Integral Calculus

Date : 06-11-20

Reduction formulae:

A reduction formula is a formula which connects an integral, which can't otherwise be integrated, (as for example $x^n e^{ax}$, $\tan^3 x$, $(x^2+a^2)^{n/2}$, $\sin^3 x$, $\cos^3 x$ etc) with other integral of the same type but of lower degree. It is generally obtained by applying the rule of integration by parts.

1) Reduction formula for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$, n being a +ve integer

a) Let $I_n = \int \sin^n dx$

$$= \int \sin^{n-1} x \sin x\,dx$$

Integrating by parts we get,

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos(-\cos x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

or $(1+n-1)I_n = -\sin^{n-1} x\cos x + (n-1) I_{n-2}$

$$\Rightarrow I_n = \frac{-\sin^{n-1}\cos x}{n} + \frac{n-1}{n} I_{n-2}$$

b) Let $I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$

$$= \ldots\ldots = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

To evaluate $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ and $\int_0^{\pi/2} \cos^n x\,dx$

we have $\int \sin^n x\,dx = -\frac{\sin^{n-1}\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n x\,dx = -\left[\frac{\sin^{n-1}\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx \quad \text{—— (1)}$$

Putting $n = n-2$ in (1) we get

$$\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx = \frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4} x\,dx$$

Substituting this value in ① we get,

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \int_0^{\frac{\pi}{2}} \sin^{n-4} x\,dx$$

$$= \frac{n-1}{2} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \int_0^{\frac{\pi}{2}} \sin^{n-6} x\,dx \quad \text{(II)}$$

and so on

Now, two cases arise, n is even or odd.

Case I n is odd.

In this case by the repeated application of reduction formula ① the last integral of (II) is

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

Hence when n is odd, from (II) we ~~get~~ have

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}$$

Case II when n is even.

In this case the last integral of (II) is $\int_0^{\frac{\pi}{2}} \sin^0 x\,dx = \frac{\pi}{2}$

Hence from (II) we get

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \cdot \frac{\pi}{2}$$

If we evaluate $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$, we get the same result.

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

Eg - i) Evaluate (i) $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$

Solⁿ $I_7 = \frac{(7-1)(7-3)(7-3)}{7(7-2)(7-4)(7-6)} = \frac{2\times 4\times \cancel{6}^{2}}{7\times5\times\cancel{3}\times1} = \frac{16}{35}$

ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx$

Solⁿ $I_8 = \frac{(8-1)(8-3)(8-5)(8-7)}{8\times(8-2)(8-4)(8-6)} \times \frac{\pi}{2}$

$$= \frac{7\times5\times\cancel{3}\times1}{8\times\cancel{6}_{2}\times4\times2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}$$

2) Reduction formula for $\int x^n e^{ax}\,dx$

Let $I_n = \int x^n e^{ax}\,dx$

Integrating by parts we get,

$$I_n = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a}\,dx$$

or $$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Ex: Find $\int x^3 e^{ax}\,dx$

$$I_3 = \int x^3 e^{ax}\,dx = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^2}\left[x\frac{e^{ax}}{a} - \frac{1}{a}\int e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^2}\left[\frac{x e^{ax}}{a} - \frac{1}{a} e^{ax}\right]$$

$$= \frac{e^{ax}}{a^4}\left(a^3x^3 - 3x^2a^2 + 6ax - 6\right)$$

3) Reduction formula for $\int \tan^n x\,dx$

Let $$I_n = \int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

Also taking limit from 0 to $\pi/4$ we get

$$\int_0^{\pi/4} \tan^n x\,dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1} - I_{n-2}$$

Ex: Evaluate i) $\int \tan^5 x\,dx$

Solⁿ $$I_5 = \int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \int \tan^3 x\,dx$$

$$= \frac{\tan^4 x}{4} - \left[\frac{\tan^2 x}{2} - \int \tan x\,dx\right]$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x|$$

ii) $\int \tan^4 x\,dx = \dfrac{\tan^3 x}{3} - \int \tan^2 x\,dx$

$= \dfrac{\tan^3 x}{3} - \int(\sec^2 x - 1)\,dx$

$= \dfrac{\tan^3 x}{3} - \tan x + x$

Note: Similarly reduction formula for $\int \cot^n x\,dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x$

Ex If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, P.T.

i) $I_n + I_{n-2} = \dfrac{1}{n-1}$

ii) $n(I_{n-1} + I_{n+1}) = 1$

Solⁿ $I_n = \int_0^{\pi/4} \tan^n x\,dx = \left[\dfrac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\,dx$

$I_n = \dfrac{1}{n-1} - I_{n-2}$

$\therefore I_n + I_{n-2} = \dfrac{1}{n-1}$ — (i)

Replacing n by $n+1$, $I_{n+1} + I_{n-1} = \dfrac{1}{n}$ — (ii)

4) Reduction formula for $\int \sec^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$

Let $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \tan x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n+2)\int \sec^{n-2} x\,dx$

or $(1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or $I_n = \dfrac{1}{n-1} \sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$

ie. $\int \sec^n x\,dx = \dfrac{1}{n-1}\sec^{n-2} x \tan x + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$

Similarly. $\int \operatorname{cosec}^n x\,dx = -\dfrac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx$

Ex Evaluate

a) $\int \sec^3 x\,dx$

Solⁿ $I_3 = \int \sec^3 x\,dx = \dfrac{1}{2}\sec x \tan x + \dfrac{1}{2}\int \sec x\,dx$

$= \dfrac{\sec x \tan x}{2} + \dfrac{1}{2}\log|\sec x + \tan x|$

b) $\int_0^{\pi/4} \sec^3 x \, dx$

$= \left[\frac{\sec x \tan x}{2}\right]_0^{\pi/4} + \frac{1}{2}\left[\log|\sec x + \tan x|\right]_0^{\pi/4}$

$= \frac{1}{2}(\sqrt{2} - 0) + \frac{1}{2}\left[\log|\sqrt{2}+1| - \log 1\right]$

$= \frac{1}{2}\left(\sqrt{2} + \log(\sqrt{2}+1)\right)$

5) Reduction formula for $\int (x^2+a^2)^n \, dx$

Let $I_n = \int (x^2+a^2)^n \, dx$

Integrating by parts taking 1 as second factor

$I_n = (x^2+a^2)^n x - \int n(x^2+a^2)^{n-1} . 2x . x \, dx$

$= x(x^2+a^2)^n - 2n \int (x^2+a^2)^{n-1}(x^2+a^2-a^2) \, dx$

$= x(x^2+a^2)^n - 2n \int (x^2+a^2)^n dx + 2n a^2 \int (x^2+a^2)^{n-1} dx$

or $(1+2n) I_n = x(x^2+a^2) + 2na^2 I_{n-1}$

$\therefore I_n = \frac{x(x^2+a^2)^n}{1+2n} + \frac{2na^2}{1+2n} I_{n-1}$

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$I_{m,n} = \int \sin^m x \cos^n x \, dx$

$= \int \sin^{m-1} x \sin x \cos^n x \, dx$ (Int by parts)

$= \sin^{m-1} x \left(-\frac{\cos^{n+1} x}{n+1}\right) - \int (m-1)\sin^{m-2} x \cos x \left(-\frac{\cos^{n+1} x}{n+1}\right) dx$

$= -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} \int \sin^{m-2} \cos^{n+2} x \, dx$

$= -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} \int [\sin^{m-2} x \cos^n x (1-\sin^2 x) dx$

$= -\frac{1}{n+1} \sin^{m-1} \cos^{n+1} x + \frac{m-1}{n+1}\left[\int \sin^{m-2} x \cos^n x \, dx - \int \sin^m x \cos^n x \, dx\right]$

or $\left(1+\frac{m-1}{n+1}\right) I_{m,n} = -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \, dx$

or $I_{m,n} = -\frac{1}{m+n} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$

or $I_{m,n} = -\frac{1}{m+n} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}$ —— (1)

To show that $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \dfrac{(m-1)(m-3)\ldots (n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots} \times \dfrac{\pi}{2}$

if both m & n are even.

From ①, $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = -\dfrac{1}{m+n} \left[\sin^{m-1} x \cos^{n+1} x\right]_0^{\frac{\pi}{2}} + \dfrac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x \, dx$

or $I_{m,n} = \dfrac{m-1}{m+n} I_{m-2,n}$ — (ii)

Replacing m by m-2 in (ii) we get,

$I_{m-2,n} = \dfrac{m-3}{m+n-2} I_{m-4,n}$

Thus, $I_{m-4,n} = \dfrac{m-5}{m+n-4} I_{m-6,n}$ and so on.

$\therefore I_{m,n} = \dfrac{(m-1)(m-3)(m-5)\ldots}{(m+n)(m+n-2)(m+n-4)} I_{m-6,n}$ — (iii)

Case I Let m be even.

$\Rightarrow$ Then $I_{m,n} = \dfrac{(m-1)(m-3)(m-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} I_{0,n}$ — (iv)

where $I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$= \dfrac{(n-1)(n-3)\ldots 2}{n(n-2)(n-4)\ldots 3}$ for n odd

$= \dfrac{(n-1)(n-3)\ldots 1}{n(n-2)(n-4)\ldots 2} \dfrac{\pi}{2}$ for n even.

So, when m & n both even

$I_{m,n} = \dfrac{\{(m-1)(m-3)\ldots\}\{(n-1)(n-3)\ldots 1\}}{\{(m+n)(m+n-2)\ldots(n+2)\}\, n(n-2)\ldots 2} \dfrac{\pi}{2}$ — (v)

~~So, coh~~

If m is even and n is odd,

(iv) $\Rightarrow I_{m,n} = \dfrac{\{(m-1)(m-3)\ldots 1\}\{(n-1)(n-3)\ldots 2\}}{(m+n)(m+n-2)(m+n-4)\ldots(n+2)\,n(n-2)\ldots 3}$ — (vi)

Case II Let m be odd

Then $I_{m,n} = \dfrac{(m-1)(m-3)\ldots}{(m+n)(m+n-2)\ldots} I_{1,n}$

where $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx = -\left[\dfrac{\cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} = \dfrac{1}{n+1}$

$\therefore I_{m,n} = \dfrac{\{(m-1)(m-3)\ldots 2.1\}\{(n-1)(n-3)\ldots\}}{\{(m+n)(m+n-2)\ldots(n+1)\}\{(n-1)(n-3)\ldots\}}$

Rule $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \dfrac{\{(m-1)\cdots 2\}\{(n-1)\cdots 2\}}{(m+n)\cdots 2}$

at the end write $\frac{\pi}{2}$ if m, n both are even, otherwise no $\frac{\pi}{2}$

Eg: Integrals

i) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx = \dfrac{5.3.1.\,3.1}{10.8.6.4.2} \cdot \dfrac{\pi}{2} = \dfrac{3\pi}{512}$

ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta = \dfrac{6.4.2.4.2}{12\times 10.8.6.4.2} = \dfrac{1}{120}$

iii) $\int_0^{\infty} \dfrac{t^6 \, dt}{(1+t^2)^{\cancel{7}\,8}}$ Put $t = \tan\theta$ $\quad t=0 \Rightarrow \theta = 0$

$\therefore dt = \sec^2\theta d\theta \quad t = \infty \Rightarrow \theta = \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} \dfrac{\tan^6\theta \sec^2\theta \, d\theta}{\sec^{14}\theta}$

$= \int_0^{\frac{\pi}{2}} \dfrac{\tan^6\theta}{\sec^{12}\theta} d\theta = \int_0^{\frac{\pi}{2}} \dfrac{\sin^6\theta}{\cos^6\theta} \cos^{12}\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^6\theta \, d\theta$

$= \cdots\cdots = \dfrac{5\pi}{2048}$

iv) $\int_0^{\infty} \dfrac{\quad}{(1+x^2)^{9/2}}$

v) $\int_0^{\frac{\pi}{6}} \cos^4 3\theta \sin^3 6\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \cos^4 3\theta \, (2\sin 3\theta \cos 3\theta)^3 d\theta$

$= 8\int_0^{\frac{\pi}{6}} \cos^7 3\theta \sin^3 3\theta \, d\theta$ Put $3\theta = x$

then $d\theta = \frac{1}{3} dx$

$\Rightarrow x = 0$

$= \cdots = \dfrac{1}{15}$ $\quad \theta = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{2}$

7) Reduction formula for

a) $\int x^2 \sin mx \, dx$

b) $\int x^3 \cos mx \, dx$

c) $\int \cos^m x \sin nx \, dx$

Let $I_n = \int x^n \sin mx\, dx$

$= x^n\left(-\frac{\cos mx}{m}\right) - \int n x^{n-1}\left(-\frac{\cos mx}{m}\right)dx$

$= -\frac{1}{m}x^n \cos mx + \frac{n}{m}\int x^{n-1}\cos mx\, dx$

$= -\frac{1}{m}x^n \cos mx + \frac{n}{m}\left[x^{n-1}\frac{\sin mx}{m} - \int (n-1)x^{n-2}\frac{\sin mx}{m}dx\right]$

$= -\frac{1}{m}x^n \cos mx + \frac{n}{m^2}x^{n-1}\sin mx - \frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\, dx$

Or $I_n = -\frac{1}{m}x^n \cos mx + \frac{n}{m^2}x^{n-1}\sin mx - \frac{n(n-1)}{m^2} I_n$

similarly, b) $\int x^n \cos mx\, dx$

$= \frac{1}{m}x^n \sin mx + \frac{n}{m^2}x^{n-1}\cos mx - \frac{n(n-1)}{m^2} I_{n-2}$

9) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1}x(-\sin x)\cdot\left(-\frac{\cos nx}{n}\right)dx$

$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x \sin x \cos nx\, dx$

Since $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

$\therefore I_{m,n} = -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1}x\,(\sin nx \cos x - \sin(n-1)x)dx$

$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^m x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1}x \sin(n-1)x\, dx$

or $\left(1+\frac{m}{n}\right) I_{m,n} = -\frac{1}{n}\cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$

ie. $I_{m,n} = \frac{-1}{m+n}\cos^m x \cos nx + \frac{m}{m+n} I_{m-1,n-1}$

Ex. Show that $\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx = \frac{m}{m+n}\int_0^{\frac{\pi}{2}} \cos^{m-1}x \cos(n-1)x\, dx$

Hence deduce that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

Let $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx$

$= \left[\cos^m x \frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} m\cos^{m-1}x(-\sin x)\frac{\sin nx}{n}dx$

$= 0 + \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1}x \sin x \sin nx\, dx$

$= \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1}x\,(\cos(n-1)x - \cos nx \cos x)dx$

$$= \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \cos(n-1)x\,dx - \frac{m}{2}\int_0^{\frac{\pi}{2}} \cos^m x \cos^n x\,dx$$

or $\left(1+\frac{m}{n}\right) I_{m,n} = \frac{m}{n}\int_0 \ldots$

or $I_{m,n} = \frac{m}{m+n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \cos(n-1)x\,dx$

when $m=n$, $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,dx = \frac{n}{2n}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x\,dx$

or $I_{n,n} = \frac{1}{2} I_{n-1,n-1}$

Replacing n by $n-1$ we get

$I_{n-1,n-1} = \frac{1}{2} I_{n-2,n-2}$

Parallaly, $I_{n-2,n-2} = \frac{1}{2} I_{n-3,n-3} = \ldots = \frac{1}{2} I_{0,0}$

$\therefore I_{n,n} = \frac{1}{2^n} I_{00} = \frac{1}{2^n}\int_0^{\frac{\pi}{2}} (\cos x)\,dx = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}$

Ex Find $\int_0^{\frac{\pi}{2}} \cos^5 x \sin nx\,dx$

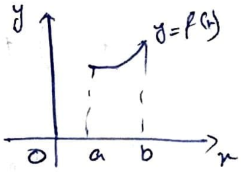
From (c) we get $I_{m,n} = -\left[\frac{\cos^m x \cos nx}{m+n}\right]_0^{\frac{\pi}{2}} + \frac{m}{m+n} I_{m-1,n-1}$

Date 13-11-20

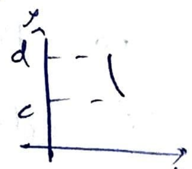
Application of definite integrals to evaluate surface area and Volumes of solids of revolution

Area of plane curves

i) The area enclosed by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ & $x=b$ is $\int_a^b y\,dx$

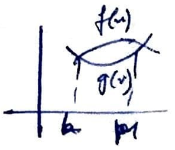


ii) The area enclosed by the curve $x=f(y)$, the y-axis and the abscissa $y=c$ & $y=d$ is $\int_c^d x\,dy$

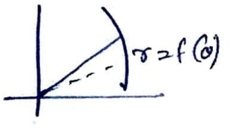


iii) The area between the curve $y=f(x)$ & $y=g(x)$ is

$\int_a^b [f(x)-g(x)]\,dx$

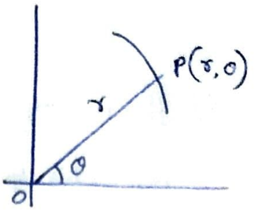


iv) The area bounded by the curve $r=f(\theta)$, radius vector $\theta=\alpha$ & $\theta=\beta$ is $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$



Polar co-ordinates –

If P be any pt on x-y plane, then its position can be indicated in polar form by setting



i) It's distance OP = r called the radius vector

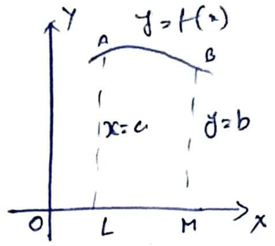
ii) ∠θ is called the vertical angle of P

iii) The line OX is called the initial line the fixed Pt. O is called the pole and (r,θ) are called polar co-ordinates of P

If (x,y) be the cartesian co-ordinates of P, then $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$, $\theta = \tan^{-1}\frac{y}{x}$

Length of curves



i) Length of arc of the curve $y = f(x)$ between the pts whose ordinates are a & b is

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

ii) If $x = \phi(y)$ then $s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

iii) If $x = f_1(t)$, $y = f_2(t)$ then $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

iv) In polar co-ordinates $s = \int_{\alpha_1}^{\alpha_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

or $s = \int_{r_1}^{r_2} \sqrt{1+\left(\frac{r\,d\theta}{dr}\right)^2}\, dr$

Curve Tracing →

In many practical applications a knowledge about the shapes of given eq^n is desirable.

