SINUSOIDAL STEADY-STATE ANALYSIS

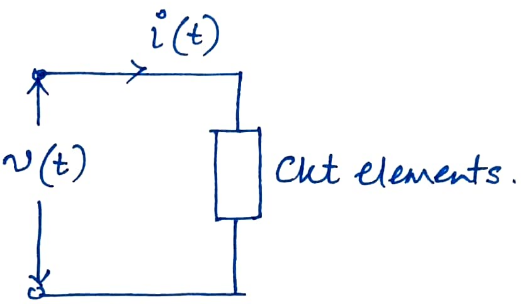
When sinusoidal excitation is applied to a circuit, the response has two parts :

(i) Natural response or Transient response

(ii) Steady-state response (Forced response)

At this stage, we are not interested to transient response.

If we apply a sinusoidal voltage in a circuit (linear) then the steady-state response $i(t)$ has the same frequency with the voltage but with some changes in phase.



Let $v(t) = V_m \sin \omega t$ then

$$i(t) = I_m \sin(\omega t \pm \varphi)$$

There are three basic electrical circuit elements : Resistance, Inductance and Capacitance. Let us establish voltage-current relationships of all these three elements.

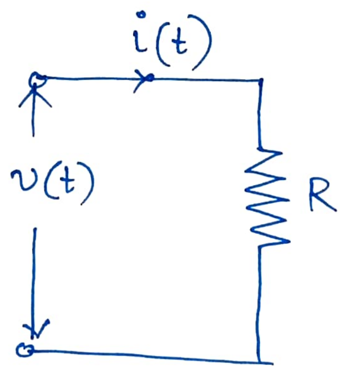
1. <u>Resistance (R)</u>

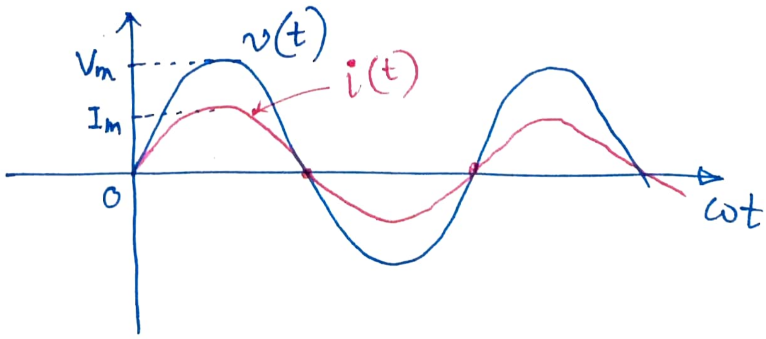
$$v(t) = V_m \sin \omega t$$

$$i(t) = \frac{v(t)}{R}$$

$$= \frac{V_m}{R} \sin \omega t$$

$$= I_m \sin \omega t \quad \text{where } I_m = \frac{V_m}{R}$$



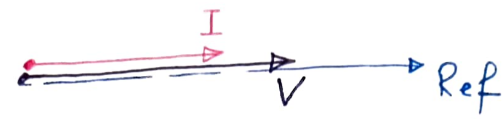


In phasor notation

$$\bar{V} = V\angle 0^\circ \qquad R: \text{scalar} \qquad V = \text{RMS value}.$$

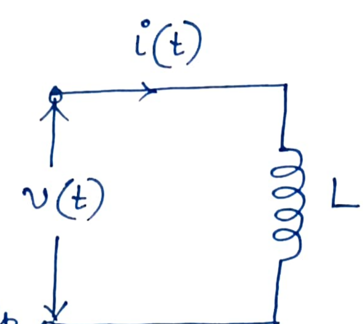
$$\bar{I} = \frac{V\angle 0}{R}$$

$$= \frac{V}{R}\angle 0^\circ = I_m\angle 0^\circ$$



So the phase difference between voltage and current is zero, i.e. both voltage and current are in phase.

2. Inductance (L)



In case of inductance L, the voltage-current relationship is given by $v_L(t) = L\frac{di(t)}{dt}$.

