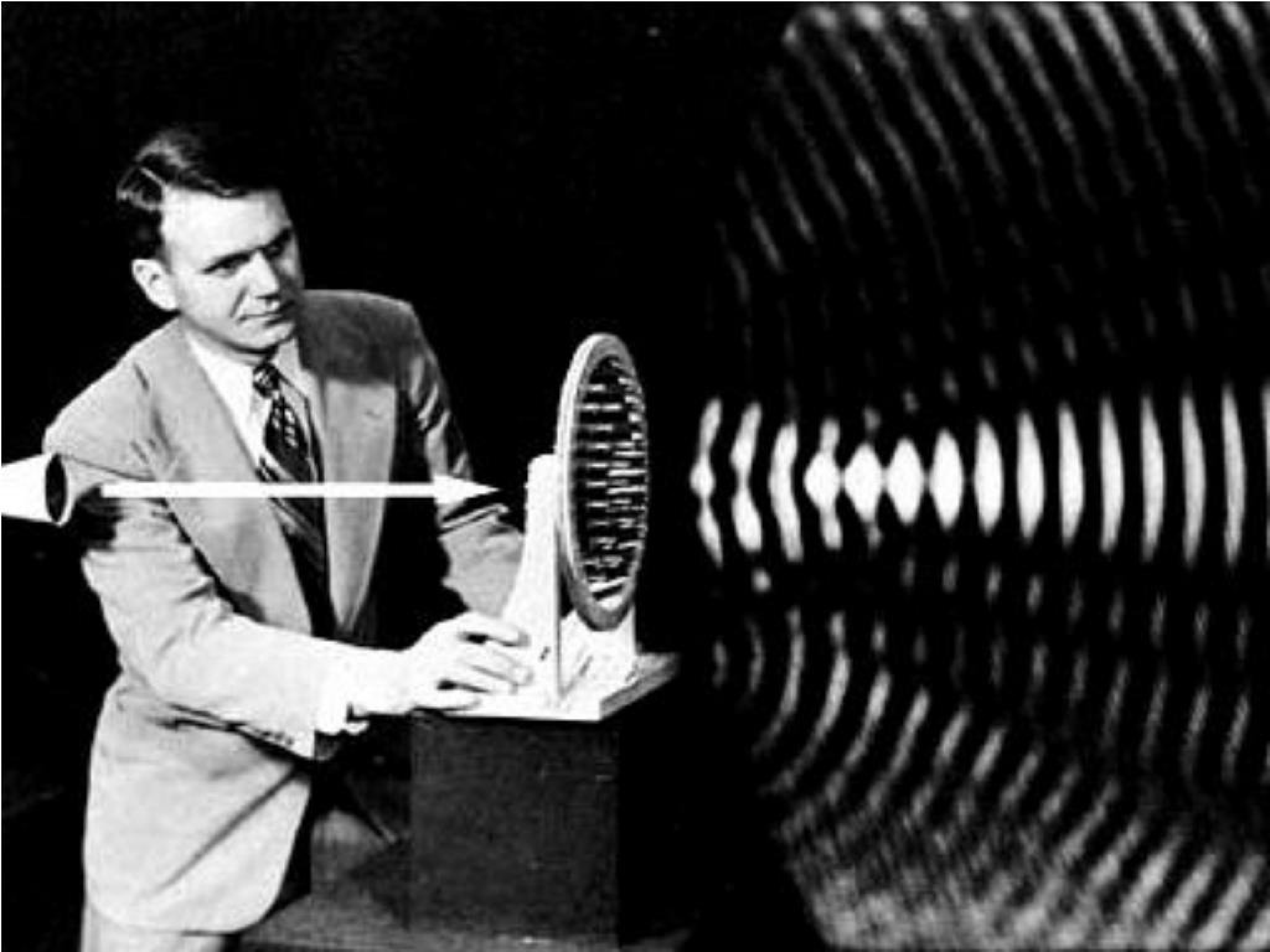
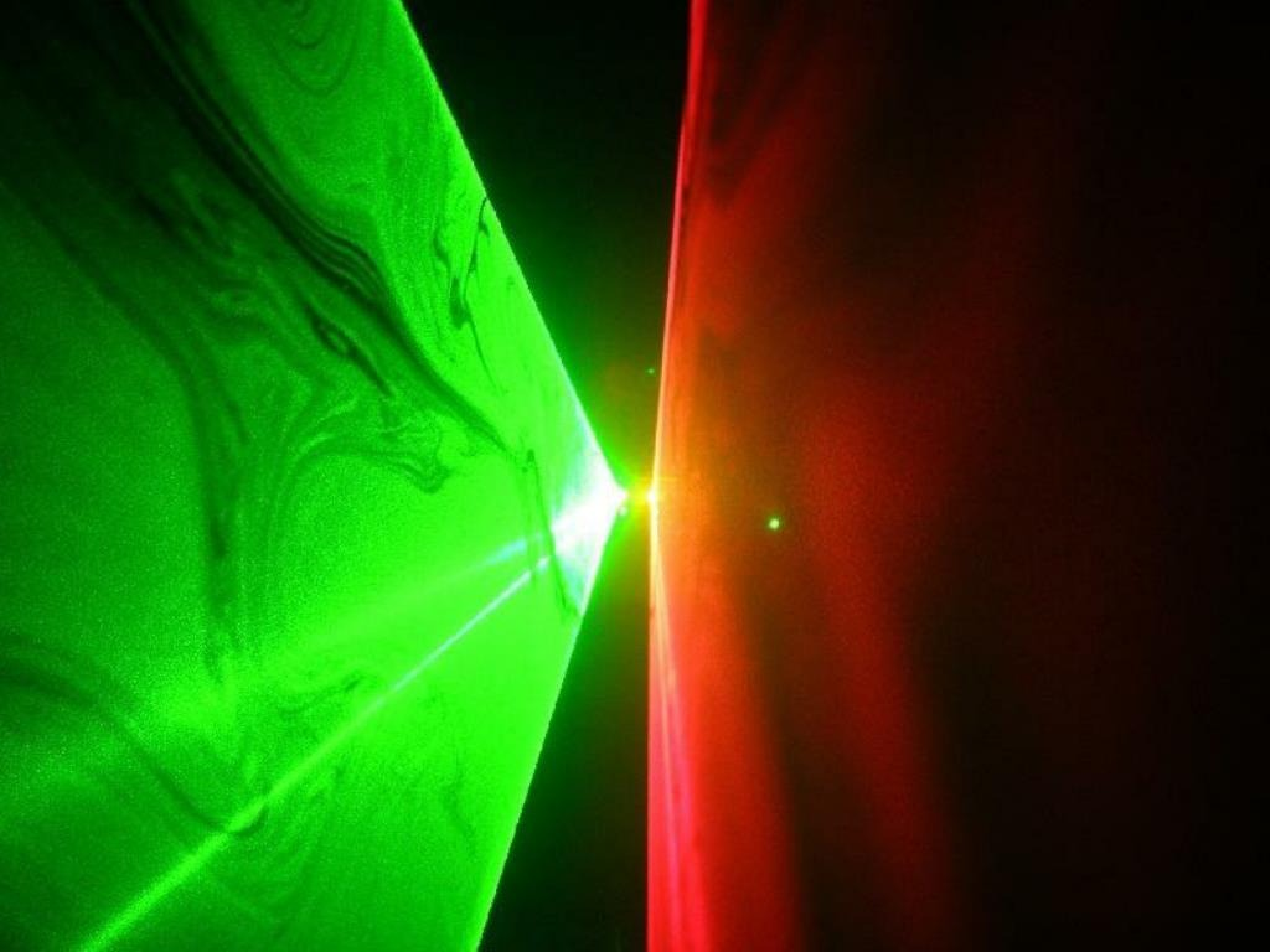
Chapter 4: *Wave equations*

What is a wave?



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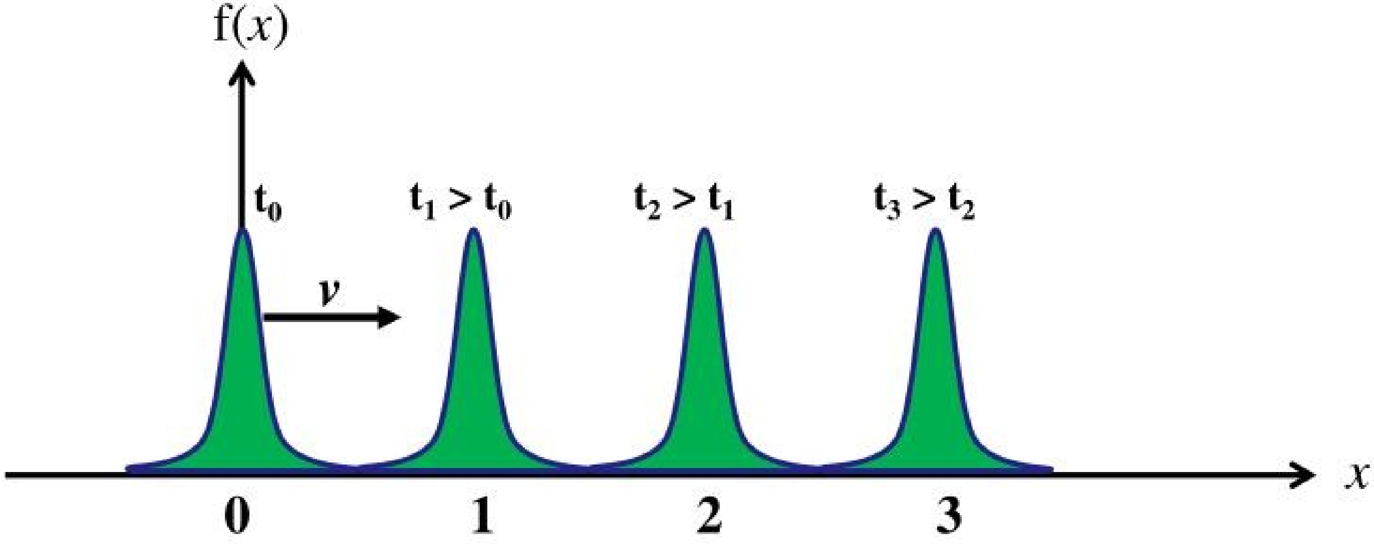
what is a wave?

anything that moves

This idea has guided my research:
for matter, just as much as for radiation, in particular light, we must introduce at one and the same time the corpuscle concept and wave concept.

