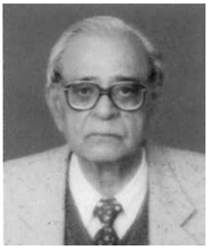
SIGNALS AND SYSTEMS

SECOND EDITION

About the Authors



I J Nagrath is Adjunct Professor BITS, Pilani, from where he retired in July 1999 as Professor of Electrical Engineering and Deputy Director. He is now actively engaged in writing books related to his long teaching and research experience. He obtained his BE with Honours in Electrical Engineering from Birla Engineering College in 1951, and MS from the University of Wisconsin, USA, in 1956.

He has co-authored several successful books which include *Electric Machines,* 2/e, *Modern Power System Analysis, Power System Engineering, System Modeling and Analysis* and *Electrical Machines,* 2/e (TMH Outline Series) and has authored *Basic Electrical Engineering* (all published by TMH). His other publications include books on Control System Engineering Electronics: Analog and Digital. He has also published several research papers in prestigious national and international journals.

S N Sharan, a PhD holder in Energy from IIT Delhi, has been associated with BITS, Pilani, for a very long period and has served in different capacities, such as Group Leader, and Electrical and Electronics Engineering Coordinator for the Centre for Renewable Energy and Environment Development. For the last couple of years he is the Director of Greater Noida Institute of Technology, Greater Noida. He has executed several sponsored research and development-based projects in the areas of Electronics and Energy and has also been involved in a couple of consultancy-based projects. He has published a little over 30 research and review papers in journals of international and national repute.

Rakesh Ranjan received his BE in Electrical Engineering from MIT Muzaffarpur, ME from BIT, Mesra, Ranchi, and PhD from BITS, Pilani.

Dr Ranjan has more than 18 years of teaching and research experience at Indian and foreign universities. He has served as Lecturer at BITS, Pilani, as an Assistant Professor and the HOD at Defence University College, Addis Ababa, Ethiopia, and as Faculty of Engineering and Technology, Multimedia University, Malaysia, Melaka Campus. Dr Ranjan joined the Institute of Technology and Management, Gurgaon, as Professor and was promoted to Dean-Academics. He also served as Principal from 2005–2009. Currently, he is Founder Director-Principal of International Institute of Technology and Business at Sonipat, Haryana.

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I J Nagrath
Adjunct Professor and former Deputy Director
BITS, Pilani

S N Sharan
Director, Greater Noida Institute of Technology
Greater Noida, UP

Rakesh Ranjan
Director, Principal
International Institute of Technology and Business
Sonipat, Haryana

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Contents

Preface

This book is designed for the undergraduates of Electrical, Electronics, Communication and Instrumentation Engineering. The coverage is in accordance with the syllabus of Signals and Systems offered as a course at various universities in India as well as abroad at the third year of a bachelor's degree course. All the topics in this book can be covered in one semester and in the same order.

A subject of this kind can be best learnt through exercises and worked-out examples, illustrating the key concepts. Therefore, there is a profusion of illustrative examples at the end of every topic and chapter-end examples which interlink the concepts covered so far. This book deals with continuous and discrete-time signals and linear time invariant (LTI) systems. It ends in the filtering theory and design of filters. The treatment is simple yet the rigour of the discipline is maintained throughout.

Since the publication of the book in 2001, revision had become due. The contents of this edition have been worked out and written after a study of the latest courseware of Signals and Systems of various universities, reviewer's comments and suggestions, and the authors' own vertical study and replanning. The major additions, deletions and alterations carried out are listed as under.

- The introductory chapter now covers all aspects of signals and systems classification and signal transformation. All other topics of the first edition have been shifted to appropriate later chapters.
- The basic design of the next two chapters (2 and 3) on the analysis of continuous and discrete-time systems has been maintained. The treatment of both time and frequency domain is placed in the same chapter, so that the comparison and applications of both domains are comprehensible for the reader.
- In Chapter 2, emphasis has been increased on complex Fourier series.
- In Chapter 3, discrete-time Fourier series have now been included. Also added are the structures of digital systems.
- The older p-operator and E-operator methods of system analysis have been deleted.
- The closed form of Discrete Fourier Transform along with all its properties has now been included preceding the Fast Fourier Transform.

- A new chapter on sampling has been added which deals with discrete-time filtering of continuous-time signals and also decimation and interpolation. Specifically, two flow charts have been devised sequentially to mechanize all the transformations involved.
- A chapter on assorted topics is now included spanning frequency response block diagrams and signal flow graphs.
- The chapter on MATLAB applications of signals and systems has been updated and expanded.
- The chapter on Random Processes has been deleted as it does not form part of the course in any university.

Chapter 1 introduces the topics of signals and systems giving classification of these and their mathematical modeling as seen from various aspects. Simple illustrative examples from various disciplines are included. Concepts of Homogeneity, Superposition and Linearity are exposed. A glimpse of Frequency Spectrum, Energy and Power Signals is also given. In fact, this chapter gives an overview of all that is to follow in later chapters. For better content organization, topics on State Variable, Frequency Response, etc., are deleted from the first chapter (added in Ch 6 and 7) and the portion on Classifications of Signals, Systems and Signal Properties has been strengthened by further expansion.

Chapters 2 and **3** are devoted to analysis of discrete and continuous-time signals. Time domain and frequency-domain techniques are covered in the same chapter, one following the other. Both these chapters begin with convolution of continuous-time and discrete-time signals.

Chapters 2 introduces the Fourier series method of representing continuous-time periodic signals with emphasis on exponential form of Fourier series. As a natural sequence, follow the Fourier transform and the Laplace transform for dealing with aperiodic signal and systems analysis. All the properties of the transforms are proved and look-up tables developed.

Chapter 3 is devoted to discrete-time signals and systems, beginning with discrete-time Fourier series. The Z-transform technique of discrete-time signals and systems is given exhaustive treatment.

The discrete-time Fourier form in closed form with properties, tables of pairs and analysis of discrete-time systems now form part of Chapter 4. In the discrete-time domain, signal processing requires explicit computation of the signal's Fourier transform either way for which the Fast Fourier Transform technique is retained as it was.

Chapter 5 is a new chapter on Sampling for which details have been given earlier in this write-up.

Chapter 6 is also a new chapter on assorted topics like Frequency Response, Bode Plots, Block Diagrams and Signal Flow Graphs.

Chapter 7 deals with state-variable formulation of both types of systems and methods of their close-form solution in time and frequency domains. Power of the discrete form in computer solutions and conversion of continuous form to discrete for this purpose are included.

Chapter 8 discusses the issues of stability for continuous as well as discrete-time LTI systems.

Chapter 9 presents the fundamental aspects of digital filter designs and their synthesis. This chapter begins with various analog filter approximations, e.g., Butterworth, Chebyshev, Elliptic, etc., and then takes up some important digital filter design techniques like Finite Impulse Response (FIR), and Infinite Impulse Response (IIR). The chapter ends with a detailed discussion on network synthesis of FIR as well as IIR filters.

Chapter 10 discusses MATLAB Tools for Signal Analysis. New commands and several more analysis and design examples are added.

The revised edition now has over 150 Solved Examples, and over 200 Exercise Problems. Apart from these, there are three appendices.

Appendix I is on Partial Fraction Expansion, and **Appendix II** discusses Matrix Analysis and Matrix Operations. Finally, **Appendix III** provides answers to the problems given in the chapters.

A **Bibliography** is added in the end which mentions an exhaustive list of books which students can read to enhance their knowledge.

The website of the book can be accessed at http://www.mhhe.com/nagrath/ss2e and contains the following material:

- Interactive Quiz for **students**
- Solution Manual (for selected problems) and PowerPoint Slides with figures from the text for **instructors**

The authors would also like to thank the reviewers who took out time to review the book and gave valuable suggestions for enhancing the book. Their names are given below.

Dinesh Chandra	*JSS Academy of Technical Education* *Noida, Uttar Pradesh*
Sarika Sachdeva	*Greater Noida Institute of Technology* *Greater Noida, Uttar Pradesh*
Nandita Pradhan	*Galgotia's College of Engineering and Technology* *Greater Noida, Uttar Pradesh*
Sikha Tripathi	*EEE Group, Faculty Division II* *BITS Pilani, Rajasthan*
Zia Abbasi	*Aligarh Muslim University* *Aligarh, Uttar Pradesh*
Smriti Agarwal	*Ramswaroop College of Engineering* *Lucknow, Uttar Pradesh*

Sugata Munshi	*Jadavpur University* *Kolkata, West Bengal*
Varsha Shah	*Rizvi College of Engineering* *Mumbai, Maharashtra*
Vikram M Gadre	*Indian Institute of Technology Bombay* *Mumbai, Maharashtra*
Mehul Raval	*Sarvajanik College of Engineering and Technology* *Surat, Gujarat*
Shamla Mathur	*Osmania University* *Hyderabad, Andhra Pradesh*

The authors welcome any constructive criticism of the book and will be grateful for any appraisal by the readers.

I J Nagrath
S N Sharan
Rakesh Ranjan

Introduction to Signals and Systems 1

Introduction

The term *system* is used in all areas of science and technology and it has also become commonplace in economics, social sciences, humanities and many other areas of study. This book will deal only with the physical systems which are well defined and governed by immutable laws of nature. Out of several definitions of the term *system*, we reproduce here the one that suits our needs the most.

A system is a collection of components wherein individual components are constrained by the connecting inter-relationships such that the system as a whole fulfills some specific functions in response to varying demands.

Varying demand is a function of one or more parameters and is called the **signal**. This signal carries the demand information in various physical forms like a current, voltage, mechanical movement, etc. The response of a system to the input signal is the output signal. The system indeed is then a set of cause-effect relationships. This is represented as a **block diagram** in Fig. 1.1(a). It is evident from this diagram that there is another input signal to the system which, in fact, is an unwanted signal called **disturbance**. This signal could also arise internally. In general, a system may have several inputs and outputs as shown in Fig. 1.1(b). Here, a thick arrow represents multiple input/output signals.

Not only the response signal, but even the signals inside the system structure may also be of interest to the designer/user of the system. Further, it is usually convenient and possible to divide the system into several subsystems and these are linked together by cause-effect relationships. The environment in which the system is embedded may

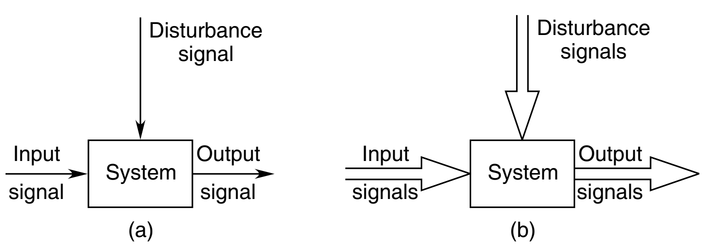


Fig. 1.1 *Block diagram of a system and signals*

also produce interactive effects on the system performance. For example, in a closed cooling/heating system, the environment would produce a disturbance effect (signal) on the system and similarly an electronic circuit would be affected by environmental temperature.

The system (subsystem) boundary and cause-effect relation can be modelled mathematically or algorithmically but it is not possible in case of **nonphysical** systems such as management systems. In such systems the cause-effect signals are not unidirectional and there may be several ill-defined interactions between subsystems constituting the system and also the environment may affect it in several ways and so sufficient clarity may not emerge. A model of a physical system, however well-defined, is subject to the approximations made in applying the physical laws to the system components (elements). Therefore, the model always has to be validated for known data or by testing its prototype. This would be all the more necessary for the models of nonphysical systems.

In analysis or design of a system, there is a need to examine the system response to all possible inputs. As this is not a feasible task, we need to select some typical ideal forms of inputs. Both dynamic and steady-state response of the system would be of interest to the designer and analyst. There are situations wherein we need to set up a system to suitably modify the signal, build missing/hazy information or reject some unwanted frequencies in the system.

1.1 EXAMPLES ON SIGNALS AND SYSTEMS

So far we have introduced some general ideas about signals and systems. Now we will support these ideas with the following examples.

Example 1.1 *Measurement of acceleration of a moving vehicle*

Let us consider a system for the measurement of acceleration of a moving vehicle shown in Fig. 1.2.

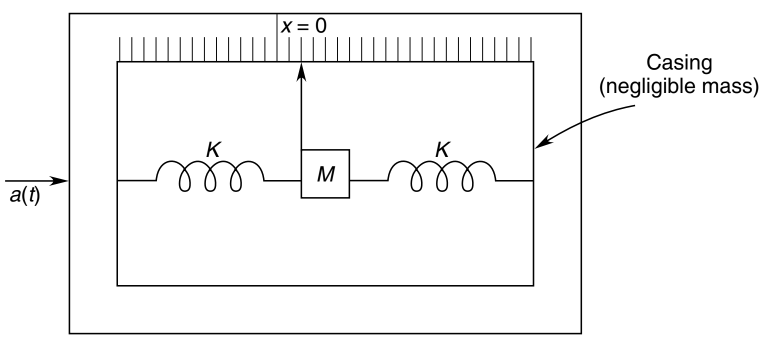


Fig. 1.2 *A moving-vehicle system*

Let

- $a(t)$ = vehicle acceleration
- K, M = spring constant and mass (to which pointer is attached)
- x = displacement of the pointer from its equilibrium position

The force on the mass caused by input acceleration signal is balanced by the spring force, i.e.,

$$M\,a(t) = Kx \tag{i}$$

or

$$x(t) = (M/K)a(t) \tag{ii}$$

It may be noted that the mass of the casing has been considered negligible and therefore does not absorb any force. In this idealized system, the system responds instantaneously and output $x(t)$ is a replica of $a(t)$ as per Eq. (ii) and is therefore, a measure of acceleration. It may be noted that x will have an upper limit because of the limited capacity of spring w.r.t. stretching or compression.

Example 1.2 *Measurement of velocity of a moving vehicle by means of the accelerometer defined in Example 1.1.*

Velocity being an integral of acceleration, can be measured by integrating x. The output signal of the accelerometer is converted into an electrical signal by means of a linear potentiometer (centre tapped) as shown in Fig. 1.3.

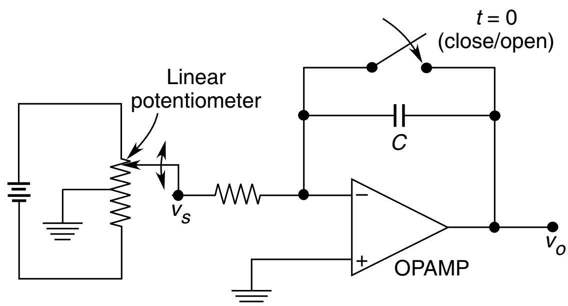


Fig. 1.3 *OPAMP integrator*

The potentiometer output v_s is then proportional to x and is fed to an OPAMP integrator (as in Fig. 1.3). The output signal (which is in voltage form) is

$$v_0(t) \propto \int_0^t a(t)\ dt$$

The accelerometer and the OPAMP circuit taken together form a system whose purpose is to process the input acceleration (mechanical) signal to an output voltage signal, which is a measure of velocity from $t = 0$ onwards.

This system has a limitation as v_o in an OPAMP cannot be allowed to exceed a specified value. To keep the integrator in the linear region, the input signal has to be suitably scaled down.

Instead of using the analog method, we could carry out the integration of $a(t)$ numerically from its value recorded from the accelerometer. The graphs of $a(t)$, $a(nT)$ are shown in Fig. 1.4. Here, $a(nT)$ is the signal value at $t = nT$, T being the sampling period. Velocity is the area underneath the curve. The graph between adjacent values under which the area is shaded is assumed to be linear.

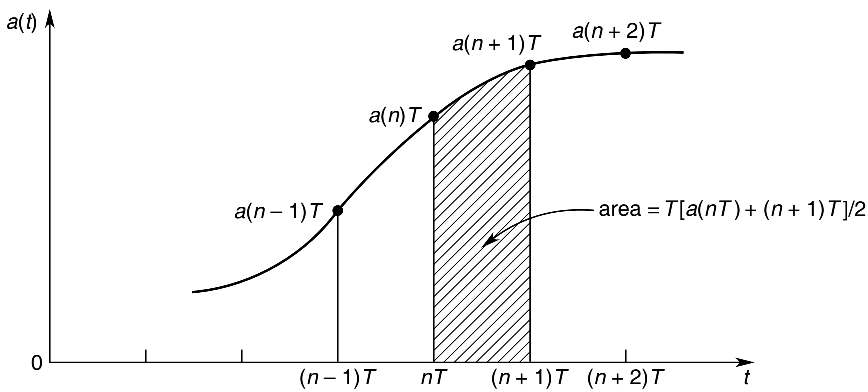


Fig. 1.4 *Computation of velocity by numerical integration*

If T is short enough compared to the variations in $a(t)$, this area represents the actual area quite accurately. We shall discuss later in Chapter 7 that sampling frequency is a certain value compared to the signal bandwidth (band of frequencies contained in the signal).

The velocity v up to $(n + 1)T$ is then computed by recursive relationship (trapezoidal rule) as

$$v\left[(n+1)T\right] = v(nT) + T\left[\frac{a(nT) + a(\overline{n+1}T)}{2}\right]$$

Such computation could be carried online by means of a Digital Signal Processing (DSP) chip.

Example 1.3 *Automatic control systems: tank level control*

The distinguishing feature of automatic control systems is the principle of feedback. Consider a simple system shown in Fig. 1.5. Here, an attempt is made to hold the tank level h within reasonable, acceptable limits even when the outlet flow through valve V_1 varies. This can be achieved by irregular and intermittent manual adjustment of the inlet flow rate by valve V_2.

The scheme of Fig. 1.6 provides automatic control of the level of the tank. It can maintain the desired tank level within quite accurate tolerances even when

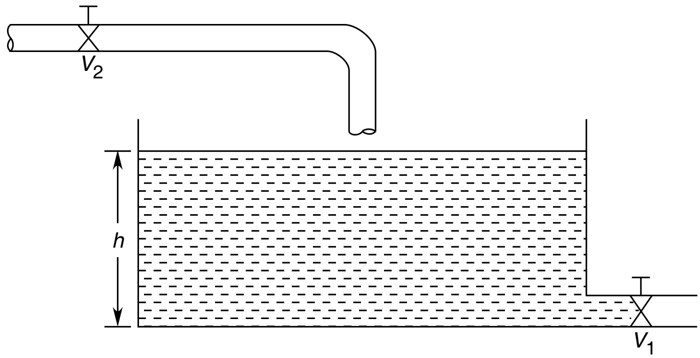


Fig. 1.5 *Tank level control system*

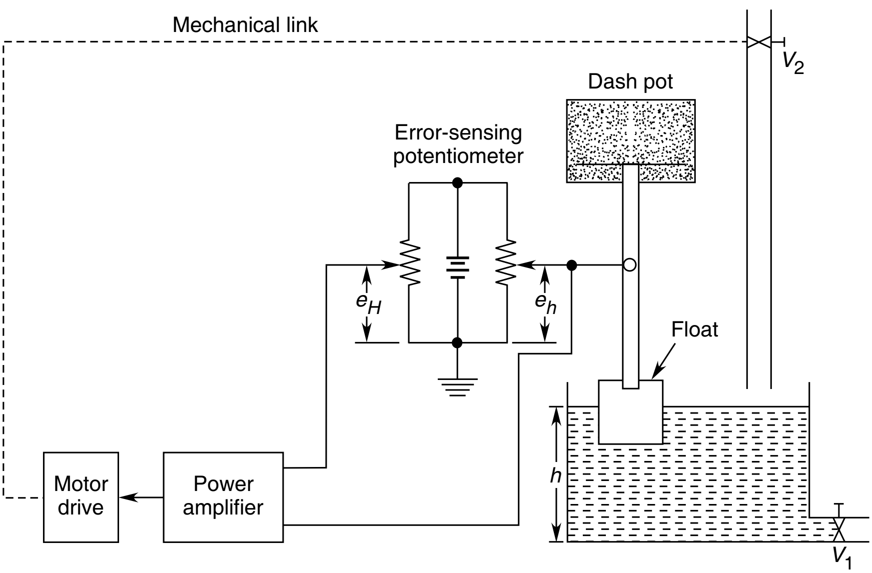


Fig. 1.6 *Automatic tank-level control system*

the output flow rate through valve V_1 is varied. This system differs from the one illustrated in Fig. 1.5, in terms of the feedback action. Here, the tank level is sensed by a float and converted to an electrical signal by means of a potentiometer. The potentiometer pair acts as an error detector which compares the signal e_h corresponding to the output response (the head h) with the reference input e_H (at steady state, h = desired head H and $e_h = e_H$). Any difference between these two signals constitutes an error or an actuating signal, which actuates the control elements (the power amplifier and motor). The control elements in turn alter the conditions in the plant (controlled member–valve V_2) in a manner so as to reduce the original error ($H - h$). Such systems are called **closed-loop** systems or automatic control systems. Systems which do not have any feedback comparison (e.g., the system shown in Fig. 1.5) are called **open-loop** systems.

Figure 1.7 gives the general block diagram of a closed-loop system. Observe that each block has a cause-effect relationship and the signal flow is unidirectional (indicated by arrows).

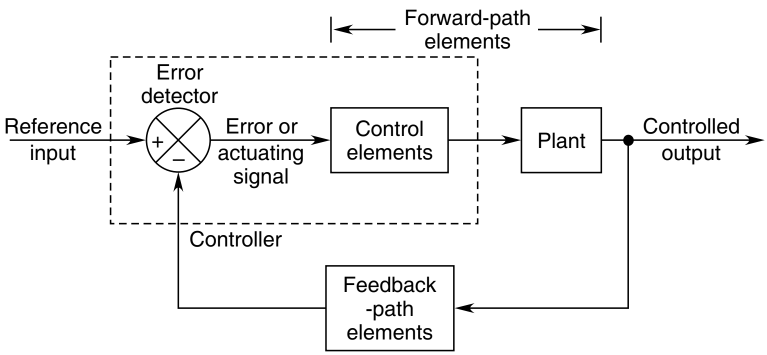


Fig. 1.7 *General block diagram of an automatic control system*

Feedback offers certain distinct advantages and is therefore, inherent to most physical and nonphysical systems though not always in the uniquely identified form as in Fig. 1.7.

Example 1.4 *Automatic aircraft-landing system*

This single example demonstrates many of the concepts of signals and systems, such as subsystem, signal processing, signal modification and feedback control loop to achieve desired performance, etc. Some of these concepts have been illustrated by earlier examples.

The aircraft-landing system consists of three basic subsystems, i.e., the aircraft, the radar unit and the controlling unit as shown in Fig. 1.8(a). For illustrative purpose, we shall consider only the lateral control system shown in Fig.1.8 (b), where aircraft lateral position signal $y(t)$ is first sampled by switch unit with sampling time $T = 0.05$ s, to $y(nT)$, where $n = 0,1,2....$Observe that $y(t)$ is a continuous time signal and $y(nT)$ is a discrete time signal. The digital controller processes the sampled values and generates the bank command at discrete intervals, i.e., the bank command is updated (signal modification) at every $T = 0.05$ s. The bank command signal is made continuous by data hold to produce $\phi(t)$ which is fed to the aircraft lateral adjustment system and the lateral position $y(t)$ changes accordingly. The control system thus brings $y(t)$ close within the value commanded by the radar unit (or applied externally by the pilot) for safe landing of the aircraft.

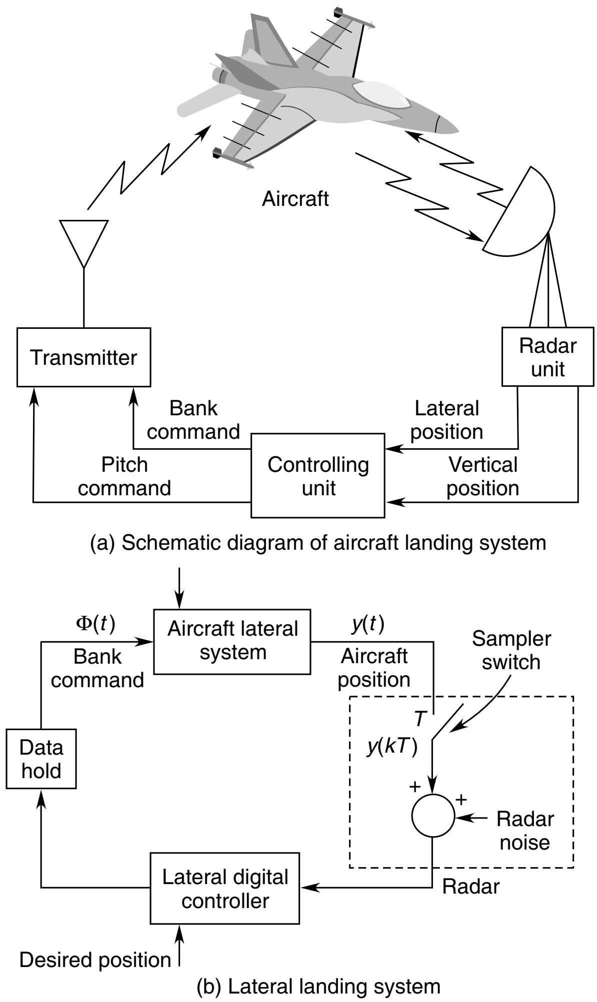


Fig. 1.8 *Automatic aircraft-landing system*

Example 1.5 *Satellite-communication system*

The most important application of signals and systems is in speech and image processing, where restoration and enhancement of signals is a great challenge. This importance is brought out by the satellite-communication system, illustrated in its most basic form in Fig. 1.9. Here, the signal is transmitted from one station and received at the other station via the satellite. The receiving station receives very weak and poor quality (noisy) signal.

Let d_1, d_2 be the respective distances between stations 1 and 2 from the satellite. Then

$\tau_1 = d_1/c =$ time in seconds of em-wave travel between Station 1 and satellite

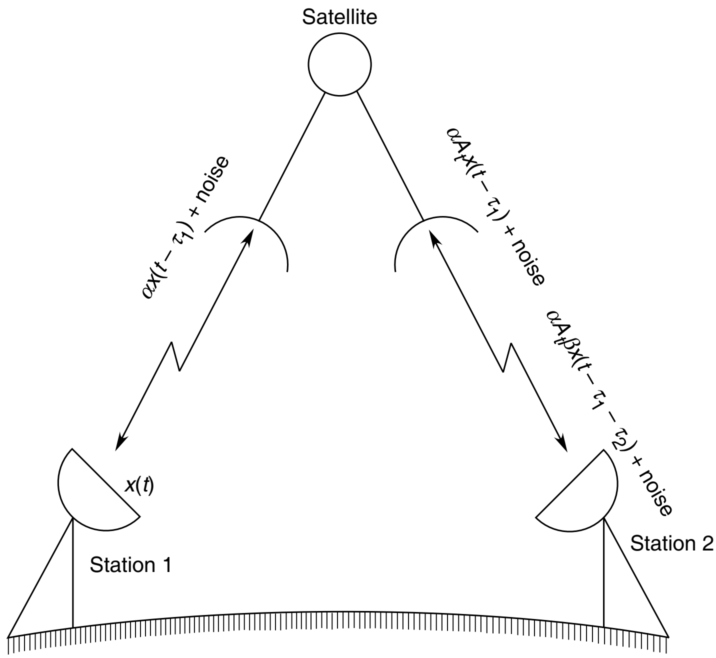


Fig. 1.9 *Satellite-communication link*

and

$\tau_2 = d_2/c =$ time in seconds of em-wave travel between Station 2 and satellite

where

$c =$ speed of em-wave

The signal is carried over a narrow band of em-waves. The central frequency of an em-wave from each station to the satellite is, say, f_1 and it is f_2 from the satellite to any of the stations. It is assumed that $f_1 > f_2$. This change in frequency occurs in the satellite repeater called *transponder*. By this arrangement, the transponder generates smaller frequency for sending signal on to stations which leads to reduction in its weight (payload of the satellite). The em-wave carrying the signal gets attenuated during travel because of the following points.

- characteristics (spreading)
- travel in the atmospheric part of the distance

Let these attenuations be α and β respectively for d_1 and d_2 for travel (either way). During travel, the em-wave picks noise caused by the em disturbances present in atmosphere and in space.

With reference to Fig. 1.9 if a wave packet $x(t)$ is transmitted from Station 1, then on reaching the satellite it can be expressed as

$$\alpha x(t - \tau_1) + \text{noise} \tag{i}$$

and on reaching the receiving station, it becomes

$$\alpha A_t \beta x(t - \tau_1 - \tau_2) + \text{noise} \tag{ii}$$

where

A_t = gain in the transponder which can only partially make up for the net attenuation.

The received signal is then enhanced and restored by noise reduction to stringent level. This example is an excellent illustration of signal enhancement and restoration.

1.2 CLASSIFICATION OF SIGNALS

In general, the independent variable signals can be functions of time or any variable other than time. However, we shall deal with signals which are functions of time. Such signals are broadly classified as continuous-time and discrete-time signals.

Continuous-time Signals

A signal which is uniquely defined at all 'time' as an independent variable, for a certain time domain except for discontinuities at denumberable set of points is known as continuous-time signal. An illustration of such a signal with a single discontinuity at $t = t_2$ in the time domain $t_1 \leq t \leq t_3$ is shown in Fig. 1.10.

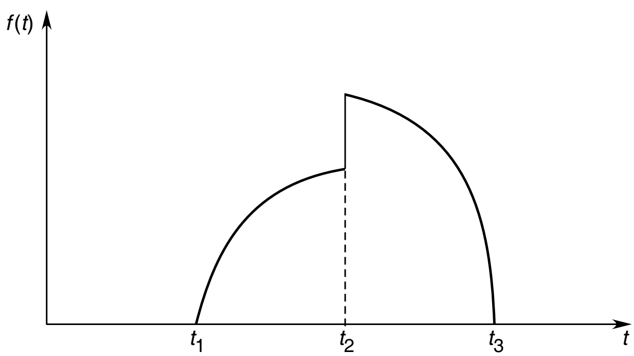


Fig. 1.10 *Continuous-time signal*

Several examples of signals and systems have been given in Section 1.1. Consider the speed measuring system of Examples 1.1 and 1.2 comprising accelerometer and integrator which are shown in block form in Fig. 1.11(a). In this system, both $a(t)$ and $v_o(t)$ are continuous as shown in Fig. 1.11(b). Such a system is called a continuous-time system.

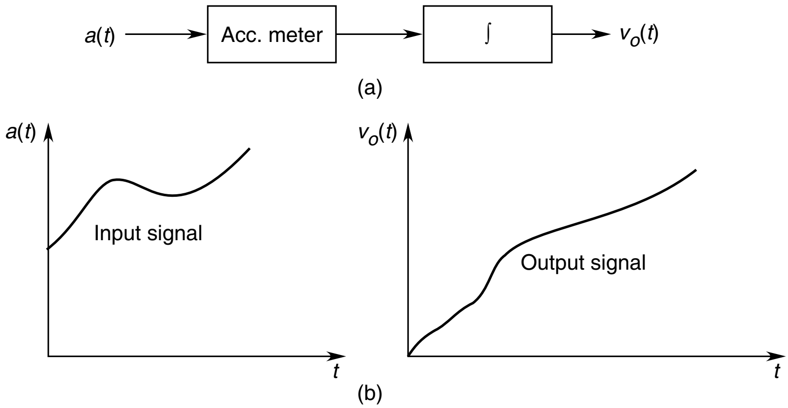


Fig. 1.11 *Continuous-time system*

There is a variety of (wave forms) input and output signals. If the system is to be simulated or tested in prototype stage, it would be meaningless and also infeasible to test it for all possible input signals. Therefore, certain ideal test signals which characterize a general signal are used. This topic will come up later in the text.

Discrete-time Signals

Discrete-time signals are defined only at discrete values of independent variable, time t. The interval between signal values is often same, but it is not always so. An example of such a kind of signal is money withdrawal from bank a/c, i.e., one may withdraw money fortnightly, monthly or with no fixed sequence, as shown in Fig. 1.12.

Mathematically, $f(n)$ as sequence of withdrawal is

$$f(n) = \{f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)\}$$

where $f(0), f(3), f(4), f(6)$ are zero, since money is not withdrawn.

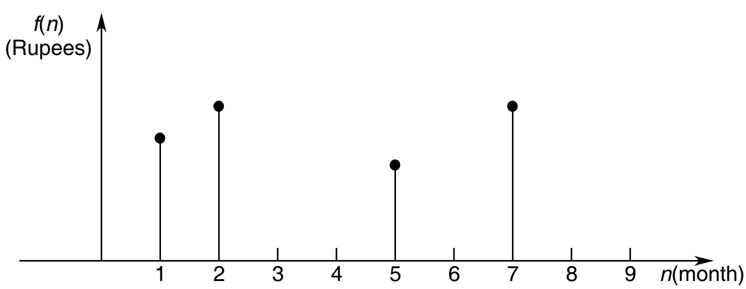


Fig. 1.12 *Discrete-time signal (called a sequence)*

A discrete sequence could also be specified by a rule for calculating the discrete values. For example,

$$f = \{1, 1/2, 1/4, \ldots + (1/2)^k + \ldots\}$$

It could be equivalently specified as

$$f(k) = \begin{cases} (1/2)^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

A discrete sequence could also exist for $k < 0$.

Quite often in physical systems, continuous-time signals have to be converted to discrete-time signals and at the output end of the signal they have to be converted back to continuous form. Most of the signal processing is off/on-line; computation can then be accomplished by digital computers, microprocessors or microcontrollers depending on the size of data and computational effort.

Systems whose input and output are characterized by discrete sequences are known as discrete-time systems.

Deterministic/Stochastic Signals

Deterministic signals are characterized by the fact that their behaviour is fully known at all times, i.e., these can be described as functions of time with certainty. In real systems, we are uncertain to varying degrees about the values of parameters, measurements, expected inputs and disturbances. The inputs and disturbances can vary randomly and can also contain unwanted random components called noise. In many practical applications, the uncertainties and noise can usually be neglected. Therefore, we proceed as if all quantities have definite values that are known precisely at all times. This is called the deterministic approach to system analysis.

There are occasions where the deterministic approach fails because the system and signal data are essentially nondeterministic. For example, in a radar-tracking system, the position and speed of the target to be tracked are not deterministic but vary in general in a random fashion. Such signals are called stochastic signals, and systems where these signals are present are known as stochastic systems. Their behaviour can be determined to a considerable degree if various statistical properties (mean, variance, etc.) of the signals and noise are known. This is called the *stochastic approach to systems*.

1.3 TRANSFORMATION OF INDEPENDENT VARIABLE

The purpose of a system is to transform the input signal so as to produce the desired output. For example, in an aircraft the pilot's signals are transformed to electrical and mechanical signals which in turn cause the adjustment of engine thrust and position of aircraft surfaces like rudder or ailerons. The result is that the aircraft requires the desired speed and direction. These are then maintained by the autopilot (a central system).

One of the rudimentary signal transformation is with the independent variable (t or n), which will now be illustrated. This type of transformation plays important role in introducing some basic properties of the systems to be taken up in this chapter, Section 1.8.

The independent variable of a signal $x(t)$ can be transformed to the general form

$$x(t) \to x(\beta t + \alpha) \tag{1.1}$$

This general transformation of the independent variable preserves the form of $x(t)$ except that it is linearly compressed for $|\beta| > 1$ and linearly expanded for $|\beta| < |$. It is time reversed for $\beta < 0$ (negative). The signal is shifted in time for $\alpha \neq 0$. These three cases will now be illustrated.

1. Time Shift

Continuous Time Case

$$\beta = 1,\ \alpha = \pm\, t_0$$
$$x(t) \to x(t + t_0);$$

signal is advanced in time (shift left); $x(t)$ values now occur earlier in time by t_0

$$x(t) \to x(t - t_0);\ \text{the signal is delayed in time by } t_0$$

The delayed signal is sketched in Fig. 1.13 wherein signal value $x(0)$ is now found at $x(t_0)$. This type of time delay occurs in systems with *transportation lag*.

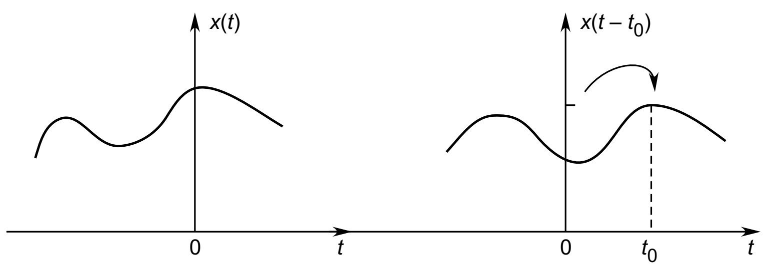


Fig. 1.13 *Delayed signal*

Discrete-time Case

$$x(n) \to x(n + n_0);\ \text{signal advanced by } n_0$$
$$x(n - n_0);\ \text{signal delayed by } n_0$$

A signal advanced by $n_0 = 2$ is illustrated in Fig. 1.14.

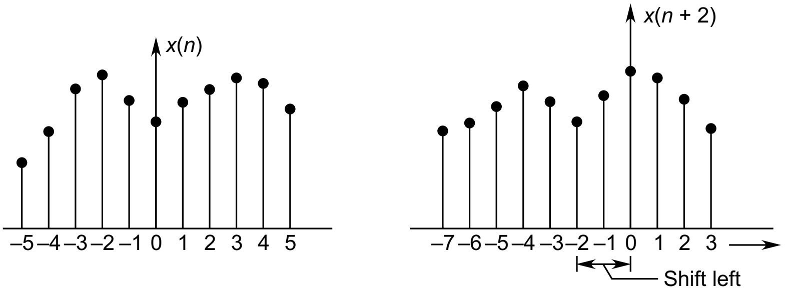


Fig. 1.14 *Signal advanced by $n_0 = 2$*

2. Time Reversal-Reflection

Continuous Time Case

$$x(t) \to x(-t) \quad \beta = -1, \alpha = 0$$

The signal gets reflected about x-axis ($t = 0$ axis). The reflection is illustrated in Fig. 1.15.

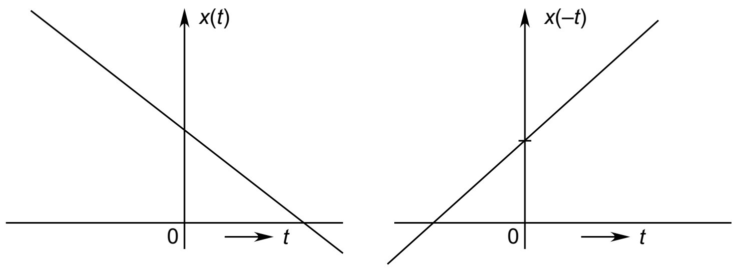


Fig. 1.15 *Reflection (CT)*

Discrete-time Case

$$x(n) \to x(-n)$$

Illustrated in Fig. 1.16.

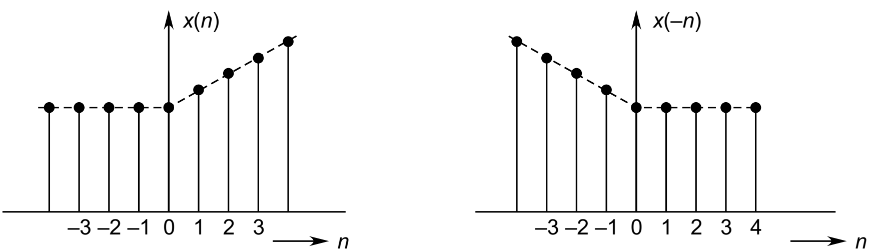


Fig. 1.16 *Reflection (DT)*

3. Time Scaling

Continuous Time Case

$$\alpha = 0$$
$$x(t) \to x(\beta t);\ \beta > 1 \text{ time compression}$$
$$\beta < 1 \text{ time expansion}$$

Illustrated in Fig. 1.17.

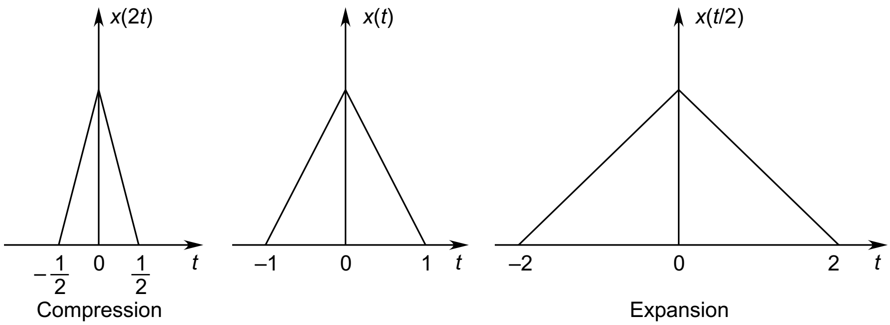


Fig. 1.17 *Time scaling*

Discrete-time Case

In general, time scaling is not possible in uniformly spaced sampling. Only advance/delay and reflection.

General Independent Variable Transformation Procedure (CT)

$$x(t) \to x(\beta t + \alpha)$$

1. $\alpha < 0$; delay $x(t)$ by $|\alpha|$
   $\alpha > 0$; advance $x(t)$ by $-\alpha$
2. $|\beta| > 1$; compress signal of step 1 by $\dfrac{1}{|\beta|}$
   $|\beta| < 1$; expand signal of step 1 by $\dfrac{1}{|\beta|}$
3. $\beta > 0$; no change
   $\beta < 0$; reflect signal of step 2 about x-axis ($t = 0$ axis)

The transformations are to be carried out in the above order.

Example 1.6 *For the CT signal x(t) as in Fig. 1.18(a), determine (a) x(t + 1), (b)* $x\left(\frac{3}{2}t\right)$, *(c)* $x\left(\frac{3}{2}t + 1\right)$, *and (d) x(–t + 1)*

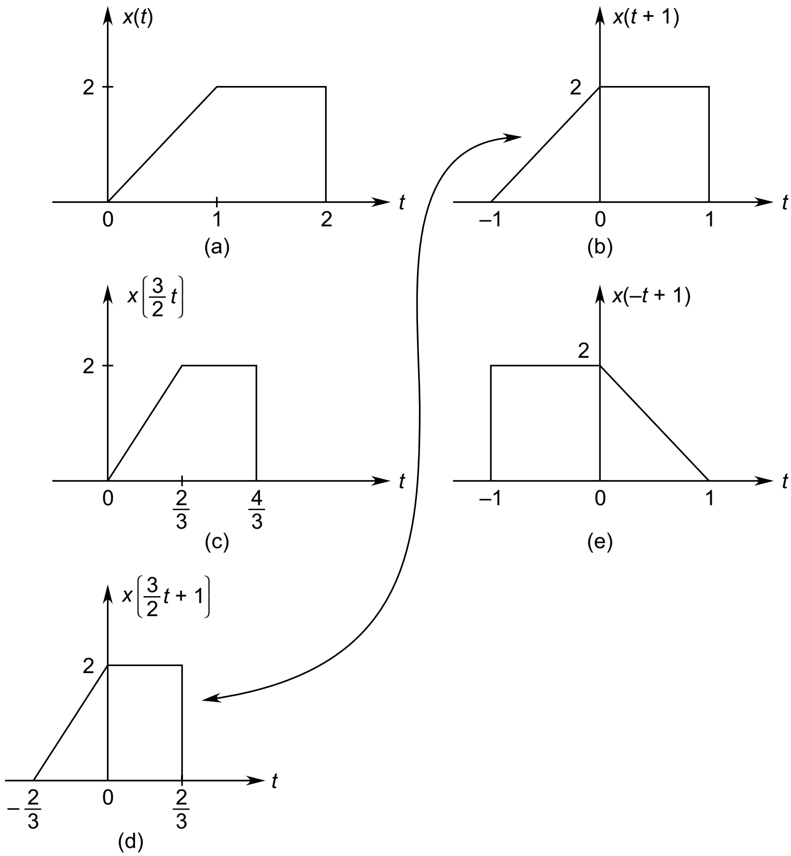


Fig. 1.18

Solution

(a) $x(t + 1)$ is $x(t)$ advanced by 1 as sketched in Fig. 1.18(b). Value of $x(t)|_{t=2}$ $= x(2)$ is now found at $x(t+1)\big|_{t=2-1=1} = x(2)$

(b) $x\left(\dfrac{3}{2}t\right)$ is $x(t)$ compressed by (2/3) as sketched in Fig. 1.8(c).

(c) For $x\left(\dfrac{3}{2}t+1\right)$, we proceed from $x(t + 1)$ and then compress by (2/3) as shown in Fig. 1.18(d). *This operation ordering is necessary*. The value of $x(t)_{t=2}$ is now found at $x\left(\dfrac{3}{2}t+1\right)\bigg|_{t=(2-1)\times\frac{2}{3}=\frac{2}{3}} = x(2).$

(d) $x(t + 1)$ is reflected to give $x(-t + 1)$ as shown in Fig. 1.18(e)

Example 1.7 *A signal $x(t) = 0$ for $t < 3$. Find the range of t for which the following signals are zero.*

(a) $x(-1-t)$ (b) $x(2-t)$
(c) $x(-1-t) + x(2-t)$ (d) $x(3t)$
(e) $x(t/3)$

Solution We use the transformation rules enunciated earlier in this section.

(a) $x(-1-t)$

$x(t-1)$ is $x(t)$ delayed by 1

$\therefore$ $x(t-1) = 0$ for $t < 4$

$x(-t-1)$, reflect $x(t-1)$

So $x(-t-1) = 0$ for $t > -4$

or directly

$$(-t-1) < 3 \Rightarrow t > -4$$

(b) $x(2-t)$

$x(t+2)$ is $x(t)$ advanced by 2

so $x(t+2) = 0$ for $t < 1$

$x(-t+2)$ is $x(t+2)$ reflected

so $x(-t+2) = 0$ for $t > -1$

or directly

$$(-t+2) < 3 \Rightarrow t > -1$$

(c) Both conditions of parts (a) and (b) are to apply. So that the condition is $t > -1$.

(d) $x(3t)$ is $x(t)$ compressed by 1/3. So $x(3t) = 0$ for $t < \dfrac{3}{3}$ or $t < 1$.

(e) $x(t/3)$ is $x(t)$ expanded 3 times. So $x(t/3) = 0$ for $x < 9$.

1.4 ENERGY AND POWER IN SIGNALS

Signals in physical systems have energy and power associated with them. For example, in a resistor R,
Power,

$$p = vi = \frac{v}{R} v$$

For convenience, we take $R = 1$, so

$$p = v^2 \tag{1.2}$$

The energy consumed from time t_1 to t_2 is

$$E_{12} = \int_{t_1}^{t_2} p \, dt \tag{1.3}$$

The average power in this period is

$$P_{12} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} p \, \mathrm{d}t \tag{1.4}$$

The terminology of energy and power of a signal can be extended to any type of continuous time (CT) and discrete-time (DT) signal. Thus, for a

CT signal

$$E_{12} = \int_{t_1}^{t_2} |x(t)|^2 \mathrm{d}t \tag{1.5}$$

wherein we have absolute value $|x(t)|$ as $x(t)$ in general is complex. Similarly, for a DT signal

$$E_{12} = \sum_{n_1}^{n_2} |x(n)|^2 \tag{1.6}$$

The average signal power is given by a CT signal

$$P_{12} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, \mathrm{d}t \tag{1.7}$$

and for a DT signal

$$P_{12} = \frac{1}{n_2 - n_1 + 1} \sum_{n_1}^{n_2} |x(n)|^2 \tag{1.8}$$

Let us extend the interval of integration to infinity, i.e., $-\infty < t < +\infty$ and $-\infty < n < +\infty$. We then have

CTS

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t \tag{1.9}$$

DTS

$$E_\infty = \lim_{N \to \infty} \sum_{-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{1.10}$$

The integral of Eq. (1.9) and summation of Eq. (1.10) will converge only for finite energy signal, i.e., $E_\infty < \infty$.

The time-averaged power in a CT signal is expressed as

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \tag{1.11}$$

and in DT signal

$$P_{\infty} = \lim_{N \to \infty} \frac{1}{2N+1} \int_{-N}^{N} |x(n)|^2 \tag{1.12}$$

Two types of signals can be identified:

Finite energy signal $\quad E_{\infty} < \infty$

Finite power signal $\quad P_{\infty} < \infty$

A finite energy signal will have

$$P_{\infty} = \lim_{T \to \infty} \frac{E_{\infty}}{2T} = 0$$

The same condition hold good for a DT signal.

It is to record here that a signal may not be either finite energy or finite power. For example, signal $x(t) = t$.

1.5 PERIODIC SIGNALS

A periodic signal has the property that

$$x(t) = x(t+T) \tag{1.13}$$

which means that $x(t)$ is unaffected by time shift T. If no such value of T exists, the signal is *aperiodic*.

If Eq. (1.13) holds then

$$x(t) = x(t+mT);\ m = \text{integer} \tag{1.14}$$

The smallest value of T for which Eq. (1.13) holds, is called the *fundamental period* or just *period* and is labelled as T_0.

A periodic signal is graphically drawn in Fig. 1.19 for illustration.

Discrete Case

A discrete periodic signal has the property that

$$x(n) = x(n+N) \tag{1.15}$$

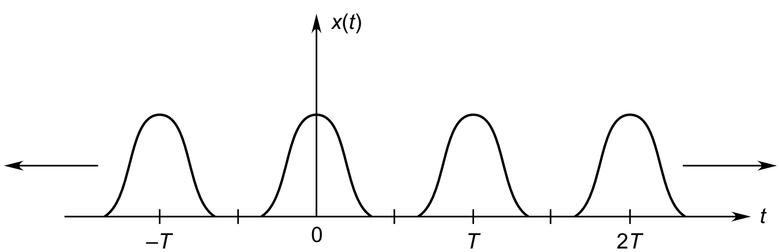


Fig. 1.19 *Periodic signal ($T = T_0$)*

If this condition is met then

$$x(n) = x(n + N) = x(n + 2N) \tag{1.16}$$

The smallest value of N for which Eq. (1.15) holds is the *fundamental period* or just *period* of the signal, denoted as N_0.

The discrete periodic signal is illustrated in Fig. 1.20 for $N_0 = 3$.

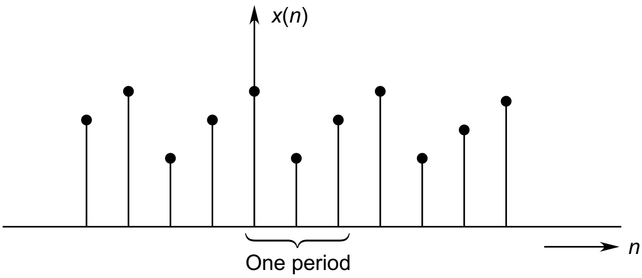


Fig. 1.20 *Periodic discrete signal*

The most important and commonly encountered periodic signal is the sine wave.

Sine Wave Signal

It is expressed as

$$x(t) = A \sin\left(2\pi \frac{t}{T_0} + \theta\right) \tag{1.17}$$

where A = amplitude, T_0 = period and θ = phase angle

The signal frequecy is $f_0 = \dfrac{1}{T_0}$ hertz (cycle/second)

So we can write

$$x(t) = A \sin(2\pi f_0 t + \theta) \tag{1.18}$$

$$\omega_0 = 2\pi f_0 = \text{frequency in rad/second}$$

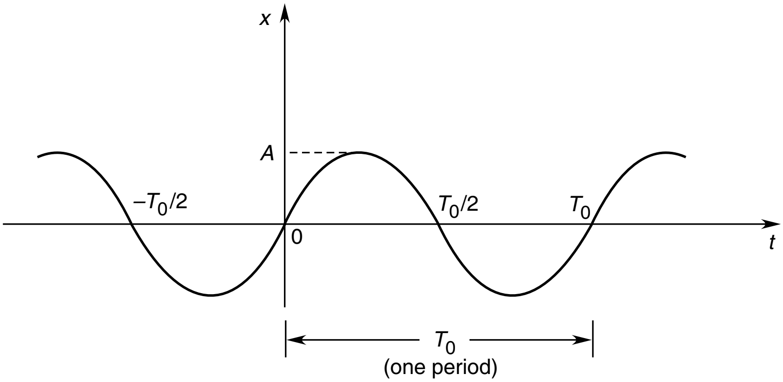
$$x(t) = A \sin(\omega_0 t + \theta) \tag{1.19}$$

We can check the periodicity of Eq. (1.17) by replacing t by $(t + T_0)$, which then gives

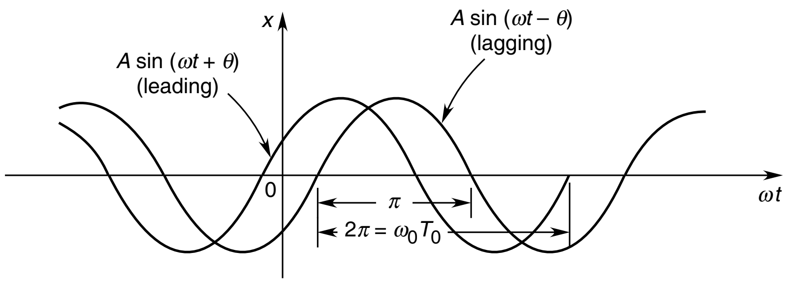
$$\begin{aligned} x(t + T_0) &= A \sin\left(2\pi \frac{t + T_0}{T_0} + \theta\right) \\ &= A \sin\left(2\pi \frac{t}{T_0} + \theta + 2\pi\right) \\ &= A \sin\left(2\pi \frac{t}{T_0} + \theta\right) = x(t) \end{aligned}$$

The sine wave can be drawn with t as an independent variable as in Fig. 1.21(a), (Eq. 1.17) or (ωt) as an independent variable as in Fig. 1.21(b), Eq. (1.19).

Figure 1.21(b) shows sine waves which are *lagging*/*leading* the sine wave of Fig. 1.21(a).



(a) Sine wave



(b) Lagging and leading sine waves

Fig. 1.21

The discrete sinusoidal signal will be discussed in detail in Section 1.6.

Example 1.8 *To determine time period of signals*

1. $x(t) = \cos\left(6t + \dfrac{\pi}{4}\right)$

$$2\pi f = 6,\ T = \frac{1}{f} = \frac{2\pi}{6} = \frac{\pi}{3}$$

2. $x(t) = e^{j\left(\frac{\pi}{2}t - 1\right)}$

$$\frac{\pi}{2} = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\pi} \times 2 = 4$$

3. $x(t) = e^{j(\pi t-1)} = e^{-j}\, e^{j\pi t}$

$$\pi T = 2\pi,\ T = \frac{2\pi}{\pi} = 2$$

4. $x(t) = \left[\cos\left(t - \frac{\pi}{3}\right)\right]^2 = \frac{1}{2}\left[1 + \cos\left(2t - \frac{2\pi}{3}\right)\right]$

$$2 = \frac{2\pi}{T} \quad \text{or} \quad T = \pi$$

The first term is constant (dc).

Not periodic, discontinuity at $t = 0$ and signal reversal.

Example 1.9 *Determine which of the following discrete-time signals are periodic and what is their fundamental period.*

(a) $x(n) = \sin\left(\frac{5\pi}{7}n + 1\right)$ (b) $x(n) = \cos\left(\frac{n}{6} - \pi\right)$

(c) $x(n) = \cos\left(\frac{\pi}{2}n^2\right)$ (d) $x(n) = \cos\left(\frac{\pi}{3}n\right)\cos\left(\frac{2\pi}{3}n\right)$

(e) $x(n) = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{2}n - \frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{8}n + \frac{\pi}{3}\right)$

Solution

(a) $\frac{5\pi}{7}N = 2\pi k$

$$N = \frac{2\times 7}{5}\cdot k$$
$$= 14 \quad \text{for} \quad k = 5$$

Periodic, period $N = 14$

(b) $\frac{N}{6} = 2\pi k$

$$N = 2\pi k, \text{ not integer, nonperiodic}$$

(c) $\frac{\pi}{2}(N)^2 = 2\pi k$

$$N^2 = 4\,k,\ N = 2 \quad \text{for} \quad k = 1$$

Periodic, period $N = 2$

(d) $x(n) = \frac{1}{2}\left[\cos(\pi\, n) + \cos\left(\frac{\pi}{3}n\right)\right]$

$$\pi N = 2\pi k$$
$$N = 2k = 2 \quad \text{for} \quad k = 1$$

Note: This is the smallest value.

(e) $\left(\dfrac{\pi}{8}\right) N = 2\pi, N = 16$

The first two terms are harmonics.

1.6 EVEN (SYMMETRIC)/ ODD(ANTI-SYMMETRIC) SIGNALS

A real-valued continuous-time signal $f(t)$ can be classified as an even signal if it satisfies the following relation [Fig. 1.22(a)].

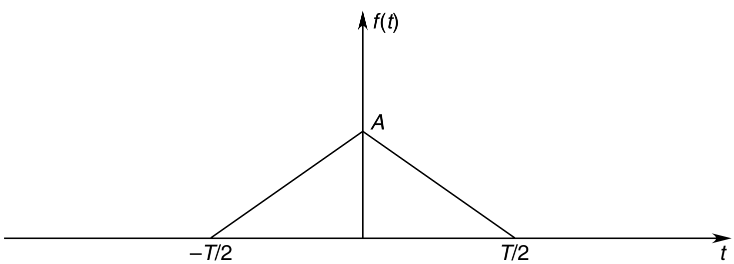
$$f(t) = f(-t) \tag{1.20}$$

If $f(t)$ satisfies the following relation then such a signal is said to be an odd signal [Fig. 1.13(b)].

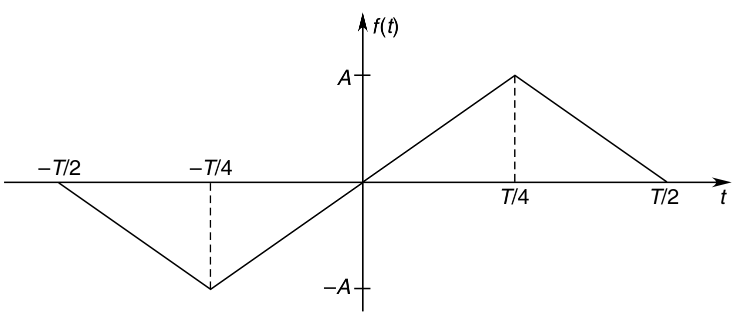
$$f(t) = -f(-t) \tag{1.21}$$

Any arbitrary signal can be expressed as the sum of its even and odd components, i.e.,

$$f(t) = f_e(t) + f_o(t) \tag{1.22}$$



(a) Even (symmetric) signal



(b) Odd (anti-symmetric) signal

Fig. 1.22 *Even and odd continuous-time signals*

where

$$f_e(t) = \frac{1}{2}[f(t)+f(-t)] \tag{1.23}$$

and

$$f_o(t) = \frac{1}{2}[f(t)-f(-t)] \tag{1.24}$$

In a similar fashion, an arbitrary discrete-time signal $f(n)$ can be expressed as

$$f(n) = f_e(n)+f_o(n) \tag{1.25}$$

where

$$f_e(n) = \frac{1}{2}[f(n)+f(-n)];\ \text{Fig. 1.22(a)} \tag{1.26}$$

and

$$f_o(n) = \frac{1}{2}[f(n)-f(-n)];\ \text{Fig. 1.22(b)} \tag{1.27}$$

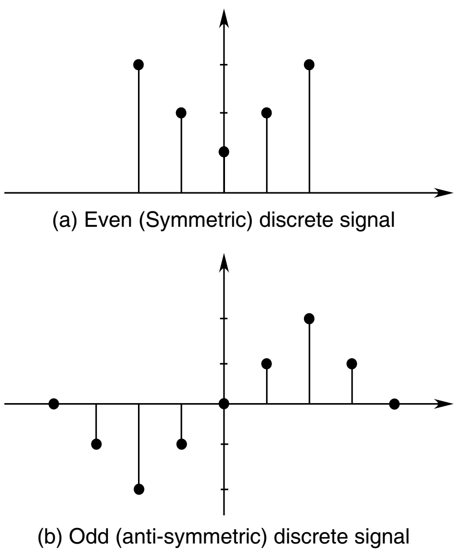


Fig. 1.23 *Even and odd discrete-time signals*

1.7 EXPONENTIAL AND SINUSOIDAL SIGNALS

Continuous-time Real and Complex Exponential Signals

A continuous-time exponential function is expressed in the general form as

$$x(t) = A\, e^{\alpha t} \tag{1.28}$$

This function (signal) has the advantage that its differential and integral are also exponential functions.

Real Exponential

In this case, α is a real quantity. If $\alpha > 0$, it is a growing exponential as sketched in Fig. 1.24(a). It could represent a chain reaction in atomic explosion and in complex chemical processes. In systems, in general, it represents an unstable behaviour, which must be avoided.

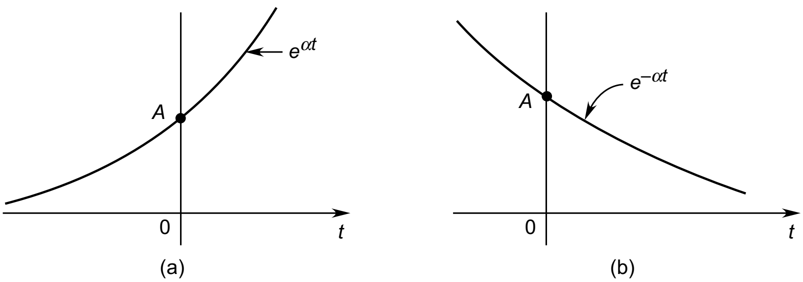


Fig. 1.24 *Growing and decaying exponentials*

For $\alpha < 0$, it is a decaying exponential which exhibits the dynamic behaviour of a stable linear system. It is sketched in Fig. 1.24(b).

Because of the importance of this decaying exponential in our study, we will pursue it further. We can write $\alpha = 1/\tau$, so

$$x(t) = A\, e^{-t/\tau} \tag{1.29}$$

τ has the significance of *time constant* which governs the decay in dynamical systems.

Periodic Complex Exponential and Sinusoidal Signal

For $\alpha = j\omega_0$, pure imaginary in Eq. 1.28, we get the complex exponential.

$$x(t) = e^{j\omega_0 t};\ \text{magnitude } A = 1 \tag{1.30}$$

We will now show that it is a periodic signal and determine its period.

$$\begin{aligned} x(t+T) &= e^{j\omega_0(t+T)} \\ &= e^{j\omega_0 t}\, e^{j\omega_0 T} \end{aligned}$$

For the signal to be periodic,

$$e^{j\omega_0 T} = 1 \tag{1.31}$$

which gives the periodicity condition

$$\omega_0 T = 2\pi \quad \text{as} \quad e^{j2\pi} = 1 \tag{1.32}$$

If $\omega_0 = 0$, $x(t) = 1$ and period T can have any value (degenerate case), the smallest value of T, the fundamental period, is given by

$$T_0 = \frac{2\pi}{|\omega_0|};\ \text{as } \omega_0 \text{ can be } \pm\omega_0 \tag{1.33}$$

Thus, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same period.

Sinusoidal Signal

Consider the exponential

$$x(t) = e^{j\omega_0 t}\, e^{j\phi}; \text{ where } e^{j\phi} \text{ is a complex number}$$
$$= e^{j(\omega_0 t+\phi)}$$

By Euler's theorem,

$$A\, e^{j(\omega_0 t+\phi)} = A\,[\cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi)]$$
$$A\, e^{-j(\omega_0 t+\phi)} = A\,[\cos(\omega_0 t + \phi) - j\sin(\omega_0 t + \phi)]$$

Adding, we get

$$A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j(\omega_0 t+\phi)} + \frac{A}{2} e^{-j(\omega_0 t+\phi)} \tag{1.34}$$

Similarly, by substracting we get

$$A\sin(\omega_0 t + \phi) = \frac{A}{2} e^{j(\omega_0 t+\phi)} - \frac{A}{2} e^{-j(\omega_0 t+\phi)} \tag{1.35}$$

The above equations establish the important link between sinusoidal and exponential signals. From Euler's theorem, we can express a sinusoidal signal as

$$A\cos(\omega_0 t + \phi) = A\, Re\,[e^{j(\omega_0 t+\phi)}] \tag{1.36a}$$

and

$$A\sin(\omega_0 t + \phi) = A\, Im\,[e^{j(\omega_0 t+\phi)}] \tag{1.36b}$$

where *Re* is the real part of and *Im* is the imaginary part of $[e^{j(\omega_0 t+\phi)}]$

Line Spectra

$$A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\omega_0 t}\, e^{j\phi} + \frac{A}{2} e^{j\omega_0 t}\, e^{j\phi} \tag{1.37}$$

The left side of this equation contains the information of magnitude A and phase angle ϕ at frequency ω_0. It can be expressed as single-line spectra in a frequency domain as in Fig. 1.25(a) and (b).

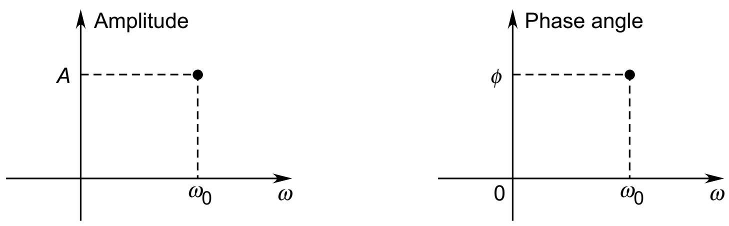


Fig. 1.25 *Line spectra—single sided*

In the right side of Eq. (1.37), the same information is in the form of two-side line spectra as shown in Fig. 1.26.

$$\omega_0 \rightarrow \frac{A}{2}, \phi$$

$$-\omega_0 \rightarrow \frac{A}{2}, -\phi$$

Here, we have introduced the concept of *negative frequency* $(-\omega_0)$

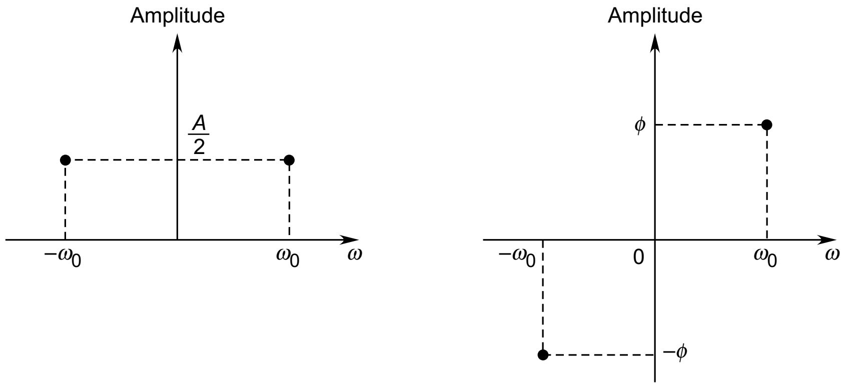


Fig. 1.26 *Double-sided line spectra*

Discrete-time Real and Complex Exponentials

A discrete complex exponential signal or sequence has the form

$$x(n) = k\,\alpha^n \tag{1.38}$$

or
$$x(n) = k\,e^{\beta n};\; e^{\beta} = \alpha \tag{1.39}$$

where k and α in general are complex.

Real Discrete Exponential Signal

From Eq. (1.39), the following cases arise

1. $\alpha > 1 \rightarrow$ growing exponential
2. $0 < \alpha < 1 \rightarrow$ decaying exponential
3. $-1 < \alpha < 0 \rightarrow$ decaying alternating exponential
4. $\alpha < -1 \rightarrow$ growing alternating exponential

For illustration, cases 2 and 3 are sketched in Fig. 1.27(a) and (b) respectively.

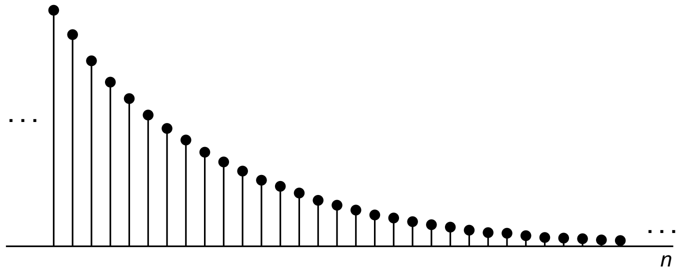
Discrete-time Sinusoidal Signal

Let $\beta = j\omega_0$ and $k = 1$ in Eq. (1.39). We get

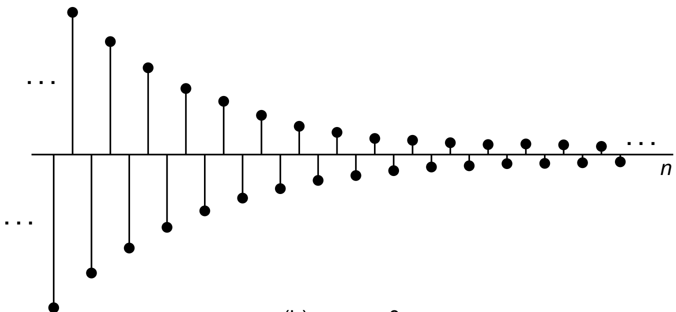
$$x(n) = e^{j\omega_0 n};\; \omega_0 = \frac{2\pi}{N},\; N = \text{period} \tag{1.40}$$

Using Euler's relationship,

$$e^{j\omega_0 n} = \cos\,\omega_0 n + j\,\sin\,\omega_0 n \tag{1.41}$$



(a) $0 < \alpha < 1$



(b) $- < \alpha < 0$

Fig. 1.27 *Real discrete exponential signal*

As in the continuous case, it then follows that

$$x(n) = \cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi}\, e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi}\, e^{-j\omega_0 n} \tag{1.42}$$

A typical plot of discrete sinusoidal signal is provided in Fig. 1.28. It is to be noticed that it is a periodic signal with period $N = 12$.

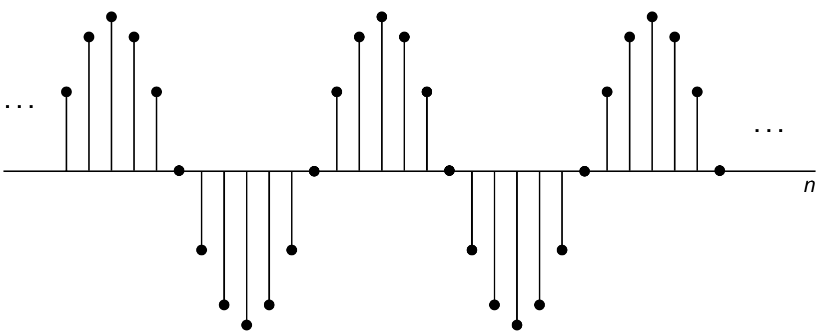


Fig. 1.28 *Plot of $x(n) = \cos(2\pi n/12)$*

The properties of the discrete sinusoid are quite different from the continuous sinusoid. These properties follow from the discrete exponential of Eq. (1.40). Let ω_0 be changed to $(\omega_0 + 2\pi)$. Then

$$x(n) = e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}\, e^{j2\pi}$$
$$= e^{j\omega_0 n} \quad \text{as} \quad e^{j2\pi} = 1 \tag{1.43}$$

Thus, the signal frequency remains at ω_0 even after increasing ω_0 to $(\omega_0 + 2\pi)$. In a continuous case, ω_0 increase means increase in signal frequency. In fact, discrete-case frequency changes in the range $0 \le \omega_0 \le 2\pi$ or any other 2π range, say, $-\pi \le \omega_0 \le \pi$.

Further, the periodicity implied by Eq. (1.43) does not increase continuously over 2π span of ω_0. In fact, N (period) at $\omega_0 = 0$ corresponds to dc as $e^{j\omega_0 N} = 1$ for all N. As ω_0 increases N (period) reduces (frequency of discrete oscillations increases) till it is minimum at $\omega_0 = \pi$ as

$$e^{j\pi n} = (-1)^n, \text{ discrete period } N = 2$$

which means that the signal reverses at every discrete count, and oscillation frequency is maximum.

As ω_0 increases further, N (period) reduces and at $\omega_0 = 2\pi$, the condition of $\omega_0 = 0$ recurs. Thus, the discrete oscillating frequency decreases from $\omega_0 = 2\pi$, $N = 2$) to $\omega_0 = 2\pi$, the dc condition, $x(n) = 1$ for all n.

$x(n) = \cos(0_n) = 1$ for all n; period undefined (Fig. 1.29a).

	N (period)
$= \cos(\pi n/8)$	16
$= \cos(\pi n/4)$	8
$= \cos(\pi n/2)$	4
$= \cos(\pi n)$	2
$= \cos(3\pi n/2) = \cos(2\pi n - 3\pi n/2) = \cos(\pi n/2)$	4
$= \cos(7\pi n/4) = \cos(2\pi n - 7\pi n/4) = \cos(\pi n/4)$	8
$= \cos(15\pi n/8) = \cos(2\pi n - 15\pi n/8) = \cos(\pi n/8)$	16
$= \cos(2\pi n) = \cos(0_n) = 1$ for all n, Fig. 1.29(a).	

The signal oscillations for $\omega_0 = \pi/4$ are shown in Fig. 1.29(b) which will repeat at $\omega_0 = 7\pi/4$.

Periodicity of Discrete Exponential

$$x(n) = e^{j\omega_0 n}$$

For its period to be $N > 0$,

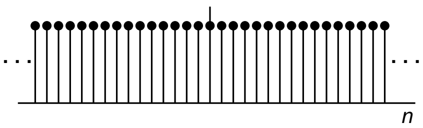
$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

which implies

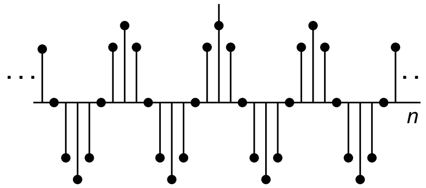
$$e^{j\omega_0 N} = 1$$

Therefore, for periodicity

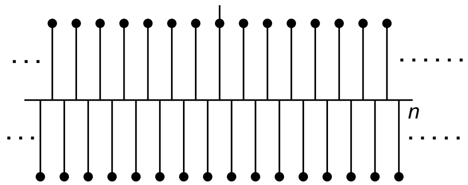
$$\omega_0 N = 2m\pi;\ m = \text{integer}$$



(a) cos (0*n*) = cos (2π*n*)



(b) cos (π*n*/4) = cos (7π*n*/4)



(c) cos (π*n*)

Fig. 1.29

or $$\frac{\omega_0}{2\pi} = \frac{m}{N} = \text{rational number}$$

The period is then

$$N = \frac{2\pi m}{\omega_0} \tag{1.44}$$

This result is different from the continuous case ($e^{j\omega_0 t}$) which is periodic for every value of ω_0.

1.8 SOME EXAMPLES

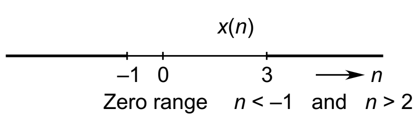
Example 1.10 *x(n) is a signal which is zero for n < –1 and n > 3. Determine values of n for which the following transformed signals are zero.*

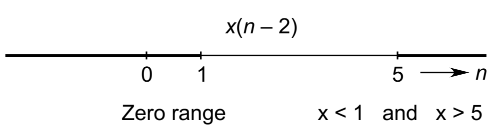
(a) x(n – 2) *(b) x(n + 3)*

(c) x(–n + 1) *(d) x(–n)*

Solution

(a) $x(n - 2)$ is $x(n)$ shifted right by 2





Directly,

$$n-2<-1 \quad \text{or} \quad n<1$$
$$n-2>3 \quad \text{or} \quad n>5$$

(b) $n+3<-1$ or $n<-4$

$$n+3>3 \quad \text{or} \quad n>0$$

(c) $-n+1<-1$ or $-n<-2$ or $n>2$

$-n+1>3$ or $-n>-2$ or $n<2$

(d) $-n<-1$ or $n>1$

$-n>3$ or $n<-3$

Example 1.11 *Determine which of the following signals is periodic and determine its period.*

(a) $x_1(t) = je^{j8t}$ *(b)* $x(t) = e^{(-1+j)t}$

(c) $x_1(n) = e^{j5\pi n}$ *(d)* $x_2(n) = e^{j2/5(n+1/3)}$

Solution

(a) $je^{j8t} = e^{j\left(8t+\frac{\pi}{2}\right)}$

$$\omega_0 = 8 = \frac{2\pi}{T}$$

or
$$T(\text{period}) = \frac{\pi}{4}$$

(b) $e^{(-1+j)t}$

$$(-1+j) = \omega_0 = \frac{2\pi}{T}$$

T cannot be complex. So, it is nonperiodic.

(c) $e^{j5\pi n} = e^{j(4\pi+\pi)n} = e^{j\pi n}$

$$\omega_0 = \frac{2\pi}{N} = \pi$$

or
$$N = 2 \text{ periodic}$$

(d) $e^{j2/5(n+1/3)}$

$$\omega_0 = \frac{2\pi}{N} = \left(\frac{2}{5}\right)k$$

N cannot be integer, nonperiodic.

Example 1.12 *Express the real part in the following signals in the form* $A\,e^{-\alpha t}\cos(\omega t + t)$ *where* A, α, ω *and* ϕ *are real and* $-\pi < \phi \le \pi$.

(a) $\sqrt{2}\,e^{j\pi/4}\cos(5t+2\pi)$ *(b)* $j\,e^{(-2+j100)t}$

Solution

(a) $\sqrt{2}\, e^{j\pi/4} \cos(5t + 2\pi)$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right]\cos 5t$$

$$= (1 + j)\cos 5t$$

Real part is $\cos 5\,t$.

(b) $j\, e^{(-2+j100)t}$

$$e^{-2t}\, je^{j100t} = e^{-2t} j\,(\cos 100\,t + j\sin 100\,t)$$

Real part is

$$-e^{-2t}\sin 100\,t$$

Example 1.13 *Determine which of the following signals is periodic and what is its fundamental period.*

(a) $x(t) = \cos\left(6t + \dfrac{\pi}{3}\right)$ (b) $x(t) = e^{j(\pi t - 1)}$

(c) $x(t) = \left[\cos\left(2t + \dfrac{\pi}{3}\right)\right]^2$ (d) $x(n) = \cos\left(\dfrac{6\pi}{7}n + 1\right)$

(e) $x(n) = \sin\left(\dfrac{n}{8} - \pi\right)$ (f) $x(n) = 2\cos\left(\dfrac{\pi}{4}n\right)\cos\left(\dfrac{\pi}{3}n\right)$

Solution

(a) $\omega_0 = \dfrac{2\pi}{T} = 6 \quad \text{or} \quad T = \dfrac{\pi}{3}$ period

(b) Complex number, so nonperiodic.

(c) $x(t) = \left[\cos^2\left(2t + \dfrac{\pi}{3}\right) - 1\right] + 1$

$$= \cos\left(4t + \frac{2\pi}{3}\right) + 1$$

$$\omega_0 = \frac{2\pi}{T} = 4 \quad \text{or} \quad T = \frac{\pi}{2}, \text{ periodic}$$

(d) $\omega_0 N = 2\pi k$

$$\frac{6\pi}{7}N = 2\pi k;$$

$$k = 3$$

$$N = 21, \text{ periodic}$$

(e) $\omega_0 N = 2\pi k$

$$\frac{1}{8}N = 2\pi k$$

$$\frac{k}{N} = \frac{\pi}{4}$$

Nonperiodic

(f) $2\cos\left(\frac{\pi}{4}n\right)\cos\left(\frac{\pi}{3}n\right) = \cos\left(\frac{7\pi}{12}n\right) - \cos\left(\frac{\pi}{12}n\right)$

$$\frac{\pi}{12}N = 2\pi k$$

$$N = 24; \text{ fundamental period}$$

The first term is seventh harmonic.

1.9 CLASSIFICATION OF SYSTEMS

Systems can be classified into the following six forms.

1. Lumped and Distributed Parameter Systems

A system is a collection of individual elements connected in a particular configuration. It is said to be a lumped parameter system, if a disturbance initiating at any point in the system propagates instantaneously to every point in the system. This assumption is valid if the largest physical dimension of the system is small as compared to the wavelength of the highest significant frequency present in the signal. Lumped parameter systems can be modelled by ordinary differential equations.

A distributed parameter system takes a finite amount of time for a disturbance at one point to be propagated to another point. It means that we have to deal with the space variable in addition to the independent time variable. The equations describing distributed parameter systems are therefore, partial differential equations.

All systems are, in fact, distributed parameter systems to some extent. We approximate a system to an equivalent system consisting of lumped elements. In a lumped model, energy is considered to be stored or dissipated in distinct isolated elements such as inductors, capacitors, resistors, masses, springs and dampers, etc.

Lumping is very commonly used in engineering systems. In electronic circuits and systems, its validity is based on the fact that the physical dimensions of the components are far smaller than the wavelength of the signal frequencies (even for 10 MHz the wavelength is about 10 m; compare this with the dimensions of transistors and other components). Of course where transmission lines are concerned, lumping when used has to make adhoc allowance for distributive effects. In microwaves and optical circuits, travelling wave phenomenon has to be considered.

In civil structures and mechanical systems with large distributed masses, finite element method is employed to account for distribution. Similar techniques are employed in chemical systems wherever lumping with adhoc adjustment of elements and parameters does not prove satisfactory.

2. Time-invariant (Fixed) and Time-varying

In a time-invariant system, the system characteristics remain fixed with time, while the characteristics of a time-varying system change with time. To express it mathematically, if the input to a system is shifted in time as $r(t \pm \tau)$, its output shifts by the same time, and hence such a system is also called shift-invariant.

Consider the following two input-output relations.

$$y(t) = c\, r(t) + r^3(t) \tag{i}$$

and

$$y(t) = c\, t\, r(t) + r^3(t) \tag{ii}$$

Let the input be shifted by τ to $r(t \pm \tau)$.

The output of the first system is then given as

$$c\, r(t \pm \tau) + r^3\,(t \pm \tau) = y(t \pm \tau) \tag{iii}$$

This means that it is a time-invariant system.

The output of the second system is given as

$$c\, t\, r(t \pm \tau) + r^3\,(t \pm \tau) \neq y(t \pm \tau) \tag{iv}$$

This system is therefore, time-variant, i.e. it is time-varying. This is due to the presence of multiplier t in the first term of the relationship.

For testing a discrete-time system, we shift input $r(n)$ in time as $r(n \pm n_0)$.

Example 1.14 *Is the following signal time-invariant?*

$$y = x(t/2)$$

Solution

$$y_1 = x_1(t) = x(t/2);\ x(t) \text{ expanded two times}$$
$$x_2(t) = x(t - t_0)$$
$$y_2 = x_2(t/2);\ x_2(t) \text{ expanded two times}$$

It is $x(t)$ expanded two times and delayed by $2\, t_0$.

It means

$$y_2(t) = y_1(t - 2\, t_0) \neq y_1(t - t_0)$$

So it is not time invariant.

Let $t_0 = 1$. The corresponding signals are drawn in Fig. 1.30 for a pulse signal of width 2 and height 2.

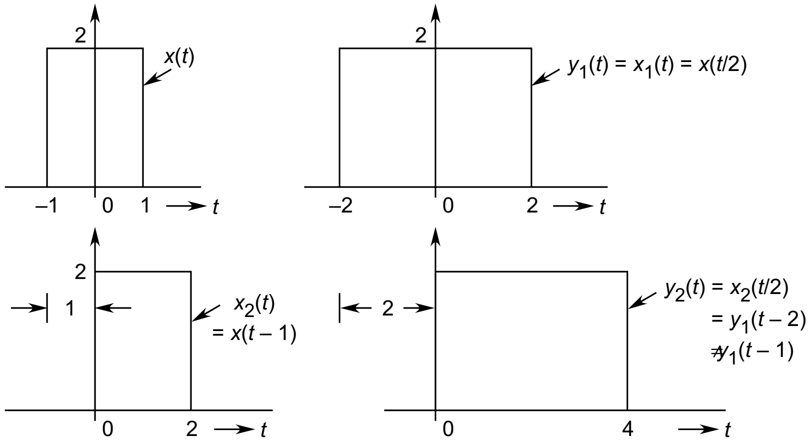


Fig. 1.30

Conclusion: Signal is not time-invariant.

Example 1.15 *Consider the discrete time system*

$$y(n) = nx(n)$$

Check if it is time-varying.

Solution Let us apply the test for time-invariance.

$$x(n) \to x(n - n_0)$$

Then

$$y(n) = nx(n - n_0) \neq y(n - n_0)$$

So, it is a time-varying system because of the multiplier n with $x(n)$.

Example 1.16 *Check if the system*

$$y(n) = r^2 (n - 2)$$

is time-invariant.

Solution An arbitrary input $r_1(n)$

$$y_1(n) = r_1^2(n - 2)$$

A time-shifted input $r_2(n) = r_1(n - n_0)$ would produce

$$y_2(n) = r_2^2(n - 2) = r_1^2(n - n_0 - 2) = y_1(n - n_0)$$

The system is therefore time-invariant.

3. Causal and Noncausal Systems

In causal or nonanticipating systems, output at any time depends only on past and present input and not on future inputs. On the other hand, in a noncausal or anticipating system the output at any time depends not only on past and present input but also on inputs in future.

Let us examine the output–input relationships for two systems.

$$y(t) = r(t-1);\ t > t_0 + 1 \qquad \text{(i)}$$

$$y(t) = r(t+1);\ t > t_0 \qquad \text{(ii)}$$

For the system described by Eq. (i), output at any time depends only at input that occurred unit time earlier. On the other hand, for system of Eq. (ii) the output at any time depends on the input that would occur unit time later. So Eq. (i) describes a causal system and Eq. (ii) is that of a noncausal system.

Consider the example of discrete time system

$$y(n) = x(n) - x(n-1) \qquad \text{(iii)}$$

$$y(n) = x(n) - x(n+1) \qquad \text{(iv)}$$

The system of Eq. (iii) is causal as the output depends on past values, while the system of Eq. (iv), is noncausal as its output depends on future values.

Causality is an intrinsic property of every physical system. All commercial systems in perfect competition involved in production of goods are anticipatory systems. The present production level is estimated for a future date anticipating the market demand at that time (in future). In this example the input/output signals are essentially discrete.

Example of a noncausal averaging system is

$$y(n) = \frac{1}{2M+1}\sum_{k=-M}^{+M} x(n-k) \qquad \text{(v)}$$

4. Static and Dynamic Systems (Memory and Memoryless System)

If the output of a causal system depends on the present as well as past input received till that time, it is called a dynamic system. A dynamic system exhibits the characterization of 'memory' as it remembers the input received earlier. Other kinds of systems where output signal only depends upon the present input are memoryless, instantaneous or static systems.

Let us consider two elemental circuits. A resistor as shown in Fig. 1.31(a) and a capacitor as shown in Fig. 1.31(b). Each of which is excited by a current $i(t)$, which is then the input.

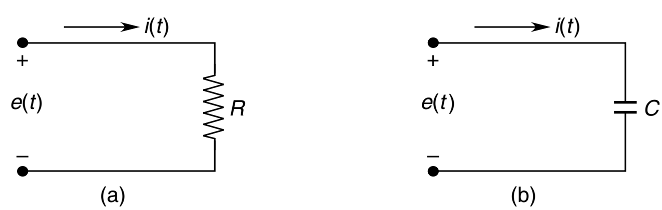


Fig. 1.31 *(a) Resistor (b) Capacitor*

In case of resistor the output $e(t)$ is given as

$$e(t) = R\,i(t) \qquad \text{(i)}$$

This output depends only upon the input at that time, so the resistor represents memoryless or static system.

In case of the capacitor, output is expressed as

$$e(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \qquad \text{(ii)}$$

It is immediately seen that the output depends on all past inputs (integral of $i(t)$) from $t = -\infty$ to t. So this is a dynamic system or memory system.

Consider the discrete-time system

$$y(n) = [r(n) + r^2(n)]^2 \qquad \text{(iii)}$$

The output $y(n)$ depends only on the instantanious value of $r(n)$. So it is a memoryless system.

Identity System It is defined as

$$y(t) = x(t)$$
$$y(n) = x(n)$$

These are memoryless systems

A discrete *accumulator or* summer is

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

It has to store past value of $x(k)$ including $x(n)$ so it is a memory system.

The accumulator equation can be expressed as

$$y(n) = \sum_{k=-\infty}^{n-1} x(n) + x(n)$$

or

$$y(n) = y(n-1) + x(n)$$

which is a causal difference equation.

Delay

$$y(t) = x(t - t_0)$$
$$y(n) = x(n - n_0)$$

It is a memory system as the system has to remember the past values of $x(t)$/$x(n)$.

The systems of all the above examples are causal systems. All memoryless system are causal.

5. Linear and Nonlinear Systems

A system is linear if it satisfies the following two conditions.

Consider a system with input $r(t)$ and output $y(t)$, i.e.,

$$r(t) \rightarrow y(t)$$

(i) *Homogeneity* If the input $r(t)$ is scaled by a factor, the output is scaled by the same factor, i.e.,

$$r_1(t) = \alpha r(t) \rightarrow y_1(t) = \alpha y(t) \tag{1.45}$$

This property of a system is known as homogeneity.

As a consequence of homogeneity,

if
$$\alpha = 0, \text{ then}$$
$$\alpha\, r(t) \rightarrow \alpha\, y(t)$$
$$0\, r(t) \rightarrow 0\, y(t) = 0$$

It means zero input produces zero output, provided the initial conditions are zero.

(ii) *Superposition* If a system is given two inputs simultaneously, the output is the sum of individual outputs caused by the application of each input separately. Let

$$r_1(t) \rightarrow y_1(t)$$
$$r_2(t) \rightarrow y_2(t)$$

If

$$r(t) = r_1(t) + r_2(t)$$

then

$$y(t) = y_1(t) + y_2(t) \tag{1.46}$$

(iii) *Linearity* A system that satisfies both homogeneity and superposition is said to be linear. By combining the two properties, we express linearity in the following manner.

If

$$r(t) = \alpha r_1(t) + \beta r_2(t)$$

then

$$y(t) = \alpha y_1(t) + \beta y_2(t) \tag{1.47}$$

Thus

Linearity = homogeneity and superposition

A linear system posseses additional property of decomposition, which is now defined.

(iv) *Decomposition* In a linear system the response to any input can be decomposed into two additive parts, i.e.

Linear system response to an input

= zero-state response + zero-excitation response (1.48)

Illustrative example follows. Consider a system, described by a first-order differential equation with constant coefficients.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = r(t) \tag{i}$$

where

$$y(t) = \text{output}$$
$$r(t) = \text{input}$$

For inputs $r_1(t)$ and $r_2(t)$, we can write

$$\frac{dy_1}{dt} + y_1 = r_1 \tag{ii}$$

$$\frac{dy_2}{dt} + y_2 = r_2 \tag{iii}$$

By doing so we can avoid writing function t repeatedly. Multiplying Eq. (ii) by scalar α and Eq. (iii) by scalar β and then adding them, we get

$$\frac{d}{dt}(\alpha y_1 + \beta y_2) + \alpha y_1 + \beta y_2 = \alpha r_1 + \beta r_2 \tag{iv}$$

Therefore

$$\alpha_1 r_1 + \beta r_2 \quad \Rightarrow \quad \alpha y_1 + \beta y_2$$

The system thus satisfies both the conditions of linearity and is hence linear. Let us now check whether the response of this system can be decomposed or not?

Let zero-state (zero initial condition) response be y_r and zero-input response by y_0. Then from Eq. (i)

$$\frac{dy_r}{dt} + y_r = r; \quad \text{zero initial condition} \tag{v}$$

$$\frac{dy_0}{dt} + y_0 = 0; \quad \text{zero input} \tag{vi}$$

Adding, Eq. (v) and Eq. (vi) we have

$$\frac{d}{dt}(y_r + y_0) + (y_r + y_0) = r \tag{vii}$$

which means that the system response is

$$y = y_r + y_0 \tag{viii}$$

As per Eq. (viii), the system response has been decomposed as expected.

Note: The response of a linear system can be decomposed. However, any system whose response can be decomposed need not be linear unless the two decomposed components individually satisfy the conditions of linearity.

In view of the properties of a linear system expressed above, we can apply the **superposition integral** (to be explained later), which leads to linear system analysis for a general input.

Example 1.17 *Consider the system described by the following differential equation.*

$$\frac{dy(t)}{dt} + ty(t) = r(t)$$

Check its linearity.

Solution For two inputs

$$\frac{dy_1}{dt} + ty_1 = r_1 \tag{i}$$

$$\frac{dy_2}{dt} + ty_2 = r_2 \tag{ii}$$

Multiplying Eq. (i) and (ii) by scalars α and β respectively and then adding and reorganizing them, we have

$$\frac{d}{dt}(\alpha y_1 + \beta y_2) + t(\alpha y_1 + \beta y_2) = \alpha r_1 + \beta r_2$$

This shows that

$$\alpha r_1 + \beta r_2 \xrightarrow{\text{Response}} \alpha y_1 + \beta y_2$$

Therefore, the system is linear and obeys the superposition law. But observe that it is a time-varying system.

Example 1.18 *Consider yet another system described by the following differential equation.*

$$y(t)\frac{dy(t)}{dt} + y(t) = r(t)$$

Check its linearity.

Solution For two inputs

$$y_1\frac{dy_1}{dt} + y_1 = r_1(t) \tag{i}$$

$$y_2\frac{dy_2}{dt} + y_2 = r_2(t) \tag{ii}$$

Multiplying Eq. (i) and (ii) by scalars α and β respectively and then adding them, we get

$$\left(\alpha y_1 \frac{dy_1}{dt} + \beta y_2 \frac{dy_2}{dt}\right) + (\alpha y_1 + \beta y_2) = \alpha r_1 + \beta r_2 \tag{iii}$$

Linearity requires that $y_1 \to \alpha y_1$ and $y_2 \to \beta y_2$ and in that case the result of Eq. (iii) should have been

$$\left(\alpha y_1 \frac{d\alpha y_1}{dt} + \beta y_2 \frac{d\beta y_2}{dt}\right) + (\alpha y_1 + \beta y_2) \neq \alpha r_1 + \beta r_2 \tag{iv}$$

But Eq. (iii) is different from Eq. (iv) hence the system in question is nonlinear.

The linearity of discrete-time systems can be tested on similar lines and will be taken up after modelling of discrete-time systems.

8. Stable and Unstable Systems

It is seen above that the response of a linear system can be subdivided into two parts. The system stability for these two parts is defined below.

(i) *Zero-input Response* If a system is brought to any particular initial condition (or state) and its response decays continuously to zero state, the system is said to be stable of a particular kind called **asymptotically stable**. On the other hand, if the response grows without any limit it is an **unstable system**. A gravity-driven pendulum always returns to its vertical (down) state (which is zero state or equilibrium state), therefore, it is a stable system. Though the response is oscillatory, it continuously decays to zero state. Consider now a vertical pendulum in vertical (top) state. It is theoretically in equilibrium state. But the slightest shift away from this position causes it to move *sideways downwards* with a limit (till it hits the horizontal ground). This simple system is unstable but the limit is imposed by the practical reality.

(ii) *Zero-state Response* If a system is excited with bounded input

$$|r(t)| \leq \partial_1 \leq \infty$$

and its output is also bounded, it is said to be **bounded-input bounded-output (BIBO)** stable.

A Linear time-Invariant (LTI) system, which is BIBO stable, is also asymptotically stable and vice versa, but this is not true for nonlinear systems.

The definitions and conclusions defined above also apply to discrete-time LTI system.

A practical system will be of use only if it is stable. An unstable system must be stabilized by feedback and compensation techniques. System stability will be discussed in detail in Chapter 8.

Example 1.19 *For the continuous time systems with input x(t) and output y(t) as listed below. Find out for each system whether the system satisfies the properties of (i) memoryless, (ii) time-invariant, (iii) linearity, or (iv) causality.*

(a) $y(t) = x(t-1) + x(1-t)$ (b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$

(c) $y(t) = [\sin(3t)]\, x(t)$ (d) $y(t) = \dfrac{dx}{dt}$

(e) $y(t) = x(2t)$ (f) $y(t) = \begin{cases} 0 & ; x(t) < 0 \\ x(t) + x(t-1) & ; x(t) > 0 \end{cases}$

Solution

(a) $y(t) = x(t-1) + x[-(t-1)]$
 If we take signal $\alpha x_1(t)$ and $\beta x_2(t)$, then $y(t) = y_1(t) + y_2(t)$. So the system is linear.

(b) Integration is a linear operation.

(c) $y(t)$ is related only to the values of $x(t)$ at that instant. So it is memoryless.

(d) Differentiation is linear operation. The output dependent on the last past increment in $x(t)$, so it is causal. Any shift in time for $x(t)$ appears in $y(t)$, so it is time-invariant.

(e) Linear but not time invariant.

Input $x_1(t)$, output $y_1(t) = x_1(2t)$, $x(t)$ compressed to 1/2 in time.

Input $x_2 = x_1(t - t_0)$, output $y_2(t) = x_2(t)$ compressed to 1/2.

Therefore, $y_2(t) = y_1(t - t_0/2)$. System is not time-invariant.

(f) Linear and causal.

Output $y(t)$ depends on past input.

1.10 SYSTEMS MODELLING

For analysis and design of systems, it is essential to build a representative model of the system. For this, it is required to identify the system as an interconnection of the basic elements. Using the elemental laws and the laws governing their interconnections, a suitable **model** of the system can be built. As approximations and idealizations are needed to identify the elemental form of the system; model is only accurate to that extent. To what accuracy the model should be built is an engineering decision. Large complex systems may need to be divided into subsystems whose models are then identified. Therefore, simulation and prototype testing may be needed when the system is put together or results of system analysis are put to any practical use. Different types of models are described as follows.

Mathematical Models

As per the system model discussed above, we can write a set of equations which are then the model of the system. This has been illustrated in most of the previous examples.

When only the terminal properties of a system are required, the input–output description of the system in the form of difference or differential equations linking the input and output variables would suffice. In the transform domain, the transfer function is a convenient and powerful input–output model. Whenever the internal as well as terminal behaviour of a system is required, the state-variable description of the system provides a suitable model. Of course, in the case of networks, the loop/nodal method of analysis also provides us with the internal behaviour of the system (current/voltage of each branch of the network).

The following are the two mathematical models of continuous-time LTI (linear time-invariant) systems.

Differential Equation Models Let us consider a simple mechanical system comprising three basic elements, mass, spring and damper (viscous friction) as shown in Fig. 1.32.

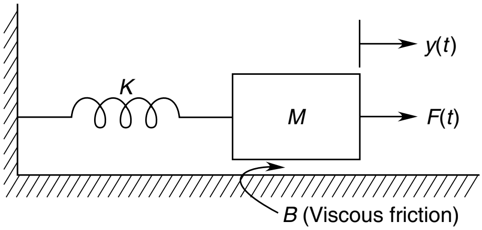


Fig. 1.32 *Simple mechanical system*

From the laws of the elements and their interconnection, we can write

$$M\frac{\mathrm{d}^2 y}{\mathrm{d}t} + B\frac{\mathrm{d}y}{\mathrm{d}t} + Ky = F(t) \tag{i}$$

This second-order linear differential equation describes the input [$F(t)$] and output [$y(t)$] relationship. The system is linear time-invariant as M and B are independent of time.

A linear system, in general, could be described by an n^{th} order differential equation given below.

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots\, a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_m\frac{\mathrm{d}^m r}{\mathrm{d}t}$$
$$+ b_{m-1}\frac{\mathrm{d}^{m-1} r}{\mathrm{d}t^{m-1}} + \cdots + b_1\frac{\mathrm{d}r}{\mathrm{d}t} + b_0 r;\ m < n \tag{1.49}$$

If a's and b's are functions of independent variable, time t, the system is time-variant. If a's and b's are constants, it is a linear time-invarient (LTI) continuous-time system. The order of the system is defined as the highest order of the derivative of the output $y(t)$ necessary to characterize the system.

State-variable Model Consider the simple example of a capacitor shown in Fig. 1.33, excited from a current source $i(t)$ and the system (capacitor) response is its voltage $e(t)$. We can write the differential equation of the system as

$$i(t) = C\frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{i}$$

After integration, we get

$$e(t) = \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)\,\mathrm{d}\tau + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau$$
$$= e(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,\mathrm{d}\tau \tag{ii}$$

The initial value of capacitor voltage $e(t_0)$ contains the history of the current that has flown through it from $-\infty$ to t_0. From another view point, the capacitor voltage at t_0 and, in fact, at any time t is the **state** of the system (in this case just a capacitor).

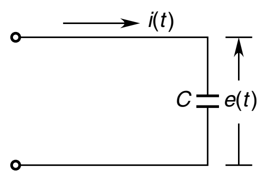


Fig. 1.33 *Capacitor*

Let us then define the state of the system as

$$x = e(t) \tag{iii}$$

From Eq. (i), we can write

$$\dot{x} = \frac{1}{C} i(t) \tag{iv}$$

Eq. (iv) is a first-order differential equation. The input $i(t)$ is usually written as

$$r(t) = i(t)$$

Then, Eq. (iv) takes the following form.

$$\dot{x} = \frac{1}{C} r \tag{v}$$

A formal definition of the state of system is as follows.

The state of a dynamical system is a minimal set of variables such that their knowledge at $t = t_0$ together with the knowledge of inputs for $t > t_0$ completely determines the behaviour of a system for $t > t_0$.

This definition is not restrictive and applies for linear time-varying and also nonlinear systems. It could easily be extended to discrete-time systems.

In a higher-order system the minimal set of state variables equals the order of the system and the first-order state equations are as many as the order of the system.

The following are the two mathematical models of discrete-time LTI systems.

Difference Equation Models Let us consider a bank saving plan which pays α per cent interest rate per month. If

$$\begin{aligned} y(0) &= \text{Initial deposit money of a person} \\ y(k-1) &= \text{Balance at the end of } (k-1) \text{ the month} \\ r(k) &= \text{Deposit or withdrawal during } k\text{th month (for } T = \text{one month)} \end{aligned}$$

The money balance equation at the end of kth month is then

$$y(k) = y(k-1) + \alpha/100\, y(k-1) + r(k) \tag{i}$$

In Eq. (i) $r(k)$ will be positive for deposit and negative for withdrawal. Rearranging Eq. (i), we get

$$y(k) = (1 + \alpha/100)\, y(k-1) + r(k) \tag{ii}$$

$$= \beta y(k-1) + r(k);\ \beta = (1 + \alpha/100) \tag{iii}$$

This equation can be rewritten as

$$y(k+1) = \beta y(k) + r(k+1) \tag{iv}$$

The banking system response $y(k)$ given in Eq. (iv) is first-order difference equation and the balance for $k \geq 1$ may be taken for the initial deposit money $y(0)$ and the input sequence signal $\{r(0), r(1), r(2), \ldots\}$.

The concepts of linearity (homogeneity + superposition) apply equally for linear discrete-time systems. We shall illustrate this through Example 1.20.

Example 1.20 *Consider a discrete-time described by the following difference equation*

$$y(k) = \frac{1}{2}r(k) + r(k-1) \tag{i}$$

For inputs $r_1(k)$ and $r_2(k)$, we can write the following equations by using Eq. (i)

$$y_1(k) = \frac{1}{2}r_1(k) + r_1(k-1) \tag{ii}$$

$$y_2(k) = \frac{1}{2}r_2(k) + r_2(k-1) \tag{iii}$$

Let the following input be applied to the system.

$$\alpha r_1(k) + \beta r_2(k)$$

The corresponding output as obtained from Eq. (i) is

$$\begin{aligned} y(k) &= \frac{1}{2}[\alpha r_1(k) + \beta r_2(k)] + [\alpha r_1(k-1) + \beta r_2(k-1)] \\ &= \left[\frac{1}{2}\alpha r_1(k) + \alpha r_1(k-1)\right] + \left[\frac{1}{2}\beta r_2(k) + \beta r_2(k-1)\right] \\ &= \alpha y_1(k) + \beta y_2(k) \end{aligned} \tag{iv}$$

So this discrete-time system is linear.

Example 1.21 *Check the linearity of the system*

$$y(n) = r^2(n-1)$$

Solution Let the input be

$$r_3(n) = \alpha r_1(n) + \beta r_2(n)$$

α, β are arbitrary constants

$$r_1(n) \to y_1(n) = r_1^2(n-1)$$
$$r_2(n) \to y_2(n) = r_2^2(n-1)$$

The output for input $r_3(n)$ is

$$\begin{aligned} y_3(n) &= r_3^2(n-1) = [\alpha r_1(n-1) + \beta r_2(n-1)]^2 \\ &= \alpha^2 r_1^2(n-1) + \beta^2 r_2^2(n-1) + 2\alpha\beta r_1(n-1) r_2(n-1) \\ &\neq \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

The system is therefore nonlinear.

In general, discrete-time system could be described by an nth order difference equation given below.

$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \dots + a_1 y(k+1) + a_0 y(k) \\ &\quad = b_m r(k+m) + \dots + b_1 r(k+1) + b_0 r(k) \end{aligned} \tag{1.50}$$

In the above equations, for an LTI non-anticipatory system a_i's and b_i's are constants and k, m, n are integers with $m \leq n$.

As in the case of a linear differential equation, Eq. (1.45), this difference equation has a solution which can be decomposed in the following form.

$$y(k) = y_0(k) + y_r(k)$$

where

$$y_0(k) = \text{zero-input response}$$
$$y_r(k) = \text{zero-state response}$$

State-variable Model The definition of state given in the preceding discussion for continuous-time is equally valid for discrete-time systems. Consider the example of the banking system presented above. The money balance (Eq. iv) can be written in state-variable form by defining

$$x_1(k) = y(k); \text{ state variable}$$

Then, Eq. (iv) being of first-order can now be expressed in terms of the single state variable identified above. Thus

$$x_1(k+1) = \beta x_1(k) + r(k+1) \tag{v}$$

1.11 INVERTABILITY

A system is said to be invertable if a distinct input produces a distinct output, which means that the systems, inverse exists. Therefore, a system and its inverse in tandem reproduce the input at the output as shows in Fig. 1.34 for a discrete case.

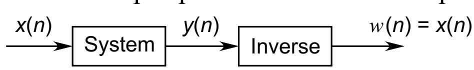


Fig. 1.34 *System and its inverse*

Example 1.22

1. $y(n) = \sum_{k=-\infty}^{n} x(k)$ System

   $w(n) = y(n) - y(n-1)$ Inverse

   The reader may check.
2. $y(t) = 4x(t)$ System

   $w(t) = \frac{1}{4} y(t) = x(t)$ Inverse
3. $y(t) = x^2(t)$ System

   $w(t) = \pm\sqrt{y(t)}$, sign ambiguous

   Inverse does not exist.

Example 1.23 *Determine the inverse of the following systems, if it is invertable. If not find two input signals to produce the same output*

(a) $y(t) = x(t-2)$ (b) $y(t) = \sin(x(t))$

(c) $y(n) = x(1-n)$ (d) $y(n) = \begin{cases} x(n-2) & n \geq 2 \\ 0 & n = 0 \\ x(n) & n \leq -2 \end{cases}$

(e) $y(n) = x(2n)$ (f) $= \int_{-\infty}^{t} x(\tau)d\tau$

(g) $y(t) = \dfrac{dx}{dt}$ (h) $y(t) = x(2t)$

Solution

(a) $x(t)$ has been delayed to $y(t) = x(t-2)$. If $y(t)$ is advanced by 2, then $y(t+2) = x(t)$. So it is invertable.

(b) Not invertable.

$$\sin(x(t) + 2\pi) = \sin(x(t)) = y(t), \text{ not distinct}$$

(c) $y(n) = x(1-n)$
Inverse is $y(n-1)$

(d) Inverse is

$$x(n+2);\ n \geq 2$$
$$x(n);\ n \leq -2$$

(e) Inverse is $x(n/2)$.

(g) Not invertable. Integration is not the inverse as limits cannot be defined. $x(t) = \text{constant}, y = 0$

(h) $x(2t)$ compressed $x(t)$ by 1/2. So inverse is $x(t/2)$ which expands it back to $x(t)$.

1.12 SOME IDEAL SIGNALS

Some basic ideal signals need to be introduced here to facilitate the concepts to the exposed in the chapters that follow.

Singularity Functions—Continuous Time

Unit Step It represents a sudden change as indicated in Fig. 1.35(a). It is mathematically defined as

$$u(t) = \begin{cases} 1;\ t > 0 \\ 0;\ t < 0;\ \text{undefined at } t = 0 \end{cases} \tag{1.51}$$

A unit step occuring at $t = \Delta t$, sketched in Fig. 1.35(b), is expressed in the following manner.

$$u(t - \Delta t) = \begin{cases} 1;\ t > \Delta t \\ 0;\ t < \Delta t \end{cases} \tag{1.52}$$

It is drawn in Fig. 1.35(b).

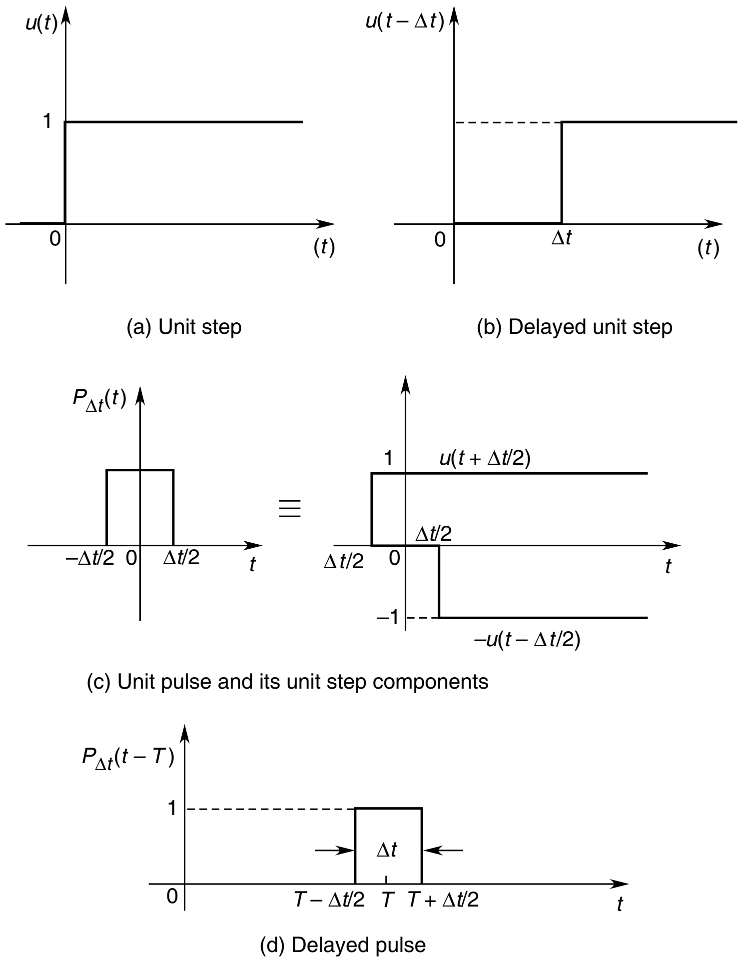


Fig. 1.35 *Unit pulses*

Unit Pulse (Width Δt) It is sketched in Fig. 1.35(c) and is mathematically defined at

$$P_{\Delta t}(t) = \begin{cases} 0 < -\,\Delta t/2 \\ 1; -\,\Delta t/2 < t < +\,\Delta t/2 \\ 0; t > +\,\Delta t/2 \end{cases} \tag{1.53}$$

We can write a pulse in terms of step functions as (Fig. 1.35(c))

$$P_{\Delta t}(t) = u(t + \Delta t/2) - u(t - \Delta t/2)$$

A delayed unit pulse occurring at time $t = T$ sketched in Fig. 1.35(d) is expressed in the following manner.

$$P_{\Delta t}(t-T) = \begin{cases} 0; t < (T - \Delta t/2) \\ 1; (T - \Delta t/2) < + < (T + \Delta t/2) \\ 0; t > (T + \Delta t/2) \end{cases} \tag{1.54}$$

Unit Impulse (δ-function) Consider a pulse occuring at $t = 0$ of height $1/\Delta$ and duration Δ as shown in Fig. 1.36(a). As we let $\Delta \to 0$, the pulse area remains 1 unit, and it occurs at $t = 0$. Such a function is known as impulse or δ - function and is mathematically expressed in the following form

$$\delta(t) = 0, t \neq 0 \tag{1.55a}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.55b}$$

Pictorial representation of an impulse is sketched in Fig. 1.36(b)

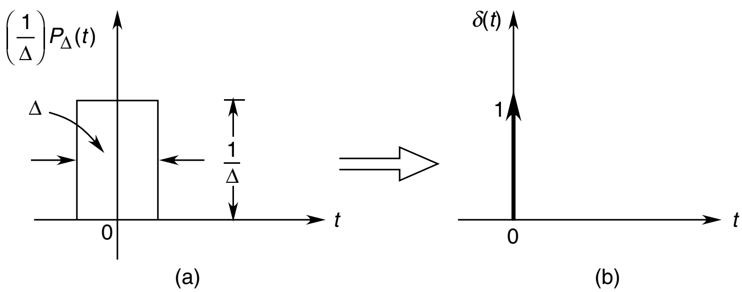


Fig. 1.36 *Unit impulse*

Properties of Unit Impulse The following are the properties of unit impulse.

1. (i) $f(t)\,\delta(t) = f(0)\,\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0) \tag{1.56}$$

(ii) $f(t)\,\delta(t - t_0) = f(t_0)$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\, dt = f(t_0); \text{ shifting or sampling property} \tag{1.57}$$

2. $\delta(at) = \dfrac{1}{|a|}\,\delta(t)$ (1.58)

Proof

We have from Eq. (1.55b)

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Let

$$t \to at;\ \mathrm{d}t \to a\ \mathrm{d}t$$

Then, we can write left side as

$$a \int_{-\infty}^{\infty} \delta(at)\ \mathrm{d}t;\ a > 0 \tag{1.59a}$$

and

$$a \int_{-\infty}^{\infty} \delta(at)\ \mathrm{d}t;\ a < 0 \tag{1.59b}$$

In the combined form, we can write Eqs. (1.59a and 1.59b) as

$$|a| \int_{-\infty}^{\infty} \delta(at)\ \mathrm{d}t \tag{1.59c}$$

We, therefore, get

$$|a| \int_{-\infty}^{\infty} \delta(at)\ \mathrm{d}t = \int_{-\infty}^{\infty} \delta(t)\ \mathrm{d}t$$

which implies

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

3. $\delta(t) = \dot{u}(t)$ = derivative of unit step (1.60)

Consider

$$\int_{-\infty}^{\infty} f(t)\frac{\mathrm{d}u(t)}{\mathrm{d}t}\ \mathrm{d}t = f(t)\ u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t$$

$$= f(\infty) - \int_{0}^{\infty} \frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t$$

$$= f(\infty) - f(t)\Big|_{0}^{\infty}$$

$$= f(\infty) - f(\infty) + f(0)$$

$$= f(0)$$

This is the property (1.(i)) of $\delta(t)$. Hence the result.

From Eq. (1.60), we can write

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau;\ \text{running integral}$$

As $\delta(\tau)$ is impulse at $\tau = 0$, the running integral is zero for $t < 0$ and unity for $t > 0$

Let $\sigma = t - \tau$; then

$$u(t) = \int_{-\infty}^{0} \delta(t - \sigma)\,(-d\sigma)$$

$$= \int_{0}^{\infty} \delta(t - \sigma)d\sigma$$

$\delta(t - \sigma)$ is an impulse at $\sigma = t$, so running integral is zero for $t < 0$ and unity for $t > 0$.

4. $\delta(-t) = \delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.61}$$

Let

$$t \to -t,\ dt \to -\,dt$$

$$-\int_{\infty}^{-\infty} \delta(-t)\, dt = \int_{-\infty}^{\infty} \delta(-t)\, dt = 1$$

Hence the result.

Example 1.24 *Sketch the following pulse*

$$P_{\Delta t}(2t + 6),\ \ \Delta t = 1s$$

Solution Rewriting the given pulse expression, we get

$$p_{\Delta t}(2(t + 3)),\ \ \Delta t = 1s$$

We find that the pulse occurs at $t = -3$ and because of multiplier 2, gets compressed in time by a factor of 2.

Its width, therefore, is

$$\Delta t/2 = 0.5\text{s}$$

The pulse is sketched in Fig. 1.37.

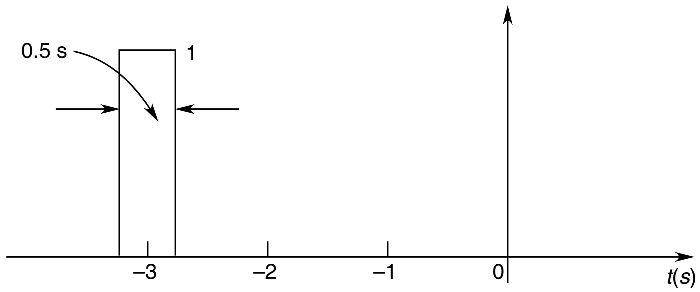


Fig. 1.37

Discrete-time Unit Impulse and Unit Step Sequence

A discrete-time unit impulse or unit sample is defined as

$$\delta(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \tag{1.62}$$

It is shown in Fig. 1.38(a). A delayed unit impulse

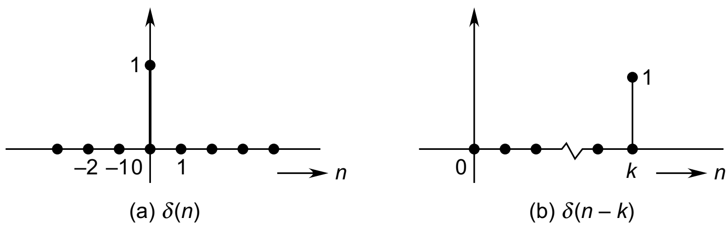


Fig. 1.38

is expressed as

$$\delta(n-k) = \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases} \tag{1.63}$$

which is shown in Fig. 1.38(b).

A discrete-time unit step is defined as

$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} \tag{1.64}$$

and is shown in Fig. 1.39.

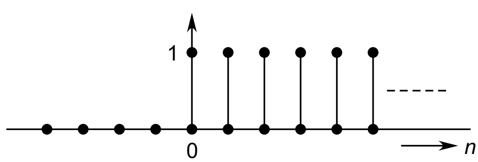


Fig. 1.39

A discrete unit impulse can be related to a unit step as

$$\delta(n) = u(n) - u(n-1); \text{ find difference} \tag{1.65}$$

This corresponds to differentiation of unit step in a continuous-time case.

A discrete-time unit step can be expressed as a summation of impulses

$$u(n) = \sum_{k=-\infty}^{0} \delta(n+k) = \sum_{k=0}^{\infty} \delta(n-k) \tag{1.66}$$

It is easily seen that $u(n)$ is a sum of impulses $\delta(n)$, $\delta(n-1)$, … which corresponds to Fig. 1.39.

Sampling Property of Unit Impulse

$$x(n)\,\delta(n) = x(0)\,\delta(n)$$

In general,

$$x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$$

The impulse extracts the signal value at the recurrence of impulse.

This property would play an important role in our study of signals and systems.

1.13 ENERGY AND POWER SIGNALS

Suppose $e(t)$ is a voltage across a resistor producing current $i(t)$. Then instantaneous power per ohm is

$$P(t) = \frac{e(t)i(t)}{R(1\Omega)} = i^2(t) \tag{1.67}$$

Integrating over the interval $-\infty$ to $+\infty$, total energy and average power on per ohm basis are defined respectively as the following limits.

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)\,\mathrm{d}t \text{ J} \tag{1.68}$$

and

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\,\mathrm{d}t \text{ W} \tag{1.69}$$

For an arbitrary signal $f(t)$, which may in general be complex, total energy normalized to per unit resistance is defined in the following equation.

$$E \stackrel{\Delta}{=} \lim_{T\to\infty} \int_{-T}^{T} |f(t)|^2\,\mathrm{d}t \tag{1.70}$$

The average power normalized to per unit resistance is defined in the following equation.

$$P \stackrel{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2\,\mathrm{d}t \tag{1.71}$$

Based on the definitions of Eqs. (1.65) and (1.66), the following classes of signals are defined.

1. $f(t)$ is an energy signal if and only if $0 < E < \infty$ so that $P = 0$.
2. $f(t)$ is a power signal if and only if $0 < P < \infty$ and $E = \infty$.
3. Signals not satisfying either of these two properties are neither energy nor power signals.

Energy and Power Spectral Densities The energy and power expressions of Eqs. (1.70) and (1.71) can be rewritten in the following forms respectively.

$$E = \int_{-\infty}^{\infty} G(\omega)\,\mathrm{d}\omega \tag{1.72}$$

and
$$P = \int_{-\infty}^{\infty} S(\omega)\, d\omega \tag{1.73}$$
where

$G(\omega)$ = energy spectral density (for energy signal); and
$S(\omega)$ = power spectral density (for power signal)

A sinusoidal signal has a two-sided spectrum as shown in Fig. 1.40 with a total power of $A^2/2$. Half of this power, i.e., $A^2/4$ associated with its power density spectrum is plotted in Fig. 1.40. It can be expressed mathematically as

$$S(\omega) = (A^2/4)\,[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{1.74}$$

Following is the total power (integrated).

$$P = 2 \times \frac{A^2}{4} = \frac{A^2}{2} \tag{1.75}$$

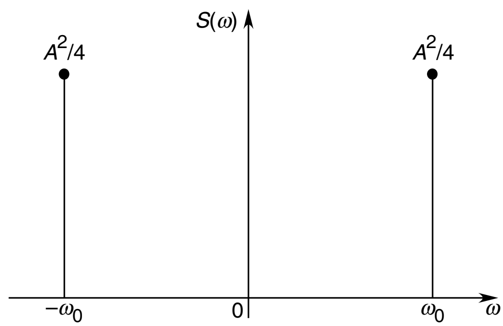


Fig. 1.40 *Two-sided spectrum of sinusoidal signal*

The concept of energy and power density will acquire a wider meaning with signals having continuum of frequencies. These aspects will be elaborated in Chapter 2.

Example 1.25 *Consider the following signal*

$$f_1(t) = A\, e^{-\alpha t}\, u(t),\ \alpha > 0$$

Then as per definition (1.70)

$$E = \lim_{T \to \infty} \int_0^T A^2 e^{-2\alpha t}\, dt$$
$$= \frac{A^2}{2\alpha}$$

Therefore $f_1(t)$ is an energy signal.

Let

$$f_2(t) = f_1(t)\Big|_{\alpha \to 0} = A\, u(t)$$

It is easy to see that this signal has $E = \infty$. Let us calculate its power.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-0}^{T} A^2 u^2(t)\, dt$$
$$= \frac{A^2}{2}$$

Example 1.26 *Consider a sine wave*

$$x(t) = A \cos(\omega_0 t + \theta_0)$$

As the wave exists for infinite time, its $E = \infty$. Let us now find its normalized power.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos^2(\omega_0 t + \theta_0)\, dt$$
$$= \lim_{T \to \infty} \frac{A^2}{2T} \int_{-T}^{T} \left[\frac{1}{2} + \frac{1}{2}\cos 2(\omega_0 t + \theta_0)\right] dt$$

The second integral term will work out to be zero. Thus

$$P = \frac{A^2}{2}$$

For the discrete-time case, the signal energy is

$$E \triangleq \lim_{N \to \infty} \int_{N=-\infty}^{N=+\infty} |f(n)|^2$$

and the signal power is

$$\triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{+N} |f(n)|^2$$

Additional Examples

Example 1.27 *Evaluate the following integrals:*

(a) $\displaystyle\int_{-\infty}^{20} (5 + \cos t)\, \delta(t - 10)\, dt$

(c) $\displaystyle\int_{-\infty}^{\infty} \cos 2\pi t\, \delta(t - 2)\, dt$

(b) $\displaystyle\int_{-\infty}^{\infty} (t - 2)^2\, \delta(t - 2)\, dt$

(d) $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha t^2}\, \delta(t - 10)\, dt$

Solution

(a) $$\int_{-\infty}^{20} (5 + \cos t)\, \delta(t - 10)\, dt = \int_{-\infty}^{20} (5 + \cos 10)\, \delta(t - 10)\, dt$$
$$= (5 + \cos 10); \text{ as the impulse occurs at } t = 10$$

(b) $\int\limits_{-\infty}^{\infty} (t-2)^2\, \delta(t-2)\, \mathrm{d}t = \int\limits_{-\infty}^{\infty} (2-2)^2\, \delta(t-2)\, \mathrm{d}t = 0$

(c) $\int\limits_{-\infty}^{\infty} \cos 2\pi t\, \delta(t-2)\, \mathrm{d}t = \int\limits_{-\infty}^{\infty} \cos 4\pi\, \delta(t-2)\, \mathrm{d}t = \cos 4\pi = 1$

(d) $\int\limits_{-\infty}^{\infty} e^{-\alpha t^2}\, \delta(t-10)\, \mathrm{d}t = \int\limits_{-\infty}^{\infty} e^{-100\alpha}\, \delta(t-10)\, \mathrm{d}t = e^{-100\alpha}$

Example 1.28 *Sketch the following pulse signals:*

(a) $P_1[(t-2)/3]$ *(b)* $P_1[(t-2)/2] + P_5(t-1)$

Solution

(a) $P_1\,[(t-2)/3]$

Pulse width = 1s
Amplitude = 1
Time of occurrence = 2s
Pulse width is enlarged to = 3 × 1 = 3s

The pulse is sketched in Fig. 1.41(a).

(b) $P_1[(t-2)/2] + P_5\,(t-1)$

This signal is sketched in Fig. 1.41(b).

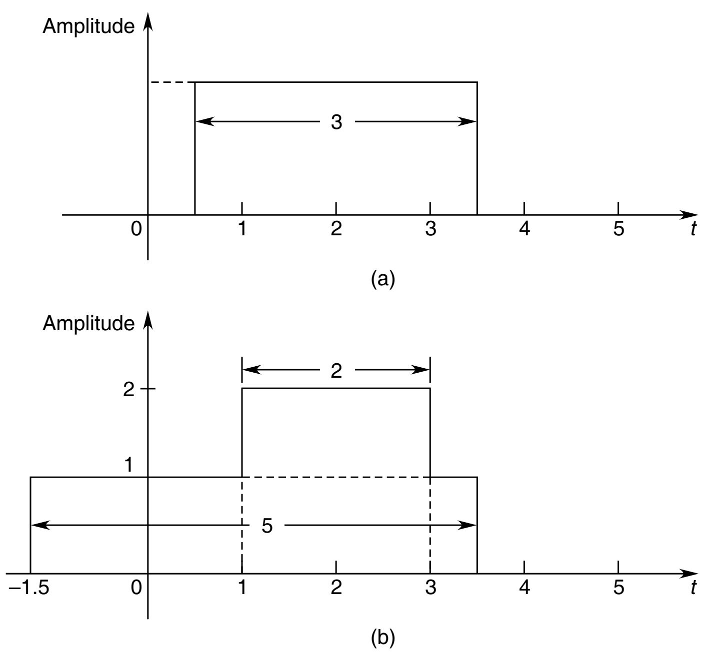


Fig. 1.41

Example 1.29 *Consider a sinusoidal signal*

$$\sin \omega_0 t;\ \text{period } T_0 = 2\pi/\omega_0$$

It is sampled with period T_0. Find the condition to be satisfied by T_0 and T_s for the sampled signal to be periodic.

For a sequence f(k) periodicity is defined as

$$f(k) = f(k + N) = f(k + 2N) = \ldots = f(k + nN)$$

where N and n are integers. The signal's fundamental period is then

$$N \text{ samples for } n = 1$$

Solution Substituting $t = kT$ in the sinusoidal function

$$\sin \omega_0 t \to \sin \omega_0 kT \tag{i}$$

For the discrete signal to be periodic

$$\sin \omega_0 kT = \sin \omega_0 (kT + NT) \tag{ii}$$
$$= \sin (\omega_0 kT + \omega_0 NT)$$
$$= \sin \omega_0 kT \cos \omega_0 NT + \cos \omega_0 kT \sin \omega_0 NT \tag{iii}$$

The equality in Eq. (iii) is only possible, if

$$\cos \omega_0 NT = 1 \text{ and } \sin \omega_0 NT = 0$$

or

$$\omega_0 NT = 2\pi n \quad \text{or} \quad \frac{\omega_0 T}{2\pi} = \frac{n}{N} \tag{iv}$$

The discrete signal would be periodic, if and only if $\omega_0 T/2\pi$ is a rational number (N and n are integers).

In other words $\sin \omega_0 kT$ is periodic, if and only if

$$n = \frac{\omega_0 NT}{2\pi} = \text{an integer} \tag{v}$$

Consider some illustrations.

1. $\sin 2k$; $\omega_0 = 2$, $T = 1$

   $$\frac{\omega_0 T}{2\pi} = \frac{1}{\pi}; \text{ not a rational number}$$

   This sequence will not repeat itself.

2. $\sin 0.1\pi k$; $\omega_0 = 0.1\pi$, $T = 1$

   $$N_1 = \frac{2\pi n}{\omega_0 T} = \frac{2\pi n}{0.1\pi} = 20n; \text{ periodic}$$
   $$= 20 \text{ samples per period for } n = 1$$

3. $\sin 0.3\pi k$; $\omega_0 \pi$, $T = 1$

   $$N_2 = \frac{2\pi n}{\omega_0 T} = \frac{2\pi n}{0.3\,\pi} = \frac{20n}{3}$$
   $$N_2 = 20 \text{ samples per period for } n = 3$$

Remark: In cases (2) and (3) above, samples/period are the same but the rate of change is 3 times in Illustration 3 as compared to Illustration 2. Sampled frequency does not have the same meaning as in continuous-time case. The reader should plot sampled sine waves given in illustrations 2 and 3 and observe the difference.

Example 1.30 *Find the integral of a unit-step function.*

Solution

$$r(t) = \int_0^t u(t)\mathrm{d}t$$

$$\int_0^t \mathrm{d}t = t;\ t > 0$$

$$= t\,u(t)$$

This signal is called a **ramp** and is sketched in Fig. 1.42.

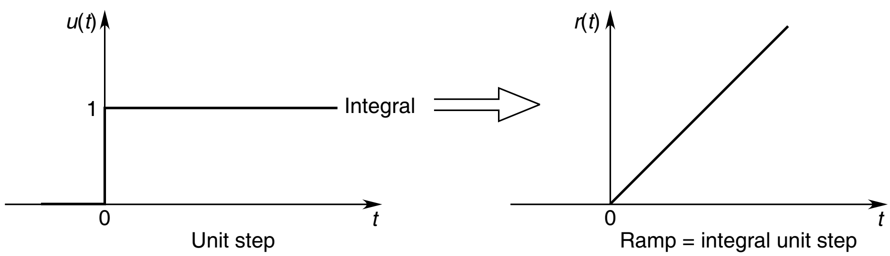


Fig. 1.42

Example 1.31

(a) *Sketch two nonsinusoidal periodic signals.*

(b) *State the condition for the sum of two sine waves to be periodic.*

(c) *Illustrate the answer of part (b) with two examples.*

Solution

(a) A triangular and a rectified sine wave are sketched in Fig. 1.43(a) and (b) respectively as examples of periodic signals.

(b) For the sum of two or more sinusoidal signals to be periodic, the ratio of their periods or frequencies should be a rational number, i.e., it should be the ratio of two integers.

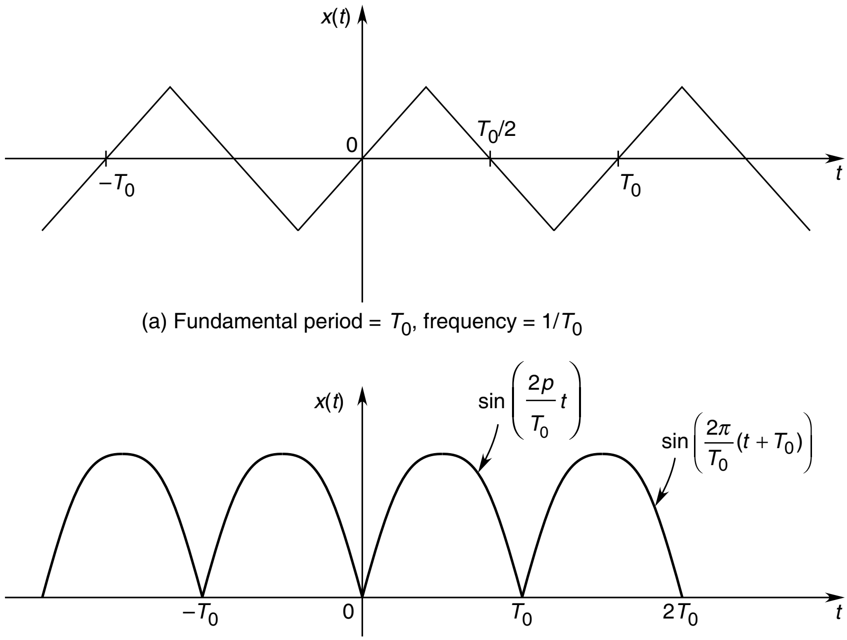
(c) (i) $\sin 100\pi t + \sin 300\pi t$

$\sin 100\pi t \rightarrow 100\pi = 2\pi f_1$

or

$$f_1 = 50 \text{ Hz},\ T_1 = 1/f_1 = 0.02 \text{ s}$$

$$\sin 300\pi t \rightarrow 300\pi = 2\pi f_2$$



(a) Fundamental period = T_0, frequency = $1/T_0$

(b) Fundamental period = T_0, frequency = $1/T_0$

Fig. 1.43

$$f_2 = 150 \text{ Hz}, \; T_2 = 1/150 \text{ s}$$
$$f_2/f_1 = 150/50 = 3$$

Therefore, the sum signal is periodic. Fundamental period $T_0 = T_1 = 0.02$ s or $f_0 = 50$ Hz. In one period of the sum signal there is one wave of first term and three waves of the second term.

