Izvodi

Definicija. Neka je funkcija f definisana i neprekidna u okolini tačke a. Prvi izvod funkcije f u tački a je

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Prvi izvod funkcije f u tački x:

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}.$$

Izvodi višeg reda funkcije f u tački x:

$$(f(x))^{(n)} = ((f(x))^{(n-1)})', \quad n \in \mathbb{N}, \qquad (f(x))^{(0)} = f(x).$$

Pravila diferenciranja:

1. $(f(x) + g(x))' = f'(x) + g'(x)$;

2. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$;

3. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$;

4. $(f\left(g(x)\right))' = f'\left(g(x)\right) g'(x)$.

Lajbnicova formula (n–ti izvod proizvoda):

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

Izvod inverzne funkcije:

$$\Big(f^{-1}(y)\Big)' = \frac{1}{f'(x)}.$$

Logaritamski izvod funkcije $f(x) = \varphi(x)^{\psi(x)}$:

$$\begin{aligned}
\log f(x) &= \log \varphi(x)^{\psi(x)} = \psi(x) \log \varphi(x) \\
(\log f(x))' &= (\psi(x) \log \varphi(x))' \\
\frac{1}{f(x)} f'(x) &= \psi'(x) \log \varphi(x) + \psi(x) \frac{1}{\varphi(x)} \varphi'(x) \\
f'(x) &= \varphi(x)^{\psi(x)} \Big(\psi'(x) \log \varphi(x) + \frac{\psi(x)}{\varphi(x)} \varphi'(x)\Big).
\end{aligned}$$

Izvodi parametarski definisane funkcije $\begin{cases} x = \varphi(t), \\ y = \psi(t) \end{cases}$:

$$y'_x = \frac{dy}{dx} = \frac{\dfrac{d\psi}{dt}}{\dfrac{d\varphi}{dt}} = \frac{\psi'_t(t)}{\varphi'_t(t)} = \frac{\dot{\psi}(t)}{\dot{\varphi}(t)},$$

$$\begin{aligned}
y''_x &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\psi'_t(t)}{\varphi'_t(t)}\right) = \frac{d}{dt}\left(\frac{\psi'_t(t)}{\varphi'_t(t)}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{\psi'_t(t)}{\varphi'_t(t)}\right)\frac{1}{\dfrac{dx}{dt}} \\
&= \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{(\varphi'(t))^3} = \frac{\ddot{\psi}(t)\dot{\varphi}(t) - \dot{\psi}(t)\ddot{\varphi}(t)}{(\dot{\varphi}(t))^3}.
\end{aligned}$$

Diferencijal:

$$df(x) = f'(x)dx, \qquad d^n f(x) = f^{(n)}(x) dx^n, \quad n \in \mathbb{N}.$$

Napomena. U celokupnom izlaganju podrazumeva se $\log = \log_e$.

Tablica izvoda:

$$
\begin{aligned}
&(x^n)' = nx^{n-1}; && (\arcsin x)' = \frac{1}{\sqrt{1-x^2}};\\
&(e^x)' = e^x; && (\arccos x)' = -\frac{1}{\sqrt{1-x^2}};\\
&(a^x)' = a^x \log a; && (\arctan x)' = \frac{1}{1+x^2};\\
&(\log x)' = \frac{1}{x}; && (\mathrm{arccot}\ x)' = -\frac{1}{1+x^2};\\
&(\sin x)' = \cos x; && (\sinh x)' = \cosh x;\\
&(\cos x)' = -\sin x; && (\cosh x)' = \sinh x;\\
&(\tan x)' = \frac{1}{\cos^2 x}; && (\tanh x)' = \frac{1}{\cosh^2 x};\\
&(\cot x)' = -\frac{1}{\sin^2 x}; && (\coth x)' = -\frac{1}{\sinh^2 x}.
\end{aligned}
$$

Zadaci

1. Po definiciji odrediti $f'(1)$ ako je:

a) $f(x) = \sqrt{x} + 2$; **b**) $f(x) = \sin x$; **c**) $f(x) = e^x$.

Rešenje: a) Kako je $f(1) = 3$, to je

$$
\begin{aligned}
f'(1) &= \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\\
&= \lim_{x\to 1} \frac{x-1}{(x-1)\big(\sqrt{x}+1\big)} = \frac{1}{2}.
\end{aligned}
$$

b) Slično određujemo

$$\begin{aligned}
f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{\sin x-\sin 1}{x-1} = \lim_{x\to 1}\frac{\sin(x-1+1)-\sin 1}{x-1}\\
&= \lim_{x\to 1}\frac{\sin(x-1)\cos 1+\cos(x-1)\sin 1-\sin 1}{x-1}\\
&= \lim_{x\to 1}\Big(\frac{\sin(x-1)}{x-1}\cos 1+\frac{\cos(x-1)-1}{x-1}\sin 1\Big)\\
&= \lim_{x\to 1}\Big(\frac{\sin(x-1)}{x-1}\cos 1-\frac{1-\cos(x-1)}{(x-1)^2}(x-1)\sin 1\Big).
\end{aligned}$$

Kako je

$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=1,\qquad \lim_{x\to 1}\frac{1-\cos(x-1)}{(x-1)^2}=\frac{1}{2},\qquad \lim_{x\to 1}(x-1)=0,$$

dobija se

$$f'(1)=\cos 1.$$

c) S obzirom na poznatu graničnu vrednost

$$\lim_{x\to 1}\frac{e^{x-1}-1}{x-1}=\log e=1,$$

važi sledeće:

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{e^x-e}{x-1}=\lim_{x\to 1}\frac{e(e^{x-1}-1)}{x-1}=e.$$

2. Odrediti izvode sledećih eksplicitno zadatih funkcija:

a) $y=\cos\pi x\,\sqrt{1+\sin^2\pi x}$; **b)** $y=\dfrac{\sin(\log x)+\cos(\log x)}{x}$;

c) $y=\arctan\dfrac{e^{2x}-1}{e^{2x}+1}$; **d)** $y=\dfrac{e^{\sin x}+e^{\cos x}}{\sin x-\cos x}$

e) $y=\log\left(\log(xe^x)\right)$; **f)** $y=\log\big(\cos x+\sqrt{\cos^2 x+1}\big)$;

g) $y=\log\dfrac{\sin x+\cos x}{\sin x-\cos x}$; **h)** $y=\arcsin\sqrt[3]{1-x^2}$.