Louis de Broglie

Waves move



A wave is…

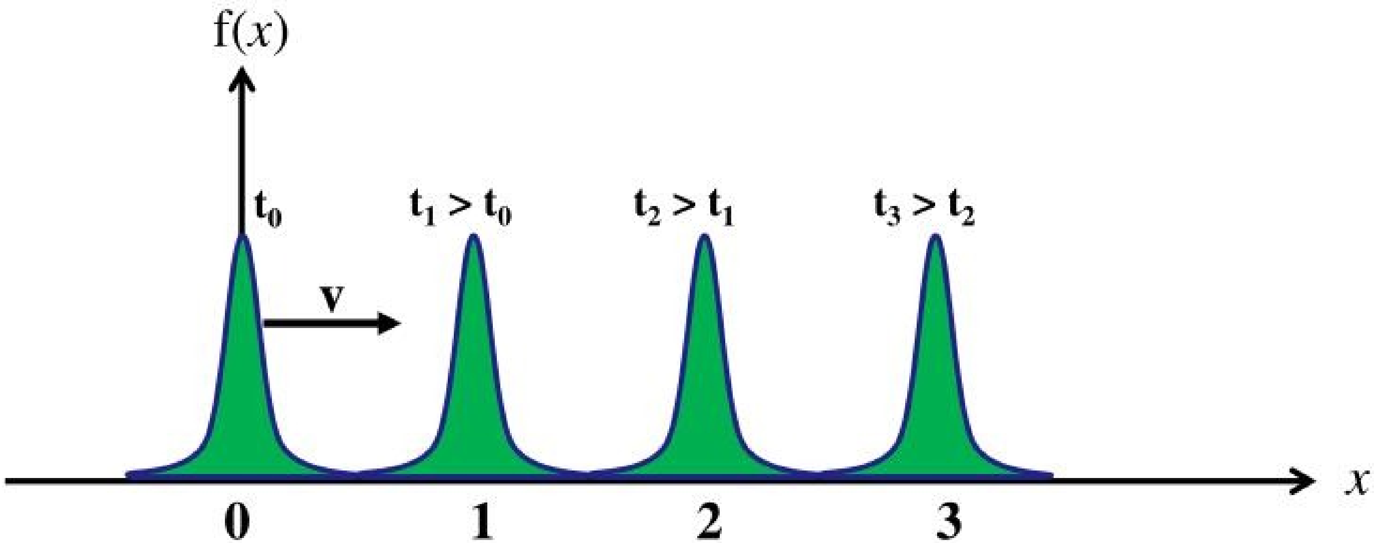
a disturbance in a medium

-propagating in space with velocity v

-transporting energy

-leaving the medium undisturbed

1D traveling wave



To displace $f(x)$ to the right: $x \rightarrow x\text{-}a$, where $a > 0$.

Let $a = vt$, where v is **velocity** and t is **time**

-displacement increases with time, and

-pulse maintains its shape.

So $f(x - vt)$ represents a rightward, or forward, propagating wave.

$f(x+vt)$ represents a leftward, or backward, propagating wave.

The wave equation

Let $y = f(x')$, where $x' = x \pm vt$. So $\frac{\partial x'}{\partial x} = 1$ and $\frac{\partial x'}{\partial t} = \pm v$

Now, use the chain rule: $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial x'}\frac{\partial x'}{\partial x}$ $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x'}\frac{\partial x'}{\partial t}$

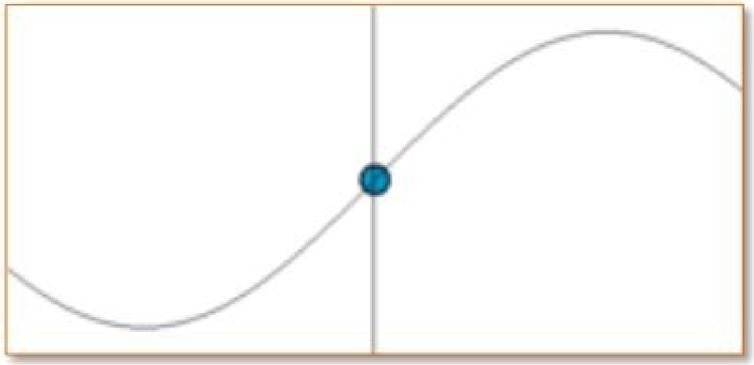
So $\frac{\partial y}{\partial x} = \frac{\partial f}{\partial x'} \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$ and $\frac{\partial y}{\partial t} = \pm v\frac{\partial f}{\partial x'} \Rightarrow \frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x'^2}$

Combine to get the 1D differential wave equation:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

works for anything that moves!

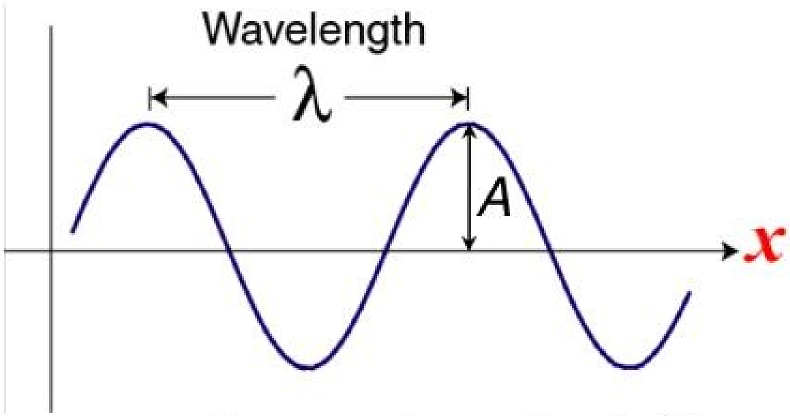
Harmonic waves



- periodic (smooth patterns that repeat endlessly)
- generated by undamped oscillators undergoing harmonic motion
- characterized by sine or cosine functions
  - -for example, $y = f(x - vt) = A\sin[k(x - vt)]$
  - -complete set (linear combination of sine or cosine functions can represent **any** periodic waveform)

Snapshots of harmonic waves

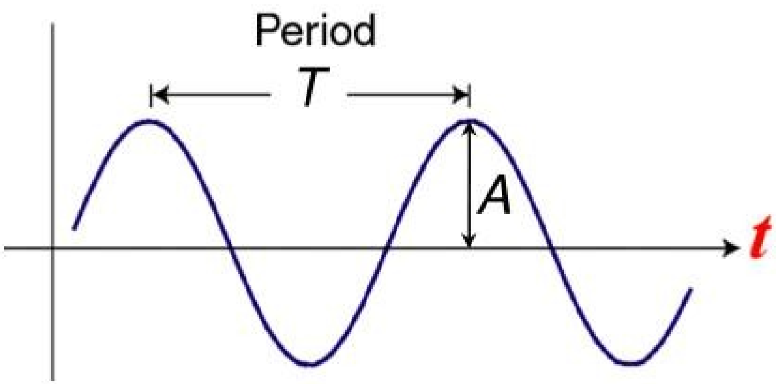
at a fixed time:



propagation constant: $k = 2\pi/\lambda$

wave number: $\kappa = 1/\lambda$

at a fixed point:



frequency: $\nu = 1/T$

angular frequency: $\omega = 2\pi\nu$

Note: $v \neq \nu$

v = **velocity (m/s)**

ν = **frequency (1/s)**

$$\boxed{v = \nu\lambda}$$

The phase of a harmonic wave

The **phase, φ,** is everything inside the sine or cosine (the argument).

$$\boxed{\begin{gathered} y = A\sin[k(x \pm vt)] \\ \varphi = k(x \pm vt) \end{gathered}}$$

For constant phase, $d\varphi = 0 = k(dx \pm v\,dt)$

$$\frac{dx}{dt} = \pm v$$

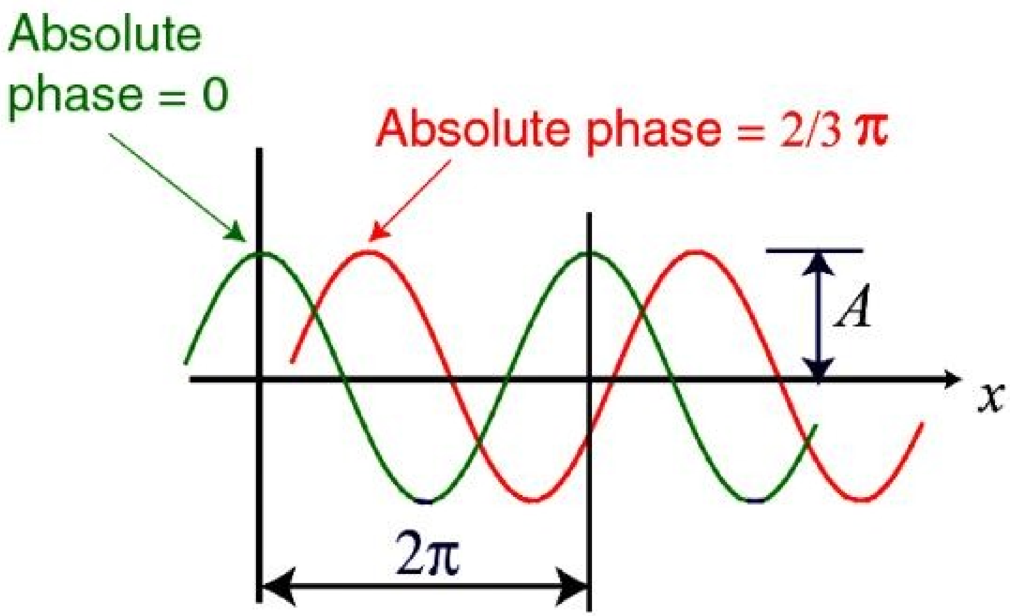
which confirms that v is the wave velocity.

