Diffl eqns of 1st order:

Defn: A d.e. is an eqn involving differentials or diffl coeffs. Thus

(1) $\frac{dy}{dx} = x^2 - 1$, (2) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$,

(3) $(x^2+y^2-2y)\,dx = (x^2+3x+y)\,dy$

(4) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ etc are all d.eqns.

(i) Diffl eqns which involve only one indep. var and the diffl coeffs w.r.t. it are called ordinary d.eqns.

The eqns (1) ~~to~~ (3) are all o.d. eqns.

(ii) Eqn (4) which involve more than one indep. var is called p.d.e.

(iii) The order of a d.e. is the order of the highest ordered derivative occuring in the d.e.

Eqn (2) is a 2nd order d.e.

§ Linear Eq^{n}s : A d.e. is said to be linear if the dep. var. and its differential coefs. occur only in the Ist degree and not multiplied together.

Thus the standard form of a linear eq^{n} of the Ist order, commonly known as Leibnitz's linear eq^{n} is $\frac{dy}{dx} + Py = Q \longrightarrow (1)$

where P and Q are f^{n}s of x.

To solve the eq^{n}, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y\left(e^{\int P dx} P\right) = Q e^{\int P dx}$$

i.e, $$\frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating both sides we get,

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

as the reqd. sol^{n}.

Obs: The factor $e^{\int P dx}$ is called the Integrating factor (I.F.) of the linear eq^{n} (1).

Imp Remember that $I.F. = e^{\int P dx}$

and sol^{n} is $y(I.F.) = \int Q (I.F.) dx + c$

Ex1 Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2 \longrightarrow (1)$

Solⁿ $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$

Which is Leibnitz's eqⁿ.

Here $P = -\frac{1}{x+1}$

$\int P dx = -\int \frac{1}{x+1} dx = -\log(x+1) = \log(x+1)^{-1}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$.

Thus the solⁿ of (1) is

$$y(I.F.) = \int e^{3x}(x+1)(I.F.)dx + c$$

or, $\frac{y}{x+1} = \int e^{3x}(x+1)\frac{1}{x+1}dx + c$

$= \int e^{3x} dx + c = e^{3x}/3 + c.$

Or, $y = \left(\frac{1}{3}e^{3x} + c\right)(x+1)$. //

Ex. Solve the following d. eqⁿs:

1. $\cos^2 x \frac{dy}{dx} + y = \tan x$
2. $x\log x \frac{dy}{dx} + y = \log x^2$
3. $\frac{dy}{dx} = x^3 - 2xy$, if $y=2$ when $x=1$.
4. $(1+x^3)\frac{dy}{dx} + 6x^2 y = e^x$

Solⁿ 1. $\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$

or, $\frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x$

It is Leibnitz's eqⁿ.

I.F. $= e^{\int \sec^2 x\, dx} = e^{\tan x}$

$\therefore$ Solⁿ is $\quad y.e^{\tan x} = \int e^{\tan x} \tan x \sec^2 x\, dx$

put $\tan x = t$
$\sec^2 x\,dx = dt$.

$\therefore \int e^{\tan x} \tan x \sec^2 x\,dx = \int e^t\, t\,dt = te^t - \int e^t dt = e^t(t-1) + c$

$\therefore \; y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

or, $y = (\tan x - 1) + c e^{-\tan x}$ //

2. $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\log x^2}{x\log x} = \frac{2}{x}$

I.F. $= e^{\int \frac{1}{x\log x}dx} = \cancel{e^{(\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot x\,dx}} = e^{\frac{x^2}{2}\log x - x}$

$\therefore$ ~~Sol^n is~~ $\cancel{y.(e^{\frac{x^2}{2}\log x - x}) = \int \frac{2}{x}}$

put $\log x = t$

$\therefore \frac{1}{x}dx = dt$

$\therefore$ I.F. $= e^{\int \frac{dt}{t}} = e^{\log t} = t = \log x$

$\therefore$ Sol^n is $\quad y.\log x = \int \log x \frac{2}{x}dx = 2\int t\,dt$

$= 2\frac{t^2}{2} + c$

$= (\log x)^2 + c$

$\therefore y = \log x + c\log x^{-1}$. //

4. $(1+x^3)\frac{dy}{dx} + 6x^2 y = e^x$

or, $\frac{dy}{dx} + \frac{6x^2 y}{1+x^3} = \frac{e^x}{1+x^3}$

$P = \frac{6x^2}{1+x^3}$, I.F. $= e^{\int \frac{6x^2}{1+x^3}dx} = e^{2\log(1+x^3)}$

