Numerička integracija- Njutn-Koutsove kvadraturne formule

I Uopštena trapezna formula

Interval integracije $[a, b]$ deli se sa $n+1$ ekvidistantnih tačaka $x_j = a + jh,\ j = 0, 1, \ldots, n$, na n podintervala jednake dužine $h = \dfrac{b-a}{n}$. Označimo $f_j = f(x_j),\ j = 0, 1, \ldots, n$.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right) + R_{n+1}(f),$$
$$R_{n+1}(f) = -\frac{b-a}{12} f''(\xi) h^2, \qquad \xi \in (a, b).$$

II Uopštena Simpsonova formula

Interval integracije $[a, b]$ deli se sa $2n+1$ ekvidistantnih tačaka $x_j = a + jh,\ j = 0, 1, \ldots, 2n$, na $2n$ podintervala jednake dužine $h = \dfrac{b-a}{2n}$. Označimo $f_j = f(x_j),\ j = 0, 1, \ldots, 2n$.

$$\begin{aligned}\int_a^b f(x)\,dx = \frac{h}{3}&\left(f_0 + 4(f_1 + \cdots + f_{2n-1}) + 2(f_2 + f_4 + \cdots + f_{2n-2}) + f_{2n}\right)\\ &+ R_{2n+1}(f),\end{aligned}$$
$$R_{2n+1}(f) = -\frac{b-a}{180} f^{(4)}(\xi) h^4, \qquad \xi \in (a, b).$$

II Rungeova ocena greške

Za Njutn–Koutsovu formulu $I(f) = I(f;h) + R(f;h)$ Rungeova ocena greške je

$$R(f;h) = \frac{I(f;h) - I(f;2h)}{2^k - 1}.$$

- Trapezna formula: $k = 2$
- Simpsonova formula: $k = 4$

Ako je $\varepsilon > 0$ zadata tačnost, primenjuje se ista formula $I(f;h)$ sa korakom h_0, $h_1 = h_0/2$, $h_2 = h_1/2, \ldots h_j = h_{j-1}/2$, dok ne bude ispunjen uslov

$$|R(f;h_j)| < \varepsilon.$$

Tada je približna vrednost integrala

$$I(f) = I(f,h_j) + \frac{I(f;h_j) - I(f;h_{j-1})}{2^k - 1}.$$

ZADACI

Zadatak 1. Pomoću uopštene trapezne formule izračunati približnu vrednost određenog integrala

a) $\displaystyle\int_{-1}^{1} (x^4 + 2)\, dx;$ **b)** $\displaystyle\int_{0}^{1} \arctan x\, dx;$

c) $\displaystyle\int_{0}^{\pi/2} \sqrt{1 - 0.5\sin^2 x}\, dx;$

deljenjem intervala integracije na 5, 10 i 20 delova redom i proceniti grešku metoda.
Sa koliko decimala treba računati da greška zaokruživanja ne utiče na tačnost?

Rešenje:

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}(f_0+2(f_1+f_2+...f_{n-1})+f_n) - \frac{(b-a)^3}{12\,n^2} f''(\xi), \quad \xi\in(a,b)$$

$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(f_0+2(f_1+f_2+...f_{n-1})+f_n) - \frac{b-a}{12}\ h^2\ f''(\xi), \quad \xi\in(a,b)$$

$$R_{n+1} \le -\frac{(-a+b)^3 M_2}{12\,n^2}, \qquad M_2 = \max\{|f''(x)|,\ a\le x\le b\}$$

a. $f(x)=x^4+2$, Tacna vrednost: $I_1=\int_{-1}^{1}(x^4+2)\ dx = \frac{22}{5} = \frac{2}{5} + 2\times 2.0$

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$f(x)=2+x^4$

$f''(x)=12\,x^2$, $M_2=\max\{|f''(x)|,\ a\le x\le b\} = 12.$

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n=5

$I \approx 0.2\left(f[-1.]+2\left(f[-0.6]+f[-0.2]+f[0.2]+f[0.6]\right)+f[1.]\right)$

$$x_k=\begin{pmatrix}-1.\\-0.6\\-0.2\\0.2\\0.6\\1.\end{pmatrix}, \qquad f_k=f(x_k)=\begin{pmatrix}3.\\2.1296\\2.0016\\2.0016\\2.1296\\3.\end{pmatrix}$$