Absolute phase

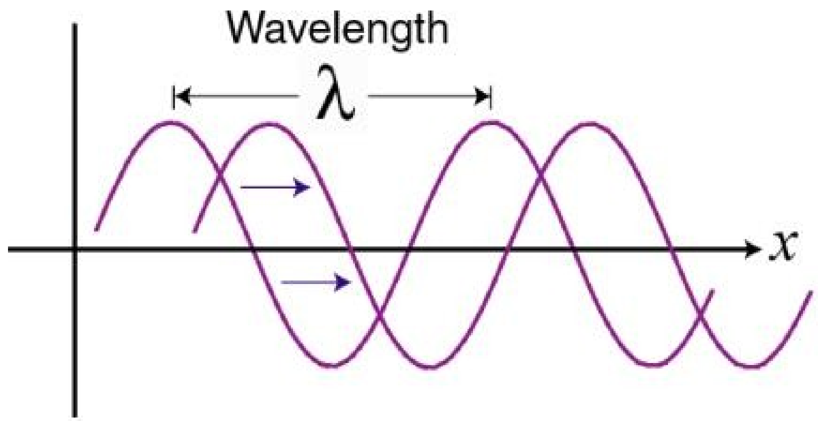
$$y = A \sin[k(x \pm vt) + \varphi_0]$$

A = amplitude

φ_0 = initial phase angle (or absolute phase) at x = 0 and t =0



How fast is the wave traveling?



The **phase velocity** is the wavelength/period: $v = \lambda/T$

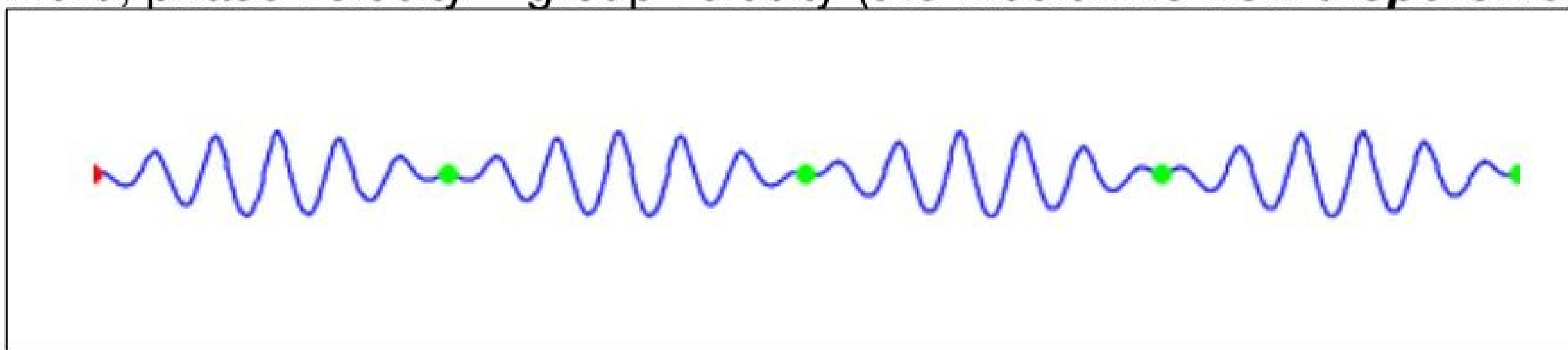
Since $\nu = 1/T$: $\boxed{v = \lambda\nu}$

In terms of k, $k = 2\pi/\lambda$, and
the angular frequency, $\omega = 2\pi/T$, this is: $\boxed{v = \omega/k}$

Phase velocity vs. Group velocity

$$v_{phase} = \frac{\omega}{k} \qquad\qquad v_{group} = \frac{d\omega}{dk}$$

Here, phase velocity = group velocity (the medium is ***non-dispersive***).



In a ***dispersive*** medium, the phase velocity ≠ group velocity.

works for any periodic wave??

Mexican waves in an excitable medium

The stimulation of this concerted motion among expectant spectators is explained.

The Mexican wave, or *La Ola*, which rose to fame during the 1986 World Cup in Mexico, surges through the rows of spectators in a stadium as those in one section leap to their feet with their arms up, and then sit down again as the next section rises to repeat the motion. To interpret and quantify this collective human behaviour, we have used a variant of models that were originally developed to describe excitable media such as cardiac tissue. Modelling the reaction of the crowd to attempts to trigger the wave reveals how this phenomenon is stimulated, and may prove useful in controlling events that involve groups of excited people.

Using video recordings, we analysed 14 waves in football stadia holding over 50,000 people. The wave (Fig. 1) usually rolls in a clockwise direction and typically moves at a speed of about 12 metres (or 20 seats) per second and has a width of about 6–12 m (corresponding to an average width of 15 seats). It is generated by no more than a few dozen people standing up simultaneously, and subsequently expands through the entire crowd as it acquires a stable, near-linear shape. (For details and interactive simulations, see http://angel.elte.hu/wave.)

Figure 1 The Mexican wave, or *La Ola*, sweeping through a crowd of spectators. A few dozen fans leap up with their arms raised and then sit down as people in the next section jump to their feet to repeat and propagate the motion.

Because of the relative simplicity of the

were originally created to describe processes such as forest fires or wave propagation in heart tissue, can be generalized to include human social behaviour.

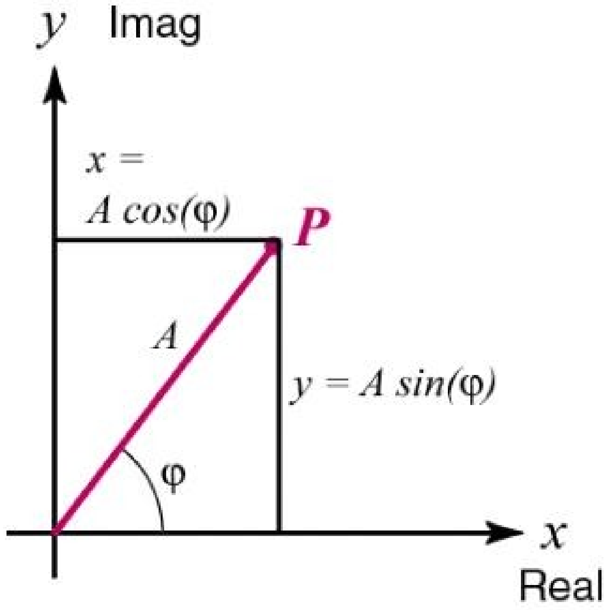
We developed two mathematical simulation models, one minimal and one more

set of internal rules to pass through the active (standing and waving) and refractory (passive) phases before returning to its original resting (excitable) state.

The simpler model distinguishes only three states (excitable, active and passive)

Complex numbers make it less complex!

$P = (x,y)$



x: **real** and y: **imaginary**

$$
\begin{aligned}
P &= x + i\,y \\
&= A\cos(\varphi) + i\,A\sin(\varphi)
\end{aligned}
$$

where $i = \sqrt{-1}$

Euler's formula

"one of the most remarkable formulas in mathematics"

Links the trigonometric functions and the complex exponential function

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$$

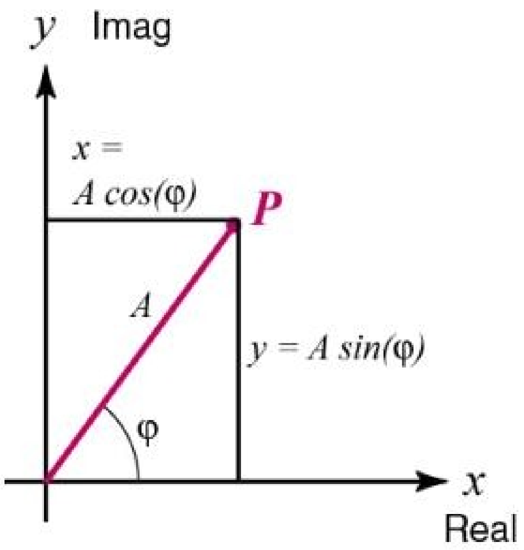
so the point, $P = A\cos(\varphi) + i\,A\sin(\varphi)$, can also be written:

$$P = A\exp(i\varphi) = A\,e^{i\varphi}$$

where

A = Amplitude

φ = Phase



Harmonic waves as complex functions

Using Euler's formula, we can describe the wave as

$$\boxed{\tilde{y} = Ae^{i(kx-\omega t)}}$$

so that $y = \mathrm{Re}(\tilde{y}) = A\cos(kx - \omega t)$

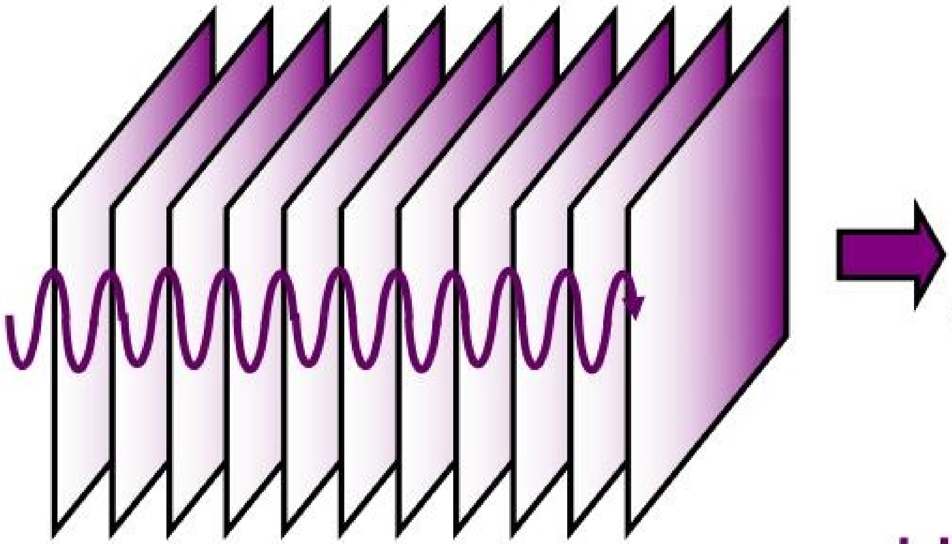
$$y = \mathrm{Im}(\tilde{y}) = A\sin(kx - \omega t)$$

Why?

