Interpolacija funkcija

Teorema. *Neka je funkcija* $f(x)$ *definisana na skupu* $X \subseteq [a, b]$ *i neka su date različite tačke* $x_i \in X, \quad i = 0, 1, \ldots, n$. *Tada postoji jedinstven polinom*

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

takav da je ispunjen uslov

$$P_n(x_i) = f(x_i), \qquad i = 0, 1, \ldots, n.$$

- čvorovi interpolacije:

$$x_i \in X \subseteq [a, b], \quad f_i = f(x_i), \qquad i = 0, 1, \ldots, n$$

- interpolacioni polinom:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

I Lagranžov interpolacioni polinom

$$P_n(x) = \sum_{k=0}^{n} f_k L_k(x),$$

$$L_k(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)},$$

$$k = 0, 1, 2, \ldots, n.$$

II Njutnov interpolacioni polinom sa podeljenim razlikama

$$
\begin{aligned}
P_n(x) = f(x_0) &+ (x-x_0)[x_0,x_1;f] + \cdots \\
&+ (x-x_0)(x-x_1)\cdots(x-x_{n-1})[x_0,x_1,\ldots,x_n;f],
\end{aligned}
$$

gde su podeljene razlike definisane sa

$$
\begin{aligned}
[x_0,x_1;f] &= \frac{f(x_1)-f(x_0)}{x_1-x_0}, \\
[x_0,x_1,x_2;f] &= \frac{[x_1,x_2;f]-[x_0,x_1;f]}{x_2-x_0}, \\
&\vdots \\
[x_0,x_1,\ldots,x_n;f] &= \frac{[x_1,x_2\ldots,x_n;f]-[x_0,x_1\ldots,x_{n-1};f]}{x_n-x_0}.
\end{aligned}
$$

III Prvi Njutnov interpolacioni polinom

Neka su čvorovi ekvidistantni, tj. $x_k = x_0 + kh, \;\; k = 0,1,\ldots,n.$

$$
P_n(x) = f_0 + \frac{\Delta f_0}{h}(x-x_0) + \cdots + \frac{\Delta^n f_0}{n!h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1}),
$$

gde su prednje razlike definisane sa

$$
\Delta f_0 = f_1 - f_0, \quad \Delta^2 f_0 = \Delta(\Delta f_0), \quad \ldots, \quad \Delta^n f_0 = \Delta(\Delta^{n-1} f_0).
$$

Smenom $\dfrac{x-x_0}{h} = p$:

$$
P_n(x) = f_0 + \Delta f_0 p + \frac{\Delta^2 f_0}{2!}p(p-1)\cdots + \frac{\Delta^n f_0}{n!}p(p-1)\cdots(p-n+1).
$$

IV Drugi Njutnov interpolacioni polinom

Neka su čvorovi ekvidistantni, tj. $x_k = x_0 + kh, \;\; k = 0,1,\ldots,n.$

$$
P_n(x) = f_n + \frac{\nabla f_n}{h}(x-x_n) + \cdots + \frac{\nabla^n f_n}{n!h^n}(x-x_n)(x-x_{n-1})\cdots(x-x_1)
$$

gde su zadnje razlike definisane sa

$$\nabla f_n = f_n - f_{n-1}, \quad \nabla^2 f_n = \nabla(\nabla f_n), \quad \ldots, \quad \nabla^n f_n = \nabla(\nabla^{n-1} f_n).$$

Smenom $\dfrac{x - x_n}{h} = p$:

$$P_n(x) = f_n + \nabla f_n p + \frac{\nabla^2 f_n}{2!} p(p+1) \cdots + \frac{\nabla^n f_n}{n!} p(p+1) \cdots (p+n-1).$$

VI Greška interpolacionih polinoma

Teorema. *Neka je* $f \in C^{n+1}[a,b]$, $a \leq x_0 < x_1 < \cdots < x_n \leq b$ *i* $P_n(x)$ *interpolacioni polinom funkcije* $f(x)$ *sa čvorovima* x_i, $i = 0, 1, \ldots, n$. *Tada postoji* $\xi \in (a,b)$ *tako da je*

$$R_n(f;x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega(x),$$

gde je $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.

