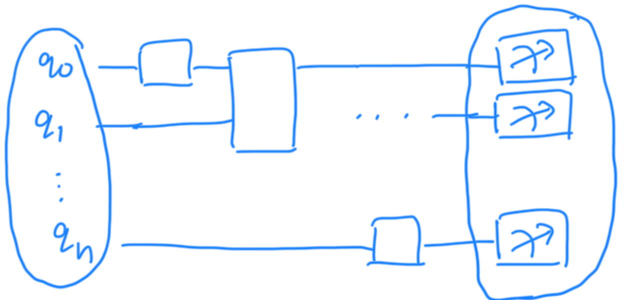
MEASUREMENT ERROR MITIGATION

General quantum circuit:



State Preparation → Measurement

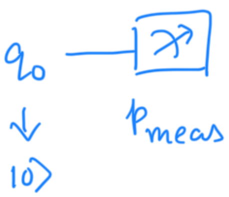
$\underbrace{\text{State Preparation} \quad \text{Measurement}}$ State Preparation And Measurement (SPAM)

SP vs. M cannot be distinguished → treat them together

SPAM error ~ $O(10^{-2})$

Mitigation mainly deals with expectation value.

Say $\rho = |0\rangle\langle 0|$ want to calculate exp val of Z.



Ideally $\langle Z \rangle = \mathrm{Tr}\{\rho . Z\} = \mathrm{Tr}\{|0\rangle\langle 0| Z\}$

$= \mathrm{Tr}\{\langle 0|Z|0\rangle\} = 1$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

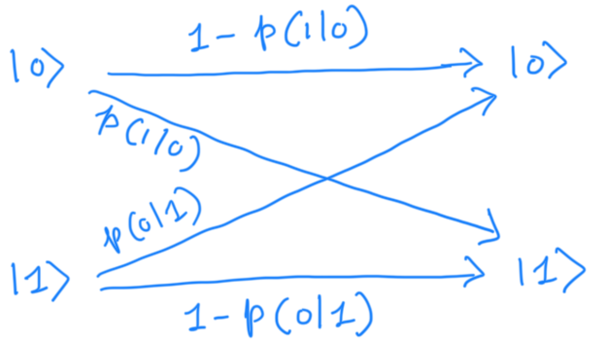
In reality: $|0\rangle\langle 0| \rightarrow (1-p_{meas})|0\rangle\langle 0| + p_{meas}|1\rangle\langle 1|$

$$= \rho_{meas}$$

$$\langle z \rangle_{noisy} = \mathrm{Tr}\{\rho_{meas}\, z\} = (1-p_{meas})\langle 0|z|0\rangle + p_{meas}\langle 1|z|1\rangle$$

$$= (1-p_{meas}) - p_{meas} = 1 - 2.p_{meas}$$

Often $p(0|1) \neq p(1|0)$ | $1 \to 0$ more likely due to damping as well



Assignment matrix
$$A = \begin{array}{c c} & \begin{array}{cc} |0\rangle & \quad |1\rangle \end{array} \to \text{prepared} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{bmatrix} 1-p(1|0) & p(0|1) \\ p(1|0) & 1-p(0|1) \end{bmatrix} \end{array}$$
$\downarrow$ measured

A is a stochastic matrix where the columns sum up to 1.

I/p = $|0\rangle$ $\quad \vec{p}_{ideal} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\boxed{\vec{p}_{noisy} = A . \vec{p}_{ideal}}$$

$$\begin{bmatrix} 1-p(1|0) & p(0|1) \\ p(1|0) & 1-p(0|1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-p(1|0) \\ p(1|0) \end{bmatrix}$$

$$\boxed{\vec{p}_{ideal} = A^{-1} \cdot \vec{p}_{noisy}}$$

$$A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} c & e \\ d & f \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} ac+bd & ae+bf \\ (1-a).c+(1-b).d & (1-a).e+(1-b).f \end{bmatrix}$$

$$ac + bd = 1$$

$$c - ac + d - bd = 0 \quad \Rightarrow \quad c+d = ac+bd = 1$$

$$ae + bf = 0$$

$$e - ae + f - bf = 1 \quad \Rightarrow \quad e+f = 1.$$

$\therefore$ Columns of A^{-1} still add up to 1.

