MATEMATIČKI METODI

Zadaci sa računskih vežbi

Rešeni zadaci

Gausove kvadraturne formule

1. zadatak

Odrediti čvorove x_k, težinske koeficijente A_k i ostatak $R(f)$ u Gausovoj kvadraturnoj formuli sa Čebiševljevom težinom

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx = A_1 f(x_1) + A_2 f(x_2) + R(f)$$

a zatim primenom ove formule približno izračunati integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\, dx$$

Napomena: Čebiševljevi polinomi mogu da se predstave kao $T_n(x) = \cos(n \arccos x)$.
Prva tri člana niza su: $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

Rešenje

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$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx = A_1 f(x_1) + A_2 f(x_2) + R_2(f)$$

čvorovi: nule polinoma $T_2(x)$.

Težinski koeficijenti: $A_k = 2 \frac{[\![T_1]\!]^2}{T_1(x_k)\, T_2'(x_k)}$, $k=1,2$.

Ostatak: $R_2(f) = \frac{[\![T_2]\!]^2}{4!} f^{(4)}(\xi)$, $\xi \in (-1,1)$.

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I Čebiševljevi polinomi:

$T_0(x)=1$,

$T_1(x)=x$,

$T_2(x)=2x^2-1$.

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II Odredjivanje parametara kvadraturne formule

čvorovi: nule polinoma $T_2(x)$.

$-1+2x^2=0$

$$\left\{\left\{x \to -\frac{1}{\sqrt{2}}\right\}, \left\{x \to \frac{1}{\sqrt{2}}\right\}\right\}$$

$x1 = -\frac{1}{\sqrt{2}} = -0.707107, \quad x2 = \frac{1}{\sqrt{2}} = 0.707107$

Težinski koeficijenti: $A_k = 2\,\frac{[\![T_1]\!]^2}{T_1(x_k)\,T_2{}'(x_k)}$

$[\![T_1]\!]^2 = (T_1, T_1) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_1(x)\,T_1(x)\,dx = \frac{\pi}{2}$

$Q_1(x_1) = -\frac{1}{\sqrt{2}}, \quad Q_1(x_2) = \frac{1}{\sqrt{2}}$

$Q_2{}'(x) = 4\,x, \quad Q_2{}'(x_1) = -2\sqrt{2}, \quad Q_2{}'(x_2) = 2\sqrt{2}$

$A1 = \frac{\pi}{2} = 1.5708, \quad A2 = \frac{\pi}{2} = 1.5708$

Napomena: Drugi način za određivanje koeficijenata je rešavanje sistema jednačina iz uslova maksimalnog stepena tačnosti.

Ostatak: $R_2(f) = \frac{[\![T_2]\!]^2}{4!} f^{(4)}(\xi), \quad \xi \in (-1, 1).$

$[\![T_2]\!]^2 = (T_2, T_2) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_2(x)\,T_2(x)\,dx = \frac{\pi}{2}$

$R_2(f) = \frac{\pi}{48} f^{(4)}(\xi) = 0.0654498\, f^{(4)}(\xi), \quad \xi \in (-1, 1).$

Kvadraturna formula:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx = \frac{\pi}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2} f\left(\frac{1}{\sqrt{2}}\right) + 0.0654498\, f^{(4)}(\xi), \quad \xi \in (-1, 1).$$

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III Primena kvadraturne formule

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\frac{1}{\sqrt{1+x^2}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, f(x)\,dx$$

$f(x) = \frac{1}{\sqrt{1+x^2}}$

$f(x1) = \sqrt{\frac{2}{3}}, \quad f(x2) = \sqrt{\frac{2}{3}}$

$f^{(4)}(x) = \frac{9 + 24\,x^2\,(-3 + x^2)}{(1 + x^2)^{9/2}}$

$|f^{(4)}(\xi)| \le \max_{-1 \le x \le 1} |f^{(4)}(x)| = 9.$

$|R_2(f)| \le 0.589049$

$I \approx \frac{\pi}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2} f\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{2}{3}}\,\pi = 2.5651$

