Određeni integrali

Osnovne osobine određenog integrala:

1^0 $\displaystyle\int_a^a f(x)dx = 0\,,$

2^0 $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx\,,$

3^0 $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,,$

4^0 $\displaystyle\int_a^b \big(\alpha f(x) + \beta g(x)\big)dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx\,,$

5^0 $\displaystyle\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$ gde je $F'(x) = f(x)$,

6^0 Ako je f parna funkcija $\big(f(x) = f(-x),\ \forall x \in \mathbb{R}\big)$, tada je

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx,$$

7^0 Ako je f neparna funkcija $\big(f(x) = -f(-x),\ \forall x \in \mathbb{R}\big)$, tada je

$$\int_{-a}^a f(x)dx = 0,$$

8^0 Ako je f periodična funkcija sa periodom T $\big(f(x+T) = f(x),\ \forall x \in \mathbb{R}\big)$,

tada je $$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx.$$

Čuvajmo drveće. Nemojte štampati ovaj materijal, ukoliko to nije neophodno.

Osnovne metode integracije:

METOD SMENE

$$\int_a^b f(x)dx = \left| \begin{array}{ll} x = u(t) & t = \varphi(x) \\ \alpha = \varphi(a) & \beta = \varphi(b) \end{array} \right| = \int_\alpha^\beta f(u(t))u'(t)dt.$$

METOD PARCIJALNE INTEGRACIJE

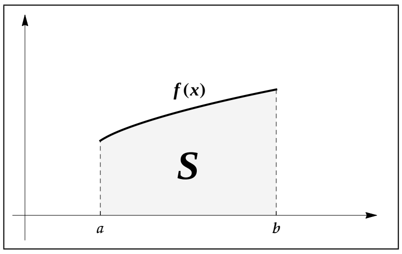
$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu.$$

Primena određenog integrala:

POVRŠINA RAVNE FIGURE

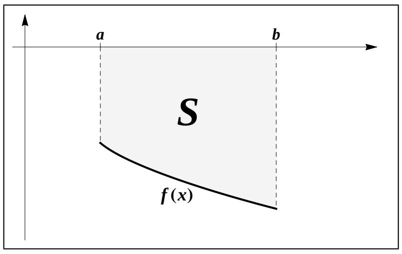
Neka je $f(x) \geq 0$ za $x \in [a, b]$. Površina krivolinijskog trapeza ograničenog krivom $y = f(x)$, pravama $x = a,\ x = b$ i $x-$osom, iznosi

$$S = \int_a^b f(x)dx.$$

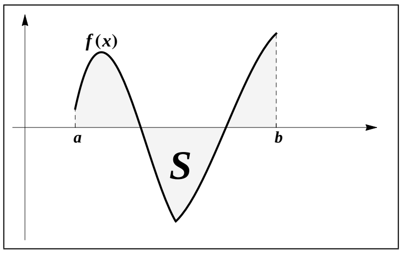


Ukoliko je $f(x) \leq 0$ za $x \in [a, b]$, tada je

$$S = -\int_a^b f(x)dx.$$



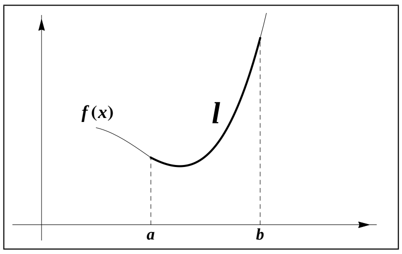
$$S = \int_a^b |f(x)|dx.$$



DUŽINA LUKA KRIVE

Dužina luka krive $f(x)$ od tačke na grafiku sa apscisom a do tačke na grafiku sa apscisom b, iznosi

$$l = \int_a^b \sqrt{1+\big(f'(x)\big)^2}\,dx.$$

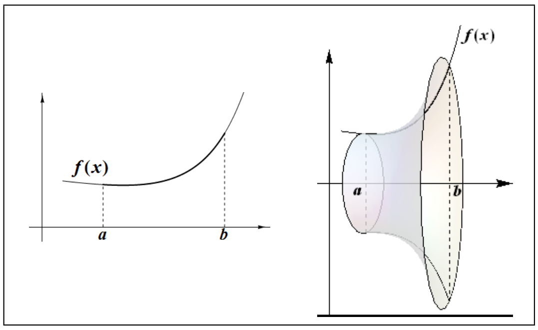


POVRŠINA I ZAPREMINA ROTACIONOG TELA

Površina i zapremina tela nastalog rotacijom dela luka krive $f(x)$ oko koordinatnih osa izračunavaju se sa:

$$V_x = \pi \int_a^b f(x)^2 dx,$$

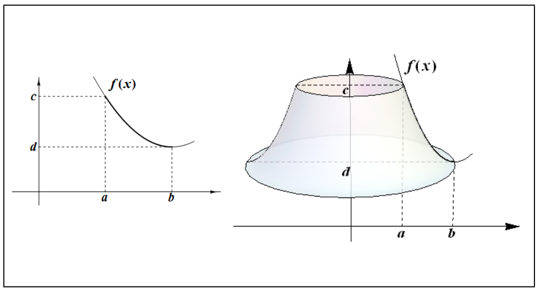
$$P_x = 2\pi \int_a^b f(x)\sqrt{1+\big(f'(x)\big)^2}dx.$$



$$V_y = 2\pi \int_a^b x f(x) dx$$

$$= \pi \int_c^d x(y)^2 dy,$$

$$P_y = 2\pi \int_c^d x(y)\sqrt{1+\big(x'(y)\big)^2}\,dy.$$



Zadaci

1. Odrediti sledeće integrale $(a>0)$:

1^0 $\displaystyle\int_0^1 \sqrt[3]{x}(x+1)^3 dx$ $\quad 2^0$ $\displaystyle\int_1^3 \frac{dx}{x^2}$ $\quad 3^0$ $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

