Design of Digital filters

→ Discrete time systems.

→ z - transform

Q. why digital filters? why not analog filters?

→ Difference eq^n

→ frequency related operations (filtering)

→ Design ~~specifi~~ flexibility

Disadvantage

→ Speed limit (cause of ADC / DAC)
(conversion time   settling time)

→ Finite wordlength.

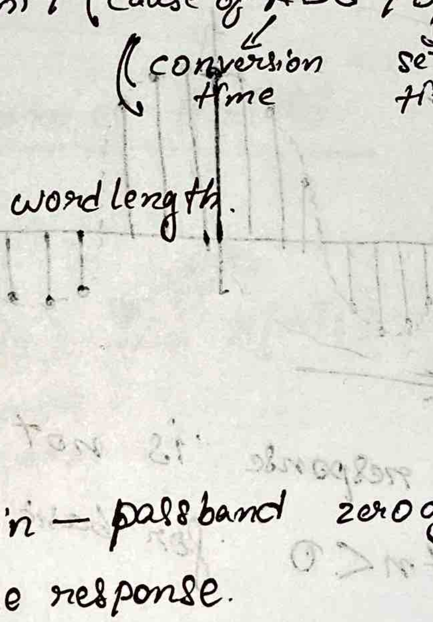
In Signal processing

Ideal filter

→ constant gain – passband   zero gain – stop band

→ Linear phase response.

Magnitude response.



$$|H(\omega)| = \begin{cases} 1 & \forall\ \omega_c < \omega < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$$h(n) = IFT\,[H(\omega)]$$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n}\,d\omega$$

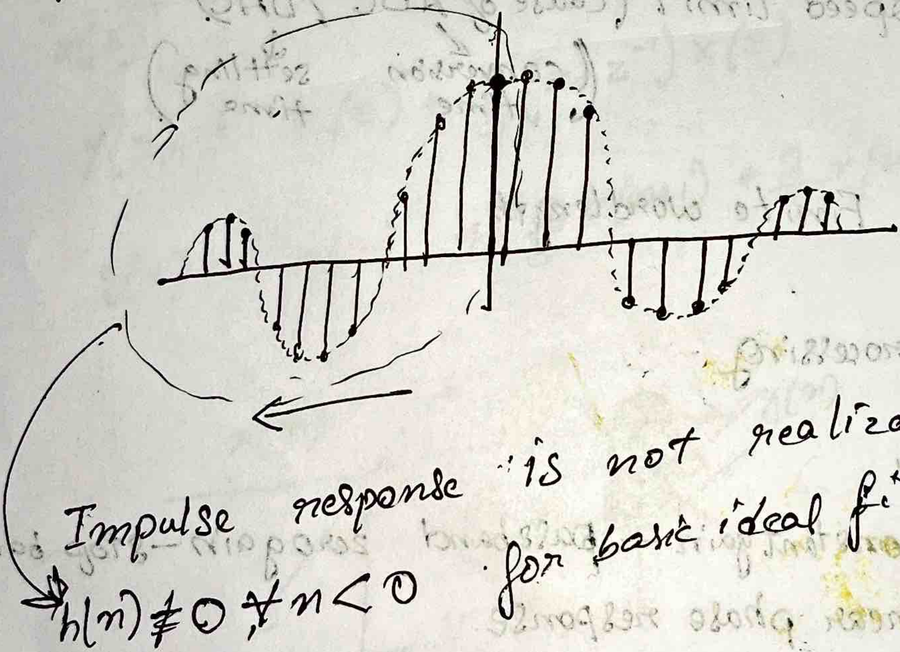
$n = 0 \qquad h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot d\omega = \frac{1}{2\pi}\left[2\omega_c\right] = \frac{\omega_c}{\pi}$

$n \neq 0 \qquad h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$= \frac{\sin \omega_c n}{\pi n} \rightarrow \text{sinc pulse.}$$

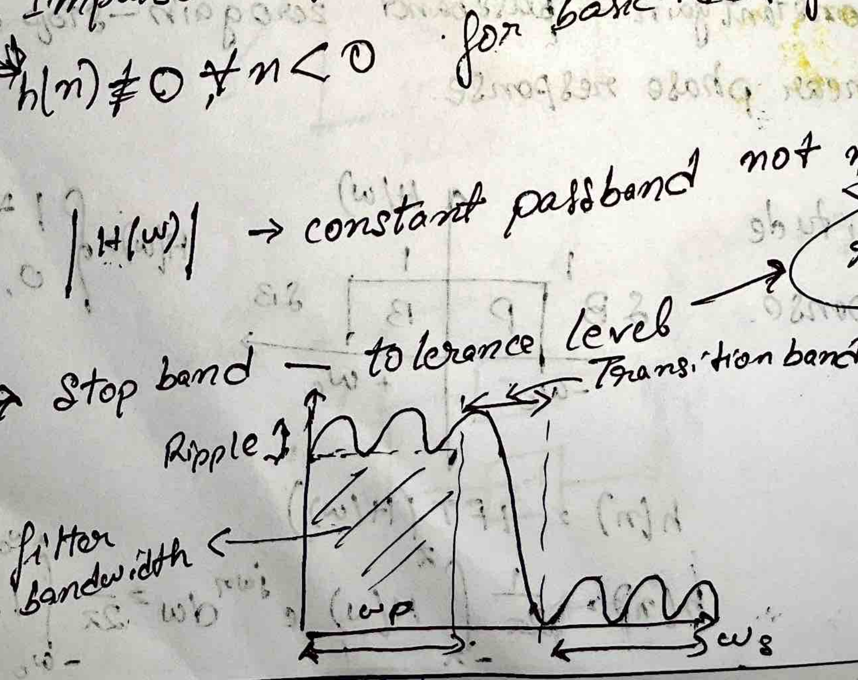
For causal filter $h(n) = 0 \ \forall\, n < 0$

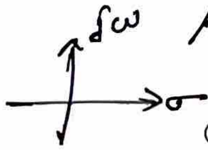


Impulse response is not realizable as $h(n) \neq 0 \ \forall\, n < 0$ for basic ideal filter.