Math is easier with exponential functions than with trigonometry.

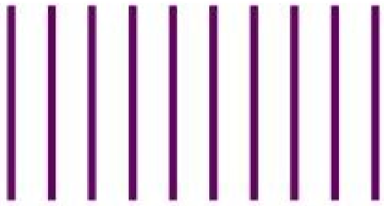
Plane waves

A plane wave's contours of maximum field, called **wavefronts**, are planes. They extend over all space.



The wavefronts of a wave sweep along at the speed of light.

- **equally spaced**
- **separated by one wavelength apart**
- **perpendicular to direction or propagation**

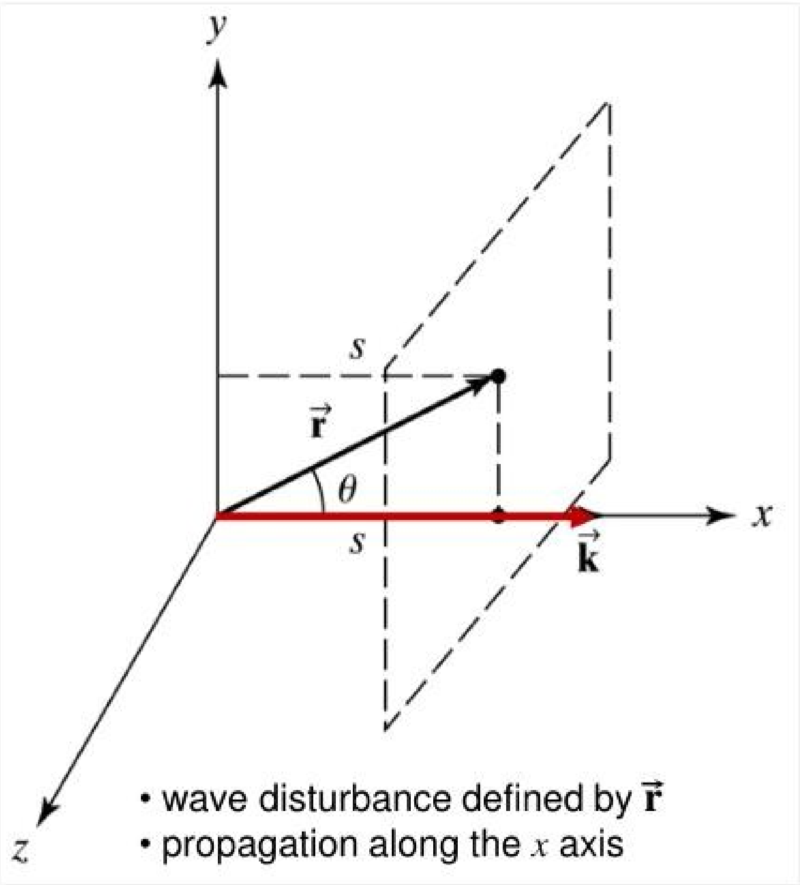


Usually we just draw lines; it's easier.

Wave vector $\vec{\mathbf{k}}$

represents direction of propagation

$$\Psi = A\sin(kx - \omega t)$$



Consider a snapshot in time, say $t = 0$:

$$\Psi = A\sin(kx)$$

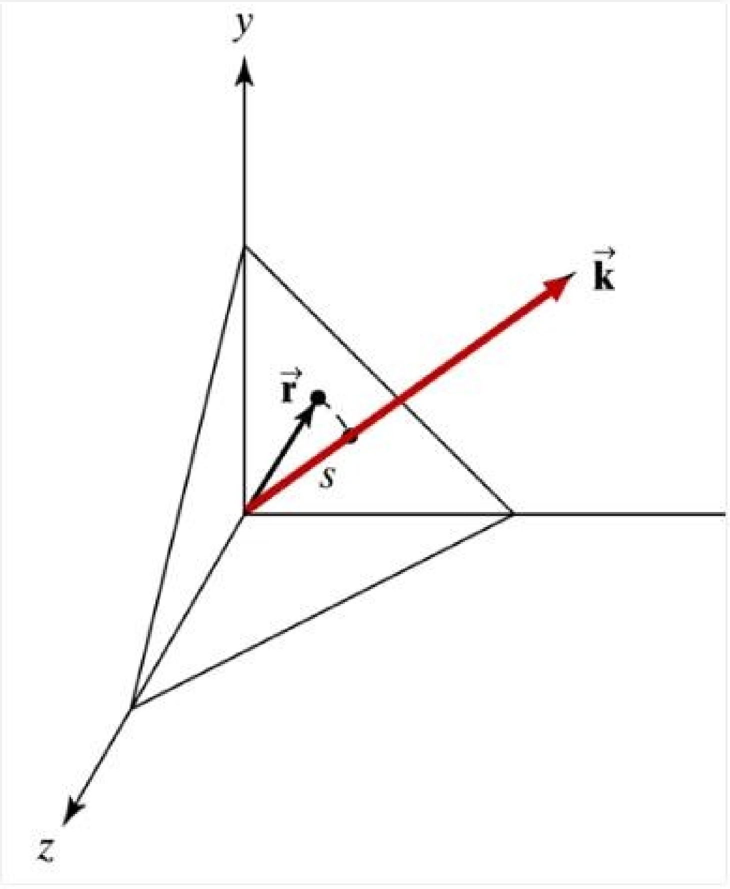
$$\Psi = A\sin(kr\cos\theta)$$

If we turn the propagation constant ($k = 2\pi/\lambda$) into a vector, $kr\cos\theta = \vec{\mathbf{k}}\cdot\vec{\mathbf{r}}$

$$\boxed{\Psi = A\sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)}$$

General case

arbitrary direction



$$\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} = xk_x = yk_y + zk_z$$
$$= kr\cos\theta$$
$$\equiv ks$$

In complex form:

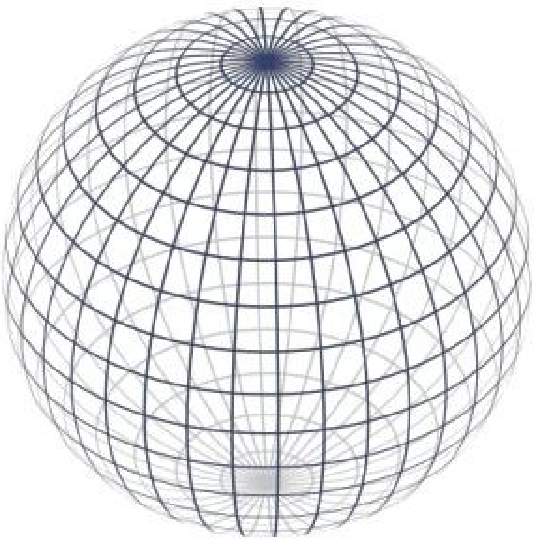
$$\Psi = Ae^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)}$$

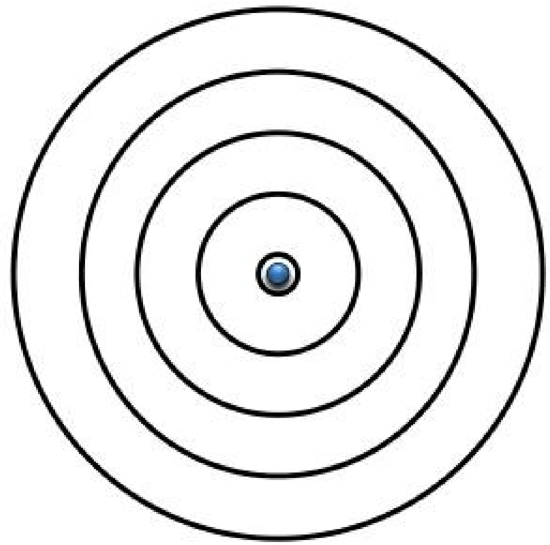
$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

