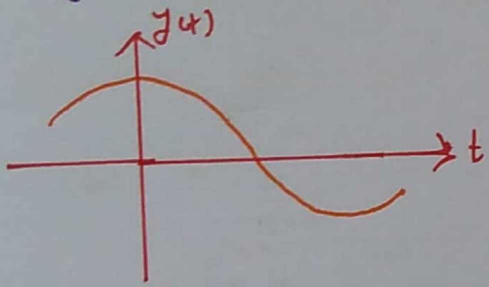
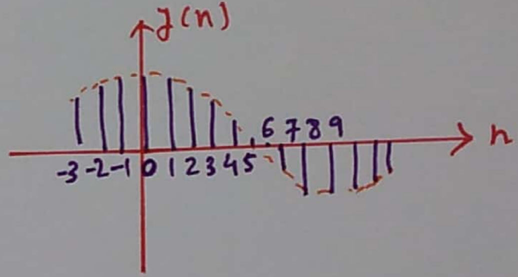
Z - Transform

$f(t) \rightarrow$ freq. $\rightarrow$ Fourier transform
Laplace transform

- Analysis of discrete time LTI systems can be done by using z - transforms

Continous	Discrete
- Time is continuous	- Time is discrete
- Signal is defined for all values of time t.	- Signal is defined for discrete interval of time t.
- Independent variable is time t.	- Independent variable is sample, denoted by n.
- graphical representation	- graphical representation





Z-transform and Inverse Z transform

- The Z-transform of a discrete time signal $x(n)$ is represented by $X(z)$

$$x(n) \xleftrightarrow{ZT} X(z)$$

- $$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

- $x(t) \longrightarrow$ freq $\longrightarrow$ F.T., L.T.
  $x(n) \longrightarrow$ freq $\longrightarrow$ D.F.T, Z.T

- $$\text{F.T.}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- $$\text{D.F.T.}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

- Relation betn DFT & Z.T.

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\boxed{z = re^{j\omega}}$$

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \underline{x(n)\, r^{-n}}\; \underline{e^{-j\omega n}}$$

$$= \text{DFT}[x(n)\, r^{-n}]$$

$$\text{DFT}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, \underline{e^{-j\omega n}}$$

- Inverse Z-transform

$\Rightarrow X(z) = \text{DFT}[x(n)\, r^{-n}]$

$\Rightarrow \text{IDFT}[X(z)] = x(n)\, r^{-n}$

$\Rightarrow x(n) = r^{n}\,[\text{IDFT}(X(z))]$

$= r^{n}\left[\frac{1}{2\pi}\int X(z)\, e^{j\omega n}\, d\omega\right]$

$= \frac{1}{2\pi}\int X(z)\,(re^{j\omega})^{n}\, d\omega$

$\rightarrow z = re^{j\omega}$

$\Rightarrow z = re^{j\omega}$

$\Rightarrow dz = re^{j\omega}\, j\, d\omega$

$\Rightarrow dz = jz\, d\omega$

$\Rightarrow d\omega = \frac{dz}{jz}$

$\Rightarrow x(n) = \frac{1}{2\pi}\int X(z)\, z^{n}\, \frac{dz}{jz}$

$\Rightarrow \boxed{x(n) = \frac{1}{2\pi j}\int X(z)\, z^{n-1}\, dz}$

Z-transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$ find $X(z)$ of given sequence.

$$x(n) = \{\overset{x(-2)}{5}, \overset{x(-1)}{3}, \underset{\uparrow}{\overset{x(0)}{-3}}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2}\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$ Find Z-transform of $x(n)$.

$$x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4}\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Examples of Inverse Z transform

Find the Signal $x(n)$ for which the z transform is

$$X(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$$

- $x(n) = z^{-1}[x(z)] = \boxed{\frac{1}{2\pi j}\int x(z)\, z^{n-1}\, dz}$

$$x(z) = \underset{x(-4)}{4z^4} - \underset{x(-3)}{1z^3} + \underset{x(-2)}{0z^2} - \underset{x(-1)}{3z} + \underset{x(0)}{0z^0} + \underset{x(1)}{4z^{-1}} + \underset{x(2)}{3z^{-2}}$$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-4}^{2} x(n)\, z^{-n}$$

- $\boxed{x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}}$

If $x(z) = 2z^2 + 3$, find $x(n)$

- $x(z) = \underset{-2}{2z^2} + \underset{-1}{0 \times z^1} + \underset{0}{3 \times z^0}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-2}^{0} x(n)\, z^{-n}$$

- $x(n) = \{2, 0, \underset{\uparrow}{3}\}$

If $x(z) = 3 + 2z^{-1} + 3z^{-3}$, find $x(n)$

- $x(z) = \underset{x(0)}{3 \times z^0} + \underset{x(1)}{2 \times z^{-1}} + \underset{x(2)}{0 \times z^{-2}} + \underset{x(3)}{3 \times z^{-3}}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{3} x(n)\, z^{-n}$

- $x(n) = \{\underset{\uparrow}{3}, 2, 0, 3\}$

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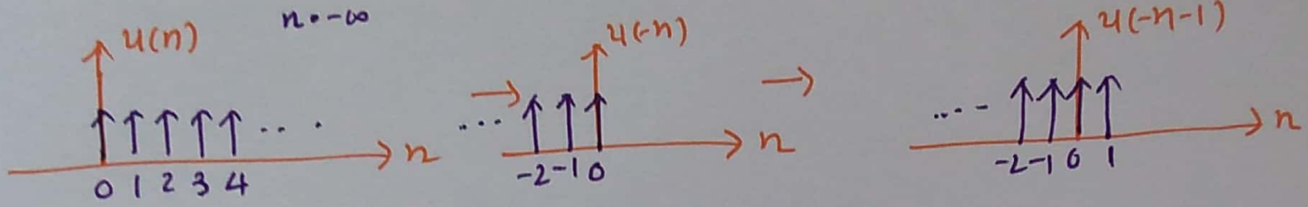
Z transform of standard basic signal.

- Find the Z-transform of the sequence

$$x(n) = a^{-n} u(-n-1)$$

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(-n-1) z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{1} a^{-n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n}$$

- reversal of limits by $n = -n$

$$X(z) = \sum_{n=1}^{\infty} (az)^{n}$$

$$= \sum_{n=0}^{\infty} (az)^{n} - (az)^{0}$$

$$= \sum_{n=0}^{\infty} (az)^{n} - 1$$

- If $\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$

- $$X(z) = \frac{1}{1-az} - 1$$

- $$X(z) = \frac{1-1+az}{1-az} = \boxed{\frac{az}{1-az}}$$

$x(n) = a^{-n} u(-n-1)$

$$x(n) = \left(\frac{1}{3}\right)^{-n} u(-n-1)$$

$$X(z) = \frac{(1/3) z}{1-(1/3) z}$$

$$= \boxed{\frac{z}{3-z}}$$

Z - Transform of Standard basic Signals

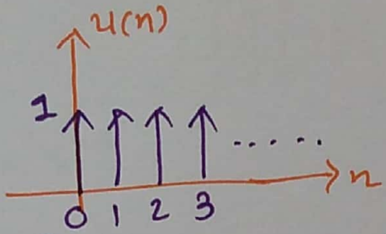
- Find The Z - transform of the signals $a^n u(n)$ & $a^{-n} u(n)$

- $x(n) = a^n u(n)$

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$



- $X(z) = \sum_{n=0}^{\infty} (a^n)(z^{-n})$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

