Quantum Error Correction (S. N. Bose)

stabilizers : A set of operators $S_1, S_2, \ldots, S_m \in \{I, X, Z\}^{\otimes n}$

i) $S_i |\psi\rangle = |\psi\rangle \quad \forall i$

ii) if $\mathcal{E}$ is an error, $\exists j$ such that $S_j(\mathcal{E}|\psi\rangle) = -\mathcal{E}|\psi\rangle$

iii) for $\mathcal{E}_1 \neq \mathcal{E}_2$, $\exists j \neq k$ such that $S_j(\mathcal{E}_1|\psi\rangle) \neq S_k(\mathcal{E}_2|\psi\rangle)$

iv) $\forall i, j \quad [S_j, S_k] = 0$

If a QECC has n-qubits and m-stabilizers then the number of logical qubits is $(n-m)$.

- Hamming Code

$$\mathcal{G} = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\underbrace{}_{I_k} \quad \underbrace{}_{P_{n-k}}$$

$\mathcal{G}_{k \times n}$

$m_{1 \times k} \cdot \mathcal{G}_{k \times n} = c_{1 \times n}$

$$H_{(n-k) \times n} = \left[P_{n-k}^T \;\vdots\; I_{n-k} \right]$$

$$= \left(\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$\mathcal{G} \cdot H^T = 0$ and for any valid codeword y, $H \cdot y^T = 0$

e.g. $m = (1001)$

$c = (1\,0\,0\,1\,1\,0\,0) = y$

$m \cdot G = (1\,0\,0\,1\,\ldots)$

$$H \cdot y^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_e = (1\,1\,0\,0\,1\,1\,0\,0)$$

$$H \cdot y^T = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \neq 0$$

<u>CSS code</u>: Take the parity check matrix H. replace 0 with I, 1 with X/Z stabilizers: S_1 : X X X I X I I

$$\left.\begin{array}{l} S_2 : X\,X\,I\,X\,I\,X\,I \\ S_3 : X\,I\,X\,X\,I\,I\,X \\ S_4 : Z\,Z\,Z\,I\,Z\,I\,I \\ S_5 : Z\,Z\,I\,Z\,I\,Z\,I \\ S_6 : Z\,I\,Z\,Z\,I\,I\,Z \end{array}\right\}$$

7 qubits, 6 stabilizers $\Rightarrow$ 1 logical qubit.

$[[7,1,3]]$ steane code

<u>Conditions for error correction</u>

- <u>Necessary</u>: $\langle \bar{j} | \mathcal{E}_b^\dagger \mathcal{E}_a | \bar{i} \rangle = 0$, $i \neq j$ i.e. errors should take orthogonal codewords to orthogonal subspaces.

- <u>Sufficient</u>: $\langle \bar{j} | \mathcal{E}_b^\dagger \mathcal{E}_a | \bar{i} \rangle = C_{ba} \cdot \delta_{ij}$

$\langle \bar{i} | \mathcal{E}_b^\dagger \mathcal{E}_a | \bar{i} \rangle$

$C_{ba} = \langle i | E_b^\dagger E_a | i \rangle$

this requirement includes degenerate codes.

↳ does not depend on the codeword i so correction is possible w/o knowing anything about the codeword

Optimal QECC :

Each qubit can be affected by X, Y, Z errors.

For n-qubits : $3n$ possibilities.
↳ for both $|0\rangle_L$ & $|1\rangle_L$.

$$2(3n+\overset{\text{no error}}{1}) \le 2^n$$

↳ min value of n is 5.

Laflamme code:

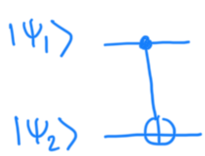
S_1 : I X Z Z X
S_2 : X I X Z Z
S_3 : Z X I X Z
S_4 : Z Z X I X

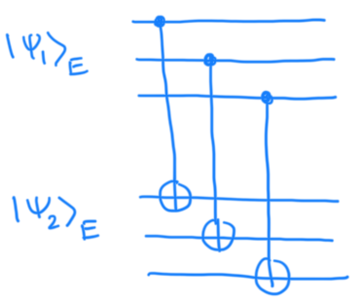
Logical Operators :

e.g. 3-qubit code : $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\psi\rangle_E = \alpha|000\rangle + \beta|111\rangle$

$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$

↳ $X_L = X \otimes X \otimes X \qquad X_L|\psi\rangle_E = \alpha|111\rangle + \beta|000\rangle$



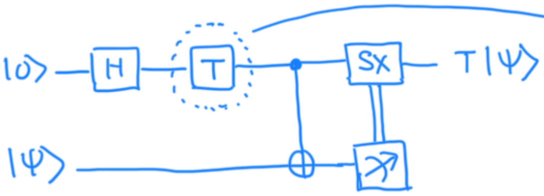


Transversal gates

Not all gates are transversal

Not all gates are transversal : Clifford + T → not transversal

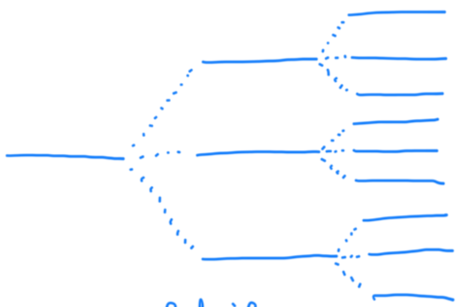
T gate implementation for steane code



→ low-fidelity T gate

Prob. of failure of Shor code : $1 - \left[(1-p)^9 + \binom{9}{1} p (1-p)^8\right] \approx 36p^2$

