Quantum Phase Estimation (QPE)

$$N = 2^n$$

Recap: $|x\rangle = |x_1 x_2 x_3 \dots x_n\rangle$

$$|\tilde{x}\rangle = QFT\,|x\rangle = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^1}}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^2}}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^n}}|1\rangle\right)$$

$U|\psi\rangle = \boxed{\lambda_\psi |\psi\rangle}$ — $|\psi\rangle$ is an eigenstate of U with eigenvalue λ_ψ

Given $|\psi\rangle$, U $\rightarrow$ estimate λ_ψ

$$\left.\begin{matrix} \lambda_\psi |\psi\rangle \\ |\psi\rangle \end{matrix}\right\} \text{ measure it - same statistics}$$

global phase

$H = H^\dagger$

$H^2 = I$

CLAIM: If U is unitary, then $\lambda_\psi = e^{i\theta_\psi}$

$$U = \sum_i \lambda_i |i\rangle\langle i| \qquad U^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|$$

$$\langle i|j\rangle = 1 \text{ if } i = j$$
$$= 0 \quad o/w$$

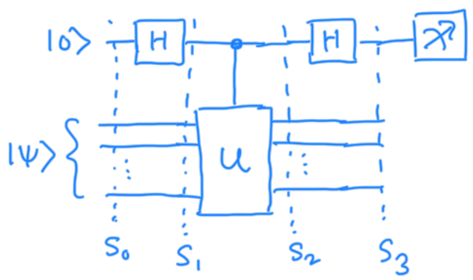
$$UU^\dagger = I \Rightarrow \sum_{i,j} \lambda_i \lambda_j^* |i\rangle\langle i|j\rangle\langle j|$$

$$= \sum_i \lambda_i \lambda_i^* |i\rangle\langle i| = \sum_i |\lambda_i|^2 |i\rangle\langle i| = I$$

we know: $\sum_i |i\rangle\langle i| = I \Rightarrow |\lambda_i|^2 = 1 \quad \forall i$

$$|e^{i\theta_\psi}|^2 = 1 \qquad \lambda_i = e^{i\theta_\psi} \leftarrow \text{estimate this}$$

TRICK:



$S_0:\ |0\rangle|\psi\rangle \qquad S_1:\ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)$

$S_2:\ \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle . e^{i\theta_\psi}|\psi\rangle)$

$$S_4 : \frac{1}{\sqrt{2}} \cdot \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} |\psi\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}} e^{i\theta_\psi} |\psi\rangle \right]$$

$$= \frac{1}{2} \left[|0\rangle \left(|\psi\rangle + e^{i\theta_\psi} |\psi\rangle \right) + |1\rangle \left(|\psi\rangle - e^{i\theta_\psi} |\psi\rangle \right) \right]$$

$$= \frac{1}{2} \left[|0\rangle (1 + e^{i\theta_\psi}) + |1\rangle (1 - e^{i\theta_\psi}) \right] |\psi\rangle$$

Measure : $|0\rangle$. w.p. $\left| \frac{1}{2} (1 + e^{i\theta_\psi}) \right|^2 = \left| \frac{1}{2} + \frac{e^{i\theta_\psi}}{2} \right|^2$

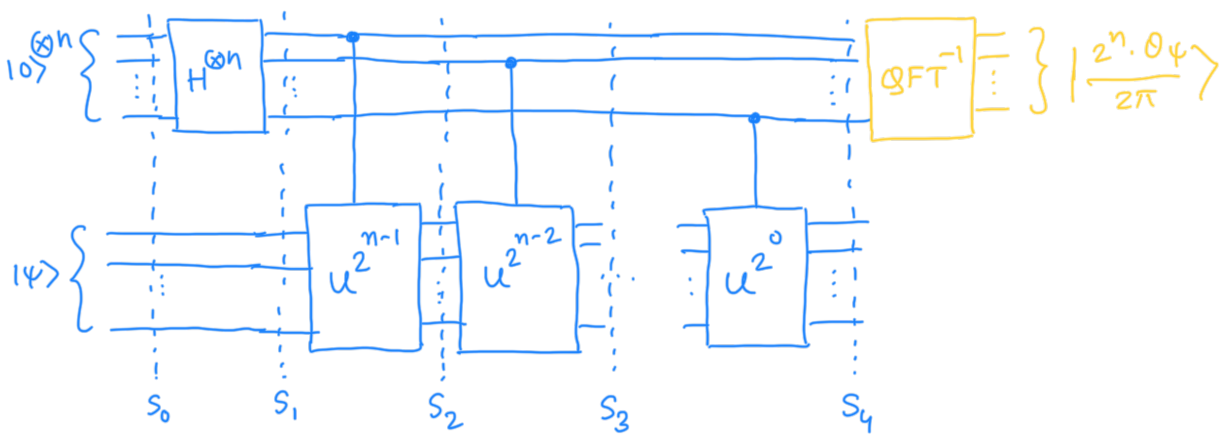
$|1\rangle$ w.p. $\left| \frac{1}{2} (1 - e^{i\theta_\psi}) \right|^2 = \left| \frac{1}{2} - \frac{e^{i\theta_\psi}}{2} \right|^2$