Rešenje: a) Najpre koristimo pravilo za izvod proizvoda, a zatim za izvod složene funkcije:

$$\begin{aligned}
y' &= \left(\cos \pi x \ \sqrt{1+\sin^2 \pi x}\right)' \\
&= (\cos \pi x)' \sqrt{1+\sin^2 \pi x} + \cos \pi x \left(\sqrt{1+\sin^2 \pi x}\right)' \\
&= -\pi \sin \pi x \ \sqrt{1+\sin^2 \pi x} + \cos \pi x \ \frac{1}{2\sqrt{1+\sin^2 \pi x}} \left(1+\sin^2 \pi x\right)' \\
&= -\pi \sin \pi x \ \sqrt{1+\sin^2 \pi x} + \cos \pi x \ \frac{1}{2\sqrt{1+\sin^2 \pi x}} \ 2\pi \sin \pi x \ \cos \pi x \\
&= \frac{-\pi \sin \pi x \left(1+\sin^2 \pi x - \cos^2 \pi x\right)}{\sqrt{1+\sin^2 \pi x}} = \frac{-2\pi \sin^3 \pi x}{\sqrt{1+\sin^2 \pi x}}.
\end{aligned}$$

b) Za određivanje izvoda primenjujemo pravilo za izvod količnika, a posle toga pravilo za izvod zbira i izvod složene funkcije:

$$\begin{aligned}
y' &= \left(\frac{\sin(\log x) + \cos(\log x)}{x}\right)' \\
&= \frac{(\sin(\log x) + \cos(\log x))' \, x - (\sin(\log x) + \cos(\log x)) \, (x)'}{x^2} \\
&= \frac{(\cos(\log x)(\log x)' - \sin(\log x)(\log x)') \, x - (\sin(\log x) + \cos(\log x))}{x^2} \\
&= \frac{\left(\cos(\log x)\dfrac{1}{x} - \sin(\log x)\dfrac{1}{x}\right) x - (\sin(\log x) + \cos(\log x))}{x^2} \\
&= \frac{\cos(\log x) - \sin(\log x) - \sin(\log x) - \cos(\log x)}{x^2} = \frac{-2\sin(\log x)}{x^2}.
\end{aligned}$$

c) U ovom slučaju najpre primenjujemo pravilo za izvod složene funkcije, a zatim za izvod količnika, zbira i ponovo složene funkcije:

$$\begin{aligned}
y' &= \left(\arctan \frac{e^{2x}-1}{e^{2x}+1}\right)' = \frac{1}{1+\left(\dfrac{e^{2x}-1}{e^{2x}+1}\right)^2} \ \left(\frac{e^{2x}-1}{e^{2x}+1}\right)' \\
&= \frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}+1\right)^2 + \left(e^{2x}-1\right)^2} \ \frac{(e^{2x}-1)'(e^{2x}+1) - (e^{2x}-1)(e^{2x}+1)'}{(e^{2x}+1)^2}
\end{aligned}$$

$$= \frac{2e^{2x}(e^{2x}+1)-(e^{2x}-1)2e^{2x}}{e^{4x}+2e^{2x}+1+e^{4x}-2e^{2x}+1} = \frac{2e^{2x}(e^{2x}+1-e^{2x}+1)}{2(e^{4x}+1)}$$

$$= \frac{2e^{2x}}{e^{4x}+1}.$$

d) $y' = \left(\dfrac{e^{\sin x}+e^{\cos x}}{\sin x-\cos x}\right)'$

$$= \frac{\left(e^{\sin x}+e^{\cos x}\right)'(\sin x-\cos x)-\left(e^{\sin x}+e^{\cos x}\right)(\sin x-\cos x)'}{(\sin x-\cos x)^2}$$

$$= \frac{\left(e^{\sin x}\cos x+e^{\cos x}(-\sin x)\right)(\sin x-\cos x)-\left(e^{\sin x}+e^{\cos x}\right)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{\left(e^{\sin x}+e^{\cos x}\right)(\sin x\cos x-\cos x-\sin x)-\left(\cos^2 xe^{\sin x}+\sin^2 xe^{\cos x}\right)}{(\sin x-\cos x)^2}$$

$$= \frac{\cos xe^{\sin x}-\sin xe^{\cos x}}{\sin x-\cos x}-\frac{(\sin x+\cos x)\left(e^{\sin x}+e^{\cos x}\right)}{(\sin x-\cos x)^2}.$$

e) $y' = (\log(\log(xe^x)))' = \dfrac{1}{\log(xe^x)}\ (\log(xe^x))' = \dfrac{1}{\log(xe^x)}\ \dfrac{1}{xe^x}\ (xe^x)'$

$$= \frac{1}{\log(xe^x)}\ \frac{1}{xe^x}\ (e^x+xe^x) = \frac{1+x}{x\log(xe^x)}.$$

f) $y' = \left(\log\left(\cos x+\sqrt{\cos^2 x+1}\right)\right)'$

$$= \frac{1}{\cos x+\sqrt{\cos^2 x+1}}\ (\cos x+\sqrt{\cos^2 x+1})'$$

$$= \frac{1}{\cos x+\sqrt{\cos^2 x+1}}\ \left(-\sin x+\frac{1}{2\sqrt{\cos^2 x+1}}(\cos^2 x+1)'\right)$$

$$= \frac{1}{\cos x+\sqrt{\cos^2 x+1}}\ \left(-\sin x+\frac{1}{2\sqrt{\cos^2 x+1}}(-2\sin x\cos x)\right)$$

$$= \frac{1}{\cos x+\sqrt{\cos^2 x+1}}\ \frac{-\sin x\sqrt{\cos^2 x+1}-\sin x\cos x}{\sqrt{\cos^2 x+1}}$$

$$= \frac{1}{\cos x + \sqrt{\cos^2 x + 1}} \; \frac{-\sin x\left(\sqrt{\cos^2 x + 1} + \cos x\right)}{\sqrt{\cos^2 x + 1}}$$

$$= -\frac{\sin x}{\sqrt{\cos^2 x + 1}}.$$

g) $y' = \left(\log \dfrac{\sin x + \cos x}{\sin x - \cos x}\right)' = (\log(\sin x + \cos x) - \log(\sin x - \cos x))'$

$$= \frac{1}{\sin x + \cos x}(\cos x - \sin x) - \frac{1}{\sin x - \cos x}(\cos x + \sin x)$$

$$= \frac{-(\cos x - \sin x)^2 - (\sin x + \cos x)^2}{(\sin x + \cos x)(\sin x - \cos x)}$$

$$= \frac{-\cos^2 x + 2\sin x \cos x - \sin^2 x - \sin^2 x - 2\sin x \cos x - \cos^2 x}{\sin^2 x - \cos^2 x}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{-\cos 2x} = \frac{2}{\cos 2x}.$$

h) $y' = \left(\arcsin \sqrt[3]{1 - x^2}\right)' = \dfrac{1}{\sqrt{1 - \left(\sqrt[3]{1 - x^2}\right)^2}} \left(\sqrt[3]{1 - x^2}\right)'$

$$= \frac{1}{\sqrt{1 - \sqrt[3]{(1 - x^2)^2}}} \; \frac{1}{3\sqrt[3]{(1 - x^2)^2}} \; (-2x)$$

$$= \frac{-2x}{3\sqrt[3]{(1 - x^2)^2} \; \sqrt{1 - \sqrt[3]{(1 - x^2)^2}}}.$$

3. Odrediti izvode sledećih implicitno zadatih funkcija:

a) $xy + \arctan y = x$; **b**) $\cos(x + y) + \sin(1 + xy) = \dfrac{y}{x}$;

c) $e^{\cos y} = \log\left(x^2 + y^2\right)$; **d**) $\tan(xy) = \arctan(x + y)$;

e) $(x + y)^2 = \cos(xy)$; **f**) $xy - \log(x + y) = 0$;

g) $\dfrac{xy + x + y}{y - x} = \sqrt{1 + \dfrac{y}{x}}$; **h**) $x^y = y^x$.