$= (1+x^3)^2$

$\therefore$ Sol^n is $y(1+x^3)^2 = \int (1+x^3)^2 \frac{e^x}{1+x^3}dx = \int (1+x^3)e^x dx$

$= e^x(x^3 - 3x^2 + 6x - 6)$

5. $\frac{dy}{dx} = \frac{x \cancel{+} y\cos x}{1+\sin x}$ $\qquad \frac{dy}{dx} + Py = Q$

6. $dr + (2r\cot\theta + \sin 2\theta)\, d\theta = 0 \Rightarrow \frac{dr}{d\theta} \# 2r\cot\theta \pm \sin 2\theta$

7. $ye^y dx = (y^3 + 2xe^y) dy \Rightarrow \frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y}$

8. $e^{-y}\sec^2 y\, dy = dx + x\, dy$ //

§ Bernoulli's eqn:

The eqn $\frac{dy}{dx} + Py = Qy^n \longrightarrow (1)$

where P, Q are f^ns of x, is reducible to the Leibnitz's linear eqn and is usually called the Bernoulli's eqn.

To solve (1) divide both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \longrightarrow (2)$$

put $y^{1-n} = z$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (2) becomes, $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

or, $\frac{dz}{dx} + P(1-n)z = Q(1-n)$

which is Leibnitz's linear in z and can be solved easily. //.

Ex 1 Solve $x\frac{dy}{dx} + y = x^3y^6$

Dividing both sides by xy^6,

$$y^{-6}\frac{dy}{dx} + \cancel{x}\frac{y^{-5}}{x} = x^2$$

put $y^{-5} = z$ so that $-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore -\frac{1}{5}\frac{dz}{dx} + \frac{1}{x}z = x^2$

or, $\frac{dz}{dx} - \frac{5}{x}z = -5x^2$

which is Leibnitz's linear in z.

$$I.F. = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = x^{-5}$$

$\therefore$ sol^n is $z x^{-5} = \int -5x^2 \cdot x^{-5}\,dx$

$$= -5\int x^{-3}\,dx$$

$$\text{or, } y^{-5}x^{-5} = -5\,\frac{x^{-2}}{-2} + c$$

Dividing by $x^{-5}y^{-5}$ we get,

$$1 = \frac{5}{2}x^3y^5 + c\,x^5 y^5$$

$$= x^3y^5\left(\frac{5}{2} + cx^2\right) \;//$$

Ex Solve the following eq^{n}s:

1. $\frac{dy}{dx} + y\tan x = y^3\sec x$
2. $r\sin\theta - \frac{dr}{d\theta}\cos\theta = r^2$
3. $2xy' = 10x^3y^5 + y$
4. $(x^3y^2 + xy)dx = dy$
5. $\frac{dy}{dx} = \frac{x^2 + \cancel{y^2+1}}{2xy}$.

Sol^n ① $y^{-3}\frac{dy}{dx} + y^{-2}\tan x = \sec x$

Let $y^{-2} = z$, then $-2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ ① ~~be~~comes, $-\frac{1}{2}\frac{dz}{dx} + z\tan x = \sec x$

or, $\frac{dz}{dx} - 2z\tan x = -2\sec x$ $\boxed{\text{form } \frac{dz}{dx} + Pz = Q}$

which is linear.

$$I.F. = e^{\int -2\tan x\,dx} = e^{-2\log\sec x} = \sec^{-2}x$$

$\therefore$ sol^n is $z.\sec^{-2}x = -2\int \sec x \sec^{-2}x\,dx + c$

$$= -2\int \sec^{-1}x\,dx = -2\int \cos x\,dx + c$$

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$$-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$$

$$\frac{dy}{dx} + Py = Q$$

$$y.\,I F = \int Q\, I.F.\, dx$$

or, $\frac{dr}{d\theta} - \tan\theta\, r = -\sec\theta\, r^2$

Which is linear.