(ii) $2 \sin 120\pi t + 5 \sin 317t$

$$\sin 120\pi \to f_1 = \frac{120\pi}{2\pi} = 60\text{Hz}, \; T_1 = 1/60 \text{ s}$$
$$\sin 317t \to f_2 = \frac{317}{2\pi}\text{Hz}, \; T_2 = \frac{2\pi}{317}\text{s}$$
$$f_2/f_1 = \frac{\omega_0 T}{2\pi} \times \frac{1}{60} = 0.841;$$

Since 0.841 is not a rational number, the sum is not periodic.

Example 1.32 *Given the following signals*

1. $2\cos 3\pi t + 3\sin 6\pi t$ *2.* $2\sin 2t + 3\cos \pi t$

3. $e^{-5t}u(t)$ *4.* $e^{t}u(t)$

(a) Identify the periodic signals and their fundamental periods

(b) *Identify energy signals and calculate their energies*

(c) *Identify power signals and calculate their average power*

Solution

(a)

1. $2 \cos 3\pi t + 3 \sin 6\pi t$

$$f_1 = \frac{3\pi}{2\pi} = 3/2 \text{ Hz}$$

$$f_2 = \frac{6\pi}{2\pi} = 3 \text{ Hz}$$

$$f_2/f_1 = 3 \div 3/2 = 2$$

As 2 is a rational number hence it is a periodic signal

Fundamental period = 1/(3/2) = 2/3 s

All other signals are aperiodic.

(b)

3. $e^{-5t}\, u(t)$

$$E = \int_{-\infty}^{\infty} (e^{-5t})^2\, u^2(t) \mathrm{d}t$$

$$= \int_0^{\infty} e^{-10t}\, \mathrm{d}t = \left.\frac{e^{-10t}}{-10}\right|_0^{\infty}$$

$$= -0 + \frac{1}{10} = 0.1\text{W}$$

It is an energy signal

4. $e^t\, u(t)$

$$E = \infty$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_0^T e^{2t}\, \mathrm{d}t = \infty$$

It is neither an energy nor a power signal.

(c)

1. $2 \cos 3\pi t + 3 \sin 6\pi t$

It is a power signal. Two-sided power spectral lines and their powers are as given under.

$\pm 3\pi$; power at each line $= (2)^2/4 = 1$ W

$\pm 6\pi$; power at each line $= (3)^2/4 = 9/4 = 2.25$ W

Total average power $= 2 \times 1 + 2 \times 2.25 = 6.5$ W

2. $2 \sin 2t + 3 \cos \pi t$

It is a power signal.

Total average power $= (2)^2/2 + (3)^2/2$

$= 2 + 4.5 = 6.5$ W

Example 1.33 *For the circuit of Fig. 1.44, write the describing differential equation in input–output form.*

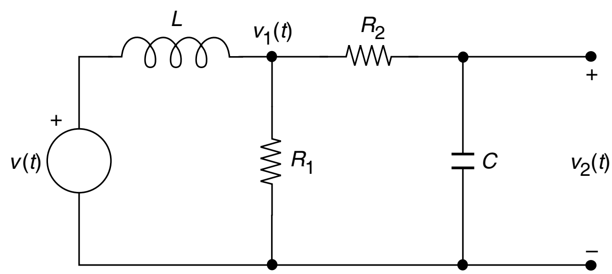


Fig. 1.44

Solution Applying KCL at the two nodes, we get

$$\frac{1}{L}\int_{-\infty}^{t}[v_1(\tau)-v(\tau)]\,\mathrm{d}\tau+\frac{1}{R_1}v_1(t)+\frac{1}{R_2}[v_1(t)-v_2(t)]=0 \tag{i}$$

and

$$\frac{1}{R_2}[v_2(t)-v_1(t)]+\frac{\mathrm{d}v_2(t)}{\mathrm{d}t}C=0 \tag{ii}$$

Differentiating Eq. (i) once, we get (not writing t within brackets)

$$\frac{1}{L}(v_1-v)+\frac{1}{R_1}\frac{\mathrm{d}v_1}{\mathrm{d}t}+\frac{1}{R_2}\left[\frac{\mathrm{d}v_1}{\mathrm{d}t}-\frac{\mathrm{d}v_2}{\mathrm{d}t}\right]=0 \tag{iii}$$

Substituting for v_1 from Eq. (ii) in Eq. (iii) and reorganizing it, we get the input-output (de) as

$$LC\left(\frac{R_1+R_2}{R_1}\right)\frac{\mathrm{d}^2v_2}{\mathrm{d}t^2}+(R_2C+L/R_1)\frac{\mathrm{d}^2v_2}{\mathrm{d}t}+v_2=v \tag{iv}$$

Example 1.34 *A bank issues a loan of Rs 1 00 000. It is to be returned in ten years in equal monthly instalments. The yearly interest rate is 15 per cent on the unpaid balance amount. Calculate the monthly instalments and the total amount received by the bank at the end of ten years.*

Solution

Let

$P(k)$ = unpaid balance at the end of kth month
x = monthly instalments (equal)
r = interest rate = 15 per cent yearly or 1.25 per cent monthly
$P(0)$ = Rs 1 00 000
$P(120)$ = Rs 0

The difference equation for the loan scheme is evolved below.

$$P(k + 1) = P(k)\,(1 + r) - x;\ k > 0 \tag{i}$$

Using this difference equation, we write the successive equations for $k = 1, 2, \ldots, k$ to find the solution for $P(k)$ in terms of initial value $P(0)$ and installments x. Thus

$$\begin{aligned}
P(1) &= (1 + r)\,P(0) - x \\
P(2) &= (1 + r)\,P(1) - x \\
&= (1 + r)^2\,P(0) - [1 + (1 + r)]x \\
P(3) &= (1 + r)^3\,P(0) - [1 + (1 + r) + (1 + r)^2]x \ldots \\
P(k) &= (1 + r)^k\,P(0) - [1 + (1 + r) + (1 + r)^2 \ldots + (1 + r)^{k-1}]x
\end{aligned}$$

or

$$P(k) = (1 + r)^k\,P(0)\left[\frac{(1+r)^k - 1}{r}\right]x \tag{ii}$$

Substituting values of $k = 120$ and $P(0) = 10^5$ in Eq. (ii), we get

$$P(120) = 0 = (1 + 0.0125)^{120} \times 10^5 - \left[\frac{(1+0.0125)^{120} - 1}{r}\right]x$$

Solving for x gives us,

$$\begin{aligned}
x &= \frac{(1.0125)^{120} \times 0.0125}{(1.0125)^{120} - 1} \\
&= \text{Rs } 1\ 613.37
\end{aligned}$$

$$\begin{aligned}
\text{Total amount received by the bank } &= 1\ 613.37 \times 120 \\
&= \text{Rs } 1\ 93\ 604.40
\end{aligned}$$

Example 1.35 *On Discrete-time Signals*

1. Let us consider a continuous-time sinusoidal signal

$$x_a(t) = A\cos(2\pi F_0 t + \theta) \tag{i}$$

It is sampled at rate $F_s = 1/T$. The resulting discrete-time signal is

$$x(n) = x_a(nT) = A\cos(2\pi f_0 n + \theta) \tag{ii}$$

where

$$f_0 = F_0/F_s = \text{relative frequency of the sinusoid.}$$

If

$$-F_s/2 \le F_0 \le F_s/2 \tag{iii}$$

or

$$-1/2 \le f_0 \le 1/2 \tag{iv}$$

then in this range of frequencies there is one-to-one correspondence between $x_a(t)$ and $x(n)$. It means that if $x(n)$ is known, we can construct $x_a(t)$.

This can also be stated as 'the sampling frequency must be at least twice the frequency of the analog sinusoidal signal which is sampled'.

Consider now the following sinusoids whose frequency lies outside the range specified above.

$$x_a(t) = A \cos (2\pi F_k t + \theta) \qquad \text{(v)}$$

where

$$F_k = F_0 + kF_s; \;\; k = \pm 1, \pm 2,... \qquad \text{(vi)}$$

If these sinusoids are sampled at rate F_s, then

$$x(n) = x_a(nT) = A \cos \left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta \right) \qquad \text{(vii)}$$

$$= A \cos (2\pi n F_0/F_s + \theta + 2\pi kn)$$

$$= A \cos (2\pi f_0 n + \theta) \qquad \text{(viii)}$$

Thus, the sampled signals are identical to $x(n)$ of Eq.(ii) with frequency $f_0 = F_0/F_s$. The sampled signals with frequencies $(f_0 + k)$; $k = \pm 1, \pm 2,...$ are the **aliases** of the signal with frequency f_0 (satisfying the inequality of Eq. (iv)). This phenomenon is known as **aliasing**. According to Eq. (viii) an infinite number of continuous-time sinusoids can be represented by the same discrete-time signal (i.e. by the same set of samples). It means that given an $x(n)$, there exists an ambiguity as to which continuous-time signal, $x_a(t)$, these values represent.

2. Discrete-time signals whose frequencies are separated by integer multiples of 2π are identical.

Proof Consider the sinusoid

$$\cos [\omega_0 + 2\pi) n + \theta] = \cos (\omega_0 n + 2\pi n + \theta) = \cos (\omega_0 n + \theta)$$

As a result, all sinusoidal sequences of the following kind

$$x_k(n) = A \cos (\omega_k n + \theta); \;\; k = 0, 1, 2,...$$

where

$$\omega_k = \omega_0 + 2k\pi, \; -\pi \le \omega_0 \le \pi$$

are indistinguishable, i.e. identical. On the other hand any two sinusoids with frequencies in the range $-\pi \le \omega \le \pi$ or $-\frac{1}{2} \le f \le \frac{1}{2}$ are distinct.

- The highest rate of oscillation in a discrete-time sinusoid is attained when

$\omega = \pi (\text{or} -\pi)$ or $f = \frac{1}{2}\left(\text{or} -\frac{1}{2}\right)$, **fundamental range**.

For illustration consider

$$x(n) = \cos \omega_0 n$$

We take values of $\omega_0 = 0,\ \pi/8,\ \pi/4,\ \pi/2,\ \pi;\ f \Rightarrow f_0 = 0\ \dfrac{1}{16}, \dfrac{1}{8}, \dfrac{1}{4}, \dfrac{1}{2}$ which results in periodic samples of $N = \infty$ (undefined), 16, 8, 4, 2 as depicted in Fig. 1.45 which illustrate the statement made above.

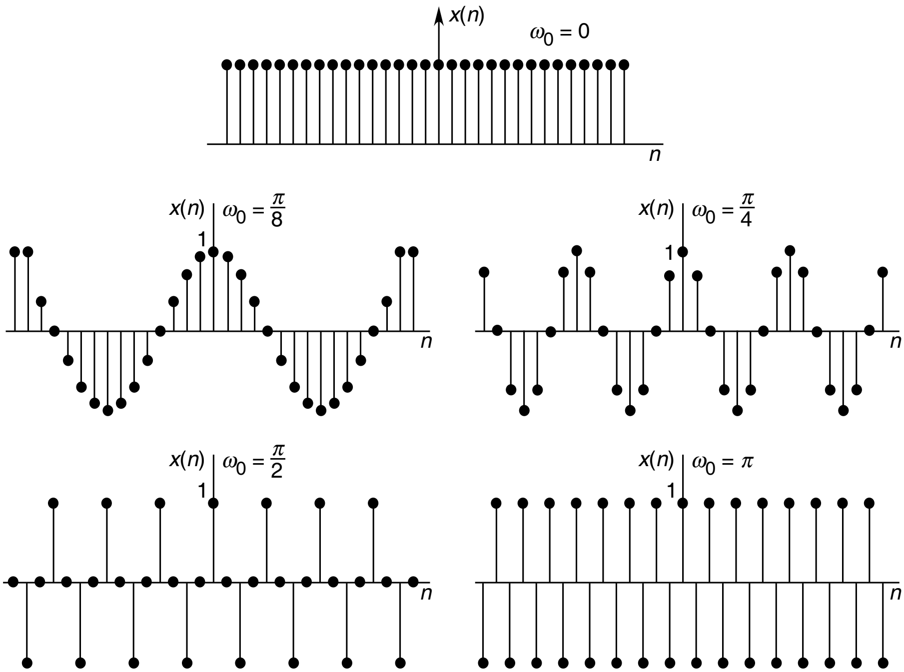


Fig. 1.45 *Signal $x(n) = \cos \omega_0 n$ for various values of the frequency ω_0*

Example 1.36 *Consider an analog signal*

$$x_a(t) = 5 \cos 100\pi t$$

(a) Determine the minimum sampling rate to avoid aliasing.

(b) If sampling rate F_s = 200 Hz, what is the discrete-time signal after sampling?

(c) Find the discrete-time signal after the signal given is sampled at 75 Hz.

(d) What is the frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

Solution

(a) To avoid aliasing $F_s = 50 \times 2 = 100$ Hz (minimum).

(b) $F_s = 200$ Hz. This gives

$$x(n) = 5 \cos \frac{100\pi}{200} n = 5 \cos \frac{\pi}{2} n \qquad \text{(i)}$$

(c) $F_s = 75$ Hz. This gives

$$\begin{aligned} x(n) &= 5\cos\frac{100\pi}{75}n = 5\cos\frac{4\pi}{3}n \\ &= 5\cos\left(2\pi - \frac{2\pi}{3}\right)n \\ &= 5\cos\frac{2\pi}{3}n \qquad \text{(ii)} \end{aligned}$$

(d) For $F_s = 75$ Hz.

$$F = fF_s = 75f$$

The frequency of the sinusoid in part (c) is $f = \frac{1}{3}$.

Hence

$$F = 25 \text{ Hz}$$

Example 1.37 *Let x(n) denote system input and y(n) denote system output. The input–output relationships of two systems are*

System 1 $\quad y_1(n) = x_1(n-2) + \frac{1}{2}x_1(n-3) \qquad$ *(i)*

System 2 $\quad y_2(n) = 2x_2(n) + 4x_2(n-1) \qquad$ *(ii)*

The two systems are connected in tandem with System 2 following System 1. Determine the overall input-output relationship.

Solution The two systems are connected as

$$\xrightarrow{x_1(n)} \boxed{\text{Sys 1}} \xrightarrow[x_2(n)]{y_1(n)} \boxed{\text{Sys 2}} \xrightarrow[y_1(n)]{}$$

Obviously,

$$\begin{aligned} y_1(n) &= x_2(n) \\ y_2(n) &= 2x_2(n) + 4x_2(n-1) \\ &= 2y_1(n) + 4y_1(n-1) \\ &= 2\left[x_1(x-2) + \frac{1}{2}x_1(n-3)\right] + 4\left[x_1(n-3) + \frac{1}{2}x_1(n-4)\right] \\ &= 2x_1(n-2) + 5x_1(n-3) + 2x_1(n-4) \end{aligned}$$

Hence, the overall system is

$$y(n) = 2x(n-2) + 5x(n-3) + 2x(n-4)$$

The reader should check that if the order of the system interconnected is reversed, the overall input–output relationship will remain the same. This is a consequence of the fact that both the systems are linear.

Example 1.38 *Consider the discrete signal x(n) of Fig. 1.46. Sketch the following signals:*

(a) x(n – 3) *(b) x(2 – n)* *(c) x(3n)*

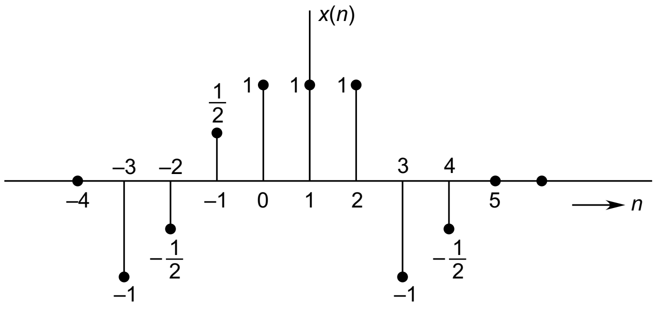


Fig. 1.46

Solution

(a) $x(n-3)$ is $x(n)$ delayed by 3. So

$$x(4) = \frac{1}{2} \text{ shifts to } x(7) = -\frac{1}{2}$$
$$x(-3) = -1 \text{ shifts to } x(0) = -1$$

(b) $x(n+2)$ is $x(n)$ advanced by 2
$x(-n+2)$ is its reflection about $n = 0$

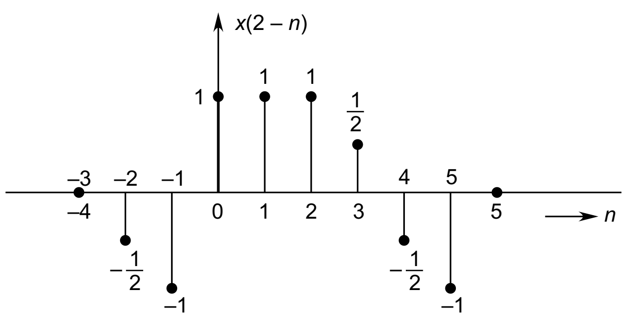


Fig. 1.47(a)

(c) $x(3n)$ is every 3rd $x(n)$

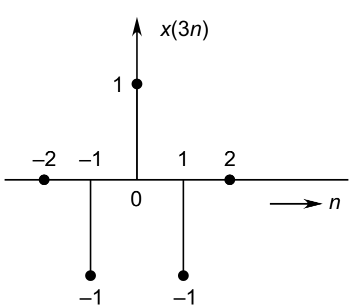


Fig. 1.47(b)

Problems

1.1 A system is classified broadly into static, dynamic, time-invariant, linear, causal, etc. Classify each of the systems described by the input–output relationship given below and justify your answer.

(i) $y(t) = e^{r(t)}$

(ii) $y(t) = dr(t)/dt$

(iii) $y(t) = \sin 6t$

(iv) $y(t) = [\sin 6t]\, r(t)$

(v) $y(t) = \begin{cases} 0 & t < 0 \\ r(t) + r(t-100); & t \ge 0 \end{cases}$

(vi) $y(t) = \begin{cases} 0 & T\,(t) < 0 \\ r(t) + r(t-100); & r\,(t) \ge 0 \end{cases}$

(vii) $y(t) = x(t/2)$

(viii) $dy(t)/dt + 2y^2(t) = r(t)$

(ix) $y(t)(dy/dt) + 2y^2(t) = r(t)$

(x) $y(k) = r(2k)$

(xi) $y(k) = kr(k)$

(xii) $y(k) = \begin{cases} r(k) & k > 1 \\ 0 & k = 0 \\ r(k+1), & k < 1 \end{cases}$

(xiii) $y(k) = \begin{cases} 0 & k = 0 \\ r(k) & \text{(otherwise)} \end{cases}$

(xiv) $y(k + 1) + 5y(k) = r(k)$

(xv) $y(k + 1)\, y(k) + 5y(k) = r(k + 1)$

1.2 Verify that a capacitor excited by $e(t)$ is a time-invariant causal system.

1.3 (a) Justify that series interconnection of linear, time-invariant systems yields linear, time-invariant systems.

(b) Three systems described by input–output relationships given below are connected in series. Find the input–output relationship of the overall interconnected system and check for its linearity and time invariance.

$$y_1(k) = \begin{cases} r(k/2) & \text{for } k = \text{even} \\ 0 & \text{for } k = \text{odd} \end{cases}$$

$$y_2(k) = r(k) + 1/2\, r(k-1) + 1/2\, r(k-2)$$

$$y_3 = r(2k)$$

1.4 Prove with the help of input–output relationship

$$y(t) = \begin{cases} \mu^2\,(t)/u(t-1) & \text{if } u(t-1) \ne 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

that the homogeneity property does not imply the additive property.

1.5 For the system shown in Fig. P1.5, check the linearity of input (r)–output (y) relationship.

1.6 Consider an electrical system shown in Fig. P1.6.

Write the differential equation relating $e(t)$ to input $e_1(t)$.

1.7 Consider the mass-spring-damper system shown in Fig. P1.7. Obtain the differential equation model relating $y(t)$ to $F(t)$.

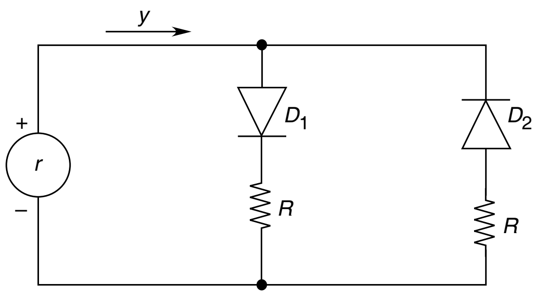


Fig. P1.5

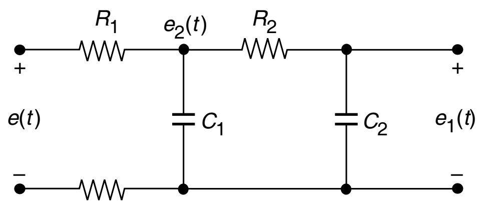


Fig. P1.6

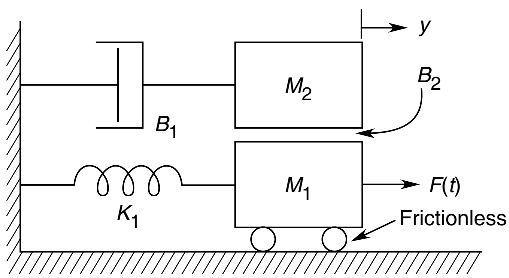


Fig. P1.7

1.8 Mr David has purchased a car for Rs 4 00 000 on an instalment basis of Rs 10 000 monthly, with the initial payment of Rs 1 00 000. The interest rate is 16 per cent compounded monthly. Formulate the difference equation model and determine the payment pattern of Mr David.

1.9 A bank pays 3 per cent interest per month on the money that is in the account for a month or less and 4 per cent per month on the money that is in the account for more than a month. If there are no withdrawals and that the interest is compounded monthly, write the equation relating the monthly bank balance and the deposits.

1.10 Let $P(k)$ be the population of a country at the beginning of the kth year. The birth and death rates during any year are b per cent and d per cent respectively. If $m(k)$ is the total migrants leaving the country during the kth year, write the difference equation for the population growth of the country.

1.11 A differentiator and an amplifier with gain proportional to time are connected in tandem as shown in Fig. P1.15 Show that the system composed of these two linear subsystems is linear. Would the output remain the same if the two subsystems are interchanged?

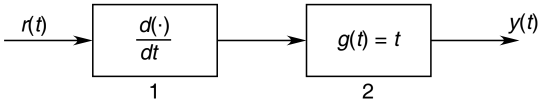


Fig. P1.11

1.12 Raw data can be 'smoothed' by replacing the present data sampled with the weighted average value of the data and the past two samples. The averaging weights are [1, 2, 2]. Set up the difference equation for this scheme.

1.13 Which of the properties hold for the following system—memoryless time invariant, linear, causal.

(a) $y(n) = x(-n)$ (b) $y(n) = x(n-1) + 2\,x(n-4)$

(c) $y(n) = nx(n-2)$ (d) $y(n) = x(2n+1)$

1.14 Find the inverse of the following systems where feasible:

(a) $y(n) = n\,x(n)$ (b) $y(n) = x(2-n)$

(c) $y(n) = x(3n)$ (d) $x(n) = \begin{cases} x(n/2), & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

1.15 A discrete-time system has output

$$y(n) = x(n)\,x(n-3)$$

(a) Is it a memoryless system? (b) Determine $y(n)$ for $x(n)\ A\ \delta(n)$

(c) Is the system invertible?

1.16 Which of these signals is periodic? Find the fundamental period.

(a) $x(n) = \cos\left(\dfrac{\pi}{4}n\right) + \sin\left(\dfrac{\pi}{8}n\right) - 2\cos\left(\dfrac{\pi}{2}n + \dfrac{\pi}{3}\right)$

(b) $x(n) = \cos\left(\dfrac{n}{6} - \dfrac{\pi}{4}\right)$

(c) $\sin\left(\dfrac{\pi}{3}n\right)\sin\left(\dfrac{\pi}{4}n\right)$

1.17 For the continuous-time signal shown in Fig. P1.17, sketch the following signals:

(a) $x(t+1)$

(b) $x(2-t)$

(c) $x\left(4 - \dfrac{t}{2}\right)$

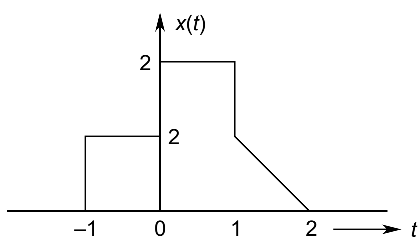


Fig. P1.17

1.18 A continuous-time linear system has the following input–output relation:

Input $x(t) = e^{\pm j3t}$

Output $y(t) = A\,e^{\pm j3t}$

Determine the system output for inputs

(a) $x_1(t) = \cos(3t)$

(b) $x_2(t) = \cos 3\left(t - \dfrac{1}{3}\right)$

Analysis of LTI Continuous-time Systems —Time Domain and Frequency Domain

2

Introduction

In this chapter methods of analysis of LTI continuous-time systems are discussed. First, time domain method is discussed and frequency domain analysis method is introduced later. In this chapter some of the important mathematical techniques, e.g., Convolution integral, Fourier series, Fourier-transform, Laplace-transform, their properties and their application in time and frequency domain analysis of linear time-invariant (LTI) systems including ideal filters are also discussed.

Fourier series is found to be suitable for the representations and analysis of periodic signals, also called power signals, whereas Fourier transform technique is best suited for analysis and characterization of non-periodic signals, also called energy signals. Fourier transform, however, can also be employed, under certain approximation, to analyse periodic signals. Power spectrum and energy spectrum are required to fully understand certain classes of signals (e.g., audio signal) which may either be periodic or non-periodic. Therefore, these two aspects have also been dealt with. Finally, to illustrate these techniques and their domain of applications, several examples have been included in this chapter. Laplace transform is a sort of generalization of Fourier transform and can be used to analyse larger class of signals and has therefore been discussed in details.

Mathematical model relating the output $y(t)$ to the input signal $r(t)$ of a LTI continuous–time system has been introduced in Chapter 1. The general nth order differential equation of LTI continuous system of Eq. (1.45) is reproduced below.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y$$
$$= b_m\frac{d^m r}{dt^m} + b_{m-1}\frac{d^{m-1}r}{dt^{m-1}} + \ldots + b_1\frac{dr}{dt} + b_0 r;\ n > m \tag{2.1}$$

Before studying the methods of determining system response, certain elementary continuous-time signals already introduced in Chapter 1 will now be discussed from application view point. These signals are helpful in obtaining system response to general signals. These elementary signals are broadly characterized under the following two heads:

(i) Singularity functions
(ii) Exponential functions

2.1 PROPERTIES OF ELEMENTARY SIGNALS

Singularity Functions

These function do not possess finite derivatives of all orders for all times. Some of the important singularity function are described here. These along with their properties have already been introduced in Section 1.12 of Chapter 1. Their parallel in discrete-time will be treated in Chapter 3.

Unit Step Function u(t) Mathematical definition and graphical representation of unit step function is given in Eq. (2.2) and Fig. 2.1 respectively.

$$u(t) = \begin{cases} 0\,; t < 0 \\ 1\,; t > 0 \end{cases} \tag{2.2}$$

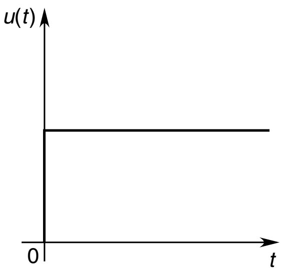


Fig. 2.1 *Unit step*

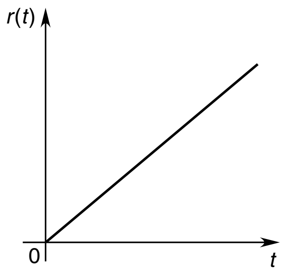


Fig. 2.2 *Unit ramp*

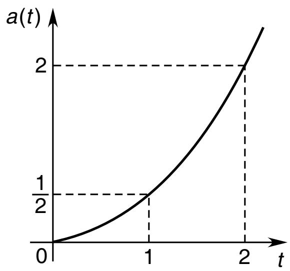


Fig. 2.3 *Unit parabolic function*

Unit Ramp Function r(t) Unit ramp function is defined as

$$r(t) = \begin{cases} 0 \text{ for } t \le 0 \\ t \text{ for } t > 0 \end{cases} \tag{2.3a}$$

It is immediately observed that $r(t)$ is an integral of unit step function over the time $-\infty$ to t, i.e.,

$$r(t) = \int_{-\infty}^{t} u(t)\, dt$$

or

$$r(t) = t\, u(t) \tag{2.3b}$$

Unit ramp is shown in Fig. 2.2.

Unit Parabolic Function a(t) Unit parabolic function is defined as

$$a(t) = \begin{cases} 0 \text{ for } t \le 0 \\ \dfrac{1}{2}t^2 \text{ for } t > 0 \end{cases} \tag{2.4a}$$

It is the double integration of unit step function, i.e.

$$a(t) = \iint u(t)\,\mathrm{d}t$$

or

$$a(t) = \tfrac{1}{2}\, t^2\, u(t) \tag{2.4b}$$

The graphical representation of unit parabolic function is shown in Fig. 2.3.

Unit Pulse Function p(t) Pulse signal is commonly used as test signals in electronic systems. An elementary pulse signal of unit height and width τ centred at $t = 0$ is defined as

$$p_\tau(t) = \begin{cases} 0\,;\, t < -\tau/2 \\ 1\,;\, -\tau/2 < t < \tau/2 \\ 0\,;\, t > \tau/2 \end{cases} \tag{2.5a}$$

Breakup of this pulse in terms of unit steps can be expressed as

$$p_\tau(t) = u(t + \tau/2) - u(t - \tau/2) \tag{2.5b}$$

Graphical representation of a pulse as defined in Eq. (2.5a) and its unit-step breakup of Eq. (2.5b) are drawn in Fig. 2.4(a) and (b) respectively. A pulse delayed by time kT is drawn in Fig. 2.4(c).

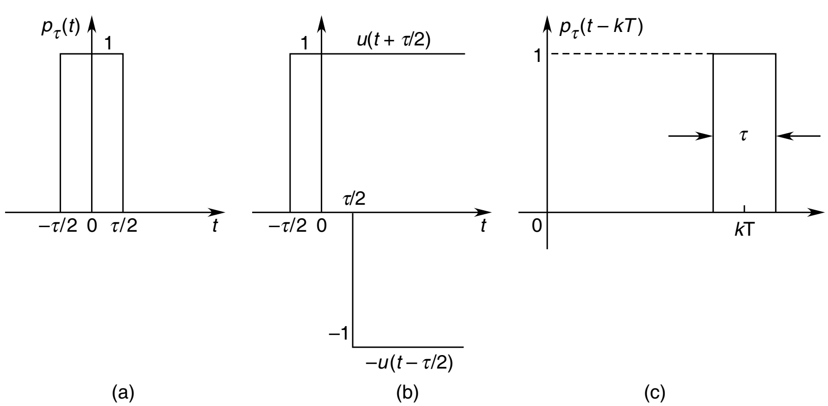


Fig. 2.4 *Pulse signal*

Unit Impulse Function δ(t) This is an important function defined by Dirac. It has already been introduced in Section 1.12 of Chapter 1 along with its properties. Only certain aspects will be reproduced here.

$$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t) \tag{2.6}$$

Because of discontinuity in $u(t)$ at $t = 0$, this derivative is not defined properly. It may be interpreted by approximating unit step $u(t)$ by $u_A(t)$ as shown in Fig. 2.5 which does not have a discontinuity at $t = 0$.

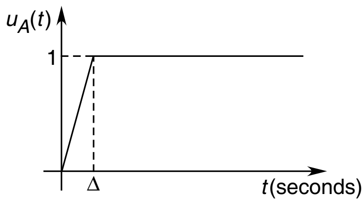


Fig. 2.5 *Unit step*

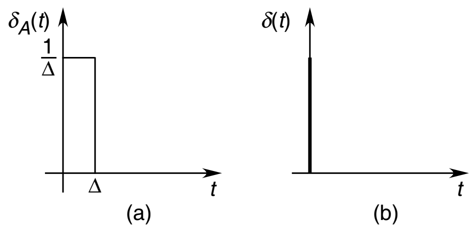


Fig. 2.6 *Unit impulse*

The signal in $u_A(t)$ rises to unity in time Δ. This signal generates unit step as

$$u(t) = \operatorname*{Lt}_{\Delta \to 0} u_A(t) \tag{2.7}$$

Let

$$\delta_A(t) = \frac{\mathrm{d}}{\mathrm{d}t} u_A(t) \tag{2.8}$$

which is shown in Fig. 2.6(a). In the limit as $\Delta \to 0$ it yields the impulse function.

$$\delta(t) = \operatorname*{Lt}_{\Delta \to 0} \delta_A(t) \tag{2.9}$$

Observe that $\delta(t)$ has unit area concentrated at $t = 0$. It is indicated by a thick line of unit height at $t = 0$ as shown in Fig. 2.6(b). The above account leads to the following definition of delta function

$$\delta(t) = 0;\ t \neq 0 \tag{2.9b}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = \operatorname*{Lt}_{\Delta \to 0} \int_0^{\Delta} \delta_A(t)\,\mathrm{d}t \tag{2.9c}$$
$$= 1$$

Properties of Unit Impulse Function

Multiplication of an impulse with any function.

If a function $f(t)$, which is continuous at $t = 0$, is multiplied by an impulse $\delta(t)$, then

$$f(t)\,\delta(t) = f(0)\delta(t) \tag{2.10a}$$

Similarly

$$f(t)\,\delta(t - T) = f(T)\delta(t - T) \tag{2.10b}$$

provided $f(t)$ is continuous at $t = T$.

Sifting Property (Sampling Property) Integrating Eq. (2.10a) and (2.10b), we get

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\mathrm{d}t = f(0)\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = f(0)$$

as
$$\left(\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1\right) \tag{2.11a}$$

Similarly,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\mathrm{d}t = f(T) \tag{2.11b}$$

Equations (2.11a) and (2.11b) are important results and may be expressed as follows.

The area under the product of $f(t)$ and $\delta(t)$ is equal to the value of that function at the instant where the unit impulse is shifted. Integral in (2.11b) is known as **convolution.** It will be generalized in Section 2.2.

Exponential Function

Exponential function e^{st} (already discussed at length in Section 1.5, Chapter 1) is a commonly used elementary function, where s is complex parameter called **complex frequency** (shown in Fig. 2.7). All practical signals can be expressed in terms of the following exponential function.

$$e^{st} = e^{(\sigma + j\omega)t} \tag{2.12a}$$

Signals in systems are always real (non-complex) and can be expressed as sum of complex conjugate exponentials. Combining these exponentials we can write

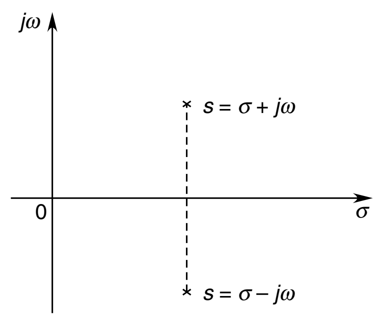


Fig. 2.7 *Complex s-plane;* $s = \sigma + j\omega$

$$2\frac{1}{2j}\left[e^{(\sigma+j\omega)t} - e^{(\sigma-j\omega)t}\right] = e^{\sigma t}\,(e^{j\omega t} - e^{-j\omega t})$$
$$= e^{\sigma t}\sin\omega t \tag{2.12b}$$

A phase angle can also be built on the basis of Eq. (2.12b) by writing

$$e^{j(\omega t+\theta)} = e^{j\omega t}e^{j\theta} \text{ in place of } e^{j\omega t}$$

By taking positive or negative value of the real part σ and for the complex part taking $\omega = 0$ and $\omega > 0$ in Eq. (2.12b), wave forms of Fig. 2.8 can be generated which exhibit typical nature. LTI continuous-time systems have a combination of these wave forms in their inputs and response terms. Similar wave forms are also there in discrete-time systems.

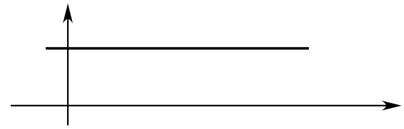
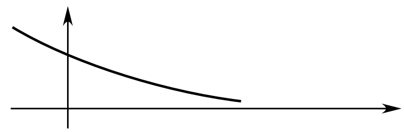
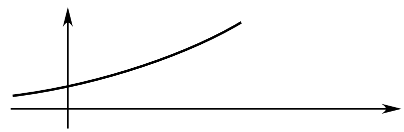
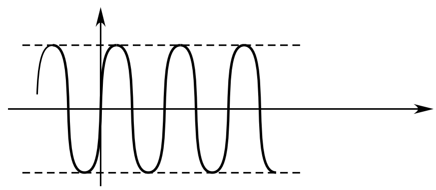
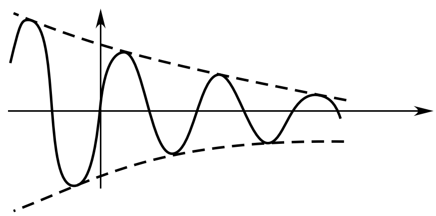
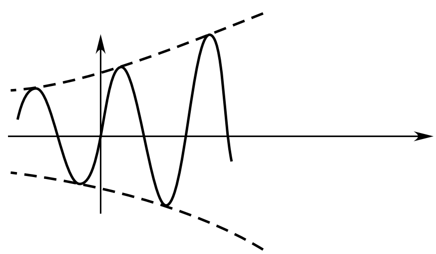
Value of s	*The signal*
(a) 0	
(b) $-\sigma$	
(c) $+\sigma$	
(d) $\pm j\omega$	
(e) $\sigma \pm j\omega,\ \sigma < 0$	
(f) $\sigma \pm j\omega,\ \sigma > 0$	

Fig. 2.8 *Wave forms*

Representation of Signals by Impulse Functions

Consider a signal $r(t)$ which will be first approximated as a set of pulses of duration T as shown in Fig. 2.9. The function can then be expressed as follows.

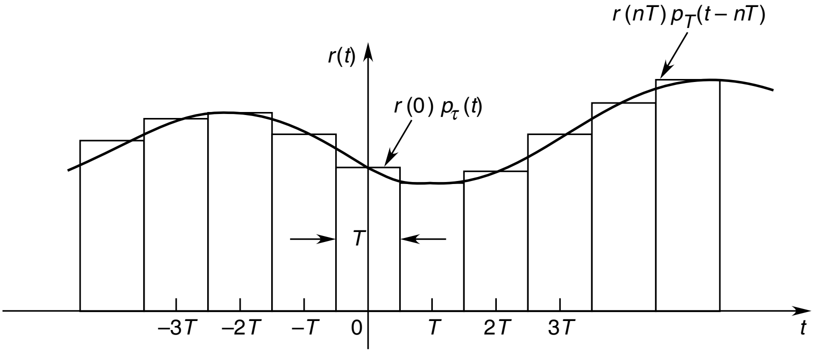


Fig. 2.9 *Signal's approximation as pulses*

$$r(t) \approx \sum_{n=-\infty}^{\infty} r(nT)p_T(t-nT) \tag{2.13}$$

where $r(nT)$ is weight of signal $r(t)$ at $t = nT$. Obviously the approximation is better for smaller value of T.

Equation (2.13) can be written in the following form.

$$r(t) \approx \sum_{n=-\infty}^{\infty} r(nT)\,T[p_T(t-nT)/T]$$

The term within large brackets is a pulse of height $1/T$ and width T and so its area is one unit. As T progressively reduces to $T \to 0$, pulse signal is approximated as impulse

$$\underset{T\to 0}{\text{Lt}}\,[p_T(t-nT)/T] \to \delta(t-nT)$$

The signal $r(t)$ is now expressed as

$$r(t) \approx \sum_{n=-\infty}^{\infty} r(nT)\,T\delta(t-nT) \quad (2.14)$$

The impulse approximation of signal $r(t)$ given in Eq. (2.14) is valid provided the pulse width T is very small and the sampling frequency $f_s = 1/T$ is high compared to frequencies present in the signal.

2.2 LINEAR CONVOLUTION INTEGRAL

Consider an LTI continuous-time system with

$h(t)$ = impulse response of the system
$r(t)$ = input signal
$y(t)$ = system output

Writing the impulse response of the system in progressive form, we get

$$\delta(t) \to h(t)$$
$$\delta(t - nT) \to h(t - nT)$$
$$r(nT)\,T\delta(t - nT) \to r(nT)\;T\,h(t - nT) \tag{2.15}$$

Using Eqs (2.14) and (2.15), the complete response as sum of responses of individual impulses can be written as

$$y(t) \approx \sum_{n=-\infty}^{\infty} r(nT)\,T\,h(t - nT) \tag{2.16}$$

Letting $T \to 0$, the summation of Eq. (2.16) takes the form of the integral with $nT \to \lambda$ (continuous variable) and $T \to \mathrm{d}\lambda$. Thus

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)\; h(t - \lambda)\mathrm{d}\lambda \tag{2.17}$$

This integral is the linear convolution integral (or just convolution). It is written symbolically as

$$y(t) = r(t) * h(t) \tag{2.18a}$$

Also

$$y(t) = h(t) * r(t)\;;\ \text{commutative property} \tag{2.18b}$$

Commutative property follows easily by letting $(t - \lambda) = \tau$ in Eq. (2.17). For a causal system $h(t)$ appears after $r(t)$ is applied at $t = 0$. Thus

$$y(t) = \int_0^t r(\lambda)h(t - \lambda)\mathrm{d}\lambda;\ t \ge 0 \tag{2.19}$$

Observe that $h(t - \lambda) = 0$ for $\lambda > t$

Properties of Convolution Integral

Some of the important properties of convolution integral are given below.

(a) *Commutative Property* If $r_1(t)$ and $r_2(t)$ are continuous time signals, then

$$r_1(t)*r_2(t) = r_2(t) * r_1(t) \tag{2.20}$$

(b) *Distributive Property*

$$r_1(t)*[r_2(t) + r_3(t)] = r_1(t)*r_2(t) + r_1(t)*r_3(t) \tag{2.21}$$

where $r_3(t)$ is another signal.

(c) *Associative Property* This property states that

$$r_1(t)*[r_2(t)*r_3(t)] = [r_1(t)*r_2(t)]*r_3(t) \tag{2.22}$$

Example 2.1 *Prove that*

(i) $r(t)*\delta(t) = r(t)$ (ii) $r(t)*\delta(t-\lambda) = r(t-\lambda)$
(iii) $r(t-\lambda_1)*\delta(t-\lambda_2) = r(t-\lambda_1-\lambda_2)$ (iv) $\delta(t-\lambda_1)*\delta(t-\lambda_2) = \delta(t-\lambda_1-\lambda_2)$

Solution

(i) By the definition of convolution

$$r(t)*\delta(t) = \int_{-\infty}^{\infty} r(\lambda)\,\delta(t-\lambda)\mathrm{d}\lambda \tag{i}$$

Using the sampling property of Eq. (2.11b), we get

$$r(t)*\delta(t) = r(t) \tag{ii}$$

Equation (ii) reveals that convolution of any function with unit impulse results in the function itself.

Similarly other results can be proved.

Convolution integral was presented above in term of impulse response. We will now express this integral in terms of step response.

Step Response, System Response and Superposition Integral

Step response of a system is easily obtained by applying a sudden change in system input. Compared to this an impulse is not easily generated and has to be approximated as a short duration pulse. Let us find the step response in terms of the impulse response.

$$\begin{aligned} g(t) &= u(t) * h(t) \\ &= \int_{-\infty}^{\infty} u(\lambda)\,h(t-\lambda)\mathrm{d}\lambda \\ &= \int_0^t h(t-\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{2.22}$$

as $\qquad h(t-\lambda) = 0 \quad$ for $\lambda > t$; as $h(t)$ is causal,

let $\qquad t-\lambda = \tau$, then

$$\begin{aligned} g(t) &= -\int_{\tau}^{0} h(\tau)\,\mathrm{d}\tau \\ &= \int_0^t h(\lambda)\,\mathrm{d}\lambda;\ \tau \to t \end{aligned} \tag{2.23}$$

It is observed from this equation that the step response is the integral of its unit impulse response. Let us now find $y(t)$ in terms of a system's step response $g(t)$. As per Eq. (2.17).

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)h(t-\lambda)\mathrm{d}\lambda \tag{2.24}$$

Integrating by parts and using Eq. (2.23), we get

$$y(t) = r(\lambda)\, g(t-\lambda)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \dot{r}(\lambda)\, g(t-\lambda)\, d\lambda$$

But $g(t) = 0$ for $t \le 0$ (causal system) and it is assumed that $r(t)$ and $g(t)$ are zero at $t = \infty$. Then

$$y(t) = \int_{-\infty}^{\infty} \dot{r}(\lambda) g(t-\lambda)\, d\lambda \tag{2.25a}$$

$$= \dot{r}(t) * g(t) \tag{2.25b}$$

Equation (2.26) relates the output response in terms of convolution of step response and the derivative of input signal. This convolution is known as superposition integral. For causal system the superposition integral becomes

$$y(t) = \int_0^{\infty} \dot{r}(\lambda)\, g(t-\lambda)\, d\lambda \tag{2.26}$$

It may be noted that this method is rather unattractive. This is because in this method for calculating $y(t)$ one requires derivative of $r(t)$ which generates noise.

Procedure of Evaluating Convolution Integral

The two procedures of evaluating convolution integral are given below.

(a) *Graphical Solution of Convolution Integral* Visual grasp and graphical interpretation are very helpful in sampling, filtering and other related problems. This is because for many signals, exact mathematical models are not available and they are described only graphically or by numerical data. Therefore, graphical and numerical solutions are the only choices. We shall now present steps involved in convolving signals $r_1(t)$ and $r_2(t)$ graphically (or numerically).

(i) Take a mirror image of signal $r_2(\lambda)$ about the vertical axis to obtain $r_2(-\lambda)$.
(ii) Shift $r_2(-\lambda)$ forward by t to obtain $r_2(t-\lambda)$.
(iii) Multiply $r_1(t)$ and $r_2(t-\lambda)$ and compute the area under $r_1(\lambda)\, r_2(t-\lambda)$ to obtain $y(t)$.
(iv) Repeat the above steps for various values of t to obtain the complete signal $y(t)$.

These steps of graphical convolution are illustrated with the help of following example.

Example 2.2 *Obtain the following convolution graphically and analytically.*

$$y(t) = r_1(t) * r_2(t)$$

where

$$r_1(t) = e^{-\alpha t}\, u(t),\ \alpha > 0$$

and

$$r_2(t) = u(t)$$

Solution Since these are causal functions, convolution integral has the following form.

$$y(t) = \int_0^t r_1(\lambda) r_2(t-\lambda)\, d\lambda$$

$r_1(\lambda)$ and $r_2(-\lambda)$ are sketched in Figs 2.10(a) and (b).

Graphical steps for one value of t depicted in Figs 2.10 (b), (c) and (d) are repeated for various values of t to obtain $y(t)$, which is plotted in Fig. 2.10(e).

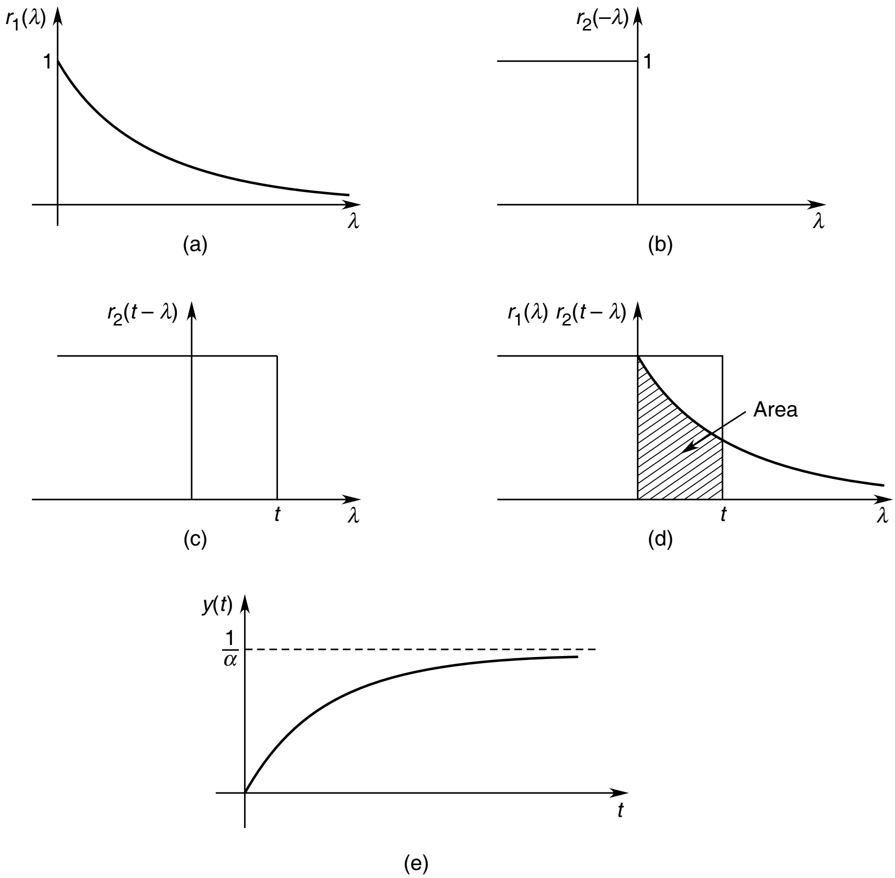


Fig. 2.10 *Steps in evaluation of convolution integral*

Since the given functions are analytical, the convolution integral can be carried out by mathematical integration as below.

$$y(t) = \int_0^{\infty} r_1(\lambda) r_2(t-\lambda)\, d\lambda$$

$$= \int_0^{t} e^{-\alpha t}\, dt\ ;\ t > 0$$

$$= (1/\alpha)\,(1 - e^{-\alpha t})\, u(t)$$

Example 2.3 *Obtain the graphical convolution of the signals shown in Fig. 2.11 (a) and (b).*

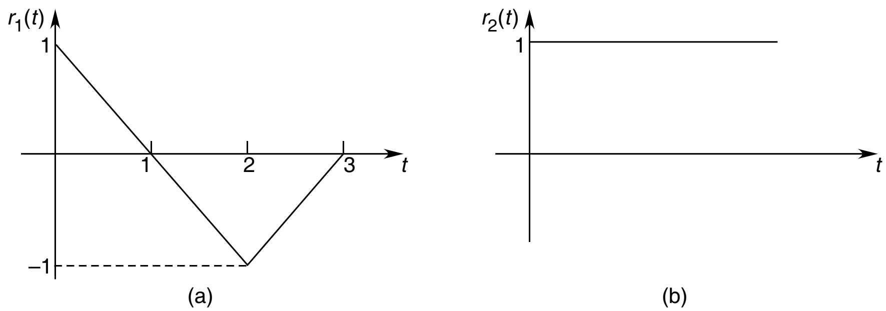


Fig. 2.11

Solution The convolution integral is written as

$$y(t) = \int_0^{t} r_1(\lambda)\, r_2\,(t-\lambda)\, d\lambda$$

Graphical steps for computing $y(t)$ are shown in Fig. 2.12(a) and (b).

The shaded area in Fig. 2.12 (b) may be computed analytically.

For $0 < t \leq 2$

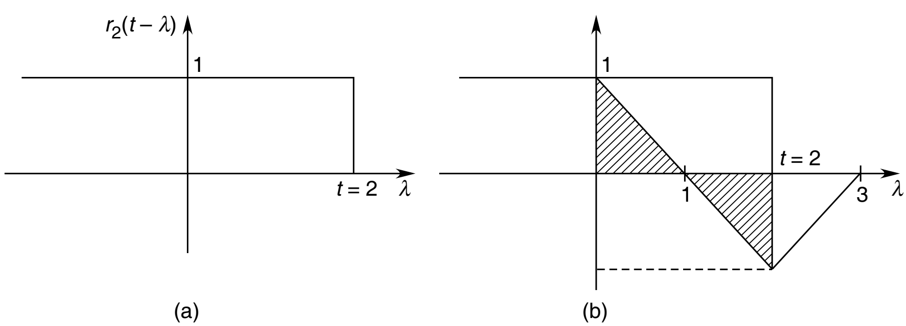


Fig. 2.12

$$y_1(t) = \int_0^t (1-\lambda)\, d\lambda$$

$$= (2t - t^2)/2$$

For $\quad 2 < t \le 3$

$$y_2(t) = y_1(2) + \int_0^t (-3+\lambda)\, d\lambda$$

$$= -3t + \frac{1}{2}t^2 \Big|_2^t = 4 - 3t + \frac{1}{2}t^2$$

For $\quad t \ge 3$

$$y_3(t) = y_2(3) = -\frac{1}{2}$$

(b) *Numerical Convolution* Numerical computation is carried out by approximating both the functions by pulse train of period T.

Consider a causal system

$$y(t) = r(t)*h(t);$$

where $h(t)$ = impulse response of the system, $r(t)$ = input

or
$$y(t) = \int_0^t r(\lambda)h(t-\lambda)\, d\lambda$$

Replacing $r(\lambda)d\lambda$ by pulse train, $Tr(nT)$, the above integral is approximated as the following summation

$$y(t) = \operatorname*{Lt}_{T\to 0} \sum_{n=0}^{t} Tr(nT)h(t-nT) \tag{2.27}$$

At $\quad t = kT$

$$y(kT) = \operatorname*{Lt}_{T\to 0} \sum_{nT=0}^{kT} Tr(nT)\, h(kT-nT)$$

We write $kT \to k$ and $nT \to n$. Then, we get

$$y(k) = \operatorname*{Lt}_{T\to 0} T\sum_{n=0}^{k} r(n)h(k-n) \tag{2.28}$$

This is the discrete-time convolution to be discussed in Chapter 3.

For noncausal systems the summation in Eq. (2.28) is modified and we get

$$y(k) = \operatorname*{Lt}_{T\to 0} T\sum_{n=-\infty}^{\infty} r(n)h(k-n) \tag{2.29}$$

For numerical solution of Eq. (2.29) a suitable small value of T is used. How to choose this value will be discussed in Section 2.12. We shall now illustrate the process by an example.

Example 2.4 *Carry out numerical convolution $y(t) = r_1(t)*r_2(t)$ where*

$$r_1(t) = tu(t)$$
$$r_2(t) = 2\,u(t)$$

Choose sampling period $T = 0.2$ s.

Solution Substituting values in Eq. (2.30) (i)

For $k = 0, 1, 2 \ldots$

$$y(0) = Tr_1(0)r_2(0) \tag{ii}$$

$$y(1) = T\sum_{n=0}^{1} r_1(n)\, r_2(1-n)$$
$$= T\{r_1(0)r_2(1) + r_1(1)r_2(0)\} \tag{iii}$$

$$y(2) = T\sum_{n=0}^{2} r_1(n)r_2(2-n)$$
$$= T\{r_1(0)r_2(2) + r_1(1)r_2(1) + r_1(2)r_2(0)\} \tag{iv}$$

Similarly other sequence components of $y(k)$ may be obtained.

Discritising the given function by letting $t = kT$, we have

$$\{r_1(kT)\} = kT \text{ or } \{r_1(k)\} = k,\ \ k > 0 \tag{v}$$

and

$$\{r_2(k)\} = 2\{u(k)\} = 2;\ k \geq 0 \tag{vi}$$

Substituting these values in Eqs (ii) to (iv), we get

$$y(0) = 0.2\ \{0\} = 0$$
$$y(1) = 0.2\ \{0 + 1 \times 2\} = 0.4$$
$$y(2) = 0.2\ \{0 + 1 \times 2 + 2 \times 2\} = 1.2$$

So,

$$\{y(k)\} = \{0, 0.4, 1.2\ \ldots\}$$

Above convolution is pictorially shown in Fig. 2.13.

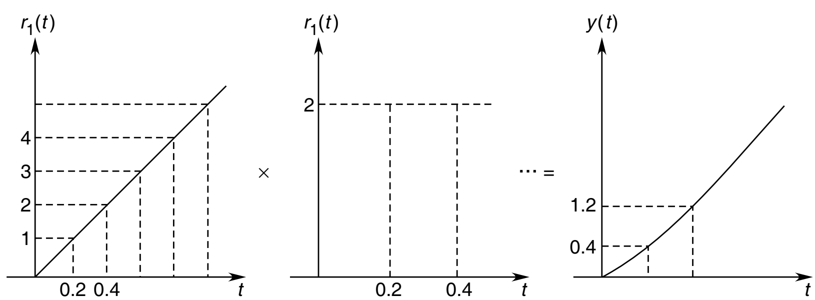


Fig. 2.13

2.3 RESPONSE OF CAUSAL LTI SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS

As already presented in Eq. (2.1), LTI systems are described by differential equations with *constant coefficients*. The classical approach to solving these equations is to obtain the *homogeneous (natural) response* and the *particular integral*. As we will be studying powerful techniques of solution later in this chapter, we shall illustrate the classical method by a simple example.

Consider a first-order system described by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = x(t) \tag{i}$$

$$x(t) = k\,e^{-3t}\,u(t), \text{ i.e., for } t \ge 0 \tag{ii}$$

the solution will comprise two parts

$$y(t) = y_h + y_p$$

where y_h = homogeneous solution, natural response

y_p = particular solution, forced response

Natural response (zero input)

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \tag{iii}$$

We assume a solution of the form

$$y_h = A\,e^{st} \tag{iv}$$

Substituting in Eq. (iii)

$$As\,e^{st} + 2A\,e^{st} = 0$$

$$A\,(s+2)\,e^{st} = 0$$

which yields

$$s = -2$$

Thus, $$y_s = A\,e^{-2t} \tag{v}$$

Particular solution (forced response)

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = k\,e^{-3t} \tag{vi}$$

The response y_p must have the same form as the input. So we assume

$$y_p = Y\,e^{-3t} \tag{vii}$$

Substituting in Eq. (vi)

$$3\,Y\,e^{3t} + 2\,Y\,e^{3t} = k\,e^{3t}$$

which yields

$$Y = \frac{k}{5}$$

Thus, $$y_p = \left(\frac{k}{5}\right)e^{3t} \tag{viii}$$

The complete solution is then

$$y(t) = \underbrace{A\,e^{-2t}}_{\text{Natural response}} + \underbrace{\left(\frac{k}{5}\right)e^{-3t}}_{\text{Forced response}};\ t > 0, \text{ causal system} \tag{ix}$$

The unknown constant A would be determined from the initial condition $y(t)|_{t=0} = y(0)$. If we assume $y(0) = 0$, then

$$0 = A + \frac{k}{5}$$

$$0 = -\frac{k}{5}$$

Then $$y(t) = \frac{k}{5}(e^{-3t} - e^{-2t};\ t > 0 \tag{x}$$

On similar lines we could obtain the solution of any higher order system for general exponential inputs. However, we shall not pursue this any further.

Response of CT-LTI Systems to Complex Exponentials

Complex exponentials offer the great advantage that a broad useful class of signals can be constructed as linear combination of exponentials. The response of an LTI system to a complex exponential has complex exponential form. Therefore, the response of an LTI system to any signal is obtained as a sum of its response to its component exponentials.

Response of an LTI system to complex exponential e^{st} is a complex exponential itself with change in amplitude (complex in general). It means that if

$$\left.\begin{array}{ll} \text{Input is} & e^{st} \\ \text{Output is} & H(s)e^{st} \end{array}\right\}$$

e^{st} is known as *eigen function* and the multiplier $H(s)$ is the *eigen value* of the system.

Consider an LTI system with impulse response $h(t)$ and input $x(t) = e^{st}$. The output is given by the convolution integral as

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, \mathrm{d}\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, \mathrm{d}\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-\tau s}\, d\tau}_{H(s)} \tag{2.30}$$

Assuming that the integral converges, we get

$$y(t) = \underbrace{H(s)}_{\text{Eigen value (in general complex).}}\ \underbrace{e^{st}}_{\text{Eigen function}} \tag{2.31}$$

2.4 FOURIER SERIES AND ITS PROPERTIES

A periodic signal has the property that

$$f(t) = f(t + nT),\ n = 0, \pm 1, \pm 2$$

where T is the fundamental period; it is the smallest value of T for which the above equation is valid; (see Section 2).

In the earlier part of this chapter we introduced the time domain tool of superposition wherein the signal is shown to be a continuum of impulses and the system output (response) is the sum (integral) of responses to unit impulses. A periodic signal is a specific signal associated with a frequency $f_0 = \dfrac{1}{T_0}$. It can be considered as an infinite sum of sinusoidal (periodic) signals of frequencies which are integral multiples of the fundamental frequency $(0, \pm 1, \pm 2 \ldots)$. The system response is then the sum of the responses to these frequencies (called harmonics). The system response to any sinusoidal signal can be obtained easily by methods to be discussed later.

The sinusoidal frequencies that make up the periodic signal are called its Fourier series. For the Fourier series of a periodic signal to exist, it must satisfy the *Dirichlet conditions* as listed below.

(i) Within one period there can only be a finite number of maxima and minima,

(ii) Number of discontinuities are finite; and

(iii) Discontinuities are bounded, which means that

$$\int_0^T |f(t)|\,\mathrm{d}t < \infty$$

Trigonometric Form of Fourier Series

A general periodic signal can be represented by an infinite sum series of sine and cosine functions, i.e.

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \tag{2.32}$$

where

a_k, b_k are constants with $k = 0, 1, 2 \ldots \infty$

and $\omega_0 = \dfrac{2\pi}{T}$; fundamental frequency

T = fundamental period of the signal (2.33)

In the Fourier series expression of Eq. (2.32)

$a_0 = dc$ term (constant)

(a_1, b_1) are the constants associated with the fundamental frequency which is the same as the signal frequency.

(a_k, b_k); $k \geq 2$ are the constants associated with the harmonics of the periodic signal.

The constants a_k, b_k are determined using trigonometric integrals evaluated over one period. These trigonometric integrals are as follows.

(i) $$\int_0^T \sin k\omega_0 t \, \mathrm{d}t = 0 \tag{2.34}$$

(ii) $$\int_0^T \cos k\omega_0 t \, \mathrm{d}t = 0 \tag{2.35}$$

(iii) $$\int_0^T (\sin n\omega_0 t \cos m\omega_0 t) \, \mathrm{d}t = 0 \tag{2.36}$$

(iv) $$\int_0^T (\sin n\omega_0 t \sin m\omega_0 t) \, \mathrm{d}t = 0; \text{ for } n \neq m \tag{2.37}$$

(v) $$\int_0^T (\cos n\omega_0 t \cos m\omega_0 t) \, \mathrm{d}t = 0; \text{ for } n \neq m \tag{2.38}$$

We now proceed to evaluate constants a_0, a_k and b_k.

Evaluation of a_0 (the dc term) Integrating both sides of Eq. (2.32) over one full period, we have

$$\int_0^T f(t) \, \mathrm{d}t = \int_0^T a_0 \mathrm{d}t + \int_0^T \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \, \mathrm{d}t \tag{2.39}$$

Using the integrals given in Eqs (2.34) and (2.35), Eq. (2.39) gets reduced to

$$\int_0^T f(t) \, \mathrm{d}t = a_0 T \rightarrow a_0 = \frac{1}{T} \int_0^T f(t) \, \mathrm{d}t \tag{2.40}$$

Also

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\omega_0 t) \, d(\omega_0 t); \text{ radian form} \tag{2.41}$$

Evaluation of a_k To determine constants a_k, multiply both sides of Eq. (2.32) by $\cos n\omega_0 t$ and integrate over one period, i.e.,

$$\int_0^T f(t) \cos n\omega_0 t \, \mathrm{d}t = \int_0^T a_0 \cos n\omega_0 t \, \mathrm{d}t$$

$$+ \sum_{k=1}^{\infty} \left[\int_0^T (a_k \cos k\omega_0 t \cos n\omega_0 t) \mathrm{d}t \right.$$

$$\left. + \int_0^T (b_k \sin k\omega_0 t \cos n\omega_0 t) \mathrm{d}t \right] \tag{2.42}$$

Using the results of Eqs (2.34) to (2.38), we find that all the right hand terms integrate to zero except the a_k term for $k = n$. This reduces the Eq. (2.42) to Eq. (2.42a) (changing index n to k).

$$\int_0^T f(t)\cos k\omega_0 t \,\mathrm{d}t = \int_0^T a_k \cos^2 k\omega_0 t \,\mathrm{d}t$$

$$= a_k T/2$$

or

$$a_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t \,\mathrm{d}t \tag{2.43}$$

Also

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t) \cos k\, \omega_0 t \,\mathrm{d}(\omega_0 t); \text{ radian form} \tag{2.44}$$

Evaluation of b_k The constants b_k can be determined by multiplying both sides of Eq. (2.32) with $\sin n\omega_0 t$ and integrating over the period 0 to T, we have

$$\int_0^T f(t)\sin n\omega_0 t \,\mathrm{d}t = \int_0^T \left[a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)\right]\sin n\omega_0 t \,\mathrm{d}t$$

Using the trigonometric integral identities on the lines similar to the above, the expression reduces to

$$\int_0^T f(t)\sin k\omega_0 t \,\mathrm{d}t = b_k(T/2)$$

or

$$b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t \,\mathrm{d}t \tag{2.45}$$

Also

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t) \sin k\omega_0 t \,\mathrm{d}(\omega_0 t); \quad \text{radian form} \tag{2.46}$$

The coefficients of Fourier series a_0, a_k and b_k can be determined from the above results.

Example 2.4 *Consider the signal $f(t)$ defined as*

$$f(t) = \sin\left(\frac{\omega_0 t}{2}\right);\ 0 \le t \le T = \frac{2\pi}{\omega_0}$$

The signal is periodic with period T. Find its Fourier series. To begin with sketch the signal.

Solution From the expression of the periodic signal the signal is sketched in Fig. 2.14. It can be seen that it is a full-wave rectified sinusoidal wave.

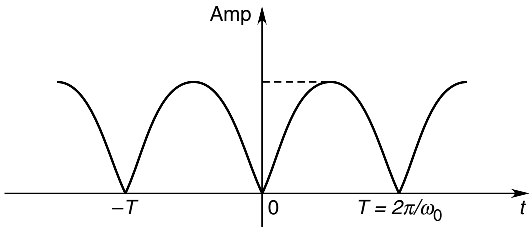


Fig. 2.14

In order to find the Fourier series of the signal given, we first evaluate the coefficients a_0, a_k and b_k in the following manner.

$$a_0 = \frac{1}{T}\int_0^T \sin(\omega_0 t/2)$$

$$= \frac{1}{T}\left[\frac{-\cos(\omega_0 t/2)}{\omega_0/2}\right]_0^T = \frac{2}{\pi} \tag{i}$$

$$a_k = \frac{2}{T}\int_0^T \sin(\omega_0 t/2)\cos k\omega_0 t \, \mathrm{d}t$$

$$= \frac{1}{T}\int_0^T [\sin\{(\omega_0 t/2) + k\omega_0 t\} + \sin\{(\omega_0 t/2) - k\omega_0 t\}]\,\mathrm{d}t$$

$$= \frac{1}{T}\left[\frac{-\cos\{(1/2+k)\omega_0 t\}}{(1/2+k)\omega_0}\right]_0^T + \frac{1}{T}\left[\frac{-\cos\{(1/2-k)\omega_0 t\}}{(1/2-k)\omega_0}\right]_0^T$$

$$= \frac{1}{(1+2k)\pi}[-\cos\{(1+2k)\pi\}+1] + \frac{1}{(1-2k)\pi}[-\cos\{(1-2k)\pi\}+1] \tag{ii}$$

In Eq. (ii)

$$\cos\{(1+2k)\pi\} = -1 = \cos\{(1-2k)\pi\}; \text{ for all values of } k$$

Thus

$$a_k = \frac{2}{(1+2k)\pi} + \frac{2}{(1-2k)\pi} = \frac{4}{\pi(1-4k^2)} \tag{iii}$$

Evaluating b_k by Eq. (2.45), it is found that $b_k = 0$; for all values of k.

The Fourier series of the given periodic signal is, finally, obtained in the following form.

$$f(t) = \frac{2}{\pi} - \frac{4}{3\pi}\cos\omega_0 t - \frac{4}{15\pi}\cos 2\omega_0 t - \frac{4}{35\pi}\cos 3\omega_0 t$$

$$- \frac{4}{63\pi}\cos 4\omega_0 t - \frac{4}{99\pi}\cos 5\omega_0 t \tag{iv}$$

Exponential Fourier Series

This is the most widely used form of Fourier series and this involves complex exponential functions. Exponential Fourier series are the equivalent form of Trigonometric Fourier series and can be obtained as follows.

In order to derive Exponential Fourier series from Trigonometric Fourier series, let us express cos $k\omega_0 t$ and sin $k\omega_0 t$ in Eq. (2.32) as

$$\cos k\omega_0 t = \frac{1}{2}[e^{jk\omega_0 t} + e^{-jk\omega_0 t}] \tag{2.47}$$

$$\sin k\omega_0 t = \frac{1}{2j}[e^{jk\omega_0 t} - e^{-jk\omega_0 t}] \tag{2.48}$$

Substituting Eqs (2.47) and (2.48) into Eq. (2.32), we get

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[\left(\frac{a_k - jb_k}{2}\right)e^{jk\omega_0 t} + \left(\frac{a_k + jb_k}{2}\right)e^{-jk\omega_0 t}\right] \tag{2.49}$$

Let

$$F_0 = a_0 \tag{2.50}$$

and

$$F_k = \frac{a_k - jb_k}{2} \tag{2.51}$$

so that

$$F_k^* = \frac{a_k + jb_k}{2} \tag{2.52}$$

Substituting Eqs (2.50), (2.51) and (2.52) into Eq. (2.49) yields the following equation.

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} F_k^* e^{-jk\omega_0 t} \tag{2.53}$$

We now proceed to determine the coefficients F_0 and F_k.

Using Eq. (2.40), we get

$$F_0 = a_0 = \frac{1}{T}\int_0^T f(t)\mathrm{d}t \tag{2.54}$$

Substituting the result of Eqs (2.43) and (2.45) for a_k and b_k respectively in Eq. (2.49) yields the following equation.

$$\begin{aligned}
F_k &= \frac{1}{2}\left[\frac{2}{T}\left\{\int_0^T f(t)\cos k\omega_0 t\,\mathrm{d}t - j\int_0^T f(t)\sin k\omega_0 t\,\mathrm{d}t\right\}\right] \\
&= \frac{1}{T}\left[\int_0^T f(t)\cos k\omega_0 t\,\mathrm{d}t - j\int_0^T f(t)\sin k\omega_0 t\,\mathrm{d}t\right] \\
&= \frac{1}{T}\left[\int_0^T f(t)\{\cos k\omega_0 t - j\sin k\omega_0 t\}\mathrm{d}t\right]
\end{aligned}$$

Thus

$$F_k = \frac{1}{T}\int_0^T f(t)\,e^{-jk\omega_0 t}\,\mathrm{d}t \qquad \textit{Analysis equation} \tag{2.55}$$

Integration could be over any period T.

Also

$$F_k = \frac{1}{2\pi}\int_0^{2\pi} f(\omega_0 t)e^{-jk\omega_0 t}\,\mathrm{d}(\omega_0 t);\ \text{radian form} \tag{2.56}$$

Using Eq. (2.55) the coefficient of the Exponential Fourier series, F_k, can be determined.

It is easily seen from Eq. (2.55) that

$$F_{-k}^* = \frac{1}{T}\int_0^T f(t)\,e^{-k\omega_0 t}\,\mathrm{d}t = F_k \tag{2.57}$$

We can now write Eq. (2.53) in the following form.

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t} + \sum_{k=-1}^{-\infty} F_{-k}^* e^{jk\omega_0 t} \tag{2.58}$$

Further using the result of Eq. (2.57), we get

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t} + \sum_{k=-1}^{-\infty} F_k e^{jk\omega_0 t}$$

or

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} \qquad \textit{Synthesis equation} \tag{2.59}$$

To Summarise The Fourier series of a continuous-time period is periodic signal in exponential form are found by the pair of equations

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} F_k e^{jk(2\pi/T)t} \tag{2.60}$$

$$F_k = \frac{1}{T}\int_T f(t)\,e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T}\int_T f(t)\,e^{-jk(2\pi/T)t}\,\mathrm{d}t \tag{2.61}$$

Example 2.5 *Find the Fourier series of the signal shown in Fig. 2.15 using the exponential form.*

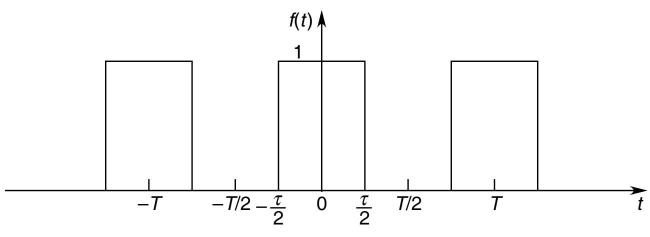


Fig. 2.15 *Sketch of periodic square pulse signal*

Solution In one period – $T/2$ to $T/2$ the signal is expressed as

$$f(t) = 1, -\tau/2 \le t \le \tau/2 \tag{i}$$
$$= 0;\ \text{otherwise}$$

From Eq. (2.55)

$$F_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} 1.e^{-jk\omega_0 t}\,\mathrm{d}t;\ \omega_0 = \frac{2\pi}{T} \tag{ii}$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0\tau/2} - e^{jk\omega_0\tau/2}}{-jk\omega_0}\right]$$

$$= \frac{\tau}{T}\left(\frac{\sin k\omega_0\tau/2}{k\omega_0\tau/2}\right) \tag{2.60a}$$

The term within the bracket of Eq. (iii) is in the form of the following *sinc function*.

$$\operatorname{sinc}\phi = \frac{\sin\phi}{\phi} \tag{2.61a}$$

Then

$$F_k = \frac{\tau}{T}\operatorname{sinc}\{(k\omega_0\tau)/2\} \tag{2.62}$$

The plot of a sinc function is given in Fig. 2.16(a). F_k as per Eq. (2.62) is a *discrete sinc function*. The number of F_k's in range of one π are given as

$$k\omega_0\tau/2 = \pi$$

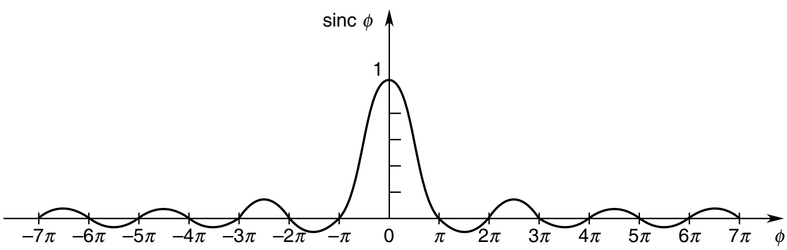
or

$$k = \left(\frac{T}{\tau}\right) \tag{2.63a}$$

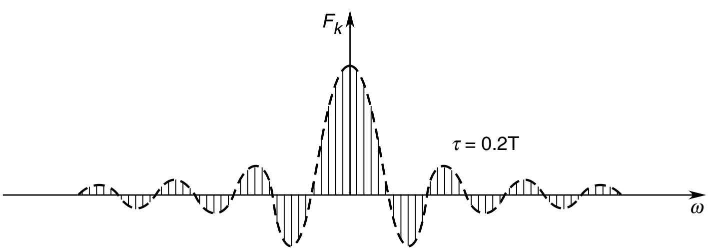
Spacing between the F_k's is given as

$$\omega_0 = \frac{2\pi}{T} \tag{2.63b}$$

Thus F_k's in π range increase as τ reduces with fixed T and the spacing increases as T is reduced with fixed τ. A typical plot of F_k's is shown in Fig. 2.16(b). This is referred to as a **line specturm**



(a) sinc function plot



(b) Fourier coefficient of rectangular pulse train; spacing $\omega_0 = 2\pi/T$

Fig. 2.16

Gibb's Phenomenon

Gibbs discovered that for a periodic signal with discontinuities, if the signal is reconstructed by adding the Fourier series, overshoots appear around the edges. These decay outwards in a damped oscillatory manner away from the edges. This is illustrated in Fig. 2.17.

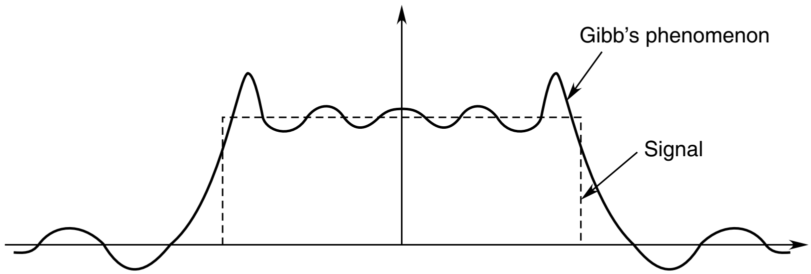


Fig. 2.17 *Illustration of Gibb's phenomenon*

Overshoots, according to Gibbs, are found to be at least 10 per cent irrespective of the number of terms in the Fourier series. It has also been observed that as more number of terms in the series are added, overshoots get sharper but adjoining oscillation amplitude reduces. The sum thus gets closer to the actual waveform except at the edges, as mentioned above.

Periodic Signal Symmetries and Fourier Coefficients

A periodic signal usually possesses various types of symmetries. By identifying one or more of these symmetries, one can achieve simplification in computing the harmonic coefficients. Following are some of the important symmetries that may be associated with periodic signals.

(i) Even symmetry : $f(t) = f(-t)$
(ii) Odd symmetry : $f(t) = -f(-t)$
(iii) Half-wave (rotational) symmetry : $f(t) = -f(t + T/2)$

These symmetries are graphically illustrated in Figs 2.18 (a), (b) and (c) respectively.

1. *Periodic Function even Symmetry; $f_e(t)$* Equation (2.32) gets reduced to the following form.

$$f_e(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \tag{2.64}$$

$$a_k = \frac{4}{T}\int_0^{T/2} f_e(t) \cos k\omega_0 t \, dt \tag{2.65}$$

Sine terms are absent in the Fourier series as these have odd symmetry (i.e., $b_k = 0$; for all values of k).

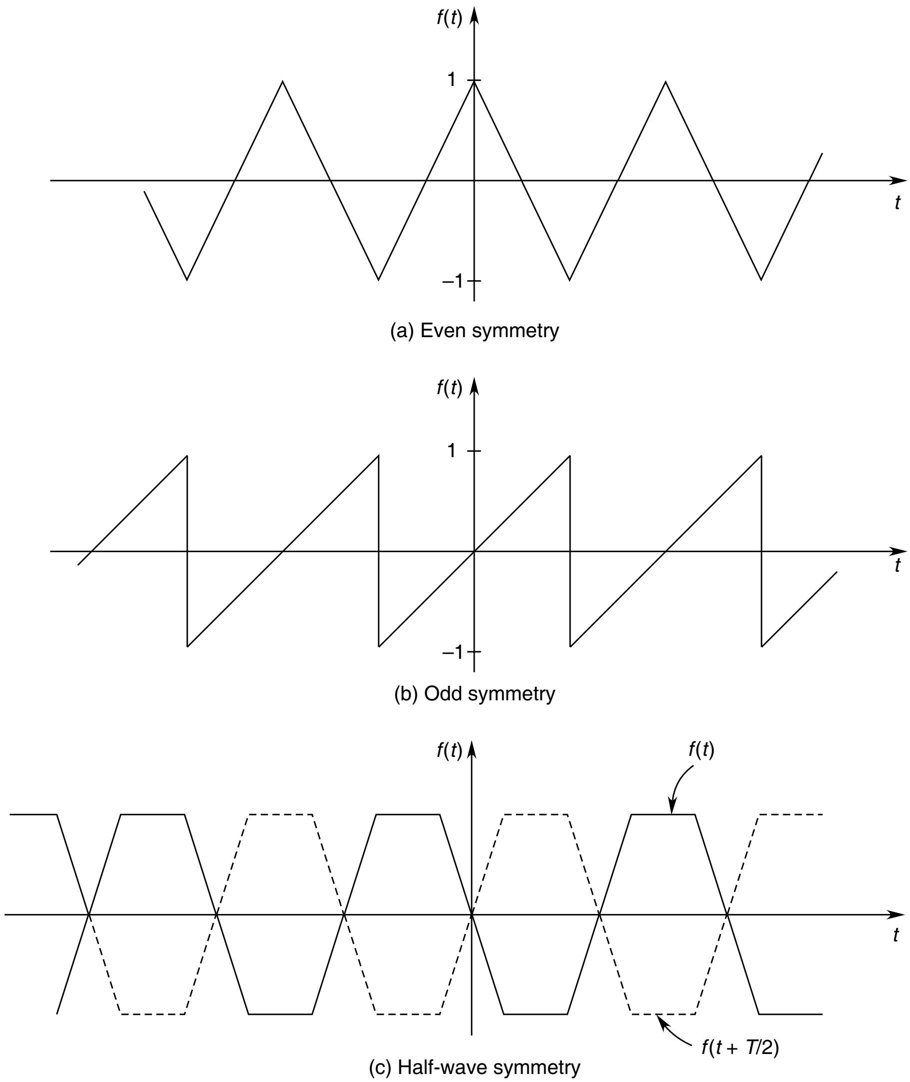


Fig. 2.18 *Illustration of various symmetries of a periodic signal*

2. *Periodic Function Odd Symmetry; $f_o(t)$* As cosine is an even symmetry function, the corresponding terms will be absent in the Fourier series, which takes the following form.

$$f_0(t) = a_0 + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \tag{2.66}$$

$$b_k = \frac{4}{T}\int_0^{T/2} f_0(t) \sin k\omega_0 t \, \mathrm{d}t \tag{2.67}$$

It should be noted that a periodic function can always be resolved into even and odd parts, i.e.,

$$f(t) = \left[\frac{f(t)+f(-t)}{2}\right]+\left[\frac{f(t)-f(-t)}{2}\right] \tag{2.68}$$
$$= f_e + f_0$$

The first term of Eq. (2.68) forms the even part and second term forms the odd part of $f(t)$.

3. *Periodic Function Half-wave Symmetry* In this case, Eq. (2.32) is reduced to

$$f(t) = \sum_{k=1}^{\infty} [a_k \cos k\omega_0 t + b_k \sin k\omega_0 t] \tag{2.69}$$

where

$$k = \text{even terms that are absent}$$

Then

$$a_k = \frac{4}{T}\int_0^{T/2} f(t) \cos k\omega_0 t \,\mathrm{d}t;\ k \text{ odd only} \tag{2.70}$$

and

$$b_k = \frac{4}{T}\int_0^{T/2} f(t) \sin k\omega_0 t \,\mathrm{d}t;\ k \text{ odd only} \tag{2.71}$$

It can be noted that half-wave symmetry is a kind of odd symmetry which means that only certain waveform having odd symmetry also possess half-wave symmetry.

The Power Spectrum of a Periodic Signal

It has been shown in Section 1.13 through Example 1.26 that a sinusoidal signal is a power signal. In general a periodic signal is a power signal. The power associated with a periodic signal is given in the following equation.

$$P = \frac{1}{T}\int_0^T f^2(t)\mathrm{d}t \tag{2.72}$$

But $\quad f^2(t) = f(t) f^*(t);\quad f(t)$ may be complex

In terms of exponential series, (Eq. 2.59)

$$f^*(t) = \sum_{k=-\infty}^{\infty} F_k^* \mathrm{e}^{-jk\omega_0 t}$$

Substituting in Eq. (2.77), we have

$$P = \frac{1}{T}\int_0^T f(t)\left(\sum_{k-\infty}^{\infty} F_k^* e^{-jk\omega_0 t}\right)\mathrm{d}t$$

Interchanging the order of summation and integration, we have

$$P = \sum_{k=-\infty}^{\infty} F_k^* \left[\frac{1}{T} \int_0^T f(t)\, e^{-jk\omega_0 t}\, dt \right] \tag{2.73}$$

The term within brackets is recognized as F_k. Then

$$P = \sum_{k=\infty}^{\infty} F_k^* F_k = \sum_{k=-\infty}^{\infty} |F_k|^2 \tag{2.74}$$

or

$$P = F_0 + 2 \sum_{k=1}^{\infty} |F_k|^2 \tag{2.75}$$

This is the *Parseval's theorem* according to which the average power of a periodic signal is the sum of powers associated with each component of Fourier series in frequency domain. It is to be noted that this is the sum of infinite terms.

Power associated with each frequency component of the given periodic signal is called **power density,** whereas the collection of power components $|F_k|^2$ as a function of frequency component $k\omega_0$ is called the **power density spectrum**. A plot of power density spectrum of a periodic (continous-time) signal is drawn in Fig. 2.19.

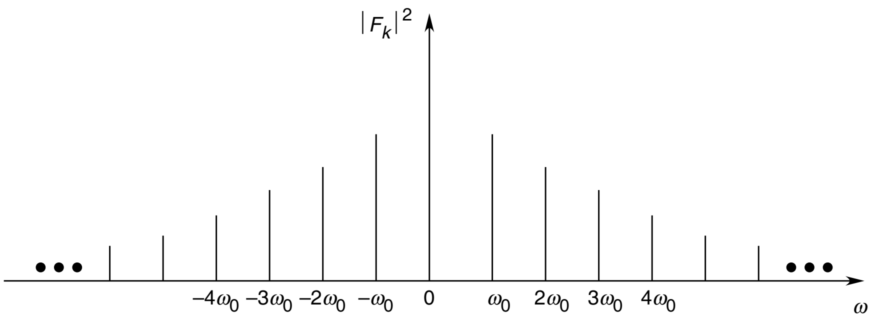


Fig. 2.19 *Power density spectrum of a continuous-time periodic signal*

Example 2.6 *Calculate the average power for the following signal.*

$$f(t) = 2\cos(10^4 \pi t)\sin(2 \times 10^4 \pi t)$$

Solution As $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$, we have

$$\begin{aligned} f(t) &= 2 \sin(2 \times 10^4 \pi t) \cos(10^4 \pi t) \\ &= \sin(3 \times 10^4 \pi t) + \sin(10^4 \pi t) \\ &= \frac{1}{2j}[e^{j3\times10^4\pi t} - e^{-j3\times10^4\pi t}] + \frac{1}{2j}[e^{j10^4\pi t} - e^{-j10^4\pi t}] \end{aligned}$$

It is found that the Fourier coefficients are

$$F_1 = \frac{1}{2j} \qquad F_{-1} = -\frac{1}{2j}$$

$$F_3 = \frac{1}{2j} \qquad F_{-3} = -\frac{1}{2j}$$

By Parseval's theorem

$$P = 2\left|\frac{1}{2j}\right|^2 + \left|\frac{1}{2j}\right|^2 = 1 \text{ W}$$

Response of LTI System to Periodic Inputs

The periodic input signal can be expressed in the form of Fourier series as

$$x(t) = \sum_{k=-\infty}^{+\infty} F_k e^{jk\omega_0 t} \tag{2.76}$$

As each component is an eigen function, the output as per Eq. (2.31) can be written as ($s = e^{jk\omega_0}$)

$$y(t) = \sum_{k=-\infty}^{+\infty} F_k H(e^{jk\omega_0})\, e^{jk\omega_0 t} \tag{2.77}$$

Thus the response (steady state) to a periodic signal is the superposition of the responses to the fundamental and harmonics. Let $\omega = k\omega_0$, $k = 0, \pm 1, \pm 2, \ldots$.

All we need to do is to find the steady-state sinusoidal response at frequency ω.

Example 2.7 *Consider the RC circuit of Fig. 2.20. It is excited by sinusoidal voltage $e_i = E_i \cos \omega t$. Find the steady-state output e_o.*

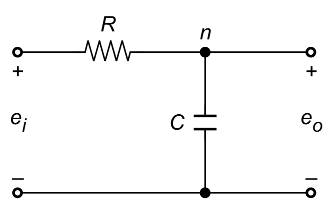


Fig. 2.20

Solution Sinusoidal and exponential functions are interchangeable in the sense that they are interconvertable. We can therefore write

$$e_i = Re[E_i\, e^{j\omega t}] = Re[E_i e^{j0}\, e^{j\omega t}]$$

$$= Re[\overline{E}_i \cdot e^{j\omega t}];\ \overline{E}_i \text{ is a } phasor. \tag{i}$$

Writing KCL at node n

$$C\frac{de_o}{dt} + \frac{e_o - e_i}{R} = 0$$

or

$$Rc\frac{de_o}{dt} + e_o = e_i \tag{ii}$$

In phasor form

$$e_o = Re[\overline{E}_o\, e^{j\omega t}] \tag{iii}$$

$$\frac{de_o}{dt} = \frac{d}{dt}[Re(e^{j\omega t})]$$

Interchanging $\frac{d}{dt}$ and Re operations, we get

$$\frac{de_o}{dt} = Re[\overline{E}_o \, j\omega \, e^{j\omega t}]$$

Equation (ii) can now be written in Re form as

$$Re[(j\omega Re + 1)\ \overline{E}_o\ e^{j\omega t}] = Re[\overline{E}_i \cdot e^{j\omega t}] \tag{v}$$

It means that the terms within Re operation must be equal. Thus,

$$(1 + j\omega Rc)\ \overline{E}_o e^{j\omega t} = \overline{E}_i e^{j\omega t} \tag{vi}$$

The exponential $e^{j\omega t}$ cancels out. Therefore,

$$(1 + j\omega Rc)\ \overline{E}_o = \overline{E}_i \tag{vii}$$

or

$$\overline{E}_o = \left[\frac{1}{1 + j\omega Rc}\right]\overline{E}_i \tag{viii}$$

or

$$\overline{E}_o = A(\omega)\ e^{j\theta}\ \overline{E}_i \tag{ix}$$

where

$$A(\omega) = \frac{1}{\sqrt{(1 + (\omega Rc)^2}} \tag{x}$$

$$\theta(\omega) = -\tan^{-1}(\omega Rc) \tag{xi}$$

We write

$$\overline{H}(\omega) = A(\omega)\ e^{j\theta} = A(\omega)\ \angle\theta(\omega)$$

and so

$$\overline{E}_o = \overline{H}(\omega)\ \overline{E}_i \tag{xii}$$

$\overline{H}(\omega)$ is called the transfer function or *frequency response function*. It transforms input phasor $\overline{E}_i$ to output phasor $\overline{E}_o$. It has two components: $A(\omega)$, the *amplitude-response function* and $\theta(\omega)$, in the *phase-response function*.

Observation

From Eqs. (x) and (xi)

$$\omega = 0 \quad A(0) = 1, \quad \theta(0) = 0$$
$$\omega \to \infty \quad A(\infty) \to 0, \quad \theta(\infty) \to -90^\circ$$

This is because at $\omega = 0$ (dc) e_i appears directly across C in steady state and at $\omega = \infty$, the capacitor acts as a short circuit. This indeed is the property of a low-pass filter.

We can write the output in time domain form from Eq. (ix) as

$$e_o = A(\omega)\ E_i \cos(\omega t + \theta) \tag{xiii}$$

Properties of Fourier Series

Fourier series have some important properties which are useful in some cases in reducing the complexity of deriving the Fourier series of periodic signals. These properties can be deduced from the properties of the Fourier transforms, the topic of the next section. We list here a few often-needed properties.

$$\text{Signal } x(t), \text{ period } T = \frac{2\pi}{\omega_0} \longrightarrow \text{Fourier series coefficients } F_k$$

$$\text{Signal } y(t), \text{ period } T = \frac{2\pi}{\omega_0} \longrightarrow \text{Fourier series coefficients } G_k$$

	Signal	*Fourier series coefficient*
Linearity	$Ax(t) + By(t)$	$AF_k + BG_k$
Time shifting	$x(t - t_0)$	$F_k\, e^{-j\omega_0 t_0}$
Differentiation	$\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$	$jk\omega_0 F_k$
Integration	$\int_{-\infty}^{t} x(t)\,\mathrm{d}t$ with condition $F_0 = 0$	$\left(\dfrac{1}{jk\omega_0}\right)F_k$
Real and even signal		F_k real and even
Real and odd signal		F_k pure imaginary and odd

Example 2.8 *Consider the signal*

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\,\omega_0 t + \frac{\pi}{4}\right)$$

Determine its Fourier series coefficients in the form of exponential series.

Solution We will proceed directly rather than use the general relationship. Writing sinusoidal functions in exponential form,
we have

$$\sin \omega_0 t = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] \tag{i}$$

$$2 \cos \omega_0 t = [e^{j\omega_0 t} + e^{-j\omega_0 t}] \tag{ii}$$

$$\cos\left(2\,\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2}\left[e^{j2\omega_0 t} e^{j\frac{\pi}{4}} + e^{-j2\omega_0 t}\, e^{-j\frac{\pi}{4}}\right]$$

$$= \frac{\sqrt{2}}{4}(1 + j1)\, e^{j2\omega_0 t} + \frac{\sqrt{2}}{4}(1 - j1)\, e^{-j2\omega_0 t} \tag{iii}$$

(i), (ii) and (iii) we can write

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t}$$

$$+ \frac{\sqrt{2}}{4}(1 + j)\, e^{j2\omega_0 t} + \frac{\sqrt{2}}{4}(1 - j)\, e^{-j2\omega_0 t} \qquad \text{(iv)}$$

By inspection of Eq. (iv) we write down the Fourier series coefficients as

$$F_0 = 1$$

$$F_1 = 1 + \frac{1}{2j},\ F_{-1} = 1 - \frac{1}{2j}$$

$$F_2 = \frac{\sqrt{2}}{4}(1 + j),\ F_{-2} = \frac{\sqrt{2}}{4}(1 - j)$$

Example 2.9 *The non-zero Fourier series coefficients in exponential form of a continuous-time periodic signal f(t) with fundamental time period T = 8 are*

$$F_1 = F_{-1}^{*} = 2,\ F_3 = F_{-3}^{*} = 4j$$

Determine f(t) in sinusoidal form

Solution

$$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4};\ \text{Fundamental frequency}$$

Only fundamental frequency and third harmonic are present in the signal

$$f(t) = \left[2e^{j\frac{\pi}{4}t} + 2e^{-j\frac{\pi}{4}t}\right] + \left[4je^{j\frac{3\pi}{4}t} - 4je^{-j\frac{3\pi}{4}t}\right]$$

$$= 4\cos\frac{\pi}{4}t - 8\sin\frac{3\pi}{4}t$$

$$= 4\cos\frac{\pi}{4}t + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

Example 2.10 *A continuous-time periodic signal has*

$$x(t) = \begin{cases} -2 & -1 \le t \le 0 \\ +2 & 0 \le t \le 1 \end{cases}$$

Its fundamental frequency is $\omega_0 = \pi$. Calculate the Fourier series coefficients F_k.

Solution

Time period $\qquad T = \dfrac{2\pi}{\omega_0} = 2$

Thus one period is from –1 to 1.

$$F_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$F_k = \frac{1}{2}\int_{-1}^{0} (-2)\, e^{-jk\pi t}\, dt + \frac{1}{2}\int_{0}^{1} 2\, e^{-jk\pi t}\, dt$$

or

$$F_k = \frac{1}{jk\pi}\cdot e^{-jk\pi t}\Big|_{-1}^{0} - \frac{1}{jk\pi}\, e^{-jk\pi t}\Big|_{0}^{1}$$

$$F_k = \frac{1}{jk\pi}\{(1 - e^{-jk\pi}) - (e^{-jk\pi} - 1)\}$$

$$= \frac{2}{jk\pi}(1 - e^{-jk\pi}) = \frac{2}{jk\pi} e^{-jk(\pi/2)}\, (e^{jk(\pi/2)} - e^{-jk(\pi/2)}$$

or

$$F_k = \frac{4}{k\pi} e^{-jk(\pi/2)} \sin(k\pi/2) = -j\frac{4}{k\pi}$$

$$F_0 = 0$$

$$F_k\,(\text{odd}) = \text{pure imaginary}$$

$$F_k\,(\text{even}) = 0$$

Observation

$x(t)$ real and odd $\Rightarrow F_k$ pure imaginary and odd.

2.5 FOURIER TRANSFORM

According to Fourier, a nonperiodic or aperiodic signal can be approximated as a special case of periodic signal with infinite period. It is quite obvious that as the period of a periodic signal increases, the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period tends to infinity, the frequencies become continuous. As a result of the formation of the continuum of frequency components, Fourier sum series is replaced by an integral.

The exponential Fourier series and its coefficients, according to Eqs (2.60) and (2.61) respectively are as follows.

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}\,;\ 0 \le t \le T \tag{2.77}$$

and

$$F_k = \frac{1}{T}\int_0^T f(t) e^{-jk\omega_0}\, dt \tag{2.78}$$

However, in case of a periodic signal, as $T \to \infty$; $\omega_0 = \dfrac{2\pi}{T} \to 0$ and the product $(k\omega_0)$ approaches a continuous frequency variable ω. It may be noted that

$\omega_0 = \dfrac{2\pi}{T} \to 0$ also implies that $F_k \to 0$ provided

$$\int_{-\infty}^{\infty} \left| f(t)e^{-jk\omega_0 t} \right| \mathrm{d}t \le \int_{-\infty}^{\infty} |f(t)| \mathrm{d}t \tag{2.79}$$

Noting that $\dfrac{\Delta(k\omega_0)}{\omega_0} = 1$ because $\Delta(k\omega_0) = \omega_0$ (in the limit increment $\Delta(k\omega_0)$ is ω_0), and so we can write Eq. (2.77) in the following form.

$$f(t) = \sum_{k\omega_0=-\infty}^{\infty} \frac{F_k}{\omega_0} e^{jk\omega_0 t} \Delta(k\omega_0) \tag{2.80}$$

In the limit, this summation can be written in integral form with the recognition that $\Delta(k\omega_0) \to \mathrm{d}\omega$ and so $k\omega_0 \to \omega$. Then

$$f(t) = \int_{-\infty}^{\infty} \tilde{F}_k(\omega)\, e^{j\omega t}\, \mathrm{d}\omega \tag{2.81}$$

where

$$\tilde{F}_k(\omega) \triangleq \frac{F_k}{\omega_0}$$

Using the same argument, Eq. (2.78) can be written in the following form.

$$\tilde{F}_k(\omega) \triangleq \frac{F_k}{\omega_0} = \frac{1}{\omega_0 T} \int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}\, \mathrm{d}t \tag{2.82}$$

Here, the integral limits have been changed from 0 to $T \Rightarrow -T/2$ to $T/2$. As $T = 2\pi/\omega_0$, Eq. (2.82) can be expressed as

$$\begin{aligned} \tilde{F}_k(\omega) &= \frac{1}{\omega_0(2\pi/\omega_0)} \int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)e^{-k\omega_0 t}\; \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)e^{-jk\omega_0 t}\, \mathrm{d}t \end{aligned} \tag{2.83}$$

As $\omega_0 \to 0$, $\pm\pi/\omega_0 \to \pm\infty$, $k\omega_0 \to \omega$. Then

$$\tilde{F}_k(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$$

For convenience, $\tilde{F}_k$ (ω) may be written as $F(\omega)$ such that Eqs. (2.81) and (2.83) are respectively written as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, \mathrm{d}\omega \tag{2.84}$$

and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, \mathrm{d}t \tag{2.85}$$

where $\dfrac{1}{2\pi}$ has been shifted from $F(\omega)$ to $f(t)$.

These two equations form the Fourier transform pair and are denoted as

$$f(t) \leftrightarrow F(\omega)$$

Transform operator $\int_{-\infty}^{\infty} f(.)e^{-j\omega t}\,\mathrm{d}t$ is symbolized as

$$\mathcal{F}\{f(t)\} = F(\omega) \tag{2.86}$$

And its inverse is symbolized as

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t)$$

In Eq. (2.84), let us replace ω by $-\omega$, It can then be written in the following form.

$$F(-\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,\mathrm{d}t = F^*(\omega) \tag{2.87}$$

This result implies that

1. $|F(\omega)| = |F(-\omega)|$, which means that the magnitude of the Fourier spectrum is an even function.
2. $\angle F(\omega) = -\angle F(-\omega)$ which implies that the angle of the Fourier spectrum is an odd function.

Note: $F(\omega)$ could also be written as $F(j\omega)$.

Example 2.11 *Obtain the Fourier transform of the following:*

(a) Impulse function Fig. 2.20(a).
(b) Exponentially decaying function Fig. 2.20(b).

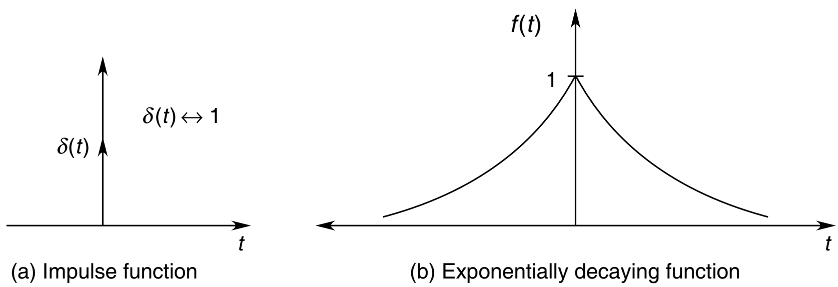


Fig. 2.20 *Some basic pulse functions*

Solution

(a) $f(\omega) = f\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,\mathrm{d}t = 1$

Hence the pair

$$\delta(t) \leftrightarrow 1$$

(b) Exponentially decaying function (Fig. 2.20(b)).

It can be mathematically expressed as

$$f(t) = e^{-|t|/\tau} \tag{i}$$

Its Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} e^{-|t|/\tau} e^{-j\omega t}\, \mathrm{d}t \tag{ii}$$

$$= \int_{-\infty}^{0} e^{(t/\tau)-j\omega t}\, \mathrm{d}t + \int_{0}^{\infty} e^{(-t/\tau)-j\omega t} \tag{iii}$$

$$= \frac{e^{(t/\tau)-j\omega t}\Big|_{-\infty}^{0}}{(1/\tau)-j\omega} + \frac{e^{(-t/\tau)-j\omega t}\Big|_{0}^{\infty}}{(-1/\tau)-j\omega} \tag{iv}$$

$$= \frac{1}{(1/\tau)-j\omega} + \frac{1}{(1/\tau)+j\omega} \tag{v}$$

$$F(\omega) = \frac{2\tau}{1+\omega^2\tau^2} \tag{vi}$$

The plot of $|F(\omega)|$ with ω is shown in Fig. 2.21.

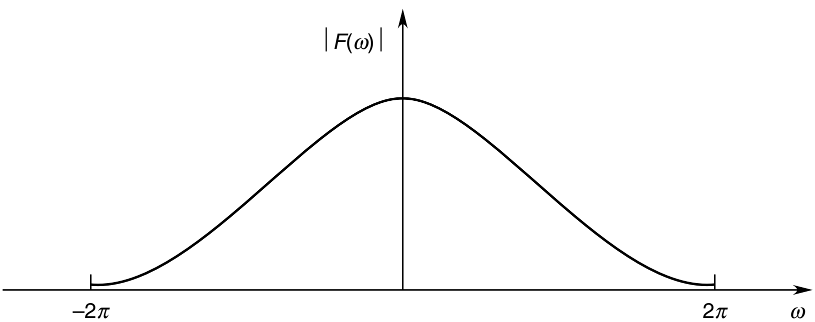


Fig. 2.21

Energy Spectral Density

The energy E of an aperiodic signal $f(t)$ in the frequency domain is obtained, as follows.

$$E = \int_{-\infty}^{\infty} |f(t)|^2\, \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} f^*(t) f(t) \mathrm{d}t \tag{2.88}$$

Expressing $f(t)$ in terms of its inverse Fourier transform, Eq. (2.88) takes the following form.

$$E \triangleq \int_{-\infty}^{\infty} f^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{-j\omega t} \mathrm{d}\omega \right] \mathrm{d}t \tag{2.89}$$

Interchanging variables, we can write in the following form.

$$E \triangleq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}\mathrm{d}t\right]\mathrm{d}\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\mathrm{d}\omega$$

Hence

$$E = \int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\mathrm{d}\omega \tag{2.90}$$

This is the *Parseval's theorem for the Fourier Transform.*

As $F(\omega)$ is an even function, we can express Eq. (2.90) as

$$E = \frac{1}{\pi}\int_{0}^{\infty} |F(\omega)|^2\mathrm{d}\omega \tag{2.91}$$

The frequency function within the integral of Eq. (2.91) can be interpreted as the energy density function, symbolized as

$$S(\omega) = \frac{|F(\omega)|^2}{\pi} \tag{2.92}$$

We can then express the signal energy as

$$E = \int_{-\infty}^{\infty} S(\omega)\mathrm{d}\omega \tag{2.93}$$

Energy in an infinitesimally small frequency band $\Delta\omega$ can be expressed as

$$S(\omega)\Delta\omega = \frac{|F(\omega)|^2}{\pi}\Delta\omega \tag{2.94}$$

Example 2.12 *Obtain the Fourier transform spectrum $G_T(\omega)$ of the rectangular pulse (also called gate function) defined as follows:*

$$g_T(t) = \begin{cases} 1; |t| \le \dfrac{T}{2} \\ 0;\ \textit{otherwise} \end{cases}$$

The gate function is sketched in Fig. 2.22.

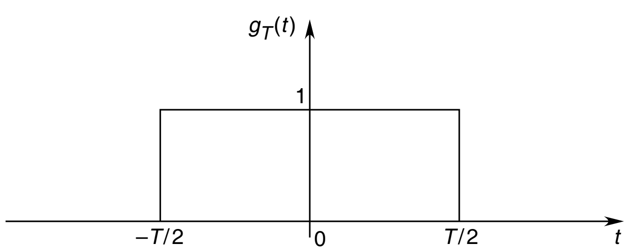


Fig. 2.22 *Gate function*

Solution The Fourier transform of the gate function, $g_T(t)$ is obtained using Eq. (2.85) as

$$G_T(\omega) = \mathcal{F}[g_T(t)] = \int_{-\infty}^{\infty} g_T(t)e^{-j\omega t}\mathrm{d}t \tag{i}$$

$$= \int_{-T/2}^{T/2} e^{-j\omega t}\mathrm{d}t = \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega} \tag{ii}$$

or

$$G_T(\omega) = T\,\frac{\sin(\omega T/2)}{\omega T/2} = T \operatorname{sinc}(\omega T/2) \tag{iii}$$

Note: $g_T(t)$ is a gate function but G_T is a sinc function.

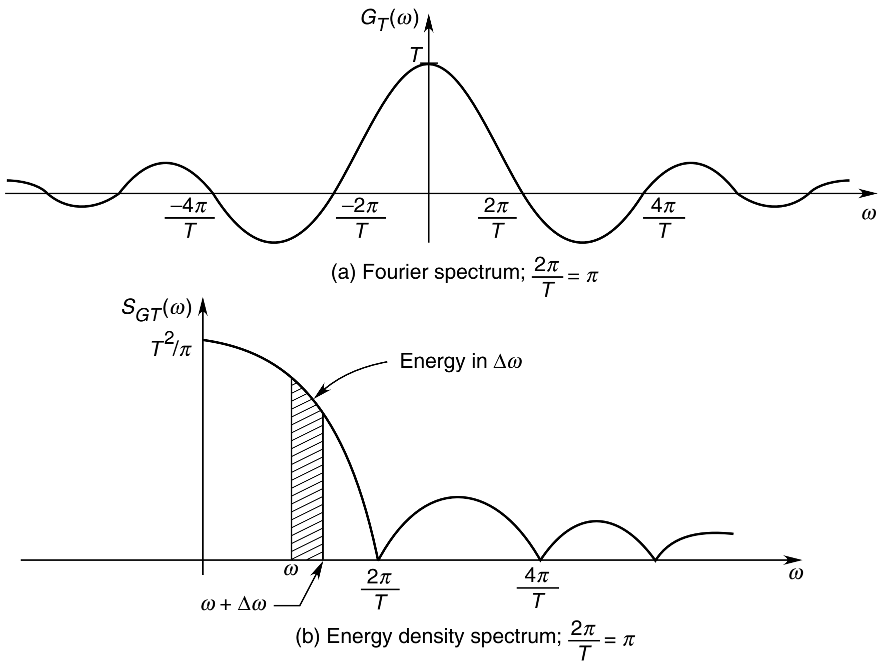
Hence, the pair

$$g_T(t) \leftrightarrow T \operatorname{sinc}(\omega T/2)$$

The spectrum $G_T(\omega)$ is shown graphically in Fig. 2.23(a). It can be noted that $G_T(\omega)$ is real and even function of ω.

The energy density spectrum of the gate function is obtained using Eq. (2.92) which yields the following result.

$$S_{GT}(\omega) = \frac{T^2}{\pi} \operatorname{sinc}^2(\omega T/2) \tag{iv}$$



(a) Fourier spectrum; $\frac{2\pi}{T} = \pi$

(b) Energy density spectrum; $\frac{2\pi}{T} = \pi$

Fig. 2.23 *Spectra of Gate Function*

The plot of the energy density spectrum of the gate function and $S_{GT}(\omega)$ is shown in Fig. 2.23(b).

It may be noted from Example 2.11(b) and 2.12 that $f(t)$ and $g_T(t)$ are even functions and their Fourier transforms are real functions. That is, the phase spectrum $\varphi(\omega)$ of even functions is a zero function.

2.6 PROPERTIES OF FOURIER TRANSFORM

1. *Linearity*

If

$$f_1(t) \leftrightarrow F_1(\omega)$$
$$f_2(t) \leftrightarrow F_2(\omega)$$

then

$$af_1(t) + bf_2(t) \leftrightarrow \mathrm{a}F_1(\omega) + \mathrm{b}F_2(\omega) \tag{2.95}$$

a and b are constants.

2. *Symmetry (Duality)*

If

$$f(t) \leftrightarrow F(\omega)$$

then

$$F(t) \leftrightarrow 2\pi f(-\omega) \tag{2.96}$$

Proof

By definition

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}\mathrm{d}\omega$$

Let $t \to -\omega$ and $\omega \to x$. Then

$$\begin{aligned} 2\pi f(-\omega) &= \int_{-\infty}^{\infty} F(x)e^{-jxt}\mathrm{d}x \\ &= \mathcal{F}[F(t)] \end{aligned}$$

Example 2.13 *As per Example 2.12, the Fourier transform of the gate functions is*

$$g_T(t) \leftrightarrow T\, sinc\left(\frac{\omega T}{2}\right) \tag{i}$$

Using symmetry property, find its corresponding Fourier pair. Check the result by direct derivation.

Solution Using the symmetry property we can write the new pair as

$$T\,\mathrm{sinc}\left(\frac{tT}{2}\right) \leftrightarrow 2\pi g_T(-\omega) \tag{ii}$$

As the Fourier transform is an even function of ω, the pair of Eq. (ii) can be written as

$$T \operatorname{sinc}\left(\frac{tT}{2}\right) \leftrightarrow 2\pi g_T(\omega)$$

Let $T \rightarrow \omega_0$ then

$$\omega_0 \operatorname{sinc}\left(\frac{\omega_0 t}{2}\right) \leftrightarrow 2\pi(\omega) \tag{iii}$$

where

$$\omega_0 = 2\pi/T$$

It may be observed that T is the gate width in time domain and ω_0 is the gate width in frequency. The new result by means of direct derivation is as follows.

Gate function in frequency domain is expressed mathematically as

$$g_{\omega_0}(\omega) = \begin{cases} 1\,; |\omega| \leq \omega_0/2 \\ 0\,; \text{otherwise} \end{cases} \tag{iv}$$

$$\begin{aligned} g_T(t) &= \mathcal{F}^{-1}[g_{\omega_0}(\omega)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{\omega_0}(\omega)e^{j\omega t}\mathrm{d}\omega \qquad\qquad \text{(v)} \\ &= \frac{1}{2\pi}\int_{-\omega_0/2}^{\omega_0/2} e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\omega_0/2}^{\omega_0/2} \\ &= \frac{1}{2\pi jt}\left(e^{j\omega_0 t/2} - e^{-j\omega_0 t/2}\right) \\ &= \frac{\sin(\omega_0 t/2)}{\pi t} = \frac{\omega_0}{2\pi}\,\frac{\sin(\omega_0 t/2)}{(\omega_0 t/2)} \end{aligned}$$

Thus, we have the Fourier transform pairs

$$g_T(t) \leftrightarrow T \operatorname{sinc}(\omega T/2) = G_T(\omega) \tag{2.97}$$

$$G_T(t) = (\omega_0/2\pi) \operatorname{sinc}(\omega_0 t/2) \leftrightarrow g_{\omega_0}(\omega) \tag{2.98}$$

Note: G_T stands for sinc function.

The graphical representation of these two pairs illustrating the use of symmetry property is drawn in Fig. 2.24(a) and (b).

3. *Scaling*

If

$$f(t) \leftrightarrow F(\omega)$$

then for a real constant *'a'*

$$f(at) \leftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \tag{2.99}$$

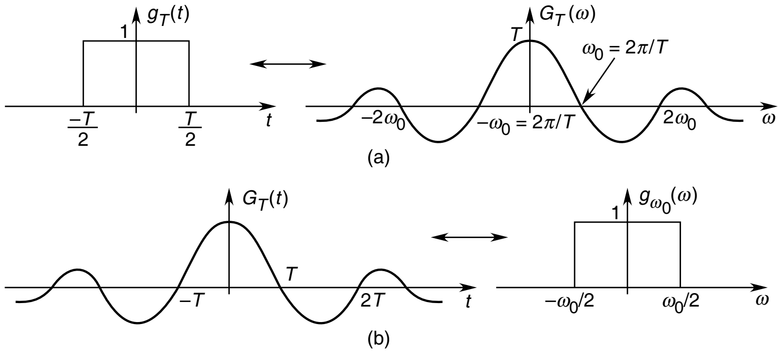


Fig. 2.24 *Illustration of symmetry property*

Proof

We have

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,\mathrm{d}t \tag{i}$$

Let $at = x$, then $a\mathrm{d}t = \mathrm{d}x$.

Substituting these in Eq. (i) we get (assuming $a > 0$)

$$\mathcal{F}[f(at)] = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j(\omega/a)x}\,\mathrm{d}x$$

$$= \frac{1}{a}F\left(\frac{\omega}{a}\right) \tag{ii}$$

For $a < t_0$, we let $x = -|a|t$. Then

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(-|a|t)\, e^{-j\omega t}\,\mathrm{d}t$$

$$= \frac{1}{-|a|}\int_{+\infty}^{-\infty} f(x)\, e^{j\omega x/|a|}\,\mathrm{d}x$$

$$= \frac{1}{|a|}\int_{+\infty}^{\infty} f(x)\, e^{j\omega x/|a|}\,\mathrm{d}x$$

$$= \frac{1}{|a|}F\left(\frac{-\omega}{|a|}\right) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \tag{iii}$$

Hence, from Eqs (ii) and (iii)

$$f(at) \leftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

4. *Time-shifting*

If

$$f(t) \leftrightarrow F(\omega) \tag{100}$$

then

It means that shift in time domain implies shift in phase $(-\omega t_0)$ in Fourier transform

Proof

$$f[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}\,\mathrm{d}t$$

Let

$$(t-t_0) = x \quad \text{or} \quad t = t_0 + x \quad \text{and} \quad \mathrm{d}t = \mathrm{d}x$$

Then

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(x)e^{-j\omega(t_0+x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,\mathrm{d}x$$

$$= e^{-j\omega t_0}F(\omega)$$

For example consider

$$\delta(t) \leftrightarrow 1$$

Then by the time-shifting property

$$\delta(t-t_0) \leftrightarrow e^{-j\omega t_0} \tag{iv}$$

Using the symmetry property

$$e^{-j(t-t_0)} \leftrightarrow 2\pi\delta(-\omega - t_0)$$

$$e^{-j(t-t_0)} \leftrightarrow 2\pi\delta(\omega - t_0)\text{; evenness of FT}$$

Let

$$t_0 \to \omega_0$$

$$e^{-j\omega t_0} \leftrightarrow 2\pi\delta(\omega - \omega_0) \tag{v}$$

Let

$$\omega_0 = 0,$$

then

$$1 \leftrightarrow 2\pi\delta(\omega) \tag{2.101}$$

5. *Frequency-shifting*

If

$$f(t) \leftrightarrow F(\omega)$$

then

$$f(t)\, e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0) \tag{2.102}$$

This implies that a shift in frequency causes the time signal to be shifted in phase by $(\omega_0 t)$.

Proof

$$\mathcal{F}[f(t)]\, e^{j\omega_0 t} = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}\,\mathrm{d}t$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\,\mathrm{d}t$$

$$= F(\omega - \omega_0)$$

It should be noted that multiplication of the signal $f(t)$ by $e^{j\omega_0 t}$ has a great significance in Communication Engineering. If $f(t)$ is considered as some information signal, containing relatively lower frequencies and $e^{j\omega_0 t}$ the carrier signal with $f_0 = \omega_0/2\pi$ of the order of Giga Hertz (GHz), then $f(t)\, e^{j\omega_0 t}$ is known as **amplitude modulated signal**. Since $e^{j\omega_0 t}$ is not a real function hence it can not be generated. Thus, in practice, amplitude modulation is achieved by multiplying $f(t)$ by a sinusoidal function, say cos $\omega_0 t$.

6. *Time-convolution*

If

$$f_1(t) \leftrightarrow F_1(\omega)$$

and

$$f_2(t) \leftrightarrow F_2(\omega)$$

then

$$f_1(t) * f_2(t) \leftrightarrow F_1(\omega)\, F_2(\omega) \tag{2.103}$$

Proof

$$\begin{aligned}
\mathcal{F}[f_1(t) * f_2(t)] &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau] e^{-j\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t}\,\mathrm{d}\tau\right] \\
&= \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega t} F_2(\omega)\mathrm{d}\tau \\
&= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega t}\mathrm{d}\tau \\
&= F_2(\omega) F_1(\omega)
\end{aligned}$$

7. *Frequency convolution*

If

$$f_1(t) \leftrightarrow F_1(\omega)$$

and

$$f_2(t) \leftrightarrow F_2(\omega)$$

then

$$f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi}[F_1(\omega) * F_2(\omega)] \tag{2.104}$$

Proof

We can write

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(u) F_2(\omega - u)\mathrm{d}u$$

Taking the inverse Fourier transform, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega) * F_2(\omega) e^{j\omega t}\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(u) F_2\,(\omega-u)\mathrm{d}u\right] e^{j\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} F_1(u)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2\,(\omega-u) e^{-j\omega t}\mathrm{d}t\right]\mathrm{d}u \qquad \text{(ii)}
\end{aligned}$$

Using frequency shifting property

$$f_2(t) e^{jut} \leftrightarrow F_2(\omega - u)$$

Equation (ii) can be written in the following form.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega) * F_2(\omega) e^{j\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} F_1(u) f_2(t) e^{jut}\mathrm{d}u$$

$$= 2\pi f_2(t)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)e^{jut}\,\mathrm{d}u$$

$$= 2\pi f_2(t)\,f_1(t)$$

Hence

$$F_1(\omega) * F_2(\omega) \leftrightarrow 2\pi f_1(t) f_2(t)$$

or

$$\frac{1}{2\pi}[F_1(\omega) * F_2(\omega)] \leftrightarrow f_1(t) f_2(t)$$

8. *Time Differentiation and Integration*

Differentiation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\,\mathrm{d}\omega$$

Differentiating both sides with respect to time,

$$\dot{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega\, F(\omega)]e^{j\omega t}\,\mathrm{d}\omega$$

We immediately recognize the pair

$$\dot{f}(t) \leftrightarrow j\omega F(\omega) \tag{2.105a}$$

In general

$$f^{n}(t) \leftrightarrow (j\omega)^{n}\, F(\omega) \tag{2.105b}$$

Integration

Let

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\,\mathrm{d}t$$

Differentiating,

$$\dot{g}(t) = f(t)$$

$\therefore$

$$j\omega G[\omega] = F(\omega)$$

or

$$G(\omega) = \left(\frac{1}{j\omega}\right) F(\omega) \tag{2.106}$$

For $G(\omega)$ to exist

$$\lim_{t\to\infty} g(t) = 0$$

or

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = 0$$

Since

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\,\mathrm{d}t$$

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = F(0) = 0$$

If this condition is not met, $g(t)$ is not an energy function and so $G(\omega)$ in Eq. (2.106) will have in addition a δ-function; see Table 2.1.

9. *Frequency Differentiation and Integration*

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

Differentiating with respect to ω

$$\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{\infty} (-jt)f(t)e^{-j\omega t}\,\mathrm{d}t$$

$$(-jt)\,f(t) \leftrightarrow F'(\omega) \tag{2.107}$$

In general

$$(-jt)^n f(t) \leftrightarrow F^{(n)}(\omega) \tag{2.108}$$

Similarly, within an additive constant

$$\frac{f(t)}{-jt} = \int_{0}^{\omega} F(\omega')\mathrm{d}\omega' \tag{2.109}$$

As an example consider

$$\mathrm{A}\,e^{-\alpha t}\,u(t) \leftrightarrow \frac{\mathrm{A}}{(\alpha + j\omega)}$$

Using Eq. (2.106), we can write

$$\mathrm{A}\,\frac{1}{(-jt)^n}\,e^{-\alpha t}\,u(t) \leftrightarrow \frac{\mathrm{A}\,n!}{(\alpha + j\omega)^{n+1}}$$

Some of the important properties of the Fourier transform are summarized in Table 2.1.

Table 2.1 *Properties of the Fourier Transform*

1.	Transformation	$f(t) \leftrightarrow F(\omega)$
2.	Linearity	$a_1f_1(t) + a_2f_2(t) \leftrightarrow a_1F_1(\omega) + a_2F_2(\omega)$
3.	Symmetry	$F(t) \leftrightarrow 2\pi f(-\omega)$
4.	Scaling	$f(at) \leftrightarrow \frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
5.	Time delay	$f(t-t_0) \leftrightarrow e^{-j\omega t_0}\,F(\omega)$
6.	Modulation (frequency shifting)	$e^{j\omega_0 t}f(t) \leftrightarrow F(\omega-\omega_0)$
7.	Time convolution	$f_1(t) * f_2(t) \leftrightarrow F_1(\omega)\,F_2(\omega)$
8.	Frequency convolution	$f_1(t)f_2(t) \leftrightarrow \frac{1}{2\pi}\,[F_1(\omega) * F_2(\omega)]$
9.	Time differentiation	$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t) \leftrightarrow (j\omega)^n\,F(\omega)$
10.	Time integration	$\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau \leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
11.	Frequency differentiation	$-jt\,f(t) \leftrightarrow \frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}$
12.	Frequency integration	$\frac{f(t)}{-jt} \leftrightarrow \int F(\omega')\mathrm{d}\omega'$
13.	Reversal	$f(-t) \leftrightarrow F(-\omega)$

Example 2.14 *Find the Fourier transform of a one-sided exponential pulse is sketched in Fig. 2.25.*

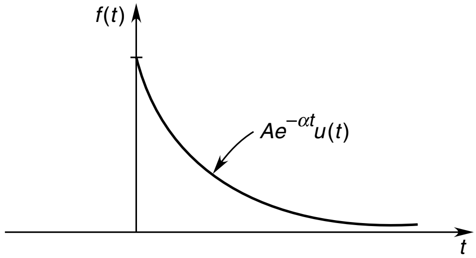


Fig. 2.25 *Exponential pulse*

Solution The Fourier transform of this function is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

$$= \int_{-\infty}^{\infty} Ae^{-\alpha t}u(t)e^{-j\omega t}\,\mathrm{d}t$$

$$= A\int_{0}^{\infty} e^{-(\alpha+j\omega)t}\,\mathrm{d}t$$

$$= \left.\frac{Ae^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right|_0^{\infty}$$

$$= \frac{A}{\alpha+j\omega} \qquad \text{(i)}$$

Its amplitude and phase spectra are given as

$$|F(\omega)| = \frac{A}{(\alpha^2+\omega^2)^{1/2}} \qquad \text{(ii)}$$

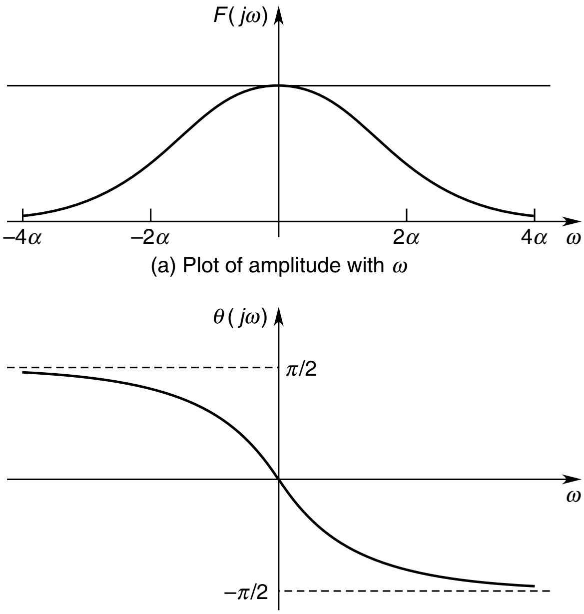
$$\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{\alpha}\right) \qquad \text{(iii)}$$

These spectra are plotted in Figs 2.26(a) and (b).

Observe that $|F(\omega)|$ is an even function and $\theta(\omega)$ is an odd function.

Example 2.15 *Obtain Fourier transform of the triangular function (shown in Fig. 2.27), mathematically described as*

$$f(t) = \begin{cases} A\left[1-\dfrac{|t|}{T}\right] & ;|t| \le T \\ 0 & ;|t| > T \end{cases}$$



(a) Plot of amplitude with ω

(b) Plot of phase angle with ω

Fig. 2.26 *Amplitude and Phase spectra of exponential pulse*

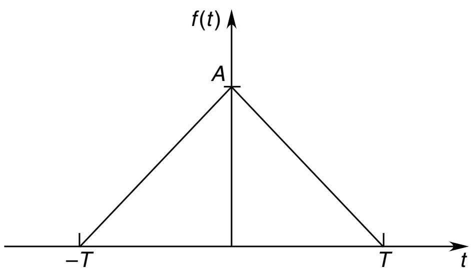


Fig. 2.27 *Triangular function*

Solution

$$\mathcal{F}[f(t) = t\int_{-\infty}^{\infty} A\left[1-\frac{|t|}{T}\right]e^{-j\omega t}\,\mathrm{d}t \quad ;|\,t\,| < T$$

We now proceed to evaluate right hand side of above expression.

$$\int_{-T}^{0} A\left[1+\frac{t}{T}\right]e^{-j\omega t} + \int_{0}^{T} A\left[1-\frac{t}{T}\right]e^{-j\omega t}\mathrm{d}t$$

or

$$\int_{-T}^{0} Ae^{-j\omega t}\mathrm{d}t + \int_{-T}^{0}\frac{A}{T}te^{-j\omega t}\mathrm{d}t + \int_{0}^{T} Ae^{-j\omega t}\mathrm{d}t - \int_{0}^{T}\frac{A}{T}te^{-j\omega t}\mathrm{d}t$$

or

$$A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T}^{0} + \frac{A}{T}\left[\frac{e^{-j\omega t}}{(-j\omega)^2}(-j\omega t - 1)\right]_{-T}^{0} + A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{T}$$

$$-\frac{A}{T}\left[\frac{e^{-j\omega t}}{(-j\omega)^2}(-j\omega t - 1)\right]_{0}^{T}$$

or

$$\frac{-A}{j\omega} + \frac{Ae^{j\omega T}}{j\omega} + \left[-\frac{1}{(j\omega)^2} - \left\{\frac{Te^{j\omega T}}{j\omega} - \frac{e^{j\omega T}}{(j\omega)^2}\right\}\right]$$

$$-\frac{Ae^{-j\omega T}}{j\omega} + \frac{A}{j\omega} - \frac{A}{T}\left[\left\{\frac{Te^{-j\omega T}}{-j\omega} - \frac{e^{-j\omega T}}{(j\omega)^2}\right\} + \frac{1}{(j\omega)^2}\right]$$

Combining exponential terms into sine and cosine and then simplifying trigonometrically, we get

$$\frac{2A}{\omega^2 T}(1 - \cos \omega T) = AT\,\frac{\sin^2(\omega T/2)}{(\omega T/2)^2} = AT\,\mathrm{sinc}^2(\omega T/2)$$

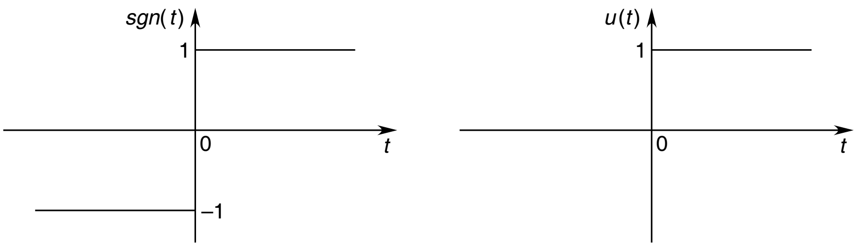
which is the desired Fourier transform.

Example 2.16 *The signum function, denoted by sgn(t), is defined as*

$$sgn(t) = \begin{cases} -1 & ; \quad t < 0 \\ 0 & ; \quad t = 0 \\ 1 & ; \quad t > 0 \end{cases}$$

Its graph is shown in Fig. 2.28(a).

(a) Obtain its Fourier transform and plot its amplitude and phase spectrum.
(b) Using the result of part (a), obtain the Fourier transfer of unit step function of Fig. 2.28(b).



(a) Signum function (b) Unit step function

Fig. 2.28 *Some basic functions*

Solution (a) The Fourier transform of *sgn(t)* can be easily obtained using the time-differential property of the Fourier transform, according to which

$$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega) \qquad \text{(i)}$$

Differentiation of sgn(t) w.r.t. time gives

$$\frac{d}{dt}\{\text{sgn}(t)\} = 2\delta(t); \quad \text{at } t = 0 \text{ there is a change } -1 \text{ to } 1\ (= 2) \text{ in zero time} \qquad \text{(ii)}$$

$$\mathcal{F}\left[\frac{d}{dt}\text{sgn}(t)\right] = j\omega\,\text{sgn}(\omega) \qquad \text{(iii)}$$

or

$$\mathcal{F}[2\delta(t)] = j\omega\,\text{sgn}(\omega)$$

or

$$2 = j\omega\,\text{sgn}(\omega) \qquad \text{(iv)}$$

Thus, we get the following result.

$$\text{sgn}(\omega) = \frac{2}{j\omega} \qquad \text{(v)}$$

The plots of magnitude and phase of sgn(ω) are shown in Figs 2.29 (a) and (b) respectively.

(b) The Fourier transform of a unit-step function can be immediately obtained using above result.

The unit-step function is defined in terms of signum function as

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\text{sgn}(t)$$

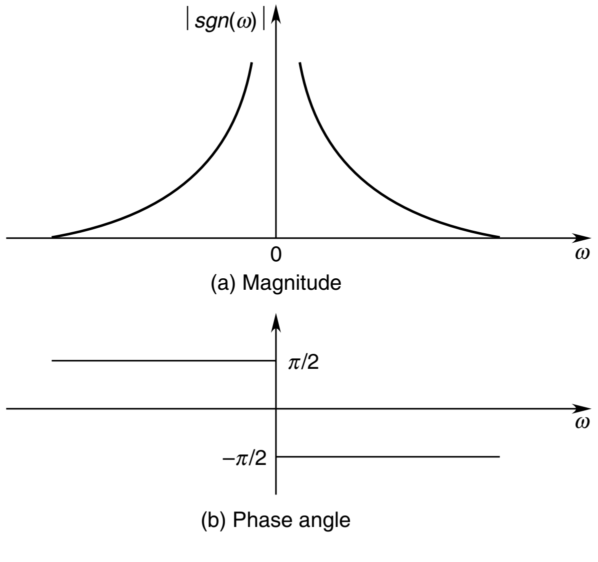


Fig. 2.29 *Plot of magnitude and phase of sgn(ω) with ω*

so

$$\mathcal{F}[u(t)] = \frac{1}{2}\mathcal{F}[1] + \frac{1}{2}\mathcal{F}[sgn(t)]; \text{ see footnote}$$
$$= \pi\delta(\omega) + \frac{1}{2}$$

i.e.

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

The presence of high frequencies in sgn(t) and $u(t)$ is due to a sharp jump (discontinuity) in the time domain at $t = 0$. At all times other than $t = 0$, the two signals are simply dc signals and therefore have only zero frequency.

2.7 TABLES OF FOURIER TRANSFORM PAIRS

Rather than finding the Fourier transform and its inverse abinitio, it is helpful to use table of transform pairs. Tables 2.2 and 2.3 list the transform pairs into two parts; the energy signals and power signals.

Table 2.2 *Some useful transform pairs of energy signals*

	Time function, f(t)	*Fourier Transform, F(ω)*
1.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
2.	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$
3.	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(a+j\omega)^n}$
4.	$g_T(t) = \begin{cases} 1; \lvert t\rvert < \frac{T}{2} \\ 0; \text{otherwise} \end{cases}$	$T\,\text{sinc}\left(\frac{\omega T}{2}\right)$
5.	$\frac{\omega_0}{2\pi}\,\text{sinc}\left(\frac{\omega_0 t}{2}\right)$	$G_{\omega_0}(\omega) = \begin{cases} 1, & \lvert\omega <\rvert \frac{\omega_0}{2} \\ 0, & \text{otherwise} \end{cases}$
6.	$\left\{A\left(1-\frac{\lvert t\rvert}{T}\right); \lvert t\rvert < T\right.$	$AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$
7.	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
8.	$\frac{1}{a^2+t^2}$	$\frac{\pi}{2}e^{-a\lvert\omega\rvert}$
9.	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
10.	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

* $\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega-\omega_0)$; Table 2.3

Let $\omega_0 = 0$, then

$$\mathcal{F}[1] = 2\pi\delta(\omega)$$

Table 2.3 *Some useful transform pairs of power signals*

	Time Function, f(t)	*Fourier Transform, F(ω)*
1.	1	$2\pi\delta(\omega)$
2.	$\delta(t)$	1
3.	$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$
4.	$\text{sgn}(t)$	$\dfrac{2}{j\omega}$
5.	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
6.	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
7.	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
8.	$\sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t}$	$2\pi \sum_{n=-\infty}^{\infty} F_n\, \delta(\omega - n\omega_0)$
9.	$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta\,(t - kT)$	$\omega_0\, \delta_{\omega_0}(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta\,(\omega - n\omega_0);\ \omega_0 = \dfrac{2\pi}{T}$
10.	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \dfrac{j\omega}{\omega_0^2 - \omega^2}$
11.	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \dfrac{\omega^2}{\omega_0^2 - \omega^2}$

2.8 FOURIER TRANSFORM OF PERIODIC SIGNALS

We now consider the Fourier transform of signals which are not absolutely integrable and signals which have infinite discontinuities. Periodic signals belong to this class, i.e., they are not absolutely integrable as these extend over all time, i.e, $t = \infty$ to $t = -\infty$ (These are power signal as explained in Section 1.4). Impulse train is the kind of signal which has infinite discontinuities.

