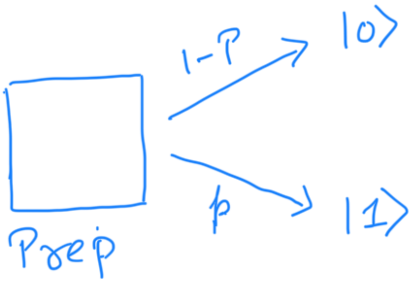
$|\psi\rangle \rightarrow$ vector in Hilbert space $\Rightarrow$ Limitation

I) 

$$\Bigg\} \cancel{X} \begin{pmatrix} \sqrt{1-p} \\ \sqrt{p} \end{pmatrix} = \underbrace{\sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle}_{\text{Superposition}}$$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle_{AB} + \frac{1}{\sqrt{2}}|11\rangle_{AB}$$

$|A\rangle =$

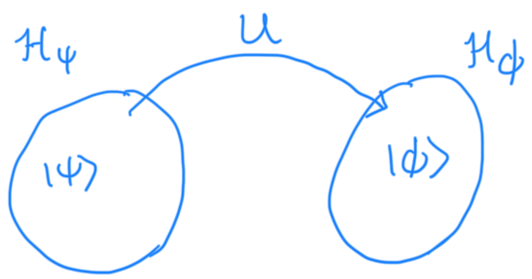
$|B\rangle =$

|| w.p. $1-p$ $|0\rangle\langle 0|$

w.p. p $X|0\rangle\langle 0|X^{\dagger}$ $\rightarrow$ error/noise

II) $U|\psi\rangle = |\phi\rangle \rightarrow$ vector in Hilbert space.

$U \rightarrow$ operator acting on the Hilbert space



Density matrix : A quantum state ρ is an operator acting on the Hilbert space.

unitary $\boxed{U = \alpha_1|0\rangle\langle 0| + \alpha_2|0\rangle\langle 1| + \alpha_3|1\rangle\langle 0| + \alpha_4|1\rangle\langle 1|}$ ✓

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \xrightarrow{\alpha_1} \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$✓\ |0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\alpha_2} \begin{pmatrix} 0 & \alpha_2 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{\alpha_3} \begin{pmatrix} 0 & 0 \\ \alpha_3 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\alpha_4} \begin{pmatrix} 0 & 0 \\ 0 & \alpha_4 \end{pmatrix}$$

$$+ \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

1) $Tr(\rho) = 1.$

$|0\rangle \Rightarrow |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.$

$\rho_\psi = |\psi\rangle\langle\psi| = (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|)$

$= |a|^2 |0\rangle\langle 0| + ab^* |0\rangle\langle 1| + ba^* |1\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$

$|0\rangle \xrightarrow{\text{measure}} 0$ w.p. 1

$\boxed{\begin{aligned} |\psi\rangle &= a|0\rangle + b|1\rangle \\ &0 \text{ w.p. } |a|^2 \\ &1 \text{ w.p. } |b|^2 \end{aligned}}$

$\begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$

what is the probability of outcome $|0\rangle\langle 0|$?

what is the probability of outcome $|1\rangle\langle 1|$?

$Tr(\rho_\psi) = |a|^2 + |b|^2 = 1$

2) $\boxed{\rho \geq 0}$ → each eigenvalue of ρ is ≥ 0

Spectral decomposition : $U \to e_0, |e_0\rangle$

$e_1, |e_1\rangle$

$U = e_0|e_0\rangle\langle e_0| + e_1|e_1\rangle\langle e_1| \Rightarrow$ for any U.

quantum state $\left[\rho_\psi = \overset{\text{prob.} \geq 0}{\psi_0} \underbrace{|\psi_0\rangle\langle\psi_0|}_{\text{state}} + \psi_1 |\psi_1\rangle\langle\psi_1|\right]$

$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \longrightarrow$ $|0\rangle\langle 0|$ w.p. $\frac{1}{2}$

$|1\rangle\langle 1|$ w.p. $\frac{1}{2}$

w.p. $\psi_0 \rightarrow |\psi_0\rangle\langle\psi_0|$

w.p. $\psi_1 \rightarrow |\psi_1\rangle\langle\psi_1|$

Evolution : $\rho_\psi = \boxed{|\psi\rangle\langle\psi|}$ ✓

$$\rho_\phi = U|\psi\rangle\langle\psi|U^\dagger = U\rho_\psi U^\dagger = |\phi\rangle\langle\phi|$$

