# Aproksimacija funkcija

### I Hermitov interpolacioni polinom

Zadati podaci:

- čvorovi $x_i, \ i = 0, 1, 2, \ldots, m$
- vrednosti funkcije $f(x_i), \ i = 0, 1, 2 \ldots, m$
- vrednosti izvoda $f'(x_i), \ f''(x_i), \ldots, f^{(k_i-1)}(x_i), \quad i = 0, 1, 2 \ldots, m$

  $k_i -$ višestrukost čvora $x_i$

*Hermitov interpolacioni polinom* je polinom stepena $n$ koji zadovoljava uslove

$$H_n^{(j)}(x_i) = f^{(j)}(x_i), \ j = 0, 1, \ldots, k_i - 1, \ i = 0, 1, \ldots, m.$$

Stepen $n$ polinoma $H_n(x)$ jednak je $\quad n = k_0 + k_1 + \cdots + k_m - 1.$

Oblik Hermitovog interpolacionog polinoma:

$$H_n(x) = P_m(x) + (x - x_0)(x - x_1) \cdots (x - x_m) H_k(x),$$

gde je $P_m(x)$ Lagranžov interpolacioni polinom.

### II Srednje–kvadratna aproksimacija

*Najbolja srednje–kvadratna aproksimacija* funkcije $f(x)$ u skupu polinoma stepena ne većeg od $n$ na intervalu $(a, b)$ sa težinom $p(x)$ je

$$Q^*(x) = a_0^* Q_0(x) + a_1^* Q_1(x) + \cdots + a_n^* Q_n(x),$$

gde su $\{Q_0(x), Q_1(x), \ldots, Q_n(x)\}$ polinomi ortogonalni u odnosu na skalarni proizvod

$$(\varphi, \psi) = \int_a^b p(x)\varphi(x)\psi(x)\, dx,$$

a koeficijenti su određeni sa $a_k^* = \dfrac{(f, Q_k)}{(Q_k, Q_k)}, \qquad k = 0, 1, \ldots, n.$

Veličina najbolje aproksimacije je

$$\|\delta_n^*\| = \sqrt{(f, f) - \sum_{k=0}^{n} \frac{(f, Q_k)^2}{(Q_k, Q_k)}}.$$

### III Gram–Šmitov postupak

Pospupak kojim se od linearno nezavisnog sistema polinoma $\{1, x, x^2, \ldots\}$ formira sistem ortogonalnih polinoma $\{Q_0(x),\ Q_1(x),\ Q_2(x), \ldots\}$.

$$\begin{aligned}
Q_0(x) &= 1,\\
Q_1(x) &= x - \frac{(x\ , Q_0)}{(Q_0, Q_0)} Q_0(x),\\
Q_2(x) &= x^2 - \frac{(x^2, Q_0)}{(Q_0, Q_0)} Q_0(x) - \frac{(x^2, Q_1)}{(Q_1, Q_1)} Q_1(x),\\
&\ \ \vdots
\end{aligned}$$

## ZADACI

**Zadatak 1.** Odrediti Hermitov interpolacioni polinom na osnovu podataka

| $x_k$ | $-1$ | $0$ | $2$ |
|---|---|---|---|
| $f(x_k)$ | $0$ | $-7$ | $3$ |
| $f'(x_k)$ | $-8$ | $-5$ | $55$ |
| $f''(x_k)$ | | $10$ | |

**Rešenje:**

Na osnovu $3+3+1=7$ datih podataka moze se formirati polinom $H_6(x)$ stepena $dg(H_6(x))=6$.
On se trazi u obliku

$$H_6(x) = P_2(x) + (x+1)\,x\,(x-2)\,H_3(x),$$

gde je: $P_2(x)$ interpolacioni polinom, npr. Lagranžov, sa zadatim čvorovima, ne uzimajući u obzir njihovu višestrukost;
$H_3(x)$ privremeno nepoznati polinom stepena
$dg(H_3(x)) = dg(H_6(x)) - dg((x+1)\,x\,(x-2)) = 6-3=3.$

$$P_2(x) = 0\,\frac{(x-0)(x-2)}{(-1-0)(-1-2)} - 7\,\frac{(x+1)(x-2)}{(0+1)(0-2)} + 3\,\frac{(x+1)(x-0)}{(2+1)(2-0)}$$
$$= \tfrac{7}{2}\left(x^2-x-2\right) + \tfrac{1}{2}\left(x^2+x\right) = 4x^2-3x-7,$$

$$H_6(x) = 4x^2-3x-7+\left(x^3-x^2-2x\right)H_3(x);$$
$$H_6(-1)=0, \quad H_6(0)=-7, \quad H_6(x)=3.$$

Diferenciranjem :

$$H'_6(x) = \left(4x^2-3x-7\right)' + \left(x^3-x^2-2x\right)' H_3(x) + \left(x^3-x^2-2x\right) H'_3(x)$$
$$= 8x-3+\left(3x^2-2x-2\right)H_3(x) + \left(x^3-x^2-2x\right)H'_3(x);$$

$$H''_6(x) = 8 + \left(3x^2-2x-2\right)' H_3(x) + \left(3x^2-2x-2\right) H'_3(x) + \left(x^3-x^2-2x\right)' H'_3(x) + \left(x^3-x^2-2x\right) H''_3(x)$$
$$= 8 + (6x-2)\,H_3(x) + 2\left(3x^2-2x-2\right)H'_3(x) + \left(x^3-x^2-2x\right)H''_3(x).$$

Zbog uslova $H'_6(x_i) = f'(x_i)$, $i=1,2,3$, $H''_6(0) = f''(0)$ vazi sledece :

$$H'_6(-1) = -8-3+(3+2-2)\,H_3(-1) = -8 \quad \Rightarrow \quad H_3(-1)=1;$$
$$H'_6(0) = -3+(-2)\,H_3(0) = -5 \quad \Rightarrow \quad H_3(0)=1;$$
$$H'_6(2) = 16-3+(12-4-2)\,H_3(-1) = 55 \quad \Rightarrow \quad H_3(-1)=7;$$

$$H''_6(0) = 8+(-2)\,H_3(0)+2\,(-2)\,H'_3(0) = 8-2-4\,H'_3(0) = 10 \quad \Rightarrow \quad H''_3(0) = -1;$$

