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PROJECTO MEFT

Characteristic Critical Collapse with Null Infinity

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Contents

1 Introduction

General Relativity is a successful theory that explains gravity as the curvature of a four dimensional spacetime manifold. This is the essence behind Einstein Field Equations (EFE), a set of non-linear partial differential equations relating the geometry of spacetime with its matter content.

Exact solutions for the EFE can only be found under simplifying assumptions, such as imposing symmetries on the system. Alternatively, one can use numerical methods to solve these equations. This is the essence of the field of Numerical Relativity.

Critical phenomena in gravitational collapse were first discovered by Choptuik, who studied the dynamical evolution of families of initial data parameterized by a single parameter p. Such system will have two end-states: if initial data is dense enough, a black hole will form, or if it is weaker it can disperse to flat spacetime. Moreover, Choptuik found a very interesting interesting characteristic of critical phenomena: just at the threshold of black hole formation the dynamics of the system become relatively simple and universal in some aspects, despite the complicated form of the EFE [4]. The study of critical collapse is very relevant as critical solutions can be helpful in studying the cosmic censorship conjecture, the long-unsolved problem of General Relativity [7].

The aim of this project is to study critical collapse of a spherically symmetric scalar field using null coordinates. This allows us to study the process from the point of view of an observer located at infinity $\mathscr{I}^+$. Different matter models will be investigated and compared with previous works (see Sections 3 and 5).

Finally, the code developed will be based on a Characteristic Initial Value Problem, in which the initial data is specified on a null hypersurface. Our spacetime is then foliated on a family of non-intersecting null hypersurfaces. We aim to compare the obtained results with an alternative approach to the same physical problem, based on the truncation of Cauchy slices.

2 Scientific Background

In this section, the necessary background for the study of critical phenomena in gravitational collapse is provided. Firstly, we'll go through the main features of the critical collapse; moving on to a description of the geometry of our problem and we finish this section by formulating the problem equations.

2.1 Critical Collapse

Gravitational collapse consists in the process by which a spherical shell of matter contracts under the influence of its own gravitational field. This mechanism is believed to be behind the structure formation of our universe and was first studied by Oppenheimer and Snyder in 1939, some time after the theory of General Relativity was developed [1].

The dynamics of gravitational collapse are described by Einstein's equations in spherical symmetry coupled to a massless scalar field. Such system will have two end-states: if the initial data is dense enough, it can collapse and form a Schwarzschild black hole, or if it is weaker, it can disperse to flat spacetime. The so-called critical phenomena occurs when attracting and repulsive forces are almost in balance. In this case, we say the initial configuration is in the threshold of black hole formation [8].

Critical phenomena were first discovered by Choptuik [2] in numerical simulations of a spherical scalar field. The phenomena studied by Choptuik has correspondence to Type II phase transitions in statistical mechanics and thus are usually referred to as critical phenomena of Type II. In this work, we are interested in this type of collapse, since it is characteristic of the phenomena we want to study (massless fields and electromagnetic waves in vacuum).

At the threshold between black hole formation and dispersion, Type II solutions are characterized by features of critical collapse such as:

- Universality;
- Self-similarity;
- Scaling.

2.1.1 Universality

General relativity can be treated as a dynamical system of infinite dimensions [8]. Initial data sets correspond to a point in phase space. This data set will be evolved using Einstein equations, as discussed in Section 4. Solution curves of the evolution are represented in Figure 1, where the system ends up either in an end-state of dispersion or formation of a black hole. The two possible end-states of the evolution correspond to basins of attraction in the phase space: Minkowski spacetime is an attractive fixed point and all the possible final black holes form an half-line of attractive fixed points [8].