Rešenje: a) Imajući u vidu da je y zavisno promenljiva, tj. funkcija nezavisno promenljive x, tražimo izvod leve i desne strane jednakosti i dobijamo:

$$\begin{aligned}(xy + \arctan y)' &= (x)', \\ (xy)' + (\arctan y)' &= 1, \\ y + xy' + \frac{1}{1+y^2}\, y' &= 1.\end{aligned}$$

Rešavanjem dobijene jednačine po y' imamo:

$$\begin{aligned}(y + xy')(1+y^2) + y' &= 1 + y^2, \\ \big(x(1+y^2) + 1\big)\, y' &= (1+y^2)(1-y) \\ y' &= \frac{(1+y^2)(1-y)}{x(1+y^2)+1}.\end{aligned}$$

b) Opisanim postupkom dobijamo y' kroz sledeći niz jednakosti:

$$\begin{aligned}(\cos(x+y) + \sin(1+xy))' &= \left(\frac{y}{x}\right)', \\ -\sin(x+y)(x+y)' + \cos(1+xy)(1+xy)' &= \frac{y'x - y}{x^2}, \\ -\sin(x+y)(1+y') + \cos(1+xy)(y+xy') &= \frac{xy'-y}{x^2}, \\ -x^2\sin(x+y)(1+y') + x^2\cos(1+xy)(y+xy') &= xy' - y, \\ \big(x^3\cos(1+xy) - x^2\sin(x+y) - x\big)\, y' &= x^2\sin(x+y) - x^2 y\cos(1+xy) - y.\end{aligned}$$

Konačno je $\quad y' = \dfrac{x^2\sin(x+y) - x^2 y\cos(1+xy) - y}{x^3\cos(1+xy) - x^2\sin(x+y) - x}.$

c)

$$\begin{aligned}e^{\cos y} &= \log\big(x^2+y^2\big), \\ (e^{\cos y})' &= \big(\log\big(x^2+y^2\big)\big)', \\ e^{\cos y}(\cos y)' &= \frac{1}{x^2+y^2}\big(x^2+y^2\big)',\end{aligned}$$

$$
\begin{aligned}
e^{\cos y}(-\sin y)y' &= \frac{1}{x^2+y^2}(2x+2yy'),\\
-\big(x^2+y^2\big)\sin y\ e^{\cos y}y' &= 2x+2yy',\\
-\left(2y+\big(x^2+y^2\big)\sin y\ e^{\cos y}\right)y' &= 2x,\\
y' &= -\frac{2x}{2y+\big(x^2+y^2\big)\sin y\ e^{\cos y}}.
\end{aligned}
$$

d)

$$
\begin{aligned}
\tan(xy) &= \arctan(x+y),\\
(\tan(xy))' &= (\arctan(x+y))',\\
\frac{1}{\cos^2(xy)}(y+xy') &= \frac{1}{1+(x+y)^2}(1+y'),\\
\big(1+(x+y)^2\big)(y+xy') &= \cos^2(xy)(1+y'),\\
\left(x\left(1+(x+y)^2\right)-\cos^2(xy)\right)y' &= \cos^2(xy)-y\left(1+(x+y)^2\right),\\
y' &= \frac{\cos^2(xy)-y\left(1+(x+y)^2\right)}{x\left(1+(x+y)^2\right)-\cos^2(xy)}.
\end{aligned}
$$

e)

$$
\begin{aligned}
(x+y)^2 &= \cos(xy),\\
\left((x+y)^2\right)' &= (\cos(xy))',\\
2(x+y)(1+y') &= -\sin(xy)(y+xy'),\\
\left(2(x+y)+x\sin(xy)\right)y' &= -2(x+y)-y\sin(xy),\\
y' &= -\frac{2(x+y)+y\sin(xy)}{2(x+y)+x\sin(xy)}.
\end{aligned}
$$

f)

$$
\begin{aligned}
xy-\log(x+y) &= 0,\\
(xy-\log(x+y))' &= 0,\\
y+xy'-\frac{1}{x+y}(1+y') &= 0,\\
(x+y)(y+xy')-(1+y') &= 0,
\end{aligned}
$$

$$(x(x+y)-1)\,y' = 1 - y(x+y),$$
$$y' = \frac{1-y(x+y)}{x(x+y)-1}.$$

g)
$$\frac{xy+x+y}{y-x} = \sqrt{1+\frac{y}{x}},$$
$$\left(\frac{xy+x+y}{y-x}\right)' = \left(\sqrt{1+\frac{y}{x}}\right)',$$
$$\frac{(y+xy'+1+y')(y-x)-(xy+x+y)(y'-1)}{(y-x)^2} = \frac{1}{2\sqrt{\dfrac{x+y}{x}}}\ \frac{y'x-y}{x^2},$$
$$\frac{(y^2+2y)-(x^2+2x)y'}{(y-x)^2} = \frac{1}{2}\sqrt{\frac{x}{x+y}}\ \frac{xy'-y}{x^2},$$
$$\left(\frac{1}{2x}\sqrt{\frac{x}{x+y}} + \frac{x(x+2)}{(y-x)^2}\right)y' = \frac{y(y+2)}{(y-x)^2} + \frac{y}{2x^2}\sqrt{\frac{x}{x+y}},$$
$$\left(x\sqrt{\frac{x}{x+y}} + \frac{2x^3(x+2)}{(y-x)^2}\right)y' = \frac{2x^2y(y+2)}{(y-x)^2} + y\sqrt{\frac{x}{x+y}}.$$

Konačno, dobijamo
$$y' = \frac{2x^2y(y+2)+y(y-x)^2\sqrt{\dfrac{x}{x+y}}}{x(y-x)^2\sqrt{\dfrac{x}{x+y}}+2x^3(x+2)}.$$

h) Najpre logaritmujemo, a zatim diferenciramo jednakost:
$$\log x^y = \log y^x,$$
$$y\log x = x\log y,$$
$$(y\log x)' = (x\log y)',$$
$$y'\log x + y\frac{1}{x} = \log y + x\frac{1}{y}y'.$$

Traženi izvod y' se dobija rešavanjem dobijene jednačine:

$$\begin{aligned} xyy'\log x + y^2 &= xy\log y + x^2y', \\ (xy\log x - x^2)y' &= xy\log y - y^2, \\ y' &= \frac{y(x\log y - y)}{x(y\log x - x)}. \end{aligned}$$

4. Odrediti izvode sledećih funkcija:

a) $y = x^{\log x}$; **b**) $y = (\arctan x)^x$;

c) $y = \left(\dfrac{x^2}{x^2+1}\right)^{\tan x}$; **d**) $y = (\sin x)^{1+\cos^2 x}$.

Rešenje: a) Kako je funkcija zadata u obliku stepena u kome i osnova i izložilac zavise od nezavisno promenljive x, najpre logaritmujemo jednakost i dobijamo

$$\log y = \log x^{\log x} = \log x \ \log x, \qquad \text{tj.} \qquad \log y = (\log x)^2.$$

