Module - 1

Vector Calculus

Vector function -

If to each value of a scalar variable t, there corresponds a value of vector $\vec{r}$, then $\vec{r}$ is called a vector function of the scalar variable t and we write $\vec{r} = \vec{r}(t)$ or $\vec{r} = \vec{f}(t)$

Eg - $\vec{r} = t\hat{i} - 2t\hat{j} + t^2\hat{k} = \vec{r}(t)$

if $t = 1$, $\vec{r}(1) = \hat{i} - 2\hat{j} + \hat{k}$

if $t = 4$, $\vec{r}(4) = 4\hat{i} - 8\hat{j} + 16\hat{k}$

Derivative of vector function w.r.t. a scalar -

Let, $\vec{r} = \vec{f}(t)$ be a vector function of scalar t. Let δt be the small increment in t and $\delta\vec{r}$ be the corresponding increment in $\vec{r}$.

$$\therefore \vec{r} + \delta\vec{r} = \vec{f}(t+\delta t)$$

$$\Rightarrow \delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)$$

$$\Rightarrow \frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

Corresponding increment in $\vec{r}$

$$\Rightarrow \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

Some rules of vector differentiation -

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vector function of a scalar t and ϕ be a scalar function of (t), then.

1) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

2) $\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\frac{d}{dt}\vec{b} + \frac{d}{dt}\vec{a}.\vec{b}$ [write as written]

3) $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$

4) $\frac{d}{dt}(\phi\vec{a}) = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$

5) $\frac{d}{dt}[\vec{a}.\vec{b}.\vec{c}] = \left[\frac{d\vec{a}}{dt}.\vec{b}.\vec{c}\right] + \left[\vec{a}.\frac{d\vec{b}}{dt}.\vec{c}\right] + \left[\vec{a}.\vec{b}.\frac{d\vec{c}}{dt}\right]$

6) $\frac{d}{dt}[\vec{a}\times(\vec{b}\times\vec{c})] = \frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right)$

Proves -

2) $LHS = \frac{d}{dt}(\vec{a}.\vec{b})$

$$= \lim_{\delta t\to 0} \frac{(\vec{a}+\delta\vec{a})(\vec{b}+\delta\vec{b})-\vec{a}.\vec{b}}{\delta t}$$

$$= \lim_{\delta t\to 0} \frac{\vec{a}.\vec{b} + \vec{a}.\delta\vec{b} + \delta\vec{a}.\vec{b} + \delta\vec{a}.\delta\vec{b} - \vec{a}.\vec{b}}{\delta t}$$

$$= \lim_{\delta t\to 0} \frac{\vec{a}.\delta\vec{b} + \delta\vec{a}.\vec{b} + \delta\vec{a}.\delta\vec{b}}{\delta t}$$

$$= \lim_{\delta t\to 0} \vec{a}.\frac{\delta\vec{b}}{\delta t} + \lim_{\delta t\to 0}\frac{\delta\vec{a}}{\delta t}.\vec{b} + \lim_{\delta t\to 0}\frac{\delta\vec{a}}{\delta t}.\cancel{\delta\vec{b}}^{\to 0}$$

$$= \vec{a}.\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}.\vec{b}$$

$= RHS$

5) $LHS = \frac{d}{dt}[\vec{a}.\vec{b}.\vec{c}]$

$$= \frac{d}{dt}\{\vec{a}.(\vec{b}\times\vec{c})\}$$

$$= \vec{a}.\frac{d}{dt}(\vec{b}\times\vec{c}) + \frac{d\vec{a}}{dt}.(\vec{b}\times\vec{c})$$

$$= \vec{a}\left(\vec{b}\times\frac{d\vec{c}}{dt} + \frac{d\vec{b}}{dt}\times\vec{c}\right) + \frac{d\vec{a}}{dt}(\vec{b}\times\vec{c})$$

$$= \vec{a}.\left(\vec{b}\times\frac{d\vec{c}}{dt}\right) + \vec{a}.\left(\frac{d\vec{b}}{dt}\times\vec{c}\right) + \frac{d\vec{a}}{dt}.(\vec{b}\times\vec{c})$$

$$= \left[\vec{a}.\vec{b}.\frac{d\vec{c}}{dt}\right] + \left[\vec{a}.\frac{d\vec{b}}{dt}.\vec{c}\right] + \left[\frac{d\vec{a}}{dt}.\vec{b}.\vec{c}\right]$$

$= RHS$

Theorem -

Differentiation of a constant vector is zero vector.

Proof - Let $\vec{r}$ be a constant vector of scalar t.

$\therefore \vec{r} = \vec{f}(t)$

$$\therefore \vec{r} + \delta\vec{r} = \vec{f}(t+\delta t) = \vec{f}(t)$$

$$\Rightarrow \delta\vec{r} = \vec{f}(t)$$

$$\therefore \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{f}(t) - \vec{f}(t)}{\delta t}$$

$$= \vec{0}$$

Theorem -

If $\vec{F}(t)$ has a constant magnitude, then $\vec{F}$ and $\frac{d\vec{F}}{dt}$ are perpendicular.