<u>But</u> : Inverse of a stochastic matrix is not stochastic

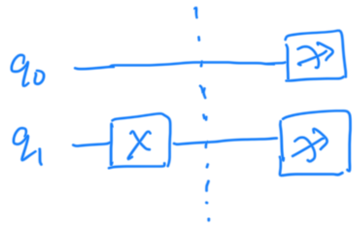
$\Rightarrow$ all the entries of A^{-1} are not positive

$\Rightarrow$ $A^{-1} \cdot p_{noisy}$ is not a probability

- negative values are acceptable if only expval is required

- if probability distribution is needed, one can obtain it via normalization / using some distance metric

- <u>Example</u> :



$$\vec{p}_{ideal} = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We want to calculate $\langle IZ \rangle$, $\langle ZI \rangle$, $\langle ZZ \rangle$

<u>Ideal values</u> :

$$\langle IZ \rangle = \langle 01 | IZ | 01 \rangle = -1$$

$$\langle ZI \rangle = \langle 01 | ZI | 01 \rangle = 1$$

$$\langle ZZ \rangle = \langle 01 | ZZ | 01 \rangle = -1$$

$$\left.\begin{aligned} p(0|1) &= 0.2 \\ p(1|0) &= 0.1 \end{aligned}\right\}$$ these values are ~10x higher than original

$$A = A_1 \otimes A_2 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \otimes \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.81 & 0.18 & 0.18 & 0.04 \\ 0.09 & 0.72 & 0.02 & 0.16 \\ 0.09 & 0.02 & 0.72 & 0.16 \\ \text{[illegible]} & 0.08 & 0.08 & 0.64 \end{bmatrix}$$

<u>Assumption</u> :

Measurement noise is uncorrelated

$0.01 \quad 0.08$

$$\vec{p}_{noisy} = A \cdot \vec{p}_{ideal} = \begin{array}{cc} \begin{bmatrix} 0.18 \\ 0.72 \\ 0.02 \\ 0.08 \end{bmatrix} & \begin{array}{c} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{array} \end{array}$$

Note: although I am calculating $\vec{p}_{noisy}$ as $A \cdot \vec{p}_{ideal}$, we actually obtain it from experiment

$$\langle IZ \rangle = -0.6$$
$$\langle ZI \rangle = 0.8$$
$$\langle ZZ \rangle = -0.48$$

← higher wt. observables incur more noise

$$A^{-1} = \begin{bmatrix} 1.306 & -0.326 & -0.326 & 0.082 \\ -0.163 & 1.469 & 0.041 & -0.367 \\ -0.163 & 0.041 & 1.469 & -0.367 \\ 0.02 & -0.184 & -0.184 & 1.653 \end{bmatrix}$$

$$\vec{p}_{ideal} = A^{-1} \cdot p_{noisy} = \begin{bmatrix} 0.0004 \\ 0.9998 \\ 0.0002 \\ -0.00032 \end{bmatrix}$$

<u>Mitigated expvals</u>:

$$\langle IZ \rangle = -0.9989$$
$$\langle ZI \rangle = 1.00032$$
$$\langle ZZ \rangle = -0.9992$$

- <u>Some notes on variance</u>

Say my observable is $O = \sum_{z \in \{0,1\}^n} O(z) |z\rangle\langle z|$

<u>Experiment</u> : Prepare and measure

Say s_i is the outcome in i-th experiment

Ideal mean $\mu = \frac{1}{M} \cdot \sum_{i=1}^{M} O(s_i)$

M → Total no. of trials

Hoeffding's inequality : $|\mu - \mathrm{Tr}\{\rho . O\}| \leq \delta$ w.p. $\geq 2/3$

if $M = 4/\delta^2$

<u>Noisy scenario</u> : Measurements are no longer projectors

POVM $\Pi_y = \sum_x \langle y|A|x\rangle |x\rangle\langle x|$

$\langle y|A|x\rangle$ ↑ Probability of observing outcome y given true outcome is x.

