5. zadatak

Dat je sistem jednačina

$$x1 = 0.5\, x1 + x2 + 2,$$
$$x2 = -1.25\, x1 - 1.5\, x2.$$

Ako je moguće rešiti ga metodom proste iteracije, odrediti prva tri člana iterativnog niza.

Rešenje

Sistem može da se predstavi u obliku $x = Bx + \beta$, gde je

$$B = \begin{pmatrix} 0.5 & 1 \\ -1.25 & -1.5 \end{pmatrix}, \quad \beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Dovoljan uslov za konvergenciju metoda proste iteracije je $[\![B]\!] < 1$,
gde je $[\![B]\!]$ proizvoljna **norma** matrice B.

$$[\![B]\!]_1 = \max\{0.5 + 1.25,\ 1 + 1.5\} = 2.5 > 1$$
$$[\![B]\!]_2 = \sqrt{0.5^2 + 1.25^2 + 1^2 + 1.5^2} = 2.25 > 1$$
$$[\![B]\!]_\infty = \max\{0.5 + 1,\ 1.25 + 1.5\} = 2.75 > 1$$

Dovoljan uslov za konvergenciju nije ispunjen, pa ispitujemo sopstvene vrednosti matrice B.

$$P(\lambda) = \det\begin{pmatrix} 0.5 - \lambda & 1 \\ -1.25 & -1.5 - \lambda \end{pmatrix} = \lambda^2 + \lambda + 0.5,$$

$P(\lambda) = 0$ za $\lambda_{1,2} = -0.5 \pm 0.5\, i$, $|\lambda_{1,2}| = 0.7 < 1$ $\Rightarrow$ metod proste iteracije konvergira.

Primena metoda :

```
b = {2, 0};
B = {{0.5, 1}, {-1.25, -1.5}};
x0 = b;
x1 = B.x0 + b;
Print["x0=", MatrixForm[x0], ", x1=",
  MatrixForm[x1], "   greska:   ", Norm[x1 - x0, Infinity]];
```

x0= $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, x1= $\begin{pmatrix} 3. \\ -2.5 \end{pmatrix}$ greska: 2.5

```
x0 = x1;
x1 = B.x0 + b;
Print["          x2=", MatrixForm[x1], "   greska:   ", Norm[x1 - x0, Infinity]];
```

x2= $\begin{pmatrix} 1. \\ 0. \end{pmatrix}$ greska: 2.5

```
x0 = x1;
x1 = B.x0 + b;
Print["          x3=", MatrixForm[x1], "   greska:   ", Norm[x1 - x0, Infinity]];
```

x3= $\begin{pmatrix} 2.5 \\ -1.25 \end{pmatrix}$ greska: 1.5

Posle 20 iteracija :

$$x20 = \begin{pmatrix} 2. \\ -1.00098 \end{pmatrix}, \quad [\![x20 - x19]\!]_\infty = 0.002.$$

6. zadatak

Jakobijevim metodom rešiti sistem jednačina

$$4x1 + 4x2 + x3 = 4,$$
$$2x1 + 4x2 - x3 = 6,$$
$$x1 + 2x2 + 4x3 = -6$$

sa tačnošću 10^{-3}.

Rešenje

$$\begin{pmatrix} 4.\,x1 + 4.\,x2 + 1.\,x3 \\ 2.\,x1 + 4.\,x2 - 1.\,x3 \\ 1.\,x1 + 2.\,x2 + 4.\,x3 \end{pmatrix} = \begin{pmatrix} 4. \\ 6. \\ -6. \end{pmatrix}$$

$$\begin{pmatrix} 4.\,\lambda & 4. & 1. \\ 2. & 4.\,\lambda & -1. \\ 1. & 2. & 4.\,\lambda \end{pmatrix}, \qquad P(\lambda) = 0. - 28.\,\lambda + 64.\,\lambda^3$$

$\{\{\lambda \to -0.661438\}, \{\lambda \to 0.\}, \{\lambda \to 0.661438\}\}$

Kako je $|\lambda_{1,2,3}| < 1$, ispunjen je uslov za konvergenciju metoda.

iterativna matrica: $B = \begin{pmatrix} 0. & -1. & -0.25 \\ -0.5 & 0. & 0.25 \\ -0.25 & -0.5 & 0. \end{pmatrix}$

$$Ax = b \quad \longrightarrow \quad x = Bx + \beta$$

$$4\,x1 + 4\,x2 + x3 = 4$$
$$2\,x1 + 4\,x2 - x3 = 6$$
$$x1 + 2\,x2 + 4\,x3 = -6$$

$$x1 = \frac{1}{4}(4 - 4\,x2 - x3)$$
$$x2 = \frac{1}{4}(6 - 2\,x1 + x3)$$
$$x3 = \frac{1}{4}(-6 - x1 - 2\,x2)$$

$$x1^{(k+1)} = \frac{1}{4}\left(4 - 4\,x2^{(k)} - x3^{(k)}\right)$$
$$x2^{(k+1)} = \frac{1}{4}\left(6 - 2\,x1^{(k)} + x3^{(k)}\right)$$
$$x3^{(k+1)} = \frac{1}{4}\left(-6 - x1^{(k)} - 2\,x2^{(k)}\right)$$
$$\left(x1^{(0)}, x2^{(0)}, x3^{(0)}\right) = (1, 1.5, -1.5)$$

```
(* Primena metoda *)
beta = {1, 1.5, -1.5};
B = {{0, -1, -0.25}, {-0.5, 0, 0.25}, {-0.25, -0.5, 0}};
x0 = beta;
x1 = B.x0 + beta;
Print["x0=", MatrixForm[x0], ", x1=",
  MatrixForm[x1], "    greska:   ", Norm[x1 - x0, Infinity]];
```