Soln $\vec{F}$ has constant magnitude

$$|\vec{F}(t)| = \text{const.}$$

Now, $\vec{F}(t).\vec{F}(t) = |\vec{F}(t)|^2 = \text{const.}$

$$\Rightarrow \vec{F}.\vec{F} = \text{const.}$$

$$\Rightarrow \frac{d}{dt}(\vec{F}.\vec{F}) = 0$$

$$\Rightarrow \vec{F}.\frac{d}{dt}\vec{F} + \frac{d\vec{F}}{dt}.\vec{F} = 0$$

$$\Rightarrow \vec{F}.\frac{d\vec{F}}{dt} = 0$$

$$\therefore \vec{F} \perp \frac{d\vec{F}}{dt}$$

Eg - If $\vec{F}$ has a const. direction them $\vec{F} \times \frac{d\vec{F}}{dt} = 0$

Soln Let $\hat{G}(t)$ be the unit vector in the direction of $\vec{F}$ and

So $\vec{F}(t) = |\vec{F}(t)| . \hat{G}(t)$

$$\Rightarrow \vec{F}(t) = f(t)\,\hat{G}(t)$$

where, Let $|\vec{F}(t)| = f(t)$

$$\therefore \frac{d}{dt}\vec{F}(t) = \frac{d}{dt}[f(t).\hat{G}(t)]$$

$$= f(t)\frac{d}{dt}\hat{G}(t) + \frac{d}{dt}f(t).\hat{G}(t)$$

$$\Rightarrow \frac{d}{dt}\vec{F}(t) = \frac{d}{dt}f(t).\hat{G}(t) \quad \left[\because \frac{d}{dt}\hat{G}(t) = 0\right]$$

$$\Rightarrow \vec{F} \times \frac{d}{dt}\vec{F}(t) = \vec{F} \times \frac{d}{dt} f(t) . \hat{G}(t)$$

$$= f(t) . \hat{G} \times \left(\frac{d}{dt} f(t) . \hat{G}(t)\right)$$

$$= f(t) . \frac{d f(t)}{dt} \cancel{(\hat{G} \times \hat{G})}^{\,0}$$

$$\therefore \vec{F} \times \frac{d}{dt}\vec{F}(t) = 0$$

 # Eg – If $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$, where $\vec{a}$ & $\vec{b}$ are constant vector and ω is a constant, then prove that

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2 \vec{r} \quad \text{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = -\omega \vec{a} \times \vec{b}$$

Sol^n Given $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$

$$\frac{d\vec{r}}{dt} = \vec{a}\cos\omega t . \omega + \vec{b}(-\sin\omega t) . \omega$$

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\omega^2 \sin\omega t - \omega^2 \vec{b}\cos\omega t$$

$$= -\omega^2 \vec{r}$$

And, $\vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a}\sin\omega t + \vec{b}\cos\omega t) \times (\vec{a}\omega\cos\omega t - \vec{b}\omega\sin\omega t)$

$$= \omega\left[(\vec{a}\sin\omega t + \vec{b}\cos\omega t) \times (\vec{a}\cos\omega t) - (\vec{a}\sin\omega t + \vec{b}\cos\omega t) \times (\vec{b}\sin\omega t)\right]$$

$$= \omega\left[(\vec{b} \times \vec{a})\cos^2\omega t - (\vec{a} \times \vec{b})\sin^2\omega t\right]$$

$$= \omega\left[-(\vec{a} \times \vec{b})\cos^2\omega t - (\vec{a} \times \vec{b})\sin^2\omega t\right]$$

$$= -\omega(\vec{a} \times \vec{b})(\cos^2\omega t - \sin^2\omega t)$$

$$= -\omega(\vec{a} \times \vec{b})$$

 # Eg – If $\frac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$ and $\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$, prove that

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{\omega} \times (\vec{u} \times \vec{v})$$

Sol^n $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$

$$= \vec{u} \times [\vec{\omega} \times \vec{v}] + [\vec{\omega} \times \vec{u}] \times \vec{v}$$

$$= \vec{u}[\vec{\omega} \times \vec{v}] - \vec{v}[\vec{\omega} \times \vec{u}]$$

$$= (\vec{u}\cdot\vec{v})\vec{w} - (\vec{u}\cdot\vec{w})\cdot\vec{v} - \left[(\vec{v}\cdot\vec{u})\cdot\vec{w} - (\vec{v}\cdot\vec{w})\cdot\vec{u}\right]$$

$$= (\vec{v}\cdot\vec{w})\vec{u} - (\vec{u}\,\vec{w})\vec{v}$$

$$= \vec{w} \times (\vec{u} \times \vec{v})$$

Gradient of a Scalar –

Let $\phi(x, y, z)$ be a scalar function, then the vector $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called gradient of the scalar ϕ and is denoted as grad ϕ or $\nabla\phi$.

