Assignment on
Successive Derivative

Date: 22.11.20

Q1: If $y = \frac{1}{ax+b}$, what will be the nth derivative

Soln $y = \frac{1}{ax+b}$

$y^1 = \frac{-a}{(ax+b)^2}$

$y^2 = \frac{1.2a^2}{(ax+b)^3}$

$y^3 = \frac{-1.2.3.a^3}{(ax+b)^4}$

In this way, we find that,

$y^n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

Q2: If $y = \log(ax+b)$. what will be nth derivative

Soln $y = \log(ax+b)$

$y^1 = \frac{a}{ax+b}$

$y^2 = \frac{-1a^2}{(ax+b)^2}$

$y^3 = \frac{1.2\,a^3}{(ax+b)^3}$

In this way, we find that,

$y^n = \frac{(-1)^{n+1}(n-1)!\,a^n}{(ax+b)^n}$

Q3: If $y = a^{mx}$, what will be nth derivative.

Soln $y = a^{mx}$

$y^1 = ma^{mx}\log a$

$y^2 = (m\log a)^2 a^{mx}$

$y^3 = (m\log a)^3 a^{mx}$

In this way, we find that

$y^n = (m\log a)^n a^{mx}$

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Q4 If $y = \sin(ax+b)$, then what is y_n?

Solⁿ $y = \sin(ax+b)$

$\therefore y_1 = a\cos(ax+b)$

$= a\sin\left(\frac{\pi}{2} + ax+b\right)$

$\therefore y_2 = a.a\cos\left(\frac{\pi}{2} + ax+b\right)$

$= a^2\sin\left\{\frac{\pi}{2} + \left(\frac{\pi}{2} + ax+b\right)\right\}$

$= a^2\sin\left(2.\frac{\pi}{2} + ax+b\right)$

$y_3 = a^2.a\cos\left(2.\frac{\pi}{2} + ax+b\right)$

$= a^3\cos\left(2.\frac{\pi}{2} + ax+b\right)$

$= a^3\sin\left\{\frac{\pi}{2} + \left(2.\frac{\pi}{2} + ax+b\right)\right\}$

$= a^3\sin\left(3.\frac{\pi}{2} + ax+b\right)$

. . . .

$y_n = a^n\sin\left(n.\frac{\pi}{2} + ax+b\right)$

Q5 If $y = \cos(ax+b)$, then what is y_n.

Solⁿ $y = \cos(ax+b)$

$\therefore y_1 = -a\sin(ax+b)$

~~$= -a\sin\left(\frac{\pi}{2} + ax+b\right)$~~

$= -a\cos\left(\frac{\pi}{2} + ax+b\right)$

$\therefore y_2 = (-a).(-a)\sin\left(\frac{\pi}{2} + ax+b\right)$

$= a^2\cos\left(2.\frac{\pi}{2} + ax+b\right)$

$y_3 = (-a).a^2\sin\left(2.\frac{\pi}{2} + ax+b\right)$

$= -a^3\cos\left(\frac{\pi}{2} + \left(2.\frac{\pi}{2} + ax+b\right)\right)$

$= -a^3\cos\left(3.\frac{\pi}{2} + ax+b\right)$

. . . .

$y_n = (-1)^n a^n\cos\left(n.\frac{\pi}{2} + ax+b\right)$

Q6 If $y = (ax+b)^m$, then find y_n.

Solⁿ $y = (ax+b)^m$

$\therefore y_1 = am(ax+b)^{m-1}$

~~$y_2 = a^2m^2(a$~~

$y_2 = a^2 . m.(m-1)\ (ax+b)^{m-2}$

$y_3 = a^2 . m(m-1).(m-2)\ (ax+b)^{m-3}$

$\cdots\cdots\cdots$

$y_n = \cancel{a^n (m!-n!)(ax+b)^n}\ a^n m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{m-n}$

Q7. If $y = e^{ax}\sin(bx+c)$, then $y_n = r^n e^{ax}\sin\cancel{(ax+b)}(bx+c+n\phi)$,
where $r = \sqrt{(a^2+b^2)}$, and $\phi = \tan^{-1} b/a$

Proof $y = e^{ax}\sin(bx+c)$

$y_1 = e^{ax}\cos(bx+c).b + ae^{ax}\sin(bx+c)$

$= be^{ax}\cos(bx+c) + ae^{ax}\sin(bx+c)$

Let, $a = r\cos\phi$, $b = r\sin\phi$

$a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$

$\tan\phi = b/a \Rightarrow \phi = \tan^{-1} b/a$

$\therefore y_1 = r\sin\phi\, e^{ax}\cos(bx+c) + r\cos\phi\, e^{ax}\sin(bx+c)$

$= re^{ax}\sin\{\phi + (bx+c)\}$

$= re^{ax}\sin(bx+c+\phi)$

$y_2 = r\{e^{ax}\cos(bx+c+\phi).b + ae^{ax}\sin(bx+c+\phi)\}$

$= r\{e^{ax}\, r\sin\phi\cos(bx+c+\phi) + r\cos\phi\, e^{ax}\sin(bx+c+\phi)\}$

$= r^2 e^{ax}\sin\{\phi + (bx+c+\phi)\}$

$= r^2 e^{ax}\sin(bx+c+2\phi)$

$y_3 = r^2\{e^{ax}\cos(bx+c+2\phi).b + ae^{ax}\sin(bx+c+2\phi)\}$

$= r^2\{r\sin\phi\, e^{ax}\cos(bx+c+2\phi) + r\cos\phi\, e^{ax}\sin(bx+c+2\phi)\}$

$= r^3 e^{ax}\sin\{\phi + (bx+c+2\phi)\}$

$= r^3 e^{ax}\sin(bx+c+3\phi)$

$\cdots\cdots\cdots$

$\therefore y_n = r^n e^{ax}\sin(bx+c+n\phi)$

Where, $r = \sqrt{a^2+b^2}$ and ~~to~~ $\phi = \tan^{-1} b/a$

~~Q8: If $y = \left(\frac{1}{ax+b}\right)$, $y_n = (-1)^n n!$~~

Q8: Show that $D^n\left[e^{ax}\cos(bx+c)\right] = (a^2+b^2)^{n/2} e^{ax}\cos\left(bx+c+n\tan^{-1}b/a\right)$

(same with Q no 7)

Q9: Find the nth derivative of $\cos x \cos 2x \cos 3x$

Soln Let $y = \cos x \cos 2x \cos 3x$

$$= \frac{1}{2}\cos 2x\,(2\cos x \cos 3x)$$

$$= \frac{1}{2}\cos 2x\,(\cos 4x + \cos 2x)$$

$$= \frac{1}{2}(\cos 2x \cos 4x + \cos^2 2x)$$

$$= \frac{1}{4}(2\cos 2x \cos 4x + 2\cos^2 2x)$$

$$= \frac{1}{4}(\cos 6x + \cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)$$

$$\therefore y_n = \frac{1}{4}\left[2^n\cos\left(2x+\frac{n\pi}{2}\right) + 4^n\cos\left(4x+\frac{n\pi}{2}\right) + 6^n\cos\left(6x+\frac{n\pi}{2}\right)\right]$$

Q10: Find the nth derivative of $\sin x \sin 2x \sin 3x$

Soln Let $y = \sin x \sin 2x \sin 3x$

$$= \frac{1}{2}\sin 2x\,(2\sin x \sin 3x)$$

$$= \frac{1}{2}\sin 2x\,(\cos 4x - \cos 2x)$$

$$= \frac{1}{2}(\sin 2x \cos 4x - \sin 2x \cos 2x)$$

$$= \frac{1}{4}(2\sin 2x \cos 4x - 2\sin 2x \cos 2x)$$

$$= \frac{1}{4}(\sin 6x + \sin 2x - \sin 4x)$$

$$= \frac{1}{4}(\sin 2x - \sin 4x + \sin 6x)$$

$$\therefore y_n = \frac{1}{4}\left[2^n\sin\left(2x+\frac{n\pi}{2}\right) + 4^n\sin\left(4x+\frac{n\pi}{2}\right) + 6^n\sin\left(6x+\frac{n\pi}{2}\right)\right]$$

Eg1 → Find the nth derivative of $e^{2x}\cos^2 x \sin x$

Solⁿ Let $y = e^{2x}\cos^2 x \sin x$

$\therefore y = \frac{1}{2} e^{2x}(2\cos x \sin x)\cos x$

$= \frac{1}{2} e^{2x} \sin 2x \cos x$

$= \frac{1}{4} e^{2x} (2\sin 2x \cos x)$

$= \frac{1}{4} e^{2x} (\sin 3x + \sin x)$

$y = \frac{1}{4}\left[e^{2x}\sin 3x + e^{2x}\sin x\right]$

$y_n = \frac{1}{4}\left[(2^2+3^2)^{n/2} e^{2x}\sin(3x + n\tan^{-1} 3/2) + (2^2+1^2)^{n/2} e^{2x}\sin(x + n\tan^{-1} 1/2)\right]$