In cartesian co-ordinate

i) Symmetry:

a) If the eq^n of the curve involves even power of y then the curve is symmetrical about the x-axis. Eg: $y^2 = 4ax$ is symmetrical about the x-axis

b) If the eq^n of the curve involves even power of x then the curve is symmetrical about y-axis. Eg $x^2 = 4ay$

c) If the eq^n of d curve remains unaltered when x & y are interchanged, then the curve is symmetrical about the line $y = x$. Eg $x^3 + y^3 = 3axy$.

d) If $f(-x, -y) = f(x)$, then the curve is symmetrical in the opposite quadrant. Eg: $xy = c^2$, $x^2 + y^2 = a^2$.

ii) Origin:

If the eq^n of the curve doesn't contain any const. term, then the curve passes through the origin.

Eg: $x^3 + y^3 = 3axy$

iii) Tangents to the curve at the origin:

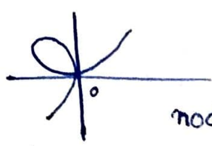
If the eq^n of the curve passes through the origin, the tangents are obtained by equating to zero to the lowest degree terms in the eq^n of the curve.

Eg $x^2 = 4ay$. lowest degree term $4ay = 0$, ie $y = 0$ is the tangent to the curve at the origin

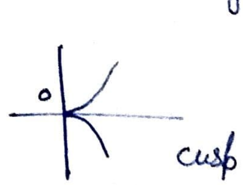
Some important points arise:

If there are two tangents at the origin, then the origin is a double point.

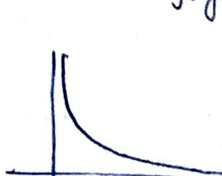
1) When the two tangents are real and distinct, then the origin is a node.



2) If the tangents are real and coincident then the origin is cusp



3) If the tangents are imaginary, the origin is a conjugate point or isolated point.



iv) Region: Points of intersection

Eg 1 Trace the curve $x^{2/3}+y^{2/3}=a^{2/3}$

Sol^n We write it as $(x^{1/3})^2+(y^{1/3})^2=(a^{1/3})^2$ — ①

~~(i) Eq^n of ① con~~

① Symmetry: Eq^n ① contains even powers of x as well as y. So the curve is symmetrical about the co-ordinate axes.

ii) Origin: As the curve contains a const. term, it does not pass through the origin. (No origin so no tangent)

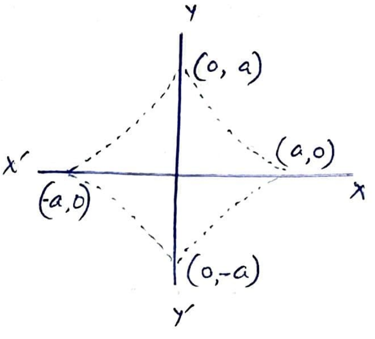
iii) Region: Points of intersection

Put $x=0$ in ① Then $y^2=a^2 \Rightarrow y=\pm a$

Also put $y=0$ in ① then $x^2=a^2 \Rightarrow x=\pm a$

Thus the point of intersection are $(0,a)$, $(0,-a)$, $(a,0)$, $(-a,0)$

The shape of curve is as shown.



Eg 2 - Trace the curve $x^3+y^3=3axy$

Sol^n i) Symmetry: If x & y are interchanged the curve remains unchanged. Thus the curve is symmetrical about the line $y=x$

ii) Origin: The curve passes through the origin.

iii) Tangents to the origin: Equating the lowest degree term to zero we get $3xy=0$, ie, $x=0$, $y=0$.

$\therefore$ Tangents at the origin are $x=0$, $y=0$

Thus the origin is a node (since the tangents are real & distinct.

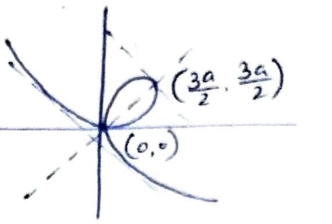
iv) Region: Putting $y=x$ in the given eq^n we get

$2x^3=3ax^2 \Rightarrow x=\frac{3a}{2}$ & $x=0$

Also, $2y^3=3ay^2 \Rightarrow y=\frac{3a}{2}$ & $y=0$

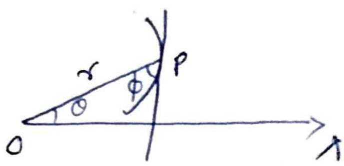
Thus the line $y = x$ meet the curve in two pts $(0,0)$ & $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Hence the shape of the curve is as shown in the fig



Curve tracing in polar co-ordinates.

The general eqn of the curve in polar co-ordinates (r, θ) in the explicit form is $r = f(\theta)$ or, $\theta = f(r)$ and in the implicit form is $f(r, \theta) = 0$



Procedure to trace a curve

i) Symmetry :

a) If the eqn of the curve doesn't change when θ is replaced by $-\theta$, the curve is symmatrical about initial line.

b) If the eqn doesn't change when r is replaced by $-r$ the curve is symmetrical about the pole and the pole is the centre of the curve

c) If $f(-r, -\theta) = f(r, \theta)$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$

ii) Pole: If $r = 0$ at $\theta = \alpha$(const). then the ~~line~~ curve passes through the pole and the ~~#~~ tangent at the pole is $\theta = \alpha$.

iii) Region extent: Find the region where the curve does not exist.

iv) Points of intersection : Points of intersection of the curve with the initial line and the line $\theta = \frac{\pi}{2}$ are obtained by putting $\theta = 0$ & $\theta = \frac{\pi}{2}$ respectively in the polar eqn.

~~v) Distance of tangent~~

v) Direction of tangent: The tangent at (r, θ) on the curve is obtained by $\tan\phi = \frac{r}{r_1} = \frac{r}{dr/d\theta}$. where ϕ is the angle between radius vector & tangent.

Eg 1 : Trace the curve $r = a(1-\cos\theta)$ {cardioid}

Sol^n (i) Symmetry: $\cos(-\theta) = \cos\theta$. Hence the curve is symmetrical about the initial line.

(ii) Pole: $r = 0 \Rightarrow \cos\theta = 1 \pm \cos\theta$ $\therefore$ $\theta = 0$. Hence the curve passes through the pole.

(iii) Points of intersection: The curve cuts the line $\theta = \pi$ at $(2a, \pi)$

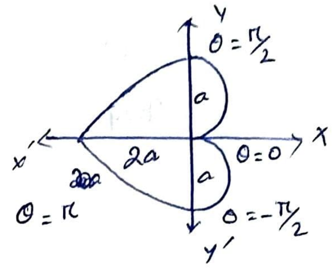
(iv) Tangent: $\tan\phi = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \dfrac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\theta/2$

$\therefore \phi = \dfrac{\theta}{2} = \dfrac{\pi}{2}$

Thus at the pt $a = \pi$, the tangent to the curve is perpendicular to the radius vector

(v) = Region:

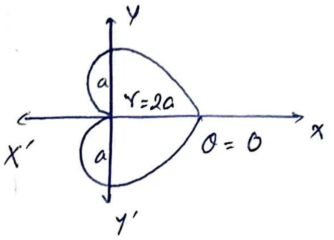
θ	0	$\pi/2$	π	$-\pi/2$
r	0	a	$2a$	a



Eg: 2 Trace $r = a(1+\cos\theta)$

pts

θ	0	$\pi/2$	$-\pi/2$	π
r	$2a$	a	a	0



Date 23-11-20

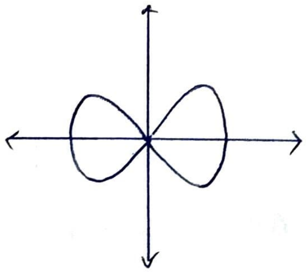
Limniscate:- It's eq^m $r^2 = a^2\cos 2\theta$

or $(x^2+y^2)^2 = a^2(x^2-y^2)$

$r = 0 \Rightarrow \cos 2\theta = 0$

$2\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \ldots$

It consists of two equal loops each symmetry about the initial line, which devides each loops in equal halves, $OA = OA' = a$



Tangents at origin are $y = \pm x$. for the upper half of the straight hand loop, ϕ varies from 0 to $\frac{\pi}{4}$.

Some examples of area and length of curves.

Illustration 1 $\rightarrow$ Find area bounded by curve $y^2 = 9x$ and $x^2 = 9y$

Sol^n $y^2 = 9x$ — (i) and $x^2 = 9y$ — (ii)

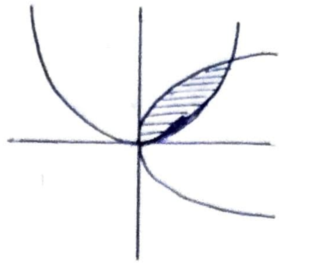
$\Rightarrow y^4 = 9^2x^2$

$\Rightarrow y^4 = 9^2 . 9y \Rightarrow y = 9$ (ii) $\Rightarrow x = 9$. Also $x = 0, y = 0$

So points of contact are $(0,0)$ and $(9,9)$

~~Req to~~ Req area $(OCADO)$ = area $(OCAB)$ − area $(ODAB)$

$$= \int_0^9 y_1\,dx - \int_0^9 y_2\,dx$$



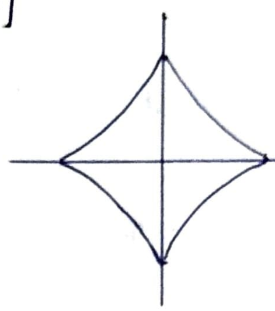
$$= \int_0^9 \left(3\sqrt{x} - \frac{x^2}{9}\right)dx$$

$$= 27$$

Illustration 2 – Find area of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a\cos^3 t$, $y = a\sin^3 t$. $\left[\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1\right]$

Solⁿ The given curve is –

$$(x^2)^{1/3} + (y^2)^{1/3} = (a^2)^{1/3}$$



Reqd area = 4 × area OAB

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^{\pi/2} a\sin^3 t\,(-3a\sin t \times \cos^2 t)\,dt$$

$$= -12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$$

$$= -12a^2 \times \frac{3\times1\times1}{6\times4\times2} \times \frac{\pi}{2}$$

$$= \frac{3}{8}\pi a^2$$

And $\frac{3}{8}\pi ab$ for $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, where $x = a\cos^3 t$, $y = b\sin^3 t$

Illustration 3 — Find area of a loop of the curve $r = a\sin 3\theta$

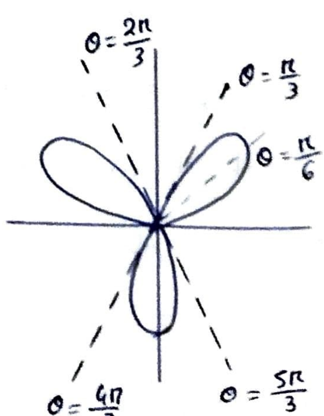
Solⁿ One curve is unsymmatric about initial lines

We have, $r = 0 \Rightarrow \sin 3\theta = 0$

$$\Rightarrow 3\theta = 0, \pi$$

$$\Rightarrow \theta = 0, \frac{\pi}{3} \rightarrow \text{Values for which } r \text{ is } 0$$

One loop lies between 0 & $\frac{\pi}{3}$



∴ Area of 1st loop $= \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2\sin^2 3\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{3}} (1-\cos 6\theta)\,d\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{3}} d\theta - \frac{a^2}{4}\int_0^{\frac{\pi}{3}} \cos 6\theta\,d\theta$$

$$= \frac{a^2}{4}\left(\frac{\pi}{3}\right) - \frac{a^2}{4}\left[\frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2\pi}{12} - 0$$

$\therefore$ The required area $= 3 \times \frac{a^2}{12}\pi = \frac{a^2\pi}{4}$

Illustration 4 – Find area of one loop of $r = a\cos 4\theta$

As $\cos(-4\theta) = \cos 4\theta$

$\therefore$ The graph is symmetric about the initial line $(\theta = 0)$

For $r = 0 \Rightarrow \cos 4\theta = 0 \Rightarrow 4\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{8}, -\frac{\pi}{8}$

$\therefore$ Area of the loop $= 2\int_0^{\frac{\pi}{8}} r^2 d\theta = 2\int_0^{\frac{\pi}{8}} a^2\cos^2 4\theta \, d\theta = \frac{\pi a^2}{6}$

Sums on arc length –

Illustration 1 – Find arc length of parabola $y^2 = 4ax$ intercepted by the line $3y = 8x$

Sol^n $3y = 8x \Rightarrow y = \frac{3}{8}x$ —— ①

Also, $y^2 = 4ax$

$$\Rightarrow \left(\frac{8}{3}x\right)^2 = 4ax \Rightarrow x\left(\frac{64}{9}x - 4a\right) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad \frac{64}{9}x - 4a = 0 \Rightarrow x = \frac{9}{16}a$$

$\therefore$ ① $\Rightarrow y = \frac{3}{2}a$

$\therefore$ Points of intersection are $(0,0)$, $\left(\frac{9}{16}a, \frac{3}{2}a\right)$

Required arc length intercepted by the line and the given parabola is

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For $x = a\tan^2\theta$

$\Rightarrow dx = 2a\tan\theta\sec^2\theta\, d\theta$

for $x = 0, \theta = 0$

$x = \frac{9a}{16}, \theta = \tan^{-1}\sqrt{\frac{9}{16}}$

$= \tan^{-1}(3/4)$

$y^2 = 4ax$

$\Rightarrow 2y\frac{dy}{dx} = 4a$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

$$= \int_0^{\frac{9a}{16}} \sqrt{1 + \frac{4a^2}{y^2}}\, dx$$

$$= \int_0^{9a/16} \sqrt{1 + \frac{4a^2}{4ax}}\, dx$$

$$= \int_0^{9a/16} \sqrt{\frac{x+a}{x}}\, dx$$

$$= \int_0^{\tan^{-1}(3/4)} \sqrt{\frac{(\tan^2\theta + 1)a}{a\tan^2\theta}}\, 2a\tan\theta\sec^2\theta\, d\theta$$

$$= 2a \int_0^{\tan^{-1}(3/4)} \sec^3\theta \, d\theta$$

$$= 2a\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta \, d\theta\right]_0^{\tan^{-1}(3/4)}$$

$$= 2a\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\log(\sec\theta + \tan\theta)\right]_0^{\tan^{-1}(3/4)}$$

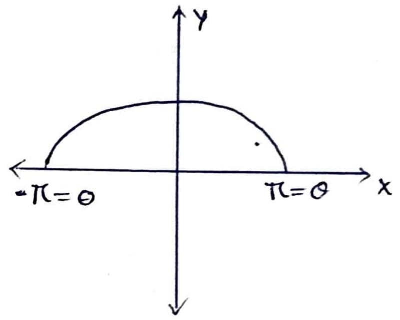
$$= a\sec\left(\tan^{-1}\frac{3}{4}\right)\frac{3}{4} + a\left(\log \sec \tan^{-1}(3/4) + 3/4\right)$$

<u>Illustration 2</u> — Find length of arc of the parabola $x^2 = y$. From vertex to the point $(1,1)$. (Given, $\log(2+\sqrt{5}) = \ln 5$

<u>Illustration 3</u> — Find the length of arc of cycloid $x = a(\theta + \sin\theta)$ and $y = (1+\cos\theta)$

(Hint: If $x = f_1(t)$, $y = f_2(t)$ the length of the curve is

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt)$$



<u>Sol^n</u>

θ	0	π	$-\pi$
x	0	$a\pi$	$-a\pi$
y	$2a$	0	0

$$\therefore \frac{dx}{d\theta} = a(1+\cos\theta) \qquad \frac{dy}{d\theta} = -a\sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2\cos^2(\theta/2)$$

$$\therefore \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2a\cos(\theta/2)$$

$$\therefore S = 2\int_0^{\pi} 2a\cos(\theta/2) = 4a\left(\frac{\sin\theta/2}{1/2}\right)_0^{\pi}$$

$$= 8a(1-0) = 8a$$

~~Eg 1 — Find the vol^m and surface are~~

Date 27-11-20

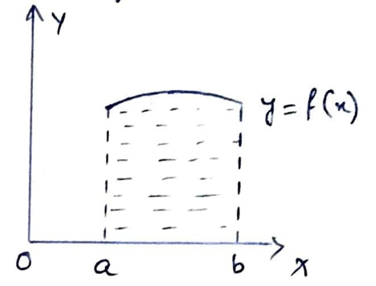
Volumes of Solids of Revolutions

Solid of revolution - If a plane area is revolved about a fixed line in its own plane then the body so generated by the revolution of the plane area is called a solid of revolution.