We know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$x(n) = a^n u(n)$

$y(n) = a^{-n} u(n)$

- $Y(z) = Z.T.[y(n)]$

$$= \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(n) z^{-n}$$

- $Y(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n}$

$$= \sum_{n=0}^{\infty} (az)^{-n}$$

- We know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

- $Y(z) = \frac{1}{1-(az)^{-1}} = \frac{1}{1-1/az}$

$$= \boxed{\frac{az}{az-1}}$$

$y(n) = a^{-n} u(n)$

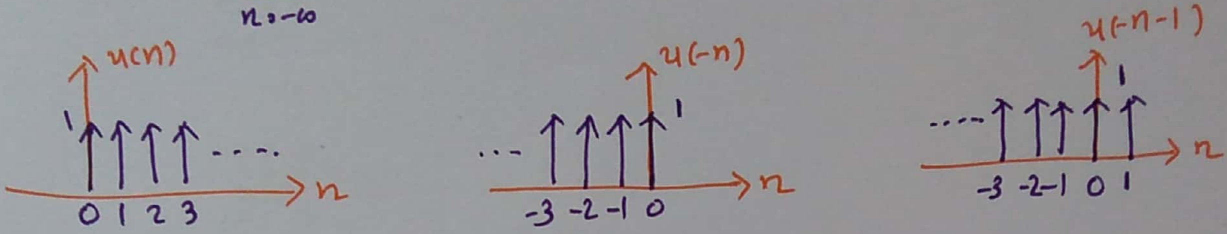
Z- Transform of standard basic signal

Find the Z- transform of the Sequence $x(n) = -a^n u(-n-1)$

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

- To swept limits place $n = -n$.

$$X(z) = -\sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= -\sum_{n=1}^{\infty} (za^{-1})^n$$

$$= -\left[\sum_{n=0}^{\infty} (za^{-1})^n - 1 \right]$$

$$= -\left[\frac{1}{1-za^{-1}} - 1 \right]$$

$$= -\left[\frac{a}{a-z} - 1 \right]$$

$$= -\left[\frac{\cancel{a}-\cancel{a}+z}{a-z} \right]$$

$$= \boxed{\frac{z}{z-a}}$$

$\uparrow$ $x(n) = -a^n u(-n-1)$

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$a^{-n} u(n) \xleftrightarrow{ZT} \frac{az}{az-1}$$

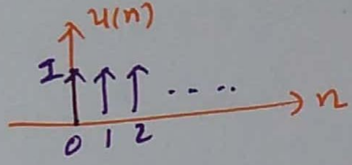
$$-a^{-n} u(-n-1) \xleftrightarrow{ZT} \frac{az}{az-1}$$

Z- Transform of unit step function

- $x(n) = u(n)$
- $X(z) = Z.T\,[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$



$$\rightarrow X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$\rightarrow \sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$$

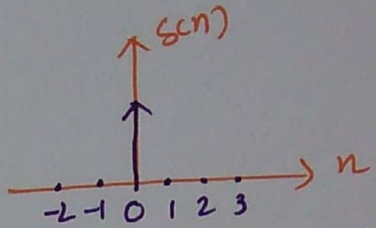
$$\rightarrow X(z) = \frac{1}{1-z^{-1}} = \frac{1}{1-1/z}$$

$$= \boxed{\frac{z}{z-1}}$$

Z-Transform of Unit Impulse function

- $x(n) = \delta(n)$

- $X(z) = Z.T.[x(n)]$



$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

$$= \ldots + \delta(-2) z^{2} + \delta(-1) z^{1} + \underline{\delta(0)} z^{0} + \delta(1) z^{-1} + \delta(2) z^{-2} + \ldots$$

$$= 0 \times z^{2} + 0 \times z^{1} + 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \ldots$$

$$= \boxed{1}$$

Z Transform of Cosine Signal

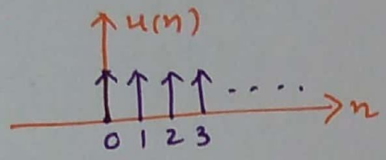
Find Z - Transform of $\cos(n\omega)\, u(n)$

- $x(n) = \cos(n\omega)\, u(n)$

- $X(z) = \text{Z.T.}\,[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \cos(n\omega)\, \underline{u(n)}\, z^{-n}$$



- $$X(z) = \sum_{n=0}^{\infty} \cos(n\omega)\, z^{-n}$$

- $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

- $$X(z) = \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2} \right] z^{-n}$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - e^{j\omega}/z} + \frac{1}{1 - e^{-j\omega}/z} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\omega}} + \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2} \left[\frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{z^2 - z e^{j\omega} - z e^{-j\omega} + 1} \right]$$

$$= \frac{1}{2} \left[\frac{2z^2 - 2z\,[e^{-j\omega} + e^{j\omega}]/2}{z^2 - 2z\,[e^{j\omega} + e^{-j\omega}]/2 + 1} \right]$$

$$= \frac{1}{\cancel{2}} \left[\frac{\cancel{2}z^2 - \cancel{2}z \cos\omega}{z^2 - 2z\cos\omega + 1} \right]$$

$$\boxed{= \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}} \leftarrow \cos(n\omega)\, u(n)$$

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Z- Transform of Cosine Signal

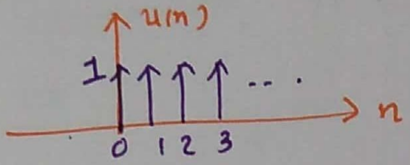
Find Z- Transform of $a^n \cos(n\omega)\, u(n)$

- $x(n) = a^n \cos(n\omega)\, u(n)$

- $X(z) = \text{Z.T.}\,[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n \cos(n\omega)\, \underline{u(n)}\, z^{-n}$$



- $X(z) = \sum_{n=0}^{\infty} a^n \cos(n\omega)\, z^{-n}$

- $\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$

- $X(z) = \sum_{n=0}^{\infty} a^n z^{-n} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2} \right]$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} a^n z^{-n} e^{j\omega n} + \sum_{n=0}^{\infty} a^n z^{-n} e^{-j\omega n} \right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (a z^{-1} e^{j\omega})^n + \sum_{n=0}^{\infty} (a z^{-1} e^{-j\omega})^n \right]$$

$$= \frac{1}{2}\left[\frac{1}{1 - \frac{a e^{j\omega}}{z}} + \frac{1}{1 - \frac{a e^{-j\omega}}{z}} \right]$$

$$= \frac{1}{2}\left[\frac{z}{z - a e^{j\omega}} + \frac{z}{z - a e^{-j\omega}} \right]$$

$$= \frac{1}{2}\left[\frac{z^2 - a z e^{-j\omega} + z^2 - a z e^{j\omega}}{z^2 - a z e^{j\omega} - a z e^{-j\omega} + a^2} \right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2az[e^{-j\omega} + e^{j\omega}]/2}{z^2 - 2az\,[e^{-j\omega} + e^{j\omega}]/2 + a^2} \right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2} z^2 - \cancel{2} a z \cos\omega}{z^2 - 2az\cos\omega + a^2} \right]$$

$$= \boxed{\frac{z^2 - az\cos\omega}{z^2 - 2az\cos\omega + a^2}}$$

$x(n) = a^n \cos(n\omega)\, u(n)$

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Z - Transforms of fundamental Signals