Let us assume that $i(t) = I_m \sin \omega t$

$$\therefore \; v_L(t) = v(t) = L\frac{d}{dt}(I_m \sin \omega t)$$

$$= \omega L I_m \cos \omega t$$

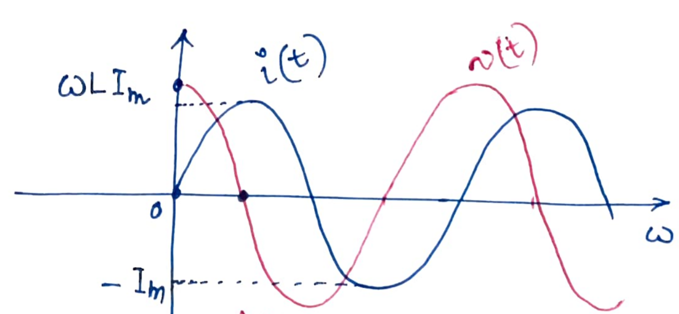
$$= \omega L I_m \sin(\omega t + 90^\circ)$$

$$= V_m \sin(\omega t + 90^\circ)$$

where $V_m = \omega L I_m$

$$\Rightarrow \frac{V_m}{I_m} = \omega L = \text{inductive reactance}$$

$$= X_L \; (\Omega)$$

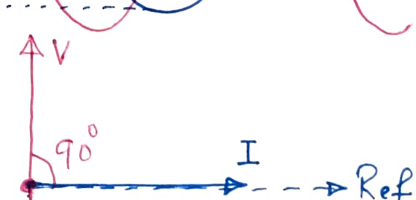


In phasor notation

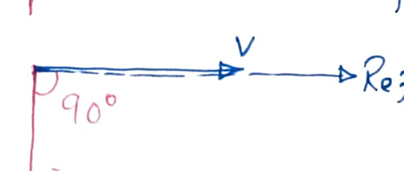
$$\bar{I} = I\angle 0^\circ$$

$$\bar{V} = \bar{V}\angle 90^\circ = \sqrt{2}\,\omega L I_m\angle 90^\circ$$

$$= \sqrt{2}\, V_m\angle 90^\circ$$



OR



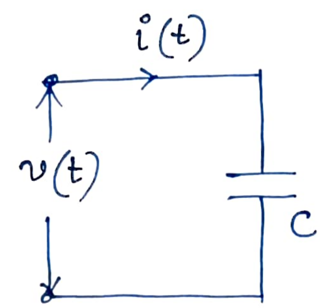
In case of inductance, the current is lagging behind the voltage by 90°.

In complex notation if $\bar{I} = I\angle 0$ then $\bar{V} = V\angle 90^\circ$
$= jV$

$$\therefore \frac{\bar{V}}{\bar{I}} = \frac{V\angle 90^\circ}{I\angle 0^\circ} = \frac{jV}{I} = \frac{j\sqrt{2}\omega L I_m}{\sqrt{2} I_m} = j\omega L$$
$$= jX_L$$

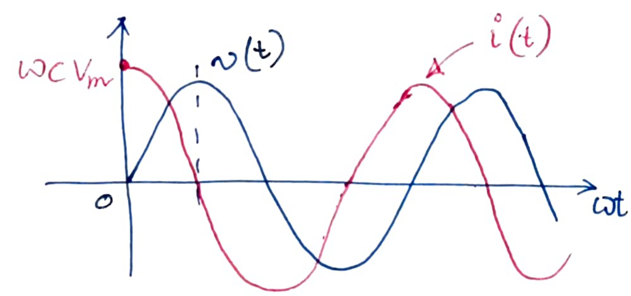
$X_L = 2\pi f L$ where f: frequency in Hz.

3. Capacitance



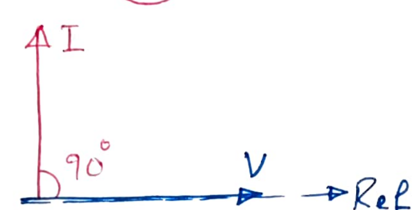
Let $v(t) = V_m \sin \omega t$

$$\therefore i(t) = C \cdot \frac{dv(t)}{dt}$$
$$= C \cdot \frac{d}{dt}\left[V_m \sin \omega t\right]$$
$$= C \cdot V_m \omega \cos \omega t$$
$$= \omega C V_m \sin(\omega t + 90^\circ)$$
$$= I_m \sin(\omega t + 90^\circ)$$



where $I_m = \omega C V_m$

$$\Rightarrow \frac{V_m}{I_m} = \frac{1}{\omega C} = \text{capacitive reactance}$$
$$= X_c$$