Ocena greške:

$$|R_n(f;x)| = |f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\omega(x)|, \qquad M_{n+1} = \max_{a \leq x \leq b} |f^{(n+1)}(x)|.$$

Teorema. *Neka je* $P_n(x)$ *Njutnov interpolacioni polinom sa podeljenim razlikama funkcije* $f(x)$ *sa čvorovima* x_i, $i = 0, 1, \ldots, n$. *Tada važi*

$$R_n(f;x) = f(x) - P_n(x) = \omega(x)[x_0, x_1, \ldots, x_n, x; f]$$

gde je $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.

Teorema. *Neka je* $P_n(x)$ *prvi Njutnov interpolacioni polinom funkcije* $f(x)$ *sa čvorovima* $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$. *Tada važi*

$$|R_n(f;x)| = |f(x) - P_n(x)| \leq \frac{\max |\Delta^{n+1} f|}{(n+1)!} |p(p-1) \cdots (p-n)|,$$

$$|R_n(f;x)| = |f(x) - P_n(x)| \leq \frac{\max |\nabla^{n+1} f|}{(n+1)!} |p(p+1) \cdots (p+n)|.$$

ZADACI

Zadatak 1. Aproksimirati funkciju $f(x) = e^x$ na segmentu $[0,\ 0.5]$ Lagranžovim interpolacionim polinomom i proceniti grešku na osnovu podataka u tačkama
a) $x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.5;$
b) $x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.5,\ x_4 = 0.4.$
Aproksimirati $f(0.3) = e^{0.3}$ dobijenim interpolacionim polinomom i proceniti grešku.

Rešenje: a) Na osnovu podataka u 3 čvora formiramo interpolacioni polinom stepena 2.

Vrednosti funkcije u čvorovima predstavljene su u sledećoj tabeli.

k	0	1	2
x_k	0.0	0.2	0.5
f_k	1.0000	1.2214	1.6487

Lagranžov interpolacioni polinom sa datim čvorovima je

$$P_2(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x),$$

gde je

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.2)(x-0.5)}{(-0.2)\cdot(-0.5)} = \frac{(x-0.2)(x-0.5)}{0.1},$$
$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.0)(x-0.5)}{0.2\cdot(-0.3)} = -\frac{x(x-0.5)}{0.06},$$
$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.0)(x-0.2)}{0.5\cdot 0.3} = \frac{x(x-0.2)}{0.15}.$$

$$\begin{aligned} P_2(x) &= 1.0000\ \frac{(x-0.2)(x-0.5)}{0.1} - 1.2214\ \frac{x(x-0.5)}{0.06} + 1.6487\ \frac{x(x-0.2)}{0.15} \\ &= 0.6348x^2 + 0.9801x + 1. \end{aligned}$$

Procena greške:

$$|R_2(f;x)| = |f(x) - P_2(x)| \leq \frac{M_3}{3!}|(x-x_0)(x-x_1)(x-x_2)|,$$
$$M_3 = \max_{0\leq x\leq 0.5} |f'''(x)|.$$

$$f'(x) = f''(x) = f'''(x) = e^x, \qquad M_3 = \max_{0\leq x\leq 0.5} |e^x| = e^{0.5} = \sqrt{e} < 2,$$

$$|R_2(f;x)| = |f(x) - P_2(x)| \leq \frac{2}{6}|x(x-0.2)(x-0.5)| = \frac{1}{3}|x(x-0.2)(x-0.5)|.$$

Specijalno, za $x = 0.3$:

$$f(0.3) \approx P_2(0.3) = 1.3512,$$
$$|R_2(f;0.3)| = |f(0.3) - P_2(0.3)| \leq \frac{1}{3}|0.3 \cdot 0.1 \cdot (-0.2)| = 0.002.$$

Kako je $f(0.3) = e^{0.3} = 1.34986\ldots$, prava greška je

$$f(0.3) - P_2(0.3) = e^{0.3} - P_2(0.3) = -0.0013.$$

b) Na osnovu podataka u 4 čvora formiramo interpolacioni polinom stepena 3.