$\rightarrow \ |H(\omega)| \rightarrow$ constant passband not necessary. ripples

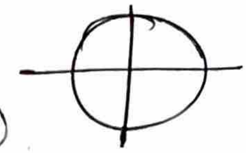
$\rightarrow$ Stop band — tolerance level





Analog filter $\longrightarrow$ Digital filter

s-plane   z-plane



unit circle

$$\boxed{H_a(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}} \Rightarrow \text{Analog system}$$

$$H_a(s) = \int h(t) e^{-st} dt$$

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt} \quad \Big| \text{ stability}$$

Impulse Invariance Method.

Analog filter Transfer function $H_a(s)$ $\xrightarrow{ILT}$ $h(t)$ impulse response $\xrightarrow{\text{sampling } t = nT_s}$ $h(nT_s) \longrightarrow H(z)$ Digital filter Transfer f^n

Analog frequency (Ω)

(1) $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - P_k}$

$h(t) = ILT[H_a(s)]$

(2) $h(t) = \sum_{k=1}^{N} A_k e^{P_k t}$

(3) $h(nT_s) = \sum_{k=1}^{N} A_k e^{P_k n T_s}$

Z-transform.

(4) $H(z) = \sum_{n=0}^{\infty} h(nT_s) z^{-n}$

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} A_k e^{p_k n T_s} z^{-n}$$

$$= \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left(e^{p_k T_s} \cdot z^{-1}\right)^n.$$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{p_k T_s} \cdot z^{-1}}}$$

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T_s} z^{-1}}$$

*

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T_s} z^{-1}}$$

$$\frac{s + a}{(s+a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT_s} (\cos bT_s) z^{-1}}{1 - 2e^{-aT_s} (\cos bT_s) z^{-1} + e^{-2aT_s} z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \longrightarrow \frac{e^{-aT_s} (\sin bT_s) z^{-1}}{1 - 2e^{-aT_s} (\cos bT_s) z^{-1} + e^{-2aT_s} z^{-2}}$$

Q. $H_a(s) = \frac{2}{(s+1)(s+2)}$, $f_s = 5 Hz$.

→ Sir Pdf (Sol^n).

Bilinear Transformation Method.

Problems in impulse invariance method.

→ Aliasing effect present

→ many to one mapping

→ Not suitable for high pass and band reject filter.

→ Only poles of the system can be mapped.

$$f_s = \frac{1}{T_s}$$

$(n-1)\,T_s$

nT_s

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

$H(s) = \frac{1}{s}$

$$\frac{Y(s)}{X(s)} = \frac{1}{s}$$

$$X(s) = sY(s)$$

$$y(t) = \int x(t)$$

$$\int_{(n-1)T_s}^{nT_s} \frac{d}{dt}\, y(t) = \int_{(n-1)T_s}^{nT_s} x(t)\,dt$$

$$\Rightarrow y(t)\Big|_{(n-1)T_s}^{nT_s} = \int_{(n-1)T_s}^{nT_s} x(t)\,dt$$

By trapezoidal rule

T_s → sampling period → very small

Trapezoidal rule → Area of integral can be approx by mean height of $x(t)$ b/w 2 limits & later multiplying by width.

$$\Rightarrow y(nT_s) - y[(n-1)T_s] = T_s\left[\frac{x(nT_s) + x(n-1)T_s}{2}\right]$$

$y(nT_s) \approx y(n)$ [∵ T_s is very small]

$y(n-1)T_s \approx y(n-1)$

$$\Rightarrow y(n) - y(n-1) = \frac{T_s}{2}\left[x(n) + x(n-1)\right]$$

Taking z-transform.

$$Y(z) - z^{-1}Y(z) = \frac{T_s}{2}\left[X(z) + z^{-1}X(z)\right]$$

$$\Rightarrow Y(z)\left[1 - z^{-1}\right] = \frac{T_s}{2} X(z)\left[1 + z^{-1}\right]$$

$$\boxed{X(z) = \frac{2}{T_s}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right] Y(z)}$$ Digital.

$$\boxed{X(s) = s\, Y(s)}$$ Analog.

$$s \longrightarrow \frac{2}{T_s}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

$\Omega \rightarrow$ Analog f^n

$\omega \rightarrow$ Digital "

$$s = \sigma + j\Omega = \frac{2}{T_s}\left[\frac{z - 1}{z + 1}\right]$$

$$\Rightarrow \sigma + j\Omega = \frac{2}{T_s}\left[\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right]$$

To solve

(1) $\dfrac{re^{-j\omega} + 1}{re^{-j\omega} + 1}$

(2) $e^{j\omega} = \cos\omega + j\sin\omega.$

$$\boxed{\sigma = \frac{2}{T_s}\left[\frac{r^2 - 1}{r^2 + 2r\cos\omega + 1}\right]}$$

$$\boxed{\Omega = \frac{2}{T_s}\left[\frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1}\right]}$$

when $r = 1$

$$\Omega = \frac{2}{T_s}\left[\frac{2\sin\omega}{1+2\cos\omega+1}\right]$$

$$= \frac{2}{T_s}\left[\frac{\sin\omega}{1+\cos\omega}\right]$$

$$= \frac{2}{T_s}\left[\frac{2\sin\omega/2\cos\omega/2}{\cancel{1}+2\cos^2\omega/2 - \cancel{1}}\right]$$

$$* \boxed{\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)}$$

$$* \boxed{\Omega = \frac{2}{T_s}\tan\left(\frac{\Omega T_s}{2}\right)}$$

$$\boxed{\Omega \to \frac{\omega}{T_s}}$$

$$\frac{\Omega T_s}{2} = \tan\frac{\omega}{2}.$$

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T_s}{2}\right)}$$

Q. An analog filter has following transfer f^n

$$\boxed{H(s) = \frac{1}{s+1}}$$

Def^n the transfer function of digital filter $H(z)$ using BLT. Also write the difference eq^n of digital filter.