In phasor notation

$\bar{V} = V\angle 0^\circ$

$\bar{I} = I\angle 90^\circ = \sqrt{2}\omega C I_m \angle 90^\circ = \sqrt{2} I_m \angle 90^\circ$

In case of capacitance, the current is leading the voltage by 90°

In complex notation, if $\bar{V} = V\angle 0$ then $\bar{I} = I\angle 90^\circ$
$= jI$

$$\therefore \frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle 90^\circ} = \frac{V}{jI} = \frac{\sqrt{2}V_m}{j\sqrt{2}\omega C V_m} = \frac{1}{j\omega C}$$
$$= \frac{1}{j\cancel{X_c}} = -jX_c$$

In summary,

R : voltage and current both are in phase

L : current is lagging behind voltage by 90°

C : current is leading the voltage by 90°

$X_L = \omega L = 2\pi f L$ = inductive reactance (Ω)

$X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$ = capacitive reactance (Ω)

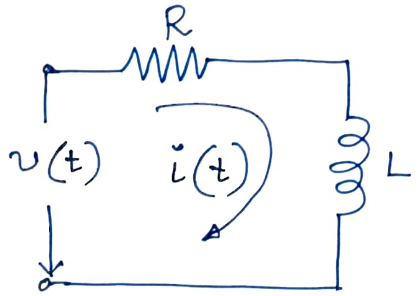
In complex notation,

Inductive reactance $= jX_L = j\omega L$

Capacitive " $= -jX_C = -j\dfrac{1}{\omega C}$

Electrical ckt

1. RL ckt



$\bar{V} = \bar{V}_R + \bar{V}_L$

$= \bar{I}R + \bar{I}\bar{X}_L$

$= \bar{I}R + j\omega L\bar{I}$

$= (R + j\omega L)\bar{I}$

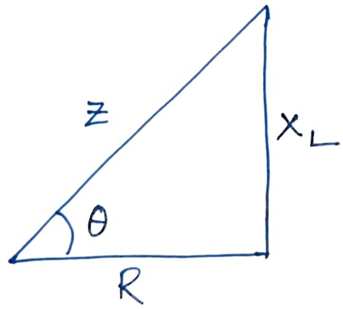
$= \bar{Z}\bar{I}$ where $\bar{Z} = R + j\omega L$ = impedance of the ckt

In polar form, $\bar{Z} = Z\angle\theta$

where $Z = |Z| = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + (\omega L)^2}$

$\theta = \tan^{-1}\left(\dfrac{\omega L}{R}\right)$

Impedance triangle



It must be observed that impedance is a complex number and not a phasor.

We have seen that $\bar{I} = \frac{\bar{V}}{\bar{Z}}$

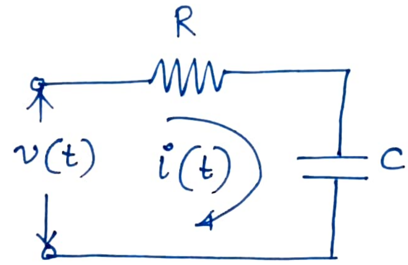
Here $\bar{I} = \frac{\bar{V}}{\bar{Z}} = \bar{Y}\bar{V}$ where $\bar{Y} = \frac{1}{\bar{Z}}$ = admittance ($\mho$)

$$\therefore \bar{Y} = \frac{1}{\bar{Z}} = \frac{1}{R+jX_L} = \frac{R}{R^2+X_L^2} - j\frac{X_L}{R^2+X_L^2}$$

$$= G - jB$$

where G = conductance and B = susceptance

2. RC ckt

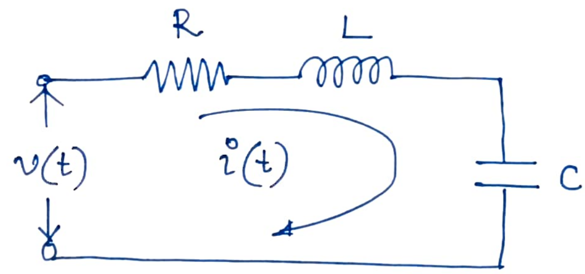


$$\bar{V} = \bar{V}_R + \bar{V}_C$$
$$= \bar{I}R + \bar{I}\bar{X}_C$$
$$= (R - jX_C)\bar{I}$$
$$= \bar{Z}\bar{I}$$ where $\bar{Z} = R - jX_C$ = impedance of the ckt