4^0 $\displaystyle\int_2^{65} \frac{dx}{\sqrt[3]{x-1}\big(\sqrt[6]{x-1}+\sqrt{x-1}\big)}$ $\quad 5^0$ $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$ $\quad 6^0$ $\displaystyle\int_2^4 \sqrt{\frac{x+2}{x-1}}\,dx$

Rešenje: 1^0 $\displaystyle\int_0^1 \sqrt[3]{x}(x+1)^3 dx = \int_0^1 x^{1/3}(x^3+3x^2+3x+1)dx$

$$= \int_0^1 (x^{10/3}+3x^{7/3}+3x^{4/3}+x^{1/3})dx = \Big(\frac{x^{13/3}}{13/3}+3\cdot\frac{x^{10/3}}{10/3}+3\cdot\frac{x^{7/3}}{7/3}+\frac{x^{4/3}}{4/3}\Big)\Big|_0^1$$

$$= \frac{3}{13}+\frac{9}{10}+\frac{9}{7}+\frac{3}{4}.$$

2^0 $\displaystyle\int_1^3 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3}+1 = \frac{2}{3}.$

3^0 $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \arctan x\Big|_0^1 = \frac{\pi}{4}.$

4^0
$$\int_2^{65} \frac{dx}{\sqrt[3]{x-1}\big(\sqrt[6]{x-1}+\sqrt{x-1}\big)} = \left|\begin{array}{ll} t^6 = x-1 & dx = 6t^5dt \\ \hline 2\mapsto \sqrt[6]{2-1}=1 & 65\mapsto\sqrt[6]{65-1}=2 \end{array}\right|$$
$$= 6\int_1^2 \frac{t^2}{1+t^2}\,dt = 6\int_1^2 \frac{t^2+1-1}{1+t^2}\,dt = 6\int_1^2 dt - 6\int_1^2 \frac{dt}{1+t^2}$$
$$= 6t\Big|_1^2 - 6\arctan t\Big|_1^2 = \frac{3\pi}{2}+6(1-\arctan 2).$$

5^0
$$\int_0^a \sqrt{a^2-x^2}\,dx = \left|\begin{array}{ll} x=a\sin t & dx=a\cos t\,dt \\ 0\mapsto 0 & a\mapsto \pi/2 \end{array}\right| = a\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2 t}\,\cos t\,dt$$
$$= a^2\int_0^{\pi/2}\cos^2 t\,dt = a^2\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt = \frac{a^2}{2}t\Big|_0^{\pi/2} - \frac{a^2}{4}\sin 2t\Big|_0^{\pi/2} = \frac{a^2\pi}{4}.$$

6^0 $$\int_2^4 \sqrt{\frac{x+2}{x-1}}\,dx = \left| \begin{array}{ll} t=\sqrt{\frac{x+2}{x-1}} & dx=-\frac{6t\,dt}{(t^2-1)^2} \\ 2\mapsto 2 & 4\mapsto\sqrt{2} \end{array} \right| = -6\int_2^{\sqrt{2}} \frac{t^2}{(t^2-1)^2}\,dt$$
$$= 6\int_{\sqrt{2}}^2 \frac{t^2}{(t^2-1)^2}\,dt = \left| \begin{array}{ll} u=t & dv=\frac{t\,dt}{(t^2-1)^2} \\ du=dt & v=-\frac{1}{2(t^2-1)} \end{array} \right|$$
$$= -\frac{3t}{t^2-1}\Big|_{\sqrt{2}}^2 + 3\int_{\sqrt{2}}^2 \frac{dt}{t^2-1} = 3\sqrt{2}-2+\frac{3}{2}\log\left|\frac{t-1}{t+1}\right|\Big|_{\sqrt{2}}^2$$
$$= 3\sqrt{2}-2+3\log\frac{\sqrt{2}+1}{\sqrt{3}}.$$

2. Odrediti sledeće integrale $(n\in\mathbb{N})$:

1^0 $\displaystyle\int_0^\pi \sin x\sin 3x\,dx$ 2^0 $\displaystyle\int_0^{\pi/2}\cos x\sin\frac{x}{2}\,dx$ 3^0 $\displaystyle\int_{\pi/6}^{\pi/3}\frac{dx}{\sin^2 x\cos^4 x}$

4^0 $\displaystyle\int_0^{\pi/6}(2x-1)\sin 3x\,dx$ 5^0 $\displaystyle\int_{-\pi}^{\pi}x^2\cos nx\,dx$ 6^0 $\displaystyle\int_0^{\pi/4}\frac{\sin x\cos x\,dx}{2\sin^2 x-5\cos^2 x}$

Rešenje: 1^0 $$\int_0^\pi \sin x\sin 3x\,dx = \frac{1}{2}\int_0^\pi(\cos 2x-\cos 4x)dx$$
$$= \frac{1}{4}\sin 2x\Big|_0^\pi - \frac{1}{8}\sin 4x\Big|_0^\pi = 0.$$

