More examples on linear eq^n of 1st order.

(1) Solve $x\cos x \frac{dy}{dx} + (x\sin x + \cos x)y = 1$

Sol^n The given eq^n is $\frac{dy}{dx} + \left(\frac{x\sin x}{x\cos x} + \frac{\cos x}{x\cos x}\right)y = \frac{1}{x\cos x}$

or, $\frac{dy}{dx} + \left(\frac{\sin x}{\cos x} + \frac{1}{x}\right)y = \frac{1}{x\cos x}$

Here I.F. $= e^{\int\left(\frac{\sin x}{\cos x} + \frac{1}{x}\right)dx}$

$= e^{(-\log(\cos x) + \log x)} = e^{\log\frac{x}{\cos x}} = \frac{x}{\cos x}$

∴ Sol^n is $y \cdot \frac{x}{\cos x} = \int \frac{1}{x\cos x}\frac{x}{\cos x}dx + c = \int \sec^2 x\, dx + c$

or, $y = \frac{\cos x}{x}(\tan x + c)$

or, $xy = \sin x + c\cos x$ //

(2) Solve $(x+y+1)dy = dx$.

$\boxed{\int u\,dv = uv - u'v_1 + u''v_2 - \cdots}$

We write it as $\frac{dx}{dy} = x + y + 1$

or, $\frac{dx}{dy} - x = y + 1$

which is linear 1st order d. eq^n.

I.F. $= e^{\int -dy} = e^{-y}$

∴ Sol^n is $x \cdot e^{-y} = \int \underbrace{e^{-y}}_{v}\underbrace{(y+1)}_{u}dy = \int \underbrace{(y+1)}_{u}\underbrace{e^{-y}dy}_{dv}$

$= (y+1)e^{-y}(-1) - 1\cdot e^{-y} = -2e^{-y} - ye^{-y} + c$

∴ General sol^n is $x = -2 - y + ce^{y}$

or, $x + y + 2 = ce^{y}$

H.W. Solve

① $x\frac{dy}{dx}+y=xy^2$ Ans: $xy(c-\log x)=1$

② $2x^2\frac{dy}{dx}=xy+y^2$. Ans: $x-y=cy\sqrt{x}$.

(3) $(x^2y^3+2xy)dy=dx$ Ans: $\frac{1}{x}=-\frac{1}{2}(y^2-1)+e^{-y^2}c$.

④ $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$. Ans: $2x=\cancel{\log f+}$ $\log y(1+2cx^2)$.

ch: Exact differential eqns

The differential of a fn $f(x,y)$ is given by

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \longrightarrow (I)$$

consider $M(x,y)dx+N(x,y)dy=0 \longrightarrow (II)$

Suppose $\frac{\partial f}{\partial x}=M(x,y)$ & $\frac{\partial f}{\partial y}=N(x,y)$.

Then (I) $\Rightarrow$ $df=M(x,y)dx+N(x,y)dy=0$

i.e, $df=0$ Int $\Rightarrow$ $f(x,y)=c$.

$\therefore$ (II) is said to be an exact diff^l eqn.

(i) Necessary condn for exactness

we have $\frac{\partial f}{\partial x}=M(x,y)$ & $\frac{\partial f}{\partial y}=N(x,y)$

$$\therefore \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial x}$$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, which is the nec. condn for the eqn (II) i.e, $Mdx+Ndy=0$ to be exact.

(ii) Method of finding the soln to an exact diff^l eqn:

To solve $Mdx+Ndy=0$, find $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given eqn is exact.

The solⁿ is then $\int M(x,y)\,dx + \int N(x,y)\,dy = c$ — (1)

In the 1st integral treating y as const. and in the sec. integral take only those terms in N which donot contain x.

i.e, Solⁿ is $\int\limits_{(y\text{ const})} M\,dx + \int\limits_{(\text{terms without } x)} N\,dy = c$.

Ex1 Solve $(ax+hy+g)dx + (hx+by+f)dy = 0$.

Here $M = ax+hy+g$, $N = hx+by+f$

$\frac{\partial M}{\partial y} = h$ $\qquad$ $\frac{\partial N}{\partial x} = h$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The eqn is exact.

Solⁿ is $\int\limits_{(y\text{ const})} (ax+hy+g)\,dx + \int\limits_{(\text{terms wt. } x)} (hx+by+f)\,dy = c$

or, $\cancel{\frac{ax^2}{2} + gx + hyx + b\frac{y^2}{2} + fy = c}$.

or, $\frac{ax^2}{2} + hyx + gx + \cancel{hxy} + b\frac{y^2}{2} + fy = c$ //

Ex2 Solve $e^y dx + (xe^y + 2y)dy = 0$

Here $M = e^y$, $N = xe^y + 2y$

$\frac{\partial M}{\partial y} = e^y$, $\frac{\partial N}{\partial x} = e^y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. eqn is exact.

∴ Solⁿ is $\int\limits_{y\text{ const}} e^y dx + \int\limits_{(\text{terms wt. } x)} (xe^y + 2y)dy = c$

or, $e^y x + \cancel{2}\frac{y^2}{\cancel{2}} = c$ //

Ex3. Solve: (i) $(3x^2y + \frac{y}{x})dx + (x^3 + \log x)dy = 0$

(ii) $(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0$.

(iii) $(x^2 - ay)\,dx = (ax - y^2)\,dy$

(iv) $(2x^2 + 6xy - y^2)\,dx + (3x^2 - 2xy + y^2)\,dy = 0$

(v) $\frac{2x}{y^3}\,dx + \frac{y^2 - 3x^2}{y^4}\,dy = 0.$

Sol^n (v) Here $M = \frac{2x}{y^3}$, $N = \frac{1}{y^2} - \frac{3x^2}{y^4}$

$\frac{\partial M}{\partial y} = \frac{-6x}{y^4}$, $\frac{\partial N}{\partial x} = \frac{-6x}{y^4}$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So the eq^n is exact.

