Correlation [basics, function, types & Importance]

- Correlation is similarity of two signals.
- If correlation is zero, then there is no similarity in between two signal.
- If we have two signals $x_1(t)$ & $x_2(t)$, then correlation in bet.^{n} two signals.

$$R(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2^*(t)\, dt$$

- Types - There are two types

① Auto Correlation - signal $x(t)$, shifting parameter τ

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

It is used to find similarities bet.^{n} signal $x(t)$ & shifted signal $x(t-\tau)$

② Cross Correlation. - Signals $x_1(t)$ & $x_2(t)$, shifting parameter τ

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(t)\, dt$$

It used to find similarities in bet.^{n} two different signals $x_1(t)$ & $x_2(t)$

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Auto Correlation of Energy Signal.

- It is similarity of signal $x(t)$ with it's shifted version $x(t-\tau)$.
- If there is no similarity in betn $x(t)$ & $x(t-\tau)$ then Auto Correlation will be zero.
- For Energy Signal, Energy of signal is finite

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

- $$R_{11}(\tau) = \int_{-\infty}^{\infty} x(t) \, x^*(t-\tau) \, dt$$

$$= \int_{-\infty}^{\infty} x(t-\tau) \, x^*(t) \, dt$$

Here τ = Shifting parameter / Scanning Parameter / Searching Parameter / Delay Parameter.

- For Real signal $x(t)$

$$R_{11}(\tau) = \int_{-\infty}^{\infty} x(t) \, x(t-\tau) \, dt$$

Auto Correlation properties for Energy Signal.

Property-1 - ACF for energy signal exhibits Conjugate Symmetry.

$$\boxed{R(\tau) = R^*(-\tau)}$$

Property-2 - ACF for energy Signal is energy at $\tau = 0$

$$\boxed{R(0) = E}$$

Property-3 - ACF for energy signal is max at $\tau = 0$.

$$\boxed{R(0) \geq R(\tau)}$$

Property-4 - Fourier Transform of Auto Correlation function for energy signal is Energy Spectral density.

$$\boxed{R(\tau) \xleftrightarrow{FT} \psi(\omega)}$$

Engineering
Funda

Property-1 Auto Correlation for Energy signal. [property & proof]

- ACF exhibits conjugate symmetry.

$$R(\tau) = R^*(-\tau)$$

Proof

- $R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$

- $R^*(\tau) = \left[\int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt\right]^*$

$= \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt$

- $R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$

- $t+\tau = y \Rightarrow t = y-\tau \Rightarrow dt = dy$

- limits of y $-\infty$ to ∞.

- $R^*(-\tau) = \int_{-\infty}^{\infty} x^*(y-\tau)\, x(y)\, dy$

$= \int_{-\infty}^{\infty} x(y)\, x^*(y-\tau)\, dy.$

- $\boxed{R^*(-\tau) = R(\tau)}$

Property-2 Auto Correlation for Energy Signal [property & Proof]

- ACF for energy signal is energy at $\tau = 0$

$$\boxed{R(0) = E}$$

Proof

$\rightarrow R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$

$\rightarrow E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$

If $\tau = 0$

$\rightarrow R(0) = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$

$= \int_{-\infty}^{\infty} |x(t)|^2\, dt$

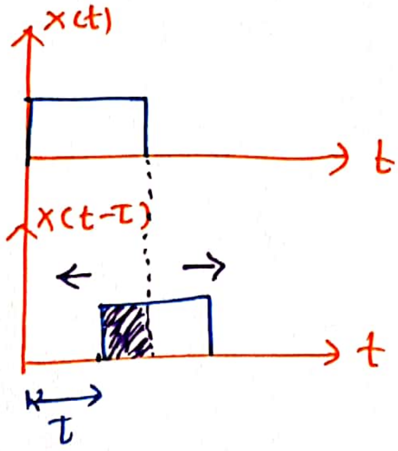
$\rightarrow \boxed{R(0) = E}$

$\rightarrow$ In graph of $R(\tau) \rightarrow \tau$, at $\tau = 0$, $\boxed{R(0) = E}$