$T_s = 1$ sec

$$\rightarrow \quad s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right].$$

$$= 2\left(\frac{z-1}{z+1}\right)$$

$$H(z) = \frac{1}{2\left(\frac{z-1}{z+1}\right)+1}$$

$$= \frac{z+1}{3z-1}$$

$$\frac{Y(z)}{X(z)} = \frac{z+1}{3z-1} = \frac{1+z^{-1}}{3-z^{-1}}$$

$$3Y(z) - Y(z)z^{-1} = X(z) + z^{-1}X(z)$$

IZT $\quad 3y(n) - y(n-1) = x(n) + x(n-1)$

$$y(n) = \frac{1}{3}y(n-1) + \frac{1}{3}x(n-1) + \frac{1}{3}x(n)$$

Q. $\quad H(s) = \dfrac{3}{(s+2)(s+3)} \qquad T_s = 0.1\,sec$

Find $H(z)$ using BLT.

Q. Design a single pole low pass filter with a 3-dB bandwidth of 0.2π rad. The transfer f^n of analog filter is of the form

$H(s) = \dfrac{\Omega_c}{s+\Omega_c}$ where Ω_c is 3dB bandwidth of analog filter.

→ $\omega_c = 0.2\pi$

$$\Omega_c = \frac{2}{T_s}\tan\left(\frac{\omega_c}{2}\right)$$

$$\Omega_c = \frac{2}{T_s}\tan(0.1\pi).$$

$$\Omega_c = \frac{0.65}{T_s}$$

$$H(s) = \frac{0.65/T_s}{s + 0.65/T_s}$$

$$s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$$

$$H(z) = \frac{0.65/T_s}{\frac{2}{T_s}\left[\frac{z-1}{z+1}\right] + 0.65/T_s}$$

$$= \frac{0.65}{\frac{2z-2}{z+1} + 0.65}$$

$$= \frac{0.245(z+1)}{z - 0.509}$$

Q. The transfer f^n of analog filter is $H(s) = \frac{s+0.1}{(s+0.1)^2+16}$ BLT obtain the transfer f^n of digital filter using which is resonant at $\omega_r = \pi/2$. ← Digital f^n.

$$\rightarrow \quad H(s) = \frac{s+0.1}{(s+0.1)^2 + 4^2}$$

$$(s+0.1)^2 + 4^2 = 0$$

$$(s+0.1)^2 = -4^2$$

$$s = -0.1 \pm 4j$$

$$\Omega = 4 \text{rad}$$

$$\Omega = \frac{2}{T_s} \tan\left(\frac{\omega}{2}\right)$$

$$= \frac{2}{T_s} \tan\left(\pi/4\right) \Rightarrow T_s = \frac{2}{4} = 0.5 \text{rad/sec}$$

$$s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right] \Rightarrow H(s).$$

Q. The analog transfer f^n of low pass filter is $H(s) = \frac{1}{s+2}$ and its bandwith is 1 rad/sec. Design a digital filter using BLT method whose cut-off freq is 20π and sampling time is 0.0167 sec by considering wrapping effect.

$\underline{\omega_c = 1}$

$\Omega \rightarrow \Omega^*$

$$\Omega^* = \boxed{\frac{2}{T_s}} \tan\left(\frac{\omega T_s}{2}\right)$$

$H(s) \rightarrow H(s)^*$

$$s^* = \frac{s}{\Omega^*}$$

$\rightarrow$

$$H(s) = \frac{1}{s+2}.$$

$$\Omega^* = \frac{2}{0.0167} \tan\left(\frac{20\pi \times 0.0167}{2}\right)$$

$$= 69.31.$$

$$H^*(s) = \frac{1}{\frac{s}{\Omega^*}+2} = \frac{1}{\frac{s}{69.31}+2} = \frac{69.32}{s+138.62}$$

$\omega = \omega_c$

$s \rightarrow s/\Omega$

$$s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right].$$

$$H(z) = \frac{69.31}{\frac{2}{0.0167}\left[\frac{z-1}{z+1}\right]+138.62}$$

$$= \frac{z+1}{3.73z+0.27}$$

Butterworth Filter Approximation

$$\boxed{|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$$

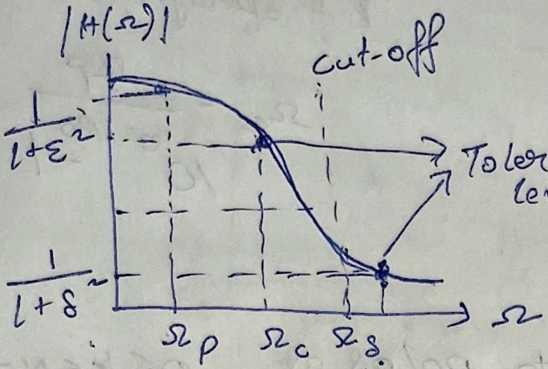
Ω_p = pass band edge frequency

Ω_c = cut-off freq.

ε = Ripple related parameter in pass band

δ = ripple related parameter in ~~pass b~~ stop band

$N \rightarrow$ order



Design Steps

Step-1 For given specification of digital; obtain equivalent analog filter

Step-2 Calculate order of filter (N)

$$N = \frac{1}{2}\left\{\frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}\right\}$$

or

$$N = \frac{1}{2}\left\{\frac{\log\left[\left(1/\delta_s^2\right)-1\right]}{\log\left(\Omega_s/\Omega_c\right)}\right\}$$

$A_p \rightarrow$ Pass band attenueation

$A_s \rightarrow$ Stop band attenueation.