2^0 $$\int_0^{\pi/2}\cos x\sin\frac{x}{2}\,dx = \frac{1}{2}\int_0^{\pi/2}\Big(\sin\frac{3x}{2}-\sin\frac{x}{2}\Big)dx = \cos\frac{x}{2}\Big|_0^{\pi/2} - \frac{1}{3}\cos\frac{3x}{2}\Big|_0^{\pi/2}$$
$$= \frac{2}{3}(\sqrt{2}-1)\ .$$

3^0 $$\int_{\pi/6}^{\pi/3}\frac{dx}{\sin^2 x\cos^4 x} = \int_{\pi/6}^{\pi/3}\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\,\frac{dx}{\cos^2 x} = \left| \begin{array}{l} \frac{1}{\cos^2 x} = 1+\tan^2 x \\ \frac{1}{\sin^2 x} = \frac{1+\tan^2 x}{\tan^2 x} \end{array} \right|$$
$$= \int_{\pi/6}^{\pi/3}\Big(2+\tan^2 x+\frac{1}{\tan^2 x}\Big)d(\tan x)$$
$$= \Big(2\tan x+\frac{1}{3}\tan^3 x-\frac{1}{\tan x}\Big)\Big|_{\pi/6}^{\pi/3} = \frac{80\sqrt{3}}{27}.$$

4^0 $$\int_0^{\pi/6}(2x-1)\sin 3x\,dx = \left|\begin{array}{ll} u=2x-1 & dv=\sin 3x \\ du=2dx\,dx & v=-\frac{1}{3}\cos 3x \end{array}\right|$$
$$=\frac{1-2x}{3}\cos 3x\Big|_0^{\pi/6}+\frac{2}{3}\int_0^{\pi/6}\cos 3x\,dx=-\frac{1}{3}+\frac{2}{9}\sin 3x\Big|_0^{\pi/6}=-\frac{1}{9}.$$

5^0 $$\int_{-\pi}^{\pi} x^2\cos nx\,dx = 2\int_0^{\pi} x^2\cos nx\,dx = \left|\begin{array}{ll} u=x^2 & dv=\cos nx\,dx \\ du=2x\,dx & v=\frac{1}{n}\sin nx \end{array}\right|$$
$$=\frac{2}{n}x^2\sin nx\Big|_0^{\pi}-\frac{4}{n}\int_0^{\pi} x\sin nx\,dx=-\frac{4}{n}\int_0^{\pi} x\sin nx\,dx$$
$$=\left|\begin{array}{ll} u=x & dv=\sin nx\,dx \\ du=dx & v=-\frac{1}{n}\cos nx \end{array}\right|=\frac{4}{n^2}x\cos nx\Big|_0^{\pi}-\frac{4}{n^2}\int_0^{\pi}\cos nx\,dx$$
$$=(-1)^n\frac{4\pi}{n^2}-\frac{4}{n^3}\sin nx\Big|_0^{\pi}=(-1)^n\frac{4\pi}{n^2}.$$

6^0 $$\int_0^{\pi/4}\frac{\sin x\cos x\,dx}{2\sin^2 x-5\cos^2 x}=\left|\begin{array}{ll} t=2\sin^2 x-5\cos^2 x & dt=14\sin x\cos x\,dx \\ 0\mapsto -5 & \pi/4\mapsto -3/2 \end{array}\right|$$
$$=\frac{1}{14}\int_{-5}^{-3/2}\frac{dt}{t}=\frac{1}{14}\log|t|\Big|_{-5}^{-3/2}=-\frac{1}{14}\log\frac{10}{3}.$$

3. Odrediti sledeće integrale:

1^0 $\displaystyle\int_0^{\pi}\sin^2 x\,dx$	2^0 $\displaystyle\int_0^{\pi}\frac{\sin^2 x}{2(1+\cos x)}\,dx$
3^0 $\displaystyle\int_0^{\pi}\sqrt{\sin x-\sin^3 x}\,dx$	4^0 $\displaystyle\int_{-\pi/2}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx$
5^0 $\displaystyle\int_0^{\pi/2}\frac{dx}{(2+\cos x)(3+\cos x)}$	6^0 $\displaystyle\int_0^{1007\pi}\sqrt{1-\cos 2x}\,dx$

Rešenje: 1^0 $$\int_0^{\pi}\sin^2 x\,dx=\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx=\frac{1}{2}x\Big|_0^{\pi}-\frac{1}{4}\sin 2x\Big|_0^{\pi}=\frac{\pi}{2}.$$

2^0 $$\int_0^\pi \frac{\sin^2 x}{2(1+\cos x)}\,dx=\int_0^\pi \frac{\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}\,dx=\int_0^\pi \sin^2\frac{x}{2}\,dx=\int_0^\pi \frac{1-\cos x}{2}\,dx=\frac{\pi}{2}\ .$$