Dobijen je skup podataka za odredjivanje polinoma $H_3(x)$ :

| $x$ | $-1$ | $0$ | $2$ |
|---|---|---|---|
| $H_3(x)$ | $1$ | $1$ | $7$ |
| $H'_3(x)$ | | $-1$ | |

Odredjujemo : $$H_3(x) = P_2^{\,1}(x) + (x+1)\,x\,(x-2)\,H_0(x),$$
gde je : $P_2^{\,1}(x)$ interpolacioni polinom, npr. Lagranžov, sa zadatim čvorovima, ne uzimajući u obzir njihovu višestrukost;

$H_0(x) = C$ privremeno nepoznati polinom stepena

$$dg\,(H_0(x)) = dg\,(H_3(x)) - dg\,((x+1)\,x\,(x-2)) = 3-3 = 0.$$

$$P_2^{\,1}(x) = 1\,\frac{(x-0)(x-2)}{(-1-0)(-1-2)} + 1\,\frac{(x+1)(x-2)}{(0+1)(0-2)} + 7\,\frac{(x+1)(x-0)}{(2+1)(2-0)}$$
$$= \tfrac{1}{3}\left(x^2-2x\right) - \tfrac{1}{2}\left(x^2-x-2\right) + \tfrac{7}{6}\left(x^2+x\right) = x^2+x+1,$$

$$H_3(x) = x^2+x+1+\left(x^3-x^2-2x\right)C;$$
$$H_3(-1)=1,\quad H_3(0)=1,\quad H_3(2)=7.$$

Diferenciranjem :

$$H'_3(x) = \left(x^2+x+1\right)' + \left(x^3-x^2-2x\right)'C = 2x+1+\left(3x^2-2x-2\right)C;$$
$$H'_3(0) = 1+(-2)\,C = -1 \quad\Rightarrow\quad C = 1.$$

$$H_3(x) = x^2+x+1+\left(x^3-x^2-2x\right) = x^3-x+1.$$

Konačni oblik polinoma $H_6(x)$ je :

$$H_6(x) = 4x^2-3x-7+\left(x^3-x^2-2x\right)H_3(x)$$
$$= 4x^2-3x-7+\left(x^3-x^2-2x\right)\left(x^3-x+1\right)$$
$$= x^6-x^5-3x^4+2x^3+5x^2-5x-7.$$

**Zadatak 2.** Odrediti Hermitov interpolacioni polinom na osnovu podataka

| $x_k$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(x_k)$ | $a$ | $b$ | $c$ |
| $f'(x_k)$ | $A$ | $B$ | $C$ |

**Rešenje:**

Na osnovu 6 datih podataka može se formirati polinom stepena 5 :

$$H_5(x) = P_2(x) + (x-1)(x-2)(x-3)\,H_2(x),$$

gde je : $P_2(x)$ interpolacioni polinom, npr. Lagranžov, sa zadatim čvorovima, ne uzimajući u obzir njihovu višestrukost;

$H_2(x)$ privremeno nepoznati polinom stepena

$$\mathrm{dg}(H_2(x)) = d\,g(H_5(x)) - \mathrm{dg}((x-1)(x-2)(x-3)) = 5-3 = 2.$$

$$P_2(x) = a\,\frac{(x-2)(x-3)}{(1-2)(1-3)} + b\,\frac{(x-2)(x-3)}{(2-1)(2-3)} + c\,\frac{(x-1)(x-2)}{(3-1)(3-2)}$$
$$= \left(\tfrac{a}{2} - b + \tfrac{c}{2}\right)x^2 + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right)x + (3a - 6b + c),$$

$$H_5(x) = \left(\tfrac{a}{2} - b + \tfrac{c}{2}\right)x^2 + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right)x + (3a-6b+c) + \left(x^3 - \mathbf{6}\,x^2 + \mathbf{11}\,x - \mathbf{6}\right)H_2(x).$$

Diferenciranjem :

$$H'_5(x) = 2\left(\tfrac{a}{2} - b + \tfrac{c}{2}\right)x + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right) + \left(3x^2 - 6x + 11\right)H_2(x) + \left(x^3 - 6x^2 + 11x - 6\right)H'_2(x).$$

Zbog uslova $H'_5(x_i) = f'(x_i)$, $i = 1, 2, 3$, važi sledeće :

$$H'_5(1) = 2\left(\tfrac{a}{2} - b + \tfrac{c}{2}\right) + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right) + (3-6+11)\,H_2(1) = A$$
$$\Rightarrow \quad H_2(1) = \tfrac{1}{16}\,(2A + 3a - 6b + c);$$

$$H'_5(2) = 4\left(\tfrac{a}{2} - b + \tfrac{c}{2}\right) + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right) + (12-12+11)\,H_2(2) = B$$
$$\Rightarrow \quad H_2(2) = \tfrac{1}{22}\,(2B + a - 2b - c);$$

$$H'_5(3) = 6\left(\tfrac{a}{2} - b + \tfrac{c}{2}\right) + \left(-\tfrac{5a}{2} + 5b - \tfrac{3c}{2}\right) + (27 - 18 + 11)\, H_2(3) = C$$

$$\Rightarrow \quad H_2(3) = \tfrac{1}{40}\,(2C - a + 2b - 3c).$$

Dobijen je skup podataka za odredjivanje polinoma $H_2(x)$ :

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $H_2(x)$ | $\frac{1}{16}(2A + 3a - 6b + c)$ | $\frac{1}{22}(2B + a - 2b - c)$ | $\frac{1}{40}(2C - a + 2b - 3c)$ |

$H_2(x)$ odredjujemo kao interpolacioni polinom, npr. Lagranžov, sa zadatim čvorovima.

$$H_2(x) = \tfrac{1}{16}(2A + 3a - 6b + c)\frac{(x-2)(x-3)}{(1-2)(1-3)} + \tfrac{1}{22}(2B + a - 2b - c)\frac{(x-1)(x-3)}{(2-1)(2-3)}$$
$$+ \tfrac{1}{40}(2C - a + 2b - 3c)\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$H_2(x) = \tfrac{1}{32}(2A + 3a - 6b + c)(x-2)(x-3) - \tfrac{1}{22}(2B + a - 2b - c)(x-1)(x-3)$$
$$+ \tfrac{1}{80}(2C - a + 2b - 3c)(x-1)(x-2)\ .$$

ili, u drugom obliku :

$$H_2(x) = (x-1)(x-2)(x-3)\left(\frac{2A + 3a - 6b + c}{32(x-1)} - \frac{2B + a - 2b - c}{22(x-2)} + \frac{2C - a + 2b - 3c}{80(x-3)}\right)$$

