A Brief Introduction to Quantum Channels

(Primarily follows from a course by Prof. Aram Harrow)

Ritajit Majumdar

Research Scientist
IBM Quantum

Table of contents

Stinespring representation

Unitary dynamics

Physical dynamics is described by unitary operators.

$$|\psi\rangle \to U|\psi\rangle$$

Henceforth, quantum state $\Rightarrow$ density matrix ρ

- For pure states $\rho = |\psi\rangle\langle\psi|$
  - $\rho \to U|\psi\rangle\langle\psi| U^\dagger = U\rho U^\dagger$
- For mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
  - $\rho \to \sum_i p_i U|\psi_i\rangle\langle\psi_i| U^\dagger = U(\sum_i p_i |\psi_i\rangle\langle\psi_i|)U^\dagger = U\rho U^\dagger$

Unitary dynamics

Physical dynamics is described by unitary operators.

$$|\psi\rangle \to U|\psi\rangle$$

Henceforth, quantum state $\Rightarrow$ density matrix ρ

- For pure states $\rho = |\psi\rangle\langle\psi|$
  - $\rho \to U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger$
- For mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
  - $\rho \to \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U(\sum_i p_i|\psi_i\rangle\langle\psi_i|)U^\dagger = U\rho U^\dagger$

Conclusion: *It is not necessary to know the exact preparation of the density matrix in order to determine its dynamics.*

Shortcomings of unitary dynamics

Unitary operator preserves the dimension of the system

- Not possible to add a system: $\rho \to \rho \otimes \sigma$

Shortcomings of unitary dynamics

Unitary operator preserves the dimension of the system

- Not possible to add a system: $\rho \to \rho \otimes \sigma$
- Or discard a subsystem: $\rho_{AB} \to Tr_B\{\rho_{AB}\} = \rho_A$

Shortcomings of unitary dynamics

Unitary operator preserves the dimension of the system

- Not possible to add a system: $\rho \to \rho \otimes \sigma$
- Or discard a subsystem: $\rho_{AB} \to Tr_B\{\rho_{AB}\} = \rho_A$

- $Tr_B\{\rho_{AB}\} = \sum_b (I_A \otimes \langle b|_B)\rho_{AB}(I_A \otimes |b\rangle_B)$.
- Example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}(|0_A 0_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|)$$

$$Tr_B\{\rho_{AB}\} = \frac{1}{2}(|0_A\rangle\langle 0_A| \, Tr(|0_B\rangle\langle 0_B|) + |0_A\rangle\langle 1_A| \, Tr(|0_B\rangle\langle 1_B|) + |1_A\rangle\langle 0_A| \, Tr(|1_B\rangle\langle 0_B|) + |1_A\rangle\langle 1_A| \, Tr(|1_B\rangle\langle 1_B|))$$

Shortcomings of unitary dynamics

Unitary operator preserves the dimension of the system

- Not possible to add a system: $\rho \rightarrow \rho \otimes \sigma$
- Or discard a subsystem: $\rho_{AB} \rightarrow Tr_B\{\rho_{AB}\} = \rho_A$

- $Tr_B\{\rho_{AB}\} = \sum_b (I_A \otimes \langle b|_B)\rho_{AB}(I_A \otimes |b\rangle_B)$.
- Example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}(|0_A 0_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|)$$

$$Tr_B\{\rho_{AB}\} = \frac{1}{2}(|0_A\rangle\langle 0_A|\, Tr(|0_B\rangle\langle 0_B|) + |0_A\rangle\langle 1_A|\, Tr(|0_B\rangle\langle 1_B|) + |1_A\rangle\langle 0_A|\, Tr(|1_B\rangle\langle 0_B|) + |1_A\rangle\langle 1_A|\, Tr(|1_B\rangle\langle 1_B|))$$

$$= \frac{1}{2}(|0_A\rangle\langle 0_A|\langle 0_B|0_B\rangle + |0_A\rangle\langle 1_A|\langle 0_B|1_B\rangle + |1_A\rangle\langle 0_A|\langle 1_B|0_B\rangle + |1_A\rangle\langle 1_A|\langle 1_B|1_B\rangle)$$

Shortcomings of unitary dynamics

Unitary operator preserves the dimension of the system

- Not possible to add a system: $\rho \to \rho \otimes \sigma$
- Or discard a subsystem: $\rho_{AB} \to Tr_B\{\rho_{AB}\} = \rho_A$

