Rita Herfi

COMP SCI 536 FINAL - Problem B - 1D Wave Equation

Introduction

This project discusses a given problem of the one dimensional wave equation. It uses specific methods that I learned in class for Computational Science 536. Some things mentioned are the 1D wave equation itself, its boundary conditions, its initial conditions provided, nondimensionalization, the finite difference scheme, including centered differences for the second derivative, and the given conditions of the problem in order to solve the equation. Some of the problems are solved using a code given to us by the instructor and slightly modified by me to meet the project's expectations.

Problem Description

Assume we have a string, whose mass/length is a constant ρ, with a function that describes its displacement in space and time u(x, t). The string is perfectly elastic in the horizontal direction, such that its horizontal tension is some constant $\overline{T}$. We only concern ourselves with deflection in the vertical direction. This string is initially displaced from its horizontal equilibrium by a function and with boundary conditions given by Condition A or B as u0(x, 0).

1. Derive the second-order partial differential equation that describes the system in terms of u(x, t).

Result for 1

Given: ρ is a constant. (Mass/Length)

u(x,t) a function that describes its placement in space and time,

$\overline{T}$ is a constant. (Horizontal Tension)

$$\Sigma F_y = -T(x,t)sin(\theta(x,t)) + T(x+\Delta x,t)sin(\theta(x+\Delta x,t))$$

$$\Sigma F_x = -T(x,t)cos(\theta(x,t)) + T(x+\Delta x,t)cos(\theta(x+\Delta x,t))$$

From Newton's Second Law: $(F = m \cdot a)$

$$\sum_i \overline{F}_i = m \cdot \frac{\partial^2 \overline{u}}{\partial t^2} \Rightarrow \sum_i [F_x, F_y]_i = m \cdot \frac{\partial^2 \overline{u}}{\partial t^2}$$

$$\Sigma F_x = 0 \Rightarrow T(x,t)cos(\theta(x,t)) = T(x+\Delta x,t)cos(\theta(x+\Delta x,t)) = \overline{T}*$$

*since we know $\overline{T}$ is a constant.

Newton's Second Law gives:

$$\sum_i (F_y) = m \cdot \frac{\partial^2 \overline{u}}{\partial t^2}$$

$$T(x+\Delta x,t)sin(\theta(x+\Delta x,t)) - T(x,t)sin(\theta(x,t)) = m \cdot \frac{\partial^2 \overline{u}}{\partial t^2}$$

↳

$$- T(x,t)cos(\theta(x,t)) + T(x+\Delta x,t)cos(\theta(x+\Delta x,t))tan(\theta(x+\Delta x,t)) = \rho\,\Delta x \cdot \frac{\partial^2 u}{\partial t^2}$$

↳ $$- \overline{T} \cdot \frac{\partial\ u(x,t)}{\partial x} + \overline{T} \cdot \frac{\partial\ u(x+\Delta x,t)}{\partial x} = \rho\Delta x \frac{\partial^2 u}{\partial t^2}$$

↳ $$\frac{\left(\frac{\partial\ u(x+\Delta x,t)}{\partial x} - \frac{\partial\ u(x,t)}{\partial x}\right)}{\Delta x} = \rho \cdot \frac{\partial^2 u}{\partial t^2}$$

↳ $$\lim_{\Delta x \to 0} \left[\frac{\left(\frac{\partial\ u(x+\Delta x,t)}{\partial x} - \frac{\partial\ u(x,t)}{\partial x}\right)}{\Delta x} \right] = \frac{\rho}{\overline{T}} \cdot \frac{\partial^2 u}{\partial t^2}$$

↳ $$\frac{\partial^2 \overline{u}}{\partial x^2} = \frac{\rho}{\overline{T}} \cdot \frac{\partial^2 \overline{u}}{\partial t^2} \to \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{\overline{T}} \cdot \frac{\partial^2 u}{\partial t^2} \to \frac{\partial^2 u}{\partial t^2} = \frac{\overline{T}}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\overline{T}}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}$$

■

2. Scale the independent and state variables and show the equation and boundary conditions after non-dimensionalization. Explain the non-dimensionalization steps.

Result for 2

We know the units for the variables ρ and $\overline{T}$ are as follows:

$[\rho] = \frac{M}{L}$, $[\overline{T}] = \frac{M \cdot L}{T^2}$ we want $\left[\frac{\overline{T}}{\rho}\right]$.

$\frac{\overline{T}}{\rho} = \frac{\frac{M \cdot L}{T^2}}{\frac{M}{L}} = \frac{M \cdot L}{T^2} \cdot \frac{L}{M} = \frac{L^2}{T^2} \Rightarrow \therefore \left[\frac{\overline{T}}{\rho}\right] = \frac{L^2}{T^2}$.

Thus we have our units. Now we want to nondimensionalize. Let us introduce a variable C where the units of C are $\frac{L}{T}$. Since the units of $\left[\frac{\overline{T}}{\rho}\right]$ are $\frac{L^2}{T^2}$ to make C = $\left[\frac{\overline{T}}{\rho}\right]$, we have to match the units. Simply squaring C on both sides gives us $C^2 = \frac{L^2}{T^2}$ which the makes it equal to $\left[\frac{\overline{T}}{\rho}\right]$, but $C^2 = \left[\frac{\overline{T}}{\rho}\right]$ not C. So to solve backwards, $C = \sqrt{\left[\frac{\overline{T}}{\rho}\right]}$. Thus it is correct that $C^2 = \left[\frac{\overline{T}}{\rho}\right]$ which makes it nondimensional. So we can now replace $\left[\frac{\overline{T}}{\rho}\right]$ with C^2 , which gives $\frac{\partial^2 u}{\partial t^2} = \frac{\overline{T}}{\rho} \cdot \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^2 u}{\partial t^2} = C^2 \cdot \frac{\partial^2 u}{\partial x^2}$.

Now, $\frac{\partial^2 u}{\partial t^2} = C^2 \cdot \frac{\partial^2 u}{\partial x^2}$ for $0 < x < l$ and $0 < t < t_f$. where x and l are the length of the wave and t, t_f are the time of the wave (t_f is the final time).