Sinusoidal Signal

Consider the following sinusoidal signal expressed in form of exponentials.

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}; \text{ Period } T = 2\pi/\omega_0 \tag{2.108}$$

Taking the Fourier transform

$$\mathcal{F}[\cos \omega_0 t] = \mathcal{F}\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right] \tag{2.109}$$

We have

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

This result can also be proved directly

$$\mathcal{F}^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}e^{j\omega_0 t}\text{; where } \omega \to \omega_0$$

Hence (in general)

$$e^{\pm j\omega_0 t} \leftrightarrow 2\pi\delta(\omega \mp \omega_0) \tag{2.110}$$

Taking the Fourier transform of right hand side of Eq. (2.109), we can write

$$\mathcal{F}[\cos\omega_0 t] = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

or

$$\cos\omega_0 t \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{2.111}$$

We see that the Fourier transform of $\cos\omega_0 t$ has two impulses located at $\omega = \pm\omega_0$ on the frequency spectrum.

Similarly we can write for the transform of $\sin\omega_0 t$.

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \leftrightarrow j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)] \tag{2.112}$$

The Fourier spectra of cosine and sine waves respectively are sketched in Fig. 2.30.

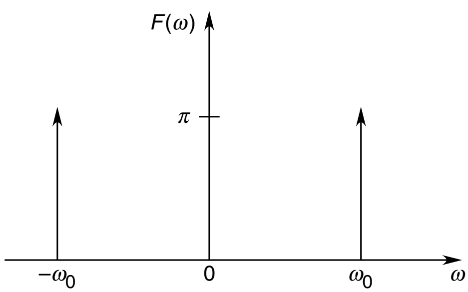


Fig. 2.30 *Fourier spectra of $\cos\omega_0 t$ and $\sin\omega_0 t$*

Example 2.17 *Find the Fourier transform of $f(t)\cos\omega_0 t$ using the convolution property.*

Solution

Let

$$f(t) \leftrightarrow F(\omega) \tag{i}$$

We know that

$$\cos\omega_0 t \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{ii}$$

According to the frequency convolution property of the Fourier transform

$$f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2\pi}[F(\omega) * \pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}]$$

i.e.,

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}[F(\omega) * \pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}]$$

$$\frac{1}{2}\left[\int_{-\infty}^{\infty} F(u)\,\delta(\omega-\omega_0-u)\mathrm{d}u + \int_{-\infty}^{\infty} F(u)\,\delta(\omega+\omega_0-u)\mathrm{d}u\right] \qquad \text{(iii)}$$

Using the sampling property of impulse function; at $u = \omega - \omega_0$, we have

$$\int_{-\infty}^{\infty} F(u)\,\delta(\omega-\omega_0-u)\mathrm{d}u = F(\omega-\omega_0) \qquad \text{(iv)}$$

Similarly, at $u = \omega + \omega_0$

$$\int_{-\infty}^{\infty} F(u)\,\delta(\omega+\omega_0-u)\mathrm{d}u = F(\omega+\omega_0) \qquad \text{(v)}$$

Thus, substituting Eqs (iv) and (v) into (iii) we obtain

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$$

i.e.

$$f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}\,[F(\omega-\omega_0)+F(\omega+\omega_0)] \qquad \text{(vi)}$$

It is observed that multiplying $f(t)$ with $\cos\omega_0 t$ causes the spectrum $F(\omega)$ to shift to $(\omega - \omega_0)$ and $(\omega + \omega_0)$ as illustrated is Figs 2.31 (a) and (b).

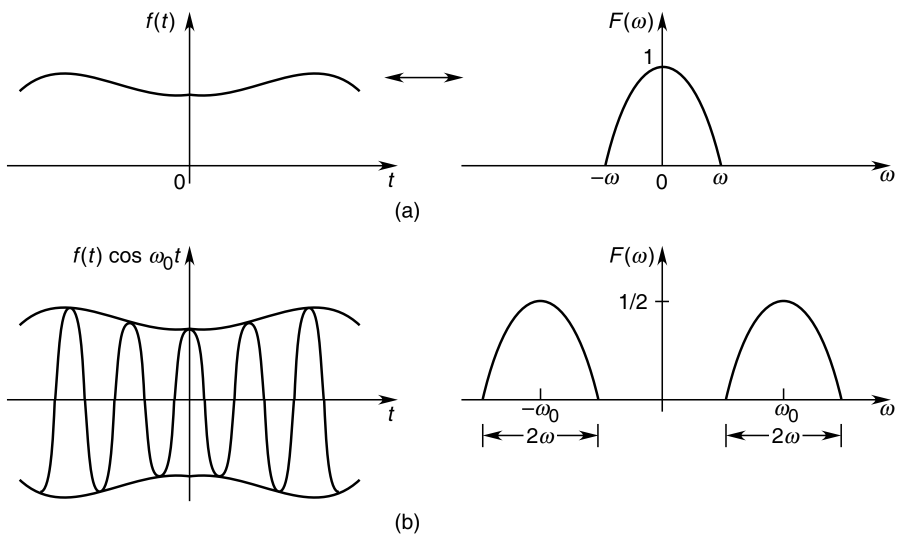


Fig. 2.31 *Frequency shifting property*

Impulse Train

Let us now proceed to obtain the Fourier transform of an impulse train of unit strength shown in Fig. 2.32 (a) expressed mathematically as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT); \text{ period } T \tag{2.113}$$

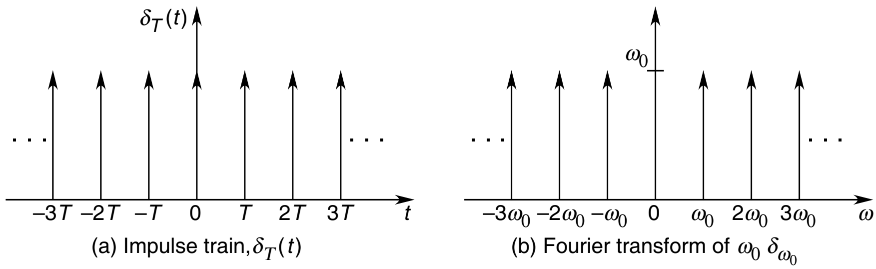


Fig. 2.32 *Periodic impulse train and its Fourier Transform*

It can be written in the form of its exponential Fourier series as

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t};\ \omega_0 = 2\pi/T \tag{2.114}$$

The Fourier series coefficients are obtained by integrating over one period. Thus

$$\begin{aligned} F_n &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}\, \mathrm{d}t \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, \mathrm{d}t = \frac{1}{T} \end{aligned} \tag{2.115}$$

We can then write

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{2.116}$$

Taking the Fourier transform and using Eq. (2.116), we get

$$\begin{aligned} \mathcal{F}[\delta_T(t)] &= \mathcal{F}\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\right] \\ &= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \end{aligned} \tag{2.117}$$

Hence we get the following pair

$$\delta_T(t) \leftrightarrow \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \tag{2.118a}$$

or

$$\delta_T(t) \leftrightarrow \omega_0\, \delta_{\omega 0} \tag{2.118b}$$

It is seen from Eq (2.118) that the Fourier transform is a impulse train in frequency domain with impulse of strength ω_0 and period $\omega_0 = 2\pi/T$ as shown in Fig. 2.32 (b).

Fourier Series from Fourier Transform

We started with the Fourier series and generalized it to the Fourier transform. We shall now demonstrate the reverse process.

Consider a periodic function $f(t)$ whose basic segment over one period (T) is defined in the following manner.

$$f_0(t) = \begin{cases} f(t) & ; |t| < T/2 \\ 0 & ; |t| > T/2 \end{cases} \tag{2.119}$$

Then $f(t)$ can be expressed in terms of $f_0(t)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f_0(t - nT) = \delta_T(t) * f_0(t) \tag{2.120}$$

Now

$$F_0(\omega) = \mathcal{F}[f_0(t)] = \int_{-T/2}^{T/2} f(t) e^{-j\omega t} \mathrm{d}t \tag{2.121}$$

Taking the Fourier transform of Eq. (2.120)

$$\begin{aligned} F(\omega) &= \omega_0\, \delta_{\omega_0} F_0(\omega) \\ &= \omega_0 F_0(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \end{aligned}$$

Using the sampling property of an impulse, we can write

$$F(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} F_0(n\omega_0)\delta(\omega - n\omega_0); \omega_0 = 2\pi/T \tag{2.122}$$

Defining

$$F_n = \frac{1}{T} F_0(n\omega_0) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, \mathrm{d}t \tag{2.123}$$

Taking the inverse Fourier transform of Eq. (2.123), we get

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\, \mathrm{d}t;\ \omega_0 = 2\pi/T \tag{2.124}$$

which is the exponential form of the Fourier series.

Example 2.18 *Find the Fourier transform of a periodic pulse train shown in Fig. 2.33. For A = 1, d = 1/16 and T = 1/4. What is percentage of total power contained within the first zero crossing of the spectrum envelope of f(t)?*

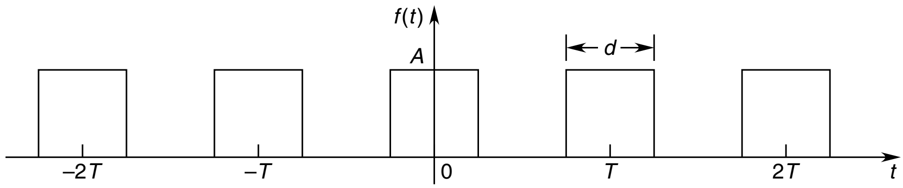


Fig. 2.33 *Sketch of pulse train*

Solution The Fourier series representation of $f(t)$ given is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad ; \omega_0 = 2\pi/T \tag{i}$$

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,\mathrm{d}t$$

$$= \frac{Ad}{T}\operatorname{sinc}\left(\frac{n\pi d}{T}\right) \tag{ii}$$

We know that

$$e^{jn\omega_0 t} \leftrightarrow 2\pi\delta(\omega - n\omega_0) \tag{iii}$$

The Fourier transform of $f(t)$ in Eq. (i) is, therefore

$$\mathcal{F}[f(t)] = \frac{2\pi Ad}{T}\sum_{n=-\infty}^{\infty}\operatorname{sinc}\left(\frac{n\pi d}{T}\right)\delta(\omega - n\omega_0);\ \omega_0 = 2\pi/T \tag{iv}$$

For the given values of A, d and T,

$$F_n = \frac{1}{4}\operatorname{sinc}\left(\frac{n\pi}{4}\right) \tag{v}$$

The spectrum has its first zero crossing (for form of sinc function see Fig. 2.16a) for $n = 4$ between the two first zero crossings there are 7 impulses corresponding to $n = -3\pi/4,\ -\pi/2,\ -\pi/4,\ 0,\ \pi/4,\ \pi/2,\ 3\pi/4$. The power within these two zero crossing is given by

$$P_{n=4} = |F_0|^2 + 2\,(|F_1|^2 + |F_2|^2 + |F_3|^2)$$

$$= \left(\frac{1}{4}\right)^2 + 2\left\{\operatorname{sinc}^2\left(\frac{\pi}{4}\right) + \operatorname{sinc}^2\left(\frac{\pi}{2}\right) + \operatorname{sinc}^2\,(3\pi/4)\right\}$$

$$= 0.226 \text{ units (normalized)}$$

Total power in $f(t)$ is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} A^2\,\mathrm{d}t = \frac{1}{T}\int_{-d/2}^{d/2} A^2\,\mathrm{d}t$$

$$= \frac{A^2 d}{T} = 4 \times \frac{1}{16} = 0.25 \text{ units}$$

Percentage of power up to first zero crossing of the spectrum is 90.4.

Observation

The first zero crossing (f_{zc}) is given by

$$\frac{n\pi d}{T} = \pi$$

or

$$n = \frac{T}{d} \tag{vi}$$

Harmonic separation between impulses is

$$\omega_0 = \frac{2\pi}{T} \tag{vii}$$

So the first zero crossing occurs at frequency

$$\omega(f_{zc}) = n\omega_0 = \frac{T}{d}\frac{2\pi}{T} = \frac{2\pi}{d} \tag{viii}$$

As T is increased with fixed T/d, the impulses get closer but number of impulses to f_{zc} remains the same.

For the given values

$$n = \frac{T}{d} = 4$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

$$\omega(f_{zc}) = n\omega_0 = 32\pi$$

2.9 IDEAL LOW-PASS FILTER

An ideal low-pass filter, shown in Fig. 2.34 is defined as

$$H_{\ell p}(j\omega) = \begin{cases} 1; & |\omega| \le \omega_c \\ 0; & |\omega| > \omega_c \end{cases} \tag{2.125}$$

The impulse response of the ideal low-pass filter can be written in the following form

$$h_{\ell p}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\ell p}(j\omega)e^{j\omega t}\mathrm{d}\omega$$

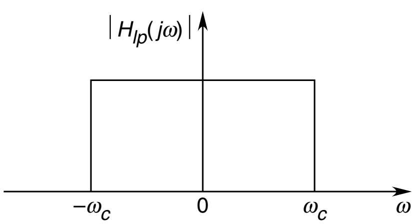


Fig. 2.34 *Characteristics of low-pass filter*

or

$$
\begin{aligned}
h_{\ell p}(t) &= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\omega_c}^{\omega_c} \\
&= \frac{1}{2\pi jt}\left[e^{j\omega_c t} - e^{j\omega_c t}\right] \\
&= \frac{2j\sin\omega_c t}{2\pi jt} = \frac{\omega_c}{\pi}\,\frac{\sin\omega_c t}{\omega_c t}
\end{aligned}
$$

or

$$
h_{\ell p}(t) = \frac{\omega_c}{\pi}\,\frac{\sin\omega_c t}{\omega_c t} = \frac{\omega_c}{\pi}\,\text{sinc}\,(\omega_c t) \tag{2.126}
$$

It is clearly seen from Eq. (2.126) that $h_{\ell p}(t)$ has non-zero values for $t < 0$, as shown in Fig. 2.16(a). Thus $h_{\ell p}$(t) can be stated as non-causal and hence is physically not realizable. In fact, a realizable form of low-pass filter is the one which introduces time delay so that $h_{\ell p}(t)$ does not have values for $t < 0$.

2.10 FREQUENCY-DOMAIN ANALYSIS OF SYSTEMS

Let us consider a linear time-invariant (LTI) system with its impulse response $h(t)$, as shown in Fig. 2.35. If we apply an arbitrary input $r(t)$ to this LTI system then the response of the system is obtained as

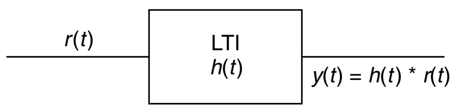


Fig. 2.35 *Linear time-invariant system*

$$
y(t) = h(t) * r(t) \tag{2.127}
$$

provided, the initial state of the system is zero.

Fourier transform of Eq. (2.127), according to the time-convolution property, is given as

$$
Y(\omega) = H(\omega)\,R(\omega) \tag{2.128}
$$

where

$$
\mathcal{F}[y(t)] = Y(\omega);\ \mathcal{F}[h(t)] = H(\omega)
$$

and

$$
\mathcal{F}[r(t)] = R(\omega)
$$

The Fourier transform of the impulse response of the system $H(\omega)$ is also interpreted as the *system transfer function*. It is assumed that impulse response of the system does not contain any term that rises to infinity. In other words, the LTI system is assumed to be stable.

How the Fourier-transform can be applied to obtain system response is illustrated Example 2.19.

Example 2.19 *Find the response e(t) of the electric circuit shown in Fig. 2.36, for following the input*

$$i(t) = e^{-t}u(t)$$

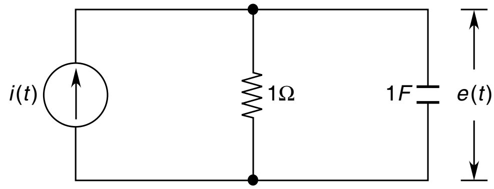


Fig. 2.36 *Parallel RC circuit*

Solution The differential equation describing the circuit of Fig. 2.36 is given as

$$i(t) = e + \frac{de}{dt} \tag{i}$$

Taking the Fourier transform of Eq. (i), we get

$$\begin{aligned} I(\omega) &= E(\omega) + j\omega E(\omega) \\ &= (1 + j\omega)\, E(j\omega) \end{aligned} \tag{ii}$$

The system function $H(\omega)$, therefore, is

$$H(\omega) = \frac{1}{1 + j\omega} \tag{iii}$$

The response $E(\omega)$ can, therefore be obtained as follows.

$$E(\omega) = H(\omega)\, I(\omega) \tag{iv}$$

We have the input

$$i(t) = e^{-t}\, u(t)$$

whose Fourier transform is

$$\begin{aligned} I(\omega) &= \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt \\ &= \left[\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right]_0^{\infty} \end{aligned}$$

or

$$I(\omega) = \frac{1}{1 + j\omega} \tag{v}$$

Substituting Eq. (v) in (iv) we get

$$E(\omega) = \frac{1}{(1 + j\omega)} \times \frac{1}{(1 + j\omega)} = \frac{1}{(1 + j\omega)^2} \tag{vi}$$

Taking the inverse Fourier transform of Eq. (vi) yields the following result.

$$e(t) = t\,e^{-t}\,u(t) \tag{vii}$$

RC Filter

A simple circuit of Fig. 2.37 acts as a filter (nonideal).

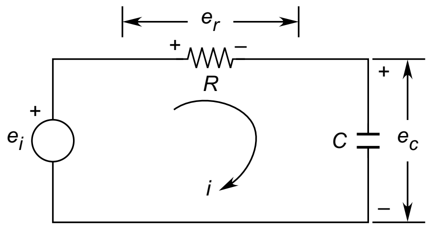


Fig. 2.37

Low-pass Filter

The output is taken across the capacitor, i.e., e_c.

$$i = C\frac{\mathrm{d}e_c}{\mathrm{d}t}$$

$$Ri + e_c = e_i$$

$$RC\frac{\mathrm{d}e_c}{\mathrm{d}t} + e_c = e_i\text{, filter differential equation} \tag{i}$$

Consider a simple input $e_i = e^{j\omega t}$. The output is then given as

$$E_c(j\omega) = H(j\omega)\,e^{j\omega t} \tag{ii}$$

where $\quad H(j\omega)$ = transfer function

Substituting $E_c(j\omega)$ for e_c in Eq. (i)

$$RC\frac{d}{\mathrm{d}t}[H(j\omega)\,e^{j\omega t}] + H(j\omega)\,e^{j\omega t} = e^{j\omega t}$$

$$j\omega RC\,H(j\omega)\,e^{j\omega t} + H(j\omega)\,e^{j\omega t} = e^{j\omega t}$$

$e^{j\omega t}$ cancels out and we get

$$H(j\omega) = \frac{1}{1 + j\omega RC} \tag{iii}$$

$$= A(\omega)\,\angle\phi(\omega) \tag{iv}$$

where

$$A(\omega) = \frac{1}{\sqrt{1+(\omega RC)^2}};\ \phi(\omega)\,\tan^{-1}\,\omega CR$$

with $\quad -\infty \le \omega \le \infty$

The *frequency response* plots of $A(\omega)$ and $\phi(\omega)$ are drawn in Figs 2.37(a) and (b)

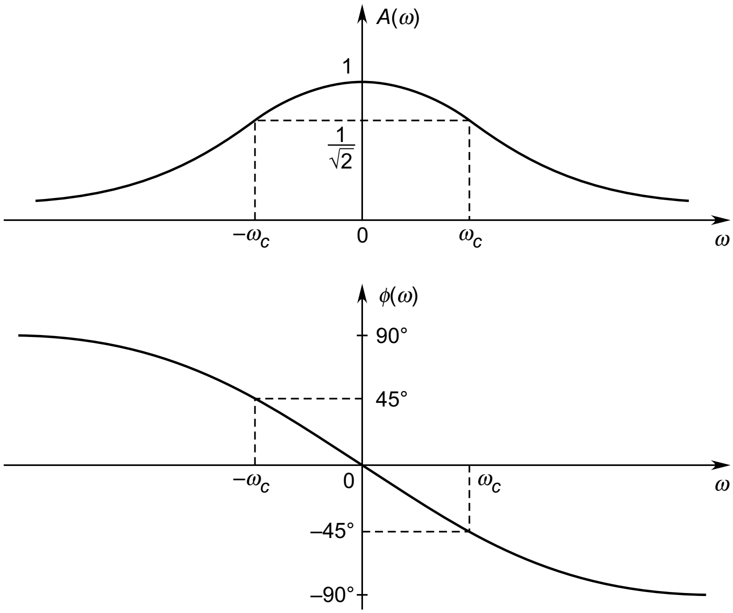


Fig. 2.38 *Frequency response-low-pass filter.*

Observation

At $\qquad \omega = 0 \qquad A(0) = 1, \qquad \phi(0) = 0$

At $\qquad \omega = \dfrac{1}{RC} = \omega_c \qquad A(\omega_c) = \dfrac{1}{\sqrt{2}}, \qquad \phi(\omega_c) = \angle -45°$

$$\omega >> \omega_c \qquad A \to 0 \qquad \phi \to -90°$$

These low-pass characteristics; $0 - \omega_c$ are passed and $\omega > \omega_c$ are cut-off. The frequency $\omega_c = 1/RC$ is called the *break* or *corner frequency*. Also, it is to be noted that $RC = \tau$, the network time constant.

Notice the cut-off is not sharp like an ideal filter in Fig. 2.34 which is not realizable.

High-pass Filter

In the circuit of Fig. 2.37, the output e_r is taken across the resistance.

$$i = \frac{e_r}{R}$$

$$e_r + \frac{1}{c}\int \frac{e_r}{R}\,\mathrm{d}t = e_i$$

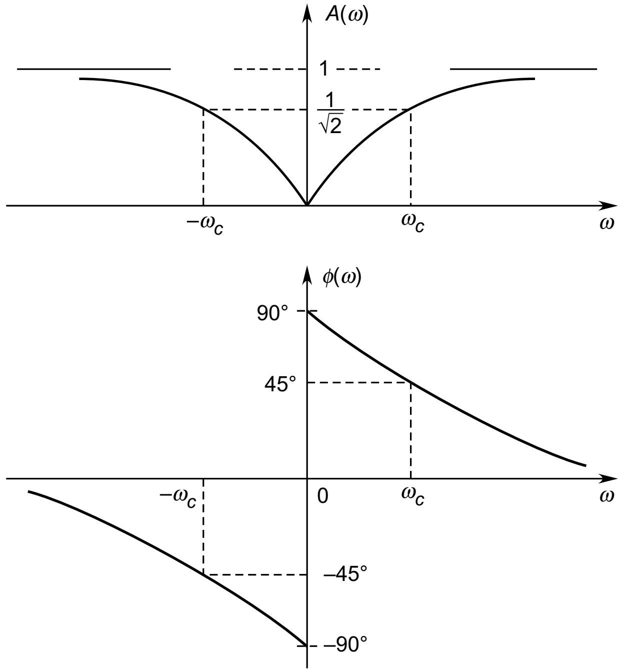


Fig. 2.39 *Frequency response*

Taking derivative

$$\frac{de_r}{dt} + \frac{1}{CR}\, e_r = \frac{de_i}{dt}\text{; differential equation of filter} \qquad \text{(i)}$$

Let $e_i = e^{j\omega t}$, then $E_r(j\omega) = H(j\omega)\, e^{j\omega t}$. Substituting in Eq. (i)

$$j\omega\, H(j\omega)\, e^{j\omega t} + \frac{1}{CR}\, H(j\omega)\, e^{j\omega t} = j\omega\, e^{j\omega t}$$

which yields

$$H(j\omega) = \frac{j\omega CR}{1 + j\omega CR} \qquad \text{(ii)}$$

$$A(\omega) = \frac{\omega CR}{\sqrt{1 + (\omega CR)^2}}$$

$$\phi(\omega) = \pm 90 \mp \tan^{-1} \omega RC\text{; for } \omega > 0 \text{ and } \omega < 0.$$

Corner frequency, $\omega_c = 1/RC$. $A(\omega)$ and $\phi(\omega)$ are plotted in Fig. 2.39.

Observation

$$\omega < \omega_c,\ A < \frac{1}{\sqrt{2}};\ \phi > 45^\circ$$

$$\omega = \omega_c,\ A = \frac{1}{\sqrt{2}};\ \phi = 45^\circ$$

$$\omega >> \omega_c;\ A \to 1;\ \phi \to 0^\circ$$

High-pass characteristic; no sharp cut-in like in ideal high-pass shown in Fig. 2.40.

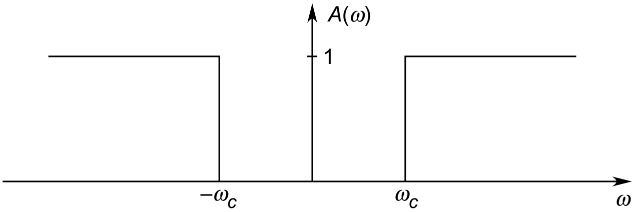


Fig. 2.40 *Ideal high-pass characteristic*

Note: Filters can be devised which are much closer to the ideal filter compared to simple RC filter discussed above.

2.11 FOURIER ANALYSIS OF SAMPLED SIGNALS

One of the important roles of Fourier analysis concerns sampling of signals. When a continuous signal is sampled at uniformly or randomly-spaced intervals, then discrete pieces of such signal are obtained. Signal obtained in the sampled form, offer great deal of advantages in its processing and transmission. It is obvious that more the number of samples, more accurately the signal can be reconstructed by interpolation. It will soon be clear as to what should be the appropriate sampling rate so as to reconstruct the exact signal, provided signal is band-limited. For the purpose of analysis we will assume here an ideal sampler, as shown in Fig. 2.41.

Figure 2.41 shows an arbitrary input $f(t)$, followed by an ideal sampler $\delta_T(t)$ that generates impulse train of strength unity and finally the sampled signal $f_s(t)$ which contains sequence of impulses of strength equal to the signal value at the instant of sampling.

Mathematically, the sampling process can be stated to be equivalent to the multiplication of signal $f(t)$ by impulse train $\delta_T(t)$ of strength unity with sampling

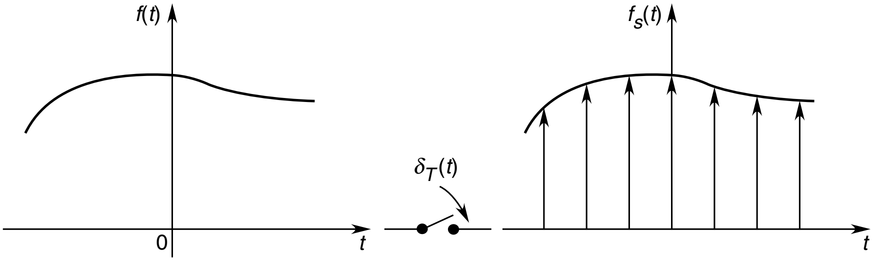


Fig. 2.41 *Ideal impulse sampling*

period T, resulting in following form.

$$f_s(t) = f(t)\ \delta_T(t) \tag{2.129}$$

where, the impulse train $\delta_T(t)$ is defined as

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.130}$$

We, therefore, have

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) \tag{2.131a}$$

The frequency spectrum of $f_s(t)$ is obtained using the frequency convolution property of the Fourier transform. We, thus, have

$$F_s(\omega) = \frac{1}{2\pi}\left[F(\omega) * \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)\right];\ \omega_0 = \frac{2\pi}{T} \tag{2.131b}$$

From this expression it can be easily seen that the spectrum of sampled signal $F_s(\omega)$ is an **aliased** version of the spectrum of the original signal $F(\omega)$, i.e. it is the sum of $F(\omega)$ repeating itself periodically with period ω_0 and scaled by a factor $\frac{1}{T}$. The mathematical process of frequency convolution is shown in Fig. 2.42 (a)–(f) for a band-limited signal $f(t)$ which has no frequency components higher than ω_m.

In order to ensure that the shape of the original signal spectrum $F(\omega)$ be preserved in $F_s(\omega)$, the spectrum of the sampled signal, it is necessary that $F(\omega)$ repeats with period ω_0 without any overlap between adjoining spectra. This requirement imposes following condition, i.e.,

$$\omega_0 \geq 2\ \omega_m \tag{2.132}$$

i.e.

$$\frac{2\pi}{T} \geq 2(2\pi f_m)$$

or

$$T \leq \frac{1}{2f_m} \tag{2.133}$$

Equation (2.133) is the mathematical statement of a well-known theorem, called **Shanon's theorem**. This theorem is described in following words.

A band-limited signal with no frequency component above f_m Hz is uniquely preserved in its sampled version provided the sampling is carried out at uniform intervals of at the most $\frac{1}{2f_m}$ seconds apart.

If the signal $f(t)$ is sampled with interval $T > \frac{1}{2f_m}$ then adjoining spectra of $F_s(\omega)$ overlap and then it is no longer possible to reconstruct $F(\omega)$ from $F_s(\omega)$ accurately. This is illustrated in Fig. 2.43.

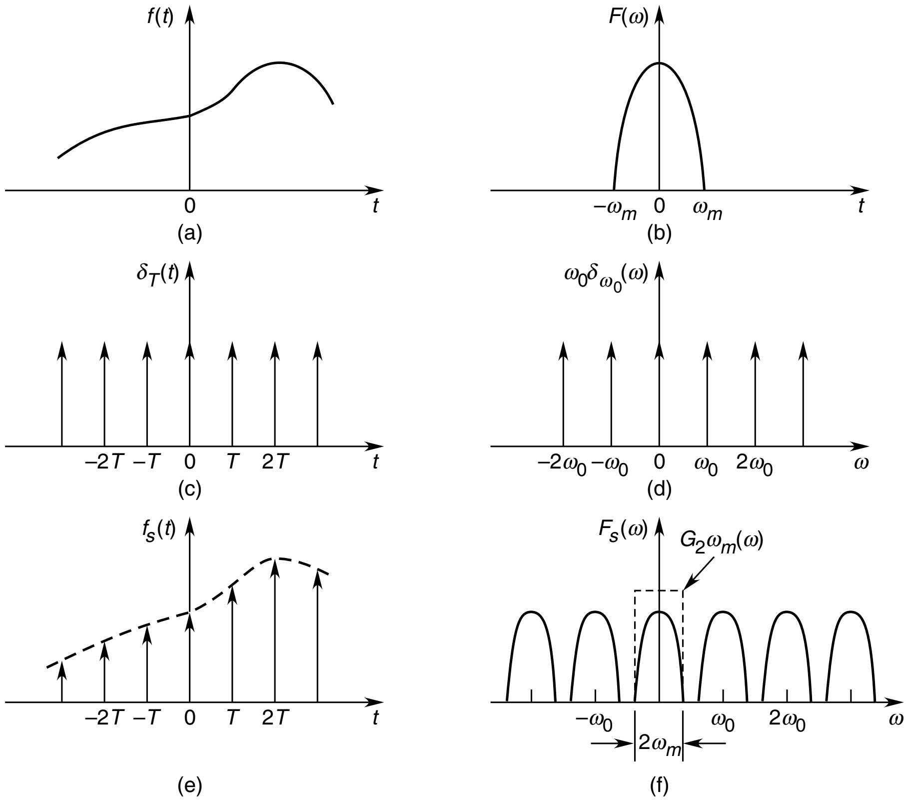


Fig. 2.42 *Illustration of frequency convolution*

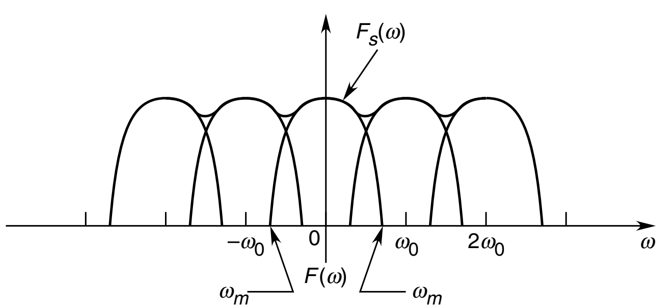


Fig. 2.43 $T > \dfrac{1}{2f_m}$, *Illustration of aliasing effect*

Recovering $f(t)$ from $f_s(t)$

The original signal $f(t)$ can easily be recovered from $f_s(t)$ by passing $f_s(t)$ through a low-pass filter with cut-off frequency ω_m, as shown in Fig. 2.44.

We assume that the sampling rate satisfies the following critical condition.

$$T = \frac{1}{2f_m} \quad \Rightarrow \quad \frac{2\pi}{\omega_0} = \frac{1}{2f_m}$$

i.e.,

$$\omega_0 = 4\pi f_m = 2\omega_m$$

In such a situation, repeated spectra will just touch each other.

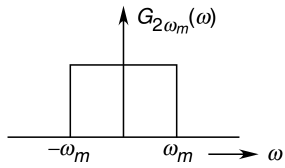


Fig. 2.44 *Low-pass filter*

From Eq. (2.131b), we have

$$F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$$

For $\omega_0 = 2\omega_m$, it can written as

$$F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - 2n\omega_m) \tag{2.134a}$$

The desired characteristics of low-pass filter for reconstructing the original signal is

$$G_{2\omega_m}(\omega) = \begin{cases} 0; |\omega| > \omega_m \\ 1; |\omega| < \omega_m \end{cases}$$

The spectrum of the reconstructed signal is given by

$$\frac{1}{T}F(\omega) = F_s(\omega) = F_s(\omega)G_{2\omega_m}(\omega) \tag{2.134b}$$

Taking the inverse Fourier transform of Eq. (2.134b), we get

$$\begin{aligned} f(t) &= Tf_s(t)\frac{\omega_m}{\pi}\ \text{sinc}\ (\omega_m t) \\ &= \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)\ \text{sinc}\ (\omega_m t) \\ &= \sum_{n=-\infty}^{\infty} f(nT)\ \text{sinc}\ [\omega_m(t - nT)] \end{aligned} \tag{2.135}$$

Equation (2.135) represents the original signal in terms of shifted sinc function. It is a kind of Fourier series in sinc function rather than in sinusoidal functions. Reconstruction of the signal $f(t)$ is shown in Fig. 2.45. It is also stated before that exact reconstruction of the signal from its sampled version is possible only if

(i) the signal is band-limited; and
(ii) the sampling interval is $T < 1/2f_m$.

It should, however, be noted that a finite duration signal is not band-limited and vice-versa. A finite duration band-limited signal is obtained by passing the

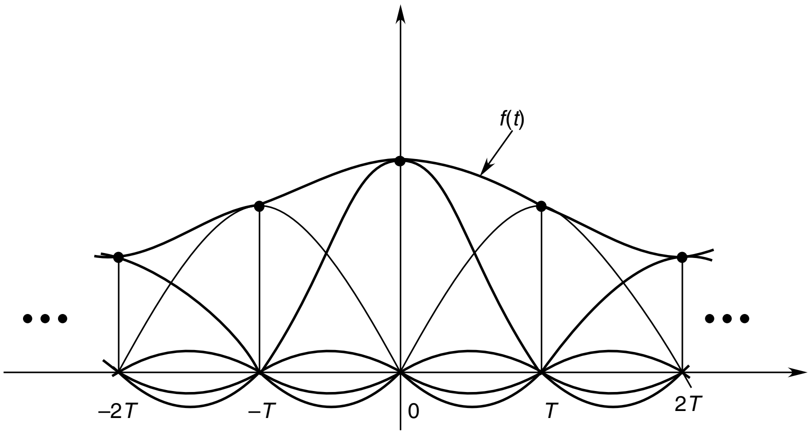


Fig. 2.45 *Reconstruction of the signal f(t)*

infinite duration band-limited signal through gate-function $G_T(t)$, also called window function, i.e.,

$$f_1(t) = G_T(t)\,f(t) \tag{2.136a}$$

Taking the Fourier transform, we get

$$F_1(\omega) = \frac{T}{2\pi}\ \text{sinc}\left(\frac{T\omega}{2}\right) F(\omega) \tag{2.136b}$$

Though $F(\omega)$ is band-limited, $F_1(\omega)$ is not because of convolution with sinc function which extends to infinite frequencies. The exact signal reconstruction after sampling is therefore, not possible for $f_1(t)$. The reconstruction error can, however, be considerably reduced in practice by choosing ω_m in a way that signal energy contained in frequencies above ω_m is negligible. The corresponding sampling interval must of course be $T < 1/2\,f_m$.

Hilbert Transform

While introducing Hilbert transform we shall recall certain results.

From Example 2.16.

$$sgn\,(t) \leftrightarrow sgn\,(\omega) = \frac{2}{j\omega}$$

By symmetry (duality) property

$$\frac{2}{jt} \leftrightarrow 2\pi\, sgn\,(-\omega)$$

or

$$\frac{1}{\pi t} \leftrightarrow j\, sgn\,(-\omega) = -j\, sgn\,(\omega)$$

as $sgn\,(\omega)$ is an odd function.

The Hilbert transform of $f(t)$ is defined as

$$\hat{f}(t) = \frac{1}{\pi f} * f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\lambda)}{t-\lambda}\, d\lambda \tag{2.137a}$$

Taking Fourier transform

$$\mathcal{F}[\hat{f}(t)] = \mathcal{F}\left(\frac{1}{\pi t}\right) F(\omega)$$

$$= -j\, sgn(\omega)\, F(\omega) \tag{2.137b}$$

This transform preserves the magnitude of $F(\omega)$ but changes its phase by $\frac{\pi}{2}$;

$$-\frac{\pi}{2} \text{ for } \omega < 0,\ \frac{\pi}{2} \text{ for } \omega < 0$$

Application

Analytical Signal Let $z(t)$ be a complex-valued analytical signal such that

$$z(t) = x(t) + jy(t)$$

The signal is such that its Fourier transform

$$\therefore \qquad Z(\omega) = X(\omega) + jY(\omega)$$

is single (one) sided transform, i.e., $z(\omega) = 0$ for $\omega < 0$

Therefore, $X(\omega)$ and $Y(\omega)$ cannot be specified independently. It requires that

$$Y(\omega) = j\, X(\omega) \text{ for } Z(\omega) = 0 \text{ for } \omega < 0$$

For $\qquad Y(\omega) = -jX(\omega)$

$$Z(\omega) = 2\,X(\omega),\ Z(\omega) \text{ doubles for } \omega > 0$$

We can then express

$$X(\omega) = -j \text{ sgn } \omega\, X(\omega) \text{ for all } \omega \tag{2.138}$$

which gives

$$Z(\omega) = \begin{cases} 2\,X(\omega) & ;\ \omega > 0 \\ 0 & ;\ \omega < 0 \end{cases}$$

In time domain; inverse Fourier transform of Eq. (2.138)

$$y(t) = \hat{x}(t);\ [Z(\omega) = 0 \text{ for } \omega < 0];\ \text{Case 1}$$

and $\qquad y(t) = -\hat{x}(t);\ [Z(\omega) = 0 \text{ for } \omega < 0];\ \text{Case 2}$

The sign of j should reverse in Case 2.

Causal System If the system is also LTI, its impulse response $h(t)$ characterises the system and $h(t) = 0$ for $t = 0$. For such a system, the real and imaginary parts of $H(\omega)$ cannot be specified independently but are the Hilbert transform of each other.

Proof

Let us divide $h(t)$ into even and odd parts.

$$h_e(t) = \frac{1}{2}[h(t) + h(-t)] \tag{i}$$

$$h_0(t) = \frac{1}{2}[h(t) - h(-t)] \tag{ii}$$

$$h(t) = h_e(t) + h_0(t) \tag{iii}$$

In order for $h(t)$ to be zero for $t < 0$

$$h_0(t) = \begin{cases} \frac{1}{2} h_e(t) & ; t > 0 \\ -\frac{1}{2} h_e(t) & ; t < 0 \end{cases} \tag{iv}$$

$$= sgn\ (t)\ h_e(t) \tag{2.139a}$$

Substituting $h_0(t)$ in Eq. (iii) results in

$$h(t) = h_e(t) + sgn\ (t)\ h_e(t) \tag{2.139b}$$

Fourier transforming,

$$H(\omega) = H_e(\omega) + \frac{2}{j\omega} * H_e(\omega)$$

$$= H_e(\omega) - j\,2\pi\left[\frac{1}{\pi\omega} * H_e(\omega)\right]$$

Thus
$$H(\omega) = H_e(\omega) - j\,2\pi\ \hat{H}_e(\omega) \tag{2.139c}$$

where $\hat{H}_e(\omega)$ is this Hilbert trnasform of $H_e(\omega)$.

2.12 THE LAPLACE TRANSFORM

In this chapter we have presented several methods of signal representation and system response determination. These are:

- The signal is represented by short duration pulses which in the limit are impulses. The system output is then the sum of impulse responses of the system. In continuous time, the output is obtained by the linear convolution of the signal input and system's impulse response. The process, though elegant, is computationally cumbersome.
- The classical method of solving differential equations governing LTI systems is presented through a simple example. It is not pursued further as more powerful methods of solving in terms of transforms are now available.
- Fourier transform is a powerful method which converts the signal into the form of continuum of frequencies and the system's impulse response to the Fourier transform function. The convolution modifies to product of Fourier transform of input and the Fourier transform function. The time-domain form of the output can be obtained by table lookup.

The limitation of the Fourier transform is that it does not converge for certain signals met in practice. However, it is the most powerful tool for investigating the system's spectral response to audio, video and other similar signals ($-\infty < t\ \infty$) and therefore highly useful in filter design.

The convergence limitation of the Fourier transform is overcome in the Laplace transform by introducing the convergent real exponential e^{-st} in the transform integral. Thus, $e^{-j\omega t}$ in Fourier transform becomes e^{-st} where $s = -\sigma + j\omega$ while the lower limit of the Laplace integral is $-\infty$, for causal signals we can use one sided Laplace transform with lower integral limit as zero.

The one-sided (unilateral) Laplace transform, transforms a signal from time domain to s-domain. It is a powerful tool in arriving at the closed form solution of linear integro-differential equation of continuous-time systems. It will soon be obvious that time domain and frequency domain (s-domain) are dual of each other and offer the same insight into the system behaviour.

2.13 ONE-SIDED UNILATERAL LAPLACE TRANSFORM

The one-sided Laplace transform of causal signals ($f(t) = 0$ for $t < 0$) is defined as

$$F(s) = \pounds[f(t)]$$

$$= \int_{0^-}^{\infty} f(t)e^{-st}\mathrm{d}t \tag{2.140}$$

where

$s = \sigma + j\omega$ (complex frequency variable)
$\pounds$ = Laplace transform notation

The lower limit of integral is 0^- instead of 0 to avoid any confusion in case $f(t)$ is an impulse at the origin. In that case 0^- ensures the inclusion of $\delta(t)$ in the integration. However, for simplificity, now we write 0 instead of 0^-, but, 0^- is used wherever it is required.

It is evident from the definition that one-sided Laplace transform is applicable for causal signals only. For noncausal signals, the bilateral Laplace transform of Section 2.17 can be used.

The **inverse Laplace transform** of $f(t)$ is $F(s)$ and is given by the following complex domain integral:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{-st}\mathrm{d}s \tag{2.141}$$

Symbolically, the transform pair may be written as

$$F(s) = \pounds[f(t)] \tag{2.142a}$$
$$f(t) = \pounds[F(s)] \tag{2.142b}$$

Also as

$$f(t) \leftrightarrow F(s) \tag{2.143}$$

The Laplace transform has the property of uniqueness and so the inverse transformation need not be carried out by the complex-domain integration of Eq. (2.141). A lookup table of the Laplace transformation can be built up for a set of simple functions and then the inverse transform of $F(s)$ can be found by breaking it up into simple factors by partial fraction method and looking up the transform table.

For existence of the Laplace transform of $f(t)$, the integral of Eq. (2.140) must converge, for which the necessary condition is

$$\int_0^\infty |f(t)|e^{-\sigma t}\mathrm{d}t < \infty \tag{2.144}$$

where

$$\sigma = \mathrm{Re}\,(s) > 0.$$

Physical limitations virtually ensure that all signals in physical systems to be Laplace transformable and so the convegnerce of integral of Eq. (2.144) is always guaranteed. Proof. of Eq. (2.144) is given below.

Proof

As per Eq. (2.140)

$$F(s) = \int_0^\infty f(t)e^{-\mathrm{st}}\mathrm{d}t$$

Let the function of interest be

$$f(t) < \mathrm{R\,e}^{\alpha t},\ t > 0$$

where R = real positive number.

Therefore

$$|F(s)| < \int_0^\infty \mathrm{R}e^{\alpha t}e^{-\mathrm{st}}\mathrm{d}t$$

$$< \frac{\mathrm{R}}{(\alpha - s)}e^{(\alpha - s)}\Big|_0^\infty$$

The above integral will converge only if

$$\mathrm{Re}(s) = \sigma > \alpha$$

Example 2.20 *Find the Laplace transform of*

(i) $\delta(t)$ *(ii)* $u(t)$ *(iii)* $e^{-\alpha t}u(t)$

Solution

(i) As per definition,

$$\pounds\,[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\mathrm{d}t$$
$$= \int_{0^-}^{\infty} \delta(t)\left(e^{-st}\Big|_{t=0}\right)\mathrm{d}t$$
$$= \int_{0^-}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

or

$$\delta(t) \leftrightarrow 1 \tag{2.145}$$

(ii)

$$\pounds\,[u(t)] = \int_{0}^{\infty} u(t)e^{-st}\,\mathrm{d}t$$
$$= \int_{0}^{\infty} e^{-st}\,\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_{0}^{\infty}$$
$$= 1/s\ ;\ \mathrm{Re}\ s > 0 \tag{2.146}$$
$$u(t) \leftrightarrow 1/s$$

(iii)

$$\pounds\,[e^{-\alpha t}u(t)] = \int_{0}^{\infty} e^{-\alpha t}e^{-st}\,\mathrm{d}t$$
$$= \int_{0}^{\infty} e^{-(\alpha+s)}\mathrm{d}t = \frac{1}{s+\alpha} \tag{2.147}$$
$$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{s+\alpha}$$

It is easily observed from Eq (2.144) and (2.147) that the region of convergence of the integral is

$$\mathrm{Re}(s+\alpha) > 0$$

or

$$(\sigma+\alpha) > 0$$

or

$$\sigma > -\alpha \tag{2.148}$$

The region of convergence is shown in Fig. 2.46.

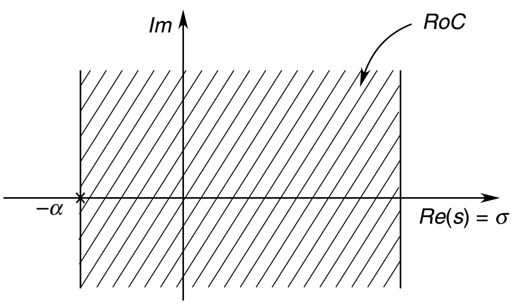


Fig. 2.46 *Region of convergence (RoC) in s-plane*

2.14 TABLE OF LAPLACE TRANSFORM

Some of the transform pairs have already been obtained in the examples earlier. In Table 2.4 some commonly used transform pairs are presented.

Table 2.4 *Table of Laplace Transform pairs*

	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
6.	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
7.	$t^n e^{-\alpha t}u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$
8a.	$\cos \omega_0 t\, u(t)$	$\frac{s}{s^2+\omega_0^2}$
8b.	$\sin \omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
9a.	$e^{-\alpha t}\cos \omega_0 t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$
9b.	$e^{-\alpha t}\sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$

2.15 PROPERTIES OF LAPLACE TRANSFORM

Properties of the Laplace transform are helpful in obtaining Laplace transform of composite functions and in the solution of linear integro-differential equations. Some properties are proved below and other useful properties are presented in Table 2.5.

Table 2.5 *Properties of Laplace Transform*

Operation	*f(t)*	*F(s)*
Addition	$f_1(t) + f_2(t)$	$F_2(s) + F_2(s)$
Scalar multiplication	$\alpha f(t)$	$\alpha F(s)$
Time differentiation	$\dfrac{\mathrm{d}f}{\mathrm{d}t}$	$sF(s) - f(0^-)$
	$\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2}$	$s^2F(s) - sf(0^-) - f^1(0^-)$
	$\dfrac{\mathrm{d}^n f}{\mathrm{d}t^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f^1(0^-)\ldots.$ $- sf^{(n-2)}(0^-) - f^{(n-1)}(0^-)$
Time integration	$\int_{0^-}^{t} f(t)\,\mathrm{d}t$	$\dfrac{1}{s}F(s)$
	$\int_{-\infty}^{t} f(t)\,\mathrm{d}t$	$\dfrac{1}{s}F(s) + \dfrac{1}{s}\int_{-\infty}^{0^-} f(t)\,\mathrm{d}t$
Time shift	$f(t - t_0)u(t - t_0)$	$F(s)e^{-st_0};\ t_0 \geq 0$
Frequency shift	$f(t)e^{\alpha t}$	$F(s - \alpha)$
Frequency differentiation	$- tf(t)$	$\dfrac{\mathrm{d}F(s)}{\mathrm{d}s}$
Frequency integration	$\dfrac{f(t)}{t}$	$\int_{s}^{\infty} F(s)\,\mathrm{d}s$
Scaling	$f(\alpha t),\ \alpha \geq 0$	$\dfrac{1}{\alpha}F\left(\dfrac{s}{\alpha}\right)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Frequency convolution	$f_1(t)f_2(t)$	$F_1(s) * F_2(s)$
Initial value	$f(0^-)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$

Linearity

If

$$f_1(t) \leftrightarrow F_1(s)$$

and

$$f_2(t) \leftrightarrow F_2(s)$$

Then

$$a\,f_1(t) + b\,f_2(t) \leftrightarrow a\,F_1(s) + bF_2(s) \tag{2.149}$$

where a and b are constants.

Proof

$$£[(af_1(t) + bf_2(t)] = a\int_0^{\infty} f_1(t)e^{-\mathrm{st}}\mathrm{d}t + b\int_0^{\infty} f_2(t)e^{-\mathrm{st}}\mathrm{d}t$$

$$= a\,F_1(s) + b\,F_2(s)$$

Frequency Shift

If

$$f(t) \leftrightarrow F(s)$$

then

$$f(t)\,e^{\alpha t} \leftrightarrow F(s - \alpha) \tag{2.150}$$

Proof

$$£[f(t)e^{\alpha t}] = \int_0^{\infty} f(t)e^{\alpha t}e^{-\mathrm{st}}\mathrm{d}t$$

$$= \int_0^{\infty} f(t)e^{-(s-a)t}\mathrm{d}t$$

$$= F(s - \alpha)$$

Time Shift

If

$$f(t) \leftrightarrow F(s)$$

then

$$f(t - t_0)u(t - t_0) \leftrightarrow F(s)\,e^{-\mathrm{st}_0} \tag{2.151}$$

Proof

$$£[f(t - t_0)u(t - t_0)] = \int_0^{\infty} f(t - t_0)u(t - t_0)e^{-\mathrm{st}}\mathrm{d}t$$

$$= \int_{-t_0}^{\infty} f(\lambda)u(\lambda)e^{-s(\lambda + t_0)}\mathrm{d}\lambda \tag{i}$$

where

$$(t - t_0) = \lambda$$

As $f(\lambda)$ is causal, the lower limit in integral of Eq. (i) can be changed to 0. Thus

$$£[f(t - t_0)u(t - t_0)] = e^{-st_0}\int_{t_0}^{\infty} f(\lambda)e^{-s\lambda}\mathrm{d}\lambda$$

$$e^{-\mathrm{st}_0}\;F(s) \tag{ii}$$

Time Differentiation

$$£\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] \leftrightarrow sF(s) - f(0^-) \tag{2.151a}$$

and, in general

$$£\left[\frac{d^n f(t)}{dt}\right] \leftrightarrow s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f^1(0^-) \ldots - f^{(n-1)}(0^-) \tag{2.152}$$

Proof

$$£\left[\frac{df(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-s\lambda} dt$$

Integrating by parts, we get

$$£\left[\frac{df(t)}{dt}\right] = f(t)e^{-st} \Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(s)e^{-st} dt \tag{i}$$

Existence of $F(s)$ gurantees $f(t)\, e^{-st}|_{t=\infty} = 0$

Hence Eq. (i) becomes

$$£\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-) \tag{ii}$$

Time Integration

$$£\left[\int_0^t f(\tau)d\tau\right] \leftrightarrow \frac{F(s)}{s} \tag{2.153}$$

and

$$£\left[\int_{-\infty}^t f(\tau)d\tau\right] \leftrightarrow \frac{F(s)}{s} + \frac{\int_{-\infty}^0 f(\tau)d\tau}{s} \tag{2.154}$$

Proof

Consider first the case of Eq. (2.153). By definition

$$£\left[\int_0^t f(\tau)d\tau\right] = \int_0^{\infty}\left[\int_0^t f(\tau)d\tau\right]e^{-st}dt$$

Integrating right-hand side by parts, we get

$$\frac{e^{-st}}{-s}\int_0^t f(\tau)d\tau\Bigg|_0^{\infty} + \frac{1}{s}\int_0^{\infty} f(t)e^{-st}dt \tag{i}$$

Existence of $F(s)$ ensures that

$$f(t)e^{-st}\Big|_{t=\infty} = 0$$

Therefore

$$e^{-st}\int_0^t f(\tau)d\tau\Big|_{t=\infty} = 0 \tag{ii}$$

From Eqs. (i) and (ii), we write

$$£\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s) \tag{iii}$$

Consider now the case of Eq. (2.154). We can write the integral in two parts, i.e.

$$\int_{-\infty}^{t} f(\tau)\mathrm{d}\tau = \left(\int_{-\infty}^{0} f(\tau)\mathrm{d}\tau\right)u(t) + \int_{0}^{t} f(\tau)\mathrm{d}\tau \tag{iv}$$

Laplace transforming yields

$$£\left[\int_{-\infty}^{t} f(\tau)\mathrm{d}\tau\right] = \frac{\int_{-\infty}^{0} f(\tau)\mathrm{d}\tau}{s} + \frac{F(s)}{s} \tag{v}$$

Time Convolution

$$[f_1(t)*f_2(t)] \leftrightarrow F_1(s)\,F_2(s) \tag{2.155}$$

where

$$f_1(t) \leftrightarrow F_1(s)$$
$$f_2(t) \leftrightarrow f_2(s)$$

Proof

$$[f_1(t)*f_2(t)] = \int_0^{t} f_1(\lambda)\, f_2\,(t-\lambda)\mathrm{d}\lambda \tag{i}$$

As $f_1(t)$ and $f_2(t)$ are causal, upper limits in Eq. (i) can be changed from t to ∞. This is because $f_2(t) = 0$, $t < 0$ or $f_2\,(t-\lambda) = 0$, $\lambda > t$. Thus

$$f_1(t)*f_2(t) = \int_0^{\infty} f_1(\lambda)\, f_2\,(t-\lambda)\mathrm{d}\lambda \tag{ii}$$

Taking the Laplace transform

$$£[f_1(t)*f_2(t)] = \int_0^{\infty}\left[\int_0^{\infty} f_1(\lambda)\, f_2\,(t-\lambda)\mathrm{d}\lambda\right]e^{-st}\,\mathrm{d}t \tag{iii}$$

Let $t - \lambda = \eta \rightarrow \mathrm{d}t = \mathrm{d}\eta$. By interchanging order of integrations, we can write Eq. (iii) as

$$£[f_1(t)*f_2(t)] = \int_0^{\infty} f_1(\lambda)\left[\int_{-\lambda}^{\infty} f_2(\eta)e^{-s\eta}\,\mathrm{d}\eta\right]e^{-s\lambda}\mathrm{d}\lambda \tag{iv}$$

It then follows from Eq. (iv) that

$$£[f_1(t)*f_2(t)] = F_1(s)\,F_2(s)$$

Initial-value Theorem

$$f(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\; sF(s) \tag{2.156}$$

Proof

From Eq. (2.151)

$$£\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = sF(s) - f(0^-)$$

or

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \int_{0^-}^{0^+} \frac{df(t)}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st} dt; \quad e^{-st}\Big|_{t=0} = 1$$

$$= f(t)\Big|_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st} dt$$

$$sF(s) = f(0^+) + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st} dt \tag{2.157}$$

Taking limit on both sides of Equation (2.157) for $s \to \infty$, we get

$$\underset{s\to\infty}{\text{Lt}}\, sF(s) = f(0^+)$$

Existence of the Laplace transform of $df(t)/dt$ ensures that

$$\underset{s\to\infty}{\text{Lt}} \int_{0^+}^{\infty} \frac{df(t)}{dt} e^{-st} = 0$$

Final-value Theorem

If

$$f(t) \leftrightarrow F(s)$$

and

$$\frac{df}{dt} \leftrightarrow sF(s) - f(0^-)$$

Then

$$\underset{s\to 0}{\text{Lt}}\, sF(s) = f(\infty) \tag{2.158}$$

Proof

Taking limit of Eq. (2.157) as $s \to \infty$ we write

$$\underset{s\to 0}{\text{Lt}}\,[sF(s)] = f(0^+) + \underset{s\to 0}{\text{Lt}} \int_{0^+}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= f(0^+) + \int_{0^+}^{\infty} \frac{df(t)}{dt} dt$$

$$= f(0^+) + f(t)\Big|_{0^+}^{\infty}$$

$$= f(0^+) + f(\infty) - f(0^+)$$

$$= f(\infty)$$

Note: These and other properties of the Laplace Transform are listed in Table 2.5.

Example 2.21 *Find the Laplace transform of cos ωt and sin ωt.*

Solution Using the frequency shift property

$$e^{-j\omega_0 t} u(t) \leftrightarrow \frac{1}{s + j\omega_0} \tag{i}$$

and

$$e^{+j\omega_0 t}u(t) \leftrightarrow \frac{1}{s - j\omega_0} \qquad \text{(ii)}$$

Adding and substracting Eqs.(i) and (ii), we get

$$\cos \omega_0 t\, u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2} \qquad \text{(iii)}$$

and

$$\sin \omega_0 t\, u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} \qquad \text{(iv)}$$

Example 2.22 *Find the Laplace transform of signal f(t) defined by the following equation.*

$$\frac{d^2 f(t)}{dt^2} = \delta(t) - \delta(t-2) \qquad \text{(i)}$$

Solution Following is the Laplace transform of Eq. (i)

$$£\left[\frac{d^2 f(t)}{dt^2}\right] = £[\delta(t) - \delta(t-2)]$$

$$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt} = 1 - e^{-2s} \qquad \text{(ii)}$$

Letting

$$f(0^-) = \frac{df(0^-)}{dt} = 0$$

Eq. (ii) can then be written as

$$s^2F(s) = 1 - e^{-2s}$$

or

$$F(s) = \frac{1 - e^{-2s}}{s^2} \qquad \text{(iii)}$$

Example 2.23 *Find the Laplace transform of the following signal.*

$$f(t) = \frac{d}{dt}[e^{-at}\, u(t)] \qquad \text{(i)}$$

Solution

$$f(t) = -\alpha e^{-\alpha t} u(t) + e^{-\alpha t}\delta(t) \qquad \text{(ii)}$$

$$£ f(t) = -\alpha \int_{0^-}^{\infty} e^{-\alpha t}\, e^{-st} dt + \int_{0^-}^{\infty} e^{-\alpha t}\, e^{-st}\delta(t)\, dt \qquad \text{(iii)}$$

$$F(s) = \frac{-\alpha}{s+\alpha} + \int_{0^-}^{\infty} \delta(t)\, dt \qquad \text{(iv)}$$

$$= \frac{-\alpha}{s+\alpha} + 1 = \frac{s}{s+\alpha} \qquad \text{(v)}$$

Example 2.24 *Find $f(0^+)$ of the signal whose Laplace transform is*

$$F(s) = \frac{(s+1)}{(s+3)(s+2)}$$

Solution From initial-value theorem

$$f(0^+) = \operatorname*{Lt}_{s-\infty} sF(s) = \operatorname*{Lt}_{s\to\infty} \frac{(s+1)}{(s+3)(s+2)} = 1$$

Example 2.25 *In the circuit shown in Fig. 2.47 find $i(\infty)$ for zero initial condition.*

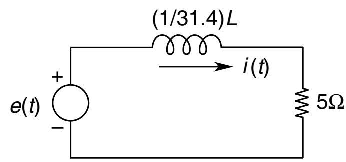


Fig. 2.47

Solution Applying the KVL

$$\frac{1}{31.4}\frac{di(t)}{dt} + 5i(t) = e(t) \tag{i}$$

Taking the Laplace transform of Eq. (i), we have

$$(1/31.4)\, sI(s) + 5\, I(s) = E(s)$$

For unit step input, $E(s) = 1/s$

$$(s + 157)\, I(s) = 31.4/s$$

or

$$I(s) = 31.4/s(s + 157) \tag{ii}$$

Using the final value theorem

$$\begin{aligned} i(\infty) &= \operatorname*{Lt}_{s\to\infty} sI(s) \\ &= \operatorname*{Lt}_{s\to\infty} s\,[31.4/s(s+157)] \\ &= 31.4/157 = 1/5\text{A} \end{aligned}$$

It can be directly seen from the circuit that as $t \to \infty$, $i(t)$ becomes a constant value such that $di(t)/dt = 0$ in Eq.(i), which gives

$$i(\infty) = u(t)/5 = 1/5\text{A}$$

2.16 INVERSE LAPLACE TRANSFORM

As per the definition of inverse Laplace transform,

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}\, ds \tag{2.159}$$

Finding the inverse Laplace transform involves complex integration, which is cumbersome. However, using the uniqueness property of the Laplace transform, inverse transform can be found by looking up Table 2.5. A rational $F(s)$ can be first broken up into simple factors by partial fractioning. This procedure is demonstrated by some examples.

Example 2.26 *Obtain the inverse Laplace transform of*

(a) $F(s) = \dfrac{s^2 + 2s + 1}{s(s+1)(s+2)}$ (b) $F(s) = \dfrac{3s+3}{(s+1)(s^2+4s+5)}$

(c) $F(s) = \dfrac{1}{s^2(s+1)}$

Solution

(a)
$$F(s) = \frac{s^2 + 2s + 1}{s(s+1)(s+2)} \tag{i}$$

Partial fractioning gives

$$F(s) = \frac{0.5}{s} + \frac{1}{s+1} - \frac{0.5}{s+2} \tag{ii}$$

Using Table 2.5, we get

$$f(t) = 0.5u(t) + e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) \tag{iii}$$

(b)
$$F(s) = \frac{2s+3}{(s+1)(s^2+4s+5)} \tag{i}$$

$$= \frac{2s+3}{(s+2+j1)(s+2-j1)(s+1)} \tag{ii}$$

Partial fractioning gives Eq. (iii)

$$F(s) = \frac{1}{4}\left[\frac{-1+j3}{s+2+j1} + \frac{-1-j3}{s+2-j1}\right] + \frac{1}{2}\frac{1}{(s+1)} \tag{iii}$$

Taking the inverse Laplace transform

$$f(t) = \frac{1}{4} \times 2\text{Re}[-1+j3)e^{-2t}e^{-jt}] + \frac{1}{2}e^{-t};\ t \geq 0$$

$$= 2e^{-2t}\,\text{Re}[(-1+j3)\,(\cos t - j\sin t] + \frac{1}{2}e^{-t}$$

$$= \frac{1}{2}[(3\sin t - \cos t)\,e^{-2t} + e^{-t}]\,\alpha(\text{t}) \tag{iv}$$

(c)
$$F(t) = \frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \tag{i}$$

$$f(t) = \pounds^{-1}[F(s)] = (t - 1 + e^{-t})\,u(t) \tag{ii}$$

2.17 TWO-SIDED (BILATERAL) LAPLACE TRANSFORM

Two-sided (bilateral) Laplace transform of signal $f(t)$ is defined in the following form

$$F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,\mathrm{d}t \tag{2.160}$$

where $s = \sigma + j\omega$, which is the complex frequency variable as defined earlier. The **inverse** Laplace transform of $F(s)$ is defined as

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F_b(s)e^{st}\,\mathrm{d}s \tag{2.161}$$

Equations (2.160) and (2.161) form the bilateral Laplace transform pair can handle causal and noncausal signals defined over $-\infty \le t \le \infty$. Symbolically this transform pair is written as follows.

$$F_b(s) = \pounds_b\,[f(t)]$$
$$f(t) = \pounds_b^{-1}\,[F(s)]$$

Also

$$f(t) \leftrightarrow F_b(s)$$

where

$\pounds_b$ = bilateral Laplace transform notation.

Convergence

For the bilateral transform to converge, the following integral should be finite.

$$\int_{-\infty}^{\infty} |\,f(t)\,|e^{-st}\,\mathrm{d}t \tag{2.162}$$

The functions of interest to us are of **exponential order,** which means that they satisfy the following condition.

$$|f(t)\,| < \begin{cases} \mathrm{R}e^{\alpha t}, & t > 0 \\ \mathrm{R}e^{\beta t}, & t < 0 \end{cases} \tag{2.163}$$

where

R = real positive number

α, β = real

Writing the definition of Eq. (2.160) in positive and negative time parts we get,

$$F_b(s) = \int_{-\infty}^{0} f(t)e^{-st}\,\mathrm{d}t + \int_{0}^{\infty} f(t)e^{-st}\,\mathrm{d}t$$

Using the inequality of Eq.(2.163), we can write

$$|F_b(s)| < \left[\int_{-\infty}^{0} \mathrm{R}\,\mathrm{e}^{(\beta - s)t}\,\mathrm{d}t + \int_{0}^{\infty} \mathrm{R}\,\mathrm{e}^{(\alpha - s)t}\,\mathrm{d}t\right]$$
$$< \left[\frac{\mathrm{R}}{(\beta - s)}e^{(\beta - s)t}\,\Big|_{-\infty}^{0} + \frac{\mathrm{R}}{(\alpha - s)}e^{(\alpha - s)t}\,\Big|_{0}^{\infty}\right] \tag{2.164}$$

From Eq.(2.164), it is obvious that for both these integrals to converge in the region the following condition should be fulfilled.

$$\alpha < \mathrm{Re}(s) < \beta \tag{2.165}$$

This is illustrated in Fig. 2.49. It is easy to see that $F_b(s)$ converges only if $\beta > \alpha$, otherwise the convergence strip will not exit.

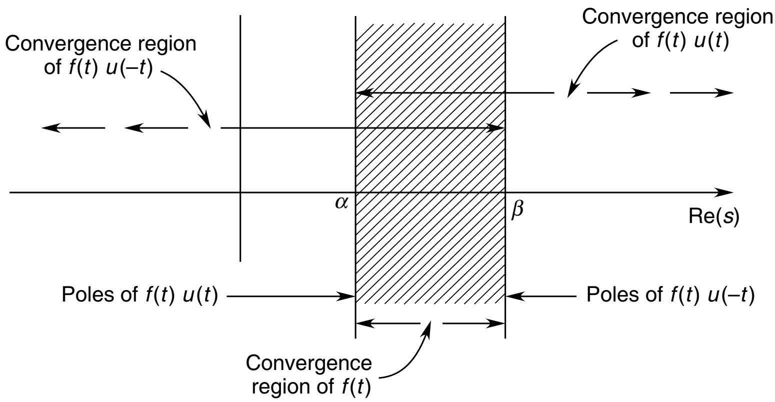


Fig. 2.49 *Region of convergence (RoC)*

Certain important results easily followed from Fig. 2.49.

- F(s) converges in the strip shown in Fig. 2.49 as per Eq. (2.165).
- Poles of $f(t)\ u(-t) \rightarrow$ lie to the right of Re(s) = β-line
- Poles of $f(t)\ u(t) \rightarrow$ lie to the left of Re(s) = α-line
- Information on regions of convergence is needed to identify $f(t)\ u(-t)$ and $f(t)\ u(t)$.

Remark: The bilateral transform is the general form of the Laplace transform as it is applicable for noncausal signals. For causal signal, it reduces to the unilateral transform. In fact, as we shall see soon, the bilateral transform of a noncausal signal is obtained by ingenuous use of unilateral form.

In the bilateral transform definition of Eq. (2.160), if we let $\sigma = 0$, it results in the Fourier transform

$$F(j\omega) = F_e(s)|_{s=j\omega} = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, \mathrm{d}t \tag{2.167}$$

Transform and Its Inverse

Any noncausal function can be split in the following manner.

$$f_1(t) = f_1(t) + f_2(t) \tag{2.168}$$

where

$f_1(t) = f(t)\ u(t)$; positive time part of $f(t)$

$f_2(t) = f(t)\ u(-t)$; negative time part of $f(t)$

This split is illustrated in Fig. 2.50.

Converting Eq. (2.168) to transform form, we get the following equation.

$$F_b(s) = \int_{-\infty}^{0} f_2(t)e^{-st}\,dt + \int_0^{\infty} f_1(t)e^{-st}dt$$

Let $\tau = -t$ in first part. Then

$$F_b(s) = \int_0^{\infty} f_2(-\tau)e^{s\tau}\,d\tau + \int_0^{\infty} f_1(t)e^{-st}dt$$

$$= F_2(-s) + F_1(s) \tag{2.169}$$

where

$$F_1(s) = £[f_1(t)]$$
$$F_2(s) = £[f_2(-t)]$$

Here $f_2(-t)$ is the mirror image of $f_2(t)$ on to positive time and is shown in Fig. 2.50. Equation (2.165) gives us the method of finding bilateral Laplace transform and its inverse, provided the region of convergence is specified. The procedure will be illustrated by two examples.

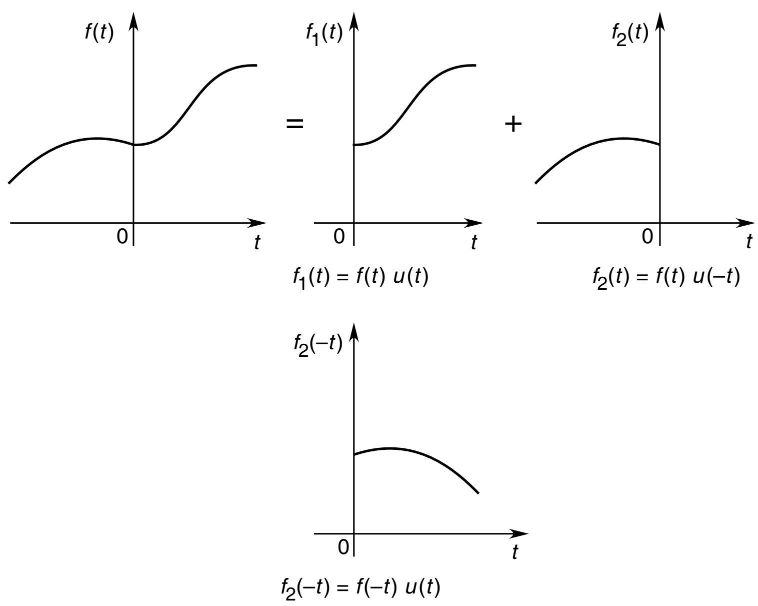


Fig. 2.50 *Splitting of a noncausal signal*

Example 2.27 *Find the two-sided Laplace transform and region of convergence of signal*

$$f(t) = e^{3t}u(-t) + e^{2t}u(t) \tag{i}$$

Solution

$$f_1(t) = e^{2t}\, u(t)$$

$$F_1(s) = \frac{1}{s-2};\ \text{Re}(s) > 2 \qquad \text{(ii)}$$

$$f_2(t) = e^{3t}\, u(-t)$$

or

$$f_2(-t) = e^{-3t}\, u(t)$$

$$F_2(s) = \frac{1}{s+3};\ -3 < \text{Re}(s) \qquad \text{(iii)}$$

$$F_2(-s) = \frac{1}{-s+3};\ \text{Re}(s) < 3 \qquad \text{(iv)}$$

From Eqs (ii) and (iii), we get

$$\begin{aligned} F(s) &= F_1(s) + F_2(-s) \\ &= \frac{1}{s-2} - \frac{1}{s-3} \\ &= \frac{-1}{(s-2)(s-3)};\ 2 < \text{Re}(s) < 3 \end{aligned}$$

Observe that the limits of convergence are contributed by $F_1(s)$ and $F_2(-s)$.

Example 2.28 *Obtain the inverse Laplace transform of*

$$F(s) = \frac{5}{(s-3)(s+2)}$$

If the region of convergence is

(a) $-2 < Re(s) < 3$ *(b)* $Re(s) > 3$ *(c)* $Re(s) < -2$

Solution

$$F(s) = \frac{1}{s-3} - \frac{1}{s+2}$$

Poles are at $s = +3$ and -2.

(a) Since region of convergence is $-2 < \text{Re}(s) < 3$ the pole at $s = -2$ which is on the $s = -2$ line, contributes positive time function.

$$F_1(s) = \frac{1}{s+2}$$

or

$$f_1(t) = £^{-1}\, \frac{1}{s+2} = e^{-2t}\, u(t)$$

Similarly $s = 3$ is just on $s = 3$ line, contributes negative time function

$$F_2(-s) = \frac{1}{s-3}$$

$$F_2(s) = -\frac{1}{s+3}$$

Taking the Laplace inverse

$$f_2(-t) = -e^{-3t}u(t)$$
$$f_2(t) = -e^{3t}u(-t)$$

Hence

$$f(t) = f_1(t) + f_2(t) = e^{-2t}u(t) - e^{3t}u(-t)$$

(b) Since region of convergence is beyond $s = 3$, both the poles, lie to the left of $s = 3$ line. Therefore, both the poles contribute only positive time functions. Thus

$$f(t) = (e^{-2t} - e^{3t})\,u(t)$$

(c) In this case, region of convergence is to the left of $s = -2$ line and all the poles fall in that region only. Therefore, these contribute only negative-time functions. Thus

$$F_2(-s) = \frac{1}{s-3} - \frac{1}{s+2};\ F_1(s) = 0$$

$$F_2(s) = \frac{1}{s-2} - \frac{1}{s+3}$$

$$£^{-1}F_2(s) = f_2(-t) = (e^{2t} - e^{-3t})\,u(t)$$

$$f_2(t) = (e^{-2t} - e^{3t})\,u(-t)$$

Hence

$$f(t) = f_2(t) = (e^{-2t} - e^{3t})\,u(-t)$$

2.18 LAPLACE TRANSFORM OF LTI DIFFERENTIAL EQUATION

As an illustration consider translational mechanical system shown in Fig. 2.51. The system behaviour is described by the following equation.

$$M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = r \qquad \text{(i)}$$

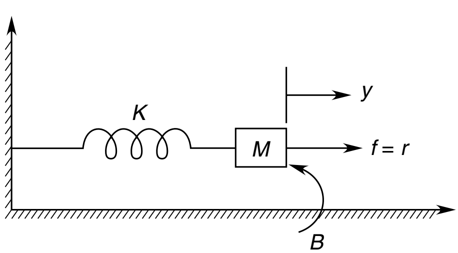


Fig. 2.51 *A translational mechanical system*

Also for mathematical convenience of manipulation assume

$$M = 3\text{kg},\ B = 4\ \text{N/(m/s) and } K = 1\ \text{N/m}$$

Taking the Laplace transform of Eq. (i), we get

$$M[s^2\, Y(s) - sy(0^-) - y(0)] + B[sY(s) - y(0^-)] + KY(s) = R(s) \tag{ii}$$

Substituting values of M and B in Eq. (ii) and rearranging, we get

$$3\,[s^2\, Y(s) - sy(0^-) - (0^-)] + 4[sY(s) - y(0^-)] + KY(s) = R(s) \tag{iii}$$

$$(3s^2\ 4s + 1)Y(s) = 3sy(0^-) + 3(0^-) + 4y(0^-) + R(s)$$

or

$$Y(s) = \frac{(3s+4)y(0^-) + 3\dot{y}(0^-)}{3s^2 + 4s + 1} + \frac{1}{3s^2 + 4s + 1}R(s)$$

$$Y(s) = Y_0(s) + Y_r(s) \tag{iv}$$

where

$Y_0(s)$ = zero-input response

and

$Y_r(s)$ = zero-state response

From the zero-state response

$$\frac{Y_r(s)}{R(s)} = \frac{1}{3s^2 + 4s + 1}$$

This indeed is the transfer function of the system.

System Transfer Function

In s-domain the transfer function of a system is defined as

$$\text{Transfer function } H(s) = \frac{\text{Zero-state output in } s\text{-domain}}{\text{Input in } s\text{-domain}} \tag{2.170}$$

The transfer function is represented in block diagram form in Fig. 2.52. This transfer function concept assumes zero-state (zero initial condition). With this understanding output is now labelled as $Y(s)$.

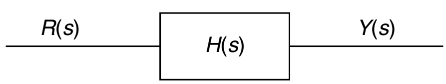


Fig. 2.52 *Transfer function representation*

If the input to a system is an impulse signal, the corresponding output response is given by

$$Y(s)\,\big|_{R(s) = £\,[\delta(t)] = 1} = H(s) = \text{transfer function}$$

This infact is the impulse response of the system in s-domain. In time domain the impulse responce is then given by

$$h(t) = £^{-1}[H(s)] \tag{2.171}$$

The output is given by convolution

$$y(t) = h(t) * r(t)$$

Taking its Laplace transform, we get

$$£[y(t)] = £[h(t) * r(t)]$$

or

$$Y(s) = H(s)R(s) \tag{2.172}$$

We notice that convolution operation in time-domain is reduced to simple multiplication, which is the power of the Laplace transformation.

2.19 LAPLACE TRANSFORM SOLUTION OF LTI CONTINUOUS-TIME SYSTEMS

For illustration we shall consider the simple system of Fig. 2.51 and find its complete response to unit step input with the system having initial conditions.

Zero-input Response We can write the zero-input response from Section 2.18, Eq. (iv) as

$$Y_0(s) = \frac{(3s+4)\,y(0^-) + 3\dot{y}(0^-)}{3s^2 + 4s + 1} \tag{i}$$

Let $y(0^-) = 0$ and $\dot{y}(0^-) = 2$, Then

$$Y_0(s) = \frac{6}{3s^2 + 4s + 1}$$

$$= 3\left[-\frac{1}{s+1} + \frac{1}{s+1/3}\right] \tag{ii}$$

Taking the Laplace inverse we get the following equation.

$$Y_0(t) = 3(-e^{-t} + e^{-t/3})\,u(t) \tag{iii}$$

Zero-state Response Again, this can be written from Section 2.18 as

$$Y_r(s) = \frac{1}{3s^2 + 4s + 1}\,.R(s);\ R(s) = \frac{1}{s} \tag{iv}$$

Then

$$Y_r(s) = \frac{1}{s(3s^2 + 4s + 1)} = \frac{1/3}{s(s+1)\left(s + \frac{1}{3}\right)}$$

The values of s at which $Y_r(s)$ becomes infinite are its *poles*. These are $s = 0$, $s = -1$ and $s = -\frac{1}{3}$. The pole $s = 0$ is contributed by the input $R(s) = \frac{1}{s}$ while $s = -1$ and $s = -\frac{1}{3}$ are system poles ($H(s)$). By partial fractioning

$$H(s) = \underbrace{\frac{1}{s}}_{\substack{\uparrow \\ \text{Input pole}}} + \underbrace{\underbrace{\frac{0.5}{s+1}}_{\uparrow} + \underbrace{\frac{1.5}{s+1/3}}_{\uparrow}}_{\text{System poles}} \tag{v}$$

Taking the Laplace inverse, we get

$$Y_r(t) = (1 + 0.5\, e^{-t} - 1.5\, e^{-t/3})\, u(t) \quad \text{(vi)}$$

Complete Response

$$\begin{aligned} y(t) &= y_0(t) + y_r(t) \\ &= 3(-e^{-t} + e^{-t/3})u(t) + (1 + 0.5e^{-t} - 1.5e^{-t/3})\, u(t) \\ &= \underset{\substack{\uparrow \\ \text{Steady-state response}}}{u(t)} + \underbrace{(1.5e^{-t/3} - 2.5e^{-t})\, u(t)}_{\substack{\uparrow \\ \text{Transient response}}} \end{aligned} \quad \text{(vii)}$$

The transient response decays for zero as $t \to \infty$; i.e.,

$$y(\text{steady states}) = y(\infty) = u(t)$$

Example 2.29 *In the circuit of Fig. 2.53, the switch S is closed at t = 0. Determine the currents $i_1(t)$ and $i_2(t)$.*

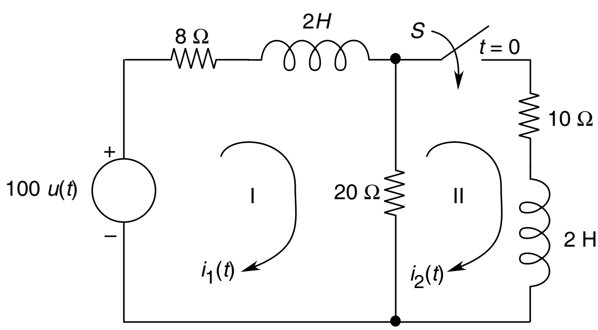


Fig. 2.53

Solution Applying the KVL to loops I and II

$$8\, i_1(t) + \frac{2\mathrm{d}i_1(t)}{\mathrm{d}t} + 20[i_1(t) - i_2(t)] = 100 \quad \text{(i)}$$

$$10\, i_2(t) + \frac{2\mathrm{d}i_2(t)}{\mathrm{d}t} + 20[i_2(t) - i_1(t)] = 0 \quad \text{(ii)}$$

Taking the Laplace transform of Eqs (i) and (ii), we get

$$8I_1(s) + 2sI_1(s) - 2i_1(0^-) + 20\,[I_1(s) - I_2(s)] = \frac{100}{s} \quad \text{(iii)}$$

$$20\,[I_2(s) - I_1(s)] + 10I_2(s) + 2sI_2(s) = 0 \quad \text{(iv)}$$

Before the switch is closed the left part of the circuit has reached steady-state with inductance acting as a short circuit. Therefore,

$$i_1(0^-) = \frac{100}{28} = 3.57\, A$$

Substituting this value in Eq. (iii) and rearranging both Eqs (iii) and (iv), we have

$$(2s + 28)I_1(s) - 20I_2(s) = \frac{100}{s} + 7.14 \tag{v}$$

$$-20I_2(s) + (2s + 30)I_2(s) = 0 \tag{vi}$$

From Eq. (vi), we have

$$I_1(s) = \frac{s+15}{10} I_2(s) \tag{vii}$$

Substituting this in Eq. (v), solving for $I_2(s)$ and factorizing its denominator, we get

$$I_2(s) = \frac{5(100 + 7.14s)}{s(s+24.5)(s+4.5)} \tag{viii}$$

By partial fractioning, we can write

$$I_2(s) = \frac{4.54}{s} - \frac{0.77}{s+24.5} - \frac{3.77}{s+4.5} \tag{ix}$$

Taking the Laplace inverse on both sides

$$i_2(t) = \left\lfloor 4.54 - 0.77e^{-24.5t} - 3.77e^{-4.5t} \right\rfloor u(t) \tag{x}$$

Now

$$I_1(s) = \frac{(s+15)(7.14s+100)}{2s(s+4.5)(s+24.5)} \tag{xi}$$

or

$$I_1(s) = \frac{6.8}{s} + \frac{0.73}{s+24.5} - \frac{3.96}{s+4.5} \tag{xii}$$

Taking the inverse Laplace transform, we get

$$i_1(t) = \left\lfloor 6.8 - 3.96e^{-4.5t} + 0.73e^{-24.5t} \right\rfloor u(t) \tag{xiii}$$

and

$$i_2(t) = \left\lfloor 4.54 - 3.77e^{-4.5t} - 0.77e^{-24.5t} \right\rfloor u(t) \tag{xiv}$$

Note: The reader may check these results by finding $i_1(\infty)$ and $i_2(\infty)$ and also $i_1(0)$ and $i_2(0)$ and calculating these values directly. The inductors act as short circuits at $t = \infty$. At $t = 0$, $i_2(0) = 0$ as the switch is open and $i_1(0)$ can be calculated by shorting the inductor as loop *I* has reached steady-state before the switch is closed.

First-order System

The RC series circuit of Fig. 2.54 is excited by a voltage impulse. We have to determine the expression for the output $e_0(t)$ and discover its qualities.

$$Ri + e_o = \delta(t)$$

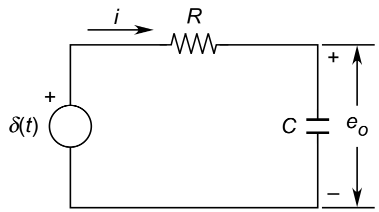


Fig. 2.54

$$i = C\frac{de_0}{dt}$$

Then

$$RC\frac{de_0}{dt} + e_o = \delta(t);\ \text{zero initial conditions} \quad \text{(i)}$$

Taking this Laplace transform, we have

$$(RCs + 1)\,E_o(s) = 1$$

or

$$E_o(s) = \frac{1/\tau}{(s+1/\tau)},\ \tau = RC \quad \text{(ii)}$$

Taking inverse Laplace transform, we get

$$e_o(t) = \frac{1}{\tau}e^{-t/\tau}\,u(t)\ t > 0 \quad \text{(iii)}$$

The output voltage is plotted in Fig. 2.55 from which we find

at $t = 0$, the voltage $e_0\,(0) = \dfrac{1}{\tau}$

at $t = \infty$, $e_o(\infty) = 0$

The decay of e_0 is governed by $\tau = 1/RC$

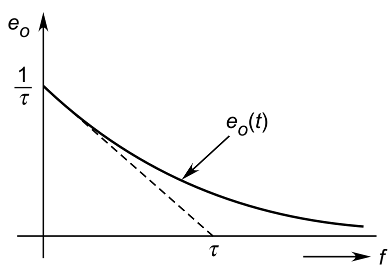


Fig. 2.55

Time Constant

$\tau = 1/RC$ is known as the time constant of the circuit. The initial slope of e_0 is

$$\left.\frac{de_o}{dt}\right|_{t=0} = -\frac{1}{\tau^2}$$

In time τ at initial slope $e_o(0)$ reduces from $\frac{1}{\tau}$ to zero.

Certain Observation As the circuit is impulse excited

The impulse response

$$e_o(t) = h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

$$E_o(s) = H(s), \text{ system transfer function}$$

Pole of $H(s)$ is $s = -1/\tau$, which determines the system response. Observe that pole, $p = 1/\tau$

Second-order System

Consider the mass-friction-spring system of Fig. 2.51 exited by force $f = \delta(t)$, unit impulse. Its describing differential equation is

$$M\frac{d^2 y}{dt^2} + B\frac{dy}{dt} + k\,y = \delta(t) \tag{i}$$

Taking its Laplace transform with zero initial condition

$$(Ms^2 + Bs + K)\;Y(s) = 1$$

Its transform function is

$$H(s) = \frac{1}{Y(s)} = \frac{1}{Ms^2 + Bs + K} \tag{ii}$$

To determine its impulse response $h(t)$, we organize it in standard second-order form

$$H(s) = \frac{(1/M) = \dfrac{1}{K}\left(\dfrac{K}{M}\right)}{s^2 + \left(\dfrac{B}{M}\right)s + \dfrac{K}{M}} \tag{iii}$$

Let
$$\frac{K}{M} = \omega_n^2,\; 2\zeta\omega_n = \left(\frac{B}{M}\right)$$

Then
$$\omega_n = \sqrt{\frac{K}{M}},\; \zeta = \frac{1}{2}\left[\frac{B}{M}\cdot\sqrt{\frac{M}{K}}\right]$$

$$\omega_n = \textit{natural frequency};$$

$$\zeta = \frac{1}{2}\frac{B}{\sqrt{MK}};\; \textit{damping factor}$$

Leaving out the scale factor $(1/K)$, Eq. (iii) is in standard form is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\,\omega_n\,s + \omega_n^2} \tag{iv}$$

Underdamped Case (ζ < 1)

The poles of $H(s)$ are

$$\alpha_1, \alpha_2 = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\;\;\omega_n \text{ for } \zeta < 1$$

The inverse of $H(s)$ can be found by the method of partial fractioning using residues. We would use here item 9b of Table 2.4 reproduced below.

$$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \leftrightarrow e^{-\alpha t}\sin \omega_0 t\, u(t) \tag{v}$$

Let us write Eq. (iv) in form of Eq. (v)

$$\frac{\omega_n^2}{(s+\zeta\omega_n)^2+(1-\zeta^2)\omega_n^2}$$

or

$$\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)\frac{\sqrt{1-\zeta^2}\,\omega_n}{(s+\zeta\,\omega_n)^2+(1-\zeta^2)\,\omega_n^2}$$

Then

$$\alpha = -\zeta\,\omega_n,\ \omega_0 = \left(\sqrt{1-\zeta^2}\right)\omega_n$$

Therefore

$$h(t) = \left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)\cdot\left[e^{-\zeta\,\omega_n t}\sin\sqrt{1-\zeta^2}\,\omega_n\, t\right] u(t)\zeta < 1 \tag{vi}$$

where

$$\omega_n\sqrt{1-\zeta^2} = \omega_d = \text{damped natural frequency.}$$

The impulse response $h(t)/\omega_n$ is plotted for various values of ζ in Fig. 2.56.

Critically Damped Case

From Eq. (iv)

$$H(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \tag{vii}$$

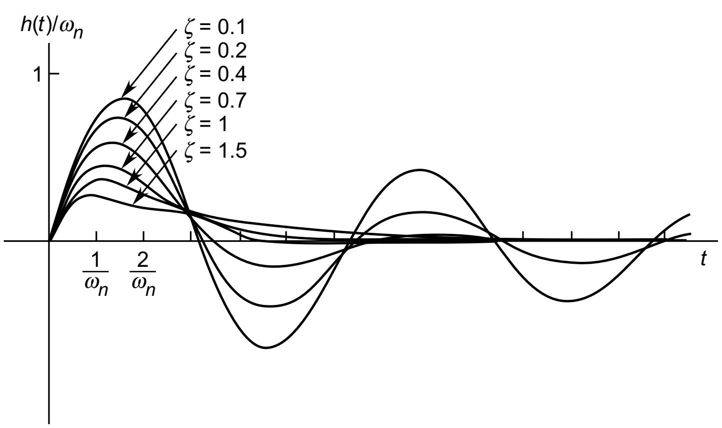


Fig. 2.56 *Impulse response of second-order underdamped system*

As per item 6 of Table 2.4,

$$h(t) = \omega_n^2\, t\, e^{-\omega_n t}\, u(t) \tag{viii}$$

The response decays monotonically.

Overdamped Case ($\zeta > 1$)

The poles α_1, α_2 we real unequal and so $h(t)$ is the sum of two decaying exponentials.

Note: Let the reader find the response of this system to unit-step input $u(t)$ and plot the same for various values of ζ using MATL AB. As $u(t) = \int_0^\infty \delta(t)\, dt$, the unit step response is the integral of impulse response.

Nature of System Response The impulse response of a system characterizes the nature of system's response. It is seen from Eq. (2.171) that

$$h(t) = £^{-1}[H(s)]$$

where $\qquad H(s)$ = system transfer function

which can be written as ratio of polynomials of s, i.e.,

$$H(s) = \frac{N(s)}{D(s)}; \text{ order of } N(s) \text{ less than order of } D(s)$$

The system's characteristic equation is

$$D(s) = 0$$

where *roots* are the *poles* of the system. Thus it is the poles of the system which determine the nature of the system's impulse response.

The response terms contributed by various types of poles is summarized in Table 2.6 and are graphically presented in Fig. 2.56.

Table 2.6 *Response terms contributed by various types of system poles*

	Type of roots	*Nature of response terms contributed*
(i)	Single root at $s = \sigma$	$Ae^{\sigma t}$
(ii)	Roots of multiplicity m at $s = \sigma$	$(A_1 + A_2 t + \ldots + A_m t^{m-1})e^{\sigma t}$
(iii)	Complex conjugate root pair at $s = \sigma \pm j\omega$	$Ae^{\sigma t} \sin(\omega t + \theta)$
(iv)	Complex conjugate root pairs of multiplicity m at $s = \sigma \pm j\omega$	$[A_1 \sin(\omega t + \theta_1) + A_2 t \sin(\omega t + \theta_2) + \ldots + A_m t^{m-1} \sin(\omega t + \theta_k)]e^{\sigma t}$
(v)	Single complex conjugate root pair on the $j\omega$-axis (i.e. at $s = \pm j\omega$)	$A \sin(\omega t + \theta)$
(vi)	Complex conjugate root pair of multiplicity m on the $j\omega$-axis	$A_1 \sin(\omega t + \theta_1) + A_2 t \sin(\omega t + \theta_2) + \ldots + A_m t^{m-1} \sin(\omega t + \theta_k)$
(vii)	Single root at origin (i.e. at $s = 0$)	A
(viii)	Roots of multiplicity m at origin	$(A_1 + A_2 t + \ldots + \mathrm{A}_m t^{m-1})$

Stability The location of system poles and nature of response decides the stability of the system.

It is easily concluded from Fig. 2.57 and Table 2.6 that a system is asymptotically stable if its poles are located in the left half of the s-plane, i.e. it has poles with negative real parts. Further details on system stability and stability criterion is discussed in Chapter 8.

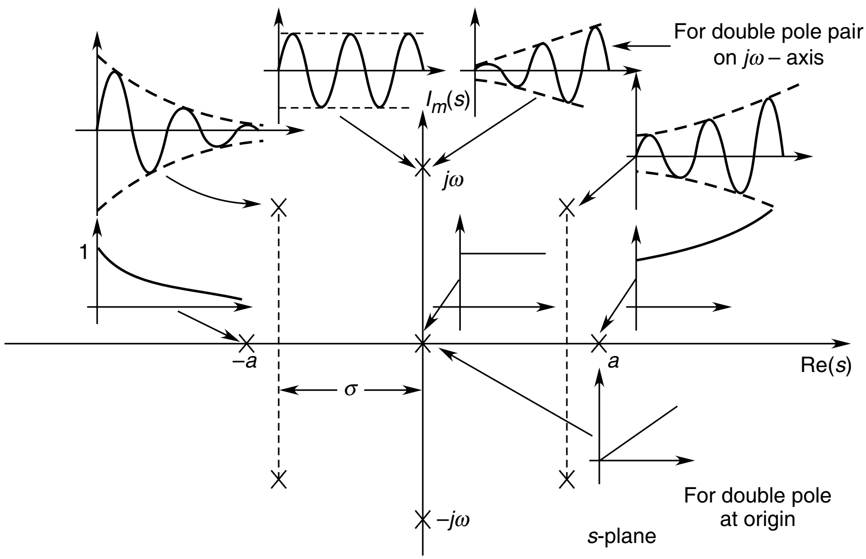


Fig. 2.57 *Nature of system response*

2.20 SINUSOIDAL STEADY-STATE RESPONSE OF LTI SYSTEM-FREQUENCY RESPONSE

In steady state, sine wave exists for all time ($t = -\infty$ to $+\infty$). So for steady-state solution we need two-sided transforms. In frequency domain, the best suited transform is the Fourier transform.

Given the Laplace transfer function $H(s)$ of an LTI system, we can obtain the Fourier transfer function as

$$H(s)|_{s=j\omega} = H(j\omega) = H(\omega) \tag{2.169}$$

As $H(\omega)$ is complex (in general), we can express it as

$$H(\omega) = |H(\omega)| \angle H(\omega) = |H(\omega)| \angle \phi_h \tag{2.170}$$

Let the system input be

$$A \cos(\omega_0 t + \phi_0) = Re[A\, e^{j\phi_0}\, e^{j\omega_0 t}] \tag{2.171}$$

where $\overline{A} = A\, e^{j\phi_0}$ is phasor input

$e^{j\omega_0 t}$ is the eigen function

$$\pounds\{Re[\overline{A}\, e^{j\omega_0 t}] = Re[\overline{A}\, \delta(\omega - \omega_0)]$$

As *Re* operation can be carried out in end, we drop *Re* at this stage. The system output is

$$Y(\omega) = H(\omega)\ \overline{A}\ \delta(\omega - \omega_0)$$

$$\begin{aligned} &= H(\omega)\ \overline{A}\ \delta(\omega - \omega_0) \\ &= |H(\omega_0)|\ e^{j\phi_h} A\ e^{j\phi_0}\ \delta(\omega - \omega_0) \end{aligned} \tag{2.172}$$

Taking inverse Fourier transform

$$y(t) = \underbrace{|H(\omega_0)|\, e^{j\phi_h}}_{\text{System's eigen value}} \qquad \underbrace{A\, e^{j\phi_0} e^{j\omega_0 t}}_{\text{Eigen function}} \tag{2.173}$$

In real time

$$\begin{aligned} y(t) &= Re[|H(\omega_0)|\ A\ e^{j(\phi_0+\phi_h)}\ e^{j\omega_0 t}] \\ &= A\ |H(\omega_0)|\ \cos\,(\omega_0 t + \phi_0 + \phi_h) \end{aligned} \tag{2.174}$$

Conclusion We find that in steady state the output is sinusoidal input modified by the system as

Magnitude	$A\,|H(\omega_0)|$
Phase	$\angle\phi_0 + \phi_h$

$A|H(\omega_0)\ e^{j(\phi_0+\phi_h)}$ is output phasor from which we can construct the time function form as

$$Re[A|H(\omega_0)|\ e^{j(\phi_0+\phi_h)}\ e^{j\omega_0 t}]$$

Frequency Response It is the variation of system magnitude and phase with frequency, i.e.,

$$H(\omega) = |H(\omega)|\ \angle H(\omega)$$

It is the plots

$$|H(\omega)| \text{ vs. } \omega \ \text{ and } \ \angle H(\omega) \text{ vs. } \omega$$

We need the plots for $+\omega$ only. Because of symmetry of $H(\omega)$, the plots for $-\omega$ are not necessary.

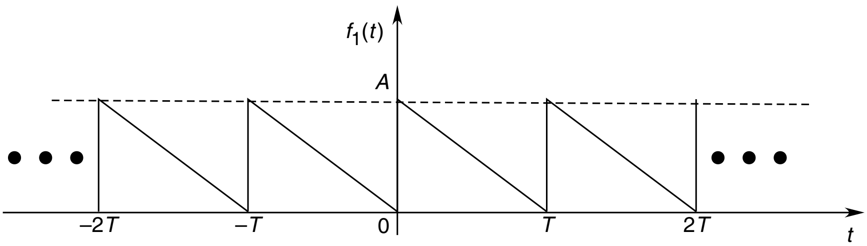
Additional Examples

Example 2.30 *The exponential Fourier series coefficients for the signal $f_1(t)$ shown in Fig. 2.58(a) are given by $F_n = \alpha_n + j\beta_n$.*

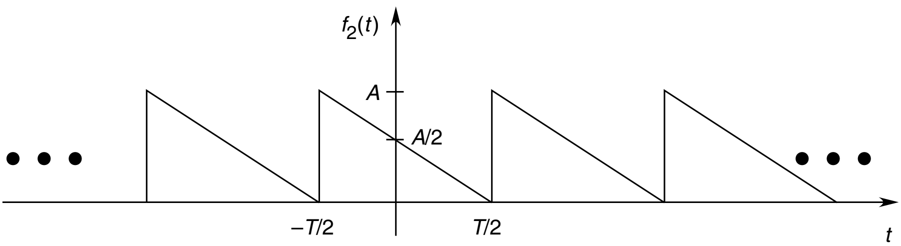
Determine Fourier coefficients F' for the signal shown in Fig. 2.58 (b).

Solution Signal $f_1(t)$ is mathematically represented by the following equation.

$$f_1(t) = \frac{A}{T}t;\ 0 \le t \le T \tag{i}$$



(a) Sketch of $f_1(t)$



(b) Sketch of $f_2(t)$

Fig. 2.58

Similarly, signal $f_2(t)$ is expressed as by the following equation.

$$f_2(t) = \frac{A}{T}\left(\frac{T}{2} - t\right); -\frac{T}{2} \le t \le \frac{T}{2} \tag{ii}$$

The Fourier coefficients of signal $f_2(t)$ is obtained as

$$F'_n = \frac{1}{T}\int_{-T/2}^{T/2} \frac{A}{T}\left(\frac{T}{2} - t\right) e^{-jn\omega_0 t}\, \mathrm{d}t$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} \frac{A}{T}\frac{T}{2} e^{-jn\omega_0 t}\, \mathrm{d}t - \int_{-T/2}^{T/2} \frac{A}{T} t\, e^{-jn\omega_0 t}\, \mathrm{d}t\right]$$

$$= \frac{A}{2T}\int_{-T/2}^{T/2} e^{-jn\omega_0 t}\, \mathrm{d}t - \frac{1}{T}\int_{-T/2}^{T/2} \frac{\mathrm{A}}{T} t\, e^{-jn\omega_0 t}\, \mathrm{d}t$$

$$= \frac{A}{2T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T/2}^{T/2} - F_n$$

or

$$F'_n = \frac{A}{2T}\frac{\sin(n\omega_0 T/2)}{n\omega_0} - \alpha_n - j\beta_n \tag{iii}$$

or

$$F'_n = \frac{A}{4}\ \mathrm{sinc}\,(n\pi) - \alpha_n - j\beta_n \tag{iv}$$

Example 2.31 *A periodic function f (t), expressed in Trigonometric Fourier series has four terms $f_0(t)$, $f_1(t)$, $f_2(t)$ and $f_3(t)$. The characteristics of each of these terms is shown in Figs 2.59 (a), (b), (c) and (d) respectively. Determine the exponential Fourier series and the Fourier coefficients of the periodic signal f (t).*

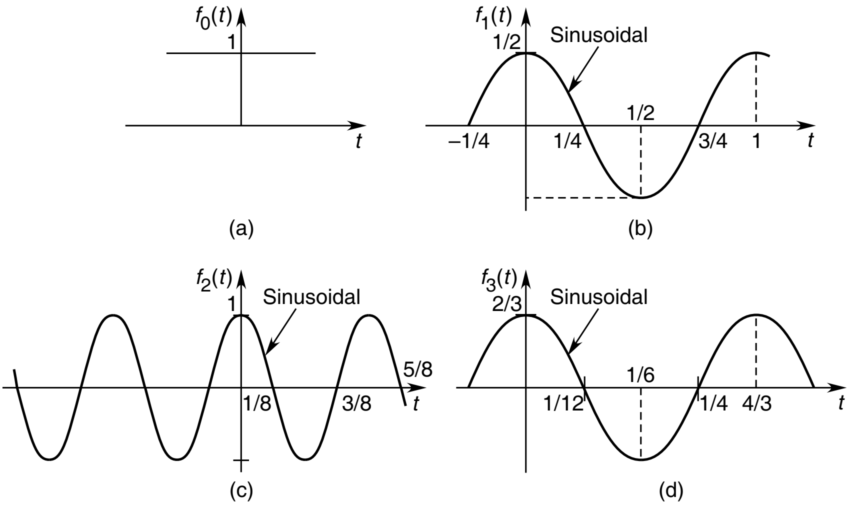


Fig. 2.59 *Characteristics of $f_0(t)$, $f_1(t)$, $f_2(t)$ and $f_3(t)$*

Solution $f_0(t), f_1(t), f_2(t)$ and $f_3(t)$ are defined as follows.

$$f_0(t) = 1$$

$$f_1(t) = \frac{1}{4}\{e^{j2\pi t} + e^{-j2\pi t}\};\ T = 1,\ \omega_0 = 2\pi$$

$$f_2(t) = \frac{1}{2}\{e^{j4\pi t} + e^{-j4\pi t}\};\ T = \frac{1}{2};\ \omega_0 = 4\pi$$

and

$$f_3(t) = \frac{1}{3}\{e^{j6\pi t} + e^{-j6\pi t}\};\ T = \frac{1}{3},\ \omega_0 = 6\pi$$

Adding $f_1(t), f_2(t)$ and $f_3(t)$, we have the exponential Fourier series of $f(t)$ as

$$f(t) = \frac{1}{4}\{e^{j2\pi t} + e^{-j2\pi t}\} + \frac{1}{2}\{e^{j4\pi t} + e^{-j4\pi t}\} + \frac{1}{3}\{e^{j6\pi t} + e^{-j6\pi t}\} \qquad \text{(i)}$$

The Fourier coefficients are

$$F_0 = 1,\ F_1 = \frac{1}{4},\ F_2 = \frac{1}{2} \text{ and } F_3 = \frac{1}{3}$$

Example 2.32 *A LTI filter has the frequency response $|H(\omega)|$ as shown in Fig. 2.60. Determine the filter output corresponding to the following input.*

$$r(t) = e^{j2t}$$

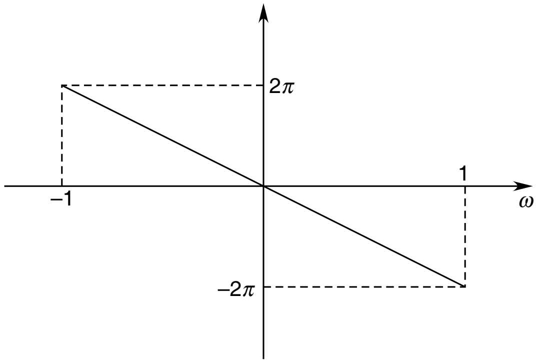


Fig. 2.60 *Frequency response of $H(\omega)$*

Solution The frequency response of the LTI filter, shown in Fig. 2.59, is given as

$$\begin{aligned} |H(\omega)| &= -2\pi\omega;\ |\omega| < 1 \\ &= 0, \text{ otherwise} \end{aligned} \tag{i}$$

Input to the filter is

$$r(t) = e^{j2t} \tag{ii}$$

Taking the Fourier transform, we get

$$R(\omega) = 2\pi\delta(\omega - 2) \tag{iii}$$

Filter output

$$\begin{aligned} Y(\omega) &= -4\pi^2\omega\delta(\omega - 2);\ |\omega| < 1 \\ &= 0, \text{ otherwise} \end{aligned}$$

Example 2.33 *A signal $f(t) = A$ sin ωt, $\omega_0 = 2\pi/T$ with T being the time period, is passed through a full-wave rectifier. Find the spectrum of the output wave-form.*

Solution The signal A sin $\omega_0 t$ after passing through a full-wave rectifier is modified in the following form (shown in Fig. 2.61a).

$$f'(t)\ A \sin \omega_0' t \tag{i}$$

With $\omega_0' = \dfrac{2\pi}{T_1}$, $T_1 = T/2$ is the time period of the signal $f'(t)$, which means that

$$\omega_0' = 2\omega_0 \tag{ii}$$

The spectrum of $f'(t)$, i.e., $F'(\omega)$ is given by

$$F'(\omega) = \mathcal{F}[f'(t)]$$

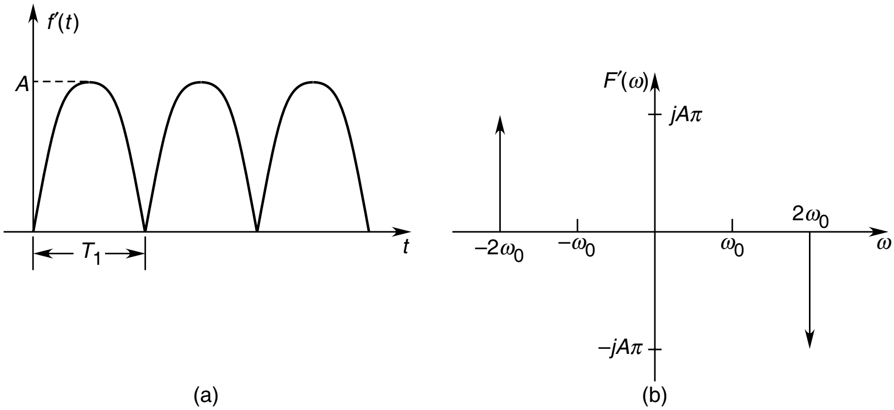


Fig. 2.61

$$= \mathcal{F}[A \sin \omega_0' t] = A\,[\mathcal{F}\{\sin \omega_0' t\}]$$

i.e.

$$F'(\omega) = jA\pi[\delta(\omega - \omega_0') - \delta(\omega - \omega_0')]$$

or

$$F'(\omega) = jA\pi[\delta(\omega - 2\omega_0) - \delta(\omega - 2\omega_0)] \qquad \text{(iii)}$$

The spectrum of the output signal is drawn in Fig. 2.61(b).

Example 2.34 *If the Fourier transform of a signal h(t) is H(ω), prove that*

$$\Delta T_1 . \Delta\omega_1 = 1$$

where

$$\Delta T_1 = \frac{\int_{-\infty}^{\infty} h(t)\,\mathrm{d}t}{h(0)}$$

and

$$\Delta\omega_1 = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\mathrm{d}\omega}{H(0)}$$

Solution

$$\Delta T_1 \Delta\omega_1 = \frac{\int_{-\infty}^{\infty} h(t)\,\mathrm{d}t}{h(0)} \cdot \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\mathrm{d}\omega}{H(0)} \qquad \text{(i)}$$

Following is the Fourier transform of $h(t)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\,\mathrm{d}t \qquad \text{(ii)}$$

At

$$\omega = 0;\ H(0) = \int_{-\infty}^{\infty} h(t)\,\mathrm{d}t$$

By the following inverse Fourier transform

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,\mathrm{d}\omega \qquad \text{(iii)}$$

At

$$\omega = 0\;;\; h(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\mathrm{d}\omega \qquad \text{(iv)}$$

Substituting Eqs (iv) and (v) in Eq. (i), we get

$$\Delta T_1 . \Delta\omega_1 = \frac{H(0)}{h(0)} \quad \frac{h(0)}{H(0)} = 1 \qquad \text{(v)}$$

Example 2.35 *A system is governed by the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y(t) = r(t) \qquad \textit{(i)}$$

Find the output if following is the input, by using the Fourier transform technique.

$$r(t) = F_n e^{jn\omega_0 t} \qquad \textit{(ii)}$$

Solution

Let

$$y(t) \leftrightarrow Y(\omega) \qquad \text{(iii)}$$

Fourier transforming the given differential equation, we get

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \leftrightarrow j\omega\, Y(\omega) \qquad \text{(iv)}$$

thus

$$\mathcal{F}\left[\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t)\right] = \mathcal{F}[r(t)]$$

or

$$j\omega\, Y(\omega) + 2Y(\omega) = R(\omega)$$

or

$$Y(\omega)\,[2 + j(\omega)] = R(\omega) \qquad \text{(v)}$$

$$R(\omega) = \mathcal{F}[F_n e^{jn\omega_0 t}] = 2\pi F_n \delta(\omega - n\omega_0) \qquad \text{(vi)}$$

From Eqs (v) and (vi)

$$Y(\omega) = \frac{R(\omega)}{2 + j\omega} = \frac{2\pi F_n \delta(\omega - n\omega_0)}{2 + j\omega}$$

Taking the inverse Fourier transform

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi F_n}{2 + j\omega}\,\delta(\omega - n\omega_0)e^{j\omega t}\mathrm{d}\omega \qquad \text{(vii)}$$

By the sampling property of δ-function the above integral yields

$$y(t) = \frac{F_n e^{jn\omega_0 t}}{(2 + jn\omega_0)}$$

Example 2.36 *Express the signal r(t) shown in Fig. 2.62 in terms of ideal functions.*

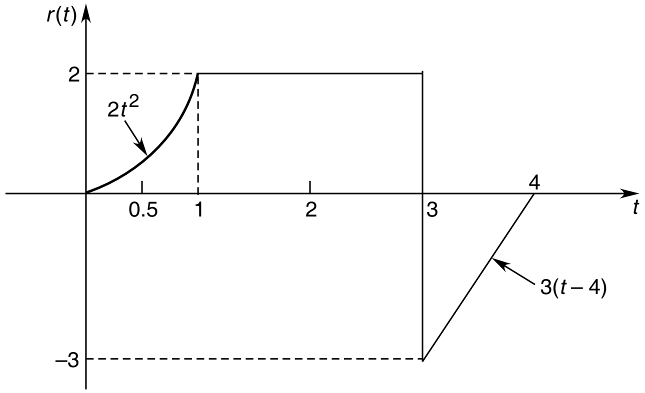


Fig. 2.62

Solution Piecewise representation of $r(t)$ is as under

$$r(t) = \begin{cases} r_1(t) = 2t^2 & 0 \le t < 1 \\ r_2(t) = 2 & 1 < t < 3 \\ r_3(t)\ 3(t-4) & 3 < t < 4 \end{cases}$$

The graphical representation of these pieces is drawn in Figs 2.63 (a), (b) and (c).

$$r_1(t) = 2t^2[u(t) - u(t-1)] \qquad \text{(i)}$$
$$r_2(t) = 2[u(t-1) - u(t-3)] \qquad \text{(ii)}$$
$$r_3(t) = 3(t-4)\ [u(t-3) - u(t-4)] \qquad \text{(iii)}$$

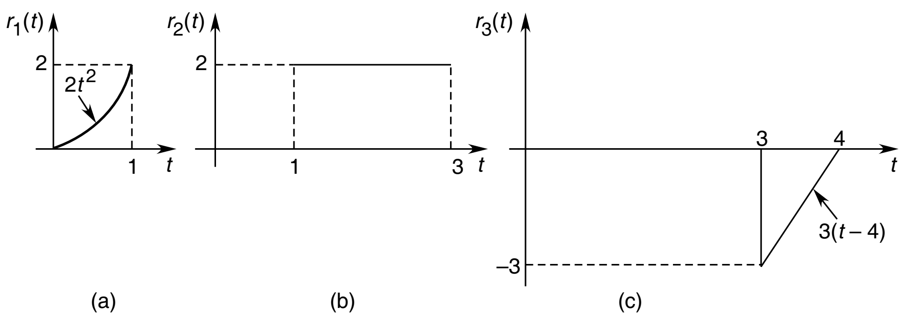


Fig. 2.63

The signal $r(t)$ can then be expressed in terms of step functions as

$$\begin{aligned} r(t) &= r_1(t) + r_2(t) + r_3(t) \\ &= 2t^2[u(t) - u(t-1)] + 2[u(t-1) - u(t-3)] \\ &+ 3(t-4)\,[u(t-3) - u(t-4)] \end{aligned} \tag{iv}$$

Example 2.37 *Find the derivative of the signal of Fig. 2.61 (Example 2.36) and sketch the same.*

Solution Taking first derivative of $r(t)$ expressed in Eq. (v) of Example 2.37, we get

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}r_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}r_2(t) + \frac{\mathrm{d}}{\mathrm{d}t}r_3(t) \tag{i}$$

Finding the drivatives term by term, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}r_1(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[2t^2\,(u(t) - u(t-1))] \\ &= 4t[u(t) - u(t-1)] + 2t^2\,[\delta(t) - \delta(t-1) \end{aligned} \tag{ii}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}r_2(t) = 2[\delta(t-1) - \delta(t-3)] \tag{iii}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}r_2(t) = 3[u(t-3) - u(t-4)] + 3(t-4)\,[\delta(t-3) - \delta(t-4)] \tag{iv}$$

Using the property of $\delta(t)$ given in Eq. (2.10a), we have

$$r(t)\,\delta(t) = r(0)\,\delta(t) \tag{v}$$

Equations (ii), (iii) and (iv) then get modified as

$$\frac{\mathrm{d}}{\mathrm{d}t}r_1(t) = 4t[u(t) - u(t-1)] + 0 - 2\delta(t-1) \tag{vi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}r_2(t) = 2[\delta(t-1) - \delta(t-3)] \tag{vii}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}r_3(t) = 3[u(t-3) - u(t-4)] - 3\delta(t-3) + 0 \tag{viii}$$

Adding the three derivative terms, we get the following result.

$$\frac{\mathrm{d}}{\mathrm{d}t}r(t) = 4t[u(t) - u(t-1)] + 3[u(t-3) - u(t-4) - 5\,\delta(t-3) \tag{ix}$$

Sketch of the $\frac{\mathrm{d}}{\mathrm{d}t}r(t)$ versus t is shown in Fig. 2.64.

Example 2.38 *Find the general expression for the Laplace transform of periodic functions (signals) with periodicity T, sketched in Fig. 2.65. These signals are commonly encountered in electrical and electronic circuits.*

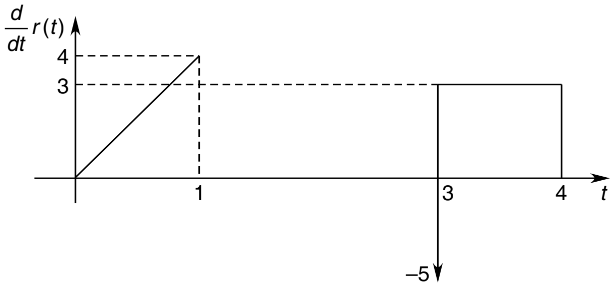


Fig. 2.64

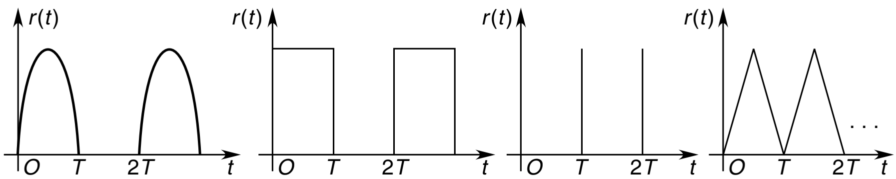


Fig. 2.65

Solution As periodic signals are of great importance in electrical engineering, we shall first derive a general result.

$$£[r(t)] = \int_0^\infty r(t)e^{-st}\,dt \tag{i}$$

when $[r(t)]$ is a periodic signal with periodicity T.

The signal in succeeding periods is generated from the first period by use of shifting property.

$$£[r(t)] = \int_0^T r(t)e^{-st}\,dt + \int_0^T r(t+T)e^{-s(t+T)}\,dt + \int_0^T r(t+2T)e^{-s(t+2T)}\,dt + ...$$

$$= \int_0^T r(t)e^{-st}\,dt + e^{-sT}\int_0^T r(t)e^{-st}\,dt + e^{-2sT}\int_0^T r(t)e^{-st}\,dt + ... \tag{ii}$$

On account of periodicity

$$r(t) = r(t+T) = r(t+2T) = \dots$$

Reorganizing Eq. (ii) we have

$$R(s) = [1 + e^{-sT} + e^{-2sT} + \dots]\int_0^T r(t)e^{-st}\,dt \tag{iii}$$

But

$$1 + e^{-sT} + e^{-2sT} + ... = \frac{1}{1-e^{-sT}} \tag{iv}$$

Then from Eqs (iii) and (iv), we get

$$R(s) = \frac{\int_0^T r(t)e^{-st}\,\mathrm{d}t}{1-e^{-sT}} \tag{v}$$

where $\int\limits_0^T r(t)e^{-st}\,\mathrm{d}t$ = Laplace transform of the first periodic signal

We conclude from Eq. (v) that

Laplace transform of a periodic signal

$$= \frac{\text{Laplace transform of first period of the signal}}{1-e^{-sT}} \tag{vi}$$

The reads may now find the Laplace transform of the give periodic

Example 2.39 *Find $r_1(t) * r_2(t)$ for the signals shown in Fig. 2.66.*

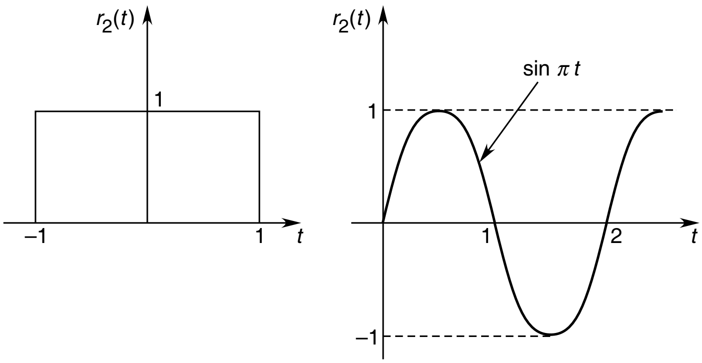


Fig. 2.66

Solution From Eq. (2.17)

$$r_1(t) * r_2(t) = \int\limits_{-\infty}^{\infty} r_1(\lambda)\, r_2(t-\lambda)\,\mathrm{d}\lambda \tag{i}$$

The convolution process of Eq. (i) is represented graphically in Fig. 2.67 at time t but integration is not yet carried out.

With reference to Fig. 2.66 and integrating over time period where both signals are non-zero, we get

$$r_1(t) * r_2(t) = \int\limits_{-1}^{t} 1 \sin \pi (t-\lambda)\,\mathrm{d}\lambda$$

$$= \frac{1}{\pi}\,[1-\cos \pi (t+1)] \tag{ii}$$

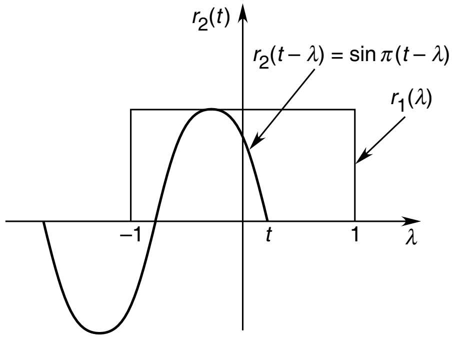


Fig. 2.67

The wave form of the convolved signal is drawn in Fig. 2.68.

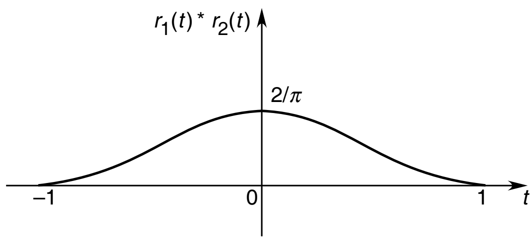


Fig. 2.68

Example 2.40 *The network shown in Fig. 2.69(a) is excited by a voltage pulse as shown in Fig. 2.69. Find the expression for the current i(t) using the Laplace transform. Assume all initial conditions are zero.*

Solution From Fig. 2.69 for the first period

$$v_1(t) = u(t) - u(t - \Delta) \tag{i}$$

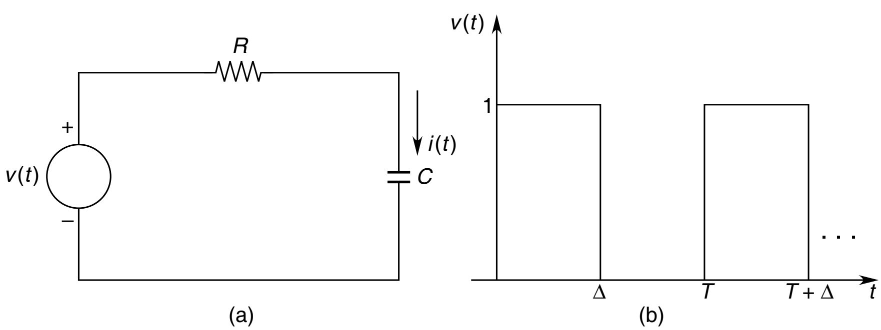


Fig. 2.69

Taking the Laplace transform of Eq. (i), we get

$$V_1(s) = \frac{1}{s} - \frac{e^{-s\Delta}}{s} = \frac{1 - e^{-s\Delta}}{s} \tag{ii}$$

Equation (ii) is the Laplace transform of the first pulse set of a periodic signals. Using the result of Example 2.38, the Laplace transform of the complete periodic signal is

$$V(s) = \frac{1 - e^{-s\Delta}}{s(1 - e^{-s\mathrm{T}})} \tag{iii}$$

We can write the KVL equation in s-domain from the circuit of Fig. 2.69(a) as

$$\begin{aligned} V(s) &= RI(s) + \frac{1}{\mathrm{C}s} I(s) \\ &= \left(\frac{RCs + 1}{Cs}\right) I(s) \end{aligned}$$

Then

$$I(s) = \left(\frac{CsV(s)}{(RCs + 1)}\right) \tag{iv}$$

Substituting for $V(s)$ from Eq. (iii) in Eq. (iv), we get

$$\begin{aligned} I(s) &= \frac{Cs}{(RCs + 1)} \cdot \frac{1 - e^{-s\Delta}}{s(1 - e^{-sT})} \\ &= \frac{1}{R} \cdot \frac{1 - e^{-s\Delta}}{\left(s + \frac{1}{RC}\right)(1 - e^{-sT})} \end{aligned}$$

Let us select $R = 1\,\Omega$ and $C = 1$F. We also carry out partial fractioning. Then the above expression in right hand side is

$$I(s) = \frac{1}{R} \frac{1 - e^{-s\Delta}}{(s + 1)(1 - e^{-sT})}$$

or

$$I(s) = \left[\frac{A(s)}{s + 1} + \frac{B(s)}{1 - e^{-sT}}\right] \tag{v}$$

We can derive the following from Eq. (v)

$$\begin{aligned} A(s) &= (s + 1)\, I(s)\big|_{s=-1} \\ &= \left.\frac{1 - e^{-\Delta s}}{1 - e^{-sT}}\right|_{s=-1} = \frac{1 - e^{\Delta}}{1 - e^{T}} \end{aligned} \tag{vi}$$

$B(s)$ is the response of the circuit to the first period of the periodic signal. It can be found from Eqs (v) and (vi) as shown below.

$$B(s) = \left[I(s) - \frac{A(s)}{(s+1)} \right] (1 - e^{-sT})$$

$$= \left[\frac{1 - e^{-s\Delta}}{(s+1)(1 - e^{-sT})} - \frac{1 - e^{\Delta}}{1 - e^{T}} \cdot \frac{1}{s+1} \right] (1 - e^{-sT})$$

$$= \frac{1}{s+1} - \frac{e^{\Delta s}}{s+1} - \left(\frac{1 - e^{\Delta}}{1 - e^{T}} \right) \cdot \frac{1 - e^{-sT}}{s+1} \qquad \text{(vii)}$$

Let

$$\frac{1 - e^{\Delta}}{1 - e^{T}} = K = \text{constant. Then}$$

$$B(s) = \frac{1}{s+1} - \frac{e^{-\Delta s}}{s+1} - \frac{K}{s+1} + \frac{Ke^{-Ts}}{s+1} \qquad \text{(viii)}$$

Taking the inverse Laplace transform of Eq. (viii), we get

$$B(t) = e^{-t}\,u(t) - e^{-(t-\Delta)}\,u(t-\Delta) - Ke^{-t}\,u(t) + Ke^{-(t-T)}\,u(t-T)$$
$$= (1-K)e^{-t}\,u(t) - e^{-(t-\Delta)}\,u(t-\Delta) + Ke^{-(t-T)}\,u(t-T) \qquad \text{(ix)}$$

The current $i(t)$ can now be expressed as

$$i(t) = K\,e^{-t}\,u(t) + (1-K)\,e^{-t}\,u(t) - e^{-(t-\Delta)}\,u(t-\Delta) + K\,e^{-(t-T)}\,u(t-T)$$

or $$i(t) = A(t) + B(t)$$
$$i(t) = e^{-t}\,u(t) - Ke^{-(t-\Delta)}\,u(t-\Delta) + K\,e^{-(t-T)}\,u(t-T)$$

Example 2.41 In the circuit of Fig. 2.70, the switch S has been open for long time and is closed at $t = 0$. For $e(t) = 3u(t)$, find $e(t)$, $t > 0$.

Solution Applying the KVL to the circuit of Fig. 2.70 after S is closed.

$$e(t) = 3\,u(t) = 2\,i(t) + 2\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} \qquad \text{(i)}$$

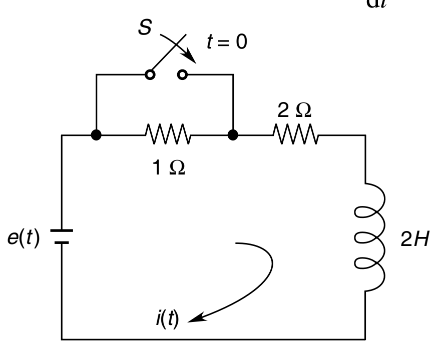


Fig. 2.70

Taking the Laplace transform of Eq. (i), we get

$$\frac{3}{s} = 2I(s) + 2\{sI(s) - i(0^-)] \tag{ii}$$

where $i(0^-)$ = initial condition

$$= \frac{3}{(1+2)} = 1\text{A}$$

From Eq. (ii), we now get

$$3 = 2sI(s) + 2s^2 I(s) - s$$

Rearranging we get

$$I(s) = \frac{s+3}{2s(s+1)} \tag{iii}$$

Partial fractioning of Eq. (iii), yields the following result.

$$I(s) = \frac{1}{2}\left[\frac{3}{s} - \frac{2}{s+1}\right] \tag{iv}$$

Taking the inverse Laplace transform of Eq. (iv), we get

$$i(t) = (1.5 - e^{-t})\, u(t) \tag{v}$$

$$i(\text{steady-state}) = i(t = \infty) = 1.5\,\text{A}$$

Example 2.42 *Find the complete response of the circuit shown in Fig. 2.71 when the input signal $r(t) = V_m \cos \omega t$.*

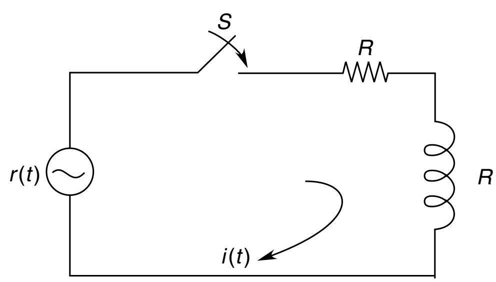


Fig. 2.71

Solution Applying the KVL after the switch S is closed at $t = 0$.

$$V_m \cos \omega t u(t) = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t};\ t > 0 \tag{i}$$

Taking the Laplace transform of Eq. (i), we get

$$\frac{sV_m}{s^2 + \omega^2} = I(s)\,(R + sL)$$

or

$$I(s) = \frac{sV_m}{(R+sL)(s^2+\omega^2)}$$

$$= \frac{V_m}{L} \cdot \frac{1}{(s+R/L)} \cdot \left(\frac{s}{s^2+\omega^2}\right) \tag{ii}$$

Partial fractioning Eq. (ii),

$$I(\mathrm{s}) = \frac{A}{s+R/L} + \frac{B}{s-j\omega} + \frac{B^*}{s+j\omega} \tag{iii}$$

$$A = \frac{V_m R}{R^2+\omega^2 L^2} = -\frac{V_m}{Z}\cos\phi$$

(refer impedance triangle of Fig. 2.72)

$$B = \frac{V_m}{L}\left.\frac{s}{(s+R/L)(s+j\omega)}\right|_{s=j\omega} = \frac{V_m}{2Z}e^{-j\phi} \tag{iv}$$

where

$\phi = \tan^{-1}\omega L/R$

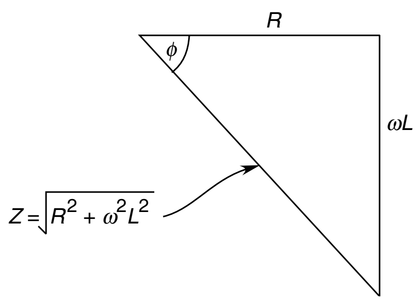


Fig. 2.72 *Impedance triangle*

Taking the Laplace inverse of Eq. (iii), we get

$$i(t) = -\frac{V_m}{Z}\cos\phi\, e^{-(R/L)t} + 2\,\mathrm{Re}\left[\frac{V_m}{2Z}e^{j(\omega t-\phi)}\right]$$

or

$$i(t) = \frac{V_m}{Z}[\cos(\omega t-\phi) - \cos\phi\, e^{-(R/L)t}];\ t>0 \tag{v}$$

At steady-state the second term (transient) of Eq. (v) decays and the steady-state term remains, i.e.

$$i(t=\infty) = \frac{V_m}{Z}\cos(\omega t-\phi) \tag{vi}$$

Observe that the response is cosinusoidal with the same frequency as the input, except that it has a magnitude determined by the **circuit impedance** and has a phase angle (lagging) given by $\tan^{-1}\omega L/R$.

Problems

2.1 Expand the periodic signal $f(t)$ shown in Fig. P-2.1 into the exponential Fourier series. What percentage of the total power is contained up to $n = 3$. Also write the expression for the Fourier spectrum of the periodic wave.

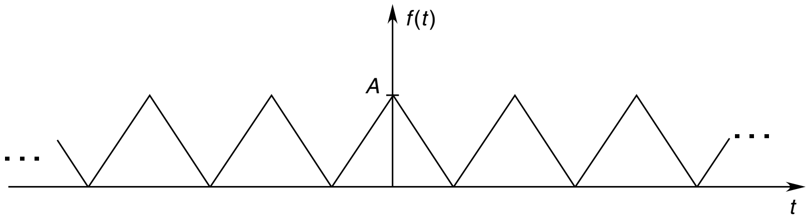


Fig. P-2.1

2.2 Determine the coefficient in the complex Fourier series for the signal shown in Fig. P-2.2.

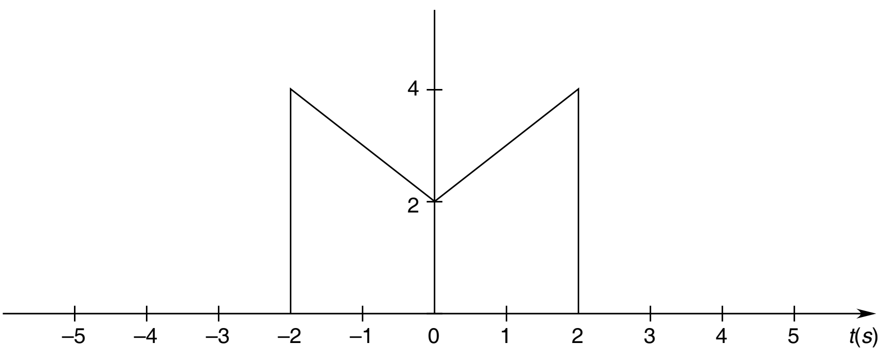


Fig. P-2.2

2.3 If $f(t) = 2 + 3 \cos (10\pi t + 30°) + 4 \cos (20\pi t + 60°) + \cos (30\pi t + 90°)$, then find

(a) the average value of $f(t)$
(b) the effective value of $f(t)$
(c) fundamental period of $f(t)$; and
(d) the value of $f(t)$ at $t = 0.05$s.