The phase space is then split in two halves, separated by a critical surface. A point that starts in this surface will remain in it throughout the evolution. Thus, the critical surface is itself a dynamical

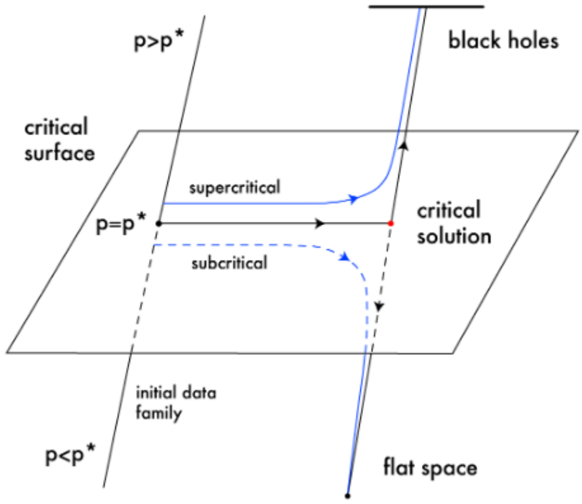


FIGURE 1: Phase space of near-critical evolutions. Reproduced from [8]

system with one dimension less. The critical solution, called a Choptuon, is an attracting fixed point that lies in the critical surface. This solution is then an attractor of codimension one.

For initial data close to the critical surface, the trajectory defined moves close to the surface for some time. The closer the initial data is to the critical plane, the longer the evolution curve will remain close to the critical plane. In the time the evolution spends in the vicinity of the universal critical solution, all details of the initial data are forgotten, except for the distance from the threshold of black hole formation. That is, all near-critical solutions look the same near the center of implosion. This is what is meant by universality of the near-critical behaviour. After some time, as the mode transverse to the surface grows, the curve of evolution drifts away either towards the flat spacetime or black hole end-states.

2.1.2 Self-similarity

A self-similar solution is a solution which is similar to itself when appropriately scaling the independent and dependent variables. Self-similarity can be observed when initial data is fine-tuned so that its end-state is either a very small black hole, or slightly below that. That is, carefully chosen initial data will result in a solution curve near the critical plane, which shows self-similarity [8].

A spacetime (M, g) is discretely self-similar (DSS) if it admits a discrete diffeomorphism $\Phi\Delta : M \to M$ which leaves the metric invariant up to a constant scale factor such that

$$(\Phi_\Delta^*)^n g|_p = e^{2n\Delta} g|_p, \forall p \in M, \tag{1}$$

where $n \in \mathbb{N}$ and Δ is a dimensionless real constant factor [8]. The critical solution discussed in Section 2.1.1 has been found to be DSS, that is, it reproduces itself in echoes. Thus, the critical solution is scale invariant by a factor of $e^{2n\Delta}$, where Δ is a universal echoing period independent of the initial data. Hence, by fine-tuning the initial data, our self-similar solution ϕ will verify

$$\phi^*(r, t) = \phi^*(e^{n\Delta} r, e^{n\Delta} t). \tag{2}$$

In a similar way, continuous self-similarity (CSS) means that there exists a (generating) vector field ξ parameterized by Δ [8]. Thus $\xi = \frac{d}{d\Delta}\Phi_\Delta|_{\Delta=0}$ and ξ obeys the conformal Killing equation with a conventional constant on the right hand side such that

$$\mathcal{L}_{\xi} g_{\mu\nu} = 2 g_{\mu\nu}. \tag{3}$$

Such spacetimes admitting a one-parameter (Δ) family of a diffeomorphism $\Phi\Delta$ is said to be CSS. Although in this work we are in fact interested in DSS, the simpler case of a CSS collapse can sometimes be useful in the study of critical behaviour [9].

Finally we note that the Choptuon is by definition discretely self-similar everywhere, but the critical solutions we'll find in our simulations will only verify self-similarity in a region around the origin, falling off smoothly at null-infinity — just like our initial data. This is consistent with an object sending signals to null-infinity, which is the scenarios we want to study [8].

