§ Equations solvable for x

If the given eqⁿ on solving for x takes the form $x = f(y, p) \longrightarrow (1)$, then differentiation w.r.t. y gives an eqⁿ of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new d.e. in y and p. Let its solⁿ be $F(y, p, c) = 0 \longrightarrow (2)$

The elimination of p from (1) & (2) gives the reqd solⁿ. The complete primitive or general solⁿ can be obtained in the parametric form $x = u(p)$ and $y = v(p)$ where p is the parameter.

Ex 1 Solve $p^3 - 4xyp + 8y^2 = 0$.

Solving for x it takes the form

$$4xyp = p^3 + 8y^2 \quad \text{or,} \quad 4x = \frac{p^2}{y} + 8\frac{y}{p} \longrightarrow (i)$$

Diffᵍ w.r.t. y we get,

$$4\frac{dx}{dy} = \frac{y\,2p\,dp/dy - p^2}{y^2} + 8\frac{(p.1 - y\,dp/dy)}{p^2}$$

$$\text{or,}\quad \frac{4}{p} = \frac{2p}{y}\frac{dp}{dy} - \frac{p^2}{y^2} + \frac{8}{p} - \frac{8y}{p^2}\frac{dp}{dy}$$

$$\text{or,}\quad -\frac{4}{p} + \frac{p^2}{y^2} = 2\left(\frac{p}{y} - 4\frac{y}{p^2}\right)\frac{dp}{dy}$$

$$\text{or,}\quad \frac{-4y^2 + p^3}{py^2} = 2\,\frac{p^3 - 4y^2}{yp^2}\frac{dp}{dy}$$

$$\text{or,}\quad \frac{1}{py^2} = \frac{2}{yp^2}\frac{dp}{dy}$$

$$\text{or}\quad p = 2y\frac{dp}{dy} \Rightarrow \frac{2\,dp}{p} = \frac{dy}{y}$$

Integration $\Rightarrow$ $2\log p = \log y + \log c$

or, $p^2 = yc \Rightarrow p = \sqrt{yc}$

Substituting this value in (i) we get,

$4x = \frac{yc}{y} + \frac{8y}{\sqrt{yc}}$

or, $x = \frac{c}{4} + 2\sqrt{\frac{y}{c}}$ //

Ex 2 Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$

Sol We have, $\tan^{-1} p = x - \frac{p}{1+p^2}$

or, $x = \tan^{-1} p + \frac{p}{1+p^2}$

Differentiating w.r.t. y we get,

$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p.2p\frac{dp}{dy}}{(1+p^2)^2}$

or, $\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2-2p^2)\,dp/dy}{(1+p^2)^2}$

$= \left\{\frac{1}{1+p^2} + \frac{1-p^2}{(1+p^2)^2}\right\}\frac{dp}{dy}$

$= \frac{1+p^2+1-p^2}{(1+p^2)^2}\frac{dp}{dy} = \frac{2\,dp/dy}{(1+p^2)^2}$

or, $dy = \frac{2\,dp.p}{(1+p^2)^2} = \frac{2p\,dp}{(1+p^2)^2}$ | put $1+p^2 = t$ $\therefore 2p\,dp = dt$

$= \frac{dt}{t^2}$

Int. $\Rightarrow$ $y = -\frac{1}{t} + c$, or, $y + \frac{1}{1+p^2} = c$

$\therefore$ $x = \frac{p}{1+p^2} + \tan^{-1} p$ and $y = c - \frac{1}{1+p^2}$ gives the parametric solⁿ of the given d.eqⁿ. //

Ex 3 Solve $x = py - p^2$

Sol^n Diff^g w.r.t. y we get,

$$\frac{dx}{dy} = p + \frac{dp}{dy}y - 2p\frac{dp}{dy}$$

or, $\frac{1}{p} = p + (y-2p)\frac{dp}{dy}$

or, $\frac{1}{p} - p = (y-2p)\frac{dp}{dy}$

or, $\frac{dy}{dp}\left(\frac{1}{p} - p\right) = y - 2p$

or, $\frac{dy}{dp} - \frac{y}{\frac{1}{p}-p} = \frac{-2p}{\frac{1}{p}-p}$ which is 1st order ~~&~~ linear d.e. in y.