Our two boundary conditions are: $u(0, t) = 0$ and $u(l, t) = 0$ for $0 < t < t_f$ and our two initial conditions are: $u(x, 0) = f(x)$ for $0 \leq x \leq l$

$$u_t(x, 0) = g(x) \text{ for } 0 < x < l$$

We need to nondimensionalize the wave equation now.

Start with letting $\hat{u} = \frac{u}{\overline{U}}$; $\hat{t} = \frac{t}{\left(\frac{l}{c}\right)}$; $\hat{x} = \frac{x}{l}$. This is similar to what we were doing earlier to make the equation unitless. Next we replace the partial derivatives of t and x to substitute into the equation where: $\frac{\partial}{\partial t} = \frac{\partial \hat{t}}{\partial t}\frac{\partial}{\partial \hat{t}} = \frac{c}{l}\frac{\partial}{\partial \hat{t}}$ and

$\frac{\partial}{\partial x} = \frac{\partial \hat{x}}{\partial x}\frac{\partial}{\partial \hat{x}} = \frac{1}{l}\frac{\partial}{\partial \hat{x}}$.

Then $\frac{\partial^2 u}{\partial t^2} = \frac{\partial \hat{t}}{\partial t}\frac{\partial}{\partial \hat{t}}\left[\frac{\partial \hat{t}}{\partial t}\frac{\partial}{\partial \hat{t}}\left(\overline{U} \cdot \hat{u}\right)\right] = \overline{U}\left(\frac{c}{l}\right)^2 \frac{\partial^2 \hat{u}}{\partial \hat{t}^2}$ and

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \hat{x}}{\partial x}\frac{\partial}{\partial \hat{x}}\left[\frac{\partial \hat{x}}{\partial x}\frac{\partial}{\partial \hat{x}}\left(\overline{U} \cdot \hat{u}\right)\right] = \overline{U}\left(\frac{1}{l}\right)^2 \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$ thus

$\frac{\partial^2 u}{\partial t^2} = C^2 \cdot \frac{\partial^2 u}{\partial x^2} \Rightarrow \overline{U}\left(\frac{c}{l}\right)^2 \frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = C^2 \overline{U}\left(\frac{1}{l}\right)^2 \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} \Rightarrow \frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$ and $\frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$

for $0 < \hat{x} < 1$ and $0 < \hat{t} < \frac{t_f}{\left(\frac{l}{c}\right)}$ with boundary conditions $\hat{u}(0, \hat{t}) = 0$ and

$\hat{u}(1, \hat{t}) = 0$ and initial conditions $\hat{u}(x, 0) = \frac{f(\hat{x})}{\overline{U}}$ for $0 \leq \hat{x} \leq 1$,

$\widehat{u_t}(x, 0) = \frac{l}{\overline{U} \cdot c} g(x)$ for $0 < \hat{x} < 1$. The result $\frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$ shows it is non dimensional since all of the unit variables ended up canceling out.

3. Solve the equation numerically. Use centered finite differences for the second derivative. Explain what considerations you take. Show in detail how you solve this problem numerically (as we did in class and for the midterm).

Result for 3

$\frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$ let $\hat{u}(\hat{x}, \hat{t}) \rightarrow U_i^j$ to start using the finite difference scheme to solve the equation. We needed to non dimensionalize in order to use this method. We will be using centered differences in this case for both sides of the equation because we are handling two second order derivatives, it will bring them down a degree in order to be able to be used in the function later. First: $\frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{\Delta t^2}$, next $\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$. Now we substitute:

$\frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} \Rightarrow \frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{\Delta t^2} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$. Let’s do some algebra to simplify.

$U_i^{j+1} - 2U_i^j + U_i^{j-1} = \left(\frac{\Delta t^2}{\Delta x^2}\right)\left(U_{i1}^j - 2U_i^j + U_{i-1}^j\right)$ let $r = \frac{\Delta t}{\Delta x}$

$$U_i^{j+1} - 2U_i^j + U_i^{j-1} = r^2\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right)$$

$$U_i^{j+1} = r^2\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right) + 2U_i^j - U_i^{j-1}$$

$$U_i^{j+1} = r^2 U_{i+1}^j + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^j - U_i^{j-1}.$$

There is a small issue with this equation. We do not know what U_i^{j+1} or U_i^{j-1} is for the first step at least. So we have to manipulate the problem in order to see what we can do. Let’s use another finite difference method for U_i^{-1} where the first derivative $\frac{\partial u}{\partial t} = \frac{u^{j+1} - u^{j-1}}{2\Delta t} = 0$. Then we have $u^{j+1} = u^{j-1}$ for the first step, we will substitute it back.

$$U_i^{j+1} = - U_i^{j-1} + \left[r^2 U_{i+1}^j + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^j\right]$$

$$U_i^{j+1} = - U_i^{j+1} + \left[r^2 U_{i+1}^j + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^j\right]$$

$$2U_i^{j+1} = \left[r^2 U_{i+1}^{j} + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^{j}\right]$$

$U_i^{j+1} = \frac{1}{2}\left[r^2 U_{i+1}^{j} + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^{j}\right]$ so at j = 0:

$U_i^{1} = \frac{1}{2}\left[r^2 U_{i+1}^{0} + 2\left(1 - r^2\right)U_i^0 + r^2 U_{i-1}^{0}\right]$ for step 1. We can proceed to use the original equation we solved for $\left(U_i^{j+1} = r^2 U_{i+1}^{j} + 2\left(1 - r^2\right)U_i^j + r^2 U_{i-1}^{j} - U_i^{j-1}\right)$ for the rest of the values since we know the first step. Now we solve the problem numerically. To do so, we put the equation in a matrix and fill in the values we already know. For condition A, which is $|x|$ is a piecewise function. If the result is less than $\frac{1}{2}$ the value is 1, if the result is equal to $\frac{1}{2}$ then it will simply be $\frac{1}{2}$. Otherwise the result is 0. We also know for this condition, the initial conditions where the first and last value are equal to 0.