Prob. of failure : $cp^2 = \frac{1}{c}(cp)^2$ $\quad | \; cp^2 < p$ if $c < 1/p$



fails if 2 errors occur $\quad \frac{(cp)^{2^1}}{c}$

fails if 4 errors occur. $\quad \frac{1}{c}(cp)^{2^2}$

K levels of concatenation

$$P(\text{fail}) = \frac{1}{c} \cdot (cp)^{2^K}$$

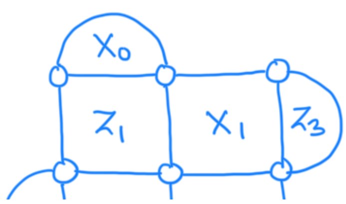
Resource increase : $O(d^K)$

Reduction of error : $O(p^{2^K})$

If desired accuracy is ϵ, we need.

$$\frac{1}{c} \cdot (cp)^{2^K} \leq \epsilon \quad \Rightarrow \quad K \geq \log_2 \left(\log_{cp} (c\epsilon) \right)$$

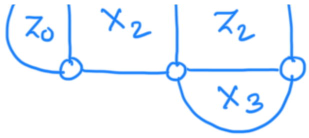
- Surface Code :

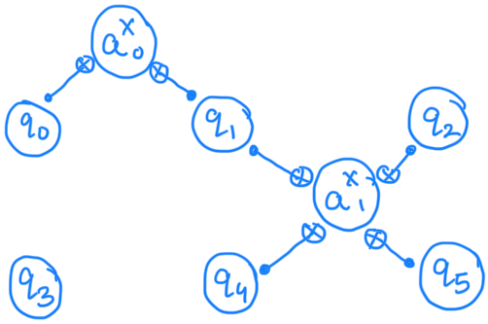


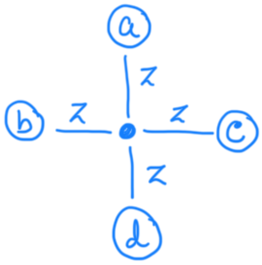
$X_0 \equiv X_0 X_1 I_2 I_3 \dots I_8$

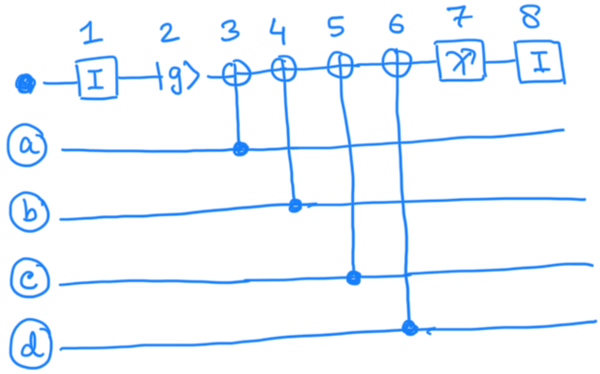
$X_1 \equiv I_0 X_1 X_2 I_3 X_4 X_5 I_6 I_7 I_8$

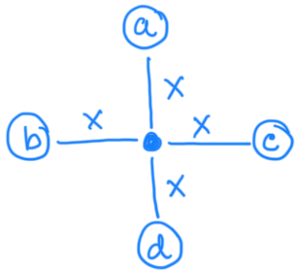
$\vdots$

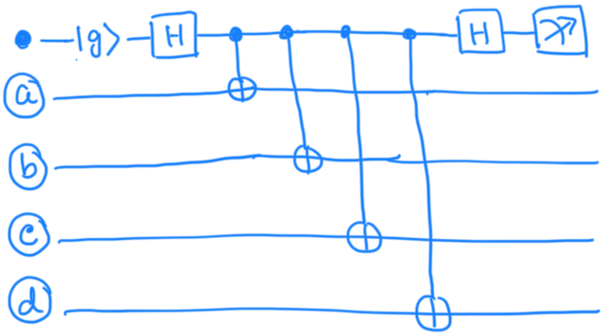




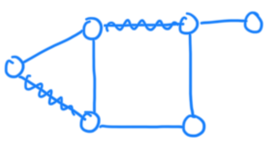




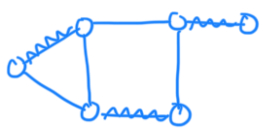


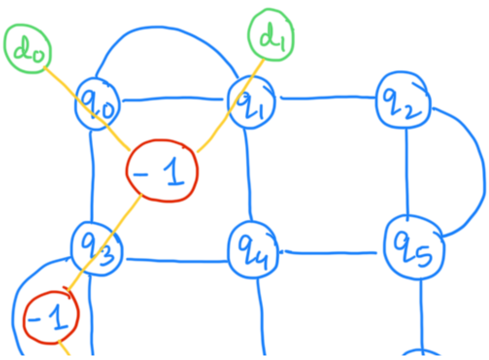


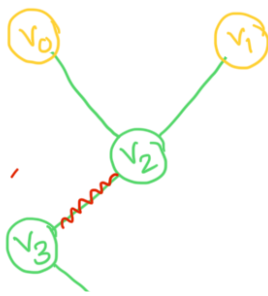
Matching : $\mathcal{G} = (V,E)$, $M \subseteq E$ s.t. each vertex appears in at most one edge



A perfect matching is one that covers every vertex of the graph







q_6
q_7
q_8
d_2

v_4