$= \frac{1}{4}\left[13^{n/2} e^{2x}\sin(3x + n\tan^{-1} 3/2) + 5^{n/2} e^{2x}\sin(x + n\tan^{-1} 1/2)\right]$

Eg2 – Find the nth derivative of $\frac{1}{x^2+a^2}$

Solⁿ $y = \frac{1}{x^2+a^2}$

$= \frac{1}{(x+ia)(x-ia)}$

Now, Let $\frac{1}{(x+ia)(x-ia)} = \frac{A}{x+ia} + \frac{B}{x-ia}$

$\Rightarrow \frac{1}{(x+ia)(x-ia)} = \frac{A(x-ia)+B(x+ia)}{(x+ia)(x-ia)}$

$\Rightarrow 1 = A(x-ia) + B(x+ia)$ —— ①

Let $x = ia$,

① $\Rightarrow 1 = B(ia+ia) = 2iaB$

$\Rightarrow B = 1/2ia$

Again Let, $x = -ia$

① $\Rightarrow 1 = A(-ia-ia)$

$= -2iaA$

$\Rightarrow A = -1/2ia$

$\therefore \frac{1}{(x+ia)(x-ia)} = -\frac{1}{2ia}\cdot\frac{1}{(x+ia)} + \frac{1}{2ia}\cdot\frac{1}{(x-ia)}$

$$\frac{1}{(x+ia)(x-ia)} = +\frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$$y = \frac{1}{x^2+a^2}$$

$$= \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$$\therefore y_n = \frac{1}{2ia}\left\{\frac{(-1)^n n!\, 1^n}{(x-ia)^{n+1}} - \frac{(-1)^n n!\, 1^n}{(x+ia)^{n+1}}\right\}$$

$$= \frac{(-1)^n n!}{2ia}\left\{\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right\}$$

Put $x = r\cos\theta$ and $a = r\sin\theta$

$$\sqrt{x^2+a^2} = r \;\&\; \theta = \tan^{-1}(a/x)$$

$$\therefore y_n = \frac{(-1)^n n!}{2ia}\left\{\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right\}$$

$$= \frac{(-1)^n n!}{2ia}\left\{\frac{1}{r^{n+1}}(\cos\theta - i\sin\theta)^{-(n+1)} - \frac{1}{r^{n+1}}(\cos\theta + i\sin\theta)^{-(n+1)}\right\}$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\Big[\{\cos[-(n+1)]\theta - i\sin(-(n+1))\theta\} - \{\cos(-(n+1))\theta + i\sin(-(n+1))\theta\}\Big]$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\Big[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\}\Big]$$

$$\therefore y_n = \frac{(-1)^n n!}{2ia r^{n+1}}\{2i\sin(n+1)\theta\}$$

$$= \frac{(-1)^n n!}{a r^{n+1}}\sin(n+1)\theta$$

Let's put $\frac{1}{r} = \frac{\sin\theta}{a} \Rightarrow \left(\frac{1}{r}\right)^{n+1} = \frac{\sin^{n+1}\theta}{a^{n+1}}$

$$y_n = \frac{(-1)^n n!}{a.\, a^{n+1}}\sin^{(n+1)}\theta \sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{a^{n+2}}\sin(n+1)\theta \sin^{(n+1)}\theta$$

<u>Leibnitz theorem</u> – (Theorem for the nth derivative of the product of two functions)

<u>Statement</u> – If u & v be two function of x possessing derivatives of the nth order, then

$$(uv)_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \dots + {}^nC_r u_{n-r}v_r + \dots {}^nC_n u v_n$$

Ex Find the nth derivative of $e^{x}(2x+3)^3$

Let, $e^x = u$ and $(2x+3)^3 = v$

If $u = e^x$, then $u_n = e^x$

$v = (2x+3)^3$

$\therefore v_1 = 3(2x+3)^2 \cdot 2$

$= 6(2x+3)^2$

$\therefore v_2 = 6 \cdot 2 \cdot 2(2x+3)$

$= 24(2x+3)$

$v_3 = 24 \cdot 2 = 48$

$v_4 = v_5 = \dots 0$

$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + {}^nc_3 u_{n-3} v_3$

$= e^x (2x+3)^3 + {}^nc_1 e^x . 6(2x+3)^2 + {}^nc_2 e^x . 24(2x+3) + {}^nc_3 e^x . 48$

$(uv)_n = \{e^x (2x+3)^3\}_n$

$= e^x (2x+3)^3 + n . e^x . 6(2x+3)^2 + \frac{n(n-1)}{2!} e^x . 24(2x+3) + \frac{n(n-1)(n-2)}{3!} e^x . 48.$

$= e^x \left[(2x+3)^3 + 6n(2x+3)^2 + 12n(n-1)(2x+3) + 8n(n-1)(n-2)\right]$

Ex If $y = \tan^{-1} x$, prove that, $(1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

Proof - $y = \tan^{-1} x$

$y_1 = \frac{1}{1+x^2}$

$\Rightarrow (1+x^2) y_1 = 1$ —— (1)

Differentiating above with help of Leibnitz's theorem for nth derivative. So that $y_{n+1}(1+x^2) + {}^nc_1 y_{(n+1)-1} \cdot 2x + {}^nc_2 y_{(n+1)-2} \cdot 2 = 0$

$\Rightarrow (1+x^2) y_{n+1} + n y_n \cdot 2x + \frac{n(n-1)}{2!} y_{n-1} \cdot 2 = 0$

$\therefore (1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$

Ex If $y = a\cos(\log x) + b\sin(\log x)$, then prove that

i) $x^2 y_2 + x y_1 + y = 0$

ii) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

Proof $y = a\cos(\log x) + b\sin(\log x)$

$y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again derivating the above once, we get

$y_1 + xy_2 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$

$\Rightarrow x(y_1 + xy_2) = -[a\cos(\log x) + b\sin(\log x)]$

$\Rightarrow x^2y_2 + xy_1 = -y$

$\Rightarrow x^2y_2 + xy_1 + y = 0$ [(I) is proved]

Again using Leibnitz theorem for nth derivative of product of functions.

$(y_{n+2}x^2 + {}^nC_1\, y_{n+1}\cdot 2x + {}^nC_2\, y_n\cdot 2) + (y_{n+1}x + {}^nC_1\, y_n\cdot 1) + y_n = 0$

$\therefore y_{n+2}x^2 + n\, y_{n+1}\cdot 2x + \frac{n(n-1)}{2!}y_n\cdot 2 + (y_{n+1}x + n\,y_n) + y_n = 0$

$\therefore x^2y_{n+2} + 2nx\,y_{n+1} + n(n-1)y_n + x\,y_{n+1} + n\,y_n + y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)x\,y_{n+1} + n^2y_n - \cancel{ny_n} + \cancel{ny_n} + y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0$ [(II) is proved]

Find nth dr. of $\{e^{x}(2x+3)^3\}^3$

Eg- If $y = (\sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Hence find $(y_n)_0$

Soln $y = (\sin^{-1}x)^2$

$y_1 = 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$

$\therefore y_1(\sqrt{1-x^2}) = 2\sin^{-1}x$

$\Rightarrow y_1^2(1-x^2) = 4(\sin^{-1}x)^2$

$\Rightarrow y_1^2(1-x^2) = 4y$ —— (1)

Again differentiating (1), we get

$2y_1y_2(1-x^2) + y_1^2(-2x) = 4y_1$

$\Rightarrow 2y_2(1-x^2) + 2xy_1 = 4$

$\Rightarrow y_2(1-x^2) - xy_1 = 2$ —— (ii)

Now, applying Leibnitz's theorem for nth derivatives on (ii), we get

$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}x + {}^nC_1 y_{n \cdot 1}\} = 0$

$\Rightarrow (1-x^2)\, y_{n+2} + n.(-2x)\, y_{n+1} - 2.\dfrac{n(n-1)}{2!} y_n - xy_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - n(n-1)\, y_n - x y_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - n\{n^2 y_n + \cancel{ny_n} - x y_{n+1} - \cancel{ny_n} = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - n^2 y_n = 0$ —— (iii)

Which is the required results.

