SIDIS cross section in TMD factorisation

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April 2, 2021

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1 Structure of the observable

In this document we report the relevant formulas for the computation of semi-inclusive deep-inelastic scattering (SIDIS) multiplicities under the assumption that the (negative) virtuality of the Q^2 of the exchanged vector boson is much smaller than the Z mass. This allows us to neglect weak contributions and write the cross section in TMD factorisation as:

$$\frac{d\sigma}{dxdQdzdq_T} = \frac{4\pi\alpha^2 q_T}{zxQ^3} Y_+ H(Q,\mu) \sum_q e_q^2 \int_0^\infty db\, b J_0\left(bq_T\right) \overline{F}_q(x,b;\mu,\zeta_1) \overline{D}_q(z,b;\mu,\zeta_2)\,, \tag{1}$$

with $\zeta_1\zeta_2 = Q^4$ and:

$$Y_+ = 1 + (1-y)^2 = 1 + \left(1 - \frac{Q^2}{xs}\right)^2\,, \tag{2}$$

where s is the squared center of mass energy. The single TMDs are evolved and matched onto the respective collinear functions as usual:

$$\overline{F}_q(x,b;\mu,\zeta) = xF_q(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \rightarrow \mu,\zeta;b) \sum_j \int_x^1 dy\, \mathcal{C}_{qj}(y;\mu_0,\zeta_0) \left[\frac{x}{y} f_j\left(\frac{x}{y},\mu_0\right)\right]\,, \tag{3}$$

and:

$$\overline{D}_i(z,b;\mu,\zeta) = z^3 D_q(z,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \rightarrow \mu,\zeta;b) \sum_j \int_z^1 dy\, \left[y^2 \mathbb{C}_{qj}(y;\mu_0,\zeta_0)\right] \left[\frac{z}{y} d_j\left(\frac{z}{y},\mu_0\right)\right]\,. \tag{4}$$

Notice that here we limit to the case $Q \ll M_Z$ such that we can neglect the contribution of the Z boson and thus the electroweak couplings are given by the squared electric charges.

As usual, low-q_T non-perturbative corrections are taken into account by introducing the monotonic function $b_*(b)$ that behaves as:

$$\lim_{b\rightarrow 0} b_*(b) = b_{\rm min} \quad \text{and} \quad \lim_{b\rightarrow \infty} b_*(b) = b_{\rm max}\,. \tag{5}$$

This allows us to replace the TMDs in Eq. (1) with their "regularised" version:

$$\begin{aligned} \overline{F}_q(x,b;\mu,\zeta) &\rightarrow \overline{F}_q(x,b_*(b);\mu,\zeta) f_{\rm NP}(x,b,\zeta)\,, \\ \overline{D}_q(z,b;\mu,\zeta) &\rightarrow \overline{D}_q(z,b_*(b);\mu,\zeta) D_{\rm NP}(z,b,\zeta)\,, \end{aligned} \tag{6}$$

where we have introduced the non-perturbative functions $f_{\rm NP}$ and $D_{\rm NP}$. It is important to stress that these functions further factorise as follows:

$$\begin{aligned} f_{\rm NP}(x,b,\zeta) &= \widetilde{f}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,, \\ D_{\rm NP}(z,b,\zeta) &= \widetilde{D}_{\rm NP}(x,b)\exp\left[g_K(b)\ln\left(\frac{\zeta}{Q_0^2}\right)\right]\,. \end{aligned} \tag{7}$$

The common exponential function represents the non-perturbative corrections to TMD evolution and the specific functional form is driven by the solution of the Collins-Soper equation where Q_0 is some initial scale. Finally the set of non-perturbative functions to be determined from fits to data are $\widetilde{f}_{\rm NP}$, $\widetilde{D}_{\rm NP}$, and $g_K(b)$. It is worth noticing that by definition

$$f_{\rm NP}(x,b,\zeta) = \frac{\overline{F}_q(x,b;\mu,\zeta)}{\overline{F}_q(x,b_*(b);\mu,\zeta)}\,, \tag{8}$$

and similarly for $D_{\rm NP}$. Therefore, one has a partial handle on the b-dependence of these functions from the region in which b is small enough to make both numerator and denominator perturbatively computable. Making use of Eq. (7) and setting $\zeta_1=\zeta_2=Q^2$ allows us to rewrite Eq. (1) as:

$$\begin{aligned} \frac{d\sigma}{dxdQdzdq_T} &= \frac{4\pi\alpha^2 q_T}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2 \\ &\times \int_0^\infty db\, J_0\left(bq_T\right)\, b\overline{F}_q(x,b_*(b);\mu,Q^2)\overline{D}_q(z,b_*(b);\mu,Q^2)f_{\rm NP}(x,b,Q^2)D_{\rm NP}(z,b,Q^2)\,. \end{aligned} \tag{9}$$

