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# 1 PDF matching conditions

If the (Zero Mass) Variable Flavour Number Scheme (ZM-VFNS) at NNLO is considered, matching conditions for the PDFs and the coupling constant at the heavy quarks thresholds ( $m_c^2$ ,  $m_b^2$  and  $m_t^2$ ) must be implemented. This is due to the fact that we are working with an “effective theory” where before a certain threshold, say  $m_h^2$ , the heavy quark flavour  $h$  is treated as infinitely massive, while after the crossing of the same threshold the same flavour is treated as massless. This results in discontinuities of PDFs and coupling constant in correspondence of the thresholds. Such discontinuities can be evaluated in perturbation theory, and in particular one can see that they start at NNLO, so that PDFs and  $a_s$  are continuous at LO and NLO [1].

The discontinuity of the PDF  $l$  (in the Mellin space) of a light quark<sup>(1)</sup> just beyond the threshold  $m_h^2 (= m_c^2, m_b^2, m_t^2)$ , where the effective flavour number passes from  $N_f$  to  $N_f + 1$ , is given as a function of the same PDF just before the threshold by the following relation [2]:

$$l^{(N_f+1)}(N, m_h^2) = [1 + a_s^2(m_h^2) A_{qq,h}^{NS,(2)}(N)] l^{(N_f)}(N, m_h^2). \quad (1)$$

with  $l = u, \bar{u}, d, \bar{d}, \dots$ , while the gluon distribution function is given by:

$$g^{(N_f+1)}(N, m_h^2) = [1 + a_s^2(m_h^2) A_{gg,h}^{S,(2)}(N)] g^{(N_f)}(N, m_h^2) + a_s^2(m_h^2) A_{gg,h}^{S,(2)}(N) \Sigma^{(N_f)}(N, m_h^2) \quad (2)$$

and, in the end, the sum of heavy quark  $h$  and its anti-quark  $\bar{h}$ , which are going to be produced after the threshold  $m_h^2$ , is:

$$(h^{(N_f+1)} + \bar{h}^{(N_f+1)})(N, m_h^2) = a_s^2(m_h^2) [\tilde{A}_{hq}^{S,(2)}(N) \Sigma^{(N_f)}(N, m_h^2) + \tilde{A}_{hg}^{S,(2)}(N) g^{(N_f)}(N, m_h^2)]. \quad (3)$$

Of course, we have  $h = \bar{h}$ .

Now, since:

$$\Sigma^{(N_f+1)} = \sum_{l=1}^{N_f} (l^{(N_f+1)} + \bar{l}^{(N_f+1)}) + (h^{(N_f+1)} + \bar{h}^{(N_f+1)}) \quad (4)$$

we find that the matching condition for the singlet is:

$$\begin{aligned} \Sigma^{(N_f+1)}(N, m_h^2) &= [1 + a_s^2(m_h^2) A_{qq,h}^{NS,(2)}(N)] \Sigma^{(N_f)}(N, m_h^2) + \\ &a_s^2(m_h^2) [\tilde{A}_{hq}^{S,(2)}(N) \Sigma^{(N_f)}(N, m_h^2) + \tilde{A}_{hg}^{S,(2)}(N) g^{(N_f)}(N, m_h^2)] \end{aligned} \quad (5)$$

so that, from eqs. (2) and (5):

$$\begin{aligned} \begin{pmatrix} \Sigma^{(N_f+1)} \\ g^{(N_f+1)} \end{pmatrix} &= \begin{pmatrix} 1 + a_s^2[A_{qq,h}^{NS,(2)} + \tilde{A}_{hq}^{S,(2)}] & a_s^2 \tilde{A}_{hg}^{S,(2)} \\ a_s^2 A_{gg,h}^{S,(2)} & 1 + a_s^2 A_{gg,h}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(N_f)} \\ g^{(N_f)} \end{pmatrix} \\ &= \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2 A_{qq,h}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2 \begin{pmatrix} \tilde{A}_{hq}^{S,(2)} & \tilde{A}_{hg}^{S,(2)} \\ A_{gg,h}^{S,(2)} & A_{gg,h}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(N_f)} \\ g^{(N_f)} \end{pmatrix} \end{aligned} \quad (6)$$

where we have omitted all the dependencies. So we have obtained the matching conditions for the singlet and the gluon distribution functions.

Now we consider the other distributions. The valence distribution  $V$  for  $N_f + 1$  active (light) flavours just beyond the threshold  $m_h^2$  is defined as:

$$V^{(N_f+1)} = \sum_{l=1}^{N_f} (l^{(N_f+1)} - \bar{l}^{(N_f+1)}) + (h^{(N_f+1)} - \bar{h}^{(N_f+1)}) \quad (7)$$

---

<sup>1</sup>Note that the light quarks run from 1 to  $N_f$ .

but, since  $h = \bar{h}$ , the last term vanish and we are left with:

$$V^{(N_f+1)} = \sum_{l=1}^{N_f} (l^{(N_f+1)} - \bar{l}^{(N_f+1)}) =$$

$$[1 + a_s^2 A_{qq,h}^{NS,(2)}] \sum_{l=1}^{N_f} (l^{(N_f)} - \bar{l}^{(N_f)}) = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V^{(N_f)}.$$
(8)

So, this is the matching condition for the valence distribution  $V$ .

Now we consider the valence distribution  $V_3$  and  $V_8$  which are both composed only by light quarks, namely:

$$V_3 = (u - \bar{u}) - (d - \bar{d}) \quad \text{and} \quad V_8 = (u - \bar{u}) + (d - \bar{d}) - 2(s - \bar{s})$$
(9)

so that, in these cases, the matching conditions work as in the case of  $V$ , i.e.:

$$V_{3,8}^{(N_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V_{3,8}^{(N_f)}.$$
(10)

The same holds for  $T_3$  and  $T_8$ , which are defined as:

$$T_3 = (u + \bar{u}) - (d + \bar{d}) \quad \text{and} \quad V_8 = (u + \bar{u}) + (d + \bar{d}) - 2(s + \bar{s})$$
(11)

so:

$$T_{3,8}^{(N_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] T_{3,8}^{(N_f)}.$$
(12)