$U|\psi\rangle \rightarrow |\phi\rangle$

$\langle\psi|U^\dagger \rightarrow \langle\phi|$

$U|\psi\rangle \xrightarrow[\text{conjugate}]{\text{complex}} \langle\psi|U^\dagger$

$$\boxed{\begin{aligned} Tr(AB) &= Tr(BA) \\ Tr(ABC) &= Tr(CAB) \end{aligned}}$$

Measurement : $\rho = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$

$\langle 0|\rho|0\rangle = |a|^2 \rightarrow$ Prob. of outcome $|0\rangle\langle 0|$ ✓

$(1 \quad 0)$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$M_0 = \left(|0\rangle\langle 0|\right)^\dagger = |0\rangle\langle 0|$

$\rightarrow$ Hermitian

$Tr\{\underbrace{\langle 0|\rho|0\rangle}_{\text{scalar}}\} = \boxed{Tr\{\underbrace{|0\rangle\langle 0|\rho}_{\text{matrix}}\}}$ $\rightarrow$ prob. of obtaining M_0

$\langle 1|\rho|1\rangle = |b|^2$

$= Tr\{\langle 1|\rho|1\rangle\} = Tr\{|1\rangle\langle 1|\rho\}$

$M_1 = |1\rangle\langle 1|$ ✓

$\rightarrow$ prob of obtaining M_1

$M_\psi = |\psi\rangle\langle\psi|$ $\quad M_\phi = |\phi\rangle\langle\phi|$ ✗

$M_+ = |+\rangle\langle +|$ $\quad M_- = |-\rangle\langle -|$ ✓

$\hookrightarrow P(M_+) = Tr\{|+\rangle\langle +|\rho\}$

$\hookrightarrow P(M_-) = Tr\{|-\rangle\langle -|\rho\}$

$$\mathrm{Tr}\{|0\rangle\langle 0|\rho\} + \mathrm{Tr}\{|1\rangle\langle 1|\rho\} = 1.$$

$$\Rightarrow \mathrm{Tr}\{|0\rangle\langle 0|\rho + |1\rangle\langle 1|\rho\} = 1$$

$$\Rightarrow \mathrm{Tr}\{(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{I})\rho\} = 1 \qquad | \ \mathrm{Tr}(\rho) = 1$$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = I \qquad | \ \underline{M_0 + M_1 = I}$$

$$\mathrm{Tr}\{(|+\rangle\langle +| + |-\rangle\langle -|)\rho\} = 1 \qquad | \ \underline{M_+ + M_- = I}$$

$\boxed{\text{Measurement is resolution of identity}}$

if $\underline{M_\psi + M_\phi = I}$ $\rightarrow$ then this is a valid measurement basis

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$$\underbrace{M_{x_1} + M_{x_2}}_{\text{these two may not be orthogonal}} + (I - M_{x_1} - M_{x_2}) = I$$

$\underline{\text{POVM}}$ → ~~measurement~~

$|0\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

M_0 $\qquad M_+$

$M_0 + M_+ + \underbrace{(I - M_0 - M_+)}_{\Downarrow}$

<u>Tensor product</u>: $\rho_1 \quad \rho_2$

$$\rho_1 \otimes \rho_2$$

$$|\psi\rangle = \sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle$$

$$|\psi\rangle\langle\psi| = (1-p)\,|0\rangle\langle 0| \qquad |0\rangle\langle 1|$$

$$|1\rangle\langle 0| \qquad |1\rangle\langle 1|$$

<u>Noise</u>

Prep $\xrightarrow{1-p} |0\rangle$

Prep $\xrightarrow{p} |1\rangle$

$p = 0.9$

Initial state: $\rho = \underline{(1-p)}\boxed{|0\rangle\langle 0|} + p\boxed{|1\rangle\langle 1|}$

$$= \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$

$\Rightarrow$ classical

$\frac{1}{3}$ $H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\{ = \begin{pmatrix} \frac{1}{2} & 0 \\ & 1 \end{pmatrix}$

