Ex 3 Solve the following:

(i) $2y\,dx + (2x\log x - xy)\,dy = 0$

Here $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x} = f(x)$

------ Solⁿ is $2y\log x - \frac{y^2}{2} = c$ //

(ii) $y(2x^2 - xy + 1)\,dx + (x-y)\,dy = 0$

Solⁿ Here $M = 2yx^2 - xy^2 + y$, $N = x - y$

$\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1$, $\frac{\partial N}{\partial x} = 1$, not exact.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x^2 - 2xy + 1 - 1}{x-y} = \frac{2x(x-y)}{x-y}$

$\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

Multiplying the given d.e. by e^{x^2} we get,

$e^{x^2}(2yx^2 - xy^2 + y)\,dx + e^{x^2}(x-y)\,dy = 0$, which is exact.

$\therefore$ Solⁿ is $\underset{(y\ \text{const})}{\int e^{x^2}(2yx^2 - xy^2 + y)\,dx} + \underset{(\text{terms w.r.t. } x)}{\int e^{x^2}(x-y)\,dy} = c$

$\cancel{\therefore \text{ or, } (2yx^2 - xy^2 + y)\frac{e^{x^2}}{2x} - \int (4xy - y^2)\frac{e^{x^2}}{2x} = c}$

or, $\int y e^{x^2}(2xy - xy + 1)\,dx + 0 = c$

put $x^2 = t$

then $2x\,dx = dt$.

$\therefore$ L.H.S. $= y\left[\int e^t\sqrt{t}\,dt - \int e^t \frac{1}{2}\,dt\, y + \int e^t \frac{1}{2\sqrt{t}}\,dt\right]$

$= y\left[\sqrt{t}\,e^t - \int \cancel{\frac{1}{2\sqrt{t}} e^t\,dt} - \frac{1}{2}y\,e^t + \int \cancel{\frac{1}{2\sqrt{t}} e^t\,dt}\right]$

$= y e^t\left(\sqrt{t} - \frac{1}{2}y\right) = y e^{x^2}\left(x - \frac{1}{2}y\right)$

$\therefore$ Sol^n is $e^{x^2}(\cancel{xy}\ xy - \frac{1}{2}y^2) = c$ //

(iii) $(x-y)dx - dy = 0$, $y(0) = 2$.

Hint/ Here $M = x-y$, $N = -1$, I.F. $= e^{x}$ $\left(\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right)$

Sol^n is $(x-1)e^{x} - ye^{-x} = c$

put $x=0$, $y=2$ so that $c = -3$.

$\therefore$ Sol^n is . . . . //

(iv) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Hint $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{2}{y}$, I.F. $= \ldots$

Sol^n is $x^3y^2 + \frac{x^2}{y} = c$.

(v) $\cancel{\frac{dy}{dx}}$ H.W. $y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$

Hint Homogeneous eq^n.

(IV) Integrating factor found by inspection

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$

(iv) $\frac{-x\,dy + y\,dx}{y^2} = d\left(\frac{x}{y}\right)$

(v) $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\frac{y}{x}\right]$

(vi) $\frac{x\,dy - y\,dx}{x^2-y^2} = d\left[\frac{1}{2}\log\frac{x+y}{x-y}\right]$.

are some integrable combinations.

Ex 1 Solve $y(2xy+e^x)\,dx = e^x\,dy$

Sol^n We write it as $2xy^2dx + ye^x dx - e^x dy = 0$

or, $2x\,dx + \dfrac{ye^x dx - e^x dy}{y^2} = 0$

or, $2x\,dx + d\left(\dfrac{e^x}{y}\right) = 0$

Int. $\Rightarrow$ $x^2 + \dfrac{e^x}{y} = c$, reqd. sol^n //.