The remaining valence distribution  $V_{15}$ ,  $V_{24}$  and  $V_{35}$  are defined as:

$$\begin{aligned} V_{15} &= (u - \bar{u}) + (d - \bar{d}) + (s - \bar{s}) - 3(c - \bar{c}) \\ V_{24} &= (u - \bar{u}) + (d - \bar{d}) + (s - \bar{s}) + (c - \bar{c}) - 4(b - \bar{b}) \\ V_{35} &= (u - \bar{u}) + (d - \bar{d}) + (s - \bar{s}) + (c - \bar{c}) + (b - \bar{b}) - 5(t - \bar{t}) \end{aligned}$$
(13)

and since in each one of them the heavy quarks appear always as difference between quark and anti-quark, they cancel exactly. For example, at the  $m_b^2$  threshold,  $V_{15}$  is entirely composed by light quarks so there is no problem, while  $V_{24}$  and  $V_{35}$  have also a  $b$ -quark contribution, given by  $-4(b - \bar{b})$  and  $(b - \bar{b})$  respectively (of course, the  $t(\bar{t})$  distribution is zero). Anyway, this terms give no matching condition since the  $b$  contribution is exactly equal to the  $\bar{b}$  contribution, so that they cancel. So:

$$V_{15,24,35}^{(N_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] V_{15,24,35}^{(N_f)}.$$
(14)

In the end, to deal with  $T_{15}$ ,  $T_{24}$  and  $T_{35}$ , we have to specify the threshold. Indeed, in these cases the heavy quark contribution does not cancel. Their definition is:

$$\begin{aligned} T_{15} &= (u + \bar{u}) + (d + \bar{d}) + (s + \bar{s}) - 3(c + \bar{c}) \\ T_{24} &= (u + \bar{u}) + (d + \bar{d}) + (s + \bar{s}) + (c + \bar{c}) - 4(b + \bar{b}) \\ T_{35} &= (u + \bar{u}) + (d + \bar{d}) + (s + \bar{s}) + (c + \bar{c}) + (b + \bar{b}) - 5(t + \bar{t}). \end{aligned}$$
(15)

Just before the threshold  $m_c^2$  we have only 3 active light flavours ( $u$ ,  $d$  and  $s$ ), while just beyond  $m_c^2$  we have 4 active flavours and among them the flavour  $c$  is considered to be heavy. Of course, we have no  $b$  and  $t$  contribution (so  $T_{24}$  and  $T_{35}$  are equal). So the matching conditions are:

$$T_{15}^{(4)} = [1 + a_s^2 A_{qq,c}^{NS,(2)}] \underbrace{\sum_{l=u,d,s} (l^{(3)} + \bar{l}^{(3)})}_{\Sigma^{(3)}} - 3a_s^2 [\tilde{A}_{cq}^{S,(2)} \Sigma^{(3)} + \tilde{A}_{cg}^{S,(2)} g^{(3)}] =$$
(16)

$$\left( 1 + a_s^2 [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] \quad -3a_s^2 \tilde{A}_{cg}^{S,(2)} \right) \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}$$

while:

$$T_{24,35}^{(4)} = \begin{pmatrix} 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2\tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}. \quad (17)$$

We can put the above relation in a matricial form:

$$\begin{pmatrix} T_{15}^{(4)} \\ T_{24}^{(4)} \\ T_{35}^{(4)} \end{pmatrix} = \begin{pmatrix} 1 + a_s^2[A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2\tilde{A}_{cg}^{S,(2)} \\ 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2\tilde{A}_{cg}^{S,(2)} \\ 1 + a_s^2[A_{qq,c}^{NS,(2)} + \tilde{A}_{cq}^{S,(2)}] & a_s^2\tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}. \quad (18)$$

But now it is easy to generalize. At  $m_b^2$ ,  $T_{15}$  does not contain, so:

$$T_{15}^{(5)} = [1 + a_s^2 A_{qq,b}^{NS,(2)}] T_{15}^{(4)} \quad (19)$$

while:

$$\begin{pmatrix} T_{24}^{(5)} \\ T_{35}^{(5)} \end{pmatrix} = \begin{pmatrix} 1 + a_s^2[A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] & -4a_s^2\tilde{A}_{bg}^{S,(2)} \\ 1 + a_s^2[A_{qq,b}^{NS,(2)} + \tilde{A}_{bq}^{S,(2)}] & a_s^2\tilde{A}_{bg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}. \quad (20)$$

Finally, at  $m_t^2$  we have:

$$T_{15}^{(6)} = [1 + a_s^2 A_{qq,t}^{NS,(2)}] T_{15}^{(5)} \quad (21)$$

$$T_{24}^{(6)} = [1 + a_s^2 A_{qq,t}^{NS,(2)}] T_{25}^{(5)}$$

and:

$$T_{35}^{(6)} = \begin{pmatrix} 1 + a_s^2[A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] & -5a_s^2\tilde{A}_{tg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(5)} \\ g^{(5)} \end{pmatrix}. \quad (22)$$

An explicit calculation for the coefficients  $A^{(2)}$  in the  $x$ -space can be found in [hep-ph/9612398]. Anyhow, that calculation is performed more generally in the case  $m_h^2 \neq \mu_F^2$ . This results in extra-terms proportional to  $\ln(m_h^2/\mu_F^2)$ , which vanish if, as we do, one takes the factorisation scale  $\mu^2$  coinciding with the scale of the process  $Q^2$ . Moreover, in that case also NLO ( $\propto a_s$ ) appear in the matching conditions.

To summarise the PDF matching conditions at the threshold  $m_h^2$ , we have that:

- singlet and gluon couple as follows:

$$\begin{pmatrix} \Sigma^{(N_f+1)} \\ g^{(N_f+1)} \end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2 A_{qq,h}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2 \begin{pmatrix} \tilde{A}_{hq}^{S,(2)} & \tilde{A}_{hg}^{S,(2)} \\ A_{gq,h}^{S,(2)} & A_{gg,h}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(N_f)} \\ g^{(N_f)} \end{pmatrix} \quad (23)$$

- from eqs. (8), (10), (14) and (12), one can see that  $V$ ,  $V_{3,8,\dots,35}$  and  $T_{3,8}$  behave in the same way, i.e.:

$$P^{(N_f+1)} = [1 + a_s^2 A_{qq,h}^{NS,(2)}] P^{(N_f)} \quad \text{with} \quad P = V, V_3, \dots, V_{35}, T_3, T_8 \quad (24)$$

- $T_{15}$ ,  $T_{24}$  and  $T_{35}$  have different matching conditions depending on the threshold. In particular: for  $m_h^2 = m_c^2$  they are given by eq. (18), for  $m_h^2 = m_b^2$  they are given by eqs. (19) and (20) and for  $m_h^2 = m_t^2$  they are given by eqs. (21) and (22)

In the following Sections we will discuss how to write the evolution kernels in the presence of the matching conditions. We will explicitly consider only the forward evolution, i.e. the final scale  $Q^2$  greater than the initial one  $Q_0^2$ . Anyway the backward evolution ( $Q_0^2 > Q^2$ ) can be easily obtained from the forward one. In fact, given the evolution kernel  $\Gamma$ , the following relation holds:

$$\Gamma(Q^2, Q_0^2) \Gamma(Q_0^2, Q^2) = 1 \implies \Gamma(Q^2, Q_0^2) = \Gamma^{-1}(Q_0^2, Q^2). \quad (25)$$

so, if  $Q^2 > Q_0^2$  the code computes directly  $\Gamma(Q^2, Q_0^2)$ , else if  $Q_0^2 > Q^2$  the code evaluates first the forward evolution  $\Gamma(Q_0^2, Q^2)$  and then, to get  $\Gamma(Q^2, Q_0^2)$ , it calculates  $\Gamma^{-1}(Q_0^2, Q^2)$ .