Konačni oblik polinoma $H_5(x)$ je :

$$H_5(x) = \left(\frac{a}{2} - b + \frac{c}{2}\right)x^2 + \left(-\frac{5a}{2} + 5b - \frac{3c}{2}\right)x + (3a - 6b + c) + (x-1)(x-2)(x-3)\,H_2(x)$$

$$H_5(x) = \left(\frac{a}{2} - b + \frac{c}{2}\right)x^2 + \left(-\frac{5a}{2} + 5b - \frac{3c}{2}\right)x + (3a - 6b + c)$$
$$+ (x-1)^2(x-2)^2(x-3)^2\left(\frac{2A + 3a - 6b + c}{32(x-1)} - \frac{2B + a - 2b - c}{22(x-2)} + \frac{2C - a + 2b - 3c}{80(x-3)}\right)$$

**Zadatak 3.** Posmatrano je kretanje automobila duž pravog puta. Podaci o njegovom položaju i brzini u startu i posle 3, 5 i 8 sekundi su dati u sledećoj tabeli:

| vreme (s) | 0 | 3 | 5 | 8 |
|---|---|---|---|---|
| pređeni put (m) | 0 | 69 | 117 | 190 |
| brzina (m/s) | 23 | 24 | 25 | 22 |

**a)** Primenom interpolacionog polinoma odrediti položaj automobila (pređeni put) i brzinu posle 6 s.
**b)** Da li je automobil prekoračio dozvoljenu brzinu od 90km/h?

**Rešenje:**

# Srednje-kvadratna aproksimacija funkcija

## 1. zadatak

Aproksimirati funkciju $f(x) = x^4$ srednje–kvadratnom aproksimacijom u skupu polinoma stepena ne većeg od 2 na intervalu $(-1,1)$ sa Čebiševljevom težinom $p(x) = \frac{1}{\sqrt{1 - x^2}}$ i odrediti veličinu najbolje aproksimacije.

### Rešenje

I Poznati su Čebisevljevi ortogonalni polinomi:

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = -1 + 2x^2$

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Skalarni proizvod: $(\varphi,\psi) = \int_a^b p(x)\,\varphi(x)\,\psi(x)\,dx = \int_{-1}^{1} \frac{\varphi[x]\ \psi[x]}{\sqrt{1-x^2}}\,dx$

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II Odredjivanje aproksimacionog polinoma

$$f(x) \approx Q^*(x) = \frac{(f,\, Q_0)}{(Q_0,\, Q_0)}\ Q_0(x) + \frac{(f,\, Q_1)}{(Q_1,\, Q_1)}\ Q_1(x) + \frac{(f,\, Q_2)}{(Q_2,\, Q_2)}\ Q_2(x)$$

$$(f,Q_0) = \int_a^b p(x)\,f(x)\,Q_0(x)\,dx = \int_{-1}^{1} x^4 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{3\pi}{8}$$

$$(Q_0,Q_0) = \int_a^b p(x)\,Q_0(x)\,Q_0(x)\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \pi$$

$$(f,Q_1) = \int_a^b p(x)\,f(x)\,Q_1(x)\,dx = \int_{-1}^{1} x^4 \frac{x}{\sqrt{1-x^2}}\,dx = 0$$

$$(Q_1,Q_1) = \int_a^b p(x)\,Q_1(x)\,Q_1(x)\,dx = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$$

$$(f,Q_2) = \int_a^b p(x)\,f(x)\,Q_2(x)\,dx = \int_{-1}^{1} x^4 \frac{2x^2-1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{4}$$

$$(Q_2,Q_2) = \int_a^b p(x)\,Q_2(x)\,Q_2(x)\,dx = \int_{-1}^{1} \frac{\left(2x^2-1\right)^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$$

$f(x) = x^4$

$$f(x) \approx Q^*(x) = \frac{3}{8} + \frac{1}{2}\left(-1 + 2x^2\right)$$

$$= -\frac{1}{8} + x^2$$

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III Veličina najbolje aproksimacije

$$[\![\delta_2]\!]^2 = (f,f) - \frac{(f,\, Q_0)^2}{(Q_0,\, Q_0)} - \frac{(f,\, Q_1)^2}{(Q_1,\, Q_1)} - \frac{(f,\, Q_2)^2}{(Q_2,\, Q_2)}$$

$$[\![\delta_2]\!]^2 = \frac{\pi}{128} = 0.0245437$$

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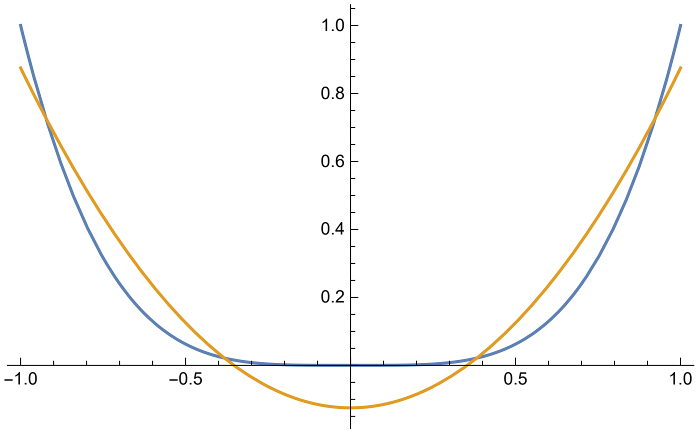
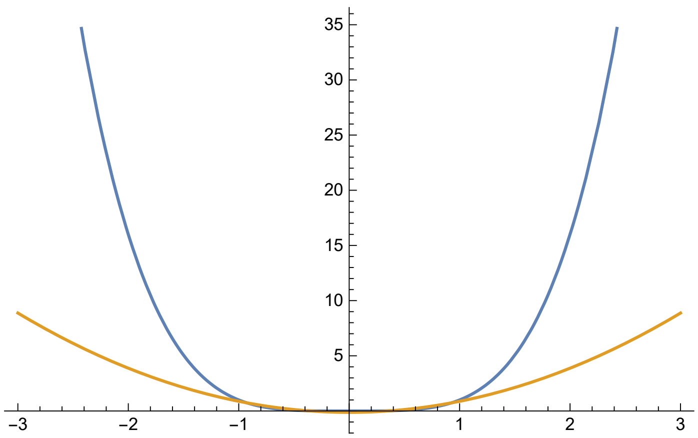
IV Uporedjivanje funkcija





## 2. zadatak

Aproksimirati funkciju $f(x)=e^x$ srednje–kvadratnom aproksimacijom u skupu polinoma stepena ne većeg od 2 na segmentu $[-1,1]$ sa težinom $p(x)=1$
i odrediti veličinu najbolje aproksimacije.