Solve:

(2) $x\,dy - y\,dx + a(x^2+y^2)\,dx = 0$

(3) $x\,dx + y\,dy = \dfrac{a^2(x\,dy - y\,dx)}{x^2+y^2}$

(4) $y(y^3-x)\,dx + x(y^3+x)\,dy = 0$

Sol^n $y^4dx - xy\,dx + xy^3dy + x^2dy = 0$

We can write it into the form

$y^3(y\,dx + x\,dy) + x(x\,dy - y\,dx) = 0$

or, $y^3\,d(xy) + x\,d\left(\dfrac{y}{x}\right).x^2 = 0$

or, $d(xy) + d\left(\dfrac{y}{x}\right)\left(\dfrac{x}{y}\right)^3 = 0$

or, $d(xy) + \left(\dfrac{y}{x}\right)^{-3} d\left(\dfrac{y}{x}\right) = 0$

Int. $\Rightarrow$ $xy + \left(\dfrac{y}{x}\right)^{-2}\Big/-2 = c$

or, $xy - \frac{1}{2}\left(\dfrac{y}{x}\right)^{-2} = c$ //

(5) $(y+x)\,dy = (y-x)\,dx$

Sol^n $y\,dy + x\,dx + (x\,dy - y\,dx) = 0$

or, $\frac{1}{2}d(x^2+y^2) + \dfrac{x\,dy - y\,dx}{x^2+y^2}(x^2+y^2) = 0$

or, $\frac{1}{2}\dfrac{d(x^2+y^2)}{x^2+y^2} + d\tan^{-1}\left(\dfrac{y}{x}\right) = 0$

Int. $\Rightarrow$ $\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\dfrac{y}{x} = c$ //.

$\therefore$ I. F. $= e^{\int \frac{1}{y} dy} = e^{\log y} = y.$

~~Solⁿ is~~ Mult. throughout by y we get,

$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$$

which is exact.

$\therefore$ Solⁿ is $\int\limits_{y\,\text{const}} (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$

or, $\frac{x^2}{2}y^4 + y^2x + 2\frac{y^6}{6} = c$ //

§ Eqⁿs of the 1st order and higher degree:

As $\frac{dy}{dx}$ will occur in higher degrees, it is convenient to denote $\frac{dy}{dx}$ by p. Such eqⁿs are of the form $f(x, y, p) = 0$.

Three cases arise for discussion.

Case I: Eqⁿs solvable for p:

A d.e. of the 1st order but of the nth degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0, \longrightarrow (1)$$

where $P_1, P_2 \ldots P_n$ are fⁿs of x and y.

Splitting up the l.h.s. of (1) into n linear factors, we have $[p - f_1(x,y)][p - f_2(x,y)] \cdots [p - f_n(x,y)] = 0$.

Equating each of the factors to zero,

$$p = f_1(x,y),\ p = f_2(x,y) \ldots p = f_n(x,y).$$

Solving each of these eqⁿs of the 1st order and 1st degree, we get the solⁿs

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0 \ldots F_n(x, y, c) = 0.$$

These n solⁿs constitute the general solⁿ of (1).

Otherwise the general sol^n of ① may be written as

$F_1(x,y,c).F_2(x,y,c)\cdots F_n(x,y,c)=0$. //

Ex1. Solve $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$.

It is $p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}$

or, $p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)=1$.

Factorising $\left(p+\frac{y}{x}\right)\left(p-\frac{x}{y}\right)=0$

Thus we have

$p+\frac{y}{x}=0$ & $p-\frac{x}{y}=0$

or, $\frac{dy}{dx}+\frac{y}{x}=0$ or, $\frac{dy}{dx}-\frac{x}{y}=0$

or, $xdy+ydx=0$ or, $xdx-ydy=0$

or, $d(xy)=0$ Int, $\frac{x^2}{2}-\frac{y^2}{2}=c$

Int, $xy=c$. or, $x^2-y^2=c$.

Thus reqd sol^n is $(xy-c)(x^2-y^2-c)=0$ //

Ex2 Solve the following eqns:

i) $y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-x=0$,

ii) $p(p+y)=x(x+y)$

iii) $y=x\left[p+\sqrt{1+p^2}\right]$

iv) $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0$

v) $p^2+2py\cot x=y^2$.