★ Volume (In cartesian form)

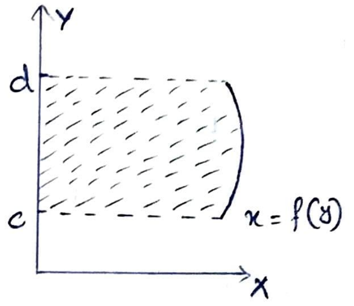
i) Volume of the solid generated by revolution of the area by the curve $y=f(x)$ about x-axis is

$$V=\pi\int_a^b y^2\,dx$$



ii) Volume of solid generated by revolution of the area bounded by the curve $x=f(y)$ about the y-axis is.

$$V=\pi\int_c^d x^2\,dy$$



In parametric form —

i) If the curve be given by parametric eqn $x=\phi(t)$, $y=\psi(t)$, then the volume of solid generated about x-axis of area is

$$V=\pi\int_a^b y^2\frac{dx}{dt}\,dt,$$

about y-axis is

$$V=\pi\int_c^d x^2\frac{dy}{dt}\cdot dt$$

In polar coordinates →

If the eqn of the generating curve is given in polar ~~for~~ co-ordinates $r=f(\theta)$ and the curve revolves about the initial line (x-axis), the volume generated is

$$V=\pi\int_{\theta=\alpha}^{\beta} y^2\frac{dx}{d\theta}\,d\theta$$

similarly about y axis, $V=\pi\int_{\alpha}^{\beta} x^2\frac{dy}{d\theta}\,d\theta$

Alternate method -

If curve is $r = f(\theta)$ and radius vectors are $\theta = \theta_1, \theta = \theta_2$ the volume of solid

i) About the initial line $\theta = 0$ (x-axis) is

$$\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

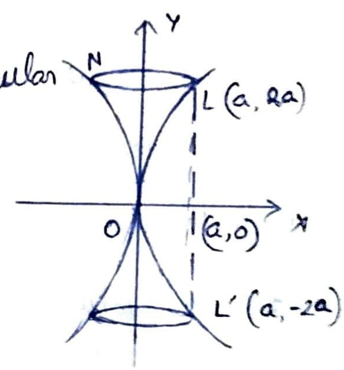
ii) About the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

iii) About any line ($\theta = \alpha$) is

$$\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin(\theta - \alpha)\, d\theta$$

Eg-1 Evaluate the volume of the solid generated by revolving the part of parabola $y^2 = 4ax$ bounded by the latus rectum about the tangent at the vertex.



Sol^n The latus rectum is the line perpendicular to the axis of the parabola & passing through the focus $S(a,0)$.

$y^2 = 4ax$ is the parabola.

LL' is the latus rectum.

The co-ordinates of L are $(a, 2a)$

The limits of the area OLN are $y = 0$ & $y = 2a$

$x = 0$ is the equation of the tangent at vertex O.

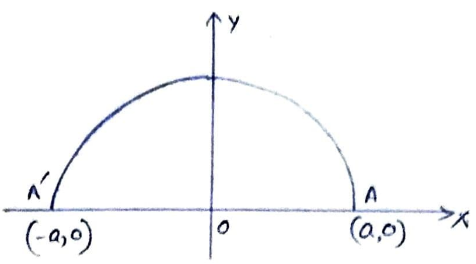
Required vol^m $= 2\int_{0}^{2a} \pi x^2 dy = 2\pi \int_{0}^{2a} \frac{y^4}{16a^2} dy$

$= \frac{\pi}{8a^2}\left(\frac{y^5}{5}\right)_0^{2a} = \frac{\pi}{8 \times 5a^2} \times 32a^5 = \frac{4}{5}\pi a^3$

Eg2- Show that volume of a sphere of radius a is $\frac{4}{3}\pi a^3$

Sol^n The sphere is generated by revol^n of a semi-circular area about its bonding diameter. The eq^n of the generating circle of radius a & centre at the origin is

$x^2+y^2=a^2$



AA' is the bonding diameter. The semicircle is symmatrical about y-axis

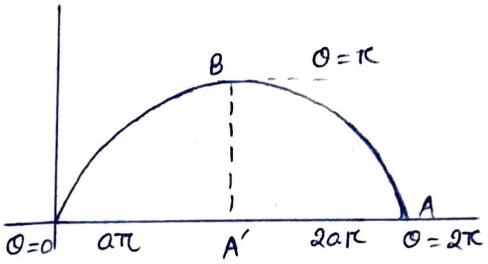
In the 1st quadrant x varies from 0 to a

$\therefore$ Req. volume $= 2\int_0^a \pi y^2 dx = 2\pi \int_0^a (a^2 - x^2) dx$

$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi a^3$

Eg-3 - Find the volume of solid formed by revolving the cycloid $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$ about it's bases.

θ	0	π	2π
x	0	$a\pi$	$2a\pi$
y	0	$2a$	0



The area OBA is revolved about +ve base ie, the x-axis for OBA, a varies from 0 to 2π and that B, $\theta = \pi$. The cycloid is symmatrical about BA'

$\therefore$ Required volume $= 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta$

$= 2\pi \int_0^{\pi} a^2 (1-\cos\theta)^2 \, a(1-\cos\theta) d\theta$

$= 2\pi a^3 \int_0^{\pi} (1-\cos\theta)^3 d\theta$

$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2 \theta/2\right)^3 d\theta$

$= 16\pi a^3 \int_0^{\pi/2} \sin^6 \phi \cdot 2d\phi$ $\qquad \frac{\theta}{2} = \phi$, $d\theta = 2d\phi$

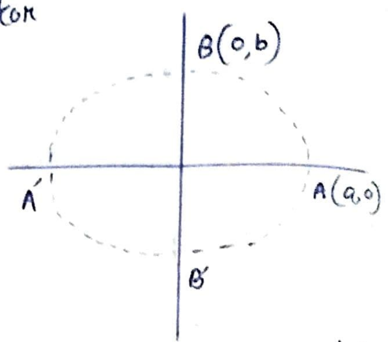
$= 32\pi a^3 \, \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2}$

$= 5\pi^2 a^3$

Eg-4 - Find the volume of solid generated by ~~to~~ revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis

Soln The solid is generated by revolving the area $ABA'OA$ about the ~~ab~~ x-axis.

The ellipse is symmatrical about y-axis. For the portion of the curve lying in 1st quadrent varies from o to a.



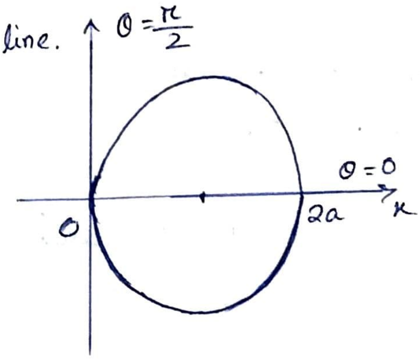
$$V = 2\pi \int_0^a y^2 dx = \ldots \ldots = \frac{4}{3}\pi(ab^2)$$

HW

Eg 5 – Find the volume of solid cylinder generated by revolution of cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, ~~of~~ $0 \le \theta \le \pi$

i) about the base, ii) About the x-axis.

Eg-6 – Find the volume of solid generated by +ve revolution of $r = 2a\cos\theta$ about +ve initial line.



Sol^n The curve is $r = 2a\cos\theta$

$\theta = \frac{\pi}{2} \Rightarrow r = 2a.0 = 0$

$\theta = 0 \Rightarrow r = 2a$

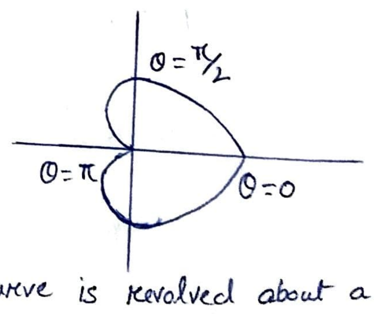
It is a circle passing through the pole. It is symmatrical about the initial line.

For the upper half of the circle ~~a~~ varies from 0 to $\frac{\pi}{2}$

$$\therefore \text{Req vol}^m = \frac{2}{3}\int_0^{\frac{\pi}{2}} \pi r^3 \sin\theta \, d\theta = \frac{2}{3}\pi\int_0^{\frac{\pi}{2}} 8a^3 \cos^3\theta \, \cancel{d} \sin\theta \, d\theta$$

$$= 16/3 \, \pi a^3 \left[-\frac{\cos^4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{4}{3}\pi a^3$$

Eg-7 → The cardioid $r = a(1+\cos\theta)$ revolves around the initial line. Find the volume of solid that generated.

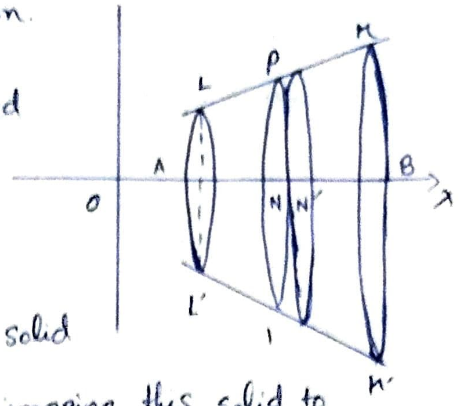


Sol^n $\text{Vol}^m = \frac{2}{3}\int_0^{\pi} \pi r^3 \sin\theta \, d\theta$

$$= \frac{2}{3}\pi \int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

Surface of revolution – If a plane curve is revolved about a fixed line lying on its own plane, then the surface, generated by the parameter of the of the curve is called a surface of revolution.

Axis of revolution — The fixed st line, say AB, about which +ve area revolves is called axis of revolution.



Let a curve LM, $y = f(x)$ be rotated about the x axis so as to form of a solid revolution.

Let us consider LL' M'N of this solid bounded by $x = x_1$, $x = x_2$ can be imagine this solid to be devided between infinity of infinity thin circular slices by planes perpendicular to the axis of revolution. If PN & P'N' be two adjacent ordinates of the curve where the co-ordinates of P & P' are (x, y), $(x + \Delta x, y + \Delta y)$, respectively.

The volume of corresponding slice, which has its thickness Δx, is equal to $\pi y^2 \Delta x$.

Hence, the total volume of the solid considered (bounded by $x = x_1$ & $x = x_2$) is given by

$$V = \lim_{\Delta x \to 0} \sum \pi y^2 \Delta x = \pi \int_{x_1}^{x_2} y^2 dx$$

Again, if Δs be the elt. of length PP' as being the arc length measured upto P forms on y fixed point on the curve LM, the surface area of ring shapped element generated by rotating PP' is ultimately $2\pi y . \Delta s$, Hence the radius SA is given by.

$$S = \lim_{\Delta s \to 0} \sum 2\pi y . \Delta s = 2\pi \int_{s_1}^{s_2} y ds$$

(s_1, s_2 being the values of s for the points L, M)

$$= 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_{y_1}^{y_2} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Surface area in parametric area.

Suppose $x = f(t)$, $y = \phi(t)$, t being variable parameter then the surface of solid of revolution about x-axis is

$$S = \int_{t_1}^{t_2} 2\pi y \frac{ds}{dt}\, dt \quad \text{where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

In polar form –

If $r = f(\theta)$ then the curved surface generated by the revolution of about the initial line of the arc interpretted by between the radius vectors $\theta = \alpha$ and $\theta = \beta$ is

$$S = \int_{\alpha}^{\beta} 2\pi (r\sin\theta) \frac{ds}{d\theta}\, d\theta$$

where,

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

In some casses, we may use the formula

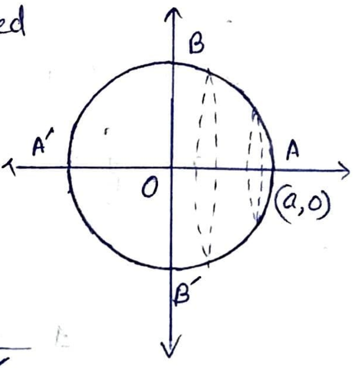
$$S = \int 2\pi y \frac{ds}{dr}\, dr \quad \text{where } \frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$$

$x = r\cos\theta \qquad y = r\sin\theta$

Date 04-12-20

Eg 1 – Find the volume & surface area of the sphere generated by the revolution of the circle $x^2 + y^2 = a^2$ about the x-axis.

Sol^n Let the quadrent OAB be rotated about x-axis, symmatrical about y-axis.



$\therefore$ Surface area $= 2\int_0^a 2\pi y\, ds$

$x^2 + y^2 = a^2$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$$= 4\pi \int_0^a y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_0^a y\sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= 4\pi \int_0^a \sqrt{y^2 + x^2}\, dx$$

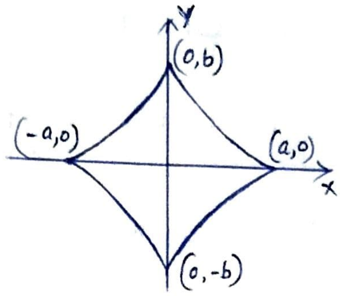
$$= 4\pi \int_0^a a\, dx$$

$$= 4\pi a [x]_0^a = 4\pi a^2$$

Eg 2 - Find the surface area of the surface generated by revolving about the y-axis the part of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, that lies in the 1st quadrant.

Sol^n The shape of the astroid

$(x^{1/3})^2 + (y^{1/3})^2 = (a^{1/3})^2$ as shown.



In parametric form, let

$x = a\cos^3\theta \quad , \; y = a\sin^3\theta$

Here, $S = 2\pi \int_0^{\pi/2} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$, in the 1st quadrant.

$= 2\pi \int_0^{\pi/2} a\sin^3\theta \sqrt{9a^2(\cos^4\theta \sin^2\theta + \sin^4\theta \cos^2\theta)}$

$= 2\pi a . 3a \int_0^{\pi/2} \sin^3\theta \cos\theta \sin\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$

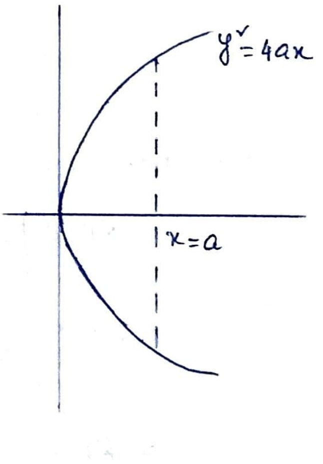
$= 6\pi a^2 \int_0^{\pi/2} \sin^4\theta \cos\theta\, d\theta$

$= 6\pi a^2 \left[\frac{\sin^5\theta}{5}\right]_0^{\pi/2}$

$= \frac{6}{5}\pi a^2$

Eg 3 - Find the volume and surface generated by revolving the parabola $y^2 = 4ax$ about the x-axis bounded by the section $x = a$.

Sol^n The required volume $= \pi \int_0^a y^2 dx$



$= \pi . 4a \int_0^x x\, dx$

$= 2\pi a^3$ cubic unit

Surface area $= 2\pi \int_0^a y\, ds = 2\pi \int_0^a y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

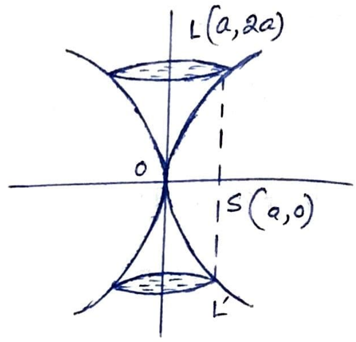
$= 2\pi \int_0^a \sqrt{y^2 + 4a^2}\, dx$

$= \frac{8}{3}\pi a^2 (2\sqrt{2} - 1)$

~~Eg 4 - Find the volume and surface generated by revolving the parabola $y^2 = 4ax$ about the x-axis bounded by the section $x = a$~~

Eg 4 – Find the volume and surface area of the solid of revolution generated by revolving the parabola $y^2 = 4ax$ bounded by the latus rectum about it's tangent at the vertex.

Sol^n Volume $= V = 2\int_0^{2a} \pi x^2 dy = \frac{4}{5}\pi a^3$



Surface area $= 2 \times 2\pi \int_0^{2a} x\, ds$

$$= 4\pi \int_0^{2a} x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 4\pi \int_0^{2a} x \sqrt{1+\frac{y^2}{4a^2}}\, dy$$

or $S = 4\pi \int_0^{2a} x \sqrt{1+\frac{4ax}{4a^2}}\, dy = \frac{4}{2a}\pi \int_0^{2a} x\sqrt{4a^2+y^2}\, dy$

$$= \frac{2}{3}\pi \int_0^{2a} \frac{y^2}{4a}\sqrt{4a^2+y^2}\, dy$$

$$= \frac{\pi}{2a^2}\int_0^{\pi/4} 2a^2\tan^2\theta \sqrt{4a^2+4a^2\tan^2\theta} \times 2a\sec^2\theta\, d\theta$$

$$= 8\pi a^2 \int_0^{\pi/4} \tan^2\theta \sec^3\theta\, d\theta$$

$$= 8\pi a^2 \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \times \frac{1}{\cos^3\theta}\, d\theta$$

$$= 8\pi a^2 \int_0^{\pi/4} (\sec^2\theta - 1)\sec^3\theta\, d\theta$$

$$= 8\pi a^2 \left[\int_0^{\pi/4} \sec^5\theta\, d\theta - \int_0^{\pi/4}\sec^3\theta\, d\theta\right]$$

Reduction Formula–

$$8\pi a^2 \left[\left\{\frac{1}{4}\left[\sec^3\theta \tan\theta\right]_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4} \sec^3\theta\, d\theta - \int_0^{\pi/4}\sec^3\theta\, d\theta\right]\right.$$

$$= 8\pi a^2 \left[\frac{1}{4}(\sqrt{2})^3 - 0 - \frac{1}{4}\int_0^{\pi/4}\sec^3\theta\, d\theta\right]$$

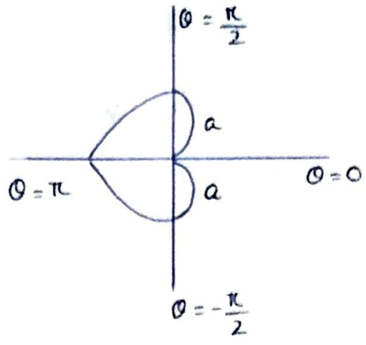
$$\cancel{= 8\pi a^2 \left[\frac{1}{4} \times 2^{3/2} - \frac{1}{4^2}(\sqrt{2}-0)\right.}$$

$$= 8\pi a^2 \left[\frac{1}{4}\times 2^{3/2} - \frac{1}{4}\left\{\frac{1}{2}\left[\sec\theta\tan\theta\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\theta\, d\theta\right\}\right]$$

$$= 8\pi a^2 \left[\frac{\sqrt{2}}{2} - \frac{1}{8}\left\{\sqrt{2} - 0 - \left[\log|\sec\theta + \tan\theta|\right]_0^{\pi/4}\right]\right.$$

$$= 8\pi a^2 \left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} - \frac{1}{8}\log(\sqrt{2}+1)\right]$$

Eg-5- If the cardioid $r = a(1-\cos\theta)$ is rotated about the initial line, find the volume and the area of the solid of required volume generated.



