mu_b}\right) + \mathcal{O}(\alpha_s^4) \,, \tag{1.13}$$

and:

$$\left(\frac{\alpha_s(\mu')}{4\pi}\right)^3 = \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4) \,. \tag{1.14}$$

Therefore, using Eq. (1.7) up to three loops, one finds:

$$\begin{aligned} \gamma_K(\alpha_s(\mu')) &= \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)\gamma_K^{(0)} + \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2 \left[-2\beta_0\gamma_K^{(0)} \ln\frac{\mu'}{\mu_b} + \gamma_K^{(1)}\right] \\ &+ \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^3 \left[4\beta_0^2\gamma_K^{(0)} \ln^2\frac{\mu'}{\mu_b} + \left(-2\beta_1\gamma_K^{(0)} - 4\beta_0\gamma_K^{(1)}\right)\ln\frac{\mu'}{\mu_b} + \gamma_K^{(2)}\right] + \mathcal{O}(\alpha_s^4) \,. \end{aligned} \tag{1.15}$$

Using Eq. (1.15) in the integral in the r.h.s. of Eq. (1.8) computed between μ_b and μ_0, one finds:

$$\begin{aligned}\int_{\mu_b}^{\mu_0}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu')) &= \int_0^{\ln\frac{\mu_0}{\mu_b}} d\ln\frac{\mu'}{\mu_b}\gamma_K(\alpha_s(\mu'))\\ &= \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)\gamma_K^{(0)}\ln\frac{\mu_0}{\mu_b}+\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2\left[-\beta_0\gamma_K^{(0)}\ln^2\frac{\mu_0}{\mu_b}+\gamma_K^{(1)}\ln\frac{\mu_0}{\mu_b}\right]\\ &+ \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^3\left[\frac{4}{3}\beta_0^2\gamma_K^{(0)}\ln^3\frac{\mu_0}{\mu_b}+\left(-\beta_1\gamma_K^{(0)}-2\beta_0\gamma_K^{(1)}\right)\ln^2\frac{\mu_0}{\mu_b}+\gamma_K^{(2)}\ln\frac{\mu_0}{\mu_b}\right]+\mathcal{O}(\alpha_s^4)\,.\end{aligned} \tag{1.16}$$

Replacing $\alpha_s(\mu_b)$ with $\alpha_s(\mu_0)$ through the expansion:

$$\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)=\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)+2\beta_0\ln\frac{\mu_0}{\mu_b}\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^2+\left(2\beta_1\ln\frac{\mu_0}{\mu_b}+4\beta_0^2\ln^2\frac{\mu_0}{\mu_b}\right)\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^3+\mathcal{O}(\alpha_s^4)\,, \tag{1.17}$$

the final result is:

$$K(\mu_0)=\sum_{n=0}^{2}\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{n+1}\sum_{m=0}^{n+1}K^{(n,m)}\ln^m\left(\frac{\mu_0}{\mu_b}\right)\,, \tag{1.18}$$

where we read off:

$$\begin{aligned}&K^{(0,1)}=-\gamma_K^{(0)}\,,\\ &K^{(1,1)}=-\gamma_K^{(1)}+2\beta_0K^{(0,0)}\,,\quad K^{(1,2)}=-\beta_0\gamma_K^{(0)}\\ &K^{(2,1)}=-\gamma_K^{(2)}+2\beta_1K^{(0,0)}+4\beta_0K^{(1,0)}\,,\quad K^{(2,2)}=-\beta_1\gamma_K^{(0)}-2\beta_0\gamma_K^{(1)}+4\beta_0^2K^{(0,0)}\,,\quad K^{(2,3)}=-\frac{4}{3}\beta_0^2\gamma_K^{(0)}\end{aligned} \tag{1.19}$$

A further important property of the anomalous dimensions can be derived by considering the fact that the crossed double derivarives of F must be equal:

$$\frac{\partial}{\partial\ln\mu}\frac{\partial\ln F}{\partial\ln\sqrt{\zeta}}=\frac{\partial}{\partial\ln\sqrt{\zeta}}\frac{\partial\ln F}{\partial\ln\mu}\,. \tag{1.20}$$

Using Eqs. (1.2) and (1.6) leads to the following additional differential equation:

$$\frac{\partial\gamma}{\partial\ln\sqrt{\zeta}}=-\gamma_K(\alpha_s(\mu))\,. \tag{1.21}$$

Using the point $\zeta=\mu^2$ as a boundary condition, the solution of the equation above is:

$$\gamma(\mu,\zeta)=\gamma(\mu,\mu^2)-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\,. \tag{1.22}$$

It turns out that $\gamma(\mu,\mu^2)$ has a purely perturbative expansion:

$$\gamma(\mu,\mu^2)\equiv\gamma_F(\alpha_s(\mu))=\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1}\gamma_F^{(n)}\,, \tag{1.23}$$

where $\gamma_F^{(n)}$ are again numerical coefficients that are given in Eq. (58) of Ref. [3] and Eq. (D.7) of Ref. [2] (Eq. (D.8) reports the coefficients for the gluon anomalous dimension). The two sets of coefficients differ by a minus sign due to the different definition of the constant (non-logarithmic) term of the RGE anomalous dimension. Therefore:

$$\gamma(\mu,\zeta)=\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\,. \tag{1.24}$$

Finally, plugging Eqs. (1.8) and (1.24) into Eq. (1.5), one finds:

$$F(\mu,\zeta)=\exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu'))-\gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}F(\mu_0,\zeta_0)\,. \tag{1.25}$$

Comparing Eq. (1.25) to Eq. (1.1) allows one to give an explicit expression to the evolution kernel:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]=\exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu'))-\gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}. \tag{1.26}$$

Eq. (1.25) has been obtained evolving the TMD distribution F first in the ζ direction (Eq. (1.3)) and then in the μ direction (Eq. (1.4)). However, it is easy to verify that exchanging the order of the evolutions leads to the exact same result, Eq. (1.25). In particular, the following relation holds:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta)\right]R\left[(\mu_0,\zeta)\leftarrow(\mu_0,\zeta_0)\right]=R\left[(\mu,\zeta)\leftarrow(\mu,\zeta_0)\right]R\left[(\mu,\zeta_0)\leftarrow(\mu_0,\zeta_0)\right]=R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]. \tag{1.27}$$

This is a direct consequence of the independence of evolution kernel R in Eq. (1.26) from the path $\mathcal{P}$ followed to connect the points (μ_0,ζ_0) to the point (μ,ζ):

$$R\left[(\mu,\zeta)\underset{\mathcal{P}}{\leftarrow}(\mu_0,\zeta_0)\right]\equiv R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]. \tag{1.28}$$

Another important piece of information comes from the fact that, for small values of b_T, the TMD F can be matched onto the respective collinear distribution f (a PDF or a FF) through the perturbative coefficients C[2]:

$$F(\mu,\zeta)=C(\mu,\zeta)\otimes f(\mu), \tag{1.29}$$

so that:

$$F(\mu,\zeta)=\exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu'))-\gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}C(\mu_0,\zeta_0)\otimes f(\mu_0). \tag{1.30}$$

Exactly as in the case of K, for $\mu_0=\sqrt{\zeta_0}=\mu_b$ the matching function admits the expansion:

$$C(\mu_b,\mu_b^2)=\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n C^{(n,0)}, \tag{1.31}$$

where the coefficients $C^{(n,0)}$ are functions of x only. In order to be able to compute the function C for generic values of the scales μ and ζ, evolution equations can be derived. Deriving Eq. (1.29) with respect to μ and ζ gives:

$$\begin{aligned}\frac{\partial F}{\partial\ln\sqrt{\zeta}}&=\frac{\partial C}{\partial\ln\sqrt{\zeta}}\otimes f(\mu),\\ \frac{\partial F}{\partial\ln\mu}&=\frac{\partial C}{\partial\ln\mu}\otimes f(\mu)+C(\mu,\zeta)\otimes\frac{\partial f}{\partial\ln\mu}=\left[\frac{\partial C}{\partial\ln\mu}+C(\mu,\zeta)\otimes 2P(\alpha_s(\mu))\right]\otimes f(\mu).\end{aligned} \tag{1.32}$$

In the r.h.s. of the second equation I have used the DGLAP equation:

$$\frac{\partial f}{\partial\ln\mu}=2P(\alpha_s(\mu))\otimes f(\mu). \tag{1.33}$$

One can also take the derivative of Eq. (1.25) and, using Eq. (1.29), the result is:

$$\begin{aligned}\frac{\partial F}{\partial\ln\sqrt{\zeta}}&=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]F(\mu,\zeta)=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]C(\mu,\zeta)\otimes f(\mu),\\ \frac{\partial F}{\partial\ln\mu}&=\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]F(\mu,\zeta)=\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]C(\mu,\zeta)\otimes f(\mu).\end{aligned} \tag{1.34}$$

Equating Eq. (1.32) and Eq. (1.34) and dropping the distribution f, the evolution equations for C are obtained:

$$\begin{aligned}\frac{\partial C}{\partial\ln\sqrt{\zeta}}&=\left[K(\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]C(\mu,\zeta),\\ \frac{\partial C}{\partial\ln\mu}&=\left\{\left[\gamma_F(\alpha_s(\mu))-\gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]\Delta-2P(\alpha_s(\mu))\right\}\otimes C(\mu,\zeta),\end{aligned} \tag{1.35}$$

[2] A sum over flavours is understood. As a matter of fact, the matching function C has to be regarded as a matrix in flavour space multipling a vector of collinear PDFs/FFs.

where the Δ is a matrix in flavour space whose components are defined as:

$$\Delta_{ij}(x) = \delta_{ij}\delta(1-x)\,, \tag{1.36}$$

being i and j flavour indices. The equations above can be solved to determine the evolution of the matching function C. If one assumes $\mu_0 = \sqrt{\zeta_0} \simeq \mu_b$ in Eq. (1.25), the matching function C can be reliably expanded as:

$$C(\mu_0, \mu_0^2) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^n \sum_{m=0}^{2n} C^{(n,m)} \ln^m\left(\frac{\mu_0}{\mu_b}\right)\,. \tag{1.37}$$

The coefficient functions $C^{(n,m)}$ have been computed for both PDFs and FFs in Ref. [2]. The same functions have been computed also in Ref. [1] and reported in Ref. [3]. The authors of the latter paper have verified the equality of the two sets of functions.