$|\bar{Z}| = \sqrt{R^2+X_C^2}$, $\theta = \tan^{-1}\left(\frac{-X_C}{R}\right) = -\tan^{-1}\frac{X_C}{R}$

$$\therefore \bar{Z} = Z\angle -\tan^{-1}\left(\frac{X_C}{R}\right)$$

3. RLC series ckt



In phasor notation

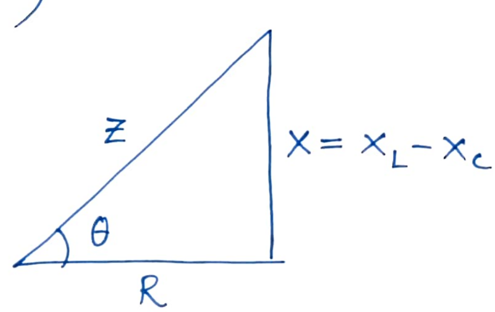
$$\bar{V} = \bar{V}_R + \bar{V}_L + \bar{V}_C$$
$$= \bar{I}R + (jX_L)\bar{I} - (jX_C)\bar{I}$$
$$= [R + j(X_L - X_C)]\bar{I}$$ (assume that $X_L > X_C$)

$\therefore$ Impedance, $\bar{Z} = R + j(X_L - X_C)$

$$Z = |Z| = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X^2}$$

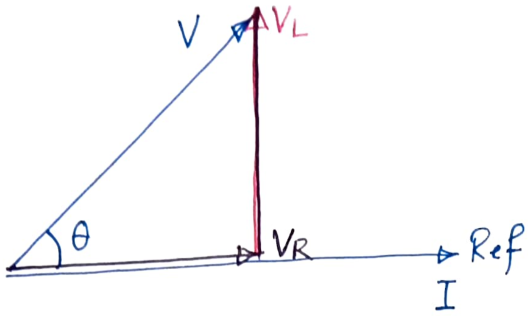
$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Impedance triangle