Diferenciranje dobijene jednakosti daje:

$$\begin{aligned} (\log y)' &= \left((\log x)^2\right)', \\ \frac{1}{y}y' &= 2\log x \ \frac{1}{x}, \\ y' &= 2\frac{y}{x}\log x, \\ y' &= 2x^{\log x - 1}\log x. \end{aligned}$$

b) Sličnim postupkom dobijamo:

$$\begin{aligned} y &= (\arctan x)^x, \\ \log y &= \log(\arctan x)^x = x\log(\arctan x), \\ (\log y)' &= (x\log(\arctan x))', \\ \frac{y'}{y} &= \log(\arctan x) + x\frac{1}{\arctan x} \ \frac{1}{1+x^2}, \end{aligned}$$

$$y' = y\left(\log(\arctan x) + \frac{x}{\arctan x}\ \frac{1}{1+x^2}\right),$$

$$y' = (\arctan x)^{x-1}\left(\arctan x\ \ \log(\arctan x) + \frac{x}{1+x^2}\right).$$

c) $$y = \left(\frac{x^2}{x^2+1}\right)^{\tan x},$$

$$\log y = \tan x \log\Big(\frac{x^2}{x^2+1}\Big),$$

$$(\log y)' = \left(\tan x \log\Big(\frac{x^2}{x^2+1}\Big)\right)',$$

$$\frac{y'}{y} = \frac{1}{\cos^2 x}\log\Big(\frac{x^2}{x^2+1}\Big) + \tan x\ \frac{x^2+1}{x^2}\ \frac{2x}{(x^2+1)^2},$$

$$\frac{y'}{y} = \frac{1}{\cos^2 x}\log\Big(\frac{x^2}{x^2+1}\Big) + \frac{2}{x\,(x^2+1)}\ \ \tan x,$$

$$y' = y\left(\frac{1}{\cos^2 x}\log\Big(\frac{x^2}{x^2+1}\Big) + \frac{2\tan x}{x\,(x^2+1)}\right),$$

$$y' = \left(\frac{x^2}{x^2+1}\right)^{\tan x}\left(\frac{1}{\cos^2 x}\log\Big(\frac{x^2}{x^2+1}\Big) + \frac{2\tan x}{x\,(x^2+1)}\right).$$

d) $$y = (\sin x)^{1+\cos^2 x},$$

$$\log y = \big(1+\cos^2 x\big)\log(\sin x),$$

$$(\log y)' = \Big(\big(1+\cos^2 x\big)\log(\sin x)\Big)',$$

$$\frac{y'}{y} = -2\cos x\sin x\log(\sin x) + \big(1+\cos^2 x\big)\,\frac{1}{\sin x}\cos x,$$

$$y' = y\left(-2\sin x\cos x\log(\sin x) + \big(1+\cos^2 x\big)\ \frac{\cos x}{\sin x}\right),$$

$$y' = \cos x\ (\sin x)^{\cos^2 x}\left(1+\cos^2 x - 2\sin^2 x\log(\sin x)\right).$$

5. Odrediti izvode sledećih parametarski zadatih funkcija:

a) $\begin{cases} x = te^t, \\ y = \arctan t; \end{cases}$ **b**) $\begin{cases} x = \sqrt{t}\cos t, \\ y = \sqrt{t}\sin t; \end{cases}$

c) $\begin{cases} x = t^3 + 1, \\ y = t^3 + t + 1; \end{cases}$ **d**) $\begin{cases} x = t\log t, \\ y = \dfrac{\log t}{t}. \end{cases}$

Rešenje: a) Kako je

$$x'(t) = (te^t)' = e^t + te^t = (1+t)e^t,$$
$$y'(t) = (\arctan t)' = \frac{1}{1+t^2},$$

to je

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{\dfrac{1}{1+t^2}}{(1+t)e^t} = \frac{e^{-t}}{1+t+t^2+t^3}.$$

b) $\quad x = \sqrt{t}\cos t, \qquad y = \sqrt{t}\sin t,$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{\left(\sqrt{t}\sin t\right)'}{\left(\sqrt{t}\cos t\right)'} = \frac{\dfrac{1}{2\sqrt{t}}\sin t + \sqrt{t}\cos t}{\dfrac{1}{2\sqrt{t}}\cos t - \sqrt{t}\sin t} = \frac{\sin t + 2t\cos t}{\cos t - 2t\sin t}.$$

c) $\quad x = t^3 + 1, \qquad y = t^3 + t + 1,$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{\left(t^3+t+1\right)'}{\left(t^3+1\right)'} = \frac{3t^2+1}{3t^2} = 1 + \frac{1}{3t^2}.$$

d) $\quad x = t\log t, \qquad y = \dfrac{\log t}{t},$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{\left(\dfrac{\log t}{t}\right)'}{(t\log t)'} = \frac{\dfrac{\dfrac{1}{t}\,t - \log t}{t^2}}{\log t + t\,\dfrac{1}{t}} = \frac{1-\log t}{t^2(1+\log t)}.$$

6. Odrediti vrednosti $y'(x)$ i $y'(x_0)$ ako je funkcija $y(x)$ zadata sa:

$$\textbf{a)}\quad y(x) = \arctan\left(\frac{\log x}{x}\right), \qquad x_0 = \frac{1}{e};$$

$$\textbf{b)}\quad xy - \sqrt{xy^2+6} = 0, \qquad x_0 = 3,\ y(x_0) = 1;$$

$$\textbf{c)}\quad y = (2+\cos x)^x, \qquad x_0 = 0;$$

$$\textbf{d)}\quad \begin{cases} x = t - \sin t, \\ y = 1 - \cos t, \end{cases} \qquad x_0 = \pi,\ y(x_0) = 2.$$

Rešenje: a) Funkcija $y(x) = \arctan\left(\dfrac{\log x}{x}\right)$ je diferencijabilna na $(0, +\infty)$ i u svakoj tački tog intervala je

$$y'(x) = \frac{1-\log x}{x^2+\log^2 x}.$$

Za $x_0 = 1/e = e^{-1}$ važi

$$y'(x_0) = y'\left(\frac{1}{e}\right) = \frac{1-\log e^{-1}}{e^{-2}+(\log e^{-1})^2} = \frac{1-(-1)}{\dfrac{1}{e^2}+(-1)^2} = \frac{2e^2}{1+e^2}.$$

b) Primenjujući postupak za određivanje izvoda implicitno zadate funkcije opisan u zadatku **3.** nalazimo izvod funkcije $y(x)$ u proizvoljnoj tački iz oblasti definisanosti:

$$y'(x) = \frac{y\left(y - 2\sqrt{xy^2+6}\right)}{2x\left(\sqrt{xy^2+6}-y\right)}.$$

Zato je

$$y'(x_0) = \frac{y(x_0)\left(y(x_0) - 2\sqrt{x_0\ (y(x_0))^2+6}\right)}{2x_0\left(\sqrt{x_0\, y(x_0)^2+6}-y(x_0)\right)},$$

tj.,

$$y'(3) = \frac{1\cdot\left(1-2\sqrt{3\cdot 1^2+6}\right)}{2\cdot 3\left(\sqrt{3\cdot 1^2+6}-1\right)} = -\frac{5}{12}.$$

c) Kao u zadatku **4.**, jednakost $y = (2+\cos x)^x$ logaritmujemo, a zatim diferenciramo:

$$\begin{aligned} \log y &= x\log(2+\cos x), \\ \frac{y'}{y} &= \log(2+\cos x) + x\frac{-\sin x}{2+\cos x}, \end{aligned}$$

odakle dobijamo

$$y'(x) = (2+\cos x)^{x-1}\left((2+\cos x)\log(2+\cos x) - x\sin x\right).$$

Specijalno,

$$y'(0) = (2+\cos 0)^{-1}\left((2+\cos 0)\log(2+\cos 0) - 0\cdot\sin 0\right) = \log 3.$$

d) Izvod parametarski zadate funkcije $x = t-\sin t$, $y = 1-\cos t$ u proizvoljnoj tački je

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t}.$$

Da bismo odredili vrednost patametra t_0 tako da je $x(t_0) = x_0$, $y(t_0) = y(x_0)$, rešavamo sistem jednačina

$$\begin{aligned} t - \sin t &= \pi, \\ 1 - \cos t &= 2. \end{aligned}$$

Iz druge jednačine zaključujemo da je $\cos t = -1$, tj. $t = \pi + 2k\pi$, $k \in \mathbb{Z}$. Zamenom u prvoj jednačini dobijamo

$$\pi + 2k\pi - \sin(\pi + 2k\pi) = \pi,$$

odakle je $2k\pi = 0$, tj. $k = 0$. Prema tome, $t_0 = \pi$, pa je

$$y'(x_0) = \frac{\sin t_0}{1-\cos t_0} = 0.$$

7. Odrediti $y''(x)$ i $y''(0)$ ako je funkcija $y(x)$ zadata eksplicitno:

a) $y = e^{\tan x} + \dfrac{1}{\cos x}$; **b**) $y = \arctan\sqrt{\dfrac{1+x}{1-x}}$.