For the sake of completeness, we rederive here the functions $C^{(n,m)}$, for $m \neq 0$, up to $n = 2$. This is done by solving the equations in Eq. (1.35) to evolve the coefficient C between (μ_b, μ_b^2) and (μ_0, ζ_0) and, under the assumption that $\mu_0 \simeq \sqrt{\zeta_0} \simeq \mu_b$, expand the solution to $\mathcal{O}(\alpha_s^2)$. Contrary to the usual assumption, we take $\mu_0 \neq \sqrt{\zeta_0}$ but in order to simplify the notation we introduce the following definitions:

$$a_b = \frac{\alpha_s(\mu_b)}{4\pi}\,, \quad a_0 = \frac{\alpha_s(\mu_0)}{4\pi}\,, \quad L_\mu = \ln\frac{\mu_0}{\mu_b}\,, \quad L_\zeta = \ln\frac{\sqrt{\zeta_0}}{\mu_b}\,, \tag{1.38}$$

and we set $C^{(0,0)} = \Delta \equiv 1$ keeping in mind that any scalar carries the flavour and x structure of Δ. Due to the fact the evolution kernel of the first equation in Eq. (1.35) is in dependent of ζ, the evolution of C in ζ reads:

$$\begin{aligned} C(\mu_b, \zeta_0) &= \exp\left[K(\mu_b) L_\zeta\right] C(\mu_b, \mu_b^2) \\ &= 1 + a_b\left[C^{(1,0)} + K^{(0,0)} L_\zeta\right] + a_b^2\left[C^{(2,0)} + \left(K^{(0,0)} C^{(1,0)} + K^{(1,0)}\right) L_\zeta + \frac{1}{2}\left(K^{(0,0)}\right)^2 L_\zeta^2\right] + \mathcal{O}(\alpha_s^3) \end{aligned} \tag{1.39}$$

Now we need to solve also the second equation in Eq. (1.35) to evolve C from (μ_b, ζ_0) to (μ_0, ζ_0). The solution is:

$$C(\mu_0, \zeta_0) = \exp\left[\sum_{n=0}^{\infty} \int_{\mu_b}^{\mu_0} \frac{d\mu'}{\mu'} \left(\frac{\alpha_s(\mu')}{4\pi}\right)^{n+1} \left\{\left[\gamma_F^{(n)} - \gamma_K^{(n)} \ln\frac{\sqrt{\zeta_0}}{\mu'}\right]\Delta - 2P^{(n)}\right\}\right] \otimes C(\mu_b, \zeta_0) \tag{1.40}$$

Finally, we need to plug Eq. (1.39) into Eq. (1.40), replace $\alpha_s(\mu_b)$ with $\alpha_s(\mu_0)$ using Eq. (1.17), and truncate the result to $\mathcal{O}(\alpha_s^2)$. We refrain from writing the full expression because too lengthy that however can be written in a compact form as:

$$C(\mu_0, \zeta_0) = \sum_{n=0}^{2} a_0^n \sum_{m=0}^{2n} \sum_{k=0}^{m} \bar{C}^{(n,m-k,k)} L_\mu^{m-k} L_\zeta^k + \mathcal{O}(\alpha_s^3)\,, \tag{1.41}$$

with:

$$C^{(n,m)} = \sum_{k=0}^{m} \bar{C}^{(n,m-k,k)}\,. \tag{1.42}$$

In order to simplify the structure, one may take:

$$L_\mu = L_\zeta = L_0\,, \tag{1.43}$$

and accounting for the fact that $K^{(0,0)} = 0$, we have:

$$\begin{aligned} C(\mu_0, \mu_0^2) &= 1 + a_0\left[C^{(1,0)} + \left(\gamma_F^{(0)} - 2P^{(0)}\right) L_0 - \frac{1}{2}\gamma_K^{(0)} L_0^2\right] \\ &+ a_0^2\Bigg[C^{(2,0)} + \left(\gamma_F^{(1)} + K^{(1,0)} - 2P^{(1)} + \left(\gamma_F^{(0)} + 2\beta_0\right) C^{(1,0)} - 2C^{(1,0)} \otimes P^{(0)}\right) L_0 \\ &+ \left(\beta_0\gamma_F^{(0)} + \frac{1}{2}\left(\gamma_F^{(0)}\right)^2 - 2\left(\beta_0 + \gamma_F^{(0)}\right) P^{(0)} + 2P^{(0)} \otimes P^{(0)} - \frac{1}{2}\gamma_K^{(1)} - \frac{1}{2}\gamma_K^{(0)} C^{(1,0)}\right) L_0^2 \\ &+ \left(-\frac{1}{2}\gamma_K^{(0)}\gamma_F^{(0)} + \gamma_K^{(0)} P^{(0)} - \frac{2}{3}\beta_0\gamma_K^{(0)}\right) L_0^3 + \frac{1}{8}\left(\gamma_K^{(0)}\right)^2 L_0^4\Bigg] + \mathcal{O}(\alpha_s^3)\,. \end{aligned} \tag{1.44}$$

In order to use Eq. (1.30) in phenomenological applications, one needs to define the values of both the initial and final pairs of scales, (μ_0, ζ_0) and (μ, ζ). As discussed above, in the $\overline{\rm MS}$ renormalisation scheme the natural value for the initial scales is $\mu_0 = \sqrt{\zeta_0} = \mu_b = 2e^{-\gamma_E}/b_T$. This particular choice is such that all logarithms of the initial scales nullify. However, in order to estimate higer-order corrections, one can displace the initial scales around the central value by a modest factors C_i^μ and C_i^ζ such that $(\mu_0, \zeta_0) = (C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2)$. To implement these variations, one needs to used Eq. (1.18) for the Collins-Soper evolution kernel K and Eq. (1.41) for the matching functions C.

Now we discuss the final scales μ and ζ. First it is important to stress that the choice of ζ in the computation of physical observables is in fact *immaterial*. The reason is that a physical observable that takes place at the typical hard scale Q is proportional to *two* TMD distributions:

$$\sigma(Q) \propto H\left(Q,\mu\right) F_1(\mu,\zeta_1) F_2(\mu,\zeta_2) \tag{1.45}$$

where H is the appropriate hard function and with the constraint $\zeta_1\zeta_2 = Q^4$. Using the evolution kernel R, we find:

$$\sigma(Q) \propto H\left(Q,\mu\right) R\left[(\mu,\zeta_1) \leftarrow (\mu_0,\zeta_0)\right] R\left[(\mu,\zeta_2) \leftarrow (\mu_0,\zeta_0)\right] F_1(\mu_0,\zeta_0) F_2(\mu_0,\zeta_0)\,. \tag{1.46}$$

But using Eq. (1.26):

$$\begin{aligned}
&R\left[(\mu,\zeta_1) \leftarrow (\mu_0,\zeta_0)\right] R\left[(\mu,\zeta_2) \leftarrow (\mu_0,\zeta_0)\right] \\
&= \exp\left\{ K(\mu_0) \ln\frac{\sqrt{\zeta_1\zeta_2}}{\zeta_0} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[2\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln\frac{\sqrt{\zeta_1\zeta_2}}{\mu'^2} \right] \right\} \\
&= \exp\left\{ 2K(\mu_0) \ln\frac{Q}{\sqrt{\zeta_0}} + 2\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln\frac{Q}{\mu'} \right] \right\}\,,
\end{aligned} \tag{1.47}$$

where in the second line we have used the constraint $\zeta_1\zeta_2 = Q^4$. It thus turns out that the combination of the evolution kernels does not depend on the scales ζ_1 and ζ_2. Therefore, any choice for these scales that obeys the constraint above leads to *exactly* the same results. This being clear, the somewhat obvious choice is $\zeta_1 = \zeta_2 = Q^2$ and one does not need to consider any variations.

Finally, we consider the scale μ. The natural choice is $\mu = Q$, however, in order to estimate higer-order corrections, one may consider variations around this value by a modest factor C_f, $\mu = C_f Q$. This variations generate logarithms of μ/Q in the perturbative expansion of the hard factor H that can be computed by requiring that the physical observable σ is independent of μ. Specifically, one requires:

$$\frac{d\sigma}{d\ln\mu} = 0\,, \tag{1.48}$$

that, using Eq. (1.46), translates into the following evolution equation for H:

$$\frac{d\ln H}{d\ln\mu} = -2\gamma_F(\alpha_s(\mu)) - 2\gamma_K(\alpha_s(\mu)) \ln\frac{\mu}{Q}\,. \tag{1.49}$$

whose solution for the evolution between Q and μ is:

$$H(Q,\mu) = \exp\left\{ -2\int_Q^\mu \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) + \gamma_K(\alpha_s(\mu')) \ln\frac{\mu'}{Q} \right] \right\} H(Q,Q)\,. \tag{1.50}$$

Knowing that:

$$H(Q,Q) \quad = \quad 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s(Q)}{4\pi}\right)^n H^{(n)}\,, \tag{1.51}$$

and assuming that Q and μ are not too far apart, Eq. (1.50) can be reliably expanded and truncated to $\mathcal{O}(\alpha_s^2)$. Defining:

$$a_\mu = \frac{\alpha_s(\mu)}{4\pi} \quad \text{and} \quad L_Q = \ln\frac{\mu}{Q}\,, \tag{1.52}$$

we find:

$$\begin{aligned} H(Q,\mu) &= 1 + a_\mu \left[H^{(1)} - 2\gamma_F^{(0)} L_Q - \gamma_K^{(0)} L_Q^2\right] \\ &+ a_\mu^2 \left[H^{(2)} + \left(-2\gamma_F^{(1)} + 2\beta_0 H^{(1)} - 2\gamma_F^{(0)} H^{(1)}\right) L_Q \right. \\ &+ \left(-\gamma_K^{(1)} - 2\beta_0 \gamma_F^{(0)} + 2\left(\gamma_F^{(0)}\right)^2 - \gamma_K^{(0)} H^{(1)}\right) L_Q^2 \\ &+ \left. \left(-\frac{2}{3}\beta_0 \gamma_K^{(0)} + 2\gamma_F^{(0)}\gamma_K^{(0)}\right) L_Q^3 + \frac{1}{2}\left(\gamma_K^{(0)}\right)^2 L_Q^4 \right] + \mathcal{O}(\alpha_s^3)\,. \end{aligned} \tag{1.53}$$

The formula above allows us to vary the final renormalisation scale μ around the natural value Q. To summarise, there three possible variations that allow one to estimate higher-order corrections. The first two are associated to the variation of the initial scales μ_0 and $\sqrt{\zeta_0}$ around the natural scale μ_b and they are parametrised by the modest factors C_i^μ and C_i^ζ, such that $L_\mu = \ln C_i^\mu$ and $L_\zeta = \ln C_i^\zeta$. The third variation is associated to the final renormalisation scale μ around the hard scale Q and is parameterised by the factor C_f, such that $L_Q = \ln C_f$. In conclusion, the cross section formula with generic scales is proportional to:

$$\sigma(Q) \propto H\left(Q, C_f Q\right) \left(R\left[(C_f Q, Q^2) \leftarrow (C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2)\right]\right)^2 F_1(C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2) F_2(C_i^\mu \mu_b, (C_i^\zeta \mu_b)^2)\,. \tag{1.54}$$

2 Non-perturbative component

In the previous section, I have considered the evolution of TMDs and thus I concentrated on their dependence on the renormalisation and rapidity scales μ and ζ, leaving aside the dependence on b_T. The computation of the rapidity evolution kernel in the $\overline{\text{MS}}$ scheme has led to the introduction of the scale:

$$\mu_b = \frac{2e^{-\gamma_E}}{b_T}\,, \tag{2.1}$$

This scale, within a modest factor, provides a natural choice for the evolution initial scales μ_0 and $\sqrt{\zeta_0}$ that prevents the appearance of large logarithms. Crucially, the strong coupling α_s has to be computed in the vicinity of the scale μ_b. Therefore, if the impact parameters b_T becomes large, $\alpha_s(\mu_b)$ may potentially become very large invalidating any perturbative calculation. Since the computation of q_T dependent observables requires Fourier transforming TMDs, they need to be accessed also at large values of b_T where the perturbative computation of the previous section breaks down. To overcome this limitation, it is customary to introduce an *arbitrary* scale $b_{\max}$ that denotes the maximum value of b_T at which one trusts perturbation theory. The value has to be such that:

$$\alpha_s\left(\frac{2e^{-\gamma_E}}{b_{\max}}\right) \ll 1\,. \tag{2.2}$$

Then one introduces a monotonic function b_* of b_T with the following behaviour:

$$\begin{aligned} &b_*(b_T) \simeq b_T \qquad \text{for} \quad b_T \to 0\,, \\ &b_*(b_T) \to b_{\max} \quad \text{for} \quad b_T \to \infty\,. \end{aligned} \tag{2.3}$$

A common choice is:

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}}\,, \tag{2.4}$$

but more complicated functions can be employed to have better control on the low-b_T region. Now, one writes:

$$F(x, b_T, \mu, \zeta) = \left[\frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)}\right] F(x, b_*(b_T), \mu, \zeta) \equiv f_{\text{NP}}(x, b_T, \zeta) F(x, b_*(b_T), \mu, \zeta)\,. \tag{2.5}$$