$\delta_s \rightarrow$ attenuation in stop band

If given in decibels.

$$N = \frac{1}{2}\left\{\frac{\log\left[\frac{10^{0.1 A_s(db)}-1}{10^{0.1 A_p(db)}-1}\right]}{\log\left(\Omega_s/\Omega_p\right)}\right\}$$

Step-3 Now we calculate cutt off frequency Ω_c

For Impulse invariance.

$$\Omega_c = \frac{\omega_c}{T_s}$$

If ω_c is not given

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{1/2N}}$$

For BLT

$$\Omega_c = \frac{2}{T_s}\tan\left(\frac{\omega_c}{2}\right)$$

If specification is in dB

$$\Omega_c = \frac{\Omega_p}{10^{0.1A_p}-1}$$

Step-4 We now calculate poles as $0 \leq K \leq N-1$

$$\left(P_K = \pm\Omega_c\, e^{j(N+2K+1)\pi/2N} \quad K = 0, 1, 2, \ldots N-1\right)$$

We should select poles from left side of plane only.

Step 5

$$H(s) = \frac{\Omega_c^N}{(s-P_1)(s-P_2)\ldots}$$

Q. A design filter has following specifications

Pass band freq $= 0.2\pi = \omega_p$

Stopband freq. $= 0.3\pi = \omega_s$.

What are the specifications in Analog domain if we use.

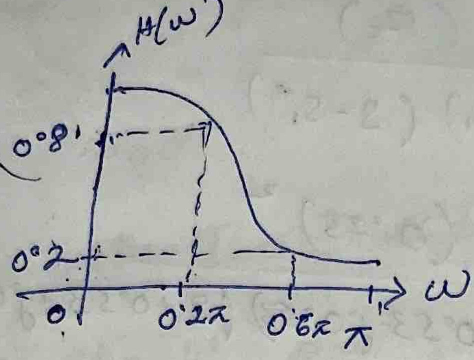
a) Impulse Invariance Method $\Omega_{s,p} = \frac{\omega_{s,p}}{T_s}$

b) BLT $\Omega_{s,p} = \frac{2}{T_s}\tan\left(\frac{\omega_{s,p}}{2}\right)$

Q. Using BLT design a butterworth filter satisfying following condⁿ.

$$0.8 \leq |H(\omega)| \leq 1 \quad \forall\ 0 \leq \omega \leq 0.2\pi$$

$$|H(\omega)| \leq 0.2 \quad \forall\ 0.6\pi \leq \omega \leq \pi$$

→ 

$A_p = 0.8$, $\omega_p = 0.2\pi$

$A_s = 0.2$, $\omega_s = 0.6\pi$

Step 1

$$\Omega = \frac{2}{T_s} \tan\left(\frac{\omega}{2}\right)$$

$$\Omega_s = \frac{2}{T_s} \tan\left(\frac{\omega_s}{2}\right) \qquad \Omega_p = \frac{2}{T_s} \tan\left(\frac{\omega_p}{2}\right)$$

$$= 2.75 \qquad\qquad = 0.65$$

Step-2

$$N = \frac{1}{2} \frac{\log\left[\left(\frac{1}{A_s^2} - 1\right) / \left(\frac{1}{A_p^2} - 1\right)\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= 1.3 \approx 2$$

Step-3

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}} = 0.75$$

Step-4

$$P_K = \pm \Omega_c e^{j(N+2K+1)\pi/2N} \qquad K = 0, 1 \quad \because N = 2$$

$$P_0 = \pm 0.75 e^{j(3\pi/4)}$$

$$= \pm 0.75 \left\{\cos\left(3\pi/4\right) + j \sin\left(\frac{3\pi}{4}\right)\right\} =$$

$= -0.53 + j0.53, \quad 0.53 - j0.53 \times$

$P_1 = -0.53 - j0.53, \quad 0.53 + j0.53 \times$

(Step 5)

$$H(s) = \frac{(\Omega_c)^N}{(s-s_1)(s-s_1^*)}$$

$$= \frac{(0.75)^2}{(s+0.53-j0.53)(s+0.53+j0.53)}$$

$$H(s) = \frac{0.56}{s^2 + 1.06s + 0.56}$$

$$s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$$

$$H(z) = \frac{0.56(z+1)^2}{6.68z^2 - 6.88z + 2.44}$$

Chebyshev Filter

→ Poles of butterworth filter lie on a circle while the pole of chebyshev filter lie on a ellipse.

→ Butterworth filter has a flat response in both PB & SB whereas chebyshev filter has ripples in pass band.

~~why should one select chebyshev over butterworth~~

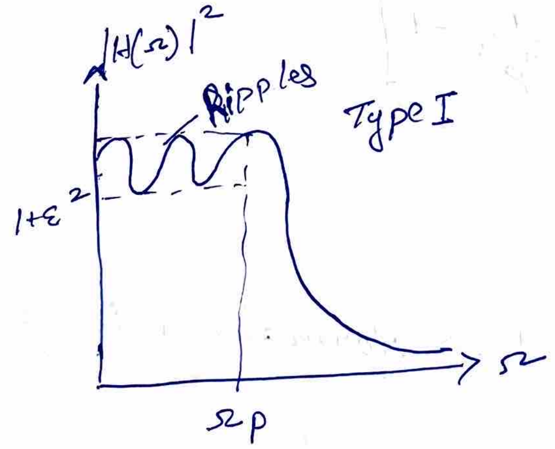
Why should one select chebyshev over butterworth..