### Rešenje

Skalarni proizvod: $(\varphi,\psi) = \int_{-1}^{1} \varphi[x]\,\psi[x]\,dx$

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I Formiranje niza ortogonalnih polinoma

$Q_0(x)=1$

$Q_1(x)=x-\frac{(x,\,Q_0)}{(Q_0,\,Q_0)}\,Q_0(x)$

$Q_2(x)=x^2-\frac{(x^2,\,Q_0)}{(Q_0,\,Q_0)}\,Q_0(x)-\frac{(x^2,\,Q_1)}{(Q_1,\,Q_1)}\,Q_1(x)$

$$(x,Q_0)=\int_a^b x\,Q_0(x)\,p(x)\,dx = \int_{-1}^{1} x\,dx = 0$$

$$(Q_0,Q_0)=\int_a^b Q_0(x)\,Q_0(x)\,p(x)\,dx = \int_{-1}^{1} dx = 2$$

$Q_1(x)=x$

$$(x^2,Q_0)=\int_a^b x^2\;Q_0(x)\,p(x)\,dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$(x^2,Q_1)=\int_a^b x^2\;Q_1(x)\,p(x)\,dx = \int_{-1}^{1} x^3 dx = 0$$

$$(Q_1,Q_1)=\int_a^b Q_1(x)\,Q_1(x)\,p(x)\,dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$Q_2(x)=-\frac{1}{3}+x^2$

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II Odredjivanje aproksimacionog polinoma

$$f(x) \approx Q^*(x)=\frac{(f,\,Q_0)}{(Q_0,\,Q_0)}\;Q_0(x)+\frac{(f,\,Q_1)}{(Q_1,\,Q_1)}\;Q_1(x)+\frac{(f,\,Q_2)}{(Q_2,\,Q_2)}\;Q_2(x)$$

$$(f,Q_0)=\int_a^b f(x)\,Q_0(x)\,p(x)\,dx = \int_{-1}^{1} e^x dx = -\frac{1}{e}+e$$

$$(Q_0,Q_0)=\int_a^b Q_0(x)\,Q_0(x)\,p(x)\,dx = \int_{-1}^{1} dx = 2$$

$$(f,Q_1)=\int_a^b f(x)\,Q_1(x)\,p(x)\,dx = \int_{-1}^{1} e^x x\,dx = \frac{2}{e}$$

$$(Q_1,Q_1)=\int_a^b Q_1(x)\,Q_1(x)\,p(x)\,dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$(f,Q_2)=\int_a^b f(x)\,Q_2(x)\,p(x)\,dx = \int_{-1}^{1} e^x\;\left(x^2-\frac{1}{3}\right)\;dx = \frac{2\left(-7+e^2\right)}{3\,e}$$

$$(Q_2,Q_2)=\int_a^b Q_2(x)\,Q_2(x)\,p(x)\,dx = \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx = \frac{8}{45}$$

$f(x) = e^x$

$$f(x) \approx Q^*(x) = \frac{1}{2}\left(-\frac{1}{e}+e\right)+\frac{3\,x}{e}+\frac{15\left(-7+e^2\right)\left(-\frac{1}{3}+x^2\right)}{4\,e}$$

$$= \frac{33}{4\,e}-\frac{3\,e}{4}+\frac{3\,x}{e}-\frac{105\,x^2}{4\,e}+\frac{15\,e\,x^2}{4}$$

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III Veličina najbolje aproksimacije

$$[\![\delta_2]\!]^2 = (f,f) - \frac{(f,\,Q_0)^2}{(Q_0,\,Q_0)} - \frac{(f,\,Q_1)^2}{(Q_1,\,Q_1)} - \frac{(f,\,Q_2)^2}{(Q_2,\,Q_2)}$$

$$[\![\delta_2]\!]^2 = -\frac{6}{e^2}-\frac{1}{2}\left(-\frac{1}{e}+e\right)^2-\frac{5\left(-7+e^2\right)^2}{2\,e^2}+\mathrm{Sinh}[2] = 0.00144057$$

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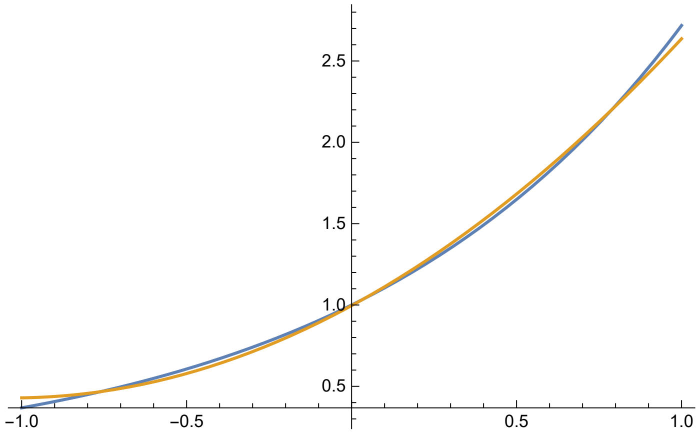
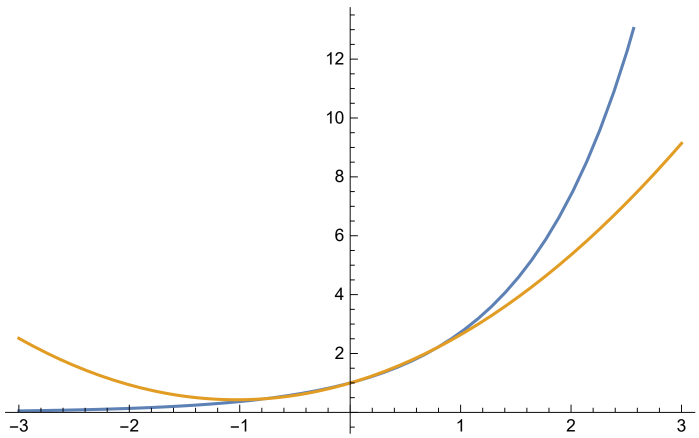
IV Uporedjivanje funkcija