Property-3 Auto Correlation for Energy Signal [property & proof]

- Auto Correlation is max at, $\tau = 0$.

$$\boxed{R(0) \geq R(\tau)}$$



Proof

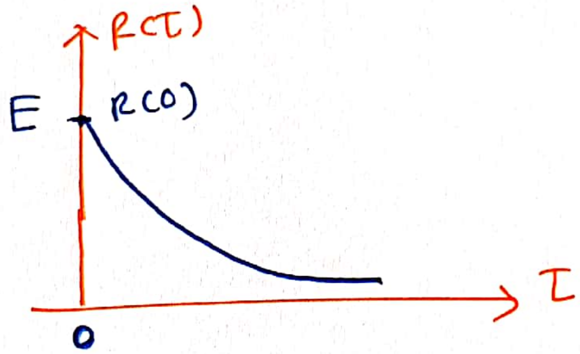
$\Rightarrow (x(t) - x(t-\tau))^2 \geq 0$

$\Rightarrow x^2(t) + x^2(t-\tau) - 2x(t)\,x(t-\tau) \geq 0$

$\Rightarrow x^2(t) + x^2(t-\tau) \geq 2x(t)\,x(t-\tau)$

$\Rightarrow \int_{-\infty}^{\infty} x^2(t)\,dt + \int_{-\infty}^{\infty} x^2(t-\tau)\,dt \geq 2\int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,dt$

$\Rightarrow E + E \geq 2R(\tau) \Rightarrow E \geq R(\tau) \Rightarrow \boxed{R(0) \geq R(\tau)}$



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Property-4 Auto Correlation for Energy Signal [property & Proof]

- Fourier Transform of ACF for Energy Signal is ESD.

$$R(\tau) \xleftrightarrow{FT} \psi(\omega)$$

Proof

Fourier transform of $x(t)$ is $x(\omega)$

$$x(t) \xleftrightarrow{FT} x(\omega)$$

$\psi(\omega) = ESD = |x(\omega)|^2$ [$x(t)$ is energy signal].

$S(\omega) = PSD = |x(\omega)|^2$ [$x(t)$ is power signal].

$$- FT[R(\tau)] = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt\; e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, e^{-j\omega t}\, e^{j\omega t}\, e^{-j\omega\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\; x(t-\tau)\, e^{-j\omega(t-\tau)}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\, d\tau$$

- $t-\tau = y \Rightarrow -d\tau = dy$.
- limits are $-\infty$ to ∞.

$$= \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] \times -\left[\int_{-\infty}^{\infty} x(y)\, e^{-j\omega y}\, dy\right]$$

$$= -X(\omega)\, X(\omega)$$

$$= |X(\omega)|^2 = \psi(\omega) = ESD.$$

Auto Correlation for periodic power signal [function & properties]

- The Auto Correlation function for periodic power signal with time T is given by.

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Properties

① ACF of power signal exhibits conjugate symmetry.

$$R(\tau) = R^*(-\tau)$$

② ACF of Power signal is Power at origin.

at $\tau = 0$, $\boxed{R(0) = P}$

③ ACF of Power signal is max at origin.

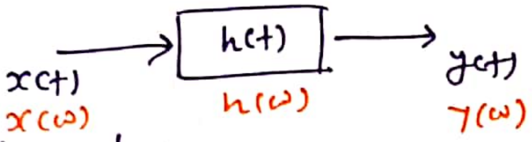
$$R(0) \geq R(\tau)$$

④ ACF of Power signal $\xleftrightarrow{FT}$ Power spectral density

$$R(\tau) \xleftrightarrow{FT} S(\omega).$$

Relation between Input Energy/power Spectral density and output Energy/power Spectral density

→ O/p Spectral density = $|H(\omega)|^2$ I/p Spectral density.