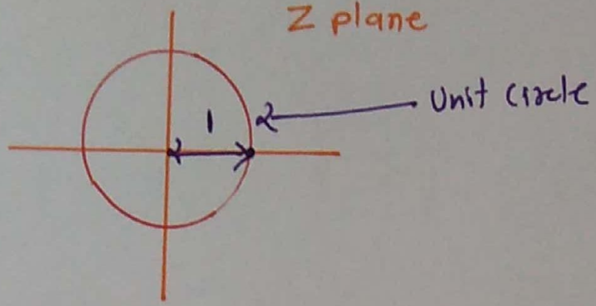
$x(n)$		$X(z)$
$\delta(n)$	$\xleftrightarrow{ZT}$	1
$u(n)$	$\xleftrightarrow{ZT}$	$\frac{z}{z-1}$ or $\frac{1}{1-z^{-1}}$
$-u(-n-1)$	$\xleftrightarrow{ZT}$	$\frac{z}{z-1}$ or $\frac{1}{1-z^{-1}}$
$a^n u(n)$	$\xleftrightarrow{ZT}$	$\frac{z}{z-a}$ or $\frac{1}{1-az^{-1}}$
$-a^n u(-n-1)$	$\xleftrightarrow{ZT}$	$\frac{z}{z-a}$ or $\frac{1}{1-az^{-1}}$
$a^{-n} u(n)$	$\xleftrightarrow{ZT}$	$\frac{az}{az-1}$ or $\frac{1}{1-(az)^{-1}}$
$-a^{-n} u(-n-1)$	$\xleftrightarrow{ZT}$	$\frac{az}{az-1}$ or $\frac{1}{1-(az)^{-1}}$
$\delta(n-m)$	$\xleftrightarrow{ZT}$	z^{-m}
$a^n \cos(\omega n)\, u(n)$	$\xleftrightarrow{ZT}$	$\frac{z^2 - az\cos\omega}{z^2 - 2az\cos\omega + a^2}$
$a^n \sin(\omega n)\, u(n)$	$\xleftrightarrow{Zt}$	$\frac{az\sin\omega}{z^2 - 2az\cos\omega + a^2}$

$u(n) - -u(-n-1)$

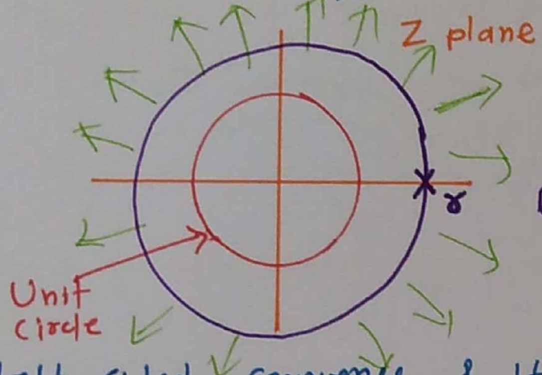
$a^n u(n) \rightarrow \frac{z}{z-a}$

Region of Convergence of Z-Transform

- The ROC in Z-tranform is indicated as circles contain unit circle



- If $x(n)$ is a right sided sequence & if $|z| = r$ in the ROC then all finite value of z, for which $|z| > r$ will also be in ROC

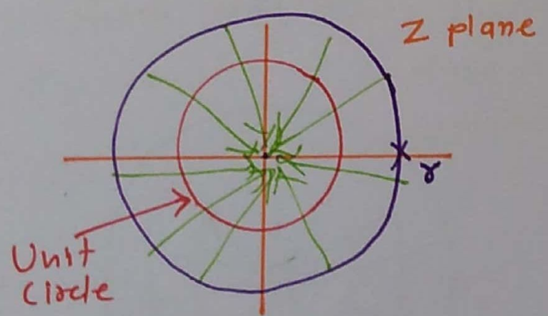


$x(n) = f(n)\,\underline{\underline{u(n)}}$

$x(n) \longleftrightarrow X(z) = \frac{z}{z-r}$

ROC : $|z| > r$

- If $x(n)$ is left sided sequence & if $|z| = r$ in the ROC, then all finite value of z, for which $|z| < r$ will also be in ROC

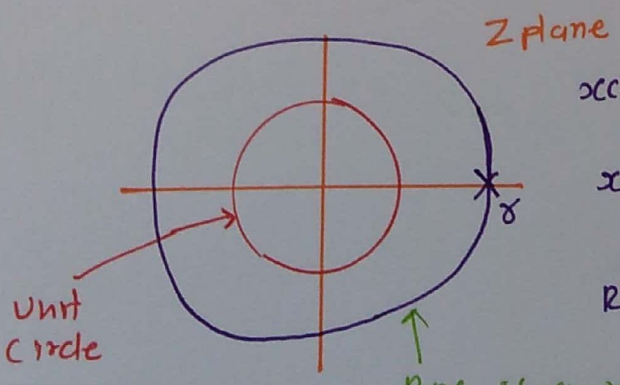


$x(n) = f(n)\,\underline{\underline{u(-n)}}$

$x(n) \longleftrightarrow X(z) = \frac{z}{z-r}$

ROC : $|z| < r$

- If $x(n)$ is two sided signal & if $|z| = r$ circle is in ROC, then the ROC will contain a ring in Z plane that include $|z| = r$



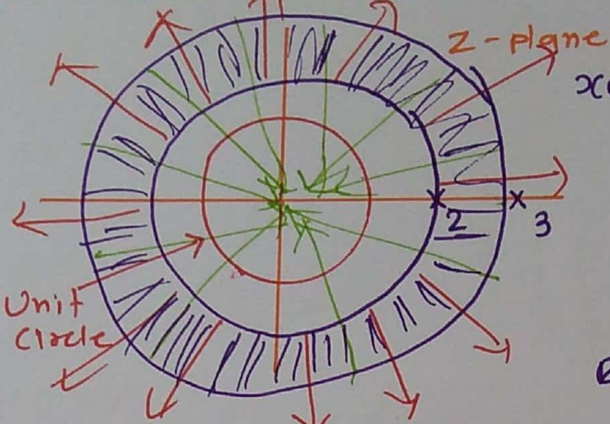
$x(n) = f_1 u(n) + f_2 u(-n)$

$x(n) \longleftrightarrow X(z) = \frac{z}{z-r}$

ROC : $|z| = r$

ROC is a ring of radius r.

- If the Z-Transform of $x(n)$ is rational, then its ROC is bounded by poles or extended to ∞



$$x(n) = 2^n u(n) - 3^n u(-n-1)$$

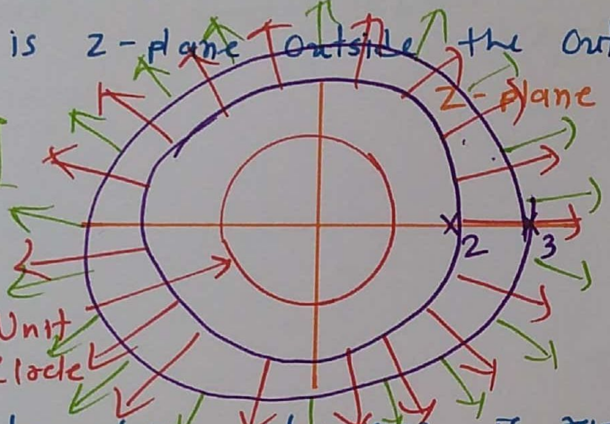
$$\frac{z}{z-2} \qquad \frac{z}{z-3}$$

$$|z| > 2 \qquad |z| < 3$$

ROC: $2 < |z| < 3$

- If the Z-transform of $x(n)$ is rational & Right sided then ROC is Z-plane outside the outer most pole.