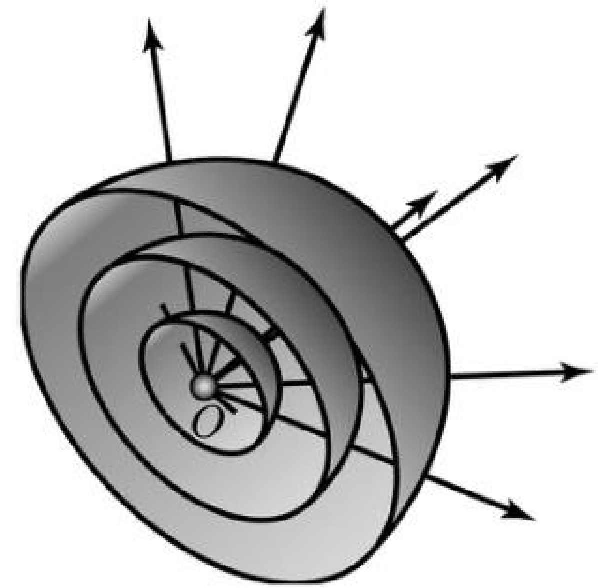
Substituting the Laplacian operator:

$$\boxed{\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}}$$

Spherical waves



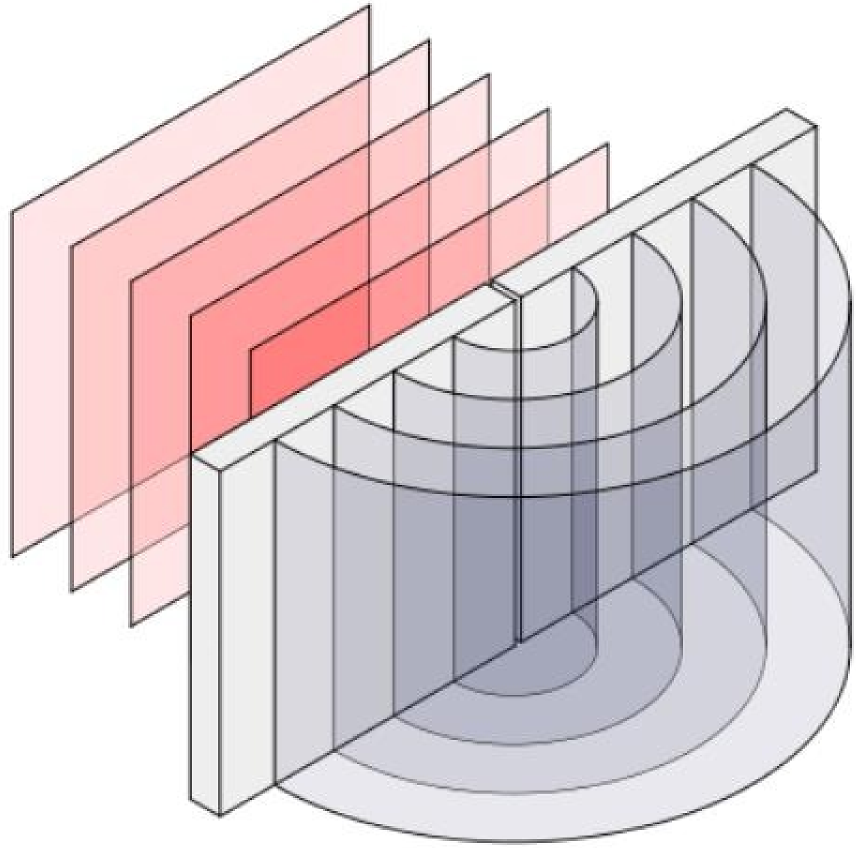




- harmonic waves emanating from a point source
- wavefronts travel at equal rates in all directions

$$\Psi = \left(\frac{A}{r} \right) e^{i\left(kr - \omega t\right)}$$

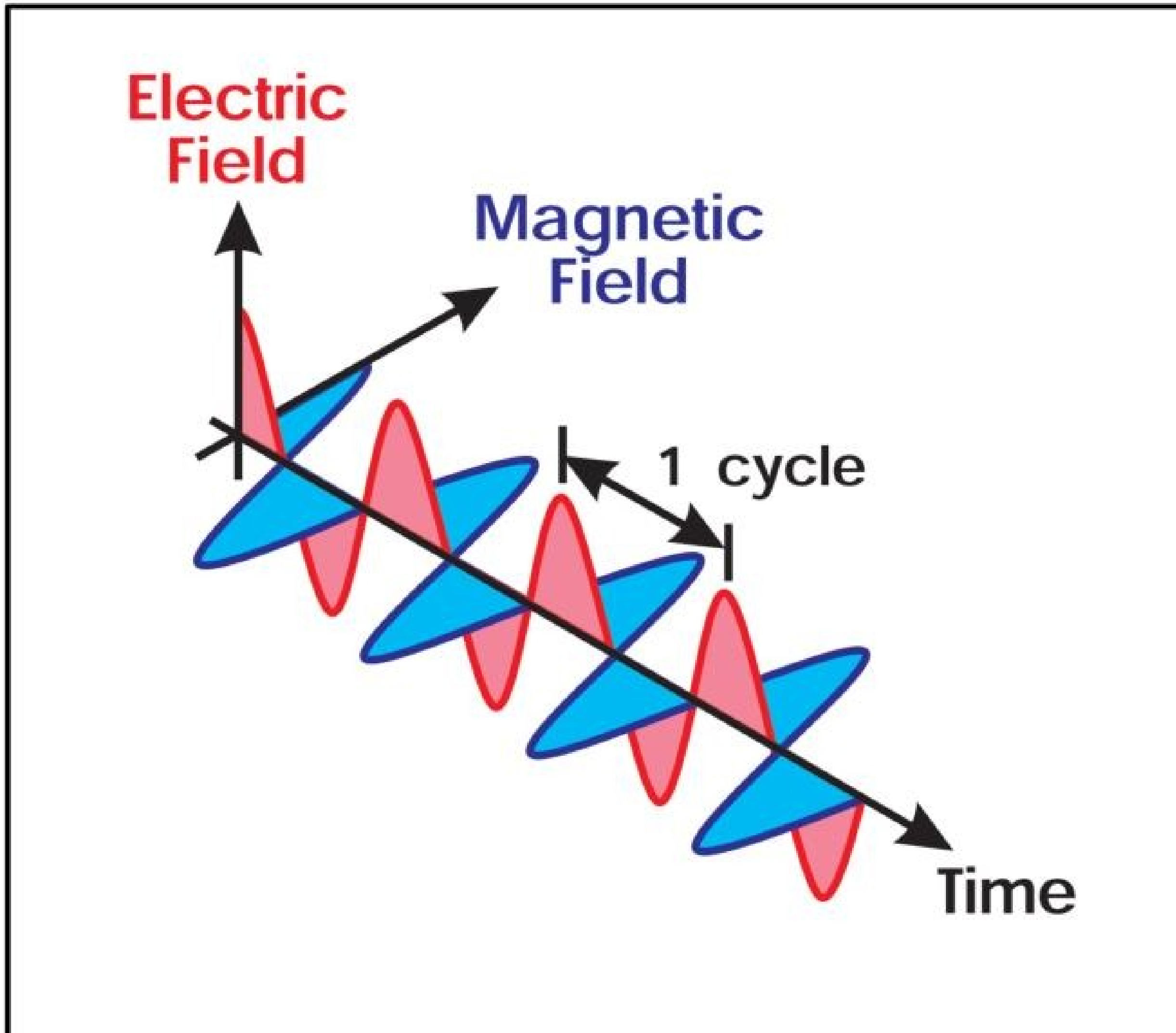
Cylindrical waves



$$\Psi = \left(\frac{A}{\sqrt{\rho}} \right) e^{i(k\rho \pm \omega t)}$$

the wave nature of light

Electromagnetic waves



Derivation of the wave equation from Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ε is the permittivity, and μ is the permeability of the medium.

As written, they assume no charges (free space).

http://www.youtube.com/watch?v=YLlvGh6aEIs

E and B are perpendicular

Perpendicular to the direction of propagation:

I. Gauss' law (in vacuum, no charges):

$$\nabla \cdot \vec{\boldsymbol{E}}(\vec{r},t) = \nabla \cdot \vec{E}_0 e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} = 0$$

$$j\vec{k} \cdot \vec{E}_0 e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} = 0$$

$$\vec{k} \cdot \vec{E}_0 = 0$$

so $\boxed{\vec{\boldsymbol{E}} \perp \vec{k}}$ *Always!*

II. No monopoles:

$$\nabla \cdot \vec{\boldsymbol{B}}(\vec{r},t) = \nabla \cdot \vec{B}_0 e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} = 0$$

$$j\vec{k} \cdot \vec{B}_0 e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} = 0$$

$$\vec{k} \cdot \vec{B}_0 = 0$$

so $\boxed{\vec{\boldsymbol{B}} \perp \vec{k}}$ *Always!*

E and B are perpendicular

Perpendicular to each other:

III. Faraday's law:

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\nabla \times \vec{E}_0 e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\left(jk_y E_{oz} - jk_z E_{oy}\right) e^{j\left(\vec{k}\cdot\vec{r}-\omega t\right)} \hat{i} = -\frac{\partial B_x}{\partial t}\hat{i}$$

$$\frac{k_y E_{oz} - k_z E_{oy}}{\omega} = B_{ox}$$

so $\boxed{\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{E}}}$

Always!

E and B are harmonic

$$\vec{E} = \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t)$$

$$\vec{B} = \vec{B}_0 \sin(\vec{k} \cdot \vec{r} - \omega t)$$

Also, at any specified point in time and space,

$$E = cB$$

where c is the velocity of the propagating wave,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \text{ m/s}$$

EM wave propagation in homogeneous media

The speed of an EM wave in free space is given by: $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = \dfrac{\omega}{k}$