2.4 Obtain the trigonometric form of the Fourier series, give the value of T and determine the average value for each of these periodic functions of time.

(a) $2.7 \sin^2 90\pi t$
(b) $2.7 \sin^3 90\pi t$
(c) $2.7 \cos 89\pi t - 1.2 \sin 90\pi t$

2.5 A periodic function is known to have even symmetry and the amplitude spectrum is shown in Fig. P-2.5.

(a) Determine the Fourier series for $f(t)$ if a_n and $b_n \geq 0$ for all n.
(b) Find the effective value of $f(t)$.
(c) Specify the instantaneous value of $f(t)$ at $t = 0.04$ s.

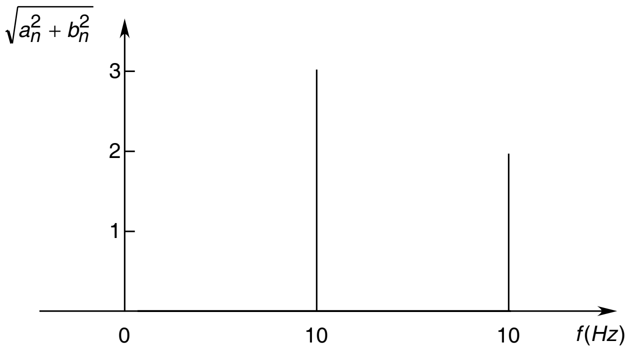


Fig. P-2.5

2.6 For each of the waveform sketched in Fig. P-2.6, state whether or not it has even symmetry, odd symmetry, half-wave symmetry or some combination of these symmetries.

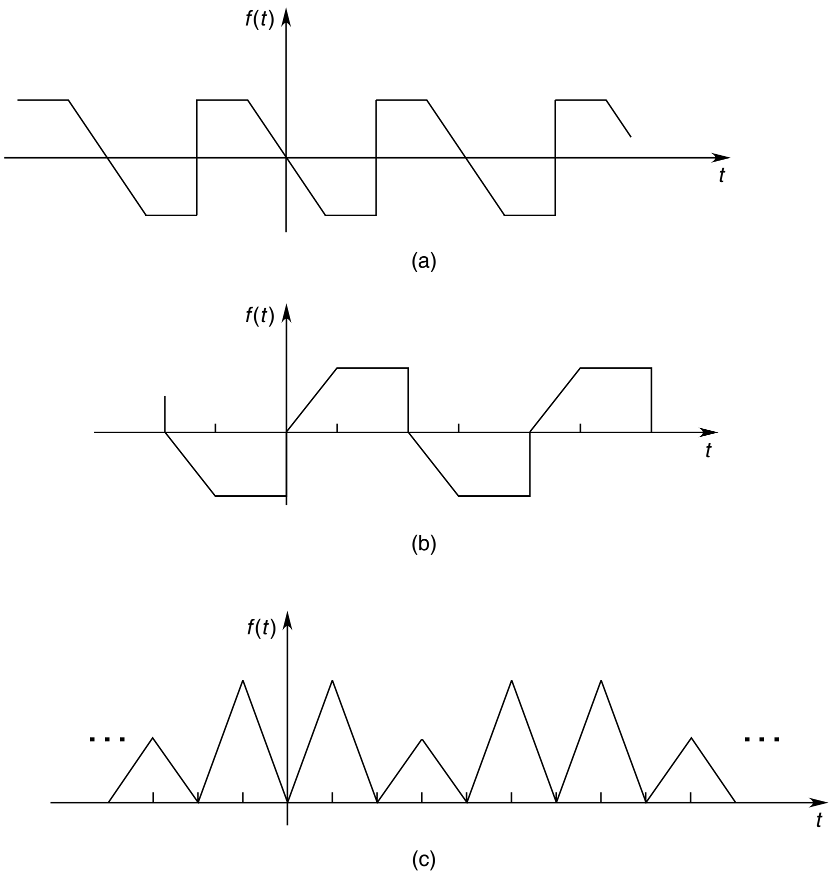


Fig. P-2.6

2.7 A continuous time periodic signal is real-valued and its fundamental period is $T = 16$. The non-zero exponential Fourier series coefficients are

$$F_i = 2j,\ F_3 = 4$$

Express $f(t)$ in cosinusoidal form

2.8 A continuous time periodic signal is

$$f(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{7\pi}{3}t\right)$$

Determine its fundamental period and frequency. Express it in exponential Fourier series form.

2.9 A continuous-time periodic signal's period is

$$f(t) = \begin{cases} 2 & 0 \le t < 1 \\ -2 & 1 \le t < 2 \end{cases}$$

Express its Fourier series coefficients in exponential form.

2.10 A periodic wave with $T = 20$ ms has complex Fourier coefficients given by $F_0 = 1$ A and $F_n = (1/n^2) - j(2/n^3)$ A for $|n| \ge 1$. A = ampere

(a) Sketch a line spectrum of $|F_n|$.
(b) What average power does this current deliver to 5-Ω resistor.

2.11 A voltage wave form $v_s(t)$ has a period of 1/12 s and is defined by $v_s = 60$ V; $0 < t < \frac{1}{96}$ s and $v_s = 0$; $\frac{1}{96} < t < \frac{1}{12}$ *s*. Find (a) F_4 (b). The voltage v_s is applied as the source in the circuit shown in Fig. P-2.11. What average power is delivered to the load?

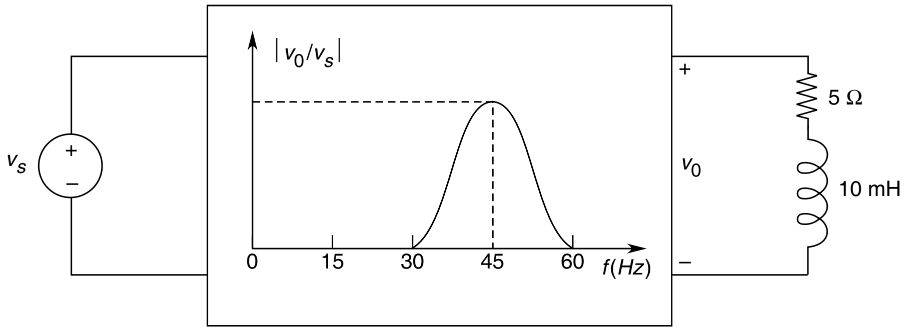


Fig. P-2.11

2.12 Deduce the Forier transform of the functions shown in Fig. P-2.12.

2.13 Deduce the Fourier transform of the function.

(a) $f(t) = e^{-t}\, u(t)$ (b) $f(t) = e^{-|t|}\, u(-t)$

2.14 Determine the Fourier transform of the given function and give the values of $|F(\omega)|$ as $\omega \to \infty$

$$f(t) = e^{-t}u(t) + \delta(t-2)$$

2.15 Determine the Fourier transform of the pulse function, shown in Fig. P-2.15

$$f(t) = \begin{cases} 1 & ; \quad |t| < 1 \\ \frac{1}{2} & ; \quad t = 0 \\ 0 & ; \quad |t| > 1 \end{cases}$$

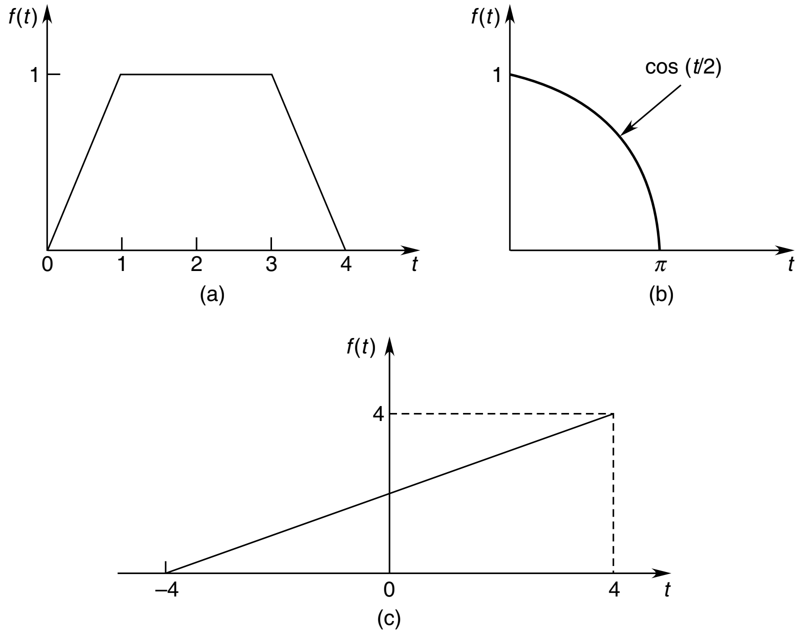


Fig. P-2.12

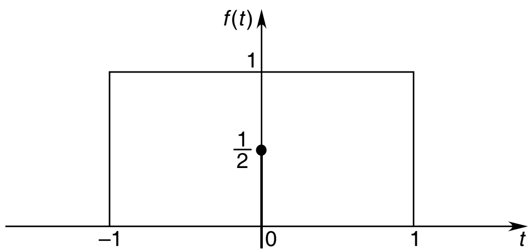


Fig. P-2.15

and show that

$$f(t) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\omega \cos\omega t}{\omega}\,\mathrm{d}\omega$$

2.16 Define the energy of a signal $f(t)$ on the interval $[t_1, t_2]$ as

$$\text{energy} = \int_{t_1}^{t_2} f^2(\tau)\,\mathrm{d}\tau$$

(a) What is the energy of the sum of two functions $f_1(t)$ and $f_2(t)$ on $[t_1, t_2]$?
(b) What is the energy of $f_1(t) + f_2(t)$ if $f_1(t)$ and $f_2(t)$ are orthogonal on (t_1, t_2)?

2.17 An electric circuit is excited by a voltage $v(t)$ as

$$v(t) = v_0 + \sum_{n=1}^{\infty} v_n \cos(n\omega_0 t + \theta_n)$$

The corresponding steady-state current is

$$i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t + \phi_n)$$

Define input power at the input terminals as

$$P = \frac{1}{T}\int_{-T/2}^{T/2} v(t)i(t)\mathrm{d}t\,;\ T = 2\pi/\omega_0$$

Show that the input power can also be written as

$$P = V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2}\cos(\theta_n - \phi_n)$$

2.18 The Fourier transform of $f(t)$ and $g(t)$ are defined below:

$$G(j\omega) = \begin{cases} \cos\omega & ; |\omega| < \pi/2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$F(j\omega) = G(j\omega - j\omega_0) + G(j\omega + j\omega_0)$$

(a) Find $g(t)$ in closed form.
(b) Find $f(t)$ in closed form.
(c) How fast should $g(t)$ be sampled for perfect reconstruction.
(d) In the demodulation scheme shown in Fig. P-2.18, can you find A, ω_1, ω_2 so that $y(t) = g(t)$.

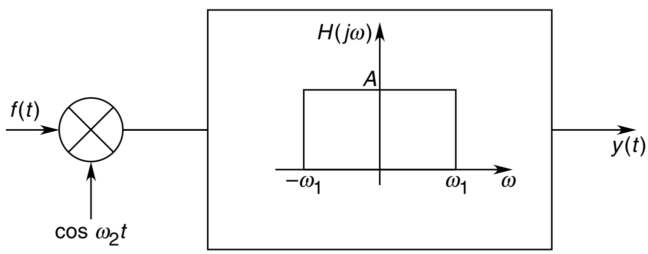


Fig. P-2.18

2.19 Find the time function $g(t)$ for the frequency spectra $G(j\omega)$ shown in Fig. P-2.19(a) and (b). Use properties of Fourier transform to help determine the time function.

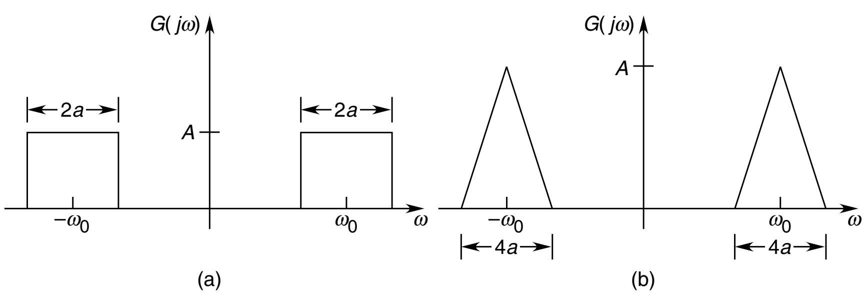


Fig. P-2.19

2.20 Sketch the frequency spectrum of $y(t)$ for the following modulation system where $f(t) = 2\cos 10t + 4\cos 20t$ and $f_m(t) = 200t$.

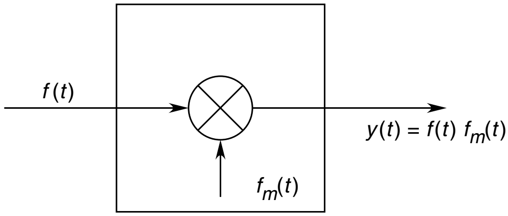


Fig. P-2.20

2.21 Use the energy theorem, i.e.,

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega$$

to find the value of

(a) $\int_{-\infty}^{\infty} \text{sinc}^2(t)\,dt$ (b) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

2.22 A time function $f(t) = \text{sinc}\, 4\pi$ is sampled every $2\pi/3$ second. Find and plot (to scale) the frequency spectrum of the sampled time function $f_s(t)$.

2.23 Plot the output wave form for an input wave form $r(t)$ shown in Fig. P-2.23. Assume the sampler to be ideal with an output $r_s(t)$ given by

$$r_s(t) = r(t)\sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} r(nT)\,\delta(t - nT)$$

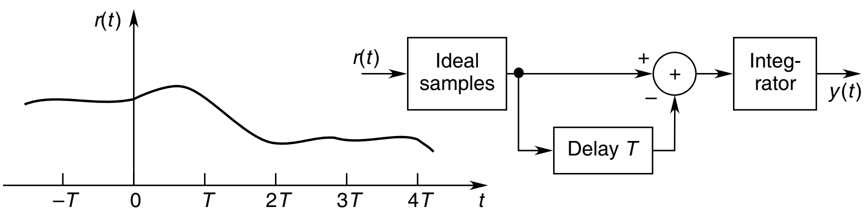


Fig. P-2.23

2.24 A band-limited signal f (t) of maximum frequency $10^3/2\pi$ Hz, modulated with an impulse train of period $T = \frac{1}{2} \times 10^{-3}$ s passed through a band-pass filter of band 2×10^3 to 24×10^3 Hz. Sketch the output spectrum and write down the time-domain expression for the output.

2.25 A signal $f(t) = t\,e^{-t}\,u(t)$ is passed through the following ideal low-pass filter.

$$H(\omega) = G_w(\omega)$$

For what value of W (filter band width W/2) is the energy contained in the filter output exactly half the input energy.

2.26 Obtain the impulse response of the circuit shown in Fig. P-2.26 using the

(a) time-domain method (b) Laplace transform method

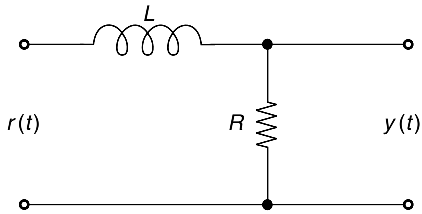


Fig. P-2.26

2.27 Obtain the impulse response of the electrical system shown in Fig. P-2.27. The buffer amplifier has infinite input impedance and zero output impedance and unit gain.

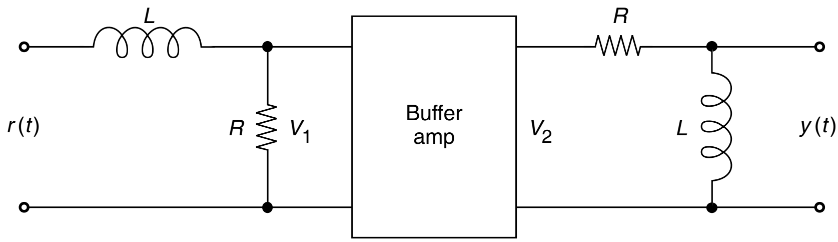


Fig. P-2.27

2.28 For the RL filter shown in Fig. P-2.28, find the impulse response. Using this response find its step response.

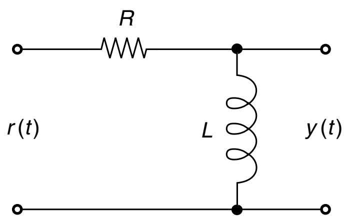


Fig. P-2.28

2.29 Obtain the pulse response of the system of Fig. P-2.29 where the unit pulse is defined as

$$r(t) = p(t-1)\ ;\ \text{pulse width} = 1\text{s}$$

Sketch the response for R/L =1.0s and 0.1s

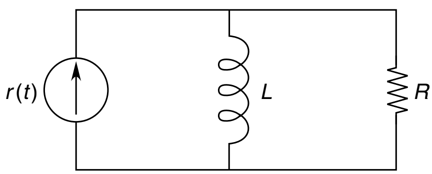


Fig. P-2.29

2.30 Consider the filter circuit of Fig.P-2.30.

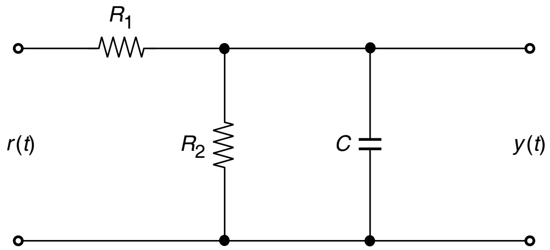


Fig. P-2.30

(a) Write the input/output relationship.
(b) Obtain its impulse response.
(c) From the result of part (b) find the step response.
(d) From the result of part (c) find the response to the unit pulse

$$r(t) = p\left[\left(t - \frac{1}{2}\right)\right] \text{; pulse width = 1s}$$

2.31 Find the Laplace transform and their region of convergence for the following signals.

(a) $(1 - e^{-3t})\, u(t)$

(b) $\delta(t) - \delta(t - 5)$

(c) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \text{otherwise} \end{cases}$

(d) $\dfrac{1}{2a^3}\,(\sin at - at \cos at)\, u(t)$

2.32 Find the Laplace transform and region of convergence of the time function

$$x(t) = e^{-5t}\, u(t) - e^{-4(t-1)}\, u(t-1)$$

2.33 Find the Laplace transform of

(a) $\cos 100\,\pi t\; u(t)$

(b) $\sin 100\,\pi t\; u(t)$

(c) $\sqrt{2} \cos (100\,\pi t - \pi/4)\; u(t)$

2.34 Obtain the Laplace transform of the triangular signal of Fig. P-2.34 using differentiation and time delay theorems.

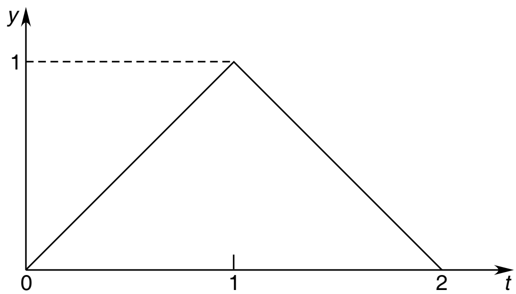


Fig. P-2.34

2.35 Consider the circuit of Fig. P-2.35. The switch has been in position ‘1’ for long time. It is thrown to position ‘2’ at $t = 0$. Solve for the current $i(t)$.

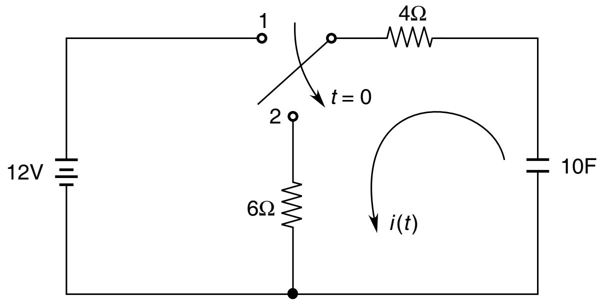


Fig. P-2.35

2.36 The circuit of Fig. P-2.36 is quiescent at $t = 0$ when it is unit impulse excited. Find $v(t)$, $t > 0$.

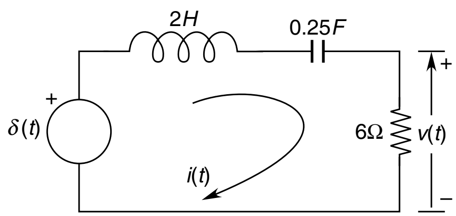


Fig. P-2.36

2.37 Find the inverse Laplace transform of

(a) $\dfrac{s+10}{s^2+8s+20}$ (b) $\dfrac{s}{s^2+6s+18}$

Hint: Instead of partial functioning use suitable theorems.

2.38 By use of appropriate theorems (if applicable) find the initial and final values of

(a) $\dfrac{s+3}{s^2+2s}$ (b) $\dfrac{s+10}{s^2+3s+2}$ (c) $\dfrac{s^2+5s+7}{s^2+3s+2}$

2.39 Obtain the inverse Laplace transform of

$$\frac{s^2\,(s+8)}{(s+2)^3\,(s+1)}$$

2.40 Using appropriate theorems find $Y(s)$ for the following system equations. Given $R(s) = s/(s + 2)$

(a) $y(t) = r(3t - 1)\,u(3t - 1)$ (c) $y(t) = r(t)*r(t)$

(b) $y(t) = e^{-2t}\,r(t)$ (d) $y(t) = r(t/2) + 3r(4t)$

2.41 Prove the following results

(a) £ $y(t/a) = aY(s)$, $a > 0$ (b) £ $[t\,y(t)] = \dfrac{dY(s)}{ds}$

2.42 Find the inverse Laplace transform of

(a) $F_1(s) = \dfrac{1-e^{-2s}}{s+4}$ (b) $F_2(s) = \dfrac{1}{(s+1)^2}$ (c) $F_3(s) = \dfrac{s+4}{s^2+10s+24}$

2.43 For initial and final values of $F_1(s)$, $F_2(s)$ and $F_3(s)$ in Problem 2.38. Verify the results using initial and final values theorems and directly from their inverse transforms.

2.44 Find the Laplace transform of the wave form shown in Fig. P-2.44.

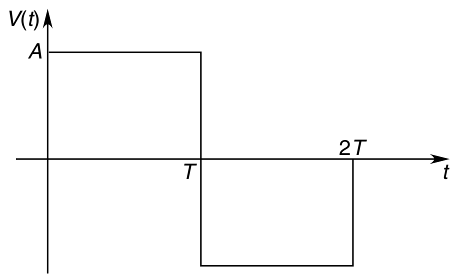


Fig. P-2.44

2.45 In the circuit of Fig. P-2.45 the switch S has been open for a long time and is closed at $t = 0$. Find $i(t)$, $t > 0$.

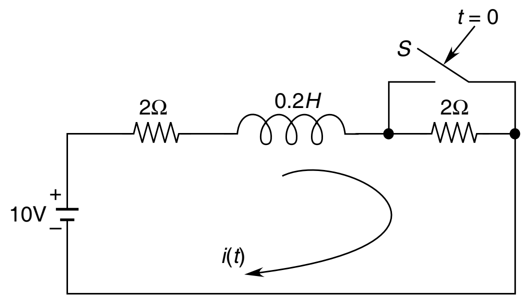


Fig. P-2.45

2.46 In the circuit of Fig. P-2.46 the switch has been in position '*a*' for a long time. It is thrown to position '*b*' at $t = 0$. Find $v(t)$, $t > 0$.

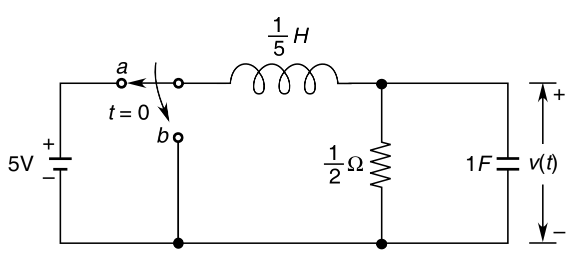


Fig. P-2.46

2.47 For the circuit shown in Fig. P-2.47, determine

(a) $V_0(s)/V_s(s)$ (b) $v_0(t)$ if $v_s(t) = u(t)$ (c) $v_0(t)$ if $v_s(t) = \cos t\, u(t)$

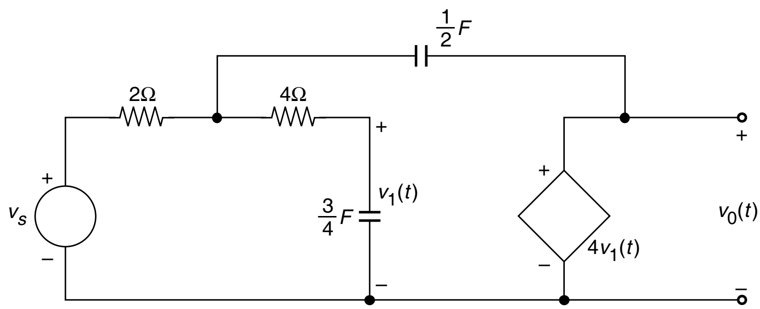


Fig. P-2.47

2.48 Find the Laplace inverse of the following

(a) $\dfrac{e^{sT}}{(s+1)^3}$ (b) $\dfrac{2}{(s^2+1)^2}$

(c) $\dfrac{1}{s^4-a^2}$ (d) $\dfrac{1}{s(s^2+a^2)^2}$

2.49 A system is described by the following differential equation.

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = r(t)$$

$$y(0) = 0,\ \dot{y}(0) = 1$$

The input $r(t)$ is sketched in Fig. P-2.49.

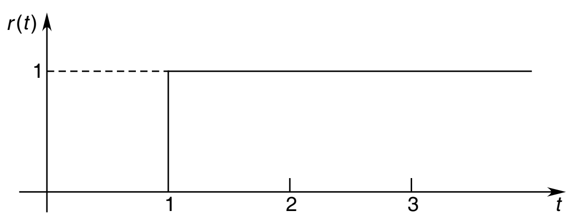


Fig. P-2.49

2.50 Find the Laplace transform of the stepped wave form sketched in Fig. P-2.50.

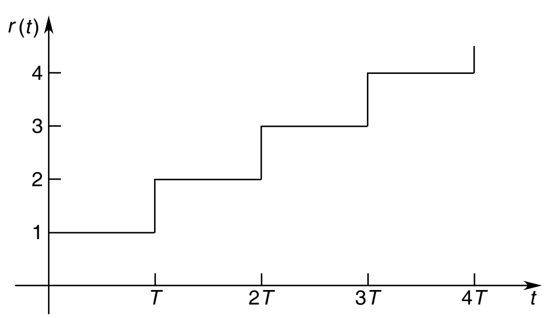


Fig. P-2.50

Analysis of LTI Discrete–Time Systems —Time Domain and Frequency Domain 3

Introduction

The concept of the discrete-time signals and systems were introduced in Chapter 1. It was said there that a discrete-time signal is natural to certain systems like financial systems, data collection, etc. In physical systems a continuous-time signal can be converted to discrete-time form so that it can be processed digitally.

In Section 2.11 it was shown through Fourier transform of sampled signals that to allow undistorted signal reconstruction, the sampling rate must satisfy Shanon's theorem. It was also shown how a simple filter can be used to reconstruct the signal from its sampled form.

Furthur, it was also presented in Chapter 1 that a discrete-time system can be modelled by a difference equation relating the output $y(k)$ to the input $r(k)$. For a LTI causal discrete-time system the difference equation model is reproduced below (see Eq. (1.50)).

$$\begin{aligned} & y(k+n) + a_{n-1}\, y(k+n-1) + \ldots + a_1\, y(k+1) + a_0\, y(k) \\ & \quad = b_m r(k+m) + \ldots + b_1 r(k+1) + b_0 r(k);\ m \le n \end{aligned} \tag{3.1a}$$

An often-used alternative form for the difference equation model is

$$\begin{aligned} & y(k) + a_{n-1}\, y(k-1) + \ldots + a_1\, y(k-\overline{n-1}) + a_0\, y(k-n) \\ & \quad = b_m r(k) + b_{m-1} r(k-1) + \ldots + b_1 r(k-\overline{m-1}) \\ & \qquad + b_0 r(k-m);\ m \le n \end{aligned} \tag{3.1b}$$

In this chapter, the time domain and frequency domain analysis of LTI discrete-time signals and systems will be discussed. In these systems, we deal with k associated with time which stands for kT, where T = sampling period. In general, k could also represent signals not associated with time.

3.1 PROPERTIES OF DISCRETE-TIME SEQUENCES

It is to be understood that input signal to a discrete-time system is essentially discrete in nature. Any discrete-time input sequence can be represented by elementary discrete sequences. Some of the useful elementary sequences are defined below.

(a) *Delta Sequence* It is defined as

$$\{\delta(k)\} = \{\ldots\ 0\ 0\ \underset{\substack{\uparrow \\ k=0}}{1}\ 0\ 0\ \ldots\} \tag{3.2a}$$

or in a compact form, it can be defined in the following manner.

$$\delta(k) = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ otherwise} \end{cases} \tag{3.2b}$$

Graphically, a delta sequence is represented in Fig. 3.1.

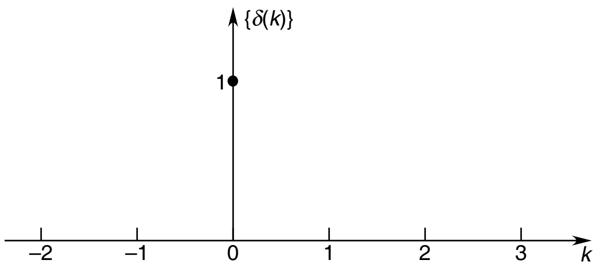


Fig. 3.1 *Delta sequence*

A delta sequence is indeed a *unit discrete impulse*.

(b) *Unit Step Sequence* It is defined as

$$\{u(k)\} = \{\ldots\ 0\ 0\ \underset{\substack{\uparrow \\ k=0}}{1}\ 1\ 1\ 1\ 1\ \ldots\} \tag{3.3}$$

or in a compact form, it can be defined in the following manner.

$$u(k) = \begin{cases} 1 \text{ for } k \geq 0 \\ 0 \text{ for } k < 0 \end{cases}$$

The graphical representation of a step sequence is drawn in Fig. 3.2.

Shift Property of Discrete-time Signals

Any discrete-time sequence can be delayed or advanced by shift operation on the sequence. This is illustrated by considering any sequence $\{r(k)\}$ represented graphically in Fig. 3.3.

(c) *Delaying a Sequence* If a sequence $\{r(k)\}$ is to be delayed by n intervals then the zero-th element of the sequence will now start at $k = n$ and other

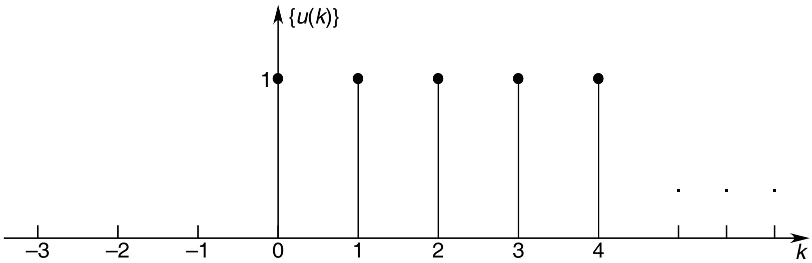


Fig. 3.2 *Step sequence*

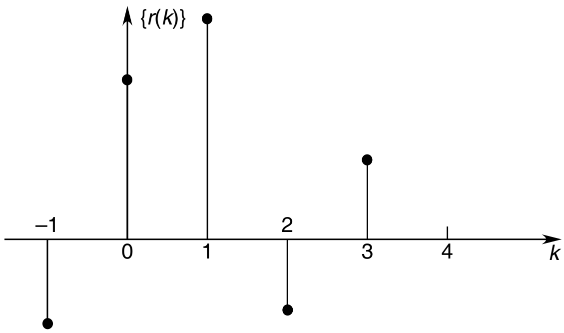


Fig. 3.3 *Shift operation*

elements will shift accordingly as shown in Fig. 3.4. This shift is represented as $\{r(k-n)\}$.

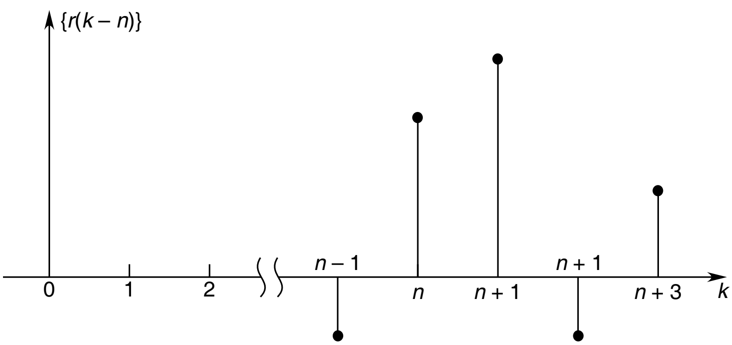


Fig. 3.4 *Delayed shift*

(d) *Advancing a Sequence* If $\{r(k)\}$ is to be advanced by n intervals then the original sequence will be available in advance by n intervals as shown in Fig. 3.5. The shifted sequence is represented as $\{r(k+n)\}$.

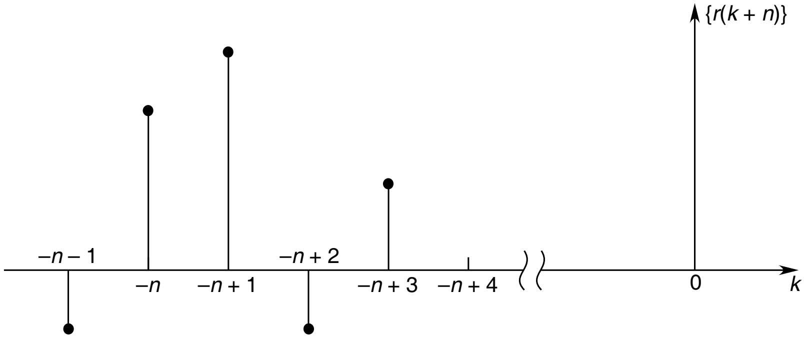


Fig. 3.5 *Advanced shift*

Representation of an Arbitrary Sequence by Elementary Sequences

Any arbitrary sequence $\{r(k)\}$ can be represented in the form of elementary sequences discussed above.

Let the arbitrary sequence be

$$\{r(k)\} = \{...r(-3)\ r(-2)\ r(-1)\ r(0)\ r(1)\ r(2)\ r(3)\ ...\} \tag{3.4}$$

Now the value of signal (sequence) $\{r(k)\}$ at the 3$^{\text{rd}}$ instant can be represented as

$$r(3)\ \{\delta(k-3)\} \tag{3.5}$$

where, the weight $r(3)$ is multiplied by shifted delta sequence to represent the element at the 3$^{\text{rd}}$ instant.

The arbitrary discrete sequence can then be represented by shifted delta sequences as

$$\{r(k)\} = \{\dots\ r(-2)\ \{\delta(k+2)\}\ r(-1)\ \{\delta(k+1)\}\ r(0)\{\delta(k)\}$$
$$r(1)\ \{\delta(k-1)\}\ r(2)\ \{\delta(k-2)\}\ \dots\}$$

or

$$\{r(k)\} = \{r(i)\}\ \{\delta(k-i)\}\} \tag{3.6}$$

This representation will be employed in finding the response of LTI discrete-time systems to arbitrary input sequences.

3.2 LINEAR CONVOLUTION

The convolution of two discrete sequences $\{r(k)\}$ and $\{h(k)\}$ is indicated by a new sequence $\{y(k)\}$ and is symbolically represented in the following way.

$$\{y(k)\} = \{h(k)\} * \{r(k)\} \tag{3.7}$$

Also,

$$\{y(k)\} = \{r(k)\} * \{h(k)\}; \text{ commutation property}$$

where

$\{h(k)\}$ = impulse response of an LTI discrete-time system

$\{r(k)\}$ = arbitrary sequence input to system

$\{y(k)\}$ = system response (output)

Mapping of $\{r(k)\}$ into $\{y(k)\}$ can be written as

$$H_D\{r(k)\} = \{y(k)\} \tag{3.8}$$

H_D is the transform operator which maps $\{r(k)\}$ into $\{y(k)\}$.

For impulse input

$$H_D\{\delta(k)\} = \{h(k)\}; \text{ impulse response} \tag{3.9}$$

The impulse response follows the properties of homogeneity and time invariance. It means that

$$H_D[K\{\delta(k)\}] = K\{h(k)\}; K = \text{constant} \tag{3.10}$$

$$H_D[\{\delta(k \pm \lambda)\}] = \{h(k \pm \lambda)\} \tag{3.11}$$

Using Eqs (3.6) and (3.8), the system response can be expressed as

$$\{y(k)\} = H_D[\{r(i)\}\{\delta(k-i)\}\}$$

As $r(i)$ is just the weight of $\{\delta(k-i)\}$, H_D operation will perform only on the inner sequence, i.e.,

$$\{y(k)\} = H_D[r(i)\{\delta(k-i)\}] = \{r(i)\{h(k-i)\}\}$$

Using the superposition theorem to get the output sequence, we add all the terms, such that we get,

$$\{y(k)\} = \sum_{i=-\infty}^{\infty} r(i)\{h(k-i)\} \tag{3.12}$$

For the k^{th} element of output sequence, all $r(i)$ weighted by $h(k-i)$ need to be added. Therefore, k^{th} element is given by

$$y(k) = \sum_{i=-\infty}^{\infty} r(i)h(k-i) \tag{3.13}$$

The expression of Eq. (3.13) is known as the convolution sum or simply *convolution*, which yields the response of an LTI discrete-time system to an arbitrary input sequence, if its impulse response is known.

For a causal system, Eq. (3.13) takes a simpler form, as input sequence $r(k)$ is applied to the system for $k \geq 0$. Thus

$$y(k) = \sum_{i=0}^{k} r(i)h(k-i) \tag{3.14}$$

Linear convolution for discrete-time systems discussed above is an important mathematical tool to obtain the zero-state system response to an arbitrary input. It is applicable to both causal and noncausal systems.

In convolution sum of Eqs (3.13) (noncausal system) and (3.14) (causal system), i is a convolution index and any index like j could be used in its place.

Example 3.1

Given $\{h(k)\} = (½)^k \{u(k)\}$

and $\{r(k)\} = \{u(k)\}$

Find $y(k)$ for $k = 2$ and 3, by the convolution sum method.

Solution

(1) From Eq. (3.14)

$$y(k) = \sum_{j=0}^{k} r(j)\,h(k-j); \text{ for } k \geq 0$$

For $k = 2$

$$y(2) = \sum_{j=0}^{2} r(j)\,h(2-j)$$

$$= r(0)\,h(2) + r(1)\,h(1) + r(2)\,h(0)$$

$$= (1/2)^2 + 1/2 + 1 = 7/4$$

Similarly, for $k = 3$

$$y(3) = \sum_{j=0}^{3} r(j)\,h(3-j)$$

$$= r(0)\,h(3) + r(1)\,h(2) + r(2)\,h(1) + r(3)\,h(0)$$

$$= (1/2)^3 + (1/2)^2 + (1/2) + 1 = 15/8$$

Algorithm for the Convolution Sum The convolution operation of Eq. (3.13) can be performed as follows.

(i) Take a mirror image of $\{h(j)\}$ to obtain $\{h(-j)\}$.
(ii) Shift $\{h(-j)\}$ to obtain $\{h(k-j)\}$.
(iii) Multiply the sequences $\{r(k)\}$ and $\{h(k-j)\}$.
(iv) Add the sequence obtained in step (iii) to get $y(k)$.
(v) Repeat steps (ii) to (iv) for various values of k to obtain $\{y(k)\}$.

All these procedures will now be demonstrated graphically by means of an example.

Example 3.2 *Solve Example 3.1 graphically.*

Solution We follow the steps given in the algorithm.

(a) Obtain the image of $\{h(j)\}$, i.e., $\{h(-j)\}$ as shown in Figs 3.6 (a) and (b).

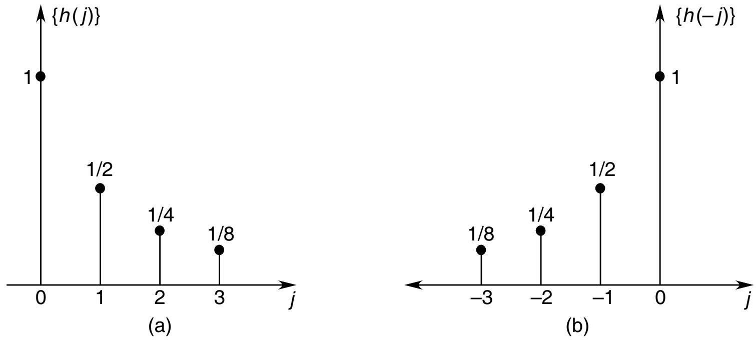


Fig. 3.6

(b) Obtain $\{h(2-j)\}$ by shifting $\{h(-j)\}$ by $k = 2$ steps as shown in Fig. 3.7 (a). The input sequence $\{r(j)\}$ is sketched in Fig. 3.7 (b).

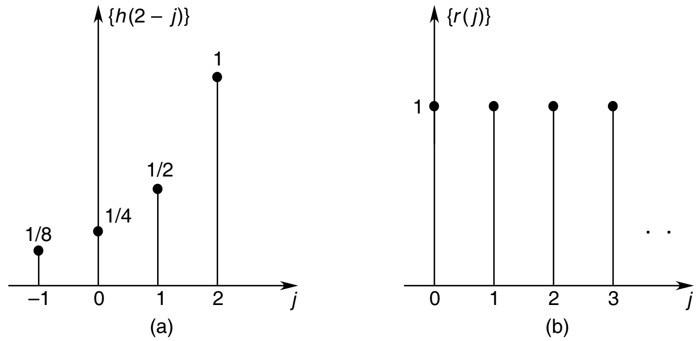


Fig. 3.7

(c) Multiply the sequence $\{r(j)\}$ and $\{h(2 - j)\}$ as in Fig. 3.8 (a). Add the resultant sequence values to obtain $y(2)$ which is shown in Fig. 3.8 (b).

(d) The convolution steps are repeated for $k = 3$.

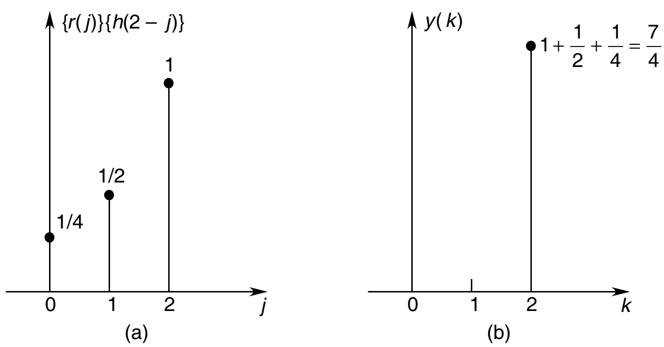


Fig. 3.8

Properties of Convolution

These properties apply to both discrete-time/continuous-time systems.

(a) *Commutative Property*

$$y = h * r$$

and

$$y = r * h \tag{3.15}$$

(b) *Distributive Property*

$$h * [r_1 + r_2] = h * r_1 + h * r_2 \tag{3.16}$$

(c) *Associative Property*

$$r_1 * (r_2 * h) = (r_1 * r_2) * h \tag{3.17}$$

In the above equations, (k) and the symbol {} have been dropped but would be used for discrete-time systems as all the signals involved are sequences.

Example 3.3 *Find the unit impulse response $h(k)$ of a system described by the following difference equation.*

$$h(k+2) - 6h(k+1) + 9h(k) + \delta(k+1) + 18\delta(k) = 0$$

Solution As the system is causal, $h(k) = 0$, $k < 0$.

Substituting values of k in ascending order in the given difference equation, we get

$$\begin{aligned}
k &= -2, & h(0) &= 6h(-1) - 9h(-2) + \delta(-1) + 18\delta(-2) = 0 \\
k &= -1, & h(1) &= 6h(0) - 9h(-1) + \delta(0) + 18\delta(-1) = 1 \\
k &= 0, & h(2) &= 6h(1) - 9h(0) + \delta(1) + 18\delta(0) \\
& & &= 6 + 18 = 24 \\
k &= 1, & h(3) &= 6h(2) - 9h(1) + \delta(2) + 19\delta(1) \\
& & &= 6 \times 24 - 9 = 135
\end{aligned}$$

Continuing the above process for $k > 1$, $\{h(k)\}$ can be computed up to any desired value of k.

3.3 DISCRETE-TIME LTI SYSTEMS DESCRIBED BY DIFFERENCE EQUATIONS

The general form of the describing difference equation with constant coefficients can be expressed as

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{3.18}$$

where $x(\cdot)$ is input and $y(\cdot)$ is output

Like in the continuous-time differential equation, the solution of the difference equation is the sum of two parts; the *particular solution* of Eq. (3.18) and the solution of the *homogeneous equation*

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \tag{3.19}$$

and constitutes the natural response of the system.

However, for the difference equation, an alternative method of solution is to rearrange Eq. (3.18) as

$$y(n) = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} b_k\, y(n-k)\right] \tag{3.20}$$

From this equation, we can obtain $y(n)$ in terms of the previous values of input and output. It is immediately seen from Eq. (3.20) that to start the solution at $n = 0$ all the input values are known but we need to know the previous output values $y(-1)\, y(-2), \ldots, y(-N)$ which indeed are the *auxiliary* or *initial conditions*.

An equation of the form (3.19) or (3.20) is a *recursive equation* as it can be solved recursively.

In our further discussion, we shall use zero auxiliary conditions, i.e., if

$$x(n) = 0 \text{ for } n < n_0 \text{ then } y(n) \text{ zero for } n < n_0.$$

Let us consider the special case of $N = 0$ in Eq. (3.20).

$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_0}\, x(n-k) \tag{3.21}$$

This is a nonrecursive equation as it needs only past and present values of input but does not require the previously computed values of $y(n)$.

By convolving input with discrete impulse $\delta(n)$, the impulse response of the system is

$$h(n) = \begin{cases} \dfrac{b_k}{a_0} & ;\ 0 \le n \le M \\ 0 & ;\ \text{otherwise} \end{cases} \tag{3.22}$$

As the system's impulse response is limited to a finite interval (0 to M), it is known as a *finite impulse response* (FIR) *system*.

Example 3.4 *Consider the difference equation*

$$y(n) - \frac{1}{3}\, y(n-1) = x(n) \tag{i}$$

Given: $$x(n) = k\,\delta(n);\ \text{discrete impulse input} \tag{ii}$$

We want to determine the sequence $y(n)$ with initial rest conditions which imply that if $x(n) = 0$ for $n \le -1$ then $y(n) = 0$ for $n \le -1$.

We write Eq.(i) in recursive form

$$y(n) = k\delta(n) + \frac{1}{3}y(n-1) \tag{iii}$$

$n = 0$ $\qquad y(0) = k \cdot 1 + 0 = k$

$n = 1$ $\qquad y(1) = 0 + \frac{1}{3}y(0);\ \delta(n) = 0 \quad \text{for } n \neq 0$

$$= \frac{1}{3}k$$

$n = 2$ $\qquad y(2) = 0 + \frac{1}{3} \cdot y(1)$

$$= \left(\frac{1}{3}\right)^2 k$$

Thus $\qquad y(n) = \left(\frac{1}{3}\right)^n k;\ n \geq 0$

or

$$y(n) = \left(\frac{1}{3}\right)^n u(k) \tag{iv}$$

If the input is unit impulse $x(n) = \delta(n)$, the system's impulse response is

$$h(n) = \left(\frac{1}{3}\right)^n u(n)$$

As $n \to \infty$, $h(\infty) \to 0$. It is a *stable* system we observe that the impulse response extends over infinity, so this is *infinite impulse response (IIR) system*.

Example 3.5 *The impulse response of a system is given by the difference equation*

$$h(k) - 6h(k-1) + 9h(k-2) - \delta(k-1) - 8\delta(k-2) = 0$$

The system is causal so that $h(k) = 0$ for $k \leq -1$

Rewriting in recursive form

$$h(k) = \delta(k-1) + 3\delta(k-2) + 6h(k-1) - 9h(k-2)$$

Substituting for k in ascending order

$k = 0$ $\qquad h(0) = 0$

$k = 1$ $\qquad h(1) = 1$

$k = 2$ $\qquad h(2) = 18 + 6h(1) = 18 + 6 = 24$

$k = 3$ $\qquad h(3) = 6h(2) - 9h(1)$

$$= 6 \times 24 - 9 \times 1 = 135$$

Thus, $h(k)$ can be computed up to any value of $k \geq 0$.

Example 3.6 *Consider the difference equation*

$$y(n) - \frac{1}{3}\,y(n-1) = x(n)$$

Find $$y(n) \text{ if } x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Proceed by finding homogenous solution and particular solution.

Solution: Homogeneous equation

$$y_h(n) - \frac{1}{3}\,y_h(n-1) = 0 \tag{i}$$

We assume the solution of the form

$$y_h(n) = A\left(\frac{1}{3}\right)^n \tag{ii}$$

Plugging in Eq. (i), we have

$$A\left(\frac{1}{3}\right)^n - A\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} = 0$$

It is satisfied.

Particular solution

$$y_p(n) - \frac{1}{3}\,y_p(n-1) = \left(\frac{1}{3}\right)^n u(n) \tag{iii}$$

The solution should have the form (same form as input)

$$y_p = B\left(\frac{1}{2}\right)^n u(n) \tag{iv}$$

Substituting in Eq. (iii),

$$B\left(\frac{1}{2}\right)^n - \frac{1}{3}\,B\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

$$B\left(\frac{1}{2}\right)^n - \frac{2}{3}\,B\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n$$

$$B - \frac{2}{3}B = 1$$

or $$B = 3$$

Then $$y_p = 3\left(\frac{1}{2}\right)^n u(n)$$

The complete solution is

$$y(n) = y_h(n) + y_p(n)$$

or $$y(n) = \left[A\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n\right] u(n) \tag{v}$$

From the given difference equation,

$$y(0) - \frac{1}{3}y(-1) = x(0)$$

or $$y(0) = x(0) = 1$$

Using this initial condition in Eq. (v),

$$y(0) = 1 = A + 3$$

or $$A = -2$$

Hence the complete solution is

$$y(n) = \left[-2\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n\right]u(n) \qquad \text{(vi)}$$

3.4 FOURIER SERIES OF DISCRETE-TIME PERIODIC SIGNALS

As defined in Chapter 1, a discrete-time periodic signal of period N is

$$x(n) = x(n + N) \qquad (3.23)$$

The smallest value of the positive integer N is the *fundamental period* and $\omega_0 = 2\pi/N$ is the *fundamental frequency*.

Consider the set of discrete-time complex exponentials

$$\phi_k(n) = e^{jk\omega_0 n} = e^{jk(2\pi/N)n};\ k = 0, \pm 1, \pm 2, \ldots \qquad (3.24)$$

$k = 0$ is the dc sequence, $k = \pm 1$ corresponds to the *fundamental frequency* and the integral multiples of the fundamental frequency are its *harmonics*.

There are only N distinct signals in the set of signals in Eq. (3.24) as

$$\begin{aligned}\phi_{k+rN}(n) &= e^{j(k+rN)(2\pi/N)n} = e^{j2\pi r}\, e^{jk(2\pi/N)n} \\ &= e^{jk(2\pi/N)n} = \phi_k(n)\end{aligned} \qquad (3.25)$$

It means that the discrete periodic terms differing in frequency by 2π are identical.

Consider now a linear combination of harmonically related periodic sequences, which has the form

$$x(n) = \sum_k A_k\, \phi_k(n) = \sum_k A_k\, e^{jk\omega_0 n} = \sum_k A_k\, e^{jk(2\pi/N)n} \qquad (3.26)$$

As the sequences ϕ_k are distinct only for (N) consecutive values of k, the sum in Eq. (3.26) need only be carried out for $k = (N)$. Thus

$$x(n) = \sum_{k=(N)} A_k\, \phi_k(n) = \sum_{k=(N)} A_k\, e^{jk\omega_0 n} = \sum_{k=(N)} A_k\, e^{jk(2\pi/N)n} \qquad (3.27)$$

where $k = (N)$ means any N consecutive values of k. For example, it could be $k = 0, 1, \ldots, N-1$ or $k = 3, 4, \ldots, N+2$.

Determination of the Fourier Series of a Discrete-time Periodic Signal

Let us write down the N values of $x(n)$ from Eq. (3.27) as

$$
\begin{aligned}
x(0) &= \sum_{k=(N)} A_k \\
x(1) &= \sum_{k=(N)} A_k\, e^{j2\pi k/N} \\
x(2) &= \sum_{k=(N)} A_k\, e^{j2\pi k(2/N)} \\
x(N-1) &= \sum_{k=(N)} A_k\, e^{j2\pi k(N-1)/N}
\end{aligned}
\tag{3.28}
$$

There are a set of N equations in N coefficients A_k. The equations being linearly independent can be solved for A_k in terms of $x(n)$.

We would however, want to obtain a closed-form result as in the case of continuous-time Fourier series. For this purpose, we need the following result:

$$
\sum_{n=(N)} e^{jk(2\pi/N)n} = \begin{cases} N & \text{for } k = 0, \pm N, \pm 2N \\ 0 & \text{Otherwise} \end{cases}
\tag{3.29}
$$

For $\quad k = 0 \quad e^{j0} = 1$

$\quad\quad k = N \quad e^{jN2\pi} = 1$

Hence, the first part of Eq. (3.29)

For $\quad k \neq N$

$$
\sum_{n=(N)} e^{j(k\omega_0)n} = \sum_{n=(N)} 1\ \angle(k\omega_0)n
$$

This is the sum of unit phasors whose angle increases progressively by $(k\omega_0)$ for each step up of n till at $n = N - 1$, the phasor sum is zero for all integral values of k $(k \neq N)$. It is illustrated for $n = 0$ to 5, in which case the angle step is $(360°/6) = 60°$.

The phasor diagram is a regular hexagon with a side length of one unit as shown in Fig. 3.9. Starting from any corner, this phasor addition is zero in six successive steps.

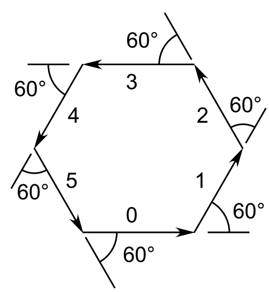


Fig. 3.9 *Phasor diagram*

Closed-form Solution

Multiply both sides of Eq. (3.27) by $e^{-jr(2\pi/N)n}$ and sum for $n = (N)$.

$$
\sum_{n=(N)} x(n)\, e^{-jr(2\pi/N)n} = \sum_{n=(N)} \sum_{k=(N)} A_k\, e^{j(k-r)(2\pi/N)n}
\tag{3.30}
$$

Interchanging order of summation on the right-hand side, gives

$$\sum_{n=(N)} x(n)\, e^{-jr(2\pi/N)n} = \sum_{k=(N)} A_k \underbrace{\sum_{n=(N)} e^{j(k-r)(2\pi/N)n}}_{\substack{= N \quad \text{for } k = r \\ = 0 \quad \text{otherwise}} \Big\} \text{ As per Eq. (3.29)}} \tag{3.31}$$

$$= NA_k$$

We then get

$$A_k = \frac{1}{N} \sum_{n=(N)} x(n)\, e^{-jk(2\pi/N)n} \tag{3.32}$$

This provides the closed-form values of the Fourier coefficients.

The discrete-time Fourier series pair is reproduced below:

Synthesis Equation

$$x(n) = \sum_{k=(N)} A_k e^{jk\omega_0 n} = \sum_{k=(N)} A_k\, e^{jk(2\pi/N)n} \tag{3.33}$$

Analysis Equation

$$A_k = \frac{1}{N} \sum_{n=(N)} x(n)\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=(N)} x(n) e^{-jk(2\pi/N)n} \tag{3.34}$$

Observation It is seen from Eq. (3.34) that unlike the continuous-time Fourier series which is an infinite series, the discrete-time Fourier series has finite number of terms $k = (N)$ beyond which the terms repeat cyclically.

From Eq. (3.34), it immediately follows that

$$A_{k+N} = A_k = A^*_{-k} \tag{3.35}$$

Some Properties of Discrete-time Fourier Series

$$x(n) \xleftrightarrow{\mathcal{F}ds} A_k$$

Linearity Property

$$a_x(n) + b_y(n) \xleftrightarrow{\mathcal{F}ds} aA_k + bB_k$$

$$x(n) \text{ real and even} \longleftrightarrow A_k \text{ real and even}$$

$$x(n) \text{ real and odd} \longleftrightarrow A_k \text{ purely imaginary and odd}$$

Time-Shifting Property

Discrete Fourier series coefficients of $x(n - n_0)$, period N

$$A_k^{n_0} = \frac{1}{N} \sum_{n=(N)} x(n - n_0)\, e^{-jk(2\pi/N)n}$$

Let $$m = n - n_0 \rightarrow n = m + n_0$$

Then $$A_k^{n_0} = \frac{1}{N} \sum_{m+n_0=(N)} x(m)\, e^{-jk(2\pi/N)\,(m+n_0)}$$

$$= e^{-jk(2\pi/N)n_0} \left[\frac{1}{N} \sum_{m=(N)} x(m) e^{-jk(2\pi/N)m} \right]$$

or $$A_k^{n_0} = e^{-jk(2\pi/N)n_0}\, A_k \tag{3.36}$$

First Difference

$$x(n) - x(n-1)$$

Using the time-shifting property, its Fourier series coefficients are

$$(1 - e^{-jk(2\pi/N)})\, A_k \tag{3.38}$$

Parseval's Theorem

$$\frac{1}{N} \sum_{n=(N)} |x(n)|^2 = \sum_{k=(N)} |A_k|^2 \tag{3.39}$$

Note: There are several other properties which need not be listed as they follow directly from the discrete-time Fourier transform (Chapter 4).

Response of Discrete-time Systems to Complex Exponential

Suppose an LTI system with impulse response $h(n)$ is excited by an input signal.

$$x(n) = z^n \tag{i}$$

where z is a complex number.

By *convolution sum*, the output is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h(k)\, z^{-k}$$

or $$y(n) = H(z) z^n \tag{ii}$$

where $$H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k} \tag{iii}$$

Observe the z^n is the *eigen function* and $H(z)$, the *eigen vector*, which modifies z^n to produce $y(n)$.

When $x(n)$ is periodic of form $x(n) = z^n$, $z = e^{j\omega}$. In terms of discrete Fourier series

$$x(n) = \sum_{k=(N)} A_k e^{jk\omega n} = \sum_{k=N} A_k z^{nk};\ z = e^{j\omega} \tag{iv}$$

From Eq. (ii) the output will be

$$y(n) = \sum_{k=(N)} A_k H(z^b) z^{kn} \tag{v}$$

The output $y_k(n)$ for each component of $x(n)$ of Eq. (iv)

$$y_k(n) = A_k H(e^{jk\omega})\, e^{j\omega nk}$$

For illustration, consider an example.

Given: System's impulse response

$$h(n) = \alpha^n u(n);\ |\alpha| < 1$$

Input

$$x(n) = \cos\left(\frac{2\pi n}{N}\right);\ \omega = 2\pi/n$$

$$= \frac{1}{2} e^{j(2\pi/N)n} + \frac{1}{2} e^{-j(2\pi/N)n};\ k = 1$$

From the impulse response ($k = 1$)

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

Summing the geometric series

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

We then get

$$y(n) = \frac{1}{2} H(e^{j2\pi/N})\, e^{j(2\pi/N)n} + \frac{1}{2} H(e^{-j2\pi/N})\, e^{-j(2\pi/N)n}$$

$$= \frac{1}{2}\underbrace{\left[\frac{1}{1 - \alpha e^{-j(2\pi/N)n}}\right] e^{j(2\pi/N)n}}_{v e^{-j\theta}} + \frac{1}{2}\underbrace{\left[\frac{1}{1 - e^{j(2\pi/N)n}}\right] e^{-j(2\pi/N)n}}_{v e^{j\theta}}$$

or $\quad y(n) = v \cos\left(\frac{2\pi}{N} n + \theta\right)$

The procedure is straightforward and applies for any value of ω. Indeed $H(e^{j\omega})$ is the *frequency response* of the *discrete-time system*.

Example 3.7 *A discrete-time signal $x(n)$ is real-valued with fundamental period $N = 5$. Its nonzero Fourier series coefficients are*

$$A_0 = 2,\ A_2 = A^*_{-2} = 2\,e^{j\pi/6},\ A_4 = A^*_{-4} = e^{-j\pi/3}$$

Express $x(n)$ in sinusoidal form.

Solution

$$x(n) = 2 + 2\,e^{j\pi/6}\,e^{j2\left(\frac{2\pi}{5}\right)n} + 2\,e^{-j\pi/6}\,e^{-j2\left(\frac{2\pi}{5}\right)n}$$

$$+ 2\,e^{-j\pi/3}\,e^{j4\left(\frac{2\pi}{5}\right)n} + e^{j\pi/3}\,e^{-j4\left(\frac{2\pi}{5}\right)n}$$

or
$$x(n) = 2 + 4\cos\left(\frac{4\pi}{5}n + \frac{\pi}{6}\right) + 2\cos\left(\frac{8\pi}{5}n - \frac{\pi}{3}\right)$$

Example 3.8 *Consider the discrete-time signal*

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\sin\left(\frac{4\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

Expand it into complex exponential and determine its Fourier coefficients from there. Also, check that (i) the coefficients are periodic, and (ii) positive and negative coefficients are complex conjugate of each other.

Solution:

$$\sin\left(\frac{2\pi}{N}\right)n = \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right]$$

$$A_1 = A_1^* = \frac{1}{2j}$$

$$\sin\left(\frac{4\pi}{N}\right)n = \frac{3}{2j}\left[e^{j2\left(\frac{2\pi}{N}\right)n} - e^{-j2\left(\frac{2\pi}{N}\right)n}\right]$$

$$A_{2,1} = A^*_{-2,1} = \frac{3}{2j}$$

$$\cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right) = \frac{1}{2}\left[e^{j2\left(\frac{2\pi}{N}\right)n}\,e^{j\frac{\pi}{2}} - e^{-j2\left(\frac{2\pi}{N}\right)n}\,e^{-j\frac{\pi}{2}}\right]$$

$$A_{2,2} = A^*_{-2,2} = \frac{1}{2}e^{j\frac{\pi}{2}} = \frac{j}{2}$$

Adding A_2, we get

$$A_2 = \frac{3}{2j} + \frac{j}{2} = \frac{-3j}{2} + \frac{j}{2} = -j = A^*_{-4}$$

Let us take $N = 3$ for illustration. Then we should show that

$$A_5 = A_{2+3} = A_2$$

The corresponding exponential term is

$$e^{j5\left(\frac{2\pi}{3}\right)n} = e^{j3\left(\frac{2\pi}{3}\right)n} \cdot e^{j2\left(\frac{2\pi}{3}\right)n} = e^{j2\left(\frac{2\pi}{3}\right)n}$$

Thus $$A_5 = A_2$$

Example 3.9 *For the discrete-time signal*

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\sin\left(\frac{4\pi}{N}\right)n + \cos\left[\left(\frac{4\pi}{N}\right) + \frac{\pi}{2}\right]$$

Find its discrete-time Fourier series coefficients.

Solution

$$A_0 = 1$$

$$A_1 \text{ as } A_{-1}$$

$$\sin\left(\frac{2\pi}{N}\right)n = \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{j\left(\frac{2\pi}{N}\right)n}\right]$$

Then

$$A_1 = -A_1^* = \frac{1}{2j}$$

$$3\sin\left(\frac{4\pi}{N}\right)n + \cos\left[\left(\frac{4\pi}{N}\right)n + \frac{\pi}{2}\right]$$

$$= \frac{3}{2j}\left[e^{j2\left(\frac{2\pi}{N}\right)n} - e^{-j2\left(\frac{2\pi}{N}\right)n}\right] + \frac{1}{2}\left[e^{j2\left(\frac{2\pi}{N}\right)n} e^{j\frac{\pi}{2}} + e^{-j2\left(\frac{2\pi}{N}\right)n} e^{-j\frac{\pi}{2}}\right]$$

which yields

$$A_2 = A_{-2}^* = \frac{3}{2j} + \frac{1}{2}e^{j\frac{\pi}{2}} = \frac{3}{2j} + j = \frac{1}{2j}$$

Example 3.10 *Given a discrete-time signal*

$$\begin{aligned} x(n) &= 1 \qquad 0 \le n \le 4 \\ &= 0 \qquad 5 \le n \le 6 \end{aligned}$$

Determine its Fourier series coefficients.

Solution It is a periodic signal with period $N = 7$

$$A_k = \frac{1}{N}\sum_{n=(N)} x(n)\, e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(i)}$$

$$= \frac{1}{7}\sum_{n=0}^{4} 1 \cdot e^{-jk\left(\frac{2\pi}{7}\right)n} \tag{ii}$$

$$= \frac{1}{7} \cdot \frac{1 - e^{-jk\left(\frac{2\pi}{7}\right)5}}{1 - e^{-jk\left(\frac{2\pi}{7}\right)}} \tag{iii}$$

$$= \frac{1}{7} \cdot \frac{e^{-jk\left(\frac{5\pi}{7}\right)}\left[e^{jk\left(\frac{5\pi}{7}\right)} - e^{-jk\left(\frac{5\pi}{7}\right)}\right]}{e^{-jk\left(\frac{\pi}{7}\right)}\left[e^{jk\left(\frac{\pi}{7}\right)} - e^{-jk\left(\frac{\pi}{7}\right)}\right]}$$

$$= \frac{1}{7} \cdot e^{-jk\left(\frac{4\pi}{7}\right)}\left[\frac{\sin\left(\frac{5k\pi}{7}\right)}{\sin\left(\frac{k\pi}{7}\right)}\right] \tag{iv}$$

Remark: In going from Eq. (ii) to Eq. (iii), we have used the sequence sum result

$$\sum_{j=0}^{k} a^j = \frac{1 - a^{k+1}}{1 - a}; \ a \neq 1$$

Example 3.11 *For periodic discrete signal x (n) of Example 3.9, determine the sequence*

$$g(n) = x(n) - x(n-1)$$

What is its period? Determine its Fourier series coefficients.

Solution It is easy to find the $g(n)$ sequence is

n	0	1	2	3	4	5	6
$g(n)$	1	0	0	0	0	-1	0

Period $N = 7$

By analysis, Eq. (3.34), it follows that

$$A_k = \frac{1}{7}\left[1 - e^{-j5\left(\frac{2\pi}{7}\right)k}\right]$$

Example 3.12 *A discrete-time periodic signal x (n) of period 8 has the following Fourier series coefficients Determine the signal x (n).*

(a) As in Fig. 3.10. *(b)* $\sin k\frac{\pi}{4} + \cos 3k\frac{\pi}{4}$

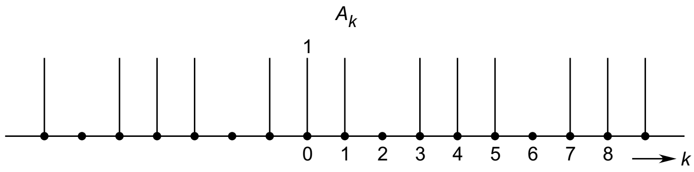


Fig. 3.10

Solution Synthesis equation is

$$x(n) = \sum_{k=(N)} A_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x(n) = \sum_{k=(8)} A_k \, e^{jk\left(\frac{\pi}{4}\right)n}$$

Substituting values

$$x(n) = 1 + 1e^{j\left(\frac{\pi}{4}\right)n} + 1\cdot e^{j\left(\frac{3\pi}{4}\right)n} + 1\cdot e^{j\left(\frac{4\pi}{4}\right)n} + 1\cdot e^{j\left(\frac{5\pi}{4}\right)n} + 1\cdot e^{j\left(\frac{7\pi}{4}\right)n}$$

$$e^{j\left(\frac{4\pi}{4}\right)n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

$$e^{j\left(\frac{\pi}{4}\right)n} + e^{j\left(\frac{7\pi}{4}\right)n} = e^{j\left(\frac{\pi}{4}\right)n} + e^{j2\pi n}\, e^{-j\left(\frac{\pi}{4}\right)n}$$

$$= e^{j\left(\frac{\pi}{4}\right)n} + e^{-j\left(\frac{\pi}{4}\right)n} = 2\cos\left(\frac{\pi}{4}n\right)$$

$$e^{j\left(\frac{3\pi}{4}\right)n} + e^{j\left(\frac{5\pi}{4}\right)n} = e^{j\left(\frac{3\pi}{4}\right)n} + e^{j2\pi}\, e^{-j\left(\frac{3\pi}{4}\right)n}$$

$$= e^{j\left(\frac{3\pi}{4}\right)n} + e^{-j\left(\frac{3\pi}{4}\right)n} = 2\cos\left(\frac{3\pi}{4}n\right)$$

We then get

$$x(n) = 1 + (-1)^n + 2\cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{3\pi}{4}n\right)$$

(b) $\sin k = \dfrac{1}{2j}\left[e^{jk\frac{\pi}{4}} - e^{-jk\frac{\pi}{4}}\right]$

Substituting in the synthesis equation,

$$x_1(n) = \sum_{k=(8)} \frac{1}{2j}\left[e^{jk\frac{\pi}{4}} - e^{-jk\frac{\pi}{4}}\right] e^{jk\frac{\pi}{4}n}$$

$$= \sum_{k=(8)} \frac{1}{2j}\left[e^{jk\frac{\pi}{4}(n+1)} - e^{-jk\frac{\pi}{4}(n-1)}\right]$$

As per the result given in Eq. (3.29),

$$\sum_{k=8} e^{jk\frac{\pi}{4}(n+1)} = 8 \text{ for } n = 7$$
$$= 0 \text{ otherwise}$$

$$\sum_{k=(8)} e^{jk\frac{\pi}{4}(n-1)} = 8 \text{ for } n = 1$$
$$= 0 \text{ otherwise}$$

So $\quad x_1(n) = -4j$ at $n = 7$ and $+4j$ at $n = 1$

This can be expressed as

$$x_1(n) = 4j\delta(n-1)\;4j\delta(n-7)$$

Similarly, it follows that (it is cos term)

Now $\quad x_2(n) = 4\delta(n-3) + 4\delta(n-5)$

$$x(n) = x_1(n) + x_2(n)$$
$$= 4j\;\delta(n-1) + 4\delta(n-3) + 4\delta(n-3) - 4j\delta(n-7)$$

Example 3.13 *The discrete counterpart of $x(t) = \sin \omega_0 t$ is*

$$x(n) = \sin \omega_0 n;\; \omega_0 = 2\pi/N$$

Find its Fourier coefficients.

Solution In this case, we can proceed directly rather than using the analysis equation. By Euler's theorem,

$$x(n) = \frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 t}$$
$$= \frac{1}{2j}e^{j(2\pi/N)n} - \frac{1}{2j}e^{-j(2\pi/N)n}$$

It has only fundamental ($k = 1$) and no harmonics. The corresponding coefficients are

$$A_1 = \frac{1}{2j};\; A_{-1} = -\frac{1}{2j}$$

The coefficients repeat, over period N. Thus,

$$A_{N+1} = A_1 = \frac{1}{2j};\; A_{-(N+1)} = -\frac{1}{2j}$$

Example 3.14 *For the discrete-time square wave (Fig. 3.11), determine the expression for its Fourier coefficients.*

Solution As per analysis equation,

$$A_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n} \tag{i}$$

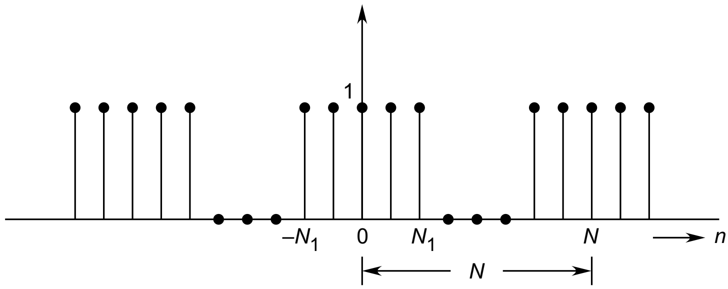


Fig. 3.11

For case of summation, we shift the origin to $-N_1$ by defining

$$m = n + N_1$$

The summation of Eq. (i) the modifies to

$$A_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)\,(m-N_1)}$$

or
$$A_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m} \tag{ii}$$

Using $\sum_{j=0}^{k} a^j = \dfrac{1-a^{k+1}}{1-a}$; $a \neq 1$ the geometric series sum, the sequence sum yields

$$A_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right] \tag{iii}$$

Digression

$$e^{ja} = \cos a + j \sin a$$
$$e^{-ja} = \cos a - j \sin a$$

Subtracting, we get

$$\sin a = \frac{e^{ja} - e^{-ja}}{2j} = \frac{e^{ja}(1 - e^{-j2a})}{2j}$$

or
$$(1 - e^{-j2a}) = 2j\, e^{-ja} \sin a$$

We can than write Eq. (iii) in sin form as

$$A_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \frac{2j\, e^{-jk2\pi(N_1+1/2)/N} \sin[2\pi k(N_1 + 1/2)/N]}{2j\, e^{-j\pi k/N} \sin(\pi k/N)}$$

If easily follows that the exponential powers add to zero, i.e., $e^0 = 1$. Then we get

$$A_k = \frac{1}{N} \cdot \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)};\ k \neq 0, \pm N, \pm 2N \tag{iv}$$

and

$$A_k = \frac{2N_1 + 1}{N};\ k = 0, \pm N, \pm 2N \quad \text{(apply L' Hospital's rule)} \tag{v}$$

Example 3.15 *The following information is known about a periodic sequence x (n).*

1. *Period N = 6*
2. $\sum_{n=0}^{5} x(n) = 2$
3. $\sum_{n=2}^{7} (-1)^n x(n) = 1$
4. *x (n) has minimum power per period.*

Determine the sequence x (n).

Solution For a periodic sequence,

$$A_k = \frac{1}{N} \sum_{n=(N)} x(n)\, e^{-jk(2\pi/N)n} \tag{i}$$

Information 2

$$\sum_{n=0}^{5} x(n) = \sum_{n=(N)} x(n) = 2;\ N = 6 \tag{ii}$$

From Eq. (i), this corresponds to $k = 0$, i.e.,

$$NA_0 = \sum_{n=(N)} x(n) = 2$$

or

$$A_0 = 2/N = 1/2 \tag{iii}$$

Information 3

$$\sum_{n=2}^{7} (-1)^n x(n) = \sum_{n=(N)} (-1)^n x(n) = 1 \tag{iv}$$

We can write

$$(-1)^n = e^{-j\pi n}$$

Comparing Eq. (iv) with Eq. (i), we have

$$e^{-j\pi n} = e^{-jk(2\pi/N)n}$$

$$\pi n = k(2\pi/6)n$$

which yields

$$k = 3$$

Then

$$A_3 = \frac{1}{6} \sum_{n=(N)} (-1)^n x(n) = \frac{1}{6};\ (N) = 2 \text{ to } 7$$

Information 4

For minimum per-period power, all other Fourier coefficients should be zero, i.e.,

$$A_1 = A_2 = A_4 = A_5 = 0$$

Hence,

$$x(n) = A_0 + A_3\, e^{-j\pi n} = \frac{1}{3} + \frac{1}{6}(-1)^n$$

Its values on one period ($n = 0$ to $n = 5$) are

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6}$$

Example 3.16 *$x(n)$ is a real odd periodic discrete signal with period $N = 7$. Three of its Fourier coefficients are*

$$A_{15} = j,\ A_{16} = 2j;\ A_{17} = 3j$$

Determine the values of

$$A_0,\ A_{-1},\ A_{-2},\ A_{-3}$$

Solution By periodicity

$$A_{15} = A(2 \times 7 + 1) = A_1 = j$$
$$A_{16} = A(2 + 7 + 2) = A_2 = 2j$$
$$A_{17} = A(2 + 7 + 3) = A_3 = 3j$$

As the signal is real and odd, the Fourier coefficients are pure imaginary and odd. By oddness property,

$$A_0 = 0,\ A_{-1} = -j,\ A_{-2} = -2j,\ A_{-3} = -3j$$

3.5 INTRODUCTION TO Z-TRANSFORM ANALYSIS OF DISCRETE-TIME SYSTEM

Z-Transform transforms a difference equation to an algebraic equation, thereby simplifying the analysis of discrete-time system behaviour in time and frequency domain. Besides this, the transform domain analysis has the following inherent advantages.

(a) The involved operation in time domain convolution is reduced to multiplication operation in transform domain.
(b) Initial conditions are directly incorporated in the solution process.
(c) Insight to the system dynamics is achieved easily.
(d) Frequency domain behaviour follows in simple steps.

However, there are certain limitations in the computer solution of the z-transform approach, giving an edge to the state-variable approach.

3.6 Z-TRANSFORM

An impulse-sampled signal,

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\ \delta(t - nT) \tag{3.40}$$

Taking the bilateral Laplace transform of Eq. (3.40), we get the following result.

$$F(s) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(nT)\, \delta(t - nT) e^{-st}\, \mathrm{d}t$$

Interchanging integration and summation yield,

$$F(s) = \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta\,(t - nT) e^{-st}\, \mathrm{d}t$$

Using the sampling property of δ-function, we get

$$F(s) = \sum_{n=-\infty}^{\infty} f(nT)\, e^{-snT}$$

Writing nT as n (count), we have

$$F(s) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-sn} \tag{3.41}$$

Defining $z = e^s$ then, we get

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} \tag{3.42}$$

$F(z)$, as defined in Eq. (3.42), is known as **bilateral *Z*-transform** (or double sided Z-transform).

If we have the sequence $f(n) = 0$; $n < 0$, called right-sided sequence then Eq. (3.42) reduces to

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} \tag{3.43}$$

It should be noted that Z-transform as given by Eq. 3.43 is a very useful form and is employed to determine response of non-relaxed causal system. This is called *single-sided Z-transform*.

The operation of obtaining the sequence $f(k)$ from $F(z)$, called the inverse Z-transform operation, is denoted as

$$\{f(k)\} = Z^{-1}[F(z)] \tag{3.44}$$

The Z-transform operation and the inverse Z-transform operation are indicated by the shorthand notation

$$f(k) \leftrightarrow F(z) \tag{3.45}$$

It is to be noted that the Z-transform of a sequence exists only if the sum-series converges. As a matter of fact, the Z-transform does not converge for all the sequences for all the values of z. For any given sequence, the set of values of z for which the Z-transform converges is called the **region of convergence** (RoC).

For the convergence of the Z-transform of a given sequence, it is necessary that

$$\sum_{k=-\infty}^{\infty} \left| f(k)\, z^{-k} \right| < \infty \tag{3.46}$$

In general, the sum-series of Eq. (3.42) converges in the annular region of the Z-plane shown in Fig. 3.12 which is defined as

$$R_{z-} < |z| < R_{z+} \tag{3.47}$$

where R_{z-}, in general, can be as small as zero and R_{z+} can be as large as infinity.

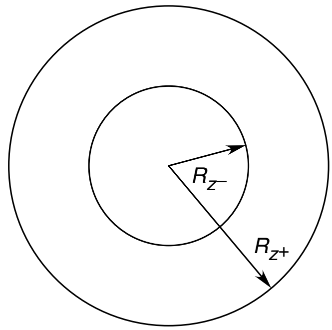


Fig. 3.12 *RoC of double-sided sequence*

The properties of RoC of the Z-transform can be easily understood by following examples. Consider a right-sided, also called causal sequence, $f(k)$, defined as

$$f(k) = \begin{cases} 0 & ; k < 0 \\ a^k & ; k \geq 0 \end{cases} \tag{3.48}$$

The Z-transform of this sequence, using Eq. (3.42) is given by

$$F(z) = \sum_{k=0}^{\infty} a^k z^{-k} \tag{3.49}$$

This sum-series converges only if

$$|az^{-1}| < 1 \quad ; \quad |z| > |a| \tag{3.50}$$

Carrying out the summation, we get

$$F(z) = \frac{1}{1 - az^{-1}} \; ; \quad |z| > |a| \tag{3.51a}$$

$$= \frac{z}{z - a} \; ; \quad |z| > |a| \tag{3.51b}$$

The RoC for this case in the complex Z-plane is shown in Fig. 3.13(a), from which one can observe the following properties of RoC of a right-sided (or causal) sequence.

(a) RoC lies outside the circle of radius $|a|$ centered at the origin in the Z-plane.

(b) The poles of $F(z)$ lie at $z = |a|$ at which $F(z) = \infty$. This means that Z-transform of a right-sided sequence does not exist at its poles.

(c) $F(z)$ of right-sided sequence exists at $z = \infty$.

(d) $F(z)$ of right-sided sequence does not exist at $z = 0$. This is because of the fact that for the right-sided sequence, the condition $|z| > |a|$ is violated, if we take $|z| = 0$.

We now consider the left-sided (or non-causal) sequence, defined as

$$f(k) = \begin{cases} b^k & ; k < 0 \\ 0 & ; k \geq 0 \end{cases} \tag{3.52}$$

Then we have

$$F(z) = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{k=1}^{\infty} b^{-k} z^k \tag{3.53}$$

which converges only if

$$|b^{-1}z| < 1 \; ; |z| < |b| \tag{3.54}$$

Thus

$$F(z) = \frac{1}{1 - bz^{-1}} \; ; |z| < |b| \tag{3.55}$$

$$= \frac{z}{z - b} \; ; |z| < |b| \tag{3.56}$$

The RoC for this case is shown in Fig. 3.13(b) from which one can observe the following properties of RoC of a left-sided (noncausal) sequence.

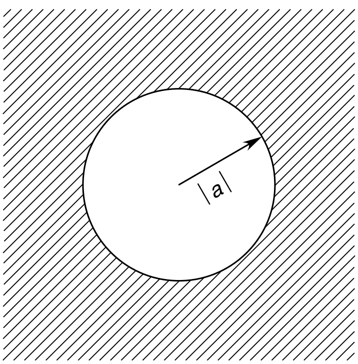
(a) RoC lies inside the circle of radius $|b|$ centered at the origin in the Z-plane.

(b) The poles of $F(z)$ lie at $z = |b|$ at which $F(z) = \infty$. This means that Z-transform of left-sided sequence does not exist at its poles.

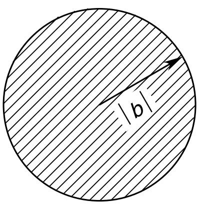
(c) $F(z)$ of the left-sided sequence exists at $z = 0$.

(d) $F(z)$ of the left-sided sequence does not exist at $z = \infty$. This is because of the fact that in this case, the condition $|z| < |b|$ is violated.

It can also be observed from the above two examples that in the case of a single-sided sequence, the Z-transform is unique and therefore, it is not necessary to mention its region of convergence.



(a) RoC right-sided sequence



(b) RoC left-sided sequence

Fig. 3.13

Let us now consider a two-sided sequence, written in the following way.

$$f(k) = \begin{cases} b^k & ; k < 0 \\ a^k & ; k \geq 0 \end{cases} \tag{3.57}$$

Combining the results obtained in Eqs (3.51(a)) and (3.55), we can write

$$F(z) = \frac{1}{1 - bz^{-1}} + \frac{1}{1 - az^{-1}} \quad ; |a| < |z| < |b| \tag{3.58a}$$

$$F(z) = \frac{z}{(z - b)} + \frac{z}{z - b} \quad ; |a| < |z| < |b| \tag{3.58b}$$

As already pointed out in Eq. (3.47), the RoC of a double-sided sequence lies within the annular region. In this example $R_{z-} = |a|$ and $R_{z+} = |b|$, as shown in Fig. 3.14.

The RoC in case of a double-sided sequence has the following properties.

(a) RoC exists only if $R_{z+} > R_{z-}$. The transform does not converge even for $R_{z+} = R_{z-}$.

(b) At poles, $F(z)$ of a double-sided sequence does not exist and RoC is bounded by poles.

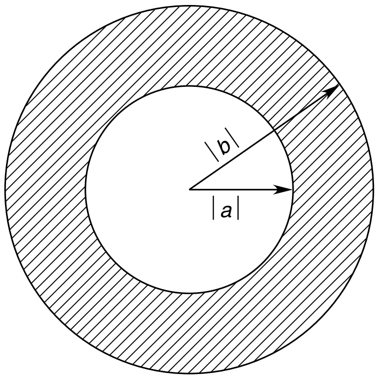


Fig. 3.14 *RoC of double-sided sequence*

Example 3.17 *Find the Z-transform of the sequences:*

(a) $\delta(k)$ *(b)* $u(k)$ *(c)* $e^{\pm \beta k}$ *;* $k \geq 0$*.*

Solution

(a) From Eq. (3.43), we get

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} \delta(k) z^{-k} \tag{i}$$

However,

$$f(k) = \delta(k) = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ otherwise} \end{cases}$$

Therefore,

$$F(z) = \delta(0)z^0 + \delta(1)z^{-1} + \delta(2)z^{-2} + ... = 1$$

i.e.

$$F(z) = 1$$

or

$$\delta(k) \leftrightarrow 1$$

(b)
$$u(k) = \begin{cases} 1 \text{ for } k \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

$$F(z) = \sum_{k=0}^{\infty} z^{-k}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

$$= \frac{z}{z-1}\ ; |z| > 1$$

(c)
$$f(k) = e^{\pm j\beta k}$$

From Eq. (3.43)

$$F(z) = \sum_{k=0}^{\infty} e^{\pm j\beta k}\, z^{-k}$$

$$= \frac{z}{z - e^{\pm j\beta}}\ ; |z| > |e^{\pm j\beta}| = 1 \qquad \text{(ii)}$$

Suppose we are interested to find out the Z-transform of $\cos\beta k$ and $\sin\beta k$. Then

$$\cos \beta k = \frac{(e^{j\beta k} + e^{-j\beta k})}{2} \qquad \text{(iii)}$$

$$\sin \beta k = \frac{(e^{j\beta k} + e^{-j\beta k})}{2j} \qquad \text{(iv)}$$

From Eq. (ii)

$$Z[\cos \beta k] = (1/2)\, [z/(z - e^{j\beta}) + z/(z - e^{-j\beta})]$$

$$= \frac{z(z - \cos \beta)}{z^2 - 2z \cos \beta + 1}\ ; |z| > 1 \qquad \text{(v)}$$

Similarly, the Z-transform of $\sin \beta k$ may be obtained as

$$Z[\sin \beta k] = \frac{z \sin \beta}{z^2 - 2z \cos \beta + 1}\ ; |z| > 1 \qquad \text{(vi)}$$

Some of the important Z-transform pairs for causal sequences are listed in Table 3.1.

3.7 Z-TRANSFORM PROPERTIES

The properties being discussed in this section are, in general, applicable to both double-sided as well as single-sided Z-transform. Wherever the properties applicable to a double-sided Z-transform differ from that of a single-sided Z-transform, it will be specified.

Table 3.1 *Z-transform pairs*

	$f(k);\ k \geq 0$	$F(z)$
1.	$\delta(k)$	1
2.	$\delta(k-n)$	z^{-n}
3.	1 or $u(k)$	$z/(z-1)$
4.	a^k	$z/(z-a)$
5.	ka^k	$az/(z-a)^2$
6.	$k^2 a^k$	$\dfrac{az(z+a)}{(z-a)^3}$
7.	$(k+1)a^k$	$z^2/(z-a)^2$
8.	$\dfrac{(k+1)(k+2)a^k}{2!}$	$z^3/(z-a)^3$
9.	$\dfrac{(k+1)(k+2)\ldots(k+n-1)a^k}{n-1!}$	$\dfrac{z^n}{(z-a)^n};\ n \geq 2$
10.	$\dfrac{a^k}{k!}$	$\exp(a/z)$
11.	$\sin \alpha k$	$\dfrac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}$
12.	$b^k \sin \alpha k$	$\dfrac{b(\sin\alpha)z}{z^2 - 2bz\cos\alpha + b^2}$
13.	$\cos \alpha k$	$\dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$
14.	$b^k \cos \alpha k$	$\dfrac{z(z-\cos\alpha)}{z^2 - 2bz\cos\alpha + b^2}$

Linearity

If

$$f_1(k) \leftrightarrow F_1(z)$$

and

$$f_2(k) \leftrightarrow F_2(z)$$

Then

$$[a f_1(k) + b f_2(k)] \leftrightarrow a F_1(z) + b F_2(z)$$

Proof

$$\begin{aligned} Z[a f_1(k) + b f_2(k)] &= a Z[f_1(k)] + b Z[f_2(k)] \\ &= a F_1(z) + b F_2(z) \end{aligned} \tag{3.59}$$

Multiplication by k

If

$$f(k) \leftrightarrow F(z)$$

then $$[k\,f(k)] \leftrightarrow -z\,\frac{\mathrm{d}F(z)}{\mathrm{d}z}$$

Proof

$$\begin{aligned}
\mathrm{Z}[kf(k)] &= \sum_{k=-\infty}^{\infty} kf(k)\,z^{-k} \\
&= -\sum_{k=-\infty}^{\infty} -f(k)\,z^{-k-1}z \\
&= -z\sum_{k=-\infty}^{\infty} f(k)\,\frac{\mathrm{d}}{\mathrm{d}z}(z^{-k}) \\
&= -z\,\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=-\infty}^{\infty} f(k)\,z^{-k} \\
&= -z\,\frac{\mathrm{d}}{\mathrm{d}z}F(z)
\end{aligned}$$

In general, $$[k^n f(k)] \leftrightarrow \left[-z\,\frac{\mathrm{d}}{\mathrm{d}z}\right]^n F(z) \tag{3.60}$$

Multiplication by a^k (Scale Change)

If

$$f(k) \leftrightarrow F(z)$$

then

$$[a^k f(k)] \leftrightarrow F(z/a)$$

Proof

$$\begin{aligned}
\mathrm{Z}[a^k f(k)] &= \sum_{k=-\infty}^{\infty} f(k)\,a^k z^{-k} \\
&= \sum_{k=-\infty}^{\infty} f(k)\,(z/a)^{-k} \\
&= F(z/a)
\end{aligned}$$

Shifting

(a) Double-sided Z-transform

Consider a double-sided sequence $f(k)$; $-\infty \le k \le \infty$.

If $$f(k) \leftrightarrow F(z)$$

then $$f(k \pm n) \leftrightarrow z^{\pm n} F(z)$$

Proof

$$Z[f(k+n)] = \sum_{k=-\infty}^{\infty} f(k+n)\,z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-(k+n)}\, z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-(k+n)}$$

Let $k + n = m$ so that for $k = \pm\,\infty$; $m = \pm\,\infty$

Then
$$Z[f(k+n)] = z^n \sum_{m=-\infty}^{\infty} f(m)\, z^{-m}$$

$$= z^n F(z) \tag{3.61}$$

Similarly, it can be shown that

$$Z[f(k-n)] = z^{-n} F(z) \tag{3.62}$$

(b) Single-sided Sequence Z-transform Let us assume a single-sided causal sequence $f(k)$; $k \geq 0$.

If
$$f(k) \leftrightarrow F(z)$$

then
$$f(k+n) \leftrightarrow z^n F(z) - \sum_{k=0}^{n-1} f(k)\, z^{n-k}$$

and
$$f(k-n) \leftrightarrow z^{-n} F(z) - \sum_{m=1}^{n} f(-m)\, z^{-n+m}$$

Proof

$$Z[f(k+n)] = \sum_{k=0}^{\infty} f(k+n)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k+n)\, z^{-(k+n)}\, z^n$$

$$= z^n \sum_{k=0}^{\infty} f(k+n)\, z^{-(k+n)}$$

Let $k + n = m$ so that at $k = 0$; $m = n$ and at $k = \infty$; $m = \infty$

Thus,

$$Z[f(k+n)] = z^n \left[\sum_{m=n}^{\infty} f(m)\, z^{-m} \right]$$

$$= z^n[\{f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots + f(n-1)z^{-(n-1)}\}$$
$$- \{f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots + f(n-1)z^{-(n-1)}\}$$

$$+ \sum_{m=n}^{\infty} f(m)\, z^{-m}\,]$$

or

$$Z[f(k+n)] = z^n \left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - \{f(0) + f(1)z^{-1} + \ldots + f(n-1)z^{-(n-1)}\} \right]$$

$$= z^n \left[F(z) - \sum_{k=0}^{n-1} f(k)\, z^{-k} \right]$$

$$= z^n F(z) - \sum_{k=0}^{n-1} f(k)\, z^{n-k} \tag{3.63}$$

We now consider the one-sided sequence $f(k-n)$; $k \geq 0$

$$Z[f(k-n)] = \sum_{k=0}^{\infty} f(k-n)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k-n)\, z^{-k+n} z^{-n}$$

$$= z^{-n} \left[\sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)} \right]$$

Let $(k-n) = m$; so that at $k = 0$; $m = -n$ and at $k = \infty$; $m = \infty$

Thus

$$Z[f(k-n)] = z^{-n} \left[\sum_{m=-n}^{\infty} f(m)\, z^{-m} \right]$$

$$= z^{-n} \left[\sum_{m=-n}^{-1} f(m)\, z^{-m} + \sum_{m=0}^{\infty} f(m)\, z^{-m} \right]$$

$$= z^{-n} \left[\sum_{m=1}^{n} f(-m)\, z^{m} + F(z) \right]$$

$$= z^{-n} F(z) + \sum_{m=1}^{n} f(-m)\, z^{-n+m} \tag{3.64}$$

If the initial condition is zero, i.e., the system is initially relaxed then

$$Z[f(k-n)] = z^{-n} F(z)$$

Initial and Final Values Theorems (Single-sided Only)

(a) Initial Value Theorem

$$Z[f(k)] = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \dots$$
$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

Taking the limit as $z \to \infty$, we get

$$\operatorname*{Lim}_{z \to \infty} F(z) = f(0) \tag{3.65}$$

(b) Final-value Theorem

According to this theorem, we have

$$\operatorname*{Lim}_{k \to \infty} f(k) = \operatorname*{Lim}_{z \to 1} \left[\left(\frac{z-1}{z} \right) F(z) \right]$$

provided the limit exists.

Proof

$$Z[f(k+1) - f(k)] = \operatorname*{Lim}_{m \to \infty} \sum_{k=0}^{m} [f(k+1) - f(k)] z^{-k}$$

Using shifting property, we get

$$z\,F(z) - z\,f(0) - F(z) = \operatorname*{Lim}_{m \to \infty} \sum_{k=0}^{m} [f(k+1) - f(k)] z^{-k}$$

Letting $z \to 1$ on both sides,

$$\operatorname*{Lim}_{z \to 1} [(z-1)\,F(z) - f(0)] = \operatorname*{Lim}_{z \to 1} \operatorname*{Lim}_{m \to \infty} \sum_{k=0}^{m} [f(k+1) - f(k)] z^{-k}$$

Interchanging the order of limits on the right-hand side, we get

$$\operatorname*{Lim}_{z \to 1} [(z-1)\,F(z)] = f(0) + \operatorname*{Lim}_{m \to \infty} \sum_{k=0}^{m} [f(k+1) - f(k)]$$
$$= f(\infty) \tag{3.66}$$

This theorem is applicable only if a finite limit exists. It implies that the poles of $F(z)$ must lie within a unit circle.

Also, note that

$$\lim_{z \to 1} \left(\frac{z-1}{z} \right) F(z) = \operatorname*{Lim}_{z \to 1} (z-1)\,F(z)$$

Discrete Convolution The Z-transform of the sequence $f(k) = f_1(k) * f_2(k)$, where * symbolizes discrete convolution, is given as

$$F(z) = F_1(z)\,F_2(z) = Z[f_1(k) * f_2(k)] \tag{3.67}$$

Proof

$$Z[f(k)] = Z[f_1(k) * f_2(k)]$$

$$= Z\left[\sum_{j=-\infty}^{\infty} f_1(j) f_2(k-j)\right]$$

$$= \sum_{k=-\infty}^{\infty} z^{-k}\left\{\sum_{j=-\infty}^{\infty} f_1(j) f_2(k-j)\right\}$$

or
$$F(z) = \sum_{j=-\infty}^{\infty} f_1(j) \sum_{k=-\infty}^{\infty} f_2(k-j)\, z^{-k}$$

$$= \sum_{j=-\infty}^{\infty} f_1(j)\, z^{-j} \sum_{k=-\infty}^{\infty} f_2(k-j)\, z^{-(k-j)}$$

Let, $k - j = m$; for $k = -\infty$, $m = -\infty$ and also for $k = \infty$; $m = \infty$

Thus,

$$F(z) = \sum_{j=-\infty}^{\infty} f_1(j)\, z^{-j} \sum_{m=-\infty}^{\infty} f_2(m)\, z^{-m} = F_1(z)\, F_2(z)$$

Example 3.18 *Consider a causal system described by the difference equation*

$$c(k+1) + \frac{1}{2}c(k) = r(k);\ c(0) = c_0 = 0$$

Let us obtain the system's impulse response, i.e.,

$$c(k+1) + \frac{1}{2}c(k) = \delta(k)$$

Taking the z-transform of both sides (use the linearity property)

$$\left(z + \frac{1}{2}\right) C(z) = 1$$

or
$$C(z) = \frac{z^{-1}}{1 + \frac{1}{2}z^{-1}} = z^{-1}\left(\frac{1}{1 + \frac{1}{2}z^{-1}}\right)$$

From Eq. (3.51),

$$\frac{1}{1 + \frac{1}{2}z^{-1}} \leftrightarrow \left(-\frac{1}{2}\right)^{k} u(k)$$

Using the shifting property,

$$z^{-1}\left(\frac{1}{1 + \frac{1}{2}z^{-1}}\right) \leftrightarrow \left(-\frac{1}{2}\right)^{k-1} u(k-1)$$

$$c(k) = \left(-\frac{1}{2}\right)^{k-1};\ k \ge 1$$

Example 3.19 *Let us solve Example 3.15 with $c_0 = 1$ and $r(k) = u(k)$.*
Taking the z-transform,

$$z[C(z) - c_0] + \frac{1}{2}C(z) = R(z)$$

or
$$C(z) = \frac{c_0}{1+\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1+\frac{1}{2}z^{-1}}R(z)$$

Now
$$R(z) = \frac{1}{1-z^{-1}} \text{ and } C_a = 1$$

Hence,

$$C(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)(1-z^{-1})}$$

The technique of inverse transforming will be presented in the next section. To determine $c(\infty)$, apply final-value theorem.

$$c(\infty) = \lim_{z\to 1}\left[\frac{(z-1)}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{z^{-1}(z-1)}{\left(1+\frac{1}{2}z^{-1}\right)(1-z^{-1})}\right]$$

$$= \frac{2}{3}$$

Note: The poles of $C(z)$ are $z = -\frac{1}{2}$, 1. Since these do not lie outside the unit circle, the final-value theorem is applicable.

Example 3.20 *Find the z-transform of the discrete camp sequence*

$$g(k) = ku(k)$$

Solution

$$\mathcal{Z}[g(k)] = \mathcal{Z}[ku(k)]$$

Using Eq. (3.60),

$$\mathcal{Z}[g(k)] = -z\frac{d}{dz}U(z) = -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)$$

$$= \frac{z^{-1}}{(1-z^{-1})^2}$$

Properties of the Z-transform are given in Table 3.2.

Table 3.2 *Z-Transform properties*

Operation	$f(k)$; $k > 0$	$F(z)$
Linearity	$a_1 f_1(k) + a_2 f_2(k)$ a_1, a_2 are constants	$a_1 F_1(z) + a_2 F_2(z)$
Multiplication	$kf(k)$	$-z \dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
Scaling	$a^k f(k)$	$F(z/a)$
Time delay	$f(k-1)$	$z^{-1}F(z)$
	$f(k-2)$	$z^{-2}F(z)$
	$f(k-m)$	$z^{-m}F(z)$
Time advance	$f(k+1)$	$zF(z) - zf(0)$
	$f(k+2)$	$z^2F(z) - z^2 f(0) - zf(1)$
	$f(k+m)$	$z^m F(z) - z^m \sum_{k=0}^{m-1} f(k) z^{-k}$
Initial value	$f(0)$	$\lim_{z\to\infty} F(z)$
Final value	$f(\infty)$	$\lim_{z\to\infty} (z-1)F(z)$
Convolution	$f_1(k) * f_2(k)$	$F_1(z)F_2(z)$

3.8 INVERSE Z-TRANSFORM

The Z-transform of a sample sequence $\{f(k)\}$ is written as,

$$F(z) = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots \tag{3.68}$$

We shall now consider the methods of recovering the sequence $\{f(k)\}$ from $F(z)$.

Direct Division

By direct division, $F(z)$ can be manipulated into a form similar to Eq. (3.68) and by comparison, the sequence $\{f(k)\}$ is determined. This method is now illustrated with examples.

Example 3.21 *Obtain the inverse Z-transform of a rational function.*

$$F(z) = \frac{z^2}{z^3 - 1.7z^2 + 0.8z + 0.1}$$

Solution $F(z)$ is expanded in power series of z^{-1} by dividing its numerator with its denominator by the method of long division.

This division gives

$$F(z) = z^{-1} + 1.7z^{-2} + 2.89z^{-3} + \ldots \tag{i}$$

Comparing Eq. (i) with Eq. (3.49), we get

$$f(0) = 0, f(1) = 1.7, f(2) = 2.89 \ldots$$

or

$$\{f(k)\} = \{0, 1.7, 2.89, \ldots\}$$

Example 3.22 *Obtain the Z-inverse of F(z) for the following,*

(i) $z/(z-0.4)$; $|z| > 0.4$ *(ii) $z/(z-0.4)$; $|z| < 0.4$*

Solution

(i) Since the region of convergence is $|z| > 0.4$, expanding $F(z)$ in negative power of z (by long division) gives

$$F(z) = 1 + 0.4z^{-1} + (0.4)^2 z^{-2} + \ldots$$

Hence the sequence is

$$\{f(k)\} = \{1, 0.4, 0.4^2, \ldots\}$$

(ii) In this case the region of convergence is $|z| < 0.4$. Now the solution of $F(z)$ is obtained by expanding it in positive power using the long division method. This gives us the following result.

$$F(z) = -0.4^{-1}z^1 - 0.4^{-2}z^2 + \ldots$$

Comparing this equation with Eq. (3.53), we can write the inverse as

$$\{f(k)\} = \{\ldots, -0.4^{-3}, -0.4^{-2}, -0.4^{-1}\}$$

Partial-fraction Method

Inversion of any rational function

$$F(z) = \frac{b_m z^m + b_{m-1}z^{m-1} + \ldots + b_0}{z^n + a_{n-1}z^{n-1} + \ldots a_0}; \; m \le n \tag{3.69}$$

can be obtained by partial fractioning it. The denominator can be written in factored form as follows.

$$D(z) = (z - p_1)(z - p_2)\ldots(z - p_n)$$

where 1, 2 are suffixes of poles of $F(z)$.

We can then write $F(z)$ of Eq. (3.69) in factored form as

$$F(z) = \frac{\alpha_1}{z - p_1} + \frac{\alpha_2}{z - p_2} + \ldots + \frac{\alpha_n}{z - p_n} \tag{3.70}$$

In case $D(z)$ has m repeated roots (poles), then these poles contribute m factors as

$$\frac{\alpha_{i1}}{(z - p_i)} + \frac{\alpha_{i2}}{(z - p_i)^2} + \ldots + \frac{\alpha_{im}}{(z - p_i)^m} \tag{3.71}$$

where the constants α_{im}'s can be found by the method given in Appendix Al.

The inverse Z-transform of $F(z)$ is then obtained by looking at Table 3.1 for the inverse transform for each factor. It is seen from Table 3.1 that the Z-transform of every sequence has a factor z in the numerator. Therefore, the partial-fraction method is applied on $F(z)/z$. Example 3.21 illustrates the process.

Example 3.23 *Find the inverse of Z-transform of F(z) for the following.*

(a) $\dfrac{3z-5}{(z-1)(z-3)}$ (b) $\dfrac{z^2-5}{(z-1)(z-2)^2}$

Solution

(a) $$F(z) = \frac{3z-5}{(z-1)(z-3)} \tag{i}$$

or

$$\frac{F(z)}{z} = \frac{(3z-5)}{z(z-1)(z-3)} \tag{ii}$$

Partial fractioning Eq. (ii) and multiplying it by z, we get

$$F(z) = -\frac{5}{3} + \frac{z}{z-1} + \frac{(2/3)z}{z-3} \tag{iii}$$

Taking z-inverse of Eq (iii), from Table 3.3, we get

$$f(k) = -(5/3)\,\delta(k) + u(k) + (2/3)(3)^k \tag{iv}$$

(b) $$F(z)\ \frac{z^2-5}{(z-1)(z-2)^2} \tag{i}$$

$$\frac{F(z)}{z} = \frac{z^2-5}{z(z-1)(z-2)^2} \tag{ii}$$

Partial fractioning $F(z)/z$ and then multiplying it by z gives the following result.

$$F(z) = -\frac{5}{4} + \frac{-4z}{z-1} + \frac{(-1/2)}{(z-2)^2} + \frac{(7/4)^2}{(z-2)} \tag{iii}$$

Taking Z-inverse, we get

$$f(k) = -(5/4)\,\delta(u) - 4u(k) - (1/2)\,k(2)^{k-1}u(k) + (7/4)2^k u(k) \tag{iv}$$

3.9 RESPONSE OF LTI DISCRETE-TIME SYSTEMS USING Z-TRANSFORM

The nth order difference equation representing linear discrete system for $m \neq n$ is reproduced as

$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \ldots + a_1 y(k+1) + a_0 y(k) \\ &= b_n r(k+m) + b_n r(k+m-1) + \ldots + b_1 r(k+1) + b_0 r(k) \end{aligned} \tag{3.72}$$

where input r is assumed to exist for $k \geq 0$.

Taking the Z-transform of Eq. (3.72) using time-shifting properties of Z-transform, we get the following equation.

$$(z^n + a_n z^{n-1} + \ldots + a_1 z + a_0)\, Y(z) = (b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0)\, R(z) \tag{3.73}$$

where initial conditions are assumed zero, i.e.

$$y(-1) = y(-2) = \ldots = y(-n) = 0$$

Equation (3.73) can be written in form of **Z-transform transfer function** as

$$H(z) = \frac{Y(z)}{R(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} \tag{3.74}$$

The Z-transfer function can also be interpreted as the zero-state response as

$$H(z) = \frac{Y_r(z)}{R(z)} = \frac{\text{zero-state response in } z\text{-form}}{\text{input in } z\text{-form}} \tag{3.75}$$

Zero-state response (in z-form) is then given as

$$Y_r(z) = H(z)\, R(z) \tag{3.76}$$

$H(z)$ of Eq. (3.74) can be written in the following form.

$$H(z) = \frac{N(z)}{D(z)} \tag{3.77}$$

where $N(z)$ and $D(z)$ are obvious from this Eq. (3.74)

The poles of $H(z)$ are the roots of characteristic equation

$$D(z) = 0 \tag{3.78}$$

In Eq. (3.75), consider the special case with input

$$r(k) = \delta(k), \text{ impulse input}$$

The impulse response $h(k)$ of the system is given as

$$Z[h(k)] = H(z)\, Z[\delta(k)] = H(z) \text{ as } Z[\delta(k)] = 1 \tag{3.79}$$

Equation (3.79) can be symbolically expressed as

$$Z\{h(k)\} \leftrightarrow H(z) \tag{3.80}$$

It is an important result relating $h(k)$ and $H(z)$.

As per Eq. (3.80),

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k} \tag{3.81}$$

and

$$h(k) = Z^{-1} H(z) \tag{3.82}$$

Zero-state Response

From the above discussion, it is obvious that with the knowledge of input and transfer function $H(z)$, zero-state response in time domain can be obtained by taking the inverse transform of Eq. (3.76). This will be illustrated by some examples.

Let us consider an LTI discrete-time system described by the following difference equation.

$$2y(k+2) + 3y(k+1) + y(k) = r(k+2) + r(k+1) - r(k)\ ;\ r(k)$$
$$r(k) = u(k) = \text{unit step} \qquad \text{(i)}$$

Taking the Z-transform of Eq. (i), we get

$$Y_r(z) = \frac{z^2+z-1}{2z^2+3z+1}R(z) \qquad \text{(ii)}$$

which gives

$$H(z) = \frac{z^2+z-1}{2z^2+3z+1} \qquad \text{(iii)}$$

Input

$$R(z) = Z[r(k)] = \frac{z}{z-1} \qquad \text{(iv)}$$

Therefore,

$$Y_r(z) = \frac{z^2+z-1}{2z^2+3z+1} \times \frac{z}{z-1} = \frac{z(z^2+z-1)}{(2z^2+3z+1)(z-1)}$$
$$= \frac{z(z^2+z-1)}{2(z+0.5)(z+1)(z-1)} \qquad \text{(v)}$$

Taking partial fractions

$$\frac{Y_r(z)}{z} = \underbrace{\frac{(-5/12)}{z+0.5} + \frac{(1/2)}{z+1}}_{\text{Poles of H}(z)\ \text{(natural mode)}} + \underbrace{\frac{1/6}{z-1}}_{\text{Input pole}} \qquad \text{(vi)}$$

The inverse Z-transform gives the zero-state time-domain response as

$$y_r(k) = \underbrace{(-5/12)(-0.5)^k + (1/2)(-1)^k}_{\text{Natural response}} + \underbrace{(1/6)(1)^k}_{\text{Forced response}} \qquad \text{(vii)}$$

Zero-input Response

Substituting $r(k) = 0$ in Eq. (i), it takes the following form.

$$2y(k+2) + 3y(k+1) + y(k) = 0 \qquad \text{(viii)}$$

or

$$y(k+2) + 1.5y(k+1) + 0.5y(k) = 0 \quad \text{(ix)}$$

Solution of this equation, $y_0(k)$, is the zero-input response. Taking the Z-transfrom, and using the shifting property to include the initial conditions, we obtain

$$\{z^2 Y_0(z) - y(0)z^2 - y(1)z\} + 1.5\{zY_0(z) - y(0)z\} + 0.5\, Y_0(z) = 0 \quad \text{(x)}$$

Rearranging Eq. (x), we have

$$\{z^2 + 1.5z + 0.5\}\, Y_0(z) = (z^2 + 1.5z)y(0) + zy(1)$$

or

$$Y_0(z) = \frac{z^2 + 1.5z}{(z^2 + 1.5z + 0.5)} y(0) + \frac{z}{(z^2 + 1.5z + 0.5)} y(1) \quad \text{(xi)}$$

or

$$\frac{Y_0(z)}{z} = \left\{\frac{z + 1.5}{(z+1)(z+0.5)}\right\} y(0) + \left\{\frac{1}{(z+1)(z+0.5)}\right\} y(1)$$

By partial fractioning, we get

$$\frac{Y_0(z)}{z} = \left\{\frac{-1}{(z+1)} + \frac{2}{(z+0.5)}\right\} y(0) + \left\{\frac{-2}{(z+1)} + \frac{2}{(z+0.5)}\right\} y(1) \quad \text{(xii)}$$

From Eq. (xii), we can write

$$Y_0(z) = \left[\frac{-z}{(z+1)} + \frac{2z}{(z+0.5)}\right] y(0) + \left[\frac{-2z}{(z+1)} + \frac{z}{(z+0.5)}\right] y(1) \quad \text{(xiii)}$$

Taking the inverse Z-tranform and combining term of $(-1)^k$ and $(-0.5)^k$, we get

$$y_0(k) = \underbrace{C_1(-0.5)^k + C_2(-1)^k}_{\substack{\text{Natural response caused} \\ \text{by initial conditions}}} \; ; k \geq 0$$

The constants C_1 and C_2 are found from initial conditions.

Complete Response

Complete response of the system is obtained by adding the zero-state and zero-input responses as found carlier.

$$\begin{aligned} y(k) &= y_r(k) + y_0(k) \\ &= (-5/12)(-0.5)^k + (1/2)(-1)^k + (1/6)(1)^k + C_1(-0.5)^k \\ &\quad + C_2(-1)^k;\ k \geq 0 \end{aligned} \quad \text{(xiv)}$$

Certain important observations can be drawn by regrouping the complete response given in Eq. (xiv) as

$$y(k) = y_1(k) + y_2(k) + y_3(k) \tag{xv}$$

where

$$y_1(k) = (-5/12)(-0.5)^k + (1/2)(-1)^k$$
$$y_2(k) = C_1(-0.5)^k + C_2(-1)^k$$
$$y_3(k) = (1/6)(1)^k$$

It is seen that response terms $y_1(k)$ and $y_2(k)$ arise due to system poles. However, $y_3(k)$ is contributed by excitation, while $y_2(k)$ is due to initial conditions. $y_1(k)$ and $y_2(k)$ taken together constitute the **transient response,** which vanishes as $k \to \infty$, if the system pole(s) lie within the unit circle in z-plane. However, in this example, there is a system pole at $z = 1$, so the term $(-1)^k$ does not decay.

$$\begin{aligned}\text{Transient response} &= y_1(k) + y_2(k) \\ &= (-5/12)(-0.5)^k + (1/2)\,(-1)^k + C_1(0.5)^k + C_2\,(-1)^k \end{aligned} \tag{xvi}$$

The third term $y_3(k)$ is called **forced response** and is contributed by the excitation pole(s). Its nature is the same as that of the forcing sequence except for the change in magnitude and phase (only in the case of oscillating input). This indeed is the **steady-state response** which exists after transient response has vanished. It is meaningful for inputs which exist for all $k \geq 0$, e.g., unit step, sinusoidal function.

Example 3.24 *Find the complete response of system whose transfer function is the following.*

$$H(z) = \frac{z^2 + z - 1}{2(z + 0.5)(z + 1)}$$

and input is $R(z) = (z + 1)/(z - 1)$

Solution

(i) *Zero-state response*

$$\begin{aligned} Y_r(z) &= H(z)\,R(z) \\ &= \frac{z^2 + z - 1}{2(z + 0.5)(z + 1)} \times \frac{z + 1}{z - 1} \end{aligned} \tag{i}$$

Taking partial fractions, we get

$$\frac{Y_r(z)}{z} = \frac{1}{z} - \frac{5}{6(z + 0.5)} + \frac{1}{3(z - 1)} \tag{ii}$$

and

$$y_r(k) = \delta(k) - (5/6)(-0.5)^k + (1/3)(1)^k \tag{iii}$$

Here we see that pole of $H(z)$ at $z = -1$ is cancelled out by the zero of the input, meaning thereby that the input does not excite the pole at -1.

(ii) *Zero-input response*

$$y_0(k) = C_1(-0.5)^k + C_2(1)^k \tag{iv}$$

(iii) *Complete response*

$$y(k) = C_1(-0.5)^k + C_2(1)^k + 2(1)^k + \delta(k) - (5/6)(-0.5)^k + (1/3)(1)^k;\ k \geq 0$$

3.10 NATURE OF RESPONSE

The nature of system response is revealed by its response to unit impulse sequence $\delta(k)$, i.e.,

$$h(k) = Z^{-1}[H(z)] = Z^{-1}\left[\frac{N(z)}{D(z)}\right] \tag{3.83}$$

It is observed from Eq. (3.83) that the nature of response is determined by the roots of the characteristic equation $D(z) = 0$. That is, the nature of response depends on the types of system poles. This is elaborated below.

Simple Real Pole

If $z = \alpha_1$ is one of the roots of characteristic equation, then

$$D(z) = (z - \alpha_1)\, D_1(z) \tag{3.84}$$

We can now write Eq. (3.83) as

$$h(k) = Z^{-1}\left[\frac{N(z)}{(z-\alpha_1)\, D_1(z)}\right]$$

The response term corresponding to the pole $z = \alpha_1$ is

$$Z^{-1}\left[\frac{1}{(z-\alpha_1)} \cdot \frac{N(\alpha_1)}{D_1(\alpha_1)}\right]$$

Looking at Table 3.1, we find

$$\frac{z}{z-\alpha_1} \leftrightarrow \alpha_1^k$$

Using the time delay property, we get the following result.

$$\frac{z}{z-\alpha_1} \leftrightarrow \alpha_1^{k-1};\ k \geq 1$$

The response term of this pole can then be expressed as

$$C(\alpha_1)^{k-1};\ k \geq 1 \tag{3.85}$$

where

$$C = \frac{N(\alpha_1)}{D_1(\alpha_1)}$$

The nature of response caused by a simple pole is given by Eq. (3.85) and the response for various locations of the poles relative to the unit circle is depicted in Fig. 3.15.

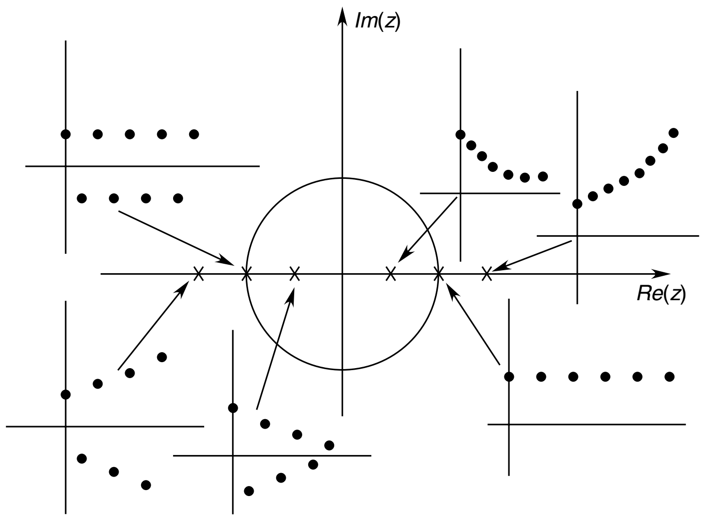


Fig. 3.15 *Response term contributed by simple real system pole*

The following general conclusions can easily be drawn.

$\lvert \alpha \rvert < 1$	(a) Positive response term (sequence) decays monotonically
(pole inside unit circle)	(b) Negative response term decays in oscillatory manner
$\lvert \alpha \rvert = 1$	(a) Positive response term constant
(pole on unit circle)	(b) Negative response term oscillatory with constant amplitude
$\lvert \alpha \rvert > 1$	(a) Positive response term grows monotonically
(pole outside unit circle)	(b) Negative response term grows monotonically in oscillatory manner

Complex Conjugate Pole Pair

If

$$\alpha_1, \alpha_2 = (p \pm jq) = re^{\pm j\theta}$$

where

$$r = \sqrt{(p^2 + q^2)};\ \theta = \tan^{-1} q/p \tag{3.86}$$

Then the response term contributed by this pole pair will be

$$A r^{k-1} e^{j(k-1)\theta} + A^* r^{k-1} e^{-j(k-1)\theta};\ k \geq 1 \tag{3.87}$$

If $A = a\, e^{j\phi}$, then the response term may be written as

$$2a\, r^{k-1} \cos[(k-1)\theta + \phi];\ k \geq 1 \tag{3.88}$$

which is a discrete sinusoid with amplitude varying with r^{k-1}.

Following conclusions are drawn from Eq.(3.88) and the kind of response terms possible are depicted in Fig. 3.16.

$r < 1$ (pole inside unit circle)	Response term is a decaying sinsoid.
$r = 1$ (pole on unit circle)	Response term is sinusoid of constant amplitude.
$r > 1$ (poles outside unit circle)	Response term is growing sinusoid.

Similar conclusions can be drawn for repeated simple pole and complex conjugate pair.

Conclusions on Stability of a Discrete-time LTI System

In Section 1.9, we have defined asymptotic stability and BIBO stability. It was projected that an LTI discrete-time system which is asymptotically stable is also BIBO stable. Examination of the nature of response of LTI discrete-time systems depicted in Figs 3.15 and 3.16 reveals that the system impulse response decays to zero or the system is asymptotically stable, if all the system poles lie within a unit circle in z-plane, i.e.,

$$|z| < 1$$

The issue of stability and testing a system's Z-transfer function for determining its stability will be discussed in detail in Chapter 8.

Characterization of LTI system with the help of Z-transform and the relationship of RoC with causality and stability of the system are illustrated in Examples 3.25 and 3.26.

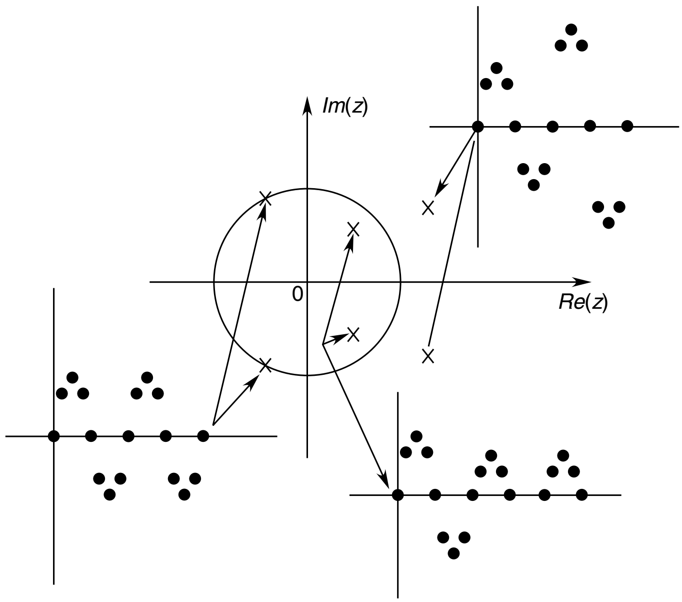


Fig. 3.16 *Response term contributed by simple complex conjugate system pole pair.*

3.10 STEADY-STATE SINUSOIDAL RESPONSE OF LTI-DT SYSTEM

Consider a discrete-time system with pulse transfer function $H(z)$. Let the input to the system be $r(n) = A\,e^{j\omega_0 n}$; $\omega_0 = 2\pi/N$, the discrete frequency. The system output is $y(n)$ as shown in Fig. 3.17.

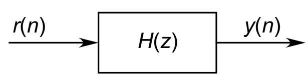


Fig. 3.17 *LTI discrete-time system*

$$Z[r(n)] = A\frac{z}{z - e^{j\omega_0}} \tag{3.89}$$

The output is then

$$Y(z) = A\,H(z)\frac{z}{z - e^{j\omega_0}} \tag{3.90}$$

For a stable system, the system poles contribute response terms which decay as $n \to \infty$. We therefore find that $y(n)|_{\text{steady-state}}$ is contributed by the excitation pole $z = e^{j\omega_0}$. Thus,

$$y(n)|_{ss} = \underset{\substack{\uparrow \\ \text{Eigen vector}}}{A\,H(e^{j\omega_0})} \qquad \underset{\substack{\uparrow \\ \text{Eigen function}}}{e^{j\omega_0 n}} \tag{3.91}$$

or

$$\begin{aligned} y(n) &= A|H(e^{j\omega_0})|\,e^{j\theta}\,e^{j\omega_0 n};\ \theta = \angle H(e^{j\omega_0}) \\ &= A|H(e^{j\omega_0})|\,e^{j(\omega_0 n + \theta)} \end{aligned} \tag{3.92}$$

In case of sinusoidal input

$$r(n) = A \cos \omega_0 n = Re\,[A e^{j\omega_0 n}] \tag{3.93}$$

Then

$$y(n) = Re\,[A|H(e^{j\omega_0})|\, e^{j(\omega_0 n+\theta)}] \tag{3.94}$$

or

$$y(n) = A|H(e^{j\omega_0})| \cos(\omega_0 n + \theta) \tag{3.95}$$

This result holds for any value of ω_0. We can thus find the discrete systems frequency response as

$$H(z)|_{z=e^{j\omega_0}} = H(e^{j\omega_0}) = |H(e^{j\omega_0})| \angle H(e^{j\omega_0}) \tag{3.96}$$

It is an important result that we can obtain the discrete-time system's frequency response by substituting $z = e^{j\omega_0}$ in its pulse transfer function.

This is similar to the continuous-time system where we get the frequency response by substituting $s = j\omega$ in its s-domain transfer function $H(s)$.

It can also be observed that $H(e^{j\omega_0})$ is periodic in ω_0 as shown below.

Let $\qquad \omega_0 = \omega_0 + N\omega_0$

Then $\qquad H(e^{j(\omega_0+N\omega_0)}) = H(e^{j\omega_0})$ as $N\omega_0 = 2\pi \qquad (3.97)$

3.11 DELAY OPERATOR AND Z-BLOCK DIAGRAMS

Delay Operator

The delay operator is of great significance in analysis and design of discrete-time or digital system.

A sequence $\{x(k)\}$ delayed by n intervals is

$$\{x\{k-n)\}$$

Taking its Z-transform, we have by the shifting property

$$Z[x(k-n)] = z^{-n} X(z); \qquad \text{for a relaxed LTI system}$$

In the Z-domain, multiplying $X(z)$ by z^{-n} yields the Z-transform of the signal-delayed n intervals. Obviously z^{-1} causes a delay of a single interval.

Z-Block Diagrams

Using the delay operator (z^{-1}), the block diagram of a difference equation can be easily drawn. Consider an nth order difference equation in delayed signal form.

$$\begin{aligned} y(k) + a_{n-1}\, y(k-1) + \ldots + a_1\, y(k-\overline{n-1}) + a_0\, y(k-n) \\ = r(k) + b_{m-1}\, r(k-1) + \ldots + b_1\, r(k-\overline{m-1}) \\ + b_0\, r(k-m);\ m \le n \end{aligned} \tag{3.98}$$

where the output is $\{y(k)$ and input is $\{r(k)\}$.

This equation can be written with output $y(k)$ on the left side as

$$y(k) = r(k) + b_{m-1}\, r(k-1) + \ldots + b_1\, r(k-\overline{m-1}) + b_0\, r(k-m)$$
$$-\, a_{n-1}\, y(k-1) - \ldots - a_1\, y(k-\overline{n-1}) - a_0\, y(k-n) \tag{3.90}$$

Its block diagram is drawn in Fig. 3.18 using the delay operator z^{-1}.

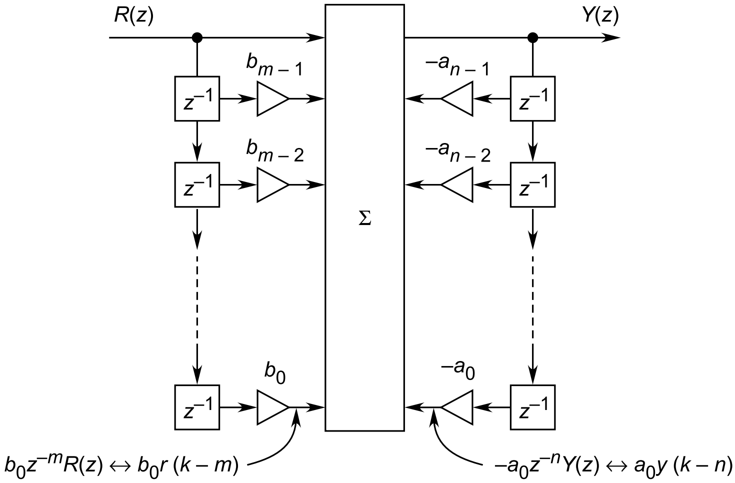


Fig. 3.18 *Block diagram of general difference equation*

This block diagram facilitates the computation of $\{y(k)\}$ for input $\{r(k)\}$ which are linked by the difference equation. The block diagram also provides the basis to realize a dedicated digital circuit of the system.

Example 3.25 *For the pulse transfer function*

$$H(z) = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

(a) write the corresponding difference equation, and

(b) draw the block diagram for H(z) in three different forms.

Solution

(a) $$H(z) = \frac{Y(z)}{R(z)} = \frac{1}{\left(1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}\right)} \tag{i}$$

or $$\left(1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}\right)Y(z) = R(z)$$

The corresponding difference equation is

$$y(k) + \frac{1}{4}y(k-1) - \frac{1}{8}y(k-z) = r(k)$$

or $$y(k) = r(k) - \frac{1}{4}y(k-1) + \frac{1}{2}y(k-2) \qquad \text{(ii)}$$

(b) I. **Direct Form**

We write the difference equations as

$$y(k) = r(k) - \frac{1}{4}y(k-1) + \frac{1}{8}y(k-2)$$

Using the delay operator (z^{-1}), the block diagram is drawn in Fig. 3.19(a).

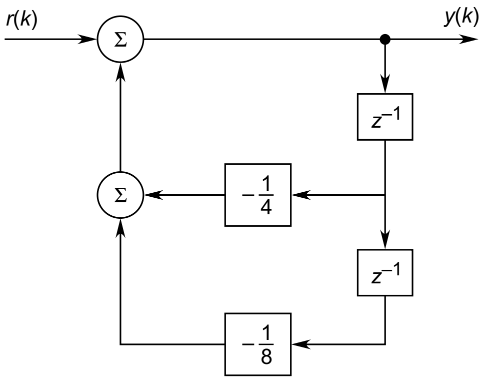


Fig. 3.19(a)

II. **Cascade Form**

Factoring the denominator, we can write

$$H(z) = \left[\frac{1}{1+\frac{1}{2}z^{-1}}\right]\left[\frac{1}{1-\frac{1}{4}z^{-1}}\right] \qquad \text{(iii)}$$

We draw the delay operator form for each component separately and connect them in cascade as shown in Fig. 3.19(b).

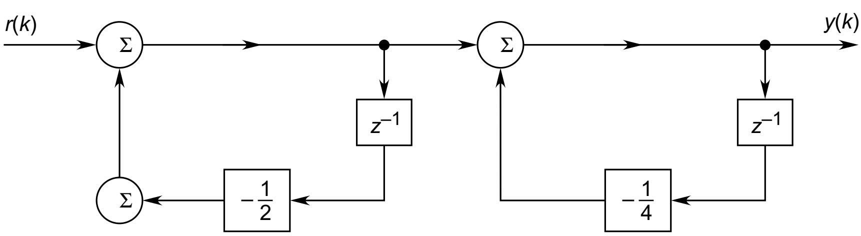


Fig. 3.19(b)

III. **Parallel Form**

Dividing $H(z)$ in partial fractions

$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}\Bigg|_{z^{-1}=-2} \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}\Bigg|_{z^{-1}=4} \frac{1}{1+\frac{1}{4}z^{-1}} \qquad \text{(iv)}$$

or $$H(z) = \frac{\frac{2}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+\frac{1}{4}z^{-1}}$$

Draw the delay operator form for each component and connect them in parallel as in Fig. 3.19(c).

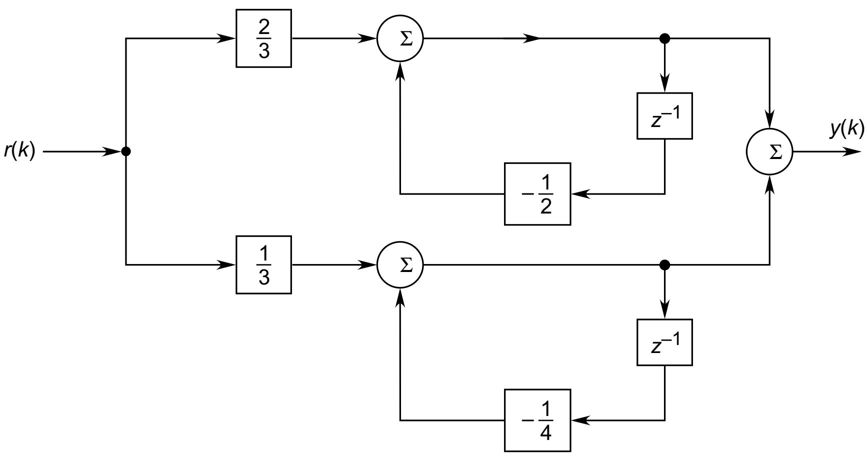


Fig. 3.19(c)

Example 3.26 *Draw the cascade-form realization of*

$$H(z) = \left[\frac{1-bz^{-1}}{1-az^{-1}}\right]$$

Solution We write

$$H(z) = H_1(z)\,H_2(z) = \frac{1}{(1-az^{-1})}\cdot(1-bz^{-1})$$

The block diagram is drawn in Fig. 3.20(a).

We can combine these into a single diagram using a single delay operator as in Fig. 3.20(b).