$x0 = \begin{pmatrix} 1 \\ 1.5 \\ -1.5 \end{pmatrix}$, $x1 = \begin{pmatrix} -0.125 \\ 0.625 \\ -2.5 \end{pmatrix}$ greska: 1.125

```
x0 = x1;
x1 = B.x0 + beta;
Print["          x2=", MatrixForm[x1], "    greska:   ", Norm[x1 - x0, Infinity]];
```

$x2 = \begin{pmatrix} 1. \\ 0.9375 \\ -1.78125 \end{pmatrix}$ greska: 1.125

```
x0 = x1;
x1 = B.x0 + beta;
Print["          x3=", MatrixForm[x1], "   greska:  ", Norm[x1 - x0, Infinity]];
```

$$\text{x3=}\begin{pmatrix} 0.507813 \\ 0.554688 \\ -2.21875 \end{pmatrix} \quad \text{greska:} \quad 0.492188$$

```
(* Jacobijev metod *)
Clear["Global`*"];
A = {{4., 4., 1.}, {2., 4., -1.}, {1., 2., 4.}};
b = {4., 6., -6.};
epsilon = 10^-3;
n = Length[A];
razlika = 10; iter = 0;
B = -A;
For[i = 1, i ≤ n, i++, B[[i, i]] = 0; b[[i]] = b[[i]] / A[[i, i]];
  For[j = 1, j ≤ n, j++, B[[i, j]] = B[[i, j]] / A[[i, i]]]
 ];
x0 = b;
Print["Pocetna vrednost:  x(0)=", x0];
While[razlika ≥ epsilon ∧ iter ≤ 100,
  x1 = B.x0 + b;
  iter = iter + 1;
  razlika = Norm[x1 - x0, Infinity];
  Print["iteracija:  k=", iter, ",    x(k)=", x1, ",    |x(k)-x(k-1)|=", razlika];
  x0 = x1];
If[iter == 100, Print["Resenje nije pronadjeno u 100 iteracija."],
  Print["x≈x(", iter, ")=", x1]];
```

```
Pocetna vrednost:  x(0)={1., 1.5, -1.5}
iteracija:  k=1,   x(k)={-0.125, 0.625, -2.5},    |x(k)-x(k-1)|=1.125
iteracija:  k=2,   x(k)={1., 0.9375, -1.78125},    |x(k)-x(k-1)|=1.125
iteracija:  k=3,   x(k)={0.507813, 0.554688, -2.21875},    |x(k)-x(k-1)|=0.492188
iteracija:  k=4,   x(k)={1., 0.691406, -1.9043},    |x(k)-x(k-1)|=0.492188
iteracija:  k=5,   x(k)={0.784668, 0.523926, -2.0957},    |x(k)-x(k-1)|=0.215332
iteracija:  k=6,   x(k)={1., 0.58374, -1.95813},    |x(k)-x(k-1)|=0.215332
iteracija:  k=7,   x(k)={0.905792, 0.510468, -2.04187},    |x(k)-x(k-1)|=0.0942078
iteracija:  k=8,   x(k)={1., 0.536636, -1.98168},    |x(k)-x(k-1)|=0.0942078
iteracija:  k=9,   x(k)={0.958784, 0.50458, -2.01832},    |x(k)-x(k-1)|=0.0412159
iteracija:  k=10,   x(k)={1., 0.516028, -1.99199},    |x(k)-x(k-1)|=0.0412159
iteracija:  k=11,   x(k)={0.981968, 0.502004, -2.00801},    |x(k)-x(k-1)|=0.018032
iteracija:  k=12,   x(k)={1., 0.507012, -1.99649},    |x(k)-x(k-1)|=0.018032
iteracija:  k=13,   x(k)={0.992111, 0.500877, -2.00351},    |x(k)-x(k-1)|=0.00788898
iteracija:  k=14,   x(k)={1., 0.503068, -1.99847},    |x(k)-x(k-1)|=0.00788898
iteracija:  k=15,   x(k)={0.996549, 0.500383, -2.00153},    |x(k)-x(k-1)|=0.00345143
iteracija:  k=16,   x(k)={1., 0.501342, -1.99933},    |x(k)-x(k-1)|=0.00345143
iteracija:  k=17,   x(k)={0.99849, 0.500168, -2.00067},    |x(k)-x(k-1)|=0.00151
iteracija:  k=18,   x(k)={1., 0.500587, -1.99971},    |x(k)-x(k-1)|=0.00151
iteracija:  k=19,   x(k)={0.999339, 0.500073, -2.00029},    |x(k)-x(k-1)|=0.000660625
x≈x(19)={0.999339, 0.500073, -2.00029}
```

Provera :

```
Solve[{4 x + 4 y + z == 4, 2 x + 4 y - z == 6, x + 2 y + 4 z == -6}, {x, y, z}]
```

$$\left\{\left\{x \to 1,\ y \to \frac{1}{2},\ z \to -2\right\}\right\}$$

7. zadatak

Gaus-Zajdelovim metodom, varijanta Nekrasova, rešiti sistem jednačina

$$\begin{aligned} 4\,x1 + 4\,x2 + x3 &= 4, \\ 2\,x1 + 4\,x2 - x3 &= 6, \\ x1 + 2\,x2 + 4\,x3 &= -6 \end{aligned}$$

sa tačnošću 10^{-3}.

