Contents

1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

In general, the evolution equation for GPDs reads:

$$\mu^2 \frac{d}{d\mu^2} f(x,\xi) = \int_{-\infty}^{+\infty} \frac{dx'}{|2\xi|} P\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) f(x',\xi)\,. \tag{2.1}$$

The GPD f and the evolution kernel P may in general be a vector and a matrix in flavour space. For now we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure. The support of the evolution kernel $P\left(\frac{x}{\xi}, \frac{x'}{\xi}\right)$ is shown in Fig. 2.1. Knowing the support of the evolution kernel, Eq. (2.1)

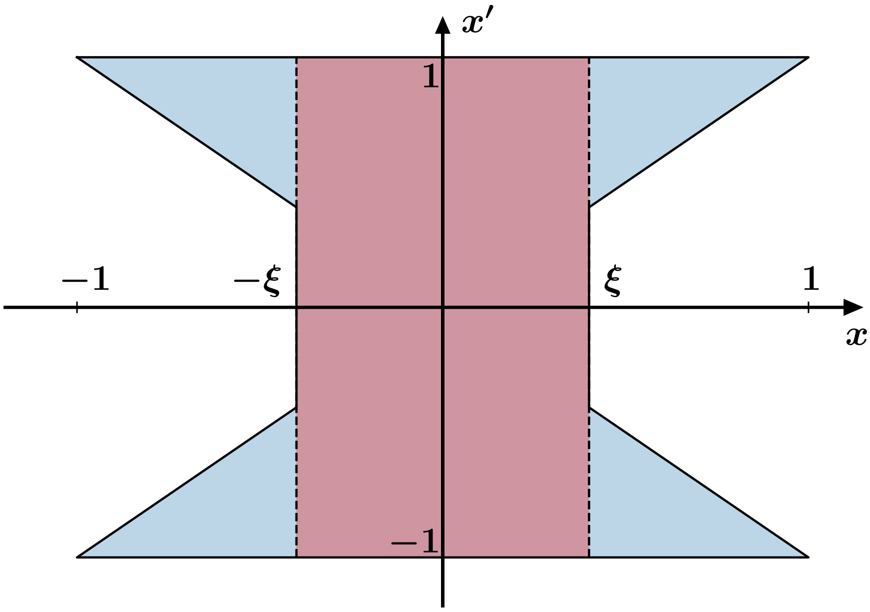

Fig. 2.1: Support domain of the evolution kernel $P\left(\frac{x}{\xi}, \frac{x'}{\xi}\right)$.

can be rearranged as follows:

$$\mu^2 \frac{d}{d\mu^2} f(\pm x,\xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \left[\frac{x'}{|2\xi|} P\left(\pm\frac{x}{\xi}, \frac{x'}{\xi}\right) f(x',\xi) + \frac{x'}{|2\xi|} P\left(\mp\frac{x}{\xi}, \frac{x'}{\xi}\right) f(-x',\xi)\right], \tag{2.2}$$

with:

$$b(x) = |x|\theta\left(\left|\frac{x}{\xi}\right| - 1\right), \tag{2.3}$$

and where we have used the symmetry property of the evolution kernels $P(y,y') = P(-y,-y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned} f^{\pm}(x,\xi) &= f(x,\xi) \mp f(-x,\xi)\,, \\ P^{\pm}(y,y') &= P(y,y') \mp P(-y,y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \frac{x'}{|2\xi|} P^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,. \tag{2.5}$$

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (henceforth DGLAP region). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (henceforth ERBL region). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ one recovers the DGLAP and ERBL equations, respectively.

For later convenience, we define the parameter:

$$\kappa(x) = \frac{\xi}{x}\,, \tag{2.6}$$

so that:

$$\frac{x'}{|2\xi|} P^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) = \frac{1}{2\kappa} \frac{x'}{x} P^{\pm}\left(\frac{1}{\kappa}, \frac{1}{\kappa}\frac{x'}{x}\right) \equiv \mathcal{P}^{\pm}\left(\kappa, \frac{x}{x'}\right)\,, \tag{2.7}$$

where the last equality effectively defines the *DGLAP-like* splitting function:

$$\mathcal{P}^{\pm}(\kappa, y) = \frac{1}{2\kappa y} P^{\pm}\left(\frac{1}{\kappa}, \frac{1}{\kappa y}\right)\,. \tag{2.8}$$

Plugging this definition into the integral in the r.h.s. of Eq. (2.5) and performing a change of variable gives:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \int_{x}^{x/b(x)} \frac{dy}{y} \mathcal{P}^{\pm}(\kappa, y)\, f^{\pm}\left(\frac{x}{y}, \xi\right)\,, \tag{2.9}$$

which (almost) has the form of a "standard" DGLAP equation. The only difference is that the upper integration bound is not one but rather $x/b(x)$ with $b(x)$ defined in Eq. (2.3). As discussed in the document *IntegralStructure.pdf*, this difference can be handled within APFEL (up to a numerical approximation to be assessed) by adjusting the integration procedure.

A crucial ingredient for an efficient implementation of GPD evolution is the availability of the DGLAP-like splitting functions $\mathcal{P}^{\pm}$ defined in Eq. (2.8) in a closed form amenable to be easily integrated as in Eq. (2.9).

2.1 On continuity of GPDs

It is well known that GPDs are required to be continuous at $x = \xi$ for factorisation to be valid [3]. It is thus interesting to consider the consequence of this constraint. To this end, let us consider the limits of Eq. (2.5) for $x \to \xi^{\pm}$:

$$\lim_{x\to\xi^+} \mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \mu^2 \frac{d}{d\mu^2} f^{\pm}(\xi,\xi) = \int_{\xi}^{1} \frac{dx'}{|2\xi|} P^{\pm}\left(\frac{\xi^+}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,, \tag{2.10}$$

and:

$$\lim_{x\to\xi^-} \mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \mu^2 \frac{d}{d\mu^2} f^{\pm}(\xi,\xi) = \int_{0}^{1} \frac{dx'}{|2\xi|} P^{\pm}\left(\frac{\xi^-}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,, \tag{2.11}$$

[1] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x) = -\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign.

where we have used the continuity of f at $x = \xi$. For the non-singlet anomalous dimension at one loop, one can easily verify that:

$$P^{\pm}\left(\frac{\xi^{+}}{\xi}, y\right) = P^{\pm}\left(\frac{\xi^{-}}{\xi}, y\right) = P^{\pm}\left(1, y\right) . \tag{2.12}$$

Taking the difference between Eqs. (2.10) and (2.11), one finds, at least for a non-singlet distribution at one loop, that:

$$\int_{0}^{\xi} dx'\, P^{\pm}\left(1, \frac{x'}{\xi}\right) f^{\pm}(x', \xi) = 0\,. \tag{2.13}$$

This appears to be some sort of (valence) sum rule that the GPDs have to fulfil.

References

[1] M. Diehl, Phys. Rept. **388** (2003) 41 doi:10.1016/j.physrep.2003.08.002, 10.3204/DESY-THESIS-2003-018 [hep-ph/0307382].

[2] J. Blumlein, B. Geyer and D. Robaschik, Nucl. Phys. B **560** (1999) 283 doi:10.1016/S0550-3213(99)00418-6 [hep-ph/9903520].

[3] A. V. Radyushkin, Phys. Rev. D **56** (1997), 5524-5557 doi:10.1103/PhysRevD.56.5524 [arXiv:hep-ph/9704207 [hep-ph]].