- $Tr_B\{\rho_{AB}\} = \sum_b (I_A \otimes \langle b|_B)\rho_{AB}(I_A \otimes |b\rangle_B)$.
- Example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \tfrac{1}{2}(|0_A 0_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|)$$

$$Tr_B\{\rho_{AB}\} = \tfrac{1}{2}(|0_A\rangle\langle 0_A| \, Tr(|0_B\rangle\langle 0_B|) + |0_A\rangle\langle 1_A| \, Tr(|0_B\rangle\langle 1_B|) + |1_A\rangle\langle 0_A| \, Tr(|1_B\rangle\langle 0_B|) + |1_A\rangle\langle 1_A| \, Tr(|1_B\rangle\langle 1_B|))$$

$$= \tfrac{1}{2}(|0_A\rangle\langle 0_A| \langle 0_B|0_B\rangle + |0_A\rangle\langle 1_A| \langle 0_B|1_B\rangle + |1_A\rangle\langle 0_A| \langle 1_B|0_B\rangle + |1_A\rangle\langle 1_A| \langle 1_B|1_B\rangle)$$

$$= \tfrac{1}{2}(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|)$$

Some terminologies

Let A and B be vector spaces, and $L(A)$ and $L(B)$ denote linear operators acting on A and B respectively. We note that density matrix is such an operator.

$L(A, B) \Rightarrow$ Linear operators from A to B.

$L(L(A), L(B)) \Rightarrow$ *"Superoperator"* (Operators acting on density matrices are superoperators)

Partial trace: $L(L(A \otimes B), L(A))$ is also a superoperator

Quantum operation $:=$ unitaries $+$ adding systems $+$ partial trace

Isometry

An isometry V is an operator

$$V \in L(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}), \text{ where } d_B \geq d_A, \text{ such that } V^\dagger V = I_{d_A}$$

$VV^\dagger = I_{d_B}$ iff $d_A = d_B \Rightarrow$ it is a unitary operator.

$$\begin{aligned} ||V|\psi\rangle|| &= \sqrt{(V|\psi\rangle)^\dagger (V|\psi\rangle)} \\ &= \sqrt{\langle\psi| V^\dagger V |\psi\rangle} \\ &= \sqrt{\langle\psi|\psi\rangle} \\ &= || |\psi\rangle || \end{aligned}$$

Therefore, Isometry preserves the norm of a quantum state.

Isometry $\Rightarrow$ Iso + metric $\Rightarrow$ same length.

Isometry (Contd.)

Consider an n-qubit system. The entire circuit can then be represented as a sequence of isometries and partial traces. Let the circuit be:

$$V^4_{n-3}\, Tr_k\, Tr_j\, V^3_{n-1}\, Tr_i\, V^2_n\, V^1_n$$

Isometry (Contd.)

Consider an n-qubit system. The entire circuit can then be represented as a sequence of isometries and partial traces. Let the circuit be:

$$V^4_{n-3}\, Tr_k\, Tr_j\, V^3_{n-1}\, Tr_i\, V^2_n\, V^1_n$$

$$= V^4_{n-3}\, Tr_k\, Tr_j\, Tr_i\, V'^3_n\, V^2_n\, V^1_n$$

Isometry (Contd.)

Consider an n-qubit system. The entire circuit can then be represented as a sequence of isometries and partial traces. Let the circuit be:

$$V_{n-3}^4 \, Tr_k \, Tr_j \, V_{n-1}^3 \, Tr_i \, V_n^2 \, V_n^1$$

$$= V_{n-3}^4 \, Tr_k \, Tr_j \, Tr_i \, V_n'^3 \, V_n^2 \, V_n^1 = Tr_k \, V_{n-2}'^4 \, Tr_j \, Tr_i \, V_n'^3 \, V_n^2 \, V_n^1$$

$$\ldots = Tr_k \, Tr_j \, Tr_i \, V_n'^4 \, V_n'^3 \, V_n^2 \, V_n^1$$

$$= Tr_{i,j,k} V_n$$

Isometry (Contd.)