The operator $\nabla \equiv \text{grad} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Eg – Evaluate grad ϕ or $\nabla\phi$ where $\phi = \log(x^2+y^2+z^2)$
$= \phi(x, y, z)$

Sol^n $\therefore$ grad $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \log(x^2+y^2+z^2)$

$$\Rightarrow \nabla\phi = \hat{i}\frac{\partial}{\partial x}\log(x^2+y^2+z^2) + \hat{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \hat{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$\Rightarrow \nabla\phi = \hat{i}\frac{1}{x^2+y^2+z^2}\cdot 2x + \hat{j}\frac{1}{x^2+y^2+z^2}\cdot 2y + \hat{k}\frac{1}{x^2+y^2+z^2}\cdot 2z$$

Grad ϕ is normal to the surface $\phi(x, y, z) = c$

Properties of gradient –

1) If ϕ is a constant, then $\nabla\phi = \vec{0}$
2) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$
3) $\nabla(c_1\phi_1 \pm c_2\phi_2) = c_1\nabla\phi_1 \pm c_2\nabla\phi_2$ (c_1 & c_2 are constants)
4) $\nabla(\phi_1\cdot\phi_2) = \phi_1\cdot\nabla\phi_2 + \phi_2\cdot\nabla\phi_1$
5) $\nabla(\phi_1/\phi_2) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}, \quad \phi_2 \neq \phi$

Prove –

LHS $= \nabla(\phi_1/\phi_2)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1/\phi_2)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1/\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1/\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1/\phi_2)$$

$$= \hat{i}\,\frac{\phi_2\frac{\partial\phi_1}{\partial x} - \phi_1\frac{\partial\phi_2}{\partial x}}{\phi_2^2} + \hat{j}\,\frac{\phi_2\frac{\partial\phi_1}{\partial y} - \phi_1\frac{\partial\phi_2}{\partial y}}{\phi_2^2} + \hat{k}\,\frac{\phi_2\frac{\partial\phi_1}{\partial z} - \phi_1\frac{\partial\phi_2}{\partial z}}{\phi_2^2}$$

$$= \frac{1}{\phi_2^2}\left[\phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) - \phi_1\left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_3}{\partial z}\right)\right]$$

$$= \frac{1}{\phi_2^2}\left[\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2\right]$$

#Eg - Find grad ϕ at $(1,-2,-1)$ if $\phi = 3x^2y - y^3z^2$

Sol^n $\nabla\phi = \hat{i}\,6xy + \hat{j}\,(3x^2 - 3y^2z) - \hat{k}\,2zy^3$

at $(1,-2,-1)$

$\nabla\phi = -12\,\hat{i} + 15\hat{j} - 16\hat{k}$

Eg - Find a unit normal to the surface $x^3+y^3+3xyz = 3$ at Point $(1,2,-1)$

Sol^n Let $\phi(x,y,z) \equiv x^3+y^3+3xyz - 3 = 0$

$\therefore \nabla\phi = (3x^2+3yz)\hat{i} + (3y^2+3xz)\hat{j} + 3xy\,\hat{k}$

at $(1,2,-1)$

$\nabla\phi = \{3+(-6)\}\hat{i} + (12-3)\hat{j} + 6\hat{k}$

$= -3\hat{i} + 9\hat{j} + 6\hat{k}$

Which is ~~the~~ normal to the given Surface

$\therefore$ The unit normal to the given surface is

~~magnitude~~ $= -\frac{3}{3\sqrt{14}}\hat{i} + \frac{9}{3\sqrt{14}}\hat{j} + \frac{6}{3\sqrt{14}}\hat{k}$

#Eg - Calculate the angle between the ~~ma~~ normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol^n Let $\phi(x,y,z) \equiv xy - z^2 = 0$

$\nabla\phi = y\hat{i} + x\hat{j} - 2z\hat{k}$

$$\nabla\phi\Big]_{(4,1,2)} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\nabla\phi\Big]_{(3,3,-3)} = 3\hat{i} + 3\hat{j} + 6\hat{k}$$

Now, angle between $\nabla\phi\Big]_{(4,1,2)}$ and $\nabla\phi\Big]_{(3,3,-3)}$ is gradient

$$\cos\theta = \left| \frac{(\hat{i}+4\hat{j}-4\hat{k}).(3\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{33}\cdot 3\sqrt{6}} \right|$$

$$= \left| \frac{3+12-24}{3\sqrt{11} \times 3 \times \sqrt{2}} \right|$$

$$= \frac{9}{9\sqrt{22}}$$

$$\cos\theta = \frac{1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{22}}$$

$$= 77.69^\circ$$

#Eg - If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then prove that

i) $\text{grad } r = \frac{\vec{r}}{r}$

Proof - Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2 \quad \text{---} (1)$$

