Contents

1 Introduction

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

2 Evolution equation

In general, the evolution equation for GPDs reads:

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-\infty}^{+\infty}\frac{dx'}{|2\xi|}\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{2.1}$$

The GPD f and the evolution kernel $\mathbb{P}$ may in general be a vector and a matrix in flavour space. For now we will just be concerned with the integral in the r.h.s. of Eq. (2.1) regardless of the flavour structure. The support of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 2.1. Knowing the support of the evolution kernel, Eq. (2.1)

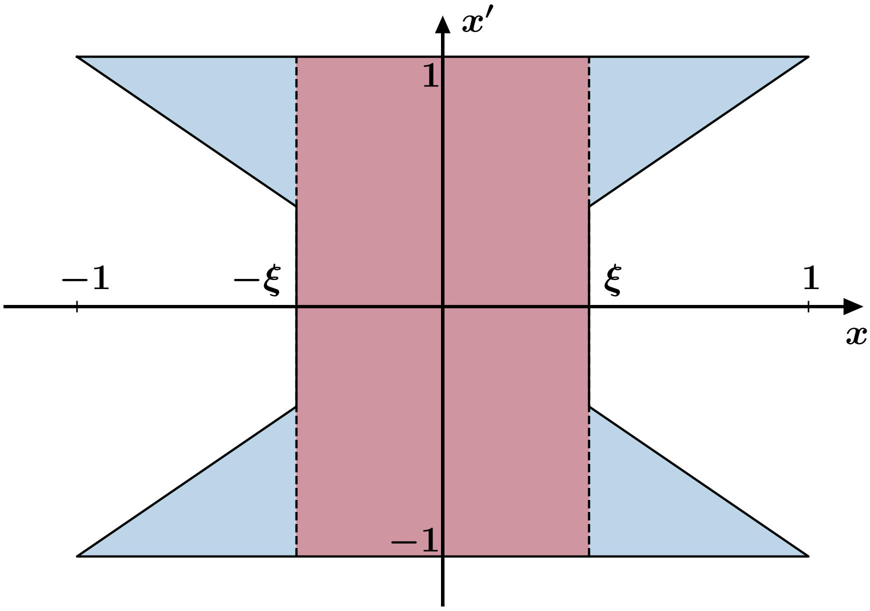

Fig. 2.1: Support domain of the evolution kernel $\mathbb{P}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

can be rearranged as follows:

$$\mu^2\frac{d}{d\mu^2}f(\pm x,\xi)=\int_{b(x)}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}\mathbb{P}\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}\mathbb{P}\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,, \tag{2.2}$$

with:

$$b(x)=|x|\theta\left(\left|\frac{x}{\xi}\right|-1\right)\,, \tag{2.3}$$

and where we have used the symmetry property of the evolution kernels $\mathbb{P}(y, y') = \mathbb{P}(-y, -y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned} f^{\pm}(x,\xi) &= f(x,\xi) \mp f(-x,\xi)\,, \\ \mathbb{P}^{\pm}(y,y') &= \mathbb{P}(y,y') \mp \mathbb{P}(-y,y')\,, \end{aligned} \tag{2.4}$$

so that the evolution equation for $f^{\pm}$ reads:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \int_{b(x)}^{1} \frac{dx'}{x'} \frac{x'}{|2\xi|} \mathbb{P}^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) f^{\pm}(x',\xi)\,. \tag{2.5}$$

In fact, the $f^{\pm}$ distributions are the GPD analogous of the $\pm$ forward distributions that can then be used to construct the usual singlet and non-singlet distributions in the QCD evolution basis. This also determines the flavour structure of the evolution kernels $\mathbb{P}^{\pm}$. Specifically, from now on we will understand that $\mathbb{P}^{+}$ is a 2×2 matrix in flavour space, while $\mathbb{P}^{-}$ is a flavour-scalar non-singlet evolution kernel.

It is relevant to observe that the presence of the θ-function in the lower integration bound b, Eq. (2.3), is such that for $|x| > |\xi|$ the evolution equation has the exact form of the DGLAP evolution equation which corresponds to integrating over the blue regions in Fig. 2.1 (henceforth DGLAP region). Conversely, for $|x| \leq |\xi|$ the lower integration bound becomes zero and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). This corresponds to integrating over the red region (henceforth ERBL region). Crucially, in the limits $\xi \to 0$ and $\xi \to \pm 1$ one recovers the DGLAP and ERBL equations, respectively.