## 3. zadatak

Aproksimirati funkciju $f(x) = \sin x$ srednje-kvadratnom aproksimacijom u skupu polinoma stepena ne viseg od 2 na segmentu $[0,2]$ sa tezinom

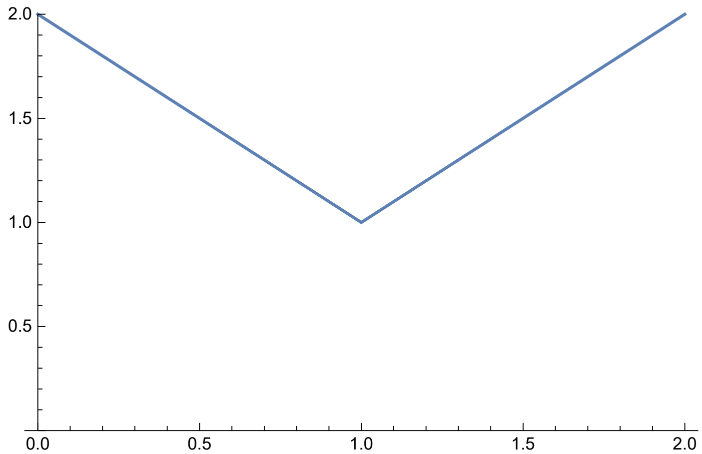
$$p(x) = \begin{cases} 2 - x, & x < 1, \\ x, & x \geq 1, \end{cases}$$

i odrediti veličinu najbolje aproksimacije.

### Rešenje

Tezina:

$$p(x) = \begin{cases} 2 - x, & x < 1, \\ x, & x \geq 1, \end{cases} \qquad [a,b] = [0,2].$$




Skalarni proizvod:

$$(\varphi,\psi) = \int_0^2 p(x)\ \varphi[x]\ \psi[x]\ dx = \int_0^1 (2-x)\ \varphi[x]\ \psi[x]\,dx + \int_1^2 x\ \varphi[x]\ \psi[x]\,dx$$

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I Formiranje niza ortogonalnih polinoma

$Q_0(x)=1$

$$Q_1(x)=x-\frac{(x,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x)$$

$$Q_2(x)=x^2-\frac{(x^2,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x)-\frac{(x^2,\ Q_1)}{(Q_1,\ Q_1)}\ Q_1(x)$$

$$(x,Q_0)=\int_a^b p(x)\ x\ Q_0(x)\,dx = \int_0^1 (2-x)\ x\ dx + \int_1^2 x\ x\ dx = 3$$

$$(Q_0,Q_0)=\int_a^b p(x)\,Q_0(x)\,Q_0(x)\,dx = \int_0^1 (2-x)\ dx + \int_1^2 x\ dx = 3$$

$Q_1(x)=-1+x$

$$(x^2,Q_0)=\int_a^b p(x)\ x^2\ Q_0(x)\,dx = \int_0^1 (2-x)\,x^2\ dx + \int_1^2 x\ x^2\ dx = \frac{25}{6}$$

$$(x^2,Q_1)=\int_a^b p(x)\ x^2\ Q_1(x)\,dx = \int_0^1 (2-x)\,x^2\,(x-1)\ dx + \int_1^2 x\ x^2\,(x-1)\ dx = \frac{7}{3}$$

$$(Q_1,Q_1)=\int_a^b p(x)\,Q_1(x)\,Q_1(x)\,dx = \int_0^1 (2-x)\ (x-1)^2\ dx + \int_1^2 x\ (x-1)^2\ dx = \frac{7}{6}$$

$$Q_2(x)=\frac{11}{18}-2\,x+x^2$$

============================================================

II Odredjivanje aproksimacionog polinoma

$$f(x)\ \approx\ Q^*(x)=\frac{(f,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x)+\frac{(f,\ Q_1)}{(Q_1,\ Q_1)}\ Q_1(x)+\frac{(f,\ Q_2)}{(Q_2,\ Q_2)}\ Q_2(x)$$

$$(f,Q_0)=\int_a^b p(x)\,f(x)\,Q_0(x)\,dx = \int_0^1 (2-x)\ \sin\ x\ dx + \int_1^2 x\ \sin\ x\ dx =$$

$$2-2\,\mathrm{Cos}[2]-2\,\mathrm{Sin}[1]+\mathrm{Sin}[2] = 2.05865$$

$$(f,Q_1)=\int_a^b p(x)\,f(x)\,Q_1(x)\,dx=\int_0^1 (2-x)\,(x-1)\sin x\;dx+\int_1^2 x\;(x-1)\sin x\;dx=$$

$$-4\,\mathrm{Cos}[1]+3\,\mathrm{Sin}[2]\;=\;0.566683$$

$$(f,Q_2)=\int_a^b p(x)\,f(x)\,Q_2(x)\,dx=\int_0^1 (2-x)\,(x^2-2x+\frac{11}{18})\sin x\;dx+\int_1^2 x\;(x^2-2x+\frac{11}{18})\sin x\;dx=$$

$$\frac{1}{18}\left(-122+122\,\mathrm{Cos}[2]+230\,\mathrm{Sin}[1]-25\,\mathrm{Sin}[2]\right)\;=\;-0.109112$$

$$(Q_2,Q_2)=\int_a^b p(x)\,Q_2(x)\,Q_2(x)\,dx=\int_0^1 (2-x)\,(x^2-2x+\frac{11}{18})^2\;dx+\int_1^2 x\;(x^2-2x+\frac{11}{18})^2\;dx=\frac{151}{540}$$

$$f(x)\;\approx\;Q^*(x)\;=\;0.686216+0.485728\,(-1+x)-0.390203\left(\frac{11}{18}-2\,x+x^2\right)$$