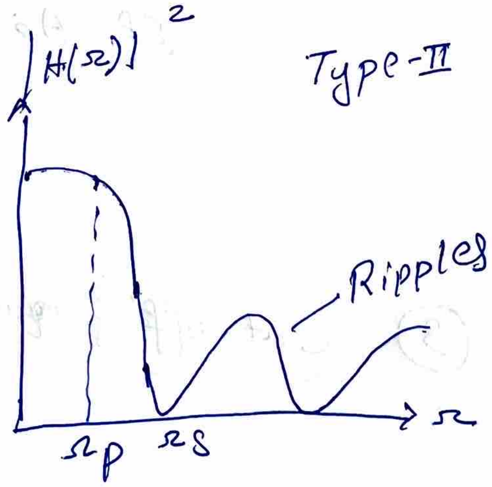
→ Chebyshev filter has better roll-off rate

eg - for same order if butterworth has say -60 dB

chebyshev will have say -80 dB

→ For sharp transitions from pass band to stop band





$$\boxed{|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N(\Omega/\Omega_p)}}$$

$C_N(\Omega/\Omega_p)$
↑
chebyshev polynomial of order N

$C_N(x)$ = chebyshev polynomial of order N

$$\boxed{C_{N+1}(x) = 2x\,C_N(x) - C_{N-1}(x).}$$

$$* \boxed{C_0(x) = 1 \quad , \quad C_1(x) = x}$$

Design.

① Frequency Response

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\Omega/\Omega_p)}$$

$C_N(\Omega/\Omega_p)$ = Chebyshev polynomial of order N

② ε

$$\varepsilon = \left[10^{0.1A_p(dB)} - 1\right]^{1/2}.$$

If A_p is not in dB

$$\varepsilon = \left[\frac{1}{A_p^2} - 1\right]^{1/2}.$$

③ Cut-off frequency

At $\Omega_c = 1 \rightarrow$ Normalized filter

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\varepsilon^2}}$$

④ Order of filter

$$|H(j\Omega)| = -20\log_{10} \varepsilon - 6(N-1) - 20\log_{10}(\Omega_s).$$

⑤ Poles

First we calculate the regular position of poles of butterworth filter $\quad \beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{1/N}$

Pole position.

$s_p = r\cos\theta_k + jR\sin\theta_k \qquad \theta = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k = 0, 1, 2, \ldots N-1$

$r = \Omega p \frac{\beta^2 - 1}{2\beta}$ minor axis of ellipse $\qquad R = \Omega p \frac{\beta^2+1}{2\beta}$ major axis

Position of poles lie on ellipse x_k & y_k $\qquad x_k = r\cos\theta_k$, $y_k = R\sin\theta_k$

Q. Design a low pass 1rad/sec bandwidth chebyshev filter with following characteristics

(i) Pass band ripple of 2dB $\qquad A_p = 2$

(ii) Cutt-off frequency of 1 rad/sec $\qquad \Omega_c = 1$

(iii) Stop band attenuation of 20dB or greater beyond 1·3rad/sec

⑥ Transfer f^n

$$H(s) = \frac{k}{(s-s_0)(s-s_1)}.$$

$$\Rightarrow H(s) = \frac{k}{s^N + b_{N-1}s^{N-1} + \cdots + b_0}$$

$$k = \begin{cases} b_0 & \forall\ n = \text{odd} \\ \frac{b_0}{\sqrt{1+\varepsilon^2}}, & \forall\ n = \text{even} \end{cases}$$

$\rightarrow$ $A_p = 2dB$, $\Omega_c = 1$ | $H(j\Omega) \longrightarrow N$

$\Omega_s = 1.3$ rad/se.

(1) $\varepsilon = \left[10^{0.1\times2} - 1\right]^{1/2} = 0.765$

$|H(\Omega)| = -20dB$, $\Omega_s = 1.3$ rad

(2) $|H(j\Omega)| = -20\log_{10}\varepsilon - 6(N-1) - 20\log_{10}\Omega_s$

$\Rightarrow -20 = -20\log_{10}(0.765) - 6(N-1) - 20\log_{10}(1.3)$

$\Rightarrow N = 3.421 \approx 4$

(3) Pole calculation.

$$\beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{1/N}$$

$= 1.31$

$r = \Omega_p \dfrac{\beta^2-1}{2\beta}$ $\qquad R = \Omega_p \dfrac{\beta^2+1}{2\beta}$

$= 1 \times \dfrac{1.31^2-1}{2\times1.31}$ $\qquad = 1.04$

$= 0.273$

$\theta_k = \dfrac{\pi}{2} + \dfrac{(2k+1)\pi}{2N}$, $\theta = 0, 1, 2, 3$

$\theta_1 = \frac{7\pi}{8}$ $\theta_2, \theta_3, \theta_0 = ---$

$S_1 = r\cos\theta_1 + jR\cos\theta_1$

$= 0.273\cos\left(\frac{7\pi}{8}\right) + j1.048\sin\left(\frac{7\pi}{8}\right)$

$= -0.25 + j0.4$

$$H(s) = \frac{K}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)}$$

↓ After solving

$$= \frac{K}{s^4 + 0.7s^3 + 1.25s^2 + 0.51s + 0.207}$$

$$K = \frac{b_0}{\sqrt{1+\varepsilon^2}} \Big|_{even}$$ b_0 = Constant term in demoninator.

$$= \frac{0.207}{\sqrt{1+0.765^2}} \approx 0.16$$

$$\therefore H(s) = \frac{0.16}{s^4 + 0.7s^3 + 1.25s^2 + 0.51s + 0.207}$$

Q. Design a chebyshev filter for following specification
$\frac{1}{2}$ dB ripple in PB $0 \le \omega \le 0.24\pi$ min 50dB
attenuation in stop band $0.35\pi \le |\omega| \le \pi$
Use BLT for T = 1 sec

