Complex differentiation –

Function of complex variable –

If x and y are real variables then $z = x+iy$ is a complex variable.

If corresponding to each value of a complex variable z in a given region R, there correspond to one or more values of another variable $\omega = u+iv$, then ω is called a function of complex variable z.

Mathematically, we can write.

$\omega = f(z)$ or $u+iv = f(x+iy)$

Eg – $f(z) = z^2$

Let $z = x+iy$

$$f(x+iy) = (x+iy)^2$$
$$= x^2 + 2xy.i + i^2y^2$$
$$= (x^2-y^2) + i.\,2xy$$

Comparing with $u+iv$, we get

$u = x^2+y^2$, $v = 2xy$

Limit of a complex function –

A function $f(z)$ tends to limit l. as z tends to z_0, along any path, then we call l is the limit of $f(z)$ and write

$$\lim_{z \to z_0} f(z) = l$$

Eg - $f(z) = \bar{z}/z$ find $\lim_{z\to 0} f(z)$

Soln Let $z = x+iy$

along x axis

$z = x$, $\bar{z} = x$

$z \to 0$, $x \to 0$

$$\therefore \lim_{z\to 0} \bar{z}/z = \lim_{x\to 0} \frac{x}{x} = 1$$

along y axis

$z = iy \quad , \quad \bar{z} = -iy$

$z \to 0 \quad . \quad y \to 0$

$$\therefore \lim_{z \to 0} \bar{z}/z = \lim_{y \to 0} \frac{-iy}{iy} = -1$$

So we can say limit does not exist.

Continuity of $f(z)$ →

A function $f(z)$ is said to be continuous at point $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivative of a complex function →

Let $\omega = f(z)$ be a complex function, then derivative of ω with respect to z is given by

$$\frac{d\omega}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} \quad (\delta \to \text{delta}, \partial \to \text{del})$$

Analytic function

If a function $f(z)$ posseses unique derivative at every points in the region R, then $f(z)$ is analytic in ~~AO~~ R.

Entire function

If a function $f(z)$ is analytic in the whole complex plane, then $f(z)$ is called entire function.

Singular point

The points in the region R, where the function $f(z)$ is not analytic are called singular points.

If $f(z)$ is entire function

⇒ $f(z)$ has zero singular points.

The necessary and sufficient condition for the function $\omega = f(z) = u(x, y) + iv(x, y)$ to be analytic are

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, are continuous in R

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$ in R

The equation (ii) are called Cauchy-Riemann (C-R) equation.

Proof <u>Necessary Part</u> -

Let $\omega = f(z) = u + iv$ is analytic in R.

$\because f'(z)$ exists all points of R.

Let $\delta x, \delta y$ be the small increment x and y and. $\delta u, \delta v, \delta z$ are the corresponding increments in (u, v, z) respectively.

$f(z) = u + iv$

$f(z+\delta z) = u + \delta u + i(v + \delta v)$

Now, $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

$= \lim_{\delta z \to 0} \frac{u + \delta u + i(v+\delta v) - u + iv}{\delta z}$

$f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + \frac{\delta v}{\delta z} \right)$ — (i)

Along x-axis -

$z = x$ and $\delta z = \delta x$ and $\delta z \to 0$ means $\delta x \to 0$

$\therefore f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{\partial z} + i \frac{\delta v}{\delta z} \right)$

$= \lim_{\delta x \to 0} \left(\frac{\partial u}{\delta x} + i \frac{\delta v}{\delta x} \right)$

$= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + i \lim_{\delta x \to 0} \frac{\delta v}{\delta x}$

$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ — (ii)

Along y axis

$z = iy$, $\delta z = i\delta y$ and $\delta z \to 0$ means $\delta y \to 0$

$\therefore f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{\partial z} + i \frac{\delta v}{\delta z} \right)$

$= \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right)$

$= \lim_{\delta y \to 0} \frac{\delta u}{i\delta y} + \lim_{\delta y \to 0} \frac{\delta v}{\delta y}$

$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ — (iii)

From (II) and (III)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

comparing real and imaginary part on both side, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Obviously $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial u}{\partial y}$ are continuous in R

Sufficient part –

Let $f(z) = u + iv$ be a complex function possess partial ~~differentiation~~ derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in the region R and satisfies the CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

we need to prove that $f(z)$ is analytic in R.

Proof – $f'(z)$ exists at any point of R

$$f'(z) = \lim_{z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$f(z) = u + iv$$

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

Taylor series

$$\left[u(x,y) + \left(\frac{\partial u}{\partial x}\cdot\delta x\right) + \left(\frac{\partial u}{\partial y}\cdot\delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x\right) + \left(\frac{\partial v}{\partial y}\cdot\delta y\right)\right]$$

Expanding in taylor's series and neglecting the higher order

$$f(z+\delta z) = \left[u(x,y) + iv(x,y)\right] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y)$$

$$\Rightarrow \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

limit $\delta z \to 0$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

So, $f'(z)$ is continuous in R as $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ are continuous in R

$\therefore$ $f'(z)$ exists.

If $f(z) = u+iv$ is an analytic function,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{or} \quad f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Laplace equation's harmonic function –

An equation of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace eq^n.