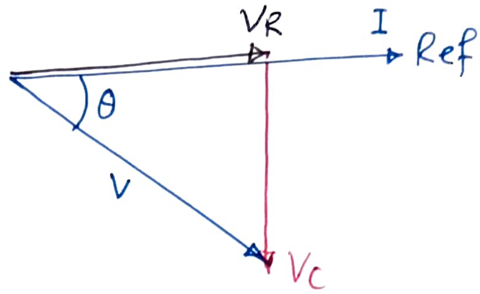


Phasor diagram

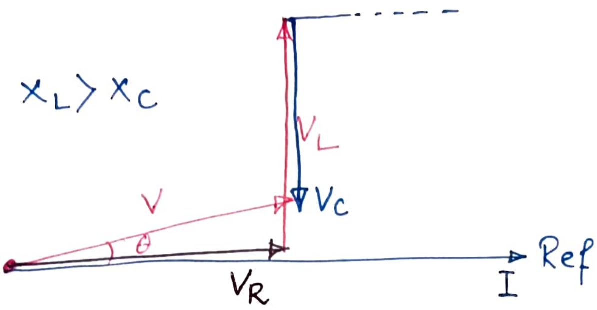
1. RL ckt



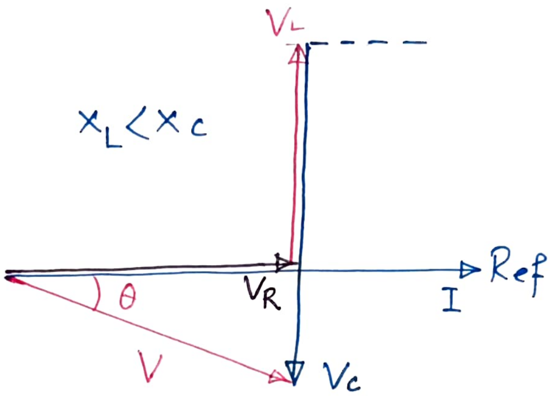
2. RC ckt



3. RLC ckt

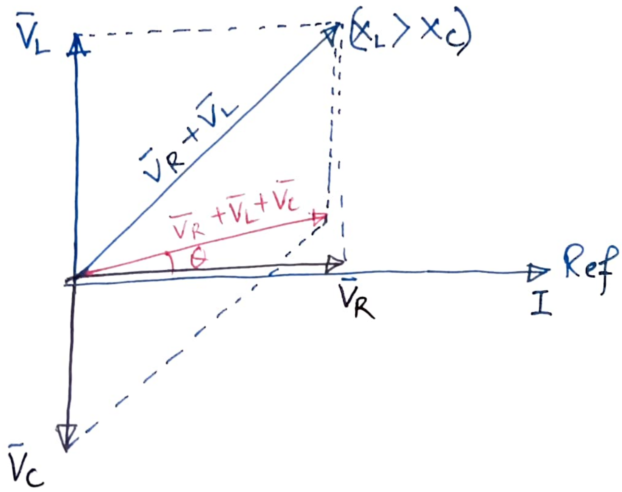


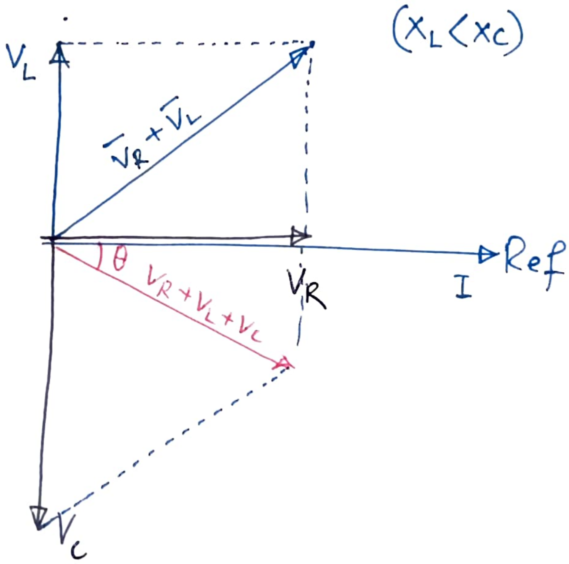
(Ckt is dominated by inductance)



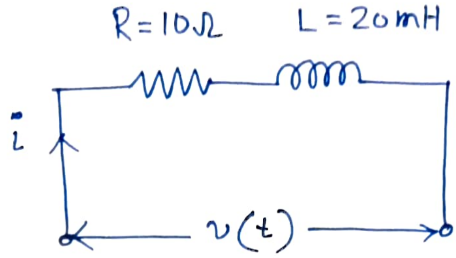
(Ckt is dominated by capacitance)

Alternative phasor diagram for RLC ckt





Ex



$i(t) = 2 \sin 500t$

$Z = \sqrt{R^2 + X_L^2}$

$= \sqrt{10^2 + (2\pi f L)^2}$

$= \sqrt{10^2 + (\omega L)^2}$

$= \sqrt{100 + (500 \times 20 \times 10^{-3})^2} = 14.14$

$\theta = \tan^{-1} \frac{\omega L}{R}$

$= \tan^{-1}\left(\frac{500 \times 20 \times 10^{-3}}{10}\right) = 45^\circ$

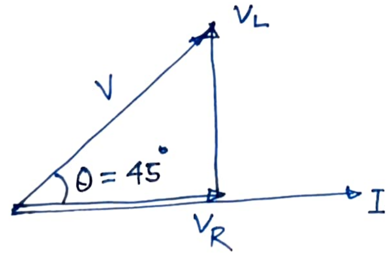
V = rms value of voltage , I = rms value of current

$\therefore V = ZI = 14.14 \times \frac{2}{\sqrt{2}} = 20V \quad [\because I_m = 2]$

$\therefore v(t) = V_m \sin(\omega t + \varphi)$

$= 20 \times \sqrt{2} \sin(500t + 45^\circ)$

$= 28.3 \sin(500t + 45^\circ)$ //



In complex notation

$\bar{I} = 2\sqrt{2} \angle 0^\circ$, $\bar{Z} = R + jX_L = 10 + j(500 \times 20 \times 10^{-3})$