→ $A_p = \frac{1}{2}$ dB $|H(\Omega)| = -50$dB $\omega_p = 0.24\pi$
$\omega_s = 0.35\pi$

$\Omega_p = \frac{2}{T_s}\tan\left(\frac{\omega_p}{2}\right)$

Frequency Transformation

When cut-off frequency $\Omega_c = 1 \rightarrow$ Normalized filter

$\Omega = 2\pi f$

Conversion table

Low pass Normalized filter

- Low pass $\quad S \rightarrow \dfrac{\Omega_c}{\Omega_{LP}} S$
- High pass $\quad S \rightarrow \dfrac{\Omega_c}{S} \Omega_{HP}$
- Band pass $\quad S \rightarrow \Omega_c \times \dfrac{S^2 + \Omega_l \Omega_u}{S(\Omega_u - \Omega_l)}$
- Band pass $\quad S \rightarrow \Omega_c \times \dfrac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_u \Omega_l}$

Sampling frequency

$f_U = \dfrac{F_U}{F_S} \quad f_L = \dfrac{F_L}{F_S}$

Q. The T.F of 1st order normalized LPF is

$$H(s) = \frac{1}{s+1}$$

Obtain the system f^n of 2nd order bandpass filter having passband from 1kHz - 2kHz.

$$\rightarrow S = \Omega_c \frac{S^2 + \Omega_l \Omega_u}{S(\Omega_u - \Omega_l)} \qquad \Omega_c = 1$$

$f_U = 2kHz \quad ; \quad f_l = 1kHz$

$\Omega_U = 2\pi f_U \qquad \Omega_l = 2\pi f_l$

FIR filter Design.

$$\text{Design} \begin{cases} \rightarrow \text{windowing} \\ \rightarrow \text{Sampling} \end{cases}$$

$$y(n) = -\cancel{\sum_{k=1}^{N} a_k y(n-k)} + \sum_{k=0}^{M} b_k x(n-k)$$

0

* Non-Recursive filter → No feedback.

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

↓ z transform.

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} b_k z^{-k}$$

Inverse Z-T

Stablity → $h(n) = \begin{cases} b_0, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$

* Inherently stable

* — → Symmetric (+ve), → Asymmetric (−ve) — Linear Phase characteristic

$$\boxed{h(n) = \pm h(M-1-n)}$$

FIR filter phase.

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi & |H(\omega)| < 0 \end{cases}$$

Q. Show that if z_1 is zero of linear phase filter → FIR then $\frac{1}{z_1}$ is also the zero of filter?

→ $H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$

$z = z_1$ $\quad H(z) \to H(z_1) = 0$

$H(z_1) = \sum_{n=0}^{M-1} h(n) z_1^{-n} = 0$ — ① given

$z = \frac{1}{z_1} \to z^{-1}$ $\quad H(z_1^{-1}) = 0$ If this is true → Prove

$H(z_1^{-1}) = \sum_{n=0}^{M-1} h(n) (z_1^{-1})^{-n} = \sum_{n=0}^{M-1} h(n) z_1^{n}$ — ②

Linear phase filter

→ $h(n) = h(M-1-n)$ $\quad$ $M-1-n = p$

$H(z_1^{-1}) = \sum_{n=0}^{M-1} h(M-1-n) z_1^{n}$

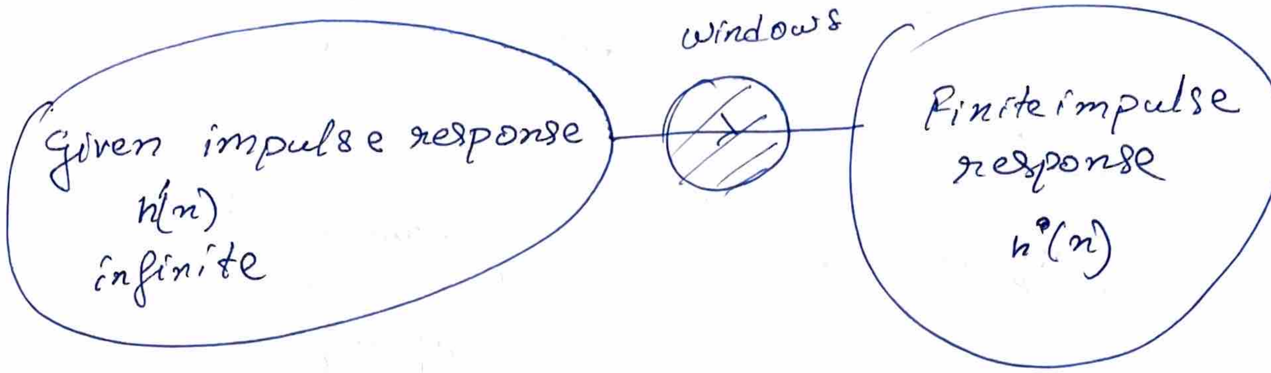
$H(z_1^{-1}) = \sum_{n=0}^{M-1} h(p) z^{M-1-p}$