Any solution or function that satisfies Laplace eq^n is called harmonic function.

$f(z) = z^2 = (x^2 - y^2) + 2ixy$

$\frac{\partial u}{\partial x} = +2x$ $\qquad$ $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = -2$ $\qquad$ So, u & v are harmonic

$\frac{\partial v}{\partial x} = 2y$ $\qquad$ $\frac{\partial^2 v}{\partial x^2} = 0$

$\frac{\partial v}{\partial y} = 2x$ $\qquad$ $\frac{\partial^2 v}{\partial y^2} = 0$

$f(z)$ is analytic iff

i) C-R equations are satisfied (must x in R)

ii) Partial derivatives is continuous in R

Prob: Form analytic function, both real and imaginary parts are harmonic.

Sol^n: $f(z) = u+iv$ be an analytic function in R

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

(ii) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

So, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ((i) w.r.t. x)

Theorem – For an analytic function, both real and imaginary parts are harmonic.

Proof Let $f(z) = u + iv$ be an analytic function in a region R

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (I)} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \text{— (II)}$$

Differentiating (I) partially wrt x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{— (III)}$$

Again differentiating (II) partially wrt y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad \text{— (IV)}$$

Now, (III) + (IV) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore u$ is harmonic.

Ex Show that the function $f(z)$ defined by $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

C-R eqn or satisfies at origin but not $f'(0)$

Soln Given $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x}$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$$

$$u = \frac{x^3 - y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

At origin, u at $(0,0)$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x^3}{x^2} \cdot \frac{1}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - (0,0)}{y}$$

$$= \lim_{y \to 0} \frac{-y^3}{y^2} \cdot \frac{1}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x^3}{x^2} \cdot \frac{1}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y^3}{y^2}\cdot\frac{1}{y} = 1$$

$\therefore$ At origin $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

ii) Now, we need to show $f'(0)$ doesn't exist.

$$\therefore f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

~~lim~~ Along x-axis

$z = x$ and $z\to 0$ meant $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{x^3(1+i)}{x^2 . x} = 1+i$$

Along y-axis

$z = iy$ and $z \to 0$ meant $y \to 0$

$$\therefore f'(0) = \lim_{y\to 0} \frac{-y^3(1-i)}{y^2 . iy} = \frac{i-1}{i} = \frac{i^2 - i}{i^2} = \frac{-1-i}{-1} = 1+i$$

Along $x = y$

$z = x + ix$, $z \to 0$, $x \to 0$

$$f'(0) = \lim_{x\to 0} \frac{x^3(1+i) - x^3(1-i)}{x^2(x+ix)}$$

$$= \lim_{x\to 0} \frac{1+i-1+i}{2(1+i)} = \lim_{x\to 0} \frac{i}{i+1} = \frac{i(1-i)}{1-i^2} = \frac{i-i^2}{1+1} = \frac{1+i}{2}$$

$\therefore f'(0)$ Does not exist.

Eg Solve that $f(z) = \begin{cases} \frac{x^3y^5(x+iy)}{x^6+y^{10}} & , z\neq 0 \\ 0 & , z = 0 \end{cases}$ is not analytic even through C-R eq^n are not satisfied at origin.

Q) Determine the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$

① Given u; find v, write $f(z)$

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}\cos 2y + 2e^{2x}x\cos 2y - 2e^{2x}y\sin 2y$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$$

$$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$= -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y)$$

As $f(z)$ is analytic, C-R eq^n must satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

So, $\dfrac{\partial v}{\partial y} = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin y)$

$$\partial v = e^{2x}(\cos 2y)\partial y + 2x\cos 2y\,\partial y - 2y\sin 2y\,\partial y$$

$$v = e^{2x}\left[\frac{\sin 2y}{2} + 2x\frac{\sin 2y}{2} - 2\int y\sin 2y\,\partial y\right]$$

$$= e^{2x}\left[\frac{\sin 2y}{2} + x\sin 2y - 2\left[\frac{\sin 2y}{4} - \frac{\cos 2y}{2}y\right]\right]$$

$$\Rightarrow v = e^{2x}(x\sin 2y + y\cos 2y) + \phi(x)$$

Now, we have to find $\phi(x)$

Differentiating partially wrt x.

$$\frac{\partial v}{\partial x} = 2e^{2x}(x\sin 2y + y\cos 2y) + e^{2x}(\sin 2y) + \phi x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\phi'(x) = 0$$

$$\Rightarrow \phi(x) = c$$

$$v = e^{2x}(x\sin 2y + y\cos 2y) + c$$

$$\omega = e^{2x}\left[(x+iy)\cos 2y + i(x+iy)\sin 2y\right] + c' \quad [c' = ic]$$

~~$= e^{2x}[z\cos 2y + iz\sin 2y] + c'$~~

$$= e^{2x}\left[z\cos 2y + iz\sin 2y\right] + c'$$

$$= ze^{2x}e^{i2y} + c'$$

$$= ze^{2(x+iy)} + c'$$

$$= ze^{2z} + c'$$

Milne - Thomson Method -

This method determines the analytic function directly if one of u & v is given.