Vrednosti funkcije u čvorovima predstavljene su u sledećoj tabeli.

k	0	1	2	3
x_k	0.0	0.2	0.5	0.4
f_k	1.0000	1.2214	1.6487	1.4918

Lagranžov interpolacioni polinom sa datim čvorovima je

$$P_3(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x) + f_3L_3(x),$$

gde je

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0.2)(x-0.5)(x-0.4)}{(-0.2)\cdot(-0.5)\cdot(-0.4)},$$
$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0.0)(x-0.5)(x-0.4)}{0.2\cdot(-0.3)\cdot(-0.2)},$$
$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0.0)(x-0.2)(x-0.4)}{0.5\cdot 0.3\cdot 0.1},$$
$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0.0)(x-0.2)(x-0.5)}{0.4\cdot 0.2\cdot(-0.1)}.$$

$$P_2(x) = 1.\ \frac{(x-0.2)(x-0.5)(x-0.4)}{(-0.2)\cdot(-0.5)\cdot(-0.4)} + 1.2214\ \frac{x(x-0.5)(x-0.4)}{0.2\cdot(-0.3)\cdot(-0.2)}$$
$$+ 1.6487\ \frac{x(x-0.2)(x-0.4)}{0.5\cdot 0.3\cdot 0.1} + 1.4982\ \frac{x(x-0.2)(x-0.5)}{0.4\cdot 0.2\cdot(-0.1)}$$
$$= 0.2202x^3 + 0.4806x^2 + 1.0021x + 1.$$

Procena greške:

$$|R_3(f;x)| = |f(x) - P_3(x)| \le \frac{M_4}{4!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|,$$
$$M_4 = \max_{0\le x\le 0.5} |f^{(4)}(x)|.$$

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = e^x, \quad M_4 = \max_{0\le x\le 0.5} |e^x| = e^{0.5} = \sqrt{e} < 2,$$

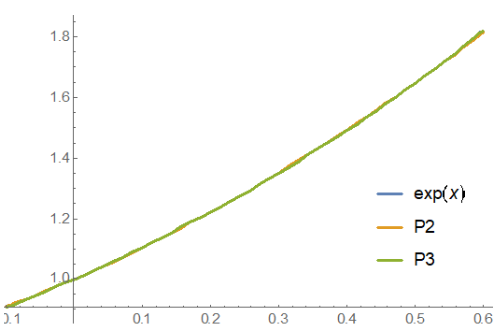
$$|R_3(f;x)| = |f(x) - P_3(x)| \le \frac{2}{24}|x(x-0.2)(x-0.5)(x-0.4)|$$
$$= \frac{1}{12}|x(x-0.2)(x-0.5)(x-0.4)|.$$

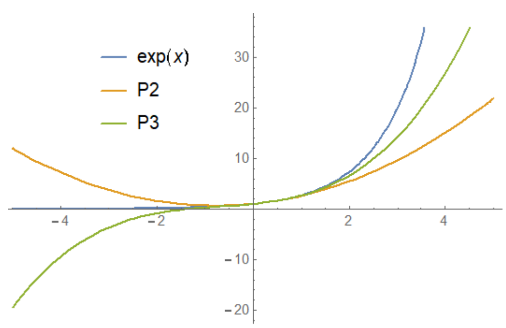
Specijalno, za $x = 0.3$:

$$f(0.3) \approx P_3(0.3) = 1.3498,$$
$$|R_3(f;0.3)| = |f(0.3) - P_3(0.3)| \le \frac{1}{12}|0.3\cdot 0.1\cdot(-0.2)(-0.1)| = 0.0002.$$