2.1.3 Scaling

An important feature of critical collapse found by Choptuik [2] is related with the mass of the black holes produced near criticality. He found that the mass of the formed black hole depends on a parameter p of a family of initial data and that this relation follows a power-law

$$M_{BH} = k|p - p^*|^{\gamma}, \tag{4}$$

where the constant k and the critical value p^* depend on the particular one-parameter family of initial data — i.e., our scalar field. On the other hand, the critical exponent γ is universal [7].

Similarly to the analysis in Section 2.1.1, the parameter p^* here will dictate the threshold of black hole formation: for a family of initial data with $p > p^*$, a black hole will form, while for $p < p^*$ the initial data is dispersed. The already mentioned interesting behaviour occurs for values of p close to p^*, for which the evolution approaches a universal solution independent of the initial shape of the data, the critical solution.

2.2 Geometric Setup

This section provides a description of the geometry of our problem. The choice of coordinates is discussed, as well as compactification.

2.2.1 Coordinates

In this work, we'll be using coordinates adapted to the null geodesics of spacetime. In fact, we're interested in studying gravitational collapse, which is described by the dynamics of spacetime in a small region around the origin, but using null coordinates will allow us to study the collapse process from the point of view of an observer located at infinity. This observer sees radiation signals emitted by the collapsing field [8].

The general form of a spherically symmetric metric takes the form

$$ds^2 = d\tau^2(t,r) + Y^2(t,r) d\Omega^2(\theta,\phi), \tag{5}$$

where $\tau^2(t,r)$ is a 2D metric of signature (-,+) and $d\Omega$ is the metric on the unit 2-sphere. Bondi coordinates are obtained by fixing $x^0 = u$, i.e. u represents a family of outgoing null geodesics. Because we're interested in spherical symmetry, $\{x^2, x^3\} = \{\theta, \phi\}$ are taken to be constant. x^1 is chosen geometrically as $x^1 = r$, such that it's a radial parameter along the null geodesics defined by u. Thus, the line element in Bondi coordinates $\{u, r, \theta, \phi\}$ can be shown to be of the form

$$ds^2 = -e^{2\beta(u,r)} \frac{V(u,r)}{r} du^2 - 2e^{2\beta(u,r)} du dr + r^2 (d\theta^2 + sin^2\theta d\phi^2), \tag{6}$$

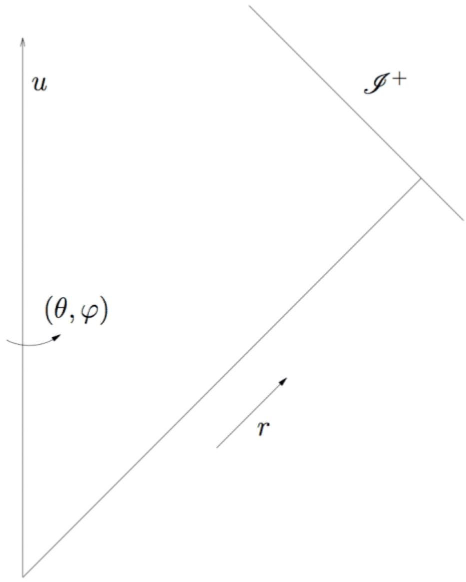


FIGURE 2: Bondi coordinates in spherical symmetry. Reproduced from [8]

in which $\beta(u, r)$ and $\frac{V(u,r)}{r}$ are smooth metric functions. The physical meaning of these functions is discussed by Pürrer [8]: β is related to the redshift between the center of symmetry and an asymptotic observer and V is the analog of a Newtonian potential. Because $u = t - r$, we can check that the metric based on Bondi coordinates is consistent with the signature (-,+,+,+).

In this work, our Gauge choice will be such that our outgoing null slices are parameterized by the proper time of an observer located at the origin, from where the signal is emanating. This simplifies the study of the region close to the center of spherical symmetry, where critical solutions show DSS, as discussed in Section 2.1.2.