Now, $z = x + iy$ $\qquad \bar{z} = x - iy$

$x = \dfrac{z+\bar{z}}{2}$ $\qquad y = \dfrac{z-\bar{z}}{2i}$ —— ①

Now, $\omega = f(z)$

$\Rightarrow f(z) = u(x,y) + iv(x,y)$ —— ⑪

$= u\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) + iv\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$ (using ①)

This is an identity. Hence, putting $\bar{z} = z$, we get

$f(z) = u(z, 0) + iv(z, 0)$ —— (III)

In an analytic function, we can replace x by z and y by 0.

Eg- Determine the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2x)$

Soln Here $e^{2x}(x\cos 2y - y\sin 2y)$ —— ①

We have the analytic function is,

$f(z) = u + iv$

$\Rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (using C-R eqn)

$\Rightarrow f'(z) = e^{2x}\cos 2y + 2e^{2x}(x\cos 2y - y\sin 2y) - i\left[e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)\right]$

(From ①)

→ f(z) is analytic, Therefore f'(z) is analytic

Replacing x by z and y by 0, we get,

$f'(z) = e^{2z} + 2e^{2z}z - i\left[e^{2z}(0)\right]$

$= e^{2z} + 2ze^{2z}$

Now, Integrating, we get

$f(z) = \dfrac{e^{2z}}{2} + 2\left[z\dfrac{e^{2z}}{2} - \dfrac{e^{2z}}{4}\right] + c$

$= ze^{2z} + c$

This is the required analytic function.

Eg- If $u - v = (x-y)(x^2+4xy+y^2)$ then, find the analytic function.

Sol^n Given, $u - v = (x-y)(x^2+4xy+y^2)$

we have, $f(z) = u + iv$ —— (I)

$\Rightarrow i.f(z) = iu - v$ —— (II)

Now, (I) + (II) $\Rightarrow f(z)(1+i) = u - v + i(u+v)$

$\Rightarrow F(z) = U + iV$ (analytic)

where, $F(z) = (1+i) f(z)$

$U = (u-v) = (x-y)(x^2+4xy+y^2)$

$V = u + v$

Now, $F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x}$

$= \frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y}$ (using C-R eq^n)

$F'(z) = (x-y)(2x+4y) + (x^2+4xy+y^2) - i\left[(x-y)(4x+2y) - (x^2+4xy+y^2)\right]$

Here $F'(z)$ is also analytic

So, replacing x by z & y by 0, we get (M-T, method)

$F'(z) = z(2z) + z^2 - i\left[z(4z) - z^2\right]$

$= 3z^2 - 3iz^2$

Integrating we get,

$F(z) = z^3 - iz^3 + c$

$\Rightarrow (1+i) f(z) = (1-i) z^3 + c$

$\Rightarrow f(z) = \frac{1-i}{1+i} z^3 + c'$

$= \frac{-2i}{2} z^3 + c'$

$= -iz^3 + c'$

HW Show that $|z|^2$ is not analytic.

Soln $z = x + iy$

$\Rightarrow |z| = \sqrt{x^2+y^2}$

$\Rightarrow |z|^2 = x^2 + y^2 + i.0$

$\therefore u = x^2+y^2, \ v = 0$

$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = 0$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

ie. C-R eqn are not satisfied.

So, $|z|^2$ is not analytic.

But C-R eqns only hold at $z=0$, ie $z = 0+i0 = 0$

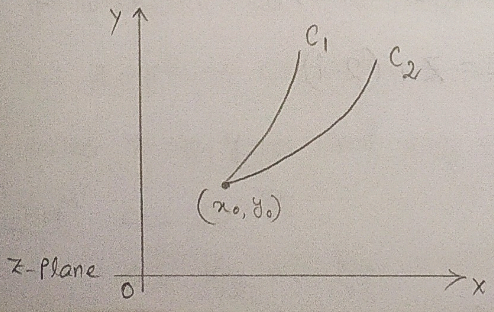
$\therefore f$ is only differentiable at $z=0$

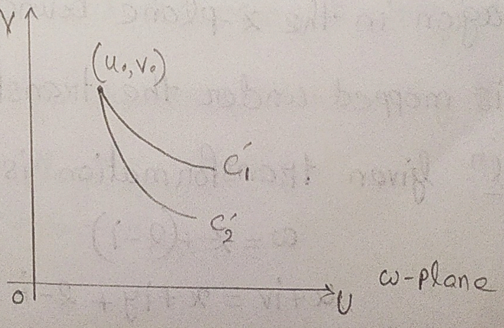
Transformation or Mapping -

A real function $y = f(x)$ or $z = f(x,y)$ can be drawn graphically and we can study various properties. However this method of graphical representation fails in case of complex function, as a complex function $\omega = f(z)$ or $u+iv = f(x+iy)$ which consists of 4 variables.

Therefore, we choose two complex planes and call them z-plane (xy-plane) and ω-plane (uv-plane). In z plane we plot $z = x+iy$ and in ω-plane we plot corresponding point $\omega = u+iv$. Thus, a complex function $\omega = f(z)$ defines a correspondence between the points of two planes.