Prava greška je

$$f(0.3) - P_2(0.3) = e^{0.3} - P_3(0.3) = 0.00006.$$





Zadatak 2. Aproksimirati funkciju $f(x) = e^x$ na segmentu $[0,\ 0.5]$ nekim od Njutnovih interpolacionih polinoma na osnovu podataka u tačkama
a) $x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.5;$
b) $x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.5,\ x_4 = 0.4.$

Rešenje: Pošto čvorovi nisu ekvidistantni, može se primeniti samo Njutnov interpolacioni polinom sa podeljenim razlikama.

a) Na osnovu podataka

k	0	1	2
x_k	0.0	0.2	0.5
f_k	1.0000	1.2214	1.6487

formiramo tablicu podeljenih razlika:

$$
\begin{array}{llll}
x_k & f_k & [\cdot,\cdot;f] & [\cdot,\cdot,\cdot;f] \\
\hline\hline
x_0 & \boxed{f_0} & & \\
 & & \boxed{[x_0,x_1;f]} & \\
x_1 & f_1 & & \boxed{[x_0,x_1,x_2;f]} \\
 & & [x_1,x_2;f] & \\
x_2 & f_2 & &
\end{array}
$$

Računamo:

$$
\begin{aligned}
[x_0,x_1;f] &= \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{1.2214-1.0000}{0.2-0.0} = 1.1070,\\
[x_1,x_2;f] &= \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{1.6487-1.2214}{0.5-0.2} = 1.4244,\\
[x_0,x_1,x_2;f] &= \frac{[x_1,x_2;f]-[x_0,x_1;f]}{x_2-x_0} = \frac{1.4244-1.1070}{0.5-0.0} = 0.6348,
\end{aligned}
$$

x_k	f_k	$[\cdot,\cdot;f]$	$[\cdot,\cdot,\cdot;f]$
0.0	$\boxed{1.0000}$		
		$\boxed{1.1070}$	
0.2	1.2214		$\boxed{0.6348}$
		1.4244	
0.5	1.6487		

Njutnov interpolacioni polinom sa podeljenim razlikama za zadate čvorove je

$$
\begin{aligned}
P_2(x) &= f_0 + (x-x_0)[x_0,x_1;f] + (x-x_0)(x-x_1)[x_0,x_1,x_2;f] \\
&= 1 + 1.1070(x-0.0) + 0.6348(x-0.0)(x-0.2) \\
&= 0.6348x^2 + 0.9801x + 1.
\end{aligned}
$$

b) Dodavanjem još jednog čvora

$$x_3 = 0.4, \quad f_3 = 1.4918,$$

dobija se interpolacioni polinom stepena 3

$$
\begin{aligned}
P_3(x) &= f_0 + (x-x_0)[x_0,x_1;f] + (x-x_0)(x-x_1)[x_0,x_1,x_2;f] \\
&\quad + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3;f] \\
&= P_2(x) + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3;f].
\end{aligned}
$$

Nova tablica podeljenih razlika je

x_k	f_k	$[\cdot,\cdot;f]$	$[\cdot,\cdot,\cdot;f]$	$[\cdot,\cdot,\cdot,\cdot;f]$
0.0	$\boxed{1.0000}$			
		$\boxed{1.1070}$		
0.2	1.2214		$\boxed{0.6348}$	
		1.4244		$\boxed{0.2202}$
0.5	1.6487		0.7228	
		1.5690		
0.4	1.4918			

$$
\begin{aligned}
P_3(x) &= P_2(x) + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3;f] \\
&= 1 + 1.1070x + 0.6348x(x-0.2) + 0.2202x(x-0.2)(x-0.5) \\
&= 0.2202x^3 + 0.4806x^2 + 1.0021x + 1.
\end{aligned}
$$

Zadatak 3. Formirati prvi i drugi Njutnov interpolacioni polinom za funkciju zadatu podacima

k	0	1	2	3
x_k	0.4	0.6	0.8	1.0
f_k	10	7	6	4

,

a zatim približno izračunati $f(0.55)$ i $f(0.85)$.