This separation, often referred to as CSS prescription after Collins, Soper, and Sterman, is advantageous because, due to the behaviour of $b_*(b_T)$, $F(x, b_*(b_T), \mu, \zeta)$ can be computed in perturbation theory for any value of b_T, while $f_{\text{NP}}(x, b_T, \zeta)$ embodies the non-perturbative dependence. It is important to stress that this separation is arbitrary and dependes of the particular choice of b_* and $b_{\max}$. Therefore, for any particular choice, only the combination in Eq. (2.5) is meaningful and it is misleading to refer to f_{NP} as to the non-perturbative

part of TMDs in a universal sense. The reason why the function $f_{\rm NP}$ in Eq. (2.5) does not depend on the renormalisation scale μ is that this dependence cancels out in the ratio. To be more specific, if one chooses $\mu_0 = \mu_b = 2e^{-\gamma_E}/b_T$ and uses Eq. (1.25), one finds:

$$f_{\rm NP}(x, b_T, \zeta) = \frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)} = \\ \exp\left\{ K(\mu_b) \ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_b}} - K(\mu_{b_*}) \ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_{b_*}}} + \int_{\mu_b}^{\mu_{b_*}} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln\frac{\sqrt{\zeta}}{\mu'} \right] \right\} \frac{F(\mu_b, \zeta_b)}{F(\mu_{b_*}, \zeta_{b_*})}\,, \tag{2.6}$$

with $\mu_{b_*} = \sqrt{\zeta_{b_*}} = 2e^{-\gamma_E}/b_*(b_T)$. It is thus apparent that the dependence on μ cancels. Due to the required behaviour of b_*, if b_T becomes small, b_T and b_* get closer and so do μ_{b_*} and μ_b as well as ζ_{b_*} and ζ_b. Therefore, from the equation above, we see that $f_{\rm NP} \to 1$ for $b_T \to 0$. Conversely, if b_T becomes large b_* saturates to $b_{\rm max}$. In this limit, we can write:

$$f_{\rm NP}(x, b_T, \zeta) \underset{b_T\to\infty}{\longrightarrow} \\ \exp\left\{ K(\mu_b) \ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_b}} - K(\mu_{\rm min}) \ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_{\rm min}}} + \int_{\mu_b}^{\mu_{\rm min}} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln\frac{\sqrt{\zeta}}{\mu'} \right] \right\} \frac{F(\mu_b, \zeta_b)}{F(\mu_{\rm min}, \zeta_{\rm min})}\,, \tag{2.7}$$

where I have defined $\mu_{\rm min} = \sqrt{\zeta_{\rm min}} = 2e^{-\gamma_E}/b_{\rm max}$. As b_T becomes large, μ_b and ζ_b become increasingly small and the exponential takes the form of an evolution kernel between two sets of scales far apart from each other. As well known, the evolution kernel suppresses the distribution for large final scales. Extrapolating, one would then expect $f_{\rm NP}$ to be exponentially suppressed as b_T becomes large.

The definition of $f_{\rm NP}$ in Eq. (2.7) allows for a more systematic study of this quantity that may help adopting an optimal parameterisation when fitting it to data. The idea is to plot the ratio in the r.h.s. of the first line of Eq. (2.7) as a function of b_T for fixed values of x and ζ. We know that for b_T larger than some value, the numerator of this ratio will become unreliable. To identify this value, one can change the definition of μ_b (as well as that for μ_{b_*}) by introducing a factor C_2:

$$\mu_b = C_2 \frac{2e^{-\gamma_E}}{b_T}\,. \tag{2.8}$$

and vary C_2 around one by, say, a factor two up and down. When the effect of varying C_2 on the ratio becomes large, one can say that non-perturbative effects are large. On the other hand, in the region of b_T where variations are small perturbation theory is still reliable. Therefore this region can be used to put some constraint on $f_{\rm NP}$.

2.1 Alternative to the CSS prescription

As discussed above, the CSS prescription has the purpose to avoid integrating the Landau pole both in the strong coupling α_s and in the collinear distributions. This is implemented by *globally* replacing b_T with b_* in Eq. (2.4) in the TMDs that in turn naturally leads to the introduction of the non-perturbative function $f_{\rm NP}$ through Eq. (2.5). Despite very simple and natural, this approach implies using b_* even where not strictly required. Specifically in the logarithms appearing explicitly in the exponent of the Sudakov form factor that will now depend on b_* rather than on b_T even though they are not problematic. Of course, the presence of $f_{\rm NP}$ will reabsorb these effects as clear from Eq. (2.5).

However, one can decide take a less invasive perspective to leave the explicit logarithms in the Sudakov unchanged while only regularising only α_s and the collinear distributions. This can be done by computing them at some modified scale $\mu_*(mu)$ su that:

$$\begin{array}{lll} \mu_*(\mu) \simeq \mu & \text{for} & \mu \to \infty\,, \\ \mu_*(\mu) \to \mu_{\rm min} & \text{for} & \mu \to 0\,. \end{array} \tag{2.9}$$

This can be easily connected to the b_* prescription as:

$$\mu_*(\mu) = \frac{b_0}{b_*\left(\frac{b_0}{\mu}\right)}\,, \tag{2.10}$$

with $b_0 = 2e^{-\gamma_E}$. Adopting the particular b_* in Eq. (2.4) and choosing $b_{\max} = b_0$ gives the surprisingly simple relation:

$$\mu_*(\mu) = \sqrt{\mu^2 + 1}\,. \tag{2.11}$$

Therefore, the alternative recipe to the CSS procedure amounts to computing α_s and collinear distributions in $\sqrt{\mu^2+1}$ rather than in μ. This suffices to avoid integrating over the Landau pole, regularising only what actually needs to be regularised and leaving the rest untouched. The one unpleasant drawback of this approach is the fact that it does not make as transparent as in CSS the introduction of the function $f_{\rm NP}$ that in any case has to be present to encapsulate non-perturbative effects.

The difference between the standard CSS prescription and its alternative discussed here is displayed in Fig. 2.1 for the q_T distribution of the Drell-Yan cross section at $\sqrt{s} = 13$ TeV, $Q = M_Z$, and $y = 0$.

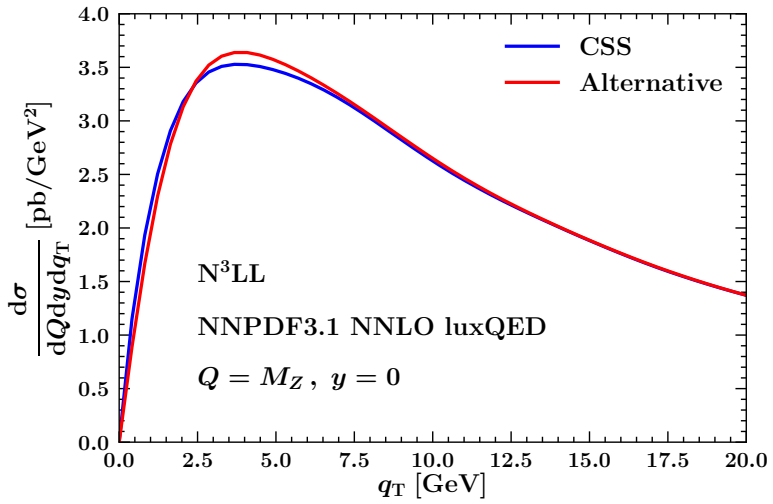

Fig. 2.1: .

3 Extraction of the singular behaviour for $q_T \to 0$

In this section we derive the expansion of the different ingredients of TMDs in order to finally extract the singular behaviour of a cross section involving TMDs for $q_T \to 0$. This will eventually serve to match the low-q_T resummed calculation to the fixed-order one valid at $q_T \simeq Q$.

3.1 Expansion of the evolution kernel

In this subsection, we work out the perturbative expansion of the evolution kernel R up to $\mathcal{O}(\alpha_s^2)$. This expansion is useful to extract the (singular) asymptotic behaviour of a cross section computed at fixed order in pQCD as q_T tends to zero. Since TMDs always appear in pairs in the computation of a cross section and that the evolution kernel is universal, it is convenient to compute the expansion of R^2. To do so, we start from Eq. (1.26) where we set $\mu_0 = \sqrt{\zeta_0} = \mu_b = 2e^{-\gamma_E}/b_T$ and $\mu = \sqrt{\zeta} = Q$, possible scale variations can be reinstated at a later stage:

$$R^2 = \exp\left\{K(\mu_b)\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left[\gamma_F(\alpha_s(\mu')) - \frac{1}{2}\gamma_K(\alpha_s(\mu'))\ln\frac{Q^2}{\mu'^2}\right]\right\}. \tag{3.1}$$

Now we use the expansions in Eqs. (1.7), (1.9) and (1.23):

$$R^2 = \exp\left\{\left[\sum_{n=0}^{\infty} a_s^{n+1}(\mu_b)K^{(n,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}a_s^{n+1}(\mu')\left(\gamma_F^{(n)} - \frac{1}{2}\gamma_K^{(n)}\ln\frac{Q^2}{\mu'^2}\right)\right]\right\}, \tag{3.2}$$

where we have defined:

$$a_s = \frac{\alpha_s}{4\pi}\,. \tag{3.3}$$

In order to carry out the perturbative expansion we need to write the argument of the exponential in terms of a common value of α_s computed at some hard scale: the natural choice in view of the matching is $\alpha_s(Q)$. This can be achieved by using the RGE for a_s at leading order:

$$\mu^2 \frac{da_s}{d\mu^2} = -\beta_0 a_s^2(\mu)\,, \tag{3.4}$$

whose solution is:

$$a_s(\mu) = \frac{a_s(Q)}{1 + a_s(Q)\beta_0 \ln(\mu^2/Q^2)} \simeq a_s(Q)\left[1 + a_s(Q)\beta_0 \ln(Q^2/\mu^2) + \mathcal{O}(a_s^2)\right]\,. \tag{3.5}$$

Plugging the expansion in the r.h.s. of Eq. (3.5) into Eq. (3.2) and retaining only terms up to $\mathcal{O}(a_s^2)$, one finds:

$$\begin{aligned} R^2 &= \exp\left\{ a_s(Q)\left[K^{(0,0)} \ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)} - \frac{1}{2}\gamma_K^{(0)} \ln\frac{Q^2}{\mu'^2}\right)\right]\right. \\ &+ a_s^2(Q)\beta_0\left[K^{(0,0)} \ln^2\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)} \ln\frac{Q^2}{\mu'^2} - \frac{1}{2}\gamma_K^{(0)} \ln^2\frac{Q^2}{\mu'^2}\right)\right] \\ &+ \left. a_s^2(Q)\left[K^{(1,0)} \ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(1)} - \frac{1}{2}\gamma_K^{(1)} \ln\frac{Q^2}{\mu'^2}\right)\right] + \mathcal{O}(a_s^3)\right\}. \end{aligned} \tag{3.6}$$