Conformal Mapping -





Consider a complex transformation $\omega = f(z)$. Suppose that under the transformations $u = u(x,y)$, $v = v(x,y)$ the point (x_0, y_0) of the z-plane is mapped into the point (u_0, v_0) in the ω-plane. While the waves C_1 and C_2 intersecting at (x_0, y_0) are mapped respectively into the curves C_1' and C_2' intersecting at (u_0, v_0).

Then if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C_1' & C_2' both in magnitude and sense, the tranformation is said to be Conformal.

A mapping or transformation which preserves the magnitude of the angles but not sense is called isogonal mapping or isogonal transformation.

Theorem – A complex transformation $\omega = f(z)$ is said to be Conformal iff $f(z)$ is analytic and $f'(z)$ is non-zero, ie $f'(z) \neq 0$ at point z_0.

Eg – $\omega = f(z) = e^z \Rightarrow$ is Conformal at everywhere.

$\omega = f(z) = z^2 \Rightarrow$ Analytic everywhere except $z = 0$.

Some standard transformations –

There are four standard transformations.

i) Translation –

$\omega = f(z) = z + b$, where b is any constant (real or complex)

This type of transformation translate the image of the z-plane but presences the shape of & size in ω-plane.

Q what are the regions of the ω-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1, y = 2$ is mapped under the transformation $\omega = z + (2 - i)$

Sol^n Given transformation is,

$$\omega = z + (2 - i)$$

$$u + iv = x + iy + 2 - i$$

$\therefore u = x+2 \quad \& \quad v = y-1 \quad \text{--- (1)}$

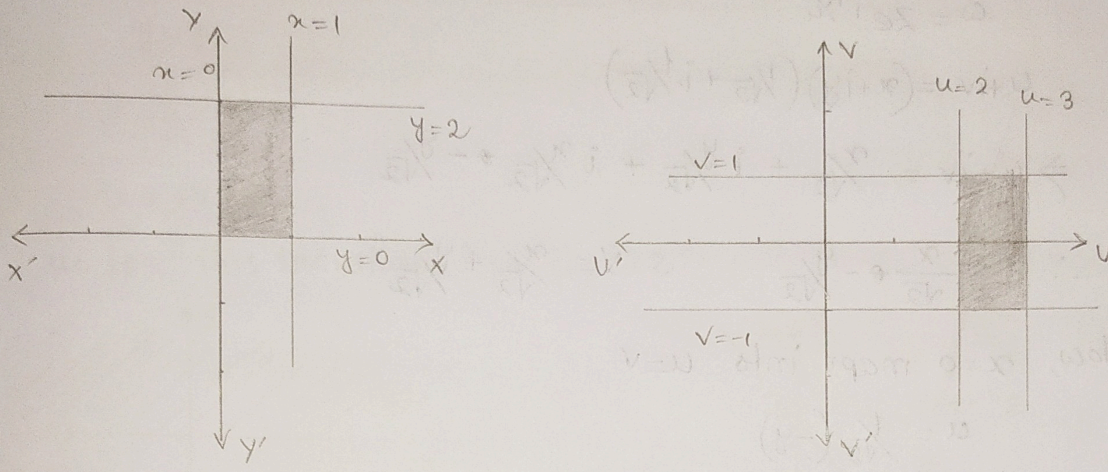
Under the transformation (1)

$x = 0$ is mapped into $u = 2$

$y = 0$ is mapped into $v = -1$

$x = 1$ is mapped into $u = 3$

$y = 2$ is mapped into $v = 1$



z-plane w-plane.

ii) Rotation and magnification -

The general form of this mapping is

$w = az$, a is any constant (real or imaginary)

If a is real, then figures in the z-plane is magnified or contracted by $|a|$ times (no rotation)

If a is imaginary, then $a = |a|e^{i\theta}$, then the image in the z-plane with magnified and contracted by $|a|$ and rotate through an angle θ. If θ is positive, rotation will be anticlockwise and if θ is negative, rotation will be clockwise.

Here, $\theta = \arg a = \tan^{-1}\left(\frac{b_2}{b_1}\right)$. $a = b_1 + ib_2$

Eg- Consider the transformation $w = ze^{i\pi/4}$ and determine the region in the z-plane corresponding to the triangular region bounded by the lines $x=0, y=0$ and $x+y=1$ in the z-plane.

Soln $w = ze^{i\pi/4}$

$a = 1/\sqrt{2} + i \cdot 1/\sqrt{2}$

$\therefore |a| = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2}$

$= 1$

$\theta = \tan^{-1}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right)$

$= 45^\circ$

$\therefore$ Anticlockwise ($\because \theta$ is +ve)

Given transformation is

$\omega = z e^{i\pi/4}$

$u + iv = (x+iy)\left(1/\sqrt{2} + i \cdot 1/\sqrt{2}\right)$

$\Rightarrow u + iv = x/\sqrt{2} + i\,y/\sqrt{2} + i\,x/\sqrt{2} - y/\sqrt{2}$

$\therefore u = \frac{x}{\sqrt{2}} - y/\sqrt{2}$ $\qquad v = x/\sqrt{2} + y/\sqrt{2}$