Differentiating (1) partially, wrt x. we get

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Parallely, $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$ — (II)

$$\text{grad } r = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r$$

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} \quad [\text{From (II)}]$$

$$= \frac{1}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\vec{r}}{r}$$

ii) $$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(n r^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(n r^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(n r^{n-1}\frac{x}{r}\right) + \hat{j}\left(n r^{n-1}\frac{y}{r}\right) + \hat{k}\left(n r^{n-1}\frac{z}{r}\right)$$

$$= n r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= n r^{n-2}\,\vec{r}$$

iii) $$\nabla\left(e^{r^2}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) e^{r^2}$$

$$= \hat{i}\frac{\partial}{\partial x}e^{r^2} + \hat{j}\frac{\partial}{\partial y}e^{r^2} + \hat{k}\frac{\partial}{\partial z}e^{r^2}$$

$$= \hat{i}\, e^{r^2}.\,2r\frac{\partial r}{\partial x} + \hat{j}\, e^{r^2}.\,2r\frac{\partial r}{\partial y} + \hat{k}\, e^{r^2}.\,2r\frac{\partial r}{\partial z}$$

$$= \hat{i}\, e^{r^2}.\,2r.\frac{x}{r} + \hat{j}\, e^{r^2}.\,2r\frac{y}{r} + \hat{k}\, e^{r^2}.\,2r.\frac{z}{r}$$

$$= 2e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \cancel{2er}\; 2e^{r^2}\,\vec{r}$$

iv) $$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

$$LHS = \nabla\left(\frac{1}{r}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= -\frac{\vec{r}}{r^3}$$

$$= RHS$$

v) $\nabla(\vec{a}\cdot\vec{r})\vec{a}$, where a is a const. vector

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ (a_1, a_2, a_3 are constants)

Now, $\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \vec{a}$$

vi) $$\nabla \log r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\log\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)\hat{i} + \frac{1}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)\hat{j} +$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)\hat{k}.$$

$$= \frac{2x}{2\left(\sqrt{x^2+y^2+z^2}\right)^2}\hat{i} + \frac{2y}{2\left(\sqrt{x^2+y^2+z^2}\right)^2}\hat{j} + \frac{2z}{2\left(\sqrt{x^2+y^2+z^2}\right)^2}\hat{k}$$

$$= \frac{1}{\left(\sqrt{x^2+y^2+z^2}\right)}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\vec{r}}{r^2} \cancel{= RHS}$$

vii) $\nabla|\vec{r}|^2 = 2\vec{r}$

$$LHS = \nabla|\vec{r}|^2$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2\vec{r} = RHS$$

viii) $\nabla f(r) = f'(r)\nabla r$

Divergence of a vector function –

The divergence of a vector function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as,

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} \quad (\text{vector } \vec{V},\ \nabla\phi \to \text{scalar})$$

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then divergence of $\vec{V} = \nabla\cdot\vec{V}$

$$\Rightarrow \text{div}\,\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$\Rightarrow \text{div}\,\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Curl of a vector function –

The curl of a vector function $\vec{V}$ is denoted by curl $\vec{V}$ and defined as Curl $\vec{V} = \nabla\times\vec{V}$

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

* If div $\vec{V} = 0$, then $\vec{V}$ is called solonoidal vector.
* If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is said to be irrotational vector otherwise it is said to be rotational.

Eg – If $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$ show that

i) div $\vec{V} = 3$ ii) Curl $\vec{V} = \vec{0}$.

Sol^n i) div $\vec{V} = \nabla\cdot\vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 1 + 1 + 1$$

$$= 3$$

ii) Curl $\vec{V} = \nabla\times\vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Eg – i) If $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, show that $\nabla\cdot\vec{v} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{v} = \vec{0}$

Proof – Given $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$\nabla\cdot\vec{v} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{3(x^2+y^2+z^2) - (x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

~~Eg – ii) If $\vec{A} = (3xz^2)\hat{i} - (yz)\hat{j} + (x+2z)\hat{k}$ find curl $\vec{A}$~~

~~Sol^n $\nabla\times\vec{A} =$~~

2nd part –
$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right] \cdots\cdots$$

$$= \hat{i}\left(\frac{-yz}{(x^2+y^2+z^2)^{3/2}} + \frac{yz}{(x^2+y^2+z^2)^{3/2}}\right) \cdots\cdots$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Eg-ii) If $\vec{A} = (3xz^2)\hat{i} - (yz)\hat{j} + (x+2z)\hat{k}$ find curl (curl $\vec{A}$)

Sol^n Curl $\vec{A} = \nabla\times\vec{A}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^2 & yz & x+2z \end{vmatrix}$$

$$= y\hat{i} + (6zx - 1)\hat{j}$$

$$\therefore \text{Curl}(\text{curl } \vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 6zx-1 & 0 \end{vmatrix}$$

$$= -6x\,\hat{i} + (6z-1)\,\hat{k}$$

Eg - iii) If $u = x^2 + y^2 + z^2$, $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$, show that $\text{div}(u.\vec{v}) = 5u$