Error mitigated mean : $\xi = \frac{1}{M} \cdot \sum_{i=1}^{M} \sum_x O(x) \langle x|A^{-1}|s_i\rangle$

$|\xi - \mathrm{Tr}\{\rho . O\}| \leq \delta$ w.p. $\geq 2/3$

if $M = \frac{4\Gamma^2}{\delta^2}$ where $\Gamma = \max_y \sum_x |\langle x|A^{-1}|y\rangle|$

e.g. $A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1.143 & -0.286 \\ -0.143 & 1.286 \end{bmatrix}$

$$\Pi_0 = \langle 0|A|0\rangle |0\rangle\langle 0| + \langle 0|A|1\rangle |1\rangle\langle 1|$$

$$|\langle 0|A^{-1}|0\rangle| = 1.143$$

$$|\langle 1|A^{-1}|0\rangle| = 0.143$$

$$\Pi_1 = \langle 1|A|0\rangle |0\rangle\langle 0| + \langle 1|A|1\rangle |1\rangle\langle 1|$$

$$|\langle 0|A^{-1}|1\rangle| = 0.286$$

$$|\langle 1|A^{-1}|1\rangle| = 1.286 \rightarrow \Gamma$$

say $\delta = 0.1 \rightarrow$ ideal $M = \dfrac{4}{0.01} = 400$

noisy $M = \dfrac{4 \times (1.286)^2}{0.01} \approx 661$

☐ Matrix-free measurement error mitigation (M3)

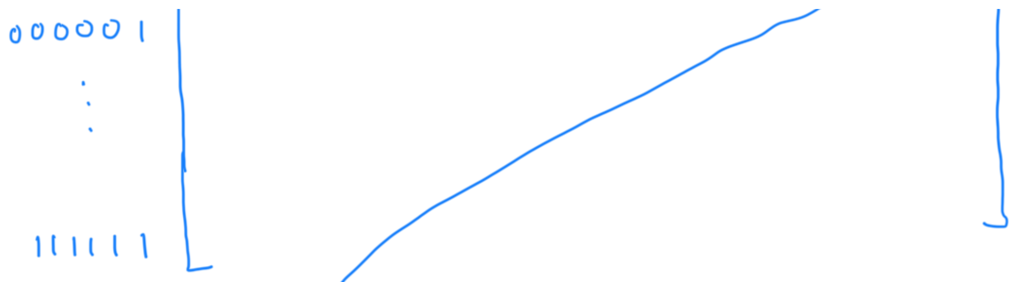
Issue with previous approach

for n qubit system : $A \sim 2^n \times 2^n$

- storing
- inverting

} costly

$$\begin{array}{c|cccc} & 000000 & 000001 & \cdots & 111111 \\ \hline 000000 & p & & & p \end{array}$$