Sol^n The curve is $r = a(1-\cos\theta)$

θ	0	$\pi/2$	$-\pi/2$	π
r	0	a	a	$2a$

Required volume about one initial line

$$= \int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$$

$$= \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} a^3 (1-\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} \left(2\sin^2 \theta/2\right)^3 2\sin\theta/2 \cos\theta/2 \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi/2} 8 . 2\sin^2 u \sin u \cos u \, 2du$$

$$= \frac{\pi . 2 \times 32 a^3}{3} \int_0^{\pi/2} \sin^7 u \cos u \, du$$

$$= \pi \times \frac{2\times 32}{3} a^3 \left(\frac{\sin^8 u}{8}\right)_0^{\pi/2} = \frac{8}{3}\pi a^3$$

Surface area $= S = 2\pi \int_0^{\pi} y \frac{ds}{d\theta} d\theta$

$$= 2\pi \int_0^{\pi} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Now, $r^2 = a^2(1-\cos\theta)^2$ $\quad \frac{dr}{d\theta} = a\sin\theta$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta$$

$$= a^2(2-2\cos\theta) = 2a^2\sin^2\theta/2$$

$$S = 2\pi \int_0^{\pi} a(1-\cos\theta)\sin\theta \; 2a\sin\theta/2 \, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\sin^2\theta/2 \; 2\sin\theta/2 \cos\theta/2 \sin\theta/2 \, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \sin^4 u \cos u \, du . 2$$

$$= 32\pi a^2 \left[\frac{\sin^5 u}{5}\right]_0^{\pi/2}$$

$$= \frac{32}{5}\pi a^2$$

Similarly, HW →

1) Find the vol^m and surface area of the cardioid $r = a(1+\cos\theta)$ about the initial line.

2) Find the vol^m of the solid generated by ~~revolution~~ revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis. Also find the surface area

Improper Integrals

An integral $\int_a^b f(x)dx$ for all x in range $[a,b]$ is said to be a person (defined) integral. But if either a or b be both are infinite or when a and b are finite but $f(x)$ becomes infinite at $x=a$ or $x=b$ or at one or more points within $[a,b]$, then the integral is called improper or infinite integral.

Kinds of improper integral –

1) Improper of integral of the 1st kind

a) $\int_a^\infty f(x)dx$

In this case are we write $\int_a^\infty f(x)dx = \lim_{x\to\infty}\int_a^x f(x)dx$

Eg 1 - $\int_0^\infty e^{-x}dx = \lim_{x\to\infty}\int_0^x e^{-x}dx = \lim_{x\to\infty}[e^{-x}]_0^x$

$= -[e^{-\infty}-e^0] = -(0-1) = 1$

Eg 2 - $\int_0^\infty e^x dx = \lim_{x\to\infty}\int_0^x e^x dx = \lim_{x\to\infty}[e^x]_0^x = \lim_{x\to\infty}(e^x - e^0) = e^\infty - 1 = \infty$

If the integral neither converses nor diverses to a definite limit. It is said to be oscillator.

ie. $\int_0^\infty x\sin x\,dx$ oscillates infinitely

b) $\int_{-\infty}^b f(x)dx$

We write as $\int_{-\infty}^b f(x)dx = \lim_{x\to\infty}\int_x^b f(x)dx$

c) $\int_{-\infty}^\infty f(x)dx$

We can write it as $\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

$$= \lim_{x\to-\infty}\int_{x}^{c} f(x)dx + \lim_{x\to\infty}\int_{c}^{x} f(x)dx$$

Eg. ① $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{c}\frac{dx}{1+x^2} + \int_{c}^{\infty}\frac{dx}{1+x^2}$

$$\lim_{x\to-\infty}\int_{x}^{c}\frac{dx}{1+x^2}dx + \lim_{x\to\infty}\int_{c}^{x}\frac{dx}{1+x^2}$$

$$= \lim_{x\to-\infty}[\tan^{-1}x]_{x}^{c} + \lim_{x\to\infty}[\tan^{-1}x]_{c}^{x}$$

$$= \tan^{-1}\infty + \tan^{-1}\infty = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

2) Test the convergene of $\int_{a}^{\infty}\frac{\sin^2 x}{x^2}dx$

Sol^n We know that $0 \le \sin x < 1$ when $x > 0$

Hence, $\int_{a}^{\infty}\frac{\sin^2 x}{x^2} \le \int_{a}^{\infty}\frac{1}{x^2}dx \le \left[-\frac{1}{x}\right]_{a}^{\infty} \le \left(-\frac{1}{\infty}+\frac{1}{a}\right) \le \frac{1}{a}$

Hence the integral is converged (as $\frac{1}{a}$ is +ve)

3) Test the convergence of $\int_{2}^{\infty}\frac{dx}{\sqrt{x^2-1}}$

Sol^n we know that $\frac{1}{\sqrt{x^2-1}} > \frac{1}{x}$ when $x \ge 2$

$\therefore \int_{2}^{\infty}\frac{dx}{\sqrt{x^2-1}} > \int_{2}^{\infty}\frac{1}{x} > [\log x]_{2}^{x}$, integral is diverging.

Date : 07-12-20

ii) Improper integrals of the 2nd kind

A definite integral $\int_{a}^{b} f(x)dx$ in which $[a,b]$ is finite but $f(x)$ is unbounded at one or more points of the interval is called an improper integral of the 2nd kind

Ex - $\int_{0}^{4}\frac{dx}{(x-2)(x-3)}$, $\int_{0}^{1}\frac{1}{x^2}dx$ etc.

In this case we define the value of the integral as follows.

a) If $f(x)$ is unbounded at $x=b$ only ie, if $f(x)\to\infty$ as $x\to b$, we define $\int_{a}^{b} f(x)dx = \lim_{\epsilon\to 0}\int_{a}^{b-\epsilon} f(x)dx$

b) If $f(x) \to \infty$ as $x \to a$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)\,dx$$

c) If $f(x) \to \infty$ as $x \to c$ only where $a<c<b$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx \right]$$

d) If $f(x)$ is unbounded at both the points a & b of the interval $[a,b]$ and is bounded at each apart of this interval, we write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a<c<b$

Eg. Evaluate

i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. Here the integral $\frac{1}{\sqrt{x}}$ becomes infinite at $x=0$

$\therefore$ We have $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon \to 0} \int_{0+\epsilon}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon \to 0} \left[2\sqrt{x}\right]_\epsilon^1$

ii) $\int_0^1 \frac{1}{\sqrt{1-x}}$. Here $\frac{1}{\sqrt{1-x}}$ becomes infinite at $x=1$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x}} = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} (1-x)^{-1/2}\,dx$$

$$= \lim_{\epsilon \to 0} \left[\frac{(1-x)^{1/2}}{1/2} \right]_0^{1-\epsilon} = \lim_{\epsilon \to 0} -2 \left[[1-(1-\epsilon)]^{1/2} - 1 \right]$$

$= 2$, ~~converging~~ converging

iii) $\int_0^1 \frac{1}{1-x}\,dx = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{dx}{1-x} = \lim_{\epsilon \to 0} \left[\log(1-x)(-1) \right]_0^{1-\epsilon}$

$$= \lim_{\epsilon \to 0} -\left[\log(1-(1-\epsilon)) - \log 1\right]$$

$= -\log 0 + 0 = -(-\infty) = \infty$, ~~dve~~ diverging

iv) $\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$, Here the integral becomes infinite at $x=0$

$$\therefore \int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}}$$

$$= \lim_{\epsilon \to 0} \int_{-1}^{0-\epsilon} x^{-2/3}\,dx + \lim_{\epsilon \to 0} \int_{0+\epsilon}^1 x^{-2/3}\,dx$$

$$= \lim_{\epsilon\to 0}\left[\frac{x^{-2/3+1}}{-\frac{2}{3}+1}\right]_{-1}^{0-\epsilon} + \lim_{\epsilon\to 0}\left[\frac{x^{-2/3+1}}{-\frac{2}{3}+1}\right]_{0+\epsilon}^{1}$$

$$= 3\lim_{\epsilon\to 0}\left[(-\epsilon)^{1/3} - (-1)^{1/3}\right] + 3\lim_{\epsilon\to 0}\left[1^{1/3} - (\epsilon)^{1/3}\right]$$

$$= 3[1+1] = 6$$

HW v) $\int_{-1}^{1}\frac{dx}{x^2}$ vi) $\int_{0}^{2a}\frac{dx}{(x-a)^2}$

* Beta and Gamma function

Beta function → The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m>0,\ n>0$ is called the Beta function and is denoted by $\beta(m,n)$

Thus $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Beta function is also called the Euleriam integral of the 1st kind.

Some properties of beta function.

i) $\beta(m,n) = \beta(n,m)$

Proof we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

By property $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$ of definite integral,

we can write $\beta(m,n) = \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx = \beta(n,m)$$

2) $\beta(m,n) = 2\int_0^{\pi/2}(\sin x)^{2m-1}(\cos x)^{2n-1}dx$

Proof Put $x = \sin^2 t$ in $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Then $dx = 2\sin t\cos t\,dt$

$x=0 \Rightarrow t=0,\ x=1 \Rightarrow t=\frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\pi/2}(\sin^2 t)^{m-1}(1-\sin^2 t)^{n-1}2\sin t\cos t\,dt$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}t\cos^{2n-1}t\,dt$$

or $B(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1}\, dx$

3) $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

<u>Proof</u> $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = \frac{y}{1+y}$. Then $1-x = 1-\frac{y}{1+y} = \frac{1}{1+y}$

When $x=0,\ y=0$

When $x \to 1,\ y \to \infty$

Also $dx = \frac{1}{(1+y)^2}\, dy$

$$B(m,n) = \int_0^{\infty} \left(\frac{y}{1+y}\right)^{m-1} \left(1-\frac{y}{1+y}\right)^{n-1} \frac{1}{(y+1)^2}\, dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}}\, dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

<u>Gamma function</u> → The definite integral $\int_0^{\infty} e^{-x} x^{n-1}\, dx$ for $n>0$ is called the gamma function and is denoted by Γn. Thus $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$

In particular $\Gamma 1 = \int_0^{\infty} e^{-x} x^0\, dx = -\left[e^{-x}\right]_0^{\infty} = 1$

1) <u>Reduction formula for Γn : $\Gamma n+1 = n\,\Gamma n$</u>

Since $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$, $\Gamma n+1 = \int_0^{\infty} e^{-x} x^{n}\, dx$

$$= \left[x^3 \frac{e^{-x}}{-1}\right]_0^{\infty} \int_0^{\infty} n x^{n-1}(-e^{-x})\, dx$$

$$= -[0] + n\int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$= n\,\Gamma n$$

We can also write $\Gamma n = \frac{\Gamma n+1}{n}$

2) Value of Γn in terms of factorial –

Using $\Gamma(n+1) = n\Gamma n$ successively we get

$\Gamma 2 = 1\Gamma 1 = 1$; $\Gamma 3 = 2\Gamma 2 = 2.1 = 2$

$\Gamma 4 = 3\Gamma 3 = 3.2! = 3!$

In generally $\Gamma(n+1) = n!$

In particular $\Gamma 1 = 0! = 1$

3) $\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

We have, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ — Put $x = y^2$ then $dx = 2y\,dy$

$$= \int_0^\infty e^{-y^2} y^{2(n-1)} . 2y\,dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

4) $\beta(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma(m+n)}$

Proof $\Gamma m \Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} dx . 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

Put $x = r\cos\theta$ $y = r\sin\theta$

$\cancel{dx = r\sin\theta}$ $\cancel{dy = r\cos\theta}$

$x^2 + y^2 = r^2$

$\cancel{\Rightarrow x\,dx + y\,dy = r\,dr\,d\theta}$

$dx\,dy = r\,dr\,d\theta$

r varies from 0 to ∞

θ " " 0 to $\pi/2$

as the region of integration Entire 1st quadrant.

$$\therefore \Gamma m \Gamma n = 4\int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2m-1+2n-1+1} dr \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr . 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= \Gamma(m+n) . \beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

Cor: Rule to evaluate $\int_0^{\pi/2} \sin^p x \cos^q x \, dx$

We know $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ —— ①

Put $2m-1 = p$, $2n-1 = q$, then $m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

& $\beta(m,n) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

① $\Rightarrow \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

In particular when $q=0$, $p=n$

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{1}{2} \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{n+2}{2}} \qquad \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \cdot \frac{\sqrt{\pi}}{2}$$

similarly $\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \frac{\sqrt{\pi}}{2}$

5) $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

Putting $m = n = \frac{1}{2}$ we get

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = \left(\Gamma\tfrac{1}{2}\right)^2$$

or $\left(\Gamma\frac{1}{2}\right)^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1} dx = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx$ —— ①

Now, putting $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta \, d\theta$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \int_0^{\frac{\pi}{2}} \sin^{-1}\theta \cos^{-1}\theta \cdot 2\sin\theta\cos\theta \, d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = 2 \cdot \frac{\pi}{2} = \pi$$

$\therefore \left(\Gamma\frac{1}{2}\right)^2 = \pi$

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$,

We can also prove that $\Gamma-\frac{1}{2} = -2\sqrt{\pi}$.

Eg-1 Express the following integrals in terms of gamma function

i) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$ limits $\rightarrow 0, \pi/2$

$dx = \frac{\cos\theta}{2x}d\theta = \frac{1}{2}\sin^{-1/2}\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{\cos\theta} \cdot \frac{1}{2}\sin^{-1}\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta = \frac{1}{2}\cdot\frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-1/2+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

ii) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^{+1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

v) $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

$$LHS = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$$

$$= \frac{\Gamma\frac{\frac{1}{2}+1}{2}}{\Gamma\frac{\frac{1}{2}+2}{2}} \frac{\sqrt{\pi}}{2} \times \frac{\Gamma\frac{-\frac{1}{2}+1}{2}}{\Gamma\frac{-\frac{1}{2}+2}{2}} \times \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{4} \times \frac{\Gamma 3/4}{\Gamma 5/4} \times \frac{\Gamma 1/4}{\Gamma 3/4}\, \pi$$

$$= \frac{\Gamma 1/4}{1/4\,\Gamma 1/4} \cdot \frac{\pi}{4} = \pi$$

Eg-2 Prove that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof Put $x^2 = t$, then $2x\,dx = dt \Rightarrow dx = \frac{1}{2x} dt = \frac{1}{2} t^{-1/2} dt$

$$\therefore \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-t}\, t^{-1/2} \frac{1}{2} dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2} \Gamma 1/2$$

$$= \frac{\sqrt{\pi}}{2}$$

Eg-3- i) show that $\Gamma\frac{7}{2} = \frac{15}{8}\sqrt{\pi}$

Sol^n $\Gamma{n+1} = n\Gamma n$

$$\therefore \Gamma\frac{7}{2} = \Gamma\frac{5}{2}+1 = \frac{5}{2}\Gamma\frac{5}{2}$$

$$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{15}{8}\sqrt{\pi}.$$

ii) Evaluate $\Gamma{-7.5}$

$$\Gamma{-7.5} = \frac{\Gamma{-7.5+1}}{-7.5} \qquad \Gamma n = \frac{\Gamma{n+1}}{n}$$

$$= \frac{\Gamma{-6.5}}{-7.5} = \frac{\Gamma{-5.5}}{(-7.5)(-6.5)} = \cdots\cdots = \frac{\Gamma{-.5+1}}{(-7.5)(-6.5)\cdots(-0.5)}$$

$$= \frac{\sqrt{\pi}}{79/8}$$

HW i) Evaluate $\Gamma{4.5}$

Eg 4 - Prove that $\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = n^x\beta(x, n+1)$

Proof: Let $\frac{t}{n} = z$ then $dt = n\,dz$

$t=0 \Rightarrow z=0 \quad t=n \Rightarrow z=1$

$$\therefore LHS = \int_0^1 (1-z)^n (nz)^{x-1} n\,dz = n^{x-1+1}\int_0^1 z^{x-1}(1-z)^n dz$$

$$= n^x\int_0^1 z^{x-1}(1-z)^{n+1-1}dz = n^x\beta(x, n+1)$$ Proved

Eg 5- Prove that $\beta(m+1, n) = \frac{m}{m+n}\beta(m,n)$

Proof LHS $= \beta(m+1,n) = \frac{\Gamma{m+1}\,\Gamma n}{\Gamma{m+n+1}} = \frac{m\Gamma m\,\Gamma n}{(m+n)\Gamma{m+n}} = \frac{m}{m+n}\beta(m,n)$