3^0 $$\begin{aligned}\int_0^\pi \sqrt{\sin x-\sin^3 x}\,dx &= \int_0^\pi \sqrt{\sin x(1-\sin^2 x)}\,dx = \int_0^\pi \sqrt{\sin x}|\cos x|\,dx\\ &= \int_0^{\pi/2}\sqrt{\sin x}\cos x\,dx-\int_{\pi/2}^{\pi}\sqrt{\sin x}\cos x\,dx=\left|\begin{matrix} t=\sin x & dt=\cos x\,dx\\ 0\mapsto 0,\ \pi/2\mapsto 1,\ \pi\mapsto 0 & \end{matrix}\right|\\ &= \int_0^1\sqrt{t}\,dt-\int_1^0\sqrt{t}\,dt = 2\int_0^1\sqrt{t}\,dt = 2\frac{t^{3/2}}{3/2}\Big|_0^1=\frac{4}{3}\ .\end{aligned}$$

4^0 $$\int_{-\pi/2}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx=\frac{4}{3}\ .$$

5^0 $$\begin{aligned}\int_0^{\pi/2}\frac{dx}{(2+\cos x)(3+\cos x)} &= \int_0^{\pi/2}\frac{(3+\cos x)-(2+\cos x)}{(2+\cos x)(3+\cos x)}\,dx\\ &= \int_0^{\pi/2}\frac{dx}{2+\cos x}-\int_0^{\pi/2}\frac{dx}{3+\cos x}=\left|\begin{matrix} t=\tan\frac{x}{2} & dx=\frac{2dt}{1+t^2}\\ 0\mapsto 0 & \pi/2\mapsto 1\end{matrix}\right|\\ &= 2\int_0^1\frac{dt}{3+t^2}-2\int_0^1\frac{dt}{4+2t^2}=\frac{2}{\sqrt3}\arctan\frac{t}{\sqrt3}\Big|_0^1-\frac{1}{\sqrt2}\arctan\frac{t}{\sqrt2}\Big|_0^1\\ &= \frac{\pi}{3\sqrt3}-\frac{1}{\sqrt2}\arctan\frac{1}{\sqrt2}\ .\end{aligned}$$

6^0 $$\begin{aligned}\int_0^{1007\pi}\sqrt{1-\cos 2x}\,dx &= \int_0^{1007\pi}\sqrt2\,|\sin x|dx = 1007\sqrt2\int_0^\pi \sin x\,dx\\ &= -1007\sqrt2\cos x\Big|_0^\pi = 2014\sqrt2\ .\end{aligned}$$

4. Odrediti sledeće integrale:

$$1^0 \int_0^{\pi/4} \frac{3\tan x+1}{3-\tan x}\,dx \qquad 2^0 \int_0^{\pi/4} \tan^2 x\,dx$$

$$3^0 \int_0^1 \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx \quad 4^0 \int_4^6 x\log\frac{2+x}{2-x}\,dx$$

$$5^0 \int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+2}\,dx \qquad 6^0 \int_0^1 \frac{e^x\sqrt{e^x+3}}{\sqrt[3]{e^x+3}}\,dx$$

Rešenje: 1^0
$$\int_0^{\pi/4} \frac{3\tan x+1}{3-\tan x}\,dx = \left|\begin{array}{ll} t=\tan x & dx=\frac{dt}{1+t^2} \\ 0\mapsto 0 & \pi/4\mapsto 1 \end{array}\right|$$
$$= \int_0^1 \frac{3t+1}{(3-t)(1+t^2)}\,dt = \int_0^1 \frac{t^2+1+t(3-t)}{(3-t)(1+t^2)}\,dt$$
$$= \int_0^1 \frac{dt}{3-t} + \int_0^1 \frac{t\,dt}{t^2+1} = -\log|3-t|\Big|_0^1 + \frac{1}{2}\log|t^2+1|\Big|_0^1$$
$$= \log\frac{3}{\sqrt{2}}\ .$$

2^0
$$\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} \frac{1-\cos^2 x}{\cos^2 x}\,dx = (\tan x - x)\Big|_0^{\pi/4} = 1-\frac{\pi}{4}\ .$$

3^0
$$\int_0^1 \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = \left|\begin{array}{ll} u=\log(x+\sqrt{1+x^2}) & dv=\frac{x\,dx}{\sqrt{1+x^2}} \\ du=\frac{dx}{\sqrt{1+x^2}} & v=\sqrt{1+x^2} \end{array}\right|$$
$$= \sqrt{1+x^2}\log(x+\sqrt{1+x^2})\Big|_0^1 - \int_0^1 dx = \sqrt{2}\log(1+\sqrt{2})-1.$$

4^0
$$\int_4^6 x\log\frac{x+2}{x-2}\,dx = \left|\begin{array}{ll} u=\log\frac{x+2}{x-2} & dv=x\,dx \\ du=\frac{4dx}{4-x^2} & v=\frac{x^2}{2} \end{array}\right| = \frac{x^2}{2}\log\frac{x+2}{x-2}\Big|_4^6 + 2\int_4^6 \frac{x^2}{x^2-4}\,dx$$
$$= 18\log 2 - 8\log 3 + 2\int_4^6 \frac{x^2-4+4}{x^2-4}\,dx$$
$$= 18\log 2 - 8\log 3 + 2x\Big|_4^6 - 2\log\frac{x+2}{x-2}\Big|_4^6 = 16\log 2 - 6\log 3 + 4.$$

5^0 $$\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+2}\,dx = \left| \begin{array}{ll} t=\sqrt{e^x-1} & 2t\,dt = e^x dx \\ 0 \mapsto 0 & \log 5 \mapsto 2 \end{array} \right| = 2\int_0^2 \frac{t^2}{t^2+3}\,dt$$
$$= 2\int_0^2 \frac{t^2+3-3}{t^2+3}\,dt = 2\Big(t-\sqrt{3}\arctan\frac{t}{\sqrt{3}}\Big)\Big|_0^2 = 4-2\sqrt{3}\arctan\frac{2}{\sqrt{3}}\ .$$

