Gradient:

If $\phi(x,y,z)$ be a scalar function of position in space (i.e coordinates of x,y and z), then its partial derivatives along the three orthogonal axes are , , .

The gradient of the scalar function ϕ is defined by,

Grad ϕ = i + j + k

Where, i + j + k = nabla

Nabla operator is a vector and when it operates with a scalar, it converts the scalar into vector.

Divergence:

If A(x,y,z) is a vector field, the scalar product of the vector operator del and A is a scalar and is called the divergence of A.

In cartesian coordinate components,

Div.A = (i + j + k) . $(iA_x + jA_y + kA_z)$

= + +

Hence div.A is a scalar function. The vector field is called solenoidal if its divergence vanishes i.e, div.A = 0

If the vector function A spreads out (diverges) from a point, then it has a positive divergence at that point and acts as a source of the field A. On the other hand, del.A is negative if the point acts as a sink of the field A.

Curl:

If A(x,y,z) is a vector field, the cross product of the operator nabla and vector A is a vector. It is denoted by nabla A (A is written with a vector sign) and known as curlA. We can obtain,

Curl A = (i + j + k) $(iA_x + jA_y + kA_z)$

Therefore, Curl A = i(+ j( ) +k( )

Irrotational Vector:

A vector is said to be irrotational if the curl of that vector is zero. i.e. A vector V is irrotational if,

$$\text{Curl } V = 0$$

Some important Vector Relationships –

1) div. ∇S , where S is a scalar field.

$= \nabla.(\nabla S)$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(\frac{\partial S}{\partial x}i + \frac{\partial S}{\partial y}j + \frac{\partial S}{\partial z}k\right)$

$= \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}$

$\therefore$ div. grad $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$

∇^2 is called Laplace operator.

2) Curl grad S , where S is a scalar field

$= \nabla \times (\nabla S)$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (\nabla S)$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(\frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j} + \frac{\partial S}{\partial z}\hat{k}\right)$

So, $(\text{Curl grad})_x S = \left(\frac{\partial^2 S}{\partial y \partial z} - \frac{\partial^2 S}{\partial z \partial y}\right)\hat{i}$ $\left[\because \hat{j}\times\hat{k} = \hat{i} \text{ and } \hat{k}\times\hat{j} = -\hat{i}\right]$

$= 0$

Similarly, $(\text{Curl grad})_y S = 0$ and $(\text{Curl grad})_z S = 0$

Therefore, Curl grad $S = \nabla \times (\nabla S) = 0$

In ~~case~~ case of a vector field V,

1) grad div V = $\nabla(\nabla . V)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right)\hat{i} + \left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right)\hat{j}$$

$$+ \hat{k}\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

2) div curl V = $\nabla . (\nabla \times V)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_y}{\partial x \partial z} + \frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_z}{\partial y \partial x} + \frac{\partial^2 V_y}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z \partial y}$$

$$= 0$$

Therefore, div. Curl V = $\nabla . (\nabla \times V) = 0$