$I_1 \approx I_T = 4.50496$

$|R_{n+1}| \le 0.32$, prava greška: $I_1-I_T = -0.10496$

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n=10

$I \approx 0.1\left(f[-1.]+\right.$
$2\left(f[-0.8]+f[-0.6]+f[-0.4]+f[-0.2]+f[0.]+f[0.2]+f[0.4]+f[0.6]+f[0.8]\right)$
$\left.+f[1.]\right)$

$$x_k = \begin{pmatrix} -1. \\ -0.8 \\ -0.6 \\ -0.4 \\ -0.2 \\ 0. \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1. \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 3. \\ 2.4096 \\ 2.1296 \\ 2.0256 \\ 2.0016 \\ 2. \\ 2.0016 \\ 2.0256 \\ 2.1296 \\ 2.4096 \\ 3. \end{pmatrix}$$

$I_1 \approx I_T = 4.42656$

$|R_{n+1}| \leq 0.08$, prava greška: $I_1 - I_T = -0.02656$

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n=20

$$I \approx 0.05 \big(f[-1.] + 2\,(f[-0.9] + f[-0.8] + f[-0.7] + f[-0.6] + f[-0.5] + f[-0.4] + f[-0.3] + f[-0.2] + f[-0.1] + f[0.] + f[0.1] + f[0.2] + f[0.3] + f[0.4] + f[0.5] + f[0.6] + f[0.7] + f[0.8] + f[0.9]) + f[1.]\big)$$

$$x_k = \begin{pmatrix} -1. \\ -0.9 \\ -0.8 \\ -0.7 \\ -0.6 \\ -0.5 \\ -0.4 \\ -0.3 \\ -0.2 \\ -0.1 \\ 0. \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 1. \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 3. \\ 2.6561 \\ 2.4096 \\ 2.2401 \\ 2.1296 \\ 2.0625 \\ 2.0256 \\ 2.0081 \\ 2.0016 \\ 2.0001 \\ 2. \\ 2.0001 \\ 2.0016 \\ 2.0081 \\ 2.0256 \\ 2.0625 \\ 2.1296 \\ 2.2401 \\ 2.4096 \\ 2.6561 \\ 3. \end{pmatrix}$$

$I_1 \approx I_T = 4.40666$

$|R_{n+1}| \leq 0.02$, prava greška: $I_1 - I_T = -0.00666$

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Posto je greška metoda $\leq$ 0.02, dovoljno je da funkciju računamo sa 2 decimale.

$n=20$

$$x_k=\begin{pmatrix} -1.0 \\ -0.90 \\ -0.80 \\ -0.70 \\ -0.60 \\ -0.50 \\ -0.40 \\ -0.30 \\ -0.20 \\ -0.10 \\ 0 \\ 0.10 \\ 0.20 \\ 0.30 \\ 0.40 \\ 0.50 \\ 0.60 \\ 0.70 \\ 0.80 \\ 0.90 \\ 1.0 \end{pmatrix}, \qquad f_k=f(x_k)=\begin{pmatrix} 3.00 \\ 2.66 \\ 2.41 \\ 2.24 \\ 2.13 \\ 2.06 \\ 2.03 \\ 2.01 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.01 \\ 2.03 \\ 2.06 \\ 2.13 \\ 2.24 \\ 2.41 \\ 2.66 \\ 3.00 \end{pmatrix}$$

$I_1 \approx I_T = 4.41$

$|R_{n+1}| \leq 0.02$, prava greška: $I_1 - I_T = 0. \times 10^{-3}$

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Zadatak 2. Pomoću uopštene Simpsonove formule izračunati približnu vrednost određenog integrala

$$\textbf{a)}\quad \int_{-1}^{1}(x^4+2)\,dx; \qquad \textbf{b)}\quad \int_{0}^{1}\arctan x\,dx;$$

$$\textbf{c)}\quad \int_{0}^{\pi/2}\sqrt{1-0.5\sin^2 x}\,dx;$$

deljenjem intervala integracije na 10 i 20 delova redom i proceniti grešku metoda.