$$=\;-0.0379695+1.26613\,x-0.390203\,x^2$$

==============================================================

III Veličina najbolje aproksimacije

$$[\![\delta_2]\!]^2\;=\;(f,f)\;-\;\frac{(f,\,Q_0)^2}{(Q_0,\,Q_0)}-\frac{(f,\,Q_1)^2}{(Q_1,\,Q_1)}-\frac{(f,\,Q_2)^2}{(Q_2,\,Q_2)}$$

$$[\![\delta_2]\!]^2\;=\;0.000561208$$

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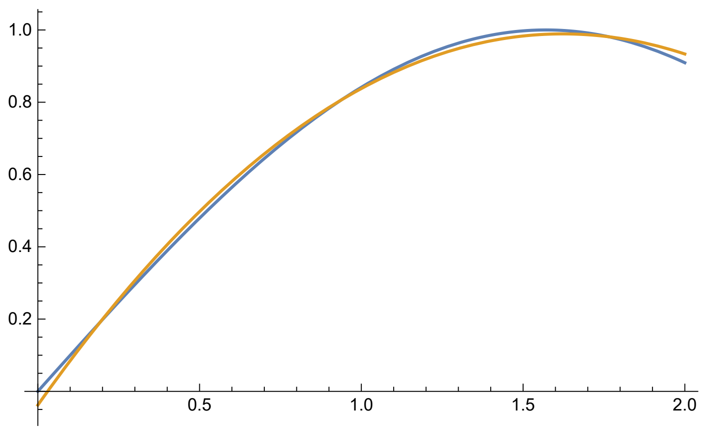
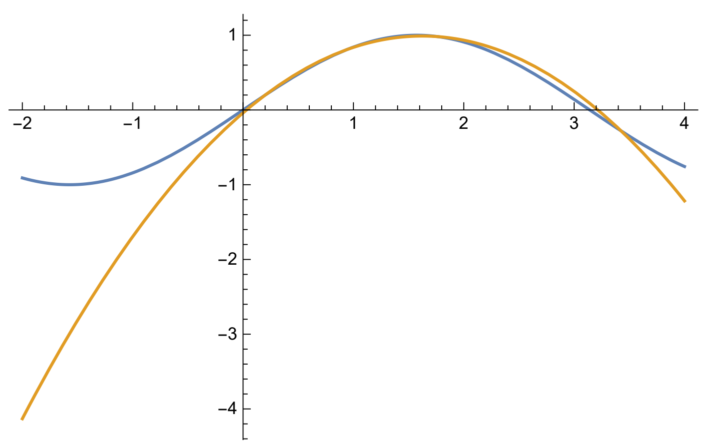
IV Uporedjivanje funkcija



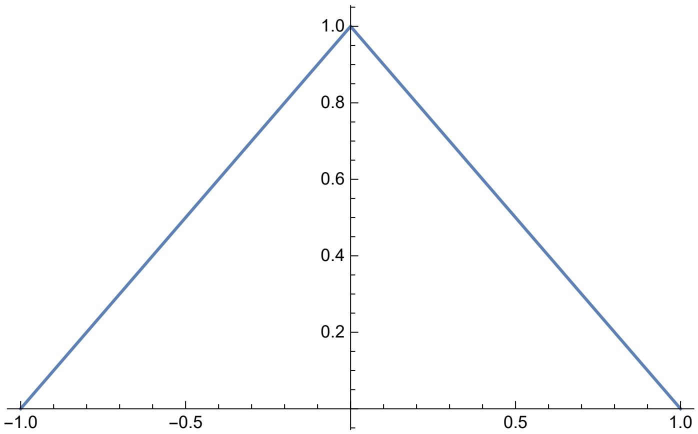



## 4. zadatak

Aproksimirati funkciju $f(x) = \frac{1}{1+x^2}$ srednje-kvadratnom aproksimacijom u skupu polinoma stepena ne većeg od 2 na segmentu $[-1,1]$ sa težinom $p(x) = 1 - |x|$
i odrediti velicinu najbolje aproksimacije.

### Rešenje

Tezina: $p(x) = 1 - |x| = \begin{cases} 1+x, & x < 0, \\ 1-x, & x \geq 0. \end{cases}$




Skalarni proizvod:

$$(\varphi,\psi) = \int_{-1}^{1} (1 - \mathrm{Abs}[x])\, \varphi[x]\, \psi[x]\, dx = \int_{-1}^{0} (1+x)\ \varphi(x)\ \psi(x)\, dx + \int_{0}^{1} (1-x)\ \varphi(x)\ \psi(x)\, dx$$

Važno: Koristiti osobine integrala parnih i neparnih funkcija na simetričnom intervalu:

Ako je $\varphi(x)$ neparna funkcija, tada je $\int_{-1}^{1} \varphi(x)\ dx = 0$.

Ako je $\varphi(x)$ parna funkcija, tada je $\int_{-1}^{1} \varphi(x)\ dx = 2\int_{0}^{1} \varphi(x)\ dx$.

==================================================================

I Formiranje niza ortogonalnih polinoma

$Q_0(x) = 1$

$$Q_1(x) = x - \frac{(x,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x)$$

$$Q_2(x) = x^2 - \frac{(x^2,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x) - \frac{(x^2,\ Q_1)}{(Q_1,\ Q_1)}\ Q_1(x)$$

$$(x, Q_0) = \int_{-1}^{1} x\ Q_0(x)\, p(x)\, dx = \int_{-1}^{0} x(1+x)\, dx + \int_{0}^{1} x(1-x)\, dx = 0$$

ili: $(x, Q_0) = \int_{-1}^{1} x\ Q_0(x)\, p(x)\, dx = 0$ kao integral neparne gunkcije.

$$(Q_0, Q_0) = \int_{-1}^{1} Q_0(x)\, Q_0(x)\, p(x)\, dx = \int_{-1}^{0} (1+x)\, dx + \int_{0}^{1} (1-x)\, dx = 1$$

ili: $(Q_0,Q_0) = \int_{-1}^{1} Q_0(x)\ Q_0(x)\, p(x)\, dx = 2\int_0^1 (1-x)\, dx = 1$ kao integral parne gunkcije.