$H(z_1^{-1}) = \sum_{n=0}^{M-1} h(p) z^{M-1} z^{-p}$

$H(z_1^{-1}) = z^{M-1} \left\{ \sum_{n=0}^{M-1} h(p) z^{-p} \right\}$

$H(z_1^{-1}) = 0$

Windowing Techniques



Causal windows $\longrightarrow$ positive

Anti-causal windows $\longrightarrow$ both +ve & -ve

① Rectangular Window

Length of window is M

$$w_R(n) = \begin{cases} 1 & \forall,\ n = 0, 1, 2, \ldots M-1 \\ 0, & \text{otherwise} \end{cases} \Bigg\} \text{causal}$$

$$h''(n) = h'(n)\, w_R(n)$$

Let the fourier transform of rectangular window is $W_R(n)$

$$W_R(n) = \sum_{n=0}^{M-1} \cancel{w_R(n)}^{\,1} e^{-j\omega n}.$$

$$W_R(n) = \sum_{n=0}^{M-1} \left(e^{-j\omega}\right)^n \qquad \sum_{K=N_1}^{N_2} a^K = \frac{a^{N_1} - a^{N_2-1}}{1-a}.$$

$$W_R(n) = \frac{1 - e^{-j\omega M}.}{1 - e^{-j\omega}.}$$

$$W_R(n) = \frac{e^{-j\omega M/2} \cdot e^{j\omega M/2} - e^{-j\omega M/2}\, e^{-j\omega M/2}.}{e^{-j\omega/2}\, e^{j\omega/2} - e^{-j\omega/2}\, e^{-j\omega/2}.}$$

$$= \frac{e^{-j\omega M/2}\left(e^{j\omega M/2} - e^{-j\omega M/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$\boxed{W_R(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \frac{\sin\left(\omega M/2\right)}{\sin\left(\omega/2\right)}}$$

2) Bartlett Window (Triangular)

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{M-1}{2}\right|}{M-1} & \forall\ n = 0, 1, \ldots M-1 \\ 0, \text{ otherwise} \end{cases}$$

Causal.

3) Blackmann Window

$$w_B(n) = 0.42 \mp 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$$

4) Hamming Window

$$w_H(n) = 0.54 \mp 0.46\cos\frac{2\pi n}{M-1}$$

5) Hanning Window

$$w_{Hn}(n) = 0.5 \mp 0.5\cos\frac{2\pi n}{M-1}$$

Windowing Techniques - Numerical Approach.

Step-1 Frequency Response of filter to be designed is given.

Step-2 We find the unit sample response $h_d(n)$ from $H_d(\omega)$ by taking inverse fourier transform.

$$\boxed{h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega}$$

Step-3 The response obtained in step-2 has infinite duration so we need to convert it into a finite duration response (M length). This is done by using appropriate window

$$\boxed{h(n) = h_d(n)\, w(n)}$$

Step-4 If required find the frequency response of FIR filter.

$$\boxed{H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}}$$

Q. Design a digital FIR filter

$$H(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\pi/4 \leq |\omega| < \pi/4 \\ 0, & \pi/4 < |\omega| < \pi \end{cases}$$

Length = 5

$\rightarrow$

$$H(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\pi/4 \leq |\omega| < \pi/4 \\ 0, & \pi/4 < |\omega| < \pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} H(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \Big| \; n = 0, 1, 2, 3, 4$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j(n-2)\omega}\, d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j(n-2)\omega}}{j(n-2)} \right|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-2)} \left[\frac{e^{j(n-2)\pi/4} - e^{-j(n-2)\pi/4}}{2j} \right]$$

$= \frac{1}{\pi(n-2)}$

$$h_d(n) = \frac{\sin(n-2)\pi/4}{\pi(n-2)}$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$h_d(0) \quad h_d(1) \quad h_d(2) \quad h_d(3) \quad h_d(4)$

$$h_d(2) = \int_{-\pi/4}^{\pi/4} \frac{1}{2\pi} \cancel{e^{j(n-2)\omega}} \, d\omega \Big|_{n=2}$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} d\omega = 1/4.$$

$h(n) = h_d(n)\,\omega(n)$

For rectangular window

$\omega(n) = 1$

$h(n) = h_d(n)$

Q. Design a symmetric FIR filter whose desired response is

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \forall\ |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

length of filter = 5 & ω_c = 1 rad/sec. Use rectangular window.

$\rightarrow$

$$h_d(n) = \frac{1}{2\pi} \int_{-1}^{1} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^{1} e^{j(n-\tau)\omega} \, d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j(n-\tau)\omega}}{(n-\tau)j} \right|_{-1}^{1}$$

$$h_d(n) = \begin{cases} \frac{\sin(n-\tau)}{\pi(n-\tau)}, & \text{when } n \neq \tau \\ \frac{1}{\pi}, & n = \tau \end{cases}$$

$$\boxed{h_d(n) = h_d(M-1-n)}$$ V.V.I

$$\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

$$n-\tau = -(M-1-n-\tau)$$

$$\therefore \tau = \frac{M-1}{2}$$

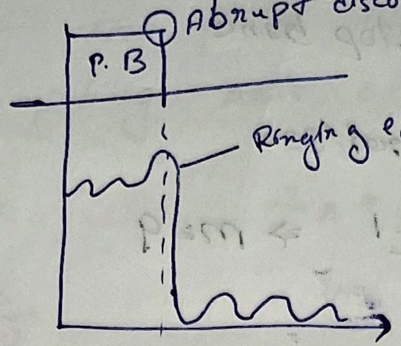
for $M = 5$

$$h_d(n) = \begin{cases} \frac{\sin\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)}, & n \neq \tau \\ 1/\pi, & n = \frac{M-1}{2} \end{cases}$$

$$h(n) = h_d(n) \cdot \cancel{w(n)} \to 1$$

For rectangular window.

Problem with Rectangular window - Gibb's phenomenon.