Consider an n-qubit system. The entire circuit can then be represented as a sequence of isometries and partial traces. Let the circuit be:

$$V_{n-3}^4 \, Tr_k \, Tr_j \, V_{n-1}^3 \, Tr_i \, V_n^2 \, V_n^1$$

$$= V_{n-3}^4 \, Tr_k \, Tr_j \, Tr_i \, V_n'^3 \, V_n^2 \, V_n^1 = Tr_k \, V_{n-2}'^4 \, Tr_j \, Tr_i \, V_n'^3 \, V_n^2 \, V_n^1$$

$$\ldots = Tr_k \, Tr_j \, Tr_i \, V_n'^4 \, V_n'^3 \, V_n^2 \, V_n^1$$

$$= Tr_{i,j,k} V_n$$

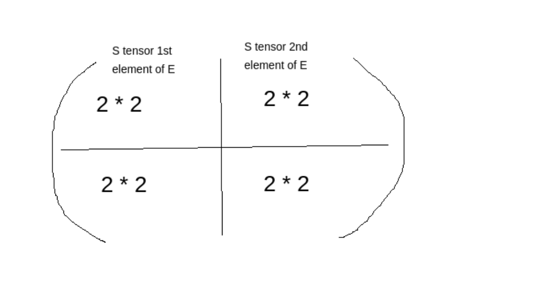
Conclusion: *Any quantum circuit can be represented as a single isometry followed by a single partial trace.*

Kraus operator representation

From Stinespring to Kraus operators

- Let S be the system of interest and E be the environment. In Stinespring representation, there is a single isometry V acting on the system $A = S \otimes E$.
- We are not always interested in the combined dynamics, and want to know only the evolution of S.

  Let $S_{2\times2}$ and $E_{2\times2}$. Then $S \otimes E$ is a 4×4 matrix, and $L(S \otimes E)$ is an $n \times 4$ operator. Take $n = 4$



From Stinespring to Kraus operators (Contd.)

Fix a basis $\{|e\rangle\}$ for E. Represent the isometry as

$$V = \sum_e V_e \otimes |e\rangle, \text{ where } V_e \in L(S, B)$$

$$\begin{aligned}
I &= V^\dagger V \\
&= (\sum_{e_1} V_{e_1}^\dagger \otimes \langle e_1|)(\sum_{e_2} V_{e_2} \otimes |e_2\rangle) \\
&= \sum_{e_1} \sum_{e_2} V_{e_1}^\dagger V_{e_2} \otimes \langle e_1|e_2\rangle \\
&= \sum_{e} V_{e}^\dagger V_{e} \otimes \delta_{e_1, e_2 = e}
\end{aligned}$$

Conclusion: *In general, individual Kraus operators are not isometries, rather they satisfy the condition*

$$\sum_e V_e^\dagger V_e = I$$

Quantum dynamics using Kraus operators

Recall that any quantum channel $\mathcal{N}$ can be represented as a single isometry followed by a single partial trace.

$$
\begin{aligned}
\mathcal{N}(\rho) &= Tr_E V \rho V^\dagger \\
&= Tr_E(\sum_{e_1} V_{e_1} \otimes |e_1\rangle)\rho(\sum_{e_2} V_{e_2}^\dagger \otimes \langle e_2|) \\
&= Tr_E(\sum_{e_1}\sum_{e_2}(V_{e_1}\rho V_{e_2}^\dagger) \otimes |e_1\rangle\langle e_2|) \\
&= \sum_{e_1}\sum_{e_2}(V_{e_1}\rho V_{e_2}^\dagger) \otimes Tr_E(|e_1\rangle\langle e_2|) \\
&= \sum_{e_1}\sum_{e_2}(V_{e_1}\rho V_{e_2}^\dagger) \otimes \langle e_1|e_2\rangle \\
&= \sum_{e} V_e \rho V_e^\dagger
\end{aligned}
$$

Kraus operators (some examples)

1. Unitary evolution: $\mathcal{N}(\rho) = U\rho U^\dagger$, single Kraus op $\{U\}$

Kraus operators (some examples)

1. Unitary evolution: $\mathcal{N}(\rho) = U\rho U^\dagger$, single Kraus op $\{U\}$
2. Unitary operators picked from a probability distribution: $\mathcal{N}(\rho) = \sum_e p_e U_e \rho U_e^\dagger$, Kraus op $\{\sqrt{p_e} U_e\}$

Kraus operators (some examples)

1. Unitary evolution: $\mathcal{N}(\rho) = U\rho U^\dagger$, single Kraus op $\{U\}$
2. Unitary operators picked from a probability distribution: $\mathcal{N}(\rho) = \sum_e p_e U_e \rho U_e^\dagger$, Kraus op $\{\sqrt{p_e} U_e\}$
3. Partial trace $Tr_B(\rho_{AB}) = \sum_b (I_A \otimes \langle b|_B)\rho_{AB}(I_A \otimes |b\rangle_B)$, Kraus op $V_B = I_A \otimes \langle b|_B$

$$\begin{aligned} \sum_b V_B^\dagger V_B &= \sum_b (I_A \otimes \langle b|_B)(I_A \otimes |b\rangle_B) \\ &= \sum_b I_A . I_A \otimes |b\rangle \langle b| \\ &= I_A \otimes \sum_b |b\rangle \langle b| = I_A \otimes I_B \end{aligned}$$