## 1.1 Matching Conditions on the Evolution Kernels: 0 Thresholds Crossing

Before to discuss the crossing of the thresholds, it would be useful to write down the evolution kernels in the “trivial” situation of no threshold crossing. There are 4 particular cases: 1)  $Q_0^2 < Q^2 < m_c^2$  with  $N_f = 3$  active flavours, 2)  $m_c^2 < Q_0^2 < Q^2 < m_b^2$  with  $N_f = 4$  active flavours, 3)  $m_b^2 < Q_0^2 < Q^2 < m_t^2$  with  $N_f = 5$  active flavours and 4)  $m_t^2 < Q_0^2 < Q^2$  with  $N_f = 6$  active flavours, which do not need the introduction of the matching conditions. So, in the following Subsections we will write down the evolution of the whole PDF set the these cases.

### 1.1.1 $Q_0^2 < Q^2 < m_c^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(3)}(Q^2) \\ g^{(3)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (26)$$

- **V:**

$$V^{(3)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(3)}(Q_0^2) \quad (27)$$

- **V<sub>3</sub> and V<sub>8</sub>:**

$$V_{3,8}^{(3)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8}^{(3)}(Q_0^2) \quad (28)$$

- **V<sub>15</sub>, V<sub>24</sub> and V<sub>35</sub>:**

$$V_{15,24,35}^{(3)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(3)}(Q_0^2) \quad (29)$$

- **T<sub>3</sub> and T<sub>8</sub>:**

$$T_{3,8}^{(3)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8}^{(3)}(Q_0^2) \quad (30)$$

- **T<sub>15</sub>, T<sub>24</sub> and T<sub>35</sub>:**

$$T_{15,24,35}^{(3)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (31)$$

### 1.1.2 $m_c^2 < Q_0^2 < Q^2 < m_b^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(4)}(Q^2) \\ g^{(4)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \quad (32)$$

- **V:**

$$V^{(4)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(4)}(Q_0^2) \quad (33)$$

- **V<sub>3</sub>, V<sub>8</sub> and V<sub>15</sub>:**

$$V_{3,8,15}^{(4)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8,15}^{(4)}(Q_0^2) \quad (34)$$

- **V<sub>24</sub> and V<sub>35</sub>:**

$$V_{24,35}^{(4)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(4)}(Q_0^2) \quad (35)$$

- **T<sub>3</sub>, T<sub>8</sub> and T<sub>15</sub>:**

$$T_{3,8,15}^{(4)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8,15}^{(4)}(Q_0^2) \quad (36)$$

- **T<sub>24</sub> and T<sub>35</sub>:**

$$T_{24,35}^{(4)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \quad (37)$$

### 1.1.3 $m_b^2 < Q_0^2 < Q^2 < m_t^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \quad (38)$$

- **V:**

$$V^{(5)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(5)}(Q_0^2) \quad (39)$$

- **V<sub>3</sub>, V<sub>8</sub>, V<sub>15</sub> and V<sub>24</sub>:**

$$V_{3,8,15,24}^{(5)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8,15,24}^{(5)}(Q_0^2) \quad (40)$$

- **V<sub>35</sub>:**

$$V_{35}^{(5)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V_{35}^{(5)}(Q_0^2) \quad (41)$$

- **T<sub>3</sub>, T<sub>8</sub>, T<sub>15</sub> and T<sub>24</sub>:**

$$T_{3,8,15,24}^{(5)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8,15,24}^{(5)}(Q_0^2) \quad (42)$$

- **T<sub>35</sub>:**

$$T_{35}^{(5)}(Q^2) = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \quad (43)$$

### 1.1.4 $m_t^2 < Q_0^2 < Q^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, Q_0^2)} \begin{pmatrix} \Sigma^{(6)}(Q_0^2) \\ g^{(6)}(Q_0^2) \end{pmatrix} \quad (44)$$

- **V:**

$$V^{(6)}(Q^2) = \Gamma^v(Q^2, Q_0^2) V^{(6)}(Q_0^2) \quad (45)$$

- **V<sub>3</sub>, V<sub>8</sub>, V<sub>15</sub>, V<sub>24</sub> and V<sub>35</sub>:**

$$V_{3,8,15,24,35}^{(6)}(Q^2) = \Gamma^-(Q^2, Q_0^2) V_{3,8,15,24,35}^{(6)}(Q_0^2) \quad (46)$$

- **T<sub>3</sub>, T<sub>8</sub>, T<sub>15</sub>, T<sub>24</sub> and T<sub>35</sub>:**

$$T_{3,8,15,24,35}^{(6)}(Q^2) = \Gamma^+(Q^2, Q_0^2) T_{3,8,15,24,35}^{(6)}(Q_0^2) \quad (47)$$

## 1.2 Matching Conditions on the Evolution Kernels: 1 Threshold Crossing

In order to implement the matching conditions in our code, we will show how to transfer them from the PDFs to the evolution kernels.

In this Section we suppose that the evolution crosses only one threshold. We will show how the matching conditions on the PDFs modify the form of the evolution kernels in the cases: 1)  $Q_0^2 < m_c^2 \leq Q^2$ , 2)  $Q_0^2 < m_b^2 \leq Q^2$  and  $Q_0^2 < m_c^2 \leq Q^2$ .

### 1.2.1 $Q_0^2 < m_c^2 \leq Q^2$

In order to evolve PDFs from the scale  $Q_0^2$  to  $Q^2$  passing through the threshold  $m_c^2$ , we have to: first evolve them from  $Q_0^2$  to  $m_c^2$ , where there are 3 active flavours, then increase the number of active flavour from 3 to 4 by imposing the matching conditions, and in the end evolve the PDFs, now having 4 active flavours, from  $m_c^2$  to the scale  $Q^2$ .

In what follows we will work only in the Mellin space, so we will drop any dependence on  $N$ .