* Type-II

Order of FIR filter:

$$M = K\left(\frac{2\pi}{\omega_2 - \omega_1}\right)$$

K is calculated from width of main lobe

width of main lobe $= K\left(\frac{2\pi}{M}\right)$

$$\omega_1 = \frac{\Omega_1}{F_s} \qquad \omega_2 = \frac{\Omega_2}{F_s}$$

Window	Transition width of main lobe	Min stopband attenuation
Rectangular	$4\pi/M+1$	-21 dB
Bartlett	$8\pi/M$	-25 dB
Hanning	$8\pi/M$	-44 dB
Hamming	$8\pi/M$	-53 dB
Blackmann	$12\pi/M$	-74 dB

Q. Design a normalized linear phase FIR filter having phase delay $\tau = 4$ & at least 40 dB attenuation in stop band

→ $\omega_c = 1$ rad/sec

$$\tau = \frac{m-1}{2} \Rightarrow m = 9$$

∴ Hanning window.

$$\omega(n) = 0.5 - 0.5 \cos \frac{2\pi n}{m-1}$$

$$\omega(n) = 0.5 - 0.5 \cos\left(\frac{\pi n}{4}\right)$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} \, d\omega$$

$$\boxed{H_d(\omega) = \begin{cases} e^{-j\omega\tau} & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}}$$

$$h(n) = h_d(n) \omega n.$$

Q. Design a ~~[illegible]~~ LPF that will have -3dB cutoff at 30π rad/sec and an attenuation of 50dB at 45π rad/sec. The filter has linear phase & uses a sampling rate of 1000 samples/sec

→ $\Omega_1 = 30\pi$ $\Omega_2 = 45\pi$ $F_s = 1000$

$\omega_1 = \frac{30\pi}{1000}$ $\omega_2 = \frac{45\pi}{1000}$

Hamming

width of lobe $= k\left(\frac{2\pi}{m}\right)$

$\frac{8\pi}{m} = k\left(\frac{2\pi}{m}\right) \Rightarrow k = 4.$

$m \geq k\left(\frac{2\pi}{\omega_2 - \omega_1}\right)$

Kaiser Window

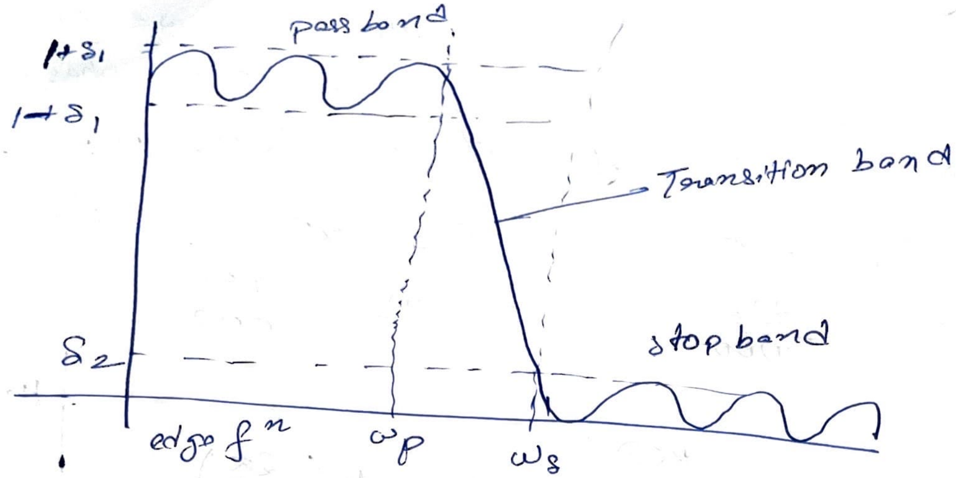
$$\omega_K(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1 - \left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}, & \forall\ 0 \leq n \leq m \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha = \frac{m}{2}$$

$$I_0(x) = 1 + \frac{0.25x^2}{(\angle 1)^2} + \frac{(0.25x^2)^2}{(\angle 2)^2} + \frac{(0.25x^2)^3}{(\angle 3)^2} + \dots$$

$I_0(\,) \rightarrow$ zero order modified Bessel's f^n

Q.



Standard MR

$$1-\delta_1 \le |H(\omega)| \le 1+\delta_1 \quad \forall \; 0 \le \omega \le \omega_p$$

$$0 \le |H(\omega)| \le \delta_2 \quad \forall \quad \omega_s \le \omega \le \pi$$

$\delta \rightarrow$ minimum of δ_1 & δ_2

transition b/w $\boxed{\Delta\omega = \omega_s - \omega_p}$

Attenuation $\boxed{A = -20\log \delta}$

$$m = \frac{A-8}{2.285\,\Delta\omega}$$

$$\beta = \begin{cases} 0.1102\,(A-8.7) & \forall \; A \ge 50 \\ 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21) & \forall \; 21 \le A \le 50 \\ 0 & \forall \; A < 21 \end{cases}$$

Q. Design a FIR filter phase filter using Kaiser window

$$0.99 \leq |H(e^{j\omega})| \leq 1.01 \quad \forall\ 0 \leq |\omega| \leq 0.19\pi$$

$$|H(e^{j\omega})| \leq 0.01 \quad \forall\ 0.21\pi \leq |\omega| \leq \pi$$

$\rightarrow$

$$1 - 0.01 \leq |H(e^{j\omega})| \leq 1 + 0.01 \quad \forall\ 0 \leq |\omega| \leq 0.19\pi$$

$$|H(e^{j\omega})| \leq 0.01 \quad \forall\ 0.21\pi \leq |\omega| \leq \pi$$

$\delta_1 = 0.01 \qquad \delta_2 = 0.01 \qquad \omega_p = 0.19\pi$

$\omega_s = 0.21\pi$

$H_d(\omega) =$

$$\delta = \min(\delta_1, \delta_2) = 0.01$$

$$\Delta\omega = \omega_s - \omega_p = 0.02\pi$$

$$A = -20 \log_{10} \delta$$

$$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$$

$$= 3.395$$

$$\omega_c = \frac{\omega_s + \omega_p}{2} \approx 0.2\pi \qquad M = \frac{A-8}{2.285\,\Delta\omega} \approx 23$$

$w(n) \rightarrow$

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-M/2\right)}, & n \neq M/2 \\ \dfrac{\omega_c}{\pi}, & n = M/2 \end{cases}$$