Quantum Fourier Transform (QFT)

QFT is effectively a change of basis

Computational basis $|x\rangle$ $\quad QFT|x\rangle = |\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i . x . y}{N}} |y\rangle$

$|x\rangle \in \{0,1\}^n \qquad N = 2^n$

$|x\rangle \in \{0,1\}^2 \quad \underbrace{00,\ 01,\ 10,\ 11}_{2^2}$

1 qubit : $\quad QFT|0\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{\frac{2\pi i . 0 . y}{2}} |y\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) = |+\rangle$

$x \in \{0,1\} \quad N = 2,\ n = 1 \qquad QFT|1\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{\frac{2\pi i . 1 . y}{2}} |y\rangle = \frac{1}{\sqrt{2}} \cdot \left(|0\rangle + e^{\pi i} . |1\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) = |-\rangle$

QFT for 1 qubit is Hadamard transform $\qquad H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$

$n = 3 : \quad |\tilde{x}\rangle = \frac{1}{\sqrt{8}} \sum_{y=0}^{7} e^{2\pi i . x . y / 2^3} |y\rangle$

$\sum_{y=1}^{8} \equiv \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_8=0}^{1} \qquad |y\rangle = |y_1 y_2 \cdots y_8\rangle = |y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_8\rangle$

$|\tilde{x}\rangle = \frac{1}{\sqrt{8}} \cdot \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_8=0}^{1} e^{2\pi i . x . y / 2^3} |y_1 y_2 \cdots y_8\rangle$

$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \cdot \sum_{y=0}^{N-1} e^{\frac{2\pi i . x . y}{N}} |y\rangle = \frac{1}{\sqrt{N}} \cdot \sum_{y=0}^{N-1} e^{\frac{2\pi i . x . y}{N}} |y_1 y_2 \cdots y_n\rangle$

$|y\rangle = |y_1 y_2 \cdots y_n\rangle = |0100110\rangle$

$1101 \equiv 2^3 . 1 + 2^2 . 1 + 2^1 . 0 + 2^0 . 1$

$= 2^{n-1} . y_1 + 2^{n-2} . y_2 + \cdots + 2^0 . y_n$

$= \sum_{k=1}^{n} y_k . 2^{n-k}$

$\frac{1}{\sqrt{N}} \cdot \sum_{y=0}^{N-1} e^{\frac{2\pi i . x}{N} \sum_{k=1}^{n} 2^{n-k} . y_k} |y_1 y_2 \cdots y_n\rangle$

$N = 2^n$

$= \frac{1}{\sqrt{N}} \cdot \sum_{y=0}^{N-1} e^{\frac{2\pi i . x}{2^n} \cdot \sum_{k=1}^{n} 2^{n-k} . y_k} |y_1 y_2 \cdots y_n\rangle$

$e^{x_1 + x_2 + x_3} = e^{x_1} . e^{x_2} . e^{x_3}$ –

$\sum_{y=0}^{N-1} \quad 2\pi i \cdot x \sum_{k=1}^{n} 2^{-k} \cdot y_k$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{y=0} e^{\;\cdots \sum_{K=1} \cdots} |y_1 y_2 \dots y_n\rangle$$

$$e^{\sum_i x_i} = \prod_i e^{x_i}$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{y=0}^{N-1} e^{2\pi i \cdot x \cdot \sum_{K=1}^{n} y_K/2^K} |y_1 y_2 \dots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{K=1}^{n} e^{2\pi i \cdot x \cdot y_K/2^K} |y_1 y_2 \dots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \dots \sum_{y_n=0}^{1} \prod_{K=1}^{n} e^{2\pi i \cdot x \cdot y_K/2^K} |y_1 y_2 \dots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \prod_{K=1}^{n} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \dots \sum_{y_n=0}^{1} e^{2\pi i \cdot x \cdot y_K/2^K} |y_1 y_2 \dots y_n\rangle$$

Take $n = 2$: $\sum_{y_1=0}^{1} \sum_{y_2=0}^{1} e^{2\pi i \cdot x \cdot y_1/2} |y_1 y_2\rangle$