Soln $\cancel{\text{div}(u.\vec{v}) = \text{div}\left[(x^3+y^2+z^2)\hat{i} + (x^2+y^3+z^2)\hat{j} + (x^2+y^2+z^3)\hat{k}\right]}$

$$\cancel{= 3x^2 + 3y^2 + 3z^2}$$

$$\text{div}(u.\vec{v}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(x^3 + xy^2 + xz^2)\hat{i} + (x^2y + y^3 + yz^2)\hat{j} + (x^2z + y^2z + z^3)\hat{k}\right]$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5(x^2 + y^2 + z^2)$$

$$= 5u$$

Properties of divergence and Curl -

i) For a constant vector $\vec{a}$, $\text{div } \vec{a} = 0$, $\text{curl } \vec{a} = \vec{0}$

ii) If $\vec{A}$ is a vector function and ϕ is a scalar function, then,

$$\text{div}(\phi\vec{A}) = \phi(\nabla\vec{A}) + (\nabla\phi)\vec{A}$$

$$= \phi(\text{div}\,\vec{A}) + (\text{grad}\,\phi)\vec{A}$$

Proof - $\text{div}(\phi\vec{A}) = \nabla(\phi\vec{A})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi\vec{A})$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \hat{i}\left(\phi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\phi}{\partial x}\vec{A}\right) + \hat{j}\left(\phi\frac{\partial\vec{A}}{\partial y} + \frac{\partial\phi}{\partial y}\vec{A}\right) + \hat{k}\left(\phi\frac{\partial\vec{A}}{\partial z} + \frac{\partial\phi}{\partial z}\vec{A}\right)$$

$$= \phi\left(\frac{\partial\vec{A}}{\partial x}\hat{i} + \frac{\partial\vec{A}}{\partial y}\hat{j} + \frac{\partial\vec{A}}{\partial z}\hat{k}\right) + \vec{A}\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \phi(\nabla\vec{A}) + (\text{grad}\,\phi).\vec{A}$$

3) $\text{Curl}(\phi \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi (\text{curl } \vec{A})$

4) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\cdot\vec{B} + (\vec{B}\cdot\nabla)\cdot\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$

<u>Proof</u> $\nabla(\vec{A}\cdot\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\vec{A}\cdot\vec{B})$

$= \sum \hat{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$ [This means $\hat{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B}) + \hat{j}\frac{\partial}{\partial y}(\vec{A}\cdot\vec{B}) + \hat{k}\frac{\partial}{\partial z}(\vec{A}\cdot\vec{B})$]

$$= \sum \hat{i}\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)$$

$$= \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} + \sum\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)\hat{i} \quad \text{—— (I)}$$

$$\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = (\vec{A}\cdot\hat{i})\cdot\frac{\partial\vec{B}}{\partial x} - \vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right)$$

$$= (\vec{A}\cdot\hat{i})\cdot\frac{\partial\vec{B}}{\partial x} + \vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = \sum(\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x} + \vec{A}\times\sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right) \quad \text{—— (II)}$$

$$\sum\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)\hat{i} = \sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}$$

$$= \left(\vec{B}\cdot\sum\hat{i}\frac{\partial\vec{A}}{\partial x}\right)\vec{A} + \vec{B}\times\sum\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right) \quad \text{—— (III)}$$

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = \left(\vec{A}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{B} + \vec{A}\times\sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right) \quad \text{—— (II)}$$

$$\sum\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right) = \sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}$$

$$= \left(\vec{B}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{A} + \vec{B}\times\sum\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right) \quad \text{—— (III)}$$

Putting (II) & (III) in (I), we get

$$\nabla(\vec{A}\cdot\vec{B}) = \left(\vec{A}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{B} + \vec{A}\times\sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right) + \left(\vec{B}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{A} + \vec{B}\times\sum\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)$$

<u>Eg</u> – Curl (grad ϕ) $= \vec{0}$

<u>Proof</u> - Curl (grad ϕ) $= \nabla\times\nabla\phi$

$$= \nabla\times\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \nabla\times\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= \vec{0}$$

Eg – div (curl $\vec{v}$) = 0

Sol^n $\nabla(\nabla \times \vec{v})$

$$= \nabla \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \nabla(0\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(0\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= 0$$

* If $\nabla \times \vec{A} = \vec{0}$, then the vector $\vec{A}$ is irrotational, then there will be a consecutive & scalar field ϕ with potential $d\phi$ and $\vec{A} = \text{grad}\,\phi$

Eg – A vector field given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Show that the ~~field~~ vector is irrotational or field is conservative and find the scalar of potential.

Sol^n Given, $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \vec{0}$$

∴ The field is conservative or we can say that vector is irrotational.