2.2.2 Compactification

In numerical studies of collapsing systems, it is often useful to perform a compactification of the radial coordinate r. We perform a coordinate transformation which maps a half-infinite domain $[0, \infty)$ to a finite one $[0, 1]$, by doing

$$x := \frac{r}{1+r}. \tag{7}$$

Points at $\mathscr{I}^+$ are then included in our grid at $x = 1$. In such manner, we can mimic observers at null-infinity which will allow us to extract global properties of our problem.

2.3 Analytic Setup

In this section, we aim to formulate the equations of the problem. In the presence of a massless scalar field ϕ, the Einstein Field Equations are given by

$$\begin{aligned} G_{ab} &= 8\pi T_{ab}, \\ T_{ab} &= \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi. \end{aligned} \tag{8}$$

In our case, the Einstein Equations give two hypersurface equations for $\beta(u,r)$ and $V(u,r)$, which will compose our evolution system [9]

$$\begin{aligned} \beta_{,r} &= 2\pi r(\phi_{,r})^2, \\ V_{,r} &= e^{2\beta}. \end{aligned} \tag{9}$$

The wave equation in vacuum takes the form

$$\Box\phi = \nabla_a \nabla^a \phi = 0, \tag{10}$$

in which the curved space d'Alembertian is written in spherical symmetry. It is often useful to introduce a rescaled field Φ,

$$\Phi = \phi r. \tag{11}$$

Note that ϕ falls of at large r. Thus, the rescaled field Φ behaves similarly to a plane wave. It is then possible to increase numerical accuracy at large distances, as the amplitude of the plane wave doesn't decrease as fast as in the original case.

3 Related Work

In this section, we aim to give an overview of relevant previous works on critical collapse. The foundations of critical phenomena in gravitational collapse were first laid by Choptuik in 1993 [2]. As already discussed, Choptuik found that at the threshold of black hole formation the dynamics of the system become simple and universal in some aspects, despite the complicated form of the EFE.

Later on, studies of global aspects of critical collapse have been carried out by Pürrer, et al. [9]. A self-gravitating massless scalar field in a spherically symmetry was evolved numerically, using a compactified grid, just like this work aims to do. Radiation quantities are investigated and they were found to reflect the DSS behaviour.The algorithm used in this work is based on the work of Gómez and Winicour [5], who propose the integration of the wave equation over the null parallelogram Σ spanned by four vertices. This will be useful in our case of a characteristic code in compactified Bondi coordinates. We look to extend this work by considering different matter models, as the Yang-Mills field.

In fact, Gundlach, et al. [6] have studied the critical collapse of two interacting matter fields in spherical symmetry, namely a scalar field and a Yang-Mills field. They find that by fine-tuning, the scalar field always dominates on sufficiently small scales. Thus, they conjecture the existence of a "quasi-discretely self-similar" solution shared by the two fields, equal to the Choptuik solution at infinitely small scales and the known Yang-Mills critical solution at infinitely large scales, with a gradual transition from one field to the other. This solution is found to act as the critical solution for any mixture of scalar field and Yang-Mills initial data. We plan to furthermore investigate the Yang-Mills critical solution, specifically by considering null infinity aspects.

Finally, in this work we're dealing with a spacetime foliation based on outgoing null hypersurfaces (see Section 4). We then perform spatial compactification of the coordinate r defined along our constant u hypersurfaces. Our goal is to compare our results with the ones obtained through a foliation based on truncated Cauchy slices. Moreover, we expect to compare the results of these two approaches with the ones of a hyperboloidal evolution of a Yang-Mills field, as carried out by Rinne [11, 12].

4 Methods

In this section, we go through essential methods that will be necessary to tackle our problem. Firstly, we'll discuss spacetime foliation and initial data and then go through the implementation of a toy model and we test for convergence. This is a starting point towards the work we will soon carry out.