→ Spectral density = $|\text{Spectrum}|^2$

→ O/p S.D. = $|Y(\omega)|^2$

→ I/p S.D. = $|X(\omega)|^2$

→ O/p of system

$$y(t) = x(t) * h(t).$$

→ Take Fourier Transform

$$\Rightarrow Y(\omega) = X(\omega)\, h(\omega)$$

$$\Rightarrow |Y(\omega)|^2 = |X(\omega)|^2\, |H(\omega)|^2$$

$$\Rightarrow \boxed{\text{O/p S.D} = \text{I/p S.D. } |H(\omega)|^2}$$

→ For energy Signal.

$$\Rightarrow \Psi_o(\omega) = \Psi_{in}(\omega)\, |H(\omega)|^2$$

→ For Power Signal

$$\Rightarrow S_o(\omega) = S_{in}(\omega)\, |H(\omega)|^2$$

Example of Auto Correlation for Energy Signal

Find Auto Correlation of $x(t) = e^{-3t} u(t)$. also Find Energy of the signal

- **Property** → $R(\tau) = x(\tau) * x(-\tau)$
  → Take Fourier Transform.
  → Inverse Fourier Transform.

→ $R(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$

→ $R(\tau) = x(\tau) * x(-\tau)$

$= e^{-3\tau} u(\tau) * e^{3\tau} u(-\tau)$

→ Take Fourier Transform

$e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$.

$e^{at} u(-t) \xleftrightarrow{FT} \frac{1}{a-j\omega}$.

→ $FT(R(\tau)) = F.T\,[\,e^{-3\tau} u(\tau) * e^{3\tau} u(-\tau)\,]$

→ $FT[R(\tau)] = FT\,[e^{-3\tau} u(\tau)]\ FT\,[e^{3\tau} u(-\tau)]$.

→ $FT[R(\tau)] = \frac{1}{3+j\omega} \times \frac{1}{3-j\omega} = \frac{1}{9-j^2\omega^2}$

→ $FT[R(\tau)] = \frac{1}{9+\omega^2}$

→ Take Inverse Fourier Transform.

$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}$

→ $R(\tau) = IFT\left[\frac{1}{9+\omega^2}\right]$

→ $R(\tau) = IFT\left[\frac{3\times 2}{3^2+\omega^2}\right] \times \frac{1}{6}$

→ $\boxed{R(\tau) = \frac{1}{6} e^{-3|t|}}$

→ For Energy of the signal From Auto Correlation

$\tau = 0$ → $R(0) = E$

→ $R(0) = E = \frac{1}{6} e^{-0} = \frac{1}{6}$ J.

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Example of Auto Correlation for Power Signal

Find Auto correlation of $x(t) = 6\cos(6\pi t + \pi/3)$. Also find power of the signal.

$V_m \rightarrow P = \frac{V_m^2}{2} = \frac{36}{2} = 18W.$

$\rightarrow R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t-\tau)\, dt$

$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 6\cos(6\pi t + \pi/3) \cdot 6\cos(6\pi t - 6\pi\tau + \pi/3)\, dt$

$= \lim_{T\to\infty} \frac{36}{T} \int_{-T/2}^{T/2} \cos(6\pi t + \pi/3)\cos(6\pi t - 6\pi\tau + \pi/3)\, dt$

$[2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$= \lim_{T\to\infty} \frac{\overset{18}{\cancel{36}}}{T} \int_{-T/2}^{T/2} \frac{\cos(12\pi t - 6\pi\tau + 2\pi/3) + \cos(6\pi\tau)}{\cancel{2}}\, dt$

$= \lim_{T\to\infty} \frac{18}{T} \int_{-T/2}^{T/2} [\underbrace{\cos(12\pi t - 6\pi\tau + 2\pi/3)}_{0} + \underline{\cos(6\pi\tau)}]\, dt$