The final step before carrying out the expansion is computing the integrals:

$$\int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_{\ln(\mu_b^2)}^{\ln Q^2} d\ln\mu'^2 \ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_0^{\ln(Q^2/\mu_b^2)} dx\, x^k = \frac{1}{k+1}\ln^{k+1}\left(\frac{Q^2}{\mu_b^2}\right)\,. \tag{3.7}$$

so that:

$$\begin{aligned} R^2 &\simeq \exp\left\{ a_s(Q)\left[\left(K^{(0,0)} + \gamma_F^{(0)}\right)L - \frac{1}{4}\gamma_K^{(0)} L^2\right]\right. \\ &+ \left. a_s^2(Q)\left[\left(K^{(1,0)} + \gamma_F^{(1)}\right)L + \left(\beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)}\right)L^2 - \frac{1}{6}\beta_0\gamma_K^{(0)} L^3\right]\right\}, \end{aligned} \tag{3.8}$$

where we have defined:

$$L \equiv \ln\frac{Q^2}{\mu_b^2}\,, \tag{3.9}$$

Eq. (3.6) can be conveniently written as:

$$R^2 = \exp\left\{\sum_{n=1}^{2} a_s^n(Q) \sum_{k=1}^{n+1} S^{(n,k)} L^k\right\}\,, \tag{3.10}$$

with:

$$\begin{aligned} &S^{(1,1)} = K^{(0,0)} + \gamma_F^{(0)}\,, \quad S^{(1,2)} = -\frac{1}{4}\gamma_K^{(0)}\,, \\ &S^{(2,1)} = K^{(1,0)} + \gamma_F^{(1)}\,, \quad S^{(2,2)} = \beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)}\,, \quad S^{(2,3)} = -\frac{1}{6}\beta_0\gamma_K^{(0)}\,. \end{aligned} \tag{3.11}$$

Eq. (3.10) can be easily expanded up to order a_s^2 as:

$$\begin{aligned} R^2 &= 1 + a_s(Q)\sum_{k=1}^{2} S^{(1,k)} L^k + a_s^2(Q)\left[\sum_{k=1}^{3} S^{(2,k)} L^k + \frac{1}{2}\left(\sum_{k=1}^{2} S^{(1,k)} L^k\right)^2\right] + \mathcal{O}(a_s^3) \\ &= 1 + a_s(Q)\sum_{k=1}^{2} S^{(1,k)} L^k + a_s^2(Q)\sum_{k=1}^{4} \widetilde{S}^{(2,k)} L^k + \mathcal{O}(a_s^3)\,, \end{aligned} \tag{3.12}$$

with:

$$\begin{aligned} \widetilde{S}^{(2,1)} &= S^{(2,1)}\,, & \widetilde{S}^{(2,2)} &= S^{(2,2)} + \frac{1}{2}\left[S^{(1,1)}\right]^2\,, \\ \widetilde{S}^{(2,3)} &= S^{(2,3)} + S^{(1,1)}S^{(1,2)}\,, & \widetilde{S}^{(2,4)} &= \frac{1}{2}\left[S^{(1,2)}\right]^2\,. \end{aligned} \tag{3.13}$$

3.2 Expansion of the DGLAP evolution

A further step towards the extraction of the singular behaviour of the resummed cross section in the limit $q_T \to 0$ requires expanding the solution of the DGLAP equation that also resums logarithms as that in Eq. (3.9). The solution of the DGLAP equation in Eq. (1.33) that evolves PDFs from the scale $\mu_0 = \mu_b$ to the scale Q can be written as([3]):

$$f(Q) = \Gamma(Q, \mu_b) \otimes f(\mu_b)\,, \tag{3.14}$$

where the evolution operator Γ obeys the equation:

$$\frac{\partial \Gamma(Q, \mu_b)}{\partial \ln Q^2} = P(Q) \otimes \Gamma(Q, \mu_b)\,, \tag{3.15}$$

with the splitting functions P having the perturbative expansion:

$$P(Q) = \sum_{n=0}^{\infty} a_s^{n+1}(Q) P^{(n)}\,. \tag{3.16}$$

The coefficients $P^{(n)}$ are functions of x (or z) only and thus (in $\overline{\text{MS}}$) P depends on Q only through the coupling a_s. We now need to expand Γ in powers of a_s up to $\mathcal{O}(a_s^2)$. To this end, we assume that Γ obeys the expansion:

$$\Gamma(Q, \mu_b) = \Delta + \sum_{n=1}^{\infty} a_s^n(Q) \sum_{k=1}^{n} L^k \Gamma^{(n,k)}\,, \tag{3.17}$$

where the coefficients $\Gamma^{(n,k)}$ are functions of x only that we need to determine up to $n = 2$. Eq. (3.15) can be solved iteratively. At $\mathcal{O}(a_s)$, considering that:

$$\frac{da_s}{d \ln Q^2} = \mathcal{O}(a_s^2)\,, \tag{3.18}$$

this gives:

$$\Gamma^{(1,1)} = P^{(0)}\,. \tag{3.19}$$

At $\mathcal{O}(a_s^2)$, this gives:

$$-L\beta_0 P^{(0)} + \Gamma^{(2,1)} + 2L\Gamma^{(2,2)} = P^{(1)} + LP^{(0)} \otimes P^{(0)}\,, \tag{3.20}$$

so that:

$$\begin{aligned} \Gamma^{(2,1)} &= P^{(1)}\,, \\ \Gamma^{(2,2)} &= \frac{1}{2}\beta_0 P^{(0)} + \frac{1}{2} P^{(0)} \otimes P^{(0)}\,. \end{aligned} \tag{3.21}$$

Putting all pieces together, we find:

$$\Gamma(Q, \mu_b) = \Delta + a_s(Q) L P^{(0)} + a_s^2(Q)\left[LP^{(1)} + \frac{L^2}{2}\left(\beta_0 P^{(0)} + P^{(0)} \otimes P^{(0)}\right)\right] + \mathcal{O}(a_s^3)\,. \tag{3.22}$$

However, the quantity we are interested in is the inverse of the operator Γ, that evolves the PDFs from the scale Q to μ_b:

$$\begin{aligned} &\Gamma(\mu_b, Q) = \Gamma^{-1}(Q, \mu_b) \\ &= \Delta - a_s(Q) L P^{(0)} - a_s^2(Q)\left[LP^{(1)} + \frac{L^2}{2}\left(\beta_0 P^{(0)} - P^{(0)} \otimes P^{(0)}\right)\right] + \mathcal{O}(a_s^3)\,. \end{aligned} \tag{3.23}$$

[3] A summation over the flavour indices is understood

3.3 Expansion of the matching coefficients at the hard scale

The final step is the expansion of the matching coefficients C defined in Eq. (1.29) for $\mu_0 = \mu_b$ around the scale Q up to order a_s^2. To do this, we use the expansion in Eq. (1.31), in which a_s is computed at μ_b, and combine it with Eq. (3.5). This yields:

$$C(\mu_b, \mu_b^2) = \Delta + a_s(Q) C^{(1,0)} + a_s^2(Q) \left[C^{(2,0)} + L\beta_0 C^{(1,0)} \right] + \mathcal{O}(a_s^3) \,. \tag{3.24}$$

3.4 Expansion of the single low-scale TMD

Before going into the more convoluted (but more relevant) case of a cross section, it is useful to compute the expansion of a single low-scale TMD, *i.e.* the convolution between matching coefficients and collinear distributions at the scale $\mu_0 = \mu_b$. This can be achieved combining Eqs. (3.23) and (3.24):

$$\begin{aligned} C(\mu_b, \mu_b^2) \otimes f(\mu_b) &= C(\mu_b, \mu_b^2) \otimes \Gamma(\mu_b, Q) \otimes f(Q) \\ &= \Big\{ \Delta + a_s(Q) \left[C^{(1,0)} - L P^{(0)} \right] \\ &+ a_s^2(Q) \Big[C^{(2,0)} + L \left(-P^{(1)} - C^{(1,0)} \otimes P^{(0)} + \beta_0 C^{(1,0)} \right) \\ &+ \frac{L^2}{2} \left(P^{(0)} \otimes P^{(0)} - \beta_0 P^{(0)} \right) \Big] \Big\} \otimes f(Q) + \mathcal{O}(a_s^3) \,. \end{aligned} \tag{3.25}$$

3.5 Expansion of the cross section

We are finally in the position to gather all pieces and write down the perturbative expansion of a cross section involving TMDs, relevant in the limit $q_T \to 0$. The only additional information required is the hard factor H that also admits a perturbative expansion. It is now opportune to reinstate the flavour indices. In general, the hard factor H has a flavour structure, meaning that it carries a pair of flavour indices. Choosing $\mu = Q$, the expansion of the hard factor reads:

$$H_{ij}(Q) = \sum_{n=0}^{\infty} a_s^n(Q) H_{ij}^{(n)} = H_{ij}^{(0)} + a_s(Q) H_{ij}^{(1)} + a_s^2(Q) H_{ij}^{(2)} + \mathcal{O}(a_s^3) \,, \tag{3.26}$$

where $H_{ij}^{(n)}$ are numerical factors. In relevant processes, such as Drell-Yan and SIDIS, it turns out that up to $\mathcal{O}(\alpha_s^2)$ the coefficients of the expansion of hard factor H are diagonal in flavour. Specifically, for scales sufficiently below the Z-mass scale, we have:

$$H_{ij}^{(n)} \propto e_i^2 \delta_{ij} \,, \quad \text{for} \quad n = 0, 1, 2 \,, \tag{3.27}$$

where e_i is the electric charge of the i-th quark flavour. For higher scales, one just needs to replace the electric charges with the effective electro-weak charges. This simplification does not hold beyond $\mathcal{O}(\alpha_s^2)$.

A physical cross section differential in q_T can finally be computed as:

$$\frac{d\sigma}{dq_T^2} \propto \sum_{ij} H_{ij}(Q) \int d^2\mathbf{b}_T e^{i\mathbf{b}_T \cdot \mathbf{q}_T} F_i(x, \mathbf{b}_T, Q, Q^2) D_j(z, \mathbf{b}_T, Q, Q^2) \tag{3.28}$$

where F_i and D_j are two different sets of TMD distributions (*e.g.* a set of PDFs and a set of FFs) indexed by a flavour index. Assuming that the flavour indices i and j run *either* on quarks *or* on the gluon and writing explicitly all perturbative factors, one has:

$$\begin{aligned} \frac{d\sigma}{dq_T^2} &\propto \sum_{ij} H_{ij}(Q) \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T \cdot \mathbf{q}_T} R^2 \left[(\mu_b, \mu_b^2) \to (Q, Q^2) \right] \\ &\times \sum_k \left[\mathcal{C}_{ik}(x; \mu_b, \mu_b^2) \otimes f_k(x, \mu_b) \right] \sum_l \left[\mathbb{C}_{jl}(z; \mu_b, \mu_b^2) \otimes d_l(z, \mu_b) \right] \\ &= \sum_{n=0}^{\infty} a_s^n(Q) \sum_{p=0}^{2n} I_p(q_T, Q) \sum_{ij,kl} B_{ij,kl}^{(n,p)}(x, z) \underset{x}{\otimes} f_k(x, Q) \underset{z}{\otimes} d_l(z, Q) \,. \end{aligned} \tag{3.29}$$

In the equation above we used different symbols to distinguish the perturbative components of the distributions F and D because in general they are different. In addition, we have defined:

$$I_p(q_T, Q) \equiv \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T\cdot\mathbf{q}_T} \ln^p\left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}}\right) = \frac{1}{2}\int_0^\infty db_T\, b_T J_0(b_T q_T) \ln^p\left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}}\right) . \tag{3.30}$$

Results for I_p have been computed up to $p = 4$ in Eq. (136) of Appendix B of Ref. [7]. Specifically, and including the trivial tranform with $p = 0$, they read:

$$\begin{aligned}
I_0(q_T, Q) &= \delta(q_T) , \\
I_1(q_T, Q) &= -\frac{1}{q_T^2} , \\
I_2(q_T, Q) &= -\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right) , \\
I_3(q_T, Q) &= -\frac{3}{q_T^2}\ln^2\left(\frac{Q^2}{q_T^2}\right) , \\
I_4(q_T, Q) &= -\frac{4}{q_T^2}\left[\ln^3\left(\frac{Q^2}{q_T^2}\right) - 4\zeta_3\right] .
\end{aligned} \tag{3.31}$$

The functions above are compute under the assumption that there is no non-perturbative component. Upon this assumption, that we will relax below, all terms with $p = 0$ will be proportional to $\delta(q_T)$. Analogous terms are *not* included in the fixed-order calculation. Therefore, when matching the resummed calculation to the fixed-order one, one should remove from Eq. (3.29) all terms with $p = 0$. Importantly, this means removing the full $\mathcal{O}(1)$ terms such that the leading-order term for $q_T > 0$ is $\mathcal{O}(a_s)$.