$Q_1(x) = x$

$$(x^2,Q_0) = \int_{-1}^{1} x^2\ Q_0(x)\, p(x)\, dx = \frac{1}{6}$$

$$(x^2,Q_1) = \int_{-1}^{1} x^2\ Q_1(x)\, p(x)\, dx = 0$$

$$(Q_1,Q_1) = \int_{-1}^{1} Q_1(x)\, Q_1(x)\, p(x)\, dx = \frac{1}{6}$$

$Q_2(x) = -\frac{1}{6} + x^2$

========================================================

II Odredjivanje aproksimacionog polinoma

$$f(x) \approx Q^*(x) = \frac{(f,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x) + \frac{(f,\ Q_1)}{(Q_1,\ Q_1)}\ Q_1(x) + \frac{(f,\ Q_2)}{(Q_2,\ Q_2)}\ Q_2(x)$$

$$(f,Q_0) = \int_{-1}^{1} f(x)\, Q_0(x)\, p(x)\, dx = \int_{-1}^{0} \frac{1}{1+x^2}(1+x)\, dx + \int_0^1 \frac{1}{1+x^2}(1-x)\, dx = \frac{1}{2}(\pi - \text{Log}[4])$$

ili: $(f,Q_0) = \int_{-1}^{1} f(x)\ Q_0(x)\, p(x)\, dx = 2\int_0^1 \frac{1}{1+x^2}(1-x)\, dx =$

$\frac{1}{2}(\pi - \text{Log}[4])$ kao integral parne gunkcije.

$$(f,Q_1) = \int_{-1}^{1} f(x)\, Q_1(x)\, p(x)\, dx = \int_{-1}^{0} \frac{x}{1+x^2}(1+x)\, dx + \int_0^1 \frac{x}{1+x^2}(1-x)\, dx = 0.$$

ili: $(f,Q_1) = \int_{-1}^{1} f(x)\ Q_1(x)\, p(x)\, dx = 2\int_0^1 \frac{1}{1+x^2} x(1-x)\, dx = 0$ kao integral neparne gunkcije.

$$(f,Q_2) = \int_{-1}^{1} f(x)\, Q_2(x)\, p(x)\, dx = \int_{-1}^{0} \frac{x^2 - \frac{1}{6}}{1+x^2}(1+x)\, dx + \int_0^1 \frac{x^2 - \frac{1}{6}}{1+x^2}(1-x)\, dx = \frac{1}{12}(12 - 7\pi + 14\,\text{Log}[2])$$

$$(Q_2,Q_2) = \int_{-1}^{1} Q_2(x)\, Q_2(x)\, p(x)\, dx = \frac{7}{180}$$

$$f(x) = \frac{1}{1+x^2}$$

$$f(x) \approx Q^*(x) = \frac{15}{7}\left(-\frac{1}{6} + x^2\right)(12 - 7\pi + 14\,\text{Log}[2]) + \frac{1}{2}(\pi - \text{Log}[4]) + \frac{3}{2}x(-2\,\text{Log}[2] + \text{Log}[4])$$

$$= -\frac{30}{7} + 3\pi - 6\,\text{Log}[2] + x^2\left(\frac{180}{7} - 15\pi + 30\,\text{Log}[2]\right)$$

$$= 0.980181 - 0.615189\, x^2$$

========================================================

III Veličina najbolje aproksimacije

$$[\![\delta_2]\!]^2 = (f,f) - \frac{(f,\ Q_0)^2}{(Q_0,\ Q_0)} - \frac{(f,\ Q_1)^2}{(Q_1,\ Q_1)} - \frac{(f,\ Q_2)^2}{(Q_2,\ Q_2)}$$

$$[\![\delta_2]\!]^2 = \frac{\pi}{4} - \frac{5}{28}(12 - 7\pi + 14\,\text{Log}[2])^2 - \frac{1}{4}(\pi - \text{Log}[4])^2 - \frac{3}{8}(-2\,\text{Log}[2] + \text{Log}[4])^2 = 0.000412363$$

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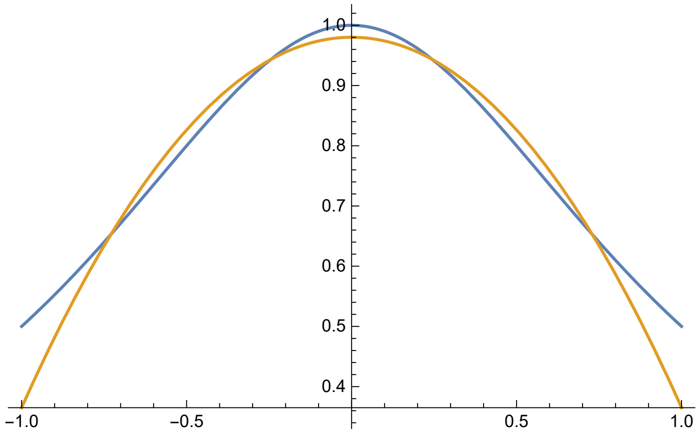
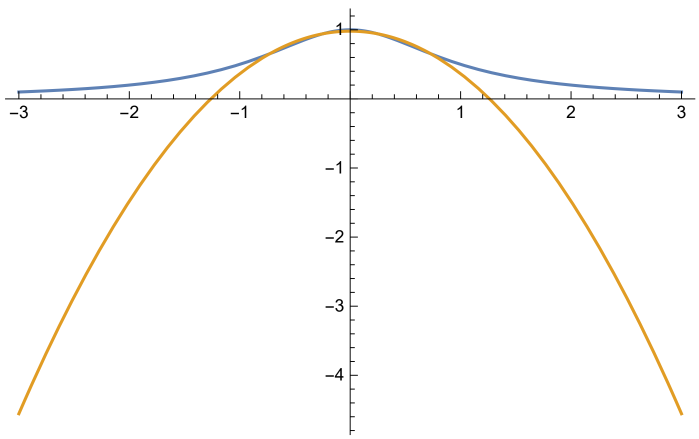
IV Uporedjivanje funkcija





## 5. zadatak

Aproksimirati funkciju f(x)=$x^2$ srednje-kvadratnom aproksimacijom u skupu polinoma stepena ne viseg od 1 na segmentu [0,π] sa tezinom p(x)= sin x.