Tačna vrednost ne može da se odredi, jer se dobija eliptički integral:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,dx = 2\,\text{EllipticK}[-1] \approx 2.62206$$

2. zadatak

Odrediti čvorove x_k, težinske koeficijente A_k i ostatak $R(f)$ u Gausovoj kvadraturnoj formuli

$$\int_0^{\pi} \sin x\, f(x)\, dx = A_1 f(x_1) + A_2 f(x_2) + R(f)$$

a zatim primenom ove formule približno izračunati integral

$$\int_0^{\pi} \sin(2x)\, e^{-3x}\, dx.$$

Rešenje

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$\int_0^{\pi} \sin(x) f(x) \, dx = A_1 f(x_1) + A_2 f(x_2) + R(f)$

čvorovi: nule polinoma $Q_2(x)$.

Težinski koeficijenti: $A_k = \dfrac{\|Q_1\|^2}{Q_1(x_k)\, Q_2'(x_k)}$, $k=1,2.$

Ostatak: $R_2(f) = \dfrac{\|Q_2\|^2}{4!} f^{(4)}(\xi)$, $\xi \in (0,\pi)$.

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I Formiranje niza ortogonalnih polinoma

Skalarni proizvod: $(\phi,\psi) = \int_0^{\pi} \sin(x)\, \phi(x)\, \psi(x)\, dx.$

$Q_0(x) = 1$

$Q_1(x) = x - \dfrac{(x, Q_0)}{(Q_0, Q_0)} Q_0(x)$

$Q_2(x) = x^2 - \dfrac{(x^2, Q_0)}{(Q_0, Q_0)} Q_0(x) - \dfrac{(x^2, Q_1)}{(Q_1, Q_1)} Q_1(x)$

$(x, Q_0) = \int_0^{\pi} \sin(x)\, x\, Q_0(x)\, dx = \pi$

$(Q_0, Q_0) = \int_0^{\pi} \sin(x)\, Q_0(x)\, Q_0(x)\, dx = 2$

$Q_1(x) = -\dfrac{\pi}{2} + x$

$(x^2, Q_0) = \int_0^{\pi} \sin(x)\, x^2\, Q_0(x)\, dx = -4 + \pi^2$

$(x^2, Q_1) = \int_0^{\pi} \sin(x)\, x^2\, Q_1(x)\, dx = \dfrac{1}{2} \pi \left(-8 + \pi^2\right)$

$(Q_1, Q_1) = \int_0^{\pi} \sin(x)\, Q_1(x)\, Q_1(x)\, dx = \dfrac{1}{2} \left(-8 + \pi^2\right)$

$Q_2(x) = 2 - \pi x + x^2$

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II Odredjivanje parametara kvadraturne formule

čvorovi: nule polinoma $Q_2(x)$.

$2 - \pi x + x^2 = 0$

$\left\{\left\{x \to \dfrac{1}{2}\left(\pi - \sqrt{-8 + \pi^2}\right)\right\}, \left\{x \to \dfrac{1}{2}\left(\pi + \sqrt{-8 + \pi^2}\right)\right\}\right\}$

$x1=\frac{1}{2}\left(\pi-\sqrt{-8+\pi^2}\right)=0.887129,\qquad x2=\frac{1}{2}\left(\pi+\sqrt{-8+\pi^2}\right)=2.25446$

Težinski koeficijenti: $A_k=\frac{[\![Q_1]\!]^2}{Q_1(x_k)\,Q_2'(x_k)}$

$[\![Q_1]\!]^2=(Q_1,Q_1)=\int_0^\pi \sin(x)\,Q_1(x)\,Q_1(x)\,dx = \frac{1}{2}\left(-8+\pi^2\right)$

$Q_1(x_1)=-\frac{1}{2}\sqrt{-8+\pi^2},\quad Q_1(x_2)=\frac{1}{2}\sqrt{-8+\pi^2}$

$Q_2'(x)=-\pi+2x,\quad Q_2'(x_1)=-\sqrt{-8+\pi^2},\quad Q_2'(x_2)=\sqrt{-8+\pi^2}$