Let's start with singlet and gluon. We have:

$$\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2) \begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(m_c^2). \quad (48)$$

From eq. (23):

$$\begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(m_c^2) = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2) \quad (49)$$

now, substituting the above relation into the eq. (48), we get:

$$\begin{aligned} \begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) &= \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2) \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \right. \\ &\quad \left. a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2). \end{aligned} \quad (50)$$

But:

$$\begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(m_c^2) = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(m_c^2, Q_0^2) \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(Q_0^2). \quad (51)$$

So, in the end:

$$\begin{aligned} \begin{pmatrix} \Sigma^{(4)} \\ g^{(4)} \end{pmatrix}(Q^2) &= \left\{ \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(Q^2, m_c^2) \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \right. \right. \\ &\quad \left. \left. a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}(m_c^2, Q_0^2) \right\} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix}(Q_0^2) \end{aligned} \quad (52)$$

Now we consider the distributions  $V$ ,  $V_{3,8}$  and  $T_{3,8}$  which evolve respectively through  $\Gamma^v$ ,  $\Gamma^-$  and  $\Gamma^+$ , but which obey the same matching conditions. So:

$$P^{(4)}(Q^2) = \Gamma^{(P)}(Q^2, m_c^2) P^{(4)}(m_c^2) \quad (53)$$

so that:

$$P = \begin{cases} V & \rightarrow \Gamma^{(P)} = \Gamma^v \\ V_{3,8} & \rightarrow \Gamma^{(P)} = \Gamma^- \\ T_{3,8} & \rightarrow \Gamma^{(P)} = \Gamma^+ \end{cases} \quad (54)$$

but:

$$P^{(4)}(m_c^2) = [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] P^{(3)}(m_c^2) \quad (55)$$

and:

$$P^{(3)}(m_c^2) = \Gamma^{(P)}(m_c^2, Q_0^2) P^{(3)}(Q_0^2) \quad (56)$$

so that:

$$P^{(4)}(Q^2) = \left\{ \Gamma^{(P)}(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^{(P)}(m_c^2, Q_0^2) \right\} P^{(3)}(Q_0^2) \quad (57)$$

Now we consider  $V_{15}$ , which before  $m_c^2$  evolves as:

$$V_{15}^{(3)}(m_c^2) = \Gamma^v(m_c^2, Q_0^2) V^{(3)}(Q_0^2) \quad (58)$$

while after  $m_c^2$  it evolves as:

$$V_{15}^{(4)}(Q^2) = \Gamma^-(Q_0^2, m_c^2) V_{15}^{(4)}(m_c^2). \quad (59)$$

From eq. (24), we find that the matching condition at  $m_c^2$  is:

$$V_{15}^{(4)}(m_c^2) = [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] V_{15}^{(3)}(m_c^2) \quad (60)$$

so:

$$V_{15}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (61)$$

Instead, for  $Q_0^2 < m_c^2 \leq Q^2$ , both  $V_{24}$  and  $V_{35}$  evolve as:

$$V_{24,35}^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (62)$$

Now, we consider  $T_{15}$ . After  $m_c^2$ , it evolves as:

$$T_{15}^{(4)}(Q^2) = \Gamma^+(Q^2, m_c^2) T_{15}^{(4)}(m_c^2). \quad (63)$$

This time the matching condition is given by the first line of eq. (18):

$$T_{15}^{(4)}(m_c^2) = \begin{pmatrix} 1 + a_s^2(m_c^2) [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2) \tilde{A}_{cg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Sigma^{(3)}(m_c^2) \\ g^{(3)}(m_c^2) \end{pmatrix}. \quad (64)$$

But:

$$\begin{pmatrix} \Sigma^{(3)}(m_c^2) \\ g^{(3)}(m_c^2) \end{pmatrix} = \begin{pmatrix} \Gamma_{qq}(m_c^2, Q_0^2) & \Gamma_{qg}(m_c^2, Q_0^2) \\ \Gamma_{gq}(m_c^2, Q_0^2) & \Gamma_{gg}(m_c^2, Q_0^2) \end{pmatrix} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (65)$$

In the end, one finds that:

$$\begin{aligned} T_{15}^{(4)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_c^2) \begin{pmatrix} 1 + a_s^2(m_c^2) [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2) \tilde{A}_{cg}^{S,(2)} \end{pmatrix} \right. \\ &\quad \left. \begin{pmatrix} \Gamma_{qq}(m_c^2, Q_0^2) & \Gamma_{qg}(m_c^2, Q_0^2) \\ \Gamma_{gq}(m_c^2, Q_0^2) & \Gamma_{gg}(m_c^2, Q_0^2) \end{pmatrix} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \end{aligned} \quad (66)$$

Now we are left only with  $T_{24}$  and  $T_{35}$ , which evolve as the single before and after the threshold, so they evolve exactly as the first line of eq. (52), i.e:

$$\begin{aligned} T_{24,35}^{(4)}(Q^2) &= \left\{ \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix} (Q^2, m_c^2) \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \right. \right. \\ &\quad \left. \left. a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \right] \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix} (m_c^2, Q_0^2) \right\} \begin{pmatrix} \Sigma^{(3)} \\ g^{(3)} \end{pmatrix} (Q_0^2) \end{aligned} \quad (67)$$

Now, let us summarise what happens to the evolution kernels, by introducing the matching conditions, if one crosses the  $m_c^2$  threshold. We remind that the matching conditions appear only from the NNLO.

• **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(4)}(Q^2) \\ g^{(4)}(Q^2) \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix}}_{(Q^2, m_c^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (68)$$



where:

$$\begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_c^2) A_{qq,c}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_c^2) \begin{pmatrix} \tilde{A}_{cq}^{S,(2)} & \tilde{A}_{cg}^{S,(2)} \\ A_{gq,c}^{S,(2)} & A_{gg,c}^{S,(2)} \end{pmatrix} \quad (69)$$

• **V:**

$$V^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (70)$$

• **V<sub>3</sub> and V<sub>8</sub>:**

$$V_{3,8}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^-(m_c^2, Q_0^2) \right\} V_{3,8}^{(3)}(Q_0^2) \quad (71)$$

• **V<sub>15</sub>:**

$$V_{15}^{(4)}(Q^2) = \left\{ \Gamma^-(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (72)$$

• **V<sub>24</sub> and V<sub>35</sub>:**

$$V_{24,35}^{(4)}(Q^2) = \left\{ \Gamma^v(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (73)$$

• **T<sub>3</sub> and T<sub>8</sub>:**

$$T_{3,8}^{(4)}(Q^2) = \left\{ \Gamma^+(Q^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^+(m_c^2, Q_0^2) \right\} T_{3,8}^{(3)}(Q_0^2) \quad (74)$$

• **T<sub>15</sub>:**

$$\begin{aligned} T_{15}^{(4)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_c^2) \left( 1 + a_s^2(m_c^2) [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] \right) - 3a_s^2(m_c^2) \tilde{A}_{cg}^{S,(2)} \right\} \times \\ &\quad \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \left\{ \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \right\} \end{aligned} \quad (75)$$

• **T<sub>24</sub> and T<sub>35</sub>:**

$$T_{24,35}^{(4)}(Q^2) = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2, m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (76)$$

Now, it is very easy to rewrite the above summary for the crossing of the remaining thresholds  $m_b^2$  ( $Q_0^2 < m_b^2 \leq Q^2$ ) and  $m_t^2$  ( $Q_0^2 < m_t^2 \leq Q^2$ ).