The integral in the r.h.s. can be numerically computed using the Ogata quadrature of zero-th degree (because J_0 enters the integral):

$$\begin{aligned} \frac{d\sigma}{dxdQdzdq_T} &\simeq \frac{4\pi\alpha^2}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2 \\ &\times \sum_{n=1}^N w_n^{(0)}\frac{\xi_n^{(0)}}{q_T}\overline{F}_q\left(x,b_*\left(\frac{\xi_n^{(0)}}{q_T}\right);\mu,Q^2\right)\overline{D}_q\left(z,b_*\left(\frac{\xi_n^{(0)}}{q_T}\right);\mu,Q^2\right) \\ &\times f_{\rm NP}\left(x,\frac{\xi_n^{(0)}}{q_T},Q^2\right)D_{\rm NP}\left(z,\frac{\xi_n^{(0)}}{q_T},Q^2\right)\,, \end{aligned} \tag{10}$$

where $w_n^{(0)}$ and $\xi_n^{(0)}$ are the Ogata weights and coordinates, respectively, and the sum over n is truncated to the N-th term that should be chosen in such a way to guarantee a given target accuracy. The equation above can be conveniently recasted as follows:

$$\frac{d\sigma}{dxdQdzdq_T} \simeq \sum_{n=1}^N w_n^{(0)}\frac{\xi_n^{(0)}}{q_T}S\left(x,z,\frac{\xi_n^{(0)}}{q_T};\mu,Q^2\right)f_{\rm NP}\left(x,\frac{\xi_n^{(0)}}{q_T},Q^2\right)D_{\rm NP}\left(z,\frac{\xi_n^{(0)}}{q_T},Q^2\right)\,, \tag{11}$$

where:

$$S\left(x,z,b;\mu,Q^2\right) = \frac{4\pi\alpha^2}{xzQ^3}Y_+H(Q,\mu)\sum_q e_q^2\left[\overline{F}_q\left(x,b_*(b);\mu,Q^2\right)\right]\left[\overline{D}_q\left(z,b_*(b);\mu,Q^2\right)\right]\,. \tag{12}$$

2 Integrating over the final-state kinematic variables

Experimental measurements of differential distributions for SIDIS production are often delivered as integrated over finite regions of the final-state kinematic phase space.

More specifically, the cross section is not integrated of the transverse momentum of the vector boson, q_T, but over the transverse momentum of the outgoing hadron, p_{Th}, that is connected to the former through:

$$p_{Th} = z q_T \,. \tag{13}$$

The integrated cross section then reads:

$$\widetilde{\sigma} = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{x_{\min}}^{x_{\max}} dx \int_{z_{\min}}^{z_{\max}} dz \int_{p_{Th,\min}/z}^{p_{Th,\max}/z} dq_T \left[\frac{d\sigma}{dx dQ dz dq_T}\right] \,. \tag{14}$$

One can exploit a property of the Bessel functions to compute the indefinite integral in q_T of the cross section. Specifically, we now compute:

$$K(x, z, Q, q_T) = \int dq_T \left[\frac{d\sigma}{dx dQ dz dq_T}\right] \,. \tag{15}$$

This is easily done by using the following property of the Bessel functions:

$$\int dx\, x J_0(x) = x J_1(x)\,, \tag{16}$$

that is equivalent to:

$$\int dq_T\, q_T J_0\left(b q_T\right) = \frac{q_T}{b} J_1\left(b q_T\right) \,. \tag{17}$$

Therefore:

$$\begin{aligned} K(x, z, Q, q_T) &= \frac{4\pi\alpha^2 q_T}{x z Q^3} Y_+ H(Q,\mu) \sum_q e_q^2 \\ &\times \int_0^\infty db\, J_1\left(b q_T\right) \overline{F}_q(x, b_*(b); \mu, Q^2) \overline{D}_q(z, b_*(b); \mu, Q^2) f_{\rm NP}(x, b, Q^2) D_{\rm NP}(z, b, Q^2) \,. \end{aligned} \tag{18}$$