Rešenje

Konvergencija :

$$P(\lambda) = \det \begin{pmatrix} 4\lambda & 4 & 1 \\ 2\lambda & 4\lambda & -1 \\ \lambda & 2\lambda & 4\lambda \end{pmatrix} = \lambda\left(-4 - 24\lambda + 64\lambda^2\right);$$

$P(\lambda) = 0$ za $\lambda_1 = 0$, $\lambda_2 = 0.5$, $\lambda_3 = -0.125$.

Kako je $|\lambda_{1,2,3}| < 1$, ispunjen je uslov konvergencije.

$$x1 = \frac{1}{4}(4 - 4\,x2 - x3),$$
$$x2 = \frac{1}{4}(6 - 2\,x1 + x3),$$
$$x3 = \frac{1}{4}(-6 - x1 - 2\,x2).$$

Formiranje metoda :

$$x1^{(k+1)} = \frac{1}{4}\left(4 - 4\,x2^{(k)} - x3^{(k)}\right)$$
$$x2^{(k+1)} = \frac{1}{4}\left(6 - 2\,x1^{(k+1)} + x3^{(k)}\right)$$
$$x3^{(k+1)} = \frac{1}{4}\left(-6 - x1^{(k+1)} - 2\,x2^{(k+1)}\right)$$
$$\left(x1^{(0)}, x2^{(0)}, x3^{(0)}\right) = (1, 1.5, -1.5)$$

Primena metoda :

```
(* Gaus-Zajdelov metod *)
Clear["Global`*"];
A = {{4., 4., 1.}, {2., 4., -1.}, {1., 2., 4.}};
b = {4., 6., -6.};
epsilon = 10^-3;
n = Length[A];
razlika = 10; iter = 0;
B = -A;
For[i = 1, i ≤ n, i++, B[[i, i]] = 0; b[[i]] = b[[i]]/A[[i, i]];
  For[j = 1, j ≤ n, j++, B[[i, j]] = B[[i, j]]/A[[i, i]]]
 ];
x0 = b;
Print["Pocetna vrednost:  x(0)=", x0];
While[razlika ≥ epsilon ∧ iter ≤ 100,
  x1 = x0;
  For[i = 1, i ≤ n, i++,
   x1[[i]] = B[[i]].x1 + b[[i]];
  ];
  iter = iter + 1;
  razlika = Norm[x1 - x0, Infinity];
  Print["iteracija:  k=", iter, ",   x(k)=", x1, ",   |x(k)-x(k-1)|=", razlika];
  x0 = x1];
If[iter == 100, Print["Resenje nije pronadjeno u 100 iteracija."],
  Print["x≈x(", iter, ")=", x1]];
```

```
Pocetna vrednost:  x(0)={1., 1.5, -1.5}
iteracija:  k=1,   x(k)={-0.125, 1.1875, -2.0625},   |x(k)-x(k-1)|=1.125
iteracija:  k=2,   x(k)={0.328125, 0.820313, -1.99219},   |x(k)-x(k-1)|=0.453125
iteracija:  k=3,   x(k)={0.677734, 0.663086, -2.00098},   |x(k)-x(k-1)|=0.349609
iteracija:  k=4,   x(k)={0.837158, 0.581177, -1.99988},   |x(k)-x(k-1)|=0.159424
iteracija:  k=5,   x(k)={0.918793, 0.540634, -2.00002},   |x(k)-x(k-1)|=0.0816345
iteracija:  k=6,   x(k)={0.95937, 0.520311, -2.},   |x(k)-x(k-1)|=0.0405769
iteracija:  k=7,   x(k)={0.979688, 0.510156, -2.},   |x(k)-x(k-1)|=0.0203185
iteracija:  k=8,   x(k)={0.989844, 0.505078, -2.},   |x(k)-x(k-1)|=0.0101555
iteracija:  k=9,   x(k)={0.994922, 0.502539, -2.},   |x(k)-x(k-1)|=0.00507822
iteracija:  k=10,   x(k)={0.997461, 0.50127, -2.},   |x(k)-x(k-1)|=0.00253905
iteracija:  k=11,   x(k)={0.99873, 0.500635, -2.},   |x(k)-x(k-1)|=0.00126953
iteracija:  k=12,   x(k)={0.999365, 0.500317, -2.},   |x(k)-x(k-1)|=0.000634765
x≈x(12)={0.999365, 0.500317, -2.}
```

MATEMATIČKI METODI

Zadaci sa računskih vežbi

Rešeni zadaci

Numerički metodi u linearnoj algebri

8. zadatak

Ispitati da li Jakobijev i Gaus-Zajdelovim metod, varijanta Nekrasova, mogu da se primene za rešavanje sistema jednačina

$$\begin{aligned} 7x_1 - 2x_2 + x_3 + 2x_4 &= 3,\\ 2x_1 + 8x_2 + 3x_3 + x_4 &= -2,\\ -x_1 + 5x_3 + 2x_4 &= 5,\\ 2x_2 - x_3 + 4x_4 &= 4, \end{aligned}$$

a zatim rešiti sistem sa tačnošću 10^{-3}. primenom svakog od metoda.