### 1.2.2 $Q_0^2 < m_b^2 \leq Q^2$

• **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \quad (77)$$

where:

$$\begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_b^2) A_{qq,b}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_b^2) \begin{pmatrix} \tilde{A}_{bq}^{S,(2)} & \tilde{A}_{bg}^{S,(2)} \\ A_{gq,b}^{S,(2)} & A_{gg,b}^{S,(2)} \end{pmatrix} \quad (78)$$

- **V:**

$$V^{(5)}(Q^2) = \left\{ \Gamma^v(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \quad (79)$$

- **V<sub>3</sub>, V<sub>8</sub> and V<sub>15</sub>:**

$$V_{3,8,15}^{(5)}(Q^2) = \left\{ \Gamma^-(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^-(m_b^2, Q_0^2) \right\} V_{3,8,15}^{(4)}(Q_0^2) \quad (80)$$

- **V<sub>24</sub>:**

$$V_{24}^{(5)}(Q^2) = \left\{ \Gamma^-(Q_0^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \quad (81)$$

- **V<sub>35</sub>:**

$$V_{35}^{(5)}(Q^2) = \left\{ \Gamma^v(Q_0^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \quad (82)$$

- **T<sub>3</sub>, T<sub>8</sub> and T<sub>15</sub>:**

$$T_{3,8,15}^{(5)}(Q^2) = \left\{ \Gamma^+(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^+(m_b^2, Q_0^2) \right\} T_{3,8,15}^{(4)}(Q_0^2) \quad (83)$$

- **T<sub>24</sub>:**

$$T_{24}^{(5)}(Q^2) = \left\{ \Gamma^+(Q^2, m_b^2) \left( 1 + a_s^2(m_b^2) [A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] \right) - 4a_s^2(m_b^2) \tilde{A}_{bg}^{S,(2)} \right\} \times$$

$$\underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \left\{ \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \right\} \quad (84)$$

- **T<sub>35</sub>:**

$$T_{35}^{(5)}(Q^2) = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \quad (85)$$

### 1.2.3 $Q_0^2 < m_t^2 \leq Q^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2) \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_t^2)} \begin{pmatrix} M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \quad (86)$$

where:

$$\begin{pmatrix} M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_s^2(m_t^2) A_{qq,t}^{NS,(2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_s^2(m_t^2) \begin{pmatrix} \tilde{A}_{tq}^{S,(2)} & \tilde{A}_{tg}^{S,(2)} \\ A_{gq,t}^{S,(2)} & A_{gg,t}^{S,(2)} \end{pmatrix} \quad (87)$$

- **V:**

$$V^{(6)}(Q^2) = \left\{ \Gamma^v(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^v(m_t^2, Q_0^2) \right\} V^{(5)}(Q_0^2) \quad (88)$$

- **V<sub>3</sub>, V<sub>8</sub>, V<sub>15</sub> and V<sub>24</sub>:**

$$V_{3,8,15,24}^{(6)}(Q^2) = \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^-(m_t^2, Q_0^2) \right\} V_{3,8,15,24}^{(5)}(Q_0^2) \quad (89)$$

- **V<sub>35</sub>:**

$$V_{35}^{(6)}(Q^2) = \left\{ \Gamma^-(Q_0^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^v(m_t^2, Q_0^2) \right\} V^{(5)}(Q_0^2) \quad (90)$$

- **T<sub>3</sub>, T<sub>8</sub>, T<sub>15</sub> and T<sub>24</sub>:**

$$T_{3,8,15,24}^{(6)}(Q^2) = \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^+(m_t^2, Q_0^2) \right\} T_{3,8,15,24}^{(5)}(Q_0^2) \quad (91)$$

- **T<sub>35</sub>:**

$$T_{35}^{(6)}(Q^2) = \left\{ \Gamma^+(Q^2, m_t^2) \left( 1 + a_s^2(m_t^2) [A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] - 5a_s^2(m_t^2) \tilde{A}_{tg}^{S,(2)} \right) \times \right. \\ \left. \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(5)}(Q_0^2) \\ g^{(5)}(Q_0^2) \end{pmatrix} \quad (92)$$

### 1.3 Matching Conditions on the Evolution Kernels: 2 Thresholds Crossing

In this Section we will discuss the case in which the evolution crosses two thresholds. Therefore there are only two situations: 1)  $Q_0^2 < m_c^2 < m_b^2 \leq Q^2$  and 2)  $Q_0^2 < m_b^2 < m_t^2 \leq Q^2$ . Anyway, there is nothing new, indeed to obtain such evolution kernels we have just to “merge” together what we have already done in the previous Section.

#### 1.3.1 $Q_0^2 < m_c^2 < m_b^2 \leq Q^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(5)}(Q^2) \\ g^{(5)}(Q^2) \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \times \right. \\ \left. \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (93)$$

- **V:**

$$V^{(5)}(Q^2) = \left\{ \Gamma^v(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^v(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (94)$$

- **V<sub>3</sub> and V<sub>8</sub>:**

$$V_{3,8}^{(5)}(Q^2) = \left\{ \Gamma^-(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^-(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^-(m_c^2, Q_0^2) \right\} V_{3,8}^{(3)}(Q_0^2) \quad (95)$$

- **V<sub>15</sub>:**

$$V_{15}^{(5)}(Q^2) = \left\{ \Gamma^-(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^-(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (96)$$

- **V<sub>24</sub>**:

$$V_{24}^{(5)}(Q^2) = \left\{ \Gamma^-(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^v(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (97)$$

- **V<sub>35</sub>**:

$$V_{35}^{(5)}(Q^2) = \left\{ \Gamma^v(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^v(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (98)$$

- **T<sub>3</sub> and T<sub>8</sub>**:

$$T_{3,8}^{(5)}(Q^2) = \left\{ \Gamma^+(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \times \right. \\ \left. \Gamma^+(m_b^2, m_c^2) [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^+(m_c^2, Q_0^2) \right\} T_{3,8}^{(3)}(Q_0^2) \quad (99)$$

- **T<sub>15</sub>**:

$$T_{15}^{(5)}(Q^2) = \left\{ \Gamma^+(Q^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^+(m_b^2, m_c^2) \times \right. \\ \left. \underbrace{\begin{pmatrix} 1 + a_s^2(m_c^2) [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] & -3a_s^2(m_c^2) \tilde{A}_{cg}^{S,(2)} \end{pmatrix}}_{(m_c^2, Q_0^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (100)$$