$\therefore$ sol^n is - - -

(vi) $\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x\sin y)\,dy = 0.$

§ Eq^{n}s reducible to exact eq^{n}s:

Sometimes a d.e. which is not exact can be made so on multiplication by a suitable factor called an integrating factor.

Methods to find an integrating factor

(I) I.F. of a homogeneous eq^n:

If $M\,dx + N\,dy = 0$ be a homo. eq^n in x & y, then $\frac{1}{Mx + Ny}$ is an I.F. $(Mdx + Ndy \neq 0)$

Ex1 Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

Sol^n Here $M = 3x^2y - y^3$, $N = -2x^2y + xy^2$

$\frac{\partial M}{\partial y} = 3x^2 - 3y^2$ $\quad \frac{\partial N}{\partial x} = -4xy + y^2$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So not exact.

Now, $\frac{1}{Mx + Ny} = \frac{1}{3x^3y^2 - xy^3 - 2x^2y^2 + xy^3} = \frac{1}{x^2y^2} = $ I.F.

Multiplying the given eq^n by the integrating factor we get, $\frac{1}{x^2y^2}(3xy^2 - y^3)\,dx + \frac{1}{x^2y^2}(-2x^2y + xy^2)\,dy = 0$

or, $\left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(-\frac{2}{y} + \frac{1}{x}\right)dy = 0$

<u>verification</u>: $\frac{\partial M}{\partial y} = -\frac{1}{x^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$, so exact.

Solⁿ is $\underbrace{\int\left(\frac{3}{x} - \frac{y}{x^2}\right)dx}_{(y\ \text{const})} + \underbrace{\int\left(-\frac{2}{y} + \frac{1}{x}\right)dy}_{(\text{terms w/o } x)} = c$

or, $\cancel{-\frac{2}{x}}\ 3\log x + \frac{y}{x} - 2\log y = c$ //

Solve the following: <u>H.W.</u>

(2) $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$, (3) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.

(II). Integrating factor of an eqⁿ of the type
$f(x,y)\,y\,dx + g(x,y)\,x\,dy = 0$.

If the eqⁿ is of this form, then $\frac{1}{Mx - Ny}$ is an I.F.
$(Mx - Ny \neq 0)$.

<u>Ex 1</u> Solve $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$.

Here $M = y + xy^2$, $N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$, so not exact.

Now $\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy + x^2y^2 - xy + x^2y^2} = \frac{1}{2x^2y^2}$

Multiplying the given eqⁿ by $\frac{1}{2x^2y^2}$ we get,

$\frac{1}{2x^2y^2}(1 + xy)\,y\,dx + \frac{1}{2x^2y^2}(1 - xy)\,x\,dy = 0$

or, $\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$

or, $\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$

which is exact.

$\therefore$ Solⁿ is $\int\left(\frac{1}{x^2y}+\frac{1}{x}\right)dx + \int\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy = c$
(y const) (terms w.r.t. x)

or, $-\frac{1}{xy} + \log x - \log y = c$. //

H.W. Solve the following :

(2) $(x^2y^2+xy+1)\,y\,dx + (x^2y^2-xy+1)\,x\,dy = 0$

(3) $y(xy+2x^2y^2)\,dx + x(xy-x^2y^2)\,dy = 0$

(III) In the eqn $Mdx + Ndy = 0$

(a) if $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ be a fn of x only $= f(x)$ say,

then $e^{\int f(x)dx}$ is an integrating factor.

(b) If $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ be a fn of y only $= F(y)$ say,

then $e^{\int F(y)dy}$ is an integrating factor.

Ex/ Solve $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$.

Here $M = xy^3+y$, $N = 2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y} = 3xy^2+1$ $\qquad \frac{\partial N}{\partial x} = 4xy^2+2$, not exact.

Now, $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \dfrac{xy^2+1}{y(xy^2+1)} = \frac{1}{y}$, a fn of y only.

$\therefore$ I.F. $= e^{\int\frac{1}{y}dy} = e^{\log y} = y$.

Multiplying the eqn by y we get.

$(xy^4+y^2)dx + 2(x^2y^3+xy+y^5)dy = 0$

$\therefore$ Solⁿ is $\int(xy^4+y^2)dx + \int 2(x^2y^3+xy+y^5)dy = c$
(y const) (terms w.r.t. x) //

Ex2 Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2y\,dy = 0$

Here $M = xy^2 - e^{\frac{1}{x^3}}$, $N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy$, not exact.

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4xy}{-x^2y} = -\frac{4}{x}$, a f^n of x only.

$\therefore$ I.F. $= e^{\int -\frac{4}{x}dx} = e^{-4\log x} = x^{-4}$

Multiplying the given eq^n throughout by x^{-4} we get,

$\frac{1}{x^4}(xy^2 - e^{\frac{1}{x^3}})dx - \frac{x^2y}{x^4}dy = 0$

or, $\left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$, which is exact now.

$\therefore$ Sol^n is $\underset{(y\text{ const})}{\int\left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}\right)dx} + \underset{(\text{terms w.o. } x)}{\int -\frac{y}{x^2}dy} = c$

or, $-y^2\frac{x^{-2}}{2} - \int\frac{e^t}{3}dt = c$

or, $-\frac{y^2}{2x^2} - \frac{1}{3}e^{\frac{1}{x^3}} = c$. //

put $\frac{1}{x^3} = t$

$\therefore -3x^{-4}dx = dt$