Rešenje: S obzirom na broj čvorova, formramo interpolacioni polinom stepena 3. Rastojanje između čvorova je $h = 0.2$.

Prvi Njutnov interpolacioni polinom:

$$P_3^{(I)}(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 f_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2).$$

Drugi Njutnov interpolacioni polinom:

$$P_3^{(II)}(x) = f_n + \frac{\nabla f_3}{h}(x - x_3) + \frac{\nabla^2 f_3}{2!h^2}(x - x_3)(x - x_2) + \frac{\nabla^3 f_3}{3!h^3}(x - x_3)(x - x_2)(x - x_1).$$

Kako je $\nabla^j f_k = \Delta^j f_{n-j}, \quad k = 0, 1, \ldots, n,$ i prednje i zadnje razlike čitaju se iz iste tablice:

$$\begin{array}{llllll}
x_0 & \boxed{f_0} & & & & \\
 & & \boxed{\Delta f_0} = \nabla f_1 & & & \\
x_1 & f_1 & & \boxed{\Delta^2 f_0} = \nabla^2 f_2 & & \\
 & & \Delta f_1 = \nabla f_2 & & \boxed{\Delta^3 f_0} = \boxed{\nabla^3 f_3} & \\
x_2 & f_2 & & \Delta^2 f_1 = \boxed{\nabla^2 f_3} & & \\
 & & \Delta f_2 = \boxed{\nabla f_3} & & & \\
x_3 & \boxed{f_3} & & & &
\end{array}$$

		f_k	Δ,∇	Δ^2,∇^2	Δ^3,∇^3
x_0	0.4	10			
			-3		
x_1	0.6	7		2	
			-1		-3
x_2	0.8	6		-1	
			-2		
x_3	1.0	4			

Prvi Njutnov interpolacioni polinom:

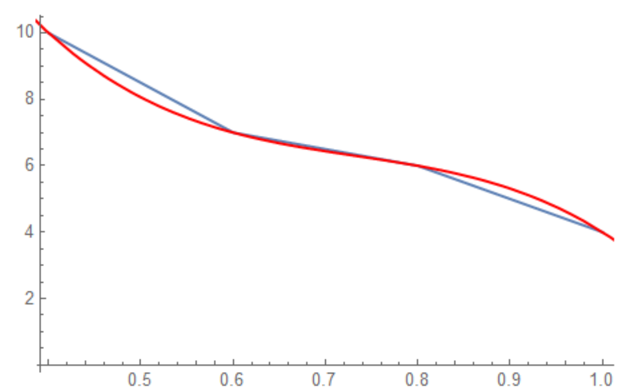
$$\begin{aligned} P_3^{(I)}(x) &= f_0 + \frac{\Delta f_0}{h}(x-x_0) + \frac{\Delta^2 f_0}{2!h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 f_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) \\ &= 10 + \frac{-3}{0.2}(x-0.4) + \frac{2}{2\cdot 0.2^2}(x-0.4)(x-0.6) + \frac{-3}{6\cdot 0.2^3}(x-0.4)(x-0.6)(x-0.8) \\ &= 34. - 105.x + 137.5x^2 - 62.5x^3 \end{aligned}$$

Drugi Njutnov interpolacioni polinom:

$$\begin{aligned} P_3^{(II)}(x) &= f_n + \frac{\nabla f_3}{h}(x-x_3) + \frac{\nabla^2 f_3}{2!h^2}(x-x_3)(x-x_2)\frac{\nabla^3 f_3}{3!h^3}(x-x_3)(x-x_2)(x-x_1) \\ &= 4 + \frac{-2}{0.2}(x-1.0) + \frac{-1}{2\cdot 0.2^2}(x-1.0)(x-0.8)\frac{-3}{6\cdot 0.2^3}(x-1.0)(x-0.8)(x-0.6) \\ &= 34. - 105.x + 137.5x^2 - 62.5x^3. \end{aligned}$$

$$f(0.55) \approx P_3^{(I)}(0.55) = 7.4453, \quad f(0.85) \approx P_3^{(II)}(0.85) = 5.7109.$$

Greška ne može da se proceni, jer ne postoji analitički zapis funkcije.