- **T<sub>24</sub>**:

$$T_{24}^{(5)}(Q^2) = \left\{ \Gamma^+(Q^2, m_b^2) \left( 1 + a_s^2(m_b^2) [A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] - 4a_s^2(m_b^2) \tilde{A}_{bg}^{S,(2)} \right) \times \right. \\ \left. \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \underbrace{\begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix}}_{(m_c^2, Q_0^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (101)$$

- **T<sub>35</sub>**:

$$T_{35}^{(5)}(Q^2) = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \end{pmatrix}}_{(Q^2, m_b^2)} \underbrace{\begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix}}_{(m_b^2, m_c^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \times \right. \\ \left. \underbrace{\begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix}}_{(m_c^2, Q_0^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (102)$$

### 1.3.2 $Q_0^2 < m_b^2 < m_t^2 \leq Q^2$

- **Singlet and gluon:**

$$\begin{aligned} \begin{pmatrix} \Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2) \end{pmatrix} &= \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix} \begin{pmatrix} M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t \end{pmatrix}}_{(Q^2, m_t^2)} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, m_b^2)} \times \\ &\quad \underbrace{\begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \left\{ \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \right\} \end{aligned} \quad (103)$$

- **V:**

$$\begin{aligned} V^{(6)}(Q^2) &= \left\{ \Gamma^v(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ &\quad \left. \Gamma^v(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \end{aligned} \quad (104)$$

- **V<sub>3</sub>, V<sub>8</sub> and V<sub>15</sub>:**

$$\begin{aligned} V_{3,8,15}^{(6)}(Q^2) &= \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ &\quad \left. \Gamma^-(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^-(m_b^2, Q_0^2) \right\} V_{3,8,15}^{(4)}(Q_0^2) \end{aligned} \quad (105)$$

- **V<sub>24</sub>:**

$$\begin{aligned} V_{24}^{(6)}(Q^2) &= \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ &\quad \left. \Gamma^-(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \end{aligned} \quad (106)$$

- **V<sub>35</sub>:**

$$\begin{aligned} V_{35}^{(6)}(Q^2) &= \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ &\quad \left. \Gamma^v(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, Q_0^2) \right\} V^{(4)}(Q_0^2) \end{aligned} \quad (107)$$

- **T<sub>3</sub>, T<sub>8</sub> and T<sub>15</sub>:**

$$\begin{aligned} T_{3,8,15}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ &\quad \left. \Gamma^+(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^+(m_b^2, Q_0^2) \right\} T_{3,8,15}^{(4)}(Q_0^2) \end{aligned} \quad (108)$$

- **T<sub>24</sub>:**

$$\begin{aligned} T_{24}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^+(m_t^2, m_b^2) \times \right. \\ &\quad \left. \underbrace{\begin{pmatrix} 1 + a_s^2(m_b^2) [A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] & -4a_s^2(m_b^2) \tilde{A}_{bg}^{S,(2)} \end{pmatrix} \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \end{aligned} \quad (109)$$

- **T<sub>35</sub>:**

$$T_{35}^{(6)}(Q^2) = \left\{ \Gamma^+(Q^2, m_t^2) \left( 1 + a_s^2(m_t^2) [A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] - 5a_s^2(m_t^2) \tilde{A}_{tg}^{S,(2)} \right) \times \right. \\ \left. \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, m_b^2)} \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(4)}(Q_0^2) \\ g^{(4)}(Q_0^2) \end{pmatrix} \quad (110)$$

#### 1.4 Matching Conditions on the Evolution Kernels: 3 Thresholds Crossing

In this Section we will discuss the case in which the evolution crosses three thresholds. Therefore there is only one situation:  $Q_0^2 < m_c^2 < m_b^2, m_t^2 \leq Q^2$

##### 1.4.1 $Q_0^2 < m_c^2 < m_b^2 < m_t^2 \leq Q^2$

- **Singlet and gluon:**

$$\begin{pmatrix} \Sigma^{(6)}(Q^2) \\ g^{(6)}(Q^2) \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(Q^2, m_t^2)} \begin{pmatrix} M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_t^2, m_b^2)} \times \right. \\ \left. \begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \right. \\ \left. \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix} \quad (111)$$

- **V:**

$$V^{(6)}(Q^2) = \left\{ \Gamma^v(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ \Gamma^v(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, m_c^2) \\ \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (112)$$

- **V<sub>3</sub> and V<sub>8</sub>:**

$$V_{3,8}^{(6)}(Q^2) = \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ \Gamma^-(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^-(m_b^2, m_c^2) \\ \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^-(m_c^2, Q_0^2) \right\} V_{3,8}^{(3)}(Q_0^2) \quad (113)$$

- **V<sub>15</sub>:**

$$V_{15}^{(6)}(Q^2) = \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\ \Gamma^-(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^-(m_b^2, m_c^2) \\ \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2) \quad (114)$$

- **V<sub>24</sub>**:

$$\begin{aligned}
V_{15}^{(6)}(Q^2) &= \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\
&\quad \Gamma^-(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, m_c^2) \\
&\quad \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2)
\end{aligned} \tag{115}$$

- **V<sub>35</sub>**:

$$\begin{aligned}
V_{15}^{(6)}(Q^2) &= \left\{ \Gamma^-(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\
&\quad \Gamma^v(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^v(m_b^2, m_c^2) \\
&\quad \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^v(m_c^2, Q_0^2) \right\} V^{(3)}(Q_0^2)
\end{aligned} \tag{116}$$

- **T<sub>3</sub>** and **T<sub>8</sub>**:

$$\begin{aligned}
T_{3,8}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\
&\quad \Gamma^+(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^+(m_b^2, m_c^2) \\
&\quad \left. [1 + a_s^2(m_c^2) A_{qq,c}^{NS,(2)}] \Gamma^+(m_c^2, Q_0^2) \right\} T_{3,8}^{(3)}(Q_0^2)
\end{aligned} \tag{117}$$

- **T<sub>15</sub>**:

$$\begin{aligned}
T_{15}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \times \right. \\
&\quad \Gamma^+(m_t^2, m_b^2) [1 + a_s^2(m_b^2) A_{qq,b}^{NS,(2)}] \Gamma^+(m_b^2, m_c^2) \\
&\quad \left. \left( 1 + a_s^2(m_c^2) [A_{qq,c}^{NS,(2)} - 3\tilde{A}_{cq}^{S,(2)}] \quad -3a_s^2(m_c^2) \tilde{A}_{cg}^{S,(2)} \right) \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix}
\end{aligned} \tag{118}$$