$K = 1$:

$$= e^{2\pi i \cdot x \cdot 0/2} |00\rangle + e^{2\pi i \cdot x \cdot 0/2} |01\rangle + e^{2\pi i \cdot x \cdot 1/2} |10\rangle + e^{2\pi i \cdot x \cdot 1/2} |11\rangle$$

$$= |00\rangle + |01\rangle + e^{i \cdot \pi \cdot x} |10\rangle + e^{i \cdot \pi \cdot x} |11\rangle$$

$x = |00\rangle = 0 \Rightarrow \overset{y_1 y_2}{|00\rangle} + |01\rangle + |10\rangle + |11\rangle$ ✓

$x = |01\rangle = 1 \Rightarrow |00\rangle + |01\rangle - |10\rangle - |11\rangle$

$$\frac{1}{\sqrt{N}} \cdot \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \dots \sum_{y_n=0}^{1} \prod_{K=1}^{n} e^{2\pi i \cdot x \cdot y_K/2^K} |y_1\rangle |y_2\rangle \dots |y_n\rangle \quad \dots (i)$$

Coefficients :

$|000\dots0\rangle = 1$

$|000\dots01\rangle = e^{2\pi i \cdot x/2^n}$

$|000\dots11\rangle = e^{2\pi i \cdot x/2^{n-1}} \cdot e^{2\pi i \cdot x/2^n}$

$\vdots$

$|111\dots1\rangle = e^{2\pi i \cdot x/2^1} \cdot e^{2\pi i \cdot x/2^2} \dots e^{2\pi i \cdot x/2^n}$

$y_j = 0 \Rightarrow e^{2\pi i \cdot x \cdot y_j/2^K} = 1.$

$y_n = 1 \Rightarrow e^{2\pi i \cdot x \cdot /2^n}$

y_n, y_{n-1}

$$\underline{\text{CLAIM}} : (i) = \frac{1}{\sqrt{N}} \cdot \left(|0\rangle + e^{2\pi i \cdot x/2^1} |1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \cdot x/2^2} |1\rangle\right) \otimes \dots \otimes \left(|0\rangle + e^{2\pi i \cdot x/2^n} |1\rangle\right)$$

Coefficient : $|00\ldots0\rangle = 1$

$$|00\ldots01\rangle = e^{2\pi i \cdot x/2^n}$$

$$|00\ldots11\rangle = e^{2\pi i \cdot x/2^{n-1}} \cdot e^{2\pi i \cdot x/2^n}$$

$$\vdots$$

$$|11\ldots1\rangle = e^{2\pi i \cdot x/2^1} \cdot e^{2\pi i \cdot x/2^2} \cdots e^{2\pi i \cdot x/2^n}$$

$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

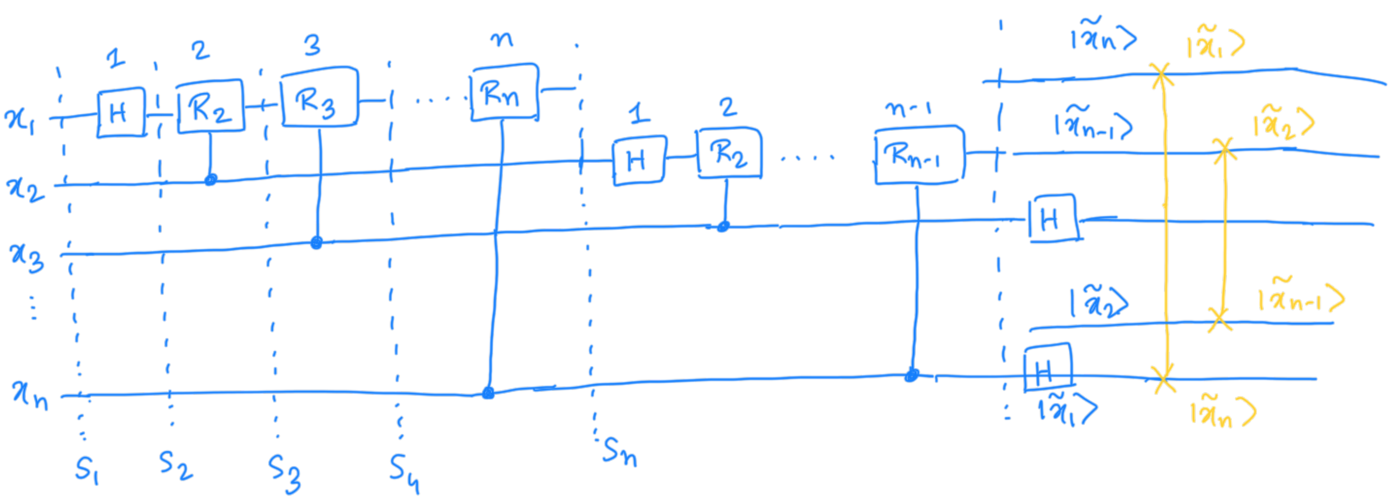
QFT $\downarrow$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^1}} |1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^2}} |1\rangle\right) \otimes \cdots \left(|0\rangle + e^{2\pi i \cdot x/2^n} |1\rangle\right)$$