$A1=-\frac{\sqrt{-8+\pi^2}}{2\left(-\frac{\pi}{2}+\frac{1}{2}\left(\pi-\sqrt{-8+\pi^2}\right)\right)}=1.,\qquad A2=\frac{\sqrt{-8+\pi^2}}{2\left(-\frac{\pi}{2}+\frac{1}{2}\left(\pi+\sqrt{-8+\pi^2}\right)\right)}=1.$

Ostatak: $R_2(f)=\frac{[\![Q_2]\!]^2}{4!}f^{(4)}(\xi),\qquad \xi\in(0,\pi).$

$[\![Q_2]\!]^2=(Q_2,Q_2)=\int_0^\pi \sin(x)\,Q_2(x)\,Q_2(x)\,dx = 40-4\pi^2$

$R_2(f)=\frac{1}{24}\left(40-4\pi^2\right)f^{(4)}(\xi)=0.0217326\ f^{(4)}(\xi),\qquad \xi\in(0,\pi).$

Kvadraturna formula:

$\int_0^\pi \sin(x)\,f(x)\,dx=2(1.f(0.887129)+1.f(2.25446))+2\cdot 0.0217326\ f^{(4)}(\xi),\qquad \xi\in(0,\pi).$

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III Primena kvadraturne formule

$I=\int_0^\pi \sin(2x)\,e^x dx=\int_0^\pi 2\sin x\ \cos x\ e^x dx=2\int_0^\pi \sin x\ f(x)\,dx$

$f(x)=\cos x\ e^{-x}$

$f(x1)=0.260133,\qquad f(x2)=-0.0662779$

$f^{(4)}(x)=-4\,e^{-x}\,\mathrm{Cos}[x]$

$|f^{(4)}(\xi)|\ \le \max_{0\le x\ge\pi}|f^{(4)}(x)|=0.268079$

$|R_2(f)|\le 0.0116521$

$I\approx 2(1.f(0.887129)+1.f(2.25446))$
$=0.387709$

Tačna vrednost: $\int_0^\pi \mathrm{Sin}[2x]\,\mathrm{Exp}[-x]\,dx=0.382714$

3. zadatak

Izračunati približnu vrednost integrala

$$\int_0^{+\infty}\frac{e^{-2x}}{\sqrt{x+10}}\,dx\ .$$

Rešenje

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Problem beskonačnog intervala integracije može da se reši
primenom Gaus-Lagerove kvadraturne formule, korišćenjem Lagerovih polinoma.

$$\int_0^{+\infty} e^{-x} f(x)\,dx = A_1 f(x_1) + \cdots + A_n f(x_n) + R_n(f)$$

čvorovi: nule polinoma $L_n(x)$.

Težinski koeficijenti: $A_k = \dfrac{[\![L_{n-1}]\!]^2}{L_{n-1}(x_k)\,L_n'(x_k)}$, $k = 1, \ldots, n$.

Ostatak: $R_n(f) = \dfrac{[\![L_n]\!]^2}{(2n)!} f^{(2n)}(\xi)$, $\xi \in (0, +\infty)$.

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Formule vaze za monicne polinome.

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Ovde će biti prikazan postupak određivanja prva tri člana niza moničnih
Lagerovih polinoma $Q_n(x)$ i primena formule sa tako određenim polinomima.

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I Formiranje niza ortogonalnih polinoma

Skalarni proizvod: $(\phi, \psi) = \int_0^{+\infty} e^{-x} \phi(x) \psi(x)\,dx$.