Now, $x=0$ maps into $u=v$

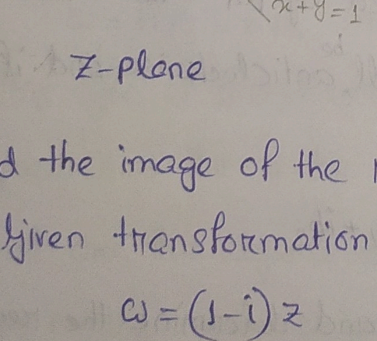
$u = 1/\sqrt{2}\,(-y)$

$v = 1/\sqrt{2}\; y$

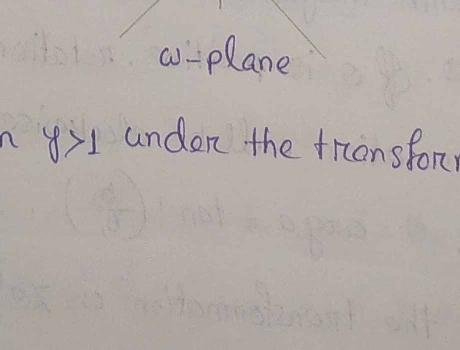
$\therefore u = -v$

Parallely $y=0$, maps into, $u=v$

For $x+y=1$ maps into $v = 1/\sqrt{2}$



Z-plane



ω-plane

Eg - Find the image of the region $y>1$ under the transformation $\omega = (1-i)z$

Sol^n Given transformation is

$\omega = (1-i)z$

$a = 1 - i$

$|a| = \sqrt{2}$

$\theta = -\pi/4$, clockwise

Now, $\omega = (1-i)z$

$\Rightarrow u + iv = (1-i)(x+iy)$

$\Rightarrow u + iv = x + iy - ix + y$

$\Rightarrow u + iv = (x+y) + i(x+y)$

$\therefore u = x+y \ \& \ v = -x+y$

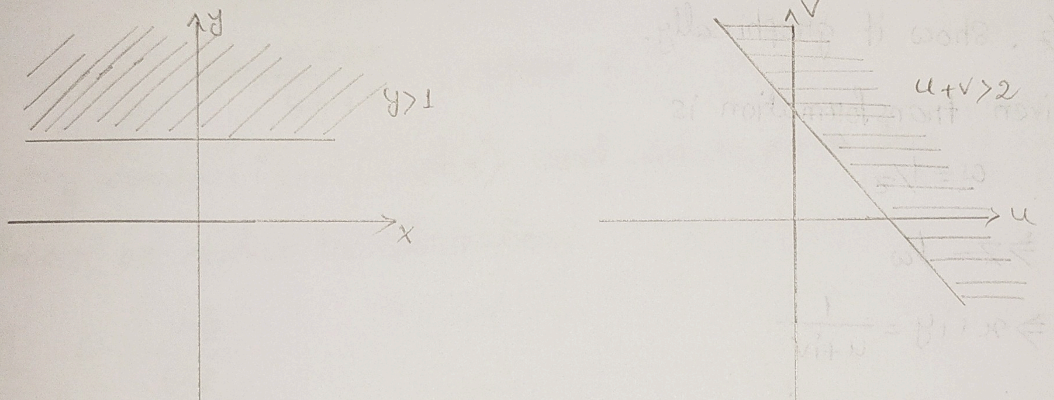
$\Rightarrow y = \frac{u+v}{2}$

$y > 1$

$\Rightarrow \frac{u+v}{2} > 1$

$\Rightarrow u + v > 2$

under this transformation $u+v > 2$



z-plane ω-plane

iii) Inversion –

This transformation maps a circle in the z-plane to a circle in the ω-plane or to a straight line if the circle in the z-plane passes through the origin.

Proof The general eq^n of any circle in the z-plane is

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let $\omega = 1/z \Rightarrow z = 1/\omega$

$\Rightarrow x + iy = \frac{1}{u+iv}$

$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$ — (II)

Putting (II) in (I), we get

$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{(u^2+v^2)} - \frac{2fv}{u^2+v^2} + c = 0$

$$\Rightarrow \frac{1}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$\Rightarrow 2gu - 2fv + 1 + c(u^2+v^2) = 0$$

If $c \neq 0$, $u^2+v^2+2g\,u/c - 2f\,v/c + 1/c = 0$ — (III)

If $c \neq 0$, the circle (1) doesn't passes through the origin and the eqn (III) represents a circle in ω-plane.

If $c = 0$, $2gu - 2fv + 1 = 0$ — (IV)

If $c = 0$, the circle (1) passes through the origin and maps into the straight line given by (IV).

Q- Find the image of the infinite strip $1/4 \leq y \leq 1/2$, under the transformation $\omega = 1/z$. Show it graphically.