Eg 6 - Prove that $\int_0^\infty e^{-9x}x^{3/2}\,dx = \frac{9}{4}\sqrt{\pi}$

Proof $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$ Put $9x = z \Rightarrow x = \frac{z}{9}$

then $9\,dx = dz$

$LHS = \int_0^\infty e^{-9x}x^{3/2}dx$ limits : $0, \infty$

$$= \int_0^\infty e^{-z}\left(\frac{z}{9}\right)^{3/2}\frac{dz}{9}$$

$$= \frac{1}{(3^2)^{3/2}}\cdot\frac{1}{9}\int_0^\infty e^{-z} z^{(5/2-1)}\,dz$$

$$= \frac{1}{279}\,\Gamma\tfrac{5}{2} = \frac{1}{27\times 9}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{27\times 12}$$

Eg 7- Evaluate $\int_0^1 x^2(1-x^2)^{7/2}\,dx$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$= \int_0^{\pi/2} \sin^2\theta(\cos^2\theta)^{7/2}\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\cos^8\theta\, d\theta$$

One type soln (Beta-gamma function)

$$= \frac{1}{2}\,\frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{8+1}{2}}{\Gamma\frac{2+8+2}{2}} = \frac{1}{2}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{9}{2}}{\Gamma 6}$$

$$= \frac{1}{2}\times\frac{\frac{1}{2}\times\Gamma\frac{1}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{5.4.3.2.1}$$

$$= \frac{7.\sqrt{\pi}}{512}$$

(2nd type soln, reduction formula)

Eg-8 Prove that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

Proof Since e^{-x^2} is an even function, $\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$

Let $x^2 = z$

then $2x\,dx = dz$

$$2\int_0^\infty e^{-z}\frac{1}{2\sqrt{z}}dz = \int_0^\infty e^{-z}z^{-1/2}dz$$

$$= \int_0^\infty e^{-z} z^{\frac{1}{2}-1}dz = \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

HW

9) Prove that $\int_0^{\pi/2}\sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$

10) Evaluate : i) $\beta\left(2.5, \frac{1}{5}\right)$. ii) $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\,dx$

iii) $\int_0^1 x^{1/2}(1-x^2)^{1/3}dx$. iv) $\int_0^\infty e^{-x^4}x^9\,dx$

(use $\beta(m,n)$, Γn formulae)

11) Show that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$

Sol^n $\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right) = \frac{2n-1}{2}\,\Gamma\left(\frac{2n-1}{2}\right)$

$$= \frac{2n-1}{2}\,\Gamma\left(\frac{2n-3}{2}+1\right) = \frac{2n-1}{2}\,\frac{2n-3}{2}\,\Gamma\left(\frac{2n-3}{2}\right)$$

$$= \frac{2n-1}{2} \times \frac{2n-3}{2} \times \frac{2n-5}{2} \cdots\cdots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)(2n-5)\ldots 5.3.1}{2^n}\sqrt{\pi} \quad \text{--- (1)}$$

Multiply N^r & D^r by $2n(2n-2)(2n-4)\ldots 4.2$.

We get $\Gamma\left(n+\frac{1}{2}\right) = \frac{2n(2n-1)(2n-2)(2n-3)\ldots 5.4.3.2.1}{2^n.2^n\, n(n-1)(n-2)\ldots 2.1}\sqrt{\pi}$

$$= \frac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}\, n!}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$$

* Inverse and Rank of a matrix

The matrix $\begin{bmatrix} 4 & 5 \\ 4 & 5 \end{bmatrix}$ is singular as $\begin{vmatrix} 4 & 5 \\ 4 & 5 \end{vmatrix} = 0$

and the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is non singular as $\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \neq 0$

Adjoint of a matrix –

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$ where A_{ij}, $i = 1, 2, 3 = j$ are cofactors of A

and we write inverse of $A = \dfrac{\text{adj } A}{|A|}$, $|A| \neq 0$

Eg-1 Find the inverse of A where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & 5 \end{bmatrix}$

Solⁿ

$$\begin{bmatrix} +\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -18 & 4 \\ -2 & 7 & -1 \\ 2 & +8 & -4 \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} -2 & -2 & 2 \\ -18 & 7 & 8 \\ 4 & -1 & -4 \end{bmatrix}$$

and $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & 5 \end{vmatrix} = 2(-2) + 1(2-20) + 3(4-0)$

$= -4 - 18 + 12$

$= -10 \neq 0$

$$\therefore A^{-1} = \frac{-1}{10}\begin{bmatrix} -2 & -2 & 2 \\ -18 & 7 & 8 \\ 4 & -1 & -4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & 2 & -2 \\ 18 & -7 & -8 \\ -4 & 1 & 4 \end{bmatrix}$$

~~A^{-1} is also~~

A^{-1} is also called reciprocal of matrix A.

Theorem – For two matrices A & B, the following are true

i) $(A^{-1})^{-1} = A$, ii) $(AB)^{-1} = B^{-1}A^{-1}$ iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

Inverse of a matrix – If a

Inverse from elementary matrices

Those elementary row transformation which reduce a given square matrix A to the unit matrix, when applied to unit matrix I, give the inverse of A.

observation - This is also known as Gauss Jordan reduction ~~mod~~ method of finding the inverse

For practical evaluation of A^{-1} the two matrices A & I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}

Eg-1 Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & 4 & -4 \end{bmatrix}$

By using Gauss jordan method.

Soln We have $\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$ $\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -3 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$ $\dfrac{R_2/2}{R_3/2}$ $\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$

$R_1 \to R_1 - R_2$, $R_3 \to R_3 + R_2$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$ $R_3/2$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

$R_1 - 6R_3$, $R_2 + 3R_3$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -\frac{5}{4} & -1/4 & 3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$

Eg- Use Gauss-Jordan method to find the inverse of the following matrices

i) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 8 & 4 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

v) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Sol^n 3) We write $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

We have $\left[\begin{array}{ccc:ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_1 \to R_1/3$ $\left[\begin{array}{ccc:ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1$ $\left[\begin{array}{ccc:ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 - R_2$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 1 \end{array}\right]$

$R_3 \times (-3)$
$R_2 \times (-1)$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & -3 \end{array}\right]$

$R_2 \to R_2 + \frac{4}{3} R_3$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & 2 & 3 & -3 \end{array}\right]$

$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & -4 \\ 2 & 3 & -3 \end{bmatrix}$

* Rank of a matrix –

Consider the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If we select any r-rows and r-columns from any matrix A, deleting all the other rows and columns, then the determinent formed by these $r \times r$ elements is called the minor of A of

Order r. Clearly there will be a number of different minors of the same order, got by deleting different rows and column's from same matrix.

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ Minors of 2nd order are

$\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} \ldots$

Difference Defination –

A matrix is said to be of rows r when

i) It has at least one - non-zero minor of order r,

ii) every minor of order higher than r vanishes.

The rows of a matrix shall be denoted by $f(A)$

Eg-1 Determine of rows of the following matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix} = 0$

$\therefore f(A) \neq 3$

Next $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4-2 = 2 \neq 0$ so $\rho(A) = 2$

Eg-2 $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ $|A| = 3(-4+4) - 1(-12+12) + 2(-6+6) = 0$

$\therefore \rho(A) \neq 3$

$\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = -4-4 \neq 0$

Elementary transformation of a matrix:

The following operations are known as elementary transformation:

1) The interchang of any two row of column's ie R_{ij} (or C_{ij}) for the interchange of i^{th} or j^{th} rows/column's

2) The multiflication of any row/colum by a non-zero number ie KR_i (KC_i) for multiflication of i^{th} row column by K.

3) The addition of a constant multiple of the elements of any row/column ie $R_i + PR_j$ (or $C_i + PC_j$)

Elementary transformations donot change either order or rank of a matrix

Equivalent matrix :

Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol $\sim$ is used for equivalence.

Eg. Determine the rank of the matrix

i) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$
$R_4 \to R_4 - R_2$ $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$R_4 \to R_4 - R_3$ $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1 - R_3$ $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} \neq 0$ $\therefore \rho(A) = 2$

ii) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 - 2R_1$
$R_3 - 3R_1$
$R_4 - 6R_1$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$

$R_4 - R_3$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

$R_4 - R_2$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 6 & -4 & -8 \end{vmatrix} = 1(-12)+2(0)+3(0) = -12 \neq 0$

$\therefore \rho(A) = 3$

HW

3) $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

4) $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

5) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & 4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

★ Normal form of a matrix.

If A is an $m \times n$ matrix and by a series of elimentary row/column operations it can be put into one of the following forms: $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $[I_r : 0]$, $[I_r]$ where I_r is the unit matrix of order r.

It is called the normal form of A.

★ Note that the rank of any identity matrix is the same as its order.

★ The rank of a zero matrix defined to be zero.

Eg- Determine the rank of the following matrices by reducing to the normal form.

1) $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$\xrightarrow[\substack{R_3 - 3R_1 \\ R_4 - 6R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \xrightarrow[\substack{C_3 + 2C_1 \\ C_4 + 4C_1}]{C_2 + C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$.

$\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \xrightarrow[R_4 - 9R_2]{R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$

$\xrightarrow[R_4/11]{R_3/11} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 6 & 4 \end{bmatrix} \xrightarrow{R_4 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_4 + 3C_2]{C_3 + 6C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{C_3/3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&0\end{bmatrix}\xrightarrow{C_4-2C_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}I_3&O_{3\times1}\\O_{1\times3}&O_{1\times1}\end{bmatrix}$$

$\therefore \rho(A) = 3$

2) ~~A=~~ $\begin{bmatrix}6&1&3&8\\4&2&6&-1\\10&3&9&7\\16&4&12&15\end{bmatrix}$

3) $A = \begin{bmatrix}2&-2&0&6\\4&2&0&2\\1&-1&0&3\\1&-2&1&2\end{bmatrix}\xrightarrow{R_{13}}\begin{bmatrix}1&-1&0&3\\4&2&0&2\\2&-2&0&6\\1&-2&1&2\end{bmatrix}\xrightarrow[R_4-R_1]{R_2-4R_1,\ R_3-2R_1}\begin{bmatrix}1&-1&0&3\\0&6&0&-10\\0&0&0&0\\0&-1&1&-1\end{bmatrix}$

$$\xrightarrow[C_4-3C_1]{C_2+C_1}\begin{bmatrix}1&0&0&0\\0&6&0&-10\\0&0&0&0\\0&-1&1&-1\end{bmatrix}\xrightarrow{R_{24}}\begin{bmatrix}1&0&0&0\\0&-1&1&-1\\0&0&0&0\\0&6&0&-10\end{bmatrix}\xrightarrow{R_4+6R_2}\begin{bmatrix}1&0&0&0\\0&-1&1&-1\\0&0&0&0\\0&0&6&-16\end{bmatrix}$$

$$\xrightarrow[C_4-C_2]{C_3+C_2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&6&-16\end{bmatrix}\xrightarrow[C_3/6]{C_2/-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&-16\end{bmatrix}\xrightarrow{C_4+16C_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix}$$

$$\xrightarrow{R_{34}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}I_3&0\\0&0\end{bmatrix}\qquad \therefore \rho(A)=3$$

4) Show that the rank of the matrix $A = \begin{bmatrix}8&1&3&6\\0&3&2&2\\-8&-1&-3&4\end{bmatrix}$ is 3

5) Show that $\rho(A)=2$ and $\rho(B)=2$ where $A = \begin{bmatrix}1&1&-1\\2&-3&4\\3&-2&3\end{bmatrix}$ and $B = \begin{bmatrix}1&-2&3\\6&12&6\\5&10&5\end{bmatrix}$ show further that that $\rho(AB)=1$ while $\rho(BA)=2$

Soln $|A| = 1(-9+8)+1(12-6)-1(-4+9) = -1+6-5=0$

Hence A is a singular matrix, so rank of $A<3$. Let us consider the 2nd order minor $\begin{vmatrix}1&1\\2&-3\end{vmatrix} = -3-2=-5\neq 0$

$\therefore \rho(A)=2$

Next $|B| = 1(60-60)-2(30-30)+3(60-60)=0$, so $\rho(B)<3$

2nd order minor $\begin{vmatrix}1&-2\\6&12\end{vmatrix} = 12+12=24=24\neq 0 \quad \therefore \rho(B)=2$

Now $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \\ 6 & 0 & 12 \end{bmatrix}$

Now, $\det(AB) = 2(0) + 0 + 4(0) = 0$

$\therefore \rho(AB) < 3$ Also every minor of order 2 is zero.

$\therefore \rho(AB) < 2$ So $\rho(AB) = 1$

Next $\rightarrow$ BA = ...

Rank of a matrix (more examples).

Eg - Find the rank of the matrix

1) $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}_{3\times 4}$

Soln Here $\rho(A) \leq 3$

$A \sim_{C_{14}} \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -4 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix}$

$\xrightarrow[\substack{C_3+4C_1 \\ C_4+2C_1}]{C_2+3C_1} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

minor $\begin{vmatrix} -4 & 3 \\ 0 & 12 \end{vmatrix} \neq 0$ $\quad \therefore \rho(A) = 2$

2) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \xrightarrow{R_4-(R_1+R_2+R_3)} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow[R_3-3R]{R_2-2R_1} \begin{bmatrix} \quad \end{bmatrix}$ . . . . $\rho(A) = 3$

3) $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ by reducing it to the normal form

Soln $A \sim_{C_{12}} \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix} \xrightarrow[\substack{R_3-3R_1 \\ R_4-4R_1}]{R_2-2R_1} \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 11 \end{bmatrix}$

$$\underset{R_4 - R_2}{\overset{R_3 - R_2}{\sim}} \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{C_3 - 3C_1 \\ C_4 - 8C_1}}{\overset{C_2 - 6C_1}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{C_4/-17}{\overset{C_2/8}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{C_4 - C_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \therefore \rho(A) = 2$$

HW Find the rank of $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Vectors

Vectors

Any ~~quality~~ quantity having n components is called a ~~veto vetor~~ vector. Thus any n numbers $x_1, x_2, \ldots x_n$ written in a particular order, denote the vetor x. $(x = (x_1, x_2 \ldots x_n))$

Linear dependence – The vectors $X_1, X_2 \ldots X_r$ are said to be linearly dependent, if there exists r numbers $\lambda_1, \lambda_2 \ldots \lambda_r$ not all zero such that $\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_n x_n = 0$ — ①

If no such numbers, other than zero exist, the vectors are said to be linearly independent.

If $\lambda_1 \neq 0$, we can write ① as

$x_1 = \mu_2 x_2 + \mu_3 x_3 + \ldots \mu_r x_r$ ~~if $\lambda_1 \neq 0$, we can write ① as~~

where $\mu_2 = -\frac{\lambda_2}{\lambda_1}$, $\mu_3 = \frac{-\lambda_3}{\lambda_1}$ etc

Then the vectors x_1 is said to be a linear combination of the vectors $x_1, x_2, \ldots x_n$

Eg 1 Are the vectors $x_1 = (1, 3, 4, 2)$, $x_2 = (3, -5, 2, 2)$ and $x_3 = (2, -1, 3, 2)$ linearly dependent? If so express one of these as a linear combination of the others.

Soln The relation $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$ — ①

ie. $\lambda_1 (1, 3, 4, 2) + \lambda_2 (3, -5, 2, 2) + \lambda_3 (2, -1, 3, 2) = 0$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$ — (I)

$3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$ — (II)

$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ — (III)

$2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$ — (IV)

solving these equations we get $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -2$ which are not all zero.

So the given vectors are linear dependent.

Also we have the relation $x_1 + x_2 - 2x_3 = 0$ (from ①)

From which we can write $x_1 = -x_2 + 2x_3$, ie. x_1 can be expressed in terms of the vectors x_2 & x_3 similarly x_2 & x_3

Eg 2 - Are the vectors below linearly dependent?

i) $(3,2,7)$; $(2,4,1)$; $(1,-2,6)$

ii) $(1,1,1,3)$; $(1,2,3,4)$; $(2,3,4,9)$

iii) $x_1 = (1,2,4)$, $x_2 = (2,-1,3)$, $x_3 (0,1,2)$, $x_4 = (-3,7,2)$ If so, find the relationship between them.

Sol^n i) $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$, where $x_1 (3,2,7)$, $x_2 (2,4,1)$, $x_3 = (1,-2,6)$

is equivalent to $\lambda_1 (3,2,7) + \lambda_2 (2,4,1) + \lambda_3 (1,-2,6) = (0,0,0)$

ie $3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ — (i)

$2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0$ — (ii)

$7\lambda_1 + \lambda_2 + 6\lambda_3 = 0$ — (iii)

$$\left.\begin{array}{l} (ii) + 2(i) \Rightarrow 8\lambda_1 + 8\lambda_2 = 0 \\ (iii) - 6(i) \Rightarrow -11\lambda_1 - 11\lambda_2 = 0 \end{array}\right\} \Rightarrow \lambda_1 = -\lambda_2$$

(iii) $\Rightarrow 6\lambda_1 + 6\lambda_3 = 0 \Rightarrow \lambda_1 = -\lambda_3$

If $\lambda_1 = 1$, then $\lambda_2 = -1 = \lambda_3$

$\therefore$ Therefore the vectors x_1, x_2, x_3 are linearly dependent

The relationship is $x_1 - x_2 - x_3 = 0$

Eg 3 Show that the vetors $x_1 (1,1,1)$, $x_2 (1,2,3)$, $x_3 = (2,3,8)$ are linearly independent.