6^0 $$\int_0^{\log 2} \frac{e^x\sqrt{e^x+3}}{\sqrt[3]{e^x+3}}\,dx = \left| \begin{array}{ll} t^6=e^x+3 & 6t^5\,dt = e^x dx \\ 0 \mapsto \sqrt[6]{4} & \log 2 \mapsto \sqrt[6]{5} \end{array} \right| = 6\int_{\sqrt[6]{4}}^{\sqrt[6]{5}} t^6 dt$$
$$= \frac{6}{7}t^7\Big|_{\sqrt[6]{4}}^{\sqrt[6]{5}} = \frac{6}{7}\big(5\sqrt[6]{5}-4\sqrt[6]{4}\big).$$

5. Izračunati vrednost integrala

$$I_1 = \int_{-\infty}^{+\infty} f(x)\cos x\,dx, \qquad I_2 = \int_{-\infty}^{+\infty} f(x)\sin x\,dx,$$

pri čemu je funkcija $f(x)$ data sa

$$f(x) = \begin{cases} 1-|x|, & -1 \le x \le 1, \\ 0, & x < -1 \vee x > 1. \end{cases}$$

Rešenje: Kako je $f(x)=0$ za $x \notin [-1,1]$, to je onda i

$$f(x)\cos x = 0,\ \ f(x)\sin x = 0 \ \text{ za } x \notin [-1,1].$$

Integral I_1 tada glasi

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} f(x)\cos x\,dx = \int_{-1}^{1} f(x)\cos x\,dx = \int_{-1}^{1} \big(1-|x|\big)\cos x\,dx \\ &= 2\int_0^1 \big(1-x\big)\cos x\,dx = 2\int_0^1 \cos x\,dx - 2\int_0^1 x\cos x\,dx \\ &= 2\sin x\Big|_0^1 - 2\int_0^1 x\cos x\,dx = \left| \begin{array}{ll} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x \end{array} \right| \\ &= 2\sin 1 - 2\Big(x\sin x\Big|_0^1 - \int_0^1 \sin x\,dx\Big) = -2\cos x\Big|_0^1 = 2(1-\cos 1). \end{aligned}$$

S obzirom da je $f(x)\sin x$ neparna funkcija, to je $I_2 = 0$.

6. Izračunati integrale

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx, \quad \text{i} \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx,$$

gde su $n, m \in \mathbb{N}$ i važi

a) $n = m = 0$; **b)** $n = m \neq 0$; **c)** $n \neq m$.

Rešenje: Označimo

$$I_{n,m} = \int_{-\pi}^{\pi} \cos nx \cos mx \, dx, \qquad J_{n,m} = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx.$$

a) $I_{0,0} = \displaystyle\int_{-\pi}^{\pi} dx = 2\pi, \qquad J_{0,0} = 0.$

b)
$$\begin{aligned} I_{n,n} &= \int_{-\pi}^{\pi} \cos^2 nx \, dx = 2\int_{0}^{\pi} \cos^2 nx \, dx = \int_{0}^{\pi} (1 + \cos 2nx) dx \\ &= \Big(x + \frac{1}{2n}\sin 2nx\Big)\Big|_0^{\pi} = \pi. \end{aligned}$$

$$\begin{aligned} J_{n,n} &= \int_{-\pi}^{\pi} \sin^2 nx \, dx = 2\int_{0}^{\pi} \sin^2 nx \, dx = \int_{0}^{\pi} (1 - \cos 2nx) dx \\ &= \Big(x - \frac{1}{2n}\sin 2nx\Big)\Big|_0^{\pi} = \pi. \end{aligned}$$

c)
$$\begin{aligned} I_{n,m} &= \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{0}^{\pi} \big(\cos(n-m)x + \cos(n+m)x\big) dx \\ &= \frac{1}{n-m}\sin(n-m)x\Big|_0^{\pi} + \frac{1}{n+m}\sin(n+m)x\Big|_0^{\pi} = 0. \end{aligned}$$

$$\begin{aligned} J_{n,m} &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{0}^{\pi} \big(\cos(n-m)x - \cos(n+m)x\big) dx \\ &= \frac{1}{n-m}\sin(n-m)x\Big|_0^{\pi} - \frac{1}{n+m}\sin(n+m)x\Big|_0^{\pi} = 0. \end{aligned}$$

7. Dat je integral $I_n = \int_1^e x(\log x)^n \, dx, \; n \in \mathbb{N}_0.$

a) Izračunati I_0 i I_1.

b) Odrediti a_n i b_n u relaciji $I_n = a_n I_{n-1} + b_n$.

c) Naći vrednost integrala I_3.