1) $$H|x_k\rangle \begin{cases} \xrightarrow{x_k = 0} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ \xrightarrow{x_k = 1} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases} \Bigg\} \quad \frac{|0\rangle + e^{\frac{2\pi i \cdot x_k}{2}} |1\rangle}{\sqrt{2}}$$

2) $$R_k |x_j\rangle = e^{\frac{2\pi i \cdot x_j}{2^k}} |x_j\rangle \quad \Bigg| \quad \begin{array}{l} x_j = 0 \to |0\rangle \\ x_j = 1 \to e^{2\pi i/2^k} |1\rangle \end{array}$$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$



$$|x_1\rangle|x_2\rangle \xrightarrow{\text{swap}} |x_2\rangle|x_1\rangle$$

S_0 : $|x_1 x_2 x_3 \ldots x_n\rangle$

S_1 : $\left(|0\rangle + e^{2\pi i/2 \cdot x_1} |1\rangle\right) \otimes |x_2\rangle |x_3\rangle \ldots |x_n\rangle$

$$S_2: \left(|0\rangle + e^{\frac{2\pi i}{2}\cdot x_1} \cdot e^{\frac{2\pi i}{2^2}\cdot x_2} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$S_3: \left(|0\rangle + e^{\frac{2\pi i}{2}\cdot x_1} \cdot e^{\frac{2\pi i}{2^2}\cdot x_2} \cdot e^{\frac{2\pi i}{2^3}\cdot x_3} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$\vdots$$

$$S_n: \left(|0\rangle + e^{\frac{2\pi i}{2}\cdot x_1} \cdot e^{\frac{2\pi i}{2^2}\cdot x_2} \cdot e^{\frac{2\pi i}{2^3}\cdot x_3} \cdots e^{\frac{2\pi i}{2^n}\cdot x_n} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$= \left(|0\rangle + e^{\frac{2\pi i}{2}\cdot x_1 + \frac{2\pi i}{2^2}\cdot x_2 + \cdots \frac{2\pi i}{2^n}\cdot x_n} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$= \left(|0\rangle + e^{\frac{2\pi i}{2^n}\cdot\left[2^{n-1}\cdot x_1 + 2^{n-2}\cdot x_2 + \cdots 2^0\cdot x_n\right]} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$= \left(|0\rangle + e^{\frac{2\pi i}{2^n}\cdot x} |1\rangle\right) \otimes |x_2 x_3 \ldots x_n\rangle$$

$$\#\text{ gates} = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$$

$$+ \frac{n}{2} \text{ swap gates}$$

$$= \frac{n(n+1)}{2} + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2} + \frac{n}{2} = \frac{n^2}{2} + n = O(n^2) \text{ gates}$$

$$2^n = N \Rightarrow n = \log N \Rightarrow O\left((\log(N))^2\right) \text{ gates}$$

Classical

Fast Fourier Transform : $O(N \log N)$

$(\log N)^2 \rightarrow$ roughly savings by a factor of $N. = 2^n$

$\downarrow$

exponential saving