4.1 Characteristic Initial Value Problem

The Characteristic Initial Value Problem plays an important role in the field of numerical relativity and our code will be based on such problem. An initial value problem consists of finding a solution of a partial differential equation when data for the unknown functions in specified on a suitable initial hypersurface [3].

The common Cauchy problem consists of an initial value problem in which the initial data is specified on a space-like hypersurface. In this case, the constraint equations are elliptic, as they specify the values of the functions on some boundary. This is analogous to solving the Poisson equation: the constraint is given on some spatial boundary and then the equation is integrated to find the value of the functions elsewhere.

In the case of a characteristic initial value problem, the data is given on one or more null hypersurfaces. In this case, the equations of the problem can be written as ordinary differential equations along characteristic curves — method of characteristics. These are often referred to as propagation equations. Thus the constraints are much simpler in the characteristic case [10].

Null geodesics are the bicharacteristics of a massless scalar field minimally coupled to gravity [8]. This essentially means that the scalar field propagates at the speed of light and thus, disturbances in our field will propagate along null geodesics. This indicates a foliation of spacetime based on a family of non-intersecting null hypersurfaces. Then our characteristic initial value problem will be solved by specifying on an initial null hypersurface the value of the field we'll be evolving.

In this work, we're then solving our set of equations based on a characteristic foliation. This is in opposition to many other works in numerical relativity which work on a $3+1$ decomposition, i.e. spacetime is foliated by spacelike hypersurfaces labelled by their time coordinate t. This is the essence of the Arnowitt-Deser-Misner formalism. The advantage of working on a characteristic foliation is not only having an efficient evolution system, but also simplifying the study of the causal structure of our solution.

4.2 Toy Model Evolution

As aforementioned, the system we will want to evolve is described by a characteristic code in compactified Bondi coordinates. We first started by implementing a Toy Model of a simpler model in $1+1$ dimensions, which we'll describe in this section. The goal of this early implementation was to build the evolution code that can later be adapted to the characteristic critical collapse we want to study. As a simpler model is easier to predict than a full General Relativity problem, we can then analyse its results and ensure that the numerical methods are built in a correct manner and work as expected.

We then want to solve for the evolution of f and g in

$$\begin{aligned}(\partial_t + \partial_x)f &= A(x,t),\\ (\partial_t - \partial_x)g &= 0.\end{aligned} \tag{12}$$

$A(x,t)$ denotes a given function which was fixed to $A(x,t) = \sin(t) = \sin(u+x)$. We will work in coordinates $\{u = t - x, x\}$, similarly to a compactified version of the Bondi coordinates $\{u, r, \theta, \phi\}$ in spherical symmetry. The evolution is performed by a foliation of spacetime in null slices of constant u, where we solve for our functions f and g. Using *Mathematica*, the evolution equations are found to be of the form

$$\partial_x f = \sin(x+u), \tag{13}$$

$$\partial_u g = \frac{1}{2}(\partial_x g(u,x)). \tag{14}$$

Our numerical grid has length 40 and is discretized in x with resolution $x_{i+1} - x_i = dx = 0.2$. The *time* increment du is set to 0.08. Our computational scheme is then the following:

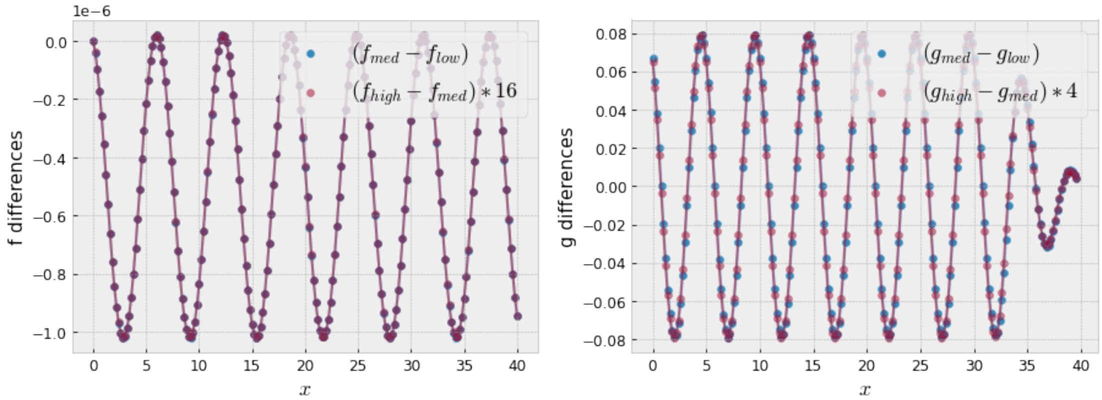


FIGURE 3: Pointwise convergence plots for f and g at last timestep of evolution.

1. Initial data for f is given at the left boundary (at $x = 0$). The value of f for an initial hypersurface of constant $u = 0$ is then computed by integrating out on x using Equation 13;
2. Initial data for g is given on the whole grid as $g(x, u = 0) = \sin(4.0\pi x/10.0)$. Note that g describes a left moving wave. Throughout the evolution, this will be our constraint on the right boundary;
3. A timestep $u_{i+1} = u_i + du$ is taken;
4. The value of f and g are calculated at every gridpoint using Equations 13 and 14 respectively;
5. Steps 3 and 4 are repeated until the evolution reaches the last timestep.

The integrations were performed using a Runge Kutta integrator of 4th order. Images of the evolution as well as the code developed (*Julia*, *Python* and *Mathematica* files) can be found on the GitHub repository.

4.3 Toy Model Convergence Tests

The first convergence test implemented is a pointwise convergence test. Using the same initial data and evolution equations, the solutions for f and g were computed at three different resolutions: the standard one, with $dx = 0.2$, a second one with resolution $dx_2 = dx/2$ and a third one with $dx_3 = dx/4$. The values of both f and g were compared at the common gridpoints.

The left side of Figure 3 shows the difference between the solution for f with medium and low resolution, $|f_{med} - f_{low}|$, as well as the same difference but for high and medium resolution $|f_{high} - f_{med}|$. The right side of this same Figure shows the same values but for g. The values shown correspond to the last timestep of the evolution.