Zadatak 4. Funkcija $f(x) = \mathrm{erf}\,(x)$ zadata je sledećim vrednostima na segmentu $[0, 2]$:

k	0	1	2	3	4	5
x_k	0.0	0.2	0.4	0.6	0.8	1.0
f_k	0.	0.222703	0.428932	0.603856	0.742101	0.842701

k	6	7	8	9	10
x_k	1.2	1.4	1.6	1.8	2.0
f_k	0.910314	0.952285	0.976348	0.989091	0.995322

Primenom odgovarajućeg interpolacionog polinoma stepena 3 odrediti $f(0.25)$ i $f(1.85)$ i proceniti grešku.

Napomena: Funkcija f (x) =erf(x) spada u specijalne funkcije i ima veliku primenu u tehnici. Definisana je na sledeći način:

$$f\,(x) = erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

Rešenje:

Pošto su čvorovi ekvidistantni, pogodni su prvi *i* drugi Njutnov interpolacioni polinom.

Prvi se koristi za početak tablice $f\,(0.25)$

Drugi se koristi za kraj tablice $f\,(1.85)$

Prvi Njutnov interpolacioni polinom:

$$PI_3(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 f_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2)$$

Greska: $$RI_3(x) = \frac{\max|\Delta^4 f|}{4!\,h^4}(x - x0)(x - x1)(x - x2)(x - x3).$$

Drugi Njutnov interpolacioni polinom:

$$PII_3(x) = f_{10} + \frac{\nabla f_{10}}{h}(x - x_{10}) + \frac{\nabla^2 f_{10}}{2!\,h^2}(x - x_{10})(x - x_9) + \frac{\nabla^3 f_{10}}{3!\,h^3}(x - x_{10})(x - x_9)(x - x_8)$$

Greska: $$RII_3(x) = \frac{\max|\Delta^4 f|}{4!\,h^4}(x - x10)(x - x9)(x - x8)(x - x7).$$

Čvorovi i prednje i zadnje razlike:

$$\begin{pmatrix} 0. \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1. \\ 1.2 \\ 1.4 \\ 1.6 \\ 1.8 \\ 2. \end{pmatrix} \begin{pmatrix} 0. & 0.222703 & -0.0170128 & -0.0132132 & 0.00622037 \\ 0.222703 & 0.20569 & -0.030226 & -0.00699283 & 0.00656665 \\ 0.428392 & 0.175464 & -0.0372189 & -0.000426184 & 0.00508459 \\ 0.603856 & 0.138245 & -0.037645 & 0.0046584 & 0.0026862 \\ 0.742101 & 0.1006 & -0.0329866 & 0.0073446 & 0.000389567 \\ 0.842701 & 0.0676132 & -0.025642 & 0.00773417 & -0.00114743 \\ 0.910314 & 0.0419711 & -0.0179079 & 0.00658673 & -0.00177594 \\ 0.952285 & 0.0240633 & -0.0113211 & 0.00481079 & 0 \\ 0.976348 & 0.0127421 & -0.00651035 & 0 & 0 \\ 0.989091 & 0.00623176 & 0 & 0 & 0 \\ 0.995322 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Prvi Njutnov interpolacioni polinom :

$$PI_3(x) = 0. + \frac{0.222703\ (0. + x)}{h} - \frac{0.00850641\ (-0.2 + x)\ (0. + x)}{h^2} - \frac{0.0022022\ (-0.4 + x)\ (-0.2 + x)\ (0. + x)}{h^3}$$

$$PI_3(x) = 0. + 1.13402\, x - 0.0474952\, x^2 - 0.275275\, x^3$$

$$PI_3(0.25) = 0.276236$$

$$RI_3(x) \le \frac{M\,\mathrm{Abs}[(-0.6 + x)\ (-0.4 + x)\ (-0.2 + x)\ (0. + x)]}{24\, h^4}, \qquad M = \frac{\max|\Delta^4 f|}{4!\, h^4} = 0.00656665$$

$$RI_3(0.25) \le 0.000112223$$

Drugi Njutnov interpolacioni polinom :