Let ϕ be the scalar of potential.

$$\vec{A} = \text{grad}\,\phi$$

$$\Rightarrow (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Comparing both sides, we get

$$\frac{\partial\phi}{\partial x} = x^2 + xy^2 \qquad \frac{\partial\phi}{\partial y} = y^2 + x^2y \qquad \frac{\partial\phi}{\partial z} = 0$$

—— (1)

Now,

For scalar of potential,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

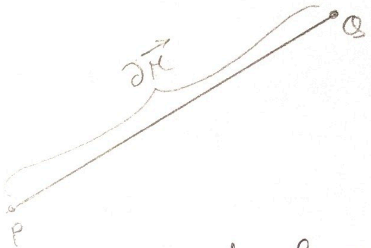
$$= (x^2 + xy^2)dx + (y^2 + x^2 y)dy + 0$$

$$d\phi = x^2 dx + \cancel{xy^2 dx} + y^2 dy + xy\underbrace{(ydx + xdy)}_{d(xy)}$$

Integrating both side, we get

$$\Rightarrow \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c$$

Directional derivative –



If ϕ be a scalar field, then $\frac{d\phi}{dr}$ is called directional derivative of ϕ at point P in the direction PQ.

The directional derivative $\frac{d\phi}{dr}$ is the resolved part of $\nabla\phi$ in the direction of an unit vector along PQ.

$$\frac{d\phi}{dr} = \frac{\partial \phi}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial \phi}{\partial y}\cdot\frac{dy}{dr} + \frac{\partial \phi}{\partial z}\cdot\frac{dz}{dr}$$

$$\Rightarrow \frac{d\phi}{dr} = \left(\frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}\right)\left(\frac{dx}{dr}\hat{i} + \frac{dy}{dr}\hat{j} + \frac{dz}{dr}\hat{k}\right)$$

$$\Rightarrow \frac{d\phi}{dr} = \nabla\phi\cdot\vec{b}, \text{ where } \vec{b} = \frac{dx}{dr}\hat{i} + \frac{dy}{dr}\hat{j} + \frac{dz}{dr}\hat{k}$$

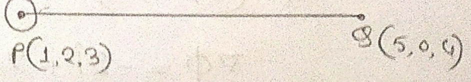
The directional derivative of ϕ in the direction PQ is given by

$$\frac{d\phi}{dr} = \nabla\phi\,\frac{\vec{b}}{|\vec{b}|}$$

$\nabla\phi$ gives the maximum rate of change of ϕ and maximum magnitude is $|\nabla\phi|$.

Eg – Find directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P(1,2,3) in the direction of PQ, where Q is the point (5,0,4)

Soln Given,



Scalar field is

$$\phi = x^2 - y^2 + 2z^2$$

Now, $\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 - y^2 + 2z^2)$

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\therefore$ At point $P(1,2,3)$

grad $\phi = 2\hat{i} - 4\hat{j} + 12\hat{k}$

Now, $\hat{b}$ is unit vector along $\vec{PQ}$

Now, $\vec{PQ} = 4\hat{i} - 2\hat{j} + 1\hat{k}$

$\therefore$ unit vector along PQ is $\hat{b}$

$$\hat{b} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$\therefore$ The directional derivative of ϕ at $P(1,2,3)$ in the direction $\vec{PQ}$ is

$$\frac{d\phi}{dr} = \nabla\phi\,\hat{b}$$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$$

$$= \frac{1}{\sqrt{21}}(8 + 8 + 12)$$

$$= \frac{28}{\sqrt{21}}$$, which is scalar.

And maximum value of directional derivative is

$$= |\nabla\phi| = \sqrt{4 + 16 + 144}$$

$$= \sqrt{164}$$

Eg- Find directional derivative for $f(x,y,z) = 2xy + z^2$ at $(1,-1,3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Sol^n Scalar field is

$$\phi = 2xy + z^2$$

Now, $\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2xy + z^2)$

$= 2y\hat{i} + 2x\hat{j} + 2z\hat{k}$

At $(1,-1,3)$

$\nabla\phi = -2\hat{i} + 2\hat{j} + 6\hat{k}$

Now,

unit vector along $\hat{i}+2\hat{j}+2\hat{k}$ is

$$\hat{b} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{9}}$$

$$= \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

The directional derivative of ϕ at $(1,-1,3)$ in the direction $(\hat{i}+2\hat{j}+2\hat{k})$ is,

$$\frac{d\phi}{dr} = \nabla\phi \cdot \hat{b}$$

$$= \frac{1}{3}(-2\hat{i}+2\hat{j}+6\hat{k}) \cdot (\hat{i}+2\hat{j}+2\hat{k})$$

$$= \frac{1}{3}(-2+4+12)$$

$$= 14/3$$

Eg- If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then prove that

i) $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0$ ~~iv) ∇~~

ii) $\nabla^2 r^n = n(n+1)r^{n-2}$

iii) $\nabla^2\left(\frac{1}{r}\right) = 0$

iv) $\nabla(r^n \cdot \vec{r}) = (n+3)r^n$

i) Proof $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = \nabla(r^{-3}\vec{r})$

$$= r^{-3}\nabla\vec{r} + \nabla r^{-3} \cdot \vec{r}$$

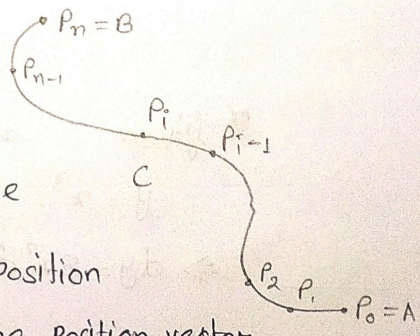
$$= r^{-3} \cdot \nabla\vec{r} + (-3)r^{-5}\vec{r}\vec{r} \quad [\because \nabla r^n = nr^{n-2}\vec{r}]$$