Rešenje

$$A = \begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix}$$

Jakobijev metod i metod Gaus – Zajdela konvergiraju ako je matrica sistema dijagonalno – dominantna.

Matrica sistema je dijagonalno dominantna, jer za dijagonalni elemente vazi :

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{4} |a_{ij}|,$$

$|a_{11}| = 7 > |-2| + 1 + 2 = 5,\ |a_{22}| = 8 > 2 + 3 + 1 = 6,\ |a_{33}| = 5 > |-1| + 2 = 5,\ |a_{44}| = 4 > 2 + |-1| = 3$

ispunjen je uslov konvergencije.

$$x1 = \frac{1}{7}(3 + 2\,x2 - x3 - 2\,x4),$$
$$x2 = \frac{1}{8}(-2 - 2\,x1 - 3\,x3 - x4),$$
$$x3 = \frac{1}{5}(5 + x1 - 2\,x4),$$
$$x4 = \frac{1}{4}(4 - 2\,x2 + x3).$$

Formiranje J akobijevog metoda :

$$x1^{(k+1)} = \frac{1}{7}\left(3 + 2\,x2^{(k)} - x3^{(k)} - 2\,x4^{(k)}\right),$$
$$x2^{(k+1)} = \frac{1}{8}\left(-2 - 2\,x1^{(k)} - 3\,x3^{(k)} - x4^{(k)}\right),$$
$$x3^{(k+1)} = \frac{1}{5}\left(5 + x1^{(k)} - 2\,x4^{(k)}\right),$$
$$x4^{(k+1)} = \frac{1}{4}\left(4 - 2\,x2^{(k)} + x3^{(k)}\right).$$

Primena metoda :

```
(* Jacobijev metod *)
Clear["Global`*"];
A = {{7, -2, 1, 2}, {2, 8, 3, 1}, {-1, 0, 5, 2}, {0, 2, -1, 4}};
b = {3, -2, 5, 4.};
epsilon = 10^-3;
n = Length[A];
razlika = 10; iter = 0;
B = -A;
For[i = 1, i ≤ n, i++, B[[i, i]] = 0; b[[i]] = b[[i]] / A[[i, i]];
  For[j = 1, j ≤ n, j++, B[[i, j]] = B[[i, j]] / A[[i, i]]]
 ];
x0 = b;
Print["Pocetna vrednost:  x(0)=", x0];
While[razlika ≥ epsilon ∧ iter ≤ 100,
  x1 = B.x0 + b;
  iter = iter + 1;
  razlika = Norm[x1 - x0, Infinity];
  Print["iteracija:  k=", iter, ",    x(k)=", x1, ",    |x(k)-x(k-1)|=", razlika];
  x0 = x1];
If[iter == 100, Print["Resenje nije pronadjeno u 100 iteracija."],
  Print["x≈x(", iter, ")=", x1]];
```

Pocetna vrednost: $x(0)=\left\{\frac{3}{7}, -\frac{1}{4}, 1, 1.\right\}$

iteracija: k=1, x(k)={-0.0714286, -0.857143, 0.685714, 1.375}, |x(k)-x(k-1)|=0.607143

iteracija: k=2, x(k)={-0.307143, -0.661161, 0.435714, 1.6}, |x(k)-x(k-1)|=0.25

iteracija: k=3, x(k)={-0.279719, -0.536607, 0.298571, 1.43951}, |x(k)-x(k-1)|=0.160491

iteracija: k=4, x(k)={-0.178686, -0.471973, 0.368253, 1.34295}, |x(k)-x(k-1)|=0.101033

iteracija: k=5, x(k)={-0.142585, -0.511291, 0.427084, 1.32805}, |x(k)-x(k-1)|=0.0588316

iteracija: k=6, x(k)={-0.157967, -0.540517, 0.440263, 1.36242}, |x(k)-x(k-1)|=0.0343671

iteracija: k=7, x(k)={-0.178019, -0.545909, 0.42344, 1.38032}, |x(k)-x(k-1)|=0.020052

iteracija: k=8, x(k)={-0.182272, -0.536826, 0.412267, 1.37881}, |x(k)-x(k-1)|=0.0111733

iteracija: k=9, x(k)={-0.17765, -0.531384, 0.41202, 1.37148}, |x(k)-x(k-1)|=0.00733498

iteracija: k=10, x(k)={-0.173964, -0.53153, 0.415878, 1.3687}, |x(k)-x(k-1)|=0.00385854

iteracija: k=11, x(k)={-0.173762, -0.53355, 0.417729, 1.36973}, |x(k)-x(k-1)|=0.00202057

iteracija: k=12, x(k)={-0.1749, -0.534425, 0.417354, 1.37121}, |x(k)-x(k-1)|=0.00147286

iteracija: k=13, x(k)={-0.175517, -0.534184, 0.416537, 1.37155}, |x(k)-x(k-1)|=0.00081677

x≈x(13)={-0.175517, -0.534184, 0.416537, 1.37155}

Formiranje Gaus – Zajdelovog metoda :