Kraus operators (some examples)

1. Unitary evolution: $\mathcal{N}(\rho) = U\rho U^\dagger$, single Kraus op $\{U\}$
2. Unitary operators picked from a probability distribution: $\mathcal{N}(\rho) = \sum_e p_e U_e \rho U_e^\dagger$, Kraus op $\{\sqrt{p_e} U_e\}$
3. Partial trace $Tr_B(\rho_{AB}) = \sum_b (I_A \otimes \langle b|_B)\rho_{AB}(I_A \otimes |b\rangle_B)$, Kraus op $V_B = I_A \otimes \langle b|_B$

$$\begin{aligned}\sum_b V_B^\dagger V_B &= \sum_b (I_A \otimes \langle b|_B)(I_A \otimes |b\rangle_B) \\ &= \sum_b I_A . I_A \otimes |b\rangle \langle b| \\ &= I_A \otimes \sum_b |b\rangle \langle b| = I_A \otimes I_B\end{aligned}$$

4. $\mathcal{N}(\rho) = |0\rangle \langle 0|$, i.e. this channel maps any quantum system to the state 0, Kraus ops: $V_0 = |0\rangle \langle 0|$, $V_1 = |0\rangle \langle 1|$

Kraus operator representation of depolarizing channel

Depolarizing Noise Model: The action of this noise model can be represented as

$$\begin{aligned}\mathcal{N}(\rho) &= (1-p)\rho + p\frac{I}{2} \\ &= (1-p)\rho + \frac{p}{4}(I\rho I + \sigma_x\rho\sigma_x + \sigma_z\rho\sigma_z + \sigma_y\rho\sigma_y)\end{aligned}$$

Note that this is equivalent to the scenario where unitary operators were selected from a probability distribution. Therefore the Kraus operators are

$$V_0 = \sqrt{(1-\tfrac{3p}{4})}I \quad V_1 = \sqrt{\tfrac{p}{4}}\sigma_x$$
$$V_2 = \sqrt{\tfrac{p}{4}}\sigma_z \quad V_3 = \sqrt{\tfrac{p}{4}}\sigma_y$$

Kraus operator representation of amplitude damping channel

The action of amplitude damping channel is

$$|0\rangle_S \otimes |0\rangle_E \to |0\rangle_S \otimes |0\rangle_E$$
$$|1\rangle_S \otimes |0\rangle_E \to \sqrt{(1-p)}\,|1\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E$$

Kraus Ops: $V_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{pmatrix} \quad V_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$

It is trivial to verify that $V_0^\dagger V_0 + V_0^\dagger V_0 = I$

Amplitude damping channel

Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then $\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Action of the amplitude damping channel

$$\begin{aligned} V_0 \rho V_0^\dagger &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{(1-p)} \\ \sqrt{(1-p)} & (1-p) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_1 \rho V_1^\dagger &= \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Amplitude damping channel (Contd.)

$$\mathcal{N}(\rho) = V_0 \rho V_0^\dagger + V_1 \rho V_1^\dagger = \frac{1}{2} \begin{pmatrix} 1+p & \sqrt{(1-p)} \\ \sqrt{(1-p)} & (1-p) \end{pmatrix}$$

Obs: As $p \to 1$, the off-diagonal terms goes to 0, i.e. the system loses its coherence and becomes classical.

Amplitude damping channel (Contd.)

$$\mathcal{N}(\rho) = V_0 \rho V_0^\dagger + V_1 \rho V_1^\dagger = \frac{1}{2} \begin{pmatrix} 1+p & \sqrt{(1-p)} \\ \sqrt{(1-p)} & (1-p) \end{pmatrix}$$

Measurement: $M_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- Prob(outcome is $|0\rangle\langle 0|$) $= Tr\{M_0 \mathcal{N}(\rho)\} = \frac{1+p}{2}$
- Prob(outcome is $|1\rangle\langle 1|$) $= Tr\{M_1 \mathcal{N}(\rho)\} = \frac{1-p}{2}$

Obs: As $p \to 1$, probability of getting output $|0\rangle\langle 0| \to 1$, i.e., the system tends to settle in its ground state.