$\xrightarrow{1/2} T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\rho_{\text{ens}} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$

$$\rho = \begin{pmatrix} \boxed{|a|^2} & ab^* \\ a^*b & \boxed{|b|^2} \end{pmatrix} \rightarrow \boxed{\text{coherence}}$$

Noise : off diagonal $\rightarrow 0$

$\boxed{\text{decoherence}}$

$$|0\rangle\langle 0| \xrightarrow{X} X|0\rangle\langle 0|X^\dagger = |1\rangle\langle 1| = \rho$$

$$Tr\{|1\rangle\langle 1|\} = 1 \quad \text{by definition}$$

$$Tr\{\rho^2\} = Tr\{|1\rangle\langle 1|1\rangle\langle 1|\} = Tr\{|1\rangle\langle 1|\} = 1$$

inner product $= 1$ for quantum state

$$\rho = 2/3\, |0\rangle\langle 0| + 1/3\, |1\rangle\langle 1|$$

$$Tr\{\rho\} = 2/3\, Tr\{|0\rangle\langle 0|\} + 1/3\, Tr\{|1\rangle\langle 1|\}$$

$$= 2/3 \cdot 1 + 1/3 \cdot 1 = 1.$$

$$Tr\{\rho^2\} = Tr\left\{\left(2/3\, |0\rangle\langle 0| + 1/3\, |1\rangle\langle 1|\right)\left(2/3\, |0\rangle\langle 0| + 1/3\, |1\rangle\langle 1|\right)\right\}$$

$$= Tr\{4/9\, |0\rangle\langle 0|0\rangle\langle 0| + \cancel{2/9\, |0\rangle\underbrace{\langle 0|1\rangle}_{\to 0}\langle 1|}$$

$$+ \cancel{2/9\, |1\rangle\underbrace{\langle 1|0\rangle}_{\to 0}\langle 0|} + \tfrac{1}{9}\, |1\rangle\langle 1|1\rangle\langle 1|\}$$

$$= Tr\{4/9\, |0\rangle\langle 0| + 1/9\, |1\rangle\langle 1|\} = 4/9 + 1/9 = 5/9.$$

$Tr(\rho^2) = 1 \Rightarrow$ Pure State $\rightarrow$ ✓ statevector

$Tr(\rho^2) < 1 \Rightarrow$ Mixed State $\rightarrow$ × statevector

$$\begin{tikzcd} |0\rangle \arrow[r, "X"] \arrow[rr, bend right, "I"'] & |1\rangle \arrow[r, "X"] & |0\rangle \end{tikzcd}$$

$$X^2 = I$$

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

2×2 will hold for $n \times n$

Pauli matrices span the 2×2 vector space.

$$= a_I I + a_X X + a_Y Y + a_Z Z$$

$$X \equiv \sigma_X$$
$$Y \equiv \sigma_Y$$
$$Z \equiv \sigma_Z$$

$$\rho = r_I \cdot I + r_X \cdot \sigma_X + r_Y \cdot \sigma_Y + r_Z \cdot \sigma_Z \quad ✓$$

$$Tr(\rho) = 1 \Rightarrow r_I \cdot Tr(I) + r_X \cdot Tr(\sigma_X) + r_Y \cdot Tr(\sigma_Y) + r_Z \cdot Tr(\sigma_Z)$$

$$= r_I \cdot 2 + r_X \cdot 0 + r_Y \cdot 0 + r_Z \cdot 0$$

$$= 2 \cdot r_I = 1 \Rightarrow r_I = 1/2$$

$$\rho = \frac{1}{2} \cdot I + r_X \cdot \sigma_X + r_Y \cdot \sigma_Y + r_Z \cdot \sigma_Z$$

$$= \frac{I + 2 \cdot r_x \cdot \sigma_x + 2 \cdot r_y \cdot \sigma_y + 2 \cdot r_z \cdot \sigma_z}{2}$$

$$= \frac{I + n_x \cdot \sigma_x + n_y \cdot \sigma_y + n_z \cdot \sigma_z}{2}$$

$$= \frac{I + \vec{n} \cdot \vec{\sigma}}{2}$$

$$\boxed{\rho = \frac{I + \vec{n} \cdot \vec{\sigma}}{2}}$$

$$n_x \cdot \sigma_x + n_y \cdot \sigma_y + n_z \cdot \sigma_z$$

$$= \begin{pmatrix} n_x & n_y & n_z \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