$$\mathbf{x1}^{(k+1)} = \frac{1}{7}\left(3 + 2\,\mathbf{x2}^{(k)} - \mathbf{x3}^{(k)} - 2\,\mathbf{x4}^{(k)}\right),$$
$$\mathbf{x2}^{(k+1)} = \frac{1}{8}\left(-2 - 2\,\mathbf{x1}^{(k+1)} - 3\,\mathbf{x3}^{(k)} - \mathbf{x4}^{(k)}\right),$$
$$\mathbf{x3}^{(k+1)} = \frac{1}{5}\left(5 + \mathbf{x1}^{(k+1)} - 2\,\mathbf{x4}^{(k)}\right),$$
$$\mathbf{x4}^{(k+1)} = \frac{1}{4}\left(4 - 2\,\mathbf{x2}^{(k+1)} + \mathbf{x3}^{(k+1)}\right).$$

Primena metoda :

```
(* Gaus-Zajdelov metod *)
Clear["Global`*"];
A = {{7, -2, 1, 2}, {2, 8, 3, 1}, {-1, 0, 5, 2}, {0, 2, -1, 4}};
b = {3, -2, 5, 4.};
epsilon = 10^-3;
n = Length[A];
razlika = 10; iter = 0;
B = -A;
For[i = 1, i ≤ n, i++, B[[i, i]] = 0; b[[i]] = b[[i]]/A[[i, i]];
  For[j = 1, j ≤ n, j++, B[[i, j]] = B[[i, j]]/A[[i, i]]]
 ];
x0 = b;
Print["Pocetna vrednost:  x(0)=", x0];
While[razlika ≥ epsilon ∧ iter ≤ 100,
  x1 = x0;
  For[i = 1, i ≤ n, i++,
   x1[[i]] = B[[i]].x1 + b[[i]];
  ];
  iter = iter + 1;
  razlika = Norm[x1 - x0, Infinity];
  Print["iteracija:  k=", iter, ",   x(k)=", x1, ",   |x(k)-x(k-1)|=", razlika];
  x0 = x1];
If[iter == 100, Print["Resenje nije pronadjeno u 100 iteracija."],
  Print["x≈x(", iter, ")=", x1]];
```

Pocetna vrednost: $x(0)=\left\{\frac{3}{7}, -\frac{1}{4}, 1, 1.\right\}$

```
iteracija:  k=1,   x(k)={-0.0714286, -0.732143, 0.585714, 1.5125},   |x(k)-x(k-1)|=0.5125
iteracija:  k=2,   x(k)={-0.296429, -0.584598, 0.335714, 1.37623},   |x(k)-x(k-1)|=0.25
iteracija:  k=3,   x(k)={-0.179624, -0.503015, 0.413584, 1.3549},   |x(k)-x(k-1)|=0.116805
iteracija:  k=4,   x(k)={-0.161346, -0.534121, 0.425769, 1.3735},   |x(k)-x(k-1)|=0.0311051
iteracija:  k=5,   x(k)={-0.177288, -0.537029, 0.415141, 1.3723},   |x(k)-x(k-1)|=0.0159419
iteracija:  k=6,   x(k)={-0.176257, -0.533151, 0.415829, 1.37053},   |x(k)-x(k-1)|=0.00387809
iteracija:  k=7,   x(k)={-0.174742, -0.533567, 0.416838, 1.37099},   |x(k)-x(k-1)|=0.00151479
iteracija:  k=8,   x(k)={-0.175137, -0.533904, 0.416575, 1.3711},   |x(k)-x(k-1)|=0.000394463
x≈x(8)={-0.175137, -0.533904, 0.416575, 1.3711}
```

9. zadatak

Gausovim i Gaus-Žordanovim metodom odrediti inverznu matricu matrice

$$A = \begin{pmatrix} 4 & -2 & 0 & 0 \\ 2 & 4 & -2 & 0 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

Rešenje

$$[A \mid I] = \left(\begin{array}{cccc|cccc} 4 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(\text{druga vrsta}) - \frac{2}{4}(\text{prva vrsta})$

{2, 4, -2, 0, 0, 1, 0, 0} - 1/2 {4, -2, 0, 0, 1, 0, 0, 0}

$\left\{0, 5, -2, 0, -\frac{1}{2}, 1, 0, 0\right\}$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 4 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & 0 & -1/2 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(\text{prva vrsta}) - \frac{-2}{5}(\text{druga vrsta})$

$(\text{treca vrsta}) - \frac{2}{5}(\text{druga vrsta})$

{4, -2, 0, 0, 1, 0, 0, 0} - (-2/5) {0, 5, -2, 0, -1/2, 1, 0, 0}
{0, 2, 4, -2, 0, 0, 1, 0} - (2/5) {0, 5, -2, 0, -1/2, 1, 0, 0}

$\left\{4, 0, -\frac{4}{5}, 0, \frac{4}{5}, \frac{2}{5}, 0, 0\right\}$

$\left\{0, 0, \frac{24}{5}, -2, \frac{1}{5}, -\frac{2}{5}, 1, 0\right\}$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 4 & 0 & -4/5 & 0 & 4/5 & 2/5 & 0 & 0 \\ 0 & 5 & -2 & 0 & -1/2 & 1 & 0 & 0 \\ 0 & 0 & 24/5 & -2 & 1/5 & -2/5 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(\text{prva vrsta}) - \frac{-4/5}{24/5}(\text{treca vrsta})$