For later convenience, we define the parameter:

$$\kappa(x) = \frac{\xi}{x}\,, \tag{2.6}$$

so that:

$$\frac{x'}{|2\xi|} \mathbb{P}^{\pm}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) = \mathrm{sign}(\xi) \frac{1}{2\kappa} \frac{x'}{x} \mathbb{P}^{\pm}\left(\frac{1}{\kappa}, \frac{1}{\kappa}\frac{x'}{x}\right) \equiv \mathrm{sign}(\xi) \mathcal{P}^{\pm}\left(\kappa, \frac{x}{x'}\right)\,, \tag{2.7}$$

where the last equality effectively defines the *DGLAP-like* splitting function:

$$\mathcal{P}^{\pm}(\kappa, y) = \frac{1}{2\kappa y} \mathbb{P}^{\pm}\left(\frac{1}{\kappa}, \frac{1}{\kappa y}\right)\,. \tag{2.8}$$

Plugging this definition into the integral in the r.h.s. of Eq. (2.5) and performing a change of variable gives:

$$\mu^2 \frac{d}{d\mu^2} f^{\pm}(x,\xi) = \mathrm{sign}(\xi) \int_{b(x)}^{1} \frac{dy}{y} \mathcal{P}^{\pm}\left(\kappa, \frac{x}{y}\right) f^{\pm}(y,\xi) = \mathrm{sign}(\xi) \int_{x}^{x/b(x)} \frac{dy}{y} \mathcal{P}^{\pm}(\kappa, y)\, f^{\pm}\left(\frac{x}{y}, \xi\right)\,, \tag{2.9}$$

with:

$$b(x) = |x| \theta\left(\frac{1}{|\kappa|} - 1\right) = |x| \theta\left(1 - |\kappa|\right)\,, \tag{2.10}$$

which (almost) has the form of a "standard" DGLAP equation. The only difference is that the upper integration bound is not one but rather $x/b(x)$ with $b(x)$. As discussed in the document *IntegralStructure.pdf*, this difference can be handled within APFEL (up to a numerical approximation to be assessed) by adjusting the integration procedure.

In the following we will assume $\xi > 0$ as, so far, this is the only experimentally accessible region. This allows us to get rid of $\mathrm{sign}(\xi)$ in Eq. (2.9). In addition, we can also restrict to positive values of x because negative values can be easily accessed by symmetry using Eq. (2.4): $f^{\pm}(-x,\xi) = \mp f^{\pm}(x,\xi)$.

3 Anomalous dimensions

A crucial ingredient for an efficient implementation of GPD evolution is the availability of the DGLAP-like splitting functions $\mathcal{P}^{\pm}$ defined in Eq. (2.8) in a closed form amenable to be easily integrated as in Eq. (2.9).

[1] Notice the seemingly unusual fact that f^{+} is defined as difference and f^{-} as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x) = -\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign.

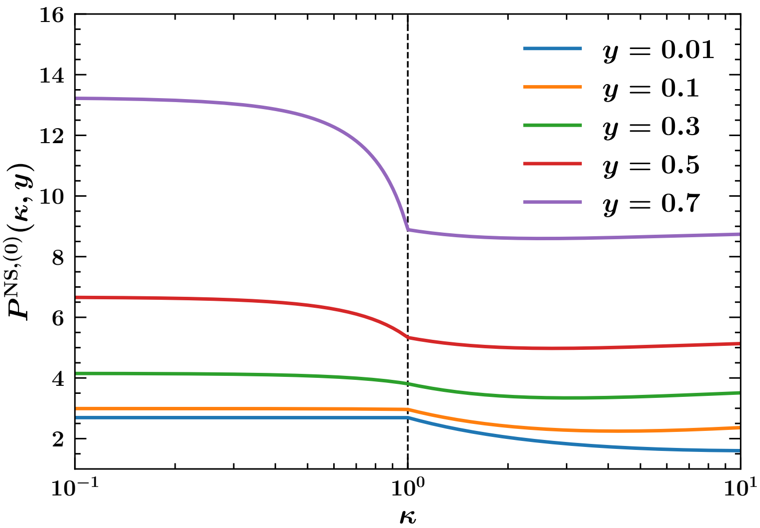

Fig. 3.1: Behaviour of the anomalous dimension $P^{\mathrm{NS},(0)}$ as a function of κ for different values of y.