The same matrix follows for condition B which is also $|x|$ but with a different answer depending on if x is less than or equal to $\frac{1}{2}$ or 0 otherwise. If it is less than or equal to $\frac{1}{2}$ it falls under the expression $cos(\pi x)^2$.

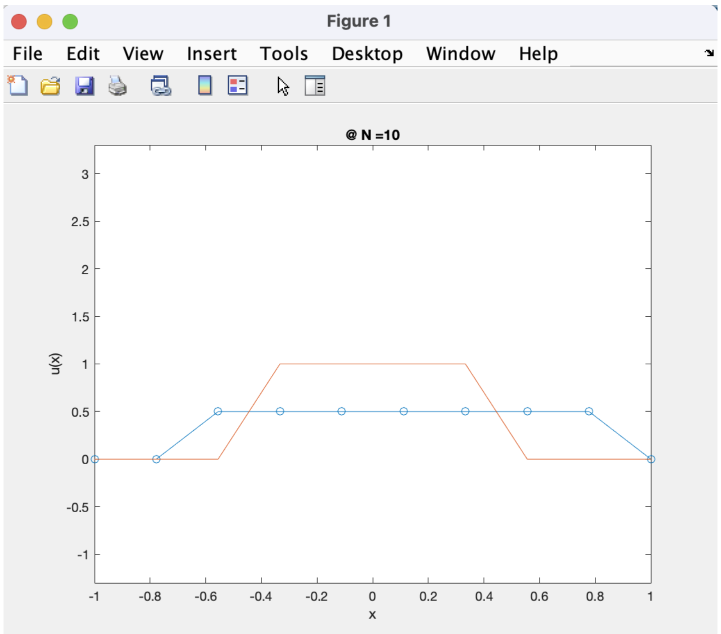
$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ r^2 & 2(1-r^2) & r^2 & \cdots & 0 & 0 \\ 0 & r^2 & 2(1-r^2) & r^2 & \cdots & 0 \\ 0 & 0 & r^2 & 2(1-r^2) & r^2 & 0 \\ \backslash & \backslash & \backslash & \backslash & \backslash & \backslash \\ 0 & 0 & \cdots & r^2 & 2(1-r^2) & r^2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 = 0 \\ u_1 \\ u_2 \\ \vdots \\ u \\ \vdots \\ u_{n-1} \\ u_n = 0 \end{bmatrix} - \begin{bmatrix} u_0 = 0 \\ u_1 \\ u_2 \\ \vdots \\ u \\ \vdots \\ u_{n-1} \\ u_n = 0 \end{bmatrix} = \begin{bmatrix} u_0 = 0 \\ u_1 \\ u_2 \\ \vdots \\ u \\ \vdots \\ u_{n-1} \\ u_n = 0 \end{bmatrix}$$

4. Solve for one complete traverse of the wave for "Condition A". Solve the problem for N sub-intervals, using N = {10, 50, 100, 500, 1000}.

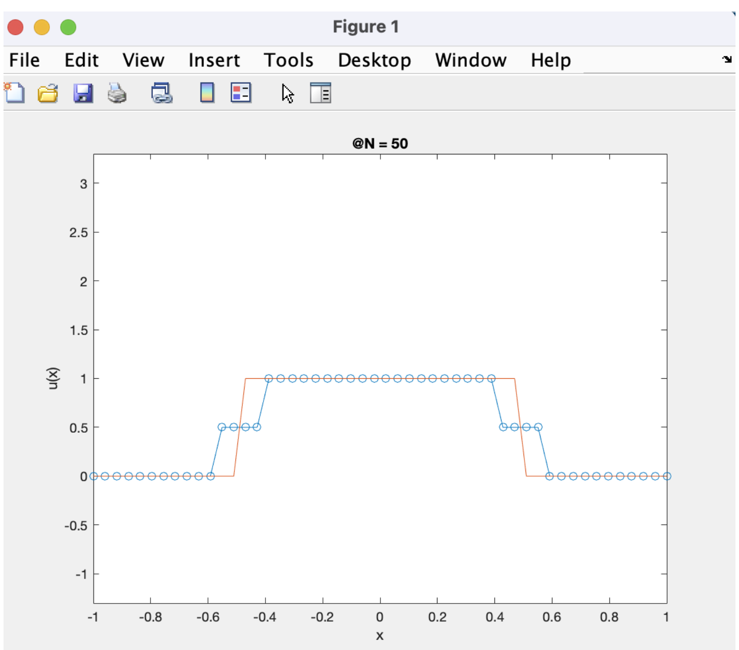
Result for 4

Here's a few visuals of what we have using the code for every amount of sub-intervals. We can see as the sub-intervals get bigger the function looks more accurate and tries to get as close to the function's shape as possible. Starting off with

N = 10:



N = 50:



N = 100:

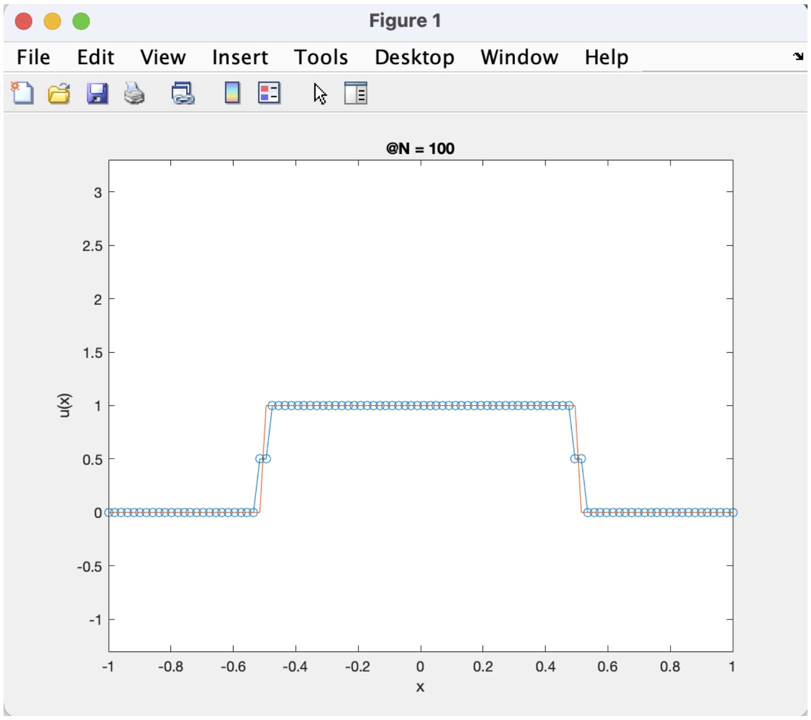


Figure 1
File Edit View Insert Tools Desktop Window Help
@N = 100
u(x)
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