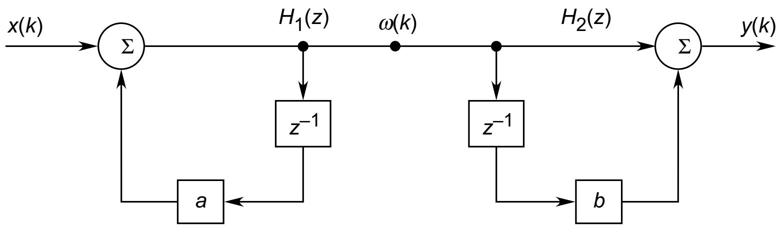


Fig. 3.20(a)

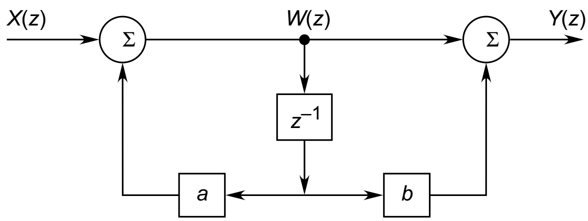


Fig. 3.20(b)

Example 3.27 *The Z-block diagram of a causal discrete system is drawn in Fig. 3.21. Write the difference equation relating output to input. Check the stability of the system.*

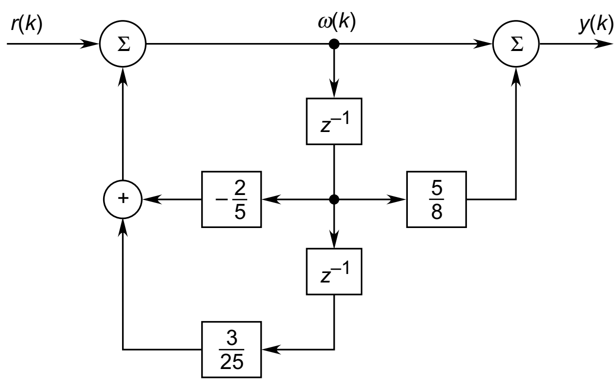


Fig. 3.21

Solution From left-side loops,

$$w(k) = r(k) - \frac{2}{5}w(k-1) + \frac{3}{25}w(k-2) \qquad \text{(i)}$$

From right-side loop,

$$y(k) = w(k) - \frac{5}{8}w(k-1) \qquad \text{(ii)}$$

Rewriting equations (i) and (ii)

$$w(k) + \frac{2}{5}w(k-1) - \frac{3}{25}w(k-2) = r(k)$$

$$w(k) + \frac{5}{8}w(k-1) = y(k)$$

Taking the *Z*-transform

$$\left(1+\frac{2}{5}z^{-1}-\frac{3}{25}z^{-2}\right)W(z) = R(z)$$

$$\left(1+\frac{5}{8}z^{-1}\right)W(z) = Y(z)$$

Dividing out, we get.

$$\frac{Y(z)}{R(z)} = \frac{\left(1+\frac{5}{8}z^{-1}\right)}{\left(1+\frac{2}{5}z^{-1}-\frac{3}{25}z^{-2}\right)} \tag{iii}$$

Its governing difference equation is

$$y(k) + \frac{2}{5}y(k-1) - \frac{3}{25}y(k-2) = r(k) + \frac{5}{8}r(k-1) \tag{iv}$$

Stability

The poles of the system are given by

$$1 + \frac{2}{5}z^{-1} - \frac{3}{25}z^{-2} = 0$$

or
$$\left(1+\frac{3}{5}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right) = 0$$

or
$$z = -\frac{2}{5},\ z = \frac{1}{5}$$

As for largest magnitude pole,

$$|z| = \frac{2}{5} < 1;\ \text{causal system}$$

The region of convergence $|z| > \frac{2}{5}$ includes the unit circle. The system is therefore stable.

Direct Form I and II Realizations

Rather than treat a general case, we shall illustrate by an example

$$H(z) = \frac{X(z)}{Y(z)} = \frac{2-0.1z^{-1}+0.2z^{-2}+0.3z^{-3}}{1-0.8z^{-1}+0.5z^{-2}+0.7z^{-3}}$$

$$= H_1(z)\,H_2(z)$$

where $H_1(z)$ is the numerator and $H_2(z)$ is the denominator. The realization $H_1(z)$ followed by $H_2(z)$ is drawn in Fig. 3.22. It is known as Direct Form I.

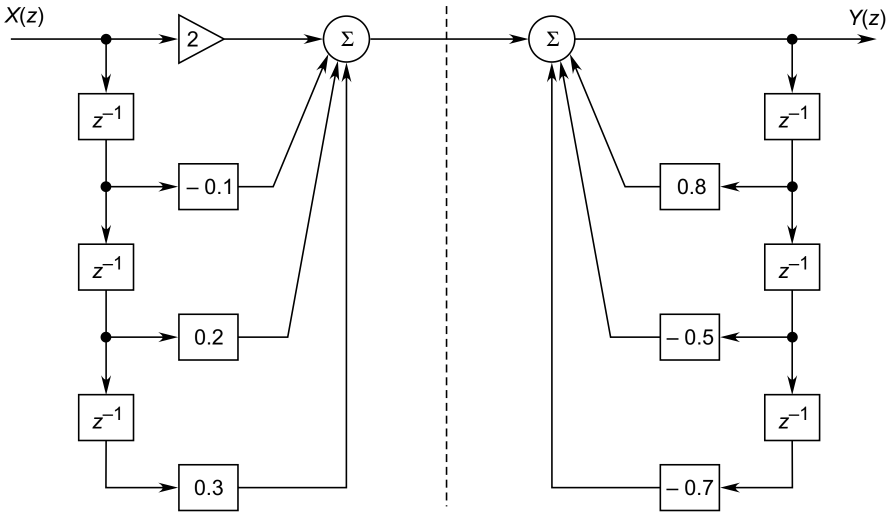


Fig. 3.22 *Direct Form I realization*

We can realize $H_1(z)$ and $H_2(z)$ in reverse order, that is, $H_2(z)$ proceeding $H_1(z)$. It means folding the Direct Form I about the dotted line. As a result, the corresponding z^{-1} pairs can be combined in one z^{-1} block. It means reducing to one-half the number of delay blocks needed for realization of $H(z)$. This is the Direct Form II drawn in Fig. 3.23.

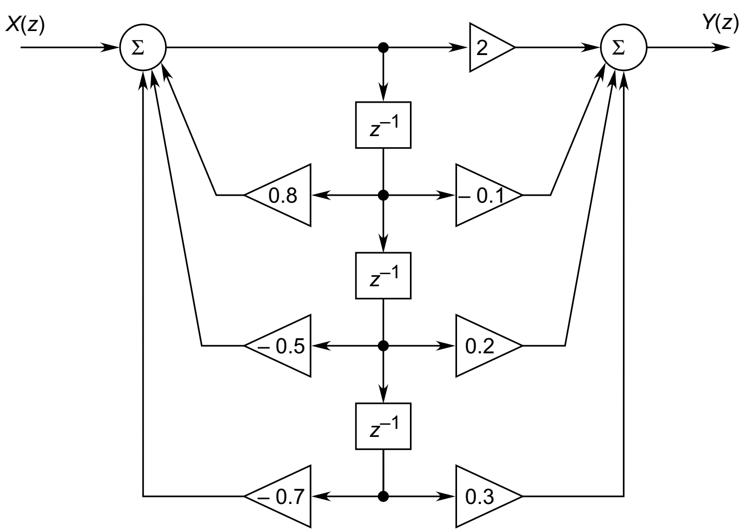


Fig. 3.23 *Direct Form II*

Realization of the Quadratic Form

Consider
$$H(z) = \frac{(1-a_j z^{-1})(1-a_j^* z^{-1})}{1-b_i z^{-1})(1-b_i^* z^{-1})}$$
$$= \frac{1-(a_j+a_j^*)z^{-1}+a_j a_j^* z^{-2}}{1-(b_i+b_i^*)z^{-1}+b_i b_i^* z^{-2}}$$

Its Direct Form II realization is drawn in Fig. 3.24.

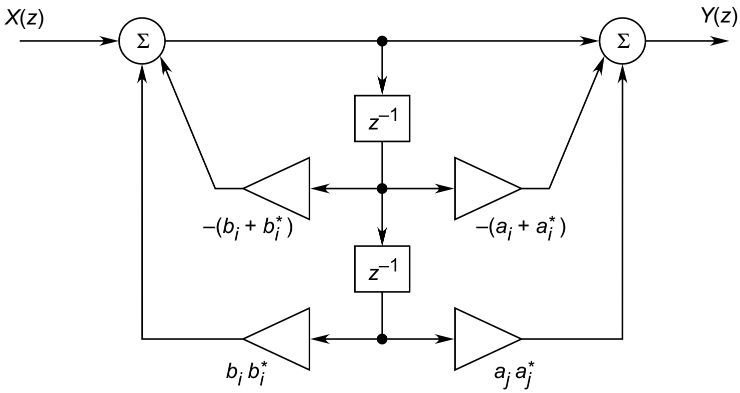


Fig. 3.24 *Quadratic-factor realization*

Discrete-time Integration

Approximate integration of a continuous-time function is carried out by discrete rectangular or trapezoidal integration.

According the rectangular method of discrete integration, the integrated value of $x(t)$ wrt time is equal to the area of rectangle, as shown by the shaded region in Fig. 3.25(a). Similarly, according to the trapezoidal method, the integrated value of $x(t)$ is equal to the area of the trapezium, as shown by the shaded region in Fig. 3.25(b).

The integrated value will be indicated as $y(kT)$. Discrete-time samples of $x(t)$ are taken at uniform time intervals T as shown in the Fig. 3.25.

Rectangular Integration From Fig. 3.25(a), we can write

$$y(kT) = y[(k-1)T] + Tx[(k-1)T] \tag{i}$$

Taking the Z-transform

$$Y(z) = z^{-1}Y(z) + Tz^{-1}X(z) \tag{ii}$$

It can be organized into the output–input form as

$$\frac{Y(z)}{X(z)} = H_1(z) = \frac{Tz^{-1}}{1-z^{-1}} \tag{iii}$$

Its z-domain block diagram is drawn in Fig. 3.25 (c).

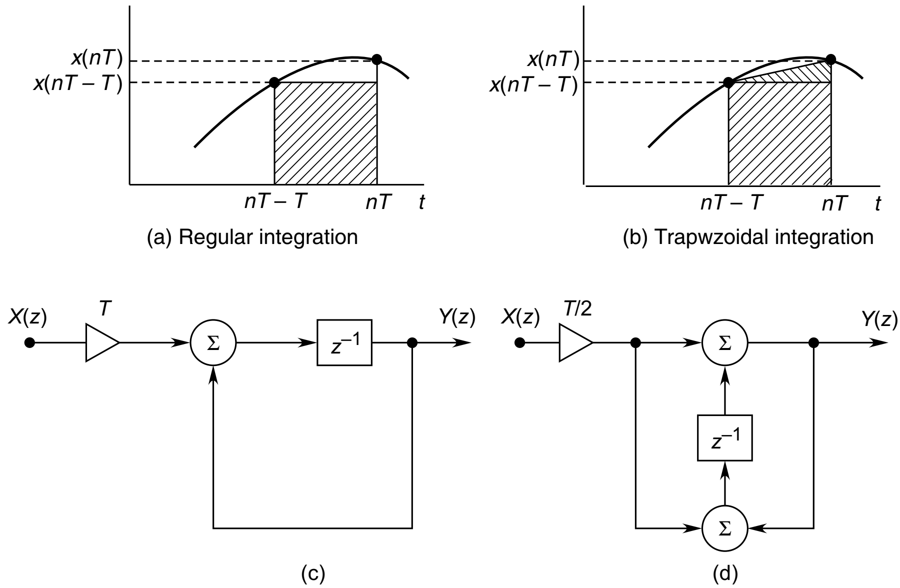


Fig. 3.25 *Discrete-time Integration*

Trapezoidal Integration

From Fig. 3.25 (b), we can write

$$y(kT) = y[(k-1)T] + T\left\{\frac{x[(k-1)T] + x(kT)}{2}\right\} \quad \text{(iv)}$$

Taking the Z-transform, we get

$$Y(z) = z^{-1}\,Y(z) + \frac{T}{2}[z^{-1}X(z) + X(z)] \quad \text{(v)}$$

Organizing in the input–output form, we get

$$\frac{Y(z)}{X(z)} = H_2(z) = \frac{T}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right] \quad \text{(vi)}$$

Its z-domain block diagram is drawn in Fig. 3.25(d).

The above results prove the two methods of computing the integral of $x(t)$, but they do not reveal much about their relative superiority. From Figs 3.25(a) and (b), it appears as if trapezoidal integration may yield better accuracy but this is misleading. The accuracy in fact depends on the frequency components of the integrand.

Frequency Response Let us write $z = e^{j\omega T}$ as

$$z = e^{j2\pi\omega/\omega_0} = e^{j2\pi r};\ \omega_0 = 2\pi/T \quad \text{(vii)}$$

$r = \omega/\omega_0$ being the normalized frequency, we subsitute in Eqs. (iii) and (vi). We get the follwoing results.

$$H_1(e^{j2\pi r}) = \frac{Te^{-j2\pi r}}{1 - e^{-j2\pi r}}; \text{ Rectangular } (R) \tag{viii}$$

and

$$H_2(e^{j2\pi r}) = \frac{T}{2}\frac{1 + e^{-j2\pi r}}{1 - e^{-j2\pi r}}; \text{ Trapezoidal } (T) \tag{ix}$$

By reorganizing, these results can be written in the following form.

$$H_1(e^{j2\pi r}) = \frac{Te^{-j\pi r}}{2j \sin \pi r}; R \tag{x}$$

and

$$H_2(e^{j2\pi r}) = \frac{T \cos \pi r}{2j \sin \pi r}; T \tag{xi}$$

As magnitude of the frequency response should be positive, both sine and cosine must be positive. This condition identifies the range of r as

$$0 \le r \le \frac{1}{2} \tag{xii}$$

Magnitude and phase as functions of frequency for the two integration methods can be written from Eq. (x) and (xi) as

Rectangular

$$\left.\begin{aligned} A_1(r) &= \frac{T}{2 \sin \pi r} \\ \phi_1(r) &= -\frac{\pi}{2} - \pi r \end{aligned}\right\} 0 \le r \le \frac{1}{2} \tag{xiii}$$

Trapezoidal

$$\left.\begin{aligned} A_2(r) &= \frac{T}{2}\frac{\cos \pi r}{\sin \pi r} \\ \phi_2(r) &= -\frac{\pi}{2} \end{aligned}\right\} 0 \le r \le \frac{1}{2} \tag{xiv}$$

The reader may plot these against r and compare the two integrations. At low frequencies $r << 1$, Eqs (x) and (xi) are approximated as

$$H_1(e^{j2\pi r}) = \frac{T}{2\pi} \cdot \frac{1}{jr} e^{-j\pi r}; \textit{ Rectangular} \tag{xv}$$

and

$$H_2(e^{j2\pi r}) = \frac{T}{2\pi} \cdot \frac{1}{jr}; \textit{ Trapezoidal} \tag{xvi}$$

Certain Observations

1. At low frequencies both integrations approach the ideal integrates, i.e., $(1/jr)$.

2. Rectangular integration has a time delay corresponding to $e^{-j\pi r}$. Trapezoidal integration has no such time delay.
3. So long as T is small, there is not much to choose between the two integrations as the observations from time and frequency domains do not coincide.

Additional Examples

Example 3.28 *Determine the RoC such that given proper rational transfer function*

$$H(z) = \frac{7z+4}{z^2 - 4z + 3}$$

represents

1. *causal system*
2. *single-sided noncausal (or left-sided) system*

What do you conclude from this example?

Solution Let us rewrite the given Z-transfer function in factored form as

$$H(z) = \frac{7z+4}{(z-1)(z-3)} \tag{i}$$

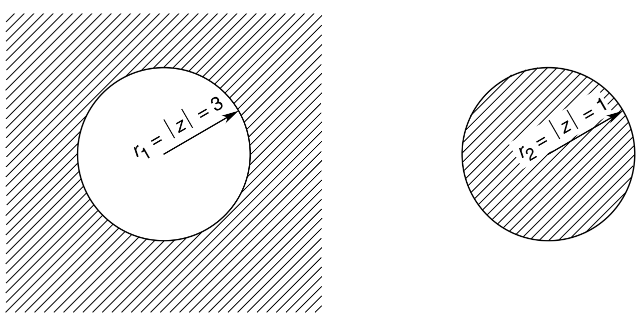
Partial fractioning $H(z)$, we obtain

$$H(z) = \frac{(-11/2)}{(z-1)} + \frac{(25/2)}{(z-3)} \tag{ii}$$

1. If the given $H(z)$ has to behave as a causal or right-sided system then RoC must lie outside a circle of radius r_1 such that this circle does not include any pole. Hence we must have

$$r_1 = |z| > 3$$

This is shown in Fig. 3.26(a).



(a) RoC of causal system; $|z| > 3$ (b) RoC of non-causal system; $|z| > 1$

Fig. 3.26

2. Similarly, for $H(z)$ to be a single-sided noncausal system, its RoC must lie within a circle of radius r_2 such that circle should not include any pole. This is only possible if $r_2 = |z| < 1$.

It is concluded from above that for a causal or right-sided system, RoC is the region in the z-plane outside the circle of radius equal to the largest magnitude pole of $H(z)$.

Similarly, for a noncausal or left-sided system, RoC is the region in the z-plane inside the circle of radius equal to the smallest magnitude pole of $H(z)$.

Example 3.29

(a) Show that the system described by the following equation is noncausal.

$$H(z) = \frac{z^3 + 2z^2 + 1.5z - 0.75}{z^2 - 4.5z + 5}$$

(b) Show that system described by the following equation is causal and unstable.

$$H(z) = \frac{2.75z - 0.75}{z^2 - 4.5z + 5}; \; RoC : |z| > 2.5$$

(c) Show that the system described by the following equation is causal and stable.

$$H(z) = \frac{z - 3.0}{z^2 - 1.25z + 0.375}; \; RoC : |z| > 0.75$$

(d) What do you conclude from the above results?

Solution

(a) According to the initial value theorem, for a causal or right-handed sequence

$$\lim_{z \to \infty} H(z) = h(0) \text{ ; finite}$$

But for given $H(z)$, we have

$$\lim_{z \to \infty} H(z) \to \frac{z^3}{z^2} \to \infty$$

Hence the given $H(z)$ represents a noncausal system. Equivalently, an improper rational function (numerator of higher degree than the denominator) represents a noncausal system or sequence.

(b) The given $H(z)$ can be rewritten as

$$H(z) = \frac{2.75z - 0.75}{(z - 2.0)(z - 2.5)}; \; \text{RoC} : |z| > 2.5$$

$$H(z) = \frac{(-9.5)}{(z-2.0)} + \frac{(12.25)}{(z-2.5)};\ \text{RoC} : |z| > 2.5 \qquad \text{(iii)}$$

Since RoC lies beyond the pole of largest magnitude (i.e., 2.5), it represents a causal system.

Taking inverse Z-transform of Eq. (iii), we obtain

$$h(k) = (-9.5)\,(2.0)^{k-1} + 12.25\,(2.5)^{k-1};\ k > 0 \qquad \text{(iv)}$$

or

$$h(k) = (-9.5)\,(2.0)^{k} + 12.25\,(2.5)^{k};\ k \geq 1 \qquad \text{(v)}$$

In order to establish whether given $H(z)$ represents stable system or not let us evaluate the expression $\sum_{k=0}^{\infty} |h(k)|$ where $h(k)$ is as given in Eq. (v). It can be easily seen that as $k \to \infty$, the expression $\sum_{k=0}^{\infty} |h(k)|$ tends towards ∞.

This clearly means that the given $H(z)$ is unstable. In this case it is important to note that RoC is $|z| > 2.5$ and hence does not include the unit circle.

(c) $$H(z) = \frac{z-30}{z^2 - 1.25z + 0.375}\quad \text{RoC} : |z| > 0.75$$

Partial fractioning this, we get

$$H(z) = \frac{10.0}{(z-0.5)} - \frac{9.0}{(z-0.75)};\ |z| > 0.75 \qquad \text{(vi)}$$

Since, RoC lies beyond the pole of largest magnitude (i.e. 0.75), $H(z)$ represents a causal system.

Again taking inverse Z-transform of Eq. (vi), we get

$$h(k) = (10.0)\,(0.5)^{k-1} - (9.0)\,(0.75)^{k-1};\ k \geq 0 \qquad \text{(vii)}$$

or

$$h(k) = (10.0)\,(0.5)^{k} - (9.0)\,(0.75)^{k};\ k \geq 1 \qquad \text{(viii)}$$

If we again evaluate the expression $\sum_{k=0}^{\infty} |h(k)|$ with $h(k)$ as given in Eq. (viii), we find that as $k \to \infty$ $\sum_{k=0}^{\infty} |h(k)|$ tends towards finite value. Hence, in this case, $H(z)$ represents a causal and stable system. We note that, in this case, the RoC includes the unit circle.

(d) From the above results, we can draw the following conclusions.

1. The necessary condition for a system to be causal is that it must be represented by a proper rational function (i.e., function with degree of numerator at least one less than the degree of denominator).

2. The sufficient conditions for a system to be causal are that it must be proper rational as well as its RoC must lie beyond the pole of largest magnitude.
3. For a system to be stable, its RoC must include the unit circle.

Example 3.30 *Find the Z-transform of $e^{-\alpha t}$ when sampled at time intervals T. What is the region of its convergence?*

Using the above result, establish the Z-transforms of the following continuous-time functions when sampled.

(i) sin bt *(ii) cos bt*

(iii) $e^{-\alpha t}$ sin bt *(iv) $e^{-\alpha t}$ cos bt*

Solution We shall first present a general result for

$$f(t) = e^{-\alpha t}\;;\; t \geq 0$$

Let $t = kT$; T being the sampling period. The sampled signal is

$$f(kT) = e^{-k\alpha T}\;;\; k \geq 0$$

The Z-transform of this sequence is given as

$$F(z) = \sum_{n=0}^{\infty} f(kT)\, z^{-k} = \sum_{n=0}^{\infty} e^{-k\alpha T} z^{-k}$$

$$= \frac{1}{1 - e^{-\alpha T} z^{-1}}\;;\; |\, e^{-\,\alpha T} z^{-1} \,| < 1$$

or

$$F(z) = \frac{z}{z - e^{-\alpha T}}\;;\; |\, e^{-\,\alpha T} \,| < |\, z \,|\;;\; \text{if } \alpha \text{ is real}$$

In case α is imaginary then $|\, e^{-\,\alpha T} \,| = 1 < |\, z \,|$

These results will be used in the solutions that follow.

(i) The sampled signal is

$$f(kT) = \sin\,(kbT) \quad ;\; k \geq 0$$

In exponential form

$$\sin\,(kbT) = \frac{e^{jkbT} - e^{-\,jkbT}}{2j}$$

Taking the Z-transform, we get

$$\mathrm{Z}[\sin\,(kbT)] = \mathrm{Z}\left[\frac{e^{jkbT} - e^{-\,jkbT}}{2j}\right]$$

$$= \frac{1}{2j}\,[\mathrm{Z}\{e^{jkbT}\} - \mathrm{Z}\{e^{-\,jkbT}\}]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{jbT}} - \frac{z}{z-e^{-jbT}}\right]$$

$$= \frac{1}{2j} \times \frac{2jz \sin bt}{z^2 - 2z\cos bT + 1}; |z| > |e^{\pm jbT}| \text{ or } |z| > 1$$

Thus,

$$Z[\sin kbT] = \frac{z \sin bT}{z^2 - 2z\cos bT + 1}; |z| > 1$$

(ii) The Z-transform of cos bt is obtained in the following manner.

$$f(kT) = \cos(kbT) = \frac{e^{jkbT} + e^{-jkbT}}{2}; k \geq 0$$

Then

$$Z[\cos kbT] = \frac{1}{2}[Z\{e^{jkbT}\} + Z\{e^{-jkbT}\}]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{jkbT}} + \frac{z}{z-e^{-jbT}}\right]; |z| > |e^{\pm jbT}| \text{ or } |z| > 1$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2z\cos bT}{z^2 - 2z\cos bT + 1}\right]; |z| > 1$$

(iii) The Z-transform of $f(t) = e^{-\alpha t}$ sin bt for $t \geq 0$ with $t = kT$ is as follows,

$$Z[f(t)] = Z\left\lfloor e^{-kbT} \sin kbT \right\rfloor$$

$$Z\left[e^{-k\alpha T}\left\{\frac{e^{jkbT} - e^{-jkbT}}{2j}\right\}\right]$$

or

$$F(z) = \frac{1}{2j}\left[\frac{z}{z-e^{-T(\alpha - jb)}} - \frac{z}{z-e^{-T(\alpha + jb)}}\right]; |e^{-\alpha T}| < |z|$$

On simplification, the result can be expressed in the following manner.

$$F(z) = \frac{1}{2j}\left[\frac{z\left\{e^{-\alpha T}(e^{jbT} - e^{jbT}\right\}}{z^2 - 2e^{-\alpha T}\{e^{-jbT} + e^{jbT}\} + e^{-2\alpha T}}\right]$$

$$= \frac{ze^{-\alpha T} \sin bT}{z^2 - 2e^{-\alpha T} z \cos bT + e^{-2\alpha T}}; |e^{-\alpha T}| < |z|$$

(iv) On similar lines, the Z-transform of $e^{-\alpha t}$ cos bt for $t \geq 0$, with $t = kT$ is obtained as

$$e^{-k\alpha T} \cos kbT \leftrightarrow \frac{ze^{-\alpha T}(ze^{\alpha T} - \cos tbT)}{z^2 - 2e^{-\alpha T} z \cos bT + e^{-\alpha T}}; |e^{-\alpha T}| < |z|$$

Example 3.31

(a) Find the Z-transforms of the following.

(i) $f(k) = k + \sin 2k\ ;\ k \geq 0$

(ii) $f(k) = k(-1)^k + k(1)^k\ ;\ k \geq 0$

(iii) $f(k) = \begin{cases} 0 & \text{for} \quad k = 1,3,5\ldots(odd) \\ 0.5k & \text{for} \quad k = 0,2,4\ldots(even) \end{cases}$

(b) Verify that

$$Z[k^2 b^k] = \frac{b(z-3b)z}{(z-b)^3}$$

Solution

(a) (i) $Z[f(k)] = Z[k] + Z[\sin 2k]$

We have $Z[k] = \sum_{k=0}^{\infty} kz^{-k} = \sum_{k=0}^{\infty} (-1)(-k)z^{-k-1}z$

or
$$= (-1)z \sum_{k=0}^{\infty} -kz^{-k-1}$$
$$= -z\frac{d}{dz}F_1(z)$$

where $F_1(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1};\ |z| > 1$

Thus

$$Z[k] = z\frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2};\ |z| > 1$$

Using the results obtained in Example 3.15, we have

$$Z[\sin 2k] = \frac{z \sin 2}{z^2 - 2z\cos 2 + 1};\ |z| > 1$$

$$Z[k + \sin 2k] = \frac{z}{(z-1)^2} + \frac{z\sin 2}{(z^2 - 2z\cos 2 + 1)};\ |z| > 1$$

(ii)
$$f(k) = k(-1)^k + k(1)^k$$
$$= k[(-1)^k + (1)^k];\ k \geq 0$$

This sequence has zero values of k-odd. So we consider even k only

Then $Z[f(k)] = \sum_{m=0}^{\infty} f(2m)z^{-2m}\ ;\ k = 2m$

$$= \sum_{m=0}^{\infty} 2m\{(-1)^{2m} + (1)^{2m}\}z^{-2m}$$

$$= \sum_{m=0}^{\infty} 2m(-1)^{2m} z^{-2m} + \sum_{m=0}^{\infty} 2m(1)^{2m} z^{-2m}$$

Since $(-1)^{2m} = (1)^{2m} = 1$

$$F(z) = 4 \sum_{m=0}^{\infty} m(z^2)^{-m}$$

Now, let $z^2 = p$, then

$$F(p) = 4\left[\sum_{m=0}^{\infty} mp^{-m}\right] = 4\left[\frac{p}{(p-1)^2}\right]; |p| > 1$$

Then

$$F(z) = 4\left(\frac{z}{z^2-1}\right)^2; |z^2| > | \to |z| > 1$$

(iii) $Z[f(k)] = \sum_{k=0}^{\infty} F(k) z^{-k}$

Separating out even and odd values,

$$F(z) = \sum_{m=0}^{\infty} f(2m) z^{-2m} + \sum_{m=0}^{\infty} f(2m+1) z^{-(2m+1)}$$

Since odd terms of the sequence $f(k)$ are zero, we have

$$F(z) = \sum_{m=0}^{\infty} 0.5 \times 2mz^{-2m} = \sum_{m=0}^{\infty} mz^{-2m}$$

$$= \sum_{m=0}^{\infty} m(z^2)^{-m}$$

Let, $z^2 = p$, so

$$F(p) = \sum_{m=0}^{\infty} mp^{-m}$$

i.e.

$$F(p) = \frac{p}{(p-1)^2}; |p| > 1$$

Then

$$F(z) = \frac{z^2}{(z^2-1)^2}; |z^2| > 1 \to |z| > 1$$

(b) In order to verify the following,

$$Z[k^2 b^k] = \frac{b(z+b)z}{(z-b)^3}$$

We express the left-hand side as

$$Z[k^2 b^k] = \sum_{k=0}^{\infty} k^2 b^k z^{-k}$$

Let

$$bz^{-1} = p$$

Then

$$Z[k^2 b^k] = \sum_{k=0}^{\infty} k^2 p^k = \sum_{k=0}^{\infty} k^2 p^{k-2}.p^2$$

$$= \sum_{k=0}^{\infty} (k^2 - k + k)\, p^{k-2}.p^2$$

$$= \sum_{k=0}^{\infty} (k^2 - k)\, p^{k-2}.p^2 + \sum_{k=0}^{\infty} kp^k$$

$$= p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left\{\sum_{k=0}^{\infty} p^k\right\} + \sum_{k=0}^{\infty} kp^k$$

$$= p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left\{\frac{p}{p-1}\right\} + \frac{p}{(p-1)^2}; |p| < 1$$

The above expression is obtained using the following results.

$$Z[1] = \frac{p}{p-1} \quad \text{and} \quad Z[k] = \frac{p}{(p-1)^2}$$

Thus,

$$Z[k^2 b^k] = p^2\left\{\frac{2}{(p-1)^3}\right\} + \left\{\frac{p}{(p-1)^2}\right\}$$

$$= \frac{2p^2}{(p-1)^3} + \frac{p}{(p-1)^2}$$

$$= \frac{b(z-3b)z}{(z-b)^3}$$

Example 3.32 *Find the inverse Z-transform of the following. Is it a positive-time function?*

$$\frac{z^3}{(z+1)}$$

Solution We have

$$F(z) = \frac{z^3}{(z+1)}$$

or

$$F_1(z) = \frac{F(z)}{z^2} = \frac{z}{(z+1)} = \frac{z}{z-(-1)}$$

$$F(z) = z^2 F_1(z)$$

$$Z^{-1}[F(z)] = Z^{-1}\left\lfloor z^2 F_1(z)\right\rfloor = z^2\left\lfloor Z^{-1}\{F_1(z)\}\right\rfloor$$

or

$$f(k) = z^2\left[Z^{-1}\left\{\frac{z}{z-(-1)}\right\}\right] = (-1)^{k+2};\ k \geq -2$$

Example 3.33

(a) If F[z] ↔ f[k] is a positive-time sequence and if we define

$$g[k] = \sum_{i=0}^{b} f[i]$$

then find out whether g[k] is a positive-time sequence or not.

(b) Show that

$$g[k] = \sum_{i=0}^{b} q[k-1]\, f[i]$$

where q[k] = unit step sequence.

(c) Show that

$$G(z) = \frac{z}{(z-1)}\, F(z)$$

Solution

(a)
$$g[k] = \sum_{i=0}^{k} f[i]$$
$$= f(0) + f(1) \ldots f(k) \quad \text{(i)}$$
$$= \text{positive-time sequence}$$

(b) Equation (i) can be written as

$$g[k] = q(k-0)\, f(0) + q(k-1)\, f(1) ... + q(k-i)\, f(i) \quad \text{(ii)}$$

$$= \sum_{i=0}^{k} q[k-i]\, f[i];\ q[k] \text{ being a step sequence} \quad \text{(iii)}$$

(c)
$$G(z) = Z[g(k)] = \sum_{k=0}^{\infty} z^{-k}\left[\sum_{i=0}^{k} q(k-i)\, f(i)\right]$$

Since the summation that immediately follows the equal sign (=) has k exten-ding up to ∞, the summation inside the bracket will have limit i from 0 to ∞. Thus, we get

$$G(z) = \sum_{k=0}^{\infty} z^{-k} \sum_{i=0}^{\infty} g(k-i)\, f(i)$$

Writing $k - i = m$, and changing the order of summation, we get

$$G(z) = \sum_{i=0}^{\infty} f(i) z^{-(i+m)} \sum_{m=-i}^{\infty} q(m)$$

$$= \sum_{i=0}^{\infty} f(i) z^{-i} \sum_{m=0}^{\infty} q(m) z^{-m}$$

The second summation will have limits from $m = 0$ to ∞. This is because for $m < 0$; $q(m) = 0$

Thus,

$$G(z) = F(z)\, Q(z)$$

It is noted that $Q(z)$ is the Z-transform of unit step sequence given by

$$Q(z) = \frac{z}{z-1}$$

Hence

$$G(z) = \frac{z}{(z-1)} F(z)$$

Example 3.34 *Find the inverse Z-transform of the following in closed form.*

(a) $$Y(z) = \frac{5}{1-0.2z^{-1}}$$

(b) $$Y(z) = \frac{z+z^{-1}}{z\left(1+\frac{1}{8}z^{-1}\right)}$$

(c) $$Y(z) = \frac{1}{(1-0.75\, z^{-1} + 0.125\, z^{-2})}$$

(d) $$Y(z) = \frac{z^{-1}}{(1-z^{-1})(1-0.2\, z^{-1})^2}$$

Solution

(a) Given $$Y(z) = \frac{5}{1-0.2z^{-1}} = \frac{5z}{z-0.2}$$

$$y(k) = Z^{-1}\left[\frac{5z}{z-0.2}\right]$$

Using the Z-transform table, we get.

$$y(k) = 5(0.2)^k \; ; k \geq 0$$

(b) Given
$$Y(z) = \frac{z + z^{-1}}{z\left(1 + \frac{1}{8} z^{-1}\right)}$$

or

$$Y(z) = \frac{z^2 + 1}{z\left(z + \frac{1}{8}\right)} = \frac{z}{\left(z + \frac{1}{8}\right)} + \frac{1}{z\left(z + \frac{1}{8}\right)}$$

We obtained the inverse of the two parts of $Y(z)$, using the Z-transform table.

$$Z^{-1}\left[\frac{z}{(z + 1/8)}\right] = \left(\frac{-1}{8}\right)^k \; ; k \geq 0$$

and

$$Z^{-1}\left[\frac{z^{-1}}{(z + 1/8)}\right] = Z^{-1}\left\{z^{-2}\left(\frac{z}{z + 1/8}\right)\right\}$$

$$= \left(-\frac{1}{8}\right)^{k-2}$$

Then

$$y(k) = \left(-\frac{1}{8}\right)^k + \left(-\frac{1}{8}\right)^{k-1}$$

$$= 65\left(-\frac{1}{8}\right)k \quad ; k \geq 0$$

(c)
$$Y(z) = \frac{1}{(1 - 0.75\, z^{-1} + 0.125\, z^{-2})}$$

or

$$Y(z) = \frac{z^2}{z^2 + 0.75\, z + 0.125}$$

Let
$$Y_1(z) = \frac{Y(z)}{z}$$

so that
$$Y_1(z) = \frac{z}{z^2 + 0.75z + 0.125}$$

or

$$Y_1(z) = \frac{z}{(z + 0.25)\,(z + 0.5)}$$

By partial fractioning, we get

$$Y_1(z) = \frac{2}{(z+0.5)} - \frac{1}{(z+0.25)}$$

$$Y(z) = \frac{2z}{(z+0.5)} - \frac{1}{(z+0.25)}$$

Therefore,

$$y(k) = 2(-0.5)^k - 3(-0.25)^k \;\; ; k \geq 0$$

(d)

$$Y(z) = \frac{z^{-1}}{(1-z^{-1})(1-0.2z^{-1})^2}$$

$$= \frac{z^2}{(z-1)(z-0.2)^2}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.2)^2}$$

$$= \frac{1.5625}{(z-1)} - \frac{1.5625}{(z-0.2)} - \frac{0.25}{(z-0.2)^2}$$

Then

$$Y(z) = (1.5625)\left[\frac{z}{(z-1)}\right] - (1.5625)\left[\frac{z}{z-0.2}\right] - 0.25\left[\frac{z}{(z-0.2)^2}\right]$$

Taking the inverse *Z*-transform, we get

$$y(k) = 1.5625(1)^k - 1.5625(0.2)^k - 0.25k(0.2)^{k-1}; k \geq 0$$

Example 3.35 *Find the solution of the following difference equations.*

$$y(k) + \sqrt{2}\, y(k-1) + y(k-2) = r(k)$$

Initial conditions: $y(-1) = 2\sqrt{2},\; y(-2) = 2;\; r(k) = u(k)$

Solution

(a) Taking the *Z*-transform on both sides of the given difference equation, we get

$$Z\left[y(k) + \sqrt{2}\, y(k-1) + y(k-2)\right] = Z[u(k)] \qquad \text{(i)}$$

Using shifting property, we can write

$$Y(z) + \sqrt{2}\,[z^{-1}Y(z) + y(-1)] + [z^{-2}Y(z) + y(-1)z^{-1} + y(-2)]$$

$$= \frac{z}{z-1} \qquad \text{(ii)}$$

Using initial conditions, we have

$$Y(z) + \sqrt{2}\,[z^{-1}Y(z) + 2\sqrt{2}] + [z^{-2}Y(z) + 2\sqrt{2}z^{-1} + 2] = \frac{z}{z-1}$$

$$Y(z) + [1+\sqrt{2}z^{-1}+z^{-2}] + (2\sqrt{2}z^{-1}+6) = \frac{z}{z-1} \qquad \text{(iii)}$$

From Eq. (iii), we can write the output as

$$Y(z) = \frac{z}{(z-1)\left(1+\sqrt{2}z^{-1}+z^{-2}\right)} - \frac{2\sqrt{2}z^{-1}+6}{1+\sqrt{2}z^{-1}+z^{-2}}$$

or

$$Y(z) = \frac{z^3}{(z-1)\left(z^2+\sqrt{2}z+1\right)} - \frac{6z^2+2\sqrt{2}z}{z^2+\sqrt{2}z+1} \qquad \text{(iv)}$$

$$= Y_1(z) - Y_2(z) \qquad \text{(v)}$$

We will now find the Z-inverse of $Y_1(z)$ and $Y_2(z)$.

From Eq. (iv) and (v), we can write

$$\frac{Y_1(z)}{z} = \frac{z^2}{(z-1)(z^2+\sqrt{2}z+1)} \qquad \text{(vi)}$$

$$\frac{Y_1(z)}{z} = \frac{z^2}{(z-1)(z^2+\sqrt{2}z+1)} \qquad \text{(vi)}$$

$$= \frac{z^2}{(z-1)\left(z+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{2}-j\frac{1}{\sqrt{2}}\right)} \qquad \text{(vii)}$$

Assuming $\frac{Y_1(z)}{z} = F_1(z)$ and partial fractioning the right-hand side of Eq. (vii), we obtain

$$\frac{Y_1(z)}{z} = F_1(z) = \frac{A}{(z-1)} + \frac{B}{\left\{z+\left(\frac{1+j}{\sqrt{2}}\right)\right\}} + \frac{B^*}{\left\{z+\left(\frac{1-j}{\sqrt{2}}\right)\right\}} \qquad \text{(viii)}$$

where

$$A = F_1(z)\,(z-1)|_{z=1} = 0.293 \qquad \text{(ix)}$$

$$B = F_1(z)\left\{z+\left(\frac{1+j}{\sqrt{2}}\right)\right\}\Bigg|_{z=-\left(\frac{1+j}{\sqrt{2}}\right)} = 0.353 - j\,0.146 \qquad \text{(x)}$$

$$B^* = F_1(z)\left\{z+\left(\frac{1-j}{\sqrt{2}}\right)\right\}\Bigg|_{z=-\left(\frac{1-j}{\sqrt{2}}\right)} = 0.353 + j\,0.146 \qquad \text{(xi)}$$

So,

$$F_1(z) = \frac{Y_1(z)}{z} = \frac{(0.293)}{(z-1)} + \frac{(0.353 - j\,0.146)}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}} + \frac{(0.353 + j\,0.146)}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}} \qquad \text{(xii)}$$

or

$$Y_1(z) = 0.293\left[\frac{z}{z-1}\right] + (0.353 - j\,0.146)$$

$$\left[\frac{z}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}}\right] + (0.353 + j\,0.146)\left[\frac{z}{\left\{z + \left(\frac{1-j}{\sqrt{2}}\right)\right\}}\right]$$

Now, converting $(a + jb)$ form into polar form, i.e., $re^{j\theta}$ form and taking inverse Z-transform, we obtain

$$y_1(k) = 0.293\,(1)^k + 0.382\,e^{-j22.47°}\left\{-\left(\frac{1+j}{\sqrt{2}}\right)^k\right\}$$

$$+\,0.382\,e^{j22.47°}\left\{-\left(\frac{1-j}{\sqrt{2}}\right)^k\right\} \quad ; k \geq 0$$

or,

$$y_1(k) = 0.293 + 0.382\,e^{-j22.47°}\,e^{j45°k} + 0.382\,e^{j22.47°}\,e^{-j45°k};\ k \geq 0$$

or

$$y_1(k) = 0.293 + 0.382\,[e^{j(45°k - 22.47°)}\,e^{-j(45°k - 22.47°)}];\ k \geq 0$$

i.e. $\quad y_1(k) = 0.293 + 0.764\cos(45°k - 22.47°);\ k \geq 0 \qquad \text{(xiv)}$

Now for the second form,

$$\frac{Y_2(z)}{z} = F_2(z) = \frac{6z + 2\sqrt{2}}{z^2 + \sqrt{2}\,z + 1} = \frac{6z + 2\sqrt{2}}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}\left\{z + \left(\frac{1-j}{\sqrt{2}}\right)\right\}} \qquad \text{(xv)}$$

By partial fractioning, we get

$$F_2(z) = \frac{Y_2(z)}{z} = \frac{A}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}} + \frac{A^*}{\left\{z + \left(\frac{1-j}{\sqrt{2}}\right)\right\}} \qquad \text{(xvi)}$$

where

$$A = F_2(z)\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}\Bigg|_{z=-\left(\frac{1+j}{\sqrt{2}}\right)} = 3 - j \tag{xvii}$$

$$A^* = F_2(z)\left\{z + \left(\frac{1-j}{\sqrt{2}}\right)\right\}\Bigg|_{z=-\left(\frac{1-j}{\sqrt{2}}\right)} = 3 + j \tag{xviii}$$

So

$$\frac{Y_2(z)}{z} = \frac{(3-j)}{\left\{z + \left(\frac{1+j}{\sqrt{2}}\right)\right\}} + \frac{(3+j)}{\left\{z + \left(\frac{1-j}{\sqrt{2}}\right)\right\}}$$

or,

$$Y_2(z) = (3-j)\left[\frac{z}{z + \left(\frac{1+j}{\sqrt{2}}\right)}\right] + (3+j)\left[\frac{z}{z + \left(\frac{1-j}{\sqrt{2}}\right)}\right] \tag{xix}$$

Taking inverse Z-transform, we get

$$y_2(k) = (3-j)\left\{-\left(\frac{1+j}{\sqrt{2}}\right)\right\}^k + (3+j)\left\{-\left(\frac{1-j}{\sqrt{2}}\right)\right\}^k \quad ; k \geq 0$$

or,

$$y_2(k) = 3\left[\left(\frac{-1-j}{\sqrt{2}}\right)^k + \left(\frac{-1+j}{\sqrt{2}}\right)^k\right] + j\left[\left(\frac{-1+j}{\sqrt{2}}\right)^k - \left(\frac{-1-j}{\sqrt{2}}\right)^k\right]; k \geq 0 \tag{xx}$$

Again converting $a + jb$ form into the polar form $re^{j\theta}$ and manipulating the expression, we get

$$y_2(k) = 6 \cos 45°\, k + 2 \sin 45°\, k;\ k \geq 0 \tag{xxi}$$

Thus, we have $y(k) = y_1(k) + y_2(k)$

i.e.

$$y(k) = 0.293 + 0.764 \cos (45°k - 22.47°) + 6 \cos 45°\, k + 2 \sin 45°\, k;\ k \geq 0 \tag{xxii}$$

Problems

3.1 Prove the following properties.

(i) $\{r(k)\}^*\{\delta(k-n)\} = \{r(k-n)\}$

(ii) $\{r(k)^*\{\delta(k)\} = \{r(k)\}$

(iii) $\{\delta(k-n)\}^*\{\delta(k-m)\} = \{\delta(k-n-m)\}$

(iv) $\{\delta(k-n)\}^*\{\delta(k-m)\} = \{\delta(k-n-m)\}$

(v) $\{u(k)\}^*\{u(k)\} = (k+1)\{u(k)\}$

(vi) $(\{u(k)\} - u\{u(k-n)\})^*\{u(k)\} = (k+1)\,\{u(k)\} - (k-n+1)\,\{u(k-n)\}$

3.2 Obtain the impulse response of cascaded time-variant discrete-time system shown in Fig. P-3.2.

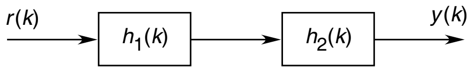


Fig. P-3.2

Also, prove that the identical impulse response will be obtained, if system blocks are reversed.

3.3 Refer to Problem 3.2 and determine $y(k)$, if

$$h_1(k) = \cos 2k$$
$$h_2(k) = \alpha^k u(k),\ |\,\alpha\,| < 1$$

and

$$r(k) = \delta(k) - \alpha\delta(k-1)$$

3.4 Determine and plot the sequence $\{r_1(k)\} = \{r(k)\} - \alpha\{r(k-1)\}$. It is given that $\{r(k)\} = \alpha^k\{u(k)\}$

3.5 Find the discrete-time convolution $\{r(k)\}^*\{h(k)\}$ of the following.

(i) $\{r(k)\} = (/2)^k\,\{u(k)\} + 3^k\,\{u(-k)\}$
$\{h(k)\} = \{u(k)\}$

(ii) $\{r(k)\} = 3^k$
$\{h(k)\} = (2/2)^k\{u(k)\}$

(iii) $\{r(k)\} = \{2\ 1\ 3 - 1\}$; 2 occurs at 0^{th} instant
$\{h(k)\} = \{3\ 5 - 2\}$; 3 occurs at 0^{th} instant

3.6 Given below are the impulse responses of certain discrete-time systems. Check if each system is causal and/or stable.

(a) $h(n) = \left(\dfrac{1}{4}\right)^n u(n)$

(b) $h(n) = (4)^n\, u(2-n)$

(c) $h(n) = \left(-\dfrac{1}{3}\right)^n u(n) + (1.02)^n\, u(n-1)$

(d) $h(n) = \left(\dfrac{1}{2}\right)^n nu(n-1)$

3.7 Find the impulse response of the first-order difference equation

$$y(n) + 3\,y(n-1) = x(n);\ x(n) \text{ causal}$$

assuming the system at rest initially.

3.8 Consider an LTI discrete-time system described by the difference equation

$$y(n) + 2\,y(n-1) = x(n) + 2\,x(n-2)$$

The input $x(n)$ is given by the data

$$x(-2) = 1,\ x(-1) = 2,\ x(0) = 4,\ x(1) = 2,\ x(2) = -2,\ x(3) = 1$$

All other values of $x(n)$ are zero.

3.9 Given

$$x(n) = \begin{cases} 1 & ; 0 \le n \le 6 \\ 0 & ; 7 \le n \le 9 \end{cases}$$

(a) What is the period of this signal? Determine its discrete Fourier series coefficients.
(b) Determine the discrete Fourier series coefficient of

$$y(n) = x(n) - x(n-1)$$

3.10 A discrete signal $x(n)$ is represented as a sequence

1 0 2 –1 1 0 2 –1 1 0 2 –1 1 02 –1 1 0 2 –1

(a) Plot the sequence and find its period
(b) Write the equations for $x(n)$ in terms of discrete Fourier series coefficients and solve them directly for A_k.
(c) Write the expression for A_k and check the result of (b).

3.11 For the signal

$$x(n) = 1 - \sin\frac{\pi n}{4} \text{ for } 0 \le n \le 3$$

determine the discrete Fourier series coefficients.

3.12 For the signal

$$x(n) = 1 - \sin\frac{\pi n}{4} \text{ for } 0 \le n \le 7$$

determine its discrete Fourier series coefficients.

3.13 Find the discrete Fourier series coefficients for the following discrete signal

$$x(n) = \sin 0.1\,\pi n;\ \text{period } N = 10$$

3.14 Obtain the discrete Fourier series for the periodic sampled signal shown in Fig. P-3.14.

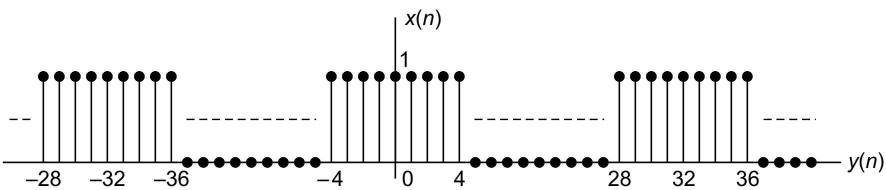


Fig. P-3.14

3.15 A discrete-time periodic signal $x(n)$ has period $N = 7$. Its non-zero Fourier coefficients are

$$A_0 = 2,\ A_2 = 2\,e^{j\pi/3},\ A_4 = e^{j\pi/5}$$

Express $x(n)$ in this form

$$x(n) = a_0 + \sum_{k=1}^{\infty} a_k \sin(\omega_k n + \phi_k)$$

3.16 A discrete-time system has impulse response

$$h(n) = \begin{cases} 1 & 0 \le n \le 3 \\ -1 & -3 < n \le -1 \end{cases}$$

The system imput is

$$x(n) = \sum_{k=-\infty}^{+\infty} \delta(n - 3k)$$

Determine its Fourier series coefficients. From there find the Fourier series coefficients of the output.

Hint: $B_k = A_k H(e^{jk\omega_0})$

3.17 Find the Z-transform the sequence ($k \ge 0$)

(i) k^2 (ii) $k\,\alpha^{k-1}\,u(k-1)$

(iii) $\cosh \alpha k$ (iv) $\alpha^k \cos k\pi$

3.18 The continuous functions ($t \ge 0$) given below are discretized by sampling mathematically at regular time intervals of width T. Find the Z-transform of the following functions after discretization.

(i) t^2 (ii) $t\,e^{-\alpha t}$

(iii) $\alpha^t e^{-\alpha t}$ (iv) $\alpha^t e^{-\alpha t} \sin \omega t\, u(t)$

3.19 Samples of a time function $x(t)$ are given below. Determine $X(z)$.

$$\begin{aligned} x(0) &= 1 \\ x(T) &= 4.7 \\ x(2T) &= x(3T) = 0 \\ x(4T) &= 0.75 \\ x(5T) &= \sqrt{2} \\ x(nT) &= 0,\ n \ge 6 \end{aligned}$$

3.20 The function $x(t) = 5p_1(t - 4)$; pulse width = I*s* is sampled. Determine the Z-transform of the sampled sequence for each of the following sampling frequencies.

(a) $f_s = 1$ Hz (b) $f_s = 0.5$ Hz

3.21 For each of the systems defined by the given difference equations, classify the system as linear or nonlinear or causal or noncausal, and shift-invariant or nonshift invariant. The system input is $r(nT)$ and the system output is $y(nT)$.

(a) $y(nT) = ar(nT) + b$; a and b are constants

(b) $y(nT) = r(nT - n_0T)$, n_0 is a constant integer

(c) $y(nT) = r(nT + T) + r(nT - T)$

3.22 Convolve the two functions $r_1(nT)$ and $r_2(nT)$ shown in Fig. P-3.22.

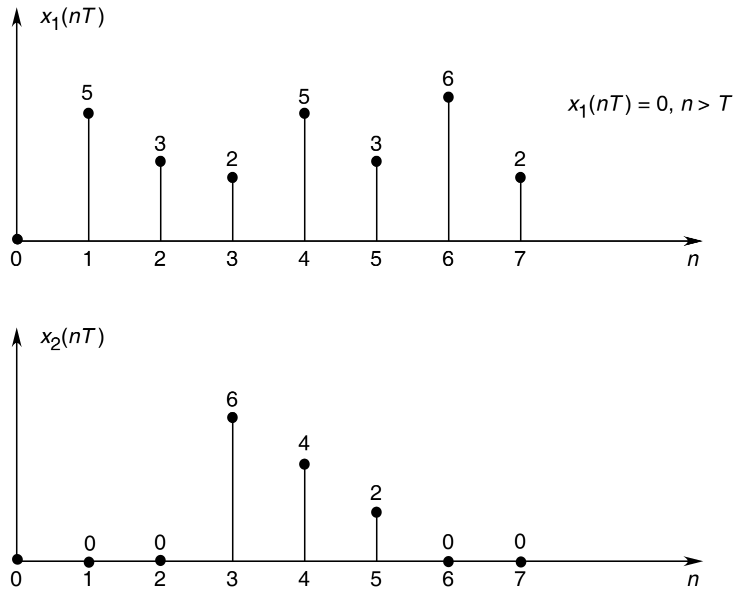


Fig. P-3.22

3.23 A linear shift-invariant system has the input $r(nT)$ and its unit pulse response $h(nT)$ as shown in Fig. P-3.23. Determine the output $y(nT)$.

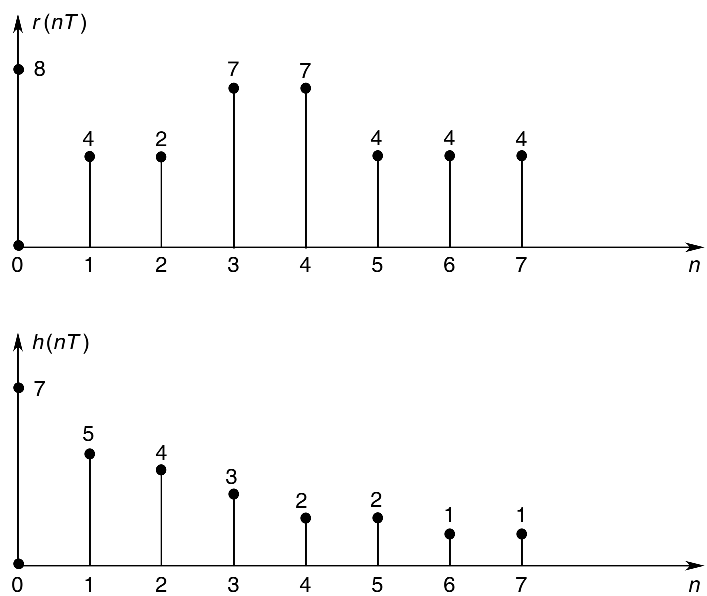


Fig. P-3.23

3.24 Find the impulse response for the discrete-time systems defined by the following equation.

(i) $3y(k+1) = y(k) + r(k+1)$

(ii) $y(k) = 3y(k-1) + r(k-1)$

(iii) $y(k+1) = -0.5y(k) + 2\delta(k+1)$

(iv) $y(k+2) + 5y(k+1) + 2y(k) = 3r(k+1) - r(k)$

3.25 Find the impulse response of the two systems connected in cascade. The individual system are defined by

$$y_1(k) - 0.8y_l(k-1) = r_1(k)$$
$$y_2(k) - 0.9y_2(k-1) = 0.\mathrm{l}r_2(k)$$

3.26 A system is described by the following difference equation.

$$y(k+1) + \alpha_1 y(k) + \alpha_2 r(k+1) + \alpha_3 r(k) = 0$$

What is its impulse response sequence? Also, prove that $h(0) = 0$ if and only if $\alpha_2 = 0$.

3.27 Solve the following difference equations. Identify which equation are homogeneous.

(i) $y(k) - 3y(k-1) + y(k-2) = 0;\ y(0) = 1,\ y(6) = 5$

(ii) $y((k+2) + 5y(k+1) + 3y(k) = u(k);\ y(0) =1,\ y(k) = 0$ for $k < 0$

(iii) $y(k+2) + 4y(k+1) + 5y(k) = 2(-3)^k u(k)$; $y(k) = 0$, for $k \le 0$.

3.28 Determine the monthly instalment plan for a debt of Rs 30 000. This is to be returned in monthly instalments with a monthly interest rate of 14 per cent on unpaid debt.

3.29 Find the Z-transform of the sequences for ($k \ge 0$)

(i) $\alpha^{k-1}u(k-1)$ (ii) $\delta(k-2) + 3^k$

(iii) α^k/k (iv) $e^{\alpha k}$

(v) $(-1/2)^k u(-k-2)$ (vi) $\cosh \alpha k$

(vii) $k^2 \alpha^k$ (viii) $k + \sin k$

3.30 Find the Z-transform (if it exists) and region of convergence of the following sequences.

(i) $(1/2)^n u(3-n)$ (ii) $\alpha_1^k u(k) - \alpha_2^k u(-k-1)$

(iii) $(1/2)^{k-1} u(k-2)$ (iv) $\delta(k+3)$

3.31 Find the inverse Z-transform of the following

(i) $1/(z+\alpha)$; $|z| > \alpha$ (ii) $1/(z-\alpha)$;$|z| < \alpha$

(iii) $z^2/(z^2-1)$ (iv) $z^2/(z^2-1)(3z+1)$

(v) $(z^2+2z+3)/(z^2-3z+2)$;$|z| < 2$

3.32 Find the unit impulse response $h(k)$ of the systems described by the following difference equations.

(i) $y(k-1) - y(k) = r(k)$

(ii) $y(k+2) + 3y(k+1) + y(k) = r(k+2) - r(k+1)$

(iii) $y(k) - 2y(k-1) + y(k-2) = r(k) + r(k-2)$

(iv) $4y(k+2) + 4y(k+1) + y(k) = r(k+1)$

3.33 Find the zero-state response of LTI discrete systems with following pulse transfer functions.

(i) $\dfrac{\alpha_1 z+\alpha_2}{(z-\alpha_1)(z-\alpha_2)}$ (ii) $\dfrac{\alpha_1(\alpha_2 z-1)}{\alpha_1 z^2-\alpha_2 z+1}$ (iii) $\dfrac{z-1}{z^2-2z+1}$

The input is a unit step sequence.

3.34 A discrete system is described by

$$y(k) + \alpha_1 y(k+1) + \alpha_2 y(k+2) = r(k+2);\ \alpha_1 = -0.8,\ \alpha_2 = 0.2$$

Find the complete response, if the input is a unit step sequence.

3.35 Obtain the complete response of the system described by the following difference equation

$$y(k+2) - 4y(k+1) - 3y(k-1) = r(k+2) - 3r(k+1)$$

Following is the input sequence.

$$r(k) = \begin{cases}(-1)^k & \text{for } k < 4\\ 0 & \text{for } k \geq 4\end{cases}$$

3.36 An LTI discrete system is described by the *Z*-transfer function

$$H(z) = \frac{z}{z-1.25};\ \text{RoC } |z| > 1.25$$

Is this system causal and stable?

3.37 Show that the system described by the transfer function is causal as well as stable.

$$H(z) = \frac{z}{z-0.5};\ \text{RoC } |z| > 0.5$$

3.38 Comment on the characteristics of the discrete system described by

$$H(z) = \frac{(z-0.2)(z-0.1)}{(z-0.5)};\ \text{RoC } |z| > 0.5$$

3.39 A discrete system is described by the difference equation

$y(n) - 0.2\,y(n-1) = r(n)$; input $r(n)$; output $y(n)$

Check by the *Z*-transform if this system is LTI and stable.

3.40 Obtain the impulse transfer function of the discrete system described by

$$y(n+1) - 0.4\,y(n) = 0.2\,r(n+1) + r(n)$$

Is this system causal?

3.41 Obtain the transfer function of the system described by

$$y(n+2) + a_1\,y(n+1) + a_0\,y(n) = \delta(n);\ \text{impulse input.}$$

3.42 Determine and sketch the Direct Form I and II realizations of the following pulse transfer functions

(a) $H(z) = \dfrac{(1-z^{-1})^2}{(1+z^{-1})^3}$ (b) $H(z) = 1 + \dfrac{0.125}{(1-0.2z^{-1})^3}$

(c) $H(z) = \dfrac{z}{1-z^{-1}} + \dfrac{1-0.4z^{-1}}{1-0.5z^{-1}}$

3.43 Determine the cascade and parallel realizations using only first-order factors for the following pulse transfer function.

(a) $H(z) = \dfrac{1-2z^{-1}}{(1-z^{-1})(1-0.2z^{-1})}$

(b) $H(z) = \dfrac{1}{(1-0.2z^{-1})(1+0.4z^{-1}-0.21z^{-2})}$

3.44 Determine the parallel realization for

$$H(z) = \frac{1-0.25z^{-2}}{(2-z^{-1})(3-2z^{-1})^2}$$

3.45 For the *Z*-block diagram drawn in Fig. P-3.45, determine its pulse transfer function and there from find its Direct Form II realization.

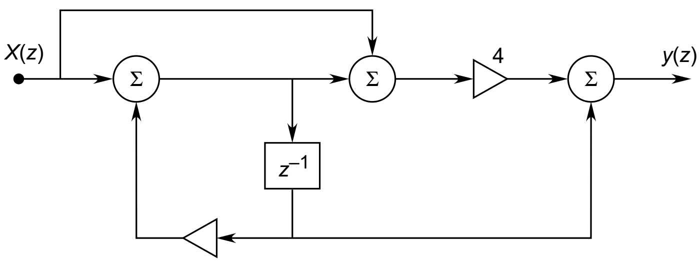


Fig. P-3.45

3.46 Realize the following using only one z^{-1} element.

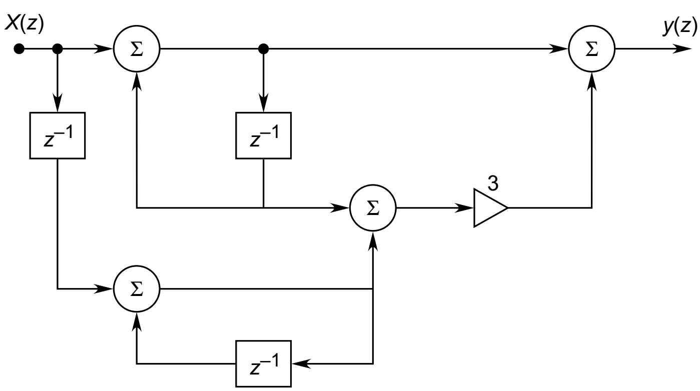


Fig. P-3.46

3.47 Consider a discrete-time system with

$$H(z) = \frac{(1-z^{-1})^3}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}$$

Draw its Z-block diagram in cascade and parallel form.

Hint: In partial fractioning, proceed by finding the partial fractions of $\dfrac{H(z)}{z}$.

3.48 For $H(z) = \dfrac{\left(1+\frac{1}{6}z^{-1}\right)(1-3z^{-1})}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{6}z^{-1}\right)}$

Draw its Z-block diagrams in the caseade and parallel forms.

3.49 For the Z-block diagram,

(a) determine the governing difference equation.

(b) check stability of the system.

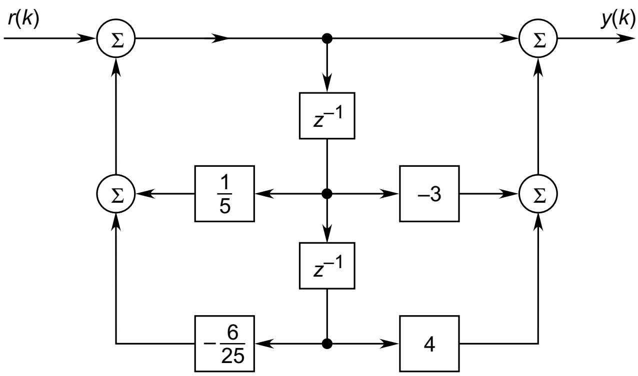


Fig. P-3.49

Hint: Label the top middle node as $w(k)$. Determine $W(z)/R(z)$ and $Y(z)/W(k)$; eliminate $W(z)$ and proceed.

Discrete Fourier Transform and Fast Fourier Transform

4

Introduction

In earlier chapters continuous and discrete-time signals and systems and system responses were studied. It was also pointed out that when a continuous signal is discretized, the discretization process must be carried out above a certain minimum sampling rate for the signal characteristics to be preserved.

Discrete-time periodic signals are conveniently represented by Discrete Fourier Series (DFS) which has been discussed in Chapter 3. As a logical step forward is the Discrete Fourier Transform (DFT) and its numerical computation by Fast Fourier Transform (FFT) algorithm.

4.1 DISCRETE FOURIER TRANSFORM (DFT)

The exponential Fourier series of a periodic signal of period T (see Eq. (2.61)) is given as

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}, \ \omega_0 = \frac{2\pi}{T} \tag{4.1}$$

The coefficients F_k are found from the relationship

$$F_k = \left(\frac{1}{T}\right) \int_T f(t)\, e^{-jk\omega_0 t}\, \mathrm{d}t \tag{4.2}$$

where $\omega_0 = 2\pi/T$ rad/s is the fundamental frequency and $\int_T (\ )$ means integrating over any one period.

A typical periodic signal is drawn in Fig. 4.1(a) indicated by the continuous line.

Let us now consider the sampled version of $f(t)$ with sampling interval Δt, so chosen that the number of samples per period is $N = T/\Delta t$, an integer. The sampled signal $f(n\Delta t) \sim f(n)$ is shown in Fig. 4.1(b). The discrete version of Eq. (4.2) is

$$F_k = \left(\frac{1}{T}\right) \sum_{n=0}^{N-1} f(n\Delta t)\, e^{-jk(2\pi/T)\,(n\Delta t)}\, \Delta t;\ k = 0, 1, 2, \ldots, (N-1) \tag{4.3}$$

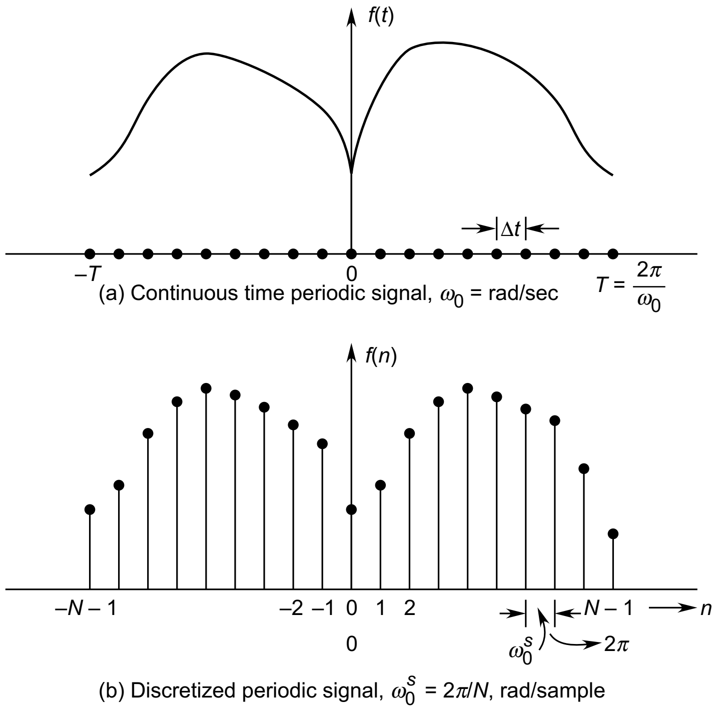


Fig. 4.1

As the discrete signal $f(n\Delta t)$ is periodic, the sum could be taken over any N consective samples say $N = 2$ to $(N + 1)$. For convenience, we choose the period as 0 to $(N - 1)$.

As $T/\Delta t = N$, Eq. (4.3) can be written in the form

$$F_k = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)\, e^{-jk(2\pi/N)n}\,;\; k = 0, 1, \ldots, (N-1) \tag{4.4}$$

Let ω_0^s (sampling frequency) = $2\pi/N$ radian. We shall for convenience write from now onwards ω_0^s as ω_0 but remember that it is not $\omega_0 = 2\pi/T$ radian per second for continuous signal, which will no longer be used. We have then the alternate form of Eq. (4.4) as

$$F_k = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)\, e^{-jk\omega_0 n}\,;\; k = 0, 1, \ldots, (N-1) \tag{4.5}$$

The corresponding discretized form of Eq. (4.1) is

$$f(n) = \sum_{k=0}^{N-1} F_k\, e^{jk(2\pi/N)n} = \sum_{k=0}^{N-1} F_k\, e^{jk\omega_0 n} \tag{4.6}$$

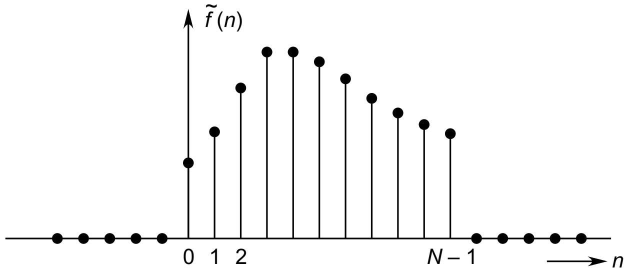


Fig. 4.2 *Discrete aperiodic signal*

In Fig. 4.1(a), $f(t)$ is periodic and so $f(n)$ in Fig. 4.1(b) is discrete periodic. Equation (4.5/4.6) is the *discrete Fourier series* (DTFS) representation of $f(n)$, with DTFS coefficients F_k. The discrete-time Fourier series has been presented at length in Section 3.4, where the symbol A_k was used in place of F_k.

From the periodic signal $f(n)$ as shown in Fig. 4.1(b), we can construct an aperiodic signal

$$f(n) = \tilde{f}(n)$$

where

$$\begin{aligned}\tilde{f}(n) &= f(n),\ n = 0, 1, \ldots, N-1 \\ &= 0 \ \text{ for } \ n < 0 \ \text{ and } \ n \geq N \end{aligned} \tag{4.7}$$

as shown in Fig. 4.2.

As Eqs. (4.5) and (4.6) are summations over one period, these are valid for aperiodic signals as well. These equations are reproduced below by replacing $f(n)$ by $\tilde{f}(n)$.

$$\begin{aligned}F_k &= \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} \tilde{f}(n)\, e^{-jk(2\pi/N)n} \\ &= \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} \tilde{f}(n)\, e^{-jk\omega_0 n};\ k = 0, 1, \ldots, (N-1)\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\tilde{f}(n) &= \sum_{k=0}^{N-1} F_k\, e^{jk(2\pi/N)n} \\ &= \sum_{k=0}^{N-1} F_k\, e^{jk\omega_0 n};\ k = 0, 1, \ldots (N-1)\end{aligned} \tag{4.9}$$

In Eqs. (4.8) F_k is the direct *Discrete Fourier Transform* (*DFT*) of the time series $f(0), f(1), \ldots, f(N-1)$. Its inverse $\tilde{f}(n)$ is given by Eq. (4.9).

The computation of DFT forward and inverse can be carried out by the above two equations. The mechanization of computation of DFT and reduction of computational effort will be taken up in Section 4.6.

4.2 CLOSED-FORM DISCRETE FOURIER TRANSFORM (DFT)

For the aperiodic signal F_k in Eq. (4.8) the sum is from $n = 0$ to $(N-1)$. As $f(n) = 0$ for $n < 0$ and $n > (N-1)$ we can write it as

$$F_k = \left(\frac{1}{N}\right) \sum_{n=-\infty}^{+\infty} f(n)\, e^{-k\omega_0 n} \tag{4.10}$$

Similarly, from Eq. (4.9), we write

$$f(n) = \sum_{k=-\infty}^{+\infty} F_k\, e^{jk\omega_0 n} \tag{4.11}$$

We now define the function

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\omega n} \tag{4.12}$$

We can then write Eq. (4.10) as

$$F_k = \left(\frac{1}{N}\right) F(e^{jk\omega_0}) \tag{4.13}$$

Substituting in Eq. (4.11) gives

$$f(n) = \sum_{k=(N)} \frac{1}{N} F(e^{jk\omega_0})\, e^{jk\omega_0 n} \tag{4.14}$$

as it is periodic in $N = 2\pi/\omega_0$. Equation (4.14) can then be written as

$$f(n) = \frac{1}{2\pi} \sum_{k=(N)} F(e^{jk\omega_0})\, e^{jk\omega_0 n}\, \omega_0 \tag{4.15}$$

As N increases, ω_0 decreases. Let $N \to \infty$. Then $\omega_0 \to \mathrm{d}\omega$ and $k\omega_0 \to \omega$. The summation of Eq. (4.15) becomes an integration,

$$f(n) = \frac{1}{2\pi} \int_{2\pi} F(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega \tag{4.16}$$

We now have the DFT pair, corresponding to equations (4.16) and (4.12)

Synthesis Equation

$$f(n) = \frac{1}{2\pi} \int_{2\pi} F(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega \tag{4.17}$$

Analysis Equation

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\omega n} \tag{4.18}$$

We can write this pair as

$$f(n) \xleftrightarrow{\text{DFT}} F(e^{j\omega}) \tag{4.19}$$

DFT equations (4.17) and (4.18) are indeed the discrete counterpart of Fourier Transform pair of Eqs. (2.84) and (2.85). Like in *FT* ($F(j\omega)$), DFT ($F(e^{j\omega})$) is the *spectrum of discrete signal* $f(n)$ and it contains the information of $f(n)$ in the form of complex frequency exponentials from which $f(n)$ can be reconstructed by Eq. (4.17). It follows from Eq. (4.18) by replacing ω with $(\omega + 2\pi)$ that DFT is periodic.

$$F(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j(\omega+2\pi)n} \quad \sum_{n=-\infty}^{+\infty} f(n) e^{-j\omega n}$$
$$= F(e^{j\omega})$$

The periodicity is inherent in DFT irrespective of the signal (aperiodic or periodic). It is of interest to note that FT of a continuous time signal is in general. aperiodic (periodic only in specific signals).

The periodicity of DFT is illustrated in Fig. 4.3 for a particular case.

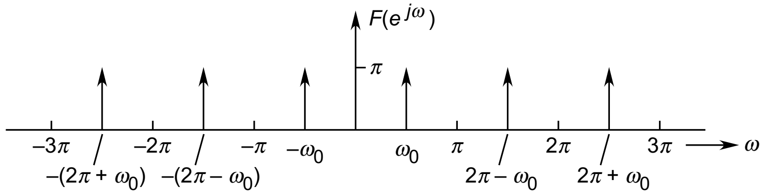


Fig. 4.3 *Repeats every 2π*

4.3 PROPERTIES OF DFT

The properties of DFT are similar to these of FT and can be proved on the same lines. Therefore, only properties will be presented here without proof. Of course, proofs are presented for certain properties for instructional purposes. Operator $\mathcal{F}_d$ is used to represent discrete time transform pair.

1. Periodicity

$$F(e^{j(\omega+2\pi)}) = F(e^{j\omega}) \tag{4.20}$$

2. Linearity

$$a\, f_1(n) + b\, F_2(n) \xleftrightarrow{\mathcal{F}_d} a\, F_1(e^{j\omega}) + b\, F_2(e^{j\omega}) \tag{4.21}$$

3. Time Shifting

$$f(n) \xleftrightarrow{\mathcal{F}_d} F(e^{j\omega})$$

$$f(n - n_0) \xleftrightarrow{\mathcal{F}_d} e^{-j\omega n_0}\, F(j\omega) \tag{4.22}$$

4. Frequency Shifting

$$e^{j\omega_0 n} f(n) \xleftrightarrow{\mathcal{F}_d} F(e^{j(\omega-\omega_0)}) \tag{4.23}$$

5. *Conjugation*

$$f(n) \xleftrightarrow{\mathcal{F}_d} F(e^{j\omega}) \tag{4.24}$$

Then

$$f^*(n) \xleftrightarrow{\mathcal{F}_d} F^*(e^{-j\omega}) \tag{4.25}$$

Special Cases

(a) $f(n)$ real

$$F(e^{j\omega}) = F^*(e^{-j\omega}) \tag{4.26a}$$

It then follows that

$Re\,[F(e^{j\omega})]$ is an even function of ω

$Im\,[F(e^{j\omega})]$ is an odd function of ω

Also,

$|F(e^{j\omega})|$ is an even function of ω and $\angle F(e^{j\omega})$ is odd function of ω.

(b) $f(n)$ real even $\rightarrow F(e^{j\omega})$, real and even (4.26b)

(c) $f(n)$ real odd $\rightarrow F(e^{j\omega})$, pure imaginary and odd (4.26c)

(d) $f(n)$ real

$$\text{Even }\{f(n)\} = f_e(n) \rightarrow Re\,[F(e^{j\omega})] \tag{4.26d}$$

$$\text{Odd }\{f(n)\} = f_0(n) \rightarrow j\,Im\,\{F(e^{j\omega})] \tag{4.26e}$$

6. Time Reversal

$$f(-n) \xleftrightarrow{\mathcal{F}_d} F(e^{-j\omega}) \tag{4.27}$$

7. Differentiation

$$[f(n) - f(n-1)] \xleftrightarrow{\mathcal{F}_d} (1 - e^{-j\omega})\, F(e^{j\omega}) \tag{4.28a}$$

Follows from time shifting property

Accumulation

$$\sum_{m=-\infty}^{n} f(n) \xleftrightarrow{\mathcal{F}_d} \frac{1}{1-e^{-j\omega}} F(e^{j\omega}) + \pi F(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \tag{4.28b}$$

The second term (impulse trains) accounts for dc or average value that could result from summation.

8. Differentiation in Frequency

$$nf(n) \xleftrightarrow{\mathcal{F}_d} j\frac{\mathrm{d}F(e^{j\omega})}{\mathrm{d}\omega} \tag{4.29}$$

9. Parseval's Relationship for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |f(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |F(e^{j\omega})|^2\, \mathrm{d}\omega \tag{4.30}$$

10. Convolution

$$y(n) = x(n) * h(n)$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}); \text{ a product} \tag{4.31}$$

11. Multiplication

$$f(n) = f_1(n)\, f_2(n) \tag{i}$$

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} f_1(n)\, f_2(n)\, e^{-j\omega n} \tag{ii}$$

As per synthesis equation

$$f_1(n) = \frac{1}{2\pi}\int_{2\pi} F_1(e^{j\theta})\, e^{j\theta n}\, d\theta \tag{iii}$$

We can then write

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} f_2(n)\left\{\frac{1}{2\pi}\int_{2\pi} F_1(e^{j\theta})\, e^{j\theta n}\, d\theta\right\} e^{-j\omega n} \tag{iv}$$

Interchanging integration and summation

$$F(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} F_1(e^{j\theta}) \underbrace{\left\{\sum_{n=-\infty}^{+\infty} f_2(n)\, e^{-j(\omega-\theta)n}\right\}}_{F_2(e^{j(\omega-\theta)})} d\theta$$

Then $$F(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} F_1(e^{j\theta})\, F_2(e^{j(\omega-\theta)})\, d\theta \tag{4.32}$$

We find that $F(e^{j\omega})$ is *periodic convolution* of $F_1(e^{j\omega})$ and $F_2(e^{j\omega})$. This property is similar to the aperiodic convolution in continuous-time Fourier Transform where the integration is from $-\infty$ to $+\infty$, Eq. (2.104). This property will find use in Chapter 5 on Sampling.

Impulse Transfer Function—Frequency Response

If $h(n)$ is the impulse response of a system, its output $y(n)$ for input $x(n)$ is

$$y(n) = x(n) * h(n)$$

By convolution property

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

where

$$H(e^{j\omega}) = \mathcal{F}_d[h(n)] \tag{4.33}$$

is the impulse transfer function or the frequency response of the system.

$H(e^{j\omega})$ at any frequency ω modifies the amplitude and phase of the input $X(e^{j\omega})$ to produce the output. For frequency section filtering, $H(e^{j\omega})$ would have the desired magnitude and phase over a range of frequencies.

For example,

Pass-band

$$\begin{aligned} |H(e^{j\omega})| &= 1;\ |\omega| < \omega_c \\ &= 0;\ |\omega| > (2\pi - \omega_c) \end{aligned}$$

Reject-band

$$\begin{aligned} |H(e^{j\omega})| &= 0;\ |\omega| < \omega_c \\ &= 1;\ |\omega| > (2\pi - \omega_c) \end{aligned}$$

It is to be noted that $H(e^{j\omega})$ is periodic with period 2π. The reader is advised to make a sketch of the above two filters.

Example 4.1 *The impulse response of a discrete-time system is*

$$h(n) = \delta(n - n_0)$$

Determine its impulse transfer function. If the input to the system is x(n), determine its output y(n).

Solution

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n - n_0)\, e^{-j\omega n}$$

By the sampling property of impulse

$$H(e^{j\omega}) = e^{-j\omega n_0} \tag{i}$$

The output is given by

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})\, X(e^{j\omega}) \\ &= e^{-j\omega n_0}\, X(e^{j\omega}) \end{aligned} \tag{ii}$$

As per time-shifting property from Eq. (ii) we get

$$y(n) = x(n - n_0)$$

Example 4.2 *An ideal discrete-time-low-pass filter has frequency response.*

$$H(e^{j\omega}) = \begin{cases} 1 & ;\ \omega < |\omega_c| \\ 0 & ;\ otherwise \end{cases}$$

Sketch the frequency response showing periodicity. Find its impulse response h(n).

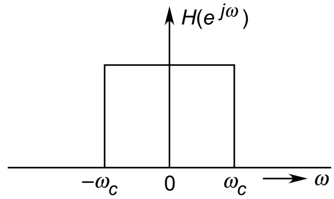


Fig. 4.4(a)

Solution $H(e^{j\omega})$ is sketch in Fig. 4.4(a).

By synthesis equation

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, \mathrm{d}\omega \\ &= \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}(e^{j\omega_c n} - e^{j\omega_c n}) \end{aligned}$$

So we get

$$h(n) = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\left(\frac{\sin \omega_c n}{\omega_c n}\right)$$

It is sketched in Fig. 4.4(b) and has the form of a discrete sinc function.

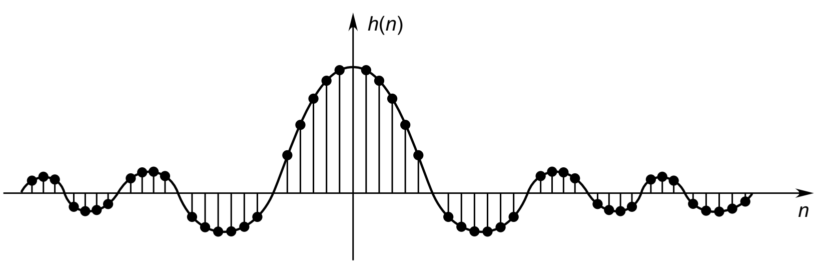


Fig. 4.4(b) *Discrete-sinc function*

We find that impulse response of ideal low-pass filter is noncausal and so not realizable. Further, its oscillatory nature is not desirable.

Example 4.3 *An ideal low-pass discrete time filter has pulse transfer function $H_{lp}(e^{j\omega})$ with magnitude unity and cut-off frequency ω_c. Write the expression for high-pass filter, derived from the low-pass centred at $\omega = \pi$. Plot both the filter functions. Determine the impulse response of the high-pass filter from the impulse response of the low-pass filter.*

Solution

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

Both these functions are sketched in Figs. 4.5 (a) and (b).

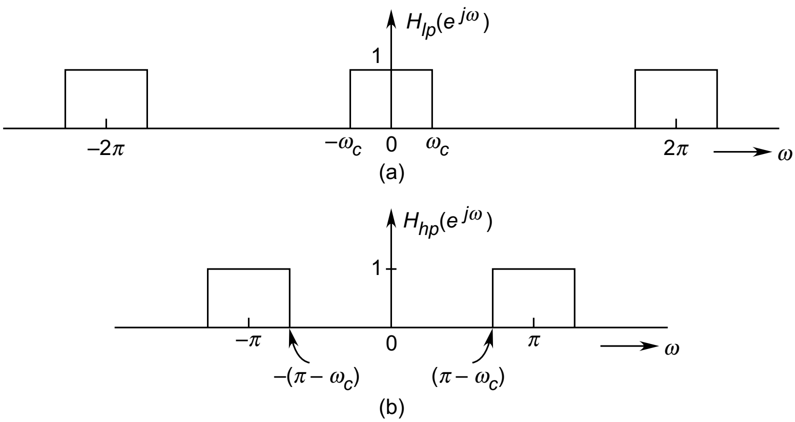


Fig. 4.5

By frequency shifting property, the impulse responses of the filters are related as

$$\begin{aligned} h_{hp}(n) &= e^{j\pi n}\, h_{lp}(n);\ \omega_0 = \pi \text{ (shift)} \\ &= (-1)^n\, h_{lp}(n) \end{aligned}$$

Example 4.4 *For the discrete signal*

$$x(n) = a^n\, u(n);\ |a| < 1$$

find its DFT.

Solution

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n\, u(n)\, e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n;\ u(n) = 0 \ \text{ for } \ n < 0 \\ &= \frac{1}{1 - ae^{-j\omega}};\ \text{summation of infinite geometric series} \end{aligned}$$

The phasor diagram of $(1 - a\,e^{-j\omega})$ is sketched in Fig. 4.6 from which the periodicity with period 2π is immediately observed.

Further,

$$\max |X(e^{j\omega})| = \frac{1}{1-a};\ \omega = 0, 2\pi,$$
$$\min |X(e^{j\omega})| = \frac{1}{1+a};\ \omega = \pi, 3\pi,$$

$\angle X(e^{j\omega})$ oscillates between $\mp \tan^{-1}\dfrac{a}{\sqrt{1-a^2}}$

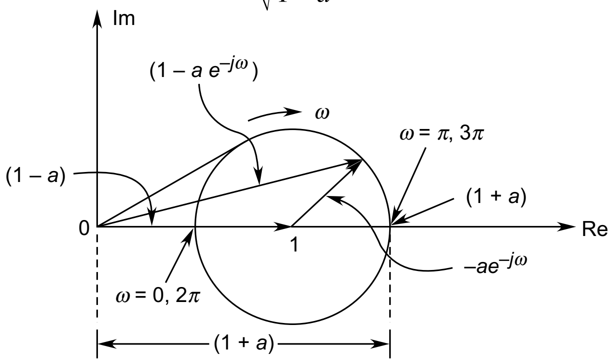


Fig. 4.6

The sketches of $|X(e^{j\omega})|$ and $\angle X(e^{j\omega})$ are drawn in Fig. 4.7.

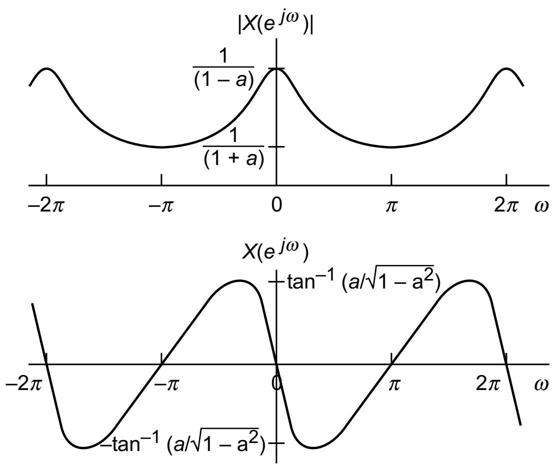


Fig. 4.7

Example 4.5 *Find the DFT for discrete rectangular pulse*

$$x(n) = 1 \text{ for } |n| \le N_1$$
$$= 0 \text{ for } |n| > N_1$$

Solution The pulse is sketched in Fig. 4.7.

$$F(j\omega) = \sum_{n=-N_1}^{N_1} 1 \cdot e^{-j\omega n}$$

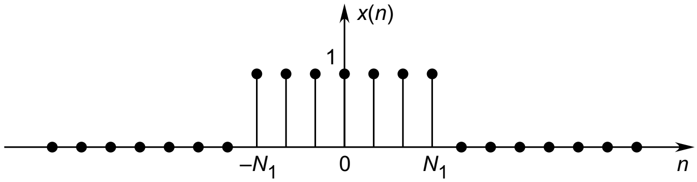


Fig. 4.7(a)

This summation can be carried on similar lines as in Example 2.18 on discrete periodic Fourier series. The result is

$$F(e^{j\omega}) = \frac{\sin \omega (N_1 + 1/2)}{\sin (\omega/2)}$$

DFT of Periodic Signals

From *FT* Table 2.3, we find that for a continuous time exponential periodic signal

$$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\, \delta(\omega - \omega_0); \text{ impulse in frequency domain}$$

As DFT is periodic, we expect the transform of a discrete periodic signal to be impulse train repeating every 2π period. Thus for

$$f(n) = e^{j\omega_0 n} \tag{4.33}$$

the discrete Fourier transform should be impulses at $\omega = \omega_0$, $\omega_0 \pm 2\pi$, …. It can be shown that for $f(n)$ of Eq. (4.33)

$$F(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l) \tag{4.34}$$

Substituting in synthesis equation

$$f(n) = \frac{1}{2\pi} \int_{2\pi} F(j\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{2\pi} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)\, e^{j\omega n}\, d\omega$$

Using sampling property impulse and integrating over 2π period containing only one impulse at $(\omega_0 + 2\pi r)$, we get

$$f(n) = e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n}$$

The impulses of $F(e^{j\omega})$ are plotted in Fig. 4.8, which are periodic with period 2π.

Using the above result, we can write down the DFT of discrete Fourier series

$$f(n) = \sum_{k=(N)} F_k\, e^{jk(2\pi/N)n} \tag{4.35}$$

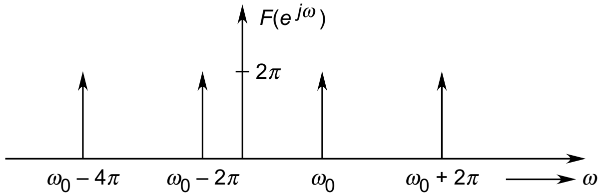


Fig. 4.8

Its DFT is

$$F(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi F_k\, \delta\left(\omega - \frac{2\pi k}{N}\right) \tag{4.36}$$

Example 4.6 *For $x(n) = \cos \omega_0 n$, find the DFT and sketch it. Given: $\omega_0 = 2\pi/3$.*

Solution

$$x(n) = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

From Eq. (4.34)

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\,\delta\left(\omega - \frac{2\pi}{3} - 2\pi l\right) + \sum_{l=-\infty}^{\infty} \pi\,\delta\left(\omega + \frac{2\pi}{3} - 2\pi l\right)$$

That is,

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{3}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{3}\right); -\pi \le \omega \le \pi$$

$X(e^{j\omega})$ is sketched in Fig. 4.9.

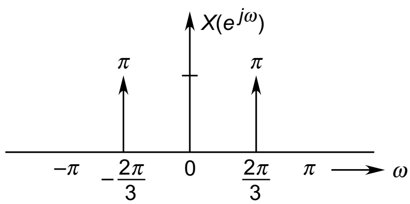


Fig. 4.9

Example 4.7 *Find the DFT of*

$$x(n) = x_1(n)\, x_2(n)$$

where

$$x_1(n) = \frac{\sin\left(\frac{\pi}{4} n\right)}{\pi n} \qquad x_2(n) = \frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n}$$

Solution As shown in Example 4.2,

$$X_1(e^{j\omega}) = \text{low-pass filter of unit amplitude and pass band} -\frac{\pi}{4} \text{ to } \frac{\pi}{4}$$

$$X_2(e^{j\omega}) = \text{low-pass filter of unit amplitude and pass-band} -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

According to multiplication property

$$X(e^{j\omega}) = X_1(e^{j\omega}) * X_2(e^{j\omega})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta, \text{ periodic convolution}$$

This convolution is easily carried out by replacing $X_1(e^{j\omega})$ by $\hat{X}_1(e^{j\omega})$ where

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), -\pi < \omega < \pi \\ 0 \end{cases}$$

Then

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

This is the convolution of a rectangular pulse with periodic pluses scaled by $(1/2\pi)$. This is illustrated in Fig. 4.10.

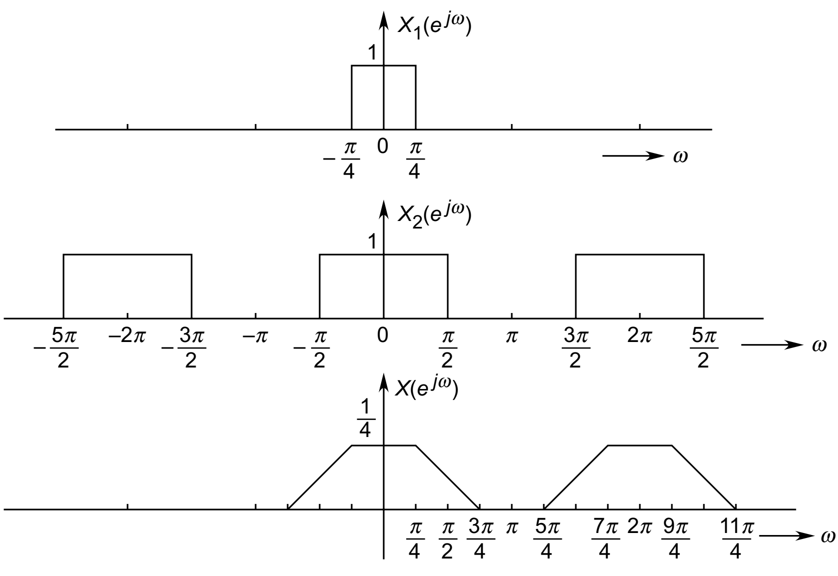


Fig. 4.10

Discrete Fourier Transform Pairs

Like in Fourier transform, the Discrete Fourier Transform (DFT) of a discrete-time signal is unique. It is therefore considerably useful in finding DFT or its inverse and good to have a list of important DFT pairs. Such a list is given in Table 4.1.

Table 4.1 *Important DFT pairs*

Signal	*Transforms*
1. $\sum_{k=(N)} F_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} F_k \delta\left(\omega - \frac{2\pi k}{N}\right)$
2. $e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$
3. $\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} [\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)]$
4. $\sin \omega_0 n$	$\frac{j}{\pi} \sum_{l=-\infty}^{+\infty} [\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)]$
5. $f(n) = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$
6. $\sum_{k=-\infty}^{+\infty} \delta(n - kN)$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$
7. $a^n u(n);\ \lvert a \rvert < 1$	$\frac{1}{1 - a\, e^{-j\omega}}$
8. $u(n)$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$
9. $\delta(n)$	1
10. $\delta(n - n_0)$	$e^{-j\omega n_0}$
11. $x(n) = \begin{cases} 1 & ;\ \lvert n \rvert \le N_1 \\ 0 & ;\ N_1 < \lvert n \rvert \le N/2 \end{cases}$	$\frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$
12. $f(n) = \begin{cases} 1 & ;\ \lvert n \rvert \le N_1 \\ 0 & ;\ \lvert n \rvert < N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$
13. $\frac{\sin W_n}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{W_n}{\pi}\right)$ $0 < W < \pi$ sinc function defined here as $\operatorname{sinc} \theta = \frac{\sin \pi\theta}{\pi\theta}$	$F(\omega) = \begin{cases} 1 & ;\ 0 \le \lvert \omega \rvert \le W \\ 0 & ;\ W < \lvert \omega \rvert \le \pi \end{cases}$ F(ω) periodic with period 2π

Example 4.8 *For the discrete functions*

$$x(n) = a^{|n|};\ |a| < 1$$

determine $X(e^{j\omega})$. Check the result against property of even/oddness.

Solution

$$x(n) = a^{|n|};\ |a| < 1 \tag{i}$$

is a real, even decaying function.

Its DFT is found from the analysis equation as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|}\, e^{-j\omega n} \tag{ii}$$

$$= \sum_{n=0}^{+\infty} a^{n}\, e^{-j\omega n} + \underbrace{\sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n}}_{\text{Let } m = -n} \tag{iii}$$

$$= \sum_{n=0}^{+\infty} (a\, e^{-j\omega})^{n} + \sum_{m=1}^{+\infty} (a\, e^{-j\omega})^{m} \tag{iv}$$

$$= \sum_{n=0}^{+\infty} (a\, e^{-j\omega})^{n} + a\, e^{j\omega}\left[\sum_{m=0}^{+\infty} (a\, e^{j\omega})^{m}\right]$$

Summing the infinite geometric series*, we get

$$X(e^{j\omega}) = \frac{1}{1 - a\, e^{-j\omega}} + \frac{a\, e^{j\omega}}{1 - a\, e^{j\omega}} \tag{v}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2} \tag{vi}$$

It is observed that as $x(n)$ is real and even, therefor $X(e^{j\omega})$ is also real and even ($\cos\omega$ is even).

Example 4.9 *Find the response y(n) of a causal LTI discrete-time system for the given*

System's impulse response,	$h(n) = \alpha^{n} u(n);\ \alpha < 1$
Input signal	$x(n) = \beta^{n} u(n);\ \beta < 1$

---

*
$$\sum_{i=0}^{k} a^{j} = \frac{1 - a^{k+1}}{1 - a},\ a \neq 1$$

$$\sum_{i=0}^{\infty} a^{j} = \frac{1}{1 - a};\ |a| < 1$$

Solution Taking DFT of the system and input (consulting Table 4.1)

$$H(e^{j\omega}) = \frac{1}{1-\alpha\, e^{-j\omega}} \tag{i}$$

$$X(e^{j\omega}) = \frac{1}{1-\beta\, e^{-j\omega}} \tag{ii}$$

As per convolution property

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})\, X(e^{j\omega}) \\ &= \frac{1}{(1-\alpha\, e^{-j\omega})\,(1-\beta\, e^{-j\omega})} \end{aligned} \tag{iii}$$

Following the routine procedure of partial fractioning, we have

$$Y(e^{j\omega}) = \left.\frac{1}{1-\beta\, e^{-j\omega}}\right|_{e^{-j\omega}=1/\alpha} \frac{1}{1-\alpha\, e^{-j\omega}} + \left.\frac{1}{1-\alpha\, e^{-j\omega}}\right|_{e^{-j\omega}=1/\beta} \cdot \frac{1}{1-\beta\, e^{-j\omega}}$$

It results in

$$Y(e^{j\omega}) = \left(\frac{\alpha}{\alpha-\beta}\right)\cdot\frac{1}{1-\alpha\, e^{-j\omega}} + \left(\frac{\beta}{\beta-\alpha}\right)\cdot\frac{1}{1-\beta\, e^{-j\omega}} \tag{iv}$$

Taking inverse DFT

$$\begin{aligned} y(n) &= \frac{\alpha}{\alpha-\beta}\,\alpha^n u(n) - \frac{\beta}{\alpha-\beta}\,\beta^n u(n) \\ &= \frac{1}{\alpha-\beta}[\alpha^{n+1}\, u(n) - \beta^{n+1}\, u(n)] \end{aligned} \tag{v}$$

It is seen from Eq. (v) that there are two response terms; one corresponding to system pole $e^{-j\omega} = 1/\alpha$ and other corresponding to excitation pole $e^{-j\omega} = 1/\beta$.

Note: Discrete system response can be obtained easily by the *Z*-transform method of Chapter 3.

It is to be noted that DFT is the way to carry out signal analysis (finding signal spectrum) and signal synthesis (from its spectrum). While the closed-form DFT theorem and results are very helpful in understanding DFT, practical discrete-time signals are synthesized and analysed by computation techniques which follows in Section 4.4.

DFT of Discrete Time Fourier Series

We can directly transform a periodic signal from its discrete-time Fourier series

$$f(n) = \sum_{k=(N)} F_k\, e^{jk(2\pi/N)n} = \sum_{k=(N)} F_k\, e^{jk\omega_0 n} \tag{4.37}$$

where (*N*) is any consective *N* terms.

From Table 4.1 entry 1,

$$F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi F_k\, \delta\left(\omega - \frac{2\pi k}{N}\right) \tag{4.38}$$

It is immediately noticed from above that $F(j\omega)$ is composed of periodic impulses of strength

$$2\pi F_{-1}, 2\pi F_0, 2\pi F_1, \ldots, 2\pi F_{N-1}$$

with spacing ω_0. These repeat on positive and negative sides of ω with period N. Thus

$$2\pi F_{-1} = 2\pi F_{-(N+1)}$$

occurring at $-\omega_0$ and $-\omega_0(N+1)$ respectively.

Example 4.10 *Consider a periodic impulse train with period N expressed as*

$$x(n) = \sum_{l=-\infty}^{\infty} \delta(n - lN)$$

The impulses occur at $n = lN$; $l = -\infty$ to ∞ (sketched in Fig. 4.11(a). Determine the DFT $X(e^{j\omega})$.

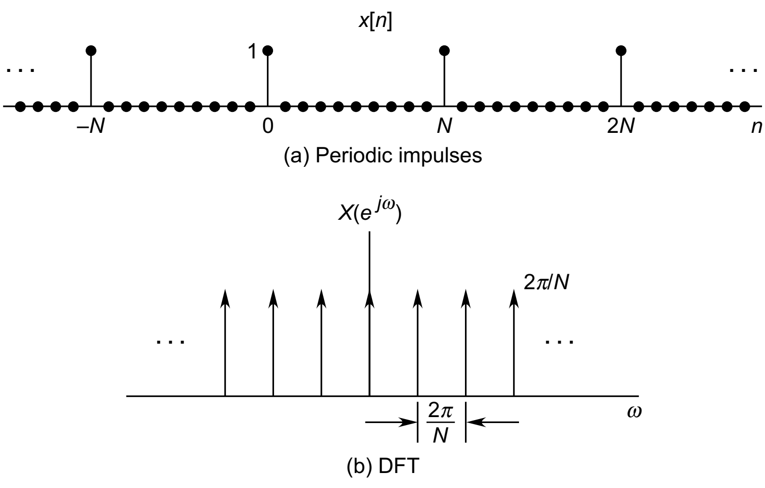


Fig. 4.11

Solution The Fourier series coefficient of $x(n)$ are found as

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-jk(2\pi/N)n}$$

where we have chosen the summation period 0 to $N - 1$. In this period the impulse occurs at $n = 0$. We then have

$$F_k = \frac{1}{N}$$

From Table 4.1 entry 1, we write

$$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{N}\right)$$

This is sketched in Fig. 4.11(b). The reader may compare Figs 4.11(a) and (b) with continuous-time Fourier transform pair of Figs 2.32(a) and (b).

Duality Between DFT and Continuous-time Fourier Series

• ***Discrete-time Fourier Transform***

Synthesis Equation

$$f(n) = \frac{1}{2\pi}\int_{2\pi} F(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega \tag{4.39}$$

Analysis Equation

$$F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n} \tag{4.40}$$

• ***Continuous-time Fourier Series***

Synthesis Equation

$$f(t) = \sum_{k=-\infty}^{\infty} F_k\, e^{jk\omega_0 t} \tag{4.41}$$

Analysis Equation

$$F_k = \frac{1}{T}\int_T f(t)\, e^{-jk\omega_0 t}\, \mathrm{d}t \tag{4.42}$$

Examination of the above two pairs of equations reveals the similarity between Eqs (4.39) and (4.42) and also between Eqs (4.40) and (4.41). Further, it is observed that $F(e^{j\omega})$ (of Eq. (4.39) is periodic in ω with period 2π. Therefore $F(e^{j\omega})$ can be split into its Fourier series whose analysis equation is (4.39) and synthesis equation is (4.40). Thus, $f(n)$ are the Fourier coefficients of $F(e^{j\omega})$ where $f(n)$ is the nth coefficient.

The above observations are the revelation of the duality between discrete-time Fourier transform and continuous-time Fourier series. This duality can be exploited to generate closed form relationship for DFT from known relationship in Fourier series and vice versa. We will not pursue this concept any further as our aim is not to generate more closed-form DFTs, but to proceed on to computational issues.

Additional Examples-I

Example 4.11 *Use the analysis equation to find the DFT of*

$$(a)^{n-n_1}\, u(n - n_1);\ |a| < 1$$

Also, write the expression for $n_1 = 1;\ a = \frac{1}{2}$.

Solution As per the analysis equation,

$$F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n}$$

$$= \sum_{n=n_1}^{\infty} (a)^{(n-n_1)}\, e^{-j\omega n}\, u(n-n_1) \qquad \text{(i)}$$

Let $m = n - n_1$; then

$$F(e^{j\omega}) = \sum_{m=0}^{\infty} (a)^m\, e^{-j\omega\,(m+n_1)} \qquad \text{(ii)}$$

$$= e^{-j\omega n_1} \sum_{m=0}^{\infty} (a)^m\, e^{-j\omega m} \qquad \text{(iii)}$$

$$= \frac{e^{-j\omega n_1}}{1 - a\, e^{-j\omega}} \qquad \text{(iv)}$$

For $n_1 = 1,\ a = \dfrac{1}{2}$

$$F(e^{j\omega}) = \frac{e^{-j\omega}}{1 - \frac{1}{2}\, e^{-j\omega}} \qquad \text{(v)}$$

Example 4.12 *Find the DFT of*

$$f(n) = \left(\frac{1}{2}\right)^{|n-n_1|}$$

Solution

$$F(e^{j\omega}) = \sum_{n=-\infty}^{n=n_1-1} \left(\frac{1}{2}\right)^{-(n-n_1)} e^{-j\omega n} + \sum_{n=n_1}^{\infty} \left(\frac{1}{2}\right)^{(n-n_1)} e^{-j\omega n} \qquad \text{(i)}$$

First term on right-hand side

$$\sum_{n=-\infty}^{n=n_1-1} \left(\frac{1}{2}\right)^{-(n-n_1)} e^{-j\omega n}$$

Change n to $-n$

$$\sum_{n=-(n_1-1)}^{+\infty} \left(\frac{1}{2}\right)^{(n+n_1)} e^{j\omega n}$$

Let $m = n + n_1$; then $n = m - n_1$. We then have

$$\sum_{m=1}^{+\infty} \left(\frac{1}{2}\right)^{m} e^{j\omega(m-n_1)} = e^{-j\omega n_1}\left[\sum_{m=1}^{+\infty} \left(\frac{1}{2}\right)^{m} e^{j\omega m}\right]$$

$$= e^{-j\omega n_1}\left[-1+\sum_{m=0}^{+\infty}\left(\frac{1}{2}\right)^m e^{j\omega m}\right]; \text{ adding and substracting } m = 0 \text{ term}$$

$$= e^{-j\omega n_1}\left[-1+\frac{1}{1-\left(\frac{1}{2}\right)e^{j\omega}}\right] = \left(\frac{1}{2}\right)\frac{e^{-j\omega(n_1-1)}}{1-\left(\frac{1}{2}\right)e^{j\omega}} \tag{ii}$$

similarly we have the result for the second term.

We then have

$$F(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{e^{-j\omega(n_1-1)}}{1-\left(\frac{1}{2}\right)e^{j\omega}} + \frac{e^{-j\omega n_1}}{1-\left(\frac{1}{2}\right)e^{-j\omega}} \tag{iii}$$

Particular case $\quad n_1 = 1, f(n) = \left(\frac{1}{2}\right)^{|n-1|}$

As per Eq. (iii),

$$F(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{1}{1-\left(\frac{1}{2}\right)e^{j\omega}} + \frac{e^{-j\omega}}{1-\left(\frac{1}{2}\right)e^{-j\omega}}$$

$$= \frac{0.75\, e^{-j\omega}}{1.25-\frac{1}{2}(e^{j\omega}+e^{-j\omega})} = \frac{0.75\, e^{-j\omega}}{1.25-\cos\omega} \tag{iv}$$

Example 4.13 *Determine the DFT of*

(a) $\delta(n-1)+\delta(n+1)$ *(b)* $\delta(n-2)-\delta(n+2)$

Solution

(a) As per analysis equation

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty}[\delta(n-1)+\delta(n+1)]\, e^{-j\omega n}$$

By sampling property of δ-function

$$F(e^{j\omega}) = e^{-j\omega} + e^{j\omega} = 2\cos\omega$$

(b) Similarly,

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty}[\delta(n-2)-\delta(n+2)]$$

$$= e^{-2j\omega} - e^{2j\omega} = 2\,j\sin 2\omega$$

Observation

(a) The signal is real and even and so the DFT is real and even.
(b) The signal is real and odd and the DFT is pure imaginary and odd.

Example 4.14 *Determine the DFT of the signals*

(a) $\cos\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)$ *(b)* $2+\sin\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)$

Solution As both the signals are periodic we first write down the DFT pair for discrete Fourier series.

$$\sum_{n=<N>} F_k\, e^{jk(2\pi/N)n} \xleftrightarrow{\mathcal{F}_d} \sum_{n=-\infty}^{+\infty} 2\pi F_k\, \delta\left(\omega-\frac{2\pi k}{N}\right) \qquad \text{(i)}$$

(a) $x(n)=\cos\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)$; range $-\pi\le\omega\le\pi$

It is a periodic signal with period

$$N=12 \text{ as } \left(\frac{\pi}{6}N\right)=2\pi$$

Writing the signal in the form of exponential Fourier series, we have

$$x(n) = \frac{1}{2}e^{j\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)}+\frac{1}{2}e^{-j\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)}$$

$$= \left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)e^{j\frac{\pi}{6}n}+\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)e^{-j\frac{\pi}{6}n} \qquad \text{(ii)}$$

Comparing with the left side of Eq. (i) that $k=1$ in Eq. (ii) and so these are only two Fourier coefficients F_1 and F_{-1} in the range $-5\le k\le 6$. It then follows that

$$F_1=\frac{1}{2}e^{j\frac{\pi}{8}} \quad \text{and} \quad F_{-1}=\frac{1}{2}e^{-j\frac{\pi}{8}}$$

Then from Eq. (i)

$$X(e^{j\omega}) = 2\pi\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)\delta\left(\omega-\frac{2\pi}{12}\right)+2\pi\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)\delta\left(\omega+\frac{2\pi}{12}\right)$$

$$= \pi\left\{e^{j\frac{\pi}{8}}\,\delta\left(\omega-\frac{\pi}{6}\right)+e^{-j\frac{\pi}{6}}\,\delta\left(\omega+\frac{\pi}{6}\right)\right\} \qquad \text{(iii)}$$

(b) $x(n)=2+\sin\left(\frac{\pi}{3}n+\frac{\pi}{6}\right)$; $-\pi\le\omega\le\pi$ (i)

Period $N=6$; $2\pi/N=\pi/3$

In exponential form of Fourier series

$$x(n) = 2+\frac{1}{2j}\left\{e^{j\frac{\pi}{6}}\,e^{j\frac{\pi}{3}n}-e^{-j\frac{\pi}{6}}\,e^{-j\frac{\pi}{3}n}\right\} \qquad \text{(ii)}$$

Therefore, the discrete Fourier series has term corresponding to $k = 1, -1$

$$F_0 = 2, F_1 = \frac{1}{2j} e^{j\frac{\pi}{6}}, F_{-1} = -\frac{1}{2j} e^{-j\frac{\pi}{6}} \tag{iii}$$

It then follows that

$$X(e^{j\omega}) = 2\delta(\omega) + \frac{1}{2j}\left\{e^{j\frac{\pi}{6}}\delta\left(\omega - \frac{\pi}{3}\right) - e^{-j\frac{\pi}{6}}\delta\left(\omega + \frac{\pi}{3}\right)\right\} \tag{iv}$$

Example 4.15 *The DFT of a discrete function f(n) is a set of impulses occurring at $\omega = 0$, magnitude 2π, $\omega = \pm\frac{\pi}{3}$ magnitude π repeating at period 2π.*

Write the expression of $F(e^{j\omega})$ and find therefore f(n).

Solution As per the statement,

$$F(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi\,\delta(\omega - 2\pi k) + \pi\delta\left(\omega - \frac{\pi}{3} - 2\pi k\right) + \pi\delta\left(\omega + \frac{\pi}{3} - 2\pi k\right) \tag{i}$$

Its inverse is obtained from the synthesis equation. We note from the expression for $F(e^{j\omega})$ that only three impulses occur in the period $-\pi$ to π i.e., for $k = 0$. Then

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[2\pi\delta(\omega) + \pi\delta\left(\omega - \frac{\pi}{3}\right) + \pi\delta\left(\omega + \frac{\pi}{3}\right)\right]e^{j\omega n}\,d\omega \tag{ii}$$

By sampling property of δ-function, we get

$$\begin{aligned} f(n) &= e^{j0} + \frac{1}{2}e^{jn\pi/3} + \frac{1}{2}e^{-jn\pi/3} \\ &= 1 + \cos(n\pi/3) \end{aligned} \tag{iii}$$

Example 4.16 *Determine the signal corresponding to the following DFTs.*

(a) $F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} (-1)^k\,\delta\left(\omega - \frac{\pi}{3}k\right)$

(b) $F(e^{j\omega}) = 1, \quad \frac{\pi}{4} \le |\omega| < \frac{3\pi}{4}$
$\quad = 0$, *elsewhere*

Solution

(a) The given $F(e^{j\omega})$ is similar to DFT of a discrete Fourier series. The pair is reproduced below.

$$\sum_{n=<N>} F_k\, e^{jk(2\pi/N)n} \xleftrightarrow{\mathcal{F}_d} 2\pi\sum_{k=-\infty}^{\infty} F_k\,\delta\left(\omega - \frac{2\pi k}{N}\right) \tag{i}$$

Given: $$F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} (-1)^k\,\delta\left(\omega - \frac{\pi}{3}k\right) \tag{ii}$$

It is periodic with fundamental period

$$\frac{\pi}{3}k = \frac{2\pi k}{N} \quad \text{or} \quad N = 6$$

and fundamental frequency is $2\pi/6 = (\pi/3)$.

Comparing with Eq. (i), its inverse is discrete Fourier series with period $N = 6$ and $F_k = \frac{1}{2\pi}(-1)^k$. Thus

$$f(n) = \frac{1}{2\pi}\sum_{k=0}^{5}(-1)^k e^{j\frac{\pi}{3}n}$$

$$= \frac{1}{2\pi}\left[1 - e^{j\frac{\pi}{3}} + e^{j\frac{2\pi}{3}} - e^{j\pi} + e^{j\frac{4\pi}{3}} - e^{j\frac{5\pi}{3}}\right] \quad \text{(iii)}$$

(b) As per analysis equation

$$f(n) = \int_{-\pi}^{\pi} F(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega$$

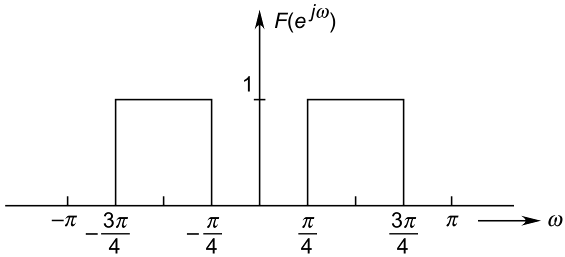


Fig. 4.12

Given: $F(e^{j\omega})$ is sketched in Fig. 4.12, from which we get

$$f(n) = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{j\omega n}\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n}\,\mathrm{d}\omega \quad \text{(i)}$$

or

$$f(n) = \frac{1}{2\pi}\left\{\frac{e^{j\omega n}}{jn}\bigg|_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + \frac{e^{j\omega n}}{jn}\bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right\}$$

$$= \frac{1}{2j\pi n}\left\{e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n} - e^{j\frac{\pi}{4}n}\right\}$$

$$= \frac{1}{\pi n}\left\{\sin\left(\frac{3\pi}{4}n\right) - \sin\left(\frac{\pi}{4}n\right)\right\} \quad \text{(iii)}$$

Example 4.17 *The DFT of a signal is sketched in the Fig. 4.13. Determine the signal.*

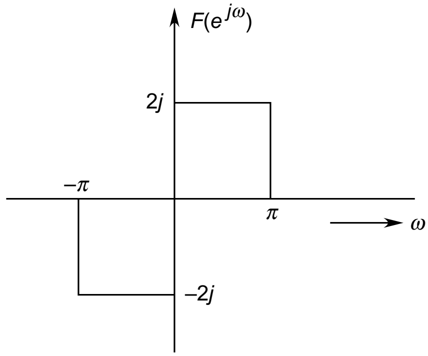


Fig. 4.13

Solution Using the analysis equation,

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{0}(-2j)\,e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{\pi}(2j)\,e^{j\omega n}\,\mathrm{d}\omega \qquad \text{(i)}$$

$$= -\frac{j}{\pi}\cdot\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{0} + \left(\frac{j}{\pi}\right)\left.\frac{e^{j\omega n}}{jn}\right|_{0}^{\pi}$$

$$= \frac{1}{\pi n}[-(1-e^{j\pi n}) + (e^{j\pi n}-1)]$$

Now,

$$(1-e^{-j\pi n}) = e^{-j\frac{\pi}{2}}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right) = 2j\,e^{-j\frac{\pi}{2}}\sin\left(n\frac{\pi}{2}\right)$$

Similarly,

$$(e^{j\pi n}-1) = 2j\,e^{j\frac{\pi}{2}n}\sin\left(n\frac{\pi}{2}\right)$$

We then get

$$f(n) = \left(\frac{2j}{\pi n}\right)\sin\left(n\frac{\pi}{2}\right)\left(e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}\right)$$

$$= -\left(\frac{4}{\pi n}\right)\sin^2\left(n\frac{\pi}{2}\right)$$

Example 4.18 *The magnitude and angle of $F(e^{j\omega})$ are*

$$|F(e^{j\omega})| = \begin{cases} 1 & ;\ 0 \le |\omega| \le \dfrac{\pi}{3} \\ 0 & ;\ \dfrac{\pi}{3} \le |\omega| \le \pi \end{cases}$$

$$\angle F(e^{j\omega}) = -\frac{3}{4}\omega$$

Determine f(n).

Solution Given,

$$F(e^{j\omega}) = 1e^{-j\frac{3}{4}\omega}\ ;\ -\frac{\pi}{3} \le |\omega| \le \frac{\pi}{3}$$

Using the synthesis equation,

$$f(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{-j\frac{3}{4}\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{j\omega\left(n-\frac{3}{4}\right)}\, d\omega$$

$$= \frac{1}{2\pi j\left(n-\frac{3}{4}\right)}\left[e^{j\frac{\pi}{3}\left(n-\frac{3}{4}\right)} - e^{-j\frac{\pi}{3}\left(n-\frac{3}{4}\right)}\right]$$

or
$$f(n) = \frac{\sin\left[\frac{\pi}{3}\left(n-\frac{3}{4}\right)\right]}{\pi\left(n-\frac{3}{4}\right)}$$

For no integral value of n, $f(n)$, can be zero except for $n = \pm\infty$.

Example 4.19 *Determine if the time-domain signals corresponding to the following DFTs are (i) real, imaginary or neither, and (ii) even odd or neither. Proceed on the basis of even/oddness properties without evaluating the inverse DFT.*

(a) $F_1(e^{j\omega}) = e^{-j\omega}\sum_{k=1}^{20} \sin(k\omega)$ (b) $F_2(e^{j\omega}) = j\sin\omega\cos 5\omega$

(c) $F_3(e^{j\omega}) = A(\omega)\, e^{jB(\omega)}$

where
$$A(\omega) = \begin{cases} 1 & 0 \le |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

$$B(\omega) = -\frac{3}{4}\omega + \pi$$

Solution

(a) $F_1(e^{j\omega}) = e^{-j\omega}\sum_{k=1}^{20} \sin(k\omega)$ (i)

It has both real and imaginary parts. So no property can be directly applied. We proceed by considering

$$Y_1(e^{j\omega}) = \sum_{k=1}^{20} \sin (k\omega) \tag{ii}$$

It is real and odd. There is no listed property. We proceed by comparison.

$F(e^{j\omega})$, pure imaginary and odd $\rightarrow f(n)$, real and odd

So $F(e^{j\omega})$, real and odd $\rightarrow f(n)$; imaginary and odd

Thus, $y_1(n)$ Eq. (ii) is imaginary and odd.

Now $$F_1(e^{j\omega}) = e^{-j\omega}\, Y_1(e^{j\omega})$$

which implies

$$f_1(n) = y_1(n-1)$$
$$f_1(n) = y_1[-(n+1)] \neq y(n) \text{ or } -y(n) \tag{iii}$$

So $f_1(n)$ is real but neither even or odd.

(b) $F_2(e^{j\omega}) = j \sin \omega \cos 5\,\omega$

$F_2(e^{-j\omega}) = -j \sin \omega \cos 5\,\omega = F(e^{j\omega})$

Therefore, $F_2(e^{j\omega})$ is pure imaginary and odd. Correspondingly, $f_3(n)$ is real and odd.

(c) $F_3(e^{j\omega}) = A(\omega)\, e^{j\left(-\frac{3}{4}\omega+\pi\right)}$

$$= -A(\omega)\, e^{-j\frac{3}{4}\omega}$$

$$|F_3(e^{j\omega})| = |-A(\omega)| = |-A(-\omega)| = |F_3(e^{-j\omega})|$$

$$\angle F_3(e^{j\omega}) = -\frac{3}{4}\omega$$

$$\angle F_3(e^{-j\omega}) = \frac{3}{4}\omega = F_3(e^{j\omega})$$

Therefore, $F_3(e^{j\omega})$ is even, but it has both real and imaginary parts. So $f_3(n)$ is neither even nor odd.

Example 4.20 *A causal LTI discrete time system has its output y(n) related to its input x(n) by the difference equation*

$$y(n) - \frac{1}{12}\, y(n-1) - \frac{1}{12}\, y(n-2) = x(n)$$

(a) Determine the frequency response $H(e^{j\omega})$ of the system.

(b) Determine the impulse response of the system.

Solution Taking DFT of the difference equation, we have

$$\left[1 - \frac{1}{12}\, e^{-j\omega} - \frac{1}{12}\, e^{-2j\omega}\right] Y(e^{j\omega}) = X(e^{j\omega}) \tag{i}$$

(a) $$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{12}e^{-j\omega}-\frac{1}{12}e^{-2j\omega}} \qquad \text{(ii)}$$

(b) As DFT of impulse $\delta(n)$ is unity, the impulse response is given by $\mathcal{F}_d^{-1}[H(e^{j\omega})]$.

Partial fractioning $H(e^{j\omega})$

$$H(e^{j\omega}) = \frac{4/7}{1-\frac{1}{3}e^{-j\omega}} + \frac{3/7}{1+\frac{1}{4}e^{-j\omega}} \qquad \text{(iii)}$$

Taking inverse DFT, we get

$$h(n) = \frac{4}{7}\left(\frac{1}{3}\right)^n u(n) + \frac{3}{7}\left(-\frac{1}{4}\right)^n u(n) \qquad \text{(iv)}$$

Example 4.21 *A causal and stable LTI discrete system produces an output of* $\left[n\left(\frac{3}{5}\right)^n u(n)\right]$ *for input* $\left[\left(\frac{3}{5}\right)^n u(n)\right]$.

(a) Determine the system's frequency response $H(e^{j\omega})$.

(b) Determine the difference equation relating output $y(n)$ to input $x(n)$.

Solution

(a) Input $$x(n) = \left(\frac{3}{5}\right)^n u(n) \xleftrightarrow{\mathcal{F}_d} \frac{1}{1-\frac{3}{5}e^{-j\omega}} = X(e^{j\omega}) \qquad \text{(i)}$$

Output $$y(n) = n\left(\frac{3}{5}\right)^n u(n)$$

Its DFT is obtained by applying frequency differentiation property to $X(e^{j\omega})$ (Eq. (4.29)).

$$Y(e^{j\omega}) = j\,\frac{dX(e^{j\omega})}{d\omega} = \frac{\left(\frac{3}{5}\right)e^{-j\omega}}{\left[1-\left(\frac{3}{5}\right)e^{-j\omega}\right]^2}$$

It then follows that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\frac{3}{5}\right)e^{-j\omega}}{1-\left(\frac{3}{5}\right)e^{-j\omega}}$$

(b) $Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$

$$\left[1-\frac{3}{5}e^{-j\omega}\right]Y(e^{j\omega}) = \left(\frac{3}{5}\right)e^{-j\omega}\,X(e^{j\omega})$$

$$Y(e^{j\omega}) - \frac{3}{5} e^{-j\omega}\, Y(e^{j\omega}) = \left(\frac{3}{5}\right) e^{-j\omega}\, X(e^{j\omega})$$

Taking inverse DFT, we get

$$y(n) - \frac{3}{5}\, y(n-1) = \left(\frac{3}{5}\right) x(n-1)$$

or
$$y(n+1) - \frac{3}{5}\, y(n) = \frac{3}{5} x(n)$$

Example 4.22 *Determine the signal corresponding to the following DFTs.*

(a) $F(e^{j\omega}) = \dfrac{e^{-j\omega} - \frac{1}{3}}{1 - \frac{1}{3} e^{-j\omega}}$ (b) $F(e^{j\omega}) = \dfrac{1 - e^{-j\omega}}{1 - \frac{1}{12} e^{-j\omega} - \frac{1}{12} e^{-2j\omega}}$

(c) $F(e^{j\omega}) = \dfrac{1 - \left(\frac{1}{5}\right)^3 e^{-3j\omega}}{1 - \left(\frac{1}{5}\right) e^{-j\omega}}$

Solution

(a) We need the DFT pair

$$a^n u(n),\ |a| < 1 \xleftrightarrow{\mathcal{F}_d} \frac{1}{1 - a e^{-j\omega}}$$

We can write the given DFT as

$$F(e^{j\omega}) = e^{-j\omega}\left[\frac{1}{1 - \frac{1}{3} e^{-j\omega}}\right] - \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

Its inverse is

$$f(n) = \left(\frac{1}{3}\right)^{n-1} u(n-1) - \frac{1}{3}\left(\frac{1}{3}\right)^{n} u(n)$$

$$= \left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n+1} u(n)$$

(b) It is convenient to write $e^{-j\omega} = s$ in the given $F(e^{j\omega})$ and divide it into partial fractions.

$$F(s) = \frac{(1-s)}{1 - \frac{1}{12} s - \frac{1}{12} s^2} = \frac{1-s}{\left(1 - \frac{1}{3} s\right)\left(1 + \frac{1}{4} s\right)}$$

By the method of residues,

$$\left.\frac{1-s}{\left(1+\frac{1}{4}s\right)}\right|_{s=3} \cdot \frac{1}{\left(1-\frac{1}{3}s\right)} + \left.\frac{1-s}{\left(1-\frac{1}{3}s\right)}\right|_{s=-4} \cdot \frac{1}{\left(1+\frac{1}{4}s\right)}$$

$$= \frac{-3/7}{\left(1-\frac{1}{3}s\right)} + \frac{15/7}{\left(1+\frac{1}{4}s\right)}$$

Thus, $$F(e^{j\omega}) = \frac{-3/7}{1-\frac{1}{3}e^{-j\omega}} + \frac{15/7}{1+\frac{1}{4}e^{-j\omega}}$$

Its inverse is

$$f(n) = -\frac{3}{7}\left(\frac{1}{3}\right)^n u(n) + \frac{15}{7}\left(-\frac{1}{4}\right)^n u(n)$$

(c) Writing $e^{-j\omega} = s$

$$F(s) = \frac{1-\left(\frac{1}{5}\right)s^3}{1-s}$$

Dividing out, we get

$$F(s) = 1 + \frac{1}{5}s + \left(\frac{1}{5}\right)^2 s^2$$

or $$F(e^{j\omega}) = 1 + \frac{1}{5}e^{-j\omega} + \left(\frac{1}{5}\right)^2 e^{-2j\omega}$$

Taking its inverse, we obtain

$$f(n) = \delta(n) + \frac{1}{5}\delta(n-1) + \left(\frac{1}{5}\right)^2 \delta(n-2)$$

We have used the DFT pair for taking inverse of $F(e^{j\omega})$

$$1 \leftrightarrow \delta(n)$$

and then used time-shifting property

$$e^{-j\omega n_0} \leftrightarrow \delta(n-n_0)$$

Example 4.23 *Given the DFT pair*

$$a^{|n|} \xleftrightarrow{\mathcal{F}_d} \frac{1-a^2}{1-2a\cos\omega+a^2} \quad is\ |a| < 1$$

By means of duality, determine the Fourier series coefficients of the continuous-time signal

$$f(t) = \frac{1}{5-4\cos(2\pi t)};\ \text{period } T = 1,\ \omega_0 = \frac{2\pi}{T} = 2\pi$$

Solution

$$\frac{1-a^2}{1-2a\cos\omega + a^2} = \frac{\dfrac{1}{a^2}-1}{\left(\dfrac{1}{a^2}+1\right)-\dfrac{2}{a}\cos\omega}$$

Comparing with the signal $f(t)$, we choose $a = \frac{1}{2}$. Then

$$\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\mathcal{F}_d} \frac{3}{5-4\cos\omega}$$

or

$$\frac{1}{3}\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\mathcal{F}_d} \frac{1}{5-4\cos\omega}$$

We write its DFT analysis equation

$$\frac{1}{5-4\cos\omega} = \sum_{n=-\infty}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{|n|} e^{-j\omega n}$$

We now use duality. Let $\omega = 2\pi t$, $n \to -k$. Then

$$\frac{1}{5-4\cos 2\pi t} = \underbrace{\sum_{k=-\infty}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{|k|} e^{j2\pi kt}}_{\text{This is the continuous-time Fourier series synthesis equation}}$$

Hence

$$F_k = \frac{1}{3}\left(\frac{1}{2}\right)^{|k|}$$

Example 4.24 *Determine the DFT of the following signals by using the properties.*

(a) $f_1(n) = f(1-n) + f(-1-n)$ *(b)* $f_2(n) = (n-1)^2 f(n)$

Solution

$$f(n) \xleftrightarrow{\mathcal{F}_d} F(e^{j\omega})$$

(a) By time-reversal property

$$f(-n) \xleftrightarrow{\mathcal{F}_d} F(e^{-j\omega})$$

By time-shifting property

$$f(-n+1) \xleftrightarrow{\mathcal{F}_d} e^{j\omega} F(e^{-j\omega})$$

$$f(-n-1) \xleftrightarrow{\mathcal{F}_d} e^{-j\omega} F(e^{-j\omega})$$

Adding

$$\begin{aligned} F_1(e^{j\omega}) &= e^{-j\omega} F(e^{-j\omega}) + e^{j\omega} F(e^{-j\omega}) \\ &= 2\, F(e^{-j\omega}) \cos \omega \end{aligned}$$

(b) $f_2(n) = n^2 f(n) - 2n\, f(n) + f(n)$

By frequency differentiation property,

$$nf(n) \leftrightarrow j\, \frac{dF(e^{j\omega})}{d\omega}$$

$$n^2 f(n) \leftrightarrow j\frac{d}{d\omega}\left[j\frac{dF(e^{j\omega})}{d\omega} \right] = -\frac{d^2 F(e^{j\omega})}{d\omega^2}$$

Then $$F_2(e^{j\omega}) = -\frac{d^2 F(e^{j\omega})}{d\omega^2} - 2j\frac{dF(e^{j\omega})}{d\omega} + F(e^{j\omega})$$

Example 4.25 *A causal, discrete LTI system having* $h(n) = \left(\frac{1}{3}\right)^n u(n)$ *is excited with following inputs.*

Determine each output:

(a) $x(n) = \left(\frac{2}{3}\right)^n u(n)$ *(b)* $x(n) = (-1)^n u(n)$

Solution

$$H(e^{j\omega}) = \mathcal{F}_d\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

(a) Input

$$X(e^{j\omega}) = \frac{1}{1-\frac{2}{3}e^{-j\omega}}$$

Output

$$Y(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{2}{3}e^{-j\omega}\right)}$$

Partial fractioning

$$\frac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)}\Bigg|_{e^{-j\omega}=\frac{3}{2}} \cdot \frac{1}{\left(1-\frac{2}{3}e^{-j\omega}\right)} + \frac{1}{\left(1-\frac{2}{3}e^{-j\omega}\right)}\Bigg|_{e^{-j\omega}=3} \cdot \frac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

We then find

$$Y(e^{j\omega}) = \frac{2}{\left(1-\frac{2}{3}e^{-j\omega}\right)} - \frac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

Taking inverse

$$y(n) = 2\left(\frac{2}{3}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n)$$

(b) Input

$$x(n) = (-1)^n u(n) = e^{j\pi n} u(n) \leftrightarrow 2\pi \sum_{k-\infty}^{\infty} \delta(\omega - \pi - 2k\pi)$$

or

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - (2k+1)\pi)$$

Output

$$Y(e^{j\omega}) = \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - (2k+1)\pi\right] \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

At

$$\omega = (2k+1)\pi \; \frac{1}{1-\frac{1}{3}e^{-j\omega}} \Rightarrow \frac{1}{1-\frac{1}{3}e^{j(2k+1)\pi}}$$

$$= \frac{1}{1-\frac{1}{3}e^{j\pi}} = \frac{3}{4}$$

Then

$$y(n) = \frac{8\pi}{3} \sum_{k=-\infty}^{\infty} \delta(\omega - (2k+1)\pi)$$

Example 4.26 *A discrete time signal is sketched in Fig. 4.14.*

(a) Calculate $F(e_0^j)$.

(b) Find $\angle F(e^{j\omega})$.

(c) Evaluate $\int_{-\pi}^{\pi} F(e^{j\omega})\, d\omega$.

(d) Determine $F(e^{j\pi})$.

(e) Evaluate $\int_{-\pi}^{\pi} |F(e^{j\omega})|^2\, d\omega$.