$$= \vec{n} \cdot \vec{\sigma}$$

$$\rho^2 = \frac{I + n_x \cdot \sigma_x + n_y \cdot \sigma_y + n_z \cdot \sigma_z}{2} \cdot \frac{I + n_x \cdot \sigma_x + n_y \cdot \sigma_y + n_z \cdot \sigma_z}{2}$$

$$= \frac{1}{4} \cdot \Big(I + n_x^2 \cdot \sigma_x^2 \; (\to I) + n_z^2 \, \sigma_z^2 \; (I) + n_y^2 \, \sigma_y^2 \; (I) +$$

$$n_x \sigma_x + n_y \cdot \sigma_y + n_z \cdot \sigma_z + \cdots \quad \text{2 times each}$$

$$+ n_x \cdot n_y \cdot \sigma_x \cdot \sigma_y + n_x \cdot n_z \cdot \sigma_x \cdot \sigma_z + \cdots \quad \text{cross terms} \Big)$$

$$\mathrm{Tr}(\rho^2) = \frac{1}{4} \cdot \Big(2 + 2 \cdot n_x^2 + 2 \cdot n_y^2 + 2 n_z^2 +$$

$$+ 0 + 0 + 0$$

$$+ 0 + 0 \cdots \Big)$$

$\mathrm{Tr}(P) = 0$

$P_i P_j = P_k$

$\mathrm{Tr}(P_k) = 0)$

$$= \frac{1}{4} \cdot (2 + 2 \cdot n_x^2 + 2 n_y^2 + 2 n_z^2)$$

Pure state : $Tr(\rho^2) = 1$.

$$\Rightarrow \frac{1}{4} \cdot \left(2 + 2 \cdot n_x^2 + 2n_y^2 + 2n_z^2\right) = 1$$

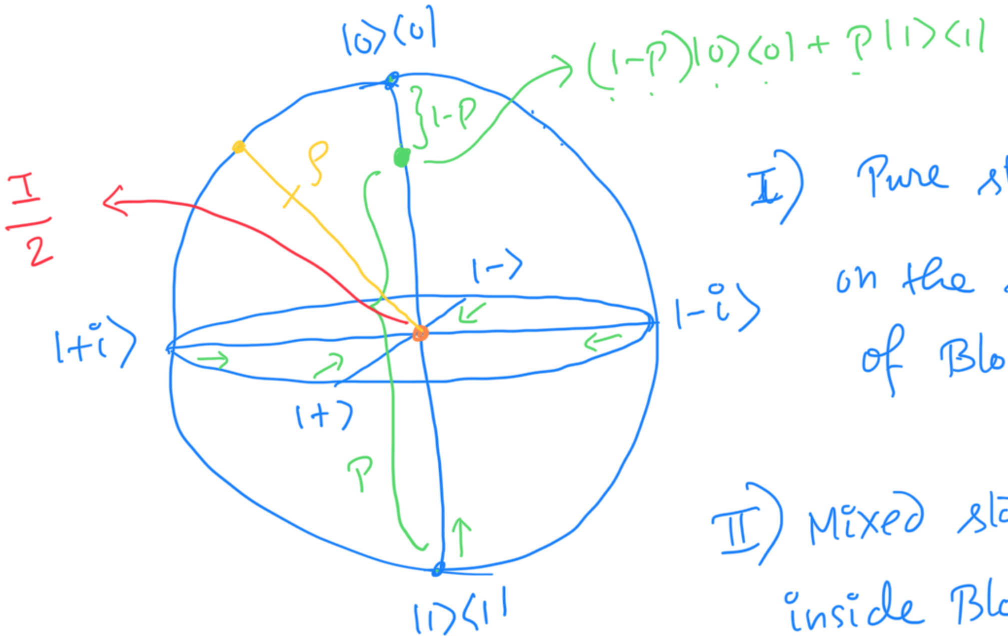
$$\Rightarrow \frac{1}{2} \cdot \left(1 + n_x^2 + n_y^2 + n_z^2\right) = 1$$

$$\Rightarrow 1 + n_x^2 + n_y^2 + n_z^2 = 2$$

$$\Rightarrow \boxed{n_x^2 + n_y^2 + n_z^2 = 1} \Rightarrow \boxed{\text{Sphere with radius } 1}$$