Rešenje: a) $I_0 = \int_1^e x dx = \left.\frac{x^2}{2}\right|_1^e = \frac{e^2-1}{2}.$

$$I_1 = \int_1^e x \log x \, dx = \left| \begin{array}{ll} u = \log x & dv = x\,dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right| = \left.\frac{x^2}{2}\log x\right|_1^e - \frac{1}{2}\int_1^e x\,dx$$
$$= \frac{e^2+1}{4}.$$

b) $$I_n = \int_1^e x(\log x)^n \, dx = \left| \begin{array}{ll} u = (\log x)^n & dv = x\,dx \\ du = n(\log x)^{n-1}\frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right|$$
$$= \left.\frac{x^2}{2}(\log x)^n\right|_1^e - \frac{n}{2}I_{n-1} = \frac{e^2}{2} - \frac{n}{2}I_{n-1},$$
$$a_n = -\frac{n}{2}, \qquad b_n = \frac{e^2}{2}.$$

c) $I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2}\Big(\frac{e^2}{2} - I_1\Big) = \frac{e^2+3}{8}$.

8. Odrediti koeficijente a_n i b_n u relaciji $I_{n+2} = a_n I_{n+1} + b_n I_n, \; n \in \mathbb{N}_0$, gde je

$$I_n = \int_0^1 (1-x^2)^{n/2} dx, \quad n \in \mathbb{N}_0.$$

Izračunati vrednost integrala I_3.

Rešenje: $I_{n+2}=\displaystyle\int_0^1(1-x^2)^{\frac{n+2}{2}}dx=\int_0^1(1-x^2)(1-x^2)^{\frac{n}{2}}dx$

$$=I_n-\int_0^1 x^2(1-x^2)^{\frac{n}{2}}dx=\left|\begin{array}{ll} u=x & dv=x(1-x^2)^{\frac{n}{2}}dx \\ du=dx & v=-\frac{1}{n+2}(1-x^2)^{\frac{n+2}{2}} \end{array}\right|$$

$$=I_n+\frac{x}{n+2}(1-x^2)^{\frac{n+2}{2}}\Big|_0^1-\frac{1}{n+2}I_{n+2}=I_n-\frac{1}{n+2}I_{n+2}.$$

$$\frac{n+3}{n+2}I_{n+2}=I_n \implies I_{n+2}=\frac{n+2}{n+3}I_n$$

$$a_n=0, \qquad b_n=\frac{n+2}{n+3}.$$

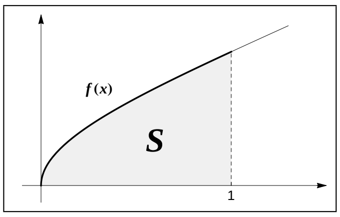
$$I_1=\int_0^1\sqrt{1-x^2}\,dx=\left|\begin{array}{ll} x=\sin t & dx=\cos t\,dt \\ 0\mapsto 0 & 1\mapsto \pi/2 \end{array}\right|=\int_0^{\pi/2}\cos^2 t\,dt$$

$$=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2t)dt=\frac{\pi}{4}.$$

$$I_3=\frac{3}{4}I_1=\frac{3\pi}{16}.$$

9. Dati geometrijsku interpretaciju određenog integrala, a zatim izračunati površinu figure ograničene lukom krive $f(x)=e^{\sqrt{x}}-e^{-\sqrt{x}}$, pravom $x=1$ i $x-$osom.

Rešenje:



$$S=\int_0^1\left(e^{\sqrt{x}}-e^{-\sqrt{x}}\right)dx=\left|\begin{array}{ll} t=\sqrt{x} & 2t\,dt=dx \\ 0\mapsto 0 & 1\mapsto 1 \end{array}\right|$$

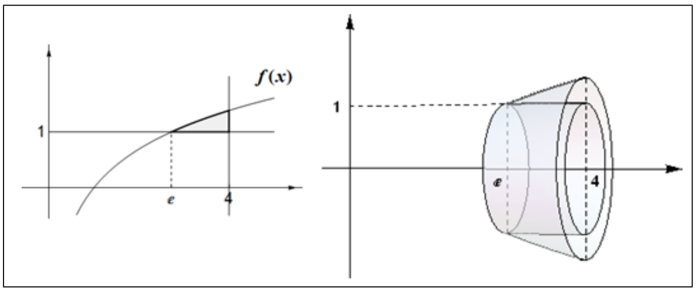
$$=2\int_0^1 t\left(e^t-e^{-t}\right)dt=\left|\begin{array}{ll} u=t & dv=\left(e^t-e^{-t}\right)dt \\ du=dt & v=e^t+e^{-t} \end{array}\right|$$

$$=2t\left(e^t+e^{-t}\right)\Big|_0^1-2\int_0^1\left(e^t+e^{-t}\right)dt$$

$$=2e+\frac{2}{e}-2\left(e^t-e^{-t}\right)\Big|_0^1=\frac{4}{e}\ .$$

10. Izračunati površinu figure ograničene linijama $y=\log x,\ y=1,\ x=4,$ kao i zapreminu tela nastalog rotacijom figure oko $x-$ose.