In order to address this problem, we need to make the flavour structure of the evolution kernels explicit. Working in the QCD evolution basis, we have:

$$\mathcal{P}^+ = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} , \tag{3.1}$$

and:

$$\mathcal{P}^- = P^{\mathrm{NS}} , \tag{3.2}$$

where P^{NS} is the appropriate evolution kernel. At one loop it turns out that all the non-singlet splitting functions are equal amongst and to $P_{qq}^{(0)}$, *i.e.* $P^{\mathrm{NS},(0)} = P_{qq}^{(0)}$.

The explicit form of the one-loop anomalous dimensions can be found, for example, in Ref. [2]. However, these functions are tricky to integrate numerically due to the presence of θ-functions. In order to define the basic steps to reduce the anomalous dimension to a suitable form, let us consider the non-singlet unpolarised anomalous dimension at one loop. Using Eq. (2.8), one finds:

$$P^{\mathrm{NS},(0)}(\kappa, y) = \begin{cases} 2C_F \left[\dfrac{1 + (1-2\kappa^2)y^2}{(1-y)(1-\kappa^2 y^2)} \right]_+ , & 0 \leq \kappa \leq 1 , \\[2ex] 2C_F \left[\dfrac{1}{1-y} + \dfrac{1}{2} \dfrac{1-\kappa}{\kappa(1+\kappa y)} \right]_+ , & 1 \leq \kappa \leq \dfrac{1}{x} . \end{cases} \tag{3.3}$$

It is interesting to take the DGLAP limit at $\kappa \to 0$:

$$P^{\mathrm{NS},(0)}(0, y) = 2C_F \left[\frac{1+y^2}{1-y} \right]_+ , \tag{3.4}$$

that coincides with the usual DGLAP splitting function. While the crossover point at $\kappa = 1$ gives:

$$P^{\mathrm{NS},(0)}(1, y) = 2C_F \left[\frac{1}{1-y} \right]_+ , \tag{3.5}$$

regardless of whether the limit is taken from the DGLAP region ($\kappa \to 1^-$) of from the ERBL region ($\kappa \to 1^+$). This tells us that $\mathcal{P}_{qq}^{+(0)}(\kappa, y)$ is a continuos function of κ. Fig. 3.1 displays the behaviour of the regular part of the anomalous dimension $\mathcal{P}_{qq}^{+(0)}$ defined in Eq. (3.3) as a function of the parameter κ for different values of the variable y. This plot shows the continuity of $\mathcal{P}_{qq}^{+(0)}$ at $\kappa = 1$. Since $P_{qq}^{(0)} = P^{\mathrm{NS},(0)}$, what we discussed above applies verbatim to $P_{qq}^{(0)}$.

Let us now turn to the remaining splitting functions $P_{qg}^{(0)}$, $P_{gq}^{(0)}$, and $P_{gg}^{(0)}$. Their explicit expressions read[2]:

$$P_{qg}^{(0)}(\kappa,y)=\begin{cases} 4n_fT_R\dfrac{1-2y+(2-\kappa^2)y^2}{(1-\kappa^2y^2)^2}\,, & 0\leq\kappa\leq 1\,,\\[2ex] 2n_fT_R\left[\dfrac{1+\kappa}{\kappa^2(1+\kappa y)}\right]\left[\dfrac{1-\kappa}{\kappa y}+\dfrac{1}{1+\kappa y}\right]\,, & 1\leq\kappa\leq\dfrac{1}{x}\,,\end{cases} \tag{3.6}$$

$$P_{gq}^{(0)}(\kappa,y)=\begin{cases} 2C_F\left[\dfrac{2-2y+(1-\kappa^2)y^2}{y(1-\kappa^2y^2)}\right]\,, & 0\leq\kappa\leq 1\,,\\[2ex] C_F\left[\dfrac{2(1+\kappa)-(1-\kappa^2)y}{\kappa y(1+\kappa y)}\right]\,, & 1\leq\kappa\leq\dfrac{1}{x}\,,\end{cases} \tag{3.7}$$