N = 500:

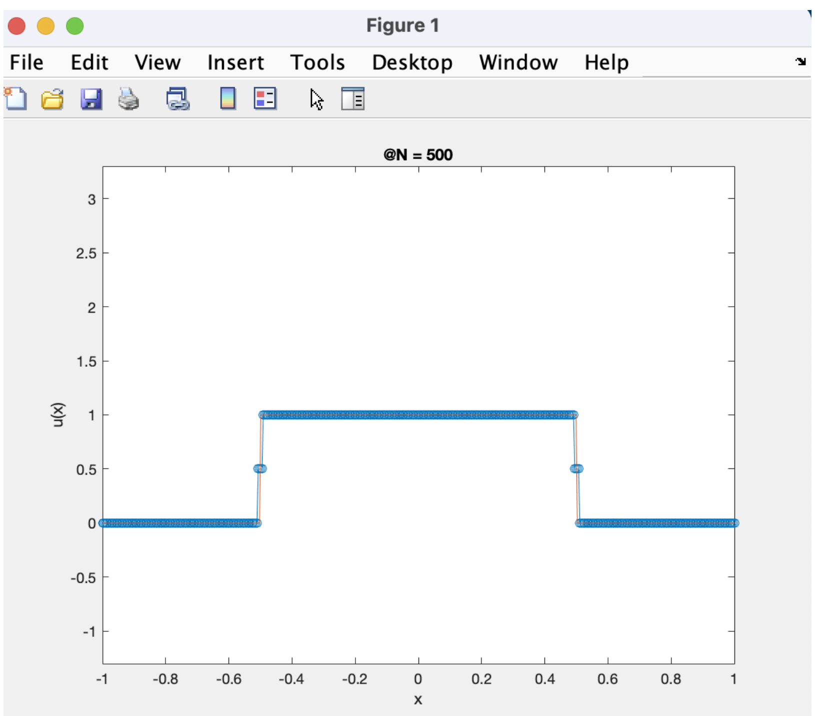


Figure 1
File Edit View Insert Tools Desktop Window Help
@N = 500
u(x)
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

N = 1000

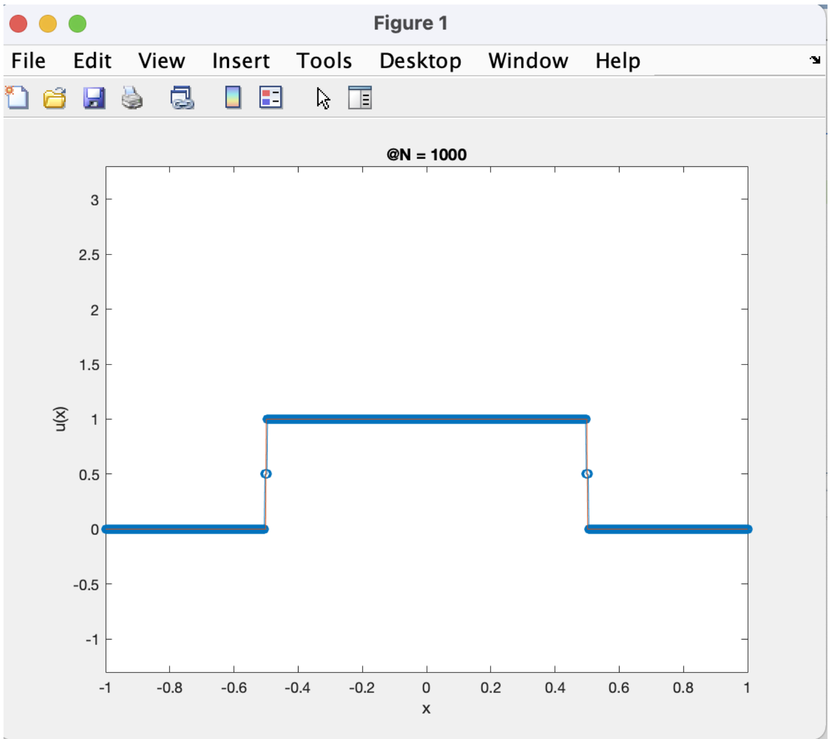


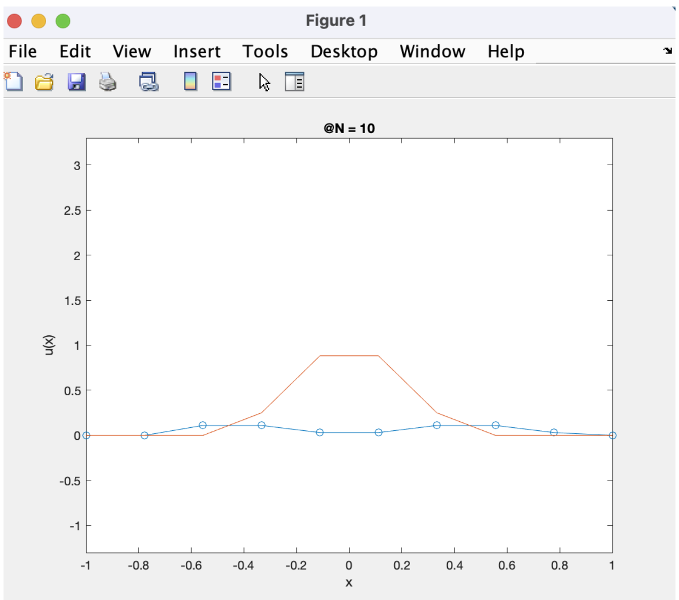
Figure 1
File Edit View Insert Tools Desktop Window Help
@N = 1000
u(x)
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

5. Solve for one complete traverse of the wave for ”Condition B” wave initial condition. Solve the problem for N sub-intervals, using N = {10, 50, 100, 500, 1000}.