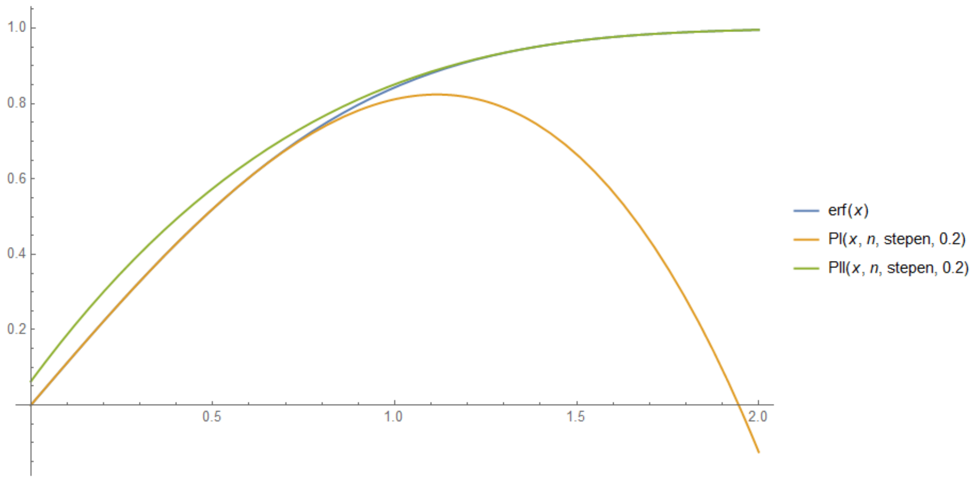
$$PII_3(x) = 0.995322 + \frac{0.00623176\ (-2. + x)}{h} - \frac{0.00325518\ (-2. + x)\ (-1.8 + x)}{h^2} + \frac{0.000801798\ (-2. + x)\ (-1.8 + x)\ (-1.6 + x)}{h^3}$$

$$PII_3(x) = 0.0627438 + 1.31058\, x - 0.622593\, x^2 + 0.100225\, x^3$$

$$PII_3(1.85) = 0.991071$$

$$RII_3(x) \le \frac{M\,\mathrm{Abs}[(-2. + x)\ (-1.8 + x)\ (-1.6 + x)\ (-1.4 + x)]}{24\, h^4}, \qquad M = \frac{\max|\Delta^4 f|}{4!\, h^4} = 0.00656665$$

$$RII_3(1.85) \le 0.000144287$$



6. zadatak

Na osnovu tri vrednosti funkcije $f(x)$: $f(a), f(b), f(c)$ ($f(a) < f(b) < f(c)$) u blizini njene nule naći približno rešenje jednačine

$$f(x)=0.$$

Rešenje

Tabela vrednosti:

x	a	b	c
f(x)	f(a)	f(b)	f(c)

Zbog $f(a) < f(b) < f(c)$ ocekuje se da je funkcija $y = f(x)$ monotona, pa ima inverznu funkciju $x = f^{-1}(y)$.