$$P_{gg}^{(0)}(\kappa,y)=\begin{cases} 4C_A\left[\dfrac{-2+y(1-y+\kappa^2(1+y))}{(1-\kappa^2y^2)^2}+\dfrac{1}{y}+\dfrac{1}{(1-y)_+}\right]+\delta(1-x)\dfrac{11C_A-4n_fT_R}{3}\,, & 0\leq\kappa\leq 1\,,\\[2ex] C_A\left[\dfrac{-1+3\kappa^2-\kappa(2+(1-\kappa)^2\kappa)y}{\kappa^3y(1+\kappa y)^2}+\dfrac{2}{y}+\dfrac{2}{(1-y)_+}\right]+\delta(1-x)\dfrac{11C_A-4n_fT_R}{3}\,, & 1\leq\kappa\leq\dfrac{1}{x}\,.\end{cases} \tag{3.8}$$

Their forward limit ($\kappa=0$) is:

$$\begin{aligned} P_{qg}^{(0)}(0,y)&=4n_fT_R\left[y^2+(1-y)^2\right]\\ P_{gq}^{(0)}(0,y)&=2C_F\left[\frac{1+(1-y)^2}{y}\right]\,,\\ P_{gg}^{(0)}(0,y)&=4C_A\left[-2+y(1-y)+\frac{1}{y}+\frac{1}{(1-y)_+}\right]+\delta(1-x)\frac{11C_A-4n_fT_R}{3}\,,\end{aligned} \tag{3.9}$$

which coincides with the usual one-loop DGLAP splitting functions. At the crossover point $\kappa=1$ the evolution kernels reduce to:

$$P_{qg}^{(0)}(1,y)=\frac{4n_fT_R}{(1+y)^2}\,, \tag{3.10}$$

$$P_{gq}^{(0)}(1,y)=\frac{4C_F}{y(1+y)}\,, \tag{3.11}$$

$$P_{gg}^{(0)}(1,y)=4C_A\left[\frac{1+y^2}{y(1+y)^2\,(1-y)_+}\right]+\delta(1-y)\frac{11C_A-4n_fT_R}{3}\,. \tag{3.12}$$

Interestingly, like $P^{\mathrm{NS},(0)}$ and $P_{qq}^{(0)}$, also $P_{qg}^{(0)}$, $P_{gq}^{(0)}$, and $P_{gg}^{(0)}$ are continuos in $\kappa=1$.

3.1 End-point (local) contributions

Some of the expressions for the anomalous dimensions discussed above contain a +-prescribed terms. It is important to treat these terms properly accounting for additional local terms stemming from the "incompleteness" of the convolution integrals. More specifically, the defintion of the +-prescription for the function g (singular in $y=1$) convoluted with a smooth test function f is:

$$\int_0^{1^+}dy\,\left[g(y)\right]_+f(y)=\int_0^{1^+}dy\,g(y)\left[f(y)-f(1)\right]\,. \tag{3.13}$$

Now, we need to work out the action of the +-prescription on integrals of the following kind (*cfr.* Eq. (2.9)):

$$I=\int_x^c dy\,\left[g(y)\right]_+f(y)\,,\quad x<1\quad\text{and}\quad c\geq 1\,. \tag{3.14}$$

[2] The expression for $P_{gq}^{(0)}$ derived from Ref. [2] seems to be wrong. I have derived the seemingly correct expression from Ref. [4] by performing the replacement $\xi\to 2\kappa y$ in Eq. (24).

In order to apply the definition of +-prescription, we manipulate the integral above as follows:

$$\begin{aligned}
I &= \int_0^{1^+} dy\ [g(y)]_+ f(y) - \int_0^x dy\, g(y)f(y) + \int_{1^+}^c dy\, g(y)f(y) \\
&= \int_0^{1^+} dy\, g(y)\,[f(y)-f(1)] - \int_0^x dy\, g(y)f(y) + \int_{1^+}^c dy\, g(y)f(y) \\
&= \int_x^c dy\, g(y)\,[f(y)-f(1)] + f(1)\,[L_1(x)+L_2(c)]
\end{aligned} \tag{3.15}$$

where for shortness we have defined:

$$L_1(x) = -\int_0^x dy\, g(y) \quad \text{and} \quad L_2(c) = \int_{1^+}^c dy\, g(y)\,. \tag{3.16}$$