$(\text{druga vrsta}) - \frac{-2}{24/5}(\text{treca vrsta})$

$(\text{cetvrta vrsta}) - \frac{2}{24/5}(\text{treca vrsta})$

$$\left\{4, 0, -\frac{4}{5}, 0, \frac{4}{5}, \frac{2}{5}, 0, 0\right\} - \frac{-4/5}{24/5}\left\{0, 0, \frac{24}{5}, -2, \frac{1}{5}, -\frac{2}{5}, 1, 0\right\}$$

$$\{0, 5, -2, 0, -1/2, 1, 0, 0\} - \frac{-2}{24/5}\left\{0, 0, \frac{24}{5}, -2, \frac{1}{5}, -\frac{2}{5}, 1, 0\right\}$$

$$\{0, 0, 2, 4, 0, 0, 0, 1\} - \frac{2}{24/5}\left\{0, 0, \frac{24}{5}, -2, \frac{1}{5}, -\frac{2}{5}, 1, 0\right\}$$

$$\left\{4, 0, 0, -\frac{1}{3}, \frac{5}{6}, \frac{1}{3}, \frac{1}{6}, 0\right\}$$

$$\left\{0, 5, 0, -\frac{5}{6}, -\frac{5}{12}, \frac{5}{6}, \frac{5}{12}, 0\right\}$$

$$\left\{0, 0, 0, \frac{29}{6}, -\frac{1}{12}, \frac{1}{6}, -\frac{5}{12}, 1\right\}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 4 & 0 & 0 & -1/3 & 5/6 & 1/3 & 1/6 & 0 \\ 0 & 5 & 0 & -5/6 & -5/12 & 5/6 & 5/12 & 0 \\ 0 & 0 & 24/5 & -2 & 1/5 & -2/5 & 1 & 0 \\ 0 & 0 & 0 & 29/6 & -1/12 & 1/6 & -5/12 & 1 \end{array}\right)$$

$$(\text{prva vrsta}) - \frac{-1/3}{29/6}(\text{cetvrta vrsta})$$

$$(\text{druga vrsta}) - \frac{-5/6}{29/6}(\text{cetvrta vrsta})$$

$$(\text{treca vrsta}) - \frac{-2}{29/6}(\text{cetvrta vrsta})$$

$$\left\{4, 0, 0, -\frac{1}{3}, \frac{5}{6}, \frac{1}{3}, \frac{1}{6}, 0\right\} - \frac{-1/3}{29/6}\left\{0, 0, 0, \frac{29}{6}, -\frac{1}{12}, \frac{1}{6}, -\frac{5}{12}, 1\right\}$$

$$\left\{0, 5, 0, -\frac{5}{6}, -\frac{5}{12}, \frac{5}{6}, \frac{5}{12}, 0\right\} - \frac{-5/6}{29/6}\left\{0, 0, 0, \frac{29}{6}, -\frac{1}{12}, \frac{1}{6}, -\frac{5}{12}, 1\right\}$$

$$\left\{0, 0, \frac{24}{5}, -2, \frac{1}{5}, -\frac{2}{5}, 1, 0\right\} - \frac{-2}{29/6}\left\{0, 0, 0, \frac{29}{6}, -\frac{1}{12}, \frac{1}{6}, -\frac{5}{12}, 1\right\}$$

$$\left\{4, 0, 0, 0, \frac{24}{29}, \frac{10}{29}, \frac{4}{29}, \frac{2}{29}\right\}$$

$$\left\{0, 5, 0, 0, -\frac{25}{58}, \frac{25}{29}, \frac{10}{29}, \frac{5}{29}\right\}$$

$$\left\{0, 0, \frac{24}{5}, 0, \frac{24}{145}, -\frac{48}{145}, \frac{24}{29}, \frac{12}{29}\right\}$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 24/29 & 10/29 & 4/29 & 2/29 \\ 0 & 5 & 0 & 0 & -25/58 & 25/58 & 10/29 & 5/29 \\ 0 & 0 & 24/5 & 0 & 24/145 & -48/145 & 24/29 & 12/29 \\ 0 & 0 & 0 & 29/6 & -1/12 & 1/6 & -5/12 & 1 \end{array}\right)$$

(prva vrsta) * 1/4
(druga vrsta) * 1/5
(treca vrsta) * 5/24
(cetvrta vrsta) * 6/29

$(1/4)\ \{4, 0, 0, 0, \frac{24}{29}, \frac{10}{29}, \frac{4}{29}, \frac{2}{29}\}$

$(1/5)\ \{0, 5, 0, 0, -\frac{25}{58}, \frac{25}{29}, \frac{10}{29}, \frac{5}{29}\}$

$(5/24)\ \{0, 0, \frac{24}{5}, 0, \frac{24}{145}, -\frac{48}{145}, \frac{24}{29}, \frac{12}{29}\}$

$(6/29)\ \{0, 0, 0, \frac{29}{6}, -\frac{1}{12}, \frac{1}{6}, -\frac{5}{12}, 1\}$