- **T<sub>24</sub>**:

$$\begin{aligned}
T_{24}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) [1 + a_s^2(m_t^2) A_{qq,t}^{NS,(2)}] \Gamma^+(m_t^2, m_b^2) \times \right. \\
&\quad \left. \left( 1 + a_s^2(m_b^2) [A_{qq,b}^{NS,(2)} - 4\tilde{A}_{bq}^{S,(2)}] \quad -4a_s^2(m_b^2) \tilde{A}_{bg}^{S,(2)} \right) \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \times \right. \\
&\quad \left. \left( \begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \underbrace{\begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right) \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix}
\end{aligned} \tag{119}$$

• **T<sub>35</sub>**:

$$\begin{aligned}
T_{35}^{(6)}(Q^2) &= \left\{ \Gamma^+(Q^2, m_t^2) \left( 1 + a_s^2(m_b^2) [A_{qq,t}^{NS,(2)} - 5\tilde{A}_{tq}^{S,(2)}] - 5a_s^2(m_t^2) \tilde{A}_{tg}^{S,(2)} \right) \times \right. \\
&\quad \underbrace{\begin{pmatrix} M_{11}^b & M_{12}^b \\ M_{21}^b & M_{22}^b \end{pmatrix} \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_b^2, m_c^2)} \\
&\quad \left. \underbrace{\begin{pmatrix} M_{11}^c & M_{12}^c \\ M_{21}^c & M_{22}^c \end{pmatrix} \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} \\ \Gamma_{gq} & \Gamma_{gg} \end{pmatrix}}_{(m_c^2, Q_0^2)} \right\} \begin{pmatrix} \Sigma^{(3)}(Q_0^2) \\ g^{(3)}(Q_0^2) \end{pmatrix}
\end{aligned} \tag{120}$$

## 2 Matching conditions with intrinsic distributions

The (quite outdated) discussion presented above concerning the PDF matching conditions is restricted to forward evolution (*i.e.*  $Q > Q_0$ ) and under the assumption that possible intrinsic heavy-quark contributions are absent.<sup>2</sup> This corresponds to assuming that heavy-quark PDFs are entirely dynamically generated at the respective threshold and are identically vanishing below threshold. While this turns out to be a good approximation for PDFs, this is not the case for fragmentation functions (FFs). Indeed, intrinsic heavy-quark FFs are sizeable. In addition, assuming that there are no heavy-quark PDFs makes backward evolution problematic. The basic problem is that the matching-function matrix is non-squared which forbids its inversion thus complicating backward evolution.

The purpose of this section is the relaxation of the no-intrinsic-PDF assumption. This will imply the introduction of new matching functions due to the presence match heavy-quark PDFs below threshold. Importantly, the perturbative expansion of these additional functions is different from zero at  $\mathcal{O}(\alpha_s)$  even choosing  $\mu_h = m_h$ , with  $\mu_h$  the matching scale, producing a discontinuity already at NLO. However, in what follows we will assume that matching conditions are not computed beyond  $\mathcal{O}(\alpha_s^2)$ , *i.e.* NNLO, and that intrinsic heavy-quark and heavy-antiquark contributions are equal below threshold. These assumptions lead to:

$$h^{(N_f)}(\mu_h) = \bar{h}^{(N_f)}(\mu_h) \quad \text{and} \quad h^{(N_f+1)}(\mu_h) = \bar{h}^{(N_f+1)}(\mu_h), \tag{121}$$

where we dropped the non-scale dependence. It immediately follows that the valence distributions  $\{V, V_3, V_8, V_{15}, V_{24}, V_{35}\}$  match multiplicatively as:<sup>3</sup>

$$V_i^{(N_f+1)}(\mu_h) = [1 + K_{il}](\mu_h) V_i^{(N_f)}(\mu_h) = [1 + K_{il}(\mu_h)] V_i^{(N_f)}(\mu_h), \quad i = 3, 8, 15, 24, 35, \tag{122}$$

with  $K_{il}$  some given function that starts at  $\mathcal{O}(\alpha_s^2)$ . This equivalently amounts to say that no non-perturbative quark-antiquark asymmetry is originally present nor it is generated by the matching procedure.<sup>4</sup>

This allows us to concentrate the discussion on the singlet sector. We start by generalising Eqs. (1),

<sup>2</sup>In fact, the discussion also assumes that the PDF matching takes place at the heavy quark-mass value. This is also a non-necessary assumption that we will however not discuss here.

<sup>3</sup>Note that “1” in Eq. (122) is to be understood as  $\delta(1-x)$  in  $x$  space.

<sup>4</sup>The assumption of absence of non-perturbative quark-antiquark asymmetry is badly violated for light-valence distributions, such as up and down in the proton. However, we will never realistically need to match PDFs or FFs at the light-quark scales. On the other hand, this assumption helps us keep the discussion general.



(2), and (3) (changing and simplifying a bit the notation) as:

$$\begin{aligned}
g^{(N_f+1)} &= [1 + K_{gg}]g^{(N_f)} + K_{gl} \sum_l l^{+(N_f+1)} + K_{gh}h^{+(N_f)}, \\
l^{+(N_f+1)} &= [1 + K_{ll}]l^{+(N_f)}, \\
h^{+(N_f+1)} &= K_{hg}g^{(N_f)} + K_{hl} \sum_l l^{+(N_f+1)} + [1 + K_{hh}]h^{+(N_f)}, \\
H^{+(N_f+1)} &= H^{+(N_f)},
\end{aligned} \tag{123}$$

where  $q^+ = q + \bar{q}$  and  $l$  runs between 1 and  $N_f$ . We remind that  $h$  is the  $(N_f+1)$ -th heavier flavour, *i.e.* the one that becomes active at the threshold under consideration. We have also introduced the “super” heavy quark flavour  $H$ , heavier than  $h$ , that is not affected by the matching at the  $N_f$ -th threshold. Note that there are two new functions coming into play, namely  $K_{gh}$  and  $K_{hh}$ , both multiplying the intrinsic heavy-quark contribution  $h^{+(N_f)}$ . Moreover, we have written Eq. (123) in such a way that the coefficients  $K$  start at least at  $\mathcal{O}(\alpha_s)$ . In fact, we have already seen that  $K_{ll} = \mathcal{O}(\alpha_s^2)$ ; the same applies to  $K_{hl}$  and  $K_{gl}$ . Defining the column vector  $P$  with components  $(g, d^+, u^+, s^+, c^+, b^+, t^+)$ , we can represent the matching in a matricial form as:

$$P^{(N_f+1)} = [\mathbb{I} + \mathbb{K}^{(N_f)}] P^{(N_f)}, \tag{124}$$

where  $\mathbb{I}$  is the  $7 \times 7$  unity matrix and  $\mathbb{K}^{(N_f)}$  are given, depending on the value of  $N_f$ , by:

$$\begin{aligned}
\mathbb{K}^{(0)} &= \begin{pmatrix} K_{gg} & K_{gh} & 0 & 0 & 0 & 0 & 0 \\ K_{hg} & K_{hh} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \mathbb{K}^{(1)} &= \begin{pmatrix} K_{gg} & K_{gl} & K_{gh} & 0 & 0 & 0 & 0 \\ 0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\ K_{hg} & K_{hl} & K_{hh} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbb{K}^{(2)} &= \begin{pmatrix} K_{gg} & K_{gl} & K_{gl} & K_{gh} & 0 & 0 & 0 \\ 0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\ K_{hg} & K_{hl} & K_{hl} & K_{hh} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \mathbb{K}^{(3)} &= \begin{pmatrix} K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gh} & 0 & 0 \\ 0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\ K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hh} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbb{K}^{(4)} &= \begin{pmatrix} K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gh} & 0 \\ 0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{ll} & 0 & 0 \\ K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hh} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \mathbb{K}^{(5)} &= \begin{pmatrix} K_{gg} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gl} & K_{gh} \\ 0 & K_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{ll} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{ll} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{ll} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{ll} & 0 \\ K_{hg} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hl} & K_{hh} \end{pmatrix} \\
& & & & & & (125)
\end{aligned}$$

The algorithm to determine the matrix  $\mathbb{K}^{(N_f)}$ , indexing the gluon distribution with 0 and the quark

distributions from 1 to 6, reads:

$$\mathbb{K}_{ij}^{(N_f)} = \begin{cases} K_{gg} & i = j = 0 \\ K_{gl} & i = 0, 1 \leq j \leq N_f \\ K_{gh} & i = 0, j = N_f + 1 \\ K_{ll} & i = j, 1 \leq i, j \leq N_f \\ K_{hg} & i = N_f + 1, j = 0 \\ K_{hl} & i = N_f + 1, 1 \leq j \leq N_f \\ K_{hh} & i = j = N_f + 1 \\ 0 & \text{elsewhere} \end{cases}. \quad (126)$$

Now one may want to rotate the matching matrices from the physical basis to the evolution basis through  $T\mathbb{K}^{(N_f)}T^{-1}$ , where  $T$  transforms the vector  $P$  into the evolution-basis vector  $E = (g, \Sigma, T_3, T_8, T_{15}, T_{24}, T_{35})$ . However, it turns out the resulting matching matrix are much more complicated and generally less sparse than those in Eq. (125). Therefore, it seems convenient to perform the matching in the physical basis.

In order to be able to perform the backward evolution, it is necessary to invert the matching matrix  $\mathbb{I} + \mathbb{K}^{(N_f)}$ . This is conveniently done perturbatively by expanding the matrix  $[\mathbb{I} + \mathbb{K}^{(N_f)}]^{-1}$  in powers of  $\alpha_s$  and truncating the expansion at the appropriate order. Given the assumption discussed above, we are not interested in going beyond  $\mathcal{O}(\alpha_s^2)$ . In addition, we know that:

$$\mathbb{K}^{(N_f)} = a_s \mathbb{K}^{(1)(N_f)} + a_s^2 \mathbb{K}^{(2)(N_f)} + \mathcal{O}(\alpha_s^3), \quad (127)$$

so that:

$$[\mathbb{I} + \mathbb{K}^{(N_f)}]^{-1} = 1 - a_s \mathbb{K}^{(1)(N_f)} - a_s^2 \left[ \mathbb{K}^{(2)(N_f)} - \left( \mathbb{K}^{(1)(N_f)} \right)^2 \right] + \mathcal{O}(\alpha_s^3). \quad (128)$$

But the matrix  $\mathbb{K}^{(1)(N_f)}$  is particularly simple because at  $\mathcal{O}(\alpha_s)$  the matching functions  $K_{ll}$ ,  $K_{hl}$ , and  $K_{gl}$  vanish. Therefore, using Eq. (126) we find:

$$\mathbb{K}_{ij}^{(1)(N_f)} = \delta_{i,0} \left( \delta_{j,0} K_{gg}^{(1)} + \delta_{j,N_f+1} K_{gh}^{(1)} \right) + \delta_{i,N_f+1} \left( \delta_{j,0} K_{hg}^{(1)} + \delta_{j,N_f+1} K_{hh}^{(1)} \right), \quad (129)$$

such that its square has the same structure and can be written as:

$$\begin{aligned} \left( \mathbb{K}_{ij}^{(1)(N_f)} \right)^2 &= \delta_{i,0} \left( \delta_{j,0} \sum_{k=g,h} K_{gk}^{(1)} K_{kg}^{(1)} + \delta_{j,N_f+1} \sum_{k=g,h} K_{gk}^{(1)} K_{kh}^{(1)} \right) \\ &+ \delta_{i,N_f+1} \left( \delta_{j,0} \sum_{k=g,h} K_{hk}^{(1)} K_{kg}^{(1)} + \delta_{j,N_f+1} \sum_{k=g,h} K_{hk}^{(1)} K_{kh}^{(1)} \right), \end{aligned} \quad (130)$$

where, in  $x$  space, the product of two  $K$  functions is to be understood as a convolution. We report below the expression in  $x$  space of the functions  $K_{gh}^{(1)}$  and  $K_{gh}^{(1)}$  for the matching of PDFs:

$$\begin{aligned} K_{gh}^{(1)}(x) &= 2C_F \frac{1 + (1-x)^2}{x} \left( \ln \frac{\mu_h^2}{m_h^2} - 1 - 2 \ln x \right), \\ K_{hh}^{(1)}(x) &= 2C_F \left[ \frac{1+x^2}{1-x} \left( \ln \frac{\mu_h^2}{m_h^2} - 1 - 2 \ln(1-x) \right) \right]_+. \end{aligned} \quad (131)$$

Considering that:

$$\begin{aligned} K_{gg}^{(1)}(x) &= -\frac{4}{3} T_R \delta(1-x) \ln \frac{\mu_h^2}{m_h^2}, \\ K_{hg}^{(1)}(x) &= 4T_R [x^2 + (1-x)^2] \ln \frac{\mu_h^2}{m_h^2}, \end{aligned} \quad (132)$$

the convolutions Eq. (130) could possibly be computed analytically. We notice that the  $\mathcal{O}(\alpha_s^2)$  correction to these functions is currently unknown, therefore we only use the  $\mathcal{O}(\alpha_s)$  expressions.

We finally comment on FFs. So far, to the best of my knowledge, none of the  $\mathcal{O}(\alpha_s^2)$  corrections to the FF matching conditions has been computed. Therefore, we must limit to  $\mathbb{K}^{(1)(N_f)}$  that is simply given by the transpose of that for PDFs.

## References

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