Sol^n (iii) $\frac{y}{x}=p+\sqrt{1+p^2}$

or, $\sqrt{1+p^2}=\frac{y}{x}-p$

Squaring, $1+p^2=\left(\frac{y}{x}\right)^2-2\frac{y}{x}p+p^2$

or, $\left(\frac{y}{x}\right)^2-2\frac{y}{x}p=1$ or, $2\frac{y}{x}p=\left(\frac{y}{x}\right)^2-1$

or $\frac{y}{x}\left(\cancel{\frac{y}{x}-2p}\right)=1$ $\Rightarrow p=\frac{\left(\frac{y}{x}\right)^2-1}{2\frac{y}{x}}$

4. $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

The eqn is $x^2p^2 + xyp - 6y^2 = 0$

or, $x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

or, $xp(xp+3y) - 2y(xp+3y) = 0$

or, $(xp+3y)(xp-2y) = 0$

$\therefore$ $xp + 3y = 0$ & $xp - 2y = 0$

or, $x\frac{dy}{dx} + 3y = 0$, $x\frac{dy}{dx} = 2y$

$x\,dy + 3y\,dx = 0$ or, $x\,dy = 2y\,dx$

or, $\frac{dy}{y} + \frac{3dx}{x} = 0$ or, $\frac{dy}{y} = 2\frac{dx}{x}$

Int, $\log y + 3\log x = \log c$ or $\log y = 2\log x + \log c$

or, $x^3y = c$, $\frac{y}{x^2} = c$

$\therefore$ Soln is $(x^3y - c)(y - cx^2) = 0$ //

Case II: Eqns solvable for y

If the given eqn on solving for y, takes the form $y = f(x,p) \rightarrow (1)$ then diffg w.r.t x gives the eqn of the form $p = \frac{dy}{dx} = \phi(x, p, \frac{dp}{dx})$

Now it may be possible to solve this new d.e. in x and p. Let its soln be $F(x,p,c) = 0 \rightarrow (2)$

The elimination of p from (1) & (2) gives the reqd soln.

In case elimination of p is not possible, then we may solve (1) & (2) for x and y and obtain $x = F_1(p,c)$, $y = F_2(p,c)$ as the reqd soln, where p is the parameter.

Ex) Solve $y - 2px = \tan^{-1}(xp^2)$

It is $y = 2px + \tan^{-1}(xp^2) \longrightarrow$ (i)

Diff'g w.r.t. x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + \frac{1}{1+x^2p^4}\cdot\left(p^2 + 2xp\frac{dp}{dx}\right)$$

or, $\cancel{p} + \cancel{\cdot} \; p + 2x\frac{dp}{dx} + \frac{p^2}{1+x^2p^4} + \frac{2xp\frac{dp}{dx}}{1+x^2p^4} = 0$

or, $p\left(1+\frac{p}{1+x^2p^4}\right) + 2x\frac{dp}{dx}\left(1+\frac{p}{1+x^2p^4}\right) = 0$

or, $\left(1+\frac{p}{1+x^2p^4}\right)\left(p+2x\frac{dp}{dx}\right) = 0$

$\Rightarrow p + 2x\frac{dp}{dx} = 0 \quad \rightarrow p\,dx + 2x\,dp = 0$

$\Rightarrow \cancel{\frac{dp}{p} + \frac{dx}{2x}} = 0 \quad \Rightarrow \int\frac{dx}{x} + \int\frac{2\,dp}{p} = 0$

or, $\log x + 2\log p = \log c$

or, $xp^2 = c$. or, $p = \sqrt{\frac{c}{x}} \longrightarrow$ (ii)

Eliminating p from (i) & (ii) we get,

$$y = 2\sqrt{\frac{c}{x}}\,x + \tan^{-1}\left(x\cdot\frac{c}{x}\right)$$

$$= 2\sqrt{cx} + \tan^{-1}c.$$

which is the req'd. general sol^n of (i). //

Obs: The factor $p/(1+p^4x^2)$ will not be considered here as it concerns 'singular sol^n' of (i) whereas we are interested only in finding general sol^n. //