$\{1, 0, 0, 0, \frac{6}{29}, \frac{5}{58}, \frac{1}{29}, \frac{1}{58}\}$

$\{0, 1, 0, 0, -\frac{5}{58}, \frac{5}{29}, \frac{2}{29}, \frac{1}{29}\}$

$\{0, 0, 1, 0, \frac{1}{29}, -\frac{2}{29}, \frac{5}{29}, \frac{5}{58}\}$

$\{0, 0, 0, 1, -\frac{1}{58}, \frac{1}{29}, -\frac{5}{58}, \frac{6}{29}\}$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6/29 & 5/58 & 1/29 & 1/58 \\ 0 & 1 & 0 & 0 & -5/58 & 5/29 & 2/29 & 1/29 \\ 0 & 0 & 1 & 0 & 1/29 & -2/29 & 5/29 & 5/58 \\ 0 & 0 & 0 & 1 & -1/58 & 1/29 & -5/58 & 6/29 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{6}{29} & \frac{5}{58} & \frac{1}{29} & \frac{1}{58} \\ -\frac{5}{58} & \frac{5}{29} & \frac{2}{29} & \frac{1}{29} \\ \frac{1}{29} & -\frac{2}{29} & \frac{5}{29} & \frac{5}{58} \\ -\frac{1}{58} & \frac{1}{29} & -\frac{5}{58} & \frac{6}{29} \end{pmatrix}$$

$X = \{\{\frac{6}{29}, \frac{5}{58}, \frac{1}{29}, \frac{1}{58}\}, \{-\frac{5}{58}, \frac{5}{29}, \frac{2}{29}, \frac{1}{29}\}, \{\frac{1}{29}, -\frac{2}{29}, \frac{5}{29}, \frac{5}{58}\}, \{-\frac{1}{58}, \frac{1}{29}, -\frac{5}{58}, \frac{6}{29}\}\};$

```
A = {{4, -2, 0, 0}, {2, 4, -2, 0}, {0, 2, 4, -2}, {0, 0, 2, 4}};
MatrixForm[A.X]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

10. zadatak

Šulcovim metodom sa tačnošću 10^{-2} odrediti inverznu matricu matrice

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

Rešenje

Šulcov metod :

$X_{k+1} = X_k\,(2\,I - AX_k\,)\quad k = 0,\ 1,\ 2,\ \ldots$
$X_0 :\quad [\![I - AX_0]\!] < 1$

Izbor startne tacke :

$$X_0 = \frac{1}{10}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix};\ AX_0 = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ 1 & 7 \end{pmatrix}$$

$$I - AX_0 = \begin{pmatrix} 0.7 & -0.1 \\ -0.1 & 0.3 \end{pmatrix},\ [\![I - AX_0]\!]_\infty = \max\{0.7 + 0.1,\ 0.1 + 0.3\} = 0.8 < 1$$

$\Rightarrow$ metod konvergira

Primena metoda :

```
Clear["Global`*"];
IM = {{1, 0}, {0, 1}};
A = {{1, 2}, {-3, 4}};
k = 0;
X0 = {{0.1, -0.1}, {0.1, 0.1}};
Print["pocetna vrednost:    X0=", MatrixForm[X0]];
X1 = X0.(2 IM - A.X0);
k = k + 1;
norma = Norm[X1 - X0];
Print["iteracija:   ", k, ",       Xk=", MatrixForm[X1], ",   greska:", norma];
```

pocetna vrednost: X0= $\begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}$

iteracija: 1,   Xk= $\begin{pmatrix} 0.18 & -0.14 \\ 0.16 & 0.12 \end{pmatrix}$,   greska:0.102333

```
X0 = X1;
X1 = X0.(2 IM - A.X0);
k = k + 1;
norma = Norm[X1 - X0];
Print["iteracija:   ", k, ",       Xk=", MatrixForm[X1], ",   greska:", norma];
```

iteracija: 2,   Xk= $\begin{pmatrix} 0.284 & -0.172 \\ 0.228 & 0.116 \end{pmatrix}$,   greska:0.127632

```
X0 = X1;
X1 = X0.(2 IM - A.X0);
k = k + 1;
norma = Norm[X1 - X0];
Print["iteracija:   ", k, ",      Xk=", MatrixForm[X1], ",   greska:", norma];
```

iteracija: 3,   Xk= $\begin{pmatrix} 0.36816 & -0.19248 \\ 0.28032 & 0.10464 \end{pmatrix}$,   greska:0.101821

```
X0 = X1;
X1 = X0.(2 IM - A.X0);
k = k + 1;
norma = Norm[X1 - X0];
Print["iteracija:   ", k, ",      Xk=", MatrixForm[X1], ",   greska:", norma];
```

iteracija: 4,   Xk= $\begin{pmatrix} 0.397607 & -0.199435 \\ 0.298521 & 0.100349 \end{pmatrix}$,   greska:0.0355691

```
X0 = X1;
X1 = X0.(2 IM - A.X0);
k = k + 1;
norma = Norm[X1 - X0];
Print["iteracija:   ", k, ",      Xk=", MatrixForm[X1], ",   greska:", norma];
```

iteracija: 5,   Xk= $\begin{pmatrix} 0.399986 & -0.199997 \\ 0.299992 & 0.100002 \end{pmatrix}$,   greska:0.00287455

```
Inverse[A] // MatrixForm
```

$$\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$$