### Rešenje

I Formiranje niza ortogonalnih polinoma

$Q_0(x) = 1$

$$Q_1(x) = x - \frac{(x,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x)$$

$$(x, Q_0) = \int_a^b x\ Q_0(x)\, p(x)\, dx = \pi$$

$$(Q_0, Q_0) = \int_a^b Q_0(x)\, Q_0(x)\, p(x)\, dx = 2$$

$$Q_1(x) = -\frac{\pi}{2} + x$$

==========================================================

II Odredjivanje aproksimacionog polinoma

$$f(x) \approx Q^*(x) = \frac{(f,\ Q_0)}{(Q_0,\ Q_0)}\ Q_0(x) + \frac{(f,\ Q_1)}{(Q_1,\ Q_1)}\ Q_1(x)$$

$$(f, Q_0) = \int_a^b f(x)\, Q_0(x)\, p(x)\, dx = -4 + \pi^2$$

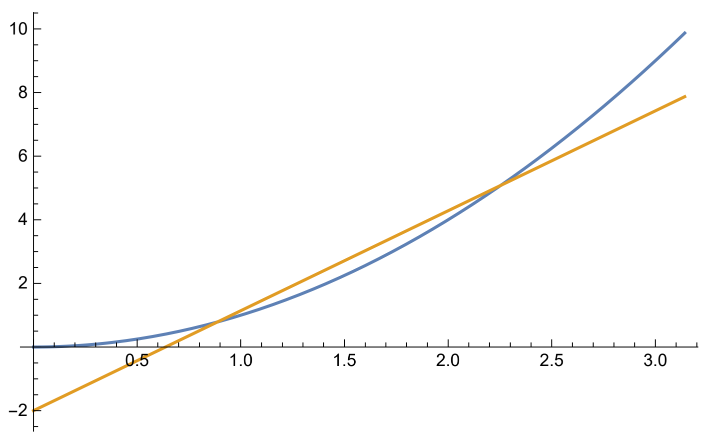
$$(f, Q_1) = \int_a^b f(x)\, Q_1(x)\, p(x)\, dx = \frac{1}{2}\pi\left(-8 + \pi^2\right)$$

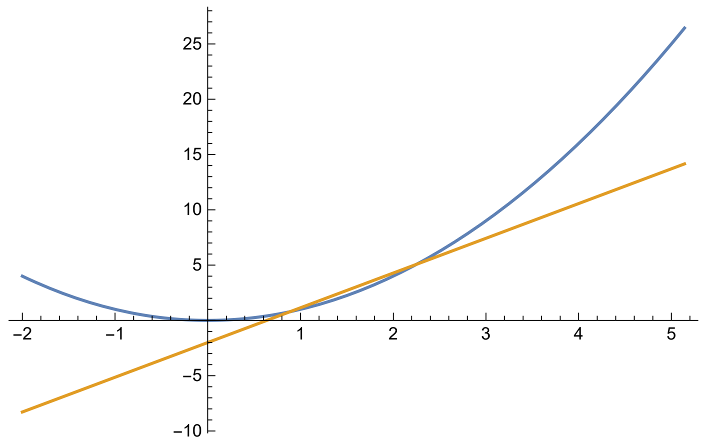
$$f(x) \approx \Phi(x) = \frac{1}{2}\left(-4 + \pi^2\right) + \pi\left(-\frac{\pi}{2} + x\right)$$

$$= -2 + \pi x$$

==========================================================

III Uporedjivanje funkcija




25
20
15
10
5
−5
−10
−2
−1
1
2
3
4
5

(3)

| $t$ | 0 | 3 | 5 | 8 |
|---|---|---|---|---|
| $s(t)$ | 0 | 69 | 117 | 190 |
| $s'(t)$ | 23 | 24 | 25 | 22 |

$4+4=8 \Rightarrow$ formiramo $H_7(x)$

$$H_7(x) = P_4(x) + x(x-3)(x-5)(x-8)H_3(x)$$

$$P_4(x) = 0 \cdot \frac{(x-3)(x-5)(x-8)}{(0-3)(0-5)(0-8)} + 69 \cdot \frac{(x-0)(x-5)(x-8)}{(3-0)(3-5)(3-8)}$$

$$+ 117 \cdot \frac{(x-0)(x-3)(x-8)}{(5-0)(5-3)(5-8)} + 190 \frac{(x-0)(x-3)(x-5)}{(8-0)(8-3)(8-5)}$$

$$= -\frac{x^3}{60} + \frac{x^2}{3} + \frac{443x}{20}$$

$$H_7(x) = -\frac{x^3}{60} + \frac{x^2}{3} + \frac{443x}{20} + (x^4 - 16x^3 + 79x^2 - 120x)H_3(x)$$

$H_7(0) = 0 \quad H_7(3) = 69$

$H_7(5) = 117 \quad H_7(8) = 190$

$$H_7'(x) = -\frac{x^2}{20} + \frac{2x}{3} + \frac{443}{20} + (4x^3 - 48x^2 + 158x - 120)$$

$$\cdot H_3(x) + (x^4 - 16x^3 + 79x^2 - 120x) \cdot H_3'(x)$$

$$H_7'(0) = \frac{443}{20} - 120\, H_3(0) = 23 \Rightarrow H_3(0) = -\frac{17}{2400}$$

$$H_7'(3) = \frac{237}{10} + 30\, H_3(3) = 24 \Rightarrow H_3(3) = \frac{1}{100}$$

$$H_7'(5) = \frac{727}{30} - 30\, H_3(5) = 25 \Rightarrow H_3(5) = -\frac{23}{900}$$

$$H_7'(8) = \frac{1457}{60} + 120 \cdot H_3(8) = 22 \Rightarrow H_3(8) = -\frac{137}{7200}$$

| $x$ | $0$ | $3$ | $5$ | $8$ |
|---|---|---|---|---|
| $H_3(x)$ | $-\frac{17}{2400}$ | $\frac{1}{100}$ | $-\frac{23}{900}$ | $-\frac{137}{7200}$ |

$$H_3(x) = P_3(x) = \frac{469x^3}{432000} - \frac{289x^2}{21600} + \frac{577x}{16000} - \frac{17}{2400}$$

$$\Rightarrow H_7(x) = \frac{469}{432000}x^7 - \frac{13284}{432000}x^6 + \frac{145110}{432000}x^5 - \frac{765224}{432000}x^4$$

$$+ \frac{1966101}{432000}x^3 - \frac{1967220}{432000}x^2 + \frac{9936000}{432000}x$$

$$H_7'(x) = \underline{\qquad}$$

$$\boxed{H_7(6) = 142.64} \quad \boxed{H_7'(6) = 25.97}$$