Rešenje: a) $\qquad y = e^{\tan x} + \dfrac{1}{\cos x},$

$$y' = e^{\tan x}\frac{1}{\cos^2 x} - \frac{1}{\cos^2 x}\,(-\sin x) = \frac{e^{\tan x} + \sin x}{\cos^2 x},$$

$$y'' = (y')' = \left(\frac{e^{\tan x} + \sin x}{\cos^2 x}\right)'$$

$$= \frac{\left(e^{\tan x}\dfrac{1}{\cos^2 x} + \cos x\right)\cos^2 x - \left(e^{\tan x} + \sin x\right)(-2\cos x \sin x)}{\cos^4 x}$$

$$= \frac{e^{\tan x} + \cos^3 x + 2\sin x\ \cos x\ e^{\tan x} + 2\sin^2 x\ \cos x}{\cos^4 x}$$

$$= \frac{e^{\tan x}\,(1 + 2\sin x\cos x) + \cos x\left(\cos^2 x + 2\sin^2 x\right)}{\cos^4 x}$$

$$= \frac{e^{\tan x}\,(1 + \sin 2x) + \cos x\left(1 + \sin^2 x\right)}{\cos^4 x},$$

$$y''(0) = \frac{e^{\tan 0}\,(1 + \sin 0) + \cos 0\left(1 + \sin^2 0\right)}{\cos^4 0} = 2.$$

b) $\quad y = \arctan\sqrt{\dfrac{1+x}{1-x}},$

$$y' = \frac{1}{2\sqrt{1-x^2}},$$

$$y'' = \left(\frac{1}{2\sqrt{1-x^2}}\right)' = \frac{1}{2}\left(\left(1-x^2\right)^{-1/2}\right)' = \frac{1}{2}\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-3/2}(-2x)$$

$$= \frac{x}{2\sqrt{\left(1-x^2\right)^3}}, \qquad\qquad y''(0) = 0.$$

8. Odrediti $y''(x)$ i $y''(x_0)$ ako je funkcija $y(x)$ zadata implicitno:

a) $\quad x^2 + 2xy + y^2 - 4x + 2y = 2, \qquad x_0 = 1;$

b) $\quad e^{xy} = x + y, \qquad x_0 = 0;$

c) $\quad \log y + \dfrac{x}{y} = 1, \qquad x_0 = 0.$

Rešenje: a) Primetimo da su jednačinom

$$x^2 + 2xy + y^2 - 4x + 2y = 2$$

implicitno zadate dve funkcije $y = y_1(x)$ i $y = y_2(x)$. Primenom postupka opisanog u zadatku **3.** dobijamo njihov prvi izvod:

$$y' = \frac{2 - x - y}{1 + x + y}.$$

Diferenciranjem dobijene jednakosti nalazimo drugi izvod obeju funkcija u proizvoljnoj tački oblasti definisanosti:

$$\begin{aligned} y'' = (y')' &= \left(\frac{2 - x - y}{1 + x + y}\right)' \\ &= \frac{(2 - x - y)'(1 + x + y) - (2 - x - y)(1 + x + y)'}{(1 + x + y)^2} \\ &= \frac{(-1 - y')(1 + x + y) - (2 - x - y)(1 + y')}{(1 + x + y)^2} \\ &= \frac{-3(1 + y')}{(1 + x + y)^2}. \end{aligned}$$

Za $x = 1$ imamo

$$y''(1) = \frac{-3(1 + y'(1))}{(1 + 1 + y(1))^2}.$$

Za određivanje $y(1)$ zamenimo $x = 1$ u jednačini

$$x^2 + 2xy + y^2 - 4x + 2y = 2$$

i dobijamo

$$1 + 2y(1) + y(1)^2 - 4 + 2y(1) = 2,$$

tj.

$$y(1)^2 + 4y(1) - 5 = 0,$$

čija su rešenja $y_1(1) = 1$ i $y_2(1) = -5$. Sada je

$$y_1'(1) = \frac{2 - 1 - y_1(1)}{1 + 1 + y_1(1)} = 0, \qquad y_2'(1) = \frac{2 - 1 - y_2(1)}{1 + 1 + y_2(1)} = \frac{6}{-3} = -2$$

i

$$y_1''(1) = \frac{-3(1+y_1'(1))}{(1+1+y_1(1))^2} = \frac{-3}{3^2} = -\frac{1}{3},$$
$$y_2''(1) = \frac{-3(1+y_2'(1))}{(1+1+y_2(1))^2} = \frac{3}{(-3)^2} = \frac{1}{3}.$$

b) Izvodi funkcije implicitno zadate sa $e^{xy} = x + y$ u proizvoljnoj tački su

$$y' = \frac{1 - ye^{xy}}{xe^{xy} - 1},$$

$$\begin{aligned}
y'' &= (y')' = \left(\frac{1 - ye^{xy}}{xe^{xy} - 1}\right)' \\
&= \frac{(1 - ye^{xy})'(xe^{xy} - 1) - (1 - ye^{xy})(xe^{xy} - 1)'}{(xe^{xy} - 1)^2} \\
&= \frac{(-y'e^{xy} - ye^{xy}(y + xy'))\,(xe^{xy} - 1) - (1 - ye^{xy})\,(e^{xy} + xe^{xy}(y + xy'))}{(xe^{xy} - 1)^2} \\
&= -e^{xy}\;\frac{(y' + y(y + xy'))\,(xe^{xy} - 1) + (1 - ye^{xy})\,(1 + x(y + xy'))}{(xe^{xy} - 1)^2} \\
&= e^{xy}\;\frac{(1 - x(e^{xy} + x - y))\,y' - 1 + y(e^{xy} - x + y)}{(xe^{xy} - 1)^2}.
\end{aligned}$$

Zamenom $x = 0$ u jednačini $e^{xy} = x + y$ dobijamo $y(0) = 1$, pa je

$$y'(0) = \frac{1 - 1\cdot e^{0\cdot 1}}{0\cdot e^{0\cdot 1} - 1} = 0,$$
$$y''(0) = e^{0\cdot 1}\;\frac{\big(1 - 0\cdot(e^{0\cdot 1} + 0 - 1)\big)\cdot 0 - 1 + 1\cdot(e^{0\cdot 1} - 0 + 1)}{(0\cdot e^{0\cdot 1} - 1)^2} = 1.$$

c) Jednačinu možemo da transformišemo u oblik $y\log y + x = y$, a zatim diferenciranjem i rešavanjem dobijene jednačine po y' dobijamo prvi izvod funkcije u proizvoljnoj tački:

$$y' = -\frac{1}{\log y}.$$

Ponovnim diferenciranjem dobijamo i drugi izvod:

$$y'' = (y')' = -\left(\frac{1}{\log y}\right)' = \frac{1}{\log^2 y}\ \frac{y'}{y} = \frac{y'}{y\log^2 y}.$$

Vrednost funkcije u tački $x = 0$ je $y(0) = e$, a vrednosti izvoda su

$$y'(0) = -1, \qquad y''(0) = -\frac{1}{e}.$$

Primetimo da se bez transformacije polazne jednačine dobijaju drugačiji oblici prvog i drugog izvoda funkcije:

$$y' = \frac{y}{x-y}, \qquad y'' = \frac{xy' - y}{(x-y)^2}.$$

Oni su ekvivalentni onima koji su prethodno dobijeni, što se može pokazati korišćenjem polazne jednačine.