The integral can again be computed using the Ogata quadrature as:

$$K(x, z, Q, q_T) \simeq \sum_{n=1}^{N} w_n^{(1)}\, S\left(x, z, \frac{\xi_n^{(1)}}{q_T}; \mu, Q^2\right) f_{\rm NP}\left(x, \frac{\xi_n^{(1)}}{q_T}, Q^2\right) D_{\rm NP}\left(z, \frac{\xi_n^{(1)}}{q_T}, Q^2\right) \,, \tag{19}$$

with S given in Eq. (12). Once K is known, the integral of the cross section over the bin $q_T \in [p_{Th,\min}/z : p_{Th,\max}/z]$ is computed as:

$$\int_{p_{Th,\min}/z}^{p_{Th,\max}/z} dq_T \left[\frac{d\sigma}{dx dQ dz dq_T}\right] = K(x, z, Q, p_{Th,\max}/z) - K(x, z, Q, p_{Th,\min}/z) \,. \tag{20}$$

This allows one to compute analytically one of the integrals that are often required to compare predictions to data.

2.1 Integrating over x, z, and Q

We now move to considering the integral of the cross section over x, z, and Q. Since these integrals usually come together with an integration in q_T, in the following we will consider the primitive function K in Eq. (19) rather than the cross section itself, that is:

$$\widetilde{K}(p_{Th}) = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{z_{\min}}^{z_{\max}} dz \int_{x_{\min}}^{x_{\max}} dx\, K(x, z, Q, p_{Th}/z) \,, \tag{21}$$

so that:

$$\widetilde{\sigma} = \widetilde{K}(p_{Th,\max}) - \widetilde{K}(p_{Th,\min}) \,. \tag{22}$$

The amount of numerical computation required to carry out the integration of a single bin is very large. Indicatively, it amounts to computing a three-dimensional integral for each of the terms of the Ogata quadrature that usually range from a few tens to hundreds. Therefore, in order to be able to do the integrations in a reasonable amount of time and yet obtain accurate results, it is necessary to put in place an efficient integration strategy. This goal can be achieved by exploiting a numerical integration based on interpolation techniques to precompute the relevant quantities. To this purpose, we first define one grid in x, $\{x_\alpha\}$ with $\alpha = 0, \dots, N_x$, one grid in z, $\{z_\beta\}$ with $\beta = 0, \dots, N_z$, and one grid in Q, $\{Q_\tau\}$ with $\tau = 0, \dots, N_Q$, each of which with a set of interpolating functions $\mathcal{I}$ associated. The grids should be such to span the full kinematic range covered by given data set. Then the value of K in Eq. (19) for any kinematics can be obtained through interpolation as:

$$
\begin{aligned}
K(x,z,Q,q_T) \quad &\simeq \quad \sum_{n=1}^{N} w_n^{(1)} S\left(x, z, \frac{\xi_n^{(1)}}{q_T}; \mu, Q^2\right) \sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} \mathcal{I}_\alpha(x)\mathcal{I}_\beta(z)\mathcal{I}_\tau(Q) \\
&\times \quad f_{\rm NP}\left(x_\alpha, \frac{\xi_n^{(1)}}{q_T}, Q_\tau^2\right) D_{\rm NP}\left(z_\beta, \frac{\xi_n^{(1)}}{q_T}, Q_\tau^2\right) .
\end{aligned} \tag{23}
$$

Once we have K in this form, the integration over x, z, and Q in Eq. (21) does not involve the non-perturbative functions $f_{\rm NP}$ and $D_{\rm NP}$ and can be written as:

$$
\widetilde{K}(p_{Th}) = \sum_{n=1}^{N}\sum_{\alpha=1}^{N_x}\sum_{\beta=1}^{N_z}\sum_{\tau=1}^{N_Q} W_{n\alpha\beta\tau}(p_{Th}) f_{\rm NP}\left(x_\alpha, \frac{z_\beta \xi_n^{(1)}}{p_{Th}}, Q_\tau^2\right) D_{\rm NP}\left(z_\beta, \frac{z_\beta \xi_n^{(1)}}{p_{Th}}, Q_\tau^2\right) , \tag{24}
$$

with:

$$
W_{n\alpha\beta\tau}(p_{Th}) = w_n^{(1)} \int_{Q_{\rm min}}^{Q_{\rm max}} dQ\, \mathcal{I}_\tau(Q) \int_{z_{\rm min}}^{z_{\rm max}} dz\, \mathcal{I}_\beta(z) \int_{x_{\rm min}}^{x_{\rm max}} dx\, \mathcal{I}_\alpha(x) S\left(x, z, \frac{z\xi_n^{(1)}}{p_{Th}}; \mu, Q^2\right) . \tag{25}
$$

Since the aim is to fit the functions $f_{\rm NP}$ and $D_{\rm NP}$ to data, one can precompute and store the coefficients W defined in Eq. (25) and compute the cross sections in a fast way making use of Eq. (24).