Amplitude damping channel (Contd.)

For a general density matrix

$$\mathcal{N}(\rho) = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{(1-p)}\rho_{01} \\ \sqrt{(1-p)}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

$\Gamma \rightarrow$ rate of spontaneous decay $\quad \Delta t \rightarrow$ time interval; $p = \Gamma \Delta t$

After n operations of the channel, $\rho_{11} \rightarrow (1-p)^n \rho_{11}$

$(1-p)^n = (1 - \Gamma \Delta t)^n = (1 - \Gamma \frac{t}{n})^n \rightarrow e^{-\Gamma t}$, where t $= n\Delta t$

$$\mathcal{N}(\rho(t)) = \begin{pmatrix} \rho_{00} + (1 - e^{-\Gamma t})\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix}$$

CPTP map representation

Axiomatic approach

Q: Is it possible that Isometry and Partial Trace do not exhaust the set of possible quantum operations?

A: *Any map* $\mathcal{N}$ *which takes a density operator* ρ *to another density operator* $\mathcal{N}(\rho)$ *is a valid quantum channel.*

Axiomatic approach

Q: Is it possible that Isometry and Partial Trace do not exhaust the set of possible quantum operations?

A: *Any map $\mathcal{N}$ which takes a density operator ρ to another density operator $\mathcal{N}(\rho)$ is a valid quantum channel.*

Properties of a density operator ρ:

1. ρ should be hermitian.
2. Trace of ρ should be equal to 1.
3. ρ should be positive semi-definite, i.e., $\rho \geq 0$.

Any map preserving the three properties should be a valid quantum channel.

Transpose of a density operator

For any density operator ρ, its transpose ρ^T is defined as:

$$\text{If } \rho = \sum_{ijkl} \lambda_{ijkl} \left|i,j\right\rangle \left\langle k,l\right|, \text{ then } \rho^T = \sum_{ijkl} \lambda_{ijkl} \left|k,l\right\rangle \left\langle i,j\right|$$

If $\rho \geq 0$, then $\rho^T \geq 0$.

Transpose of a density operator

For any density operator ρ, its transpose ρ^T is defined as:

If $\rho = \sum_{ijkl} \lambda_{ijkl} |i,j\rangle \langle k,l|$, then $\rho^T = \sum_{ijkl} \lambda_{ijkl} |k,l\rangle \langle i,j|$

If $\rho \geq 0$, then $\rho^T \geq 0$.

If $\rho = |\phi^+\rangle \langle \phi^+|$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \rho^T$$

Eigenvalues of ρ (and hence of ρ^T) are [1,0,0,0].

Partial transpose of a density matrix

Partial transpose of a density operator is defined as: If $\rho = \sum_{ijkl} \lambda_{ijkl} \left|i, j\right\rangle \left\langle k, l\right|$, then

$$\rho_B^T = \sum_{ijkl} \lambda_{ijkl} \left|i, l\right\rangle \left\langle k, j\right| \quad \rho_A^T = \sum_{ijkl} \lambda_{ijkl} \left|k, j\right\rangle \left\langle i, l\right|$$

Partial transpose of a density matrix

$$
\begin{aligned}
\rho &= \frac{1}{2}(|00\rangle\langle 00| + (|00\rangle\langle 11| + (|11\rangle\langle 00| + (|11\rangle\langle 11|) \\
\rho_B^T &= \frac{1}{2}(|00\rangle\langle 00| + (|01\rangle\langle 10| + (|10\rangle\langle 01| + (|11\rangle\langle 11|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Eigenvalues of ρ_B^T are $[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$, i.e. $\rho_B^T < 0$.

Partial transpose of a density matrix

$$
\begin{aligned}
\rho &= \frac{1}{2}(|00\rangle\langle 00| + (|00\rangle\langle 11| + (|11\rangle\langle 00| + (|11\rangle\langle 11|) \\
\rho_B^T &= \frac{1}{2}(|00\rangle\langle 00| + (|01\rangle\langle 10| + (|10\rangle\langle 01| + (|11\rangle\langle 11|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Eigenvalues of ρ_B^T are $[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$, i.e. $\rho_B^T < 0$.

Peres-Horodecki criterion: The joint density matrix of two quantum systems is separable if its partial transpose is positive.

Fine tuning property 3

Obs: There exists maps which keeps the composite system positive semi-definite, but may not keep the subsystems positive semi-definite when acting on it alone.