ε_0 = permittivity of free space, μ_0 = magnetic permeability of free space

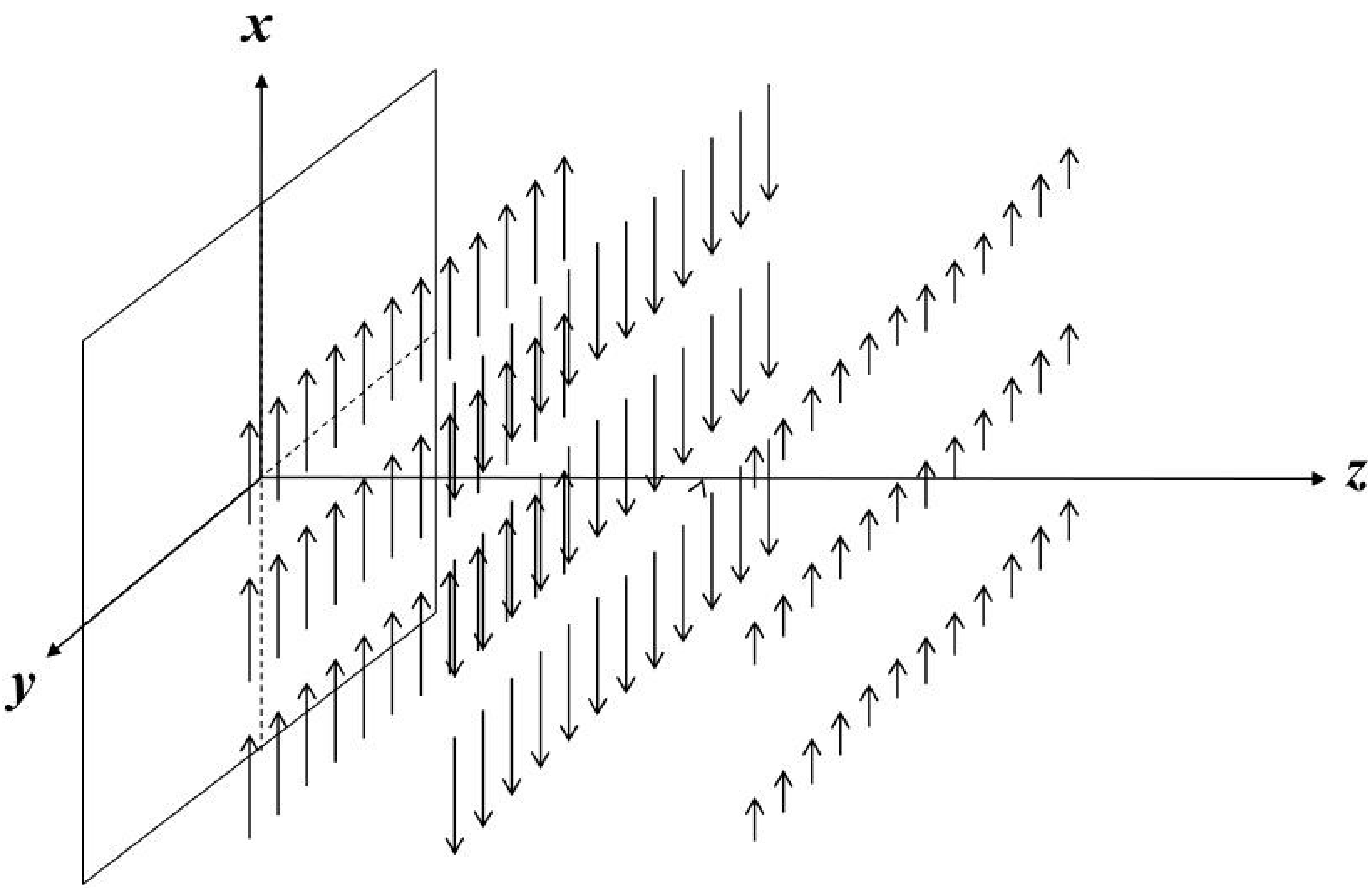
To describe EM wave propagation in other media, two properties of the medium are important, its *electric permittivity* ε and *magnetic permeability* μ. These are also complex parameters.

$\varepsilon = \varepsilon_0(1+ \chi) + i\, \sigma/\omega$ = *complex permittivity*

σ = *electric conductivity*
χ = *electric susceptibility* (to polarization under the influence of an external field)

Note that ε and μ also depend on frequency (ω)

Simple case— plane wave with $\boldsymbol{E}$ field along $\boldsymbol{x}$, moving along z:



General case— along any direction

$$\boxed{\nabla^2 \vec{\boldsymbol{E}} = \frac{1}{c^2}\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2}}$$

which has the solution: $\boxed{\vec{\boldsymbol{E}}(x, y, z, t) = \vec{\boldsymbol{E}}_0 \exp[i(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t)]}$

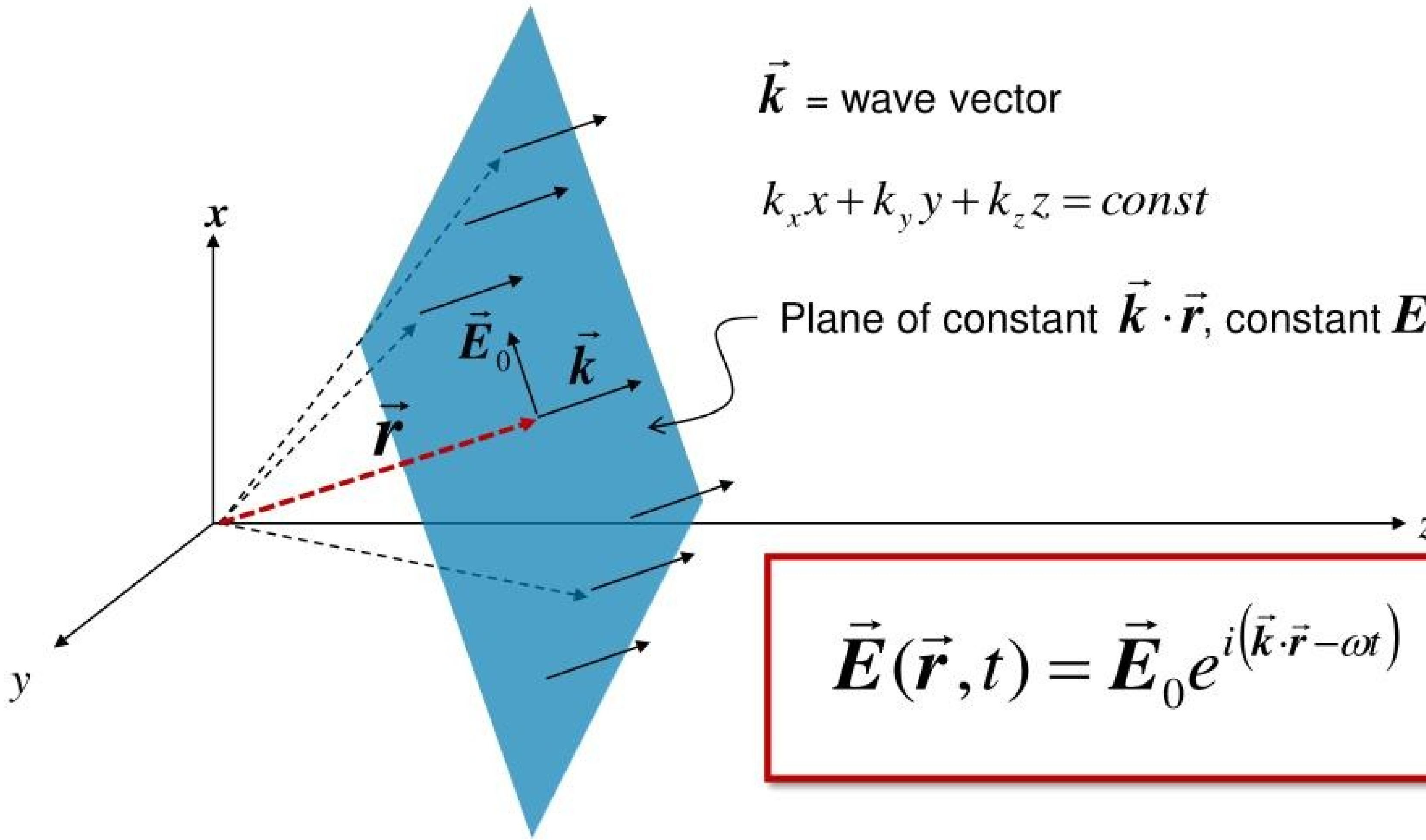
where $\vec{k} \equiv (k_x, k_y, k_z)$ $\quad \vec{r} \equiv (x, y, z)$

and $\vec{k} \cdot \vec{r} \equiv k_x\, x + k_y\, y + k_z\, z$

and $\vec{\boldsymbol{E}}_0 = (\boldsymbol{E}_{0x}, \boldsymbol{E}_{0y}, \boldsymbol{E}_{0z})$

Arbitrary direction

$$\vec{E}(x,y,z,t) = \vec{E}_0 \sin\left(\vec{k}\cdot\vec{r} - \omega t\right)$$



$\vec{k}$ = wave vector

$$k_x x + k_y y + k_z z = const$$

Plane of constant $\vec{k}\cdot\vec{r}$, constant $\boldsymbol{E}$

$$\boxed{\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}}$$

Electromagnetic waves transmit energy

Energy density (J/m^3) in an electrostatic field:

$$u_E = \tfrac{1}{2}\varepsilon_0 E^2$$

Energy density (J/m^3) in an magnetostatic field:

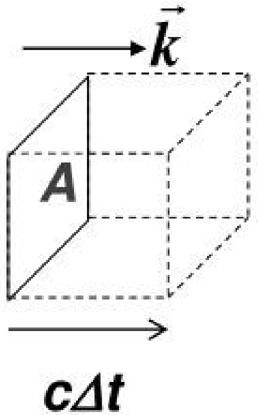
$$u_B = \frac{1}{2\mu_0} B^2$$

Energy density (J/m^3) in an electromagnetic wave is equally divided:

$$u_{total} = u_E + u_B = \varepsilon_0 E^2 = \tfrac{1}{\mu_0} B^2$$

Rate of energy transport

Rate of energy transport: Power (W)



$$P = \frac{energy}{\Delta t} = \frac{u\Delta V}{\Delta t} = \frac{uAc\Delta t}{\Delta t}$$

$$P = ucA$$

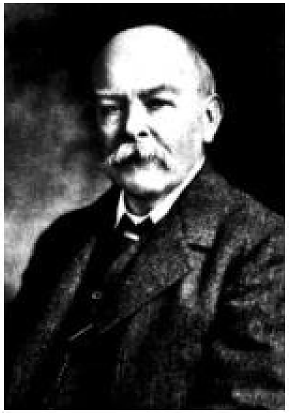
Power per unit area (W/m²):