Result for 5

Similarly, we can also see with the function for condition B, that the bigger the amount of sub-intervals gets, the more accurate the wave function. However, in the case of condition B there is quite a bit of error because the answer results in terms of $cos(\pi x)^2$, while the function is supposed to be an absolute value. Regardless of how many sub-interval we increase it to, it will never reach the actual function. It is shown starting with

N = 10



N = 50

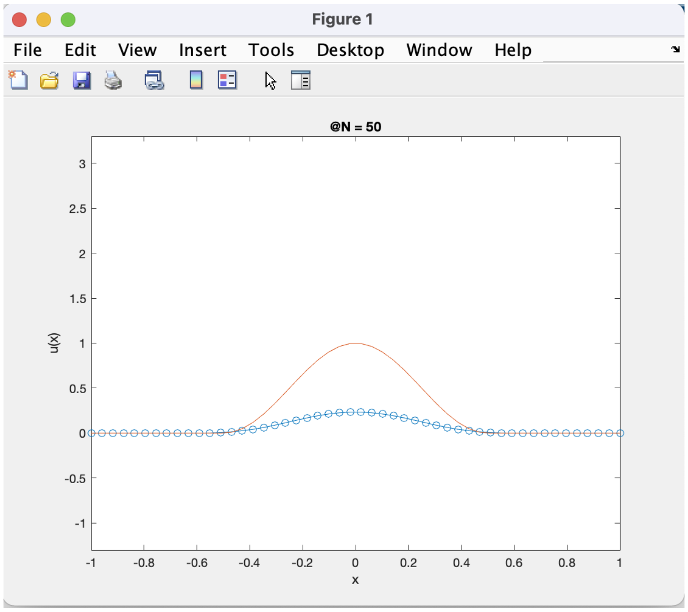


Figure 1
File Edit View Insert Tools Desktop Window Help
@N = 50
u(x)
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

N = 100

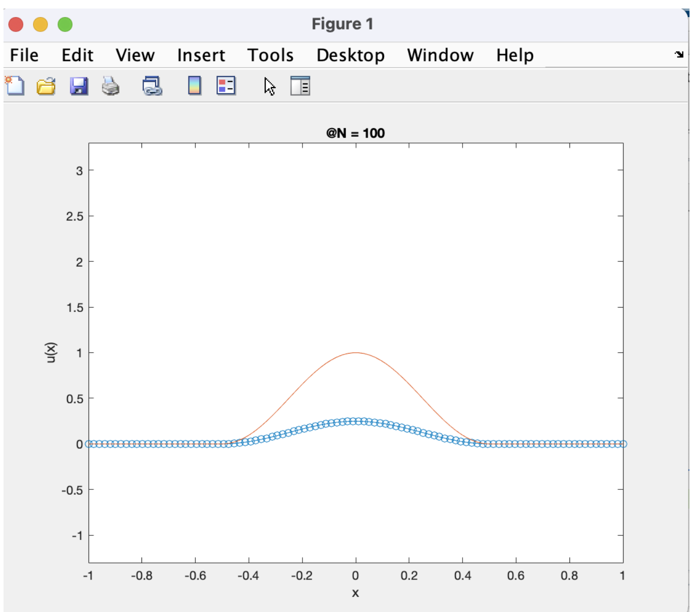
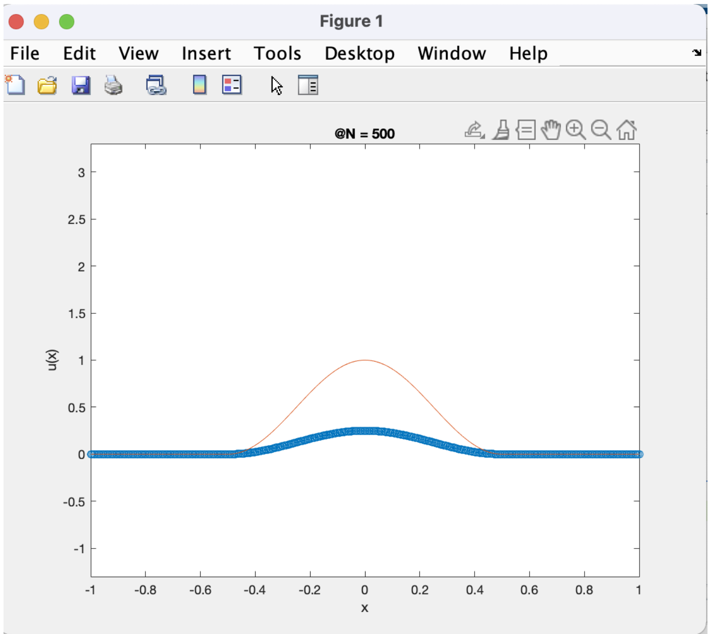
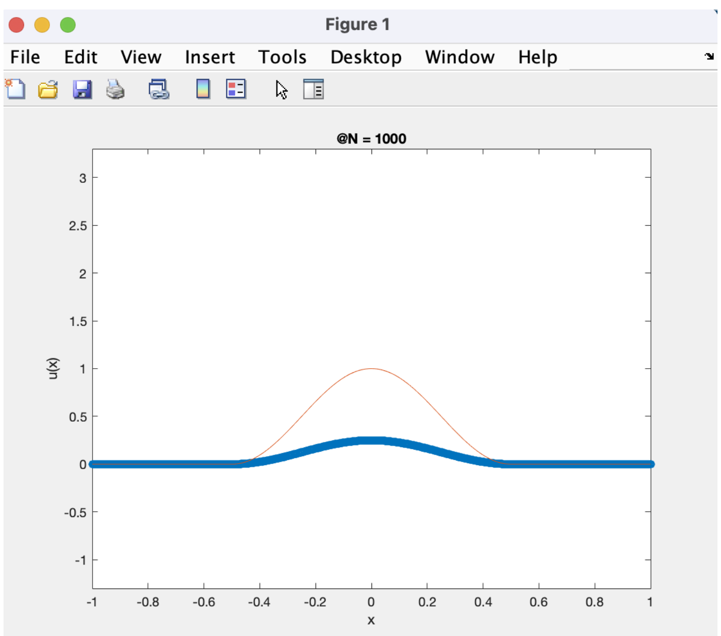