Now, putting $x=0$ in (iii), we get $(y_{n+2})_0 = n^2 (y_n)_0$ —— (⊙)

$\therefore$ we get, $(y_1)_0 = 0$, from (i)

And from (ii) $(y_2)_0 = 2$

Now putting $n = 1, 3, 5, 7, \ldots$ in (⊙), we get

$(y_1)_0 = (y_3)_0 = (y_5)_0 = \ldots = 0$, when n is odd

Again let's put, $n = 2, 4, 6 \ldots$ in (⊙), we get

$(y_4)_0 = 2^2 (y_2)_0 = 2.2^2$

$(y_6)_0 = 4^2 (y_4)_0 = 2.2^2.4^2$

$(y_8)_0 = 6^2 (y_6)_0 = 2.2^2.4^2.6^2$

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$(y_n)_0 = 2.2^2.4^2.6^2 \ldots (n-2)^2$

when n is even.

Eg- If $y = \sin(m \sin^{-1} x)$, prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$ &

$(1-x^2)\, y_{n+2} - 2(n+1)\, x y_{n+1} + (m^2 - n^2)\, y_n = 0$

Soln $y = \sin(m \sin^{-1} x)$

$y_1 = \cos(m \sin^{-1} x).\, m.\, \dfrac{1}{\sqrt{1-x^2}}$

$\therefore y_1 \sqrt{1-x^2} = \cos(m \sin^{-1} x)$

$$\Rightarrow y_2\sqrt{1-x^2} + \frac{1}{2\sqrt{1-x^2}}(-2x)y_1 = m\{-\sin(m.\sin^{-1}x)\}\, m.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_2(1-x^2)+(-x)y_1 = -m^2 y$$

$$\Rightarrow y_2(1-x^2) - xy_1 + m^2 y = 0$$

By applying Leibnitz's Theorem we get,

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)m^2 - \{y_{n+1}x + {}^nC_1\, y_n\} + y_n = 0$$

$$\Rightarrow y_{n+2} - y_{n+2}x^2 + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)m^2 - \{y_{n+1}x + ny_n\} + y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2(n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

Eg- If $y = x^n \log x$, Prove that $y_{n+1} = n!/x$

Sol^n A $\quad y_1 = x^n.\frac{1}{x} + nx^{n-1}\log x$

$$xy_1 = x^n + n(x^{n-1}\log x).x$$

$$= x^n + nx^n \log x$$

$$= x^n + ny$$

Applying Leibnitz's theorem, we get

$$y_{n+1}x + {}^nC_1\, y_n.1 = n! + ny_n$$

$$xy_{n+1} + ny_n = n! + ny_n$$

$$y_{n+1} = n!/x$$

Partial Derivative

Functions of two variables :

A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variable x and y and we write

$$z = f(x, y)$$

We often come across quantities which depend on two or more variables.

For eg – The area of a rectangle of length x and breath y is given by $A = xy$. For a given pair of values of x, y; A has definite values.

The volume of parallalopiped $(= xyh)$ depends on x, y and h.

P: $z = x^3 + y^3 - 3axy$

$$\cancel{\frac{\partial z}{\partial x} = 3x^2 - 3ay} \qquad \cancel{\frac{\partial z}{\partial y} = 3y^2 - 3ay}$$

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay \qquad \frac{\partial z}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 z}{\partial x^2} = 6x - \cancel{3a} \qquad \frac{\partial^2 z}{\partial y^2} = 6y$$

$$\frac{\partial}{\partial x}(3y^2 - 3ax) = \frac{\partial^2 z}{\partial x \partial y} = -3a$$

$$\frac{\partial}{\partial y}(3x^2 - 3ay) = \frac{\partial^2 z}{\partial x \partial y} = -3a$$

P. If $z = f(x + ct) + \phi(x - ct)$, Prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Sol^n $\frac{\partial z}{\partial t} = f'(x+ct)\,c + \phi(x-ct)(-c)$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)\,c^2 + \phi''(x-ct)\,c^2$$

& $\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \phi''(x-ct)\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Homogeneous function

An expression of the form $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$ is called a homogeneous function of degree n. This can be written as

$$x^n\left[a_0 + a_1(y/x) + a_2(y/x)^2 + \dots a_n (y/x)^n\right]$$

Thus the function $f(x,y)$ which can be expressed in the form $x^n \phi(y/x)$ is called a homogeneous function of degree n in x & y

For eg ~~$x^3 \cos y/x$~~ $x^3 \cos(y/x)$ homogeneous function of degree 3 in x & y

In general a function $f(x, y, z, t, \dots)$ is said to be a homogeneous function if it be expressed as $x^n \phi(y/x, z/x, t/x, \dots)$

Eular's theorem in homogeneous function.

If u be a homogeneous function of degree n in x & y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof Since u is a homogeneous function of degree n, then

$$u = x^n \phi(y/x) \quad \text{—— (1)}$$

Then $$\frac{\partial u}{\partial x} = n x^{n-1}\phi(y/x) + x^n \phi(y/x) \cdot y/x^2$$

$$= n x^{n-1}\phi(y/x) - y x^{n-2}\phi(y/x) \quad \text{—— (ii)}$$

Corrilary – If z is a homogeneous function of degree n in x & y, show that $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Since z is a homogeneous function of degree n in x & y, then by Eular's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{—— (1)}$$

Now, differentiating partially (1). wrt x we get,

$$x\frac{\partial^2 z}{\partial x^2} + 1.\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n.\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x} \quad \text{—— (II)}$$

Differentiating partially (I) wrt y we get

$$x\frac{\partial^2 z}{\partial y\partial x}+y\cdot\frac{\partial^2 z}{\partial y^2}+1\cdot\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \quad \text{—— (III)}$$

Multiplying (II) by x, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+xy\frac{\partial^2 z}{\partial x\partial y}=(n-1)x\frac{\partial z}{\partial x} \quad \text{—— (IV)}$$

Multiplying (III) by y, we get

$$y^2\frac{\partial^2 z}{\partial y^2}+xy\frac{\partial^2 z}{\partial y\partial x}=(n-1)y\frac{\partial z}{\partial y} \quad \text{—— (V)}$$

Adding (IV) & (V), we get –

$$x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^2 z}{\partial x\partial y}=(n-1)\left\{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right\}$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^2 z}{\partial x\partial y}=n(n-1)Z$$

Home assignment →

i) If $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

ii) If $u=x f\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$, Prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$

iii) If $u=\tan^{-1}\frac{x^3+y^3}{x+y}$, Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u$$

24-Jan

Verify Eular's theorem when i) $f(x,y)=ax^2+2hxy+by^2$

★ $u = \sin^{-1} \dfrac{x^2+y^2}{x+y}$, Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

Sol^n $u = \sin^{-1} \dfrac{x^2+y^2}{x+y}$

$$\Rightarrow \sin u = \frac{x^2+y^2}{x+y}$$

$$\Rightarrow \sin u = \frac{x^2\left(1+y^2/x^2\right)}{x\left(1+y/x\right)}$$

$$\Rightarrow \sin u = x \cdot \frac{1+\left(y/x\right)^2}{1+\left(y/x\right)}$$

$\therefore Z = \sin u$ is a homogeneous function of degree 1.

By Eular's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

$$\Rightarrow x\cos u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

★ If $\sin u = \frac{x^2 y^2}{x+y}$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

Solⁿ $\sin u = \frac{x^2 y^2}{x+y}$

$\Rightarrow \sin u = x^3 \frac{(y/x)^2}{1+(y/x)}$

$\therefore z = \sin u$ is a homogeneous function of degree 3.

By Eular's Theorem

$\frac{\partial z}{\partial x} = \cos u \frac{\partial u}{\partial x}$ $\qquad$ $\frac{\partial z}{\partial y} = \cos u \frac{\partial u}{\partial y}$

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

$\Rightarrow x\cos u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial y} = 3\sin u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

★ $u = \sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$. find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Solⁿ $\sin u = \frac{x+2y+3z}{x^8+y^8+z^8}$

$= \frac{x(1+2y/x+3z/x)}{x^8(1+y^8/x^8+z^8/x^8)}$

$= x^{-7}\frac{1+2y/x+3z/x}{1+(y/x)^8+(z/x)^8}$

$\therefore p = \sin u$ is a homogeneous function of degree -7

By Eular's theorem

$\frac{\partial p}{\partial x} = \cos u \frac{\partial u}{\partial x}$

$\frac{\partial p}{\partial y} = \cos u \frac{\partial u}{\partial y}$

$\frac{\partial p}{\partial z} = \cos u \frac{\partial u}{\partial z}$

$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = -7p$

$\Rightarrow x.\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u$.