Rešenje:

$$\int_a^b f(x)\,dx = \frac{b-a}{6n}\left(f_0+4(f_1+f_3+\ldots f_{2n-1})+2(f_2+f_4+\ldots f_{2n-2})+f_n\right)-\frac{(b-a)^5}{2880\,n^4}f^{IV}(\xi),\quad \xi\in(a,b)$$

$$h = \frac{b-a}{2n}$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f_0+4(f_1+f_3+\ldots f_{2n-1})+2(f_2+f_4+\ldots f_{2n-2})+f_n\right)-\frac{b-a}{180}h^4\,f^{IV}(\xi),\quad \xi\in(a,b)$$

$$R_{n+1} \le -\frac{(-a+b)^4 M_4}{2880\,n^4},\qquad M_4 = \max\{|f^{(4)}(x)|,\ a\le x\le b\}$$

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c. $f(x)=\sqrt{1-0.5\sin^2 x}$, Tacna vrednost $I_3=\int_0^{\pi/2}\sqrt{1-0.5\sin^2 x}\,dx$ nepoznata.

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$$f(x)=\sqrt{1-0.5\,\mathrm{Sin}[x]^2}$$

$$f^{(4)}(x)=\left(\left((116.375+213.\,\mathrm{Cos}[2x]-12.5\,\mathrm{Cos}[4x]+3.\,\mathrm{Cos}[6x]+0.125\,\mathrm{Cos}[8x])\sqrt{1-0.5\,\mathrm{Sin}[x]^2}\right)\Big/(3.+1.\,\mathrm{Cos}[2x])^4\right)$$

$$M_4=\max\{|f^{(4)}(x)|,\ a\le x\le b\} = 4.94975$$

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2n=10

$$I_3 \approx 0.0523599 \Big(f[0.] + 2 \big(f[0.314159] + f[0.628319] + f[0.942478] + f[1.25664]\big) + 4 \big(f[0.15708] + f[0.471239] + f[0.785398] + f[1.09956] + f[1.41372]\big) + f[1.5708]\Big)$$

$$x_k = \begin{pmatrix} 0. \\ 0.15708 \\ 0.314159 \\ 0.471239 \\ 0.628319 \\ 0.785398 \\ 0.942478 \\ 1.09956 \\ 1.25664 \\ 1.41372 \\ 1.5708 \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 1. \\ 0.993863 \\ 0.975835 \\ 0.947072 \\ 0.909535 \\ 0.866025 \\ 0.820211 \\ 0.776565 \\ 0.740098 \\ 0.715707 \\ 0.707107 \end{pmatrix}$$

$I_3 \approx I_S = 1.35064$

$|R_{n+1}| \leq 0.0000167413$

2n=20

$$I_3 \approx 0.0261799 \Big(f[0.] + 2 \big(f[0.15708] + f[0.314159] + f[0.471239] + f[0.628319] + f[0.785398] + f[0.942478] + f[1.09956] + f[1.25664] + f[1.41372]\big) + 4 \big(f[0.0785398] + f[0.235619] + f[0.392699] + f[0.549779] + f[0.706858] + f[0.863938] + f[1.02102] + f[1.1781] + f[1.33518] + f[1.49226]\big) + f[1.5708]\Big)$$

$$x_k = \begin{pmatrix} 0. \\ 0.0785398 \\ 0.15708 \\ 0.235619 \\ 0.314159 \\ 0.392699 \\ 0.471239 \\ 0.549779 \\ 0.628319 \\ 0.706858 \\ 0.785398 \\ 0.863938 \\ 0.942478 \\ 1.02102 \\ 1.09956 \\ 1.1781 \\ 1.25664 \\ 1.33518 \\ 1.41372 \\ 1.49226 \\ 1.5708 \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 1. \\ 0.99846 \\ 0.993863 \\ 0.986282 \\ 0.975835 \\ 0.962692 \\ 0.947072 \\ 0.929246 \\ 0.909535 \\ 0.888318 \\ 0.866025 \\ 0.843144 \\ 0.820211 \\ 0.797811 \\ 0.776565 \\ 0.757115 \\ 0.740098 \\ 0.726119 \\ 0.715707 \\ 0.70928 \\ 0.707107 \end{pmatrix}$$

$I_3 \approx I_S = 1.35064$

$|R_{n+1}| \leq 1.04633 \times 10^{-6}$

Zadatak 3. Sa zadatom tačnošću $\varepsilon = 10^{-3}$ izračunati približnu vrednost određenog integrala

$$I = \int_0^1 e^{-3x} \cos 2x \, dx$$

a) primenom uopštene trapezne formule;
b) primenom uopštene Simpsonove formule.
Uporediti potreban broj čvorova ako se tačnost smanji na $\varepsilon = 10^{-2}$.