$$\text{I.F.} = e^{\int \frac{1}{p-\frac{1}{p}}dp}$$

$$= e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$$

$\therefore$ sol^n is $y.\sqrt{p^2-1} = \int \frac{-2p}{\frac{1}{p}-1}\sqrt{p^2-1}\,dp + c$

$$= \int \frac{+2p^2}{p^2-1}\sqrt{p^2-1}\,dp + c$$

$$= \int \frac{2p^2}{\sqrt{p^2-1}}dp + c$$

$$= 2\int \frac{p^2-1+1}{\sqrt{p^2-1}}dp + c = 2\int\sqrt{p^2-1}\,dp + 2\int\frac{dp}{\sqrt{p^2-1}} + c$$

$$= 2\left[\frac{p\sqrt{p^2-1}}{2} - \frac{1}{2}\log(p+\sqrt{p^2-1})\right] + 2\log(p+\sqrt{p^2-1}) + c$$

$$= p\sqrt{p^2-1} + \log(p+\sqrt{p^2-1}) + c$$

or, $y = p + [\log(p+\sqrt{p^2-1}) + c]/\sqrt{p^2-1} \rightarrow (i)$

$\therefore x = py - p^2 = \cancel{p^2} + p[\log(p+\sqrt{p^2-1}) + c]/\sqrt{p^2-1} \cancel{-p^2} \rightarrow (ii)$

parametric sol^n //

Ex 4 Solve $y = 2px + y^2p^3$

Soln On solving for x it takes the form

$$2x = \frac{y - y^2p^3}{p}$$

Diffg w.r.t. y we get,

$$2\frac{dx}{dy} = \frac{p\{1-(2yp^3+y^2 3p^2\frac{dp}{dy})\} - (y-y^2p^3)\frac{dp}{dy}}{p^2}$$

or, $2\frac{p^2}{p} = p - p(2yp^3+3y^2p^2\frac{dp}{dy}) - (y-y^2p^3)\frac{dp}{dy}$

or, $2p - p = -2yp^4 - (3y^2p^3+y-y^2p^3)\frac{dp}{dy}$

or, $p(1+2yp^3) = -(2y^2p^3+y)\frac{dp}{dy}$

or, $p = -y\frac{dp}{dy}$ or, $\frac{dy}{y} + \frac{dp}{p} = 0$

Int. $\Rightarrow \log y + \log p = \log c$

or, $p = c/y \longrightarrow (1)$

Thus eliminating p from the given eqn we get,

$y = 2\frac{c}{y}x + y^2\frac{c^3}{y^3}$ or, $y^2 = 2cx + c^3$ //.

Ex Solve the following problems:

1. $p^3y + 2px = y$
2. $x - yp = ap^2$
3. $p^2y + 2px = y$.

§ Clairaut's Form

A d.e. of the form $y = px + f(p)$ where $p = \frac{dy}{dx}$ is called a Clairaut's eqn. The general soln of this eqn is $y = cx + f(c)$ where c is any const.

To see this we differentiate the given eqn and get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or, $p = p + (x + f'(p))\frac{dp}{dx}$

or, $\frac{dp}{dx}(x + f'(p)) = 0$

$\Rightarrow \frac{dp}{dx} = 0$ or $x + f'(p) = 0$.

$\Rightarrow dp = 0$

Int. $\Rightarrow p = c$ (say).

So given eqn $\Rightarrow y = cx + f(c)$.

And $x + f'(p) = 0$ does not give a general soln since $x = -f'(p) \Rightarrow y = -pf'(p) + f(p)$, does not contain any arbitrary const. Gives singular soln.

~~Ex~~ Solve: (1) $p = \sin(y - xp)$.

It can be written as $y - xp = \sin^{-1} p$

or, $y = xp + \sin^{-1} p$, which is Clairaut's eqn.

$\therefore$ soln is $y = cx + \cancel{\tan^{-1} c}\ \sin^{-1} c$ //

(2) $(px - y)(py + x) = a^2 p$.

Ex Obtain the complete primitives and singular sol^{n}s of the Clairaut's eq^{n}s :

(i) $y = px + \sqrt{a^2p^2 + b^2}$, (ii) $yp = p^2(x-b) + a$.

Sol^n (i) Diff^{tg} w.r.t. x we get,

$$p = p + x\frac{dp}{dx} + \frac{1 \cdot 2a^2p}{2\sqrt{a^2p^2+b^2}}\frac{dp}{dx}$$

or, $\left(x + \frac{a^2p}{\sqrt{a^2p^2+b^2}}\right)\frac{dp}{dx} = 0$

$\therefore \frac{dp}{dx} = 0$ or, $x + \frac{a^2p}{\sqrt{a^2p^2+b^2}} = 0$

$\Rightarrow p = c$

$\therefore$ complete primitive is $y = cx + \sqrt{a^2c^2+b^2}$.

and $x + \frac{a^2p}{\sqrt{a^2p^2+b^2}} = 0 \Rightarrow x = \frac{-a^2p}{\sqrt{a^2p^2+b^2}} \longrightarrow (1)$

$$\therefore (i) \Rightarrow y = \frac{-a^2p^2}{\sqrt{a^2p^2+b^2}} + \sqrt{a^2p^2+b^2}$$

$$= \frac{-a^2p^2 + a^2p^2 + b^2}{\sqrt{a^2p^2+b^2}} = \frac{b^2}{\sqrt{a^2p^2+b^2}} \longrightarrow (2)$$

From (2) & (3) eliminating p we get,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2p^2}{a^2p^2+b^2} + \frac{b^2}{a^2p^2+b^2} = 1$, which is the reqd. singular sol^n.

//