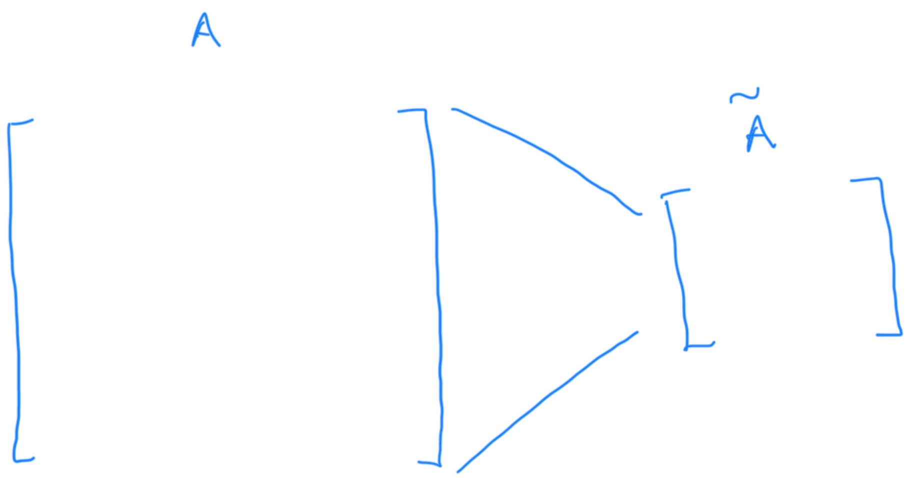


→ prepared in $|111111\rangle$ and measured as $|1000000\rangle$

- highly unlikely
- prob : $p_{meas}^{6} = 10^{-12}$

$p_{meas} \sim 10^{-2}$

Most of the transitions in the A matrix are highly unlikely.



subspace reduction

$$\tilde{A}_{row,\,col} = \begin{cases} \dfrac{A_{row,\,col}}{\displaystyle\sum_{\substack{k \in \vec{P}_{noisy} \\ d(k,col) \le D}} A_{k,\,col}} & \text{if } d(row,col) \le D \\ 0 & \text{otherwise} \end{cases}$$

the division is renormalization to ensure that $\tilde{A}$ is also stochastic

We can use iterative methods to solve this without explicitly doing matrix inversion.

e.g. build a Jacobi preconditioner : P^{-1}

$P^{-1}_{i,i} = 1/\tilde{A}_{i,i}$ $\quad$ P^{-1} is a diagonal matrix

$$\boxed{P^{-1}\tilde{A}\vec{x} = P^{-1}\cdot\vec{p}_{noisy}}$$

Creation of A-matrix :

1-qubit :

$q_0 - |0\rangle$ — [measure] $\Big\}$ $P(0|0)$, $P(1|0)$

$q_0 - |0\rangle$ — [X] — [measure] $\Big\}$ $P(0|1)$, $P(1|1)$

$$A = \begin{bmatrix} P(0|0) & P(1|0) \\ P(0|1) & P(1|1) \end{bmatrix}$$

2-qubit :

q_0 — $|0\rangle$ — [measure]

q_1 — $|0\rangle$ — [measure]

$P(00|00)$

$P(01|00)$

$P(10|00)$

$P(11|00)$

marginalization

$P(0|0)$ for q_0

$P(1|0)$ for q_0

$P(0|0)$ for q_1

$P(1|0)$ for q_1

$q_0 - |0\rangle$ — [X] — [measure]

$q_1 - |0\rangle$ — [X] — [measure]

$P(00|11)$

$P(01|11)$

$P(10|11)$

$P(11|11)$

$P(0|1)$ for q_0

$P(1|1)$ for q_0.

A_0 = A-matrix for q_0.

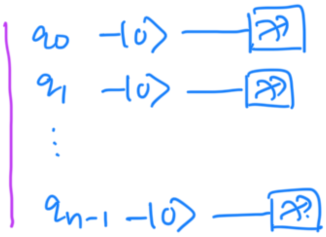
$P(0|1)$ for q_1

$P(1|1)$ for q_1

A_1 = A-matrix for q_1

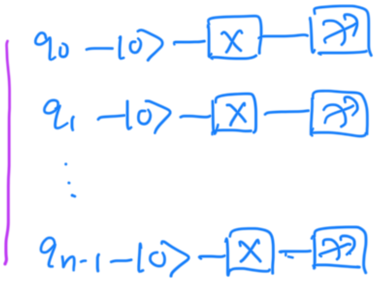
2-qubit A-matrix = $A = A_0 \otimes A_1$

n-qubit :



$\Rightarrow$ Do marginalization

always 2 calibration circuits



$\Rightarrow$ Do marginalization