9. Odrediti $y''(x)$ i $y''(x_0)$ ako je funkcija $y(x)$ zadata parametarski:

$$\textbf{a)}\ \begin{cases} x = e^{-2t}, \\ y = e^{2t}, \end{cases} \qquad x_0 = e;$$

$$\textbf{b)}\ \begin{cases} x = 2(t - \cos t), \\ y = 2(1 - \sin t), \end{cases} \qquad x_0 = \pi.$$

Rešenje: a) Prvi izvod parametarski zadate funkcije u proizvoljnoj tački je

$$y'(x) = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{(e^{2t})'}{(e^{-2t})'} = \frac{2e^{2t}}{-2e^{-2t}} = -e^{4t}.$$

Drugi izvod određujemo na sledeći način:

$$y''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-e^{4t}\right).$$

Primenjujući pravila za izvod složene funkcije:

$$\frac{d}{dx}\left(-e^{4t}\right) = \frac{d}{dt}\left(-e^{4t}\right)\frac{dt}{dx} = -4e^{4t}\frac{dt}{dx}$$

i izvod inverzne funkcije:

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{-2e^{-2t}},$$

dobijamo

$$y''(x) = -4e^{4t}\frac{1}{-2e^{-2t}} = 2e^{6t}.$$

Tačka čija je apscisa $x_0 = e$ dobija se za vrednost parametra $t_0 = -1/2$. Zato je

$$y''(e) = 2e^{-3} = \frac{2}{e^3}.$$

b) Slično kao u prethodnom zadatku određujemo:

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(2(1-\sin t))'}{(2(t-\cos t))'} = \frac{-2\cos t}{2(1+\sin t)} = -\frac{\cos t}{1+\sin t},$$

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{\cos t}{1+\sin t}\right) = \frac{d}{dt}\left(-\frac{\cos t}{1+\sin t}\right)\frac{dt}{dx} \\ &= \frac{1}{1+\sin t}\,\frac{1}{\frac{dx}{dt}} = \frac{1}{1+\sin t}\,\frac{1}{2(1+\sin t)} = \frac{1}{2(1+\sin t)^2}. \end{aligned}$$

Specijalno, $2(t-\cos t) = \pi$ se dobija za $t_0 = \pi/2$, pa je

$$y''(\pi) = \frac{1}{2(1+\sin\pi/2)^2} = \frac{1}{8}.$$

10. Dokazati da je funkcija

$$y(x) = e^{\sqrt{x}} + e^{-\sqrt{x}}$$

rešenje diferencijalne jednačine

$$xy'' + \frac{1}{2}y' - \frac{1}{4}y = 0.$$

Rešenje: Izvodi zadate funkcije su

$$y' = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}, \qquad y'' = \frac{e^{\sqrt{x}}\left(\sqrt{x}-1\right) + e^{-\sqrt{x}}\left(\sqrt{x}+1\right)}{4x\sqrt{x}}.$$

Zamenom u levoj strani jednačine dobija se

$$
\begin{aligned}
& xy'' + \frac{1}{2}y' - \frac{1}{4}y \\
&= \frac{e^{\sqrt{x}}\left(\sqrt{x}-1\right) + e^{-\sqrt{x}}\left(\sqrt{x}+1\right)}{4\sqrt{x}} + \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{4\sqrt{x}} - \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4} \\
&= \frac{e^{\sqrt{x}}\left(\sqrt{x}-1+1\right) + e^{-\sqrt{x}}\left(\sqrt{x}+1-1\right)}{4\sqrt{x}} - \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4} = 0.
\end{aligned}
$$

11. Dokazati da funkcija

$$f(x) = \log\frac{e^{2x}-1}{e^{2x}+1}$$

zadovoljava diferencijalnu jednačinu

$$2(e^{4x}+1)f'(x) + (e^{4x}-1)f''(x) = 0.$$

Rešenje: Kako je

$$f'(x) = \frac{4e^{2x}}{e^{4x}-1}, \qquad f''(x) = -\frac{8e^{2x}\left(e^{4x}+1\right)}{\left(e^{4x}-1\right)^2},$$

imamo

$$
\begin{aligned}
& 2\left(e^{4x}+1\right)f'(x) + \left(e^{4x}-1\right)f''(x) \\
&= 2\left(e^{4x}+1\right)\frac{4e^{2x}}{e^{4x}-1} - \left(e^{4x}-1\right)\frac{8e^{2x}\left(e^{4x}+1\right)}{\left(e^{4x}-1\right)^2} = 0.
\end{aligned}
$$

12. Odrediti $y^{(n)}(x)$ $(n \in \mathbb{N})$ ako je:

a) $y = \sin(ax+b)$; **b**) $y = \cos(ax+b)$;

c) $y = e^{ax+b}$; **d**) $y = \log(ax+b)$.

Rešenje: a) Imajući u vidu da je prvih nekoliko izvoda funkcije jednako:

$$
\begin{aligned}
y' &= (\sin(ax+b))' = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right), \\
y'' &= \left(a\sin\left(ax+b+\frac{\pi}{2}\right)\right)' = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+2\cdot\frac{\pi}{2}\right), \\
y''' &= \left(a^2\sin\left(ax+b+2\cdot\frac{\pi}{2}\right)\right)' = a^3\cos\left(ax+b+2\cdot\frac{\pi}{2}\right) = a^3\sin\left(ax+b+3\cdot\frac{\pi}{2}\right), \\
y^{(4)} &= \left(a^3\sin\left(ax+b+3\cdot\frac{\pi}{2}\right)\right)' = a^4\cos\left(ax+b+3\cdot\frac{\pi}{2}\right) = a^4\sin\left(ax+b+4\cdot\frac{\pi}{2}\right),
\end{aligned}
$$

možemo da pretpostavimo da je izvod reda n ($n \in \mathbb{N}$) oblika

$$y^{(n)} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right).$$

Dokaz izvodimo matematičkom indukcijom. Za $n = 1$ tvrđenje važi. Pretpostavimo da važi za neki prirodni broj k, tj. da je

$$y^{(k)} = a^k \sin\left(ax + b + \frac{k\pi}{2}\right).$$

Tada je

$$\begin{aligned} y^{(k+1)} &= \left(y^{(k)}\right)' = \left(a^k \sin\left(ax + b + \frac{k\pi}{2}\right)\right)' = a^{k+1} \cos\left(ax + b + \frac{k\pi}{2}\right) \\ &= a^{k+1} \sin\left(ax + b + \frac{k\pi}{2} + \frac{\pi}{2}\right) = a^{k+1} \sin\left(ax + b + \frac{(k+1)\pi}{2}\right), \end{aligned}$$