Q1 $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solⁿ Given $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

Partially differentiating wrt x.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y} + \frac{1}{1+(y/x)^2}\left(-y/x^2\right)$$

Partially differentiating wrt ~~x~~ y

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}$$

Now, LHS $= x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= \frac{x}{y}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} - \frac{y}{x}\cdot\frac{1}{1+(y/x)^2} - \frac{x}{y}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} + \frac{y}{x}\cdot\frac{1}{1+(y/x)^2}$$

$= 0 =$ RHS proved.

Q2 If $u = x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

Solⁿ Given, $u = x f(y/x) + g(y/x)$

Partially differentiating wrt x.

$$\frac{\partial u}{\partial x} = f(y/x) + x f'(y/x)\cdot(-y/x^2) + g'(y/x)\cdot(-y/x^2)$$

$$= f(y/x) - f'(y/x)\cdot y/x - g'(y/x)(y/x^2) \quad \text{—(i)}$$

Again Partially differentiating wrt x

$$\frac{\partial^2 u}{\partial x^2} = \cancel{f'(y/x)(-y/x^2)} + f''(y/x)(y^2/x^3) + \cancel{f'(y/x)(-y/x^2)} + g''(y/x)\cdot(y^2/x^4)$$
$$- g'(y/x)(-2y/x^3)$$

$$= f''(y/x)(y^2/x^3) + g''(y/x)\cdot(y^2/x^4) + g'(y/x)(2y/x^3)$$

Partially differentiating wrt y.

$$\frac{\partial u}{\partial y} = x f'(y/x)\frac{1}{x} + g'(y/x)(1/x)$$

$$= f'(y/x) + g'(y/x)(1/x) \quad \text{—(ii)}$$

Again Partially differentiating wrt y

$$\frac{\partial^2 u}{\partial y^2} = f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + g''\left(\frac{y}{x}\right)\left(\frac{1}{x^2}\right)$$

Now, differentiating ① wrt y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = f'(y/x)(1/x) - f''(y/x)(1/x)(y/x) - f'(y/x)(1/x) - g'(y/x)(1/x^2)$$
$$- g''(y/x)(1/x)(y/x^2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = -f''(y/x)(y/x^2) - g'(y/x)(1/x) - g''(y/x)(y/x^2)$$

Now, LHS $= x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

$$= f''(y/x)(y^2/x) + g''(y/x)(y^2/x^2) + 2g'(y/x)(y/x) - 2f''(y/x)(y^2/x)$$
$$- 2g'(y/x)(y/x) - 2g''(y/x)(y^2/x^2) + f''(y/x)(y^2/x) + g''(y/x)(y^2/x^2)$$

$= 0 =$ RHS. Proved

iii) If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$, Prove that i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$.

Sol^n Given $u = \tan^{-1}\frac{x^3+y^3}{x+y}$

$$\Rightarrow \tan u = \frac{x^3(1+(y/x)^3)}{x(1+y/x)}$$
$$= x^2 \cdot \frac{1+(y/x)^3}{1+(y/x)}$$

$\therefore$ $z = \tan u$ is a homogeneous function of degree 2.

$$\frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} \qquad \frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$\therefore x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \qquad \text{— ①}$$

Partially differentiating ① wrt x.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}(2\cos 2u - 1)$$

Multiplying with x

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x \frac{\partial u}{\partial x}(2\cos 2u - 1) \quad \text{---(ii)}$$

Partially differentiating (i) wrt y

$$x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u \frac{\partial u}{\partial y}$$

$$\Rightarrow x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}(2\cos 2u - 1)$$

Multiplying with y

$$y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = y \frac{\partial u}{\partial y}(2\cos 2u - 1) \quad \text{---(iii)}$$

Now, (ii) + (iii) $\Rightarrow$ $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$= (2\cos 2u - 1)(\sin 2u)$$

$$= 2\cos 2u \sin 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2 \sin u \cos 3u$$

Proved

Multiple Integration

$$\int_a^b f(x)\,dx$$

$$\int_x^{x'}\int_y^{y'} f(x,y)\,dx\,dy \longrightarrow \text{Double integration}$$

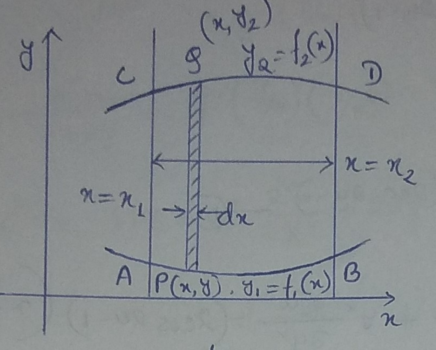


fig-1

Case 1 - when y_1, y_2 are functions of x and x_1, x_2 are constants $f(x,y)$ is first integrated wrt. y keeping x as constant between limits y_1, y_2 and then resulting expression is integrated wrt x within the limits x_1, x_2 ie

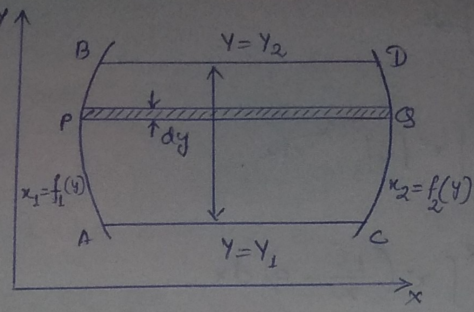
$$I = \int_{x_1}^{x_2}\left[\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x)\,dy\right]dx$$

where integration is carried out from the inner to the outer rectangle.

Fig-1 - Illustrate this process

Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and PQ is a vertical streep width dx.

The inner rectangle integrals means that the integration is along one edge of the strip PQ from P to Q. x as remaining constant while the outer rectangle integral corresponds the sliding of the edge from AC to BD thus covering the whole region of integration. ABCD abbott



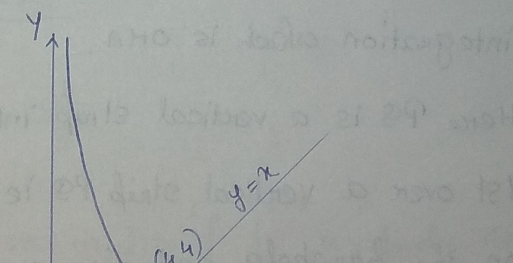
Case II when x_1, x_2 are the functions of y & y_1, y_2 are constants, $f(x,y)$ is first integrated wrt x keeping y as constant within the limits x_1, x_2 and then the resulting expression is integrated wrt y between the limits Y_1, Y_2 ie

$$I = \int_{y=y_1}^{y=y_2}\left[\int_{x=x_1}^{x=x_2} f(x,y)\,dx\right] dy$$

In figure 2 AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$ and PQ is a horizontal strip of width dy. The inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the

Eg- Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the 1st quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$

Soln The line AL is, $x=8$ intersects the hyperbola $xy=16$ at $A(8,2)$ while the line $y=x$ intersects this hyperbola at $B(4,4)$



Therefore the region of integration is OLAB. To evaluate the given integral, we devided this area into two parts OMB and MLAB.

$$\therefore \iint_R x^2\,dx\,dy = \int_{x \text{ at } O}^{x \text{ at } M}\int_{y \text{ at } P}^{y \text{ at } Q} x^2\,dx\,dy + \int_{x \text{ at } M}^{x \text{ at } L}\int_{y \text{ at } P'}^{y \text{ at } Q'} x^2\,dx\,dy$$

$$= \int_0^4\int_0^x x^2\,dy\,dx + \int_4^8\int_0^{16/x} x^2\,dy\,dx$$

$$= \int_0^4 x^2(x-0)\,dx + \int_4^8 x^2$$

$$= \int_0^4 x^2 [y]_0^x \,dx + \int_4^8 x^2 [y]_0^{16/x} \,dx$$

$$= \int_0^4 x^2(x-0)\,dx + \int_4^8 x^2\left(\frac{16}{x} - 0\right)dx$$

$$= \left.\frac{x^4}{4}\right]_0^4 + \left.\frac{16x^2}{2}\right]_4^8$$

$$= \frac{64\times 4}{4} + 8(64-16)$$

$$= \cancel{1}64 + 384$$

$$= 448$$

Eg - Evaluate

$\iint_A xy\,dx\,dy$, where A is the domain bounded by x axis, ordinate $x = 2a$ and the curve $x^2 = 4ya$

Sol^n The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at ~~x~~ $L(2a, a)$.

Fig 1 - Shows the region of integration which is OMA.