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$ is equivalent to

$\lambda_1 (1,1,1) + \lambda_2 (1,2,3) + \lambda_3 (2,3,8) = (0,0,0)$

$\Rightarrow \lambda_1 + \lambda_2 + 2\lambda_3 = 0$ — (i)

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ — (ii)

$\lambda_1 + 3\lambda_2 + 8\lambda_3 = 0$ — (iii)

(ii) - (i) $\Rightarrow \lambda_2 + \lambda_3 = 0$ — (iv)

(iii) - (i) $\Rightarrow \lambda_2 + 5\lambda_3 = 0$ — (v)

(iv) $\Rightarrow \lambda_3 = -\lambda_2$

(v) $\Rightarrow -4\lambda_2 = 0 \Rightarrow \lambda_2 = 0$

$\therefore \lambda_3 = 0 = \lambda_1$

So the vectors are linearly independent.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

If A & B are symmatric matrices prove that AB-BA is a skew symmatric matrix.

Date 04-01-2021 →

System of linear equations —

System of linear non-homogeneous equations -

Let
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

be a system of m non-homogeneous equations in unknown $x_1, x_2, \ldots x_n$. It can be written as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m\times 1}$$

Any set of values which simultaneously satisfies these equations is called a solⁿ of the system (1). When the system has one or more solutions the equations are said to be consistent otherwise they are said to be inconsistent.

Consistency of a system of linear equations →

To determine whether the equations (1) are consistent (ie possess a solⁿ) or not, we consider the ranks of the matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

As the coefficient matrix and K is called augmented matrix of eqⁿ (1)

Theorem - If the system of eqⁿ (1) inconsistent iff. then cofficient of matrix A and the augmented matrix K are of the same rank. otherwise the system is inconsistent.

Procedure to test the consistency –

Find the rank of A and aug K by reducing A to triangular form by elementary row transformation.

Let the rank of A be 'r' & rank of ~~the~~ K be 'r′'

i) If $r \neq r'$, the equations are inconsistent.

ii) If $r = r' = n$, the equations are consistent and there is a unique soln.

iii) If $r = r' < n$, the equations are consistent and there are infinite no of soln. Giving arbitary values to $n-r$ of the unknowns, we may express the other r unknowns in ~~tem~~ terms of those.

** Matrix $\begin{bmatrix} a & f & g \\ 0 & b & h \\ 0 & 0 & c \end{bmatrix}$ is upper Δ^r and matrix $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ is lower Δ^r

Eg1- Test the consistency and solve

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Soln It is of the form $AX = B$. Now,

$$\text{aug } A = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \xrightarrow[5R_3 - 7R_1]{5R_2 - 3R_1} \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & +33 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$$\xrightarrow{R_2/11} \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Now, $\rho(A) = 2$ and $\rho(\text{aug } A) = 2$

$\therefore$ The system is consistent $2 < 3$ (infinite soln)

Soln is given by $5x + 3y + 7z = 4$

$11y - z = 3$

So $y = \frac{3}{11} + \frac{z}{11}$ & $x = \frac{7}{11} - \frac{16}{11} z$

where z is a parameter

If we put $z = 0$, $x = \frac{7}{11}$, $y = \frac{3}{11}$ is a particular soln of the system.

Eg 2 - Test the consistency and solve

i) $2x - 3y + 7z = 5$; $3x + y - 3z = 13$; $2x + 19y - 47z = 32$

ii) $x + y + z = 6$; $2x + y + 3z = 13$; $5x + 2y + z = 12$; $2x - 3y - 7z = -10$

Soln ii) It is of form $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ 2 & -3 & -2 \end{bmatrix}_{4\times 3} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} \qquad B = \begin{bmatrix} 6 \\ 13 \\ 12 \\ -9 \end{bmatrix}_{4\times 1}$$

$$\text{Aug } A = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \\ 5 & 2 & 1 & 12 \\ 2 & -3 & -2 & -9 \end{array}\right] \xrightarrow[R_4 - 2R_1]{R_2 - 2R_1,\ R_3 - 5R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 7 \\ 0 & -3 & -4 & -18 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

$$\xrightarrow[R_4 - 5R_2]{R_3 - 3R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -7 & -21 \\ 0 & 0 & -9 & -27 \end{array}\right] \xrightarrow[R_4/-9]{R_3/-7} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_4 - R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* Rank of a matrix is equal to number of non-zero rows.

Here $\rho(A) = 3$ & $\rho(\text{Aug } A) = 3$ So consistent

For soln, we write

$$x + y + z = 6$$
$$-y + z = 1$$
$$z = 3$$
$$-y = -2 \Rightarrow y = 2$$
$$x + 2 + 3 = 6 \Rightarrow x = 1$$

Soln of system is $(1, 2, 3)$

Eg 3 Show that the equations $3x + 4y + 5z = a$; $4x + 5y + 6z = b$; $5x + 6y + 7z = c$ don't have a soln unless $a + c = 2b$

Soln Eqn are
$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$ in $AX = B$ form

using elementary row transformations and reducing the augmented matrix ~~Ax=B~~ [AB] to ~~at~~ echelon (upper Δ^r form) we get,

$$AB = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix} \xrightarrow[R_3 - 5/3 R_1]{R_2 - 4/3 R_1} \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -4/3 & -2/3 & \frac{3b-4a}{3} \\ 0 & -2/3 & -4/3 & \frac{3c-5a}{3} \end{bmatrix}$$

$$\xrightarrow[3R_3]{3R_2} \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & (3c-5a)-2(3b-4a) \end{bmatrix}$$

Here $\rho(A) = 2$, $\rho(AB) \neq 2$

So, the system is inconsistent. It will be consistent only when

$(3c - 5a) - 2(3b - 4a) = 0$

$\Rightarrow 3c - 6b + 3a = 0$

$\Rightarrow a + c = 2b$

Eg 4- Investigate for what values of λ & μ the simultaneous eqn $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have i) no soln, ii) a unique soln, iii) an infinite no of soln.

Soln The system is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

aug matrix $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

i) It will have no soln if $\lambda = 3$, $\mu \neq 10$

ii) A unique soln if $\lambda \neq 3$, & μ may take any values

iii) An infinite of soln if $\lambda = 3$ & $\mu = 10$

Eg-5 Test the consistency of $2x + 6y + 11 = 0$, $6x + 2y - 6z + 3 = 0$ & $6y - 18z + 1 = 0$

system of linear homogeneous equation

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\} \text{--- (ii)}$$

It is of the form $AX = 0$

To solve the system (ii) find $\rho(A)$ by reducing it to to the triangular form by elementary row operation.

i) If $\rho(A) = n$, the equations have only a trivial solⁿ

ii) If $\rho(A) < n$, the equations have an infinite no of trivial solⁿ.

Theorem – If r is the rank of the matrix A then the no of linearly independent solⁿ of m homogeneous linear eqⁿ in n unknowns is $(n-r)$

Case 1 If

Eg-1 Does the following system of equations possess a common non-zero solⁿ ?

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

Solⁿ The system can be written as $AX = 0$ ie.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{vmatrix} = -2 \neq 0$

$\therefore \rho(A) = 3 =$ no of unknowns

Hence zero solⁿ ie, $x=0, y=0, z=0$ is the only possible solⁿ of the system of equations.

Eg-2 $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ It is $AX = 0$ form

$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ \dots \end{vmatrix} = 0$ $\therefore \rho(A) < 3$

Applying Elementary row operation in A we get

$$A \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \rho(A)=2$$

Hence the system of equation possess $3-2=1$ linear independent sol^n

Now the system is equivalent to $x+3y-2z=0$

$-7y+8z=0$

$\Rightarrow y=\frac{8}{7}z, \quad x+(3.\frac{8}{7}-2)z=0$

$\Rightarrow x=-\frac{10}{7}z$

choose $z=c$, arbitary const. Then $x=-\frac{10}{7}c$, $y=\frac{8}{7}c$, $z=c$

Constitute the general sol^n of the given system. We get infinite no of sol^n ~~of the given system~~. for different values of c.

~~Eg 3~~ * Test the consistency of $2x+6y+11=0$, $6x+20y-6z \neq 3$,

$6y-18z+1=0$

~~sol^n~~ Eg 3 Solve $x+y-2z+3\omega=0$

$x-2y+z-\omega=0$

$4x+y-5z+8\omega=0$

$5x-7y+2z-\omega=0$

coefficient matrix $A=\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \xrightarrow[\substack{R_3-4R_1 \\ R_4-5R_1}]{R_2-R_1} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$

$$\xrightarrow[R_4-4R_2]{R_3-R_2} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore \rho(A)=2<4$$

So non-trivial sol^n exist in terms of $n-r=4-2=2$ variables.

choose $z=K_1$, $\omega=K_2$. Thus solving we get.

$x+y-2z+3\omega=0$ — (I)

$-3y+3z-4\omega=0$ — (II)

(II) $\Rightarrow y=+z-\frac{4}{3}\omega=K_1-\frac{4}{3}K_2$

(I) $\Rightarrow x=-y+2K_1-3K_2=K_1-\frac{5}{3}K_2$

where K_1, K_2 are arbitary Const. So we get infinite no of sol^n.

Eg-4 Determine b such that the system of homogeneous eqⁿs $2x+y+2z=0$, $x+y+3z=0$, $4x+3y+bz=0$ has

i) trivial solⁿ ii) non-trivial solⁿ. Find the non-trivial solⁿ.

characteristic eqⁿ, Eigens volumes, Eigen vectors

Definition

Let $A = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ an independent indeterminment. The matrix $(A - \lambda I)$ is called the characteristic matrix of A where I is the unit matrix of order n.

The determinent $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A. The eqⁿ $|A - \lambda I| = 0$ is called the characteristic eqⁿ of A and the roots of this eqⁿ are called the characteristic roots or characteristic values or eigen values or latent roots or probe values of the matrix A. The set of Eigen values of A is called the spectrum of A.

If λ is a characteristic root of the matrix A then $|A - \lambda I| = 0$ and the matrix $A - \lambda I$ is Singular. Therefore there exists a non-zero vector of X such that $(A - \lambda I) X = 0$ or $AX = \lambda X$. Then the vector X is called the characteristic vector or eigen vector of A corresponding to the characteristic root λ.

Theorem-1 - λ is a Characteric root of a matrix A iff $\exists$ a non-zero vector X such that $AX = \lambda X$

Theorem-2 - If X is a characteristic vector of a matrix A corresponding to the characteristic vector λ, then KX is also a characteristic vector of A corresponding to same characteristic value λ. Here K is a non-zero scaler.

Theorem-3 - If X is a characteristic vector of a matrix A, then X can't correspond to more than one characteristic values of A.

Proof Let X be a characteristic vector of a matrix A Corresponding to two characteristic values λ_1 & λ_2, then

$AX = \lambda_1 X$ & $AX = \lambda_2 X$

$\Rightarrow (\lambda_1 - \lambda_2) X = 0 \Rightarrow \lambda_1 = \lambda_2$ $\quad X \neq 0$

Eg-1 Find Eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solⁿ characteristic eqⁿ is $|A - \lambda I| = 0$ where $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

ie, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (-2-\lambda)\{(1-\lambda)(-\lambda) - 12\} + 2\{6 + 2\lambda\} - 3\{-4 + (1-\lambda)\} = 0$

$\Rightarrow (-2-\lambda)(-\lambda + \lambda^2 - 12) + (12 + 4\lambda) + (9 + 3\lambda) = 0$

$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\lambda = -3$ satisfies this eqⁿ $\therefore \lambda + 3$ is a factor

$\therefore (\lambda + 3)(-\lambda^2 + 2\lambda + 15) = 0$

or $\Rightarrow (\lambda + 3)(-\lambda^2 + 5\lambda - 3\lambda + 15) = 0$

$\Rightarrow (\lambda + 3)(\lambda(5 - \lambda) + 3(5 - \lambda)) = 0$

$\Rightarrow (\lambda + 3)(\lambda + 3)(5 - \lambda) = 0$

$\therefore \lambda = -3, -3, 5$ which all the eigen values of A. The eigen vectors of A corresponding to the eigen value -3 are given by

$(A - \lambda I) X = 0$. Here $(A + 3I) X = 0$

ie, $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ —— ① $\quad 3I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

We are to solve ①.

Here $|A + 3I| = 1(12 - 12) + 2(6 - 6) - 3(-4 + 4) = 0$

$\therefore \rho(A + 3I) \neq 3$ ie, < 3

Reducing the matrix to triangular form using elementary row transformations we get.

① $\xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \rho(A+3I) = 1$, So there are $3-1=2$ linearly independent

Sol^n The system reduces to the form $x+2y-3z=0$.

Obviously $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are two linearly

independent eigen

vectors of A corresponding to the eigen value -3. If c_1 & c_2

are scalars not both equals to zero, then $c_1x_1 + c_2x_2$ gives

all the eigen vectors of A.

Next the eigen vectors of A corresponding to the eigen value 5

are given by $(A-5I)=0$ $\quad 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

ie, $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (ii)

Now $|A-5I| = -7(20-12)+2(6+10)-3(-4-4) = 0$

$\therefore \rho(A-5I) < 3$

Again reducing $(A-5I) = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$ to triangular form

we get

$(A-5I) \xrightarrow{R_{13}} \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \xrightarrow[R_3-7R_1]{R_2+2R_1} \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}$

$\xrightarrow{R_3+2R_2} \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \therefore \rho(A-5I) = 2$

So here is $3-2=1$ linear independent Sol^n

The system can be written as $\quad x+2y+5z=0$
$\quad 8y+16z=0$

$\Rightarrow y=-2z$, $x=4z-5z=-z$

If $z=1$, $x=-1$, $y=-2$

$x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is an eigen vector of A.

Eg- Find the eigen values and corresponding eigen vectors of the matrices.

i) $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

iv) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ v) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solⁿ: i) Let $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

It's characteristic eigen is $|A-\lambda I| = 0$ ie, $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(4-\lambda) - 10 = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow (\lambda-6)(\lambda+1) = 0$

$\therefore \lambda = 6, -1$ are characteristic eigen roots of A.

Characteristic vector of corresponding to charecteristic roots is given by $(A-6I)X = 0$. ie., $\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(A-6I) \xrightarrow{R_2 - R_1} \begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix}$ $\therefore \rho(A-6I) = 1$

So there is $2-1 = 1$ independent solⁿ

The system is $5x + 2y = 0 \Rightarrow x = -\frac{2}{5}y$.

If $y = 1$, $x = -\frac{2}{5}$ $\therefore x_1 = \begin{bmatrix} -2/5 \\ 1 \end{bmatrix}$ is an eigen vector of a corresponding to 6.

Next characteristic vector corresponding to -1 is given by

$(A + 1\lambda) X = 0$

ie $\begin{bmatrix} 0 & -2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Here $\rho(A + I\lambda) = 2$ So we will get trival solⁿ

ie $x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is eigen vector of A.

Solⁿ Let $A = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

characteristic eqn is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda)^2 + 0 + 1(0-(2-\lambda)) = 0$

$\Rightarrow (2-\lambda)(\lambda^2-4\lambda+3)$

$\Rightarrow (2-\lambda)(\lambda-3)(\lambda-1) = 0$

$\lambda = 2, 3, 1$ are eigen values of A.

~~Now~~ eigen vectors corresponding to $\lambda = 2$ ~~are~~ given by $(A-2I)X = 0$

ie $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$z = 0$, $x = 0$, $y \neq 0$ $\quad$ As $\rho(A-2I) = 1$

So y can take ~~two~~ any value hence $x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$ is given by $(A-3I)X = 0$

ie $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A-3I| = \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0$

$\therefore \rho(A-3I) < 3$

Now $A - 3I \xrightarrow{R_3 + R_1} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \rho(A-3I) = 2$

So there is $3-2 = 1$ linear independent solⁿ

The system is now $\quad -x + z = 0 \quad$ } $y = 0$
$\quad y = 0 \quad$ } $\Rightarrow x = z$

If in particular, $z = 1, x = 1$.

$x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is eigen vector corresponding to eige value $\lambda = 3$

For $\lambda = 1$ eigen vector is given by $(A - \lambda I)X = 0$

ie $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $A - \lambda I \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $\rho(A - \lambda I) = 2$

System is $x + z = 0$ $y = 0 \Rightarrow x = -z$

$\therefore x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The Cayley-Hamilton Theorem -

Every square matrix satisfies it's characteristic eqn.