Requirement: A quantum map should keep both the composite system as well as its sub-systems positive semi-definite $\Rightarrow$ *Completely positive*.

Definition: *Any Completely Positive Trace Preserving (CPTP) map $\mathcal{N}$ is a valid quantum channel*.

CPTP map $\Rightarrow$ Kraus Operator

Let $\mathcal{N}$ be a CPTP map acting on a Hilbert Space S. Take another Hilbert Space R which is identical to S. Define the Bell state on Hilbert Space R+S

$$|\Omega\rangle = \sum_i |i\rangle_R \otimes |i\rangle_S$$

Construct the **Choi Operator**

$$\Lambda_{\mathcal{N}} \quad = \quad (I_R \otimes \mathcal{N}_S) |\Omega\rangle \langle\Omega|$$

CPTP map $\Rightarrow$ Kraus Operator

Let $\mathcal{N}$ be a CPTP map acting on a Hilbert Space S. Take another Hilbert Space R which is identical to S. Define the Bell state on Hilbert Space R+S

$$|\Omega\rangle = \sum_i |i\rangle_R \otimes |i\rangle_S$$

Construct the **Choi Operator**

$$\begin{aligned}
\Lambda_{\mathcal{N}} &= (I_R \otimes \mathcal{N}_S) |\Omega\rangle \langle\Omega| \\
&= (I_R \otimes \mathcal{N}_S)(\sum_{i,j} |i,i\rangle \langle j,j|) \\
&= (I_R \otimes \mathcal{N}_S)(\sum_{i,j} |i\rangle_R \langle j| \otimes |i\rangle_S \langle j|) \\
&= \sum_{i,j} |i\rangle_R \langle j| \otimes \mathcal{N}(|i\rangle_S \langle j|)
\end{aligned}$$

CPTP map $\Rightarrow$ Kraus Operator (Contd.)

Claim: Given a Choi Operator $\Lambda_{\mathcal{N}}$, the action of the CPTP map $\mathcal{N}$ on a density operator ρ is completely defined by $Tr_R\{(\rho^T \otimes I_S)\Lambda_{\mathcal{N}}\}$

Proof:

$$\begin{aligned}
Tr_R\{(\rho^T \otimes I_S)\Lambda_{\mathcal{N}}\} &= \sum_{i,j} Tr_R\{(\rho^T \otimes I_S)[\left|i\right\rangle_R \left\langle j\right| \otimes \mathcal{N}(\left|i\right\rangle_S \left\langle j\right|)]\} \\
&= \sum_{i,j} Tr_R\{\rho^T \left|i\right\rangle_R \left\langle j\right|\} \otimes \mathcal{N}(\left|i\right\rangle_S \left\langle j\right|) \\
&= \sum_{i,j} Tr_R\{\left\langle j\right| \rho^T \left|i\right\rangle\} \otimes \mathcal{N}(\left|i\right\rangle_S \left\langle j\right|) \\
&= \sum_{i,j} (\left\langle j\right| \rho^T \left|i\right\rangle) \otimes \mathcal{N}(\left|i\right\rangle_S \left\langle j\right|) \\
&= \mathcal{N}(\sum_{i,j} \rho_{i,j} \left|i\right\rangle \left\langle j\right|) = \mathcal{N}(\rho)
\end{aligned}$$

CPTP map $\Rightarrow$ Kraus Operator (Contd.)

Diagonalize $\Lambda_{\mathcal{N}} = \sum_k \lambda_k |k\rangle\langle k| = \sum_k |m_k\rangle\langle m_k|$; $|m_k\rangle = \sqrt{\lambda_k}\,|k\rangle$

Now we see,

$$\begin{aligned}
\mathcal{N}(\rho) &= Tr_R\{(\rho^T \otimes I_S)\sum_k |m_k\rangle\langle m_k|\} \\
&= \sum_k \sum_i \langle i|\,\rho^T\,|m_k\rangle\langle m_k|i\rangle \\
&= \sum_k \sum_i \langle i|\,\rho^T(\sum_j |j\rangle\langle j|)\,|m_k\rangle\langle m_k|i\rangle \\
&= \sum_{k,i,j} \langle i|\,\rho^T\,|j\rangle\langle j|m_k\rangle\langle m_k|i\rangle \\
&= \sum_{k,i,j} \langle j|\,\rho\,|i\rangle\langle j|m_k\rangle\langle m_k|i\rangle \\
&= \sum_{k,i,j} \rho_{i,j}\,\langle j_R|m_k\rangle\langle m_k|i_R\rangle
\end{aligned}$$

CPTP map $\Rightarrow$ Kraus Operator (Contd.)