$$f(x^*) = 0 \quad \Rightarrow \quad x^* = f^{-1}(0)$$

Interpolacioni polinom za $x = f^{-1}(y)$:

y	$f(a)$	$f(b)$	$f(c)$
$f^{-1}(y)$	a	b	c

$$P(y) = a\frac{(y-f(b))(y-f(c))}{(f(a)-f(b))(f(a)-f(c))} + b\frac{(y-f(a))(y-f(c))}{(f(b)-f(a))(f(b)-f(c))} + c\frac{(y-f(a))(y-f(b))}{(f(c)-f(a))(f(c)-f(b))};$$

$$x^* = f^{-1}(0) \approx P(0) = a\frac{(0-f(b))(0-f(c))}{(f(a)-f(b))(f(a)-f(c))} + b\frac{(0-f(a))(0-f(c))}{(f(b)-f(a))(f(b)-f(c))} + c\frac{(0-f(a))(0-f(b))}{(f(c)-f(a))(f(c)-f(b))};$$

$$x^* \approx \frac{a\,f(b)f(c)}{(f(a)-f(b))(f(a)-f(c))} + \frac{b\,f(a)f(c)}{(f(b)-f(a))(f(b)-f(c))} + \frac{c\,f(a)f(b)}{(f(c)-f(a))(f(c)-f(b))}.$$

7. zadatak

Na osnovu vrednosti funkcije y=f(x)

$$\begin{pmatrix} x_k \\ f_k \end{pmatrix} = \begin{pmatrix} 0.4 & 0.6 & 0.8 & 1. \\ 10 & 7 & 6 & 4 \end{pmatrix}$$

naći približno rešenje jednačine *f(x)=5*.

Rešenje

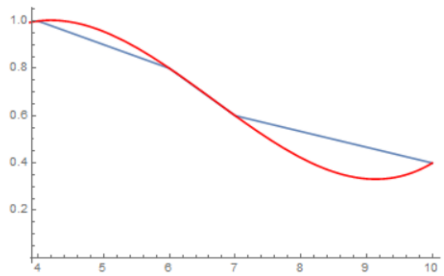
I nacin: Interpolacioni polinom inverzne funkcije; $f(x)=5 \Rightarrow x=f^{-1}(5)$.

čvorovi i podeljene razlike: $\begin{pmatrix} 10 \\ 7 \\ 6 \\ 4 \end{pmatrix} \begin{pmatrix} 0.4 & -0.0666667 & 0.0333333 & 0.0111111 \\ 0.6 & -0.2 & -0.0333333 & 0 \\ 0.8 & -0.1 & 0 & 0 \\ 1. & 0 & 0 & 0 \end{pmatrix}$

$P_3(x) = 0.4 - 0.0666667\,(-10+x) + 0.0333333\,(-10+x)\,(-7+x) + 0.0111111\,(-10+x)\,(-7+x)\,(-6+x)$

$\mathbf{P_3(x) = -1.26667 + 1.27778\,x - 0.222222\,x^2 + 0.0111111\,x^3}$

$\mathbf{x^* = f^{-1}(5) = P_3(5) = 0.955556}$



II nacin: Interpolacioni polinom funkcije; $y=f(x)\approx P(x)$, pa rešavanje jednacine $P(x)=5$.

$$P(x) = 34 - 105\,x + 137.5\,x^2 - 62.5\,x^3 = 5;$$
$$29 - 105\,x + 137.5\,x^2 - 62.5\,x^3 = 0;$$

početna vrednost: x(0)=10.

iteracija 1:	x(1)=6.90959	greška:	3.09041
iteracija 2:	x(2)=4.84862	greška:	2.06097
iteracija 3:	x(3)=3.47367	greška:	1.37495
iteracija 4:	x(4)=2.55575	greška:	0.917919
iteracija 5:	x(5)=1.94227	greška:	0.613481
iteracija 6:	x(6)=1.53188	greška:	0.410391
iteracija 7:	x(7)=1.25825	greška:	0.273634
iteracija 8:	x(8)=1.08038	greška:	0.17787
iteracija 9:	x(9)=0.977119	greška:	0.10326
iteracija 10:	x(10)=0.936295	greška:	0.0408234
iteracija 11:	x(11)=0.930246	greška:	0.00604902
iteracija 12:	x(12)=0.930126	greška:	0.000120542
iteracija 13:	x(13)=0.930126	greška:	4.69349×10^{-8}

rešenje: $\mathbf{x^{(1)} \approx x(13) = 0.930126}$