$$= r^{-3} \cdot 3 - 3 \cdot r^{-5} \cdot r^2$$

$$= 3r^{-3} - 3r^{-3}$$

$$= 0$$

Line Integral -



Consider a continuous vector function $\vec{F}(r)$ which is defined at each point of the curve in space. Divide C into n parts at the points $P_0 = A, P_1, P_2, \ldots, P_{n-1}, P_n = B$. Let their position vectors be $\vec{r_0}, \vec{r_1}, \vec{r_2}, \ldots \vec{r_n}$. Let $\vec{r}$, be the position vector of any point on the arc P_{i-1}, P_i. Now consider the sum.

$$S = \sum_{i=0}^{n} \vec{F}(r_i)\,\delta\vec{r_i}, \text{ where } \delta\vec{r_i} = \vec{r_i} - \vec{r}_{i-1}$$

The limit of the sum at $n \to \infty$ in such a way that $|\delta\vec{r_i}| \to 0$ is called the tangential line integral of $\vec{F}(r)$ along c is symbolically written as

$$\int_c \vec{F}(r)\,d\vec{r} \text{ or } \int_0 \vec{F}\,\frac{d\vec{r}}{dt}\,dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$

If $\vec{F}(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector function and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\int_c \vec{F}(r).\,d\vec{r} = \int_c (F_1 dx + F_2 dy + F_3 dz), \text{ where } d\vec{x} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

The other two types of line integrals are $\int_c \vec{F}\times d\vec{r}$ and $\int_c \phi\, d\vec{r}$, if $\vec{F}$ is replaced by a scalar function ϕ. Both are vectors.

Work $\to$ If $\vec{F}$ represents the forces acting on a particle moving along an arc AB, then the work done during small displacement $\delta\vec{r}$ is $\vec{F}.\delta\vec{r}$

The total work done by $\vec{F}$ during the displacement from A to B is given by line integral $\int_A^B \vec{F}.d\vec{r}$

Eg- i) If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$ evaluate $\int_c \vec{F}.d\vec{r}$ along the curve c in the xy plane $y = x^3$ from the point $(1,1)$ to $(2,8)$

Sol^n In the xy-plane let $\vec{r} = x\hat{i} + y\hat{j}$

Then $\vec{F}.\,d\vec{r} = \left[(5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}\right].(dx\,\hat{i} + dy\,\hat{j})$

$$= (5xy - 6x^2)dx + (2y - 4x)dy \quad \text{—— ①}$$

Given $y = x^3$

$\Rightarrow dy = 3x^2 dx$

① $\Rightarrow \vec{F}.d\vec{r} = (5x^4 - 6x^2)dx + (2x^3 - 4x)\,3x^2 dx$

$\therefore \int_c \vec{F}.d\vec{r} = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$

$= 35$

Eg:2 – A vector field is given by $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate the line integral over a circular path $x^2+y^2=a^2$, $z=0$

Sol^n We take, $\vec{r} = x\hat{i} + y\hat{j}$

$$\text{So, } \vec{F}\cdot d\vec{r} = \sin y\,dx + x(1+\cos y)\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (\sin y\,dx + x\cos y\,dy) + \int_C x\,dy$$

$$= \int_C x\sin y\,dx + \int_C x\,dy$$

For circular path

let, $x = a\cos t$, $y = a\sin t$, $z=0$

where t varies from 0 to 2π

Then, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$

~~So, dx = -a sint dt, dy~~

$$\text{So, } \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} d\{a\cos t\sin(a\sin t)\} + \int_0^{2\pi} a\cos t\,(a\cos t)\,dt$$

$$= \Big[a\cos t\sin(a\sin t)\Big]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2 t\,dt$$

$$= \Big[a\cos 2\pi\sin(a\sin 2\pi) - 0\Big] + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$

$$= 0 + \frac{a^2}{2}\left[t + \frac{\sin t}{2}\right]_0^{2\pi}$$

$$= \cancel{0}\ \frac{a^2}{2}(2\pi + 0) = \pi a^2$$

Eg 3) If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ evaluate $\int_C \vec{F}\times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t=0$ to $t = \pi/2$

Sol^n
$$\vec{F}\times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= \hat{i}(-z\,dz - x\,dy) + \hat{j}(x\,dx - 2y\,dz) + \hat{k}(2y\,dy + z\,dx)$$

Now, replace,

x by $\cos t$, dx by $(-\sin t\,dt)$

y by $\sin t$, dy by $(\cos t\,dt)$

z by $2\cos t$, dz by $(-2\sin t\,dt)$

Then integrate $\int_0^{\pi/2} \vec{F} \times d\vec{r} = \ldots\ldots$

Ans $= (2 - \pi/c)\hat{i} + (-\frac{1}{2} + \pi)\hat{j}$

Eg 4) Find the work done by the force $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in the displacement along the closed path c consisting of the segments c_1, c_2, c_3, where