Figure 1
File Edit View Insert Tools Desktop Window Help
@N = 100
u(x)
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

N = 500



N = 1000



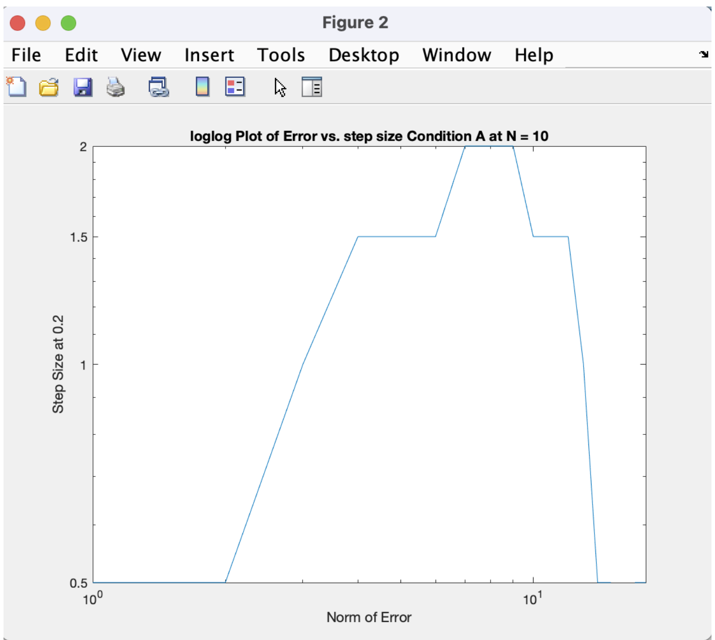
To be able to change the amount of sub-intervals you need to adjust the ‘m’ variable accordingly in the code for both condition A and B.

6. Calculate the error vectors (difference between the exact and the numerical solution) for the sub-intervals. Show the loglog plot of the $\infty -$ Norm of the error vectors versus the step sizes. Discuss the interesting features of these two measurements.

Result for 6

Using the code here are the error plots of Condition A:

N = 10



N = 50

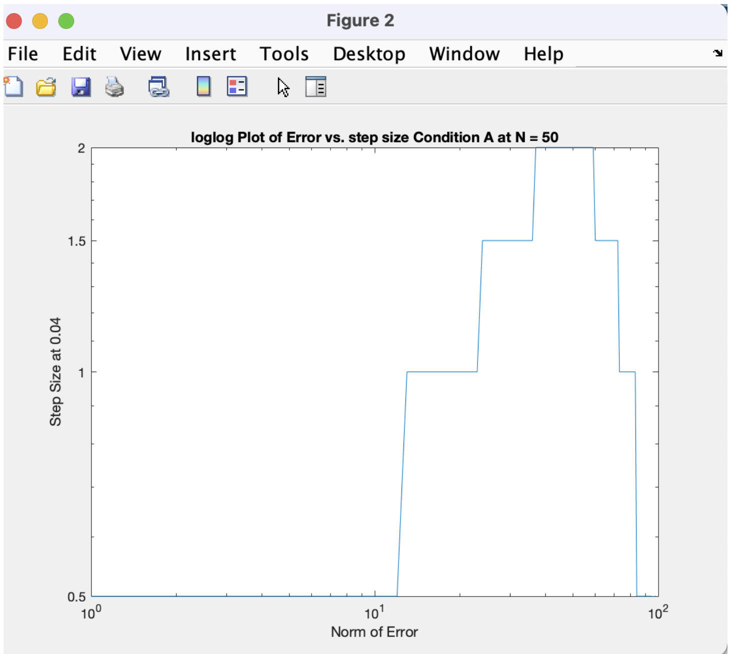


Figure 2
File Edit View Insert Tools Desktop Window Help
loglog Plot of Error vs. step size Condition A at N = 50
Step Size at 0.04
Norm of Error
2
1.5
1
0.5
10^0
10^1
10^2