The coefficients $B^{(n,k)}$ in Eq. (3.29) can be determined by using the expansions worked out above in Eqs. (3.12), (3.25), and (3.26). The leading order is obtained by just retaining the $\mathcal{O}(a_s^0)$ terms in all the expansions. This gives:

$$B_{ij,kl}^{(0,0)}(x, z) = H_{ij}^{(0)}\delta_{ik}\delta_{jl}\delta(1 - x)\delta(1 - z) . \tag{3.32}$$

The $\mathcal{O}(a_s)$ terms are obtained by combining the leading-order ones of all expansions but one. Finally organising the terms in powers of L, this yields:

$$\begin{aligned}
B_{ij,kl}^{(1,0)}(x, z) &= H_{ij}^{(1)}\delta_{ik}\delta(1 - x)\delta_{jl}\delta(1 - z) + H_{ij}^{(0)}\delta_{ik}\delta(1 - x)\mathbb{C}_{jl}^{(1,0)}(z) + H_{ij}^{(0)}\mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1 - z) , \\
B_{ij,kl}^{(1,1)}(x, z) &= H_{ij}^{(0)}S^{(1,1)}\delta_{ik}\delta(1 - x)\delta_{jl}\delta(1 - z) - H_{ij}^{(0)}\delta_{ik}\delta(1 - x)\mathbb{P}_{jl}^{(0)}(z) - H_{ij}^{(0)}\mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1 - z) , \\
B_{ij,kl}^{(1,2)}(x, z) &= H_{ij}^{(0)}S^{(1,2)}\delta_{ik}\delta(1 - x)\delta_{jl}\delta(1 - z) .
\end{aligned} \tag{3.33}$$

Finally, the $\mathcal{O}(a_s^2)$ terms are more convoluted and read:

$$
\begin{aligned}
B_{ij,kl}^{(2,0)}(x,z) &= H_{ij}^{(2)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\,,\\
&+ H_{ij}^{(1)}\left[\mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,0)}(z)\right]\\
&+ H_{ij}^{(0)}\mathcal{C}_{ik}^{(2,0)}(x)\delta_{jl}\delta(1-z)+H_{ij}^{(0)}\mathcal{C}_{ik}^{(1,0)}(x)\mathbb{C}_{jl}^{(1,0)}(z)+H_{ij}^{(0)}\delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(2,0)}(z)\,,\\
B_{ij,kl}^{(2,1)}(x,z) &= \left(H_{ij}^{(0)}\widetilde{S}^{(2,1)}+H_{ij}^{(1)}S^{(1,1)}\right)\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\\
&- H_{ij}^{(1)}\left[\mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(0)}(z)\right]\\
&+ H_{ij}^{(0)}S^{(1,1)}\left[\mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,0)}(z)\right]\\
&+ H_{ij}^{(0)}\left(-\mathcal{P}^{(1)}(x)-\sum_{\alpha}\mathcal{C}_{i\alpha}^{(1,0)}(x)\otimes\mathcal{P}_{\alpha k}^{(0)}(x)+\beta_0\mathcal{C}_{ik}^{(1,0)}(x)\right)\delta_{jl}\delta(1-z)\\
&- H_{ij}^{(0)}\mathcal{P}_{ik}^{(0)}(x)\mathbb{C}_{jl}^{(1,0)}(z)-H_{ij}^{(0)}\mathcal{C}_{ik}^{(1,0)}(x)\mathbb{P}_{jl}^{(0)}(z)\\
&+ H_{ij}^{(0)}\delta_{jl}\delta(1-z)\left(-\mathbb{P}^{(1)}(z)-\sum_{\beta}\mathbb{C}_{j\beta}^{(1,0)}(z)\otimes\mathbb{P}_{\beta l}^{(0)}(z)+\beta_0\mathbb{C}_{jl}^{(1,0)}(z)\right)\,,\\
B_{ij,kl}^{(2,2)}(x,z) &= \left(H_{ij}^{(0)}\widetilde{S}^{(2,2)}+H_{ij}^{(1)}S^{(1,2)}\right)\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\\
&+ H_{ij}^{(0)}\widetilde{S}^{(1,2)}\left[\mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,0)}(z)\right]\\
&- H_{ij}^{(0)}S^{(1,1)}\left[\mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(0)}(z)\right]\\
&+ H_{ij}^{(0)}\mathcal{P}_{ik}^{(0)}(x)\mathbb{P}_{jl}^{(0)}(z)\\
&+ H_{ij}^{(0)}\frac{1}{2}\left(\sum_{\alpha}\mathcal{P}_{i\alpha}^{(0)}(x)\otimes\mathcal{P}_{\alpha k}^{(0)}(x)-\beta_0\mathcal{P}_{ik}^{(0)}(x)\right)\delta_{jl}\delta(1-z)\\
&+ H_{ij}^{(0)}\delta_{ik}\delta(1-x)\frac{1}{2}\left(\sum_{\beta}\mathbb{P}_{j\beta}^{(0)}(z)\otimes\mathbb{P}_{\beta l}^{(0)}(z)-\beta_0\mathbb{P}_{jl}^{(0)}(z)\right)\,,\\
B_{ij,kl}^{(2,3)}(x,z) &= H_{ij}^{(0)}\widetilde{S}^{(2,3)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)-H_{ij}^{(0)}S^{(1,2)}\left[\mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1-z)+\delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(0)}(z)\right]\,,\\
B_{ij,kl}^{(2,4)}(x,z) &= H_{ij}^{(0)}\widetilde{S}^{(2,4)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\,.
\end{aligned}
\tag{3.34}
$$

3.6 A note on the logarithmic ordering

The logarithmic accuracy is driven by the evolution kernel R defined in Eq. (1.1). As it should be clear from Eq. (3.12), the evolution kernel squared R^2, also referred to as Sudakov form factor, admits the following expansion:

$$
R^2=\sum_{n=0}^{\infty}a_s^n\sum_{k=1}^{2n}\widetilde{S}^{(n,k)}L^k\,. \tag{3.35}
$$

This expansion naturally allows us to define a logarithmic ordering such that:

$$R^2 = \sum_{m=0}^{\infty} R^2_{\mathrm{N^mLL}}\,, \tag{3.36}$$

with:

$$R^2_{\mathrm{N^mLL}} = \sum_{n=[m/2]}^{\infty} \widetilde{S}^{(n,2n-m)}\, a_s^n L^{2n-m}\,, \tag{3.37}$$

where $[m/2]$ stands for "integer part of $m/2$". Now, I multiply $R^2_{\mathrm{N^mLL}}$ by a_s to a generic power p and do some manipulation defining $n+p=j$. This way, I get:

$$a_s^p R^2_{\mathrm{N^mLL}} = \sum_{j=[(m+2p)/2]}^{\infty} \widetilde{S}^{(j-p,2j-(m+2p))}\, a_s^j L^{2j-(m+2p)} \sim R^2_{\mathrm{N^{m+2p}LL}}\,, \tag{3.38}$$

where the symbol $\sim$ means that the terms at two sides have the same logarithmic accuracy. This finding is relevant in that, in order to compute a cross section, the Sudakov from factor has to be multiplied by the hard factor H and a pair of matching functions C. Both these components admit an expansion in powers of a_s without enhancing logaritms L (see Eqs. (1.31) and (3.26)). Therefore R^2 will eventually be multiplied by some power of a_s.

Suppose we compute the Sudakov form factor to leading-logarithmic (LL) accuracy ($m=0$): Eq. (3.38) tells us that, in terms of logarithmic accuracy, including $\mathcal{O}(a_s)$ corrections in H and C implies introducing next-to-next-to-leading-logarithmic (NNLL) corrections. In practice, this means that to LL and NLL the functions H and C can be taken at leading order, at NNLL we only need to introduce $\mathcal{O}(a_s)$ corrections in H and C, while at $\mathrm{N^3LL}$ we have to introduce $\mathcal{O}(a_s^2)$ corrections in H and C.

As a final remark, we notice that, as opposed to the Sudakov form factor, the evolution of collinear distributions and α_s provide resummation of single logarithms. Therefore, the inclusion of perturbative corrections to the β-functions and splitting functions should be accordingly. Specifically, the perturbative accuracy at which they are computed should be the same as that of the non-cusp anomalous dimension γ_F.

4 Non-perturbative effect on the asymptotic cross section

As discussed in Sect. 2, TMDs have a non-perturbative component that can be parameterised as in Eq. (2.5). This implies two main ingredients: the introduction of a function $b_*(b_T)$ that prevents the perturbative component to enter the non-perturbative regime and a function f_{NP} that parametrises the actual non-perturbative component. These changes leave almost unchanged the derivation of the asymptotic behavior of a cross section for $q_T \to 0$. The only change concerns the functions I_p defined in Eq. (3.30), that have to be replaced with:

$$\widetilde{I}_p(x,z,q_T,Q) \equiv \frac{1}{2}\int_0^{\infty} db_T\, b_T J_0(b_T q_T) f^{(1)}_{\mathrm{NP}}(x,b_T,Q^2) f^{(2)}_{\mathrm{NP}}(z,b_T,Q^2) \ln^p\left(\frac{b_*^2(b_T)Q^2}{4e^{-2\gamma_E}}\right)\,, \tag{4.1}$$

where $f^{(1)}_{\mathrm{NP}}$ and $f^{(2)}_{\mathrm{NP}}$ are the non-perturbative functions associated to the two TMD distributions F and D involved in the cross sections. Here we are assuming $\mu = \sqrt{\zeta} = Q$ and $\mu_0 = \sqrt{\zeta_0} = \mu_b$. In general, the integral in Eq. (4.1) cannot be evaluated analytically but one can use the Ogata quadrature method to compute it numerically.([4]) In the case $k=0$ Eq. (4.1) has an intersting consequence. Specifically, contrary to the case in which no non-perturbative contribution is introduced, $\widetilde{I}_0$ is different from zero also for $q_T > 0$. Therefore, also the terms in the expansion in Eq. (3.29) that are independent of L contribute to the cross section differential in q_T. Clearly, any non-perturbative contribution is expected to give a substantial contribution only in the vicinity of $q_T = 0$.

In this respect, one should check that the introduction of the non-perturbative components does not affect the large-q_T region. This is amounts to show that:

$$\widetilde{I}_p(x,z,q_T,Q) \underset{q_T \gtrsim Q}{\sim} I_p(q_T,Q)\,. \tag{4.2}$$

Since for small values of b_T we (must) have that $b_*(b_T) \simeq b_T$ and that $f_{\mathrm{NP}}(x,b_T,\zeta) \simeq 1$, this implies that for large values of q_T (that is the conjugate variable of b_T), $\widetilde{I}_p$ has to tend to I_p. This can be shown numerically on

[4] The convergence of the Ogata quadrature method crucially depends on the function f_{NP} being exponentially suppressed for large values of b_T, such that full integrand is dumped in this region.

a case-by-case base. In particular, we can compute the integral in Eq. (4.1) using the Ogata quadrature method and compare the results to those in Eq. (3.31). For the comparison we use Eq. (2.4) with $b_{\max} = 2e^{-\gamma_E}$ and choose as non-perturbative functions $f_{\rm NP}$ the following x-independent form:

$$f^{(1)}_{\rm NP}(b_T,\zeta) = f^{(2)}_{\rm NP}(b_T,\zeta) = \exp\left[\left(-g_1 - g_2 \ln\left(\frac{\sqrt{\zeta}}{2Q_0}\right)\right)\frac{b_T^2}{2}\right], \tag{4.3}$$

with $g_1 = 0.02$, $g_1 = 0.5$, and $Q_0 = 1.6$ GeV.