Rešenje:

$$I = \int_0^1 e^{-3x} \cos 2x \, dx = \frac{1}{13}\left(3 + \frac{-3\,\mathrm{Cos}[2] + 2\,\mathrm{Sin}[2]}{e^3}\right) = 0.242515$$

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Ukupna greška = greška metoda + greška zaokruživanja

$\epsilon = R(f) + E$

Zaokruživanjem na 3 decimale: $E = 0.5\cdot 10^{-3}$

$R(f) \le \epsilon_M = \epsilon - E = 10^{-3} - 0.5\cdot 10^{-3} = 0.5\cdot 10^{-3}$

Zaokruživanjem na 2 decimale: $E = 0.5\cdot 10^{-2}$

$R(f) \le \epsilon_M = \epsilon - E = 10^{-2} - 0.5\cdot 10^{-2} = 0.5\cdot 10^{-2}$

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a) $$\int_a^b f(x)\,dx = \frac{b-a}{2n}(f_0 + 2(f_1 + f_2 + \ldots f_{n-1}) + f_n) - \frac{(b-a)^3}{12n^2} f''(\xi), \quad \xi \in (a,b)$$

$$R(f) = -\frac{(b-a)^3}{12n^2} f''(\xi), \quad \xi \in (a,b)$$

$f(x) = e^{-3x}\,\mathrm{Cos}[2x]$, $\quad f''(x) = e^{-3x}\,(5\,\mathrm{Cos}[2x] + 12\,\mathrm{Sin}[2x])$

$$|R(f)| \le \frac{1}{12n^2}|f''(\xi)| \le \frac{M}{12n^2}, \qquad M = \max\{|f''(\xi)|, \ \xi \in (0,1)\} = 5.39679$$

$$n \ge \sqrt{\frac{M}{12\,\epsilon_M}}, \quad \epsilon_M = 0.0005 \qquad \Longrightarrow \qquad n = \left[\sqrt{\frac{M}{12\,\epsilon_M}}\right] + 1 = 30$$

$$x_k = \begin{pmatrix} 0 \\ 0.0333 \\ 0.0667 \\ 0.100 \\ 0.133 \\ 0.167 \\ 0.200 \\ 0.233 \\ 0.267 \\ 0.300 \\ 0.333 \\ 0.367 \\ 0.400 \\ 0.433 \\ 0.467 \\ 0.500 \\ 0.533 \\ 0.567 \\ 0.600 \\ 0.633 \\ 0.667 \\ 0.700 \\ 0.733 \\ 0.767 \\ 0.800 \\ 0.833 \\ 0.867 \\ 0.900 \\ 0.933 \\ 0.967 \\ 1.00 \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 1.00 \\ 0.903 \\ 0.811 \\ 0.726 \\ 0.647 \\ 0.573 \\ 0.505 \\ 0.443 \\ 0.387 \\ 0.336 \\ 0.289 \\ 0.247 \\ 0.210 \\ 0.176 \\ 0.147 \\ 0.121 \\ 0.098 \\ 0.077 \\ 0.060 \\ 0.045 \\ 0.032 \\ 0.021 \\ 0.012 \\ 0.004 \\ -0.003 \\ -0.008 \\ -0.012 \\ -0.015 \\ -0.018 \\ -0.020 \\ -0.021 \end{pmatrix}$$

$I \approx 0.243$

$R_n \le 0.000499703, \qquad I = 0.242515$

$$\epsilon_M = 0.005 \quad \Rightarrow \quad n = \left[\sqrt{\frac{M}{12\,\epsilon_M}}\,\right] + 1 = 10$$

$$x_k = \begin{pmatrix} 0 \\ 0.10 \\ 0.20 \\ 0.30 \\ 0.40 \\ 0.50 \\ 0.60 \\ 0.70 \\ 0.80 \\ 0.90 \\ 1.0 \end{pmatrix}, \qquad f_k = f(x_k) = \begin{pmatrix} 1.0 \\ 0.73 \\ 0.51 \\ 0.34 \\ 0.21 \\ 0.12 \\ 0.06 \\ 0.02 \\ -\boxed{0. \times 10^{-3}} \\ -0.02 \\ -0.02 \end{pmatrix}$$

