Combining evolution and DIS operators

1 The structure of the observables

In all cases the inclusive DIS structure functions are conveniently expressed in terms of PDF combinations in the so-called physical basis $\{q_i^{\pm}\}$, with $q_i^{\mp} = q_i \pm \overline{q}_i$, where q_i and $\overline{q}_i$ are the PDFs of the $i-th$ quark-flavour, with $i = u, d, s, c, b, t$, and its antiflavour, respectively. Schematically, a DIS structure function can be written as:

$$F = C_g g + \sum_i \left(C_i^+ q_i^+ + C_i^- q_i^- \right) , \tag{1.1}$$

being C the appropriate coefficient functions. Conversely, the evolution of PDFs is usually compute in the so-called QCD evolution basis $\{d_i^{\pm}\}$, with $d_1^+ = \Sigma$, $d_2^+ = -T_3$, $d_3^+ = T_8$, $d_4^+ = T_{15}$, $d_5^+ = T_{24}$, and $d_6^+ = T_{35}$ and $d_1^- = V$, $d_2^- = -V_3$, $d_3^- = V_8$, $d_4^- = V_{15}$, $d_5^- = V_{24}$, and $d_6^- = V_{35}$. The gluon remains unchanged.

$$g = \Gamma_{gg} g_0 + \Gamma_{gq} d_{1,0}^+ \tag{1.2}$$

while:

$$\begin{aligned} d_i^{\pm} &= \theta_{i2}\theta(Q-m_i)\Gamma^{\pm} d_{i,0}^{\pm} + \theta(m_i - Q) \left\{ \begin{array}{ll} \Gamma_{qq} d_{1,0}^+ + \Gamma_{qg} g_0 & \text{for } + \\ \Gamma^v d_{1,0}^- & \text{for } - \end{array} \right. \\ &= \left\{ \begin{array}{l} \Gamma_{qq} d_{1,0}^+ + \Gamma_{qg} g_0 \\ \Gamma^v d_{1,0}^- \end{array} \right. + \theta(Q-m_i) \left[\theta_{i2} \Gamma^{\pm} d_{i,0}^{\pm} - \left\{ \begin{array}{l} \Gamma_{qq} d_{1,0}^+ + \Gamma_{qg} g_0 \\ \Gamma^v d_{1,0}^- \end{array} \right. \right] . \end{aligned} \tag{1.3}$$

so that:

$$q_i^- = \delta_{i1} \Gamma^v d_{1,0}^- + \Gamma^{\pm} \sum_{j=2}^{6} \theta_{ji} \frac{1 - \delta_{ij} j}{j(j-1)} \theta(Q - m_j) d_{j,0}^{\pm} \tag{1.4}$$

Therefore, we need to relate these two bases. This is done through the linear transformation:

$$q_i^{\pm} = \sum_{j=1}^{6} M_{ij} d_j^{\pm} , \tag{1.5}$$

where the trasformation matrix M_{ij} can be written as:

$$\begin{aligned} M_{ij} &= \theta_{ji} \frac{1 - \delta_{ij} j}{j(j-1)} \quad j \geq 2 , \\ M_{i1} &= \frac{1}{6} , \end{aligned} \tag{1.6}$$

with $\theta_{ji} = 1$ for $j \geq i$ and zero otherwise.

Using eq. (1.5) we can make the following identifications:

$$D^{\pm} = q_{2j-1}^{\pm} \quad \text{and} \quad U^{\pm} = q_{2j}^{\pm} , \quad j = 1, 2, 3 , \tag{1.7}$$

so that we can write:

$$F^{\pm} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 \left[C_{\pm} \left(q_{2j-1}^{\pm} \pm q_{2i}^{\pm} \right) + 4P^{\pm} C_g g \right] . \tag{1.8}$$

Using the definition of M_{ij} in eq. (1.6), we can rewrite $F^{\pm}$ in terms of PDFs in the evolution basis as:

$$F^{\pm} = \sum_{i=1}^{3} \sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 F_{ij}^{\pm} , \tag{1.9}$$

with:

$$F_{ij}^{\pm} = C_g 2P^{\pm} g + C_{\pm}^{\mathrm{S}} P^{\pm} \frac{1}{6} d_1^{\pm} + C_{\pm} \sum_{k=2}^{6} \frac{\theta_{k,2j-1}(1-\delta_{2j-1,k}k) \pm \theta_{k,2i}(1-\delta_{2i,k}k)}{2k(k-1)} d_k^{\pm}\,. \tag{1.10}$$

Eq. (1.10) is valid only for F_2 and F_3. In order to obtain a similar equation also for F_3, one needs to change sign to the antiquark distributions, *i.e.* $\overline{q}_i \to -\overline{q}_i$. In the QCD evolution basis, this has the consequence of exchanging the T-like distributions with the V-like ones, that is to say $d_k^+ \leftrightarrow d_k^-$. It is the easy to see that:

$$F_3^{\pm} = \sum_{i=1}^{3} \sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 F_{3,ij}^{\pm}\,, \tag{1.11}$$

with:

$$F_{3,ij}^{\pm} = C_g 2P^{\pm} g + C_{\pm}^{\mathrm{S}} P^{\mp} \frac{1}{6} d_1^{\pm} + C_{\pm} \sum_{k=2}^{6} \frac{\theta_{k,2j-1}(1-\delta_{2j-1,k}k) \mp \theta_{k,2i}(1-\delta_{2i,k}k)}{2k(k-1)} d_k^{\pm}\,. \tag{1.12}$$

It is now useful to consider the inclusive structure functions and exploit the unitarity of the CKM matrix elements V_{UD}:

$$\sum_{i=1}^{3} |V_{2i,(2j-1)}|^2 = \sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 = 1 \quad \Rightarrow \quad \sum_{i=1}^{3}\sum_{j=1}^{3} |V_{2i,(2j-1)}|^2 = 3\,. \tag{1.13}$$

Summing over i and j in eq. (1.9) and using eq. (1.10), one obtains:

$$F^{\pm} = C_g 6P^{\pm} g + C_{\pm}^{\mathrm{S}} P^{\pm} \frac{1}{2} d_1^{\pm} + \frac{1}{2} C_{\pm} \sum_{k=2}^{6} d_k^{\pm} \sum_{l=1}^{6} (\pm 1)^{l+1} M_{lk}\,. \tag{1.14}$$

Considering separately F^+ and F^- and using eq. (**??**), one finds:

$$F^{+} = C_g 6g + C_{+}^{\mathrm{S}} \frac{1}{2} d_1^{+} \tag{1.15}$$

and:

$$F^{-} = \frac{1}{2} C_{-} \sum_{k=2}^{6} \left[\frac{P^{+}}{k-1} - \frac{P^{-}}{k} \right] d_k^{-}\,, \tag{1.16}$$

with the even/odd projectors defined as:

$$P_k^{\pm} = \frac{1 \pm (-1)^k}{2}\,. \tag{1.17}$$

It should be pointed out that such simple expressions (independent of the CMK matrix elements) is achievable only if it is possible to factorize the non-singlet coefficient functions as implicitly done in eqs. (1.10) and (1.12). In fact, this is possible only in the ZM case in which the coefficient functions of each PDF combination is the same.

For F_3 we find:

$$F_3^{+} = C_g 6g + C_{-}^{\mathrm{S}} \frac{1}{2} d_1^{-} \tag{1.18}$$

and:

$$F_3^{-} = \frac{1}{2} C_{+} \sum_{k=2}^{6} \left[\frac{P_k^{+}}{k-1} - \frac{P_k^{-}}{k} \right] d_k^{+}\,. \tag{1.19}$$