Now, we can write $|m_k\rangle = \sum_{p,l} (M_K)_{p,l} |p\rangle_R \otimes |l\rangle_S$. Then,

$$\begin{aligned}
\langle j|m_k\rangle &= \langle j|_R \big(\sum_{p,l} (M_K)_{p,l} |p\rangle_R \otimes |l\rangle_S\big) = \sum_l (M_k)_{j,l} |l\rangle_S \\
\langle m_k|i\rangle &= \big(\sum_{p,l} (M_k^\dagger)_{p,l} \langle p|_R \otimes \langle l|_S\big) = \sum_l (M_k^\dagger)_{i,l} \langle l|_S
\end{aligned}$$

CPTP map $\Rightarrow$ Kraus Operator (Contd.)

Now, we can write $|m_k\rangle = \sum_{p,l} (M_K)_{p,l} |p\rangle_R \otimes |l\rangle_S$. Then,

$$
\begin{aligned}
\langle j|m_k\rangle &= \langle j|_R \big(\sum_{p,l} (M_K)_{p,l} |p\rangle_R \otimes |l\rangle_S\big) = \sum_l (M_k)_{j,l} |l\rangle_S \\
\langle m_k|i\rangle &= \big(\sum_{p,l} (M_k^\dagger)_{p,l} \langle p|_R \otimes \langle l|_S\big) = \sum_l (M_k^\dagger)_{i,l} \langle l|_S
\end{aligned}
$$

Putting it in the previous equation:

$$
\begin{aligned}
\mathcal{N}(\rho) &= \sum_{k,i,j} \rho_{i,j} \langle j_R|m_k\rangle \langle m_k|i_R\rangle \\
&= \sum_{k,i,j,l} \rho_{i,j} (M_k)_{j,l} |l\rangle_S \langle l|_S (M_k^\dagger)_{i,l} \\
&= \sum_{k,i,j,l} (M_k)_{j,l} |l\rangle \langle j| \rho |i\rangle \langle l| (M_k^\dagger)_{i,l} \\
&= \sum_k M_k \rho M_K^\dagger
\end{aligned}
$$

Measurement as a quantum operator

Quantum measurement

A quantum measurement is a quantum channel which takes a quantum state as an input, and produces a probability distribution.

$$\mathcal{N}(\rho) = \sum_x p(x) |x\rangle \langle x|$$

Consider a quantum state $|\psi\rangle = \sqrt{p_0}\,|0\rangle + \sqrt{p_1}\,|1\rangle$. Then, after measurement in $\{|0\rangle, |1\rangle\}$ basis, we get output $|0\rangle$ w.p. p_0, or $|1\rangle$ w.p. p_1.

$$\sum_{x=0}^{1} p(x) |x\rangle \langle x| = p_0 |0\rangle \langle 0| + p_1 |1\rangle \langle 1| = \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}$$

Expectation of an observable

Let $\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$. For some observable $\mathcal{A}$, I want to find the expectation value of $\mathcal{A}$ w.r.t.ρ.

$$\begin{aligned}
Tr\{\mathcal{A}\rho\} &= Tr\{\mathcal{A}\sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|\} \\
&= \sum_n \sum_i p_i \left\langle n\right| \mathcal{A} \left|\psi_i\right\rangle \left\langle\psi_i|n\right\rangle \\
&= \sum_n \sum_i p_i \left\langle\psi_i|n\right\rangle \left\langle n\right| \mathcal{A} \left|\psi_i\right\rangle \\
&= \sum_i p_i \left\langle\psi_i\right| (\sum_n \left|n\right\rangle \left\langle n\right|) \mathcal{A} \left|\psi_i\right\rangle \\
&= \sum_i p_i \left\langle\psi_i\right| \mathcal{A} \left|\psi_i\right\rangle = \langle\mathcal{A}\rangle
\end{aligned}$$

Therefore, if $\mathcal{A}$ is a projector, such as $\mathcal{M}_0 = \left|0\right\rangle \left\langle 0\right|$, then $Tr\{\mathcal{M}_0\rho\} = \langle\mathcal{M}_0\rangle$ gives the probability of getting outcome $\left|0\right\rangle \left\langle 0\right|$.