N = 100

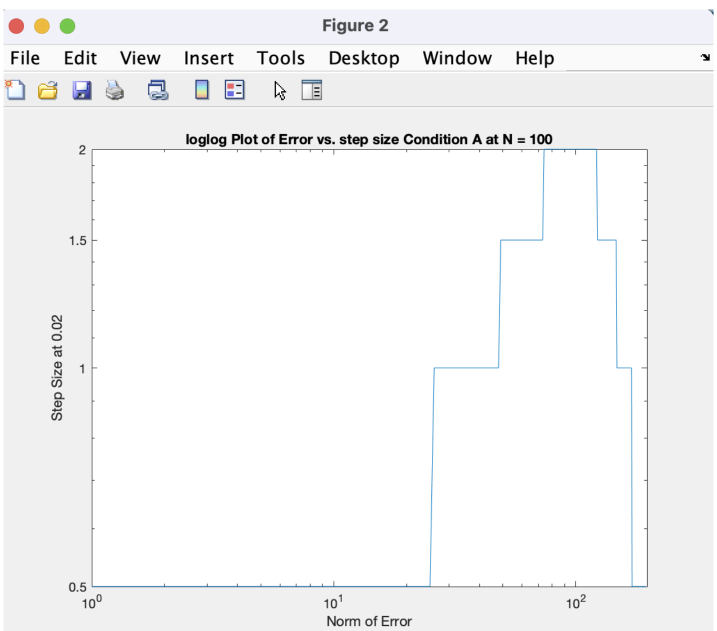
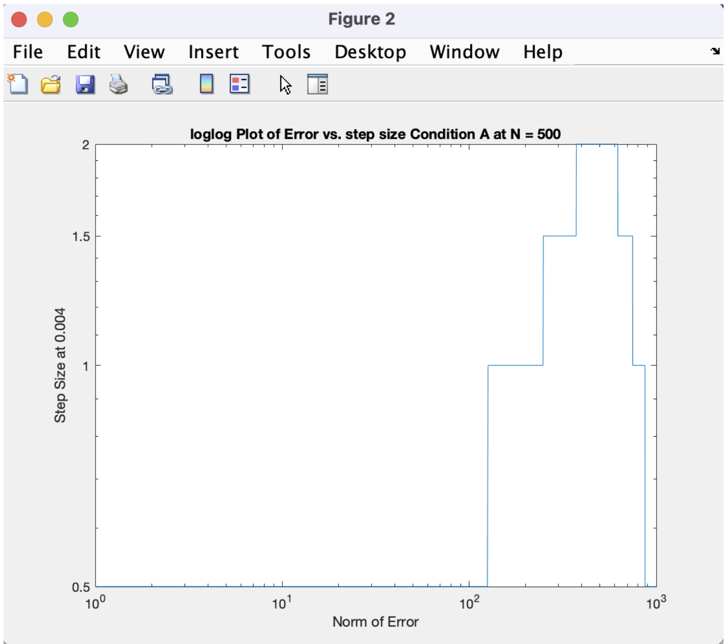
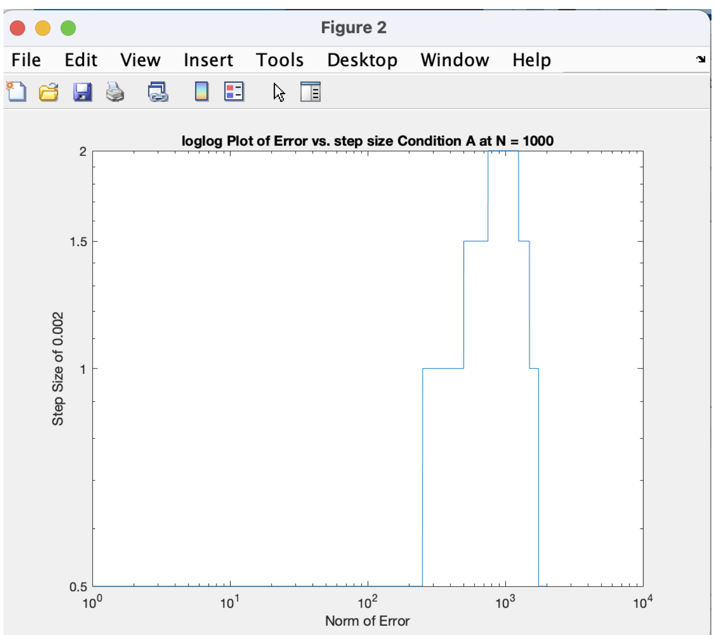


Figure 2
File Edit View Insert Tools Desktop Window Help
loglog Plot of Error vs. step size Condition A at N = 100
Step Size at 0.02
Norm of Error
2
1.5
1
0.5
10^0
10^1
10^2

N = 500



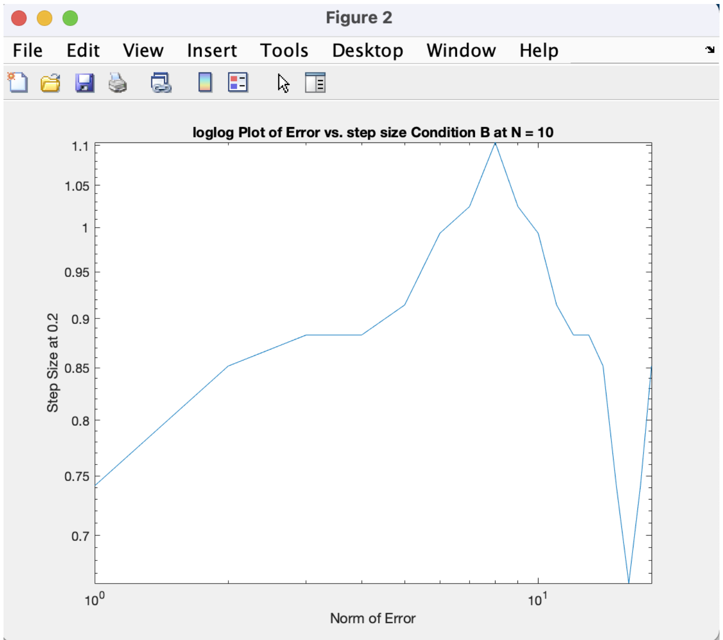
N = 1000



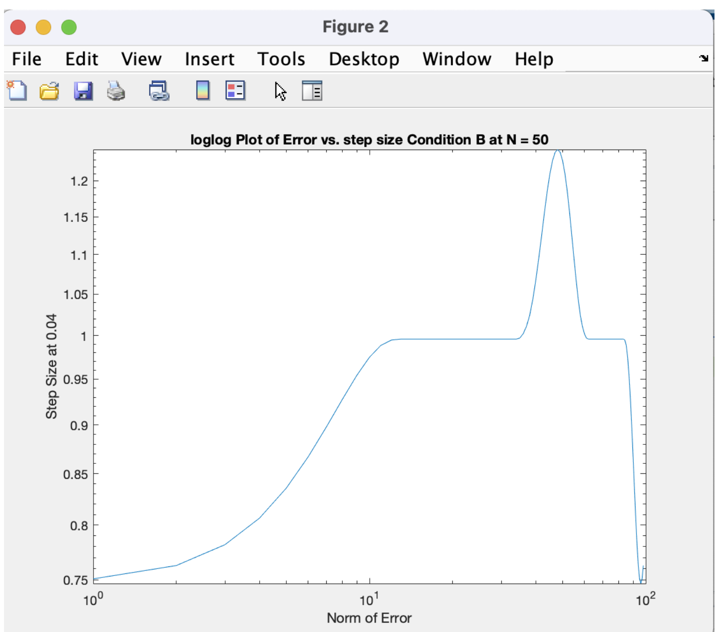
For condition A, the errors subtly also look like they are trying to conform to the original equation of $|x|$. Every time the step size gets smaller so does the width of the shape of the error. Numerically though the number tends to get bigger as N also gets bigger.