ROC: $|z| > 3$

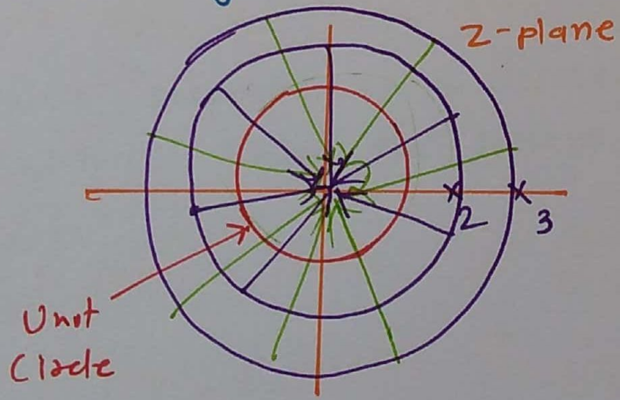


$$x(n) = 2^n u(n) + 3^n u(n)$$

$$\frac{z}{z-2} \qquad \frac{z}{z-3}.$$

$$|z| > 2 \qquad |z| > 3.$$

- If the Z-transform of $x(n)$ is rational & Left sided Then ROC is the region in Z-plane inside the inner most pole.



$$x(n) = -2^n u(-n-1) - 3^n u(-n-1)$$

$$\frac{z}{z-2} \qquad \frac{z}{z-3}.$$

$$|z| < 2 \qquad |z| < 3$$

ROC: $|z| < 2$

GATE Example of Z Transform

The Z - Transform of the System is $H(z) = \frac{z}{z-0.2}$.
If the ROC is $|z| < 0.2$, Then the Impulse response of the system is

(a) $(0.2)^n u(n)$ (b) $(0.2)^n u(-n-1)$ (c) $-(0.2)^n u(n)$ (d) $-(0.2)^n u(-n-1)$ ✓

- $H(z) = \frac{z}{z-0.2}$, ROC : $|z| < 0.2$ — Left sided Sequence.

- $\frac{z}{z-a} \longleftrightarrow a^n u(n) \longrightarrow h(n) = (0.2)^n u(n)$

- $\frac{z}{z-a} \longleftrightarrow -a^n u(-n-1) \longrightarrow h(n) = -(0.2)^n u(-n-1)$

GATE Example of Z - Transform

The ROC is $|z| < 0.2$, Then the impulse response of the system is

(a) $(0.2)^n u(n)$ (b) $(0.2)^n u(-n-1)$ (c) $-(0.2)^n u(n)$

(d) $-(0.2)^n u(-n-1)$

Initial & Final Value Theorm of Z Transform

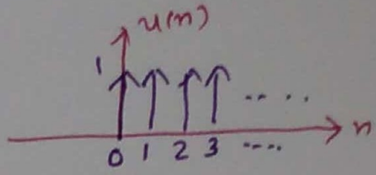
- Initial value $x(0)$ in Z Transform is given by

$$x(0) = \lim_{z \to \infty} X(z)$$

- Final value $x(\infty)$ in Z Transform is given by

$$x(\infty) = \lim_{z \to 1} (z-1)X(z)$$

- For unit step signal calculate $x(0)$ & $x(\infty)$.



$$u(n) \xleftrightarrow{ZT} \frac{z}{z-1}$$

- $X(z) = \dfrac{z}{z-1}$

- $x(0) = \lim\limits_{z \to \infty} X(z) = \lim\limits_{z \to \infty} \dfrac{z}{z-1} = \lim\limits_{z \to \infty} \dfrac{1}{1-1/z} = \dfrac{1}{1-0} = 1$

- $x(\infty) = \lim\limits_{z \to 1} (z-1)X(z) = \lim\limits_{z \to 1} \cancel{(z-1)}\left(\dfrac{z}{\cancel{z-1}}\right) = \lim\limits_{z \to 1} z = 1$

Properties of Z-Transform

- linearity property .
  - $x_1(n) \xleftrightarrow{ZT} X_1(z)$
  - $x_2(n) \xleftrightarrow{ZT} X_2(z)$

  $\Big\} \longrightarrow a_1x_1(n) + a_2x_2(n) \xleftrightarrow{ZT} a_1X_1(z) + a_2X_2(z)$

- Time shifting property
  - $x(n) \xleftrightarrow{ZT} X(z)$
  - $x(n-m) \xleftrightarrow{ZT} z^{-m}X(z)$

- Time reversal property
  - $x(n) \xleftrightarrow{ZT} X(z)$
  - $x(-n) \xleftrightarrow{ZT} X(z^{-1})$

- ~~Time~~ Scaling property
  - $x(n) \xleftrightarrow{ZT} X(z)$
  - $a^n x(n) \xleftrightarrow{ZT} X(z/a)$

- Differenciation property
  - $x(n) \xleftrightarrow{ZT} X(z)$
  - $n\,x(n) \xleftrightarrow{ZT} -z\dfrac{dX(z)}{dz}$

linearity property of Z-Transform

- If $x_1(n) \xleftrightarrow{ZT} X_1(z)$

$x_2(n) \xleftrightarrow{ZT} X_2(z)$

Then linearity property states that

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{ZT} a_1 X_1(z) + a_2 X_2(z).$$

Proof

- $x(n) = a_1 x_1(n) + a_2 x_2(n)$

- $X(z) = \text{Z.T.}[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) z^{-n} + a_2 x_2(n) z^{-n}]$$

$$= a_1 \underline{\sum_{n=-\infty}^{\infty} x_1(n) z^{-n}} + a_2 \underline{\sum_{n=-\infty}^{\infty} x_2(n) z^{-n}}$$

$$= \boxed{a_1 X_1(z) + a_2 X_2(z)}$$

Time shifting property of Z transform

- If $x(n) \xleftrightarrow{ZT} X(z)$
- The time shifting property state that

$$x(n-m) \xleftrightarrow{ZT} z^{-m} X(z)$$

$$x(n+m) \xleftrightarrow{ZT} z^{m} X(z)$$

<u>Proof</u>

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $Z.T.[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$

- If $n-m=p \Rightarrow n=p+m$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} z^{-m}$$

$$= z^{-m} \underline{\sum_{p=-\infty}^{\infty} x(p) z^{-p}}$$

$$= \boxed{z^{-m} X(z)} \xleftrightarrow{ZT} x(n-m)$$

<u>example</u> find Z-Transform of $\delta(n-k)$

$$\delta(n) \xleftrightarrow{ZT} 1$$

$$\delta(n-k) \longleftrightarrow \boxed{z^{-k}}$$

Time reversal property of Z Transform

- If $x(n) \xleftrightarrow{ZT} X(Z)$

Then time reversal property states that

$$x(-n) \xleftrightarrow{ZT} X(Z^{-1})$$

Proof

- $X(Z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

- $Z.T[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$

If $p = -n$

$$= \sum_{p=-\infty}^{\infty} x(p) Z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (Z^{-1})^{-p}$$

$$= X(Z^{-1})$$

e.g. $x(n) = a^n u(-n)$. find $X(Z)$

- $x(n) = a^n u(-n)$

$$= (1/a)^{-n} u(-n)$$

$$x(n) = (1/a)^n u(n) \xleftrightarrow{ZT} \frac{Z}{Z - 1/a}$$

$$(1/a)^{-n} u(-n) \xleftrightarrow{ZT} \frac{Z^{-1}}{Z^{-1} - 1/a} = \frac{1/Z}{1/Z - 1/a} = \boxed{\frac{a}{a-Z}}$$