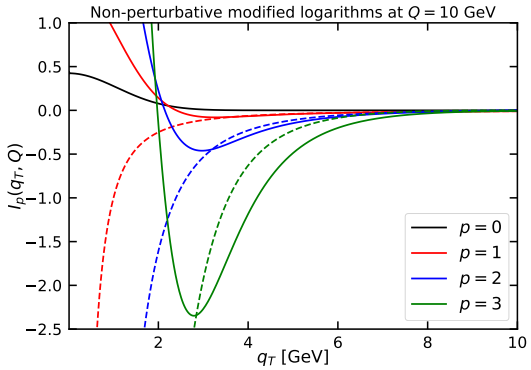

Fig. 4.1: Comparison of the integral in Eq. (4.1) for $p = 0, 1, 2, 3$ (solid lines) to the expressions in Eq. (3.31) (dashed lines) as functions of q_T at $Q = 10$ GeV. The black dashed line is not present because it corresponds to $\delta(q_T)$.

In Fig. 4.1, I compare the integral in Eq. (4.1) for $p = 0, 1, 2, 3$ (solid lines) to the expressions in Eq. (3.31) (dashed lines) at $Q = 10$ GeV as functions of q_T. Notice that the black dashed line is not present because it corresponds to $\delta(q_T)$. It is thus true that $\widetilde{I}_p$ tends to I_p as q_T increases. However, this is not enough to satisfy Eq. (4.2). As a matter of fact, if one zooms in around $q_T \lesssim Q$, one finds a substantial difference between $\widetilde{I}_p$ and I_p. In Fig 4.2 the zoom is showed for $p = 1$. Even if q_T is abundantly in the fixed-order region, the difference

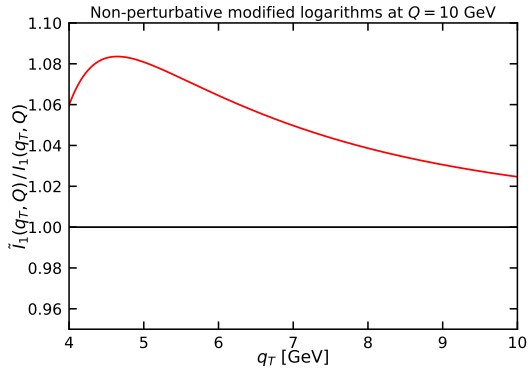

Fig. 4.2: Ratio between $\widetilde{I}_1$ and I_1 in the region of q_T close to Q.

between $\widetilde{I}_1$ and I_1 is still large. Therefore, it appears that the relation in Eq. (4.2) is not plainly fulfilled. Moreover, the discrepancy tends to become larger as p increases. Therefore, when performing the matching between resummed and fixed-order cross sections, it is very important to include non-perturbative effects in the expanded calculation in order to ensure a proper cancellation with the resummed calculation in the large-q_T region.

On the other hand, by design, the inclusion of the non-perturbative effects modifies significantly also the low-q_T region. The modification is such to prevent the cancellation at small q_T between the expansion of the resummed calculation and the fixed-order one. A possible solution is to use yet another definition of the functions I_p in the expanded calculation. These functions have to be such to tend to those in Eq. (3.31) for small values of q_T but converge to the definition in Eq. (4.1) more rapidly as q_T increases. To do this, one may

exploit the fact that power-suppressed contributions do not affect the logarithmic expansion of the resummed calculation to combine the two definitions as follows:

$$\hat{I}_p(x,z,q_T,Q) = \left[1 - \left(\frac{q_T}{Q}\right)^S\right] I_p(q_T,Q) + \left(\frac{q_T}{Q}\right)^S \widetilde{I}_p(x,z,q_T,Q)\,, \tag{4.4}$$

where the exponent S can be adjusted to make the transition from one regime to the other more or less strong. Clearly, this definition should not be pushed to values of q_T much above Q. As an example, Fig. 4.3 shows the shape of $\hat{I}_1$ defined in Eq. (4.4) for $S = 0.2$ along with its components given in Eqs. (3.31) and (4.1).

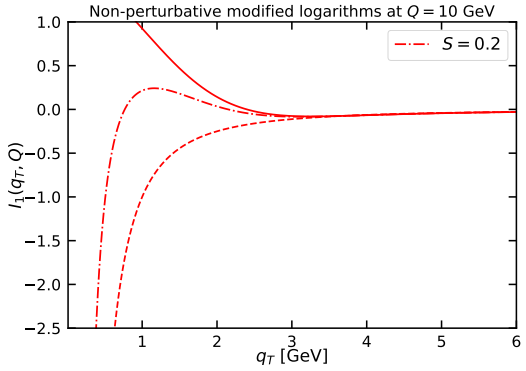

Fig. 4.3: Behaviour of $\hat{I}_1$ defined in Eq. (4.4) for $S = 0.2$ compared to the single components defined in Comparison of the integral in Eqs. (3.31) and (4.1).

Part II. Semi-inclusive deep-inelastic-scattering

5 Fixed order and asymptotic limit

In order to validate the results above, it is opportune to compare the $\mathcal{O}(a_s)$ expressions to those present in the literature. To this end, we write explicitly the expression for the SIDIS cross section differential in q_T for $q_T > 0$, *i.e.* without the $\delta(q_T)$ terms, and with no non-perturbative effect. Considering that, $S^{(1,1)} = 6C_F$, $S^{(1,2)} = -2C_F$, and $H^{(0)}_{ij} = e_i^2\delta_{ij}$, this yields:

$$\begin{aligned}
\frac{d\sigma}{dxdydzdq_T^2} &\propto a_s(Q)\sum_{ij,kl}\left[-B^{(1,1)}_{ij,kl}(x,z)\frac{1}{q_T^2} - B^{(1,2)}_{ij,kl}(x,z)\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\right]\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q)\\
&= \frac{a_s(Q)}{q_T^2}\sum_{ij,kl} e_i^2\delta_{ij}\Bigg[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\\
&+ \mathcal{P}^{(0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}^{(0)}_{jl}(z)\Bigg]\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q) \qquad (5.1)\\
&= \frac{a_s(Q)}{q_T^2}\sum_i e_i^2\Bigg[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x,Q)d_i(z,Q)\\
&+ \left(\sum_k \mathcal{P}^{(0)}_{ik}(x)\underset{x}{\otimes} f_k(x,Q)\right)d_i(z,Q) + f_i(x,Q)\left(\sum_l \mathbb{P}^{(0)}_{il}(z)\underset{z}{\otimes} d_l(z,Q)\right)\Bigg] + \mathcal{O}(a_s^2)\,.
\end{aligned}$$

This result, up to pre-factors that will be made explicit below, nicely agrees with that of, *e.g.*, Refs. [8, 9, 10].

In order to check that the matching is actually removing the double counting terms, it is instructive to derive Eq. (5.1) extracting the asymptote from the fixed-order computation at $\mathcal{O}(a_s)$. We take the expressions for the coefficient functions from Eqs. (106)-(109) of Appendix B of Ref. [10] or from Eqs. (4.6)-(4.20) of Ref. [11]. Referring to the second reference, some simplifications apply. In particular, we consider cross sections with unpolarised projectiles ($\lambda_e = 0$) on unpolarised targets ($S^\mu_\perp = 0$) and integrated over the azimuthal angles ϕ_H and ϕ_S. By doing so and after a simple manipulation, the cross section simplifies greatly and can be written in terms of structure functions as:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{2\pi\alpha^2}{xyQ^2}\left[Y_+F_{UU,T} + 2(1-y)F_{UU,L}\right] = \frac{2\pi\alpha^2}{xyQ^2}Y_+\left[F_{UU,2} - \frac{y^2}{Y_+}F_{UU,L}\right], \qquad (5.2)$$

with:

$$Y_+ \equiv 1 + (1-y)^2\,, \qquad (5.3)$$

and where we have defined the structure function:

$$F_{UU,2} \equiv F_{UU,T} + F_{UU,L}\,. \qquad (5.4)$$

Notice that, as compared to Ref. [11], we have factored out from the structure functions a factor $1/(\pi z^2)$([5]) so that they factorize as:

$$\begin{aligned}
F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)\Bigg[\hat{B}^{S,\mathrm{FO}}_{qq}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\\
&+ \hat{B}^{S,\mathrm{FO}}_{qg}(\bar{x},\bar{z},q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}^{S,\mathrm{FO}}_{gq}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\Bigg] + \mathcal{O}(a_s^2)\,.
\end{aligned} \qquad (5.5)$$

[5] The factor z^2 is the consequence of the fact that we are writing the cross section differential in q_T^2 that is the transverse momentum of the exchanged photon while in Ref. [11] the cross section is differentila in p_T^2 that is the transverse momentum of the of the outgoing hadrons. Since $p_T = zq_T$, the factor z^2 cancels.

with $S = 2, L$ and where the sum over i runs over the active quark and antiquark flavours. The explicit expressions for the coefficient functions are:

$$\begin{aligned}
\hat{B}^{2,\mathrm{FO}}_{qq}(x,z,q_T) &= 2C_F\left[(1-x)(1-z)+4xz+\frac{1+x^2z^2}{xz}\frac{Q^2}{q_T^2}\right],\\
\hat{B}^{L,\mathrm{FO}}_{qq}(x,z,q_T) &= 8C_F xz\,,\\
\hat{B}^{2,\mathrm{FO}}_{qg}(x,z,q_T) &= 2T_R\left[[x^2+(1-x)^2][z^2+(1-z)^2]\frac{1-x}{xz^2}\frac{Q^2}{q_T^2}+8x(1-x)\right],\\
\hat{B}^{L,\mathrm{FO}}_{qg}(x,z,q_T) &= 16T_R x(1-x)\,.\\
\hat{B}^{2,\mathrm{FO}}_{gq}(x,z,q_T) &= 2C_F\left[(1-x)z+4x(1-z)+\frac{1+x^2(1-z)^2}{xz}\frac{1-z}{z}\frac{Q^2}{q_T^2}\right],\\
\hat{B}^{L,\mathrm{FO}}_{gq}(x,z,q_T) &= 8C_F x(1-z)\,,
\end{aligned} \tag{5.6}$$

These expressions are enough to compute the SIDIS cross section at $\mathcal{O}(a_s)$ in the region $q_T \lesssim Q$. In order to match Eq. (5.1), one has to take the limit $q_T/Q \to 0$ and retain in the coefficient functions only the terms enhanced as $\ln(Q^2/q_T^2)$. This automatically means that $F_{UU,L}$ does not contribute in this limit because it contains no logarithmic enhancements such that:

$$F_{UU,L} \underset{q_T/Q\to 0}{\ll} F_{UU,2}\,. \tag{5.7}$$

Another crucial observation is that the δ-function in Eq. (5.5) can be expanded as follows([6]):

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-x)(1-z)}{xz}\right) \underset{q_T^2/Q^2\to 0}{\longrightarrow} \ln\left(\frac{Q^2}{q_T^2}\right)\delta(1-x)\delta(1-z)+\frac{x\delta(1-z)}{(1-x)_+}+\frac{z\delta(1-x)}{(1-z)_+}\,, \tag{5.8}$$

so that:

$$\begin{aligned}
F_{UU,2} \underset{q_T/Q\to 0}{\longrightarrow} a_s\frac{x}{q_T^2}\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\Bigg[&\hat{B}^{2,\mathrm{asy}}_{qq}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\\
+\;&\hat{B}^{2,\mathrm{asy}}_{qg}(\bar{x},\bar{z},q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}^{2,\mathrm{asy}}_{gq}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\Bigg]+\mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.9}$$

with:

$$\begin{aligned}
\hat{B}^{2,\mathrm{asy}}_{qq}(x,z,q_T) &= 2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)+\frac{1+x^2}{(1-x)_+}\delta(1-z)+\delta(1-x)\frac{1+z^2}{(1-z)_+}\right]\\
&= 2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)-3\right]\delta(1-x)\delta(1-z)+\mathcal{P}^{(0)}_{qq}(x)\delta(1-z)+\delta(1-x)\mathbb{P}^{(0)}_{qq}(z)\,,\\
\hat{B}^{2,\mathrm{asy}}_{qg}(x,z,q_T) &= 2T_R\left[x^2+(1-x)^2\right]\delta(1-z)=\mathcal{P}^{(0)}_{qg}(x)\delta(1-z)\,,\\
\hat{B}^{2,\mathrm{asy}}_{gq}(x,z,q_T) &= \delta(1-x)2C_F\left[\frac{1+(1-z)^2}{z}\right]=\delta(1-x)\mathbb{P}^{(0)}_{qg}(z)\,.
\end{aligned} \tag{5.10}$$

It is thus easy to see that we can rewrite Eq. (5.9) as:

$$\begin{aligned}
F_{UU,2} \underset{q_T/Q\to 0}{\longrightarrow} a_s\frac{x}{q_T^2}\sum_i e_i^2\Bigg[&4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x)\,d_i(z)\\
+\;&\left(\sum_{k=q,g}\mathcal{P}^{(0)}_{qk}(x)\otimes f_k(x)\right)d_i(z)+f_i(x)\left(\sum_{k=q,g}\mathbb{P}^{(0)}_{qk}(z)\otimes d_k(z)\right)\Bigg]+\mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.11}$$

[6] The proof of this relation is given in Appendix A.