Soln Given transformation is

$$\omega = 1/z$$

$$\Rightarrow z = 1/\omega$$

$$\Rightarrow x+iy = \frac{1}{u+iv}$$

$$\Rightarrow x+iy = \frac{u-iv}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \quad \text{—— (1)}$$

Now, $y = 1/4$

$$\Rightarrow -\frac{v}{u^2+v^2} = 1/4$$

$$\Rightarrow u^2+v^2+4v = 0$$

$$\Rightarrow u^2+(v+2)^2 = 4$$

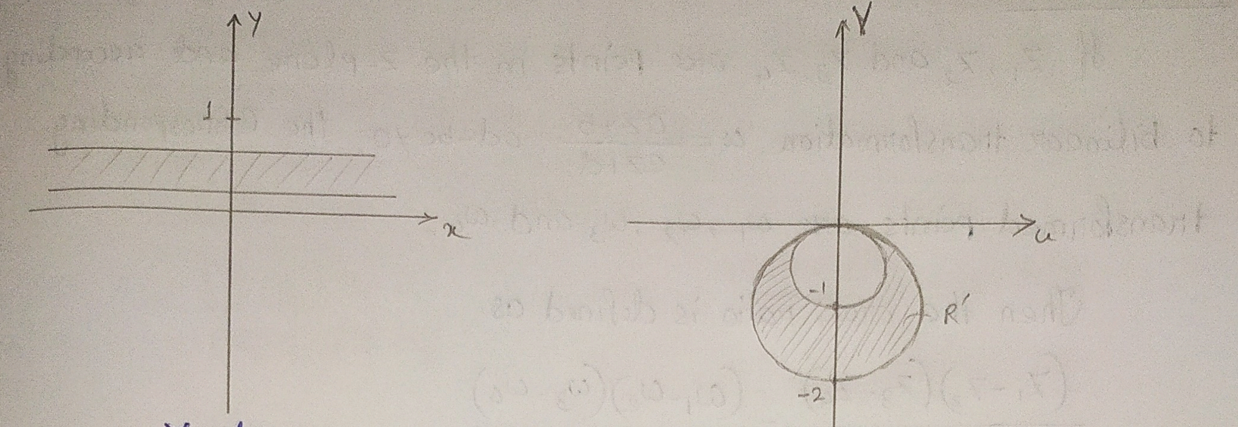
$y = 1/4$ mapped into a circle with centre $(0,-2)$ and radius 2.

Now, for $y = 1/2$

$$\Rightarrow u^2+v^2+2v = 0$$

$$\Rightarrow u^2+(v+1)^2 = 1$$

$y = 1/2$ mapped into a circle with centre $(0,-1)$ and radius 1.



Z-plane

w-plane

Hence, the infinite strips transformed into a region R′ shown in w-plane

iv) <u>Bilinear transformation</u> –

A transformation of the form,

$$\omega = \frac{aZ+b}{cZ+b}, \quad \text{where } a, b, c, d \text{ are}$$

any constant (real/complex) and $ad-bc \neq 0$ ~~is~~ ~~like~~ is called bilinear or modius transformation.

<u>Eg</u> – $\omega = \dfrac{2-Z}{i-Z}$,

Here $a=-1$ $\quad c=-1$

$b=2$ $\quad d=1$

$\therefore ad-cb = -1+2 = 1 \neq 0$

Consider a bilinear transformation $\omega = f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

$$\frac{d\omega}{dz} = \frac{(cz+d)a-(az+b)c}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cZ+d)^2} \neq 0$$

$f(z) = \dfrac{az+b}{cz+d}$ is analytic and $f'(z) \neq 0$

So, $\dfrac{az+b}{cz+d}$ is conformal

Thus bilinear transformation is conformal.

Cross Ratio -

If z_1, z_2 and z_3, z_4 are points in the z-plane and according to bilinear transformation $\omega = \frac{az+b}{cz+d}$, $ad-bc \neq 0$, the corresponding transformed points are $\omega_1, \omega_2, \omega_3$ and ω_4.

Then the cross ratio is defined as

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_2-\omega_3)(\omega_4-\omega_1)}$$

Theorem - The cross ratio is preserved under bilinear transformation.

Proof - Consider a bilinear transformation $\omega = \frac{az+b}{cz+d}$ —①

$ad-bc \neq 0$ —(ii)

Let z_1, z_2, z_3 & z_4 are four points in z-plane. According to the transformation ① let $\omega_1, \omega_2, \omega_3$ & ω_4 are the corresponding transformation is in ω-plane. We have to prove that

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_2-\omega_3)(\omega_4-\omega_1)}$$

Now, z_1, z_2, z_3 & z_4 mapped to $\omega_1, \omega_2, \omega_3$ & ω_4 and

$$\omega = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

So, $\omega_1 = \frac{az_1+b}{cz_1+d}$ $\qquad \omega_2 = \frac{az_2+b}{cz_2+d}$

$\omega_3 = \frac{az_3+b}{cz_3+d}$ $\qquad \omega_4 = \frac{az_4+b}{cz_4+d}$

Now, $$\text{RHS} = \frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_2-\omega_3)(\omega_4-\omega_1)}$$

$$= \frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right)\cdot\left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d}\right)\cdot\left(\frac{az_4+b}{cz_4+d} - \frac{az_1+b}{cz_1+d}\right)}$$

$$= \frac{[(az_1+b)(cz_2+d)-(az_2+b)(cz_1+d)][(az_3+b)(cz_4+d)-(az_4+b)(cz_3+d)]}{[(az_2+b)(cz_3+d)-(az_3+b)(cz_2+d)][(az_4+b)(cz_1+d)-(az_1+b)(cz_4+d)]}$$

$$= \frac{[z_1(ad-bc) - z_2(ad-bc)][z_3(ad-bc) - z_4(ad-bc)]}{[z_2(ad-bc) - z_3(ad-bc)][z_4(ad-bc) - z_1(ad-bc)]}$$

$$= \frac{(ad-bc)(z_1-z_2)(ad-bc)(z_3-z_4)}{(ad-bc)(z_2-z_3)(ad-bc)(z_4-z_1)}$$

$$= \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$= LHS.$

Theorem – Every bilinear transformation maps circles and straight into circles and straight lines.