$I \approx 0.24$

$R_n \le 0.00449733, \qquad I = 0.242515$

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b) $\int_a^b f(x)\,dx = \frac{b-a}{6n}(f_0+4(f_1+f_3+\ldots f_{2n-1})+2(f_2+f_4+\ldots f_{2n-2})+f_n)-\frac{(b-a)^5}{2880\,n^4}f^{IV}(\xi),\ \ \xi\in(a,b)$

$R(f)=-\frac{(b-a)^5}{2880\,n^4}f^{IV}(\xi),\ \ \xi\in(a,b)$

$f(x)=e^{-3x}\,\mathrm{Cos}[2x],\qquad f^{IV}(x)=e^{-3x}\left(-119\,\mathrm{Cos}[2x]+120\,\mathrm{Sin}[2x]\right)$

$|R(f)|\le\frac{1}{2880\,n^4}|f^{IV}(\xi)|\le\frac{M}{2880\,n^4},\qquad M=\max\{|f^{IV}(\xi)|,\ \xi\in(0,1)\}=119.$

$n\ \ge\ \sqrt[4]{\frac{M}{2880\,\epsilon_M}}\qquad \epsilon_M=0.0005\qquad\Longrightarrow\qquad n=\left[\sqrt[4]{\frac{M}{2880\,\epsilon_M}}\right]+1=4$

2n=8

$$x_k=\begin{pmatrix}0\\0.125\\0.250\\0.375\\0.500\\0.625\\0.750\\0.875\\1.00\end{pmatrix},\qquad f_k=f(x_k)=\begin{pmatrix}1.00\\0.666\\0.415\\0.238\\0.121\\0.048\\0.007\\-0.013\\-0.021\end{pmatrix}$$

$I\ \approx\ 0.243$

$R_n\ \le\ 0.000161404,\qquad I\ =0.242515$

$\epsilon_M\ =0.005$

$$n=\left[\sqrt[4]{\frac{M}{2880\,\epsilon_M}}\right]+1=2$$

2n=4

$$x_k=\begin{pmatrix}0\\0.25\\0.50\\0.75\\1.0\end{pmatrix},\qquad f_k=f(x_k)=\begin{pmatrix}1.0\\0.41\\0.12\\0.\times10^{-3}\\-0.02\end{pmatrix}$$

$I\ \approx\ 0.24$

$R_n\ \le\ 0.00258247,\qquad I\ =0.242515$

Zadatak 4. Simpsonovom formulom sa tačnošću $\varepsilon = 10^{-3}$ približno izračunati integral

$$I = \int_0^\pi \frac{1}{x + \cos x}\, dx.$$

Koristiti Rungeovu ocenu greške.

Rešenje: Integral I prvo izračunamo Simpsonovom formulom sa najvećim mogućim korakom, $h_0 = \pi/2$ (deljenjem intervala $[0, \pi]$ na dva podintervala), a zatim smanjujemo korak polovljenjem, sve dok ne postignemo željenu tačnost.

$$h_0 = \frac{\pi}{2}, \quad 2n = 2, \qquad \mathbf{x} = \begin{bmatrix} 0. \\ 1.5708 \\ 3.1416 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1. \\ 0.63662 \\ 0.46694 \end{bmatrix},$$

$$\begin{aligned} I \approx I(h_0) &= \frac{h}{3}(f_0 + 4f_1 + f_2) \\ &= 0.5236\,(1. + 4 \cdot 0.63662 + 0.46694) = 2.10142 \end{aligned}$$

$$h_1 = \frac{\pi}{4}, \quad 2n = 4, \qquad \mathbf{x} = \begin{bmatrix} 0. \\ 0.7854 \\ 1.5708 \\ 2.35619 \\ 3.1416 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1. \\ 0.67002 \\ 0.63662 \\ 0.60640 \\ 0.46694 \end{bmatrix},$$

$$\begin{aligned} I \approx I(h_1) &= \frac{h}{3}(f_0 + 4(f_1 + f_3) + 2f_2 + f_4) \\ &= 2.05403. \end{aligned}$$

Rungeova ocena greške:

$$R \approx \frac{I(h_1) - I(h_0)}{2^4 - 1} = \frac{2.05403 - 2.10142}{15} = -0.0032.$$

Kako je greška po modulu veća od zadate tačnosti, ponovo smanjujemo korak.