Here PQ is a vertical strip integrating

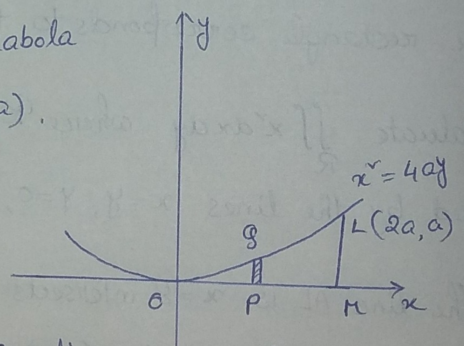


fig-1

1st over a vertical strip PQ ie. wrt y from $P(y=0)$ to ~~y~~ $Q(y = x^2/4a)$ on the parabola and then wrt x from $x=0$ to $x = 2a$, we get

$$\iint_A xy\,dx\,dy = \int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx$$

$$= \int_0^{2a} x \left|\frac{y^2}{2}\right|_0^{x^2/4a} dx$$

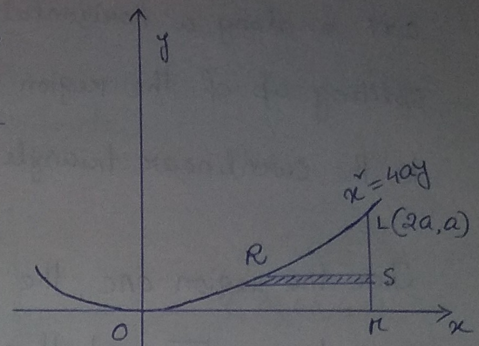
$$= \int_0^{2a} x\left(\frac{x^4}{32a^2} - 0\right)dx$$

$$= \left.\frac{x^5}{32\times 5a^2}\right|_0^{2a}$$

$$= \frac{32a^5 \times 2}{32\times 5a^2} - 0 = \cancel{\frac{a^3}{5}}\ \frac{a^4}{3}$$

other wise —

Integrating 1st over a horizontal strip RS ie wrt x from R $(x=2\sqrt{ay})$ on the parabola to S $(x=2a)$ and then wrt y from $y=0$ to $y=a$, we get



$$\iint_A xy\,dx\,dy = \int_0^a \int_{2(\sqrt{ay})}^{2a} xy\,dx\,dy$$

$$= \int_0^a y \left|\frac{x^2}{2}\right|_{2(\sqrt{ay})}^{2a} dy$$

$$= \int_0^a y\left\{\frac{(2a)^2}{2} - \frac{4ay}{2}\right\} dy$$

$$= \int_0^a y\,(2a^2 - 2ay)\,dy$$

$$= 2a\int_0^a (ay - y^2)\,dy$$

$$= 2a\left|a\frac{y^2}{2} - \frac{y^3}{3}\right|_0^a$$

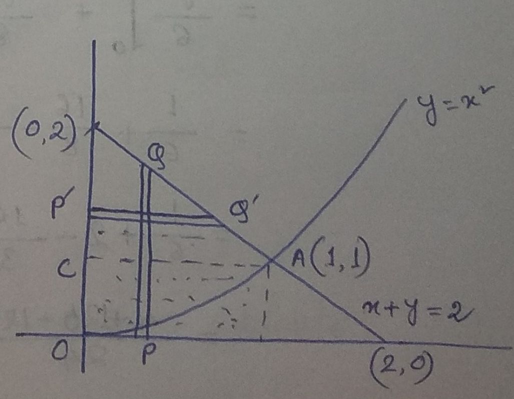
$$= 2a\left\{\left(a.\frac{a^2}{2} - \frac{a^3}{3}\right) - 0 - 0\right\}$$

$$= 2a\,\frac{3a^3 - 2a^3}{6}$$

$$= \frac{a^3}{3}.a = \frac{a^4}{3}$$

Ex Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same

Sol^n Here the integration is 1st wrt y along a vertical strip PQ which extends from P on the Parabola, $y=x^2$ to Q on the line $y=2-x$, Such a strip slides from $x=0$ to $x=1$, giving the region of integration as the curvilinear triangle OAB (shaded)



On changing the ~~orded~~ order of integration, we 1st integrate wrt x along a horizontal strip $P'Q'$ and that requires the spliting up of the region OAB into two parts by the line AC $(y=1)$ ie the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x=0$ to $x=\sqrt{y}$ and those for y are from $y=0$ to $y=1$, So the Contribution to the integration I from the region OAC is

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

For the region ABC, the limits of integration for x are from $x=0$ to $x=2-y$ and those for y are from $y=0$ to $y=2$, So the contribution to I from the region ABC is

$$\cancel{I_2 = \int_0^1 \int_0^{\sqrt{y}}} \quad I_2 = \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$\therefore I = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y \left| \frac{x^2}{2} \right|_0^{\sqrt{y}} dy + \int_1^2 y \left| \frac{x^2}{2} \right|_0^{2-y} dy$$

$$= \int_0^1 y \cdot \frac{y}{2}\,dy + \int_1^2 y \cdot \frac{(2-y)^2}{2}\,dy$$

$$= \int_0^1 y^2/2\,dy + \int_1^2 \frac{4y - 4y^2 + y^3}{2}\,dy$$

$$= \frac{y^3}{6}\Big]_0^1 + \frac{y^4}{8} - 4 \cdot \frac{y^3}{6} + 4 \cdot \frac{y^2}{4}\Big]_1^2$$

$$= \frac{1}{6} + \frac{16}{8} - \frac{32}{6} + \frac{16}{4} - \frac{1}{8} + \frac{4}{6} - \frac{4}{4}$$

$$= \frac{1}{6} + 2 - \frac{16}{3} + 4 - \frac{1}{8} + \frac{2}{3} - 1$$

$$= \frac{4 + 128 - 128 - 3 + 16}{24}$$

$$= \frac{9}{24}$$

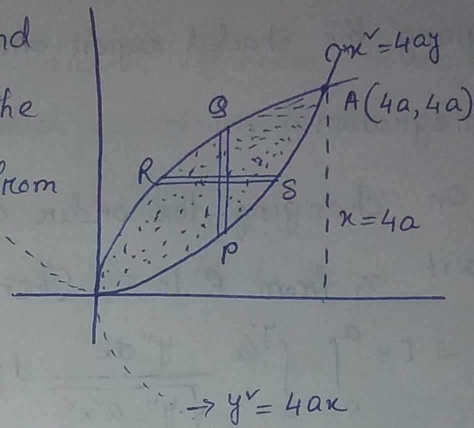
$$= \frac{3}{8}$$

change the order of the integration

Ex – Change the order of the integration in

$$I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx \quad \text{and hence evaluate.}$$

Solⁿ Here integration is 1st wrt y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then wrt x from $x = 0$ to $x = 4a$ giving the shaded portion of integration.



On changing the order of integration. We 1st integrate wrt x from $R(y^2/4a)$ to $S(2\sqrt{ay})$ then wrt y from $y = 0$ to $y = 4a$

$$\therefore I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}}$$

$$I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - y^2/4a\right) dy$$

$$= \left| 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a}$$

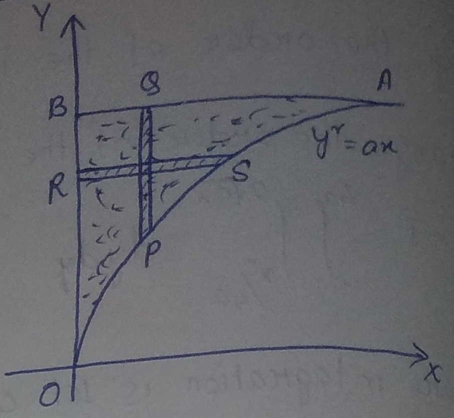
$$= \left[2\sqrt{a} \times \frac{2}{3} \times (4a)^{3/2} - \frac{(4a)^3}{12a} \right]$$

$$= \frac{32a^2 - 16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Ex Change the order of integration and hence evaluate.

$$I = \int_0^{a}\int_{\sqrt{ax}}^{a} \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}$$



Sol^n Here integration is 1st wrt y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then wrt x from $x=0$ to $x=a$ giving the shaded region OAB of integration.