$$S = uc$$

$$S = \varepsilon_0 c^2 EB$$

$$\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$$

Poynting



Poi~~nt~~ing vector

Poynting vector oscillates rapidly

$$E = E_0 \cos(\omega t) \qquad B = B_0 \cos(\omega t)$$

$$S = \varepsilon_0 c^2 E_0 B_0 \cos^2(\omega t)$$

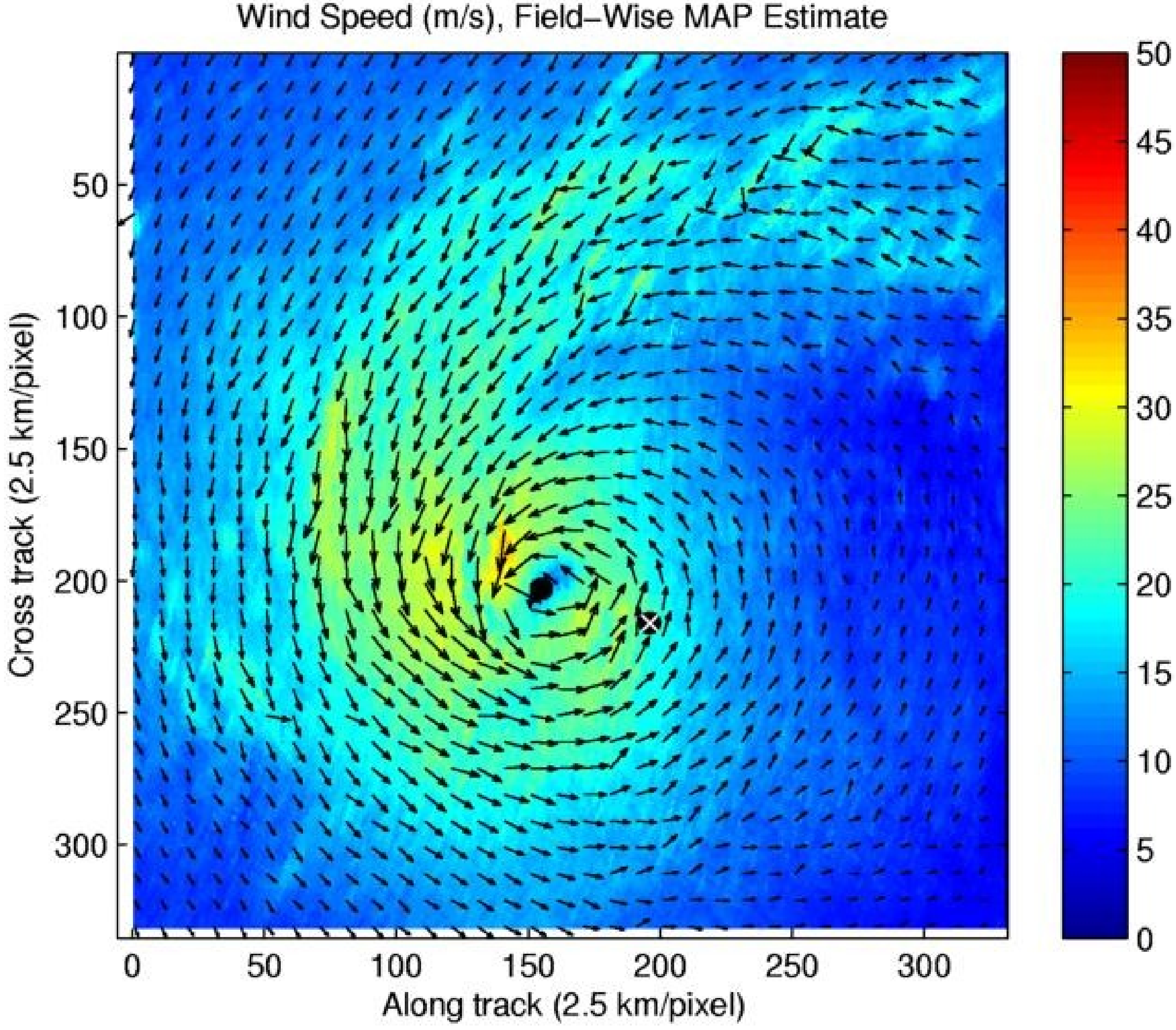
Take the time average:

$$\langle S \rangle = \varepsilon_0 c^2 E_0 B_0 \overline{\cos^2(\omega t)}$$

$$\langle S \rangle = \tfrac{1}{2} \varepsilon_0 c^2 E_0 B_0$$

$$\langle S \rangle = E_e$$ **"Irradiance" (W/m^2)**

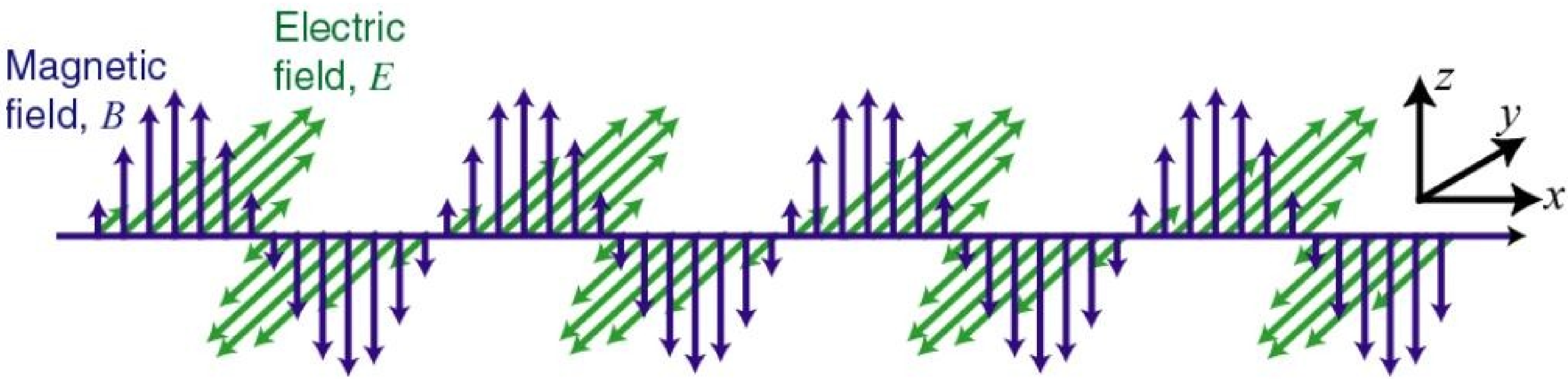
Light is a vector field



A 2D vector field assigns a 2D vector (i.e., an arrow of unit length having a unique direction) to each point in the 2D map.

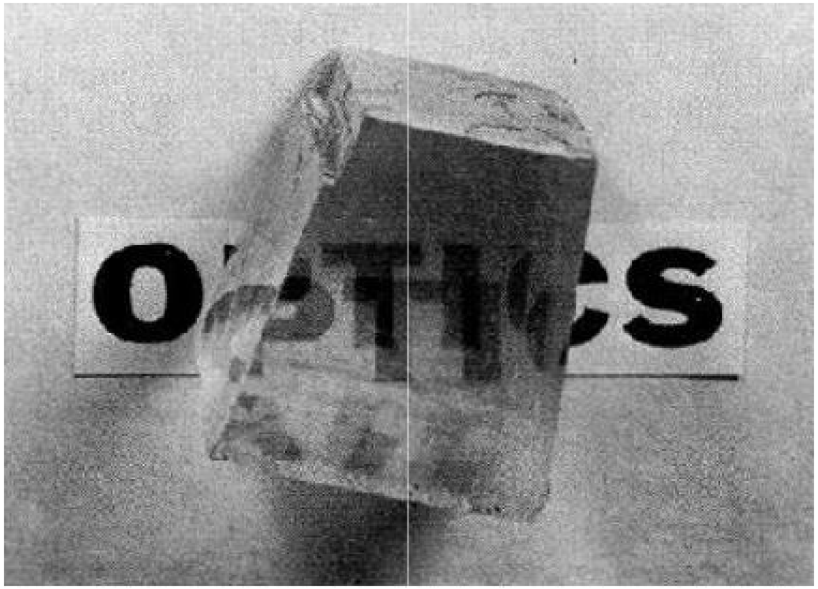
Light is a 3D vector field

A light wave has both electric and magnetic 3D vector fields:



A 3D vector field assigns a 3D vector (i.e., an arrow having both direction and length) to each point in 3D space.