$$h_2 = \frac{\pi}{8}, \quad 2n = 8, \qquad \mathbf{x} = \begin{bmatrix} 0. \\ 0.3927 \\ 0.7854 \\ 1.1781 \\ 1.5708 \\ 1.9635 \\ 2.35619 \\ 2.7489 \\ 3.1416 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1. \\ 0.75954 \\ 0.67002 \\ 0.64071 \\ 0.63662 \\ 0.63259 \\ 0.60640 \\ 0.54794 \\ 0.46694 \end{bmatrix},$$

$$\begin{aligned} I \approx I(h_1) &= \frac{h}{3}\left(f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right) \\ &= 2.04414. \end{aligned}$$

$$R \approx \frac{I(h_2) - I(h_1)}{2^4 - 1} = \frac{2.04414 - 2.05403}{15} = -0.00066.$$

Sada je greška manja od zadate tačnosti, pa za približnu vrednost integrala uzimamo

$$I \approx I(h_2) + R = 2.04414 - 0.00066 = 2.04348 \approx 2.043.$$

Zadatak 5. Odrediti

$$\int_{0.2}^{1} f(x)\, dx,$$

ako je funkcija $f(x)$ zadata tabelom

x_k	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_k	1.31	1.48	1.60	1.70	1.78	1.85	1.91	1.96	1.99

i proceniti grešku.

Rešenje:

$$h_0 = 0.2, \quad 2n = 4, \qquad \mathbf{x} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1.0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1.31 \\ 1.60 \\ 1.78 \\ 1.91 \\ 1.99 \end{bmatrix},$$

$$I \approx I(h_0) = \frac{h}{3}\left(f_0 + 4(f_1 + f_3) + 2f_2 + f_4\right) = 1.393,$$

$$h_1 = 0.1, \quad 2n = 8, \qquad \mathbf{x} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 1.0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1.31 \\ 1.48 \\ 1.60 \\ 1.70 \\ 1.78 \\ 1.85 \\ 1.91 \\ 1.96 \\ 1.99 \end{bmatrix},$$

$$I \approx I(h_0) = \frac{h}{3}\left(f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right) = 1.395$$

Moguća je samo Rungeova procena greške.

$$R \approx \frac{I(h_1) - I(h_0)}{2^4 - 1} = \frac{1.395 - 1.393}{15} = 0.0001.$$

Zadatak 6. Sa tačnošću 10^{-2} izračunati nesvojstveni integral

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}}\, dx.$$

Dati integral je nesvojstven, jer je podintegralna funkcija neograničena u okolini tačke $x = 0$.

Integral možemo da ga predstavimo u obliku

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx = \int_0^C \frac{\cos x}{\sqrt{x}}\,dx + \int_C^1 \frac{\cos x}{\sqrt{x}}\,dx = I_1 + I_2$$

pri čemu je C neki broj između 0 i 1. Pošto je integral konvergentan, tj.

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx = \lim_{c\to 0} \int_C^1 \frac{\cos x}{\sqrt{x}}\,dx \qquad \text{je konačna vrednost,}$$

mora da važi

$$\lim_{c\to 0} I_1 = \lim_{c\to 0} \int_0^C \frac{\cos x}{\sqrt{x}}\,dx = 0.$$

Zato može da se odredi $C \in (0, 1)$, tako da je

$$|\, I_1 \,| = \left| \int_0^C \frac{\cos x}{\sqrt{x}}\,dx \right| < \frac{\epsilon}{2}.$$

Ako se za tako određeno C integral I_2 izračuna primenom neke kvadraturne formule sa tačnošću $\epsilon/2$,

$$I_2 = K_n(f) + R_n(f), \qquad \left| R_n(f) \right| < \frac{\epsilon}{2},$$

tada za integral I važi :

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx = I_1 + I_2 = I_1 + K_n(f) + R_n(f),$$

$$|\, R(f) \,| = |\, I - K_n(f) \,| = \left| I_1 + R_n(f) \right| \le \left| I_1 \right| + \left| R_n(f) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Najpre određujemo C tako da je :

$$| I_1 | = \left| \int_0^C \frac{\cos x}{\sqrt{x}} dx \right| < \frac{\epsilon}{2}.$$

$$\left| \int_0^C \frac{\cos x}{\sqrt{x}} dx \right| \leq \int_0^C \frac{| \cos x |}{\sqrt{x}} dx \leq \int_0^C \frac{1}{\sqrt{x}} dx = 2\left(\sqrt{x}\right)_{x=0}^{x=C} = 2\sqrt{C} < \frac{\epsilon}{2}$$

$$\Rightarrow \; C < \frac{\epsilon^2}{16} = 6.25 \cdot 10^{-6} \;\; \Rightarrow \;\; C = 6 \cdot 10^{-6}.$$

Sada se rešava integral

$$\mathbf{I_2} = \int_C^1 \frac{\cos x}{\sqrt{x}} dx$$

na primer, uopštenom Simpsonovom formulom sa tačnošću $\frac{\epsilon}{2} = 0.5 \cdot 10^{-2}$.