The term L_1 is the usual term arising from the incompleteness of the convolution integral and is finite for $x < 1$. The term L_2 is new. In the DGLAP region ($\kappa < 1$) the upper integration bound c is equal to one so that $L_2(1) = 0$ and we recover the usual DGLAP structure. In the ERBL region ($\kappa > 1$), instead, $c = +\infty$ so that:

$$L_2(\infty) = \int_{1^+}^{\infty} dy\, g(y)\,. \tag{3.17}$$

The question now is whether the second integral in the rightmost term converges: the answer is generally not. A simple example is Eq. (3.5) that, for $\kappa = 1^+$ gives:

$$\int_{1^+}^{\infty} \frac{dy}{1-y} = -\infty\,. \tag{3.18}$$

By inspection, one finds that this divergence is logarithmic, signifying that the function g goes to zero linearly as $y \to \infty$. Of course, the presence of this divergence seems to invalidate the full evolution procedures because the derivative of GPDs with respect to μ in the ERBL region is not defined. The question is how to overcome this problem. Since the function g is the result of a perturbative calculation, the only possibility relies on imposing an appropriate constraint on test function f that guarantees the convergence of the integral in Eq. (3.14). To do so, we need to *require* that the function f is such that the last term in the r.h.s. of the first line of Eq. (3.15) converges when $c = \infty$:

$$\left| \int_{1^+}^{\infty} dy\, g(y)f(y) \right| < \infty\,. \tag{3.19}$$

As we will see in the next section, the requirement of continuity of GPDs sets the particular value of this integral at $x = \xi$ to zero. Since g decays linearly, in order for the integral to converge, also the function f needs to tend to zero as $y \to \infty$ as:

$$f(y) \underset{y\to\infty}{\longrightarrow} \frac{1}{y^a}\,, \quad a > 0\,. \tag{3.20}$$

3.2 On continuity of GPDs

It is well known that GPDs are required to be continuous at $x = \xi$ for factorisation to be valid [3]. It is thus interesting to consider the consequence of this constraint. To this end, let us consider the limits of Eq. (2.5) for $x \to \xi^\pm$:

$$\lim_{x\to\xi^+} \mu^2 \frac{d}{d\mu^2} f^\pm(x,\xi) = \mu^2 \frac{d}{d\mu^2} f^\pm(\xi,\xi) = \int_\xi^1 \frac{dx'}{2\xi} \mathbb{P}^\pm\left(\frac{\xi^+}{\xi}, \frac{x'}{\xi}\right) f^\pm(x',\xi)\,, \tag{3.21}$$

and:

$$\lim_{x\to\xi^-} \mu^2 \frac{d}{d\mu^2} f^\pm(x,\xi) = \mu^2 \frac{d}{d\mu^2} f^\pm(\xi,\xi) = \int_0^1 \frac{dx'}{2\xi} \mathbb{P}^\pm\left(\frac{\xi^-}{\xi}, \frac{x'}{\xi}\right) f^\pm(x',\xi)\,, \tag{3.22}$$

where we have used the continuity of f at $x = \xi$. As we have shown in the previous section, anomalous dimension at one loop are also continuous in $x = \xi$ (*i.e.* in $\kappa = 1$), such that:

$$\mathbb{P}^\pm\left(\frac{\xi^+}{\xi}, y\right) = \mathbb{P}^\pm\left(\frac{\xi^-}{\xi}, y\right) = \mathbb{P}^\pm(1, y)\,. \tag{3.23}$$

Taking the difference between Eqs. (3.21) and (3.22) paying attention to remove also the end point contributions, one finds, at least at one loop, that:

$$\int_0^{\xi^-} \frac{dx'}{2\xi}\,\mathbb{P}^{\pm}\left(1,\frac{x'}{\xi}\right) f^{\pm}(x',\xi) = \int_0^{\xi^-} \frac{dx'}{x'}\,\mathcal{P}^{\pm}\left(1,\frac{\xi}{x'}\right) f^{\pm}(x',\xi) = \int_{1^+}^{\infty} \frac{dx'}{x'}\,\mathcal{P}^{\pm}\left(1,x'\right) f^{\pm}\left(\frac{\xi}{x'},\xi\right) = 0\,. \quad (3.24)$$

This appears to be some sort of sum rule that the GPDs and anomalous dimensions in the ERBL region have to fullfil. We will attempt a numerical estimate of the impact of possible violations of this requirement.

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