On c_1 : $0 \leq x \leq 1$, $y = x, z = 0$

On c_2 : $0 \leq z \leq 1$, $x = 1, y = 1$

On c_3 : $1 \geq x \geq 0$, $y = x = z$

Sol^n Here $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

The work done $= \oint_c \vec{F}.d\vec{r}$

$$= \oint_c (xdx - zdy + 2ydz)$$

On c_1 : $dy = dx$, $dz = 0$, and x varies from 0 to 1. (as $z = 0$)

on c_2: $dx = 0$, $dy = 0$ & z varies from 0 to 1

On c_3: $dy = dx = dz$ and x varies from 1 to 0

$$\therefore \oint_c \vec{F}.d\vec{r} = \left(\int_{c_1} + \int_{c_2} + \int_{c_3}\right) \vec{F}.d\vec{r}$$

$$= \int_0^1 xdx + \int_0^1 2dz + \int_1^0 (xdx - xdx + 2xdx)$$

$$= \left[\frac{x^2}{2}\right]_0^1 + [2z]_0^1 + \left[\frac{2x^2}{2}\right]_0^1$$

$$= \frac{1}{2} + 2 - 1$$

$$= 3/2$$

Eg5- Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to 2

Ans 303

Eg-6 - Find the work done in moving a particle once around a circle in the x-y plane, if the circle has centre at the origin and radius 3 and if the force field is given by $\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$

Hint - In xy- plane $\Rightarrow z=0$

For the circle $x = 3\cos\theta$, $y = 3\sin\theta$ where $0 \le \theta \le 2\pi$

$\vec{r} = x\hat{i} + y\hat{j}$, $\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$

Ans is 18π

Eg-7 - If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 120xz^2\hat{k}$ evaluate $\int \vec{A}\cdot d\vec{r}$, from $(0,0,0)$ to $(1,1,1)$ along the following path c.

i) $x=t$, $y=t^2$, $z=t^3$

ii) The straight line from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

iii) The straight line joining $(0,0,0)$ and $(1,1,1)$

Sol^n Here, $\int_c \vec{A}\cdot d\vec{r} = \int \left[(3x^2+6y)dx - (14yz)dy + (20xz^2)dz\right]$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

i) Put $x=t$, $y=t^2$, $z=t^3$

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$

The points $(0,0,0)$ and $(1,1,1)$ corresponds to $t=0$ to $t=1$

$$\therefore \int_c \vec{A}\cdot d\vec{r} = \int_0^1 \left[(3t^2+6t^2)\,dt + (-14t^5)\cdot 2t\,dt + (20t^7)\,3t^2\,dt\right]$$

$$= \int_0^1 (3t^2 - 28t^6 + 60t^9)\,dt$$

$$= 5$$

ii) Along the straight line from $(0,0,0)$ to $(1,0,0)$, $y=0$, $z=0$, $dy=0$, $dz=0$ and x varies from 0 to 1.

Then the integral over this part of the path is

$$\int_{x=0}^{1} 3x^2\,dx = 1$$

Along the st line from $(1,0,0)$ to $(1,1,0)$, $x=1$, $z=0$, $dx=0$, $dz=0$ and y varies from 0 to 1.

Then the integral over this part of the path is 0.

Along the straight line from $(1,1,0)$ to $(1,1,1)$, $x=1$, $y=1$ and $dx=0$, $dy=0$, z varies from 0 to 1. Then the integrale over this part of the path is $\int_0^1 20z^2\,dz = 20\left(\frac{z^3}{3}\right)_0^1 = \frac{20}{3}$

$\therefore$ By adding we get

$$\int_C \vec{A}\cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

ii) The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric form by $x=t$, $y=t$ and $z=t$, where t varies from 0 to 1

$\because$ limits are same, so $x=t$, $y=t$ & $z=t$

$$\therefore \int_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{1} \left\{(3t^2+6t)\,dt - 14t^2\,dt + 20t^3\,dt\right\}$$

$$= \frac{13}{3}$$

H/w – 1) If $\vec{F} = xy\,\hat{i} + (x^2+y^2)\,\hat{j}$ obtain $\int_C \vec{F}\cdot d\vec{r}$ where c is the arc of the parabola $y = x^2 - 4$ from $(2,0)$ to $(4,12)$

2) If $\phi = 2xyz^2$, and c is the curve, $x=t^2$, $y=2t$, $z=t^3$ from $t=0$ to $t=1$, evaluate the live integral $\int_C \phi\, d\vec{r}$ $(\vec{r} = x\hat{i} + y\hat{j} + z\hat{k})$

Scalar fields and vector fields –

If to each point $P(x,y,z)$ of a region R in space, there correspond a unique ~~region R~~ $f(P)$, then $f(P)$ is called a scalar point function and we say that a scalar field f has been defined in R.

For eg –