Rešenje:



$$S=\int_1^4 \big(f(x)-1\big)dx=\int_1^4 \big(\log x-1\big)dx=\int_1^4 \log x\,dx-\int_1^4 dx$$

$$=\left|\begin{array}{ll} u=\log x & dv=dx \\ du=\frac{dx}{x} & v=x \end{array}\right|=x\log x\Big|_1^4-3-x\Big|_1^4=4\log 4-6.$$

$$V_x=\pi\int_1^4\big(f(x)-1\big)^2dx=\pi\int_1^4\big(\log x-1\big)^2dx=\left|\begin{array}{ll} u=\log x-1 & dv=(\log x-1)dx \\ du=\frac{dx}{x} & v=x(\log x-2) \end{array}\right|$$

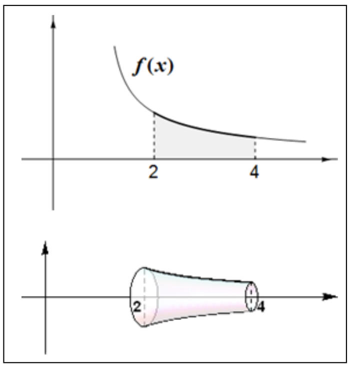
$$=\pi x(\log x-1)(\log x-2)\Big|_1^4-\pi\int_1^4(\log x-2)dx=4(\log 4-2)^2\pi-1.$$

11. Odrediti površinu figure ograničene lukom krive

$$f(x)=\frac{1}{\sqrt{x^2+x-2}},\ \ x\in[2,4]$$

i $x-$osom, a zatim izračunati zapreminu tela nastalog rotacijom tog luka oko $x-$ose.

Rešenje:



$$S=\int_2^4\frac{dx}{\sqrt{x^2+x-2}}=\int_2^4\frac{dx}{\sqrt{(x+2)(x-1)}}$$

$$=\left|\begin{array}{ll} t(x+2)=\sqrt{(x+2)(x-1)} & dx=\frac{6t\,dt}{(1-t^2)^2} \\ 2\mapsto 1/2 & 4\mapsto 1/\sqrt{2} \end{array}\right|$$

$$=2\int_{1/2}^{1/\sqrt{2}}\frac{dt}{1-t^2}=\log\frac{1+t}{1-t}\Big|_{1/2}^{1/\sqrt{2}}$$

$$=\log\left(1+\frac{2\sqrt{2}}{3}\right).$$

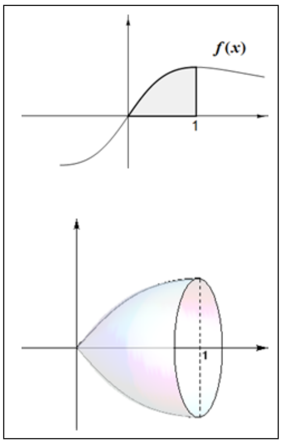
$$
\begin{aligned}
V_x &= \pi \int_2^4 \frac{dx}{x^2+x-2} = \frac{\pi}{3}\int_2^4 \frac{x+2-(x-1)}{(x+2)(x-1)}\,dx \\
&= \frac{\pi}{3}\int_2^4 \frac{dx}{x-1} - \frac{\pi}{3}\int_2^4 \frac{dx}{x+2} = \frac{\pi}{3}\log\frac{x-1}{x+2}\Big|_2^4 = \frac{\pi}{3}\log 2.
\end{aligned}
$$

12. Data je funkcija

$$f(x) = \frac{x}{1+x^2}.$$

Izračunati površinu figure ograničene krivom $f(x)$, $x-$osom i pravom $x = 1$. Izračunati zapreminu tela nastalog rotacijom date figure oko $x-$ose.

Rešenje:



$$S = \int_0^1 \frac{x\,dx}{1+x^2} = \frac{1}{2}\log(1+x^2)\Big|_0^1 = \frac{1}{2}\log 2.$$

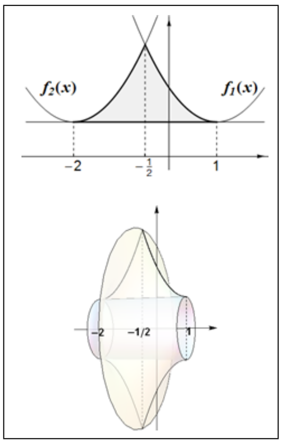
$$
\begin{aligned}
V_x = \pi\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx &= \left| \begin{array}{ll} u = x & dv = \frac{x\,dx}{(1+x^2)^2} \\ du = dx & v = \frac{-1}{2(1+x^2)} \end{array} \right| \\
&= -\pi\frac{x}{2(1+x^2)}\Big|_0^1 + \frac{\pi}{2}\int_0^1 \frac{dx}{1+x^2} \\
&= -\frac{\pi}{4} + \frac{\pi}{2}\arctan x\Big|_0^1 = \frac{\pi^2}{8} - \frac{\pi}{4}\ .
\end{aligned}
$$

13. Izračunati površinu figure ograničene lukovima krivih $f_1(x) = x^2 - 2x + 2$, $f_2(x) = x^2 + 4x + 5$ i $y = 1$. Izračunati zapreminu tela nastalog rotacijom oko $x-$ose ove figure.

Rešenje: Odredimo najpre presečnu tačku grafika krivih f_1 i f_2 :

$$x^2 - 2x + 2 = x^2 + 4x + 5 \iff x = -1/2.$$

Primetimo da je slika simetrična u odnosu na pravu $x = -1/2$.



$$
\begin{aligned}
S &= \int_{-2}^{-1/2} f_2(x)dx + \int_{-1/2}^{1} f_1(x)dx \\
&= 2\int_{-1/2}^{1} \big(x^2 - 2x + 2\big)dx \\
&= 2\Big(\frac{x^3}{3} - x^2 + 2x\Big)\Big|_{-1/2}^{1} = \frac{21}{4}\ .
\end{aligned}
$$

$$
\begin{aligned}
V_x &= 2\pi \int_{-1/2}^{1} f_1(x)^2 dx = 2\pi \int_{-1/2}^{1} \big(x^2 - 2x + 2\big)^2 dx \\
&= 2\pi \int_{-1/2}^{1} \big((x-1)^2 + 1\big)^2 dx = \frac{843\pi}{80}\ .
\end{aligned}
$$