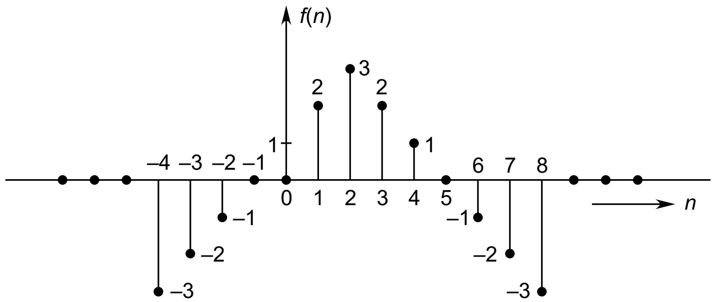


Fig. 4.14

Solution

(a) $F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} f(n)\, e^{-j\omega n}$

For $\omega = 0$

$$F(e^{j_0}) = \sum_{k-\infty}^{\infty} f(n) = -3$$

(b) The signal $y = f(n-2)$, $(f(n)$ shifted by two steps to left is even. Therefore, by the evenness property, $Y(e^{j\omega})$ is real, i.e., $\angle Y(e^{j\omega}) = 0$.

$$F(e^{j\omega}) = e^{-j2\omega}\, Y(e^{j\omega})$$
$$\angle F(e^{j\omega}) = e^{-j2\omega}$$

(c) $f(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} F(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega$

At $n = 0$

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\, \mathrm{d}\omega = 1$$

which gives

$$\int_{-\pi}^{\pi} F(e^{j\omega}) = 2\pi$$

(d) $F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} f(n)\, e^{-j\omega n}$

At $\omega = \pi$

$$F(e^{j\pi}) = \sum_{k=-\infty}^{\infty} f(n)\, e^{-j\pi n} = \sum_{k=-\infty}^{\infty} f(n)\, (-1)^n$$
$$= -3$$

(e) By Parseval's theorem

$$\int_{-\pi}^{\pi} |F(e^{j\omega})|^2 \, d\omega = \sum_{k=-\infty}^{\infty} |f(n)|^2 = -3$$

Example 4.27 *A causal LTI system is described by the following difference equation.*

$$y(n) + \frac{1}{3} y(n-1) = x(n)$$

(a) Determine its frequency response.
(b) Find the system output for the following inputs.

$$\text{(i)}\ x(n) = \left(-\frac{1}{3}\right)^n u(n) \qquad \text{(ii)}\ x(n) = \delta(n) + \frac{1}{2}\delta(n-1)$$

Solution

(a) Taking DFT of the differential equation, we get

$$\left(1 + \frac{1}{3} e^{-j\omega}\right) Y(e^{j\omega}) = X(e^{j\omega})$$

which yields

$$H(e^{j\omega}) = \frac{1}{1 + \frac{1}{3} e^{-j\omega}} \tag{i}$$

(b) (i) Input

$$X(e^{j\omega}) = \frac{1}{1 + \frac{1}{3} e^{-j\omega}}$$

Then, output $= H(e^{j\omega})\, X(e^{j\omega})$

$$Y(e^{j\omega}) = \frac{1}{\left(1 + \frac{1}{3} e^{-j\omega}\right)^2} \tag{ii}$$

Since denominator is a square term, frequency differentiation property can be used.

$$\frac{1}{1 + \frac{1}{3} e^{-j\omega}} \leftrightarrow \left(-\frac{1}{3}\right)^n u(n) \tag{iii}$$

$$j \frac{d}{d\omega}\left[\frac{1}{1 + \frac{1}{3} e^{-j\omega}}\right] \leftrightarrow n\left(-\frac{1}{3}\right)^n u(n)$$

or $$\frac{-\frac{1}{3}e^{-j\omega}}{\left(1+\frac{1}{3}e^{-j\omega}\right)^2} \leftrightarrow n\left(-\frac{1}{3}\right)^n u(n) \qquad \text{(iv)}$$

Adding Eq. (iii) and (iv)

$$\frac{1}{1+\frac{1}{3}e^{-j\omega}} + \frac{-\frac{1}{3}e^{-j\omega}}{\left(1+\frac{1}{3}e^{-j\omega}\right)^2} \leftrightarrow (n+1)\left(-\frac{1}{3}\right)^n u(n)$$

or $$\frac{1}{\left(1+\frac{1}{3}e^{-j\omega}\right)^2} \leftrightarrow (n+1)\left(-\frac{1}{3}\right)^n u(n) \qquad \text{(v)}$$

Thus $$y(n) = (n+1)\left(-\frac{1}{3}\right)^n u(n) \qquad \text{(vi)}$$

(ii) Input

$$x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$$

$$X(e^{j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

Output

$$Y(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1+\frac{1}{3}e^{-j\omega}} = -1 + \frac{2}{1+\frac{1}{3}e^{-j\omega}}$$

Taking inverse

$$y(n) = -\delta(n) + 2\left(-\frac{1}{3}\right)^n u(n)$$

4.4 COMPUTATION OF DFT-FAST FOURIER TRANSFORM (FFT)

In practice, the desired number of DFT terms is quite large may be 1000 to 10,000 and even more. The data therefore is not amenable to closed form results. For computation we go back to finite sum equations (4.8) and (4.9) which for link-up are reproduced below. Also, we begin write $f(n)$ as f_n

Analysis Summation

$$F_k = \sum_{n=0}^{n-1} f_n \, e^{-j(2\pi/N)kn}; \; k = 0, 1, \ldots (N-1) \tag{4.43}$$

Synthesis Summation

$$f_n = \left(\frac{1}{N}\right) \sum_{k=0}^{N-1} F_k \, e^{j(2\pi/N)kn}; \; n = 0, 1, 2, \ldots (N-1) \tag{4.44}$$

In these equations, we have shifted $\left(\frac{1}{N}\right)$ from F_k to f_n, the reason for this change is that it will permit use the same computation algorithm for forward and inverse DFT. Note also that we have the ordering in the exponential power.

Taking complex conjugate of Eq. (4.44), we can write

$$Nf_n^* = \sum_{k=0}^{N-1} F_k^* \, e^{-j(2\pi/N)kn}; \; n = 0, 1, \ldots, N-1 \tag{4.45}$$

This result is useful in fast computational technique. Comparison of Eqs (4.43) and (4.44) shows that the algorithm used for calculating forward transform can also be used for the inverse transform by the frequency sample F_k prior to computation. The result as obtained is then divided by N and conjugated to obtain time samples f_n.

Before proceeding further, we shall introduce the symbol

$$W_N = e^{-j2\pi/N} \tag{4.46}$$

DFT Eq. (4.43) can then be written as

$$F_k = \sum_{n=0}^{N-1} f_n \; W_N^{nk}; \; k = 0, 1, \ldots, (N-1) \tag{4.47}$$

This symbolization will prove convenient in computation and development of FFT. Observe that W_N^n is periodic with period N.

Examination of Eq. (4.43) reveals that the computation of each point of DFT requires the following.

> $(N-1)$ complex multiplication, $(N-1)$ complex additions (first term in sum involves $e^{j0} = 1$). To compute N points in DFT we then require
>
> $N(N-1)$ complex multiplication and same number of complex additions.

A complex multiplication has the following form:

$$(A + jB)(C + jD) = (AC - BD) + j(BC + AD)$$

i.e., it requires four real multiplications and two real additions.

Therefore, computation of N-point DFT requires

$4N(N-1)$ real multiplications; and
$4N(N-1)$ real additions

The total computations become prohibitive for large N, which is a practical requirement.

4.5 PROPERTIES OF DIRECT DFT

The properties of DFT in closed forms have been presented in Section 4.3. Properties in finite seqence for DFT are enumerated here:

1. *Linearity*

If

$$f(n) \leftrightarrow F(k)$$
$$g(n) \leftrightarrow G(k)$$

then

$$a\, f(n) + b\, g(n) \leftrightarrow a\, F(k) + b\, G(k) \tag{4.48}$$

where 'a' and 'b' are constants.

2. *Time Shifting or Circular Shifting*

(a)

If

$$f(n) \leftrightarrow F(k)$$

then

$$f(n-m) \leftrightarrow F(k)\, W_N^{km} \tag{4.49}$$

(b) Frequency shifting

$$f(n)\, e^{j\left(\frac{2\pi}{N}\right)nm} \leftrightarrow F(k-m)$$

or $$f(n)\, W_N^{-nm} \leftrightarrow F(k-m) \tag{4.50}$$

The meaning of circular shifting will get clarified in the convolution property.

3. *Duality*

If

$$f(n) \leftrightarrow F(k)$$

then $$(1/N)\, F(k) \leftrightarrow f(-n) \tag{4.51}$$

4. *Parseval's Theorem*

$$\sum_{n=0}^{N-1} | x(n) |^2 = (1/\mathrm{N}) \sum_{k=0}^{N-1} | \mathrm{X}(k) |^2 \tag{4.52}$$

5. *Complex Sequences*

If

$$f_1(n) \leftrightarrow F_1(k);$$
$$f_2(n) \leftrightarrow F_2(k);$$
$$f_3(n) = f_1(n) + j f_2(n)$$

then

$$F_3(n) \leftrightarrow F_1(k) + j F_2(k) \tag{4.53}$$

6. *Circular Convolution*

Convolution as defined in time domain is

$$f_3(n) = \sum_{m=0}^{N-1} f_1(m) f_2(n - m) \tag{4.54}$$

where $f_1(n)$, $f_2(n)$ and $f_3(n)$ are finite sequences of modulo N. In performing the convolution process as in Eq. (4.54), it will be imagined that these are periodic sequences; then only DFT will be applicable.

To carry out convolution as defined in Eq. (4.54), it is easy to imagine that finite sequence is spread out (uniformly) on a cylinder (or circle). As this sequence is shifted, it is equivalent to rotating the cylinder in appropriate direction. That is why this convolution is called circular. It can be illustrated by Example 4.28.

Example 4.28 *Perform circular convolution on the following sequences.*

$$f_1(n) = \{2, 1, 2, 1\} \quad \text{and} \quad f_2(n) = \{1, 2, 3, 4\}$$

Solution We refer to convolution as defined in Eq. (4.54). The sequences $f_1(m)$ and $f_2(m)$ are shown with data points spaced around a circle in Figs 4.15(a) and (b). The sequence $f_2(-m)$ or $f_2(n - m)|_{n=0}$ is obtained from $f_2(m)$ by merely reversing the sequence about $f_2(0)$ as shown in Fig. 4.15(c).

By multiplying the corresponding points of $f_1(m)$ and $f_2(-m)$ and adding gives

$$\begin{aligned} f_3(0) &= \overset{2\times 1}{f_1(0) f_2(0)} + \overset{1\times 2}{f_1(1) f_2(3)} + \overset{2\times 3}{f_1(2) f_2(2)} + \overset{1\times 4}{f_1(3) f_2(1)} \\ &= 14; \text{ by substituting values} \end{aligned}$$

Let us now find $f_3(1)$, i.e. $n = 1$. The sequence $f_1(1 - m)$ is obtained by shifting forward the sequence $f_2(-m)$ by one step which is drawn in Fig. 4.15(d). Again multiplying corresponding points in Figs 4.15(a) and (d) and adding, gives

$$\begin{aligned} f_3(1) &= f_1(0) f_2(1) + f_1(1) f_2(0) + f_1(2) f_2(3) + f_1(3) f_2(2) \\ &= 14 \end{aligned}$$

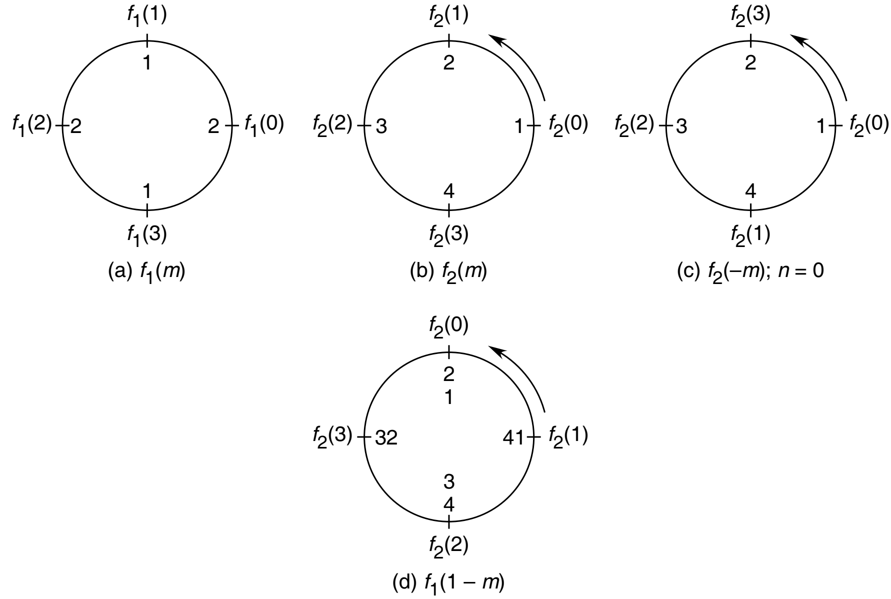


Fig. 4.15

Similarly, we can find all the four points in the convolved sequence $f_3(n)$ which is

$$f_3(n) = \{14,14,14,16\}$$

Remarks The reason why the convolution of Eq. (4.54) is called circular convolution gets clarified as the reversed second sequence is shifted forward w.r. to the first sequence. As an element first moves out of position 1, its position is taken by the last element, and so on.

Though the two sequences are of finite length, they are assumed to be periodic in a way such that the shifted sequence always remains of length N, as the shift is circular. The reader can draw a developed (linear) diagram of the imagined periodic sequence and observe the same effect.

We can now define the DFT property of circular convolution.

7. *Circular Convolution*

If

$$f_1(n) \leftrightarrow F_1(k)$$
$$f_2(n) \leftrightarrow F_2(k)$$

Then

$$f_1(n) \textcircled{N} f_2(n) \leftrightarrow F_1(k)F_2(k) \tag{4.55}$$

where $\textcircled{N}$ = symbol of circular convolution.

8. *Multiplication*

$$f_1(n)\,f_2(n) \leftrightarrow (1/N)\,F_1(k) \; (\text{N}) \; F_2(k)$$

Some More on Convolution

Consider two sequences of four unit elements each as shown in Fig. 4.16(a). Their circular convolution is given in Fig. 4.16(b). The linear convolution of $f_1(n)$ and $f_2(n)$ is shown in Fig. 4.16(c).

Notice that when we obtain linear convolution of sequence $f_1(n)$ and $f_2(n)$, each having N (= 4) samples, it leads to the sequence $f_3^{\text{LC}}(n)$ having $2N - 1$ (= 7) nonzero samples, as shown in Figs 4.16(c). One can also obtain $f_3^{\text{LC}}(n)$

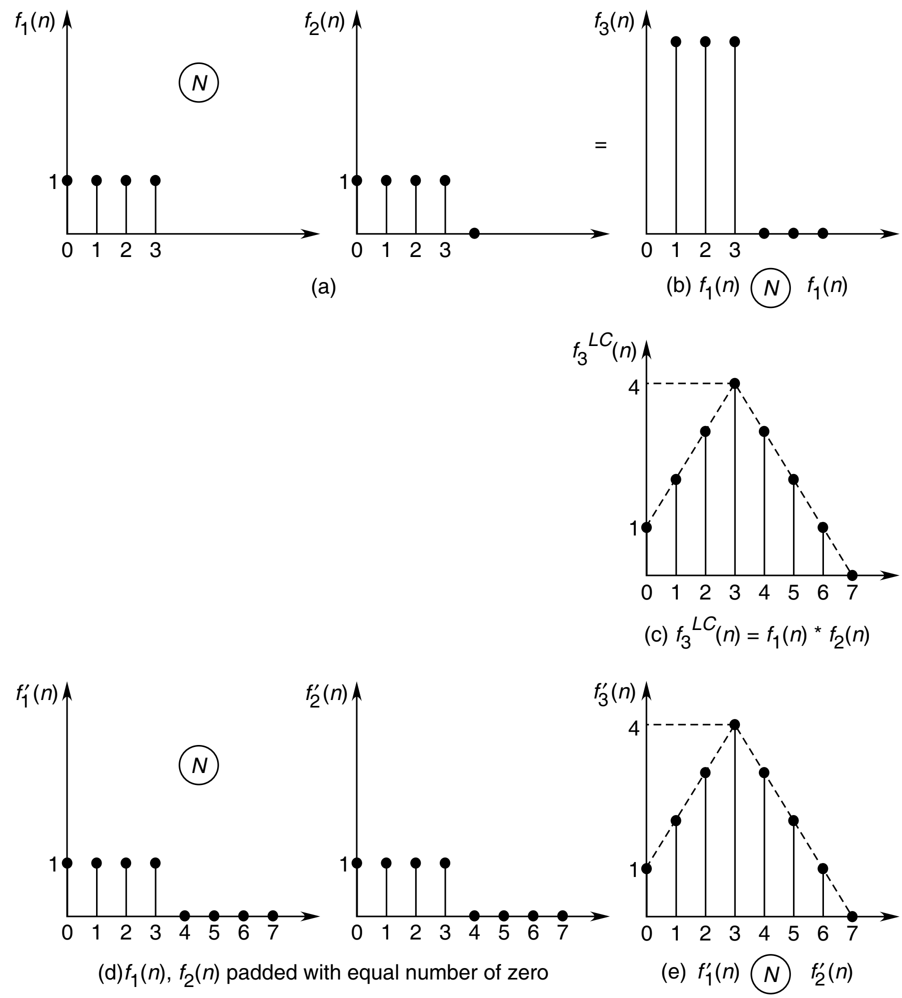


Fig. 4.16

by performing circular convolution, as shown in Fig. 4.16(e). However, before performing circular-convolution, the input sequences $f_1(n)$ and $f_2(n)$ should be modified as $f_1'(n)$ and $f_2'(n)$ respectively, where $f_1'(n)$ and $f_2'(n)$ are the sequences $f_1(n)$ and $f_2(n)$ respectively, each with $N = 4$ additional samples of zero value.

It may be seen that if one performs linear convolution on two sequences $f_1(n)$ and $f_2(n)$, as shown in Fig. 4.16(a) then it results into a sequence $f_3^{LC}(n)$, as shown in Fig. 4.16(c).

On the other hand, if we obtain sequences $f_1'(n)$ and $f_2'(n)$ from $f_1(n)$ and $f_2(n)$ by padding as many number of zeroes as the number of nonzero samples (shown in Fig. 4.16(d)) and perform circular convolution on the sequences $f_1'(n)$ and $f_2'(n)$ then it results into a sequence $f_3'(n)$ which is same as f_3^{LC}.

In Fig. 4.16(d) the sequences $f_1(n)$ and $f_2(n)$ are padded by an equal number of zeros which are then indicated as $f_1'(n)$ and $f_2'(n)$. Their circular convolution is shown in Fig. 4.16(e). It is immediately observed this is the same sequence as obtained by linear convolution of original sequences.

It can then be concluded that linear convolution can be carried by circular convolution by padding the sequences to be convolved with equal number of zeros.

Linear Convolution of Long and Short Sequences

This situation arises in filtering a very long sequence (infinite length) of say an audio or video signal, where the impulse response of the filter is known and is comparatively of very short length. In using the DFT method, the long sequence is to be stored and it's time consuming DFT needs to be calculated. In this method of filtering, no filtered point can be computed until all the input points have been collected. So there is a need to avoid such a long delay in processing the signal to produce the output. Instead we could subdivide the long sequence into short sequences and then use linear convolution. This is illustrated by Fig. 4.17.

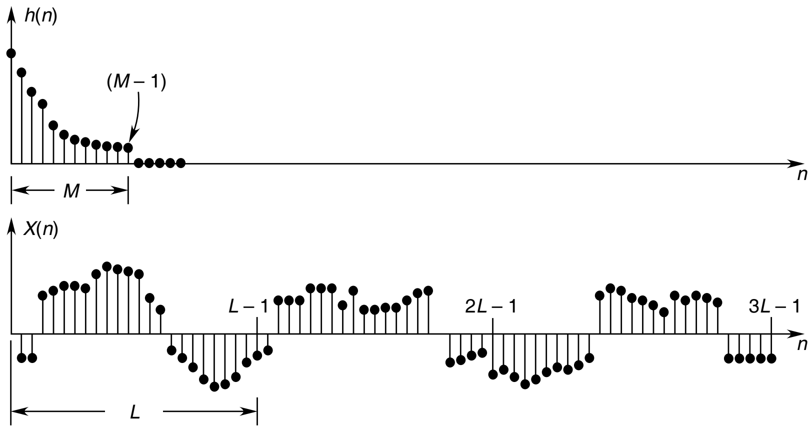


Fig. 4.17

The impulse response $h(n)$ has M points and the long input sequence $x(n)$ is subdivided in shorter duration sequences of L points. Their linear convolution would yield $(L + M - 1)$ points such that $(M - 1)$ points overlap in the initial part of the linear convolution of the next. The complete linear convolution could be obtained by adding the overlapping portion in the adjoining parts of the sections. This method is called **overlap-add** method.

Now the second method called **overlap save** will be illustrated in detail. Here linear convolution is carried out by circular convolution, which can be obtained by product of their DFTs. For circular convolution we shall add zeros to the M-length sequence to make it of length L and then carry out circular convolution of two L length sequences. But this will be used to compute L points and not $L + M - 1$ points it would otherwise generate. Consider an example with $L = 7$ and $M = 3$ as in Fig. 4.18.

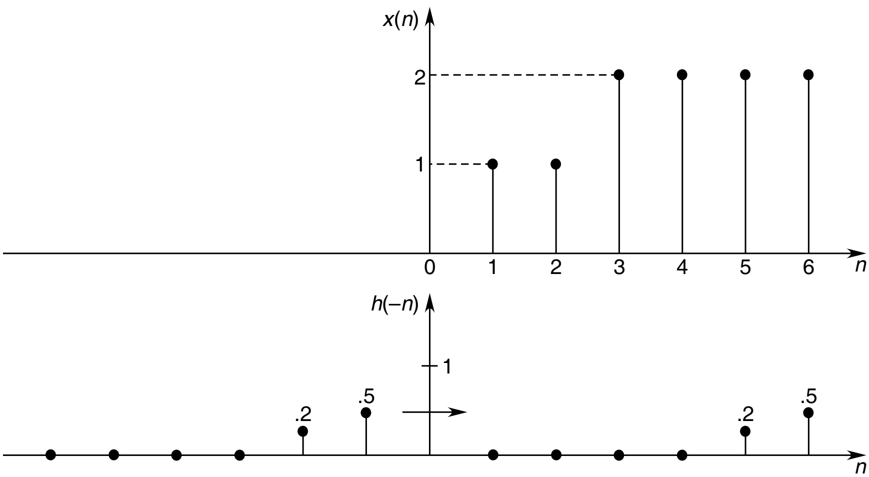


Fig. 4.18

Circular convolution will yield 7 + 3 – 1 = 9 points but will be truncated at $L = 7$ to avoid overlapping with adjoining section L. The 7 points obtained by circular convolution and linear convolution are compared in Table 4.2.

Table 4.2

	Circular	*Linear*
$M - 1 = 2$	2.4	1.0
incorrect points	1.9	1.5
$L - M + 1 = 5$	1.7	1.7
correct points	2.7	2.7
	3.2	3.2
	3.4	3.4
	2.5	2.5

It is seen that circular convolution first gives $M - 1(= 2)$ incorrect points in the beginning of the L-section. This is because partitioning of correct data has not been presented for initial $(M - 1)$ points for circular convolution. So initial $(M - 1)$ points will be rejected. Therefore, the next section will be made to overlap $(M - 1)$ points, with the proceeding section and again initial $(M - 1)$ point are rejected; that is why the name overlap reject method. This is illustrated in Fig. 4.19.

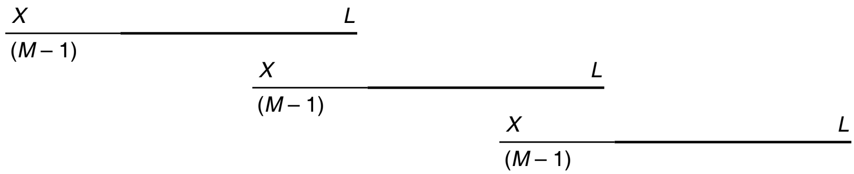


Fig. 4.19

4.6 DFT-ERRORS AND THEIR MINIMIZATION

Three commonly encountered problems in using DFT and the errors introduced by these are the following.

1. Aliasing
2. Leakage
3. Picket fence

How these errors arise in DFT computations will be illustrated here by means of a simple example. The discussion will also be suggestive of the remedial measures that can be applied. These will be summarized in the end.

Consider a simple signal cos (ωt) of infinite time length. Various processes on the signal and the corresponding Fourier Transform (FT) operations along with certain effects are shown in Fig. 4.20. These are further elaborated in Table 4.3 with windowing of some commonly used signals and their effects in frequency domain.

Picket-fence Effect

Fourier coefficients should act ideally as rectangular filters in the frequency domain. However, because the signal is passed through a window of finite width (T-seconds) their amplitude response is like the centre lobe of Fig. 4.20(b) (the presence and effect of smaller lobe is ignored in this discussion). The width of this lobe is inversely proportional to the window width.

In view of the above at the frequencies computed, the main lobes appear to be N independent filters of the form $e^{j\omega t}$ (amplitude unity) with frequency that is integral multiple by $1/T$, i.e. a response of unity at the appropriate frequency and

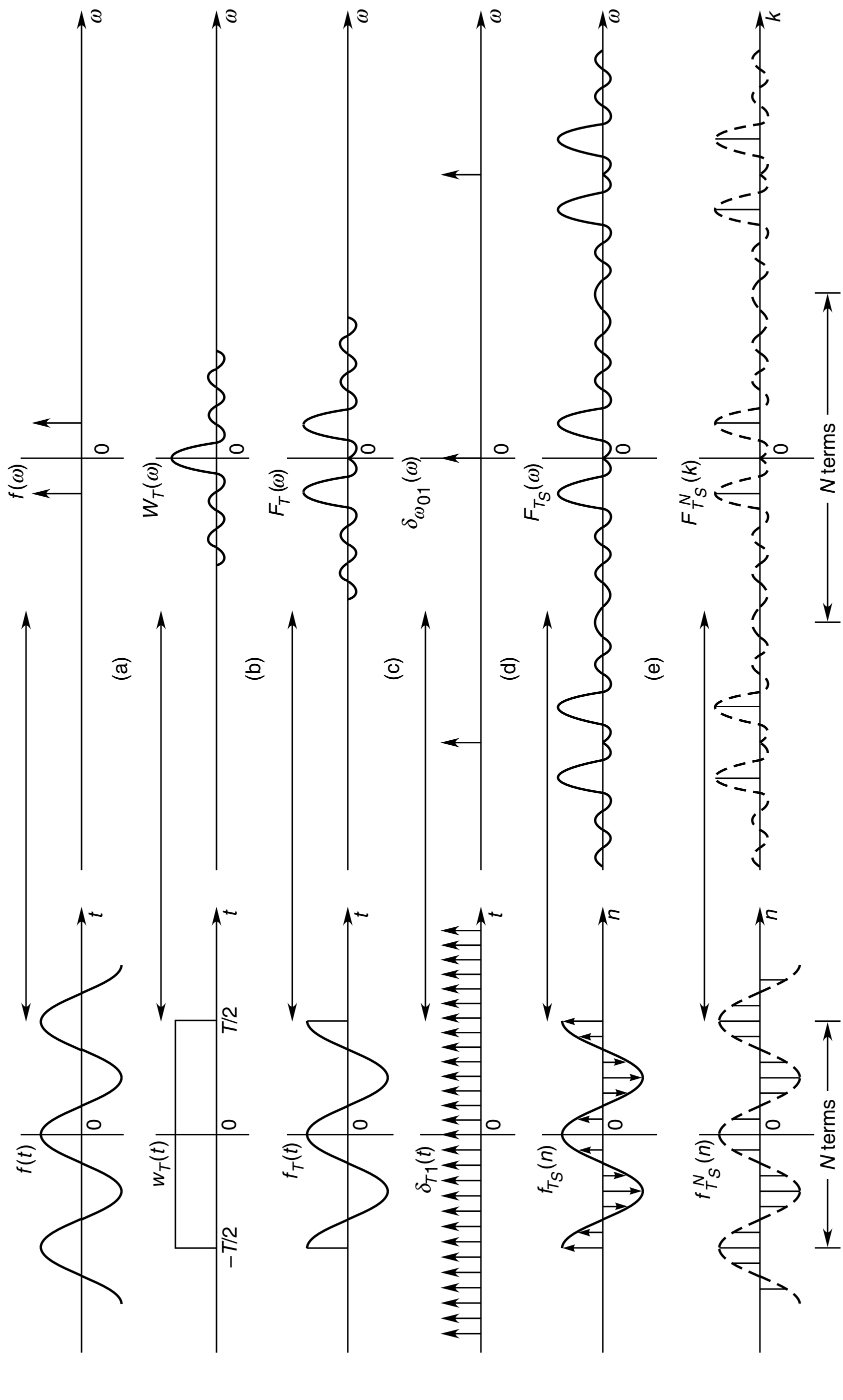

f(t)
f(ω)
w_T(t)
W_T(ω)
f_T(t)
F_T(ω)
δ_T1(t)
δ_ω01(ω)
f_TS(n)
F_TS(ω)
f^N_TS(n)
F^N_TS(k)
−T/2
T/2
0
t
ω
n
k
N terms
(a)
(b)
(c)
(d)
(e)
(f)

Fig. 4.20

Table 4.3

Time signal	*FT*	*Effects*
(a) $f(t) = \cos \omega t$	Two impulses at $\pm\ \omega$	–
(b) Time window $W_T(t)$	$W_T(\omega) =$ $T\ \text{sinc}(W_T/2)$ $\omega_o = 2/T$	–
(c) Truncated signal $f_T(t) = f(t).W_T(t)$	$F_T(\omega) = F(\omega)*W_T(\omega)$	Two frequency functions with certain overlap and side lobes. This effect is known as leakage as smaller signal peaks appear elsewhere.
Remark *As window width T is increased, the sinc function gets compressed in frequency and so the signal representation in frequency domain improves.*		
(d) Delta impulses $\delta_{T_0}(n)$; period T_1	$\omega_{01}\ \delta_{\omega 01}(\omega)$ Impulses in frequency domain; span $\omega_{01} = 2\pi/T_1$	Reducing T_1 will widen ω_{01} (better separation)
(e) Signal sampling $f_{Ts}(n) = f_T(t).\ \delta_{T_0}(n)$ $= f(t).\ W_T(t).\ \delta_{T_0}(n)$	$F_{Ts}(\omega) =$ $F_T(\omega)*\omega_{01}\ \delta_{\omega 01}(\omega)$	$F_T(\omega)$ repeats periodically, period ω_{01} adjoining $F_T(\omega)$'s overlap. This is aliasing effect (see Section 4.7)
Remark *Aliasing may be minimized if $\omega_{01} \geq$ Nyquist rate*		
(*f*) $f_{Ts}(n)$ repeating with period N and labelled as $f_{Ts}^{\ N}(n)$	$F_{Ts}^{\ N}(k)$	Picket-fence effect (for detail see Fig. 4.20.)

zero at all other harmonics as shown in Fig. 4.21(a). This response appears like a band of sine shaped filters.

The picket fence effect will show up when the signal being analyzed is not one of these orthogonal frequencies. For example, the signal between third and fourth harmonics are seen by both windows but at a value lower than one. The signal amplitude is reduced to 0.637 in both the windows. The power being square of the signal strength is reduced to 0.406 or that it is reduced by a factor of $1/0.406 \approx 2.5$. The corresponding power response is shown in Fig. 4.21(b) with the ripple-type variations. The result is that it appears as if one is viewing the true spectrum through a picket fence.

This problem is minimized by increasing the length of $f_{Ts}(n)$ with a set of samples which are identically zero, i.e., the time sequence to be analysed is

$$
\begin{aligned}
f_{Ts}^{\ N}(n) &= f_{Ts}(n) \text{ for } 0 \leq n < N \\
&= 0 \text{ for } N \leq n < 2N
\end{aligned}
$$

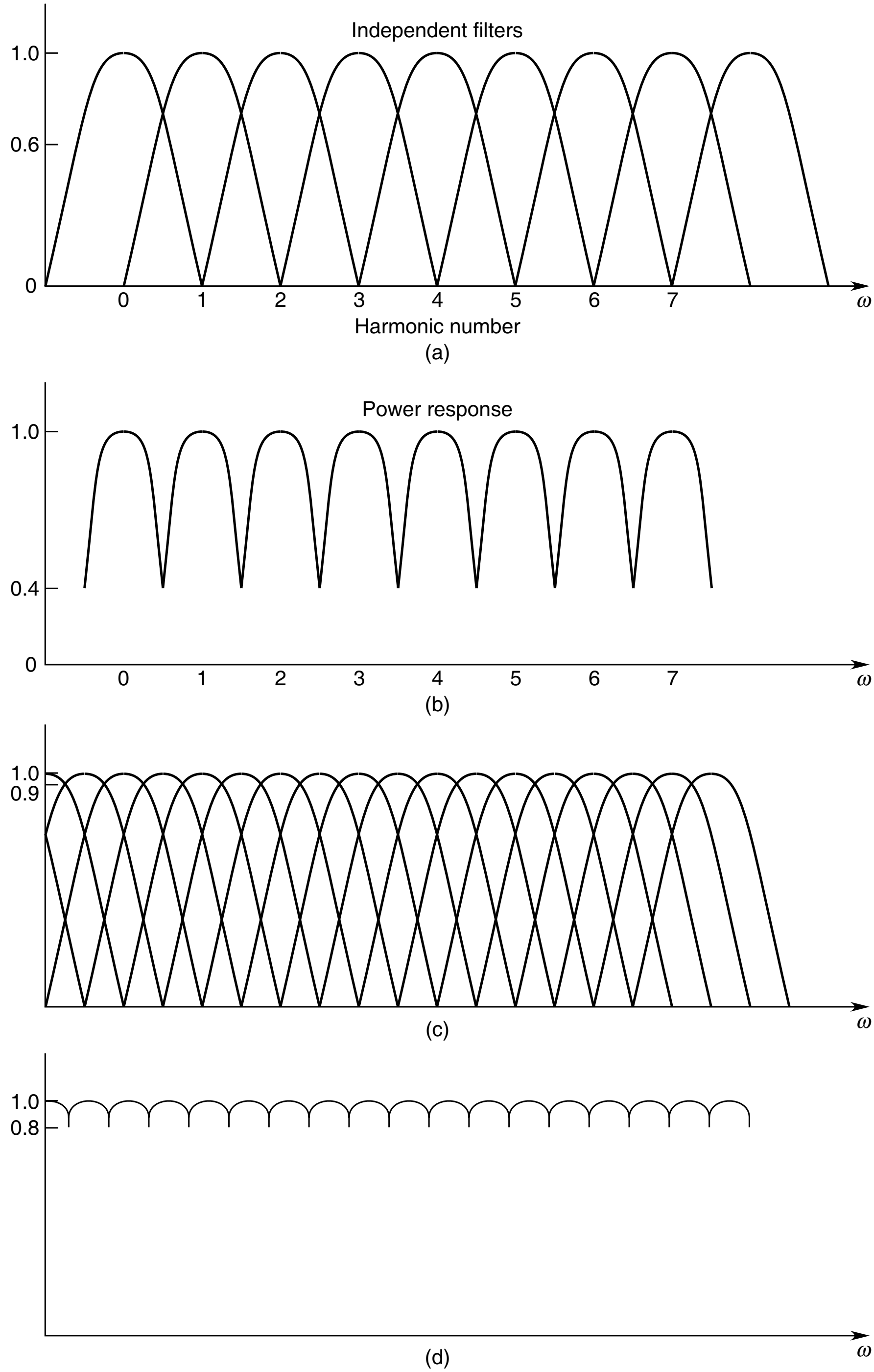

Independent filters
1.0
0.6
0
0
1
2
3
4
5
6
7
ω
Harmonic number
(a)
Power response
1.0
0.4
0
0
1
2
3
4
5
6
7
ω
(b)
1.0
0.9
ω
(c)
1.0
0.8
ω
(d)

Fig. 4.21

As shown in Fig. 4.21(c), the additional Fourier coefficients are interleaved with the original set and ripple in the power spectrum (shown in Fig. 4.21(d) varies between 1 and 0.8 only.

Remedial measures for minimizing above errors are given in Table 4.4.

Table 4.4 *Processing errors in signal and their remedies*

Nature of Error	*Remedial measures*
1. Aliasing	(i) Sampling rate should be sufficiently more than the Nyquist rate. (ii) Signal should be pre-filtered so that high frequency components having insignificant contribution to overall energy of the signal are minimized.
2. Leakage	(i) Increase the width of the window function in the time domain. (ii) Use such window functions whose spectrum have low side lobes.
3. Picket-fence	Increase the length of the function in the time domain, however the sampling rate should remain unchanged.

4.7 FAST FOURIER TRANSFORM (FFT)

It has been shown above that in computation of DFT the number of real multiplication and real additions are $4N(N-1)$ each. In practical application N has to be large, so the computations increase tremendously. For a long time, several fast algorithms which reduce the number of computations have been developed. An efficient and commonly used FFT algorithm will be presented here.

FFT Algorithm (Radix-2)

This algorithm assumes that the number of points to be calculated in DFT are a power of 2 or 2^p and hence the name being Radix-2. There are two ways of using Radix-2, decimation in time and decimation in frequency. Both the ways are discussed here.

Decimation-in-time In decimation-in-time FFT, time sequence f_n is decomposed progressively into even and odd sequences until two-point sequences are reached. This is possible as $N = 2^p$. At every stage of decomposition, the decomposed sequences are renumbered and then divided into even and odd sequences.

In order to follow the place taken by the starting sequence numbers when the sequence subdivision reaches the stage of two-point sequences, division of sequences is shown in Table 4.5, where the original numbers are preserved.

It may be noted that at each stage the sequence is decomposed into smaller sequences by replacing n with $(2n)$ and $(2n + 1)$ respectively. This decomposition is carried out till sequences of only two elements are reached.

Table 4.5 *Reordering of samples in decimation-in time FFT algorithm*

Initial location of (n) of samples			*Relocation of samples after applying algorithm*		
n in decimal numbers	*n in binary number*	*Samples in natural order*	*Samples reordered*	*Reordered locations in decimal numbers*	*Reordered location in binary numbers*
0	000	f (0)	f (0)	0	000
1	001	f (1)	f (4)	4	100
2	010	f (2)	f (2)	2	010
3	011	f (3)	f (6)	6	110
4	100	f (4)	f (1)	1	001
5	101	f (5)	f (5)	5	101
6	110	f (6)	f (3)	3	011
7	111	f (7)	f (7)	7	111

The decomposition as shown in Fig. 4.22, is carried for a finite sequence $N = 8$. The results are shown in Table 4.4. The original sequence in decimal numbers and the final sequences of two elements each also in decimal numbers are both given in the table along with their binary equivalents. It is observed that the final ordering (2 elements) in binary numbers is obtained simply by bit reversal of the starting sequence in binary. This avoids the necessity of going through the intermediate sequences as in Fig. 4.22.

DFT Eq. (4.47) is reproduced below:

$$F_k = \sum_{n=0}^{N-1} f_n\, W_N^{nk};\ k = 0,1,2,\ ...,\ N-1 \tag{4.56}$$

At the first stage of decomposition, the sequence f_n is divided into f_{2n} and f_{2n+1}; $n = 0,1,2...$; $(N/2) - 1$. Equation (4.56) can then be written in two parts in the following manner.

$$F_k = \sum_{n=0}^{(N/2)-1} f_{2n}\, W_N^{k(2n)} + \sum_{n=0}^{(N/2)-1} f_{2n+1}\, W_N^{k(2n+1)};\ k = 0,1,2,\ ...,\ N/2 - 1 \tag{4.57}$$

Now,

$$W_N^{kn} = e^{j(2\pi/N)\,(2kn)} = e^{-j(2\pi/N/2)\,(kn)}$$

or

$$W_N^{2kn} = W_{N/2}^{kn} \tag{4.58}$$

Also,

$$W_N^{k(2n+1)} = W_{N/2}^{kn}\, W_N^{k}$$

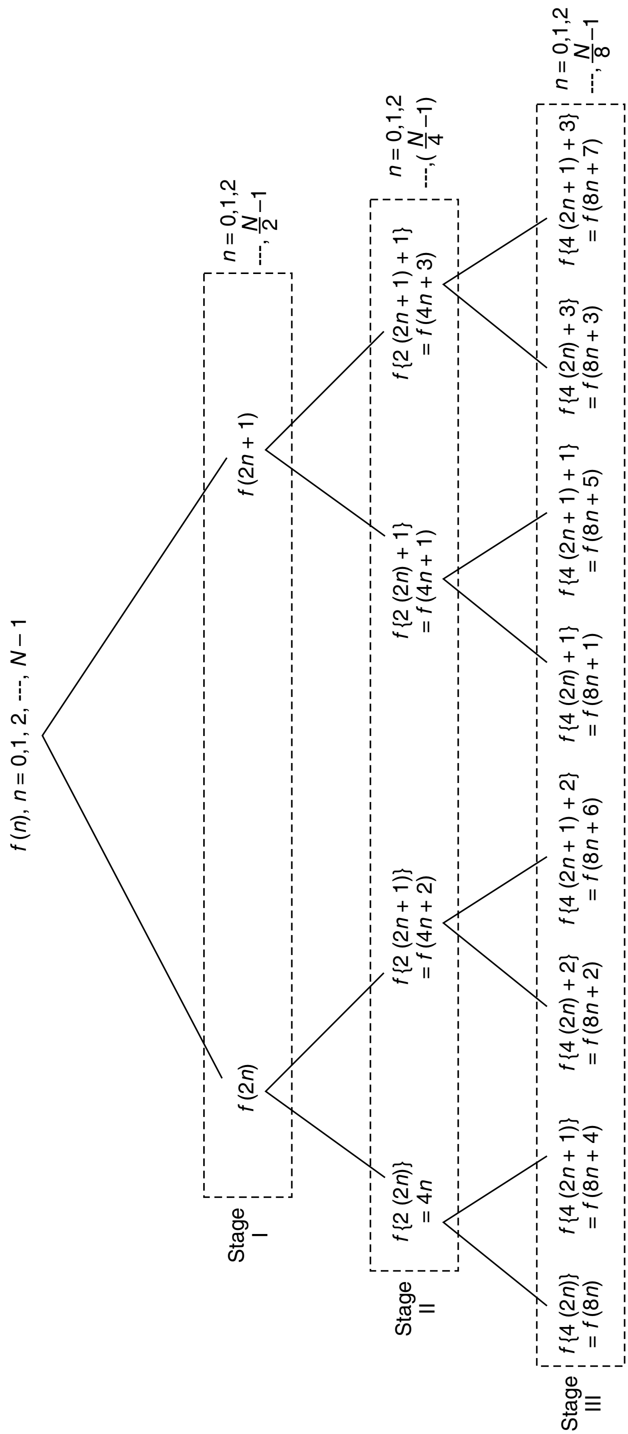


Fig. 4.22 Algorithm for decomposing the sequences

Equation (4.57) then takes the form

$$F_k = \sum_{n=0}^{(N/2)-1} f_{2n}\, W_{N/2}^{kn} + W_N^k \sum_{n=0}^{(N/2)-1} f_{2n+1}\, W_{N/2}^{kn};\ k = 0,1,2, ..., N/2 - 1 \tag{4.59}$$

Define

$$G_k = \sum_{n=0}^{(N/2)-1} f_{2n}\, W_{N/2}^{kn};\ k = 0,1,2, ..., (N/2) - 1 \tag{4.60}$$

$$H_k = \sum_{n=0}^{(N/2)-1} f_{2n+1}\, W_{N/2}^{kn};\ k = 0,1,2, ..., (N/2) - 1 \tag{4.61}$$

So

$$F_k = G_k + W_N^k\, H_k,\ k = 0,1,2, ..., N - 1 \tag{4.62}$$

The periodicity of G_k and H_k in k for two halves ($N/2$) is established below.

$$\begin{aligned} G_{k+N/2} &= \sum_{n=0}^{(N/2)-1} f_{2n} W_{N/2}^{(k+N/2)n} \\ &= \sum_{n=0}^{(N/2)-1} f_{2n} W_{N/2}^{kn}\ W_{N/2}^{Nn/2} \end{aligned}$$

But

$$W_{N/2}^{(Nn)/2} = e^{-j(2\pi/N/2)\,(Nn/2)} = 1$$

Then

$$G_{k+N/2} = \sum_{n=0}^{(N/2)-1} f_{2n} W_{N/2}^{kn} = G_k \tag{4.63}$$

Similarly, it follows that

$$H_{k+N/2} = H_k \tag{4.64}$$

Also

$$W_N^{k+N/2} = W_N^k\, W_N^{N/2} = -\, W_N^k \tag{4.65}$$

Since

$$W_N^{N/2} = e^{-j(2\pi/N)(N/2)} = e^{-j\pi} = -1$$

Thus Eq. (4.59) can be written for the two ($N/2$) half points as

$$F_k = G_k + W_N^k H_k;\ k = 0, 1, 2, .., (N/2) - 1 \tag{4.66}$$

and

$$F_{k+N/2} = G_k + W_N^{k+N/2}\, H_k;\ k = 0, 1, 2, ..., (N/2) - 1 \tag{4.67}$$

$$= G_k - W_N^k H_k;\ k = 0, 1, 2, ..., (N/2) - 1 \tag{4.68}$$

Using Eqs (4.66) – (4.68), a butterfly diagram as shown in Fig. 4.23 can be drawn for visual representation of these equations.

Equations 4.66 – 4.68 and the corresponding butterfly diagram reduce computations as these use the same G_k and H_k for computing F_k and $F_{k+N/2}$.

Computations can be further reduced by observing in Fig. 4.23 that the two branches starting at H_k have weight W_N and $-W_N$. So this weight can be calculated out leaving only 1 and – 1 as multipliers in the two branches.

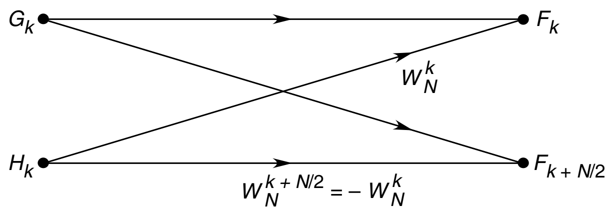


Fig. 4.23

Such a modified diagram is drawn in Fig. 4.24. Reduction in computation is obvious.

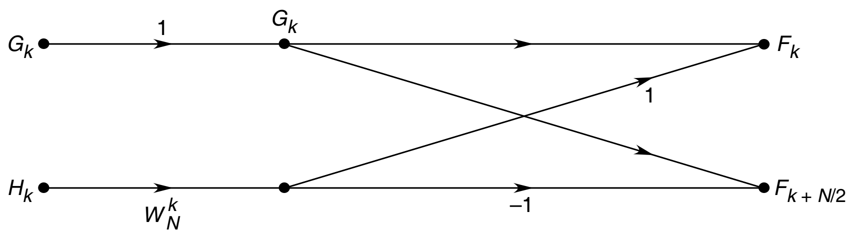


Fig. 4.24

Let us now subdivide $G(k)$ and $H(k)$ of Eqs (4.60) and (4.61) into even and odd sequences. It is sufficient to consider only $G(k)$, as the same result follows for $H(k)$.

$$
\begin{aligned}
G(k) &= \sum_{r=0}^{(N/2)-1} g(\ell)\, W_{N/2}^{rk};\ k = 0,1,2,\ ...,\ (N/2-1) \\
&= \sum_{\ell=0}^{(N/4)-1} g(2\ell)\, W_{N/2}^{2\ell k} + \sum_{\ell=0}^{(N/4)-1} g(2\ell+1)\, W_{N/2}^{(2\ell+1)k} \\
&= \underbrace{\sum_{\ell=0}^{(N/4)-1} g(2\ell)\, W_{N/4}^{\ell k}}_{G(k)\text{ (new)}} + W_{N/2}^{k} \underbrace{\sum_{\ell=0}^{(N/4)-1} g(2\ell+1)\, W_{N/4}^{\ell k}}_{H(k)\text{ (new)}} \qquad (4.69)
\end{aligned}
$$

$; k = 0, 1, 2, \ldots, (N/4) - 1$

It then follows that

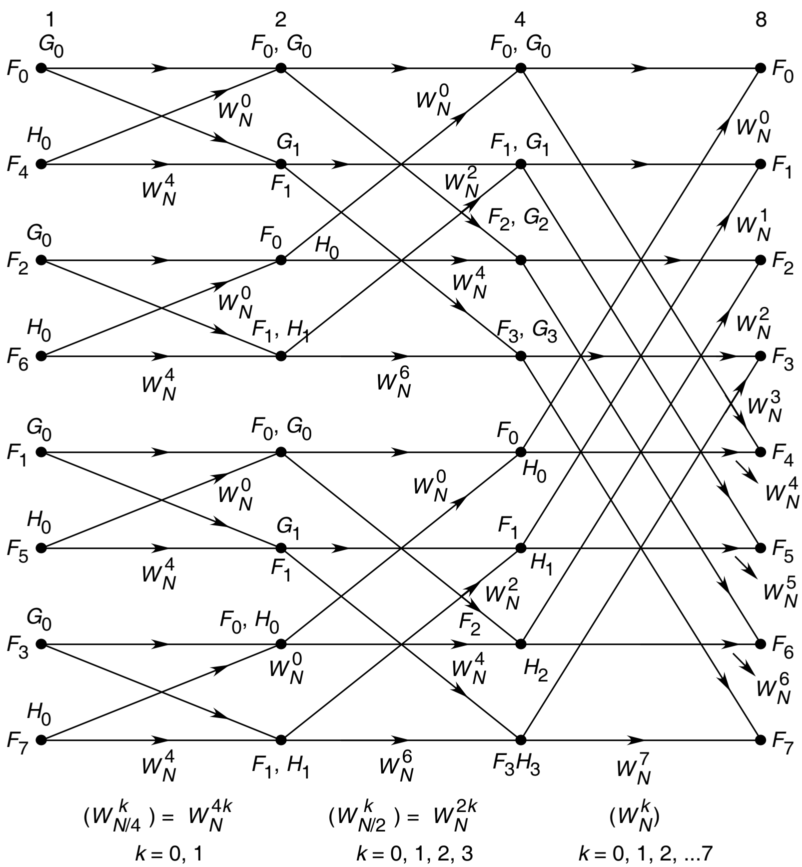
$$H(k) = \underbrace{\sum_{\ell=0}^{N/4} h(2\ell)\, W_{N/4}^{\ell k}}_{G(k)\ \text{(new)}} + W_{N/2}^{k} \underbrace{\sum_{\ell=0}^{(N/4)-1} h(2\ell+1)\, W_{N/4}^{\ell k}}_{H(k)\ \text{(new)}} \qquad (4.70)$$

$$; k = 0, 1, 2, \ldots, (N/4) - 1$$

It is observed here that at each decimation G and H have the same form as at the first decimation, i.e. Eqs (4.60) and (4.61) except that the W inside the summation has suffix as the number of sequence elements after decimation and W multiplier of H has suffix equal to number of sequence elements being decimated.

The butterfly diagram applies at every stage of decimation in the light of the above remarks.

Consider an example with $N = 8$ elements. These are progressively decimated as 4×2, 2×4, and 1×8 as shown in Fig. 4.25.



$W_N^4 = -W_N^0,\ W_N^6 = -W_N^2,\ W_N^7 = -W_N^3$

Fig. 4.25 *Successive points in N = 8 point FFT*

The input elements are arranged by bit reversal procedure as per Table 4.4. The DFT of a single element is $f_n W_1^{\,0} = f_n$ (the element itself). Further, relationship (4.58) is used to make suffixes of all W's as N.

The computational requirement, as mentioned earlier, can further be reduced if butterfly structures at each stage in the signal flow graph (Fig. 4.25) is modified in accordance with butterfly structure shown in Fig. 4.24. We, therefore, have modified signal flow graph of Fig. 4.25, as shown in Fig. 4.26.

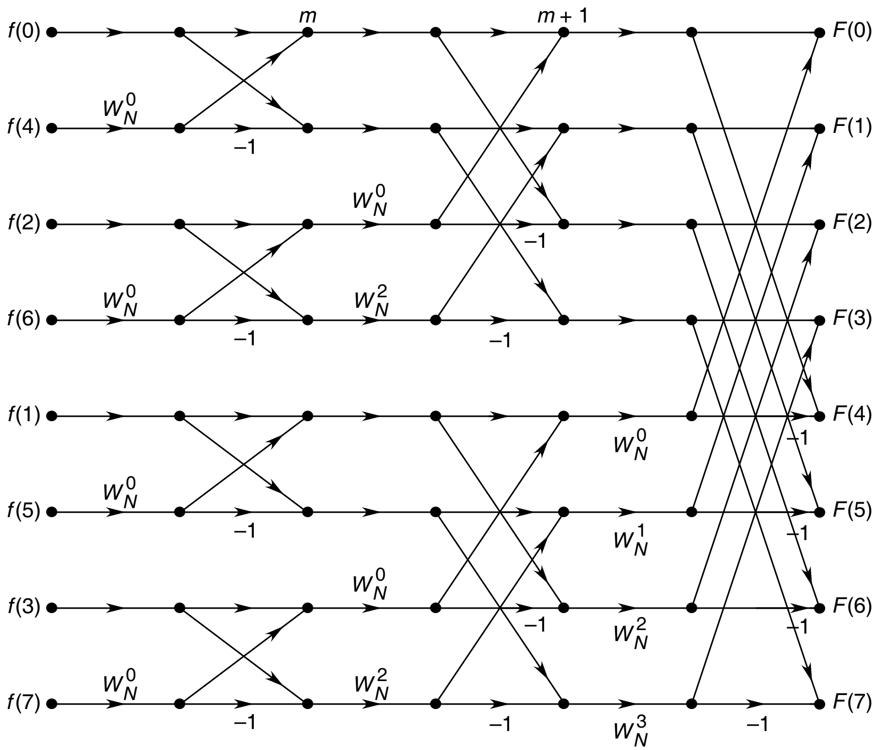


Fig. 4.26

It is clear from Fig. 4.26 that computation is carried out in three stages for $N = 8$ and each stage requires four multiplications and hence a total of 12 complex multiplications are required. In general, for an N-point sequence there are $\log_2 N$ stages and each stage requires $N/2$ complex multiplication; hence a total of $N/2 \log_2 N$ complex multiplications are required. The number of additions required, however, is $N \log_2 N$. Comparison on computation basis between the direct method and decimation-in-time FFT method is given in Table 4.6.

It is, therefore, obvious that decimation-in-time FFT algorithm achieves considerable reduction in the computation of DFT.

Table 4.6 *Order of computations of DFT in direct method and decimation-in time algorithm*

	Direct Method		*Decimation-in-time FFT algorithm*	
N	*Complex multiplications N(N – 1)*	*Complex additions N(N – 1)*	*Complex multiplications N/2 Log2 N*	*Complex additions N log2 N*
4	12	12	4	8
8	56	56	12	24
16	240	240	32	64
32	992	992	80	160
64	4032	4032	192	384

Moreover, one may think of using only two arrays of storage registers, one for data being used and one for array being computed at every stage of computation. This is called **in-place computation**.

The algorithm for computation of N-point DFT using decimation-in-time can be described in short as below.

(i) Reord the samples $f(n)$ in the manner as depicted in Table 4.2.
(ii) Carry out computation at each stage as per butterfly structure as shown in Fig. 4.27.

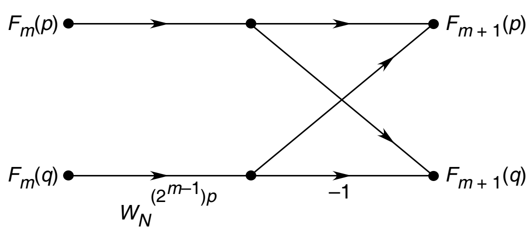


Fig. 4.27

Step (ii) of the algorithm is elaborated below.

Let

$m \rightarrow$ computational stage (we take rightmost or final stage of computation m = log N/2 and the leftmost or initial stage of computation $m = 1$).

$p, q \rightarrow$ the dual of the butterfly structure

p and q are related as

$$q = p + N/2^m$$

where p can take values 0, 1, 2, … $\left(\dfrac{N}{2^m} - 1\right)$ for a given stage of computation (m).

Let us illustrate this step of algorithm with the following illustrative example.

Consider $N = 4$, then m will take values 1 and 2 ($\log_2 4 = 2$) with $m = 1$ being the final stage of computation and $m = 2$ being the first stage of computation.

For m = 1

$$p = 0, 1, 2, \ldots \left(\frac{4}{2} - 1\right); p = 0 \text{ and } 1$$

for p = 0

$$q = 0 + 4/2 = 2$$

and

for p = 1

$$q = 1 + 4/2 = 3$$

Now, the weighting factor $W_N^{(2^{m-1})p}$ is obtained.

For $m = 1, p = 0$, weighting factor $= W_4^{\,0}$

Also for $m = 1; p = 1$, weighting factor $= W_4^{\,1}$

Thus, final stage of computation will flow as shown in Fig. 4.28.

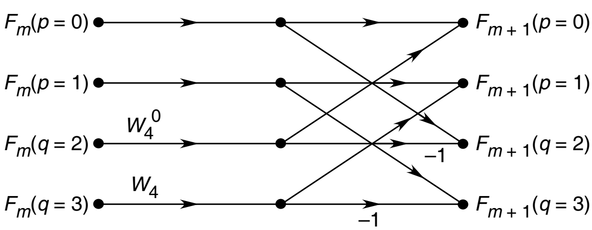


Fig. 4.28

For *m* = 2

$$p = 0, 1, 2, \ldots \left(\frac{4}{4} - 1\right); p = 0 \text{ only}$$

$$q = 0 + \frac{4}{4} = 1$$

and the weighting factor $= W_4^{(2^{2-1})p} = W_4^{\,0}; p = 0$

Thus, initial stage of computation will flow as shown in Fig. 4.29.

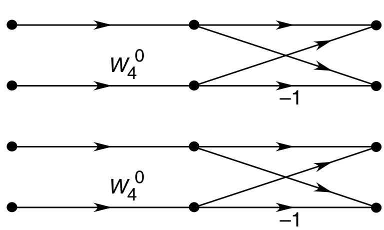


Fig. 4.29

Decimation-in-frequency FFT Algorithm (DIFFFT) In the decimation- in-frequency algorithm, the output or frequency points are regrouped or subdivided and hence it is called decimation-in-frequency FFT algorithm.

As a first step in arriving at this algorithm, the DFT expression (4.56) is divided into first half and last half in the following manner.

$$F_k = \sum_{n=0}^{(N/2)-1} f_n W_N^{nk} + \sum_{n=N/2}^{N-1} f_n W_N^{nk}$$

This can be rewritten as

$$F_k = \sum_{n=0}^{(N/2)-1} f_n W_N^{nk} + \sum_{n=0}^{(N/2)-1} f_{n+N/2} W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{(N/2)-1} f_n W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{(N/2)-1} f_{n+N/2} W_N^{nk} \tag{4.71}$$

$$= \sum_{n=0}^{(N/2)-1} [f_n + (-1)^k f_{n+(N/2)}] W_N^{nk} \tag{4.72}$$

It may be noted that the Eq. (4.72) is obtained on substituting $\left(W_N^{N/2}\right)^k = (-1)^k$ in Eq. (4.71).

Let

$$k = 2r \text{ (even)}, k = (2r + 1) \text{ (odd)}$$

F_k can then be split as

$$F_{2r} = \sum_{n=0}^{(N/2)-1} [f_n + f_{n+N/2}] W_{N/2}^{rn} ; r = 0, 1, 2, ..., (N/2) - 1 \tag{4.73}$$

and

$$F_{2r+1} = \sum_{n=0}^{(N/2)-1} [f_n - f_{n+N/2}] W_N^{(2r+1)n}$$

$$= \sum_{n=0}^{(N/2)-1} [f_n - f_{n+N/2}] W_{N/2}^{rn} W_N^n ; r = 0,1,2, ..., (N/2) - 1 \tag{4.74}$$

The flow graph for $N = 8$, obtained using decimation-in-frequency FFT algorithm is shown in Fig. 4.30. If the butterfly structures at each stage are modified as in Fig. 4.31 then the modified signal flow graph (Fig. 4.30) is as shown in Fig. 4.32. In this case also, number of multiplications and additions required are $N/2 \log_2 N$ and $N \log_2 N$ respectively (i.e., same as that required in case of decimation-in-time FFT algorithm.

In this case, one can observe from Fig. 4.31 that the flow graph is just the reversal of Fig. 4.26. The steps of the algorithm are as follows.

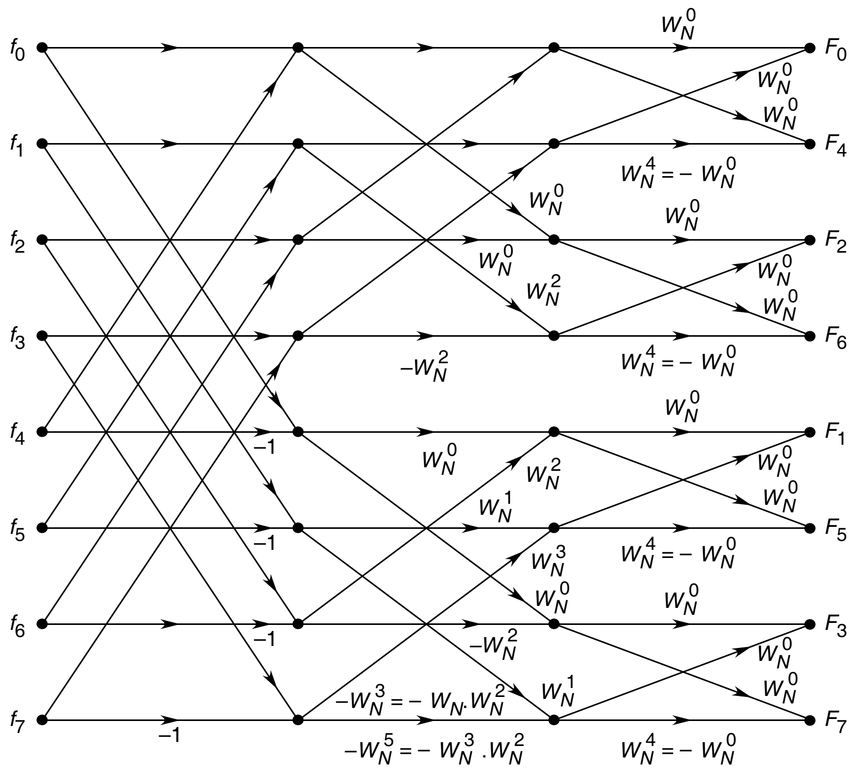


Fig. 4.30

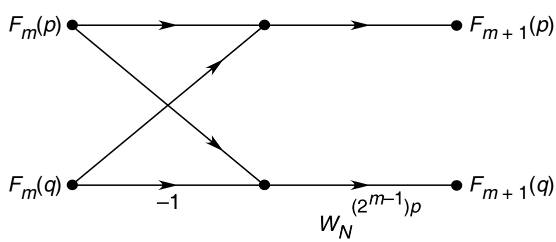


Fig. 4.31

(i) Reorganize the frequency samples $F(k)$ in the same manner as $f(n)$.

(ii) Carry out computation at each stage as per butterfly structure shown in Fig. 4.32.

Here m, p and q are defined in the same manner as in the case of decimation-in-time algorithm. However, here $m = 1$ corresponds to initial stage and $m = \log_2 N$ correspond to the final stages of computation.

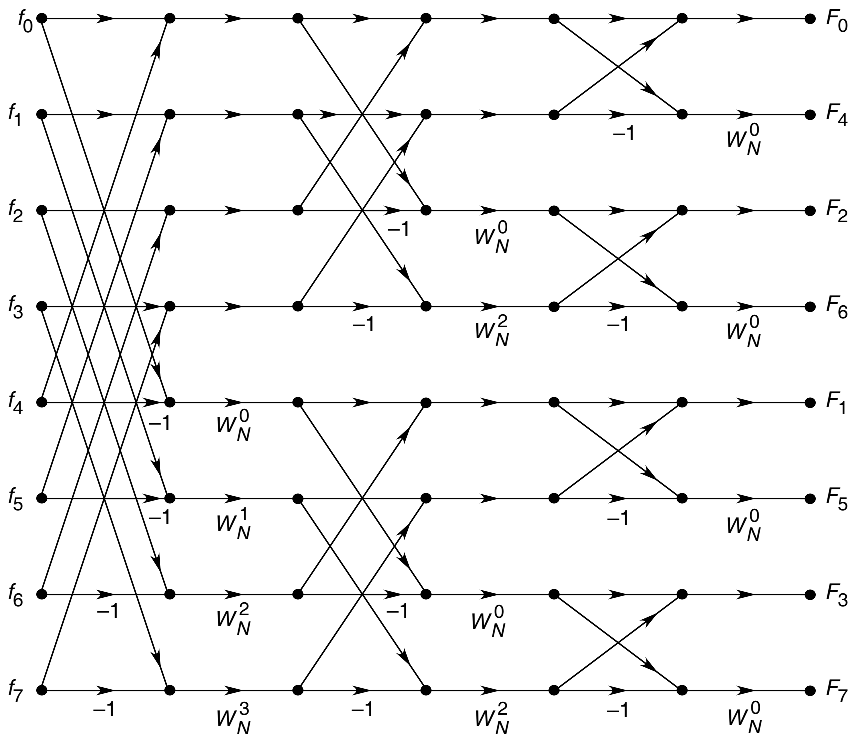


Fig. 4.32

Additional Examples-II

Example 4.29 *A 50 ms signal obtained in an experiment has no significant spectral content above 500 Hz. This signal is to be analysed spectrally with a resolution of at least 10 Hz.*

(a) What should be the number of points required in the DFT? Find the window length in the unit of time.

(b) Can we do zero padding to achieve the desired resolution? Comment.

Solution

(a) Given

$$f_m = 500 \text{ Hz} \tag{i}$$

We choose

$$f_s = 2 \times 500 = 1000 \text{ Hz} \tag{ii}$$

which gives sampling rate $T = 1/1000 = 0.001\text{s}$ (iii)

The frequency resolution is $\Delta f = \dfrac{2f_m}{N}$

i.e. $$10 = \frac{1000}{N} \quad \Rightarrow \quad N = 100 \qquad \text{(iv)}$$

So we require record-length (time domain) = $100 \times 0.001 = 0.1$s (v)

(b) Signal length = 50 ms

Record length, required (as in part (a)) = 100 ms
Zero – padding needed = 50 ms

Example 4.30 *The spectrum of a transient signal of duration 2 ms is to be determined by FFT analysis technique. To avoid aliasing, the sampling rate of 5 kHz would suffice. Spectral resolution desired is 100 Hz. Check its feasibility.*

Solution Let N be the total number of samples in the spectrum and f_m be the maximum frequency component in the spectrum. The desired spectral resolution = 100 Hz.

This requires that

$$f_s = 2f_m = N \times 100; \; N = \text{number of samples} \qquad \text{(i)}$$

Sampling period,

$$T = \frac{1}{2f_m} \quad \Rightarrow \quad T = \frac{1}{100N} \qquad \text{(ii)}$$

This gives length of the signal in time.

$$t_m = NT = 1/100 = 0.01\text{s} = 10 \text{ ms.}$$

Since, the transient signal has only 2 ms duration, the resolution of 100 Hz is not feasible.

Example 4.31 *Compute an appropriate sampling rate and DFT size $N(=2^p)$ in order to analyze a signal with no significant frequency content above 10 kHz and with a resolution of 100 Hz. Assume an arbitrarily long signal.*

Solution Given

$$fm = 10 \times 10^3 \text{ Hz}$$

Desired resolution being

$$\Delta F = 100 \text{ Hz}$$

We must have

$$N = (2 \times 10 \times 10^3)/100 = 200$$

As N must be expressable in power of 2, we choose

$$N = 256 = 2^8 \text{ (nearest)}$$

Required sampling frequency is

$$f_s = 256 \times 100 = 25\,600 \text{ Hz or } 25.6 \text{ kHz}$$

Example 4.32 *Find DFT of a time sequence f_n, if*

$$f_n = 0 \,;\, \text{for } n = 0, 4, 5$$
$$f_n = 1 \,;\, \text{for } n = 1, 2, 3$$

Solution

We have $F_k = \sum_{n=0}^{5} f_n W_N^{kn}$; for $k = 0, 1, 2, \ldots 5$ (i)

Also

$$\begin{aligned}
W_6^0 &= W_6^6 = W_6^{12} = W_6^{18} = W_6^{24} = 1 \\
W_6^1 &= W_6^7 = W_6^{13} = W_6^{19} = W_6^{25} = e^{-j\pi/3} \\
W_6^2 &= W_6^8 = W_6^{14} = W_6^{20} = e^{-j2\pi/3} \\
W_6^3 &= W_6^9 = W_6^{15} = W_6^{21} = e^{-j\pi} = -1 \\
W_6^4 &= W_6^{10} = W_6^{16} = W_6^{22} = e^{-j4\pi/3} \\
W_6^5 &= W_6^{11} = W_6^{17} = W_6^{23} = e^{-j5\pi/3}
\end{aligned}$$

Thus

$$\begin{aligned}
F_0 &= 3, F_1 = -(1 + j1.732) \\
F_2 &= 1, F_3 = -1 \\
F_4 &= -1 \quad \text{and} \quad F_5 = -1
\end{aligned}$$

Example 4.33 *Prove the duality property of DFT.*

Solution Duality property of DFT is

If

$$f_n \leftrightarrow F_k \tag{i}$$

then

$$\frac{1}{N} F_n \leftrightarrow f(-k) \tag{ii}$$

Now

$$f_n = \frac{1}{N} \sum_{n=0}^{N-1} F_k e^{j2\pi nkn/N} \tag{iii}$$

and

$$f_{-n} = \frac{1}{N} \sum_{n=0}^{N-1} F_k e^{-j2\pi k/N} \tag{iv}$$

Interchanging the indices n and k in Eq. (iv), we get

$$f_{-k} = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{-j2\pi nk/N}$$

or

$$f_{-k} \leftrightarrow \frac{1}{N} F_n \tag{v}$$

Example 4.34 *Prove the following Parseval theorem given below.*

$$\sum_{n=0}^{N-1} |f_n|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |F(k)|^2$$

Solution We can write $|f_n|^2 = f_n f_n^*$ (i)

Then

$$\sum_{n=0}^{N-1} |f_n|^2 = \sum_{n=0}^{N-1} f_n f_n^*$$

$$= \sum_{n=0}^{N-1} f_n \left[\frac{1}{N}\sum_{k=0}^{n-1} f_k^* \, e^{-j2\pi nk/N} \right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} f_n \left[\sum_{k=0}^{N-1} F_k^* \, W^{kn} \right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} F_k^* \left[\sum_{k=0}^{N-1} F_k^* \, W^{kn} \right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} F_k^* F_k = \frac{1}{N}\sum_{k=0}^{N-1} |F_k|^2$$

Problems

4.1 Determine the DFT of the following signals.

(a) $x(n) = u(n-2) - u(n-5)$

(b) $x(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$

(c) $x(n) = \left(\frac{1}{3}\right)^{|n|} u(-n-2)$

4.2 Determine the DFT of the following signals.

(a) $f(n) = 2^n \sin\left(\frac{\pi}{3}n\right)u(-n)$ (b) $f(n) = \sin\left(\frac{\pi}{2}n\right) + \cos n$

(c) $f(n) = (n-1)\left(\frac{1}{3}\right)^{|n|}$

4.3 Determine the signal corresponding to the following DFTs.

(a) $F(e^{j\omega}) = 1 + 3\,e^{-j\omega} + 2\,e^{-2j\omega} - 4\,e^{-5j\omega}$

(b) $F(e^{j\omega}) = e^{-3j\omega}$ for $-\pi \le \omega \le \pi$

4.4 Given

$$F(e^{j\omega}) = \cos^2 \omega + \sin^2 3\omega$$

determine $f(n)$.

4.5 Determine the DFT of the following signals.

(a) $f_1(n) = f(n-1) + f(-1-n)$ (b) $f_2(n) = \dfrac{f(-n)+f(n)}{2}$

4.6 The following information is known about a signal

1. $f(n) = 0$ for $n > 0$;
2. $f(0) > 0$
3. $Im\,[F(e^{j\omega})] = \sin\omega - \sin 3\,\omega$
4. $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |\,F(e^{j\omega})\,|^2\, d\omega = 6$

Determine $f(n)$.

Hint: Use property odd $[f(n)] \xleftrightarrow{\mathcal{F}_d} j\,Im\,[F(e^{j\omega})]$

4.7 Evaluate

$$\sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n$$

using DFT.

Hint: Start with $\mathcal{F}_d\,[a^n u(n)]$; $a = \dfrac{1}{3}$

4.8 A discrete system has impulse response

$$h(n) = \left(\frac{1}{3}\right)^n u(n)$$

Determine its response to the following inputs.

(a) $x(n) = (n+1)\left(\frac{1}{6}\right)^n u(n)$ (b) $x(n) = (-1)^n\, u(n)$

Hint: $(-1)^n = e^{j\pi n}$

4.9 Signals $x(n)$ and $h(n)$ have the following DFTs.

$$X(e^{j\omega}) = 3\,e^{2j\omega} + 1 - e^{-j\omega} + 2\,e^{-3j\omega}$$
$$H(e^{j\omega}) = -e^{j\omega} + 2\,e^{-2j\omega} + e^{-4j\omega}$$

Determine $\quad y(n) = x(n) * h(n)$

4.10 A causal LTI system has frequency response

$$H(e^{j\omega}) = \frac{1}{1+\frac{1}{3}e^{-j\omega}}$$

Find the response of these systems to the input having following DFTs

(a) $X(e^{j\omega}) = \dfrac{1-\frac{1}{6}e^{-j\omega}}{1+\frac{1}{3}e^{-j\omega}}$ (b) $X(e^{j\omega}) = \dfrac{1+\frac{1}{6}e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}}$

(c) $X(e^{j\omega}) = 1 + 3\,e^{-4j\omega}$

4.11 A causal LTI system is governed by the difference equation

$$y(n) + \frac{1}{3}\,y(n-1) = x(n)$$

Determine the system output for the following inputs.

(a) $x(n) = \left(-\dfrac{1}{3}\right)^n u(n)$ (b) $x(n) = n\left(\dfrac{1}{3}\right)^n u(n)$

(c) $X(e^{j\omega}) = \dfrac{1}{\left(1-\frac{1}{6}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)}$

Hint: In part (c), partial fractioning can be done. For convenience, write $e^{-j\omega} = s$.

4.12 A causal LTI system has frequency response

$$H(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$$

It is excited by the signal having DFT

$$X(e^{j\omega}) = \frac{2e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

Determine the output $y(n)$.

4.13 Determine the DFT of the following signals.

(a) $f(n) = 3^n \sin\left(\dfrac{\pi}{3}n\right)u(-n)$ (b) $f(n) = \sin\left(\dfrac{2\pi}{3}n\right) + \cos\left(\dfrac{8\pi}{3}n\right)$

(c) $f(n) = n^y\left(\dfrac{1}{3}\right)^{|n|}$

4.14 Two causal LTI systems are connected in parallel. The frequency response of the overall system

$$H(e^{j\omega}) = \frac{18 - 7e^{-j\omega}}{6 - 5e^{-j\omega} + e^{-2j\omega}}$$

If one of the systems has impulse response $h_1(n) = \left(\frac{1}{3}\right)^n u(n)$, find the impulse response of the other system.

4.15 Evaluate the inverse of the following DFTs.

(a) $F(e^{j\omega})$ is sketched in the adjoining figure. Check for symmetry properties.

(b) $F(e^{j\omega}) = e^{-j\omega/3}$ for $-\pi \le \omega \le \pi$

(c) $F(e^{j\omega}) = \dfrac{e^{-j\omega} - \frac{1}{2}}{1 - \frac{1}{3}e^{-j\omega}}$

(d) $F(e^{j\omega}) = \dfrac{1 - \left(\frac{1}{2}\right)^6 e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$

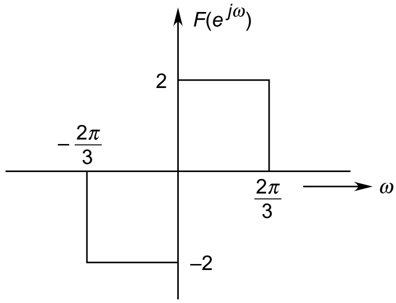


Fig. P-4.15

4.16 The frequency response of two causal LTI systems in tandem are

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{\left(1 + \frac{1}{3}e^{-j\omega}\right)}$$

(a) Determine the overall frequency response of the system.

(b) Determine the impulse response of the composite system.

4.17 Plot the complex numbers W^2, W^4 and W^8 in the complex plane for $N = 16$.

4.18 Deduce the DFT of the sequence shown in Fig. P-4.18.

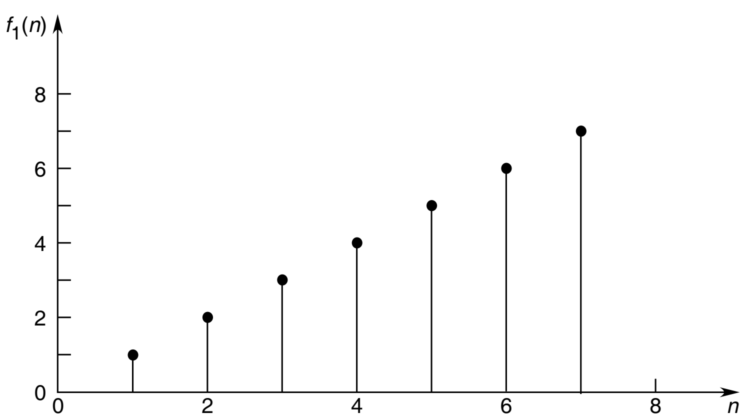


Fig. P-4.18

4.19 Compute the DFT of each of the following finite-duration sequences of length N.

(a) $f(n) = \delta(n)$

(b) $f(n) = \delta(n - m)$, for $0 < m < N$

(c) $f(n) = a^n$, for $0 \le N - 1$

4.20 Find DFT of the function shown in Fig. P-4.20 discretized with $T = 0.5$ s.

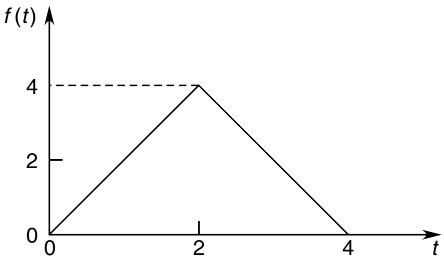


Fig. P-4.20

4.21 Deduce the DFT of the function $f(t) = 2$, for $0 \le t \le 6$; $= 0$, otherwise for the following cases.

(a) $T = 1.0$, $NT = 16$

(b) $T = 0.2$, $NT = 16$

4.22 Find the DFT of the exponential function shown in Fig. P- 4.22 for sampling periods of (i) 1.0 s and (ii) 0.5 s.

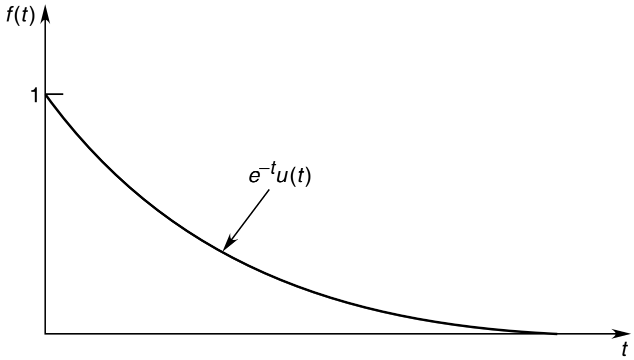


Fig. P-4.22

4.23 Show that the DFT of a finite duration sequence corresponds to samples of its z-transform on the unit circle.

4.24 If $f(n)$ denotes a finite-length sequence of length N then show that

$$f((-n))_N = f((N - n))_N$$

4.25 An analog data whose spectrum is to be analysed, is sampled at 10 kHz and the DFT of 1024 samples computed. Determine the frequency spacing between spectral samples.

4.26 Prove that each of the DFT properties listed in Section 4.5 is true.

4.27 Verify Parseval's theorem using the sequence $f(n) = \{1, -1, 4\}$.

4.28 Show that the DFT of an arbitrary periodic function $f(n)$ can be written in terms of even and odd parts

$$F(k) = R(k) + jP(k)$$
$$= F_e(k) + F_o(k)$$

4.29 A sequence $f(n)$ with DFT, $F(k)$, is padded with a zero between each member of the sequence to give a stretched sequence $\hat{f}(n)$ of period $2N$. Show that $\hat{f}(k)$ is sequential repetition of the sequence $f(k)$ of N terms.

4.30 Consider the time sequence $f(n)$; $n = 0, 1, 2, ... (N - 1)$ with $F(k)$ as its DFT. Consider a new sequence $\hat{f}(n)$ such that

$$\hat{f}(n) = f(n), \text{ for } n = 0, 1, 2, ..., (N-1)$$
$$= 0, \text{ for } n = N, (N+1), ... (2N-1)$$

Determine the DFT and relate it to $F(k)$. Show that the odd values of n give interpolation between the even values.

4.31 Use circular convolution to find the linear convolution of the two signals shown in Figs. P-4.31(a) and (b).

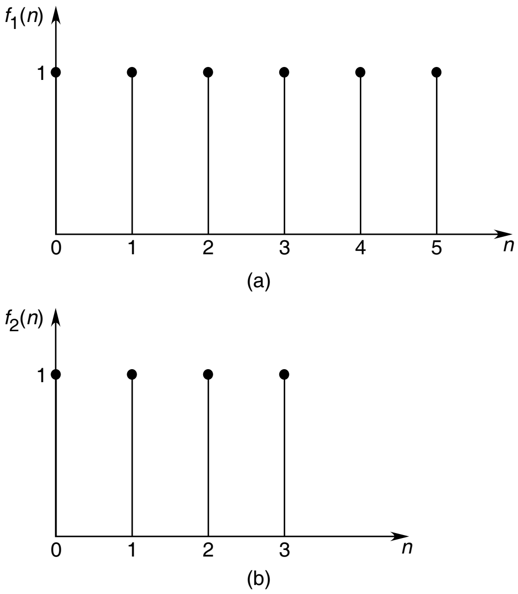


Fig. P-4.31

4.32 Find the circular convolution of the two sequences shown in Figs. P-4.32(a) and (b). Now suppose that these are simple linear sequences. Deduce their linear convolution by using the extended sequences and cyclic convolution technique.

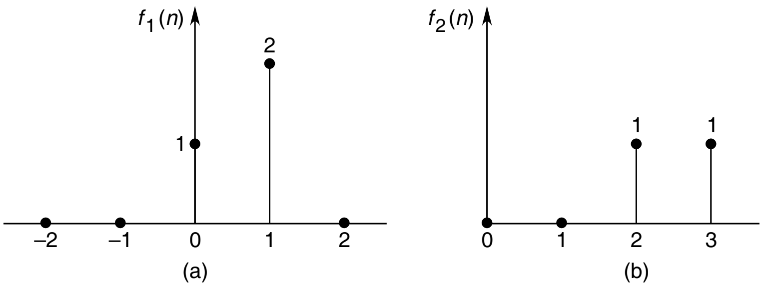


Fig. P-4.32

4.33 Deduce the linear convolution of the sequences $f_1(n) = \{2, 1, 0, 1\}$ and $f_2(n) = \{1, 2, 2\}$ employing cyclic convolution technique.

Sampling 5

Introduction

With the advent of digital circuits and wide use of computers in every walk of life, discretization of continuous-time signal is an essential requirement for the processing of a signal. Discrete-time signals are often obtained by *sampling* a continuous time signal suitable electronic hardware. The process of converting continuous-time signal to digital using form is called Analog (continuous) to Digital (C/D) conversion. These digital signals are also converted back to analog for driving the machine or understandable output (like speech, audio signal, seismic, heart, pulse, etc.) by humans. High-fidelity conversion from digital to analog (continuous) signal (D/C conversion) is only possible if samples are taken fully observing Shanon's theorem (Sampling theorem). The theorem and processes have already been dealt in Chapter 2.

With the availability and active research of fast, flexible programmable and re-programmable digital hardware and software, discrete signal processing of continuous-time signals is usually preferred. Fundamentals of sampling and related issues of signal processing will be discussed in this chapter.

5.1 REPRESENTING A CONTINUOUS-TIME SIGNAL BY IT'S DISCRETE SAMPLES (DISCRETIZATION)

Let us first consider the discretization qualitatively. If the samples of a continuous-time signal are taken at a regular interval T, any number of continuous signals can be constructed which have the same value at the kT-points. This is illustrated in Fig. 5.1 where

$$x_1(kT) = x_2(kT) = x_3(kT)$$

It can however, be shown that a continuous-time signal can be *uniquely* specified by its uniformly spaced samples provided the signal is *band-limited*, i.e., it does not have frequency components above a certain maximum frequency (ω_m) and the sampling period T is short enough in relation to ω_m. This condition is laid down as sampling theorem (Shanon's theorem) already presented in Section 2.11.

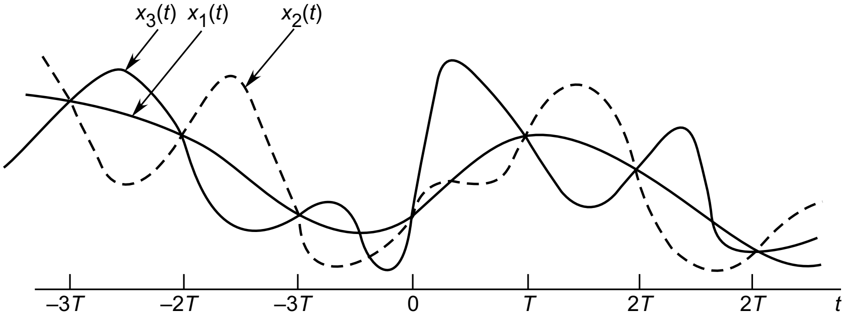


Fig. 5.1 *Three continuous-time signals with identical sample values at kT ; k = 0, ±1, ±2…*

Impulse-train Sampling

A signal in electrical voltage/current form can be sampled by electronic switching producing an output as short-duration pulses spaced by a period T.

These pulses can be approximated by impulses of strength equal to the pulse area, which is proportional to the signal strength as duration of pulse is constant for DT (discrete-time) in the switching process. This process can then be considered as *mathematical impulse train signal sampling*. We shall repeat here what has been presented in Section 2.11 but with change in symbols which will be obvious.

The impulse-train sampling is depicted in Fig. 5.2 where the signal $x(t)$ is multiplied with impulse train or the *sampling function* $p(t)$ to produce a *pulsed signal* $x_p(t)$.

$$\text{Sampling period } = T$$

$$\text{Sampling frequency } \omega_s = \frac{2\pi}{T}$$

In time domain

$$x_p(t) = x(t)\,p(t) \tag{5.1}$$

where

$$p(t) = \delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \tag{5.2}$$

We then have

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \tag{5.3}$$

The signals $x(t)$, $p(t)$ and $x_p(t)$ are sketched in Fig. 5.2.

Using the frequency convolution property of Eq. (2.104) we write the Fourier transform of Eq. (5.1) as

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

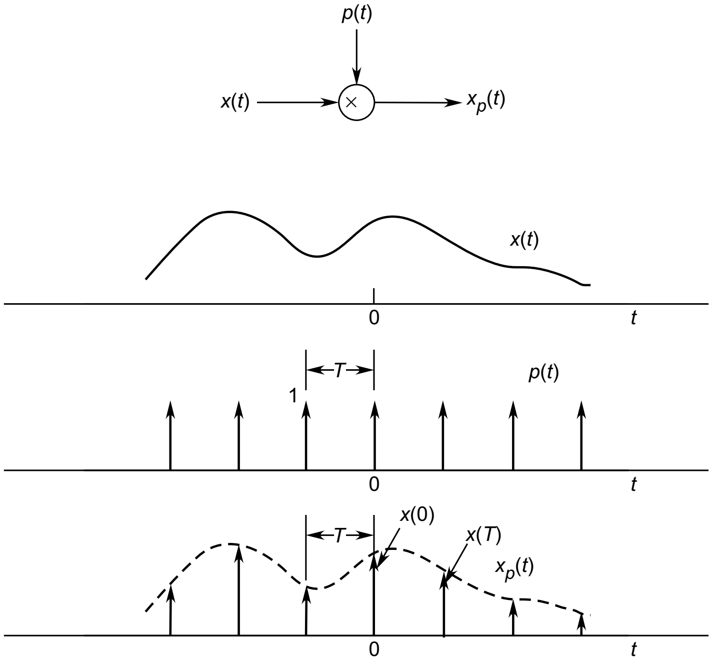


Fig. 5.2 *Impulse train sampling*

Using Eq. (2.117), we express

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega - n\omega_s);\ \omega_s = \frac{2\pi}{T} \tag{5.4}$$

Then

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)\right]$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X[j(\omega - k\omega_s)] \tag{5.5}$$

For a typical band-limited $X(j\omega)$, $P(j\omega)$ and $X_p(j\omega)$ are sketched in Fig. 5.3. It is seen from Eq. (5.5) and Fig. 5.3(c) that sampling causes the spectrum $X(j\omega)$ to be repeated around every $k\omega_s$; the aliasing effect. As T is increased, i.e., ω_s is decreased, the adjoining aliases of $X(j\omega)$ get overlapped as shown in Fig. 5.3(d). The signal can be recovered only if there is no overlap for which the following condition must be met.

$$\omega_s - \omega_m > \omega_m$$

or

$$\omega_s > 2\,\omega_m \tag{5.6}$$

This basic result is the Sampling theorem (already presented in Section 2.13) which is stated as follows.

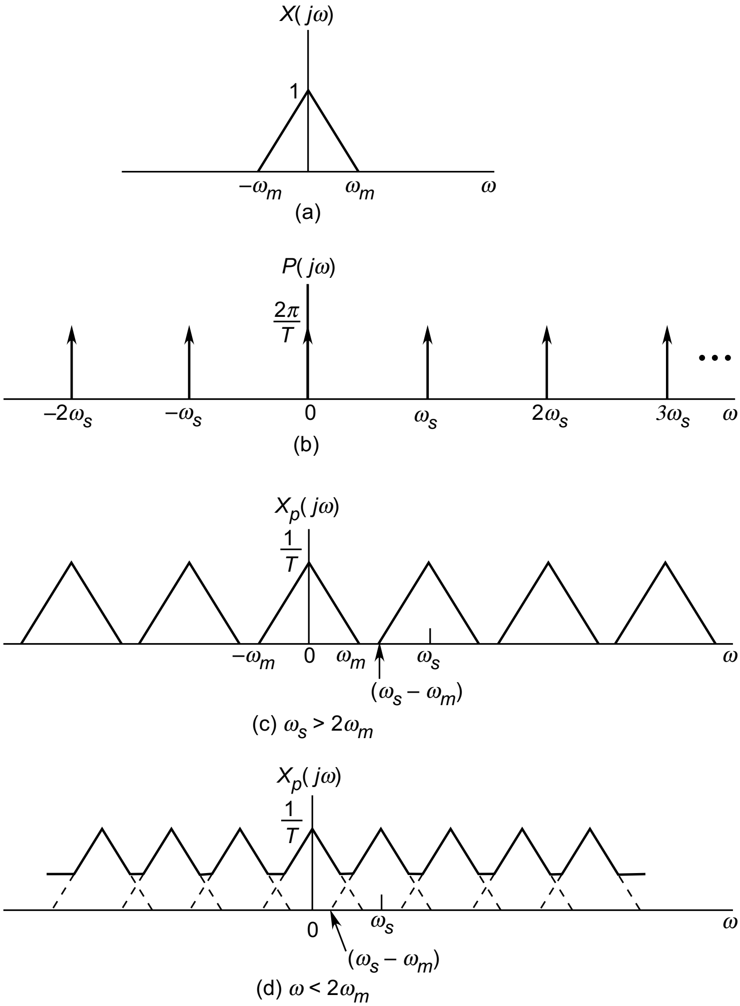


Fig. 5.3 *Effect of time-domain sampling reflected in frequency domain*

Sampling Theorem

A band-limited signal $x(t)$ with $X(j\omega) = 0$ for $|\,\omega\,| < \omega_m$ can be uniquely determined from its samples $x(nT)$; $n = 0, \pm 1, \pm 2 \ldots$ provided the sampling frequency

$$\omega_s \geq 2\,\omega_m \tag{5.7}$$

where $$\omega_s = \frac{2\pi}{T};\; T = \text{sampling period}$$

$\omega_s = 2\,\omega_m$ is also called the Nyquist ratio and ω_s is also termed as the Nyquist frequency.

Signal Reconstruction

A signal satisfying the condition (5.7) can be reconstructed from its impulsed samples by feeding these to a low-pass filter with gain T and cut-off frequency ω_c such that

$$\omega_m < \omega_c < (\omega_s - \omega_m)$$

The reconstruction process is shown diagrammatically in Fig. 5.4(a). The filtering is presented in Fig. 5.4(b) and the Fourier transform of the reconstructed signal in Fig. 5.4(c) which recovers the signal as $X_r(j\omega) = X(j\omega)$.

As an ideal filter is non-realizable, a realizable non-ideal filter closely approximating the ideal is employed in practice. This means some variation of $x_r(t)$ from $x(t)$. The choice of a non-ideal filter depends on the acceptable range of this variation.

How the signal in time domain gets reconstructed is explained by the derivation in Section 2.11, Eq. (2.135) and Fig. 2.45. It is also proved there that even with an ideal filter, an exact signal reconstruction is not possible for $x(t)$ having finite time length.

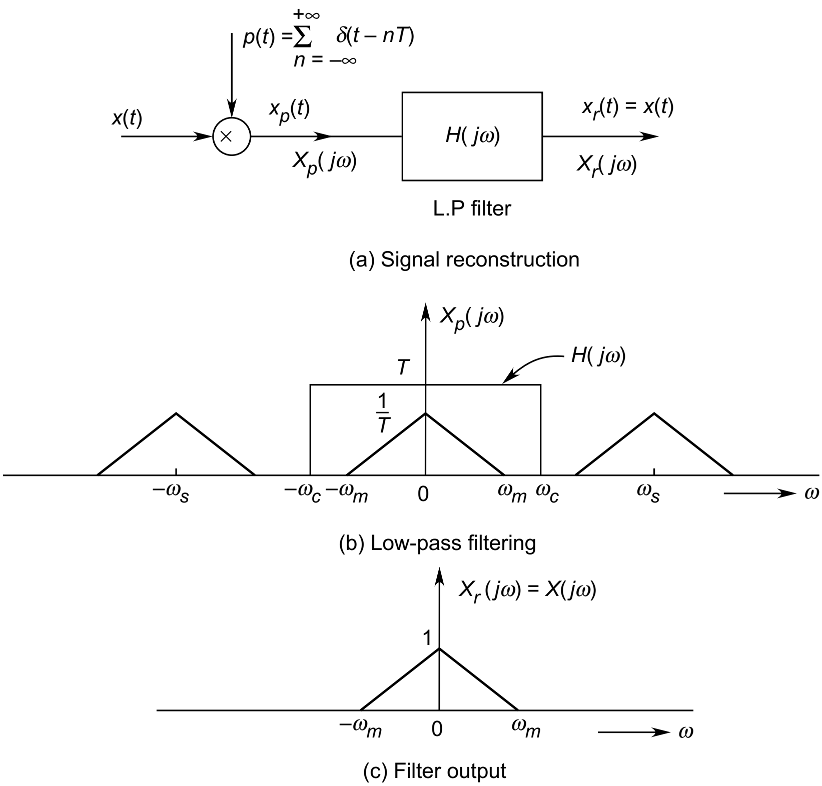


Fig. 5.4

For the purpose of theoretical study, we shall assume ideal filtering throughout in this chapter.

Zero-Order Hold Sampling

In impulse train sampling, the output impulses are approximated by short-duration pulses of large magnitude, which are difficult to generate. It is convenient to obtain signal samples by the method of zero-order hold. In this method, the signal $x(t)$ at the instant of sampling is held constant till the next instant of sampling as shown in Fig. 5.5. The reconstructing of the signal is then carried out by suitable filtering of the output $x_0(t)$.

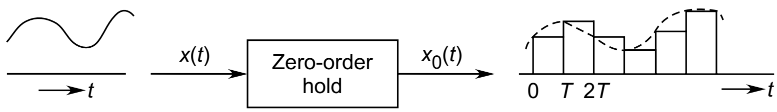


Fig. 5.5

Equivalently, the zero-order hold can be visualized as a cascade of impulse train sampler and an LTI system with rectangular impulse response as in Fig. 5.6(a).

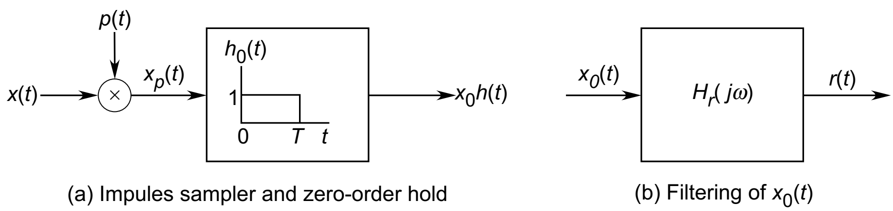


Fig. 5.6

Frequency response of zero-order hold

$$\mathcal{F}(h_0(t)) = H_0(j\omega)$$

The Fourier transform of rectangular pulse centred at the origin is derived in Example 2.12 (Eq. (iii). By a time shift of $T/2$, we get

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right] \tag{5.8}$$

For an ideal flat-response unity gain filter $H(j\omega)$ we need a reconstruction transfer function $H_r(j\omega)$ in tandem with $H_0(j\omega)$, as shown in Fig. 5.6(b) such that

$$H_0(j\omega)\, H_r(j\omega) = H(j\omega) = 1 \text{ (ideal)} \tag{5.9}$$

which gives

$$H_r(j\omega) = \left[\frac{e^{-j\omega T/2}}{\dfrac{2\sin(\omega T/2)}{\omega}}\right] \tag{5.10}$$

The ideal magnitude and phase response of $H_r(j\omega)$ with cut-off frequency $\omega_s/2$ is sketched in Fig. 5.7. As ideal $H_r(j\omega)$ cannot be realized, the output $x_r(t)$ is a very coarse approximation to $x(t)$ and it finds use in certain systems like sampled-data central systems.

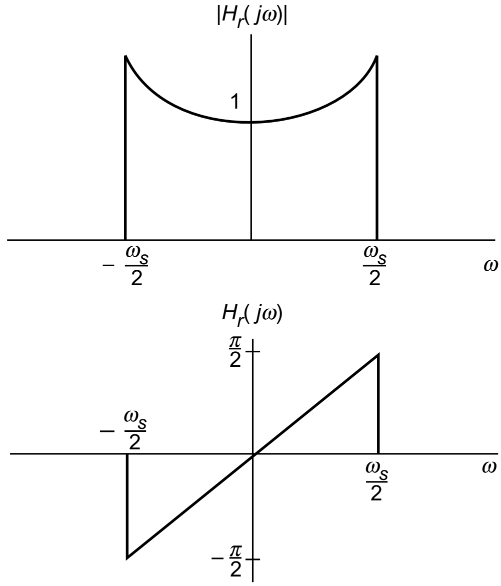


Fig. 5.7 *Magnitude and phase characteristics of ideal reconstruction filter for zero-order hold*

5.2 DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

There are considerable advantages in processing a continuous-time (CT) signal by converting it to discrete-time (DT) form and converting the results back to CT form. This is because of the availability of general purpose computers, microprocessors and special purpose digital signal processing (DSP) chips. There is also the advantage of modifying or upgrading the DT processing algorithm (software wise).

The schematic of DT processing of CT signal is presented in Fig. 5.8 where T indicates the sampling time. In digital systems, wherein analog inputs and

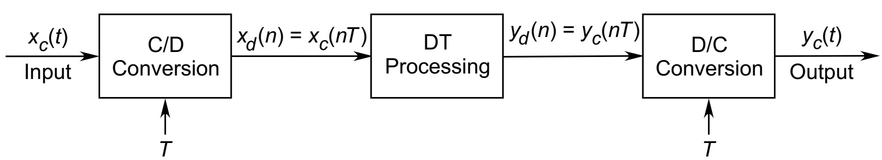


Fig. 5.8 *Stages in DT processing of CT signal*

outputs are encountered, CT/DT interconversions are carried out by Analog-to-Digital and Digital-to-Analog hardware indicated in the figure by symbols C/D and D/C.

C/D conversion is carried out in two steps—first the CT signal $x_c(t)$ is sampled by an impulse train producing the pulsed signal $x_p(t)$ from which DT sequence $x_d(n)$ is extracted such that

$$x_d(n) = x_c(nT) \tag{5.11}$$

The independent variable is n, the sequence count. These two steps are illustrated in Fig. 5.9.

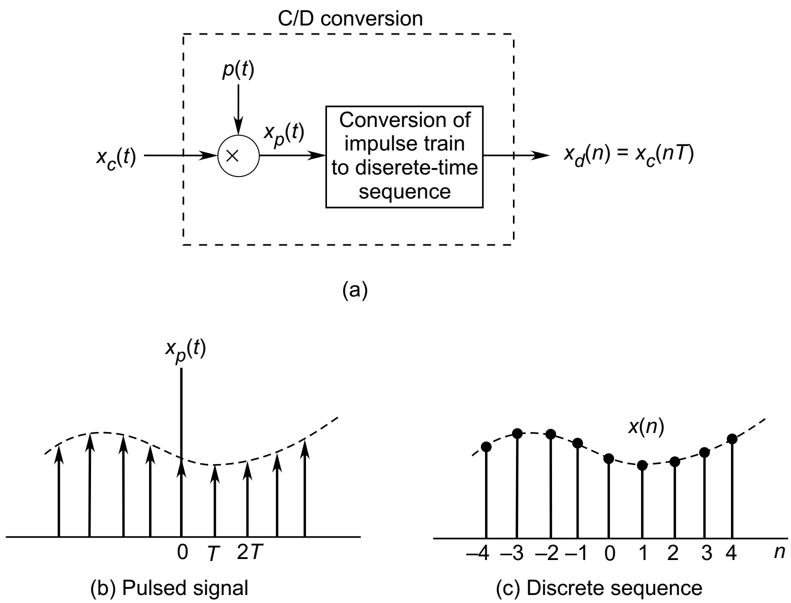


Fig. 5.9 *Steps in C/D conversion*

Our aim is to create the link between $X_c(j\omega)$, the FT of $x_c(t)$ and $X_d(e^{j\Omega})$, the DFT of $x_d(n)$.

Let us first write the expression of FT of $x_p(t)$.

$$x_p(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} x_c(nT)\, \delta(t - nT) \tag{5.12}$$

As $\quad \mathcal{F}[\delta(t - nT)] = e^{-j\omega nT}$; it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT} \tag{5.13}$$

Consider now the DFT of $x_d(n)$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d(n)\, e^{-j\Omega n} \tag{5.14}$$

As $\quad x_d(n) = x_c(nT)$, we have

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_c(nT)\, e^{-j\Omega n} \tag{5.15}$$

Comparing Eqs. (5.13) and (5.15), we get the relationship

$$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) = X_p(j\omega);\ \omega T = \Omega \tag{5.16}$$

Taking the FT of $x_p(t)$ of Eq. (5.12) using time-function product property (frequency convolution) yields.

$$X_p(j\omega) = \frac{1}{2\pi}\left\{X_c(j\omega) * \mathcal{F}\left[\sum_{k=-\infty}^{+\infty} \delta(t-kT)\right]\right\}$$

As

$$\mathcal{F}\left[\sum_{k=-\infty}^{+\infty} \delta(t-kT)\right] = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s);\ \omega_s = \frac{2\pi}{T}$$

Then

$$X_p(j\omega) = \frac{1}{2\pi}\left[X_c(j\omega) * \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)\right]$$

or

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) \tag{5.17}$$

Using this result in Eq. (5.16) gives the relationship

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T) \tag{5.18}$$

For a given $x_c(t)$ (band-limited), the relationship between $X_c(j\omega)$, $X_p(j\omega)$ and $X_d(e^{\Omega})$ is sketched in Fig. 5.10.

It is seen that $X_p(j\omega)$ is a magnitude-scaled function of $X_c(j\omega)$ repeating every $\omega_s = \dfrac{2\pi}{T}$ (relationship of Eq. (5.17) and $X_d(e^{j\Omega})$ is a frequency-scaled version of $X_p(j\omega)$ repeating at period 2π as $\Omega = \omega T$ (relationship of Eq. (5.16)). Also, $X_d(e^{j\Omega})$ can be directly related to $X_c(j\omega)$ with frequency scaling of $\Omega = \omega T$ (relationship of Eq. (5.18)).

Having established the frequency-domain relationship between $x_c(t)$ and $x_d(n)$, we now proceed to analyze the complete system of Fig. 5.8, which is redrawn in detailed form in Fig. 5.11. The frequency response of the DT filter is $H_d(e^{j\Omega})$. Its DT output $y_d(n)$ is converted to impulse train $y_p(t)$.

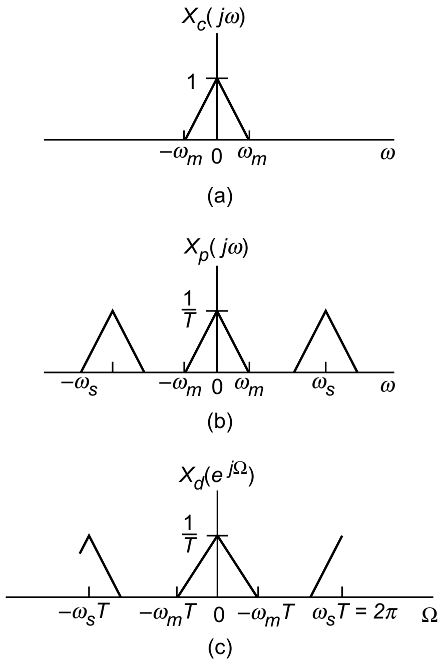


Fig. 5.10 *$X_p(j\omega)$ and $X_d(e^{j\Omega})$ for given $X_c(j\omega)$*

The impulse train $y_p(t)$ which carries the DT signal is converted to CT form $y_c(t)$ by passing it through a low-pass filter with flat gain T and frequency range $-\omega_s/2$ to $+\omega_s/2$. It is understood that impulsing frequency (ω_s) meets the condition of the sampling theorem.

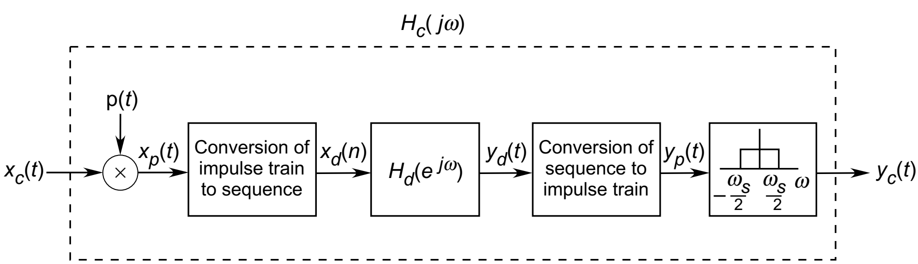


Fig. 5.11 *DT processing of CT filter*

The output of the DT filter is given by

$$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})\, X_d(e^{j\Omega}), \text{ period } 2\pi \tag{5.19}$$

For example, we take $X_d(e^{j\Omega})$ of Fig. 5.10(c) and a certain form of $H_d(e^{j\Omega})$ is superimposed on it as shown in Fig. 5.12(a); $Y_d(e^{j\Omega})$ being the multiplication of

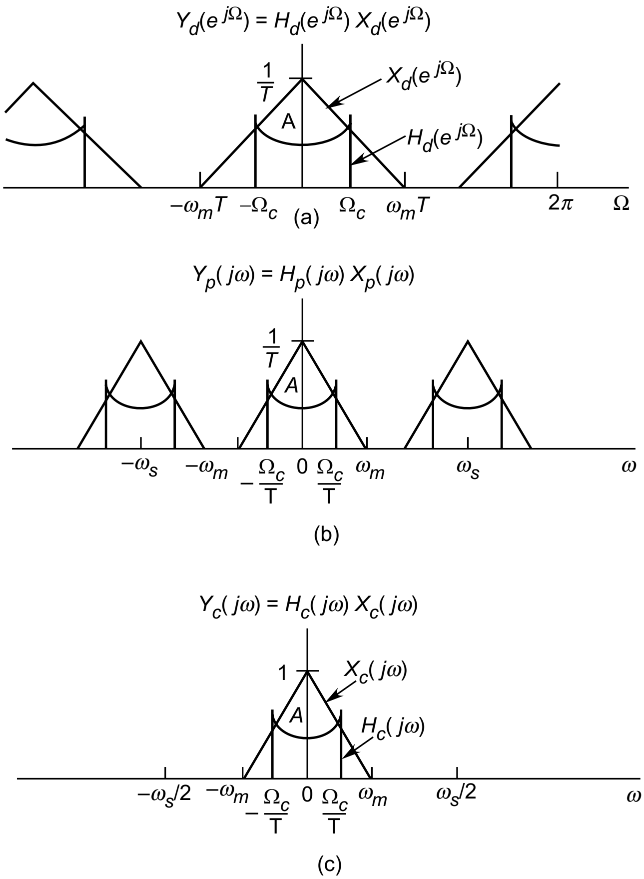


Fig. 5.12 *Stages in obtaining $Y_c(j\omega)$ from $X_d(e^{j\Omega})$*

these two. Using the relationship of Eq. (5.16), we get the impulsed output

$$Y_p(j\Omega/T) = Y_d(e^{j\Omega}) = H_d(e^{j\Omega})\, X_d(e^{j\Omega}) \tag{5.20}$$

wherein $\dfrac{\Omega}{T} = \omega$. Then,

$$Y_p(j\omega) = H_p(j\omega)\, X_p(j\omega);\ H_p(j\omega) = H_d(e^{j\omega T}) \tag{5.21}$$

which are obtained from Fig. 5.12(a) by frequency scaling $\omega = \Omega/T$ and their spectra repeats at $n\,\omega_s$; $n = 0, \pm 1, \pm 2$, as shown in Fig. 5.12(b).

The impulsed signal $y_p(t) = \mathcal{F}^{-1}[Y_p(j\omega)]$ is fed to the low-pass filter whose output is then

$$Y_c(j\omega) = H_c(j\omega)\, X_c(j\omega) \tag{5.22}$$

which is a magnitude scaled replica of $H_p(j\omega)\, X_p(j\omega)$ over the frequency range $-\,\omega_c/2$ to $+\,\omega_c/2$. In fact, we have followed the process of Fig. 5.10 in reverse.

$H_c(j\omega)$ in Eq. (5.22) can be directly obtained from $H_d(e^{j\Omega})$ by frequency transformation $\Omega = \omega T$, which is linear change in frequency. It is of course restricted to one period about zero frequency. We can express this transformation

$$H_c(j\omega) = \begin{Bmatrix} H_d(e^{j\omega T}) & ; |\omega| < \omega_s/2 \\ 0 & ; |\omega| > \omega_s/2 \end{Bmatrix} \tag{5.23}$$

where $H_c(j\omega)$ is the *equivalent continuous-time LTI system*. This *equivalence holds for band-limited input signals with sampling rate sufficiently high so as to avoid signal distortion by aliasing.*

The relationship of Eq. (5.23) is illustrated in Fig. 5.13.

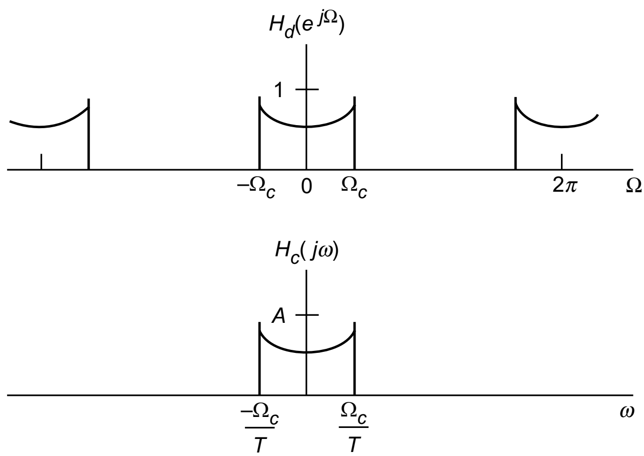


Fig. 5.13 *DT frequency response and equivalent frequency response*

The stages and transformations in DT processing of CT signal are presented in the flow chart of Fig. 5.14.

5.3 SAMPLING DISCRETE-TIME SIGNALS

We have discussed so far sampling of CT signals and its properties and applications particularly for DT processing of CT filters. We will now take up sampling of DT signals.

Impulse-Train Sampling

A discrete signal $x(n)$ can be sampled at integer multiples of the sampling period N and the intermediate samples are regarded as zero, that is

$$x_p(n) = \begin{cases} x(n) & ; \ n = kN, k = 0, \pm 1, \pm 2 \ldots \\ 0 & ; \ \text{otherwise} \end{cases} \tag{5.24}$$

	Time domain	Frequency domain
CT input	$x_c(t)$	$X_c(j\omega)$
	Sampling	$\downarrow$ $\omega_s = \frac{2\pi}{T} > 2\,\omega_m$
Impulse train	$x_p(t)$	$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$
	Conversion to discrete	$\downarrow$
Discrete input	$x_d(n) = x_c(nT)$	$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c[j(\Omega - 2\pi k)/T]$
	DT filter	$\downarrow$ $\Omega = \omega T$; period 2π $H_d(e^{j\Omega})$ =; bandwidth $-\Omega_c$ to $+\Omega_c$ (cut-off); period 2π
Discrete output	$y_d(n)$	$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})\, X_d(e^{j\Omega})$
	Conversion to impulse train	$\downarrow$
	$y_p(t)$	$Y_p(j\omega) = Y_p^{(j\Omega/T)} = H_p(j\omega)\, X_p(j\omega)$
	Conversion to CT (inter polation)	$\downarrow$ Ideal low-pass filter
CT output	$y_c(t)$	$Y_c(j\omega) = H_c(j\omega)\, X_c(j\omega)$ $= TY_p(j\omega)$ over $-\omega_c/2$ to $\omega_c/2$ = zero elsewhere

$$\begin{cases} H_c(j\omega), \text{CT filter} \\ H_c(j\omega) = \begin{cases} H_d\,(e^{j\omega T}); |\omega| < \omega_s/2 \\ 0 \qquad ; |\omega| > \omega_s/2 \end{cases} \end{cases}$$

Fig. 5.14 *Flow chart DT filtering of CT signal*

Steps in sampling of DT signal are provided in Fig. 5.15. This sampling is like CT sampling, equivalent to multiplying $x(n)$ with discrete impulses $p(n)$ at intervals of N. Thus

$$x_p(n) = x(n)\,p(n) = \sum_{k=-\infty}^{+\infty} x(n)\,\delta(n - kN) \tag{5.25}$$

Taking DFT of Eq. (5.25) using the multiplication property yields the frequency response

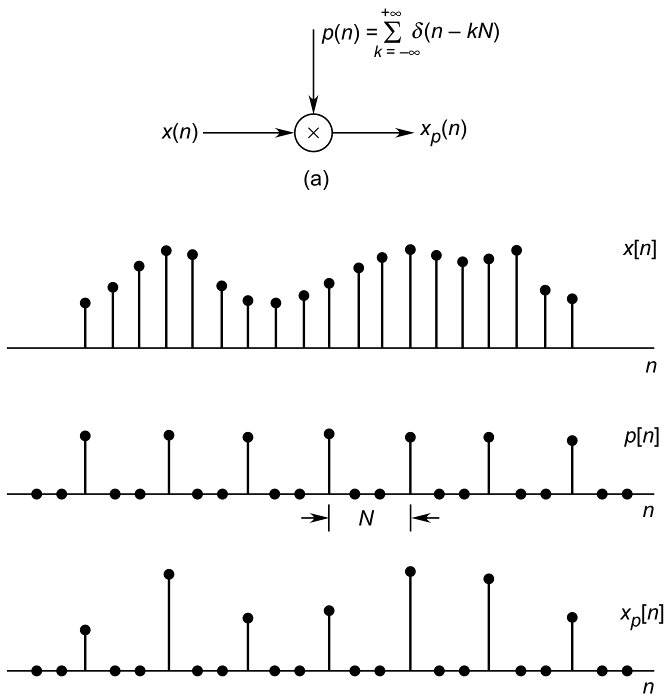


Fig. 5.15 *DT Sampling*

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\omega})\, X(e^{j(\omega-\theta)})\, d\theta \tag{5.26}$$

Now

$$P(e^{j\omega}) = \mathcal{F}_d\left[\sum_{k=-\infty}^{+\infty} \delta(n - kN)\right]$$

$$= \frac{2\pi}{N} \sum_{k-\infty}^{+\infty} \delta(\omega - k\omega_s) \tag{5.27}$$

where $\omega_s = \dfrac{2\pi}{N}$, the sampling frequency.

Substituting $P(e^{j\theta})$ in Eq. (5.26), we get

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\theta - k\omega_s)\, X(e^{j(\omega-\theta)})\, d\theta$$

Using the sampling property of impulse results in

$$X_p(e^{j\omega}) = \frac{1}{N} \sum_{k=-\infty}^{+\infty} X(e^{j(\omega-k\omega_s)}) \int_{2\pi} \delta(\theta - k\omega_s)\, d\theta$$

or
$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)}); \text{ repeats at } k\omega_s = \frac{2\pi k}{N}$$
$$k = 0, \pm 1, \ldots, (N-1) \quad (5.28)$$

Impulse-train sampling of DT signal in frequency domain is depicted in Fig. 5.16 for $\omega_s > 2\omega_m$ which avoids aliasing overlap.

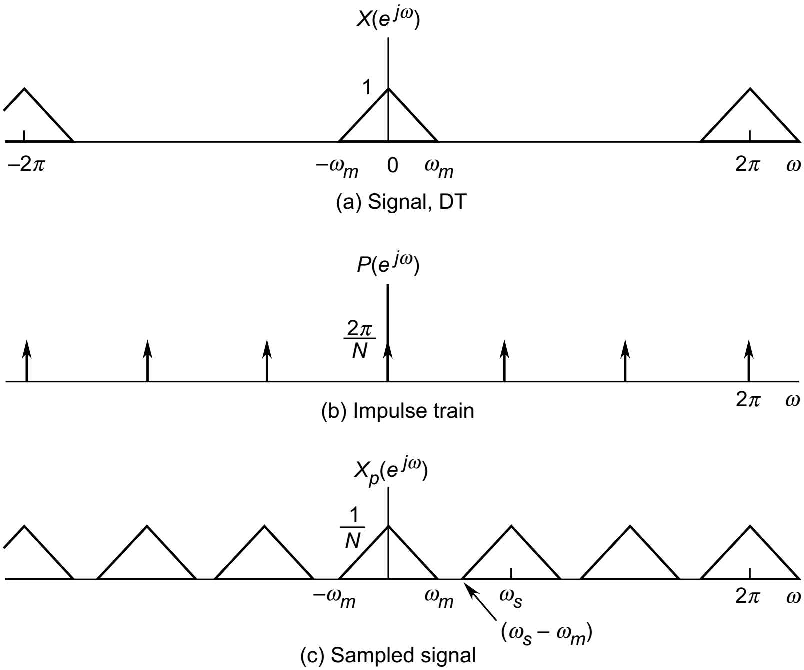


Fig. 5.16 *DT signal sampling in frequency domain*

Signal Reconstruction

The sampled signal is fed to a low-pass filter as shown stepwise in Fig. 5.17. Observe that the filter is a discrete-time filter, so its spectrum is periodic with period 2π. The reconstructed signal $X_r(e^{j\omega})$ also has period 2π. For reconstruction, the condition to be met is

$$\omega_s - \omega_m > \omega_m \quad \Rightarrow \quad \omega_s/2 > \omega_m$$

where $\omega_s/2 = \omega_c$, the cut-off frequency of the filter. If this condition is not met, aliasing overlap occurs and so $x_r(n) \neq x(n)$. However, as in the case of CT signal the recovered signal at kN is

$$x_r(kN) = x(kN);\ k = 0, \pm 1, \pm 2 \quad (5.29)$$

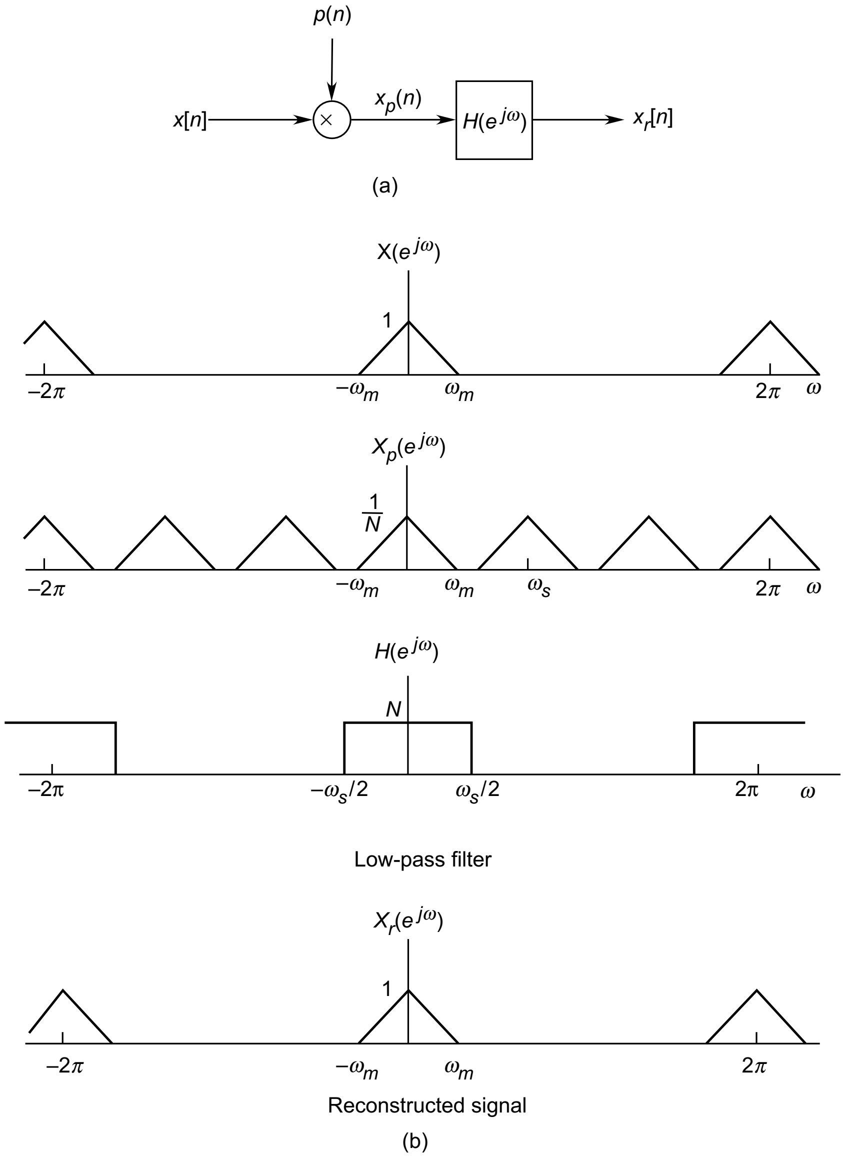


Fig. 5.17 *Signal reconstruction*

In time domain, the reconstruction of $x(n)$ by low-pass filtering of $x_p(n)$ is an *interpolation relationship* in time domain. As in case of a CT filter, the impulse response of ideal DT low-pass filter is

$$h(n) = \frac{N\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n}$$

The reconstructed signal is then

$$x_r(n) = x_p(n) * h(n)$$

which can be expressed as

$$x_r(n) = \sum_{k=-\infty}^{+\infty} x(kN)\left[\frac{N\omega_c}{\pi}\cdot\frac{\sin\omega_c(n-kN)}{\omega_c(n-kN)}\right] \tag{5.30}$$

The ideal filter has to be approximated by a realizable filter. So in general

$$x_r(n) = \sum_{k=-\infty}^{+\infty} x(kN)\, h_r(n-kN) \tag{5.31}$$

where $h_r(n)$ is the impulse response of a realizable reconstruction filter.

5.4 DISCRETE-TIME DECIMATION

Discrete-time sampling finds applications in filter design and implementation, and in communication. With reference to Fig. 5.15(b), it is seen that it is inefficient to store or transmit zeros in between samples. Instead, we replace the sampled sequence $x_p(n)$ with $x_b(n)$ which are only the N-th sampled values, that is,

$$x_b(n) = x_p(kN) = x(kN) \tag{5.31}$$

The process of retaining only N-th samples of a sequence is known as *decimation*. The sequence $x(n)$, sampled sequence $x_p(n)$ and decimated sequence $x_b(n)$ are illustrated in Fig. 5.18.

The DFT of a decimated sequence is

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b(k)\, e^{-j\omega k} \tag{5.32}$$

$$= \sum_{k=-\infty}^{+\infty} x_p(kN)\, e^{-j\omega k} \tag{5.33}$$

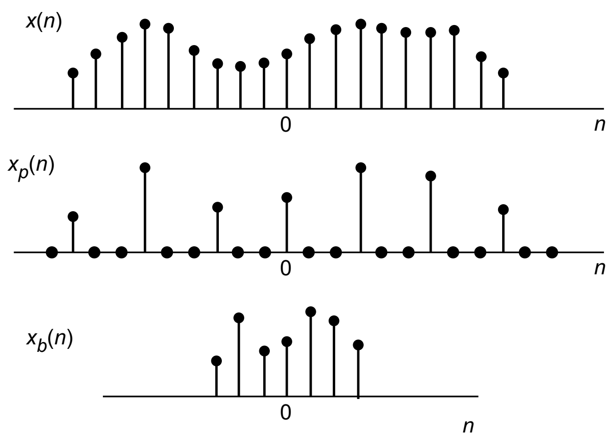


Fig. 5.18 *Stages in decimation*

Let $\quad n = kN$ or $k = n/N$. Then

$$X_b(e^{j\omega}) = \sum_{n=kN} x_p(n)\, e^{-j\omega n/N};\ k = 0, \pm 1, \pm 2, \ldots \tag{5.34}$$

As $x_p(n) = 0$ for n, and not integer multiple of N, we can write Eq. (5.34) as

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p(n)\, e^{-j(\omega/N)n} \tag{5.35}$$

The RHS of Eq. (5.35) is recognized as

$$\sum_{n=-\infty}^{+\infty} x_p(n)\, e^{-j(\omega/N)n} = X_p(e^{j(\omega/N)})$$

Thus,

$$X_b(e^{j\omega}) = X_p(e^{j(\omega/N)}) \tag{5.36}$$

From this equivalence we can find the frequency responses of $x_b(n)$ from that of $x_p(n)$. This is illustrated in Fig. 5.19. As per Eq. (5.36), this bandwidth of $-\omega_m$ to $+\omega_m$ of the pulsed signal $x_p(n)$ expands to $-N\omega_m$ to $+N\omega_m$ when decimated, but its period remains 2π. For plotting $X_p(e^{j\omega})$, see Eq. (5.28)

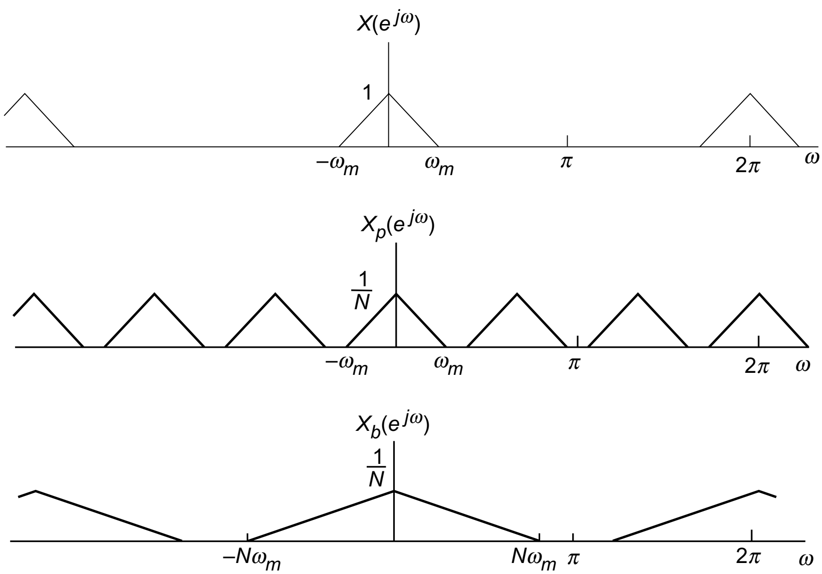


Fig. 5.19 *Deriving $X_b(e^{j\omega})$ from $X_p(e^{j\omega})$*

If a signal after decimation does not have aliasing overlap, then the original CT signal from which $x(n)$ was obtained by sampling was oversampled. It means that the sampling rate of a CT signal can be reduced without aliasing overlap. The process of decimation is therefore also referred as *down-sampling*.

Interpolation (or Up-sampling)

Starting with the decimated sequence $x_b(n)$, we can by adding $(N - 1)$ zeros between consecutive samples, construct $x_p(n)$. The signal $x(n)$ could then be recovered by feeding $x_p(n)$ to an ideal low-pass DT filter. This is the interpolation or up-sampling drawn in block diagram form in Fig. 5.20 (a). Associated sequences and the spectra for up-sampling by a factor of 2 are presented in Figs 5.20 (b) and (c) respectively.

Observe that the spectrum band decreases by a factor 2 from $X_b(e^{j\omega})$ to $X(e^{j\omega})$ because of up-sampling.

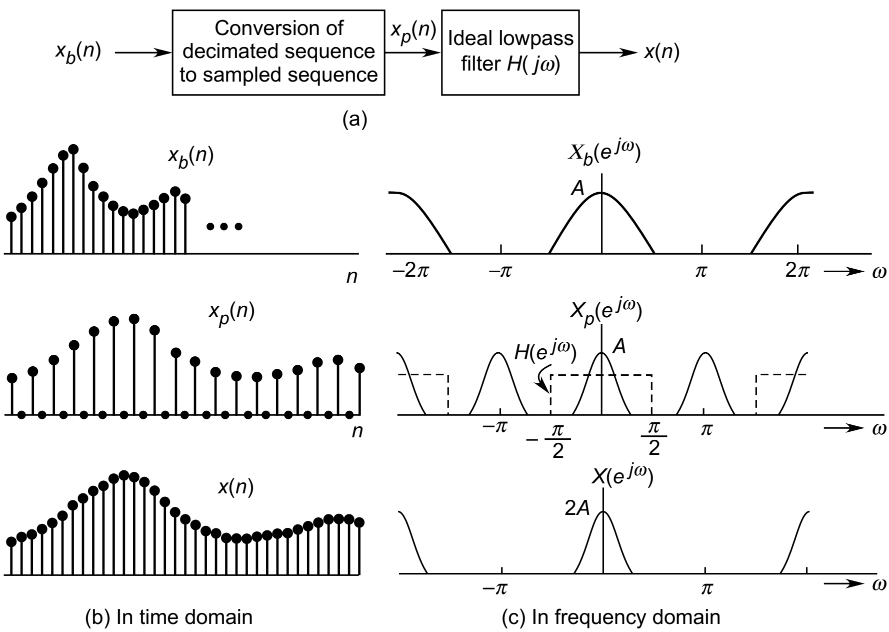


Fig. 5.20 *Processes in up-sampling*

Stagewise processes of discrete-time signal sampling, decimation and up-sampling (interpolation) are presented in Fig. 5.21.

Example 5.1 *Find the minimum sampling frequency f_s for non-zero frequency spectra $X(j\omega)$ defined between 20 kHz < $|\omega|$ < 30 kHz.*

Solution

$$\text{Bandwidth (BW) of } X(j\omega) = (30 - 20) \text{ kHz}$$
$$= 10 \text{ kHz}$$
$$\text{Minimum sampling frequency} = 2 \text{ BW}$$
$$= 20 \text{ kHz.}$$

Sampling DT Signals, Decimation and Interpolation

Note: As all signals and filter are DT, these have periodic spectra.

Time domain	Frequency domain
$x(n)$	$X(e^{j\omega})$; band limit $-\omega_m$ to $+\omega_m$
Sampling $x_p(n) = \begin{cases} x(n), n = kN \\ 0 \quad , \text{otherwise} \end{cases}$ $X_p(n) = \sum\limits_{k=-\infty}^{+\infty} \delta(n - kN)$ $X_p(n) = x(n)\, p(n)$	$X_p(e^{j\omega}) = \frac{1}{N} X(e^{j(\omega - k\omega_s)})$
Gain $\frac{1}{N}$	$\omega_s = \frac{2\pi}{N} > 2\omega_m$ repeats at $k\omega_s$, $k = 0, \pm 1, \pm 2, - - -$
Decimation $x_b(n) = x_p(kN) = x(kN)$	$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$
Up-sampling adding $(N - 1)$ zero between consecutive $x_b(n)$ sequence we get $x_p(n)$	$X_p(e^{j\omega})$ spectrum expanded N-times no change in magnitude $X_p(e^{j\omega}) = X_b(e^{j\omega N})$
Reconstruction	Spectrum compressed N-times no change in magnitude $H(e^{j\omega})$ Ideal low-pass filter
$x(n)$	$X(e^{j\omega})$ Magnitude N-times that of X_p, no change in frequency scale

Fig. 5.21

Note: As all signals and filters are DC; these have periodic spectra.

Example 5.2 *Find minimum sampling interval T_s as per sampling theorem.*

(i) $x(t) = 1 + \cos 20\pi t$ *(ii)* $x(t) = \cos(30\pi t)\,\dfrac{\sin(\pi t)}{\pi t}$

Solution

(i) From the problem, $\omega_m = 20\,\pi$

As per sampling theorem, Nyquist frequency or minimum sampling frequency

$$\omega_s \geq 2\,\omega_m$$

i.e.,

$$\frac{2\pi}{T_s} \geq 2 \times 20\,\pi$$

$$T_s \leq \frac{1}{20}\text{ s}$$

(ii) For the given $x(t) = \cos 30\pi t\, \dfrac{\sin \pi t}{\pi t} = \dfrac{1}{2\pi t}[\sin 31\pi t - \sin 29\pi t]$

$$\omega_m = 31\pi$$

$$T_s \leq \frac{1}{31}\text{ s}$$

Example 5.3 *For the signal x(t), if Nyquist rate is ω_s, determine the Nyquist rate for*

(i) $\dfrac{dx}{dt}(t)$ *(ii)* $x^2(t)$ *(iii)* $x(t)\cos \omega_0 t$

Solution

(i) Nyquist rate will be same for signal $x(t)$ and differentiation of signal w.r.t time (readers are advised to prove this proposition).