$= \lim_{T\to\infty} \frac{18}{T} \cos 6\pi\tau \int_{-T/2}^{T/2} 1\, dt$

$= \lim_{T\to\infty} \frac{18}{\cancel{T}} \cos 6\pi\tau \times \cancel{T} \Rightarrow \boxed{R(\tau) = 18\cos 6\pi\tau}$

$\rightarrow$ For Power, from Auto correlation of Power signal $\tau = 0$

$R(0) = P$

$\rightarrow R(0) = 18\cos 6\pi \times 0 = \boxed{18W}$

Cross Correlation for Energy Signal [basics, function & properties]

- It is the measure of similarity between one signal and the time delayed version of other signal.

- If we have two complex signals $x_1(t)$ & $x_2(t)$. then.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2^*(t)\, dt$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_1^*(t)\, x_2(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x_1^*(t-\tau)\, x_2(t)\, dt$$

- If we have two real signals $x_1(t)$ & $x_2(t)$. then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(t)\, dt = R_{21}(\tau)$$

- Properties

1) Cross Correlation function exhibits Conjugate Symmetry

$$R_{12}(\tau) = R_{12}^*(-\tau)$$

2) If $R_{12}(0) = 0$, then $x_1(t)$ & $x_2(t)$ are orthogonal to each other.

$$R_{12}(0) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$$

3) Correlation theorem

$$R_{12}(\tau) \xleftrightarrow{FT} X_1(\omega)\, X_2^*(\omega)$$

$$R_{21}(\tau) \xleftrightarrow{FT} X_1^*(\omega)\, X_2(\omega)$$

Cross Correlation for Power Signal [basics, function & Properties]

- If we have two signals $x_1(t)$ and $x_2(t)$ with same time period T, then the cross correlation is defined as,

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)\, x_2^*(t-\tau)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t+\tau)\, x_2^*(t)\, dt$$

$$R_{21}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1^*(t)\, x_2(t-\tau)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1^*(t-\tau)\, x_2(t)\, dt$$

- For $x_1(t)$ & $x_2(t)$ as real signals.

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)\, x_2(t-\tau)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t-\tau)\, x_2(t)\, dt = R_{21}(\tau)$$

Properties

1. It exhibits conjugate symmetry.

$$R_{12}(\tau) = R_{12}^*(-\tau) \quad \Big| \quad R_{21}(\tau) = R_{21}^*(-\tau)$$

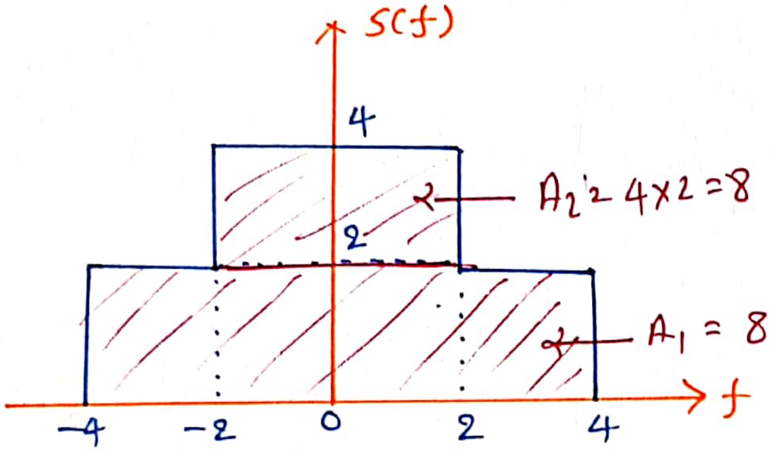
2. It does not follows commutative property.
3. If $R_{12}(0) = 0$, $x_1(t)$ & $x_2(t)$ are orthogonal to each other.
4. Correlation theorem

$$R_{12}(\tau) \xleftrightarrow{FT} X_1(\omega)\, X_2^*(\omega)$$

$$R_{21}(\tau) \xleftrightarrow{FT} X_1^*(\omega)\, X_2(\omega)$$

Example of Power from power Spectral density.