$= 10 + j10$

$= 14.14 \angle 45^\circ$

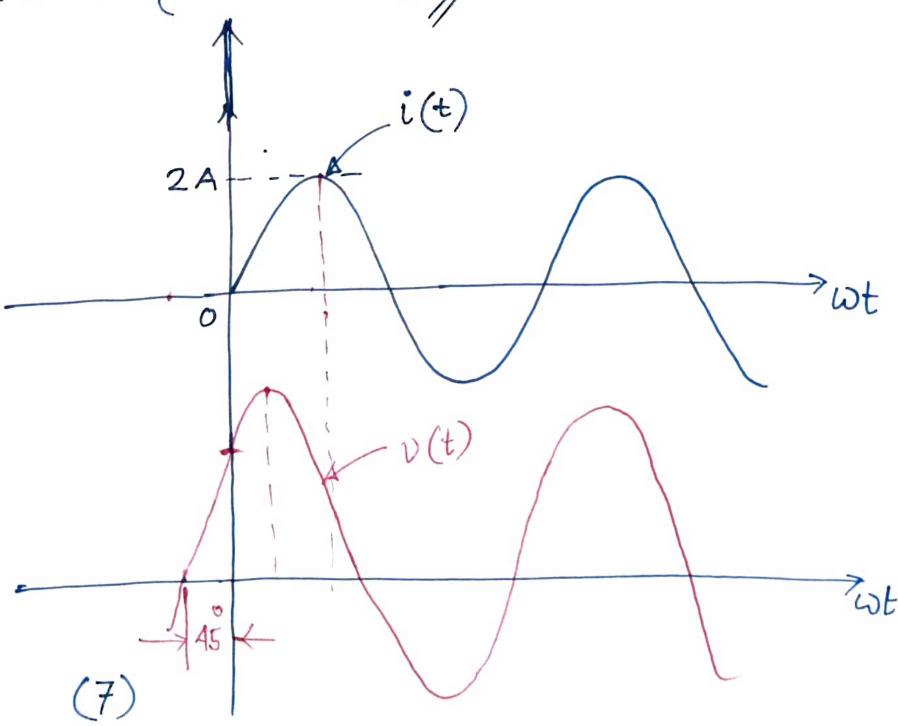
$\bar{V} = \bar{I}\bar{Z} = (2\cancel{\sqrt{2}} \angle 0^\circ)(14.14 \angle 45^\circ)$

$= 20 \angle 45^\circ$

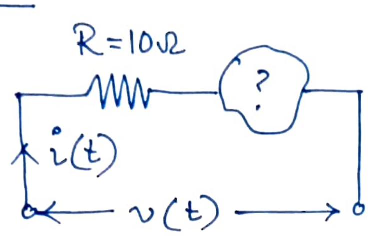
In sinusoidal waveform.

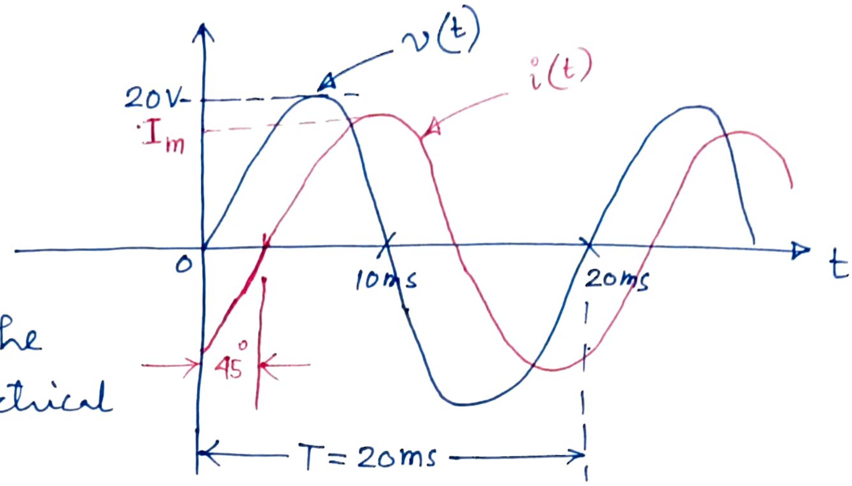
$v(t) = \sqrt{2} V \sin(500t + 45^\circ)$

$= 28.3 \sin(500t + 45^\circ)$ //



Ex





Calculate I_m and the value of unknown electrical element.