što znači da važi i za prirodan broj $k+1$. Prema tome, tvrđenje važi za svaki prirodan broj n, tj.

$$y^{(n)} = (\sin(ax+b))^{(n)} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right).$$

b) Na isti način dobijamo

$$(\cos(ax+b))^{(n)} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right) \qquad (n \in \mathbb{N}).$$

c) Kako za funkciju $y = e^{ax+b}$ važi:

$$y' = ae^{ax+b}, \quad y'' = a^2 e^{ax+b}, \quad y''' = a^3 e^{ax+b}, \ldots$$

pretpostavljamo da je

$$y^{(n)} = \left(e^{ax+b}\right)^{(n)} = a^n e^{ax+b} \qquad (n \in \mathbb{N}).$$

Tvrđenje se dokazuje matematičkom indukcijom na prethodno opisan način.

d) Za funkciju $y = \log(ax+b)$ imamo:

$$y' = \frac{a}{ax+b}, \quad y'' = -\frac{a^2}{(ax+b)^2}, \quad y''' = \frac{2a^3}{(ax+b)^3}, \quad y^{(4)} = -\frac{3 \cdot 2a^4}{(ax+b)^4}.$$

S obzirom na prva četiri izvoda pretpostavljamo da je

$$y^{(n)} = (-1)^{n-1} \frac{a^n (n-1)!}{(ax+b)^n}$$

i dokazujemo matematičkom indukcijom. Za $n = 1$ tvrđenje važi. Iz indukcijske pretpostavke da tvrđenje važi za $n = k$, tj.

$$y^{(k)} = (-1)^{k-1} \frac{a^k (k-1)!}{(ax+b)^k},$$

sledi

$$\begin{aligned} y^{(k+1)} &= \left(y^{(k)}\right)' = \left((-1)^{k-1} \frac{a^k (k-1)!}{(ax+b)^k}\right)' = (-1)^{k-1} a^k (k-1)! \left((ax+b)^{-k}\right)' \\ &= (-1)^{k-1} a^k (k-1)! \left(-ka(ax+b)^{-k-1}\right) = (-1)^k \frac{a^{k+1} k!}{(ax+b)^{k+1}}, \end{aligned}$$

što znači da važi i za $n = k+1$. Prema tome, važi

$$y^{(n)} = (\log(ax+b))^{(n)} = (-1)^{n-1} \frac{a^n (n-1)!}{(ax+b)^n} \qquad (n \in \mathbb{N}).$$

13. Ako je $n \in \mathbb{N}$, odrediti:

a) $\left(\frac{1}{2x-3}\right)^{(n)}$; **b)** $\left(\frac{2x+3}{2x-3}\right)^{(n)}$; **c)** $\left(\frac{1}{4x^2-9}\right)^{(n)}$.

Rešenje: a) Na način opisan u zadatku **12. d)** dokazujemo da je

$$\left(\frac{1}{2x-3}\right)^{(n)} = \frac{(-1)^n 2^n n!}{(2x-3)^{n+1}}.$$

b) Kako je

$$\frac{2x+3}{2x-3} = 1 + \frac{6}{2x-3},$$

važi

$$\left(\frac{2x+3}{2x-3}\right)^{(n)} = \left(1 + \frac{6}{2x-3}\right)^{(n)} = 6\left(\frac{1}{2x-3}\right)^{(n)} = \frac{(-1)^n 6 \cdot 2^n n!}{(2x-3)^{n+1}}.$$

c) Transformišemo datu racionalnu funkciju na sledeći način:

$$\frac{1}{4x^2-9} = \frac{1}{(2x+3)(2x-3)} = \frac{1}{6}\left(\frac{1}{2x-3} - \frac{1}{2x+3}\right),$$

odakle dobijamo

$$\begin{aligned}
\left(\frac{1}{4x^2-9}\right)^{(n)} &= \frac{1}{6}\left(\frac{1}{2x-3} - \frac{1}{2x+3}\right)^{(n)} = \frac{1}{6}\left(\left(\frac{1}{2x-3}\right)^{(n)} - \left(\frac{1}{2x+3}\right)^{(n)}\right) \\
&= \frac{1}{6}\left(\frac{(-1)^n 2^n n!}{(2x-3)^{n+1}} - \frac{(-1)^n 2^n n!}{(2x+3)^{n+1}}\right) \\
&= \frac{(-1)^n 2^n n!}{6}\left(\frac{1}{(2x-3)^{n+1}} - \frac{1}{(2x+3)^{n+1}}\right).
\end{aligned}$$

14. Odrediti

$$\left(\frac{1}{\sqrt{1-x}}\right)^{(n)} \qquad (n \in \mathbb{N}),$$

a zatim, koristeći dobijeni rezultat, odrediti i

$$\left(\frac{1+x+x^2}{\sqrt{1-x}}\right)^{(n)} \qquad (n \in \mathbb{N}).$$

Rešenje: Neka je $f(x) = \dfrac{1}{\sqrt{1-x}}$. Tada je:

$$\begin{aligned}
f'(x) &= \left((1-x)^{-1/2}\right)' = \frac{1}{2}(1-x)^{-3/2}, \\
f''(x) &= \frac{1}{2}\left((1-x)^{-3/2}\right)' = \frac{1}{2}\,\frac{3}{2}\,(1-x)^{-5/2}, \\
f'''(x) &= \frac{1}{2}\,\frac{3}{2}\left((1-x)^{-5/2}\right)' = \frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}(1-x)^{-7/2}.
\end{aligned}$$

Uočavanjem pravilnosti pretpostavljamo da za proizvoljno $n \in \mathbb{N}$ važi

$$f^{(n)}(x) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}(1-x)^{-(2n+1)/2}. = \frac{(2n-1)!!}{2^n\sqrt{(1-x)^{2n+1}}}.$$

Dokaz matematičkom indukcijom opisan u zadatku **12.** biće izostavljen.

Za određivanje $\left(\dfrac{1+x+x^2}{\sqrt{1-x}}\right)^{(n)}$ primenjujemo Lajbnicovu formulu

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x),$$

pri čemu je $g(x) = 1 + x + x^2$. Kako je

$$g'(x) = 1 + 2x, \qquad g''(x) = 2, \qquad g^{(k)}(x) = 0, \quad k = 3, 4, \ldots$$

u navedenoj sumi su svi sabirci za $k = 3, 4, \ldots, n$ jednaki nuli. Zato je

$$\begin{aligned}
(f(x)g(x))^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x) \\
&= \binom{n}{0} f^{(n)}(x) g(x) + \binom{n}{1} f^{(n-1)}(x) g'(x) + \binom{n}{2} f^{(n-2)}(x) g''(x) \\
&= \frac{(2n-1)!!}{2^n \sqrt{(1-x)^{2n+1}}}(1 + x + x^2) + n \frac{(2n-3)!!}{2^{n-1}\sqrt{(1-x)^{2n-1}}}(1 + 2x) \\
&\quad + n(n-1) \frac{(2n-5)!!}{2^{n-2}\sqrt{(1-x)^{2n-3}}}.
\end{aligned}$$

Sređivanjem poslednjeg izraza dobijamo

$$\left(\frac{1+x+x^2}{\sqrt{1-x}}\right)^{(n)} = \frac{3(2n-5)!!}{2^n \sqrt{(1-x)^{2n+1}}} \left((1 - 6n + 4n^2) - (2n-1)x + x^2\right).$$

15. Za $n \in \mathbb{N}$ odrediti

a) $\left(e^{3x+2}\right)^{(n)}$; **b)** $\left((3x^2 + 2x + 1)e^{3x+2}\right)^{(n)}$.