```
In[551]:= Clear["Global`*"];
IM = {{1, 0}, {0, 1}};
A = {{1, 2}, {-3, 4}};
epsilon = 10^-2;
maxiter = 20; norma = epsilon + 1;
k = 0;
X0 = {{0.1, -0.1}, {0.1, 0.1}};
Print["pocetna vrednost:    X0=", MatrixForm[X0]];
While[norma > epsilon ∧ k ≤ maxiter,
  X1 = X0.(2 IM - A.X0);
  k = k + 1;
  norma = Norm[X1 - X0];
  Print["iteracija:   ", k, ",      Xk=", MatrixForm[X1], ",   greska:", norma];
  X0 = X1
 ];
Print["Približno rešenje:     A^-1≈", MatrixForm[X1]];
```

pocetna vrednost: $X0=\begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}$

iteracija: 1, $Xk=\begin{pmatrix} 0.18 & -0.14 \\ 0.16 & 0.12 \end{pmatrix}$, greska:0.102333

iteracija: 2, $Xk=\begin{pmatrix} 0.284 & -0.172 \\ 0.228 & 0.116 \end{pmatrix}$, greska:0.127632

iteracija: 3, $Xk=\begin{pmatrix} 0.36816 & -0.19248 \\ 0.28032 & 0.10464 \end{pmatrix}$, greska:0.101821

iteracija: 4, $Xk=\begin{pmatrix} 0.397607 & -0.199435 \\ 0.298521 & 0.100349 \end{pmatrix}$, greska:0.0355691

iteracija: 5, $Xk=\begin{pmatrix} 0.399986 & -0.199997 \\ 0.299992 & 0.100002 \end{pmatrix}$, greska:0.00287455

Približno rešenje: $A^{-1}\approx\begin{pmatrix} 0.399986 & -0.199997 \\ 0.299992 & 0.100002 \end{pmatrix}$

10. zadatak

Šulcovim metodom sa tačnošću 10^{-3} odrediti inverznu matricu matrice

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

Rešenje

In[541]:=
```
Clear["Global`*"];
IM = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
A = {{1., 2, 3}, {-3, 4, 1}, {2, 2, 2}};
epsilon = 10^-3;
maxiter = 20; norma = epsilon + 1;
k = 0;
X0 = 1/100. {{-4, -2, 6}, {-5, 3, 6}, {8, -2, -6}};
Print["A=", MatrixForm[A], ",    pocetna vrednost:     X0=", MatrixForm[X0]];
While[norma > epsilon ∧ k ≤ maxiter,
  X1 = X0.(2 IM - A.X0);
  k = k + 1;
  norma = N[Norm[X1 - X0, Infinity]];
  Print["iteracija:    ", k, ",       Xk=", MatrixForm[X1], ",    greska:", norma];
  X0 = X1
 ];
Print["Približno rešenje:     A^-1≈", MatrixForm[X1]];
```

$A=\begin{pmatrix} 1. & 2 & 3 \\ -3 & 4 & 1 \\ 2 & 2 & 2 \end{pmatrix}$, pocetna vrednost: $X0=\begin{pmatrix} -0.04 & -0.02 & 0.06 \\ -0.05 & 0.03 & 0.06 \\ 0.08 & -0.02 & -0.06 \end{pmatrix}$

iteracija: 1, $Xk=\begin{pmatrix} -0.0748 & -0.0364 & 0.1128 \\ -0.0938 & 0.0554 & 0.1128 \\ 0.1508 & -0.0364 & -0.1128 \end{pmatrix}$, greska:0.14

iteracija: 2, $Xk=\begin{pmatrix} -0.131372 & -0.0607614 & 0.200152 \\ -0.165762 & 0.0951514 & 0.200152 \\ 0.268932 & -0.0607614 & -0.200152 \end{pmatrix}$, greska:0.229846

iteracija: 3, $Xk=\begin{pmatrix} -0.206276 & -0.0873852 & 0.320183 \\ -0.263229 & 0.144338 & 0.320183 \\ 0.434089 & -0.0873852 & -0.320183 \end{pmatrix}$, greska:0.311811

iteracija: 4, $Xk=\begin{pmatrix} -0.272392 & -0.102113 & 0.435332 \\ -0.353862 & 0.183583 & 0.435332 \\ 0.598271 & -0.102113 & -0.435332 \end{pmatrix}$, greska:0.294058

iteracija: 5, $Xk=\begin{pmatrix} -0.298503 & -0.10166 & 0.491636 \\ -0.39507 & 0.198226 & 0.491636 \\ 0.684769 & -0.10166 & -0.491636 \end{pmatrix}$, greska:0.143255

iteracija: 6, $Xk=\begin{pmatrix} -0.300096 & -0.100076 & 0.49986 \\ -0.399978 & 0.199958 & 0.49986 \\ 0.699624 & -0.100076 & -0.49986 \end{pmatrix}$, greska:0.0246635

iteracija: 7, $Xk=\begin{pmatrix} -0.3 & -0.1 & 0.5 \\ -0.4 & 0.2 & 0.5 \\ 0.7 & -0.1 & -0.5 \end{pmatrix}$, greska:0.000591409

Približno rešenje: $A^{-1}\approx\begin{pmatrix} -0.3 & -0.1 & 0.5 \\ -0.4 & 0.2 & 0.5 \\ 0.7 & -0.1 & -0.5 \end{pmatrix}$