If for a square matrix of order n, $|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + \dots + a_n]$

then the matrix eqn $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n I = 0$ is

satisfied by $x = A$ ie, $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots a_n I = 0$ —— (1)

Corollary - If A is non-singular, then $|A| \neq 0$

Premultiplying (1) by A^{-1} we get,

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots a_n A^{-1} = 0$$

$$A^{-1} = -\frac{1}{a_n} \left(A^{n-1} + a_1 A^{n-2} + \dots a_{n-1} I \right)$$

Eg Verify Cayley-Hamilton Thm for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Soln we have $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{vmatrix}$

Now characteristic eqn $|A - \lambda I| = 0 \rightarrow$

$$(1-\lambda)\{(1-\lambda)^2 - 3\} + 1\{2 - 3(1-\lambda)\} + 2\{9 - 2(1-\lambda)\} = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 7\lambda + 11 = 0$$

To verify that $-A^3 + 3A^2 + 7A + 11I = 0$ —— (1)

Now, $A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 42 & 31 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{bmatrix}$$

$$LHS \text{ ① } = -\begin{bmatrix} 42 & 31 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{bmatrix} + 3\begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix} + 7\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RHS$$

Hence, Caley - Hemilton theorem is varified

Multiplying ① by A^{-1} we get, $-A^2 + 3A + 7I + 11A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{11}\left[A^2 - 3A - 7I\right]$$

$$= \frac{1}{11}\left[\begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{11}\begin{bmatrix} 2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$$

Eg-2 Verify Cauly Hemilton theorem for the matrices below, and find inverse i) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Eg-3 If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & 4 & -3 \end{bmatrix}$, evaluate A^{-1}, A^{-2}, A^{-3}

Sol^n characteristic eq^n of the matrix A is $|A - \lambda I| = 0$

ie $\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$

$$\Rightarrow -\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

According to C-H theorem we get $-A^3 + 4A^2 + A - 4I = 0$ — (I)

Premultiplying (I) by A^{-1} we get, $-A^2 + 4A + I - 4A^{-1} = 0$ — (II)

$$\Rightarrow A^{-1} = \frac{1}{4}\left[-A^2 + 4A + I\right]$$

Premultiplying (II) by A^{-1} we get $-A + 4I + A^{-1} - 4A^{-2} = 0$ — (III)

$$\Rightarrow A^{-2} = \frac{1}{4}\left[-A + 4I + A^{-1}\right]$$

Premultiplying (iii) by A^{-1} we get. $-I + 4A^{-1} + A^{-2} - 4A^{-3} = 0$

$$\Rightarrow A^{-3} = \frac{1}{4}\left(-I + 4A^{-1} + A^{-2}\right)$$

— × —

Fourier Series

Eg 3 – Find the F-series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$

Sol^n Let $f(x) = x - x^2$, neither even nor odd f^n

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = +\frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi - \pi^2).0 + \frac{1}{n^2}\left\{(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi\right\}\right.$$

$$= \frac{1}{\pi . n^2}\cos n\pi\,(1 - 2\pi - 1 - 2\pi)$$

$$= -\frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(\pi - \pi^2)\cos n\pi - (-\pi-\pi^2)\cos n\pi\right] - \frac{2}{n^3}(\cos n\pi - \cos n\pi)$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi\,(\pi - \pi^2 + \pi + \pi^2)\right] = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$\therefore$ F - Series is :

$$x - x^2 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

or $$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots\right]$$

Obs Putting $x = 0$, we find

$$0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right] + 0$$

$$= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots = \frac{\pi^2}{12}$$

Eg 4 - Find the F-series corresponding to $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$. Hence Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Sol^n $a_0 = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx = \cdots = \frac{\pi^2}{6}$

$a_n = \cdots = \frac{1}{n^2}$, $b_n = 0$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

Put $x = 0$ Then $\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum \frac{1}{n^2}$

$$\Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

Eg 5 - Find the fourier series representation of $f(x) = x$, $0 < x < 2\pi$

Ans $a_0 = 2\pi$, $a_n = 0$, $b_n = -\frac{2}{n}$

Eg 6 - Given that $f(x) = x^2 + x$ for $-\pi < x < \pi$, find the F. Series of $f(x)$ deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Ans $a_0 = \frac{2}{3}\pi^2$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = \frac{2(-1)^n}{n}$

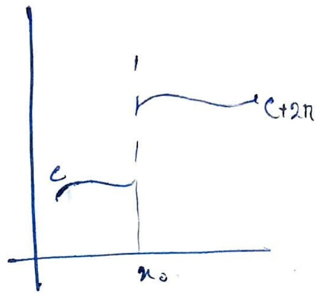
~~Eg 7 Find F-Series of $f(x) = e^{-x}$ in $0 < x < 2\pi$~~

~~Ans $a_0 = \frac{1}{\pi}(1-e^{-2\pi})$~~

Deduction Put $x = \pi$, $x = -\pi$, and add

If $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier Series

Fourier Series



Let $f(x) = f_1(x)$. $c < x < x_0$

$= f_2(x)$, $x_0 < x < c + 2\pi$

where x_0 is the point of finite discontinuity in the interval $(c, c+2\pi)$, the values of x_0. a_n, b_n are given by ~~as~~

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0-0)$ & $f(x_0+0)$ exist but are unequal.

The sum of the F-Series $= \frac{1}{2}\left[f(x_0-0) + f(x_0+0)\right]$

Eg 1 Find the F-Series ~~for~~ to represent the function $f(x)$ given by $f(x) = x$ for $0 \le x \le \pi$

$= 2\pi - x$ for $\pi \le x \le 2\pi$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Sol^n $a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx = \dots = \pi$

$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{2\pi}(2\pi - x)\cos nx\,dx$

$= \dots\dots = \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}$

$b_n = 0$

$$\therefore f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx \quad \text{——} \quad (A)$$

Put $x = \pi$ in (A) As π is the point of discontinuity for $f(x)$.

Here $f(\pi-) = \pi$ & $f(\pi+) = \pi$

By Dirichlet's theorem the sum of the series at

$x = \pi$ is $\dfrac{f(\pi-) + f(\pi+)}{2} = \dfrac{2\pi}{2} = \pi$

$$\therefore \pi = \frac{\pi}{2} + \frac{2}{\pi}\left\{\frac{-2}{1^2}\cos\pi + 0 - \frac{2}{3^2}\cos 3\pi \ldots\right\}$$

$$\Rightarrow \cancel{\frac{\pi \cdot \pi}{2}}\ \frac{\pi}{2} \cdot \frac{\pi}{2} = 2\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right\}$$

For discontinuity

Eg 8 - Find the F-series of the function

$$f(x) = -1, \quad -\pi < x < -\pi/2$$
$$= 0, \quad -\pi/2 < x < \pi/2$$
$$= 1, \quad \pi/2 < x < \pi$$

Solⁿ $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)dx + \int_{-\pi/2}^{\pi/2} 0.dx + \int_{\pi/2}^{\pi} 1.dx\right] = 0$$

$a_n = 0$, $b_n = \dfrac{2}{n\pi}\left[\cos\dfrac{n\pi}{2} - \cos n\pi\right]$

Eg 9. $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots = \dfrac{\pi^2}{8}$

Change of interval

In many engineering ~~programme~~ problem, it is desired to expand a function of in a Fourier series over an interval of $2l$ and not 2π. This can be achieved by a transformation of the variable.

Consider $f(x)$ defined in the interval $c < x < c+2l$. To change the interval into one of the length 2π, we put $z = \dfrac{\pi x}{l}$

$x = c \Rightarrow z = \dfrac{\pi c}{l} = d$, $x = c+2l \Rightarrow z = \dfrac{\pi}{l}(c+2l) = \dfrac{\pi c}{l} + 2\pi = d + 2\pi$

Thus the function $f(x)$ of period $2l$ in $(c, c+2l)$ is transformed to the function $f(z)$ say the period 2π in $(d, d+2\pi)$ and can be expressed as the F-series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

where $a_0 = \frac{1}{\pi} \int_d^{d+2\pi} F(z)\,dz$, $a_n = \frac{1}{\pi} \int_d^{d+2\pi} F(z) \cos nz\,dz$

$$b_n = \frac{1}{\pi} \int_d^{d+2\pi} F(z) \sin nz\,dz$$

Now making the inverse substitution

$$z = \frac{\pi x}{l}, \quad dz = \frac{\pi}{l}\,dx$$

when $z = d$, $x = c$, when $z = d + 2\pi$, $x = c + 2l$

the expression (1) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x\right)$$

where $a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi}{l} x\,dx$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi}{l} x\,dx$$

Cor 1 If $c = 0$, $0 < x < 2l$ &

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\,dx, \quad a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi}{l} x\,dx \text{ etc}$$

Cor 2 If $c = -l$, $-l < x < l$

$$a_0 = a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx \text{ etc.}$$

If $f(x)$ is even, $a_0 = \frac{2}{l} \int_0^l f(x)\,dx$, $a_n = \frac{2}{l} \int_0^l f(x) . \cos \frac{n\pi}{l} x\,dx$

$b_n = 0$

If $f(x)$ is odd, $a_0 = a_n = 0$ & $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x\,dx$

Eq-1 Dirichlet's condition for a F-series -

Let $f(x)$ be a function satisfying the following conditions

i) $f(x)$ is periodic, single valued and bounded.

ii) $f(x)$ has at most a finite no of maxima & minima.

iii) $f(x)$ has a finite no of discontinuities in any one period.

The above conditions are called Dirichlet's conditions.

Eg1- Expand $f(x) = e^{-x}$ as F-series in $(-1,1)$. Here $2l = 2 \Rightarrow l = 1$

$$a_0 = 1 \int_{-1}^{1} e^{-x}\,dx = \left[-e^{-x}\right]_{-1}^{1} = -e^{-1} + e = 2\sinh 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} e^{-x}\cos\frac{2\pi}{1}x\,dx = \frac{e^{-x}}{1+n^2\pi^2}\left[-1\cos n\pi x + n\pi\sin n\pi x\right]_{-1}^{1}$$

From Bernaullis formula

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

or $$a_n = \frac{1}{1+n^2\pi^2}\left[-e^{-1}\cos n\pi + e^{1}\cos n\pi + n\pi\left(-e^{-1}\sin n\pi - e^{1}\sin n\pi\right)\right]$$

$$= \frac{\cos n\pi}{1+n^2\pi^2}\left(e^{1} - e^{-1}\right) + 0 = \frac{2(-1)^n}{1+n^2\pi^2}\sinh 1$$

$$b_n = \int_{-1}^{1} e^{-x}\sin\frac{n\pi}{1}x\,dx = \left[\frac{e^{-x}}{1+n^2\pi^2}\left(-1\sin n\pi x - n\pi\cos n\pi x\right)\right]_{-1}^{1}$$

$$= \frac{1}{1+n^2\pi^2}\left[-n\pi\left(e^{-1}\cos n\pi - e^{1}\cos n\pi\right)\right]$$

$$= -n\pi\frac{\cos n\pi}{1+n^2\pi^2}\left(e^{-1} - e^{1}\right) = \frac{2n\pi(-1)^n}{1+n^2\pi^2}\sinh$$

$$\therefore f(x) = \sinh\left(\left[\frac{2}{2} + 2\sum\left\{\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi x + \frac{(-1)^n}{1+n^2\pi^2}n\pi\sin n\pi x\right\}\right]\right)$$

Half range series

Sometimes it is required to expand a function $f(x)$ in the range $(0,\pi)$ in a F-series of period 2π, or more generally in the range $(0,l)$ in a F-series of period $2l$.

If it is required to expand $f(x)$ in $(0,l)$, then it is immaterial what the function may be outside the range $0<x<l$. We are free to choose it arbitary in the interval $(-l,0)$.

If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extented f^n is even for which $b_n = 0$, so f(x) will contain only cosine terms.

If we extend f(x) by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$ and f(x) will contain only sine terms.

Examples of Dirichlet's Condition –

Eg-2- Find the F- series for the function

$$f(x) = \begin{cases} x & \text{in } 0 < x < 1 \\ 1-x & \text{in } 1 < x < 2 \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Here $2l = 2 \Rightarrow l = 1$

$$a_0 = \frac{1}{1}\left[\int_0^1 x\,dx + \int_1^2 (1-x)\,dx\right]$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left(x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{2}\left\{(2-1) - \frac{1}{2}(4-1)\right\}$$

$$= \frac{1}{2} + 1 - \frac{3}{2} = 0$$

$$a_n = \frac{1}{1}\int_0^1 x\cos n\pi x\,dx + \int_1^2 (1-x)\cos n\pi x\,dx$$

$$= \left[x\frac{\sin n\pi x}{\pi x} - 1\left(-\frac{\cos n\pi x}{\pi^2 n^2}\right)\right]_0^1 + \left[(1-x)\frac{\sin n\pi x}{n\pi} - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

or $a_n = \frac{1}{n^2\pi^2}\{(-1)^n - 1\} + \frac{1}{n^2\pi^2}\{(-1)^n - 1\}$

$$= \frac{2\{(-1)^n - 1\}}{n^2\pi^2}$$

$$b_n = \int_0^1 x\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx$$

$$= \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(1-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]$$

$$= -\frac{1}{n\pi}(\cos n\pi - 0) + 0 - \frac{1}{n\pi}(-1\cos 2n\pi - 0)]$$

$$= \frac{1}{n\pi}\left[-\cos n\pi + \cos 2n\pi\right]$$

$$= \frac{1}{n\pi}\left[-(-1)^n + 1\right]$$

$$\therefore f(x) = 0 + 2\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-(-1)^n)\sin n\pi$$

$$= \frac{2}{\pi^2}\left\{\frac{-2}{1^2}\cos\pi x + 0 - \frac{2}{3^2}\cos 3\pi x \cdots\right\} + \frac{1}{\pi}\left\{\frac{2}{1}\sin\pi x + \frac{2}{3}\sin 3\pi x + \cdots\right\} \quad \text{— (A)}$$

We now put $x=1$, since 1 is the point of discontinuity for $f(x)$. Thus the left limit $f(1^-) = 1$ and the right limit $f(1^+) = 1-1 = 0$, by Dirichlet's theorem the sum of the series at $x=1$ is $\cancel{f(1^-)+f(1^+)}$ $\frac{f(1-)+f(1+)}{2} = \frac{1}{2}$

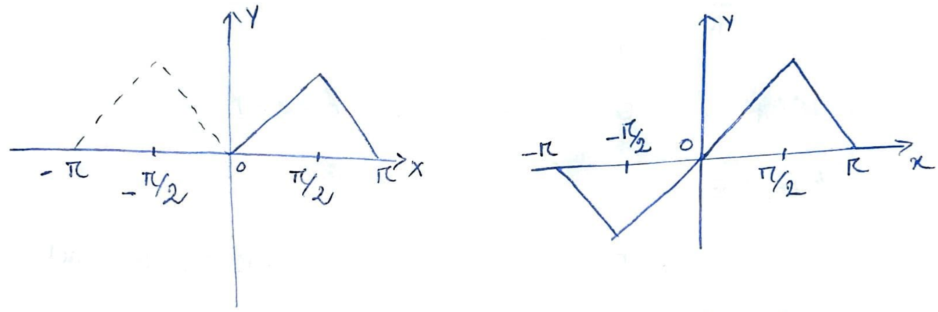
Thus, $\frac{1}{2} = \frac{2(-2)}{\pi^2}\left\{\frac{\cos\pi}{1^2} + \frac{\cos 3\pi}{3^2} + \cdots\right\} + \frac{2}{\pi}\left\{\frac{\sin\pi}{1} + \frac{\sin 3\pi}{3} + \cdots\right\}$

$= \frac{-4}{\pi^2}\left(-\frac{1}{1^2} - \frac{1}{3^2} - \cdots\right) + 0$

$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \cancel{\frac{8}{\pi^2}}\ \frac{\pi^2}{8}$

Eg $f(x) = x, \quad 0 < x < \frac{\pi}{2}$

$= \pi - x, \quad \pi/2 < x < \pi$



Hence a function $f(x)$ defined over the interval $0 < x < 1$ is capable of two distinct half range series. The half range cosine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\,dx \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\,d\ \sin\frac{2\pi}{l}x$$

corrilary – If the range is $0 < x < \pi$ then

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \ldots$

Ex 1 Find the half range F-series of $f(x) = x$ in $0 < x < 2$

Sol^n Here $b_n = \frac{2}{2}\int_0^2 x \sin\frac{n\pi x}{2}\,dx$

$= \left[x\left(-\frac{\cos n\pi}{2} \times \frac{2}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{2} \times \frac{2}{n^2\pi^2}\right)\right]$

$= \left[-\frac{2}{2\pi}(2\cos n\pi - 0) + \frac{2n}{n^2\pi^2}(\sin n\pi - 0)\right]$

$= \frac{-4}{n\pi}(-1)^n$

$= \frac{4}{n\pi}(-1)^{n+1}$

Half range sine series

$$= \frac{4}{\pi} \sum_{n=1}^{\alpha} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x$$

$$= \frac{4}{\pi} \left\{ \frac{\sin \frac{n\pi}{2}}{1} - \frac{1}{2} \frac{\sin 2\pi x}{2} + \dots \right\}$$

Eg-2 - Express $f(x) = 2$ as a half range cosine series in $0<x<2$

Soln Here $a_0 = \frac{2}{2} \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$

$$a_n = \int_0^2 x \frac{\cos n\pi}{2} x\,dx = \left[x \sin \frac{n\pi}{2} x \cdot \frac{2}{n\pi} - 1 \left(-\cos \frac{n\pi}{1} x\right)\right.$$

$$= 0 + \frac{2^2}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{2^2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore x = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\alpha} \left\{ \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{2} \right.$$

4) Obtain half range sine series of e^x in $0<x<1$

10) odd even function

Eg- Expand $f(x) = [x]$ in F- series in interval $-\pi \leq x \leq \pi$ and prove

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Soln $f(x) = |x|$ is even periodic function

So $f(x) = \frac{a_0}{2} + \sum a_m \cos n\pi$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\,dx . 2\pi$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\pi\,dx$

Eg A periodic function of period 4. $f(x) = |x|$, $-2<x<2$. Find its F- series.

Soln $f(x) = |x|$, $-2<x<2$

$$f(x) = \begin{cases} -x, & -2<x<0 \\ x, & 0<x<2 \end{cases} \quad \left.\begin{matrix} 2l = 4 \\ l = 2 \end{matrix}\right\}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_0^2 x\,dx\right] = 2$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} -x \cos \frac{n\pi x}{2}\,dx + \int_0^2 x \cos \frac{n\pi x}{2}\,dx\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Since $f(x) = |x|$ is even function, $b_n = 0$

Eg. Find half range cosine series of function

$$f(x) = \begin{cases} Kx & , 0 \leq x \leq \frac{1}{2} \\ K(1-x) & \frac{1}{2} \leq x < 1 \end{cases}$$

and hence prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Convergence of sequence and series

Sequences

Def^n A sequence is a function whose domain is the set of natural numbers. If the co-domain is the set of real no.; it is called a real sequence; if it is the set $\mathbb{C}$ of complex nos, it is called a complex sequence and likewise if it is a set of polynomials, it is a sequence of polynomials. The sequence is denoted by $\{x_n\}$

Thus ① $x_n = \frac{1}{n}$ is a sequence whose 1st, 2nd, 3rd terms are resply. $1, \frac{1}{2}, \frac{1}{3}, \dots$ This sequence is called the harmonic sequence.