On changing the order of integration we 1st integrate wrt x from R to S. Then wrt y from $y=0$ to $y=a$

$$\therefore I = \int_0^a \int_0^{y^2/a} \frac{y^2\,dx}{\sqrt{y^4 - a^2x^2}}\,dy$$

$$= \frac{1}{a}\int_0^a \int_0^{y^2/a} y^2 \frac{dx}{\sqrt{(y^2/a)^2 - x^2}}\,dy$$

$$= \frac{1}{a}\int_0^a y^2 \left| \sin^{-1}\left(\frac{x\,a}{y^2}\right)\right|_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2\,dy \left\{ \sin^{-1}\frac{y^2/a \cdot a}{y^2} - \sin^{-1} 0\right\}$$

$$= \frac{1}{a}\int_0^a y^2\,dy \left\{\sin^{-1}(1) - \sin^{-1} 0\right\}$$

$$= \frac{1}{a}\int_0^a \frac{\pi}{2} y^2\,dy$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{2a}\cdot\frac{a^3}{3}$$

$$= \frac{\pi a^2}{36}$$

Change of the variables

In double integral: Let the variables x, y be changed to the new variables, u, v by the transformations.

$$x = \phi(u,v), \quad y = \psi(u,v)$$

where $\phi(u,v)$ and $\psi(u,v)$ and have continuous 1st order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy plane. Then

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{uv}} f[\phi(u,v), \psi(u,v)]\,|J|\,du\,dv$$

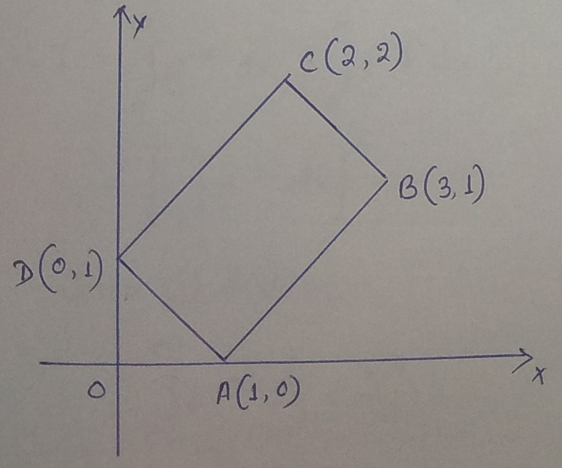
where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$ $(\neq 0)$

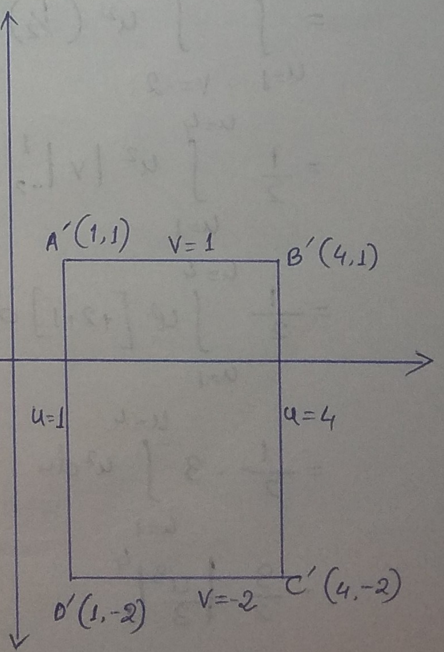
is the jacobian of transformation from (x,y) to (u,v) coordinates.

To change cartesian co-ordinates (x,y) to polar co-ordinates (r,θ), we have, $x = r\cos\theta$ $\quad y = r\sin\theta$ and $J = \dfrac{\partial(x,y)}{\partial(u,v)} = r$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example: Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the parallelogram in the xy plane. with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$ using the transformation $u = x+y$ and $v = x-2y$





Solution: The region R, ie parallelogram ABCD in the xy-planes becomes the region R', ie rectangle $A'B'C'D'$ in the uv plane as shown in the above figures by taking,

$$u = x+y \text{ — (I)} \quad \text{and} \quad v = x-2y \text{ — (II)}$$

$$\text{(I)} - \text{(II)} \Rightarrow u - v = 2y$$

$$\Rightarrow y = \tfrac{1}{2}(u-v) \text{ — (III)}$$

$$\text{(I)} \times 2 + \text{(II)} \Rightarrow 2u + v = 2x + 2y + x - 2y$$

$$\Rightarrow 3x = 2u + v$$

$$\Rightarrow x = \tfrac{1}{3}(2u+v)$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/3 & 1/2 \\ 1/3 & -1/2 \end{vmatrix}$$

$$= -\tfrac{1}{3} - \tfrac{1}{6}$$

$$= \frac{-2-1}{6} = -\tfrac{1}{2}$$

$$\therefore |J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\tfrac{1}{2} \right| = \tfrac{1}{2}$$

$$\therefore \iint_R (x+y)^2 \, dx\, dy$$

$$= \iint_{R'} u^2 \, |J| \, du\, dv$$

$$= \int_{u=1}^{u=4} \int_{v=-2}^{v=1} u^2 \left(\tfrac{1}{2}\right) du\, dv$$

$$= \frac{1}{2} \int_{u=1}^{u=4} u^2 \, |v|_{-2}^{1} \, du$$

$$= \frac{1}{2} \int_{u=1}^{u=4} u^2 [+2+1] \, du$$

$$= \frac{1}{2} \cdot 3 \int_{u=1}^{u=4} u^2 \, du$$

$$= \frac{3}{2} \left| \frac{u^3}{3} \right|_1^4$$

$$= \frac{\cancel{3}}{2} \times \frac{1}{\cancel{3}} [64-1]$$

$$= \frac{1}{2} \times 63 = \frac{63}{2}$$

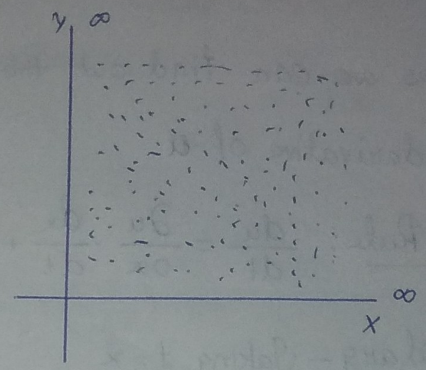
Ex- Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Hence show that, $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$

Sol^n The region of integration being the first quadrent of the xy-plane.

r varies from 0 to ∞ and θ

varies from 0 to $\pi/2$

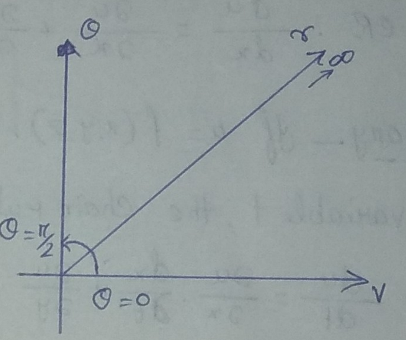


$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$



$$\therefore I = -\frac{1}{2}\int_0^{\pi/2} \left[\int_0^\infty e^{-r^2}(-2r)\,dr\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left(e^{-\infty^2} - e^0\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4} \quad \text{—— (I)}$$

$$\therefore I = \int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy$$

$$= \left\{\int_0^\infty e^{-x^2}\,dx\right\}^2 \quad \text{—— (II)}$$

From (I) & (II)

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Total Derivative

If $u = f(x,y)$ and $x = \phi(t)$, $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$

Thus we can find out the ordinary derivative $\frac{du}{dt}$ which is called total derivative of u.

Chain Rule: $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Corrilary – Taking $t = x$

CR: $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

Corrilary – If $u = f(x,y,z)$, where x, y, z are all functions of a variable t, the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

Differentiation of implicit functions:

If $f(x,y) = c$ be an implicit relation between x and y, then we can say that x is function of x itself and y canbe also expressed in terms of x, ie both x and y are functions of x, then by chain rule of total derivative,

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $f(x,y) = c$

$$\therefore \frac{df}{dx} = \frac{dc}{dx} = 0$$

$$\therefore 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}, \quad \frac{\partial f}{\partial y} \neq 0$$

Change of variable

If $u = f(x,y)$

where $x = \phi(s,t)$ and $y = \psi(s,t)$

$$\therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial u}{\partial y}\cdot\frac{dy}{ds}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Ex given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find $\frac{du}{dt}$ as a function of t. verify your result by direct substitution,

Sol^n we have,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \cos(x/y)\cdot\frac{1}{y}e^t + \cos(x/y)\left(-\frac{x}{y^2}\right)\cdot 2t$$

$$= \cos\left(e^t/t^2\right)\cdot\frac{1}{t^2}e^t + \cos\left(e^t/t^2\right)\left(-\frac{e^t}{t^4}\right)\cdot 2t$$

$$= \cos\left(e^t/t^2\right)\left[\frac{e^t}{t^2} - \frac{2e^t}{t^3}\right]$$

$$= \cos\left(e^t/t^2\right)\cdot\frac{e^t}{t^3}(t-2)$$

Also, $u = \sin(x/y) = \sin\left(e^t/t^2\right)$

$$\therefore \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\cdot\frac{t^2e^t - e^t\cdot 2t}{t^4}$$

$$= \cos\left(e^t/t^2\right)\cdot\frac{e^t}{t^3}(t-2)$$

Ex If x increased at the rate of 2 cm/s at the instant when $x = 3$cm, and $y = 1$cm at which rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

Sol^n Let $u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \quad \text{--- (1)}$$

when $x = 3$ & $y = 1$

$\frac{dx}{dt} = 2$ and u is higher.