(ii) $x^2(t)$ may be considered as two signals are multiplied

$$x_1(t) = x(t) \times x(t) \tag{i}$$

Fourier transform of Eq. (i)

$$X_1(j\omega) = \frac{1}{2\pi}\,X(j\omega) * X(j\omega) \tag{ii}$$

Thus, ω_m for Eq. (ii) will be two times to that of $X(j\omega)$. Hence

$$\text{Nyquist frequency} = 2\omega_s$$

(iii) Signal $x_1(t) = x(t)\cos \omega_0 t$ is a modulating signal with shift of spectrum by frequency ω_0. Hence, Nyquist frequency of modulated signal $x(t)$ will be $\omega_s + 2\omega_0$.

Example 5.4 *If the frequency spectrum x(ω) of real-valued band-limited signal is defined as*

$$X(j\omega) = 0 \qquad \text{for } |\,\omega\,| > \omega_m$$

prove that x(t) is

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\omega_m(t-nT)}{\omega_m(t-nT)}$$

for $T = \pi/\omega_m.$

Solution From the knowledge of Fourier transform,

$$x(t) \rightarrow X(\omega)$$

and using Eq. (5.5)

$$TX_p(j\omega) = \sum_{n=-\infty}^{\infty} X[j(\omega - n\omega_s)] \quad \text{(i)}$$

$$= X(j\omega) \text{ with imposed limits}$$

From Eq. (5.13)

$$X(j\omega) = TX_p(j\omega) = T\sum_{n=-\infty}^{\infty} x(nT)\, e^{-jnT\omega} \quad \text{(ii)}$$

or

$$X(j\omega) = \frac{\pi}{\omega_m}\sum_{n=-\infty}^{\infty} x(nT)\, e^{-jnT\omega};\ |\,\omega\,| < \omega_m \quad \text{(iii)}$$

Inverse Fourier transform of Eq. (iii)

$$x(t) = \frac{1}{2\omega_m}\int_{-\omega_m}^{\omega_m} \sum_{n=-\infty}^{\infty} x(nT)\, e^{j\omega(t-nT)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \times \frac{1}{2\omega_m}\int_{-\omega_m}^{\omega_m} e^{j\omega(t-nT)}\, d\omega$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\omega_n(t-nT)}{\omega_m(t-nT)}$$

Example 5.5 *Frequency spectrum X(jω) of a band-limited signal x(t) is shown below.*

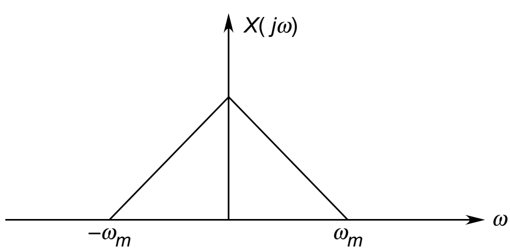


Fig. 5.22

Construct a sampled spectra if (i) $\omega_s \geq 2\omega_m$*, and (ii)* $\omega_s < 2\omega_m$*.*

Solution

(i) For $\omega_s \geq 2\omega_m$, sampling spectra $X_s(j\omega)$ will be

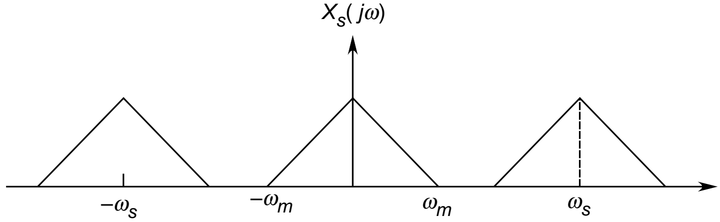


Fig. 5.23

(ii) For $\omega_s < 2\omega_m$, sampled spectra $X_s(j\omega)$

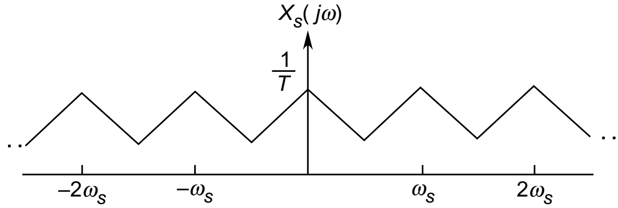


Fig. 5.24

In the above spectra $X_s(j\omega)$, the reader should notice the overlapping of spectra due to sampling frequency being smaller than the Nyquist rate.

Example 5.6 *For a real-valued continuous-time signal $x_c(t)$ the Nyquist frequency is $\omega_s = 1000\,\pi$ rad/second. Find the continuous-time frequency ω range where $X_c(j\omega)$ is zero.*

Solution As per sampling theorem

$$X_c(j\omega) = 0 \text{ for } |\,\omega\,| > \frac{\omega_s}{2}$$

Therefore

$$|\,\omega\,| > \frac{1000\,\pi}{2}$$

$$|\,\omega\,| > 500\,\pi$$

Example 5.7 *For a signal x(t) with maximum frequency 30 kHz, frequency components range 7.5 kHz $\le f \le$ 15 kHz are to be removed by ideal filter $H(e^{j\Omega})$ in between C/D and D/C conversion. Find the frequency response of an ideal filter $H(e^{j\Omega})$ to remove the above frequency components.*

Solution Using Equation 5.16

$$\Omega = \omega T = \frac{2\pi f}{f_s} \text{ (discrete frequency)}$$

$$f_s = 2 \times 30 = 60 \text{ kHz}$$

Ω corresponding to unwanted frequency range 7.5 kHz $\le f \le$ 15 kHz would be

$$\frac{\pi}{4} \le \omega \le \frac{\pi}{2}$$

Thus, the corresponding ideal filter would be a band-stop filter with the frequency response as shown below.

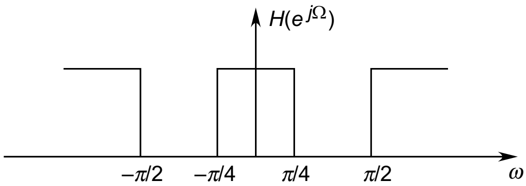


Fig. 5.25

Example 5.8 *Find the discrete-time signal $x_c(nT)$ or $x_d(n)$ if continuous-time signal $x_c(t) = A\cos(2\pi ft + \theta)$ is sampled at sampling frequency f_s (samples/second)*

Solution

$$x_c(t) = A\cos(2\pi ft + \theta)$$

$$x_c(nT) = A\cos(2\pi fnT + \theta);\ T = \frac{1}{f_s}$$

$$2\pi fT = \omega T = \Omega$$

$$\therefore \quad x_c(nT) = A\cos(\Omega n + \theta)$$

where Ω = discrete time frequency = $\dfrac{2\pi f}{f_s}$.

Example 5.9 *For the given signal*

$$x(t) = \cos(200\pi t + \theta)$$

(i) If x(t) is sampled at 250 Hz, 500 Hz and 100 Hz. At which frequency does aliasing phenomena take place?

(ii) What is the discrete-time signal $x_d(n)$ if sampling frequency is 100 Hz?

Solution

(i) To avoid aliasing, minimum sampling rate

$$f_s = 2 \times f_m \ (f_m = 200\ ;\ \text{given})$$
$$= 400 \text{ Hz}$$

Any sampling frequency more than 400 Hz will avoid aliasing and for sampling frequency less than 400 Hz, aliasing will take place. Hence, for a frequency of 500 Hz, no aliasing occurs while for 250 Hz and 100 Hz aliasing takes place.

(ii) $x_d(n) = x_c(nT) = A\cos(200\pi nT + \theta)$; $T = \dfrac{1}{f_s}$

$$x_d(n) = A\cos\left(\frac{200\pi}{100}n + \theta\right) = A\cos(2\pi n + \theta)$$

Example 5.10 *A C/D and D/C converter is interfaced with discrete-time filter with*

$$H(e^{j\omega T}) = \frac{\frac{1}{2} - e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

If the sampling frequency for C/D and D/C converters are same, f_s = 500 Hz then find the output y(n) of discrete filter if input to C/D converter is

$$x(t) = \sin 125\pi t$$

Solution For the given $x(t)$ and f_s

$$x(n) = \sin(125\,\pi\, nT);\ T = \frac{1}{f_s}$$
$$= \sin(n\pi/4)$$

Since $x(n)$ is sinusoidal sequence, $y(n)$ is also sinusoid, say

$$y(n) = A\sin(n\pi/4 + \phi);\ \Omega = \pi/4$$

where $$A = |H(j\omega)|$$

For the given $H(e^{j\omega T})$

$$|H(j\omega)|^2 = \frac{\left(\frac{1}{2} - e^{-j\omega}\right)\left(\frac{1}{2} - e^{j\omega}\right)}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{2}e^{j\omega}\right)} = 1$$

Hence, $$A = 1$$

Also, $$H(j\omega) = H(e^{j\omega T}) = \frac{\frac{1}{2} - e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

$$H(j\omega) = \frac{\frac{1}{2} - e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \times \frac{\left(1 - \frac{1}{2}e^{j\omega}\right)}{1 - \frac{1}{2}e^{j\omega}}$$

$$= \frac{\frac{1}{4} - \frac{5}{4}\cos\omega + j\frac{3}{4}\sin\omega}{\left|1 - \frac{1}{2}e^{-j\omega}\right|^2}$$

$$\angle H(j\omega) = \tan^{-1}\frac{\frac{3}{4}\sin\omega}{\frac{1}{4} - \frac{5}{4}\cos\omega}$$

or $$\phi = \angle H(j\omega)|_{\omega=\pi/4} = \tan^{-1}\frac{\frac{3}{4}\sin\frac{\pi}{4}}{\frac{1}{4} - \frac{5}{4}\cos\frac{\pi}{4}} = 40°$$

Therefore, output of discrete filter $H(e^{j\omega T})$ is

$$y(n) = \sin[(n\pi/4) + 40°]$$

Example 5.11 *If $X(e^{j\Omega})$ is the sequence in frequency domain to be interpolated by factor 2 then draw the interpolation process and find the spectrum of interpolated sequence $X_1(e^{j\Omega})$. Given*

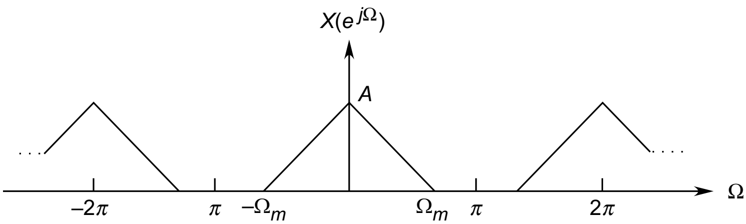


Fig. 5.26

Solution The spectrum of frequency $X(e^{j\Omega})$ after inserting 2 – 1 = 1 zero in between the original sequence is shown in Fig. 5.27.

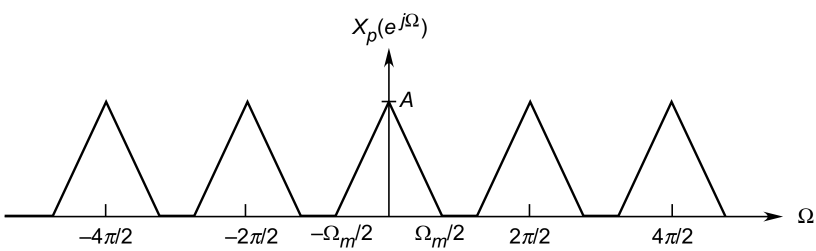


Fig. 5.27

The above sequence $X_p(e^{j\Omega})$ is fed through discrete-time ideal filter as shown in Fig. 6.28.

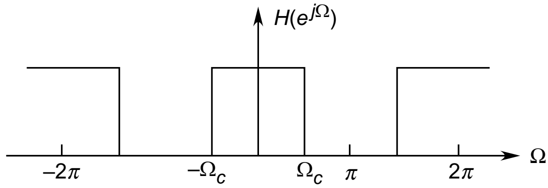


Fig. 5.28 *Discrete-time ideal filter*

Output of the above filter will be the spectrum of the interpolated sequencing as shown in Fig. 5.29.

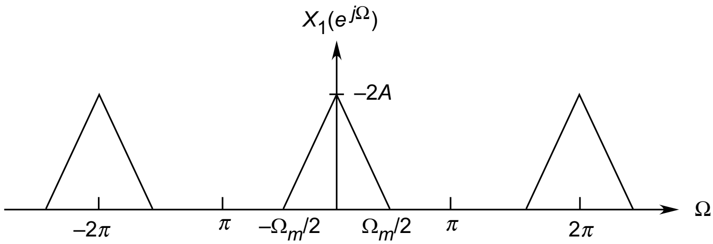


Fig. 5.29 *Interpolated sequence*

Example 5.12 *A discrete signal $x_d(n)$ has frequency transform $X_d(e^{j\Omega})$ such that*

$$X_d(e^{j\Omega}) = 0 \text{ for } \pi/4 \le |\Omega| \le \pi$$

sequence signal $x_d(n)$ is converted to continuous time by D/C conversion using sampling time $T = 10^{-3}$ second. Find the value of ω for which $X_c(j\omega)$ is guaranteed to be zero.

Solution From Eq. (5.16)

$$\Omega = \omega T$$

or
$$\omega = \Omega/T$$

For the given $X_d(e^{j\Omega})$, minimum $\Omega = \pi/4$

Therefore, $$X_c(j\omega) = 0 \quad \text{for } \omega\,(\text{min}) = \frac{\pi/4}{T} = \frac{\pi/4}{10^{-3}}$$

$$= 10^3\pi/4 = 250\,\pi \text{ rad/s}$$

Example 5.13 *A continuous-time signal $x_c(t)$ has Fourier transform which meets the condition*

$$X_c(j\omega) = 0 \text{ for } |\omega| \ge 2500\,\pi$$

If discrete-time data is given as

$$x_d(n) = x_c(n(0.4 \times 10^{-3}))$$

Its discrete Fourier transform $X_d(e^{j\omega})$ has one of the following constraints at a time. How can these constraints translate into $X_c(j\omega)$?

(a) $X_d(e^{j\omega})$ *is real*

(b) $|X_d(e^{j\omega})|_{max} = 1$ *for all* ω

(c) $X_d(e^{j\omega}) = 0$ *for* $\frac{3\pi}{4} \leq |\omega| \leq \pi$

(d) $X_d(e^{j\omega}) = X_d(e^{j(\omega-\pi)})$

Solution A per Eq. (5.18), – Ω changed to ω

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c[j(\omega - 2\pi k)/T] \tag{i}$$

(a) As $X_d(e^{j\omega})$ is the sum of replicated $X_c(j\omega)$, so if $x_d(e^{j\omega})$ is real $X_c(j\omega)$ must also be real.

(b) To meet this condition $|X_c(j\omega)|_{\max}$ must be T to cancel out $\frac{1}{T}$ scale factor in the right side of Eq. (i). As

$$x_d(n) = x_c(nT) = x_c[n(0.4 \times 10^{-3})]$$

Therefore, $T = 0.4 \times 10^{-3}$

(c) The region $\frac{3\pi}{4} \leq \omega \leq \pi$ in discrete data domain translates back into continuous-time domain frequency by replacing ω with $\frac{\omega}{T}$. Thus,

$$X_c(j\omega) = 0 \text{ for } \frac{3\pi}{4T} \leq |\omega| \leq \frac{\pi}{T},$$

i.e., $1875\,\pi \leq |\omega| \leq 2500\,\pi$

The condition $\omega \leq 2500\,\pi$ is already met by $|\omega| \geq 2500\,\pi$. The only condition to be met is

$$|\omega| \geq 1875\,\pi$$

(d) ω (in discrete-data domain) = $\frac{\omega}{T}$ (in continuous-time domain)

$$\pi \to \frac{\pi}{0.4 \times 10^{-3}} = 2500\,\pi$$

The requisite condition is then

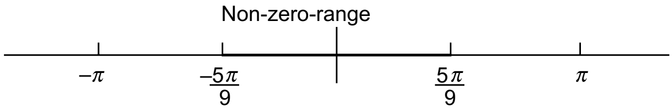
$$X_c(j\omega) = X_c(j(\omega - 2500\,\pi)$$

Example 5.14 *The signal g(n) is obtained by impulse train sampling of x(n) as*

$$g(n) = \sum_{k=-\infty}^{\infty} x(n)\,\delta(n - kN)$$

If $X(e^{j\omega}) = 0$ *for* $\frac{5\pi}{9} \le |\omega| \le \pi$, *determine the largest value of the sampling period N for no aliasing to occur in the sampled signal.*

Solution



The spectrum expands by a factor of N, but the period remains 2π. Therefore for aliasing

$$\frac{5\pi}{9}N < \left(2\pi - \frac{5\pi}{9}N\right)$$

or
$$N < \frac{18}{5}$$

$$N\,(\max) = 3$$

Example 5.15 *Consider a continuous-time band-limited differentiation* $H_c(j\omega) = j\omega$ *with a cut-off frequency of* ω_c. *Determine its corresponding discrete-time transfer function.*

Solution

$$H_c(j\omega) = \begin{cases} j\omega & ; |\omega| < \omega_c \\ 0 & ; |\omega| > \omega_c \end{cases}$$

Let the sampling frequency be

$$\omega_s = \frac{2\pi}{T} = 2\,\omega_c,\ T = \text{sampling period.}$$

To determine the discrete-time transfer function, we use the transformation $\Omega = \omega T$. Thus

$$H_c(e^{j\Omega}) = H_c(j\omega)|_{j\omega = j\Omega/T} = j\left(\frac{\Omega}{T}\right),\ |\Omega| < \pi$$

The magnitude and phase plots of $H_c(j\omega)$ and $H_d(e^{j\Omega})$ are drawn in Figs. 5.30 (a) and (b) respectively.

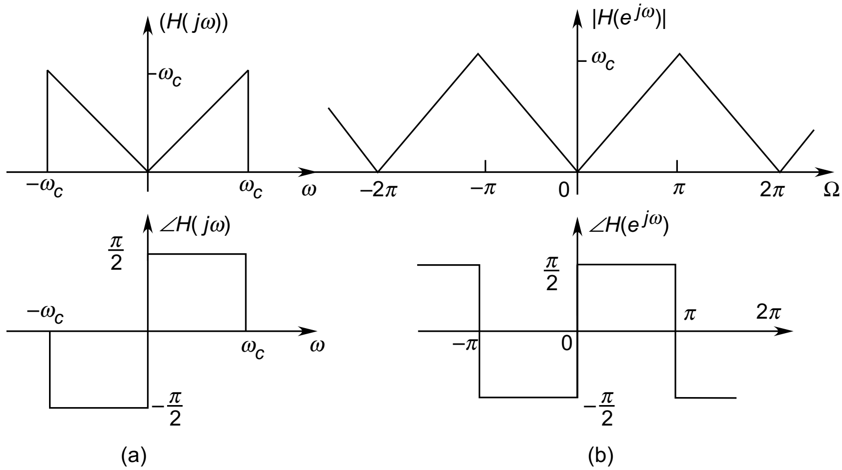


Fig. 5.30

Example 5.16 *The signal x(t) has the Fourier transforms*

$$X(j\omega) = u(\omega) - u(\omega - \omega_0)$$

What is the maximum sampling period T at which x(t) should be impulse sampled with aliasing? How the signal can be recovered?

Solution The Fourier transform of an impulse-sampled signal is given by

$$G(j\omega) = X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)] \; ; \; \omega_s = \frac{2\pi}{T}$$

The spectrum $X(j\omega)$ is plotted in the figure below.

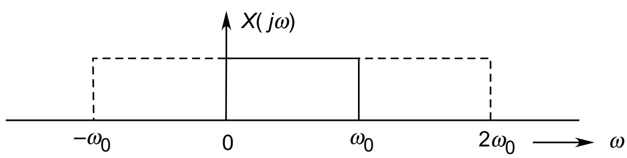


Fig. 5.31

To just avoid aliasing, the spectrum must shift by at least ω_0 as shown by the dotted spectrum. Therefore.

$$\omega_s = \omega_0 = \frac{2\pi}{T}$$

The maximum sampling period is

$$T = \frac{2\pi}{\omega_0}$$

To recover the signal, it can be passed through band-limited filter with plot gain T. Thus

$$H(j\omega) = \begin{cases} T & ;\ 0 \le |\omega| \le \omega_0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Example 5.17 *The Fourier spectrum of x(t) is*

$$X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$$

Determine the maximum sampling period to avoid aliasing.

Solution The spectrum is a zero-centred frequency pulse sketched in the figure. The minimum sampling frequency is

$$\omega_s(\text{min}) = 2\,\omega_0$$

for just avoiding aliasing. So for maximum T, the sampling period

$$\frac{2\pi}{T} = 2\,\omega_0 \text{ or } T\,(\text{max}) = \frac{\pi}{\omega_0}$$

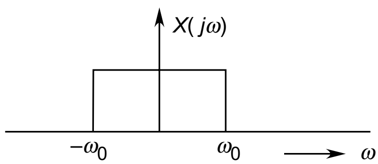


Fig. 5.32

Example 5.18 *The signal x(t) is a zero-centred unit pulse. Can it be impulse sampled with no aliasing?*

Solution The time-domain pulse is sketched in the adjoining figure. It is Fourier transforms is a sinc function which is not band-limited. So its impulse sampled signal would always have aliasing irrespective of the sampling period.

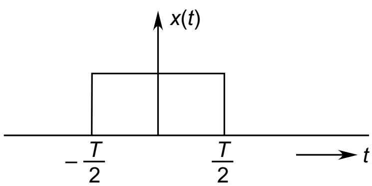


Fig. 5.33

Problems

5.1 For a real-valued signal $x(t)$ having Nyquist rate $\omega_s = 100\,\pi$, find the value of ω for which $X(j\omega)$ is guaranteed to be zero.

5.2 Find the Nyquist rate for the following signals:

(i) $x(t) = \cos 200\,\pi t + \dfrac{\sin(2000\pi t)}{200\,\pi t}$ (ii) $x(t) = 1 + \cos 20\,\pi t + \sin(10\,\pi t - 45°)$

(iii) $x(t) = \dfrac{\sin(2000\pi t)}{200\,\pi t}$ (iv) $x(t) = \left(\dfrac{\sin(2000\pi t)}{200\,\pi t}\right)^2$

5.3 If ω_s is the Nyquist rate of $x(t)$, find the Nyquist rate for

(i) $x(2t)$ (ii) $x^2(2t)$

(iii) $x(t/2)$ (iv) $x(t) * x(t)$

5.4 Determine the Nyquist rate for the signals given below if Nyquist rate for $x(t)$ is ω_s

(i) $\dfrac{dx(t)}{dt}$ (ii) $x(t)\cos\omega_0 t$

5.5 Find the Nyquist rate for

(i) $x(t) = 5G_T(t); -\dfrac{T}{2}$ to $\dfrac{T}{2}$ (ii) $x(t) = 5\,\text{sinc}\,(2t)$

5.6 Find the sampling time T so that $x(t)$ can be uniquely represented by $x_d(n)$.

(i) $x(t) = \cos 2\pi t + \sin 4\pi t$

(ii) frequency spectrum of $x(t)$ is given as in Fig. P-5.6.

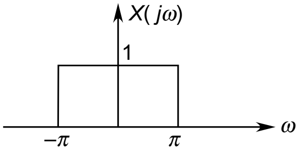


Fig. P-5.6

5.7 Two signals $x_1(t)$ and $x_2(t)$ having frequency spectrum

$$X_1(j\omega) = 0, |\omega| \geq \omega_{m1}$$
$$X_2(j\omega) = 0, |\omega| \geq \omega_{m2}$$

are multiplied to get $x_3(t)$. Find the sampling time T so that $x_3(t)$ can uniquely be recovered from impulse sample sequence of $x_3(t)$ using ideal filter.

5.8 The sampling theorem in frequency domain states that if a real signal $x(t)$ is a duration-limited signal, that is

$$x(t) = 0; |t| > T_m$$

then its Fourier transform $X(\omega)$ can be uniquely determined from its values $x\left(\dfrac{n\pi}{T_m}\right)$ at equidistant points π/T_m space apart. Then, prove that

$$X(\omega) = \sum_{n=-\infty}^{\infty} x\left(\frac{n\pi}{T_m}\right)\frac{\sin(\omega T_m - n\pi)}{\omega T_m - n\pi}$$

5.9 If $x_p(n)$ is impulse train sample of sequence $x[n]$ and if

$$X(e^{j\Omega}) = 0 \text{ for } 3\pi/4 \leq |\Omega| \leq \pi$$

then find $N_{\max}$ to avoid aliasing.

5.10 If the Fourier transform of signal $x(t)$ is $X(\omega)$

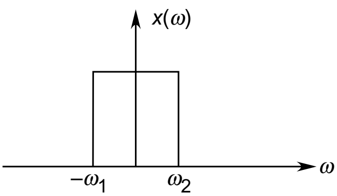


Fig. P-5.10

then draw the Fourier transform of sampled signal $X_s(\omega)$ when sampling time $T = 2\pi/10$ seconds

5.11 Frequency spectrum $X(j\omega)$ of a band-limited signal $x(t)$ is

$$X(j\omega) = 0\ |\omega| \geq \omega_0$$

Draw the sample spectrum when (i) the sampling frequency $\omega_s \geq 2\,\omega_0$, and (ii) $\omega_s < 2\,\omega_0$.

5.12 For a bandpass signal $x(t)$, the frequency spectrum is drawn in Fig. P-5.12.

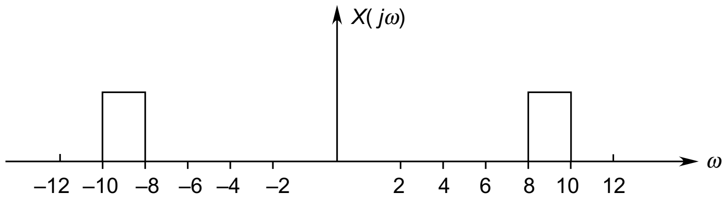


Fig. P-5.12

Sketch the sampled spectrum $X_s(\omega)$ if $\omega_s = 4$ kHz.

5.13 Find $y(n)$ output of C/D converter for the following system:

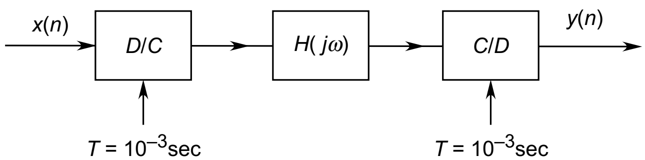


Fig. P-5.13

If the ideal low-pass filter is

$$H(j\omega) = \begin{cases} 1 & |\omega| < 1\text{kHz} \\ 0 & \text{otherwise} \end{cases}$$

then find the relationship expression between $y(n)$ to input $x(n)$.

5.14 Frequency responses of two discrete-time responses are

$$H_1(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

$$H_2(e^{j\omega}) = \frac{\frac{1}{2}+e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

Prove that magnitude and phase between the systems are equal.

5.15 The discrete-time sequence $x(n)$ is shown below:

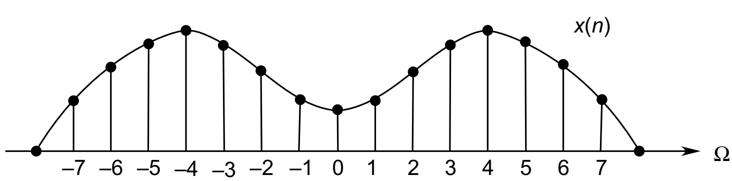


Fig. P-5.15

If $x_p(n)$ is obtained by sampling $x(n)$ with sampling period 2 and $x_d(n)$ is a decimated sequence obtained from $x(n)$ by decimating $x(n)$ by a factor of 2 such that $x_d(n) = x(2n)$ then draw the sampled and decimated sequence.

Transformed Networks; Frequency Response and Topological Models 6

Introduction

The solution of electric networks starting from their describing differential equation by the Laplace transform method has been presented in Chapter 2. In this chapter we shall convert the circuit elements (capacitance and inductance) to their Laplace transformed form on which all the network laws and theorems would apply. This considerably mechanizes the solution process and obviates the need of writing the network differential equations and their manipulation before Laplace transforming, quite a cumbersome process.

The concept of transfer function presented in Section 2.18 will now be extended to find the sinusoidal steady-state response and the complete frequency response of networks. The Bode plot representation then follows which offers the great convenience of graphical presentation of the complete frequency response over $\omega = 0$ to ∞.

The systems which can be broken up into subsystems with individual transfer functions and the subsystems that are non-loading individual subsystems can be represented topologically in the form of blocks. The signal flow among the blocks is unidirectional resulting in interconnection of blocks in the form of block diagrams, which can be topologically manipulated to find the overall transfer function. As the non-loading condition is not met by networks, the block diagram technique does not apply.

An alternative technique will then be presented which is known as the signal flow graph. It offers an advantage over the block-diagram technique that the overall transfer function can be determined by an algorithm called Mason's gain formula. It does not require any intermediate stage reduction which are needed in the block-diagram technique.

6.1 LAPLACE TRANSFORMED NETWORKS

All that is needed is to Laplace transform the governing relationships of the storage elements.

Capacitance

For a capacitance (Fig. 6.1(a))

$$v = \frac{1}{C}\int_{0^-}^{t} i(t)\,\mathrm{d}t + \underbrace{v_C(0^-)}_{\text{Initial condition}} \qquad (6.1)$$

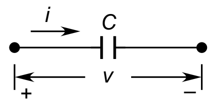


Fig. 6.1(a)

Laplace transforming, we have

$$V(s) = \frac{1}{C}\cdot\frac{I(s)}{s} + \frac{v_c(0^-)}{s}$$

or

$$V(s) = \left(\frac{1}{sC}\right) I(s) + \frac{v_c(0^-)}{s} \qquad (6.2)$$

From this result we draw the s-domain representation of Fig. 6.1(b) where we denote

$$X_c = \frac{1}{sC} = \textit{capacitive reactance}$$

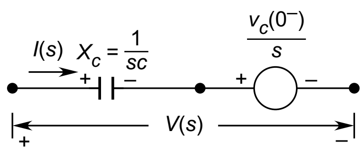


Fig. 6.1(b) *KVL representation of capacitance*

The elemental relationship of a capacitance can also be written in differential form as

$$i(t) = C\,\frac{dv(t)}{\mathrm{d}t} \qquad (6.3)$$

Laplace transforming, we have

$$I(s) = sC\,V(s) - C\,v_c(0^-) \qquad (6.4)$$

The corresponding s-domain representation is drawn in Fig. 6.1(c) where

$$B_c = sC$$
$$= \textit{capacitive suspectance}$$

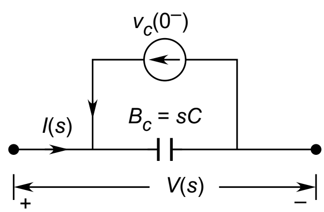


Fig. 6.1(c) *KCL representation of capacitance*

Inductance

For an inductance (Fig. 6.2(a)), the elemental relationship is

$$v(t) = L\,\frac{di}{\mathrm{d}t} \qquad (6.5)$$

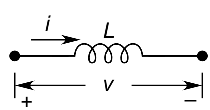


Fig. 6.2(a)

Laplace transforming, we have

$$V(s) = sLI(s) - Li(0^-) \qquad (6.6)$$

This can also be written as

$$I(s) = \frac{1}{sL} V(s) + \frac{1}{s} i(0^-) \tag{6.7}$$

In Eqs (6.6) and (6.7)

$$sL = X_L \textit{ inductive reactance}$$
$$\frac{1}{sL} = B_L \textit{ inductive susceptance}$$

The Laplace circuit equivalents of inductance are drawn in Figs 6.2 (b) and (c).

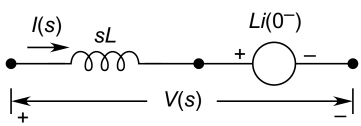


Fig. 6.2(b) *KVL representation of inductance*

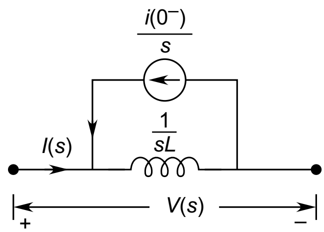


Fig. 6.2(c) *KCL representation of inductance*

Resistance

By Ohm's law

$$v = R_i \quad \Rightarrow \quad V(s) = RI(s) \tag{6.8}$$

By using the above results, any circuit can be transformed to its s-domain form. Also, the excitation is converted to its Laplace form. All the circuit theory laws and theorems apply. The result obtained in s-form is inverse Laplace transformed to bring it to time-domain form. We shall illustrate the method by examples.

Examples 6.1 *For the circuit of Fig. 6.3 after the switch is closed, find i(t) for $t \geq 0$. Given*

$$R = 4\ \Omega,\ L = 2\ H \text{ and } C = 0.125\ F$$

In case of complex conjugate roots

$$\mathcal{F}^{-1}\left[\frac{A_1}{(s+\sigma+j\omega)} + \frac{A_1^*}{(s+\sigma-j\omega)}\right] = 2\,e^{-\sigma t}(a \cos \omega t + b \sin \omega t)$$

where $$A_1 = a + jb \tag{6.9}$$

Solution The circuit of Fig. 6.3(a) is initially quiescent. So the s-domain circuit is as given in Fig. 6.3(b). The circuit *impedance* is given by

$$Z(s) = R + Ls + 1/Cs \tag{i}$$

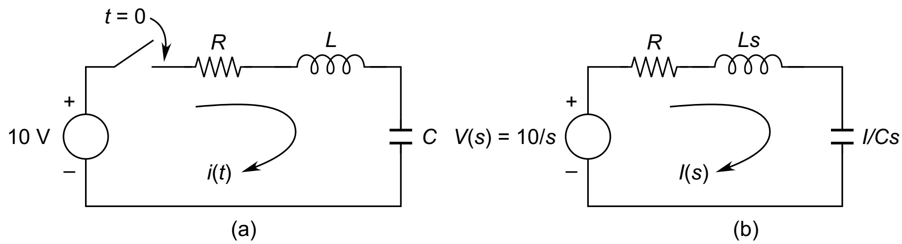


Fig. 6.3

By Ohm's law,

$$I(s) = V(s)/Z(s) = \frac{V(s)}{(R + Ls + 1/Cs)} \tag{ii}$$

Substituting values, we get

$$I(s) = \frac{10/s}{(4 + 2s + 1/0.125s)} = \frac{5}{(s^2 + 2s + 4)} \tag{iii}$$

The roots of the denominator, called the *characteristic equation*, are

$$s = -1 \pm j\sqrt{3}$$

The residue at $s = (-1 - j\sqrt{3})$ is

$$A = \left.\frac{5}{(s + 1 - j\sqrt{3})}\right|_{s=-1-j\sqrt{3}} = \frac{5}{-2j\sqrt{3}} = jb$$

Hence, by inverse Laplace transforming Eq. (iii)

$$i(t) = 2\,e^{-t}[5/2\sqrt{3})\sin\sqrt{3}\,t]u(t)$$

$$= (5/\sqrt{3})\,e^{-t}\sin\sqrt{3}\,tu(t) \tag{iv}$$

Example 6.2 *In the circuit of Fig. 6.4, the switch has been open for a long time. Find v(t) after the switch is closed at t = 0.*

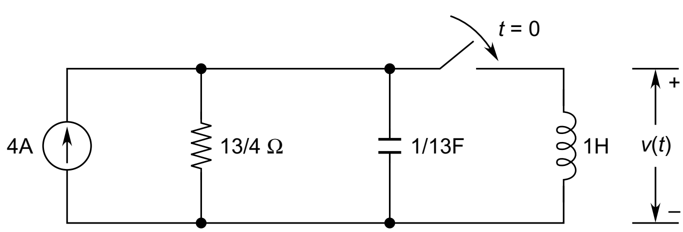


Fig. 6.4

Solution With the switch open, the capacitor gets charged fully to a voltage of

$$v_C(0) = 4 \times 13/4 = 13 \text{ V}$$

After the switch is closed at $t = 0$, the s-domain circuit is drawn as in Fig. 6.5. The parallel circuit *admittance* is

$$Y(s) = 4/13 + s/13 + 1/s = (s^2 + 4\,s + 13)/13\,s$$

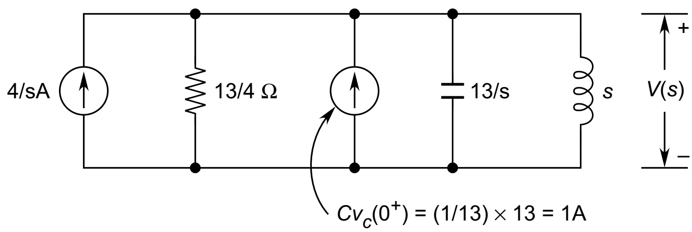


Fig. 6.5

The current input to admittance is

$$(4/s + 1) = (s + 4)s = I(s)$$

Hence,

$$V(s) = I(s)/Y(s) = 13(s + 4)/(s^2 + 4\,s + 13)$$

The characteristic equation is

$$s^2 + 4\,s + 13 = 0$$

The roots are $s = -2 \pm j3$

Residue, $$A = \left.\frac{13(s+4)}{(s+2-j3)}\right|_{s=-2-j3} = 13(1/2 + j1/3)$$

By Laplace inversion, using the result (6.9), we get

$$\begin{aligned} v(t) &= 2 \times e^{-2t}\,(13/2 \cos 3t + 13/3 \sin 3t)\,u(t) \\ &= 4.33\,e^{-2t}\,(3 \cos 3t + 2 \sin 3t)\,u(t) \\ &= 15.6\,e^{-2t} \cos (3t - 33.7°) \end{aligned}$$

Example 6.3 *For the circuit of Fig. 6.6, the switch has been in position '1' for a long time. At t = 0 the switch is thrown to position '2'. Find v(t) for t > 0.*

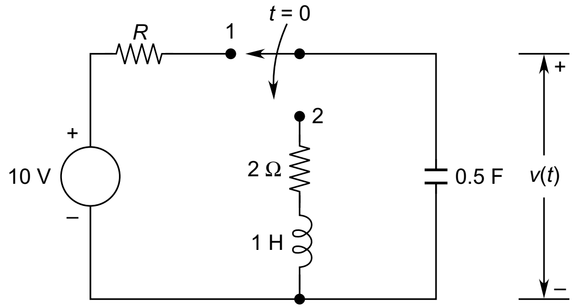


Fig. 6.6

Solution Before change over of the switch, the capacitor gets charged to a voltage of 10 V. In the s-domain this is equivalent to a current source of $Cv(0^+) = 5\text{A}$. The circuit is now drawn in the s-domain in Fig. 6.7.

$$Z(s) = \frac{(2+s)\times(2/s)}{(2+s)+(2/s)} = \frac{2(s+2)}{s^2+2s+2}$$

$$V(s) = Z(s)\times 5 = \frac{10(s+2)}{s^2+2s+2}$$

$$s^2+2s+2 = 0 \quad \Rightarrow \quad s = -1 \pm j1$$

$$A = \left.\frac{10(s+2)}{s+1+j1}\right|_{s=-(1+j1)} = 5(1+j1)$$

Hence,

$$v(t) = 10\,e^{-t}(\cos t + j\sin t)\,u(t)$$
$$= 10\sqrt{2}\,e^{-t}\cos(t-45°)\,u(t)$$

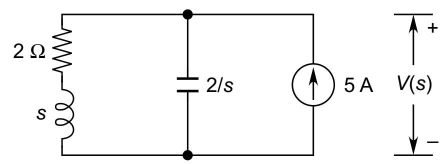


Fig. 6.7

Example 6.4 *For the circuit of Fig. 6.8 with initial conditions indicated there in, draw the s-domain circuit and write therefrom the mesh equations.*

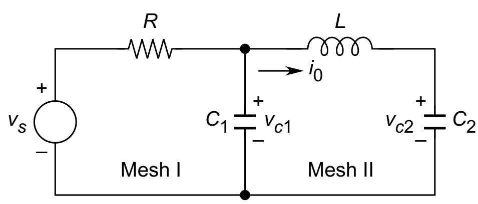


Fig. 6.8

Solution For writing the mesh equation we need Laplace representation of storage elements. The s-domain circuit is drawn in Fig. 6.9.

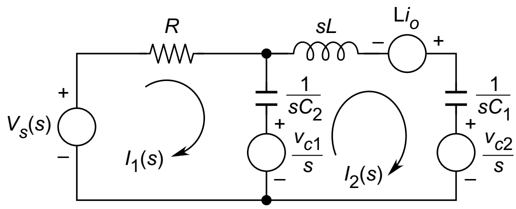


Fig. 6.9

Mesh Equations

Mesh I

$$-V_s(s) + RI_1(s) \frac{1}{sC_2}[I_1(s) - I_2(s)] + \frac{v_{c_1}}{s} = 0$$

or
$$\left(R + \frac{1}{sC_2}\right) I_1(s) - \frac{1}{sC_2} I_2(s) = V_s(s) - \frac{v_{c_1}}{s} \tag{i}$$

Mesh II

$$-\frac{v_{c_1}}{s} + \frac{1}{sC_2}[I_2(s) - I_1(s)] + sL\, I_2(s) - Li_0 + \frac{1}{sC_1} I_2(s) + \frac{v_{c_2}}{s}$$

or
$$-\frac{1}{sC_2} I_1(s) + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) I_2(s) = \frac{v_{c_1} + v_{c_2}}{s} \tag{ii}$$

6.2 FREQUENCY RESPONSE

It has been proved in Section 2.20 that the sinusoidal transfer function of a system is obtained from its s-domain transfer function as

$$H(j\omega) = H(s)|_{j\omega} = \mathcal{F}[(h(t)]$$

So $H(j\omega)$ is the Fourier transform of the system's impulse response. We can express it as

$$H(j\omega) = |\,H(j\omega)\,| \angle H(j\omega) \tag{6.10}$$

in polar form, i.e., *amplitude* and *phase shift*. We may shorten the symbols by writing

$$H(j\omega) = A(\omega) \angle\phi(\omega) \tag{6.11a}$$
$$= A(\omega)\, e^{j\phi(\omega)} \tag{6.11b}$$

$H(j\omega)$ can be directly obtained for any circuit by writing

$$sL \to j\omega L \tag{6.12a}$$
$$\frac{1}{sC} \to \frac{1}{j\omega C} \tag{6.12b}$$

Let the sinusoidal input to a system with transfer function $H(j\omega)$ be

$$x(t) = X \sin(\omega t + \theta)$$

$H(j\omega)$ modifies the magnitude and phase of the input. We can therefore express $x(t)$ as

$$\overline{X} = X \angle\theta = Xe^{j\theta} \tag{6.13}$$

where $\overline{X}$ is known as a *phasor*.

The output phasor is then given by

$$\overline{Y} = X e^{j\theta} \cdot A e^{j\phi} = A e^{j\phi}\ \overline{X} \tag{6.14}$$

The transfer function is then

$$H(j\omega) = \frac{\overline{Y}}{\overline{X}} = A\ e^{j\phi} = A\angle\phi \tag{6.15}$$

By comparison with Eq. 6.11(b)

$$|\ H(j\omega)\ | = A\ (\omega);\ \angle H(j\omega) = \phi(\omega)$$

According to Eq. (6.14), $A(\omega)$ modifies the amplitude of the input sinusoid and $\phi(\omega)$ shifts the phase of the input sinusoid. Both are functions of ω, the frequency. We refer to $A(\omega)$ as the system *gain* and $\phi(\omega)$ as the system *phase shift* (or just phase). The gain and phase shift are together referred as system's *frequency response.*

Filters

If the system (or circuit) causes desirable change in the amplitude and phase of the frequencies in this input signal, the system is referred as a *filter*. Any undesirable changes are known as *distortions*.

A filter where gain is constant and phase shift is zero is an *all-pass filter*.

At this stage we will present two simple filter examples.

Low-pass RC Filter The filter circuit is drawn in Fig. 6.10 wherein the components are represented in frequency domain. The voltage-dividing method yields the transfer function as

$$H(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C}$$

or

$$H(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau};\ \tau = RC \tag{6.16}$$

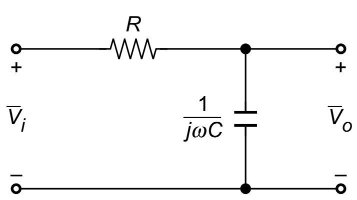


Fig. 6.10

It can expressed in magnitude and phase form as

$$H(j\omega) = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1} \omega\tau \tag{6.17}$$

$$= H(\omega)\ \angle\phi(\omega)\ \text{ or }\ H(\omega)\ e^{j\phi(\omega)}$$

Observation Magnitude and phase are functions of ω, so these are real quantities.

$$|H(j\omega)| = A(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}};$$
$$\angle H(j\omega) = \phi(\omega) = \angle -\tan^{-1}\omega\tau \tag{6.18}$$

$A(\omega)$ vs ω and $\phi(\omega)$ vs ω for the low-pass filter are drawn in Fig. 6.11. It is noticed that at the cut-off frequency ω_c, the gain reduces to $1/\sqrt{2}$ and the phase shift is 45°.

It is further observed that $\tau = RC$ is the circuit time constant which determines its impulse response. So there is a direct link between time and frequency domain.

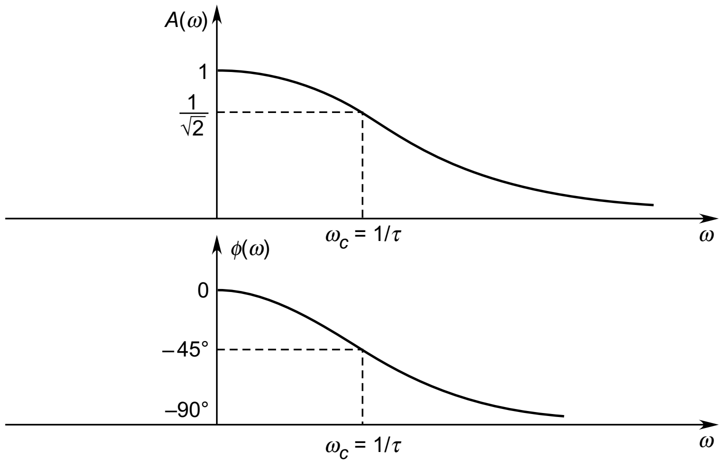


Fig. 6.11

High-pass Filter The filter circuit is drawn in Fig. 6.12. By voltage division

$$\frac{\overline{V}_o}{\overline{V}_i} = H(j\omega) = \frac{j\omega L}{R + j\omega L} = A(\omega)\,\angle\phi(\omega) \tag{6.12}$$

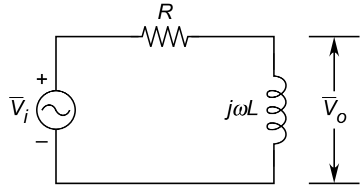


Fig. 6.12

where

$$A(\omega) = \frac{\omega L/R}{\sqrt{1+(\omega L/R)^2}} \tag{6.20a}$$

$$\phi(\omega) = 90° - \tan^{-1}\frac{\omega L}{R} \tag{6.20b}$$

The frequency of the filter is plotted in Fig. 6.12 where the *cut-off frequency* is $\omega_c = \dfrac{R}{L} = \dfrac{1}{\tau}$ (*natural frequency*) at which

$$A(\omega_c) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \phi(\omega) = -45°$$

For $\omega < \omega_c$ the frequencies are highly attenuated. Further, $\omega_c = 1/\tau$ determines the filter's impulse response in time domain.

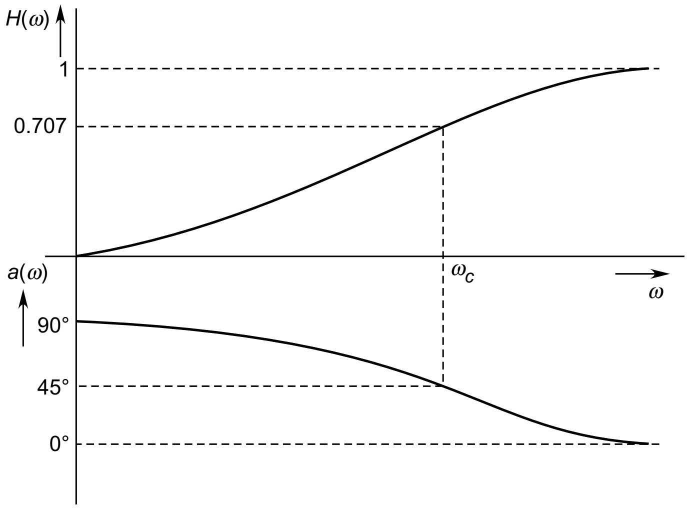


Fig. 6.13 *Frequency response of RL series network*

Ideal Filter Types

A filter is a frequency selective network which allows a certain range of frequencies of the input signal to pass and to suppress other frequencies which are not desired in the output. Filters are classified according to their frequency domain behaviour, their gain and phase response. Filters can be classified into the following types. Their ideal frequency domain behaviour in terms of gain is presented in Fig. 6.13.

(i) Low-pass; Fig. 6.14(a)

(ii) High-pass; Fig. 6.14(b)

(iii) Band-pass; Fig. 6.14(c)

(iv) Band-stop Fig. 6.14(d)

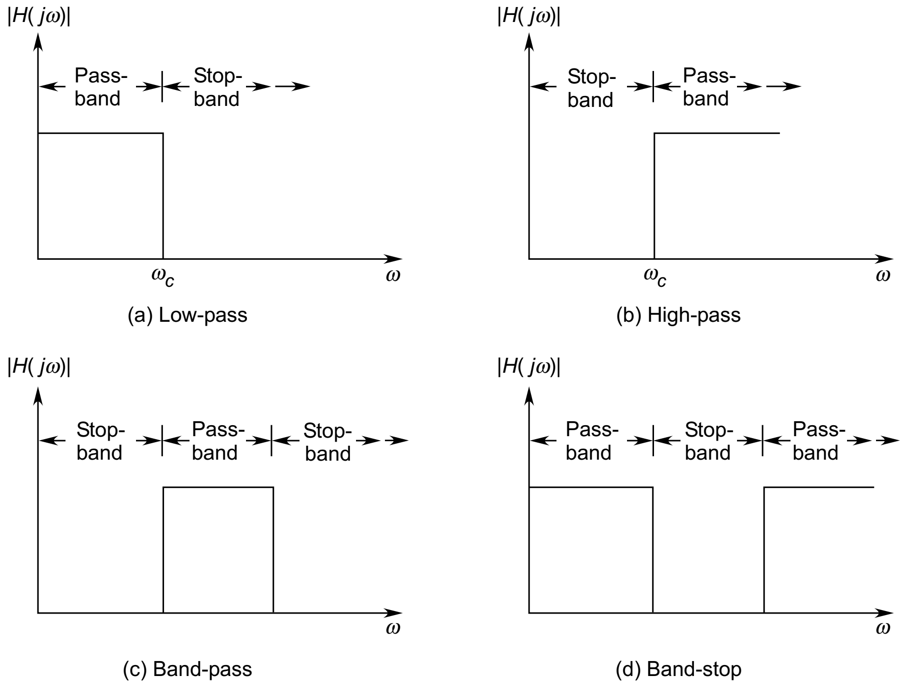


Fig. 6.14 *Ideal filter types*

The ideal filters are not physically realizable, so various approaches have been developed which closely approximate the ideal behaviour demanded. This forms the subject matter of Chapter 9 both for analog and digital filter design.

Frequency Response Discrete-time Systems

We shall consider the first-order system only, described by the difference equation

$$y(n) - a\,y(n-1) = x(n);\ |\,a\,| < 1$$

Taking this discrete Fourier transform and writing it in the transfer function form, we have

$$H(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} \tag{6.21}$$

It's impulse response is

$$h(n) = a^n\, u(n)$$

The constant 'a' has similar nature as the time constant in a first-order continuous time system. However, there is a difference. For negative values of 'a', $h(n)$ decays in an oscillatory manner. This has been demonstrated in Chapter 3.

We can write

$$H(e^{j\omega}) = \frac{1}{1 - a(\cos\omega - j\sin\omega)} \tag{6.21a}$$

where magnitude is

$$|H(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} \tag{6.21b}$$

and phase angle is

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right] \tag{6.21c}$$

The magnitude and phase versus ω plots are of course periodic with period 2π. These have to plotted computationally. There is no easy way of plotting these as for continuous-time systems. We will not present the actual plots as these do not serve much purpose. Certain conclusions are presented below.

1. For $a > 0$, peak values occur at $\omega = 0, \pm 2\pi\ldots$. The peak value increases with larger values of 'a'.
2. For $a < 0$, peak values occur at $\omega = \pm\pi$

   How to deal with discrete-time filters will be considered at length in Chapter 9.

Properties of Ideal Filter-Group Delay and Time Delay

If $x(t)$ is the input to an ideal filter, its output should be

$$y(t) = G\,x(t - \tau) \tag{6.22}$$

Only change in amplitude and time-shift is permitted.

Fourier transforming gives the frequency response of the filter as

$$\overline{H}(\omega) = Ge^{-j\omega\tau};\ \omega = 2\pi f \tag{6.23}$$

whose amplitude and phase response are

$$\left.\begin{aligned} A(\omega) &= G \\ \phi(\omega) &= -\,\omega\tau \end{aligned}\right\} \tag{6.24}$$

Instead of specifying the phase response of a filter, the *group delay* is specified, which is defined as

$$T_g(\omega) = -\frac{d}{d\omega}\phi(\omega) \tag{6.25}$$

For the ideal filter, using Eq. (6.23) and by the use of Definition (6.25), we have

$$T_g(\omega) = \tau \tag{6.26}$$

It is then concluded that for a distortionless filter, its gain and group delay are constant over the non-zero range of the input spectrum.

The concept of group delay, which is negative of the rate of change of phase of a filter with frequency, applies to a narrow band of frequencies centred at ω_0.

When a single spectral component is applied to a linear filter (LTI), the output is sinusoidal though its amplitude and phase (w.r.t reference) may change. Thus, for single frequency input, LTI filter is always distortionless.

Let the filter input be

$$r(t) = A \cos \omega t \tag{6.26}$$

The output can be written as

$$y(t) = B \cos (\omega t + \theta) \tag{6.27}$$

where

$$\theta = \text{phase shift}$$

The output can also be expressed as

$$y(t) = B \cos \omega (t - t_0) \tag{6.28}$$

where

$$t_0 = -\theta/\omega \tag{6.29}$$

which is the phase delay (time-wise). In general, the *phase delay* is written as

$$T_p(\omega) = -\frac{\phi(\omega)}{\omega} \tag{6.30}$$

where

$$\phi(\omega) = \text{filter phase shift}$$

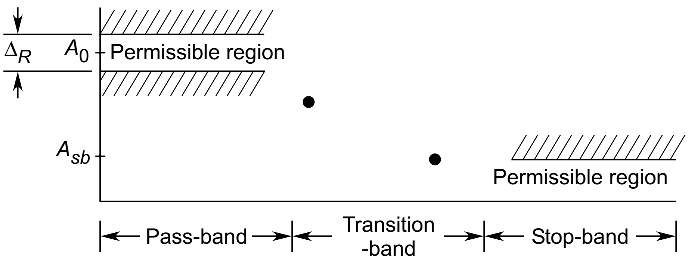
It is concluded that ideal filters have constant group and phase delays.

The concept of group delay applies equally to the discrete-time system as $e^{-j\omega n_0}$ which has a linear phase slope of $-n_0$ has associated with it's time delay of $x(n - n_0)$.

However, as explained in Section 2.9, an ideal filter is non-causal and hence its physical realization is not possible. It is therefore, necessary to approximate the transfer function or the characteristics of an ideal filter to that of a realizable filter. Deviation from the ideal amplitude characteristic is called amplitude distortion whereas deviation from the ideal (or linear) phase characteristics is called phase distortion. Filters, applicable in the voice communication are designed such that its amplitude distortion is minimal whereas phase distortion, to some extent, may be tolerated. This is because of the fact that the human ear is relatively insensitive to phase distortion. There are some other applications like video or image signal processing where linear phase characteristics are very much desirable. However, amplitude distortion may be allowed to some extent. We, therefore, observe that the characteristics of an analog filter has to be approximated such that on one hand it meets the desired specifications within given tolerance, and on the other hand it must be physically realizable.

Approximating Ideal Filters by Practical Filters

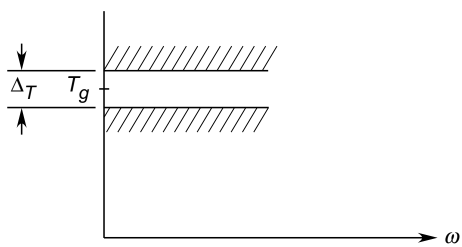
Ideal filters are not physically realizable. Therefore, some variation about the nominal amplitude (gain) A_0 has to be permitted in the pass-band to achieve realizability. Also the stop-band has to be relaxed to a transition band as shown in Fig. 6.15. The variation in the pass-band is limited to a peak-to-peak value of Δ_R. Also, in the stop-band the amplitude must be lower than a certain value of A_{sb}. Just coming out of pass-band and entering the stop-band through a transition-band, the response passes through two points shown by heavy dots. The edge of the pass-band is defined as a frequency where the amplitude attenuates to a value of $1/\sqrt{2}$ (0.707 or 3dB) of the nominal gain. This requirement could be made more stringent.



(a) Specification of amplitude response

Fig. 6.15 *Approximation bands in practical filters*

Nominal group delay T_g and allowable variation Δ_T in the pass-band are shown in Fig. 6.16.



(b) Specification of group delay

Fig. 6.16 *Approximation bands in practical filters*

6.3 TIME DOMAIN PROPERTIES OF IDEAL LOW-PASS FILTER

These properties have already been exposed in chapters 2 and 3. These are summarised here and further explored.

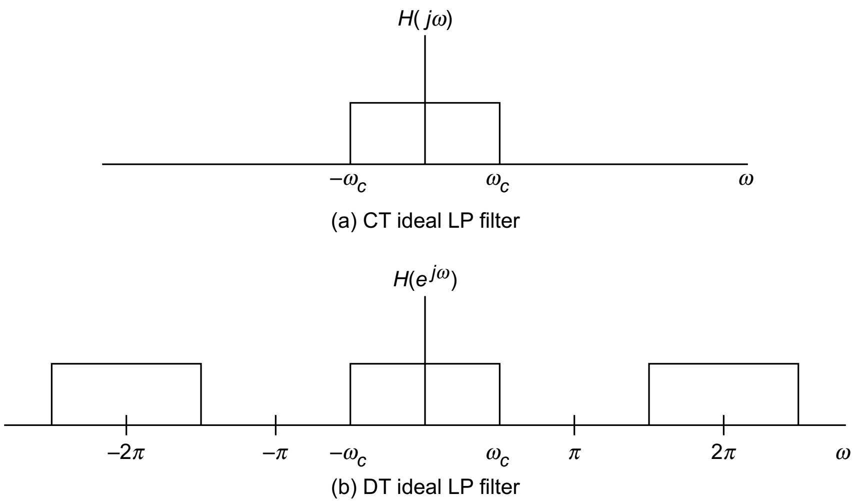
The continuous-time LP filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & ; \ |\omega| \le \omega_c \\ 0 & ; \ |\omega| > \omega_c \end{cases} \tag{6.22}$$

It has zero phase response. This frequency response is sketched in Fig. 6.17(a). It cuts off all frequencies higher than ω_c. It passes sharp frequency selectivity. Similarly, a discrete-time LP filter has

$$H(e^{j\omega}) = \begin{cases} 1 & ; \ |\omega| < \omega_c \\ 0 & ; \ \omega_c < |\omega| \le \pi \end{cases} \tag{6.23}$$

with zero phase response. It is sketched in Fig. 6.17(b) and is seen to be periodic with period 2π.



(a) CT ideal LP filter

(b) DT ideal LP filter

Fig. 6.17

The time-domain response of a filter is best revealed by its impulse response. For the filter of Eq. (6.22),

$$h(t) = \frac{\omega_c}{\pi} \text{ sinc } \omega_c t \tag{6.24}$$

and for the filter of Eq. (6.23),

$$h(n) = \frac{\omega_c}{\pi} \text{ sinc } \omega_c n \tag{6.25}$$

These are sketched in Figs 6.18 (a) and (b) respectively. It is observed that the width of the main lobe of the impulse response is proportional to $1/\omega_c$. As the band width of the filter increases, the impulse response becomes narrower and is a vice-versa *effect of phase shift*.

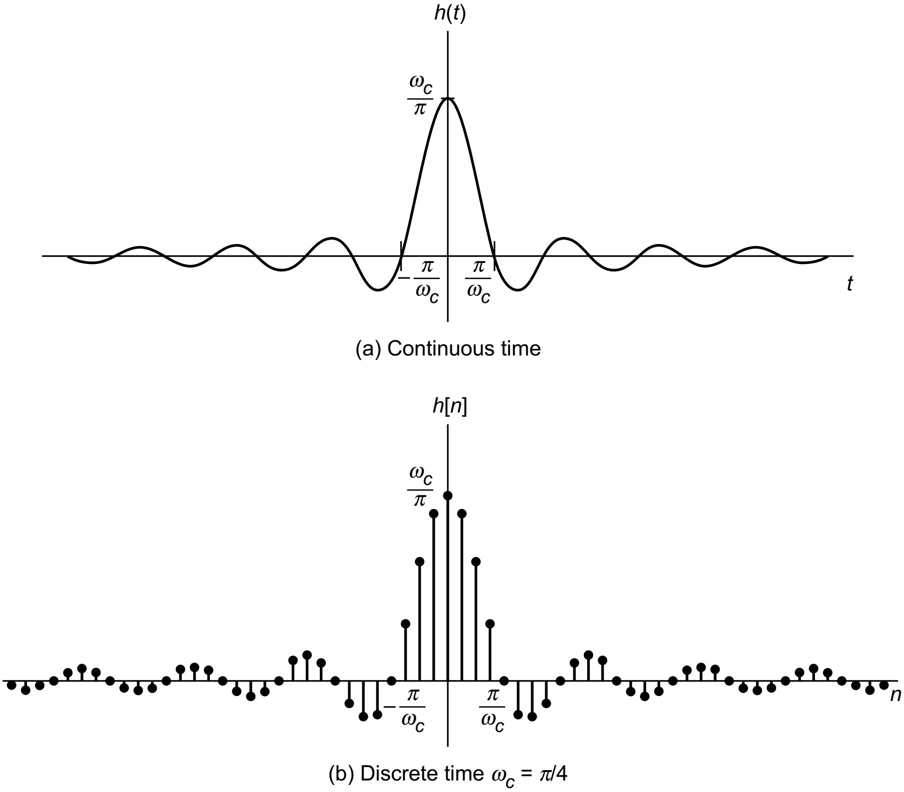


Fig. 6.18 *Impulse response LP filter*

If the phase shift is linear as in Fig. 6.19, the impulse response is time shifted by τ, the slope of the phase shift, as shown in Fig. 6.20. However, if the filter has a nonlinear phase shift, the output gets distored even though the filter gain is constant.

Example 6.5 *A continuous-time LTI has frequency response $H(j\omega) = A(\omega)\, e^{j\phi(\omega)}$; its impulse response h(t) is real. An input $x(t) = \cos(\omega_0 t + \phi_0)$ is applied to the system. Determine its output.*

Solution Writing the input in the form of complex exponentials,

$$x(t) = \frac{1}{2}\,[e^{j\omega_0 t}\, e^{j\phi_0} + e^{-j\omega_0 t}\, e^{-j\phi_0}] \tag{i}$$

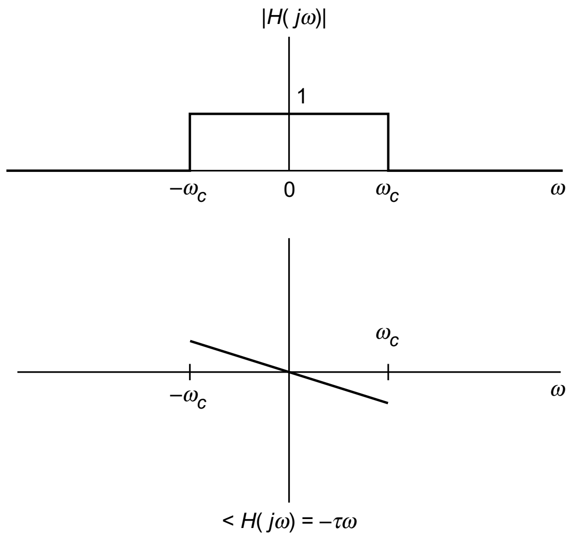


Fig. 6.19 *LP filter with linear phase shift*

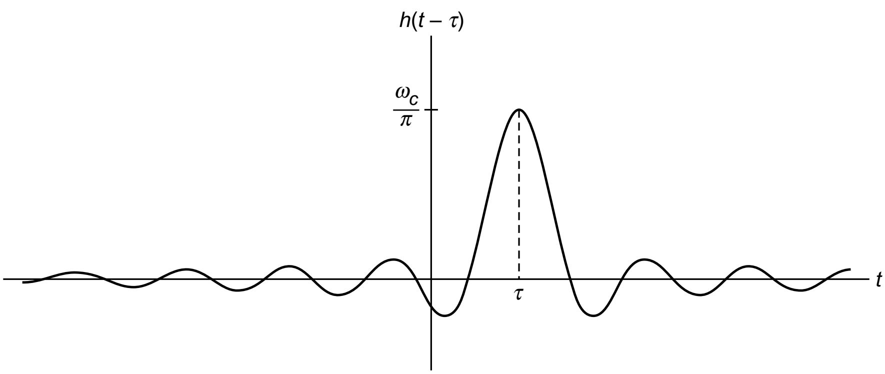


Fig. 6.20 *Shifted impulse response*

$$= \frac{1}{2}\, x_1(t) + \frac{1}{2}\, x_2(t)$$

$$H(j\omega) = A(\omega)\, e^{j\phi(\omega)}$$

The output to each input is

$$y_1(t) = \frac{1}{2}\, A(\omega_0)\, e^{j\phi(\omega_0)}\, e^{j\omega_0 t}\, e^{j\phi_0} \tag{ii}$$

$$y_2(t) = \frac{1}{2}\, A(-\omega_0)\, e^{j\phi(-\omega_0)}\, e^{-j\omega_0 t}\, e^{-j\phi_0} \tag{iii}$$

From the property of the Fourier transform

$$A(\omega_0) = A\,(-\omega_0) \text{ and } \phi(-\omega_0) = -\phi(\omega)$$

Adding Eqs (ii) and (iii)

$$y(t) = y_1(t) + y_2(t)$$

$$= \frac{1}{2} A(\omega_0)\, e^{j(\omega_0 t + \phi(\omega_0) + \phi_0)} + e^{-j(\omega_0 t + \phi(\omega_0) + \phi_0)}]$$

or

$$y(t) = A(\omega_0) \cos\,[\omega_0 t + \phi(\omega_0) + \phi_0]$$

$$= A(\omega_0) \cos\left[\omega_0\left(t + \frac{\phi(\omega_0)}{\omega_0}\right) + \phi_0\right]$$

$$= A(\omega_0)\, x(t + t_0),\ t_0 = \frac{\phi(\omega_0)}{\omega_0}$$

where

$A(\omega_0)$ is the amplitude scaling t_0 is the time shift.

Example 6.6 *A discrete-time LTI system has its transfer function $H(e^{j\omega}) = |\,H(e^{j\omega})\,|\ \angle H(e^{j\omega})$. An input $x(t) = \cos\,(\omega_0 n + \phi_0)$ is applied to it. Determine the output.*

Solution We proceed on similar lines as in Example 6.5.

For convenience we write

$$|\,H(e^{j\omega})\,| = A(\omega)\ \angle H(e^{j\omega}) = \phi(\omega)$$

The input can be expressed as

$$x(t) = \frac{1}{2}\,[e^{j(\omega_0 n)}\, e^{j\phi_0} + e^{-j(\omega_0 n)}\, e^{-j\phi_0}]$$

We multiply each eigen function with the eigen vector $A(\omega)\, e^{j\phi(\omega)}$ and add. We get

$$y(t) = \frac{1}{2}\,[A(\omega_0 n)\, e^{j(\omega_0 n + \phi(\omega_0))}\, e^{j\phi_0} + A(-\omega_0 n)\, e^{-j(\omega_0 n - \phi(-\omega_0)}\, e^{-j\phi_0}]$$

Using the symmetry property of DFT, we get

$$y(t) = A(\omega_0 n) \cos\,[\omega_0 n + \phi(\omega_0) + \phi_0]$$

$$= A(\omega_0 n) \cos\left[\omega_0\left(n - \frac{-\phi(\omega_0)}{\omega_0}\right) + \phi_0\right]$$

$$= A(\omega_0 n)\, x\left(\omega_0 n - \frac{-\phi(\omega_0)}{\omega_0}\right) = A(\omega_0 n)\, x(n - n_0)$$

where $\quad n_0 = \dfrac{-\phi(\omega_0)}{\omega_0}$ must be an integer

Therefore,

$$\phi(\omega_0) = -n_0\omega_0 \text{ or } -n\,(\omega_0 + 2k\pi) \text{ as } H(e^{j\omega}) \text{ is periodic with period } 2\pi.$$

Example 6.7 *A continuous-time causal stable LTI system has the frequency response of*

$$H(j\omega) = \frac{1 - j2\omega}{1 + j2\omega}$$

Determine

(a) $|H(j\omega)|$ *(b) Group delay* $\tau(\omega)$

Solution

(a) $$|H(j\omega)| = \frac{\sqrt{1+(2\omega)^2}}{\sqrt{1+(2\omega)^2}} = 1$$

(b) $$\angle H(j\omega) = \phi(\omega) = -\tan 2\omega - \tan^{-1} 2\omega = -2\tan^{-1} 2\omega$$

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = \frac{4}{1+4\omega^2}$$

Example 6.8 *Consider an ideal band-pass filter with frequency response*

$$H(j\omega) = \begin{cases} 1 & \omega_c \le |\omega| \le 3\omega_c \\ 0 & \text{elsewhere} \end{cases}$$

What time function must be multiplied with $\frac{\omega_c}{\pi}$ *sinc* $\omega_c t$*, so that it corresponds to h(t), impulse response of* $H(j\omega)$*?*

Solution It is seen from Fig. 6.21 that

$$H(j\omega) = H_1(j\omega - 2\omega_c) + H_1(j\omega + 2\omega_c)$$

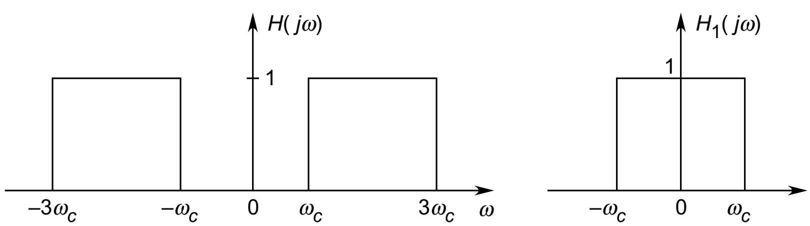


Fig. 6.21

Taking inverse Fourier transform

$$h(t) = h_1(t)\, e^{j2\omega_c t} + h_1(t)\, e^{-j2\omega_c t}$$
$$= 2\, h_1(t) \cos 2\,\omega_c t = h_1(t)\,(2\cos 2\,\omega_c t)$$

So multiplying function

$$g(t) = 2\cos 2\,\omega_c t$$

6.4 BODE PLOTS

The frequency response of a system is determined from its transfer function

$$H(s)|_{s=j\omega} = H(j\omega) = |\,H(\omega)\,| \angle\phi(\omega)$$

which comprises the plots of $|\,H(\omega)\,|$ vs ω and $\phi(\omega)$ vs ω. These plots have been presented in Figs 6.11 and 6.13 for low-pass and high-pass filters respectively. These plots are helpful in filter design.

It is conventional and convenient to use *decibel* (dB) scale for magnitude and log scale for ω.

The frequency response plots then become dB vs log ω and ϕ vs log ω, which are known as Bode plots. The decibel unit of magnitude is defined as

$$20 \log_{10} |\,H(j\omega)\,| \text{ in dB} \tag{6.24}$$

Advantages of dB-log ω Plot

1. The transfer function is the ratio of polynomials

$$H(s) = \frac{P(s)}{Q(s)}$$

   where $P(s)$ and $Q(s)$ are normally available in factored form or can be brought into factored form. The computation of $H(j\omega)$ requires conversion of each factor (which are in rectangular form) to polar form and their multiplication/division, which is quite cumbersome.

   As dB is based on $\log_{10} |\,H(j\omega)\,|$, the dB of factors converts products into additions, which are much simpler to carry out. It is further mechanized by asymptotic dB-log ω plots which are straight lines; explanation now follows.

2. log ω compresses a wide range of frequencies into a few decades. A *decade* is frequency range where end frequencies are in the ratio of 10. Thus,

   if $\omega_2/\omega_1 = 10$, then

$$\log_{10} \omega_2/\omega_1 = 1, \text{ one decade} \tag{6.25}$$

   The frequency range $(\omega_2/\omega_1) = 2$ is called an *octave*. In terms of dB, one octave is

$$20 \log_{10} \omega_2/\omega_1 = 20 \log_{10}^{2} = 6 \text{ dB} \tag{6.26}$$

   The octave scale is used in musical instruments, otherwise seldom used.

Kind of Factors of H($j\omega$)

(a) Constant gain K

(b) Poles or zeros at origin; $j\omega$

(c) Poles or zeros or real axis; $(1 + j\omega T)$

(d) Complex conjugate poles or zeros; $[1 + (2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2]$, $\zeta < 1$ where ζ = damping factor

Bode Plot of Individual Factors

Constant Gain K

$$\begin{aligned} \text{dB}(K) &= 20 \log_{10} K = \text{constant} \\ \phi(\omega) &= 0 \end{aligned}$$

On Bode magnitude plot, constant gain is a constant dB-line and its phase is zero.

Pole at Origin $\left(\dfrac{1}{j\omega}\right)$

$$\begin{aligned} \text{dB}\left(\frac{1}{j\omega}\right) &= 20 \log_{10} \left|\frac{1}{j\omega}\right| = 20 \log_{10} \frac{1}{\omega} \\ &= -20 \log \omega \\ \phi(\omega) &= \angle 1/j\omega = -90^\circ \end{aligned}$$

It is a straight line of slope –20 dB/decade. At $\omega = 1$, dB = 0. So the line passes through 0 dB at $\omega = 1$. The phase angle is constant –90°.

Zero at Origin ($j\omega$) Its dB-log ω plot is a straight line of slope +20 dB/decade passing through 0 dB at $\omega = 1$. It phase $\phi(\omega)$ = +90° constant.

The Bode plots of the factor $(j\omega)^{\pm n}$ are drawn in Fig. 6.22 for various values of n.

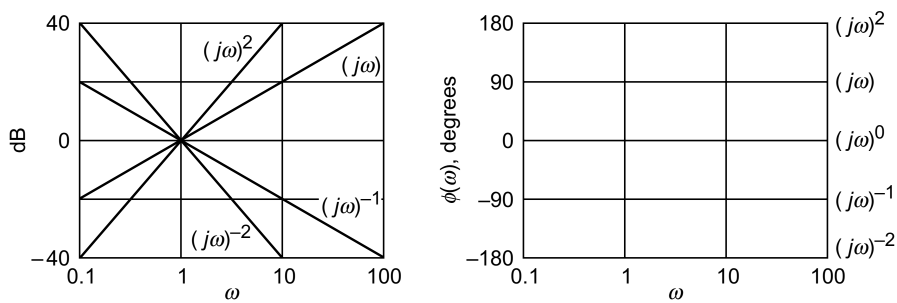


Fig. 6.22 *Bode plot of factor (jω)*

Pole (or Zero) on Real Axis Consider the pole factor

$$\frac{1}{1 + j\omega T} = \frac{1}{\sqrt{1 + \omega^2 T^2}} \angle -\tan^{-1} \omega T \tag{6.27}$$

Its dB is

$$20 \log_{10} (1 + \omega^2 T^2) = -10 \log_{10} (1 + \omega^2 T^2) \qquad (6.28)$$

Consider low and high frequency regions of this factor

- $\omega^2 T^2 << 1$ or $\omega << \dfrac{1}{T}$ (6.29)

  From Eq. (6.28), we find

  $$20 \log_{10} 1 = 0 \text{ dB} \qquad (6.30)$$

  The dB –log ω plot in this region is 0 dB line parallel to log ω– axis, called an *asymptote*.

- $\omega^2 T^2 >> 1$ or $\omega >> \dfrac{1}{T}$ (6.31)

  From Eq. (6.21), we find

  $$-10 \log_{10} \omega^2 T^2 = -20 \log_{10} \omega T \qquad (6.32)$$

  It is a straight line of slope –20 dB/decade, an asymptote, crossing the 0 dB-line at $\omega = \dfrac{1}{T}$.

The high-frequency asymptote of slope –20 dB/decade intersects the low frequency 0 dB/decade asymptote at $\omega = \dfrac{1}{T}$ as shown in Fig. 6.23(a). The frequency $\omega = \dfrac{1}{T}$ is called the *corner frequency* or *break frequency*.

The actual dB of the pole factor at the break frequency $\left(\omega = \dfrac{1}{T}\right)$ is

$$-20 \log (1 + 1) = -20 \log_{10}^{2} = -3 \text{ dB} \qquad (6.33)$$

The asymptotic plot and actual plot almost merge at $\omega = \dfrac{0.1}{T}$ and $\omega = \dfrac{10}{T}$, i.e., one decade below and one decade above the break frequency. Both the plots are drawn in Fig. 6.23(a).

Phase Plot

$$\phi(\omega) = -\tan^{-1} \omega T$$

$$\omega \to 0, \ \phi(\omega) = 0°$$

$$\omega = \frac{1}{T}, \ \phi(\omega) = -45°$$

$$\omega \to \infty, \ \phi(\omega) = -90°$$

The phase can also be approximated by an asymptote joining $\phi = 0$ at $\omega = \dfrac{0.1}{T}$ with $\phi = -90°$ at $\omega = \dfrac{10}{T}$; that is one decade lower and one decade higher than the break frequency. It passes through –45° at $\omega = \dfrac{1}{T}$.

The asymptotic and exact phase plots of the pole factor are drawn in Fig. 6.23(b). The asymptotic plot is not as good an approximation as the dB asymptotic plot. The error at frequencies $\frac{0.1}{T}$ and $\frac{10}{T}$ is 5.7°.

Zero Factor

$$(j\omega T + 1)$$

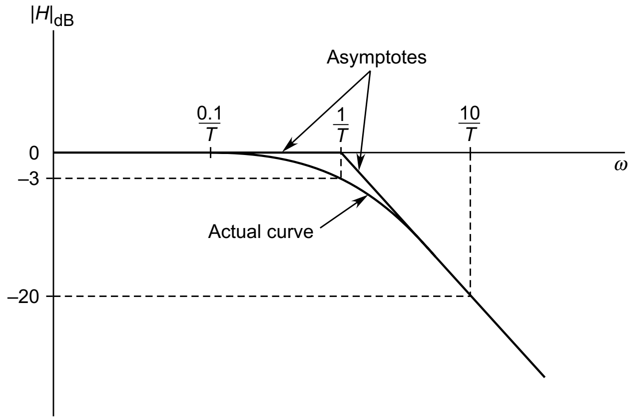
The asymptote starting at break frequency $\omega = \frac{1}{T}$ has a slope of +20 dB/decade and $\phi(\omega)$ is positive.

Pole of order n

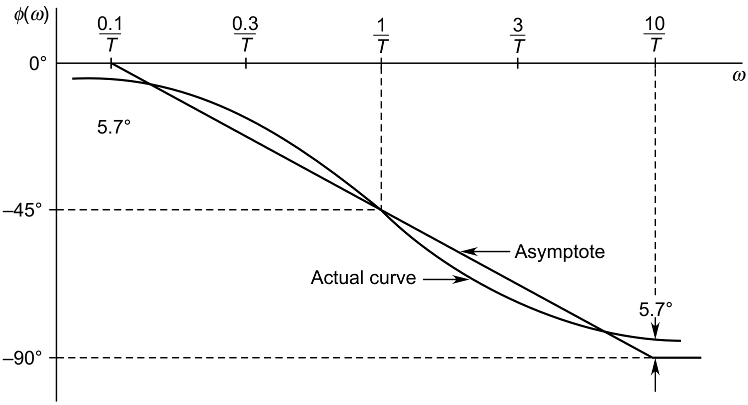
$$\frac{1}{(j\omega T + 1)^n}$$

The asymptote slope is –20 n dB/decade and $\phi(\omega)$ is $0° \to -45°\, n \to -90°\, n$.

For zero of order n , the –ive sign changes to +ive sign.



(a) Asymptotic and actual magnitude curves for a real pole factor



(b) Asymptotic and actual phase curves for a real pole factor

Fig. 6.23

Complex Conjugate Pole (Zero) Pair

The transfer function of this pair has the form

$$H(j\omega) = \frac{1}{[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]} \tag{6.34}$$

Let $\omega/\omega_n = u$, *normalized frequency*.

Then

$$H(j\omega) = \frac{1}{(1 - u^2) + j^2\zeta u} \tag{6.35}$$

Its magnitude is

$$|H(j\omega)| = [(1 - u^2)^2 + 4\zeta^2 u^2]^{-1/2} \tag{6.36}$$

Its phase angle is

$$\phi(u) = -\tan^{-1}\left(\frac{2\zeta u}{1 - u^2}\right) \tag{6.37}$$

The dB magnitude is

$$\text{dB}\,|H(ju)| = -10\log_{10}[(1 - u^2)^2 + 4\zeta^2 u^2] \tag{6.38}$$

For $u \ll 1$, $\text{dB}\,|H(j\omega)| = 0$ (6.39)

For $u \gg 1$, $\text{dB}\,|H(j\omega)| = -10\log_{10} u^4$

$$= -40\log_{10} u \tag{6.40}$$

The low-frequency asymptote is 0 dB line, while the high frequency asymptote has a slope of -40 dB/decade on the high frequency asymptote at $u = 1$, dB $= -40\ \log_{10}^{1} = 0$. It means that the high-frequency asymptote intersects the 0 dB asymptote at $u = 1$, which is the *break frequency* ($u = \omega/\omega_n = 1$ or $\omega = \omega_n$).

However, around the break frequency the difference between asymptotic and actual Bode magnitude plot is quite large and must be accounted for $\zeta \le 0.707$. The asymptotic and actual Bode plots (dB and ϕ) are drawn in Figs 6.24(a) and (b) for different values of ζ.

The maximum value of $|H(j\omega)|$ called *resonant peak* ($M_{p\omega}$) is obtained by equating it's derivative w.r.t. ω and setting it equal to zero. The *resonant frequency* is found to be

$$\omega_r = \omega_n\sqrt{1 - \zeta^2}\ ;\ \zeta < 0.707 \tag{6.41}$$

and resonant peak as

$$M_{p\omega} = |H(j\omega_r)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{6.42}$$

It is seen from Eq. (6.42) that $M_{p\omega}$ is a good measure of damping factor of a system in which a pair of complex conjugate poles dominate. This indicates the link between frequency domain and time domain response of an LTI system.

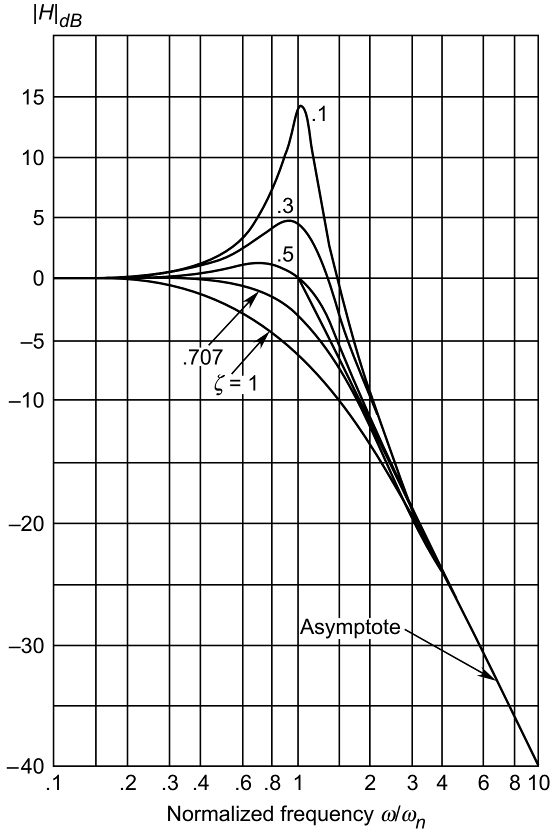


Fig. 6.24(a) *Asymptotic and actual magnitude curves for a complex pole factor*

Rules for Drawing Bode Plots

1. Record the numerator and denominator factors the break frequencies in increasing value of ω and also note down the order of each factor.
2. Separate out factor $K(j\omega)^{\pm n}$. At $\omega = 1$, from dB(K) point draw the asymptote with slope $\pm 20\, n$ dB/decade. If this factor is absent, the asymptote is a horizontal line at dB(K).
3. Locate the lowest break frequency (ω_1) on the asymptote drawn in the step 1. The next asymptote starts at this point with a slope change of ± 20 dB/decade or $\pm 20\, n$ dB/decade for higher-order factor.
4. Repeal procedure 3 in increasing values of break frequencies for all the break frequencies.
5. Note that at a break frequency corresponding to a complex conjugate pole/zero pair, the asymptote slope changes by ± 40 dB/decade.

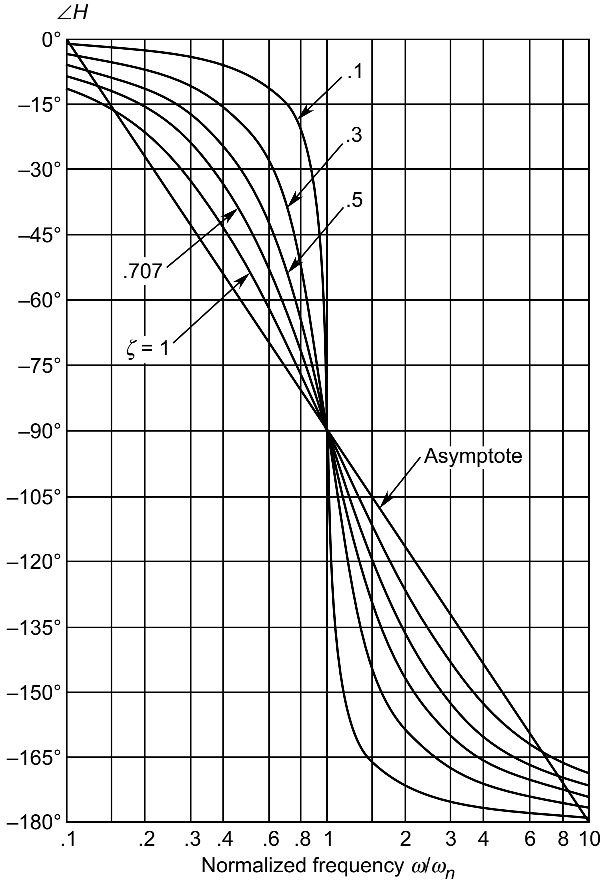


Fig. 6.24(b) *Asymptotic and actual phase curves for a complex pole factor for a complex pole factor.*

6. Apply correction of $\mp 3\,n$ dB at all real pole/zero break frequencies.

   Note: Correction at one octave below and one octave above a break frequency is $\mp 1n$ dB.

7. Corrections at complex conjugate pole/zero break frequency are obtained from Fig. 6.24(a).
8. A smooth curve is now drawn through the corrected points asymptotically to the asymptotes.

Phase Plot

Except for complex conjugate poles, phase asymptotes of all the break frequencies are added. It is sufficiently accurate for must purposes. At complex conjugate break frequency, corrections are applied as per Fig. 6.24(b).

Example 6.9 *Draw the Bode asymptotic plots for the transfer function*

$$H(j\omega) = \frac{8(1 + j\,0.1\,\omega)}{(j\omega)\,(1 + j\,0.5\,\omega)\,[1 + j\,0.6\,(\omega/50) + (j\omega/50)^2]}$$

Also, draw the corrected dB plot.

Solution We can write the given transfer function as

$$H(j\omega) = \frac{8(1 + j\omega/10)}{(j\omega)\,(1 + j\,\omega/2)\,[(1 + j\,0.6\,(\omega/50) + (j\omega/50)^2]}$$

$$\left.\left(\frac{8}{j\omega}\right)\right|_{\text{dB at }\omega=1} = 20 \log 8 = 18 \text{ dB; slope } -20 \text{ dB/decade; angle } -90°$$

At the break frequency, the j-part of each factor is unity. This gives.

Break frequencies

Numerator	$\omega_2 = 10$
Denominator	$\omega_1 = 2$, $\omega_3 = 50$ (complex conjugate case)

dB-log ω Plot (Fig. 6.25)

At $\omega = 1$, dB = 18 asymptote of slope –20 dB/decade is drawn.

At $\omega_1 = 2$ the denominator factor $(1 + j\omega/2)$ becomes effective; so the asymptote slope becomes (–20 – 20) = –40 dB/decade.

At $-\omega_2 = 10$, the numerator factor becomes effective; so the asymptotic slope changes to –40 + 20 = –20 dB/decade

At $\omega_3 = 50$ the quadratic factor in denominator becomes effective and the slope changes to –20 –40 = –60 dB/decade

Corrections

At ω_1, correction +3 dB

At ω_2, correction –3 dB

Around ω_3 correction as per Fig. 6.24 (a), corresponding to $\zeta = 0.6/2 = 0.3$. dB-plots asymptotic and corrected are drawn in Fig. 6.25.

Phase Plot

Phase asymptotes are

$\omega_1 = 2$	0.02 to 20 slope –45° decade
$\omega_2 = 10$	1 to 100 slope +45° decade
$\omega_3 = 50$	5 to 500 slope –90°/decade

The asymptotic phase slope change occurs at $\omega = 1$, $\omega = 10$ and $\omega = 50$.

The corrected phase plot is also drawn in Fig. 6.25, with phase corrections read from Fig. 6.24(b) corresponding to $\zeta = 0.3$.

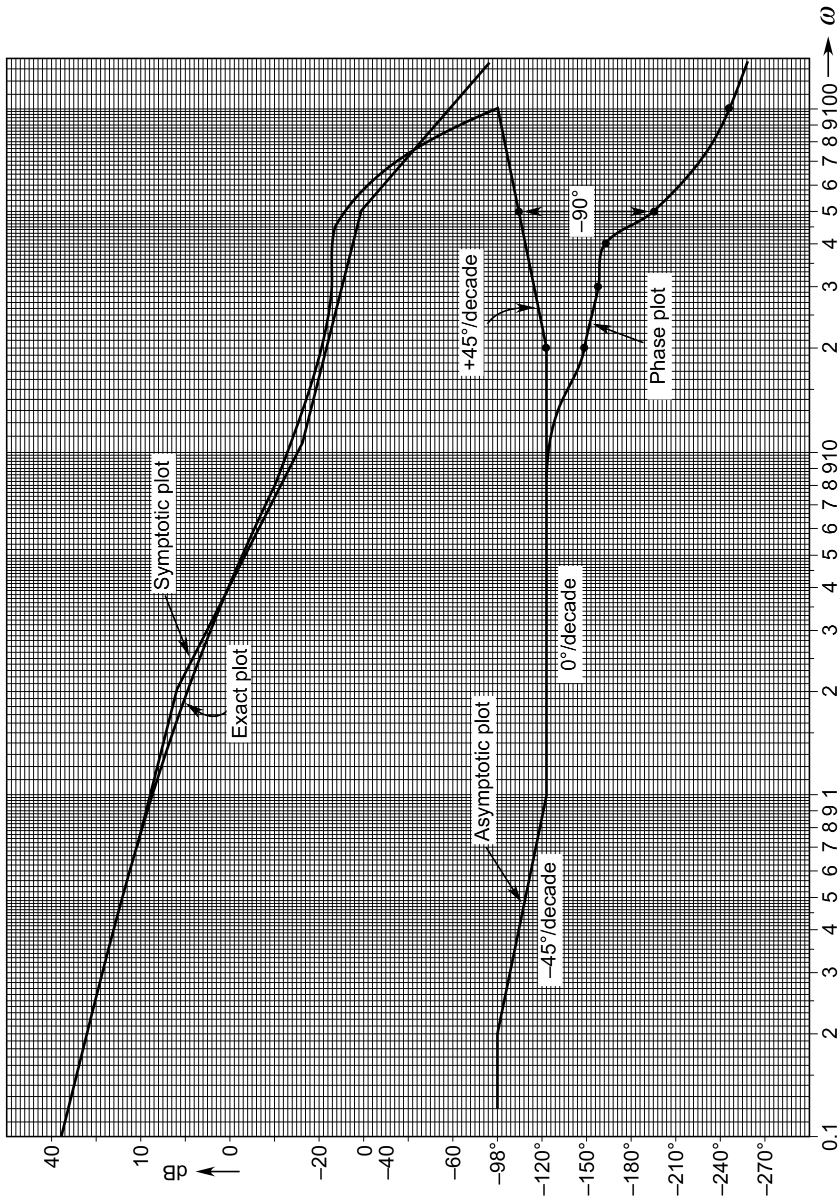

Symptotic plot
Exact plot
Asymptotic plot
–45°/decade
0°/decade
+45°/decade
–90°
Phase plot
dB
ω
40
10
0
–20
0
–40
–60
–98°
–120°
–150°
–180°
–210°
–240°
–270°
0.1
2
3
4
5
6
7
8
9
1
2
3
4
5
6
7
8
9
10
2
3
4
5
6
7
8
9
100

Fig. 6.25

Example 6.10 *Draw the asymptotic Bode plots of the transfer function*

$$H(j\omega) = \frac{10(1+j\omega/2)}{(j\omega)(1+j\omega/0.1)(1+j\omega/0.5)(1+j\omega/10)}$$

Solution The corner (break) frequencies are

Numerator $\omega_3 = 2$

Denominator $\omega_1 = 0.1,\ \omega_2 = 0.5,\ \omega_4 = 10$

Initial slope $\text{dB}\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{\omega} = -20\log\omega$; –20 dB/decade

At $\omega = 1$

$\text{dB} = 20\log 10 = 20$

Initial asymptote has slope –20 dB/decade which passes through 20 dB at $\omega = 1$.

The asymptotic dB plot is drawn to scale in Fig. 6.26.

Asymptotic phase plots

$\omega_1 = 0.1$	asymptote	$\omega = 0.01$ to 1, slope –45°/decade
$\omega_2 = 0.5$	–do–	$\omega = 0.05$ to 5, slope –45°/decade
$\omega_3 = 5$	–do–	$\omega = 0.5$ to 50, slope +45°/decade
$\omega_4 = 10$	–do–	$\omega = 1$ to 100, slope –45°/decade

Initial angle $\angle 1/j\omega = -90°$ up to 0.01

Final angle $-90° - 90° - 90° + 90° - 90° = -270°$ at $\omega = 100$

The asymptotic Bode phase plot is drawn in Fig. 6.26.

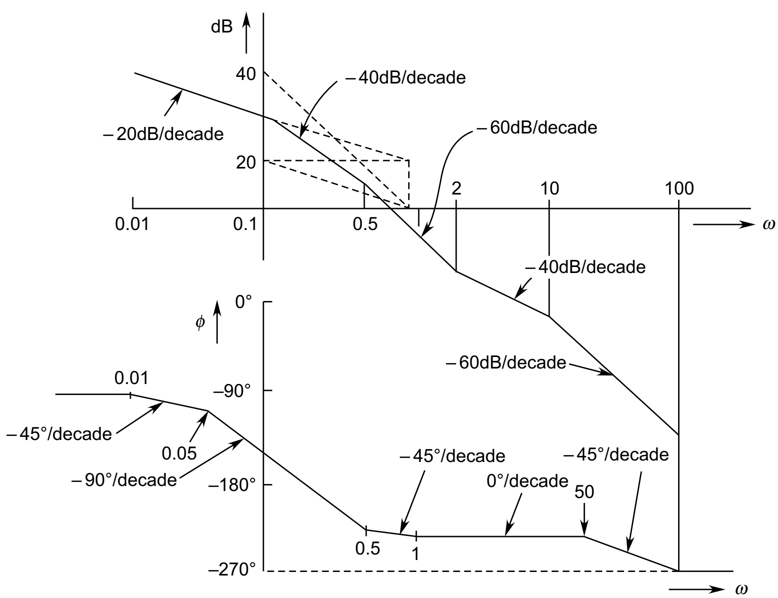


Fig. 6.26

Example 6.11 *For the circuit of Fig. 6.27 with dependent source, find the transfer function* $H(j\omega) = \dfrac{\overline{V}_0(j\omega)}{\overline{V}_i(j\omega)}$, *and draw the asymptotic dB Bode plot.*

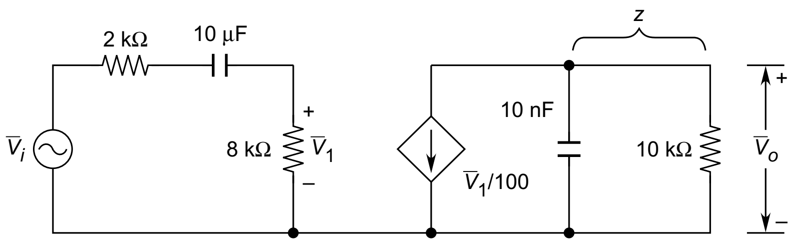


Fig. 6.27

Solution

$$\overline{V}_1 = \frac{8\times 10^3\,\overline{V}_i}{\left[(2+8)\times 10^3 + \dfrac{1}{j\omega\times 10\times 10^{-6}}\right]} \tag{i}$$

or
$$\overline{V}_1 = \frac{0.08\,j\omega}{(1+j\omega/10)} \tag{ii}$$

Output impedance

$$\overline{Z} = \frac{R}{1+j\omega RC} = \frac{10\times 10^3}{1+j\omega\times 10\times 10^{-9}\times 10\times 10^3} \tag{iii}$$

$$= \frac{10}{(1+j\omega/10^4)} \tag{iv}$$

$$\overline{V}_0 = (\overline{V}_1/100)\overline{Z} \tag{v}$$

or
$$\frac{\overline{V}_0}{\overline{V}_i} = \frac{0.08\,j\omega}{(1+j\omega/10)} \times \frac{10}{(1+j\omega/10^4)}$$

$$H(j\omega) = \frac{8\,j\omega}{(1+j\omega/10)\,(1+j\omega/10^4)} \tag{vi}$$

The corner frequencies are

$$\omega = 10,\ \omega = 10^4, \quad 20\log 8 = 18\text{ dB}.$$

The asymptotic Bode plot of $\overline{V}_0/\overline{V}_i(j\omega)$ is drawn in Fig. 6.28. This is a typical frequency response of the RC-coupled amplifier. Approximately, the half-power points (3 dB below the flat response part) are at the corner frequencies, $\omega = 10$ rad/s and $\omega = 10^4$ rad/s.

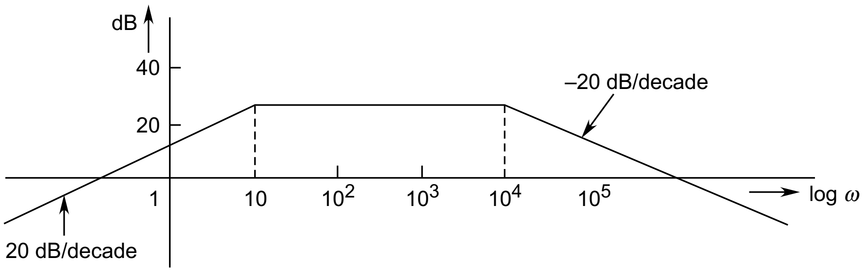


Fig. 6.28

Example 6.12 *For the dB log ω asymptotic plot of Fig. 6.29, determine the transfer function H(s). On approximate basis, find its phase angle at ω = 2 and ω = 520.*

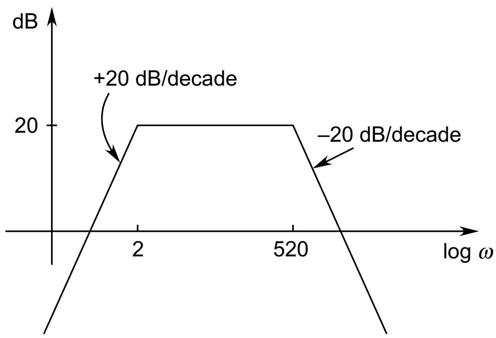


Fig. 6.29

Solution From the plot, the break frequencies are

$$\omega_1 = 2 \quad \omega_2 = 520$$

The transfer function is then

$$H(s) = \frac{K(j\omega)}{(1 + j\omega/2)(1 + j\omega/520)}$$

To find K

At $\omega = 2$ ignoring denominator factors

$$\text{dB}\,|H| = \text{dB}\,(K \times 2) = 20$$

which gives

$$K = 5$$

phase angle at $\omega = 2$

$$\phi_1 = \angle j2 - \angle(1 + j1) = 45°$$

Phase angle at $\omega = 520\ (\omega_2 > 10\ \omega_1)$

$$\phi_2 = -\angle(1 + j1) = -45°$$

Example 6.13 *For the Bode magnitude asymptotic plot of Fig. 6.30, determine the transfer function in frequency domain form.*

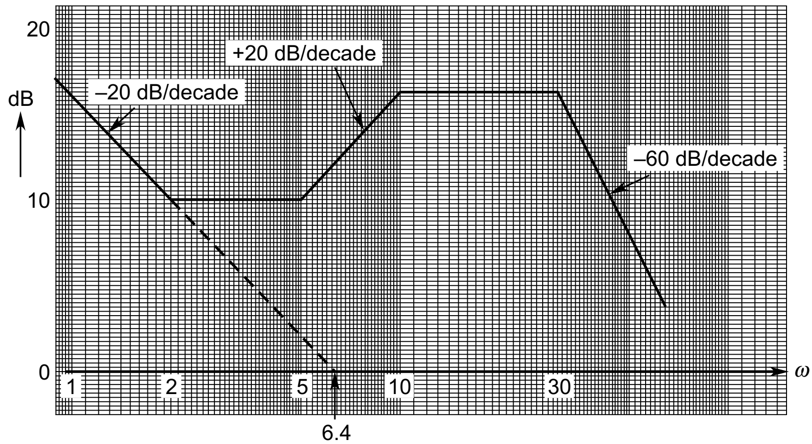


Fig. 6.30

Solution The break frequencies as found from the Bode plot are

$$2,\ 5,\ 10,\ 30$$

The initial stope is –20 dB/decade which corresponds to the term $(1/j\omega)$.

The initial –20 dB/decade asymptote when extended forward intersects the 0 dB line at $\omega = 6.4$. Therefore, $K = 6.4$ (proved below)

Following the asymptote slopes and break frequencies, the transfer function is written down as

$$H(j\omega) = \frac{6.4\,(1 + j\omega/2)\,(1 + j\omega/10)}{(j\omega)\,(1 + j\omega/5)\,(1 + j\omega/30)^3}$$

Proof

At $\omega = 1,\ |\,H(j\omega)\,| = K$

$$20\log_{10} K = 20 \times \frac{6.4}{1} \quad \text{or} \quad K = 6.4$$

6.5 BLOCK DIAGRAMS

To represent input and output of a transfer function, it is convenient to use a block for the transfer function (see Fig. 2.54) as shown in Fig. 6.31(a), wherein

$$Y(s) = H(s)\,R(s) \tag{6.43}$$

it is to be noted that the signal flow through the block is unidirectional.

An underlying assumption in block -diagram representation is that any component or another subsystem connected at the block output does not affect its transfer function.

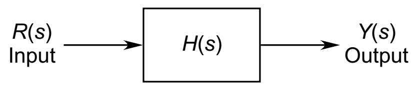


Fig. 6.31(a) *Block diagram*

A system can usually be broken down into several such systems, each represented by a block provided the conditions just mentioned above hold. Thus, the complete block diagram of a system is the interconnection of several blocks which reveal the system structure. The block diagram is very helpful in system analysis, design and restructuring.

Block-Diagram Algebra

As a system, a block diagram has several interconnected blocks, so we will use other symbols such as transfer functions like $G(s)$. Also, we may write only G and it is understood that it is a function of s.

Block in Tandem

$$X = G_1 R$$
$$Y = G_2 X = G_1 G_2 R = GR;\ G = G_1 G_2$$

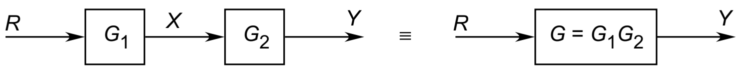


Fig. 6.31(b)

Signal Summation (or Difference)

Symbol

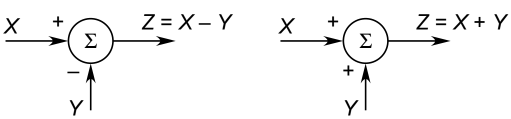


Fig. 6.31(c)

More than two signals could be added using these symbols.

Signal Take-off A signal can be taken off from a point without disturbing the origional signal. In fact, any number of signals can be taken off from a point. This is the basic property of a block diagram.

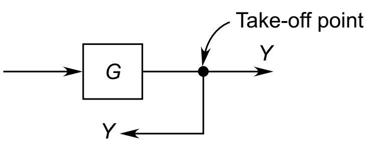


Fig. 6.31(d)

Basic Feedback Loop

Here, the output signal of a block is fed back to the input of the block directly or through another block called feedback block. The signal fed back could be added or subtracted from the input signal resulting in positive or negative feedback loop. Figure 6.32 shows a negative feedback loop, which has most practical applications in control systems and electronic feedback amplifiers.

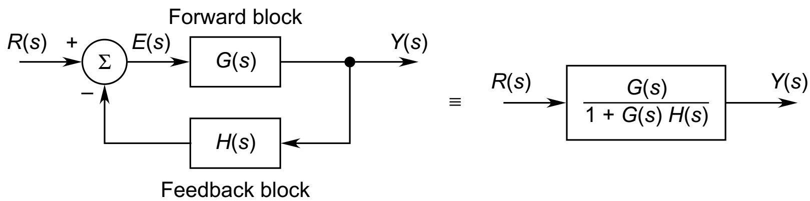


Fig. 6.32 *Basic negative feedback loop*

In Fig. 6.32, the signal

$$E(s) = R(s) - H(s)\,C(s) \tag{i}$$

The output

$$Y(s) = G(s)\,E(s) \tag{ii}$$

Substituting $E(s)$ from Eq. (i) in Eq. (ii), we get

$$Y(s) = G(s)\,R(s) - G(s)\,H(s)\,Y(s) \tag{iii}$$

This equation can be written in the form of overall transfer function

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

as shown in Fig. 6.31. Apart from other uses this relationship is very helpful in reduction of the block diagrams of complex systems.

In case of positive feedback, the loop reduces to

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)} \tag{6.45}$$

It is observed that the denominator which determines the poles of the system is

$$1 \pm G(s)\,H(s);\ + \text{ for negative feedback} - \text{ for positive feedback}$$

where $\qquad G(s)\,H(s) = \text{loop gain}$

Positive feedback is used in electronic oscillators. Oscillations occur when

$$1 - G(j\omega)\,H(j\omega) = 0 \tag{6.46}$$

such that $Y(s)/R(s)$ tends to infinite gain.

Some block-diagram reduction techniques are given in Fig. 6.33 which are self-explanatory. There are of great help in reducing any complex block dia-

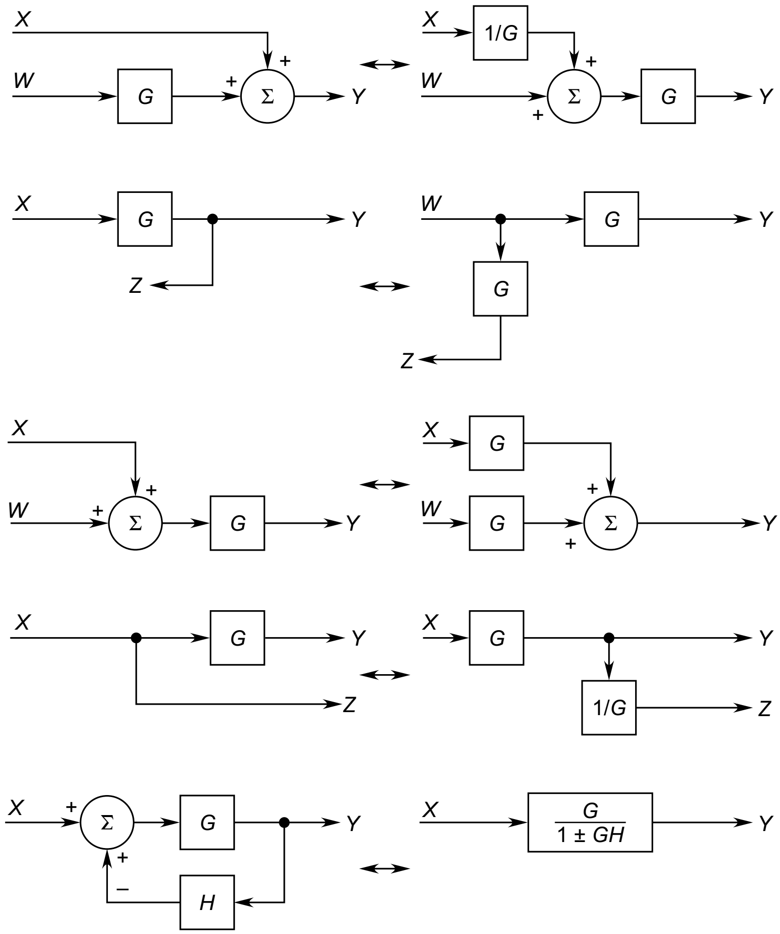


Fig. 6.33 *Block-diagram reduction techniques*

gram. As there can be more than one input and output, symbols X, Y, W, etc., are used.

Example 6.14 *For the block diagram of Fig. 6.34, find $T(s) = Y(s)/R(s)$.*

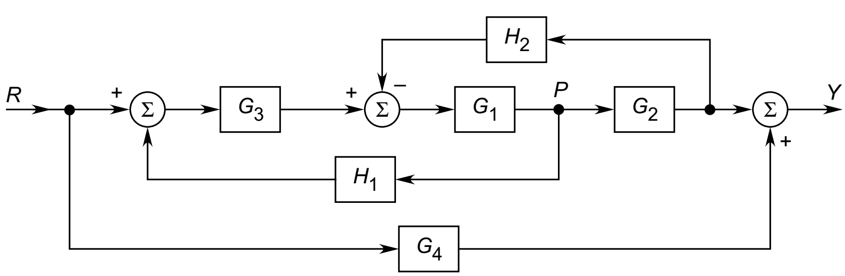


Fig. 6.34

Solution Shift the signal take-off from P to the right of G_2. The block diagram is redrawn in Fig. 6.35.

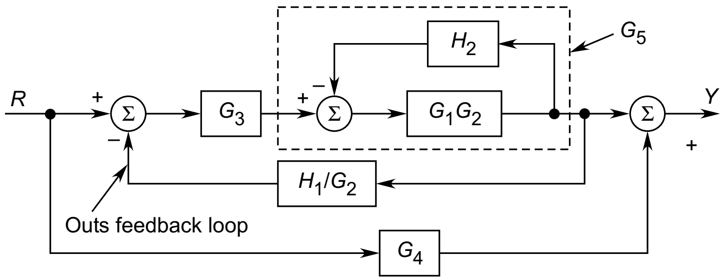


Fig. 6.35

The inner feedback loop is reduced to

$$G_5 = \frac{G_1 G_2}{1 + G_1 G_2 H_2} \tag{i}$$

The outer feedback loop gets reduced to

$$G_6 = \frac{G_3 G_5}{1 + G_3 G_5 H_1 / G_2} = \frac{G_2 G_3 G_5}{G_2 + G_3 G_5 H_1} \tag{ii}$$

This result is the block diagram of Fig. 6.36.

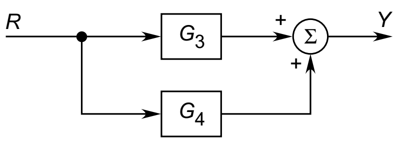


Fig. 6.36

The overall transfer function is then

$$T(s) = \frac{Y(s)}{R(s)} = G_6 + G_4 \tag{iii}$$

Substituting Eq. (i) in Eq. (ii)

$$G_6 = \frac{G_2 G_3 \dfrac{G_1 G_2}{1 + G_1 G_2 H_1}}{G_2 + G_3 H_2 \dfrac{G_1 G_2}{1 + G_1 G_2 H_1}}$$

or

$$G_6 = \frac{G_1 G_2^2 G_3}{G_2 (1 + G_1 G_2 H_1) + G_1 G_2 G_3 H_2} \tag{iv}$$

$$T(s) = \frac{G_1 G_2^2 G_3}{G_2 + G_1 G_2^2 H_1 + G_1 G_2 G_3 H_2} + G_4$$

Example 6.15 *Reduce the block diagram of Fig. 6.37 to the form of basic feedback loop by determining G(s) and H(s). Find from there the transfer function*

$$T(s) = Y(s)/R(s)$$

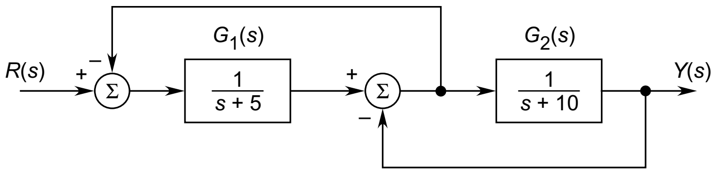


Fig. 6.37

Solution Shifting the take-off to the right of $G_2(s)$ and the feedback from $Y(s)$ to the left of $G_1(s)$, the block diagram gets modified to that of Fig. 6.38. The two feedback transfer functions can now be added to yield the diagram 6.39.

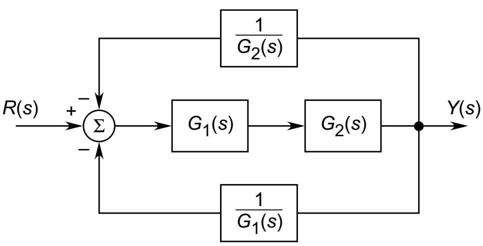


Fig. 6.38

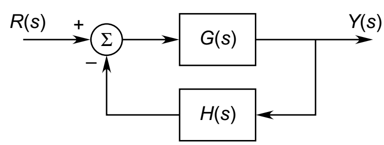


Fig. 6.39

where

$$G(s) = G_1(s)\, G_2(s) = \frac{1}{(s+5)\,(s+10)} \tag{i}$$

$$H(s) = \frac{1}{G_1(s)} + \frac{1}{G_2(s)} = (s + 5) + (s + 10) \tag{ii}$$

$$= 2\,s + 15$$

The overall transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\dfrac{1}{(s+5)\,(s+10)}}{1 + \dfrac{1}{(s+5)\,(s+10)} \times (2\,s + 15)}$$

or

$$T(s) = \frac{1}{(s+5)\,(s+10) + (2\,s+15)}$$

$$= \frac{1}{s^2 + 17\,s + 65} \tag{iii}$$

SISO and MIMO Systems

Systems whose block diagrams we considered so for are single-input single-output (SISO) kind. Systems with multiple-input and multiple-output are known as MIMO. By considering one input and one output at a time and using *superposition*, we could handle these systems through block diagrams. We shall illustrate this through two inputs and one output system, whose block diagram is drawn in Fig. 6.40.

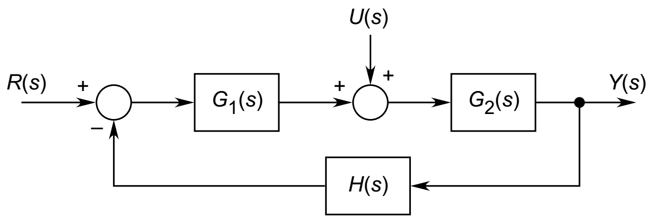


Fig. 6.40

Let $\qquad U(s) = 0$. Then the output due to input $R(s)$ is

$$Y_R(s) = \left[\frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_1(s)\,H(s)}\right] R(s) \tag{6.47}$$

Let $R(s) = 0$. The forward path transfer function is $G_2(s)$ while the negative feedback is through $G_1(s)\,H(s)$. The output due to $U(s)$ is then

$$Y_U(s) = \left[\frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}\right] U(s) \tag{6.48}$$

The output due to both inputs is then

$$\begin{aligned} Y(s) &= Y_R(s) + Y_U(s) \\ &= \left[\frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}\right] (G_1(s)\,R(s) + U(s)) \end{aligned} \tag{6.49}$$

The signal $U(s)$ in Fig. 6.40 arises from *disturbance input* (loading in control systems or noise in electronic amplifiers).

It is possible to use the *feed-forward* scheme shown in Fig. 6.41 to eliminate the contribution of $U(s)$ in the output $Y(s)$. With $R(s) = 0$, $U(s)$ signal enters the system at two points. The output is given by

$$\begin{aligned} Y_U(s) &= \frac{G_2(s)}{1 + G_1(s)G_2(s)\,H(s)} \cdot U(s) + \frac{G_c(s)\,G_1(s)\,G_2(s)}{1 + G_1(s)\,G_1(s)\,H(s)} \cdot U(s) \\ &= \frac{G_2(s)\,[1 + G_c(s)\,G_1(s)]}{1 + G_1(s)\,G_2(s)\,H(s)}\; U(s) \end{aligned} \tag{6.50}$$

For $\qquad Y_U(s) = 0$

$$1 + G_c(s)\,G_1(s) = 0$$

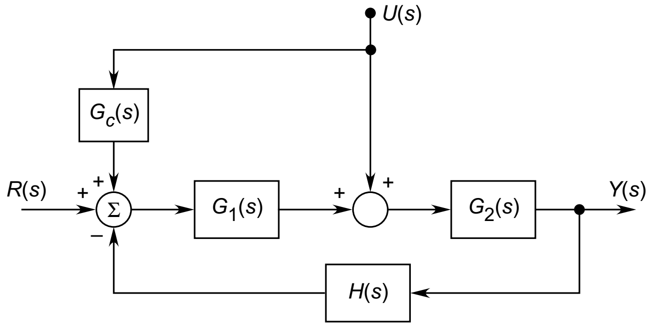


Fig. 6.41

or
$$G_c(s) = -\frac{1}{G_1(s)} \tag{6.51}$$

By introducing the feed-forward compensator as per Eq. (6.51), the effect of disturbance in the output can be eliminated.

6.6 SIGNAL-FLOW GRAPH MODELS

The block-diagram models presented in the preceding section are quite useful in determining the input–output relationships for simple systems. The block-diagram reduction method becomes quite cumbersome in a large system with no unique method of block reduction. The signal-flow graph method developed by Mason offers the great advantage of a general method for determining the input–output relationship (gain). The visual advantage of the block in system restructuring is however lost.

In a signal-flow graph, a block is represented by a directed line called a *branch* with multiplicative gain written on it. The two ends of the branch are the *nodes* as shown in Fig. 6.42 wherein x_1 and x_2 are the node variable (signals). A signal-flow graph is the interconnection of several branches at the nodes.

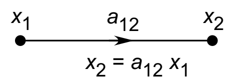


Fig. 6.42

The signal at any node in the graph is the sum of all incoming signals. All outgoing signals from a node have the value of the node variable which is unaffected by the outgoing signals (this is so in block diagrams also).

We shall present certain definitions before proceeding to the method of determing the gain between any two nodes.

- A *path* is a branch or continuous sequence of branches which are traversed from one node to another without going through the same node more than once.
- A *loop* is a closed path that starts and terminates at the same node without going through the same node more than once.

- Two loops are said to be *non-touching* if they do not have a common node.
- *Gain* of any path or loop is the product of gains of all the branches traversed.

Mason's gain Formula The relationship between an input variable and an output variable of a signal-flow graph is given by the net gain between the input and output nodes and is known as the overall gain of the system. Mason's gain formula for determination of the overall system gain is given below.

$$T = \frac{1}{\Delta}\sum_{k=1}^{N} P_k \Delta_k = \frac{X_{\text{out}}}{X_{\text{in}}} \tag{6.43}$$

where

T = gain between X_{out} and X_{in}

X_{out} = output node variable

X_{in} = input node variable

N = total number of forward paths

P_k = path gain of kth forward path

Δ = 1 – (sum of loop gains of all individual loops) + (sum of gain products of all possible combinations of two non-touching loops) – (sum of gain products of all possible combinations of three non-touching loops) + ...

Δ_k = the value of Δ for part of the graph not touching the kth for word path

Let L_m be the loop gain of the mth loop. We can then write Δ as

$$\Delta = 1 - \sum_{\text{all } m} L_m + \sum_{\substack{\text{all non} \\ \text{touching } p,q}} L_p L_q - \sum L_p L_q L_r \tag{6.48}$$

We will illustrate the use of Mason's gain formula by examples.

Example 6.16 *For the signal-flow graph of Fig. 6.43, determine Y(s)/R(s).*

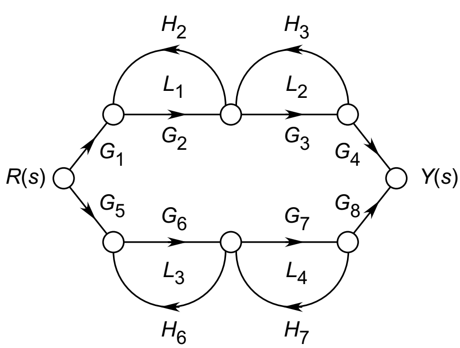


Fig. 6.43

Solution *Forward paths*

Path 1: $P_1 = G_1G_2G_3G_4$

Path 2: $P_2 = G_5G_6G_7G_8$

Loops

$$L_1 = G_2H_2, L_2 = G_3H_3, L_3 = G_6H_6, L_4 = G_7H_7$$

Two non-touching loops

$$L_1L_3, L_1L_4, L_2L_3, L_2L_4$$

There are no three non-touching loops. Then

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

Path 1 does not touch L_3, L_4.

$$\Delta_1 = 1 - (L_3 + L_4)$$

Path 2 does not touch L_1, L_2.

$$\Delta_2 = 1 - (L_1 + L_2)$$

Therefore the transfer function of the system

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

The result follows as substituting values.

Example 6.17 *For the signal-flow graph of Fig. 6.44, find Y(s)/R(s).*

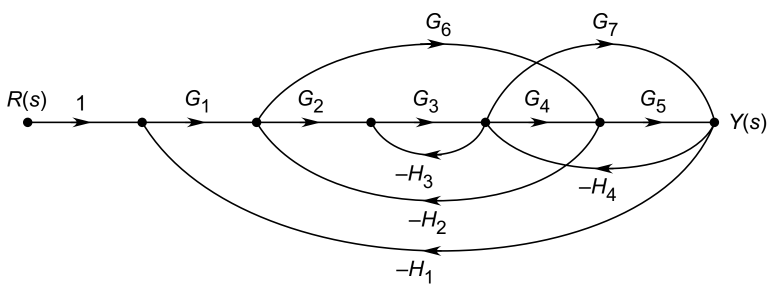


Fig. 6.44

Solution *Forward paths*

$$P_1 = G_1G_2G_3G_4G_5$$
$$P_2 = G_1G_6G_5$$
$$P_3 = G_1G_2G_3G_7$$

Individual loops

$$L_1 = -G_3 H_3$$
$$L_2 = -G_7H_4$$

$$L_3 = -G_6H_2$$
$$L_4 = -G_1G_6G_5H_1$$
$$L_5 = -G_1G_1G_3G_4G_5H_1$$
$$L_6 = -G_1G_2G_3G_7H_1$$

Two non-touching loops

$$L_1L_3,\ L_1L_4,\ L_2L_3$$

There are no three non-touching loops.

We then have

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + (L_1L_3 + L_1L_4 + L_2L_3)$$

Path 1 touches all loops, so

$$\Delta_1 = 1$$

Path 2 does not touch loop L_1, so

$$\Delta_2 = 1 - L_1$$

Path 3 touches all loops, so

$$\Delta_3 = 1$$

The overall transfer function is given by

$$\frac{Y(s)}{R(s)} = \frac{P_1 + P_2\,\Delta_2 + P_3}{\Delta}$$

6.7 OPERATIONAL AMPLIFIER (OPAM) AS CIRCUIT ELEMENT

An OPAMP is a high-gain direct coupled amplifier. In practical applications, the gain can be taken as infinite. It has input terminals *inverting* (–) and *non-inverting* (+). At both these terminals, the input impedance is very high, and can be considered infinite for practical purposes. The output terminal has a low output impedance which can be considered to be zero.

An OPAMP symbol is sketched in Fig. 6.45. While assuming idealness of input–output impedances, we shall examine the frequency sensitivity of the gain.

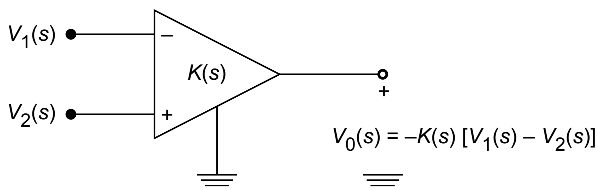


Fig. 6.45

The gain can then be represented by a single corner frequency (single time constant) as

$$k(s) = \frac{A_0}{1 + s/\omega_c} \tag{6.49}$$

where the Bode plot is sketched in Fig. 6.46.

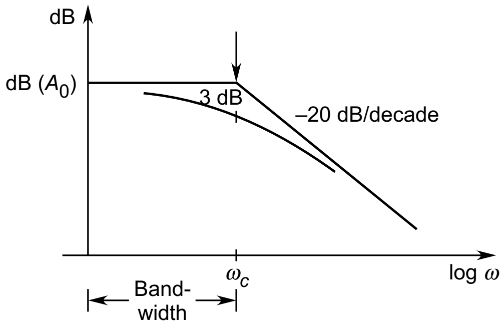


Fig. 6.46

Typical values of the two parameters are

$$A_0 = 2 \times 10^5$$
$$f_c = \omega_c/2\pi = 6 - 10 \text{ Hz}$$

It can be seen from the figure that

$$f_c = \text{bandwidth}$$

OPAMP is a versatile standard chip used in *analog signal processing,* and to provide stability it is normally connected in feedback mode.

A simple OPAMP feedback circuit is drawn in Fig. 6.47 with the non-inverting terminal grounded.

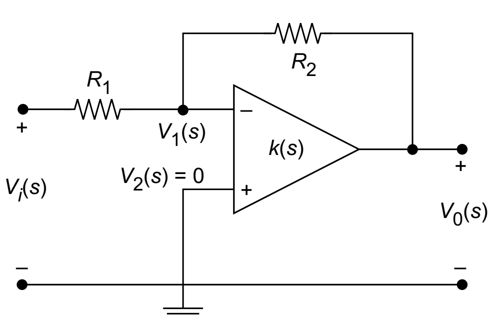


Fig. 6.47 *OPAMP in feedback mode*

As the inverting terminal does not draw any current (infinite input impedance), the KCL equation at node V_1 is

$$\frac{V_i - V_1}{R_1} + \frac{V_0 - V_1}{R_2} = 0 \tag{6.50}$$

The output voltage is

$$V_0(s) = -K(s)\,[V_1(s) - V_2(s)] = -K(s)\,V_1(s) \tag{6.51}$$

or

$$V_1(s) = -\frac{V_0(s)}{K(s)} \tag{6.52}$$

Gain Infinite

As $K \to \infty$ in Eq. (6.52)

$$V_1(s) = 0 \tag{6.53}$$

It means that the inverting terminal acts as *virtual ground*, which is an important result.

Using this result in Eq. (6.50), we get

$$\frac{V_0}{V_i} = -\frac{R_2}{R_1} \tag{6.54}$$

We find that gain with feedback is independent of OPAMP gain as OPAMP gain is very large (2×10^5).

Effect of Feedback on Bandwidth We use $V_0(s)/V_1(s) = -K(s)$ from Eq. (6.52) in Eq. (6.50) and then substitute $K(s)$ from Eq. (6.49). Further, some manipulation yields the frequency dependent transfer function

$$H(s) = \frac{V_0(s)}{V_i(s)} = -\frac{A_0/(1 + s/\omega_c)}{1 + (R_1/R_2)\,[(1 + A_0/(1 + s/\omega_c)]}$$

$$= -\frac{A_0\,\omega_c\,\beta(R_2/R_1)}{s + (1 + A_0\,\beta)\,\omega_c}\,;\ \beta = \frac{R_1}{R_1 + R_2} \tag{6.55}$$

The low-frequency gain of the amplifiers obtained by setting $s = 0$ in Eq. (6.55), which is

$$H(0) = -\frac{A_0\,\beta(R_2/R_1)}{(1 + A_0\,\beta)} \tag{6.56}$$

The bandwidth with feedback is

$$\omega_c(fb) = (1 + A_0\beta)\,\omega_c \tag{6.56}$$

Let us examine the low-frequency gain and bandwidth product of the OPAMP based feedback amplifier.

$$H(0)\,\omega_c(fb) = A_0\omega_c\,\frac{R_2}{R_1 + R_2} \approx A_0\omega_c;\ R_2 >> R_1 \tag{6.57}$$

Thus, gain-bandwidth product is nearly constant. The feedback reduces the gain but increases the bandwidth.

If we take $\dfrac{R_2}{R_1} = 10$, $\beta = \dfrac{1}{1 + R_2/R_1} = \dfrac{1}{11}$, $A_0 = 2 \times 10^5$, the bandwidth increases to

$$\omega_c(fb) = \left(1 + 2 \times 10^5 \times \frac{1}{11}\right)\omega_c$$

or

$$f_c(fb) = 0.091 \times 10^5 \times 10 \approx 90 \text{ kHz}$$

Example 6.18 *For the OPAMP feedback circuit of Fig. 6.48, show that the transfer function $V_0(s)/V_a(s)$ has the form*

$$H(s) = \frac{\omega_n^2/\beta}{s^2 + 2\alpha s + \omega_n^2} \tag{6.58}$$

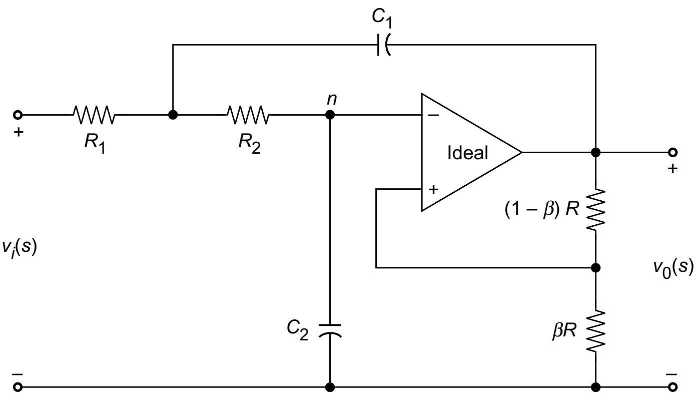


Fig. 6.48

Determine the expressions for α and ω_n.

Solution Voltage at the non-inverting terminal is $\beta V_0(s)$.

$\therefore$ voltage at the inverting terminal is also $\beta V_0(s)$.

The circuit in the s-domain can now be drawn as in Fig. 6.49.

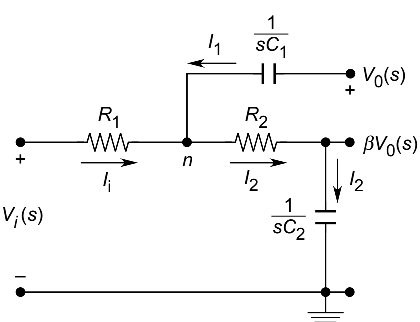


Fig. 6.49

The currents $I_1(s)$ and $I_2(s)$ can be expressed as

$$I_2(s) = \frac{V_n(s) - \beta V_0(s)}{R_2} = s\beta C_2\, V_0(s);\ \text{as the terminal of OPAMP does not draw any current} \quad \text{(i)}$$

$$I_1(s) = sC_1\,[V_0(s) - V_n(s)] \quad \text{(ii)}$$

$$I_1(s) = I_2(s) - I_1(s) = s\beta C_2 V_0(s) - sC_1[V_0(s) - V_n(s)] \quad \text{(iii)}$$

Now $V_i(s) = V_n(s) + R_1\, I_i(s)$

or
$$V_i(s) = V_n(s) + R_1[s\beta C_2 V_0(s) - sC_1 V_0(s) + sC_1 V_n(s)$$
$$= (1 + sR_1C_1)\, V_n(s) + [s\beta R_1 C_2 - sR_1C_1]\, V_0(s) \quad \text{(iv)}$$

From Eq. (i) we get

$$V_n(s) = \beta(1 + sR_2C_2)\, V_0(s) \quad \text{(v)}$$

Substituting $V_n(s)$ from Eq. (v) in Eq. (iv), we get

$$V_i(s) = [\beta(1 + sR_1C_1)\,(1 + sR_2C_2) + s\beta R_1C_2 - sR_1C_1]\, V_0(s) \quad \text{(vi)}$$

This yields the transfer function

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{\beta(1 + sR_1C_1)\,(1 + sR_2C_2) + s\beta R_1C_2 + sR_1C_1} \quad \text{(vii)}$$

This can be organized in the form

$$H(s) = \frac{\dfrac{\beta}{R_1C_1R_2C_2}}{s^2 + \left[\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1} + \dfrac{1}{R_2C_2}\left(1 - \dfrac{1}{\beta}\right)\right]s + \dfrac{1}{R_1C_1R_2C_2}} \quad \text{(viii)}$$

Let
$$\frac{1}{R_1C_1R_2C_2} = \omega_n^2 \quad \text{(ix)}$$

$$2\alpha = \frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2}(1 - 1/\beta) \quad \text{(x)}$$

Then

$$H(s) = \frac{\omega_n^2/\beta}{s^2 + 2\alpha s + \omega_n^2}$$

This is the Sallon-Kely circuit which has the feature that magnitude and angle of pole location in s-plane can be adjusted independently.