Properties of measurement operators

1. **Trace Preserving**:

   $1 = \sum_x p_x = \sum_x Tr\{\mathcal{M}_x\rho\} = Tr\{\rho\sum_x \mathcal{M}_x\}$ for all ρ

$$
\begin{aligned}
& Tr\{\rho\sum_x \mathcal{M}_x\} = 1 \\
\Rightarrow\quad & \sum_n \langle n|\,\rho\sum_x \mathcal{M}_x\,|n\rangle = 1 \\
\Rightarrow\quad & \sum_n \langle n|\sum_m p_m\,|m\rangle\langle m|\,(\sum_x \mathcal{M}_x)\,|n\rangle = 1 \\
\Rightarrow\quad & \sum_n\sum_m p_m\,\langle n|m\rangle\langle m|\,(\sum_x \mathcal{M}_x)\,|n\rangle = 1 \\
\Rightarrow\quad & \sum_n p_n\,\langle n|\,(\sum_x \mathcal{M}_x)\,|n\rangle = 1
\end{aligned}
$$

   Now, $\sum_n p_n\,\langle n|n\rangle = 1$. Therefore, $\sum_x \mathcal{M}_x = I$.

Properties of measurement operators

2. **Completely Positive**: $diag(p_x)$ is PSD $\Rightarrow$ each of $p_x \geq 0$.
   $p_x \geq 0 \Rightarrow Tr\{\rho\mathcal{M}_x\} \geq 0$, for all x.

$$
\begin{aligned}
Tr\{\rho\mathcal{M}_x\} \geq 0 \quad &\Rightarrow \quad \sum_n \langle n| \rho M_x |n\rangle \geq 0 \\
&\Rightarrow \quad \sum_n \langle n| \rho \sum_x \lambda_x |x\rangle \langle x|n\rangle \geq 0 \\
&\Rightarrow \quad \sum_x \langle x| \rho\lambda_x |x\rangle \geq 0 \\
&\Rightarrow \quad \sum_x \lambda_x \langle x| q_x |x\rangle \langle x|x\rangle \geq 0 \\
&\Rightarrow \quad \sum_x \lambda_x q_x \langle x|x\rangle \langle x|x\rangle \geq 0 \Rightarrow q_x\lambda_x \geq 0
\end{aligned}
$$

   But $q_x \geq 0$, since they are probabilities $\Rightarrow \lambda_x \geq 0$, $\forall x$.
   Therefore, $\mathcal{M}_x \geq 0$.

Generalized Measurement

- For projectors, we have the condition $\forall i, j \; P_i P_j = \delta_{ij} P_i$.
- A more generalized measurement can be operators $\mathcal{M}_0, \mathcal{M}_1, \ldots \mathcal{M}_k$, such that $\sum_x \mathcal{M}_x = I$ and $\mathcal{M}_x \geq 0$, $\forall x$.
- Let $|\psi\rangle = |0\rangle$, $|\phi\rangle = |+\rangle$.
- Define $\mathcal{M}_0 = |1\rangle\langle 1|$, $\mathcal{M}_1 = |-\rangle\langle -|$ and $\mathcal{M}_2 = I - \sum_x \mathcal{M}_x$.
- After measurement,
  1. if $\mathcal{M}_0$ triggers, then the state cannot be $|0\rangle$, must be $|+\rangle$.
  2. if $\mathcal{M}_1$ triggers, then the state cannot be $|+\rangle$, must be $|0\rangle$.
  3. if $\mathcal{M}_2$ triggers, then we have no information which can determine the state.

  Such measurement is called **Positive Operator Valued Measure** (POVM).

Generalized Measurement (Contd.)

We have shown measurement is a CPTP map. So

$$\mathcal{N}(\rho) = \sum_{e} \frac{V_e \rho V_e^\dagger}{Tr\{V_e \rho V_e^\dagger\}} \otimes \left(Tr\{V_e \rho V_e^\dagger\}\right) |e\rangle \langle e|$$

Here, $Tr\{V_e \rho V_e^\dagger\}$ gives the probability of getting outcome $|e\rangle \langle e|$. Note that,

$$Tr\{V_e \rho V_e^\dagger\} = Tr\{V_e^\dagger V_e \rho\} = Tr\{\mathcal{M}_e \rho\}$$

where, $\mathcal{M}_e = V_e^\dagger V_e \Rightarrow V_e = \sqrt{\mathcal{M}_e}$.

Given V_e, $\mathcal{M}_e$ is uniquely defined. But it is not true the other way. Therefore, after such a measurement, the post measurement state is not uniquely defined.