Increasing no decreasing ie $\frac{du}{dt} = 0$

(1) $\Rightarrow 0 = (2.1 - 6.3.1).2 + (2.3 - 3.3^2)\frac{dy}{dt}$

$\Rightarrow 32 = -21\frac{dy}{dt}$

$$\Rightarrow \frac{dy}{dt} = \frac{-32}{21} \text{ cm/s}$$

Thus y is decreasing at the rate of $\frac{32}{21}$ cm/s

Ex If $u = F(x-y, y-z, z-x)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Sol^n Let $x-y = r$ $\quad y-z = s \quad z-x = t$

So that $u = f(r, s, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{dr}{dx} + \frac{\partial u}{\partial s}\cdot\frac{ds}{dx} + \frac{\partial u}{\partial t}\cdot\frac{dt}{dx}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}(-1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{--- (I)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{dr}{dy} + \frac{\partial u}{\partial s}\cdot\frac{ds}{dy} + \frac{\partial u}{\partial t}\cdot\frac{dt}{dy}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}\cdot(1) + \frac{\partial u}{\partial t}\cdot 0$$

$$= \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \quad \text{--- (II)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{dr}{dz} + \frac{\partial u}{\partial s}\cdot\frac{ds}{dz} + \frac{\partial u}{\partial t}\cdot\frac{dt}{dz}$$

$$= \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$= \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \quad \text{--- (III)}$$

Now, (I) + (II) + (III)

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} = 0$$

Ex If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, Show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{-1}{\sqrt{1-(x-y)^2}}\cdot 12t^2$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)-8t^2(1-t^2)+16t^4(1-t^2)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)\{1-8t^2+16t^4\}}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

$$\therefore \frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$$

Ex – If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$,

Show that, $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x . \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$

Solⁿ Here z = is a composite function of u and v.

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} . e^u + \frac{\partial z}{\partial y} (-e^{-u})$$

and, $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x} (-e^{-u}) + \frac{\partial z}{\partial y} (-e^v)$$

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u + e^{-v}) + \frac{\partial z}{\partial y} (-e^{-u} + e^v)$$

$$= x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

Ex – If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. Show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$

Solⁿ The ~~define~~ given equations define w as function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial w}{\partial x} \cos\theta + \frac{\partial w}{\partial y} \sin\theta$$

$$= \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta \quad \text{— ①} \quad [\because w = f(x, y)]$$

Also, $\dfrac{\partial w}{\partial \theta} = \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial \theta} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial \theta}$

$$= \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$= -r\frac{\partial f}{\partial x}\sin\theta + r\frac{\partial f}{\partial y}\cos\theta \quad [\because w = f(x,y)] \quad \text{—— (11)}$$

$$\therefore \left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2$$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{1}{r^2}\left(-r\frac{\partial f}{\partial x}\sin\theta + r\frac{\partial f}{\partial y}\cos\theta\right)^2$$

$$\therefore \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sin\theta\cos\theta$$

$$+\frac{1}{r^2}\left[r^2\left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + r^2\left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta - 2r^2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sin\theta\cos\theta\right]$$

$$\Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\cos^2\theta + \sin^2\theta)$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Ex If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find the value of $\dfrac{dz}{dx}$ when $x = y = a$

Sol^n The given equations are of the form, $z = f(x,y)$ and $\phi(x,y) = c$

$\therefore z$ is complete function of x

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Now, $\dfrac{\partial z}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

Similarly, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

Now, $x^3+y^3+3axy = 5a^2$

As $\phi(x,y) = c$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2+3ay}{3y^2+3ax}$$

$$= -\frac{x^2+ay}{y^2+ax}$$

$$\therefore \frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(-\frac{x^2+ay}{y^2+ax}\right)$$

$$\therefore \left[\frac{dz}{dx}\right]_{\substack{x=a\\y=a}} = \frac{a}{\sqrt{a^2+a^2}} + \frac{a}{\sqrt{a^2+a^2}}\left(-\frac{a^2+a^2}{a^2+a^2}\right)$$

$$= 0$$

HW If $f(x,y)=c$, then show that $\frac{dy}{dx} = -\frac{f_x}{f_y}$

and $$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_y f_{xy} + f_{yy}f_x^2}{f_y^3}$$

Mean value theorems and expandsion of function.

Indeterminate form –

Theorem 2.6 Let f and g be functions ~~f~~ such that

i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

ii) $f'(a)$ and $g'(a)$ exist and $g'(a)\neq 0$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

Proof – Since, by hybothesis theorem, f and g are deriveable at $x=a$,

They are also continuous at $x=a$ and so,

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a)$$

Therefore by condition ①, $f(a)=g(a)=0$. On the other hand,

$$\left.\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{f(x)}{x-a} \\ g'(a) &= \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{g(x)}{x-a}\end{aligned}\right\} \text{— (II)}$$

Consequently, $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left[\frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}}\right] = \frac{\lim_{x\to a}\frac{f(x)}{x-a}}{\lim_{x\to a}\frac{g(x)}{x-a}} = \frac{f'(a)}{g'(a)} \quad \text{— (II)}$$

Theorem 2.7 (L' Hospital rule for $0/0$ form). Let f and g be functions such that i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

ii) $f'(x)$ and $g'(x)$ exist and $g'(x)\neq 0$ for $x\in(a-\partial, a+\partial), \partial>0$ except possibly at $x=a$ and

iii) ~~Then~~ $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, $g'(x) \neq 0$

Proof: Under the given hypothesis, f and g are continuous at $x=a$ and

$f(a) = \lim_{x \to a} f(x) = 0$ and $g(a) = \lim_{x \to a} g(x) = 0$

By Taylor's Theorem truncated at the first-derivative term, we have

$f(a+h) = f(a) + h\, f'(a+\theta_1 h) = h\, f'(a+\theta_1 h)$

$0 < \theta_1 < 1$ and

$g(a+h) = g(a) + h\, f'(a+\theta_2 h) = h\, g'(a+\theta_1 h)$

$0 < \theta_2 < 1$

Therefore,

$$\cancel{\frac{\lim f(x)}{\lim g(x)}} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = \lim_{h \to 0} \frac{f'(a+\theta_1 h)}{g'(a+\theta_2 h)} = \frac{f'(a)}{g'(a)}, \ g'(a) \neq 0$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Hence, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, $g'(a) \neq 0$

Obviously, this relation fails if $g'(a) = 0$. If $g'(a) = 0$ and if $f'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = +\infty \text{ or } -\infty$$

If $f'(a) = g'(a) = 0$, then by Taylor's theorem (assuming second-order derivatives of f and g exist), we have

$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a+\theta_3 h)$

$0 < \theta_3 < 1$

$= \frac{h^2}{2!} f''(a+\theta_3 h)$

and $g(a+h) = g(a) + h\, g'(a) + \frac{h^2}{2!} g''(a+\theta_4 h)$

$0 < \theta_4 < 1$

Therefore, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = \lim_{h \to 0} \frac{f''(a+\theta_3 h)}{g''(a+\theta_4 h)}$

$$= \frac{f''(a)}{g''(a)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}, \ g''(a) \neq 0$$

Hence in general, if

$f(a) = f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$

$$g(a) = g'(a) = g''(a) = \cdots\cdots = g^{(n-1)}(a) = 0$$

and $g^{(n)}(a) \neq 0$, then using Taylor's Theorem with a reminder after n terms, we have.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}, \quad g^{(n)}(a) \neq 0,$$

which is called the generalized L' Hospital rule.

General Theorems

The theorems abblicable to a class of functions are known as general theorems.