Ex Find the two elements in a series ckt, given that the current and total voltage are

$$i(t) = 10\cos(5000t - 23{\cdot}13^\circ), \quad v(t) = 50\cos(5000t + 30^\circ)$$

[Ans. $R = 3\,\Omega$, $L = 0{\cdot}8$ mH]

Ex A series ckt with $R = 2\,\Omega$ and $C = 200\,\mu F$, has a sinusoidal applied voltage with a frequency of 99·47 MHz. If the max. voltage across the capacitance is 24 V, what is the max. voltage across the series combination?

Ex. Obtain the sum of the three voltages

$$v_1(t) = 147{\cdot}3\cos(\omega t + 98{\cdot}1)\ (V) \quad v_2(t) = 294{\cdot}6\cos(\omega t - 45^\circ)$$

and $v_3(t) = 88{\cdot}4\sin(\omega t + 135^\circ)$

Ex. An RLC series ckt has a current which lags the applied voltage by 30°. The inductor voltage maximum is twice the capacitor voltage maximum, and ~~v_L~~ $v_L = 10\sin 1000t$ (V). Determine L and C, given that $R = 20\,\Omega$

[Ans. $L = 23{\cdot}1$ mH, $C = 86{\cdot}5\,\mu F$]

Series-parallel combination of R-L-C ckt

Impedance in series: $\bar{Z} = \bar{Z}_1 + \bar{Z}_2 + \bar{Z}_3 + \cdots$

Impedance in parallel: $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots$

Admittance in series : $\bar{Y} = \bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3 + \cdots$

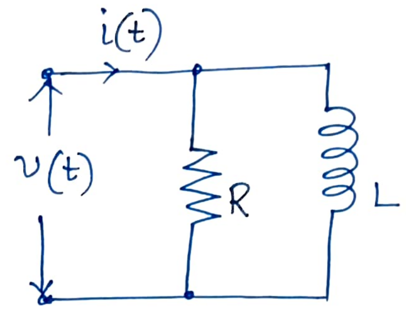
Admittance in parallel : $\frac{1}{Y} = \frac{1}{Y_1} + \frac{1}{Y_2} + \cdots$

For series combination $Z = \bar{Z}_1 + \bar{Z}_2 + \bar{Z}_3 + \cdots$ is convenient

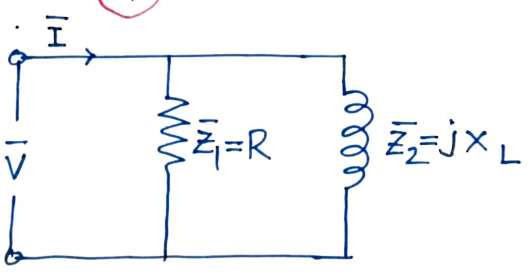
For parallel " $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots$

$\Rightarrow \bar{Y} = \bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3 + \cdots$ is convenient.