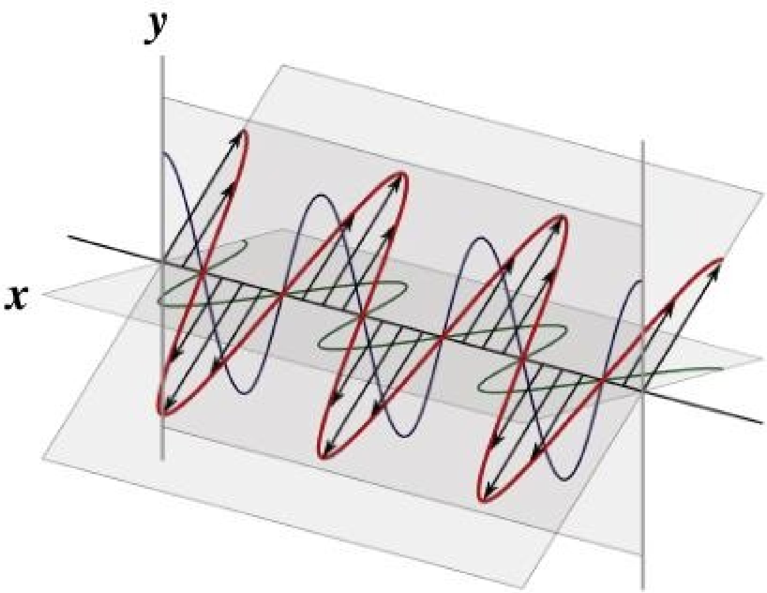
Polarization

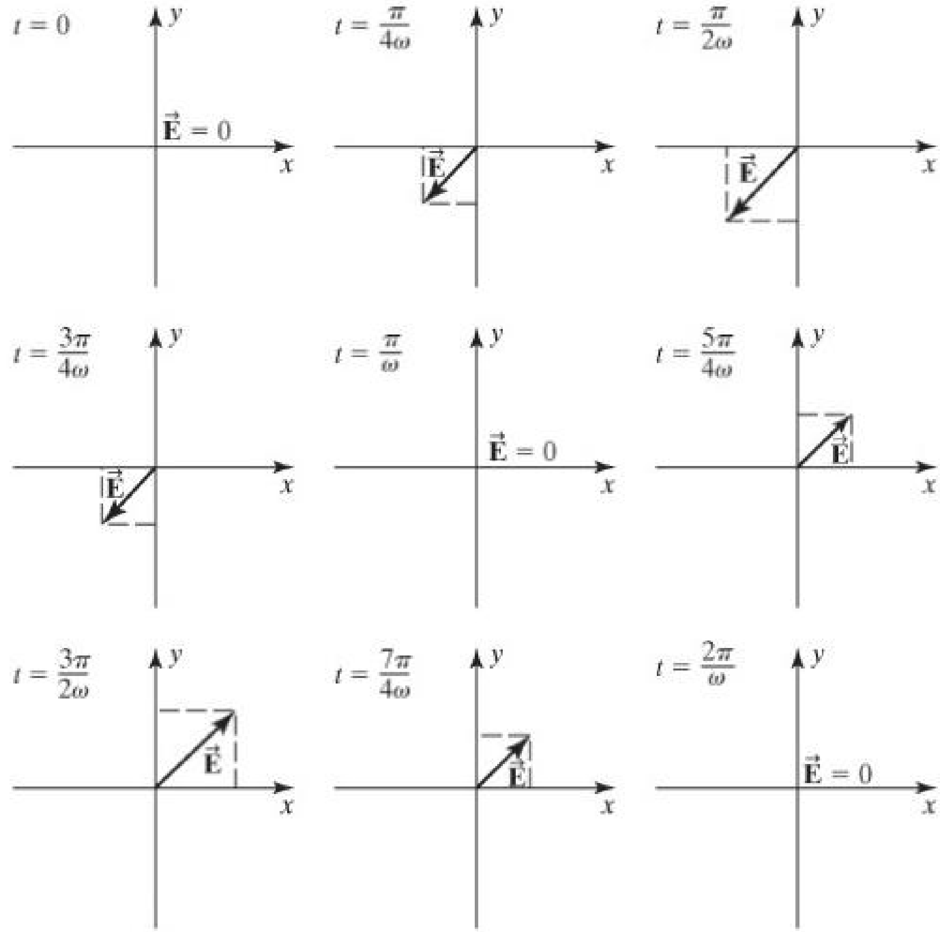


- corresponds to direction of the electric field
- determines of force exerted by EM wave on charged particles (Lorentz force)
- linear,circular, eliptical

Evolution of electric field vector

***linear* polarization**

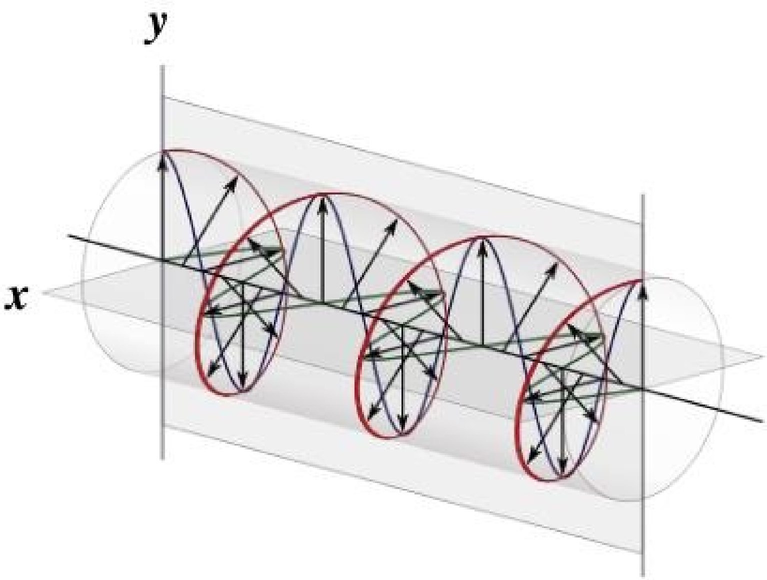


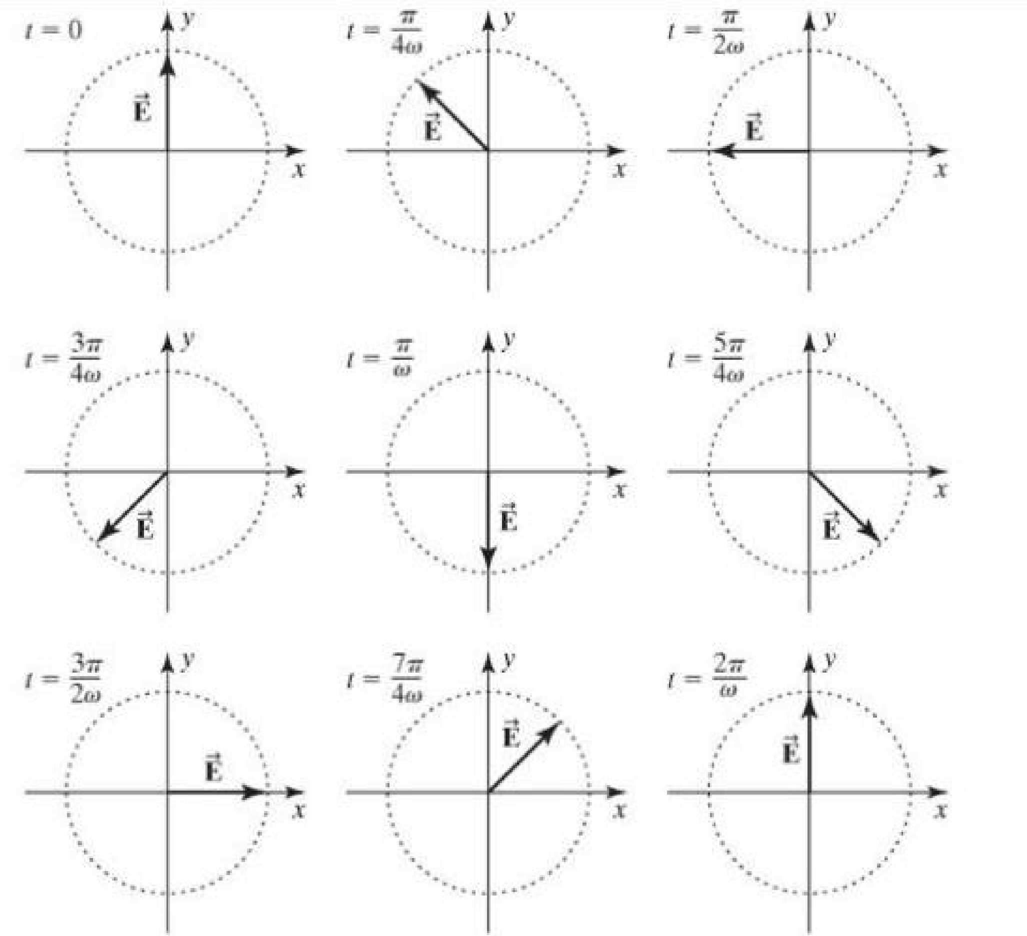


$$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$$

Evolution of electric field vector

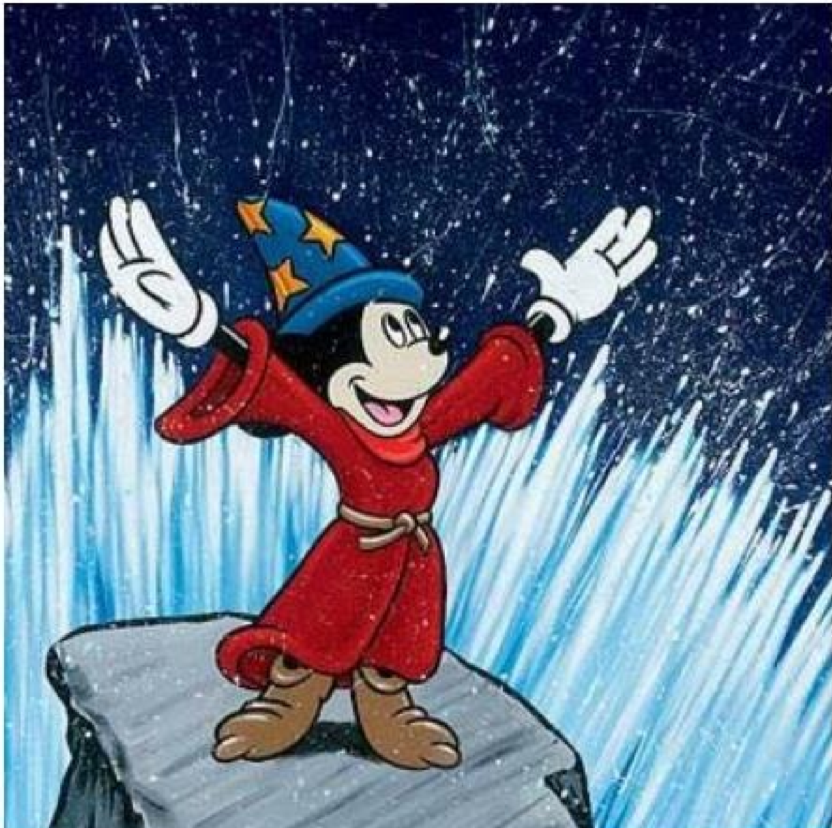
circular polarization





$$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \cos(kz - \omega t)\hat{\boldsymbol{y}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t + \pi/2)\hat{\boldsymbol{y}}$$

Exercises



You are encouraged to solve all problems in the textbook (Pedrotti[3]).

The following may be covered in the werkcollege on 14 September 2011:

Chapter 4:
3, 5, 7, 13, 14, 17, 18, 24