Therefore, up to factors, Eq. (5.11) agrees with Eq. (5.1). This confirms that the expansion of the resummed calculation, as well as the asymptotic limit of the fixed order, removes the double-counting terms when doing the matching.

In order to provide a version of Eq. (5.5) that can be readily implemented, we need to perform one of the integrals making use of the δ-function. We integrate over $\bar{x}$ so that we write:

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)=\frac{\bar{z}\bar{x}_0^2}{1-\bar{z}}\delta(\bar{x}-\bar{x}_0)\,, \tag{5.12}$$

with:

$$\bar{x}_0=\frac{1-\bar{z}}{1-\bar{z}\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.13}$$

This allows us to write:

$$\begin{aligned} F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}\bar{x}_0\left[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \left.\hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_g\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_g\left(\frac{z}{\bar{z}}\right)\right]+\mathcal{O}(a_s^2)\,, \end{aligned} \tag{5.14}$$

with:

$$z_{\max}=\frac{1-x}{1-x\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.15}$$

Now we can rewrite the cross section above in such a way that it matches that at $\mathcal{O}(a_s)$ of Ref. [12]. That would allow us to confidently use the $\mathcal{O}(a_s^2)$ calculation presented in that reference for the matching to the resummed calculation. This is made tricky by the different notation used in Ref. [12] and from the fact that in that paper the cross section is differential in a different set of variables. Specifically, we would like it to be differential in x, y, z, and q_T^2 while in Ref. [12] it is differential in x, Q^2, η, and p_T^2, where the last two are the rapidity and the transverse momentum of the outgoing hadron, respectively. Eq. (13) of Ref. [12], can be translated into our notation by noticing that:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta}=\frac{x}{zQ^2}\sum_{i,j}\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}f_i\left(\frac{x}{\bar{x}_0}\right)d_j\left(\frac{z}{\bar{z}}\right)\frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta}+\mathcal{O}(a_s^2)\,, \tag{5.16}$$

where we have exploited the δ-functions in Eqs. (18)-(20) to get rid of the integral over z.([7])

The $\mathcal{O}(a_s)$ partonic cross sections in Eqs. (18)-(20) of Ref. [12], setting $\varepsilon=0$, can be written as:

$$\frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta}=\frac{2\pi\alpha^2a_se_q^2\bar{x}_0}{xQ^4}Y_+\left[\underbrace{\left(F_{UU,M}^{ij}(\bar{x}_0,\bar{z})+\frac{3}{2}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right)}_{F_{UU,2}^{ij}}-\frac{y^2}{Y_+}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right]\,. \tag{5.17}$$

One can verify that $F_{UU,2}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{gq}(\bar{x}_0,\bar{z})$, and $F_{UU,L}^{gq}(\bar{x}_0,\bar{z})$ correspond exactly to $\hat{B}_{qq}^{2,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{gq}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, and $\hat{B}_{gq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$ in Eq. (5.6) of Ref. [11]. It is crucial to notice that the correspondence holds only if $\bar{x}_0$ defined in Eq. (5.13) as a function of $\bar{z}$ is used. As an example, taking into account the different factors (a factor 2 for $F_{UU,M}$ and a factor 4 for $F_{UU,L}$) and using our notation for the integration variables ($y\to\bar{z}$, $\rho(z=0)\to\bar{x}_0$), we read off from Eq. (18) of Ref. [12]:

$$\begin{aligned} F_{UU,M}^{qq}(\bar{x}_0,\bar{z}) &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2+2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})}\right]\,, \\ F_{UU,L}^{qq}(\bar{x}_0,\bar{z}) &= 8C_F\bar{x}_0\bar{z}\,, \end{aligned} \tag{5.18}$$

[7] Notice that the z variable of Ref. [12] does not coincide with our definition.

so that:

$$\begin{aligned} F_{UU,2}^{qq}(\bar{x}_0,\bar{z}) &= F_{UU,M}^{qq}(\bar{x}_0,\bar{z}) + \frac{3}{2}F_{UU,L}^{qq}(\bar{x}_0,\bar{z}) \\ &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2+2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})}+3\bar{x}_0\bar{z}\right] \\ &= 2C_F\left[(1-\bar{x}_0)(1-\bar{z})+4\bar{x}_0\bar{z}+\frac{1+\bar{x}_0^2\bar{z}^2}{\bar{x}_0\bar{z}}\left(\frac{\bar{x}_0\bar{z}}{(1-\bar{x}_0)(1-\bar{z})}\right)\right]. \end{aligned} \tag{5.19}$$

Using Eq. (5.13), it is easy to see that the factor in round brackets in the last line of the equation above is equal to Q^2/q_T^2. Therefore, it reduces exactly to the first relation in Eq. (5.6) of Ref. [11]. The same holds also for the two remaining partonic channels.

Putting all pieces together, Eq. (5.16) can be recast as:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{zxQ^4}Y_+\left[F_{UU,2}-\frac{y^2}{Y_+}F_{UU,L}\right], \tag{5.20}$$

with:

$$\begin{aligned} F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}\bar{x}_0\left[F_{UU,L}^{qq}(\bar{x}_0,\bar{z})f_i\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \left. F_{UU,L}^{qg}(\bar{x}_0,\bar{z})f_g\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)+F_{UU,L}^{gq}(\bar{x}_0,\bar{z})f_i\left(\frac{x}{\bar{x}_0}\right)d_g\left(\frac{z}{\bar{z}}\right)\right]+\mathcal{O}(a_s^2). \end{aligned} \tag{5.21}$$

Therefore, the structure of the observables is exactly the same. What is left to work out is the Jacobian to express the cross section differential in the same variables as in Eq. (5.2). What we need is to know is how the variables Q^2, p_T, and η are related to y, z, and q_T. The relevant relations are:

$$\left\{\begin{aligned} Q^2 &= xyS_H \\ p_T^2 &= z^2q_T^2 \\ \eta &= \frac{1}{2}\ln\left(\frac{y(1-x)S_H}{q_T^2}\right) \end{aligned}\right. \quad\Longrightarrow\quad dQ^2dp_T^2d\eta = \frac{zQ^2}{y}dydzdq_T^2. \tag{5.22}$$

where S_H is the squared center-of-mass energy of the collision, so that:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{zQ^2}{y}\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{xyQ^2}Y_+\left[F_{UU,2}-\frac{y^2}{Y_+}F_{UU,L}\right], \tag{5.23}$$

exactly like in Eq. (5.2). This confirms that the computation of Ref. [12] can be matched to the resummed calculation provided that the correct Jacobian is taken into account.

In Fig. 5.1, a comparison between the $\mathcal{O}(\alpha_s)$ computation of Ref. [12] for the differential cross section in Eq. (5.16), implemented in the computer code TIMBA, and the implementation in the APFEL++ framework [13] of the expressions in Eq. (5.14) is presented. The cross section as a function of the hadron transverse momentum p_T obtained with APFEL++ is displayed as ratios to TIMBA for representative values of the kinematic variables: the agreement between the two codes is excellent. Since we will be using TIMBA for the NLO (*i.e.* $\mathcal{O}(a_s^2)$) cross section calculation to be matched to the NNLL resummed computation, this provides a solid ground to start from.

In order to be able to compare the theoretic predictions to the data, it is useful to work out the for of the cross section differential in the variables x, Q^2, z, and p_T^2. This is easily done starting from Eq. (5.23) and considering that:

$$\eta = \ln\left(\frac{zW}{p_T}\right), \quad \text{with} \quad W=\sqrt{\frac{Q^2(1-x)}{x}}, \quad \Rightarrow \quad d\eta = \frac{dz}{z}. \tag{5.24}$$

It follows that:

$$\frac{d\sigma}{dxdQ^2dzdp_T^2} = \frac{2\pi\alpha^2}{xQ^4}Y_+\left[F_{UU,2}-\frac{y^2}{Y_+}F_{UU,L}\right], \tag{5.25}$$

with $y = Q^2/xS_H$ (see Eq. (5.22)).

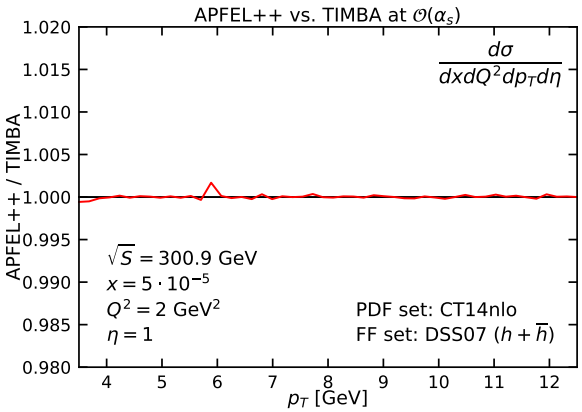

Fig. 5.1: Comparison at $\mathcal{O}(\alpha_s)$ between the TIMBA code, implementation of the results of Ref. [12], and the implementation of the expressions in Eq. (5.14) in the APFEL++ code [13].