Scaling property of Z Transform

- $x(n) \xleftrightarrow{ZT} X(z)$

Then Scaling property of Z transform states that

$a^n x(n) \xleftrightarrow{ZT} X[z/a]$

Proof

$$X(z) = Z.T.[x(n)]$$
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$
$$Z.T.[a^n (x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} x(n) (az^{-1})^{n}$$
$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{az^{-1}}\right)^{-n}$$
$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$
$$= \boxed{X(z/a)}$$

e.g. $x(n) = a^n u(n)$, find $X(z)$

$$u(n) \xleftrightarrow{ZT} z/z-1$$
$$a^n u(n) \xleftrightarrow{ZT} \frac{(z/a)}{\left(\frac{z}{a}\right)-1} = \frac{z}{z-a}$$

Differentiation property of Z Transform

- If $x(n) \xleftrightarrow{ZT} X(z)$

Then differentiation property states that

$$n\,x(n) \xleftrightarrow{ZT} -z\frac{dX(z)}{dz}$$

e.g. $X(z) = \log(1 + az^{-1})$, $|z| > |a|$, find $x(n) = ?$

$$\Rightarrow \frac{dX(z)}{dz} = \frac{d}{dz}\log(1+az^{-1})$$

$$= \frac{1}{1+az^{-1}}\left(0 + az^{-2}(-1)\right)$$

$$= \frac{-az^{-2}}{1+az^{-1}}$$

$$\Rightarrow -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}} = \frac{a}{z+a} \xleftrightarrow{I.Z.T} \underline{n\,x(n)}$$

$$\Rightarrow a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$\Rightarrow (-a)^n u(n) \xleftrightarrow{ZT} \frac{z}{z+a}$$

$$\Rightarrow (-a)^{n-1} u(n-1) \xleftrightarrow{ZT} z^{-1}\left(\frac{z}{z+a}\right) = \frac{1}{z+a}$$

$$\Rightarrow \underline{(-a)^n u(n-1)} \xleftrightarrow{ZT} \frac{a}{z+a}$$

$$\Rightarrow n\,x(n) = (-a)^n u(n-1)$$

$$\Rightarrow \boxed{x(n) = \frac{(-a)^n}{n} u(n-1)}$$

Multiplication & Convolution property of Z-Transform

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$

$x_2(n) \xleftrightarrow{ZT} X_2(z)$

- Multiplication property states that

$$x_1(n)\, x_2(n) \xleftrightarrow{ZT} X_1(z) * X_2(z)$$

- convolution property states that

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z).$$

e.g. $u(n-1) * \delta(n)$, find Z-Transform

- $x_1(n) = u(n-1) \xleftrightarrow{ZT} X_1(z) = \frac{1}{z-1}$

- $x_2(n) = \delta(n) \xleftrightarrow{ZT} X_2(z) = 1$

- $X(z) = ZT[x_1(n) * x_2(n)]$

$= X_1(z)\, X_2(z)$

$= \frac{1}{z-1}$

e.g. $u(n-2) * \delta(n-3)$, find Z-Transform.

- $x_1(n) = u(n-2) \xleftrightarrow{ZT} X_1(z) = z^{-2}\left(\frac{z}{z-1}\right) = \frac{1}{z}\left(\frac{1}{z-1}\right)$

- $x_2(n) = \delta(n-1) \xleftrightarrow{ZT} X_2(z) = z^{-1} = 1/z$

- $X(z) = ZT[x_1(n) * x_2(n)]$

$= X_1(z)\, X_2(z)$

$= \frac{1}{z}\left(\frac{1}{z-1}\right)\left(\frac{1}{z}\right)$

$= \frac{1}{z^2(z-1)}$

Example based on Initial Value theorm

The Z-Transform of a anticausal System is

$$X(z) = \frac{3-4z}{1-2z+5z^2}$$

The Value of $x(0)$ is

(a) 3 (b) 0 (c) $-4/5$ (d) $1/5$

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= \lim_{z\to\infty} \left[\frac{3-4z}{1-2z+5z^2}\right]$$

$$= \lim_{z\to\infty} \left(\frac{z}{z^2}\right)\left[\frac{3/z-4}{1/z^2-2/z+5}\right]$$

$$x(0) = 0\left[\frac{0-4}{0-0+5}\right]$$

$$\boxed{x(0) = 0}$$

Example based on Final Value theorm

The Z Transform of the Signal given by $C(z) = \frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}$

It's final Value is given by

(a) 4 (b) 0 (c) 1 (d) ∞

$$C(\infty) = \lim_{z\to 1} (z-1)\,C(z)$$

$$= \lim_{z\to 1} (z-1)\left[\frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}\right]$$

$$= \lim_{z\to 1} (z-1)\left[\frac{1/z\,(1-1/z^4)}{4(1-1/z)^2}\right]$$

$$= \lim_{z\to 1} \cancel{(z-1)}\frac{(z^4-1)}{4(z-1)^{\cancel{2}}}\,\frac{z^2}{z^5}$$

$$= \lim_{z\to 1} \frac{z^4-1}{4z^3(z-1)}$$

$$C(\infty) = \lim_{z\to 1} \frac{(z^2-1)(z^2+1)}{4z^3(z-1)}$$

$$= \lim_{z\to 1} \frac{(z+1)(z^2+1)}{4z^3}$$

$$= \frac{2\times 2}{4}$$

$$\boxed{= 1}$$

Example of Z Transform using differentiation property

e.g. Find the Inverse Z-Transform of $\log\left(\frac{1}{1-a^{-1}z}\right)$, $|z|<|a|$

↑ Left sided signal.

If $x(n) \xleftrightarrow{ZT} X(z)$

$n\,x(n) \xleftrightarrow{ZT} -z\frac{d\,X(z)}{dz}$

- $X(z) = \log\left(\frac{1}{1-a^{-1}z}\right) = \log(1-a^{-1}z)^{-1} = -\log(1-a^{-1}z)$

$$\Rightarrow \frac{dX(z)}{dz} = \frac{d}{dz}\left[-\log(1-a^{-1}z)\right]$$

$$= \frac{-1}{1-a^{-1}z}\left[0-a^{-1}(1)\right]$$

$$\Rightarrow \frac{d\,X(z)}{dz} = \frac{a^{-1}}{1-a^{-1}z}$$

$$\Rightarrow -z\frac{d\,X(z)}{dz} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{-z/a}{1-z/a} = \frac{-z}{a-z} = \frac{z}{z-a}$$

$$\Rightarrow -z\frac{dX(z)}{dz} = \frac{z}{z-a}$$

$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$ ×

$-a^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$ ✓

- take Inverse Z-Transform at both sides.