It is often the case that the integrated cross section, Eq. (14), is given within a certain acceptance region which is typically defined as:

$$
W = \sqrt{\frac{(1-x)Q^2}{x}} \geq W_{\rm min} , \quad y_{\rm min} \leq y \left(= \frac{Q^2}{sx}\right) \leq y_{\rm max} . \tag{26}
$$

These constraints can be expressed as constraints on the variable x for a fixed value of Q:

$$
x \leq \frac{Q^2}{W_{\rm min}^2 + Q^2} , \quad x \geq \frac{Q^2}{s y_{\rm max}} , \quad x \leq \frac{Q^2}{s y_{\rm min}} . \tag{27}
$$

Therefore, in order to implement the acceptance cuts in the computation of the integrated cross sections, it is enough to replace the integration bounds of the integral in x in Eq. (14) as follows:

$$
x_{\rm min} \rightarrow \overline{x}_{\rm min}(Q) = \max\left[x_{\rm min}, \frac{Q^2}{s y_{\rm max}}\right] , \quad x_{\rm max} \rightarrow \overline{x}_{\rm max}(Q) = \min\left[x_{\rm max}, \frac{Q^2}{s y_{\rm min}}, \frac{Q^2}{W_{\rm min}^2 + Q^2}\right] . \tag{28}
$$

Given the number and complexity of integrals as that in Eq. (25) to be computed, a numerical implementation requires devising a strategy that maximises the efficiency. In order to do so, we need to "unpac" the function S and perform the integrations in a way that unnecessary computations are avoided as much as possible. Taking $\mu = Q$ and $b = z\xi_n^{(1)}/p_{Th}$, using the fact that the Sudakov evolution factor for quarks R_q is universal, and dropping the unnecessary dependencies, Eq. (12) can be conveniently reorganised as:

$$
\begin{aligned}
S\left(x,z,\frac{z\xi_n^{(1)}}{p_{Th}},Q\right) &= 4\pi\frac{\alpha^2(Q)H(Q)}{Q^3}R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right),Q\right) \\
&\times \frac{1}{z}\sum_q e_q^2\overline{D}_q\left(z,b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right)\frac{Y_+(x,Q)}{x}\overline{F}_q\left(x,b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}}\right)\right)\,.
\end{aligned}
\tag{29}
$$

with:

$$
\overline{F}_q(x,b) = \sum_j\int_x^1 dy\,\mathcal{C}_{qj}(y)\left[\frac{x}{y}f_j\left(\frac{x}{y},\frac{b_0}{b}\right)\right]\,, \tag{30}
$$

and:

$$
\overline{D}_i(z,b) = \sum_j\int_z^1 dy\,\left[y^2\mathbb{C}_{qj}(y)\right]\left[\frac{z}{y}d_j\left(\frac{z}{y},\frac{b_0}{b}\right)\right]\,, \tag{31}
$$

being the initial-scale TMD distributions before Sudakov evolution. The weights in Eq. (25) can then be computed as:

$$
\begin{aligned}
W_{n\alpha\beta\tau}(p_{Th}) &= w_n^{(1)}4\pi\int_{Q_{\min}}^{Q_{\max}}dQ\,\mathcal{I}_\tau(Q)\frac{\alpha^2(Q)H(Q)}{Q^3} \\
&\times \int_{z_{\min}}^{z_{\max}}dz\,\mathcal{I}_\beta(z)R_q^2\left(b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}},Q\right),Q\right)\frac{1}{z}\sum_q e_q^2\overline{D}_q\left(z,b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}},Q\right)\right) \\
&\times \int_{\overline{x}_{\min}(Q)}^{\overline{x}_{\max}(Q)}dx\,\mathcal{I}_\alpha(x)\frac{Y_+(x,Q)}{x}\overline{F}_q\left(x,b_*\left(\frac{z\xi_n^{(1)}}{p_{Th}},Q\right)\right)\,.
\end{aligned}
\tag{32}
$$

This nesting should optimise the computation of the integrals.