$Q_0(x) = 1$

$$Q_1(x) = x - \frac{(x, Q_0)}{(Q_0, Q_0)} Q_0(x)$$

$$Q_2(x) = x^2 - \frac{(x^2, Q_0)}{(Q_0, Q_0)} Q_0(x) - \frac{(x^2, Q_1)}{(Q_1, Q_1)} Q_1(x)$$

$$(x, Q_0) = \int_0^{+\infty} e^{-x}\, x\, Q_0(x)\,dx = 1$$

$$(Q_0, Q_0) = \int_0^{+\infty} e^{-x} Q_0(x) Q_0(x)\,dx = 1$$

$Q_1(x) = -1 + x$

$$(x^2, Q_0) = \int_0^{+\infty} e^{-x} x^2\, Q_0(x)\,dx = 2$$

$$(x^2, Q_1) = \int_0^{+\infty} e^{-x} x^2\, Q_1(x)\,dx = 4$$

$$(Q_1, Q_1) = \int_0^{+\infty} e^{-x} Q_1(x) Q_1(x)\,dx = 1$$

$Q_2(x) = 2 - 4x + x^2$

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II Odredjivanje parametara kvadraturne formule

$$\int_0^{+\infty} e^{-x} f(x)\,dx = A_1 f(x_1) + A_2 f(x_2) + R_3(f)$$

čvorovi: nule polinoma $Q_2(x)$.

$2 - 4x + x^2 = 0$

$\{\{x \to 2-\sqrt{2}\}, \{x \to 2+\sqrt{2}\}\}$

x1=$2-\sqrt{2}$=0.585786,   x2=$2+\sqrt{2}$=3.41421

Težinski koeficijenti: $A_k = \dfrac{[\![Q_1]\!]^2}{Q_1(x_k)\, Q_2'(x_k)}$

$[\![Q_1]\!]^2 = (Q_1, Q_1) = \int_0^{+\infty} e^{-x} Q_1(x) Q_1(x)\, dx = 1$

$Q_1(x_1) = 1-\sqrt{2}$, $Q_1(x_2) = 1+\sqrt{2}$

$Q_2'(x) = -4+2x$, $Q_2'(x_1) = -4+2\left(2-\sqrt{2}\right)$, $Q_2'(x_2) = -4+2\left(2+\sqrt{2}\right)$

$A1 = \dfrac{1}{\left(1-\sqrt{2}\right)\left(-4+2\left(2-\sqrt{2}\right)\right)} = 0.853553$, $A2 = \dfrac{1}{\left(1+\sqrt{2}\right)\left(-4+2\left(2+\sqrt{2}\right)\right)} = 0.146447$

Ostatak: $R_2(f) = \dfrac{[\![Q_2]\!]^2}{4!} f^{(4)}(\xi)$, $\xi \in (0, +\infty)$.

$[\![Q_2]\!]^2 = (Q_2, Q_2) = \int_0^{+\infty} e^{-x} Q_2(x) Q_2(x)\, dx = 4$

$R_2(f) = \dfrac{1}{6} f^{(4)}(\xi) = 0.166667\, f^{(4)}(\xi)$, $\xi \in (0, +\infty)$.

Kvadraturna formula:

$\int_0^{+\infty} e^{-x} f(x)\, dx = 0.853553 f(0.585786) + 0.146447 f(3.41421)) + 0.166667\, f^{(4)}(\xi)$, $\xi \in (0, +\infty)$.

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III Primena kvadraturne formule

$I = \int_0^{+\infty} \dfrac{e^{-2x}}{\sqrt{x+10}}\, dx = \int_0^{+\infty} e^{-x} \dfrac{e^{-x}}{\sqrt{x+10}}\, dx = \int_0^{+\infty} e^{-x} f(x)\, dx$

$f(x) = \dfrac{e^{-x}}{\sqrt{x+10}}$

f(x1)=0.171094,   f(x2)=0.00898345

$f^{(4)}(x) = \dfrac{e^{-x}\,(200505 + 8x\,(9395 + x\,(1329 + 2x\,(42+x))))}{16\,(10+x)^{9/2}}$

$|f^{(4)}(\xi)| \le \max_{0 \le x \le +\infty} |f^{(4)}(x)| = 0.396283$

$|R_2(f)| \le 0.0660471$

I≈0.853553f(0.585786)+0.146447f(3.41421)
   =0.147353

Tačna vrednost: $\int_0^{+\infty} \dfrac{e^{-2x}}{\sqrt{x+10}}\, dx = 0.154426$

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Tačnost može da se poveća primenom Gaus–Lagerove kvadraturne formule sa 3 čvora.