$$\Rightarrow n\,x(n) = -a^n u(-n-1)$$

$$\Rightarrow \boxed{x(n) = \frac{-a^n}{n}u(-n-1)}$$

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Example of Inverse Z Transform using differentiation property

e.g. Find The Inverse Z-Transform of $\log(1-az^{-1})$

$$\text{If } x(n) \xleftrightarrow{ZT} X(z)$$

$$n x(n) \xleftrightarrow{ZT} -z\frac{dX(z)}{dz}$$

$$\Rightarrow \frac{dX(z)}{dz} = \frac{d\left(\log(1-az^{-1})\right)}{dz}$$

$$= \frac{1}{1-az^{-1}}\left[0 - az^{-2}(-1)\right]$$

$$\Rightarrow \frac{dX(z)}{dz} = \frac{az^{-2}}{1-az^{-1}}$$

$$\Rightarrow -z\frac{dX(z)}{dz} = -\cancel{z}\left[\frac{az^{-2}}{1-az^{-1}}\right] = -\left[\frac{az^{-1}}{1-az^{-1}}\right] = -\left[\frac{a/z}{1-a/z}\right]$$

$$\Rightarrow -z\frac{dX(z)}{dz} = -\frac{a}{z-a}$$

$$\Rightarrow a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$\Rightarrow a^{(n-1)} u(n-1) \xleftrightarrow{ZT} z^{-1}\left[\frac{\cancel{z}}{z-a}\right] = \left[\frac{1}{z-a}\right]$$

$$\Rightarrow -a\,a^{(n-1)} u(n-1) \xleftrightarrow{ZT} \frac{-a}{z-a}$$

→ Apply Inverse Z-Transform at both sides.

$$\Rightarrow n x(n) = -a^n u(n-1)$$

$$\Rightarrow \boxed{x(n) = -\frac{a^n}{n} u(n-1)} \xleftrightarrow{ZT} \log(1-az^{-1})$$

Example of Z Transform using Convolution property of Z transform

- Find the Convolution of $x_1(n)$ & $x_2(n)$ using property of Z-transform

$$x_1(n) = \{\underset{\uparrow}{1}, 3, 3, 1\}$$

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 2 & 4 \le n \le 5 \\ 0 & \text{else.} \end{cases}$$

$$x(n) = x_1(n) * x_2(n) \overset{ZT}{\longleftrightarrow} X_1(z)X_2(z)$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$
$$= 1\times z^{0} + 3\times z^{-1} + 3\times z^{-2} + 1\times z^{-3}$$
$$= 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$
$$= 1\times z^{0} + 1\times z^{-1} + 1\times z^{-2} + 1\times z^{-3} + 2\times z^{-4} + 2\times z^{-5}$$
$$= 1 + z^{-1} + z^{-2} + z^{-3} + 2z^{-4} + 2z^{-5}$$

- $X(z) = X_1(z)X_2(z)$

$$= [1 + 3z^{-1} + 3z^{-2} + z^{-3}][1 + z^{-1} + z^{-2} + z^{-3} + 2z^{-4} + 2z^{-5}]$$
$$= 1 + 3z^{-1} + 3z^{-2} + z^{-3} + z^{-1} + 3z^{-2} + 3z^{-3} + z^{-4} + z^{-2} + 3z^{-3} + 3z^{-4} + z^{-5}$$
$$+ z^{-3} + 3z^{-4} + 3z^{-5} + z^{-6} + 2z^{-4} + 6z^{-5} + 6z^{-6} + 2z^{-7} +$$
$$2z^{-5} + 6z^{-6} + 6z^{-7} + 2z^{-8}$$
$$= 1 + 4z^{-1} + 7z^{-2} + 8z^{-3} + 9z^{-4} + 12z^{-5} + 13z^{-6} + 8z^{-7} + 2z^{-8}$$

- from co-efficient we can get $x(n)$

- $x(n) = \{\underset{\uparrow}{1}, 4, 7, 8, 9, 12, 13, 8, 2\}$

- $x_1(n) = \{\underset{\uparrow}{1}, 3, 3, 1\}$

$x_2(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 2, 2\}$

	1	1	1	1	2	2
1	1	1	1	1	2	2
3	3	3	3	3	6	6
3	3	3	3	3	6	6
1	1	1	1	1	2	2

$$x(n) = x_1(n) * x_2(n)$$
$$= \{\underset{\uparrow}{1}, 4, 7, 8, 9, 12, 13, 8, 2\}$$

Transfer function & Impulse response using Z-Transform

Determine the transfer function and unit impulse response of system using Z transform described by eq.^{n}

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

- $H(z) = \frac{Y(z)}{X(z)}$ - $h(n) = IZT[H(z)]$

$$\Rightarrow y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

- take Z-transform of above eq.^{n}

$$\Rightarrow \cancel{X}Y(z) = \frac{1}{2}z^{-1}Y(z) + 2X(z)$$

$$\Rightarrow Y(z)(1 - \frac{1}{2}z^{-1}) = 2X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$\Rightarrow H(z) = \frac{2z}{z-1/2} = 2\left[\frac{z}{z-1/2}\right]$$

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$(1/2)^n u(n) \xleftrightarrow{ZT} \frac{z}{z-1/2}$$

- take Inverse Z Transform

$$\Rightarrow H(n) = 2(1/2)^n u(n) = \boxed{2^{(1-n)} u(n)}$$

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Impulse response of the System by Z Transform

The Sequence $x(n)$ with Z Transform $X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$ is applied as an Input to a linear time invariant system with impulse response $h(n) = 2\,\delta(n-3)$. The Output at $n=4$ is <u>0</u>

- $X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$
- $h(n) = 2\,\delta(n-3) \xleftrightarrow{ZT} H(z) = 2z^{-3}$
- $H(z) = \dfrac{Y(z)}{X(z)} \Rightarrow Y(z) = H(z)\,X(z)$

$$= 2z^{-3}[z^4 + z^2 - 2z + 2 - 3z^{-4}]$$

$$= 2z + 2z^{-1} - 4z^{-2} + 4z^{-3} - 6z^{-7}$$

→ O/p can be Calculated by given Sequence

$$y(n) = \{2, \underset{\uparrow}{0}, 2, -4, 4, \boxed{0}, 0, 0, -6\}$$

$$-1 \quad n=0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

ROC of discrete time sequence in Z Transform

1. If Sequence is purely right sided or causal, Then

   ROC : Entire Z plane except $z=0$

   $$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \xleftrightarrow{ZT} X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

2. If Sequence is purely Left sided or Anticausal, Then

   ROC : Entire Z plane except at $z=0$

   $$x(n) = \{1, 2, 3, \underset{\uparrow}{4}\} \xleftrightarrow{ZT} X(z) = z^{3} + 2z^{2} + 3z + 4$$

3. If Sequence is a two sided, Then

   ROC : Entire Z plane except at $z=0$ & $z=\infty$

   $$x(n) = \{1, 2, \underset{\uparrow}{1}, 3, 1\} \xleftrightarrow{ZT} X(z) = 2z^{2} + 2z + 1 + 3z^{-1} + z^{-2}$$

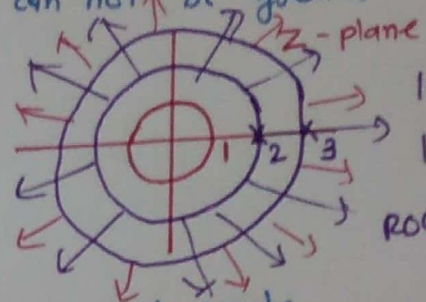
Causality & Stability check ROC of Z transform

* For Causality

1. The ROC is the exterior of the outside most pole.
2. If H(z) is expressed as ratio of polynomial in 'z', The Order of Numerator can not be greater than order of the denominator.