Eg- Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $\omega = i, 0, -i$ respectively, hence find the image of $|z| < 1$ according to this transformation.

Sol^n - Consider the points in z-plane as $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and the respective transformed points in ω-plane as $\omega_1 = i$, $\omega_2 = 0$, $\omega_3 = -i$

Consider the point z in the z-plane is transformed in the point ω in the ω-plane by this bilinear transformation.

By the preservence of cross ratio, we have

$$\frac{(z_1-z_3)(z_3-z)}{(z_2-z_3)(z_4-z)} = \frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega_2-\omega_3)(\omega_4-\omega_1)}$$

$$\Rightarrow \frac{(1-i)(-1-z)}{(i+i)(z-1)} = \frac{(i-0)(-i-\omega)}{(0+i)(\omega-i)}$$

$$\Rightarrow \frac{(1-i)(1+z)}{(i+1)(z-1)} = \frac{i+\omega}{\omega-i}$$

$$\Rightarrow \frac{(1-i)(1+z)+(1+i)(z-1)}{(1-i)(1+z)-(1+i)(z-1)} = \frac{\omega+\cancel{i}+\omega-\cancel{i}}{\cancel{\omega}+i-\cancel{\omega}+i}$$

$$\Rightarrow \frac{\cancel{1}+z-i-\cancel{iz}+z-\cancel{1}+\cancel{iz}-i}{1+\cancel{z}-\cancel{i}-iz-iz+\cancel{i}-\cancel{z}+1} = \frac{\omega}{i}$$

$$\Rightarrow \frac{2(z-i)}{2(1-iz)} = \frac{\omega}{i}$$

$$\Rightarrow \omega = \frac{1+iz}{1-iz}$$, This is the required bilinear transformation

Now, $iz + 1 = \omega - i\omega z$

$\Rightarrow z(i\omega + i) = \omega - 1$

$\Rightarrow z = \dfrac{\omega - 1}{i\omega + i}$

$\because |z| < 1 \Rightarrow \left|\dfrac{\omega - 1}{i\omega + i}\right| < 1$

$\Rightarrow |\omega - 1| < |i\omega + i|$

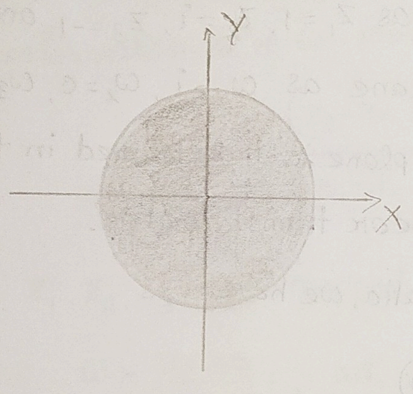
$\Rightarrow |u + iv - 1| < |i(u + iv) + i|$

$\Rightarrow \sqrt{(u-1)^2 + v^2} < \sqrt{(-v)^2 + (u+1)^2}$

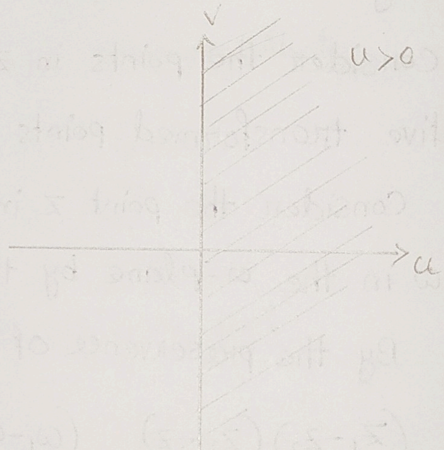
$\Rightarrow \cancel{u^2} + \cancel{1} - 2u + \cancel{v^2} < \cancel{u^2} + \cancel{v^2} + 2u + \cancel{1}$

$\Rightarrow u > 0$

Now,



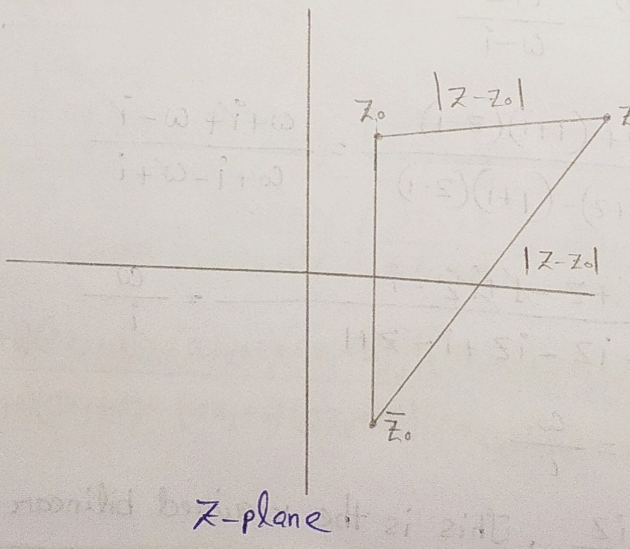
Z-plane



ω-plane

Eg- If z_0 is in the upper half of the z-plane, show that the bilinear transformation $\omega = e^{i\theta}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$. maps the upper half of the z-plane into the interior of the unit circle of ω-plane.