14. Izračunati zapreminu tela nastalog rotacijom luka krive $y(x) = \cos^2 x,\ x \in [-\pi/2, \pi/2]$ oko $x-$ose.

Rešenje:

$$
\begin{aligned}
V_x &= \pi \int_{-\pi/2}^{\pi/2} \cos^4 x\, dx = 2\pi \int_0^{\pi/2} \cos^4 x\, dx = \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x)^2 dx \\
&= \frac{\pi}{2} \int_0^{\pi/2} \Big(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\Big)dx = \frac{3\pi^2}{8}\ .
\end{aligned}
$$

15. Izračunati zapreminu tela nastalog rotacijom oko $x-$ose figure ograničene lukom krive $f(x) = x \log x$ i $x-$osom.

Rešenje: Kako je

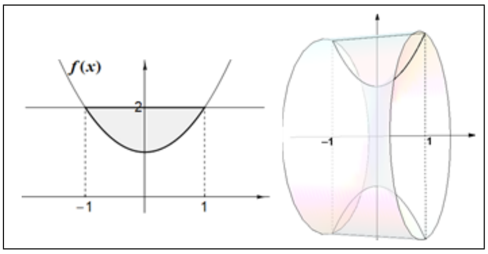
$$
\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0,
$$

i $f(1) = 0$, to posmatramo deo luka krive f za $x \in (0, 1]$.

$$
V_x = \pi \int_0^1 x^2 \log^2 x\, dx = \frac{2\pi}{27}\ .
$$

16. Izračunati zapreminu tela koje nastaje rotacijom figure ograničene delovima krivih $y = x^2 + 1$ i $y = 2$ oko $x-$ose.

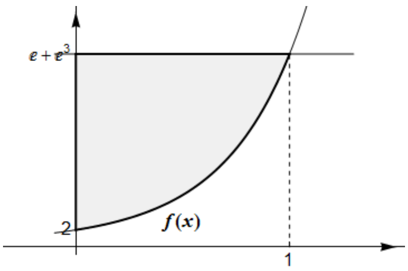
Rešenje: $V_x = \pi \int_{-1}^{1} \left(2^2 - (x^2+1)^2\right) dx = \dfrac{64\pi}{15}$.



17. Izračunati zapreminu tela nastalog rotacijom oko $x-$ose figure ograničene linijama

$$y = e^{3x} + e^x, \quad x = 0, \quad y = e^3 + e.$$

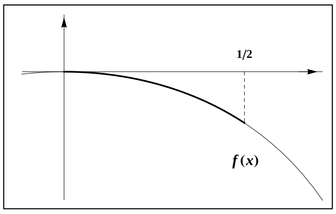
Rešenje:



$$\begin{aligned} V_x &= \pi \int_0^1 \left((e^3+e)^2 - (e^{3x}+e^x)^2\right) dx \\ &= \frac{1}{6}(7 + 3e^2 + 9e^4 + 5e^6)\pi \ . \end{aligned}$$

18. Izračunati dužinu luka krive L date sa $y(x) = \log(1-x^2)$, $x \in [0, 1/2]$.

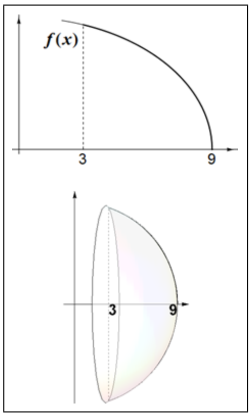
Rešenje: $y'(x) = \dfrac{2x}{x^2-1}$.



$$\begin{aligned} l &= \int_0^{1/2} \sqrt{1 + \frac{4x^2}{(x^2-1)^2}}\, dx = \int_0^{1/2} \frac{x^2+1}{1-x^2}\, dx \\ &= -\int_0^{1/2} dx + 2\int_0^{1/2} \frac{dx}{1-x^2} = \log 3 - \frac{1}{2} \ . \end{aligned}$$

19. Izračunati dužinu luka krive $y(x) = \sqrt{81 - x^2}$ za $3 \le x \le 9$, kao i površinu rotacione površi nastale rotacijom tog luka oko $x-$ose.

Rešenje: $y'(x) = -\dfrac{x}{\sqrt{81-x^2}}, \quad \sqrt{1+y'(x)^2} = \dfrac{9}{\sqrt{81-x^2}}.$



$$\begin{aligned} l &= \int_3^9 \sqrt{1+y'(x)^2}\,dx = 9\int_3^9 \frac{dx}{\sqrt{81-x^2}} \\ &= \left| \begin{array}{ll} x = 9\sin t & dx = 9\cos t\,dt \\ 3 \mapsto \arcsin\frac{1}{3} & 9 \mapsto \frac{\pi}{2} \end{array} \right| \\ &= 9\Big(\arcsin\frac{1}{3} - \frac{\pi}{2}\Big) = 9\arccos\frac{1}{3}\ . \end{aligned}$$

$$P_x = 2\pi \int_3^9 y(x)\sqrt{1+y'(x)^2}\,dx = 18\pi \int_3^9 dx = 108\pi.$$