$H_1(z) = \frac{z^2+z+1}{z^3+z+1}$ ← Causal

$H_2(z) = \frac{z^4+1}{z^3+z^2+1}$ ×



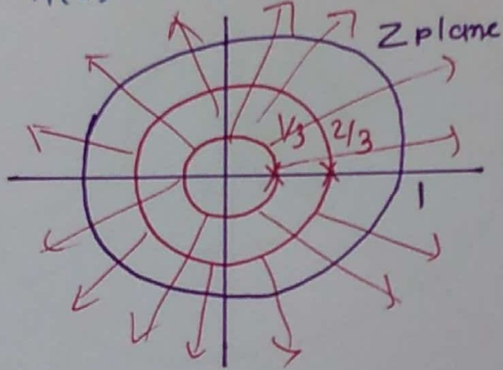
$|z| > 2$

$|z| > 3$

ROC: $|z| > 3$

* For Stability

1. System function H(z) includes unit circle.
2. A Causal LTI System is stable If and Only If all the Poles of H(z) Lies in inside the unit Circle.



$|z| > 1/3$

$|z| > 2/3$

ROC: $|z| > 2/3$

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Example of stability and causality from system response using ROC of Z Transform

- Check for stability & causality of a system with response $\left(\frac{1}{3}\right)^n u(n) + 2^n u(-n-1)$

- $h(n) = \left(\frac{1}{3}\right)^n u(n) + 2^n u(-n-1)$

$\quad\downarrow$ Right sided $\qquad \downarrow$ Left sided

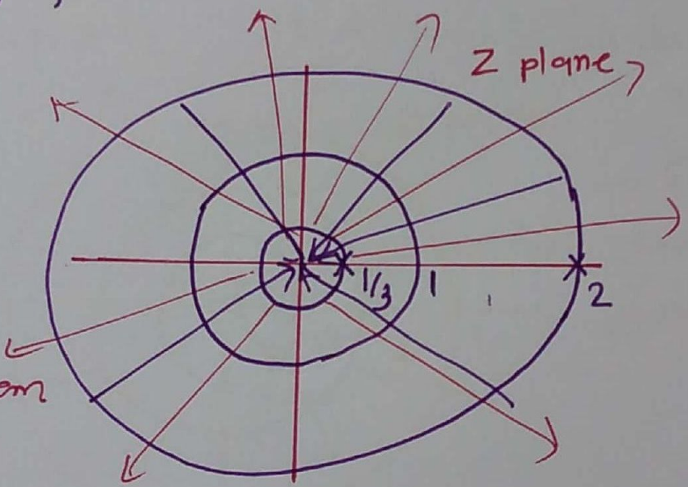
$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$

$-a^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$

- $H(z) = \frac{z}{z-1/3} - \frac{z}{z-2}$

$|z| > 1/3 \quad , \quad |z| < 2 \longrightarrow$ Common ROC $\boxed{\frac{1}{3} < |z| < 2}$

- Common ROC is not exterior to outer most pole.
- So, system is Anticausal system



- ROC includes unit circle.
- So system is stable.

Example of Stability and Causality from System response using ROC of Z Transform

- Check for stability & causality of a system with

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + 3^{-n} u(-n-1)$$

$$= \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(-n-1)$$

$\left(\frac{1}{2}\right)^n u(n)$ ↓ Right Sided

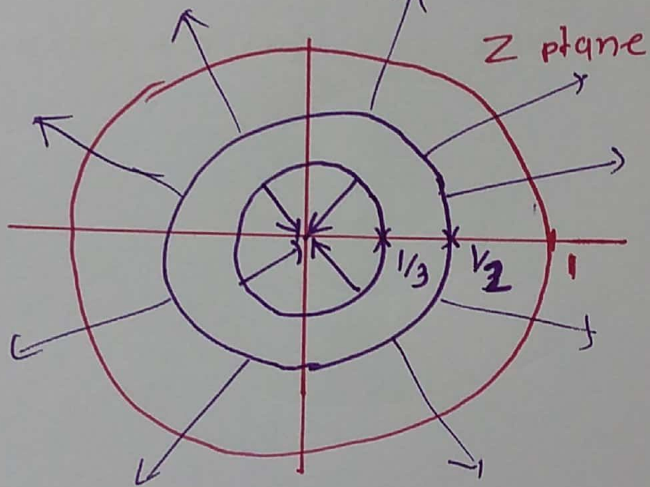
$\left(\frac{1}{3}\right)^n u(-n-1)$ ↓ Left Sided

$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$

$-a^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$

$$X(z) = \frac{z}{z-1/2} - \frac{z}{z-1/3}$$

$|z| > 1/2$ $|z| < 1/3$



→ Here there is no common ROC.

→ So stability and causality can be justified here.

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Example of ROC in Z-Transform

IF the region of convergence of $x_1(n) + x_2(n)$ is $\frac{1}{3} < |z| < \frac{2}{3}$, then the region of convergence of $x_1(n) - x_2(n)$ is

(a) $\frac{1}{3} < |z| < 3$ (b) $\frac{2}{3} < |z| < 3$ (c) $\frac{3}{3} < |z|\ 3$ (d) $\frac{1}{3} < |z| < \frac{2}{3}$ ✓

ROC: $\frac{1}{3} < |z| < \frac{2}{3}$

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$|z| > 1/3$, $|z| < 2/3$

$\downarrow$ $\downarrow$

$X_1(z) = \frac{z}{z - 1/3}$ $X_2(z) = \frac{z}{z - 2/3}$

$\downarrow$ $\downarrow$

$x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ $x_2(n) = -\left(\frac{2}{3}\right)^n u(-n-1)$

$\rightarrow x(n) = x_1(n) + x_2(n)$ $\rightarrow y(n) = x_1(n) - x_2(n)$

Stability of System from ROC of Z-Transform

A Linear discrete time system has the characteristic eq.^{n}

$$z^3 - 0.81z = 0$$

The system is

(a) Stable ✓

(b) Imaginary stable

(c) Unstable

(d) Stability can not be assumed from given Information

$\Rightarrow z^3 - 0.81z = 0$

$\Rightarrow z[z^2 - 0.81] = 0$

$\Rightarrow z[z-0.9][z+0.9] = 0$

→ $|z| = 0$, $z = 0.9$, $z = -0.9$

→ Here all poles are there inside the unit circle.

→ So system is stable.

Example of ROC of Z-Transform

Find the ROC $x(n) = 3^{|n|}$

- $x(n) = 3^{|n|}$

$$= 3^{n}u(n) + 3^{-n}u(-n-1)$$

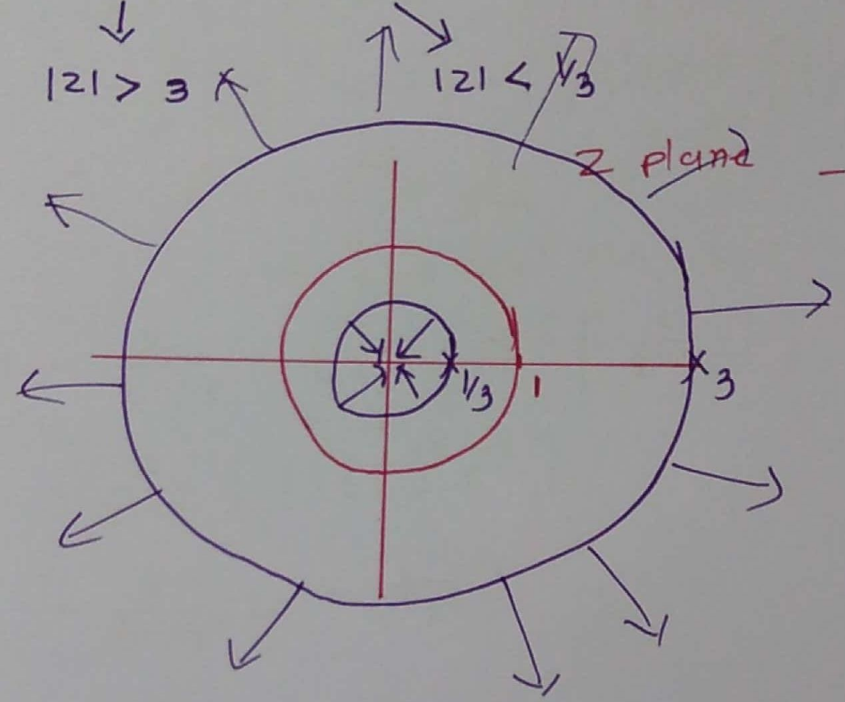
$$= 3^{n}u(n) + \left(\frac{1}{3}\right)^{n}u(-n-1)$$

$$a^{n}u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^{n}u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