Soln



Z-plane

From the figure, it is clear that

$$|z-z_0| \geq |z-z_0| \quad \text{--- (1)}$$

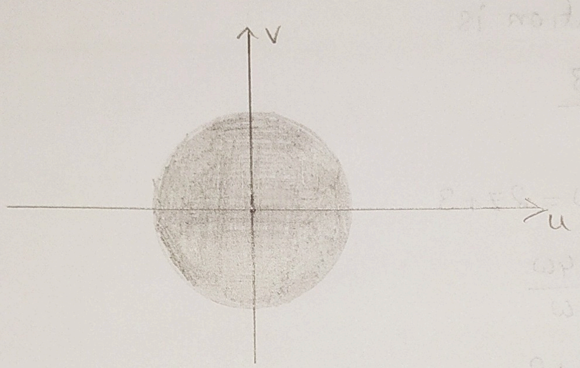
Given bilinear transformation is

$$\omega = e^{i\theta}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$$

$$\Rightarrow |\omega| = \underbrace{|e^{i\theta}|}_{1}\left|\frac{z-z_0}{z-\bar{z}_0}\right|$$

$$\Rightarrow |\omega| = \frac{|z-z_0|}{|z-\bar{z}_0|} \leq 1 \ (\text{From (1)})$$

$$\Rightarrow |\omega| \leq 1$$



ω-plane

Eg - Show that the transformation $\omega = i\frac{1-z}{1+z}$ transforms the circle $|z| = 1$ into the real axis of ω-plane and the interior of the circle $|z| < 1$ into the upper half of the ω-plane

Soln Given transformation,

$$\omega = i\,\frac{1-z}{1+z}$$

$$\Rightarrow \omega + z\omega = i - iz$$

$$\Rightarrow z(\omega + i) = i - \omega$$

$$\Rightarrow z = \frac{i-\omega}{\omega + i} \quad \text{--- (1)}$$

$$\because |z| = 1 \Rightarrow \frac{|i-\omega|}{|i+\omega|} = 1$$

$$\Rightarrow |i-(u+iv)| = |i+(u+iv)|$$

$$\Rightarrow \sqrt{(-u)^2 + (1-v)^2} = \sqrt{(u)^2 + (1+v)^2}$$

$$\Rightarrow u^2+1+v^2-2v = u^2+1+v^2+2v$$

$$\Rightarrow 4v = 0$$

$$\Rightarrow v = 0$$

Again $|z|<1 \Rightarrow \dfrac{|1-w|}{|1+w|} < 1$

$$\Rightarrow |1-(u+iv)| < |1+(u+iv)|$$

$$\Rightarrow v > 0$$

Eg Show that the transformation, $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the straight line $4u+3=0$

Soln Given transformation is

$$w = \frac{2z+3}{z-4}$$

$$\Rightarrow zw - 4w = 2z+3$$

$$\Rightarrow z = \frac{3+4w}{-2+w}$$

$$\text{or } z = \frac{4w+3}{w-2} \quad \text{--- (i)}$$

Given straight line is,

$$x^2+y^2-4x=0$$

$$\Rightarrow z.\bar z - 2(z+\bar z) = 0 \quad \text{--- (ii)}$$

using (i) in (ii), we get

$$\left(\frac{4w+3}{w-2}\right)\left(\overline{\frac{4w+3}{w-2}}\right) - 2\left[\left(\frac{4w+3}{w-2}\right)+\left(\overline{\frac{4w+3}{w-2}}\right)\right] = 0$$

$$\Rightarrow \frac{4w+3}{w-2}\cdot\frac{4\bar w+3}{\bar w-2} - 2\left(\frac{4w+3}{w-2}+\frac{4\bar w+3}{\bar w-2}\right) = 0$$

$$\Rightarrow \frac{16w\bar w+12w+12\bar w+9}{(w-2)(\bar w-2)} - 2\times\frac{4w\bar w-8w+3\bar w-6+4w\bar w+3w-8\bar w-6}{(w-2)(\bar w-2)} = 0$$

$$\Rightarrow 16w\bar w+12w+12\bar w+9-16w\bar w+10w+10\bar w+24=0$$

$$\Rightarrow 22w+22\bar w+33=0$$

$$\Rightarrow 22(w+\bar w)+33=0$$

$$\Rightarrow 22\times 2u+33=0$$

$\Rightarrow 44u + 33 = 0$

$\Rightarrow 4u + 3 = 0$