Caution: Sometimes one is tempted to write (ii) as $\frac{dy}{dx} = \sqrt{\frac{c}{x}}$

On integration, $y = 2\sqrt{cx} + c'$

Such a resolution is incorrect. //

Ex 2 Solve $y = 2px + p^{a}$ $\longrightarrow$ (i)

diff'g w.r.t. x: $\dfrac{dy}{dx} = 2\dfrac{dp}{dx}x + 2p + a p^{a-1}\dfrac{dp}{dx}$

$$\text{or, } p = 2x\frac{dp}{dx} + 2p + a p^{a-1}\frac{dp}{dx}$$

$$\text{or, } 0\,\cancel{\frac{dx}{dp}} = 2x + p\frac{dx}{dp} + a p^{a-1}$$

$$\text{or, } \frac{dx}{dp} + \frac{2x}{p} + a p^{a-2} = 0$$

$$\text{or, } \frac{dx}{dp} + \frac{2x}{p} = -a p^{a-2}$$

This is Leibnitz's linear eq^n in x & p

$$\text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2.$$

Sol^n is
$$x\cdot p^2 = -a\int p^{a}dp + c$$
$$= -a\frac{p^{a+1}}{a+1} + c$$

$$\text{or, } x = -\frac{a}{a+1}p^{a-1} + cp^{-2} \longrightarrow \text{(ii)}$$

Substituting this value of x in (i) we get,

$$y = 2p\cdot\left(-\frac{a}{a+1}p^{a-1} + cp^{-2}\right) + p^{a}$$
$$= -\frac{2a}{a+1}p^{a} + 2cp^{-1} + p^{a}$$
$$= \frac{2c}{p} + \left(1 - \frac{2a}{a+1}\right)p^{a} = \frac{2c}{p} + \frac{1-a}{a+1}p^{a} \longrightarrow \text{(iii)}$$

The eq^{n}s (ii) & (iii) taken together constitute the general sol^n of (i) with parameter p.

Ex Solve the following eq^{n}s:

1. $y = x + a\tan^{-1}p$
2. $y + px = x^4p^2$
3. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$
4. $xp^2 + x = 2yp$
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.
7. $y = 2px - p^2$

Solⁿ 1. $y = x + a\tan^{-1}p \longrightarrow (i)$

Diff. w.r.t. x, $\frac{dy}{dx} = 1 + a\frac{1}{1+p^2}\frac{dp}{dx}$

or, $p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$

or, $(p-1)(1+p^2) = a\frac{dp}{dx}$

or, $\frac{dx}{a} = \frac{dp}{(p-1)(p^2+1)}$

$\frac{1}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$

$1 = A(p^2+1) + (Bp+C)(p-1)$

$1 = A - C$

$0 = A + B$

$0 = -B + C$

$A + C = 0$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$\therefore C = A - 1 = -\frac{1}{2} = B$

or, $\frac{1}{a}dx = \left(\frac{1/2}{p-1} - \frac{\frac{1}{2}(p+1)}{p^2+1}\right)dp$

$= \frac{1}{2(p-1)} - \frac{1}{2}\frac{p}{p^2+1} - \frac{1}{2(p^2+1)}$

Int, $\frac{1}{a}x = \frac{1}{2}\log(p-1) - \frac{1}{4}\log(p^2+1) - \frac{1}{2}\tan^{-1}p$.

or, $x = \frac{a}{2}\left\{\log(p-1) - \tan^{-1}p\right\} + C - \frac{a}{4}\log(p^2+1)$

$x + c = \frac{a}{2}\left\{\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p\right\} \longrightarrow (ii)$

Substituting this value of x in (i) we get,

$y = \frac{a}{2}\log\frac{p-1}{\sqrt{p^2+1}} - \frac{a}{2}\tan^{-1}p + a\tan^{-1}p$

$= \frac{a}{2}\left\{\log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1}p\right\} + C \longrightarrow (iii)$

The eqns (ii) & (iii) taken together constitute the general solⁿ (i) with parameter p.