- Z-transform

$$X(z) = \frac{z}{z-3} - \frac{z}{z-1/3}$$



- Here there is no common ROC in given eq^{n}

Unilateral Z - Transform

$$UZT[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} \quad , \quad ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

- $ZT[a^n \underline{u(n)}] = \frac{z}{z-a} \xrightarrow{UZT} UZT[a^n u(n)] = \frac{z}{z-a}$

- $Z.T[a^{-n} u(-n)] = \frac{z^{-1}}{z^{-1}-a} = \frac{1}{1-az} \longrightarrow UZT[a^{-n} u(-n)] = x(0) z^{-0}$

$$= a^{-0}$$

$$= 1$$

- e.g. Find UZT $x(n) = \{3, 4, \underset{\uparrow}{5}, 0, 1, 2\}$

$\times \boxed{-2 \quad -1}$ $n=0 \quad 1 \quad 2 \quad 3$ ✓

$$UZT[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5 + 0 \times z^{-1} + 1 \times z^{-2} + 2 \times z^{-3}$$

$$= 5 + z^{-2} + 2z^{-3}$$

$$ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 3z^2 + 4z + 5 + z^{-2} + 2z^{-3}$$

Long division Method to get Inverse Z Transform

- Find Inverse Z Transform using Long division method

$$x[z] = \frac{z}{z - 1/2}, \quad |z| > 1 \qquad x(z) = \frac{z}{-\frac{1}{2} + z} = \frac{z}{z - 1/2}$$

- $|z| > 1$ means system is causal.
- For causal system, take desending power of z to apply long division
  , take asending " " z^{-1} " "
- If $|z| < 1$ means system is anticausal.
- For anticausal system, take asending power of z to apply long division
  , " desending " " z^{-1} " "

- For causal system take desending power of z

$$x(z) = \frac{z}{z - 1/2}$$

$$
\begin{array}{rl}
 & 1 + (0.5)z^{-1} + (0.5)^2 z^{-2} + \dots \\
z - 0.5 \,\big)\! & z \\
 & -\; z - 0.5 \\
 & \quad\; + \\
\hline
 & 0.5 \\
 & -\; 0.5 - (0.5)^2 z^{-1} \\
 & \; + \\
\hline
 & (0.5)^2 z^{-1} \\
 & -\; (0.5)^2 z^{-1} - (0.5)^3 z^{-2} \\
 & \phantom{-\; (0.5)^2 z^{-1}}\; + \\
\hline
 & (0.5)^3 z^{-2}
\end{array}
$$

→ $x(z) = 1 + 0.5 z^{-1} + (0.5)^2 z^{-2} + (0.5)^3 z^{-3} + \dots$

→ From coefficient we can have Inverse z transform.

→ $x(n) = \{ \underset{\uparrow}{1}, (\frac{1}{2}), (\frac{1}{2})^2, (\frac{1}{2})^3, \dots \}$.

$= (\frac{1}{2})^n u(n)$

Direct method or Partial Fraction method to Calculate Inverse Z Transform

- Find Inverse Z Transform of

$$H(z) = \frac{z^2 + 2z}{z^2 - 3z + 2}$$

Step-1 $\rightarrow \frac{H(z)}{z}$

Step-2 $\rightarrow$ Partial Fraction expansion

Step-3 $\rightarrow \frac{H(z)}{z} \rightarrow$

Step-4 $\rightarrow$ take IZT

$$\Rightarrow H(z) = \frac{z^2 + 2z}{z^2 - 3z + 2} = \frac{z(z+2)}{z^2 - 3z + 2}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{z+2}{z^2 - 3z + 2} = \frac{z+2}{z^2 - 2z - z + 2} = \frac{z+2}{z(z-2) - 1(z-2)}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{z+2}{(z-2)(z-1)}$$

$\rightarrow$ Partial Fraction

$$\Rightarrow \frac{z+2}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$$

$$\Rightarrow z + 2 = A(z-1) + B(z-2)$$

If $z = 2$

$\Rightarrow 2 + 2 = A(2-1)$

$\Rightarrow A = 4$

- If $z = 1$

$\Rightarrow 1 + 2 = B(1-2)$

$\Rightarrow B = -3$

$$\Rightarrow \frac{H(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1}$$

$$\Rightarrow H(z) = 4\left[\frac{z}{z-2}\right] - 3\left[\frac{z}{z-1}\right]$$

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$\Rightarrow \boxed{H(n) = 4(2^n)\,u(n) - 3\,u(n)}$$

$$H(n) = \{\underset{\uparrow}{1}, 5, 13, 29, \dots\}$$

Residue Method to get Inverse Z-Transform

e.g. $X(z) = \dfrac{z^{18}}{(z-1/2)(z-1)(z-4)}$, $X(z)$ converges for $|z| = 1$

Find $x(n=-16) = ?$

$x(n) = \sum$ Residue of $X(z) \times X(z) \times z^{n-1} \Big|_{\text{at the pole of residue}}$.

$$x(n) = \cancel{(z-1/2)}\left[\frac{z^{18}}{\cancel{(z-1/2)}(z-1)(z-4)}\right] z^{n-1}\Bigg|_{\text{at } z=1/2} +$$

$$\cancel{(z-1)}\left[\frac{z^{18}}{(z-1/2)\cancel{(z-1)}(z-4)}\right] z^{n-1}\Bigg|_{\text{at } z=1} +$$

$$\cancel{(z-4)}\left[\frac{z^{18}}{(z-1/2)(z-1)\cancel{(z-4)}}\right] z^{n-1}\Bigg|_{\text{at } z=4}$$

$$x(n) = \underbrace{\frac{z^{18} z^{n-1}}{(z-1)(z-4)}}_{\times}\Bigg|_{\text{at } z=1/2} + \underbrace{\frac{z^{18} z^{n-1}}{(z-1/2)(z-4)}}_{\checkmark}\Bigg|_{\text{at } z=1} + \underbrace{\frac{z^{18} z^{n-1}}{(z-1/2)(z-1)}}_{\times}\Bigg|_{\text{at } z=4}$$

$$= \frac{1^{18}(1)^{n-1}}{(1/2)(-3)} = \frac{1^{n-1}}{(-3/2)} = -\frac{2}{3}\, 1^{n-1}$$

$\rightarrow x(n=-16) = -\dfrac{2}{3}\, 1^{-16-1} = -\dfrac{2}{3}$