6 Integrating over q_T

It is interesting to carry out the integration over q_T of Eq. (5.5) analytically. The result can then be compared to those presented in Appendix C of Ref. [14]. Exploiting the δ-function yields:

$$\begin{aligned}\int_0^\infty dq_T^2\, F_{UU,S} &= a_s x \sum_i e_i^2 \int_x^1 \frac{d\bar{x}}{\bar{x}} \int_z^1 \frac{d\bar{z}}{\bar{z}} \left[\hat{B}_{qq}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_i\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \hat{B}_{qg}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_g\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) \\ &+ \left. \hat{B}_{gq}^{S,\mathrm{FO}}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right) f_i\left(\frac{x}{\bar{x}}\right) d_g\left(\frac{z}{\bar{z}}\right)\right] + \mathcal{O}(a_s^2)\,. \end{aligned} \tag{6.1}$$

replacing q_T^2/Q^2 in Eq. (5.6) has no effect on the coefficient functions for $F_{UU,L}$. The reason is that $F_{UU,L}$ is independent of q_T and therefore integrating over q_T has essentially the effect of removing the δ-function. On the other hand, integrating $F_{UU,2}$ gives:

$$\begin{aligned}\hat{B}_{qq}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2C_F\left[(1-x)(1-z)+4xz+\frac{1+x^2z^2}{(1-x)(1-z)}\right], \\ \hat{B}_{qg}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2T_R\left[[x^2+(1-x)^2]\left(\frac{1}{1-z}+\frac{1}{z}-2\right)+8x(1-x)\right], \\ \hat{B}_{gq}^{2,\mathrm{FO}}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2C_F\left[(1-x)z+4x(1-z)+\frac{1+x^2(1-z)^2}{z(1-x)}\right], \end{aligned} \tag{6.2}$$

Instating the plus prescription in all terms that behave as $(1-x)^{-1}$ and $(1-z)^{-1}$([8]), one can see that up to "local" terms (*i.e.* those terms proportional to the δ-function) the expressions of Appendix C of Ref. [14] are reproduced([9]). The local terms correspond to virtual corrections that thus give contributions at $q_T = 0$. As such, they should coincide with the terms proportional to $\delta(q_T)$ generated in the expansion of the resummed calculation. Specifically, they should be equal to the coefficients $B_{ij,kl}^{(n,0)}$ in the expansion in Eq. (3.29). This is evidently the case for $B_{ij,kl}^{(0,0)}$ that gives the leading-order contribution to the integrated cross section. Unfortunately, this is not true at next-to-leading order because the expressions for the local terms in Appendix C of Ref. [14] (Eqs. (C.2)-(C.4)) do not match $B_{ij,kl}^{(1,0)}$. However, when checking the expressions explicitly an interesting pattern emerges: if all the occurrences of $\ln(1-x)$ (or $\ln(1-z)$) in Eqs. (C.2)-(C.4) of Ref. [14]

[8] This step has the effect of removing the uncanceled soft divergencies.
[9] The expressions reported in Ref. [14] are those computed long time ago in Ref. [15].

are replaced with $\ln(x)$ (or $\ln(z)$) the two sets of expressions coincide. In fact, the expression to $\mathcal{O}(\alpha_s)$ that we derived above are free of large threshold logarithms associated to soft-gluon emission.

This should not be surprising because the CS equation

7 Integrating over z

In order to understand which of the two is the correct set of expressions, whether those computed in Ref. [15] or those derived here, we may try to integrate over z summing over all possible hadrons produced in the final state. By doing so, we should obtain the expressions of the hard cross sections for inclusive deep-inelastic scattering. This last set of expressions has also been computed in Ref. [15] and is very likely to be correct. The presence of terms proportional to $\ln(1-x)$ in the inclusive DIS hard cross sections suggests that the expressions reported in Ref. [14] and computed in Ref. [15] are the correct ones. To integrate the expressions in Ref. [14] we need to use the fact that the SIDIS cross section integrate in q_T for the production of a hadron h has the following structure:

$$\frac{d\sigma^h}{dxdydz} = \sum_{ij} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} C_{ij}\left(\frac{x}{\xi}, \frac{z}{\zeta}\right) d_{i/h}(\zeta) f_j(\xi)\,, \tag{7.1}$$

where $d_{i/h}$ is the fragmentation function for the parton i fragmenting into the hadron species h. What we need to compute is:

$$\frac{d\sigma}{dxdy} = \sum_h \int_0^1 dz\, z\, \frac{d\sigma^h}{dxdydz}\,, \tag{7.2}$$

where the sum over h extends over all possible hadronic species. In order to compute the integral above we need to use the following property of the Mellin convolution:

$$C(z) \otimes d(z) = \int_z^1 \frac{d\zeta}{\zeta} C\left(\frac{z}{\zeta}\right) d(\zeta) = \int_0^1 d\zeta \int_0^1 d\eta\, C(\zeta) f(\eta) \delta(z - \zeta\eta)\,. \tag{7.3}$$

By using this property we have:

$$\begin{aligned} \frac{d\sigma}{dxdy} &= \sum_{ij} \int_x^1 \frac{d\xi}{\xi} \int_0^1 d\zeta\, C_{ij}\left(\frac{x}{\xi}, \zeta\right) f_j(\xi) \sum_h \int_0^1 d\eta\, d_{i/h}(\eta) \int_0^1 dz\, z\delta(z - \zeta\eta) \\ &= \sum_j \int_x^1 \frac{d\xi}{\xi} \left[\int_0^1 d\zeta\, \zeta \sum_i C_{ij}\left(\frac{x}{\xi}, \zeta\right)\right] f_j(\xi) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, \widetilde{C}_j\left(\frac{x}{\xi}\right) f_j(\xi)\,. \end{aligned} \tag{7.4}$$

where we have defined the inclusive DIS coefficient functions as:

$$\widetilde{C}_j(\xi) \equiv \int_0^1 d\zeta\, \zeta \sum_i C_{ij}(\xi, \zeta)\,, \tag{7.5}$$

and we have used the momentum sum rule:

$$\sum_h \int_0^1 d\eta\, \eta d_{i/h}(\eta) = 1\,, \tag{7.6}$$

that essentially encodes the fact that the total probability for a parton to fragment into any hadron in one. The correctness of Eq. (7.5) can be verified explicitly by integrating over z the expressions reported in Appendix C of Ref. [14] for both structure functions F_2 and F_L. The leading order is trivial:

$$\widetilde{C}^{(0)}_{2,q}(\xi) \equiv \int_0^1 d\zeta\, \zeta \delta(1-\xi)\delta(1-\zeta) = \delta(1-\xi)\,, \tag{7.7}$$

as well know, while the gluon as well as the longitudinal coefficient functions are zero. The next-to-leading order coefficient functions are computed as:

$$\begin{aligned} \widetilde{C}^{(1)}_q(\xi) &\equiv \int_0^1 d\zeta\, \zeta \left[C^{(1)}_{qq}(\xi, \zeta) + C^{(1)}_{gq}(\xi, \zeta)\right]\,, \\ \widetilde{C}^{(1)}_g(\xi) &\equiv \int_0^1 d\zeta\, \zeta C^{(1)}_{qg}(\xi, \zeta)\,. \end{aligned} \tag{7.8}$$

For F_L they read:

$$\begin{aligned} \widetilde{C}_{L,q}^{(1)}(\xi) &= 8C_F\xi\,, \\ \widetilde{C}_{L,g}^{(1)}(\xi) &= 8T_R\xi(1-\xi)\,, \end{aligned} \tag{7.9}$$

while for F_2:

$$\begin{aligned} \widetilde{C}_{2,q}^{(1)}(\xi) &= 2C_F\left[2\left(\frac{\ln(1-\xi)}{1-\xi}\right)_+ - \frac{3}{2}\left(\frac{1}{1-\xi}\right)_+ - (1+\xi)\ln(1-\xi)\right. \\ &- \left.\frac{1+\xi^2}{1-\xi}\ln\xi + 3 + 2\xi - \left(\frac{9}{2}+2\zeta_2\right)\delta(1-\xi)\right], \\ \widetilde{C}_{2,g}^{(1)}(\xi) &= 2T_R\left[(\xi^2+(1-\xi)^2)\ln\left(\frac{1-\xi}{\xi}\right) + 6\xi(1-\xi) - 1\right]. \end{aligned} \tag{7.10}$$

These expressions coincide with those reported, *e.g.*, in Chapter 4 of Ref. [16] with the only exception of $\widetilde{C}_{2,g}^{(1)}$ where the factor 6 in front of the term $\xi(1-\xi)$, according to Ref. [16], should be a 8. This should be investigated further.

Part III. Drell-Yan production

Part IV. Appendices

A Expansion of the kinematic δ-function

In order to prove the equality in Eq. (5.8), we consider the following integral:

$$I(\varepsilon) = \int_0^1 dx \int_0^1 dy\, \delta(xy - \varepsilon) f(x, y)\,, \tag{A.1}$$

where $f(x, y)$ is a test function "well-behaved" over the integration region. Now I split the integral as follows:

$$I(\varepsilon) = \left(\int_0^1 dx \int_x^1 dy + \int_0^1 dy \int_y^1 dx \right) \delta(xy - \varepsilon) f(x, y)\,, \tag{A.2}$$

where the first term in the r.h.s. corresponds to the integral over the grey region *above* the blue line while the second over the grey region *below* the blue line in Fig. A.1.

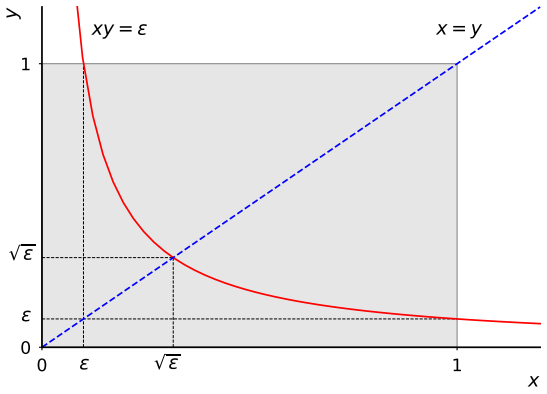

Fig. A.1: Integration region of the integral in Eq. (A.1). The integral is along the red curve defined by the δ-function.

Now I use the following equalities:

$$\delta(xy - \varepsilon) = \begin{cases} \dfrac{1}{x}\delta\left(y - \dfrac{\varepsilon}{x}\right)\theta(y - \sqrt{\varepsilon}) & \text{integral over } y, \\ \dfrac{1}{y}\delta\left(x - \dfrac{\varepsilon}{y}\right)\theta(x - \sqrt{\varepsilon}) & \text{integral over } x. \end{cases} \tag{A.3}$$

The θ-functions arise from the fact that the first integral has to be done along the upper branch on the red curve while the second along the lower branch. The two branches are joint at the point $x = y = \sqrt{\varepsilon}$ and thus the integration ranges are bounded from below by this point. Therefore, I find:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} f\left(x, \frac{\varepsilon}{x}\right) + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} f\left(\frac{\varepsilon}{y}, y\right)\,. \tag{A.4}$$

It is crucial to realise that in the first and the second integral the following conditions hold: $\varepsilon < \sqrt{\varepsilon} \leq x$ and $\varepsilon < \sqrt{\varepsilon} \leq y$, respectively. Therefore, in the limit $\varepsilon \to 0$, the arguments ε/x and ε/y of the function f will tend to zero. I now add and subtract a term proportional to $f(0, 0)$ to both integrals, so that:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x}\left[f\left(x, \frac{\varepsilon}{x}\right) - f(0,0)\right] + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y}\left[f\left(\frac{\varepsilon}{y}, y\right) - f(0,0)\right] + 2f(0,0)\underbrace{\int_{\sqrt{\varepsilon}}^1 \frac{d\xi}{\xi}}_{-\ln\sqrt{\varepsilon}}\,. \tag{A.5}$$

Finally, I take the limit for $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \int_0^1 \frac{dx}{x}\left[f(x, 0) - f(0,0)\right] + \int_0^1 \frac{dy}{y}\left[f(0, y) - f(0,0)\right] - \ln(\varepsilon)\, f(0,0)\,, \tag{A.6}$$

that I rewrite as:

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \int_0^1 dx \int_0^1 dy \left\{ \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y) \right\} f(x,y)\,. \tag{A.7}$$

Comparing the equation above with Eq. (A.1), one deduces that:

$$\delta(xy-\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y)\,. \tag{A.8}$$

Finally, substituting:

$$x \to \frac{1-x}{x}\,, \quad y \to \frac{1-z}{z}\,, \quad \text{and} \quad \varepsilon \to \frac{q_T^2}{Q^2}\,, \tag{A.9}$$

it is easy to recover Eq. (5.8). In particular, one needs to use the fact that:

$$\delta\left(\frac{1-x}{x}\right) = \delta(1-x)\,. \tag{A.10}$$

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