$I.F. = e^{\int -\tan\theta\, d\theta} = e^{-\log\sec\theta} = \cos\theta$

$\therefore$ solⁿ is $r\cos\theta = \int -\sec\theta\, r^2 \cos\theta\, d\theta + c$

$= \int r^2(-d\theta) + c$

$= r^2\theta + c$

Ans; $\frac{1}{r} = \sin\theta + c\cos\theta$.

$r^{-2}\frac{dr}{d\theta} - \tan\theta\, r^{-1} = -\sec\theta \rightarrow (1)$

put $r^{-1} = z$ so that $-r^{-2}\frac{dr}{d\theta} = \frac{dz}{d\theta}$

$\therefore$ (1) becomes,

$-\frac{dz}{d\theta} - \tan\theta\, z = -\sec\theta$

or, $\frac{dz}{d\theta} + \tan\theta\, z = \sec\theta$

Which is linear.

$I.F. = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$

$\therefore$ solⁿ is $z\sec\theta = \int \sec^2\theta\, d\theta + c$

or, $z\frac{1}{\cos\theta} = \tan\theta + c$

or, $r^{-1} = \sin\theta + c\cos\theta$ //

3. $2x\frac{dy}{dx} = 10x^3y^5 + y$ | $\frac{dy}{dx} + Py = Qy^n$

or, $\frac{dy}{dx} = 5x^2y^5 + \cancel{2}\frac{y}{2x}$

or, $\frac{dy}{dx} - \cancel{2}\frac{y}{2x} = 5x^2y^5$

or, $y^{-5}\frac{dy}{dx} - \frac{\cancel{2}}{2x}y^{-4} = 5x^2 \longrightarrow (i)$

Now put $y^{-4} = z$

so that $-4y^{-5}\frac{dy}{dx} = \frac{dz}{dx}$

$(i) \Rightarrow -\frac{1}{4}\frac{dz}{dx} - \frac{\cancel{2}}{2x}z = 5x^2$

or, $\frac{dz}{dx} + \frac{2}{x}z = -20x^2$ which is linear

$I.F. = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2$

$\therefore$ sol^n is $z.x^2 = \int -20x^2x^2\,dx + c$

$= -\cancel{20}^{4}\frac{x^5}{\cancel{5}} + c$

$\therefore y^{-4}x^2 = -4x^5 + c$

or, $x^2 + (4x^5 - c)y^4 = 0$ //

6. $\sec^2 y\frac{dy}{dx} + x\tan y = x^3$

7. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$

Solⁿ 7

we can write $\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y}\frac{1}{1+x} = (1+x)e^{x}$

or, $\cos y \frac{dy}{dx} - \sin y \frac{1}{1+x} = (1+x)e^{x} \rightarrow$ (i)

put $\sin y = z$, so that $\cos y \frac{dy}{dx} = \frac{dz}{dx}$

(i) becomes,

$$\frac{dz}{dx} - \frac{1}{1+x}z = (1+x)e^{x}$$

which is linear.

$$\text{I.F.} = e^{\int -\frac{1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1+x}.$$

$\therefore$ solⁿ is $z.\frac{1}{1+x} = \int \frac{1}{1+x}(1+x)e^{x}dx + c$

$$= e^{x} + c$$

$\therefore$ $z = (e^{x}+c)(1+x)$

or, $\sin y = (e^{x}+c)(1+x)$ //

§ Exact diff eqⁿs

① Defⁿ: A d.e. of the form $M_{(x,y)}dx + N_{(x,y)}dy = 0$ is said to be exact if its left hand member is the exact differential of some fⁿ $u(x,y)$

i.e, $du = Mdx + Ndy = 0$.

Its solⁿ is $u(x,y) = c$.

② Thᵐ: The nec. & suff. condⁿ for the d.e. $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(3). Method of solⁿ:

The solⁿ of $Mdx + Ndy = 0$ is

$$\int_{(y\ \text{const})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c,$$

provided $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Ex1. Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

It can be written as

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$$

Here $M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$.

$\therefore \dfrac{\partial M}{\partial y} = \cos x + \cos y + 1$

$\dfrac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Thus the eqⁿ is exact and its solⁿ is

$$\int_{(y\ \text{const})} M\,dx + \int (\text{terms of } N \text{ without } x)\,dy = c$$

i.e., $y\sin x + (\sin y + y)x + 0 = c$. //

Ex Solve the following eqⁿs:

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy$
2. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - c^2)y\,dy = 0$.
3. $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$