$\int_0^{+\infty} e^{-x} f(x)\, dx = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3) + R_3(f)$

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I Koristimo poznate Lagerove polinome $L_n(x)$.

Označimo sa $Q_n(x)$ monične Lagerove polinome, tj. $Q_n(x) = (-1)^n n!\, L_n(x)$.

$Q_0(x) = 1$

$Q_1(x) = -1 + x$

$Q_2(x) = 2 - 4x + x^2$

$Q_3(x) = -6 + 18x - 9x^2 + x^3$

==

II Određivanje parametara kvadraturne formule

čvorovi: nule polinoma $Q_3(x)$.

$-6 + 18x - 9x^2 + x^3 = 0$

Out[28]= {{x → 0.415775}, {x → 2.29428}, {x → 6.28995}} x1=0.415775=0.415775,   x2=2.29428=2.29428,   x3=6.28995=6.28995

Težinski koeficijenti: $A_k = \frac{[\![Q_2]\!]^2}{Q_2(x_k)\, Q_3'(x_k)}$

$[\![Q_2]\!]^2 = (Q_2, Q_2) = \int_0^{+\infty} e^{-x} Q_2(x) Q_2(x)\, dx = 4$

$Q_2(x_1) = 0.50977,\quad Q_2(x_2) = -1.9134,\quad Q_2(x_3) = 16.4036$

$Q_3'(x) = 18 - 18x + 3x^2,\quad Q_3'(x_1) = 11.0347,\quad Q_3'(x_2) = -7.50588,\quad Q_3'(x_3) = 23.4712$

A1=0.711093=0.711093,   A2=0.278518=0.278518,   A3=0.0103893=0.0103893

Ostatak: $R_3(f) = \frac{[\![Q_3]\!]^2}{6!} f^{(6)}(\xi),\quad \xi \in (0, +\infty).$

$[\![Q_3]\!]^2 = (Q_2, Q_2) = \int_0^{+\infty} e^{-x} Q_3(x) Q_3(x)\, dx = 36$

$R_3(f) = \frac{1}{20} f^{(6)}(\xi) = 0.05\, f^{(6)}(\xi),\quad \xi \in (0, +\infty).$

Kvadraturna formula:

$\int_0^{+\infty} e^{-x} f(x)\, dx = 0.711093 f(0.415775) + 0.278518 f(2.29428) + 0.0103893 f(6.28995) + 0.05\, f^{(6)}(\xi),\quad \xi \in (0, +\infty).$

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III Primena kvadraturne formule

$I = \int_0^{+\infty} \frac{e^{-2x}}{\sqrt{x+1}} dx = \int_0^{+\infty} e^{-x} \frac{e^{-x}}{\sqrt{x+1}} dx = \int_0^{+\infty} e^{-x} f(x)\, dx$

$f(x) = \frac{e^{-x}}{\sqrt{x+10}}$

f(x1)=0.204449,   f(x2)=0.0287578,   f(x3)=0.00045957

$f^{(6)}(x) = \frac{1}{64\,(10+x)^{13/2}} e^{-x} \left(93\,553\,795 + 51\,737\,340\,x + 12\,030\,300\,x^2 + 1\,503\,200\,x^3 + 106\,320\,x^4 + 4032\,x^5 + 64\,x^6\right)$

$|f^{(6)}(\xi)| \le \max_{0 \le x \le +\infty} |f^{(6)}(x)| = 0.462255$

$|R_3(f)| \le 0.0231127$

$I \approx 0.711093 f(0.415775) + 0.278518 f(2.29428) + 0.0103893 f(6.28995)$
$= 0.153397$

Tačna vrednost: $\int_0^{+\infty} \frac{e^{-2x}}{\sqrt{x+1}} dx = 0.154426$