$\Downarrow$

Bloch Sphere



I) Pure state belongs on the surface of Bloch sphere

II) Mixed state belongs inside Bloch sphere

$Tr(\rho^2) < 1$

Centre of Bloch sphere

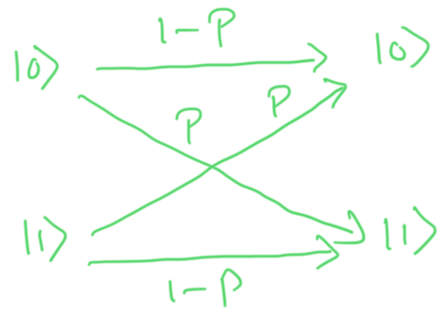
$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$

$\Downarrow$

$I/2$

$= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i|$

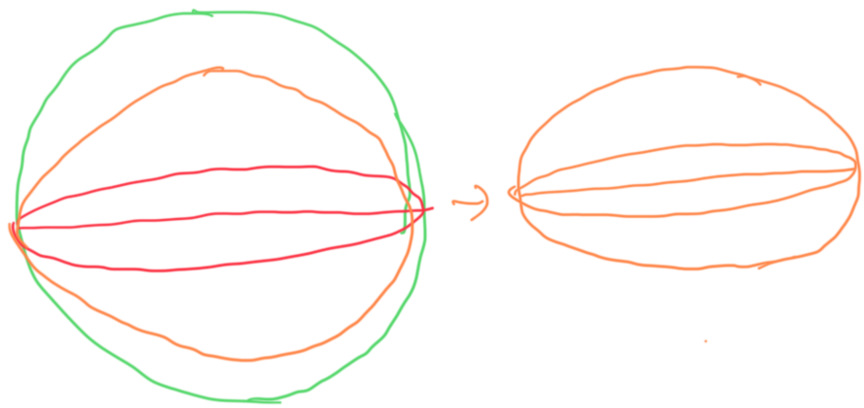
Bit flip:



Symmetric

$X|+\rangle = |+\rangle$

$X|-\rangle = |-\rangle$

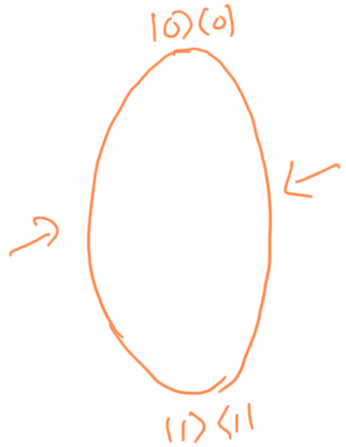


$\rho \to (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$

$= (1-p)\rho + p \cdot X\rho X^{+}$

Phase flip: $Z|0\rangle = |0\rangle$ | $Z|+\rangle = |-\rangle$

$Z|1\rangle = -|1\rangle$ | $Z|-\rangle = |+\rangle$

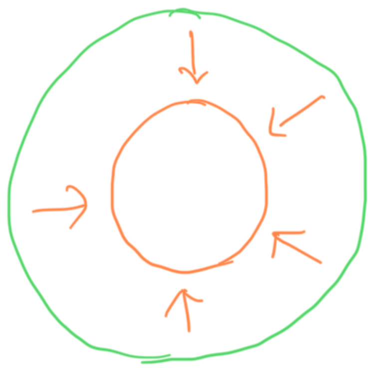


$\rho \to (1-p)|+\rangle\langle +| + p|-\rangle\langle -|$

$= (1-p)\rho + p\, Z.\rho Z^{+}$

Depolarizing noise $\to$ $P/_3\, X$, $P/_3\, Y$, $P/_3\, Z$

$p \equiv$ prob. of error



$p \to 1$

B.S $\to$ center

$$\rho \to (1-p)\rho + \underbrace{\tfrac{p}{3} \cdot X\rho X^{\dagger} + \tfrac{p}{3} Y\rho Y^{\dagger} + \tfrac{p}{3} Z\rho Z^{\dagger}}$$

$$= \boxed{(1-p)\rho + p \cdot \frac{I}{2}}$$ $\to$ Depolarizing noise ✓

$\mathcal{E} \to$ operator : $\{I, X, Y, Z\}$ span