Next are the error plots for Condition B:

N = 10



N = 50



N = 100

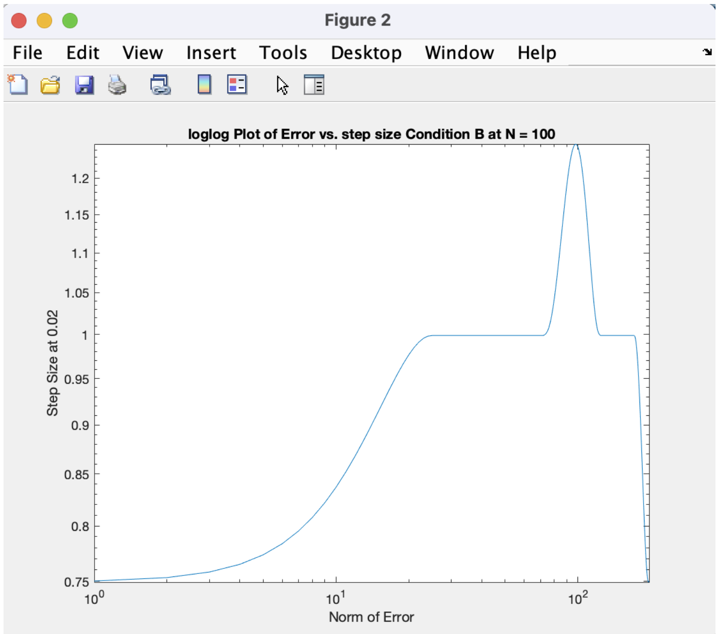


Figure 2
File Edit View Insert Tools Desktop Window Help
loglog Plot of Error vs. step size Condition B at N = 100
Step Size at 0.02
Norm of Error

N = 500

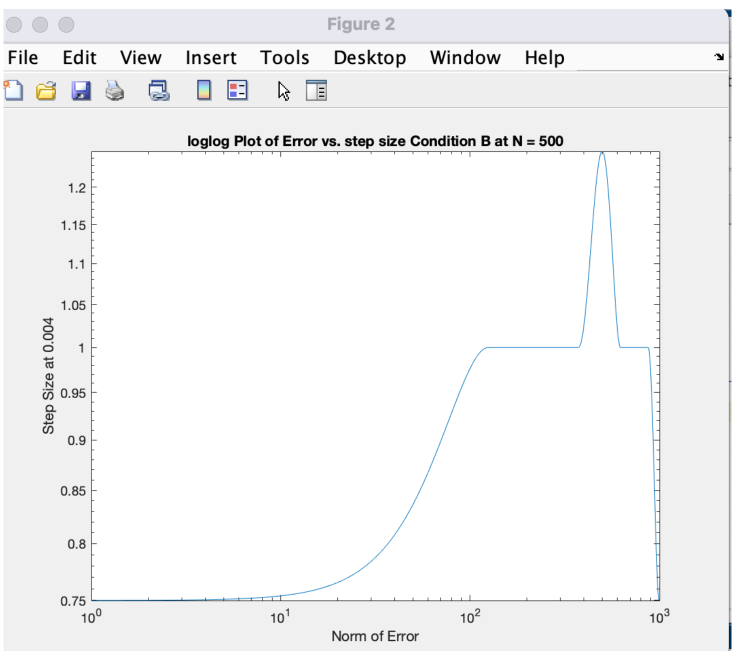
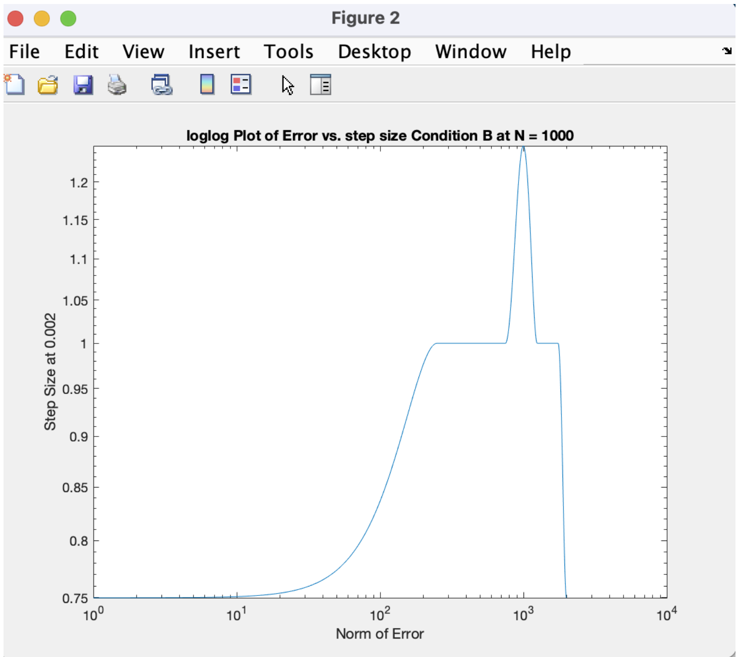


Figure 2
File Edit View Insert Tools Desktop Window Help
loglog Plot of Error vs. step size Condition B at N = 500
Step Size at 0.004
Norm of Error

N = 1000



For condition B, it similarly tries to conform to a cosine shape like its boundary condition, but the shape generally stays the same unless there are drastic changes to the sub interval. The errors also seem to roughly stay the same width with the exception of a few step sizes in the middle depending on which N.

Conclusion

In this report there were many aspects shown regarding computational science. First we solved the second order partial differential equation of the one dimensional wave, then scaled the independent and state variables and showed the equation and boundary conditions after nondimensionalization. Next we solved the equation numerically using finite differences including the centered difference scheme for the second order differential equation. Then we solved traverses of conditions A and B separately with multiple sub-intervals, and we also plotted the error with the specific step sizes according to the N sub-intervals. We came to visually see how much the wave equation is affected by its errors and by how big or small we make the sub-intervals out to be. Overall, as the sub-intervals get smaller, so does the error.