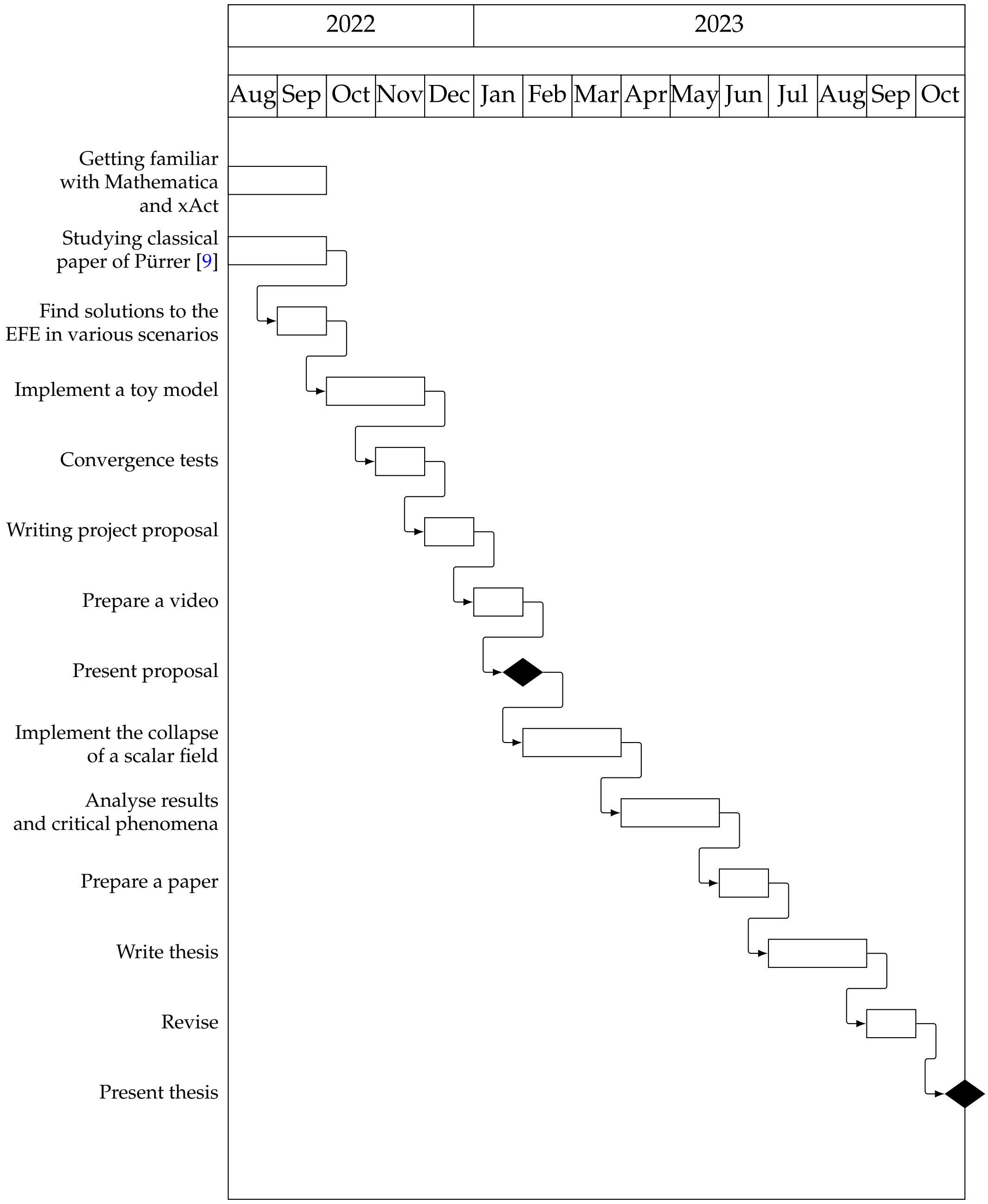
The L^2 convergence was also investigated by computing the L^2 norm of the errors plotted in Figure 3, resulting in a convergence factor Q of

$$Q_f = log_2\left(\left(\frac{\sum(f_{med} - f_{low})^2}{\sum(f_{high} - f_{med})^2}\right)^{\frac{1}{2}}\right) = 4.001, \qquad Q_g = log_2\left(\left(\frac{\sum(g_{med} - g_{low})^2}{\sum(g_{high} - g_{med})^2}\right)^{\frac{1}{2}}\right) = 1.9994. \tag{15}$$

Note that, as expected, both convergence tests indicate that f has 4th order convergence, and g has 2nd order. This is as expected, since f is computed at each timestep using a Runge Kutta integrator of 4th order (Equation 13). On the other hand, the derivative $\partial_x g$ is calculated using a finite difference method of $2nd$ order, which dominates the error and thus g has $2nd$ order convergence (Equation 14).

It was then possible to successfully construct our integration scheme and evolution code, as well as test convergence of our methods. This will allow for easier implementation of the characteristic critical collapse in the future, when we implement the system described in Section 2.3.

5 Preparatory Work and Research Plan



As discussed in Section 3, we aim to extend Pürrer's study of global aspects of critical collapse in spherical symmetry [9] by now considering a Yang-Mills field. To our knowledge such collapse hasn't been yet studied in our setup, formulated as a characteristic initial value problem, with an evolution based on null hypersurfaces foliation, and including future null infinity $\mathscr{I}^+$ in our grid.

Our goal is then to compare the obtained results with the ones obtained by implementing a foliation based on Cauchy truncated slices, which we hope will culminate in the publication of a paper in an international peer-reviewed journal.

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