Rolle's theorem –

If a function f is

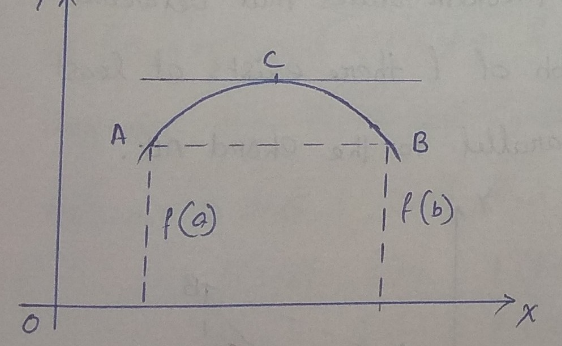
i) Continuous in the closed interval $[a, b]$

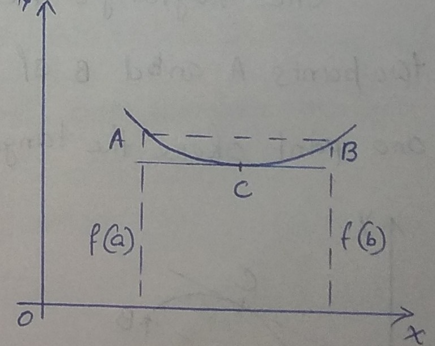
ii) Deriveable in the open interval (a, b)

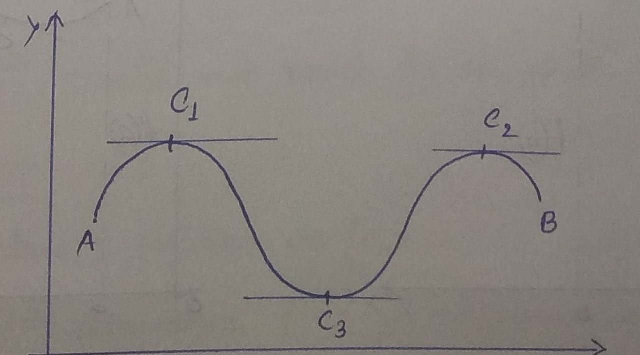
iii) $f(a) = f(b)$, then there exists at least one real number $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical interpretation –

If we draw the curve $y = f(x)$, which is continuous on $[a,b]$ and deriveable on (a,b), then the Rolle's theorem states that between two points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to the x-axis.







Algebric interpretation of Rolle's theorem –

In Rolle's theorem, $f(a) = f(b)$, so if a and b are zeros of f, then Rolle's theorem says that between two zeros a and b of f, there exists at least one zero of $f'(x)$

Lagrang's Mean value theorem –

If f is a function defined on $[a, b]$ is

i) Continuous on $[a, b]$ and

ii) Deriveable in (a, b),

Then there exists at least one real number $c \in (a, b)$,

Such that $\dfrac{f(a) - f(b)}{a-b} = f'(c)$

Note i) Lagrang's Mean value theorem is also known as First mean value theorem of the differential calculus.

ii) If we put $b = a+h$, then the number between a and b can be written as $a + \theta h$, $0 < \theta < 1$

Thus the Langrang's Mean value theorem takes the form

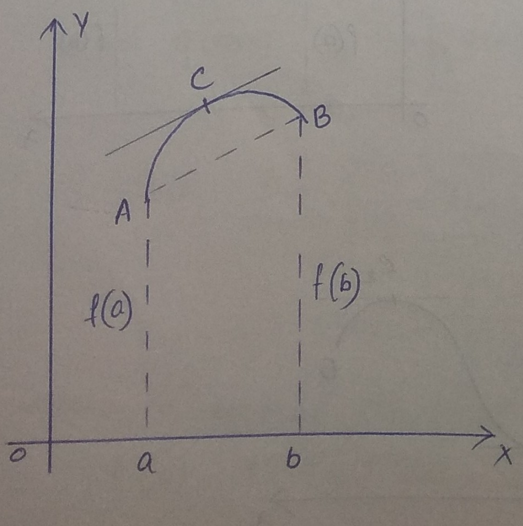
$f(a+h) - f(a) = hf'(a+\theta h)$

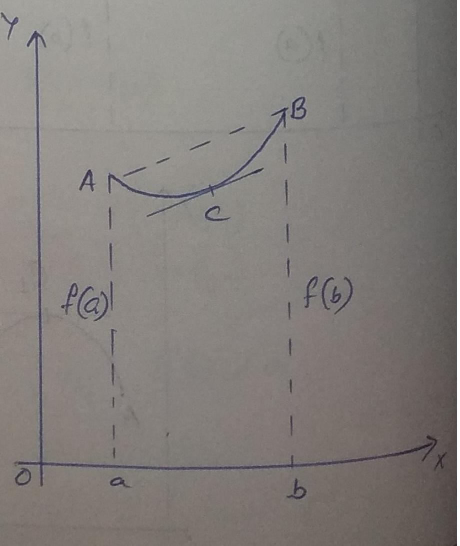
or $f(a+h) = f(a) + hf'(a+\theta h)$

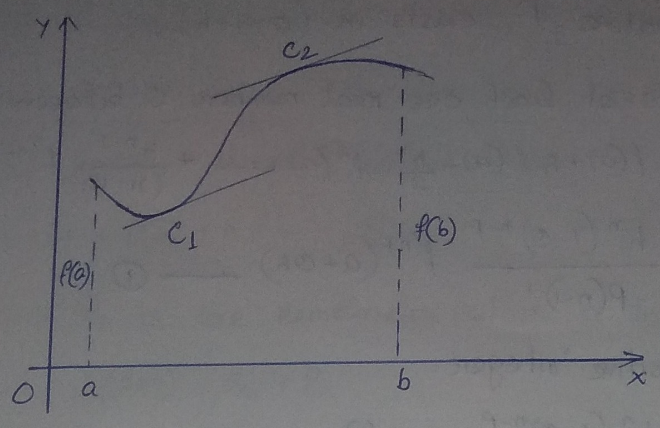
where $0 < \theta < 1$ and $h = b-a$

Geometrical interpretation –

The Lagrang's Mean value theorem states that between two points A and B of the graph of f there exists at least one point where the tangent is parallel to the chord AB.







Cauchy's Mean value theorem -

If two functions f and F defined on $[a,b]$ are i) continuous on $[a,b]$

ii) Deriveable on (a,b) and

iii) $F'(x) \neq 0$ for any $x \in (a,b)$

Then there exists a point $c \in (a,b)$ such that,

$$\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}, \quad a<c<b$$

(obviously $F(b) \neq F(a)$. otherwise F satisfies all the conditions of the Rolle's theorem and as such, $F'(c)=0$, which contradicts Condition (iii) of the statement)

Another form of the statement.

If two functions f and F are continuous in $[a, a+h]$, deriveable in $(a, a+h)$ and $F'(x) \neq 0$ for any $x \in (a, a+h)$ ~~and $F'(x) \neq 0$ for any $x \in$~~ then there ~~are~~ exists at least one ~~member~~ number θ, between 0 and 1 such that,

$$\frac{f(a+h)-f(a)}{F(a+h)-F(a)} = \frac{f'(a+\theta h)}{F'(a+\theta h)}$$

($f'(x)$ and $F'(x)$ should not vanish for the same value of x)

Taylor's theorem -

If a function f defined on $[a, a+h]$ is such that

i) The $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

ii) The nth derivative $f^{(n)}$ exists in $(a, a+h)$,

Then there exists at least one real number θ between 0 & 1

Such that, $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) +$

$$\frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h) \quad \text{—— (1)}$$

where p is a positive integer.

The term $R_n = \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$

is called Taylor's remainder after n terms and the theorem with this form of remainder is known as Taylor's theorem with Schilomilch and Roche form of remainder.

Substituting $p=1$, we get

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

which is called Cauchy's form of the remainder.

Substituting $p=n$, we get

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

which is called Lagrange's form of the remainder.

If we put $a+h = x$, then (1) becomes $f(x)$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \cdots +$$

$$\frac{(x-a)^n}{p(n-1)!} (1-\theta)^{n-p} f^{(n)}(a+\theta(x-a)) \quad \text{—— (11)}$$

Substituting $a=0$ in the last expression, we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{p(n-1)!} (1-\theta)^{n-p} f$$

which is known as Maclaurin's theorem with schlomilch and Roche form of remainder.

Substituting $p = 1$, we get

$$R_n = \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta x)$$

which is called Lagrange's form of remainder for Maclaurin's theorem.

Taylor's Infinite Series and Power Series expansion —

The Taylor's theorem asserts that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where R_n is the remainder after n terms.

Thus, $f(a+h) = S_n + R_n$

where, $S_n = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$.

Thus if $R_n \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [f(a+h) - R_n]$$

$$= f(a+h) - 0 = f(a+h)$$

and so the series.

$$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dots$$

Called Taylor's series, converges to $f(a+h)$

On the other hand if we put $a+h=x$, then

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$ which is the expansion of $f(x)$ in powers of $(x-a)$.

Maclaurin's infinite series —

If the remainder R_n in the Maclaurin's expansion of a function tends to zero, then we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

which is called Maclaurin's infinite expansion of f in power of x.

The series, $f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$ is called Maclaurin's infinite Series.