1. R-L parallel ckt



OR



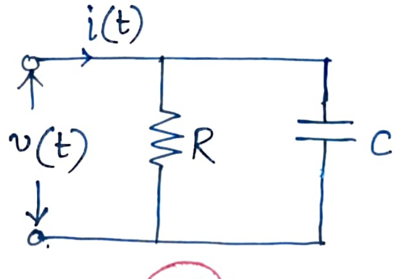
$$\bar{Z}_{eq} = Z_1 Z_2 / Z_1 + Z_2$$

$$\bar{I} = \frac{\bar{V}}{\bar{Z}_{eq}} = \bar{Y}_{eq}\bar{V}$$

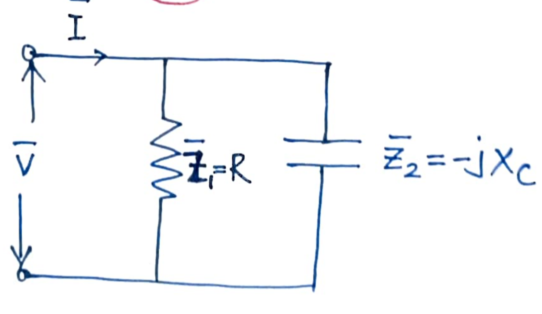
$$= (\bar{Y}_1 + \bar{Y}_2)\,\bar{V}$$

$$\bar{Y}_1 = \frac{1}{\bar{Z}_1}, \quad \bar{Y}_2 = \frac{1}{\bar{Z}_2}$$

2. R-C parallel ckt



OR

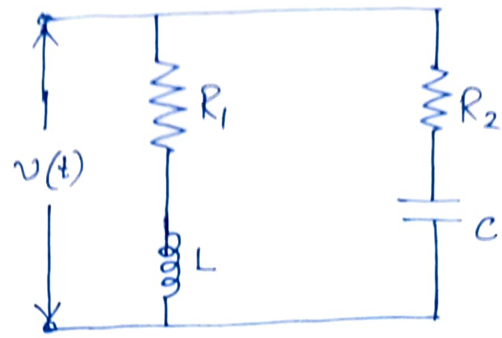


$$\bar{Z}_{eq} = \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}$$

$$\bar{Z}_1 = R \Rightarrow \bar{Y}_1 = \frac{1}{R} = G$$

$$\bar{Z}_2 = -jX_C \Rightarrow \bar{Y}_2 = \frac{1}{-jX_C} = +jB$$

3. R-L-C parallel ckt



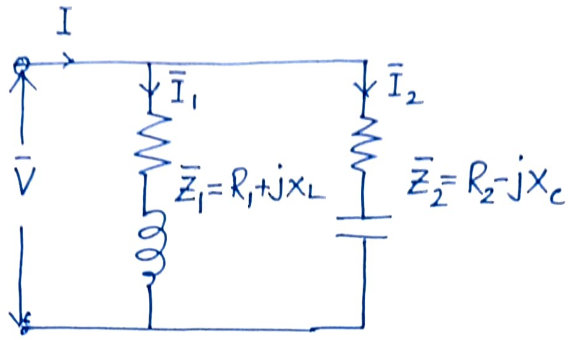
$\bar{Z}_1 = R_1 + jX_L \Rightarrow \bar{Y}_1 = \frac{1}{\bar{Z}_1} = G_1 - jB_1$

$\bar{Z}_2 = R_2 - jX_C \Rightarrow \bar{Y}_2 = \frac{1}{\bar{Z}_2} = G_2 + jB_2$

$\bar{Y}_{eq} = \bar{Y}_1 + \bar{Y}_2 = (G_1 + G_2) + j(B_2 - B_1)$

$=$ $\qquad B_2 > B_1$

OR

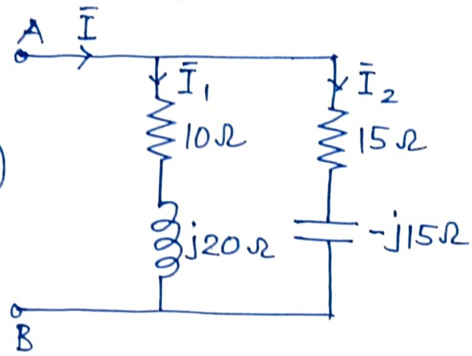


$\bar{I} = \bar{Y}_{eq}\bar{V}$

$\bar{I} = \bar{I}_1 + \bar{I}_2$

$\bar{I}_1 = \bar{V}\bar{Y}_1 \quad , \quad \bar{I}_2 = \bar{V}\bar{Y}_2$

Ex obtain $\bar{Z}_{eq}$ and $\bar{Y}_{eq}$ for the ckt of Fig.



$\bar{Z}_1 = 10 + j20 = 22.4\angle 63.43^\circ$

$\bar{Z}_2 = 15 - j15 = 21.2\angle -45^\circ$

$$Z_{eq} = \frac{\bar{Z}_1 \bar{Z}_2}{Z_1 + Z_2} = \frac{(22.4\angle 63.43^\circ)(21.2\angle -45^\circ)}{(10+j20)+(15-j15)}$$

$= 18.63\angle 7.12^\circ \ \Omega$

$Y_{eq} = \frac{1}{Z_{eq}} = 0.0537\angle -7.12^\circ$ (℧) OR (S)

mho $\qquad$ siemens

If $\bar{V} = 60\angle 0^\circ$ is applied across AB find $\bar{I}, \bar{I}_1, \bar{I}_2$