② $y_n = (-1)^n$ is a ~~te~~ sequence. It's 1st few terms are $\{-1, 1, -1, 1, \dots\}$

③ $z_n = 5$ is also a sequence each of its terms being 5. Such a sequence is called a constant sequence.

Bounded and unbounded Sequence -

A sequence $\{x_n\}$ is said to be bounded above if all its tearms are less than or equal to a real number, ie there exists $K \in \mathbb{R}$ st. $x_n \leq K \ \forall n \in \mathbb{N}$. For example, the sequence $\{\frac{1}{n}\}$ is abounded above since $\frac{1}{n} \leq 1 \ \forall n \in \mathbb{N}$. But the sequence $\{n^2\}$ is not bounded above since there exists no such real no K so that $n^2 \leq K \ \forall n$.

A sequence $\{x_n\}$ is said to be bounded below if all its teams are greater than or equal to a real no ie, there exist $K \in \mathbb{R}$ such $x_n \geq K \ \forall n \in \mathbb{N}$. The sequence $\{\frac{1}{n}\}$ is bounded below since $\frac{1}{n} \geq 0 \ \forall n$. The sequence $\{(-1)^n 5\}$ is bounded below since $(-1)^n 5 \geq (-5)$ $\forall n$, but the sequence $\{(-2)^n\}$ is not bounded below since there is no such real no K for which $K \leq (-2)^n$. A sequence is said to be bounded if it is bounded above and below.

Convergent sequence

Def^n A sequence $\{x_n\}$ is said to converge to a real no l if for every $\epsilon > 0$, $\exists$ $n_0 \in N$ such that $|x_n - l| < \epsilon$ for every $n \geq n_0$

The no l is called limit of the sequence $\{x_n\}$ ie $\lim_{n \to \infty} x_n = l$

A sequence $\{x_n\}$ is called convergent if it converges to a limit l

A sequence which converges to zero is called a null sequence

Monotone sequence –

A sequence $\{x_n\}$ is said to be monotone increasing if $x_n \leq x_{n+1}$ $\forall n \in N$: The sequence is called strictly increasing if $x_n < x_{n+1}$ $\forall n \in N$

Clearly the sequence $\{x^n\}$ is monotone (strictly) increasing since $n^2 \leq (n+1)^2$ always. The sequence $\{(-2)^n\}$ is a monotone increasing since $(-2)^2 \neq (-2)^3$

A sequence $\{x_n\}$ is said to be monotone decreasing if $x_{n+1} \leq x_n$ $\forall n \in N$; The sequence is called strictly decreasing if $x_{n+1} < x_n$ $\forall n \in N$.

The sequence $\{\frac{1}{n^2+1}\}$ is monotone (strictly) decreasing as $\frac{1}{(n+1)^2+1} \leq \frac{1}{n^2+1}$ $\forall n$. The sequence $\{-n^3\}$ is strictly decreasing as $-(n+1)^3 < -n^3$ but the sequence $\{-\frac{1}{2}\}^n$ is not monotone or strictly decreasing as $(-\frac{1}{2})^4 \nless (-\frac{1}{2})^3$

Fact 1: A sequence may or may not converge

Fact 2: If a sequence is convergent, it converges to a ~~limit~~ unique limit

Fact 3: Every convergent sequence is always bounded but not conversely

Fact 4: A monotone increasing sequence bounded above is always convergent and converges to its least upper bound.

Fact 5: A monotone decreasing sequence bounded below is always convergent and converges to its greatest lower bound.

Fact 6: Every const. ~~seq~~ sequence is convergent.

Infinite series –

An infinite series is denoted by $\sum_{n=1}^{\infty} x_n$ or simply by $\sum u_n$

The sum of the 1st n terms of this series is denoted by S_n where $S_n = (u_1 + u_2 + \dots + u_n)$ and is called the nth partial sum of the series

i) If $S_n \to s$ (a finite value) as $n \to \infty$, then the series $\sum u_n$ is said to be convergent and s is called its sum.

ii) If $S_n \to \pm\infty$ as $n \to \infty$, then the series $\sum u_n$ is called a divergent series.

iii) If S_n oscillates (finitely or infinitely) as $n \to \infty$ the finite series $\sum u_n$ is said to be oscillatory.

A divergent or oscillatory series is called non-convergent. The infinite series $\sum u_n$ is said to converge to s, if for every arbitary small +ve so ϵ, $\exists$ a +ve integer m depends on ϵ st $|s_n - s| < \epsilon \ \forall n \geq m$. For example the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} \dots$ is convergent

$$\text{as } S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \quad \text{If } n \to \infty \quad S_n \to 1$$

Hence $S = 1$

ii) The series $1 + 2 + 3 + 4 + \dots + n + \dots$ diverges to ∞

iii) The series $1 - 1 + 1 - 1 + 1 - 1 \dots$ oscillates finitely and the series $1 - 2 + 3 - 4 + \dots$ oscillatory infinitely.

Note that nature of a series is determined by the nature of the sequence of its nth partial sum.

The geometric series: $a + ax + ax^2 + ax^3 + ax^{n-1} + \dots$ $(a>0)$ is i) convergent if the common ratio x lies between -1 and 1 (ie $-1 < x < 1$) and the sum of the series is $\frac{a}{1-x}$.

ii) Divergent (to $+\infty$) if $x \geq 1$

iii) oscillates finitely if $x = -1$ and oscillates infinitely if $x < -1$

Proof The nth partial sum of the given series is

$$S_n = a + ax + ax^2 + \dots + ax^{n-1}$$

$$= a\frac{x^n - 1}{x - 1} \quad (x \neq 1)$$

i) If $-1 < x < 1$ there when $n \to \infty$, $x^n \to 0$

$$\therefore S_n = \frac{a}{1-x} \quad \text{as } n \to \infty$$

ii) If $x > 1$ then as $n \to \infty$, $x^n \to \infty$ and then $S_n \to \infty$ as $n \to \infty$

Hence the geometric series diverges to $+\infty$ if $x>1$.

~~and $S_n > na \to \infty$ as $n \to \infty$~~

~~$\therefore$ the series is diverges to $+\infty$~~

iii) If $x=-1$, the series becomes $a-a+a-a\ldots$ and

$S_n = \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ Hence the series oscillates finitely.

If S_n ~~x~~ $x<-1$, x^n oscillates infinitely between $-\infty$ and ∞

iv) when $x=1$ the series $\Rightarrow a+a+a\ldots$

and $S_n = na \to \infty$ as $n \to \infty$

$\therefore$ the series diverges to $+\infty$

Tests of convergence and divergence

Result 1: If $\sum u_n$ be a convergent series of +ve terms, then it necessarily follows that $\lim_{n\to\infty} u_n = 0$

Result 2: If $\sum u_n$ is a convergent series of +ve and decreasing terms then it necessarily follows $\lim_{n\to\infty} n u_n = 0$

1. Comparison test: Let $\sum u_n$ and $\sum v_n$ be two infinitely series of +ve terms. If $\lim_{n\to\infty} \frac{u_n}{v_n} = k$, a non-zero finite. Quantity, then the series are both convergent or both divergent (If $\lim_{n\to 0} \frac{u_n}{v_n}$ = non-zero finite quantity, then if $\sum v_n$ convergent, $\sum u_n$ is convergent and if $\sum v_n$ is divergent. $\sum u_n$ is divergent)

Ex-1 Test the convergence of the series whose nth terms is $\sqrt{n^2+1}-n$

Soln Here $u_n = \sqrt{n^2+1}-n$ and we take $v_n = \frac{1}{n}$

$$\therefore \frac{u_n}{v_n} = \frac{\sqrt{n^2+1}-n}{1/n} = \frac{n\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n}$$

$$= \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{1+1/n^2}+1}$$ As $n\to\infty$, $\frac{u_n}{v_n} \to \frac{1}{2}$

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$ and $\sum v_n = \sum \frac{1}{n}$ is divergent,

By the comparison test, $\sum u_n$ is divergent

The p-series: The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

i) converges if $p>1$

ii) diverges if $p\leq 1$

Ex-2 Test the convergence of the series $\sum u_n$ where $u_n=\sqrt{n^4+1}-\sqrt{n^4-1}$

Sol^n $u_n=\sqrt{n^4+1}-\sqrt{n^4-1} = n^2\sqrt{1+1/n^4}-n^2\sqrt{1-1/n^4}$

$= n^2\left(1+1/n^4\right)^{1/2} - n^2\left(1-\frac{1}{n^4}\right)^{1/2}$

$= n^2\left\{1+\frac{1}{2}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\frac{1}{\cancel{n^4}\, n^4}+\cdots\right\}-n^2\left\{1-\frac{1}{2.n^4}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\frac{1}{(n^4)^2}\cdots\right\}$

$= n^2\left[\left\{1+\frac{1}{2.n^4}-\frac{1}{8.n^8}+\cdots\right\}-\cancel{n^2}\left\{1-\frac{1}{2.n^4}-\frac{1}{8n^8}+\cdots\right\}\right.$

$= \left(\frac{1}{2n^2}-\frac{1}{8n^6}+\cdots\right)-\left(-\frac{1}{2n^2}-\frac{1}{8.n^6}\cdots\right)$

Consider the series $\sum V_n$ as $\sum V_n=\sum\frac{1}{n^2}$

Clearly $\lim_{n\to\infty}\frac{u_n}{V_n}=1$

Since $\sum\frac{1}{n^2}$ is a p-series with $p=2>1$, which is convergent.

by the comparison test $\sum u_n$ is ~~al~~ convergent.

HW Ex 3 Show that the series whose nth ~~#~~ term is $\frac{1}{n}\sin\frac{1}{n}$ is convergent

Ex 4 $\frac{5}{1.2.4}+\frac{7}{2.3.5}+\frac{9}{3.4.6}+\frac{11}{4.5.7}+\cdots$

Test the convergence of this series.

Ex Show that the series whose nth term is $\frac{1}{n}\sin\frac{1}{n}$ is convergent.

Sol^n Here $u_n=\frac{1}{n}\sin\frac{1}{n}$

$=\frac{1}{n}\left\{\frac{1}{n}-\frac{1}{n^3 3!}+\frac{1}{n^5 5!}-\cdots\right\}$

$=\frac{1}{n^2}-\frac{1}{n^4 3!}+\frac{1}{n^6 5!}-\cdots$

Now, we take the convergent series as $\sum V_n$ where $V_n=\frac{1}{n^2}$.
which is convergent.

$\therefore \lim_{n\to\infty}\frac{u_n}{V_n}=1$. Finite ~~oscillation~~ quantity.

Since V_n is convergent, by the comparison test, the series ~~the~~ $\sum u_n$ is also convergent.

2. Cauchy's Root test: Let $\sum u_n$ be a series of +ve terms and

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

i) If $l<1$, then $\sum u_n$ is convergent.

ii) If $l>1$, then $\sum u_n$ is divergent.

iii) If $l=1$, then ~~$\sum$~~ the test fails.

Ex 1 Test the convergence of the series.

$$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \dots + \left(\frac{n}{2n+1}\right)^n + \dots$$

Solⁿ Here $u_n = \left(\frac{n}{2n+1}\right)^n$

$$\therefore (u_n)^{1/n} = \frac{n}{2n+1} = \frac{1}{2+1/n}$$

$\therefore \lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{2} < 1$. Hence by Cauchy's root test the series is convergent.

Ex 2 Test the convergence of the series $\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots, x>0$

Solⁿ Here $\sum u_n = \sum \frac{x^n}{n^2}$

$$\therefore (u_n)^{1/n} = \frac{x}{n^{2/n}}$$

$$\therefore \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{x}{(n^{1/n})^2}$$

$$\because \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

$$\therefore \lim_{n\to\infty} (u_n)^{1/n} = x$$

It will be convergent if $x<1$, by Cauchy's root test and divergent if $x>1$

Ex 3 - Test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$$

Hint: $\sum u_n = \sum \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-n} \cdots\cdots$

3) D'Alembert's Ratio test: Let $\sum u_n$ be a series of +ve terms and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

i) If $l < 1$ then $\sum u_n$ is convergent.

ii) If $l > 1$ then $\sum u_n$ is divergent.

iii) If $l = 1$ then the test fails.

Ex 1 Prove that the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ $(x \geq 0)$ is convergent if $0 \leq x < 1$ and divergent if $x > 1$

Sol^n Here $u_n = \frac{x^n}{n}$ & $u_{n+1} = \frac{x^{n+1}}{n+1}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{n+1} \times \frac{n}{x^n} = \frac{x}{1+1/n} \longrightarrow x \text{ as } n \to \infty$$

If $0 \leq x < 1$ then $\lim_{n\to 0} \frac{u_{n+1}}{u_n} < 1$ and the series become convergent.

If $x = 1$ then D'Alembert's ratio test doesn't give any definite conclusion.

But when $x = 1$, the given series

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} + \cdots,$$ which is a divergent harmonic series.

$\therefore$ The series is convergent if $0 \leq x < 1$ and divergent if $x \geq 1$

HW: Ex 2 Show that the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.

Ex 3 Show that the series $\frac{1}{2} + \frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \cdots$ is convergent.

Sol^n Here $u_n = \frac{1}{2^{n-1}+1}$ and $u_n = \frac{1}{2^n+1}$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n-1}+1}{2^n+1} = \lim_{n\to\infty} \frac{1}{2} \times \frac{2^n+1}{2^n+1} = \lim_{n\to\infty} \frac{1/2 + 1/2^n}{1 + 1/2^n} = \frac{1}{2}$$

$$\because \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Hence by D'Alembert's ratio test the given series is convergent.

Parseval's theorem of Fourier constants -

If F-series $f(x)$ over $c < x < c+2l$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$. Then $\frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

Proof The F-series $f(x)$ is in $c < x < c+2l$ is given as $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x \right\} \quad \text{--- (1)}$$

where $a_0 = \frac{1}{l} \int_c^{c+2l} f(x)dx$,

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx;\quad b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Multiply both side ~~by~~ of ① by $f(x)$ ~~whereas~~ we have

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty} a_n f(x)\cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n f(x)\sin\frac{n\pi}{l}x \quad \text{—(ii)}$$

Next $$\int_{c}^{c+2l}[f(x)]^2 dx = \frac{a_0}{2}\int_{c}^{c+2l} f(x)\,dx + \sum a_n\int_{c}^{c+2l} f(x)\cos\frac{n\pi}{l}x\,dx + \sum b_n\int_{c}^{c+2l} f(x)\sin\frac{n\pi}{l}x\,dx$$

$$\Rightarrow \cancel{\int_{c}^{c+2l}[f(x)]^2 dx = \frac{l}{2}} = \frac{a_0}{2}la_0 + \sum a_n(la_n) + \sum b_n(lb_n)$$

$$\Rightarrow \int_{c}^{c+2l}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

which is Parseval's identity.

Parseval's ~~theorem~~ formula —

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx = \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)$$

The F-series of $f(x)$ in $(-l,l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \quad \text{—①}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx$

From ① $\Rightarrow$ $$\int_{-l}^{l}[f(x)]^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum a_n\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx + \sum b_n\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx \quad \text{—(ii)}$$

Now, $\int_{-l}^{l} f(x)\,dx = la_0$

$$\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx = la_n,\quad \int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx = lb_n$$

$$\therefore \text{(ii)} \Rightarrow \int_{-l}^{l}[f(x)]^2 dx = \frac{a_0}{2}la_0 + \sum a_n\, la_n + \sum b_n\, lb_n$$

$$= l\left[\frac{a_0^2}{2} + \sum(a_n^2+b_n^2)\right]$$

This is the Parseval's formula.

Note ① If $0<x<2l$, $\int_{0}^{2l}[f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum(a_n^2+b_n^2)\right]$

(ii) If $0<x<l$ (half range cosine series)

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum a_n^2\right]$$

(iii) If $0 < x < l$ (half range sine series)

$$\int_0^l [f(x)]^2 dx = \frac{l}{2} \sum b_n^2$$

Ex – Find F-series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

Sol^n The F-series of $f(x) = x^2$ in $(-\pi, \pi)$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2} \cos nx \quad \text{—— (1)}$$

Here $a_0 = \frac{2}{3}\pi^2$, $a_n = \frac{n(-1)^n}{n^2}$, $b_n = 0$

Now, Picard's identity is.

$$\frac{1}{l}\int_{-l}^{l} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum (a_n^2 + b_n^2)$$

$$\therefore \int_{-\pi}^{\pi} (x^2)^2 dx = \pi\left[\frac{4\pi^4}{9.2} + \sum\left(\frac{16(-1)^{2n}}{n^4} + 0\right)\right]$$

or $$\left[\frac{x^5}{5}\right]_{-1}^{\pi} = \pi\left[\frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}\right]$$

or $$\frac{1}{5}(\pi^5 + \pi^5) - \frac{2\pi^5}{9} = \pi \sum \frac{16}{n^4}$$

or $$\left(\frac{18-10}{45}\pi^5\right) = \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

or $$\frac{8}{45}\pi^4 = 16\left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right\}$$

or $$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$