Pri tome, greška zaokruživanja može da se zanemari, jer ćemo računati sa više od 3 decimale.

$$\int_a^b f(x)\, dx = \frac{b-a}{6n}\left(f_0 + 4\left(f_1 + f_3 + ... f_{2n-1}\right) + 2\left(f_2 + f_4 + ... f_{2n-2}\right) + f_n\right)$$

$$- \frac{(b-a)^5}{2880\, n^4} f^{IV}(\xi), \qquad \xi \in (a, b)$$

$$R(f) = -\frac{(b-a)^5}{2880\, n^4} f^{IV}(\xi), \quad \xi \in (a, b)$$

$$a = \frac{3}{500\,000}, \qquad b = 1,$$

$$f(x) = \frac{\operatorname{Cos}[x]}{\sqrt{x}},$$

$$f^{IV}(x) = \frac{\left(105 - 72\, x^2 + 16\, x^4\right) \operatorname{Cos}[x] + 8\, x \left(15 - 4\, x^2\right) \operatorname{Sin}[x]}{16\, x^{9/2}}$$

$$|R(f)| \leq \frac{1}{2880\, n^4} \, |f^{IV}(\xi)| \leq \frac{M}{2880\, n^4},$$

$$M = \max\{|f^{IV}(\xi)|,\ \xi \in (C,1)\} = 2.06723 \times 10^{24}$$

$$n \geq \sqrt[4]{\frac{M}{2880\, \epsilon_M}} \qquad \epsilon_M = 0.005$$

$$n = \left[\sqrt[4]{\frac{M}{2880\, \epsilon_M}}\right] + 1 = 732\,006$$

$$2n = 1\,464\,012$$

$$I_2 \approx 1.80$$

$$R_n \leq 0.00249991$$

Iz rešenja zadatka vidi se da je za postizanje tačnosti potreban veliki broj čvorova, čak $2n+1 = 1464013$, ako se primeni procena greške Simpsonove formule. Razlog tome je što $f^{iv}(x)$ ima velike vrednosti na intervalu integracije $(6 \cdot 10^{-6},\ 1)$.

Ako se koristi Rungeova ocena greške, tražena tačnost $\varepsilon/2 = 0.5 \cdot 10^{-2}$ se postiže primenom uopštene Simpsonove formule sa $2n+1 = 2049$ čvorova, tj. sa dužinom podintervala $h \approx 0.00049$. Zaista,

$$I_2(h) = 1.84729, \quad I_2(2h) = 1.90233, \qquad R \approx \frac{I_2(h) - I_2(2h)}{2^4 - 1} = -0.0037,$$

pa za rezultat može da se uzme

$$I \approx I_2 \approx I_2(h) + R = 1.8473 - 0.0037 = 1.8436.$$

Drugi način za izračunavanje nesvojstvenog integrala I je konstruisanje kvadraturne formule, na primer

$$\int_0^1 \frac{1}{\sqrt{x}}\, f(x)\, dx = A_1 f(0) + A_2 f\left(\frac{1}{2}\right) + A_3 f(1) + R_3(f).$$

Korišćenjem uslova maksimalnog algebarskog stepena tačnosti određuju se koeficijenti:

$$A_1 = \frac{4}{5}, \quad A_2 = \frac{16}{15}, \quad A_3 = \frac{2}{15},$$

tj.

$$\int_0^1 \frac{1}{\sqrt{x}}\, f(x)\, dx = \frac{4}{5}\, f(0) + \frac{16}{15}\, f\left(\frac{1}{2}\right) + \frac{2}{15}\, f(1) + R_3(f).$$

Dobijenom formulom izračunavamo integral I za $f(x) = \cos x$.

$$\int_0^1 \frac{\cos x}{\sqrt{x}}\, dx \approx \frac{4}{5}\, \cos 0 + \frac{16}{15}\, \cos 0.5 + \frac{2}{15}\, \cos 1 \approx 1.808.$$