- Figure shows PSD of $x(t)$. Find average power of $x(t)$.



→ $P = \int_{-\infty}^{\infty} S(f)\, df$

→ Area of $S(f)$ w.r.t f is avg. power.

→ Total Area $= A_1 + A_2 = 24$

→ Avg Power $P = 24$ W.

$$P = \int_{-4}^{4} S(f)\, df$$

$$= \int_{-4}^{-2} 2\, df + \int_{-2}^{2} 4\, df + \int_{2}^{4} 2\, df$$

$$= 2[f]_{-4}^{-2} + 4[f]_{-2}^{2} + 2[f]_{2}^{4}$$

$$= 2[-2+4] + 4[2+2] + 2[4-2]$$

$$= 4 + 16 + 4 = 24 \text{ W}$$

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Example of Energy & verification by Parseval's Energy theorem

Find the energy & verify Parseval's theorem for the $x(t) = e^{-4t}u(t)$

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2 dt}$$

$$= \int_{-\infty}^{\infty} |e^{-4t}u(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} e^{-8t} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-8t} dt$$

$$= \left[\frac{e^{-8t}}{-8}\right]_0^{\infty}$$

$$= \left[0 - \frac{e^{-0}}{-8}\right] = \boxed{\frac{1}{8}}$$

Application

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- $E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$
- $e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}.$
- $\boxed{X(\omega) = \frac{1}{4+j\omega}.}$
- $|X(\omega)|^2 = \frac{1}{4^2+\omega^2}$
- $E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4^2+\omega^2} d\omega$

$$\int \frac{a}{a^2+\omega^2} d\omega = \tan^{-1}\left(\frac{\omega}{a}\right).$$

- $E = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{4}{4^2+\omega^2} d\omega.$

$$= \frac{1}{8\pi}\left[\tan^{-1}\left(\frac{\omega}{4}\right)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{8\pi}\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{8\pi}\left[\pi/2 - (-\pi/2)\right] = \boxed{\frac{1}{8}}$$

Example of energy Spectral density

A filter has Input $x(t) = e^{-t}u(t)$ & the Impulse response $h(t) = e^{-2t}u(t)$. Find ESD of output.

$$\underset{X(\omega)}{x(t)} \longrightarrow \underset{h(\omega)}{\boxed{h(t)}} \longrightarrow \underset{Y(\omega).}{y(t)}$$

- O/p SD $= |H(\omega)|^2 \times$ (I/p SD)

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$h(t) \xleftrightarrow{FT} H(\omega)$$

- $e^{-at}u(t) \xleftrightarrow{FT} \dfrac{1}{a+j\omega}$

$$X(\omega) = \frac{1}{1+j\omega}.$$

$$H(\omega) = \frac{1}{2+j\omega}$$

→ I/p SD $= |X(\omega)|^2$

$$= \frac{1}{1+\omega^2}$$

→ $|H(\omega)|^2 = \dfrac{1}{4+\omega^2}$

→ O/p ESD $= |H(\omega)|^2 \times$ I/p ESD

$$= \frac{1}{4+\omega^2} \times \frac{1}{1+\omega^2}$$

Energy of Time Scaled Signal.

- A signal $x(t)$ has Energy E, calculate the energy of $x(3t)$.

- $x(t)$ with Energy E

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

- $x(at)$ with Energy E'

$$E' = \int_{-\infty}^{\infty} |x(at)|^2 \, dt$$

- If $at = s \Rightarrow a\,dt = ds$
$\Rightarrow dt = ds/a$

- limits, $-\infty$ to ∞

$$E' = \int_{-\infty}^{\infty} |x(s)|^2 \frac{ds}{a} = \frac{\int_{-\infty}^{\infty} |x(s)|^2 \, ds}{a} = \boxed{\frac{E}{a}}$$

→ for $x(3t)$, energy

$$\boxed{E' = \frac{E}{3}}$$