Rešenje: a) Izvod reda n funkcije $f(x) = e^{3x+2}$ je (videti zadatak **12. c)**)

$$f^{(n)}(x) = \left(e^{3x+2}\right)^{(n)} = 3^n e^{3x+2}.$$

b) Izvod $\left((3x^2 + 2x + 1)e^{3x+2}\right)^{(n)}$ određujemo primenom Lajbnicove formule za izvod proizvoda funkcija

$$f(x) = e^{3x+2} \qquad \text{i} \qquad g(x) = 3x^2 + 2x + 1.$$

Izvodi ovih funkcija su

$$f^{(k)}(x) = 3^k e^{3x+2}, \quad k = 0, 1, 2, \dots,$$

$$g^{(0)}(x) = 3x^2 + 2x + 1, \qquad g'(x) = 6x + 2, \qquad g''(x) = 6,$$
$$g^{(k)}(x) = 0, \quad k = 3, 4, \dots,$$

pa važi:

$$\begin{aligned}
&\left((3x^2 + 2x + 1)e^{3x+2}\right)^{(n)} = (f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x) \\
&= \binom{n}{0} f^{(n)}(x) g^{(0)}(x) + \binom{n}{1} f^{(n-1)}(x) g'(x) + \binom{n}{2} f^{(n-2)}(x) g''(x) \\
&= \binom{n}{0} 3^n e^{3x+2}(3x^2 + 2x + 1) + \binom{n}{1} 3^{n-1} e^{3x+2}(6x + 2) + \binom{n}{2} 3^{n-2} e^{3x+2} 6 \\
&= 3^{n-1} e^{3x+2} \left(3(3x^2 + 2x + 1) + 2n(3x + 1) + n(n-1)\right) \\
&= 3^{n-1} e^{3x+2} \left(9x^2 + 6(n+1)x + n^2 + n + 3\right).
\end{aligned}$$

16. Odrediti $\left(2^{3x}3^{2x}\right)^{(n)}$ $(n \in \mathbb{N})$.

Rešenje: Matematičkom indukcijom se može dokazati da za proizvoljno $n \in \mathbb{N}$ važi:

$$\left(2^{3x}\right)^{(n)} = (3 \log 2)^n 2^{3x}, \qquad \left(3^{2x}\right)^{(n)} = (2 \log 3)^n 3^{2x}.$$

Primenom Lajbnicove formule dobijamo:

$$\begin{aligned}
\left(2^{3x}3^{2x}\right)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} \left(2^{3x}\right)^{(k)} \left(3^{2x}\right)^{(n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left((3 \log 2)^n 2^{3x}\right) \left((2 \log 3)^{n-k} 3^{2x}\right) \\
&= (2 \log 3)^n 2^{3x} 3^{2x} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3 \log 2}{2 \log 3}\right)^k \\
&= (2 \log 3)^n 2^{3x} 3^{2x} \left(1 + \frac{3 \log 2}{2 \log 3}\right)^n \\
&= (2 \log 3)^n 2^{3x} 3^{2x} \frac{(2 \log 3 + 3 \log 2)^n}{(2 \log 3)^n} \\
&= 2^{3x} 3^{2x} \log^n 72.
\end{aligned}$$

Do istog rezultata se može doći i bez korišćenja Lajbnicove formule na sledeći način:

$$\left(2^{3x}3^{2x}\right)^{(n)} = \left(\left(2^3 \cdot 3^2\right)^x\right)^{(n)} = \left(72^x\right)^{(n)} = 72^x \log 72 = 2^{3x}3^{2x} \log^n 72.$$

17. Dokazati da funkcija

$$f(x) = \arctan \sqrt{\frac{1+x}{1-x}}$$

zadovoljava diferencijalnu jednačinu

$$\left(1-x^2\right)f''(x) - xf'(x) = 0$$

i odrediti $f'(0)$ i $f''(0)$. Da li se može odrediti $f^{(n)}(0)$ za proizvoljno $n \in \mathbb{N}$?

Rešenje: Izvodi date funkcije su

$$f'(x) = \frac{1}{2\sqrt{1-x^2}}, \qquad f''(x) = \frac{x}{2\left(1-x^2\right)\sqrt{1-x^2}},$$

pa je zaista

$$\left(1-x^2\right)f''(x) - xf'(x) = \left(1-x^2\right)\frac{x}{2\left(1-x^2\right)\sqrt{1-x^2}} - x\frac{1}{2\sqrt{1-x^2}} = 0.$$

Da bismo odredili $f^{(n)}(0)$ za proizvoljno $n \in \mathbb{N}$, potražimo $n$ti izvod izraza na levoj i desnoj strani jednakosti primenom Lajbnicove formule:

$$\begin{gathered}\left(1-x^2\right)f''(x) = xf'(x), \\ \left(\left(1-x^2\right)f''(x)\right)^{(n)} = \left(xf'(x)\right)^{(n)}, \\ \sum_{k=0}^{n}\binom{n}{k}\left(1-x^2\right)^{(k)}\left(f''(x)\right)^{(n-k)} = \sum_{k=0}^{n}\binom{n}{k}(x)^{(k)}\left(f'(x)\right)^{(n-k)}.\end{gathered}$$

Kako je $\quad (f''(x))^{(n-k)} = f^{(n-k+2)}(x), \ (f'(x))^{(n-k)} = f^{(n-k+1)}(x),$

$$\left(1-x^2\right)' = -2x, \qquad \left(1-x^2\right)'' = -2, \qquad \left(1-x^2\right)^{(k)} = 0, \quad k = 3, 4, \ldots$$

i

$$(x)' = 1, \qquad (x)^{(k)} = 0, \quad k = 2, 3, \ldots,$$

imamo:

$$\binom{n}{0}(1-x^2)f^{(n+2)}(x)+\binom{n}{1}(-2x)f^{(n+1)}(x)+\binom{n}{2}(-2)f^{(n)}(x)$$
$$=\binom{n}{0}xf^{(n+1)}(x)+\binom{n}{1}f^{(n)}(x),$$

tj.

$$(1-x^2)f^{(n+2)}(x)-2nxf^{(n+1)}(x)-n(n-1)f^{(n)}(x)=xf^{(n+1)}(x)+nf^{(n)}(x).$$

Za $x=0$ jednakost postaje

$$f^{(n+2)}(0)=n^2f^{(n)}(0), \qquad n\in\mathbb{N}.$$

Imajući u vidu da je $f'(0)=1/2$ i $f''(0)=0$, važi sledeće:

$$\begin{array}{lll} f'''(0)=\dfrac{1}{2}, & f^{(4)}(0)=0, & f^{(5)}(0)=3^2\cdot\dfrac{1}{2}, \\ f^{(6)}(0)=0, & f^{(7)}(0)=5^2\cdot 3^2\cdot\dfrac{1}{2}, & f^{(8)}(0)=0. \end{array}$$

Pretpostavku da je za proizvoljno $k\in\mathbb{N}$

$$f^{(2k+1)}(0)=((2k-1)!!)^2\,\frac{1}{2}, \qquad f^{(2k)}(0)=0.$$

treba dokazati matematičkom indukcujom, što prepuštamo čitaocu.