$\theta_\psi = 1^\circ \quad P(0) = 0.9999 \quad P(1) = 7.6 \times 10^{-5}$

$\theta_\psi = 10^\circ \quad P(0) = 0.9924 \quad P(1) = 0.0076$

$$U^2 |\psi\rangle = U(U|\psi\rangle) = U(e^{i\theta_\psi} |\psi\rangle) = e^{i\theta_\psi} U|\psi\rangle = e^{i\theta_\psi} \cdot e^{i\theta_\psi} |\psi\rangle$$
$$= e^{i \cdot 2\theta_\psi} |\psi\rangle$$

$$U^n |\psi\rangle = e^{i \cdot n\theta_\psi} |\psi\rangle \qquad \boxed{U^{2^x} |\psi\rangle = e^{i 2^x \theta_\psi} |\psi\rangle}$$



$S_0 : |0\rangle^{\otimes n} |\psi\rangle$

$S_1 : \left(\frac{1}{\sqrt{2}} \right)^n (|0\rangle + |1\rangle)^{\otimes n} |\psi\rangle$

$$S_2 : \left(\frac{1}{\sqrt{2}} \right)^n (|0\rangle + |1\rangle) |\psi\rangle \otimes (|0\rangle + |1\rangle)^{\otimes n-1} = \left(\frac{1}{\sqrt{2}} \right)^n \left(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle \right) \otimes (|0\rangle + |1\rangle)^{\otimes n-1}$$

$$= \left(1/\sqrt{2} \right)^n \left(|0\rangle|\psi\rangle + |1\rangle . e^{i \cdot 2^{n-1} \cdot \theta_\psi} |\psi\rangle \right) \otimes (|0\rangle + |1\rangle)^{\otimes n-1}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle + e^{i \cdot 2^{n-1} \cdot \theta_\psi} |1\rangle\right) |\psi\rangle \otimes \left(|0\rangle + |1\rangle\right)^{\otimes n-1}$$

$$S_3: \left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle + e^{i \cdot 2^{n-1} \cdot \theta_\psi} |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) |\psi\rangle \otimes \left(|0\rangle + |1\rangle\right)^{\otimes n-2}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle + e^{i \cdot 2^{n-1} \cdot \theta_\psi} |1\rangle\right) \otimes \left(|0\rangle + e^{i \cdot 2^{n-2} \cdot \theta_\psi} |1\rangle\right) |\psi\rangle \otimes \left(|0\rangle + |1\rangle\right)^{\otimes n-2}$$

$$S_4: \left(\frac{1}{\sqrt{2}}\right)^n \boxed{\left(|0\rangle + e^{i \cdot 2^{n-1} \cdot \theta_\psi} |1\rangle\right) \otimes \left(|0\rangle + e^{i \cdot 2^{n-2} \cdot \theta_\psi} |1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{i \cdot 2^{0} \cdot \theta_\psi} |1\rangle\right)} |\psi\rangle$$

QFT $|x\rangle$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \cdot \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^1}} |1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^2}} |1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{\frac{2\pi i \cdot x}{2^n}} |1\rangle\right)$$

$$e^{i \cdot 2^{n-1} \cdot \theta_\psi} = e^{i \cdot \frac{2\pi \cdot x}{2^1}} \Rightarrow x = \frac{2^n \cdot \theta_\psi}{2\pi}$$

$$e^{i \cdot 2^{n-k} \cdot \theta_\psi} = e^{i \cdot \frac{2\pi \cdot x}{2^k}} \Rightarrow x = \frac{2^n \cdot \theta_\psi}{2\pi}$$

$\rightarrow$ QFT $\left|\frac{2^n \cdot \theta_\psi}{2\pi}\right\rangle$ find θ_ψ from this

find $\lambda_\psi = e^{i\theta_\psi}$

gates : n Hadamard gates

n controlled U gates

QFT requires $O(n^2)$ gates

$O(n^2) + O(n) \equiv O(n^2)$ gates

Classically: $U|v\rangle = e^{i\theta}|v\rangle$ $\quad v = N \times 1 \quad U = N \times N$

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ u_{21} & u_{22} & \cdots & u_{2N} \\ \vdots & & & \\ u_{N1} & u_{N2} & \cdots & u_{NN} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = e^{i\theta} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Take 1st row: $u_{11} v_1 + u_{12} v_2 + \cdots + u_{1N} \cdot v_N = e^{i\theta} \cdot v_1$

$$e^{i\theta} = \frac{u_{11} \cdot v_1 + u_{12} \cdot v_2 + \cdots + u_{1N} \cdot v_N}{v_1} \Rightarrow$$

N multiplication
N-1 addition
1 division

2N elementary operations $= O(2^n)$ operations