Example 6.19 *For the circuit with OPAMP of Fig. 6.50, determine the Thevenin and Norton equivalent at terminals 'ab'.*

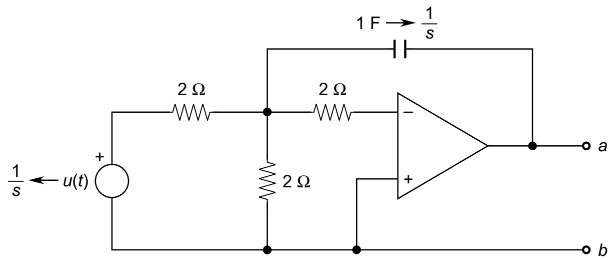
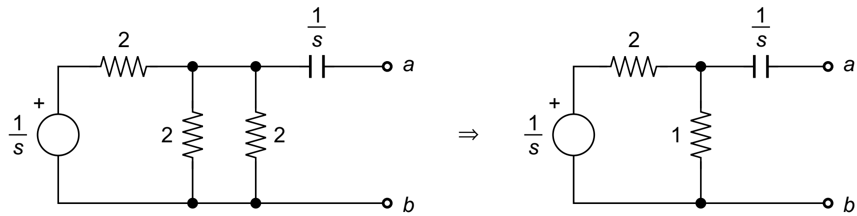


Fig. 6.50

Solution The circuit is redrawn below with the inverting terminal grounded (virtual ground).

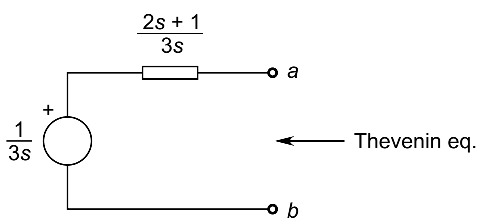


$$V_{TH} = V_{OC}\,(\text{at } `ab') = \frac{1}{3}\cdot\frac{1}{s} = \frac{1}{3s}$$

Shorting the source voltage source $\left(\frac{1}{s}\right)$

$$Z_{TH} = \frac{1\times 2}{3} + \frac{1}{s} = \frac{2s+1}{3s}$$

The Thevenin circuit is drawn below.

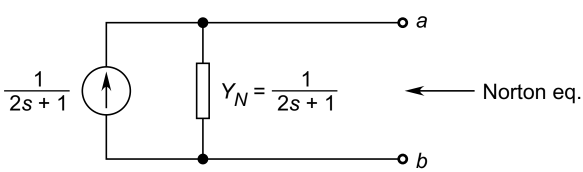


Shorting terminals *ab*, we get

$$I_{SC} = I_N = \frac{1}{3s} + \frac{2s+1}{3s} = \frac{1}{2s+1}$$

$$Y_N = \frac{1}{Z_{TH}} = \frac{3s}{2s+1}$$

The Norton circuit is drawn below.



Problems

6.1 Consider the bridged-*T* network of Fig. P-6.1. Determine the transfer function $V_0(s)/V_i(s)$. Plot the pole-zero diagram for $R_1 = 1\Omega$, $R_2 = 2\ \Omega$ and $C = 0.5$ F.

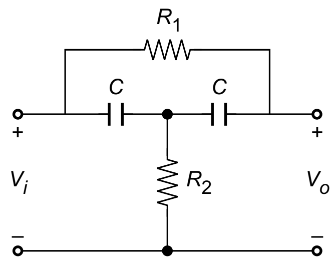


Fig. P-6.1

6.2 For the ladder network of Fig. P-6.2, determine the transfer function $E_0(s)/E_i(s)$. Show that it is a low-pass filter.

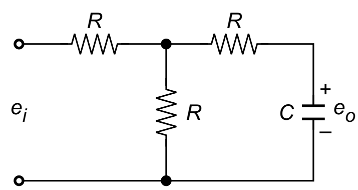


Fig. P-6.2

6.3 For the asymptotic dB-log ω plot of Fig. P-6.3, write the transfer function $H(j\omega)$. Find the angle $\angle H(j\omega)$ at $\omega = 20$ rad/s and $\omega = 800$ rad/s.

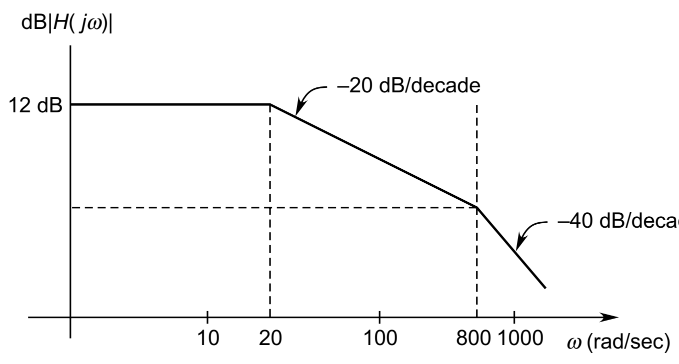


Fig. P-6.3

6.4 For the expression of $H(j\omega)$ given, find the break frequencies. Draw to scale dB-log ω plots on the semi log graph paper.

(a) $20\left(\dfrac{j\omega + 0.25}{j\omega + 50}\right)$ (b) $\dfrac{625}{(j\omega)^2 + 27.5\, j\omega + 62.5}$

6.5 The Bode magnitude asymptotic plot is drawn to scale in Fig. P-6.5. Write the expression for $H(j\omega)$. Draw the asymptotic Bode plot of its inverse.

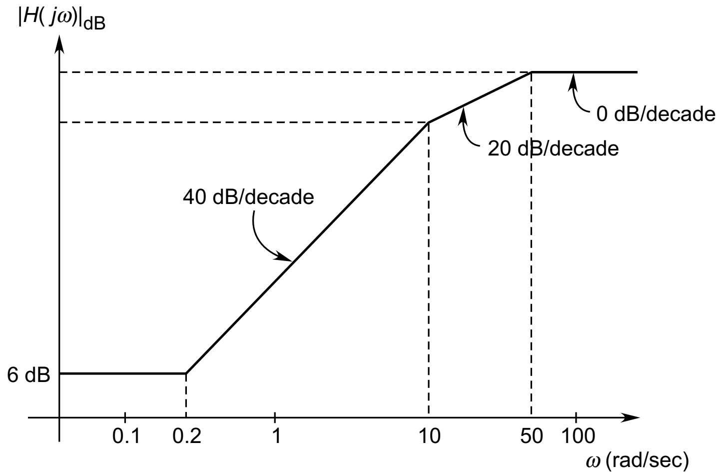


Fig. P-6.5

6.6 Consider the asymptotic dB-log ω plot of $H(j\omega)$ and $H_1(j\omega)$. If $H(j\omega) = H_1(j\omega)\,H_2(j\omega)$, determine the transfer function $H_2(j\omega)$. Draw its magnitude Bode plot.

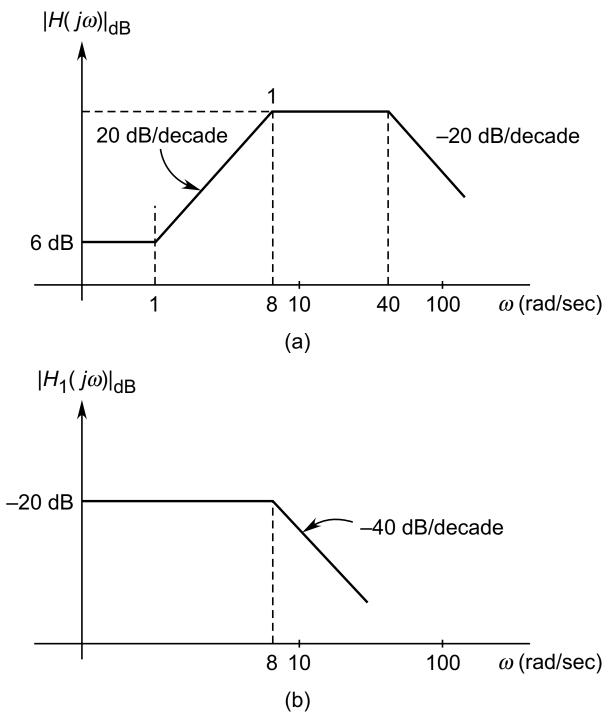


Fig. P-6.6

6.7 Determine the transfer function $Y(s)/R(s)$ for the block diagram of Fig. P-6.7.

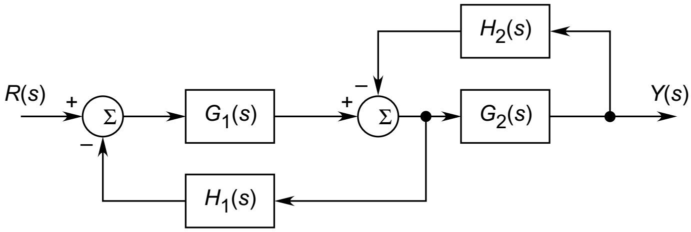


Fig. P-6.7

6.8 For the block diagram of Fig. P-6.8, determine the transfer function $Y(s)/R(s)$. Draw the corresponding signal-flow graph and verify.

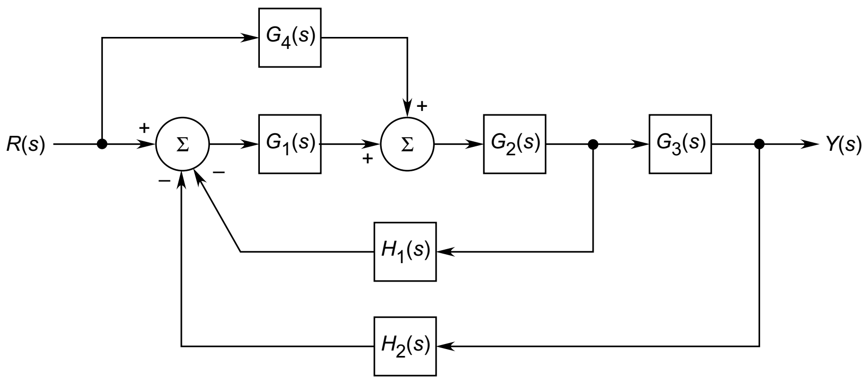


Fig. P-6.8

6.9 In the block diagram of Fig. P-6.9, $D(s)$ is the disturbance signal, and $R(s)$ is the command signal.

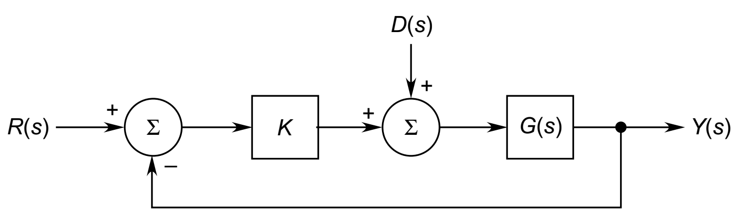


Fig. P-6.9

$$G(s) = 8/(s + 1)\,(s + 4);\; R(s) = A/s \text{ and } D(s) = B/s.$$

Show that if K is made large enough $\lim_{t \to \infty} y(t) \to A$ which means that the disturbance has negligible effect.

6.10 For the signal-flow graph of Fig. P-6.10, determine $Y(s)/R(s)$.

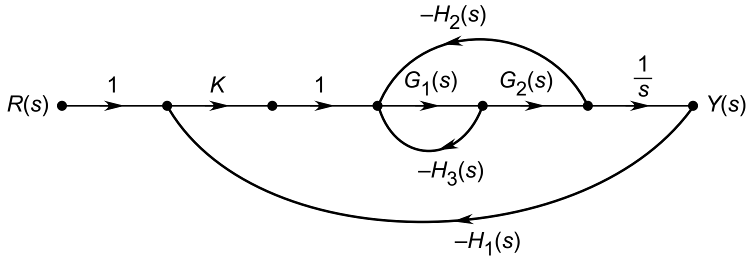


Fig. P-6.10

6.11 For the signal-flow graph of Fig. P-6.11, determine $Y_2(s)/R_1(s)$.

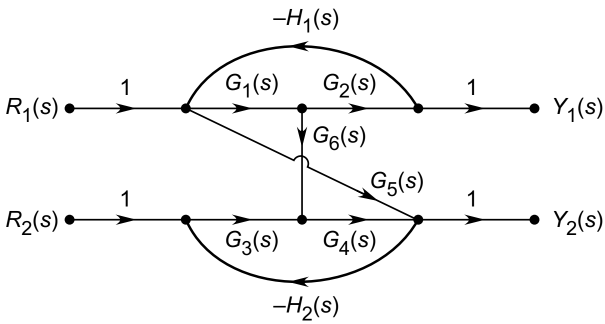


Fig. P-6.11

6.12 For the OPAMPs circuit of Fig. P-6.12, determine $V_0(s)/V_i(s)$. Given R_1 = 167 kΩ, R_2 = 240 kΩ, R_3 = 1 kΩ, R_4 = 100 kΩ, and C = 1 μF.

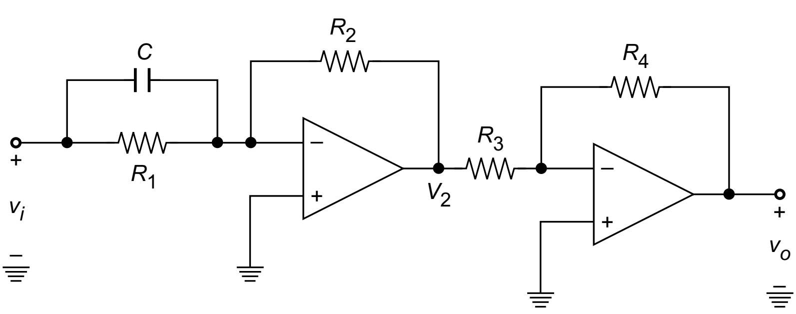


Fig. P-6.12

6.13 An OPAMP is connected as non-inverting amplifier as in Fig. P-6.13. Write its governing equation and draw its signal flow graph. Determine therefore the transfer function $\dfrac{v_0}{v_i}$.

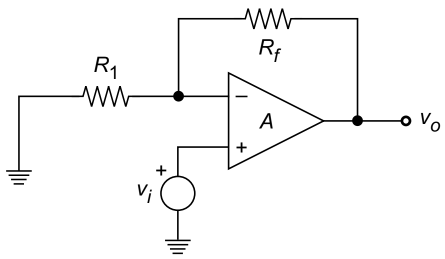
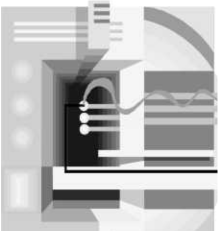


Fig. P-6.13



State Space Analysis

7

Introduction

Compared to input-output (transfer function) approach, the state variable approach yields the time behaviour of internal variables of the model. From state variable approach behaviour of signal in any internal part of the system and the output variables can be obtained. This model is ideally suited for computer solutions particularly for multi-input, multi-output (MIMO) systems.

In this chapter various formulations of state variable model and the methods of solution of state equations will be discussed.

7.1 CONCEPTS OF STATE, STATE VARIABLES AND STATE MODEL

A mathematical abstraction to represent or model the dynamics of a system utilizes three types of variables called the *input*, *the output and state variables*.

Consider a mechanical system shown in Fig. 7.1 wherein mass M is acted upon by the force $F(t)$. This system is characterized by the following relations.

$$\frac{d}{dt} v(t) = \frac{1}{M} F(t) \qquad \text{(i)}$$

$$\frac{d}{dt} x(t) = v(t) \qquad \text{(ii)}$$

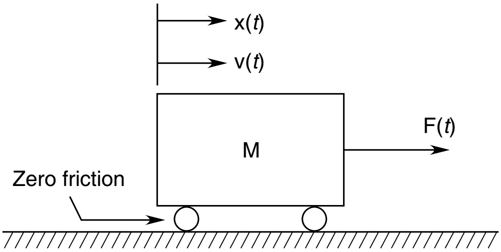


Fig. 7.1 *A simple mechanical system*

From these relations, we get

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} F(t)\mathrm{d}t = \frac{1}{M}\int_{-\infty}^{t_0} F(t)\mathrm{d}t + F(t)\mathrm{d}t\ \frac{1}{M}\int_{t_0}^{t}$$

$$= v(t_0) + \frac{1}{M}\int_{t_0}^{t} F(t)\mathrm{d}t \qquad \text{(iii)}$$

$$x(t) = \int_{-\infty}^{t} v(t)\mathrm{d}t = \int_{-\infty}^{t_0} v(t)\mathrm{d}t + \int_{t_0}^{t} v(t)\mathrm{d}t$$

$$= x(t_0) + [t - t_0]\ v(t_0) + \frac{1}{M}\int_{t_0}^{t} \mathrm{d}\tau \int_{t_0}^{t} F(t)\mathrm{d}t \qquad \text{(iv)}$$

We observe that the displacement $x(t)$ (output variable) at any time $t \geq t_0$ can be computed if we know the applied force $F(t)$ (input variable) from $t = t_0$ onwards, provided $v(t_0)$ the initial velocity and $x(t_0)$ the initial displacement are known. We may conceive of initial displacement as describing the status or state of the system at $t = t_0$. The state of the system given in Fig. 7.1 at any time t is given by the variables $x(t)$ and $v(t)$, which are called the state variables of the system.

Precise Definitions of State and State Variables

The state of a dynamical system is a minimal set of variables (known as state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

For the simple system shown in Fig. 7.1, two state variables are defined from the two first-order differential equations (i) and (ii). Let

$$x_1 = x;\ \text{position}$$
$$x_2 = v = \dot{x} = \dot{x}_1;\ \text{velocity} \qquad \text{(v)}$$

The two differential equations can then be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \dot{v} = \frac{1}{M}F(t) \qquad \text{(vi)}$$

These state equations can be expressed in the following vector-matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{M} \end{bmatrix} r;\ r = F(t) \qquad \text{(vii)}$$

where $F(t) = r(t)$ is the input.

If the desired output is velocity then

$$y = [0\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(viii)}$$

The state equation and output equation (viii) can be written in compressed form as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x} + r;\ \text{state equation} \\ y &= \mathbf{c}'\,\mathbf{x};\ \text{output equation} \end{aligned} \tag{ix}$$

where **A** is a matrix, **x** is a vector, **c′** is vector transpose and r and y are scalars. Their values are obvious from Eqs (vii) and (viii).

We can now formulate the general state variable model.

7.2 STATE-VARIABLE MODEL

The general state variable model of an n^{th} order continuous-time LTI system with m inputs and p outputs may be described by

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}r_1 + b_{12}r_2 + \ldots + b_{1m}r_m \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}r_1 + b_{22}r_2 + \ldots + b_{2m}r_m \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}r_1 + b_{n2}r_2 + \ldots + b_{nm}r_m \end{aligned} \tag{7.1}$$

Equation (7.1) is called the state equation where derivative of each state variable is described by the linear combination of system states and inputs. In vector matrix form this equation may be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{r}(t)\ ;\ \textbf{state equation} \tag{7.2}$$

where

$\mathbf{x}(t)$ is $n \times 1$ state vector

$\mathbf{r}(t)$ is $m \times 1$ input vector

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = n \times n \text{ matrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix} = n \times m \text{ matrix}$$

Similarly, the output variables are linear combinations of state variables and inputs, i.e.,

$$y_1(t) = c_{11}\, x_1(t) + \ldots + c_{1n}x_n(t) + \mathrm{d}_{11}r_1(t) + \ldots + d_{1m}r_m(t)$$
$$\vdots$$
$$y_p(t) = c_{p1}\, x_1(t) + \ldots + c_{pn}x_n(t) + dp_1r_1(t) + \ldots + \mathrm{d}_{pm}r_m(t) \tag{7.3}$$

In vector matrix form this equation may be written as

$$\boldsymbol{y}(t) = \mathbf{C}\, \mathbf{x}(t) + \mathbf{D}\, \mathbf{r}(t);\ \textbf{output equation} \tag{7.4}$$

where **C** and **D** are expressed as

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} = p \times n \text{ matrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} = p \times m \text{ matrix}$$

The state and output equations of multi-input, multi-output (MIMO) system given above in Eqs (7.2) and (7.4), are reproduced below:

$$\dot{\mathbf{x}}\,(t) = \mathbf{A}\,\mathbf{x}\,(t) + \mathbf{B}\,\mathbf{r}\,(t) \tag{7.5a}$$
$$\mathbf{y}\,(t) = \mathbf{C}\,\mathbf{x}\,(t) + \mathbf{D}\,\mathbf{r}\,(t) \tag{7.5b}$$

These equations are represented by a block diagram (Fig. 7.2) where the multiple signals are represented.

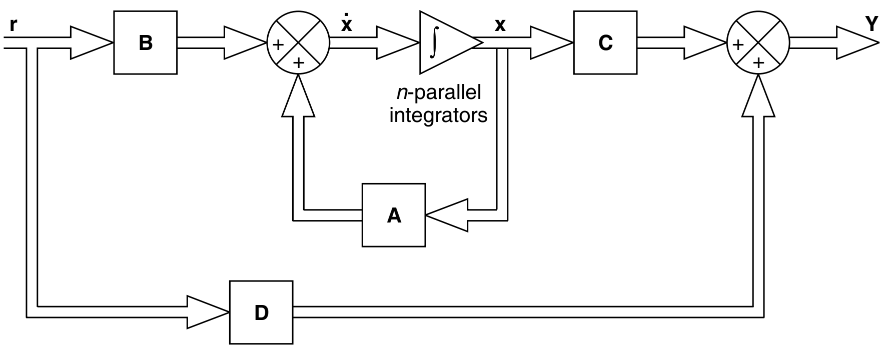


Fig. 7.2 *Block diagram of a MIMO system*

For a single-input, single-output (SISO) system Eqs (7.5a) and (7.5b) take the following form.

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\, r \tag{7.6a}$$
$$\mathbf{y} = \mathbf{c}'\,\mathbf{x} + d\, r \tag{7.6b}$$

where

r = scalar input
y = scalar output
$\mathbf{A}, \mathbf{x}$ = as defined for MIMO system
$\mathbf{b}$ = $(n \times 1)$ vector
$\mathbf{c}'$ = $(1 \times n)$, transpose of a vector
d = scalar

The block-diagram representation of a SISO system is drawn in Fig. 7.3.

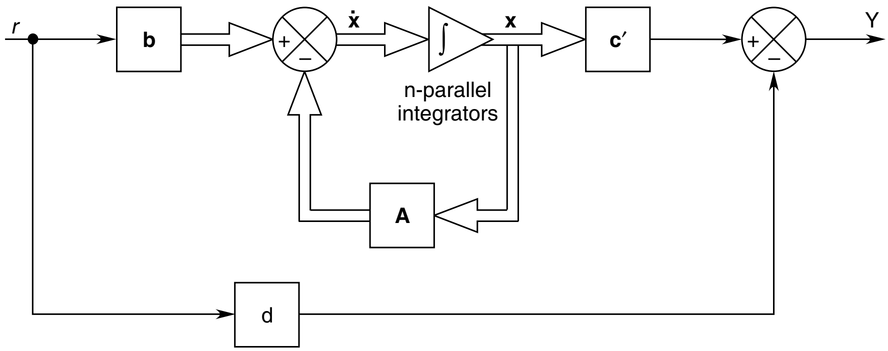


Fig. 7.3 *Block diagram of a SISO system*

Linearization of the State Equation

Consider the non-linear state equation.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{r}) \; ; n^{\text{th}} \text{ order} \tag{7.7}$$

It is assumed that the system represented by this equation is in equilibrium at $\mathbf{x_0}$, $\mathbf{r_0}$, i.e.,

$$\dot{\mathbf{x}}_\mathbf{0} = \mathbf{f}(\mathbf{x}_0, \mathbf{r}_0) = 0 \tag{7.8}$$

The state Eq. (7.7) can be linearized about this equilibrium point by expanding it into Taylor Series and neglecting the terms of second and higher order. Carrying out both these operations, we can write

$$\dot{\mathbf{x}}_i = f_i(\mathbf{x_0}, \mathbf{r_0}) + \sum_{j=1}^{n} \frac{\partial f_i\,(\mathbf{x}, \mathbf{r})}{\partial x_j}\bigg|_{\substack{\mathbf{x}=\mathbf{x_0}\\ \mathbf{r}=\mathbf{r_0}}} (x_j - x_{j0}) + \sum_{k=1}^{m} \frac{\partial f_i\,(\mathbf{x}, \mathbf{r})}{\partial r_j}\bigg|_{\substack{\mathbf{x}=\mathbf{x_0}\\ \mathbf{r}=\mathbf{r_0}}} (r_k - r_{k0}) \tag{7.9}$$

Defining

$$\tilde{x}_j = x_j - x_{j0}$$
$$\tilde{r}_k = r_k - r_{k0}$$
$$\dot{\tilde{x}}_i = \dot{x}_i - \dot{x}_{i0} = \dot{x}_i - f_i\,(\mathbf{x}_0, \mathbf{r}_0)$$

The linearized state equation can now be written as

$$\dot{\tilde{\mathbf{x}}}_i = \sum_{j=1}^{n} \frac{\partial f_i(\mathbf{x}, \mathbf{r})}{\partial x_j}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{r}=\mathbf{r}_0}} \tilde{x}_j + \sum_{k=1}^{m} \frac{\partial f_i(\mathbf{x}, \mathbf{r})}{\partial r_k}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{r}=\mathbf{r}_0}} \tilde{r}_k \tag{7.10}$$

This equation can be written in matrix form as

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\,\tilde{\mathbf{x}} + \mathbf{B}\,\tilde{\mathbf{r}} \tag{7.11}$$

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial f_1}{\partial r_1} & \dfrac{\partial f_1}{\partial r_2} & \cdots & \dfrac{\partial f_1}{\partial r_m} \\ \dfrac{\partial f_2}{\partial r_1} & \dfrac{\partial f_2}{\partial r_2} & \cdots & \dfrac{\partial f_2}{\partial r_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_n}{\partial r_1} & \dfrac{\partial f_n}{\partial r_2} & \cdots & \dfrac{\partial f_n}{\partial r_m} \end{bmatrix}$$

The output equation can be similarly linearized.

Non-uniqueness of the State Variable Model

We will now show the non-uniqueness of the state variable model in the general case for which state and output equations (Eqs (7.5a) and (7.5b)) are reproduced below.

$$\dot{\mathbf{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\mathbf{r} \tag{7.12a}$$

$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x} + \mathbf{D}\mathbf{r} \tag{7.12b}$$

Consider now another n-dimensional vector $\tilde{\mathbf{x}}$ such that

$$\boldsymbol{x} = \mathbf{P}\tilde{\mathbf{x}} \tag{7.13}$$

where $\mathbf{P}$ is any $n \times n$ non-singular constant matrix

Since $\mathbf{P}$ is a constant matrix, it follows that

$$\dot{\mathbf{x}} = \mathbf{P}\dot{\tilde{\mathbf{x}}}$$

Substituting $\mathbf{x}$ and $\dot{\mathbf{x}}$ from above transformations in Eq. (7.12a), we get

$$\mathbf{P}\dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{P}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{r}$$

Premultiplying by $\mathbf{P}^{-1}$, we obtain

$$\dot{\tilde{\mathbf{x}}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\tilde{\mathbf{x}} + \mathbf{P}^{-1}\mathbf{B}\mathbf{r} \tag{7.14a}$$

$$= \tilde{\mathbf{A}}\,\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\,\mathbf{r}$$

where

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P};\ \tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$$

From Eq. (7.12b), we have

$$\begin{aligned} y &= \mathbf{C}\mathbf{P}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{r} \\ &= \tilde{\mathbf{C}}\mathbf{P}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{r} \end{aligned} \tag{7.14b}$$

where

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P}$$

In the above transformation

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}, \text{ where } \mathbf{P} \text{ is non-singular}$$

As $\tilde{\mathbf{A}}$ is similar to **A, P** is called a **similarity transformation**.

In the above derivation, we have arrived at a new state variable model using a non-singular but non-unique state transformation matrix **P**. It is therefore concluded that the state variable model of an LTI system is non-unique. Which particular model will be used for a given system will depend upon the specific needs of the system analysis and design and may even be a matter of judgement. However, any two state variable models of a system are uniquely related.

We will now give the procedure for the formulation of two commonly used state variable models.

State Space Representation by Two Specific Types of State Variables

(a) *Physical Variables as State Variables* Physical variables of a system seem to be a natural choice as state variables, as these can be currently measured and manipulated. Already in the capacitor example of Fig. 1.15, we formulated the state model using the physical variables, of capacitor voltage (or charge). Also in the example advanced in Section 7.1 the position and velocity of the mass were selected as state variables. Here we will introduce another example to highlight the use of physical variables in form of a electric circuit as shown in Fig. 7.4.

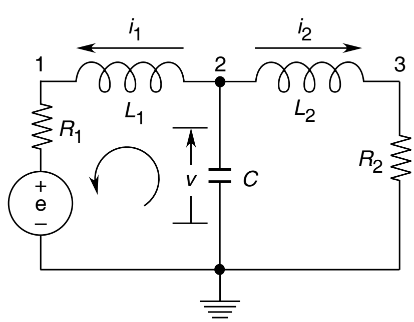


Fig. 7.4 *An RLC circuit*

This circuit is of the third order as it has three energy storage elements; two inductors and one capacitor. We choose the two inductor currents and one capacitor voltage as the state variables of the system. In terms of this choice, we have the following.

$$x_1(t) = v(t)$$

$$x_2(t) = i_1(t) \tag{i}$$
$$x_3(t) = i_2(t)$$

The differential equations governing the behaviour of the RLC network are found below:

At node 2

$$i_1 + i_2 + C\frac{\mathrm{d}v}{\mathrm{d}t} = 0$$

Round the left side mesh

$$L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + R_1 i_1 + e - v = 0$$

$$L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_2 i_2 - v = 0$$

We are interested in expressing the derivatives of the variables as linear combinations of the variables and input e. For this purpose, we rewrite these equations in the following form.

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{1}{C}i_1 - \frac{1}{C}i_2\\ \frac{\mathrm{d}i_1}{\mathrm{d}t} &= \frac{1}{L_1}v - \frac{R_1}{L_1}i_1 - \frac{1}{L_1}e\\ \frac{\mathrm{d}i_2}{\mathrm{d}t} &= \frac{1}{L_2}v - \frac{R_2}{L_2}i_2\end{aligned} \tag{ii}$$

In terms of the state variables defined above and the input $r(t) = e(t)$, we have the following state equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & -1/C & -1/C\\ 1/L_1 & -R/L_1 & 0\\ 1/L_2 & 0 & -R_2/L_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ -1/L_1\\ 0\end{bmatrix}e \tag{iii}$$

Assume that voltage across R_2 and current through R_1 are the output variables y_1 and y_2 respectively. The output equation is then given by

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & 0 & R_2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{iv}$$

This concludes the state-space representation of the RLC network. Equations (iii) and (iv) provide the state model of system.

(b) *Phase Variables as State Variables* These are not physical variables but variables defined by organizing the differential equation of an LTI system into a set of first-order differential equations. A general n^{th} order differential equation of a system is given in Eq. (1.7). We will illustrate the method by a simple case where no derivatives of the input are present. In this case we write the differential equations as

$$y^n + a_1 y^{(n-1)} + \ldots + a_{n-1}\, y + a_n\, y = b\, r \tag{7.15}$$

Define the state variables as

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\ldots\ldots \\ &\ldots\ldots \\ x_n &= y^{(n-1)} \end{aligned} \tag{7.16}$$

With these definitions, Eq. (7.15) is reduced to n first-order differential equations given below.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ \dot{x}_n &= -a_n x_1 - a_{n-1}\, x_2 - \ldots. - a_1 x_n + br \end{aligned}$$

The above equations result in the following state equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} r \tag{7.17}$$

It is easily observed from this equation that the **A** matrix can be written directly from the original differential equation (Eq. 7.15).

The output being x_1, the output equation is given by

$$y = \mathbf{c}'\,\mathbf{x} \tag{7.18}$$

where

$$\mathbf{c}' = [1\ 0\ \ldots.\ 0\ 0] = (1 \times n) \text{ transpose of vector}$$

It is easily observed that Eqs (7.16) and (7.17) have the form of general SISO equations (7.6a) and (7.6b), i.e.,

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\, r \tag{7.19a}$$

$$y = \mathbf{c}'\,\mathbf{x} \tag{7.19b}$$

It is seen that the term br is absent as no derivations of r are present in Eq. (7.15). The form of matrix **A** in this formulation of phase variables is known as the **Bush form or Companion form**.

The disadvantage of phase variables formulation is that these variables, in general, are not physical variables and are not directly available for measurement and control. In certain manipulations of the state variable form, they serve a useful purpose. Special computational algorithms are used to compute phase variables of a system and for online use, special hardware (programmable) is needed.

(c) *Canonical State Variables* In arriving at transformed state variables as in Eqs (7.14a) and (7.14b), if the transformation matrix is suitably selected, then

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}$$

can be brought into the diagonal form (or in general block diagonal form which is explained later in Section 7.4). The corresponding state variables

$$\dot{\tilde{\mathbf{x}}} = \mathrm{P}^{-1}\mathbf{x}$$

are known as canonical state variables or just canonical variables.

Discovering such a transformation and also other methods of getting into diagonal form will be elaborated in Sections 7.4 and 7.5.

7.3 TRANSFER FUNCTION FROM STATE VARIABLE MODEL AND VICE-VERSA

Apart from the internal state behaviour the state variable model yields the input-output behaviour as well. So it is possible to convert the state variable model to its transfer function form and vice-versa. We will now examine the methods of proceeding either way.

State Variable Form to Transfer Function

For simplicity and from practical application view point, we will consider the SISO case, whose state and output equations (Eq. (7.6a) and (7.6b) respectively are reproduced below.

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,r \tag{7.6a}$$

$$y = \mathbf{c}'\mathbf{x} + \mathrm{d}\,r \tag{7.6b}$$

Taking the Laplace transform of Eq. (7.6a), we get

$$\mathbf{s}\mathbf{X}(s) - \mathbf{x}(t_0) = \mathbf{A}\,\mathbf{X}(s) + \mathbf{b}\,R(s)$$

Assuming zero initial state, i.e., $\mathbf{x}(t_0) = \mathbf{0}$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{b}\,R(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\,R(s) \tag{7.20}$$

Now taking the Laplace transform of Eq. (7.6b)

$$Y(s) = \mathbf{c}'\mathbf{X}(s) + \mathrm{d}\, R(s)$$

Substituting $X(s)$ from Eq. (7.20), we get

$$Y(s) = [\mathbf{c}'\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} + \mathrm{d}]R(s) \tag{7.21}$$

From Eqs (7.20) and (7.21), we can write the transfer function as

$$T(s) = \frac{Y(s)}{R(s)} = [\mathbf{c}'(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} + d] \tag{7.22}$$

We can further rewrite this result as

$$T(s) = \frac{\mathbf{c}'\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{b}}{\det(s\mathbf{I} - \mathbf{A})} + d \tag{7.23}$$

With reference to Fig. 7.3 if there is no direct signal link from input to output (i.e., there is no feed forward), we then have

$$T(s) = \frac{\mathbf{c}'\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}}{\det(s\mathbf{I} - \mathbf{A})} \tag{7.24}$$

It is immediately observed from the transfer function of Eq. (7.24), that if its denominator is equated to zero, we then have

$$\det(s\mathbf{I} - \mathbf{A}) = 0; \textbf{ characteristic equation} \tag{7.25}$$

Also, observe from Eq. (7.23) that the characteristic equation would remain the same even if d is present. Of course it is already known from Chapter 2 that the factored form of the characteristic equation gives the system poles.

The transfer function is a **unique** model of a dynamic system, while the state variable model is **non-unique** (proved in Section 7.2). So irrespective of what state variable model of a system we start with, we will end up with the same transfer function.

In order to illustrate the use of Eq. (7.24) let us consider a simple second-order system described by the following differential equation.

$$\ddot{y} + a_1\dot{y} + a_2 = br \tag{i}$$

From Eqs (7.17) and (7.18) its phase variable formulation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} r \tag{ii}$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} [1 \;\; 0] \tag{iii}$$

The system transfer function is given by

$$T(s) = \frac{Y(s)}{R(s)} = [\mathbf{c}'\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}] \tag{iv}$$

Substituting values and first carrying out the inverse, we get

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ a_2 & s + a_1 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{Adj}(s\mathbf{I} - \mathbf{A})}{\det(\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} s + a_1 & 1 \\ a_2 & s \end{bmatrix}}{s^2 + a_1 s + a_2} \tag{v}$$

The characteristic equation of this system is

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

or

$$s^2 + a_1 s + a_2 = 0 \tag{vi}$$

The transfer function is found below.

$$T(s) = \frac{[1 \quad 0]\begin{bmatrix} s + a_1 & 1 \\ a_2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ b \end{bmatrix}}{s^2 + a_1 s + a_2}$$

$$= \frac{b}{s^2 + a_1 s + a_2} \tag{vii}$$

This result can be directly verified by taking the Laplace transform of Eq. (i).

Transfer Function to State Variable Form

(a) *Phase Variable Formulation* Let us consider the following transfer function.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \tag{7.26}$$

The above equation may be written as

$$T(s) = \frac{Y(s)}{R(s)} = \frac{X_1(s)}{R(s)} \cdot \frac{Y(s)}{X_1(s)} \tag{7.27}$$

where

$$\frac{X_1(s)}{R(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 + a_3} \tag{7.28a}$$

and

$$\frac{Y(s)}{X_1(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3 \tag{7.28b}$$

Taking the inverse Laplace transform of Eq.(7.28a), we get

$$\dddot{x}_1 + a_1 \ddot{x}_1 + a_2 \dot{x}_1 + a_3 x_1 = r \tag{7.29}$$

Let

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 = \ddot{x} \qquad (7.30)$$

then

$$\dot{x}_3 = \dddot{x}$$

Substituting results of Eq. (7.30) in Eq. (7.29), we get

$$\dot{x}_3 + a_1 x_3 + a_2 x_2 + a_3 x_1 = r$$

or

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + r \qquad (7.31)$$

The state equations (7.30) and (7.31) can be written in vector matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (7.32)$$

From Eqs. (7.28b) and (7.31), we write

$$y = b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_2 \dot{x}_1 + b_3 x_1$$
$$= b_0(-a_3 x_1 - a_2 x_2 - a_1 x_3 + r) + b_1 x_3 + b_2 x_2 + b_3 x_1$$

or

$$y = [(b_3 - a_3 b_0)\ (b_2 - a_2 b_0)\ (b_1 - a_1 b_0)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 r \qquad (7.33)$$

Equations (7.32) and (7.33) together are the phase variable state model of the transfer function of Eq. (7.26). It may be easily observed that these equations can be written directly from the transfer function.

The signal flow graph corresponding to Eqs (7.32) and (7.33) is drawn in Fig. 7.5.

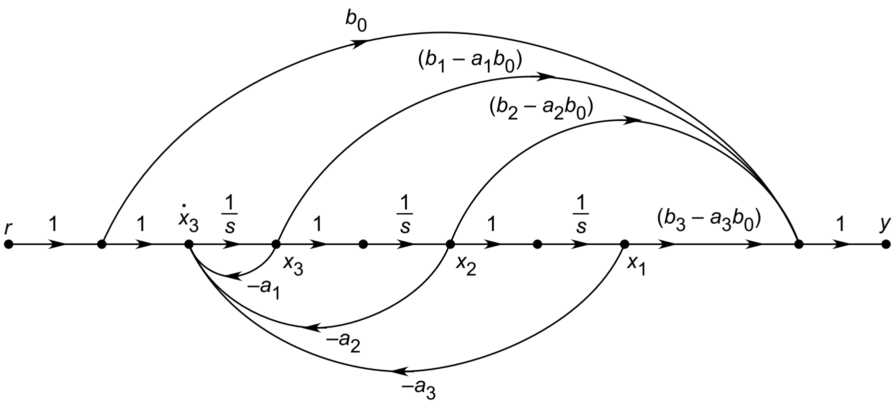


Fig. 7.5

It is observed from the state equation (7.32) that the matrix A is solely determined by the coefficient of the denominator of the transfer function, i.e., its poles. The effect of zeros appears in the output Eq. (7.33).

Let us arrive at the phase variable formulation of the transfer function of Eq. (7.26) by using Mason's gain formula. For this purpose we rewrite the transfer function in form of integrators (1/s). Thus,

$$T(s) = \frac{y(s)}{R(s)} = \frac{b_0 + b_1/s + b_2/s^2 + b_3/s^3}{1-(-a_1/s - a_2/s^2 - a_3/s^3)} \tag{7.34}$$

We identify three state variables x_1, x_2 and x_3 and draw the signal flow graph of Fig. 7.6 using the Mason's gain formula (Eq. 6.43) which is reproduced below

$$T(s) = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

where Δ, P_k and Δ_k are defined in the referred equation.

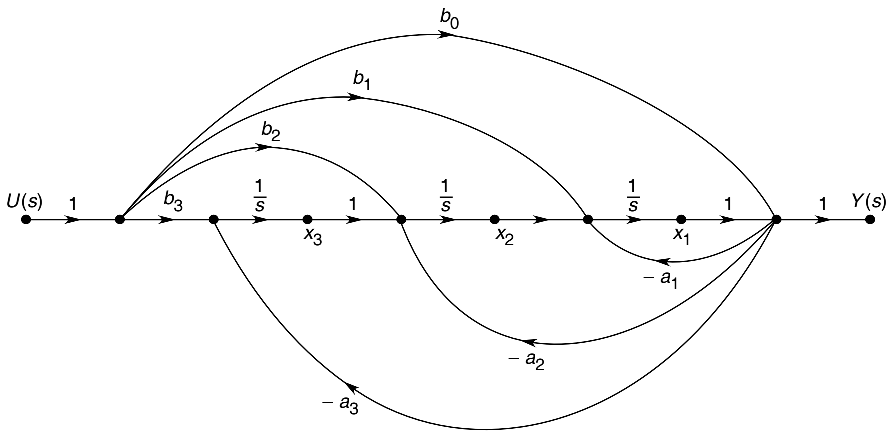


Fig. 7.6 *Signal flow graph of T(s)*

Examining Eq. (7.34), we observe that its signal flow graph consists of

(i) three feedback loops (touching each other) with gains $-a_i/s$, $-a_2/s^2$ and $-a_3/s^3$

(ii) Four forward paths which touch the loops and have gains b_0, b_1/s, b_2/s^2 and b_3/s^3

A signal flow graph configuration which satisfies the above requirements is shown in Fig. 7.6. From this figure, we have

$$y = x_1 + b_0 r$$

$$\dot{x}_1 = -a_1 x_1 + x_2 + b_1 r$$

$$\dot{x}_2 = -a_2 x_1 + x_3 + b_2 r$$
$$\dot{x}_3 = -a_3 x_1 + b_3$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} r \tag{7.35a}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 r \tag{7.35b}$$

It is observed from the state variable model of Eqs (7.35a) and (7.35b) that the matrix **A** has a column derived from the coefficients of the denominator of $T(s)$ but no coefficient from its numerator. Therefore, the matrix **A** though of different form from Eq. (7.32) has in it the information on system poles. This issue will be elaborated in Section 7.4 on Diagonalization. It may also be noticed here that input r enters the state equation through combinations of the coefficients of the numerator of $T(s)$.

Because of the non-uniqueness of the state variable model, system behaviour may be represented by different state variable models. Any two models are also uniquely related. Thus, for a given system a suitable state variable model can be determined as per requirements and needs of the design engineer.

Example 7.1 *Obtain the state variable model in phase variable form for the transfer function.*

$$T(s) = \frac{Y(s)}{R(s)} = \frac{s+3}{s^3 + 5s^2 + 8s + 3}$$

Solution

Let

$$\frac{X_1(s)}{R(s)} = \frac{1}{s^3 + 5s^2 + 8s + 4} \tag{i}$$

and

$$\frac{Y(s)}{X_1(s)} = s + 3 \tag{ii}$$

From Eqs (i) and (ii) the state variable form and output equation can be directly written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \tag{iii}$$

and the output equation can be written in the following way.

$$Y(s) = sX_1(s) + 3X_1(s)$$

Taking the Laplace inverse

$$\begin{aligned} y &= \dot{x}_1 + 3x_1 \\ &= x_2 + 3x_1 \end{aligned}$$

or

$$y = [3 \quad 1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{iv}$$

(b) *Canonical Form of State Variable Model* As already stated in Section 7.2 that in canonical state variable model, the matrix **A** has diagonal or block diagonal form. We shall now present one method of formulating this model where the denominator can be brought into factored form, i.e., its poles are known.

Consider the systems described by the following transfer function.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}; \; m, \leq n \tag{7.36}$$

Two cases identified are

* district (nonrepeated) poles only
* there could be some repeated poles

Denominator with distinct poles Equation (7.36) can be written in factored forms as

$$\frac{Y(s)}{R(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n} \tag{7.37}$$

From these factors, we can identify the state variable in s-domain as

$$X_i(s) = \frac{R(s)}{s - p_i} \quad i = 1, 2...n \tag{7.38}$$

or

$$s\text{X}_i(s) - p_i\text{X}_i(s) = R(s) \tag{7.39}$$

or

$$sX_i(s) = p_iX_i(s) + R(s)$$

From this equation we can write the time domain state variable formulation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \cdots 0 \\ 0 & p_2 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \cdots \vdots \\ 0 & 0 & 0 \quad p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} r \tag{7.40}$$

From Eqs (7.39) and (7.40), we can write the output as

$$Y(s) = K_1X_1(s) + K_2X_2(s) + ... + K_n\,X_n(s) \tag{7.41}$$

The time domain form of Eq. (7.41) is given below.

$$y = [K_1\ K_2\ ....\ K_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{7.42}$$

It is seen from Eq. (7.40) that matrix **A** in this state variable formulation is a diagonal matrix. This means that each state equation is a first-order differential equation decoupled from other state variables, whose time domain solution is easily carried out. This formulation is known as canonical form.

Repeated Poles Also Take the simple case of a transfer function with a single pole of multiplicity three. The output can then be expressed as

$$Y(s) = \frac{K\,R(s)}{(s-p_i)^3}$$

or

$$Y(s) = \frac{K_u\,R(s)}{(s-p_i)^3} + \frac{K_v R(s)}{(s-p_i)^2} + \frac{K_w R(s)}{(s-p_i)} \tag{7.43}$$

Let

$$X_{i1}(s) = \frac{R(s)}{(s-p_i)^3} = \frac{X_{i2}(s)}{(s-p_i)} \tag{7.44a}$$

$$X_{i2}(s) = \frac{R(s)}{(s-p_i)^2} = \frac{X_{i3}(s)}{(s-p_i)} \tag{7.44b}$$

$$X_{i3}(s) = \frac{R(s)}{(s-p_i)} \tag{7.44c}$$

Taking the inverse transform of Eqs (7.44a), (7.44b) and (7.44c) progressively, we can write the state variable formulation as

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \underset{\mathbf{A}}{\begin{bmatrix} p_i & 1 & 0 \\ 0 & p_i & 1 \\ 0 & 0 & p_i \end{bmatrix}}\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} + \underset{\mathbf{b}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} r \tag{7.45}$$

The output from Eq. (7.43) is now given by

$$y = [K_u\ \ K_v\ \ K_w]\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} \tag{7.46}$$

It is observed from Eq. (7.45) that while **A** is not diagonal but still is in its simplest possible form called **Jordan cannonical** form, where each of the first two state variables is coupled to another state variable, while the third state variable is decoupled.

In case of a transfer function with distinct as well as repeated poles, the Jordan block will appear corresponding to the repeated pole and the rest of the matrix will have diagonal form. For example, if the transfer function in the above example also has two distinct poles, the matrix **A** will have the following form.

Jordan block

$$\begin{bmatrix} p_{i1} & 1 & 0 & 0 & 0 \\ 0 & p_{i2} & 1 & 0 & 0 \\ 0 & 0 & p_{i3} & 0 & 0 \\ 0 & 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & 0 & p_5 \end{bmatrix} \tag{7.47}$$

There are other powerful methods of converting any state variable form to the diagonalized form. Also, we have not considered here cannonical variable formulation.

In the previous section it has been shown that the system behaviour may be represented by different state variable models. This concept is demonstrated by obtaining the diagonal form of the Example 7.1.

Example 7.2

$$T(s) = \frac{Y(s)}{R(s)} = \frac{(s+3)}{s^3 + 5s^2 + 8s + 4} = \frac{(s+3)}{(s+2)^2\,(s+1)} \tag{i}$$

As the transfer function T(s) is in factorized form, we can write it in partial fraction form as

$$\frac{Y(s)}{R(s)} = \frac{-1}{(s+2)^2} + \frac{-2}{s+2} + \frac{2}{(s+1)} \tag{ii}$$

Define

$$X_1(s) = \frac{R(s)}{(s+2)^2} = \frac{X_2(s)}{(s+2)} \tag{iii}$$

$$X_2(s) = \frac{R(s)}{(s+1)} \tag{iv}$$

$$X_3(s) = \frac{R(s)}{(s+1)} \tag{v}$$

Solution Taking the inverse Laplace transform, the state variable model in the form of Jordan block is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} r \tag{vi}$$

and the system response is

$$Y(s) = -X_1(s) - 2X_2(s) + 2X_1(s) \tag{vii}$$

or

$$y = [-1 \;\; -2 \;\; +2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{viii}$$

Equations (vi) and (viii) constitute the Jordan cannonical state model.

Examples 7.1 and 7.2 clearly demonstrate the non-uniqueness of the state variable model. This is accomplished as two different models have been constructed in these examples for the same transfer function.

The cannonical state variable model has been obtained by partial fractioning the transfer function. This is a cumbersome process particularly for higher order systems. Cannonical form can be obtained by a superior method which is presented now.

7.4 DIAGONALIZATION

Consider an n^{th} order MIMO state model

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{r} \tag{7.48a}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{r} \tag{7.48b}$$

Let us define a new state vector $\mathbf{z}$ such that

$$\mathbf{x} = \mathrm{M}\,\mathbf{z} \tag{7.49}$$

where the transformation matrix $\mathbf{M}$ is non-singular, i.e.,

$$|\mathbf{M}| \neq 0$$

From Eqs (7.48) and (7.49), we get

$$\dot{\mathbf{z}} = \mathbf{M}^{-1}\mathbf{A}\,\mathbf{M}\,\mathbf{z} + \mathbf{M}^{-1}\,\mathbf{B}\,\mathbf{r} \tag{7.50a}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{M}\,\mathbf{z} + \mathbf{D}\,\mathbf{r} \tag{7.50b}$$

The selection of $\mathbf{M}$ is such that

$$\mathbf{M}^{-1}\,\mathbf{A}\,\mathbf{M} = \text{diagonal matrix} = \Lambda$$

Let

$$\tilde{\mathbf{B}} = \mathbf{M}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{M}$$

Thus, Eqs (7.50a) and (7.50b) may be written as

$$\dot{\mathbf{z}} = \Lambda\,\mathbf{z} + \tilde{\mathbf{B}}\,\mathbf{r} \tag{7.51a}$$

$$\mathbf{y} = \tilde{\mathbf{C}}\,\mathbf{z} + \mathbf{D}\,\mathbf{r} \tag{7.51b}$$

Equations (7.51a) and (7.51b) give a different set of state variables. The transformation matrix **M** is called the **modal** matrix. We present now the method of finding the modal matrix for a given **A**.

Eigenvalues and Eigenvectors

Consider the following equation

$$\mathbf{A}\,\mathbf{x} = \mathbf{y} \tag{7.52}$$

wherein the matrix **A** transforms the $(n \times 1)$ vector **x** into $(n \times 1)$ vector **y**.

We pose overselves a question. Does there exist an **x** such that $\mathbf{y} = \lambda\ \mathbf{x}$ (λ is a constant), i.e. is the transformed vector **y** in the same direction as **x**? Such a vector is the solution of the equation.

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x} \tag{7.53}$$

or

$$(\lambda\,\mathbf{I} - \mathbf{A})\,\mathbf{x} = 0 \tag{7.54}$$

The set of homogeneous Eqs. (7.54) will have a non-trivial solution, if and only if

$$|\lambda\,\mathrm{I} - \mathbf{A}| = 0 \tag{7.55}$$

This equation in expanded form is the following polynomial.

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0 \tag{7.56}$$

The values of λ which satisfy this equation are called **eigenvalues** of the matrix **A** and the equation itself is called the **characteristic equation** corresponding to matrix **A**.

Now for $\lambda = \lambda_{\mathbf{i}}$ satisfying Eq. (7.55), we have the following equation from Eq. (7.54).

$$(\lambda_{\mathbf{i}\bar{\mathbf{I}}} - \mathbf{A})\,\mathbf{x} = 0 \tag{7.57}$$

The solution $\mathbf{x} = \mathbf{m}_i$ of Eq. (7.57) is called on **eigenvector** associated with the eigenvalue $\lambda_{\mathbf{i}}$. If the eigenvalues of **A** are all distinct (non-repeated), the rank of matrix $(\lambda\mathbf{I} - \mathbf{A})$ is $(n - 1)$. If the rank is less then $(n - 1)$, then there are some repeated eigenvalues.

Modal Matrix

The eigenvector corresponding to a particular eigenvalue is obtained by taking the cofactors of any row of $(\lambda\mathbf{I} - \mathbf{A})$, i.e.,

$$m_i = \begin{bmatrix} C_{k_1} \\ C_{k_2} \\ \vdots \\ C_{k_n} \end{bmatrix};\ k = 1 \text{ or } 2 \text{ or } \ldots, \text{ or } n \tag{7.58}$$

where C_{k_j} are the cofactors of any particular row.

Consider a matrix **A** with all distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_n}$ be the corresponding eigenvectors. Now form a matrix **M**, called **modal matrix**, by placing together these eigenvectors. Thus

$$\mathbf{M} = [\mathbf{m_1}\ \mathbf{m_2} \ldots \mathbf{m_n}] \tag{7.59}$$

Then

$$\begin{aligned}\mathbf{A\,M} &= [\mathbf{Am_1}\ \mathbf{Am_2} \ldots \mathbf{Am_n}] \\ &= [\lambda_1\mathbf{m_1}\ \lambda_2\mathbf{m_2} \ldots \lambda_n\mathbf{m_n}] \\ &= \mathbf{M}\Lambda\end{aligned}$$

where

$$\begin{aligned}\Lambda &= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= \mathbf{M}^{-1}\mathbf{A\,M}\end{aligned} \tag{7.60}$$

According to this transformation (Eq. (7.60)), the modal matrix **M** is the diagonalizing matrix. It also follows that both **A** and Λ have the same eigenvalues.

If A has distinct eigenvalues and is expressed in the following phase variable form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & & \cdots & 0 \\ \vdots & & & \cdots & \vdots \\ 0 & & & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{7.61}$$

then the modal matrix can be shown to have the following special form.

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \tag{7.62}$$

This matrix is called the **Vander Monde matrix.**

Example 7.3 Consider a matrix **A** given below.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Corresponding to this matrix, the characteristic equation is

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ -3 & \lambda & -2 \\ 12 & 7 & \lambda + 6 \end{vmatrix} = 0$$

or

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

Therefore the eigenvalues of matrix **A** are

$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$$

The eigenvector $\mathbf{m_1}$ associated with $\lambda_1 = -1$ is obtained from cofactors of the matrix

$$(\lambda_1 \mathbf{I} - \mathbf{A}) = \begin{bmatrix} -1 & -1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

The result is the following, which means the eigenvector has unique direction.

$$\mathbf{m_1} = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix} \quad \text{or} \quad \mathbf{m_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

The eigenvector m_1 could also be obtained by solving the following homogeneous equations.

$$\begin{aligned} -x_1 - x_2 &= 0 \\ -3x_1 - x_2 - 2x_3 &= 0 \\ 12x_1 + 7x_2 + 5x_3 &= 0 \end{aligned}$$

Choosing $x_1 = 1$, we get $x_2 = -1$, $x_3 = -1$, which is the same result as obtained by the method of cofactors. Solution of homogeneous equations is computationally more efficient than obtaining cofactors of a row when the dimension of **A** is large.

Similarly, eigenvectors $\mathbf{m_1}$ and $\mathbf{m_2}$ associated with $\lambda_2 = -2$ and $\lambda_3 = -3$ respectively are

$$\mathbf{m_2} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{m_3} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

The modal matrix **M** obtained by placing the eigenvectors (columns) together is given by

$$\mathbf{M} = [\mathbf{m_1} \ \vdots\ \mathbf{m_2} \ \vdots\ \mathbf{m_3}] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

From which we find

$$\mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} 9 & 5 & 2 \\ -6 & -4 & -2 \\ 5 & 3 & 2 \end{bmatrix}$$

Therefore

$$\Lambda = \mathbf{M}^{-1}\mathbf{AM} = \frac{1}{2}\begin{bmatrix} 9 & 5 & 2 \\ -6 & -4 & -2 \\ 5 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

which is a diagonal matrix with eigenvalues of **A** as its diagonal elements. In fact, Λ could be written directly without the need to compute $\mathbf{M}^{-1}\mathbf{AM}$.

Generalized Eigenvectors

Let us now consider a more general case wherein A can have repeated eigenvalues. For example consider the following matrix.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -21 & -8 \end{bmatrix}$$

A has eigenvalues $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = -3$. For λ_1

$$(\lambda_1\mathbf{I} - \mathbf{A}) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & -1 \\ 18 & 21 & 6 \end{bmatrix}$$

Taking the cofactors of the first row, we get

$$\mathbf{m_1} = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ 36 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Here we have simplified by dividing all the elements of $\mathbf{m_1}$ by 9. This is valid, as an eigenvector is non-unique within a multiplicative constant.

$$(\lambda_2\mathbf{I} - \mathbf{A}) = \begin{bmatrix} \lambda_2 & -1 & 0 \\ 0 & \lambda_2 & -1 \\ 18 & 21 & \lambda_2 + 8 \end{bmatrix}$$

The only independent eigenvector corresponding to λ_2 is given by

$$\mathbf{m_2} = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} \lambda_2(\lambda_2+8)+21 \\ -18 \\ -18\lambda_2 \end{bmatrix}_{\lambda_2=-3} = \begin{bmatrix} 6 \\ -18 \\ 54 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

Now $\mathbf{m_3}$ can be generated from the independent eigenvector $\mathbf{m_2}$ in the following manner.

$$\mathbf{m_3} = \begin{bmatrix} \dfrac{d}{d\lambda_2}C_{11} \\ \dfrac{d}{d\lambda_2}C_{12} \\ \dfrac{d}{d\lambda_2}C_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_2+8 \\ 0 \\ -18 \end{bmatrix}_{\lambda_2=-3} = \begin{bmatrix} 2 \\ 0 \\ -18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$$

The vector $\mathbf{m_3}$ is a generalized eigenvector. The modal matrix $\mathbf{M}$ is then given by

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ 4 & 9 & -9 \end{bmatrix}$$

The modal matrix $\mathbf{M}$ transforms $\mathbf{A}$ into the Jordan matrix.

$$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Jordan block

The above result could be generalized, if an eigenvalue λ_i has multiplicity q and the rank of $(\lambda_i \mathbf{I} - \mathbf{A})$ is $(n-1)$.

The matrix $\mathbf{M}^{-1}\mathbf{AM}$ will then have a $(q \times q)$ Jordan block of the following form.

$$q \times q \text{ Jordan block} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \vdots \\ q \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & \cdots & \cdots & q \end{array} \\ \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix} \end{array} \quad (7.64)$$

Complex Eigenvalues

In case there are complex conjugate eigenvalues (these always occur in pair as the matrix **A** has real entries only), the modal matrix leads to the following matrix for two real eigenvalues and one pair of complex conjugate eigenvalues.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha + j\beta \end{bmatrix} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} \tag{7.65}$$

This form as such is not much useful in practice but can be further transformed into a real matrix by the following similarity transformation.

$$\hat{\mathbf{M}}\mathbf{\Lambda}\hat{\mathbf{M}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & j & -j \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} \tag{7.66}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.5j \\ 0 & 0 & 0.5j \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix} = \overline{\mathbf{\Lambda}}$$

Companion Form Matrices The following matrices

$$\begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix} \qquad \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} \tag{7.67}$$

and their transposes

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \qquad \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} \tag{7.67b}$$

all have the following characteristic polynomial.

$$q(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

These matrices can easily be formed from the coefficients of $q(\lambda)$ and are called companion form matrices. The companion form matrices will be repeatedly dealt with later. The matrices in Eqs (7.17) and (7.35a) are in such a form.

7.5 EQUIVALENT STATE EQUATIONS

In Section 7.2, while establishing non-uniqueness of the state variable model, we have used the following non-singular transformation.

$$\mathbf{x} = \mathbf{P}\tilde{\mathbf{x}}$$

As a result the state equation given in Eqs (7.5a) and (7.5b) transforms to the following form.

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{r} \tag{7.68}$$

$$\mathbf{y} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \tilde{\mathbf{D}}\mathbf{r} \tag{7.69}$$

where

$$\tilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP},\ \tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B},\ \tilde{\mathbf{C}} = \mathbf{CP},\ \tilde{\mathbf{D}} = \mathbf{D}$$

The form of state Eq. (7.68) and (7.69) are said to be **equivalent algebraically** to state equations (7.5a) and (7.5b). Observe that the matrix **D**, called the **direct transmission part** between input and output does not undergo any change, i.e., it is not affected by equivalent transformation.

Let us now examine the characteristic equation of two equivalent forms. For the transformed form

$$\begin{aligned}\tilde{q}(\lambda) &= \det(\lambda\mathbf{I} - \tilde{\mathbf{A}}) = \det(\lambda\mathbf{PP}^{-1} - \mathbf{PAP}^{-1}) \\ &= \det(\mathbf{P})\det(\lambda\mathbf{I} - \mathbf{A})\det(\mathbf{P}^{-1}) \\ &= \det(\lambda\mathbf{I} - \mathbf{A}) = q(\lambda)\end{aligned} \tag{7.70}$$

Therefore, the equivalent forms have the same characteristic equation and hence same eigenvalues.

Consider now the transfer function matrix of transformed Eq. (7.67b) and (7.68). It is a matrix as we are now dealing with state equations of a MIMO system. By appropriate changes in Eq. (7.21) this can be easily written down as

$$\begin{aligned}\tilde{\mathbf{T}}(s) &= \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}} \\ &= \mathbf{CP}[\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{P}]\mathbf{P}^{-1}\mathbf{B} + \mathbf{D} \\ &= \mathbf{CPP}^{-1}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{PP}^{-1}\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\end{aligned} \tag{7.71}$$

Thus, the equivalent forms have the same transfer matrix. This means the uniqueness of the transfer matrix (or transfer function in SISO case).

The above results can be summarized as follows.

Any two set of equivalent state equations have

1. the same characteristic equation (and so same eigenvalues)
2. the same transfer matrix (or transfer function)
3. they are also **zero-state equivalence** (not proved here)

7.6 SOLUTION OF LTI STATE EQUATIONS

Consider the following linear time-invariant state-space equations.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{r}\,(t) \tag{7.72}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{r}(t) \tag{7.73}$$

We first define a matrix exponential function like a scalar exponential as

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2}{2!}t^2 + \ldots + \frac{\lambda^n}{n!}t^n + \ldots. \tag{7.74}$$

The corresponding matrix exponential is

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \ldots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k \tag{7.75}$$

where

$$\mathbf{A}^k = \mathbf{A}\,\mathbf{A}\,\mathbf{A}\,\ldots\,\mathbf{A}\ (k\text{-times})$$

Consider now the derivative of $e^{\mathbf{A}t}$ of Eq. (7.75).

$$\frac{d}{\mathrm{d}t}e^{\mathbf{A}t} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}t^{k-1}\mathbf{A}^k$$

$$= \mathbf{A}\left(\sum_{k=0}^{\infty} \frac{t^k}{(k-1)!}\mathbf{A}^k\right) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k\right)\mathbf{A}$$

We thus have

$$\frac{d}{\mathrm{d}t}e^{\mathbf{A}t} = \mathbf{A}\,e^{\mathbf{A}t} \tag{7.76}$$

Now to get the solution, premultiply on both sides of Eq. (7.72) by $e^{-\mathbf{A}t}$ which yields

$$e^{-\mathbf{A}t}\,\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\ \mathbf{A}\ \mathbf{x}(t) = e^{-\mathbf{A}t}\ \mathbf{B}\,\mathbf{r}(t)$$

or

$$e^{-\mathbf{A}t}\,[\dot{\mathbf{x}}(t) - \mathbf{A}\ \mathbf{x}(t)] = \frac{d}{\mathrm{d}t}\,[e^{-\mathbf{A}t}\ \mathbf{x}(t)] = e^{-\mathbf{A}t}\ \mathbf{B}\,\mathbf{r}(t)$$

Integrating w.r. t t from 0 to t, we get

$$e^{-\mathbf{A}t}\,\mathbf{x}(t)\Big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau$$

We thus get

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau$$

Multiplying both sides by $e^{\mathbf{A}t}$ and reorganizing yields

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau \tag{7.77}$$

Substituting this result in Eq. (7.73) gives (assume D to be zero)

$$\mathbf{y}(t) = \underbrace{\mathbf{C}\mathbf{e}^{\mathbf{A}t}\,\mathbf{x}(0)}_{\text{Zero-input response}} + \underbrace{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau}_{\text{Zero-state response}} \tag{7.78}$$

If the initial state is at $t = t_0$, Eqs (7.77) and (7.78) can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\,\mathbf{x}(t_0) + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau \tag{7.79}$$

and

$$y(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\,\mathbf{x}(t_0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{r}(\tau)\mathrm{d}\tau \tag{7.80}$$

Consider now only the zero-input response. Then from Eqs (7.77) and (7.79)

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}(0) \tag{7.81}$$

Also

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\,\mathbf{x}(0) \tag{7.82}$$

According to these equations the property of $e^{\mathbf{A}t}$ (of the system) is to take the system from any state $\mathbf{x}(0)$ or $\mathbf{x}(t-t_0)$ to state at t, i.e., $\mathbf{x}(t)$. Therefore, it is known as the state transition matrix and is denoted by the following symbol

$$\Phi(t) = e^{\mathbf{A}t} = \textbf{state transition matrix} \tag{7.83}$$

Also

$$\Phi(t-t_0) = e^{\mathbf{A}(t-t_0)} \tag{7.84}$$

The solution of state equations (Eqs (7.77) and (7.78) or (7.79) and (7.80)) can be rewritten in terms of this symbol. Being obvious there is no need to rewrite these.

It has been shown above that once we know the state transition matrix ($e^{\mathbf{A}t}$), we can obtain the solution of state equations using the results given in Eqs (7.77) and (7.78). Before presenting the method of solving the state transition method some of its properties are given below.

$$e^0 = \mathbf{I} \tag{7.85}$$

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}\,e^{\mathbf{A}t_2} \tag{7.86}$$

$$[e^{\mathbf{A}t}]^{-1} = e^{-\mathbf{A}t} \tag{7.87}$$

Various methods of computing the state transition matrix (in closed form or numerically) are as follows.

1. Taylor series expansion
2. By use of Cayley–Hamilton Theorem
3. By canonical transformation
4. By inverse Laplace transform (Section 7.7)

These methods are now taken up in this order.

Computation by Taylor Series Expansion

For scalar λ the Taylor series of exponential function is given as

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!} + \ldots\ldots \tag{7.88}$$

It converges for all finite λ and t. On parallel line we write the expansion for state transition matrix as

$$e^{\mathbf{A}t} = \mathbf{I} + t\,\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k\, \mathbf{A}^k \tag{7.89}$$

This series involves only matrix multiplications and additions and may converge rapidly. Therefore it is suitable for computer computation.

Example 7.4 *Find the state transition matrix for*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \tag{i}$$

Solution It can easily be derived that

$$\mathbf{A}^2 = \begin{bmatrix} -6 & -5 \\ 30 & 19 \end{bmatrix}; \mathbf{A}^3 = \begin{bmatrix} 30 & 19 \\ -114 & -65 \end{bmatrix}; \mathbf{A}^4 = \begin{bmatrix} -114 & -65 \\ 390 & 211 \end{bmatrix}$$

Therefore,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} t + \begin{bmatrix} -6 & -5 \\ 30 & 19 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 30 & 19 \\ -114 & -65 \end{bmatrix} \frac{t^3}{6} + \ldots \tag{ii}$$

$$e^{-\mathbf{A}t} = \begin{bmatrix} 1 - \dfrac{6t^2}{2!} + \dfrac{30t^3}{3!} - \dfrac{114t^4}{4!} + \ldots & t - \dfrac{5t^2}{2!} + \dfrac{19t^3}{3!} - \dfrac{65t^4}{4!} + \ldots \\ -6t + \dfrac{30t^2}{2!} - \dfrac{114t^3}{3!} + \dfrac{390t^4}{4!} + \ldots & 1 - 5t + \dfrac{19t^2}{2!} - \dfrac{65t^3}{3!} + \dfrac{211t^4}{4!} + \ldots \end{bmatrix} \tag{iii}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 63^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \tag{iv}$$

The form of Eq. (iv) is not easy to recognize. Numerical computation can be carried by MATLAB. Using a MATLAB program, it is shown here that the series does converge, as it should in case of the closed form result of Eq. (iv). To check convergence, we use the matrix form defined as

$$\|A\|_1 = \max_j \left(\sum_{i=1}^{m} |a_{ij}| \right) = \text{largest column absolute sum}$$

The results as obtained by MATLAB for $t = 1$ are plotted in Fig. 7.7. The norm peaks at 5^{th} term and goes down steeply ending with practically zero value at 10^{th} term.

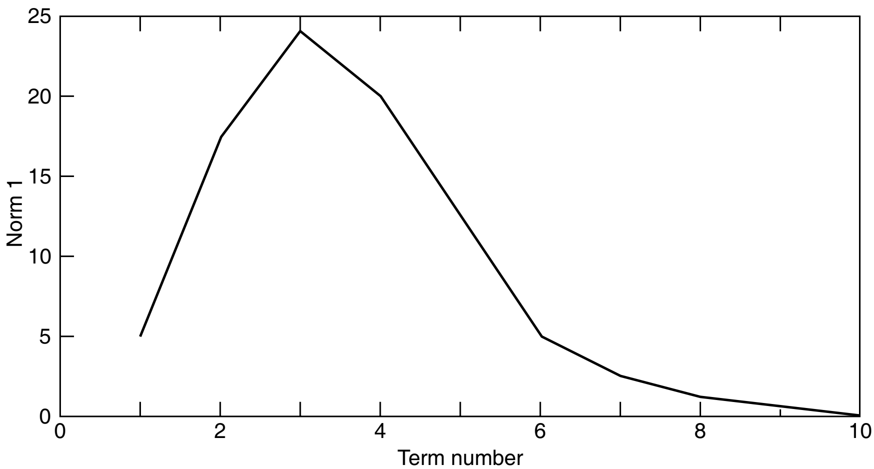


Fig. 7.7 *Convergence of Taylor series of e^{At}*

Example 7.5 *Obtain the response of the states of the following system.*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t) \qquad (i)$$

where $r(t)$ is unit step occurring at $t = 0$ and $x_0^T = [1 \ \ 0]$.

Solution The state transition matrix is given by

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \ldots$$

Substituting the values of $\mathbf{A}^n$ for $n = 1, 2, 3, \ldots$, we get

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \ldots & 0 \\ 0 & 1 - 2t + \frac{4}{2}t^2 - \frac{8}{6}t^3 + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \qquad \text{(ii)}$$

The time response of the system is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[\mathbf{x}_0 + \int_0^t e^{-A\tau}\,\mathbf{B}\tau(\tau)\,\mathrm{d}\tau\right] \tag{iii}$$

It is given that $r = 1$, $t \geq 0$. So,

$$e^{-\mathbf{A}\tau}\mathbf{B}r = \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{2\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\tau} \\ e^{2\tau} \end{bmatrix}$$

Therefore,

$$\int_0^t e^{-\mathbf{A}\tau}\,\mathrm{Brd\tau} = \begin{bmatrix} e^t - 1 \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix} \tag{iv}$$

Substituting Eqs (ii) and (iv) in Eq. (iii), we get

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} e^t - 1 \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{2}(1 - e^{-2t}) \end{bmatrix}$$

Computation by the Technique Based on the Cayley–Hamilton Theorem

The state transition matrix may be computed using the technique based on the **Cayley–Hamilton theorem**. For large systems, this method is far more convenient computationally as compared to the other methods. To begin with, let us state the Cayley–Hamilton theorem.

Every square matrix **A** satisfies its own characteristic equation. In other words, if

$$q(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \tag{7.90}$$

is the characteristics equation of **A**, then

$$q(\mathbf{A}) = \mathbf{A}^n + a_1\,\mathbf{A}^{n-1} + \ldots + a_{n-1}\,\mathbf{A} + a_n\,\mathbf{I} = \mathbf{0} \tag{7.91}$$

This equation implies that $\mathbf{A}^n$ can be written as linear combination of powers of **A**. Multiplying Eq. (7.91) by **A** yields

$$\mathbf{A}^{n+1} + a_1\,\mathbf{A}^n + a_2\,\mathbf{A}^{n-1} + \ldots + a_{n-1}\,\mathbf{A}^2 + a_0\,\mathbf{A} = \mathbf{0} \tag{7.92}$$

which implies that $\mathbf{A}^{n+1}$ can be written as a linear combination of $\{\mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^n\}$. Proceeding forward, we conclude that for any polynomial $f(\lambda)$, no matter how large its degree, $f(\mathbf{A})$ can always be written as

$$f(\mathbf{A}) = k_0\mathbf{I} + k_1\,\mathbf{A} + \ldots + k_{n-1}\,\mathbf{A}^{n-1} \tag{7.93}$$

We will now present the computational procedure. Given an $n \times n$ matrix **A** with characteristic equation of the type of Eq. (7.90) and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Consider now a matrix polynomial

$$f(\mathbf{A}) = k_0\mathbf{I} + k_1\,\mathbf{A} + k_1\,\mathbf{A}^2 + \ldots + k_n\,\mathbf{A}^n + k_{n+1}\,\mathbf{A}^{n+1} \tag{7.94}$$

It can be computed by considering the corresponding scaler polynomial

$$f(\lambda) = k_0 + k_1\lambda + k_2\lambda^2 + \ldots + k_n\lambda^n + k_{n+1}\,\lambda^{n+1} \tag{7.95}$$

If $f(\lambda)$ is divided by the characteristic polynomial $q(\lambda)$, then we have

$$\frac{f(\lambda)}{q(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{q(\lambda)}$$

or

$$f(\lambda) = \mathrm{Q}(\lambda)q(\lambda) + R(\lambda) \tag{7.96}$$

where $R(\lambda)$ is the remainder polynomial of the following form:

$$R(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \ldots + \alpha_{n-1}\,\lambda^{n-1} \tag{7.97}$$

If we evaluate $f(\lambda)$ at the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; then $q(\lambda) = 0$ and we have

$$f(\lambda_i) = R(\lambda_i) \quad \text{where} \quad i = 1, 2, \ldots, n \tag{7.98}$$

The coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, can be obtained by successively substituting $\lambda_1, \lambda_2, \ldots, \lambda_n$ in Eq. (7.98).

Substituting **A** for the variable λ in Eq. (7.96), we get

$$f(\mathbf{A}) = Q(\mathbf{A})\,q(\mathbf{A}) + R(\mathbf{A})$$

Since $q(\mathbf{A})$ is zero, it follows that

$$\begin{aligned} f(\mathbf{A}) &= R(\mathbf{A}) \\ &= \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \ldots + \alpha_{n-1}\,\mathbf{A}^{n-1} \end{aligned} \tag{7.99}$$

which is the desired result.

The formal procedure of evaluation of the matrix polynomial $f(\mathbf{A})$ is as follows.

1. Find the eigenvalues of matrix **A**.
2. If all the eigenvalues are distinct, solve n simultaneous equations given by Eq. (7.98) for the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$.

   If **A** possesses an eigenvalue λ_k of order m, then only one independent equation can be obtained by substituting λ_k into Eq. (7.98). The remaining (m – 1) linear equations, which must be obtained in order to solve for α_k's, can be found by differentiating both sides of Eq. (7.98). Since

$$\left.\frac{d^j q(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_k} = 0; j = 0,1,..., m-1, \tag{7.100}$$

it follows that

$$\left.\frac{d^j f(\lambda)}{\mathrm{d}\lambda^j}\right|_{\lambda=\lambda_k} = \left.\frac{d^j R(\lambda)}{\mathrm{d}\lambda^j}\right|_{\lambda=\lambda_k} \quad ; j = 0, 1, \ldots, m - 1 \tag{7.101}$$

3. The coefficients α_j obtained in Step 2 and Eq. (7.101) yields the required result.

Example 7.6 *Find f(A) = A^{10} for*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Solution The characteristic equation is

$$q(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = (\lambda + 1)\,(\lambda + 2) = 0$$

The matrix **A** has distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

Since **A** is of second-order, the polynomial $R(\lambda)$ will be of the following form.

$$R(\lambda) = \alpha_0 + \alpha_1\lambda$$

The coefficients α_1 are evaluated from the following equations.

$$f(\lambda_1) = \lambda_1^{10} = \alpha_0 + \alpha_1\lambda_1$$

$$f(\lambda_2) = \lambda_2^{10} = \alpha_0 + \alpha_1\lambda_2$$

The result is $\alpha_0 = -1022$, $\alpha_1 = -1023$.

From Eq. (7.99), we get

$$f(\mathbf{A}) = \mathbf{A}^{10} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$$

$$= \begin{bmatrix} -1022 & -1023 \\ 2046 & 2047 \end{bmatrix}$$

The Cayley–Hamilton technique allows us to attack the problem of computation of $e^{\mathbf{A}t}$, where **A** is a constant $n \times n$ matrix.

The following power series for the scalar $e^{\lambda t}$,

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!} + \ldots$$

converges for all finite λ and t. It follows from this result that the matrix power series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots + \frac{\mathbf{A}^n t^n}{n!} + \ldots$$

converges for all **A** and for all finite t. Therefore, the matrix polynomial $f(\mathbf{A}) = e^{\mathbf{A}t}$ can be expressed as a polynomial in **A** of degree $(n - 1)$ using the technique presented earlier. This is illustrated below with the help of an example.

Example 7.7 *Find* $f(A) = e^{At}$ *for*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Solution The characteristic equation is

$$q(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{vmatrix} = (\lambda + 1)^2 = 0$$

The matrix **A** has eigenvalues $\lambda_1, \lambda_2 = -1$.

Since **A** is of second order, the polynomial $R(\lambda)$ will be of the following form.

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda$$

The coefficients α_0 and α_1 are evaluated from (Eqs (7.98) and (7.101)

$$f(\lambda) = e^{\lambda t}$$

Then

$$f(-1) = e^{-t} = \alpha_0 - \alpha_1$$

$$\frac{d}{d\lambda} f(\lambda)\Big|_{\lambda=-1} = te^{-t} = \frac{d}{d\lambda} R(\lambda)\Big|_{\lambda=-1} = \alpha_1$$

The result is

$$\alpha_0 = (1 + t)e^{-t}, \ \alpha_1 = te^{-t}$$

From Eq. (7.99), we get

$$f(\mathbf{A}) = e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Computation by Canonical Transformation

Consider the following homogeneous state equation.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}; \ \mathbf{x}(0) = \mathbf{x}_0 \tag{7.102}$$

Assume that **A** is non-singular and has distinct eigenvalues. Transforming by means of the modal matrix, we define a new state variable such that

$$\mathbf{x} = \mathbf{M}\mathbf{z} \tag{7.103}$$

The state model now modifies to

$$\dot{\mathbf{z}} = \Lambda \mathbf{z}; \ \Lambda = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} \tag{7.104}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

In Eq. (7.104), the i^{th} state equation is

$$\dot{z}_1 = \lambda_1 z$$

which has the following solution.

$$z_i(t) = e^{\lambda_1 t} z_i(0)$$

Combining the solutions of all the states, we can write

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ \vdots \\ z_n(0) \end{bmatrix} \tag{7.105}$$

or in a compact form it can be written in the following manner

$$\mathbf{z}(t) = e^{\Lambda t}\mathbf{z}(0) \tag{7.106}$$

where

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda^2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \tag{7.107}$$

Transforming Eq. (7.107) back into the original state vector, we get

$$\mathbf{x}(t) = \mathbf{M}\mathbf{e}^{\Lambda t}\mathbf{M}^{-1}\mathbf{x_0} \tag{7.108}$$

State transition matrix is therefore, $\mathbf{e}^{\Lambda t} = \mathbf{M}\mathbf{e}^{\Lambda t}\mathbf{M}^{-1}$ (7.109)

For the case of repeated eigenvalues the solution vector of Eq. (7.105) takes the following form.

$$\mathbf{z}(t) = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2!}t^2 e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \mathbf{z}(0)$$

or

$$\mathbf{z}(t) = \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & 0 & \cdots & 0 \\ 0 & 1 & t & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} e^{\Lambda t}\mathbf{z}(0)$$

where $e^{\Lambda t}$ is given by

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_1 t} & & & & \\ & & e^{\lambda_1 t} & & & \\ & & & e^{\lambda_4 t} & & \\ & & & & \ddots & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}$$

The state variable vector **z** can then be transformed to the original state vector **x**. The state-transition matrix can then be evolved by transforming Jordan canonical form.

Example 7.8 Consider a system with the following state model.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r;\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix};\ r = \text{unit step}$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -3$.

The modal matrix is found to be

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix},\ \mathbf{M}^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{M}e^{\mathbf{\Lambda}t}\,\mathbf{M}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}\, d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}\tau}\int_0^t e^{-\mathbf{A}t}\mathbf{B}u\, d\tau \end{aligned}$$

Computing each part separately, we get

First part

$$\begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

Second part

(i)
$$\int_0^t \begin{bmatrix} 3e^{2\tau}-2e^{3\tau} & 2e^{2\tau}-2e^{3\tau} \\ -3e^{2\tau}+3e^{3\tau} & -2e^{2\tau}+3e^{3\tau} \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} 2e^{2\tau} & -2e^{3\tau} \\ -2e^{2\tau} & +3e^{3\tau} \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} e^{2t}-\frac{2}{3}e^{3t}-\frac{1}{3} \\ -e^{2t}+e^{3t} \end{bmatrix}$$

(ii)
$$\begin{bmatrix} 3e^{-2t}-2e^{-3t} & 2e^{-2t}-2e^{-3t} \\ -3e^{-2t}+3e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}\begin{bmatrix} e^{2t}-\frac{2}{3}e^{3t}-\frac{1}{3} \\ -e^{2t}+e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}+\frac{2}{3}e^{-3t}+\frac{1}{3} \\ 2e^{-2t}-e^{-3t} \end{bmatrix}$$

Hence

$$\mathbf{x}(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + \begin{bmatrix} -e^{-2t}+\frac{2}{3}e^{-3t}+\frac{1}{3} \\ e^{-2t}-e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}e^{-3t}+\frac{1}{3} \\ -e^{-3t} \end{bmatrix}$$

7.7 SOLUTION OF STATE EQUATIONS—LAPLACE TRANSFORM METHOD

Application of the Laplace transform to Eq. (7.6a) gives the following for single input single output (SISO) case.

$$s\mathbf{X}(s) - x(0) = \mathbf{A}\,\mathbf{X}(s) + \mathbf{b}\,R(s) \tag{7.110}$$

Rearranging

$$(s\mathbf{I}-\mathbf{A})\,\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{b}\,R(s)$$

or

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{x}(0)}_{\text{Zero input response}} + \underbrace{(s\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{b}\,R(s)}_{\text{Zero state response}} \tag{7.111}$$

It may be easily noted that

$$\Phi(t) = e^{\mathbf{A}t} = \pounds^{-1}\,(s\mathbf{I}-\mathbf{A})^{-1} = \textbf{state transition matrix} \tag{7.112}$$

From Eq. (7.6b) the output in s-domain is given as

$$Y(s) = \mathbf{c}'\,\mathbf{X}(s) + \mathbf{d}\,R(s)$$

Substituting for $\mathbf{X}(s)$ from Eq. (7.111) gives

$$Y(s) = \mathbf{c}'(s\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{x}(0) + [\mathbf{c}'(s\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{b} + \mathbf{d}]R(s) \tag{7.113}$$

If $x(0) = 0$ (i.e., initial zero state), this equation becomes

$$Y(s) = [\mathbf{c}'(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} + \mathbf{d}]\,R(s)$$

or

$$\frac{Y(s)}{R(s)} = H(s) = \mathbf{c}'(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} + \mathbf{d} \tag{7.114}$$

This indeed is the transfer function of a single-input, single-output (SISO) system.

Equation (7.114) may be written as

$$T(s) = \frac{\mathbf{c}'\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{b}}{\det(s\mathbf{I} - \mathbf{A})} + d$$

In many practical problems, $\mathbf{d} = 0$. Then

$$T(s) = \frac{\mathbf{c}'\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{b}}{\det(s\mathbf{I} - \mathbf{A})} \tag{7.115}$$

The characteristic equation of the transfer function is

$$q(s) = |s\mathbf{I} - \mathbf{A}| = 0 \tag{7.116}$$

Equation (7.116) gives the poles of $T(s)$.

Example 7.9 Solve Example 7.4 by the Laplace transform method.

The state transition matrix in the Laplace transform is

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

Inverting the matrix we get

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

Taking the inverse Laplace transform

$$£^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} - 6e^{-3t} & -e^{-2t} + 3e^{-3t} \end{bmatrix}$$

This verifies the solution in part (i). But inverting a matrix in Laplace form is not easy to carry out.

Example 7.10 *Find the output response of the system described by the following state variable formulation*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

to unit step input.

It is given that

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{x}^T(0) = [1\ 1]$$

Solution It is observed that this is a system with one input and two outputs. The Laplace transform of the response is obtained by modifying Eq. (7.113). That is, by changing $\mathbf{c}' \to \mathbf{C}$ and $Y(s) \to \mathbf{Y}(s)$ which means, that $\mathbf{C}$ is a matrix and the output transform is a vector. Thus

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\,R(s) \tag{i}$$

Let us first obtain the state transition matrix

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \tag{ii}$$

Substituting Eq.(ii) in Eq.(i), we get

$$\mathbf{Y}(s) = \frac{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{s^2 + 3s + 2} + \frac{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}}{s^2 + 3s + 2} \tag{iii}$$

After simplification

$$\mathbf{Y}(s) = \begin{bmatrix} \dfrac{s+4}{(s+1)(s+2)} \\ \dfrac{2}{s+2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{s(s+2)} \end{bmatrix}$$

After partial fractioning

$$\mathbf{Y}(s) = \begin{bmatrix} \dfrac{3}{s+1} - \dfrac{2}{s+2} \\ \dfrac{2}{s+2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{2s} - \dfrac{1}{s+1} + \dfrac{1}{2(s+2)} \\ \dfrac{1}{2s} - \dfrac{1}{2(s+2)} \end{bmatrix} \tag{iv}$$

Taking the Laplace inverse of Eq. (iv), we get

$$\mathbf{y}(t) = \begin{bmatrix} (3e^{-t} - 2e^{-2t})\,u(t) \\ e^{-2t}\,u(t) \end{bmatrix} + \begin{bmatrix} \left[\left(\frac{1}{2}\right) - e^{-t} + \left(\frac{1}{2}\right)e^{-2t}\right]u(t) \\ \left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)e^{-2t}\right]u(t) \end{bmatrix} \tag{v}$$

or

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right) + 2e^{-t} - \left(\frac{3}{2}\right)e^{-2t} \\ \left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)e^{-2t} \end{bmatrix} u(t) \tag{vi}$$

This is the required solution.

Example 7.11 *Reconsider the system discussed in Example 7.5 and obtain the solution by the Laplace transform method.*

Solution

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix} \qquad \text{(i)}$$

The state transition matrix in s-domain is

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{(s+1)} & 0 \\ 0 & \dfrac{1}{(s+2)} \end{bmatrix} \qquad \text{(ii)}$$

Therefore,

$$\Phi(t) = £^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\, £^{-1} = \begin{bmatrix} \dfrac{1}{(s+1)} & 0 \\ 0 & \dfrac{1}{(s+2)} \end{bmatrix} \qquad \text{(iii)}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \qquad \text{(iv)}$$

The solution of the state equation is,

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}_0 + £^{-1}[\Phi(s)\mathbf{B}R(s))]$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + £^{-1} \left\{ \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \qquad \text{(v)}$$

Simplifying the above equation we get,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}(1 - e^{-2t}) \end{bmatrix} \qquad \text{(vi)}$$

7.8 STATE EQUATIONS DISCRETE TIME-SYSTEMS

For n^{th} order LTI discrete-time (MIMO) system, the state model is

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{r}(k)\;;\; \mathbf{x}(kT) = \mathbf{x}(k) \qquad (7.117a)$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,r(k) \qquad (7.117b)$$

where

$$\begin{aligned} \mathbf{x}(k) &= n \times 1 \text{ state vector} \\ \mathbf{r}(k) &= m \times 1 \text{ input vector} \\ \mathbf{y}(k) &= p \times 1 \text{ output vector} \\ \mathbf{A} &= n \times n \text{ system matrix} \\ \mathbf{B} &= n \times m \text{ input matrix} \\ \mathbf{C} &= p \times n \text{ output matrix} \\ \mathbf{D} &= p \times m \text{ direct transmission matrix.} \end{aligned}$$

Solution of the State Equations From the state model of Eq. (7.117a) we can write for $k = 0,1,2$ ...

$$\begin{aligned} \mathbf{x}(1) &= \mathbf{A}\,\mathbf{x}(0) + \mathbf{B}\,\mathbf{r}(0) \\ \mathbf{x}(2) &= \mathbf{A}\,\mathbf{x}(1) + \mathbf{B}\,\mathbf{r}(1) \\ &= \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\,\mathbf{B}\,\mathbf{r}(0) + \mathbf{B}\,\mathbf{r}(1) \end{aligned}$$

It then follows

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}^k\mathbf{x}(0) + \mathbf{A}^{k-1}\,\mathbf{B}\,\mathbf{r}(0) + \mathbf{A}^{k-2}\,\mathbf{B}\,\mathbf{r}(1) + \ldots \\ &\quad \ldots + \mathbf{B}\,\mathbf{r}(k-1) \end{aligned} \tag{7.118}$$

Equation (7.118) can be written in summation form as

$$\mathbf{x}(k) = \underbrace{\mathbf{A}^k\,\mathbf{x}(0)}_{\text{Zero-input response}} + \underbrace{\sum_{i=0}^{k-1} \mathbf{A}^{k-(i+1)}\,\mathbf{B}\,\mathbf{r}(i)}_{\text{Zero-state response}} \tag{7.119}$$

From Eq. (7.117b) the system output can be written by substituting $x(k)$ from Eq. (7.119). Thus,

$$\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k\mathbf{x}(0) + \sum_{i=1}^{k-1} \mathbf{C}\mathbf{A}^{k-(i+1)}\,\mathbf{B}r + \mathbf{D}r(k) \tag{7.120}$$

In most cases $\mathbf{D} = 0$ in Eqs (7.119) and (7.120).

For the *state transition matrix* for convenience of writing, we may use the symbol

$$\Phi(k) = \mathbf{A}^k$$

then

$$\mathbf{A}^{k-i-1} = \Phi^{(k-i-1)}$$

The properties of the state transition matrix are:

1. $\Phi(0) = \mathbf{I}$
2. $\Phi^{-1}(k) = \Phi(-k)$
3. $\Phi(k, k_0) = \Phi(k - k_0) = \mathbf{A}^{(k-k_0)}$; $k > k_0$

Equations (7.118) and (7.119) could then be written in terms of the symbol $\Phi(k)$.

State Transition Matrix (A^k) Various methods of computing the state transition matrix ($\mathbf{A}^k$) are listed below.

1. By use of Cayley–Hamilton theorem
2. By canonical transformation
3. By inverse Z-transform

Cayley–Hamilton theorem and canonical transformation method given for LTI continuous-time system apply as it is for the discrete-time case. The method of finding the diagonalization matrix (**M**) is also the same.

The diagonalized matrix for nonrepeated eigenvalues presented in Eq. (7.107) now takes the following form.

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \tag{7.121}$$

The state $\mathbf{x}(k+1)$ in terms of state $\mathbf{x}(0)$ is now given by

$$\mathbf{x}(k+1) = \mathbf{A}^k\mathbf{x}(0) = \mathbf{M}\Lambda^k\mathbf{M}^{-1}\mathbf{x}(0) \tag{7.122}$$

For the case of repeated eigenvalues, Λ^k has the Jordan form $\mathbf{J}^k$, which is presented below for λ_1 repeated three-time and λ_4, λ_5 non-repeated.

$$\mathbf{J}^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & \frac{1}{2}k(k-1)\lambda_1^{k-2} & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_4^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_5^k \end{bmatrix} \tag{7.123}$$

The method of canonical transformation is illustrated in two examples below. For application of Cayley–Hamilton method see Example 7.23.

Example 7.12 *Consider the following system.*

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(-1)^k$$

$$x_1(0) = 1 = x_2(0)$$
$$y(k) = x_1(k)$$

Find y(k) for ≥ 1.

Solution From the given state equation, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{i}$$

The characteristic equation of the system is

$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^2 + 3\lambda + 2 = 0 \tag{ii}$$

which yields $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of $\mathbf{A}$. The modal matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \tag{iii}$$

gives the diagonalized matrix

$$\Lambda = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{iv}$$

From Eq. (7.121), we have

$$\mathbf{A}^k = \mathbf{M}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\mathbf{M}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix} \tag{v}$$

Using Eq. (7.120), we get

$$y(k) = \mathbf{c}'\mathbf{A}^k\,\mathbf{x}(0) + \sum_{i=0}^{k-1} \mathbf{c}'\mathbf{A}^{k-i-1}\mathbf{B}r(i)\ ;\ \mathbf{D} = 0 \tag{vi}$$

with $\mathbf{c}' = [1\ \ 0]$.

$\mathbf{A}^k$ has been calculated above

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad r(i) = (-1)^i$$

Using these value in Eq. (vi) yields the following result.

$$y(k) = 3\,(-1) - 2(-2)^k + \sum_{i=0}^{k-1} [(-1)^{k-i-1} - (-2)^{k-i-1}](-1)^i$$

$$= 3(-1)^k - 2(-2)^k + (-1)^{k-1} - (-2)^{k-1} \sum_{i=0}^{k-1} \left(\tfrac{1}{2}\right)^i \tag{vii}$$

Using sum of geometric series, we have

$$\sum_{i=0}^{k-1} \left(\tfrac{1}{2}\right)^i = \frac{1 - \left(\frac{1}{2}\right)^k}{1 - \frac{1}{2}} = -2\left[\left(\tfrac{1}{2}\right)^k - 1\right]$$

Therefore,

$$y(k) = 3(-1)^k - 2(-2)^k - (-1)^k + \tfrac{1}{2}(-2)^k\left[2 - 2\left(\tfrac{1}{2}\right)^k\right]$$

$$= (-1)^k - (-2)^k \tag{viii}$$

Example 7.13 *For the matrix,*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

find $\boldsymbol{A}^k$.

Solution The characteristic equation is

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} = 0 = (\lambda+1)^2 \qquad \text{(i)}$$

Therefore, the eigenvalues are $-1, -1$.

The rank of the matrix $(\lambda\mathbf{I} - \mathbf{A})$ is 1. Therefore, there is only one linearly independent eigenvector for $\lambda = -1$. This eigenvector is obtained as follows.

$$\mathbf{m}_1 = \begin{bmatrix} \lambda+2 \\ -1 \end{bmatrix}_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{(ii)}$$

The generalized eigenvector is given in the following form.

$$\mathbf{m}_2 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\begin{bmatrix} \lambda+2 \\ -1 \end{bmatrix}_{\gamma=-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(iii)}$$

Therefore the modal matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{(iv)}$$

This gives

$$\mathbf{M}^{-1}\mathbf{AM} = \mathbf{J} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \text{(v)}$$

From Eq. (7.123), we have

$$\begin{aligned} \mathbf{A}^k &= \mathbf{MJ}^k\mathbf{M}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^k & -k(-1)^k \\ 0 & (-1)^k \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ &= (-1)^k\begin{bmatrix} 1-k & -k \\ k & 1+k \end{bmatrix} \end{aligned}$$

Z-Transform Solution of the State Equations

Applying the z-transform to Eq. (7.117a) and assuming SISO system, we get

$$z\mathbf{X}(z) - zx(0) = \mathbf{A}\,\mathbf{X}(z) + \mathbf{b}\,R(z)$$

or

$$(z\mathbf{I} - \mathbf{A})\,\mathbf{X}(z) = z\mathbf{x}(0) + \mathbf{b}\,R(z)$$

The solution from $\mathbf{X}(z)$ is then written as

$$\mathbf{X}(z) = \underbrace{(z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{x}(0)}_{\text{Zero-input response}} + \underbrace{(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}\, R(z)}_{\text{Zero-state response}} \tag{7.124}$$

Taking the z-transform of Eq. (7.117b) and substituting for $\mathbf{X}(z)$ from Eq. (7.124) the system output is expressed as

$$Y(z) = \underbrace{\mathbf{c}'(z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{x}(0)}_{\text{Zero-input response}} + \underbrace{[\mathbf{c}'(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathbf{d}]R(z)}_{\text{Zero state response}} \tag{7.125}$$

If $\mathbf{x}(0) = 0$, then

$$Y(z) = [\mathbf{c}'(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathbf{d}]\, R(z)$$

This can be written in the ratio of output/input form as

$$\frac{Y(z)}{R(z)} = H(z) = \mathbf{c}'(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathbf{d} \tag{7.126}$$

Equation (7.126) is the **sampled transfer function** with the state transition matrix

$$\Phi(z) = (z\mathbf{I} - \mathbf{A})^{-1}$$

Compare Eqs (7.111), (7.113) and (7.114) with Eqs. (7.124), (7.125), (7.126) respectively. These are identical if z is replaced by s. Thus, steps to obtain the transfer function from the state equations and vice-versa will be similar.

7.9 DISCRETISATION OF CONTINUOUS-TIME STATE EQUATIONS

Method 1

Consider the following continuous-time state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\,\mathbf{r}(t) \tag{7.127}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\,\mathbf{r}(t) \tag{7.128}$$

If this set of equations is to be solved on a digital computer, it must be discretised. As

$$\dot{\mathbf{x}}(t) = \lim_{T \to 0} \frac{\mathbf{x}(t+T) - \mathbf{x}(t)}{T}$$

we can approximate this equation as

$$\mathbf{x}(t+T) = \mathbf{x}(t) + \mathbf{A}\,\mathbf{x}(t)\,T + \mathbf{B}\,r(t)T \tag{7.129}$$

If we compute $\mathbf{x}(t)$ and $\mathbf{y}(t)$ only at $t = kT$; $k = 0, 1, \ldots$, then Eqs (7.129) and (7.128) become

$$\mathbf{x}((k+1)T) = (\mathbf{I} + T\mathbf{A})\,\mathbf{x}(kT) + T\,\mathbf{B}\,\mathbf{r}\,(kT) \tag{7.130}$$

$$y(kT) = \mathbf{C}\,x(kT) - \mathbf{D}\,\mathbf{r}\,(kT) \tag{7.131}$$

Define

$$\mathbf{A}_T = T\,\mathbf{A} + \mathbf{I},\ \mathbf{B}_T = T\,\mathbf{B}$$

Letting $kT \rightarrow k$, we get Eqs (7.129) in the following form.

$$\mathbf{x}(k+1) = \mathbf{A}_T\,\mathbf{x}(k) + \mathbf{B}_T\,r(k) \tag{7.132}$$

Accuracy of this method of computation will improve as T is reduced but computational time will increase. Computational efficiency of this method is now examined by means of an example.

Example 7.14 *Consider the state variable model of a second-order system represented as*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ r = \textit{unit step}$$

(a) Find the state response $x(t)$, $t > 0$.

(b) Find the state response by using discrete-time approximation (Method 1).

Solution

(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{i}$$

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} \tag{ii}$$

$$[s\mathbf{I} - A] = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \tag{iii}$$

The inverse matrix is

$$[s\mathbf{I} = \mathbf{A}]^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix} \tag{iv}$$

$$\Delta(s) = (s+1)\,(s+2)$$

Then

$$\Phi(s) = \begin{bmatrix} \dfrac{s+3}{(s+1)\,(s+2)} & \dfrac{1}{(s+1)\,(s+2)} \\ \dfrac{-2}{(s+1)\,(s+2)} & \dfrac{s}{(s+1)\,(s+2)} \end{bmatrix} \tag{v}$$

Taking the Laplace inverse

$$e^{\mathbf{A}t} = \Phi(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} - 2e^{-2t}) \end{bmatrix};\ \text{for } t > 0 \tag{vi}$$

We know that the time response of the system is given as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}(\tau)\,d\tau\right];\ r(t) = u(t);\ \text{unit step} \tag{vii}$$

Substituting values and simplifying Eq. (vii), we get

$$\mathbf{x}(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix} \tag{viii}$$

(b) State response by discrete-time approximation (Method 1)
The discrete-time approximation of the state equation as per Eq. (7.132) is

$$\mathbf{x}(k+1) = (T\mathbf{A} + \mathbf{I})\,\mathbf{x}(k) + T\,\mathbf{B}\,r(k) \tag{ix}$$

Substituting the value of **A** and **B** matrices, we get

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ -2T & -3T+1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 2T \end{bmatrix} r(k) \tag{x}$$

For higher accuracy of the approximation of the derivative, usually time interval T is chosen less than one tenth of the smallest time constant of the system. The time constant of the system may be obtained by the knowledge of their eigenvalues, which can be obtained by solving

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$

For this system it being

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0$$

or

$$\lambda^2 + 3\lambda + 2 = 0$$

So, the eigenvalues of the matrix **A** are

$$\lambda_1 = -1,\ \lambda_2 = -2$$

Therefore, the smallest time constant is 0.5s. So the time interval is taken as T = 0.05s and solution is obtained through computer. One step of compu-tation is carried out below for $T = 0.1s$ for ease of computation.
At $k = 0$

$$\mathbf{x}(1) = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.7 \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix};\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

At $k = 1$,

$$\mathbf{x}(2) = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.7 \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.19 \\ 0.81 \end{bmatrix}$$

Similarly, the response for $k = 2,3,4...$ is computed.

This method of discretisation and computation is compared with the next method in Table 7.1 at the end of which certain observations are made.

An Efficient Method of Discretisation (Method 2)

If the input $\mathbf{r}(t)$ is generated by digital computer followed by a digital-to-analog converter, then $\mathbf{r}(t)$ will be piecewise constant. This is usually a part of a computer control. Let

$$\mathbf{r}(t) = \mathbf{r}(kT) \to \mathbf{r}(k) \quad \text{for} \quad kT \le t \le (k+1)T; \quad \text{for} \quad k = 0, 1, 2, \ldots$$

These input values change only at discrete-time instants. It easily follows from Eq. (7.79) that

$$\mathbf{x}(k) \leftarrow \mathbf{x}(kT) = e^{\mathbf{A}kT}\,\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\,\mathbf{B}\mathbf{r}(kT)\mathrm{d}\tau \tag{7.133}$$

and

$$\mathbf{x}(k+1) \leftarrow \mathbf{x}((k+1)T) = e^{\mathbf{A}(k+1)T}\,\mathbf{x}(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}\,\mathbf{B}\mathbf{r}(k)\mathrm{d}\tau \tag{7.134}$$

Equation (7.134) can be written as

$$\mathbf{x}(\mathrm{k}+1) = e^{\mathbf{A}T}\left[e^{AkT}\,\mathbf{x}(0) + \int_0^{kT} e^{A(kT-\tau)}\mathbf{B}\,\mathbf{r}(k)\mathrm{d}\tau\right] + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}\mathbf{B}\,\mathbf{r}(k)\mathrm{d}\tau$$

Substituting Eq. (7.133) in the above equation and also introducing the new variable $\alpha = kT + T - \tau$, it takes the following form.

$$\mathbf{x}(k+1) = e^{\mathbf{A}T}\,\mathbf{x}(k) + \left(\int_0^{T} e^{A\alpha}\mathrm{d}\alpha\right)\mathbf{B}\mathbf{r}(k) \tag{7.135}$$

Thus, if the input changes only at discrete-time instants kT (and remains constant at that value for period T) and the responses only at $t = kT$ are to be computed, then Eqs (7.135) and (7.128) are written as

$$\mathbf{x}\,(k+1) = \mathbf{A}_d\,\mathbf{x}(k) + \mathbf{B}_d\,\mathbf{r}(k) \tag{7.136}$$

$$\mathbf{y}(k) = \mathbf{C}_d\,\mathbf{x}(k) + \mathbf{D}_d\,r(k) \tag{7.137}$$

where

$$\mathbf{A}_d = e^{\mathbf{A}T},\ \mathbf{B}_d = \left(\int_0^{T} e^{A\tau}\mathrm{d}\tau\right)\mathbf{B},\ \mathbf{C}_d = \mathbf{C},\ \mathbf{D}_d = \mathbf{D} \tag{7.138}$$

Equations (7.136) and (7.137) are piecewise state space equations and no approximation has been made in their derivation. Therefore, these yield exact solution of Eqs (7.127) and (7.128) at $t = kT$, if the input is piecewise constant.

Computation Method We express $\mathbf{B}_d$ of Eq. (7.138) using Taylor series. Thus

$$\mathbf{B}_d = \left[\int_0^T \left(\mathbf{I} + \mathbf{A}\tau + \mathbf{A}^2\frac{\tau^2}{2!} + \dots\right)\mathrm{d}\tau\right]\mathbf{B}$$

$$= \left[T\boldsymbol{I} + \frac{T^2}{2!}\mathbf{A} + \frac{T^3}{3!} + \frac{T^4}{4!}\mathbf{A}^4 + \dots\right]\mathbf{B} \tag{7.139}$$

This power series can be computed recursively as presented earlier in Eqs (7.51). If **A** is nonsingular, then the power series can be written in the following manner.

$$\mathbf{A}^{-1}\left(T\mathbf{A} + \frac{T^2}{2!}\mathbf{A}^2 + \frac{T^3}{3!}\mathbf{A}^3 + \dots \mathbf{I} - \mathbf{I}\right)$$

$$= \mathbf{A}^{-1}\,(e^{\mathbf{A}T} - \mathbf{I})$$

Thus, we have

$$\mathbf{B}_d = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\,\mathbf{B};\ \text{if } \mathbf{A} \text{ is nonsingular} \tag{7.140}$$

The use of this formula avoids the need of computing an infinite power series.

Comparing Discretisation Methods Comparison of computation by the two methods of discretisation presented above is given in Table 7.1. These results have been obtained through the MATLAB software using appropriate menu.

Table 7.1

Time(s)		*0.2* $\mathbf{x}(t)$	*0.4* $\mathbf{x}(t)$	*0.6* $\mathbf{x}(t)$	*0.8* $\mathbf{x}(t)$
T(s)	**CT Sol.**	0.1813	0.3297	0	0.5507
		0.8187	0.6703	1	0.4493
0.2	**M-1**	0.2	0.3600	0.0	0.5904
		0.8	0.6400	1.0	0.4493
	M-2	0.1813	0.3297	0.0	0.5507
		0.8187	0.6703	1.0	0.4493
0.1	**M-1**	0.1900	0.3439	0.0	0.5695
		0.8100	0.6461	1.0	0.4305
	M-2	0.1813	0.3297	0.0	0.5501
		0.8187	0.6703	1.0	0.4493
0.05	**M-1**	0.1855	0.3366	0.0	0.5599
		0.8145	0.6634	1.0	0.4401
	M-2	0.1813	0.3297	0.0	0.5507
		0.8187	0.6703	1.0	0.4493

Legends

C.T. – Continuous time solution
M-1 – Discrete time solution (Method - 1)
M-2 – Discrete time solution (Method - 2)
$\mathbf{x}(t)$ – Time response
T(s) – Sampling time

Remarks

- The Discretisation Method-2 solution follows the CT solution for all values of T.
- The Discretisation Method-1 gives satisfactory results as sampling time is reduced but that requires considerable computational effort, also the accuracy of the results is not uniform over the time scale as is evident from the Table 7.1 and hence is not a good approximation.
- Method-2 for which means are available in MATLAB is adopted invariably for solution of state-space equation (continuous or discrete-time). The continuous-time equations are first discretised by appropriate command of MATLAB.

Additional Examples

Example 7.15 *The block diagram of a dc servo motor position control system is drawn in Fig. 7.8. What is the order of this system? Identify* ***physical variables*** *as state variable. Formulate the state variable model of the system and therefrom find its transfer function in s-domain.*

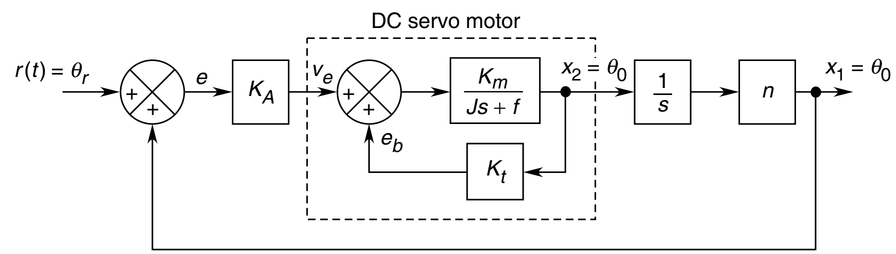


Fig. 7.8

Solution The physical variables are $x_1 = \theta_0$ and $x_2 = \theta_0$ and these are also shown in Fig. 7.8 from which, we can write the following expressions for x_1 and x_2.

$$X_1(s) = \frac{n}{s}X_2(s) \tag{i}$$

$$X_2(s) = [K_A(R(s) - X_1(s)) - K_t X_2(s)]\ \frac{K_m}{(Js + f)} \tag{ii}$$

$$Y(s) = X_1(s) \tag{iii}$$

Taking the inverse Laplace transform we get,

$$\dot{x}_1 = nx_2(t) \tag{iv}$$

$$\dot{x}_2 = -\left(\frac{K_A K_m}{J}\right) x_1(t) - \left(\frac{f + K_t K_m}{J}\right) x_2(t) + \left(\frac{K_A K_m}{J}\right) r(t) \tag{v}$$

$$y(t) = x_1(t) \tag{vi}$$

The state variable model in matrix form can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -\dfrac{K_A K_m}{J} & -\dfrac{(f + K_t K_m)}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_A K_m}{J} \end{bmatrix} r(t) \tag{vii}$$

$$y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{viii}$$

Since only two state variables are required to describe the system, it is a second order system. The transfer function of the system in s-domain is given by

$$T(s) = \frac{Y(s)}{R(s)} = [\mathbf{c}' (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + d] \tag{ix}$$

Here

$$\mathbf{c}' = [1 \;\; 0], \mathbf{b} = \begin{bmatrix} 0 \\ \dfrac{K_A K_m}{J} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & n \\ \dfrac{K_A K_m}{J} & -\dfrac{(f + K_t K_m)}{J} \end{bmatrix}, d = 0$$

Substituting the above values, we get

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -n \\ \dfrac{K_A K_m}{J} & s + \left(\dfrac{f + K_t K_m}{J}\right) \end{bmatrix} \tag{x}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s\left(s + \dfrac{f + K_t K_m}{J}\right) + \dfrac{nK_A K_m}{J}} \begin{bmatrix} s + \dfrac{f + K_t K_m}{J} & n \\ -\dfrac{K_A K_m}{J} & s \end{bmatrix} \tag{xi}$$

Substituting values in Eq. (ix), the transfer function of the dc servo motor system is found to be

$$T(s) = \frac{n K_A K_m}{Js^2 + (f + K_t K_m) s + nK_A K_m} \tag{xii}$$

Example 7.16 *A feedback system is characterized by the following closed-loop transfer function.*

$$T(s) = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$$

Draw a suitable signal flow graph and therefrom construct the state model of the system.

Solution Writing the closed loop transfer function in negative powers of s, we get

$$T(s) = \frac{s^{-1} + 3s^{-2} + 3s^{-3}}{1 + 2s^{-1} + 3s^{-2} + s^{-3}}$$

The numerator terms represent the forward path factors in Mason's signal flow gain formula. The forward path will touch all loops. A suitable signal flow graph realisation of $T(s)$ is drawn in Fig. 7.9.

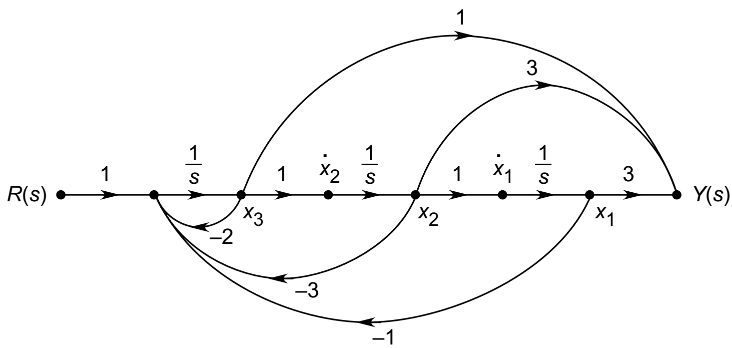


Fig. 7.9

From the signal flow graph, we obtain the following set of first order differential equations.

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = -x_1 - 3x_2 - 2x_3 + r$$

where x_1, x_2 and x_3 are the phase variables.

The output is given as

$$y = 3x_1 + 3x_2 + x_3$$

In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [3 \ \ 3 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 7.17 *Given*

$$A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}; A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

Compute $e^{\mathbf{A}t}$ *where* $A = A_1 + A_2$.

Solution

$$e^{\mathbf{A}t} = e^{(A_1 + A_2)t} = e^{A_1 t} e^{A_2 t}$$

Now,

$$e^{\mathbf{A}_1 t} = \pounds^{-1}\left[(s\mathbf{I} - \mathbf{A}_1)^{-1}\right]$$

$$= \pounds^{-1}\begin{bmatrix} \dfrac{1}{s-\sigma} & 0 \\ 0 & \dfrac{1}{s-\sigma} \end{bmatrix} = \begin{bmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{bmatrix}$$

Similarly,

$$e^{\mathbf{A}_2 t} = \pounds^{-1}\left[(s\mathbf{I} - \mathbf{A}_2)^{-1}\right]$$

$$= \pounds^{-1}\begin{bmatrix} \dfrac{s}{s^2+\omega^2} & \dfrac{\omega}{-s^2+\omega^2} \\ \dfrac{\omega}{s^2+\omega^2} & \dfrac{s}{s^2+\omega^2} \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\omega\sin\omega t \\ \omega\sin\omega t & \cos\omega t \end{bmatrix}$$

Now combining the expressions for $e^{\mathbf{A}_1 t}$ and $e^{\mathbf{A}_2 t}$, we get

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{\sigma t}\cos\omega t & -\omega e^{\sigma t}\sin\omega t \\ \omega e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}$$

Example 7.18 *The following is the state equation of a system.*

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t)$$

Find A *if the state response to two initial conditions is given as*

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \mathbf{x}(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \mathbf{x}(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

Solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}(0) \qquad \text{(i)}$$

Let

$$e^{\mathbf{A}t} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{(ii)}$$

Then

$$\mathbf{x}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x}(0) \tag{iii}$$

Using the first initial condition, we get

$$\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{iv}$$

Two equations can be written from Eq. (iv) as below

$$e^{-2t} = a_{11} - 2a_{12} \tag{v}$$

$$2e^{-2t} = -a_{21} + 2a_{22} \tag{vi}$$

Using the second initial condition, we can write

$$\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{vii}$$

This yields two equations as

$$e^{-t} = a_{11} - a_{12} \tag{viii}$$

$$e^{-t} = -a_{21} + a_{22} \tag{ix}$$

Solving Eqs. (v), (vi), (viii) and (ix) gives the following result.

$$a_{11} = 2e^{-t} - e^{-2t},\ a_{12} = e^{-t} - e^{-2t}$$

$$a_{21} = 2e^{-2t} - 2e^{-t}, a_{22} = 2e^{-2t} - e^{-t}$$

But

$$\pounds^{-1}(s\mathbf{I} - \mathbf{A})^{-1} = e^{\mathbf{A}t}$$

Substituting values, we write

$$\pounds^{-1}(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Taking the Laplace transform on both sides

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

Taking matrix inverse on both sides gives

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

From which, we get

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Example 7.19 *Consider the hydraulic system shown in Fig. 7.10. Let*

q_0, q_1, q_2 = *liquid flow rates*

A_1, A_2 = *cross-sectional area of the tanks*

h_1, h_2 = *liquid levels*

R_1, R_2 = *hydraulic resistances, controlled by valves*

Solution

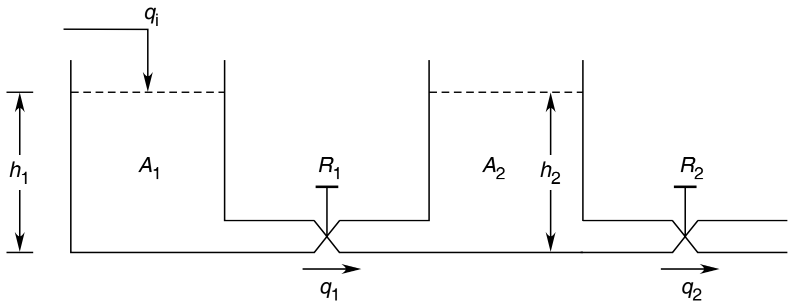


Fig. 7.10

It is assumed that q_1 and q_2 are governed by

$$q_1 = \frac{h_1 - h_2}{R_1} \quad \text{and} \quad q_2 = \frac{h_1}{R_2} \tag{i}$$

These are proportional to relative liquid levels and inversely proportional to flow resistances. The changes in the liquid levels are governed by

$$A_1 \frac{dh_1}{dt} = q_i - q_1;\ q_i = \text{input flow rate} \tag{ii}$$

$$A_2 \frac{dh_2}{dt} = q_i - q_2;\ q_i = \text{output flow rate} \tag{iii}$$

The state variables of the system can be chosen as the liquid levels, that is, $x_1(t) = h_1$ and $x_2(t) = h_2$. From Eqs (i), (ii) and (iii) we have,

$$\frac{dx_1}{dt} = \frac{q_i}{A_1} - \frac{x_1 - x_2}{A_1 R_1} \tag{iv}$$

$$\frac{dx_2}{dt} = \frac{x_1 - x_2}{A_1 R_1} - \frac{x_2}{A_2 R_2} \tag{v}$$

which can be expressed in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{A_1 R_1} & \dfrac{1}{A_1 R_1} \\ \dfrac{1}{A_2 R_1} & \dfrac{-(R_1 + R_2)}{A_2 R_1 R_2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{A_1} \\ 0 \end{bmatrix} q_i(t) \tag{vi}$$

and the output is

$$q_2 = \begin{bmatrix} 0 & \dfrac{1}{R_2} \end{bmatrix} \begin{bmatrix} x_1\ (t) \\ x_2\ (t) \end{bmatrix} \tag{vii}$$

Example 7.20 *The spread of an epidemic disease can be described by a set of differential equations. The population under study is made up of three groups x_1, x_2 and x_3, such that the group x_1 is susceptible to the epidemic disease, group x_2 is infected with the disease, and group x_3 has been removed from the initial population. The removal of x_3 will be due to immunization, death or isolation from x_1. This feedback system can be represented by the following equations.*

$$\frac{dx_1}{dt} = \alpha x_1 - \beta x_2 - u_1(t) \tag{i}$$

$$\frac{dx_2}{dt} = -\beta x_1 - \gamma x_2 - u_2(t) \tag{ii}$$

$$\frac{dx_3}{dt} = -\alpha x_1 + \gamma x_2 \tag{iii}$$

The rate at which new suceptibles are added to the population is equal to $u_1(t)$ and the rate at which new infectives are added to the population is equal to $u_2(t)$. Taking $\alpha = \beta = \gamma = 1$, find out the discrete-time transient response of the spread of disease, when the rate of new susceptibles is zero ($u_1 = 0$) and one new infective is added as the initial time only ($u_2(0) = 1$ and $u_2(k) = 0$, $k \geq 1$). Assume zero initial response, i.e. $x_1(0) = x_2(0) = x_3(0) = 0$.

Solution The characteristic equation of the above system is given by

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+1 & 1 & 0 \\ -1 & s+1 & 0 \\ -1 & -1 & s \end{vmatrix} = 0 \tag{iv}$$

$$s(s^2 + s + 2) = 0 \tag{v}$$

The time constant of the characteristic equation is $1/\zeta\omega_n = 2$ days and therefore, we will use $T = 0.2$ days.

Then the discrete-time equation of the system is given by the following.

$$\mathbf{x}(k+1) = (T\mathbf{A} + \mathbf{I})\mathbf{x}(k) + T\mathbf{B}r(k) \tag{vi}$$

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} u_2(k) \tag{vii}$$

The response at the first instant, $t = T$, is obtained from the above expression when $k = 0$ as

$$\mathbf{x}(1) = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} \tag{viii}$$

As the input $u_2(k)$ is zero for $k \geq 1$, the response at the second instant, $t = 2T$

$$\mathbf{x}(2) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.04 \\ 0.16 \\ 0.04 \end{bmatrix} \tag{ix}$$

The response at $t = 3T$ is then given by

$$\mathbf{x}(3) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} -0.04 \\ 0.16 \\ 0.04 \end{bmatrix} = \begin{bmatrix} -0.064 \\ 0.120 \\ 0.064 \end{bmatrix} \tag{x}$$

Also, the ensuing values can be readily evaluated.

Example 7.21 *An LTI system is characterized by the non–homogeneous state equation.*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} r(t);\ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(i) Compute the solution when $r(t) = 0$ by

(a) the Laplace transform method

(b) canonical transformation

(ii) Also, find the solution of the system for unit step input, i.e. $r(t) = 1$.

Solution

(a) Laplace method

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}$$

The state transition matrix in s-domain is

$$\Phi(t) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s^2 + 3s + 2)}\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{-2}{s^2+3s+2} \\ \dfrac{1}{s^2+3s+2} & \dfrac{s}{s^3+3s+2} \end{bmatrix}$$

Therefore

$$\Phi(t) = \pounds^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \pounds^{-1}\begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{-2}{s^2+3s+2} \\ \dfrac{1}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}$$

$$= \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

(b) Canonical transformation The technique for deriving the canonical form of a state variable model dealt in Section 7.3 requires that the poles of the system are known. In this example we illustrate a generalized method of canonical transformation of a state variable model. The first step is to find out the diagnolization or the modal matrix for which the eigenvalues and the eigenvectors of the system have to be evaluated.

The characteristic equation of the matrix **A** is

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & 2 \\ -1 & \lambda + 3 \end{vmatrix} = 0$$

or $(\lambda + 1)(\lambda + 2) = 0$. Therefore, the eigenvalues of matrix **A** are

$$\lambda_1 = -1, \lambda_2 = -2$$

The eigenvector $\mathbf{m}_1$ associated with $\lambda_1 = -1$ is obtained from co-factors of the first row of the following matrix

$$(\lambda\mathbf{I} - \mathbf{A}) = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

The result is

$$\mathbf{m}_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Similarly, the eigenvector $\mathbf{m}_2$ associated with $\lambda_2 = -2$ is

$$\mathbf{m}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The modal matrix **M** obtained by placing the eigenvectors (columns) together is given by

$$\mathbf{M} = [\mathbf{m}_1 \quad \mathbf{m}_2] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

from which we find

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Therefore, the diagonal matrix Λ is given by

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Then $$\mathbf{z}(t) = e^{\Lambda t}\,\mathbf{z}(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

and the solution $\mathbf{x}(t)$ is given by

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{M}e^{\Lambda t}\,\mathbf{M}^{-1}\,\mathbf{x}_0 \\ &= \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} \end{aligned}$$

(ii) For unit step input, $r(t) = 1$, the solution of the state equation is the following equation.

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{x}_0 + £^{-1}\,[\Phi(s)\,\mathbf{B}\,R(s)] \\ &= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + £^{-1}\left\{\begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{-2}{s^2+3s+2} \\ \dfrac{1}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{1}{s}\right\} \\ &= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix} \end{aligned}$$

Simplifying the above equation we get,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{3}{2} - e^{-t} + \dfrac{1}{2}e^{-2t} \\ 1 - 2e^{-t} + 2e^{-2t} \end{bmatrix}$$

Example 7.22 *A discrete-time system has the following state and output equations.*

$$\begin{aligned} x_1(k+1) &= -\frac{1}{4}x_1(k) + u(k) \\ x_2(k+1) &= -\frac{1}{8}x_1(k) - \frac{1}{8}x_2(k) + u(k) \\ y(k) &= \frac{1}{2}x_1(k) \end{aligned}$$

Find the transfer function of the above system in z-domain when $x_1(0) = 0$ and $x_2(0) = 0$.

Solution It is easily seen that

$$\mathbf{A} = \begin{bmatrix} -\dfrac{1}{4} & 0 \\ -\dfrac{1}{8} & -\dfrac{1}{8} \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \mathbf{c}' = \begin{bmatrix} \dfrac{1}{2} & 0 \end{bmatrix}$$

The state transition matrix is z-domain is

$$\Phi(z) = (z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\left(z+\frac{1}{4}\right)\left(z+\frac{1}{8}\right)} \begin{bmatrix} z+\frac{1}{8} & 0 \\ -\frac{1}{8} & z+\frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{\left(z+\frac{1}{4}\right)} & 0 \\ \dfrac{1}{8\left(z+\frac{1}{4}\right)\left(z+\frac{1}{8}\right)} & \dfrac{1}{\left(z+\frac{1}{8}\right)} \end{bmatrix}$$

The transfer function in z-domain when $\mathbf{x}(0) = 0$ is then given by

$$H(z) = \frac{Y(z)}{R(z)} = \mathbf{c}'(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

$$= [\tfrac{1}{2}\ \ 0] \begin{bmatrix} \dfrac{1}{\left(z+\frac{1}{4}\right)} & 0 \\ -\dfrac{1}{8\left(z+\frac{1}{4}\right)\left(z+\frac{1}{8}\right)} & \dfrac{1}{\left(z+\frac{1}{8}\right)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{4z+1}$$

Example 7.23 *Compute the state transition matrix using Cayley–Hamilton approach for the system defined by the following equation.*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t)$$

Solution The Laplace transform method is quite convenient for analytical work since it yields answers in closed form. However, it is not very useful for machine computation and for large systems. The Cayley–Hamilton method presented in Section 7.4 provides a simple procedure for evaluating $e^{\mathbf{A}t}$ by expressing it exactly by a finite power series in $\mathbf{A}$. The formal procedure of evaluating $e^{\mathbf{A}t}$ is given in the three steps that follow.

1. Find the n eigenvalues of matrix $\mathbf{A}$.
2. If all the eigenvalues are distinct, solve n simultaneous equations given by

$$e^{\lambda_i t} = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \ldots + \alpha_{n-1}\lambda_i^{n-1}\ i = 1, 2, \ldots, n$$

If some or all the eigenvalues are repeated, the method must be modified.

3. The coefficients α_i obtained in step 2 when substituted in the following equation give the desired result.

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} + \ldots + \alpha_{n-1} \mathbf{A}^{n-1}$$

We will now use the example to illustrate the procedure. The characteristic equation is given by

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

Matrix **A** has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Equations to be solved can be written in following manner.

$$e^{\lambda_1 t} = e^{-t} = \alpha_0 - \alpha_1$$
$$e^{\lambda_2 t} = e^{-2t} = \alpha_0 - 2\alpha_1$$

Solving for α_0 and α_1 we get $\alpha_0 = 2e^{-t} - e^{-2t}$ and $\alpha_1 = e^{-t} - e^{-2t}$. Then the state transition matrix can be written in the following way.

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} - \alpha_1 \mathbf{A}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Example 7.24 *A discrete-time system has state equation given by*

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \boldsymbol{x}(k)$$

Use the Cayley–Hamilton approach to find its state transition matrix.

Solution The characteristic equation is

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda + 5 \end{vmatrix} = (\lambda + 2)(\lambda + 3) = 0$$

Clearly, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. We have

$$\Phi(k) = \mathbf{A}^k = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

and the equations to be solved are

$$(-2)^k = \alpha_0 - 2\alpha_1$$
$$(-3)^k = \alpha_0 - 3\alpha_1$$

Solving we obtain $\alpha_0 = 3(-2)^k - 2(-3)^k$ and $\alpha_1 = (-2)^k - (-3)^k$

Then

$$\Phi(k) = \left((-2)^k - (-3)^k\right)\begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} + \left(3(-2)^k - (-3)^k\right)\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-2)^k - 2(-3)^k & (-2)^k - (-3)^k \\ -6(-2)^k + 6(-3)^k & -2(-2)^k + 3(-3)^k \end{bmatrix}$$

Problems

7.1 For the electrical and mechanical circuits shown in Figs P-7.1(a) and (b), identify the set of state variables and their state-variable description.

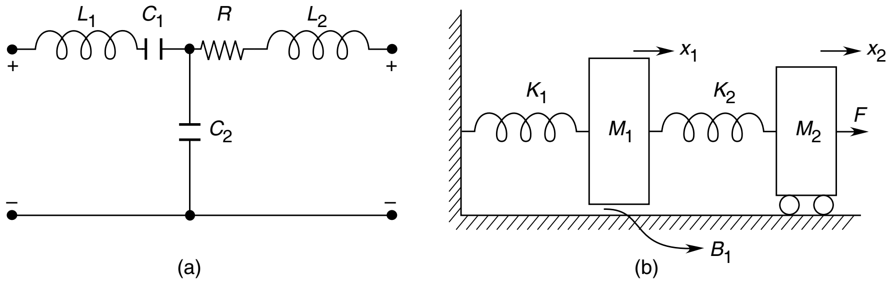


Fig. P-7.1

7.2 Obtain the state variable equation for the systems shown in Figs P-7.2(a) and (b).

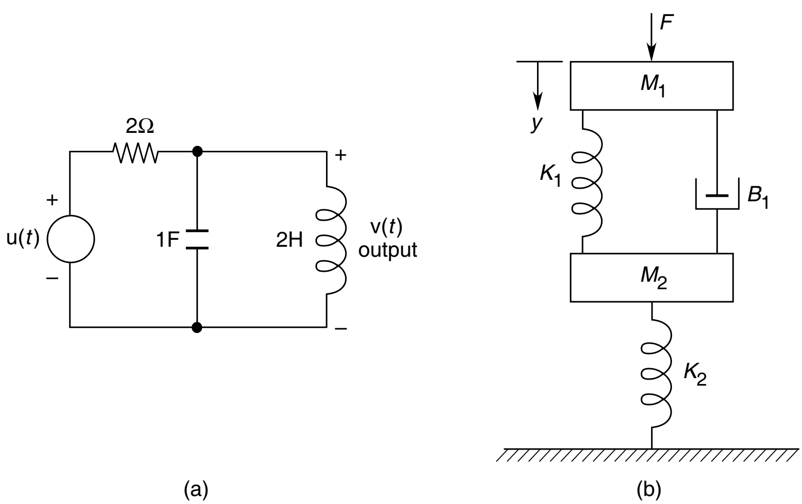


Fig. P-7.2

7.3 Construct the state variable model and signal flow graph of the state variables for a system described by the following differential equation.

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y = 4$$

7.4 A system is described by the following transfer function.

$$H(s) = \frac{Y(s)}{R(s)} = \frac{5(s+4)}{s(s+1)(s+3)}$$

(i) Determine the three different state models.

(ii) Determine the state transition matrix for one of these models.

7.5 Find the general expression for $e^{\mathbf{A}t}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

7.6 Find the general expression for $\mathbf{A}^k$.

(i) $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$ (ii) $\mathbf{A} = \begin{bmatrix} 1/2 & 0 \\ 2 & 3 \end{bmatrix}$

7.7 A system has the following state model.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Determine $\Phi(s)$ and $\Phi(t)$ for the system.

7.8 A system is described by the following state-variable model.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}r$$

$$y = [4 \quad 6 \quad 2]\mathbf{x}$$

Determine $H(s) = Y(s)/R(s)$

7.9 For a given system

$$\mathbf{A} = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix}$$

(a) Find the roots of the characteristic equation.

(b) Find the eigenvalues and eigenvectors.

7.10 A system is characterized by the following state equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r(t)$$

Compute the solution of the state vector for the input $r(t) = u(t)$ and initial condition $\mathbf{x}^T(0) = [1 \ \ 0]$.

7.11 A linear time-invariant system is described in the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r;\ r = \text{unit step}$$

$$y = [4 \quad 6 \quad 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Transform this state model into canonical form and obtain the solution for the state vector and output. Initial condition is given as $\mathbf{x}^T(0) = [0\ 0\ 1]$.

7.12 A linear time-invariant discrete-time system is characterized by the following state equation.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Compute the solution of the state equation assuming that

$$[x_1(0)\ x_2(0)] = [0 \quad 1]$$

7.13 A LTI system's state equation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k)$$

Compute the solution of the state equation for the following initial conditions.

$$[x_1(0)\ x_2(0)] = [0 \quad 1];\ r(k) \text{ unit step}$$

7.14 A LTI discrete-time system has

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ \mathbf{c}' = [0 \ \ 1] \text{ and } \mathbf{D} = [1]$$

Initial state is $\mathbf{x}^T(0) = [2 \ \ 1]$. For unit step input obtain

(a) $y(k)$ using the time domain method.
(b) $y(k)$ using the frequency domain method.

7.15 A discrete-time system has the following z-transfer function.

$$T(z) = \frac{2z^3 - 12z^2 + 13z - 6}{(z-1)^2(z-3)}$$

Determine the state model of the system in

(i) Phase variable form
(ii) Jordan canonical form

7.16 A discrete-time system is described by the following difference equa-tion

$$y(k+2) + 5y(k+1) + 6y(k) = u(k)$$

If $y(0) = y(1) = 1$ and $T = 1s$, is given

(i) Determine the state model in canonical form.

(ii) Find the state transition matrix.

(iii) Find the output $y(k)$.

7.17 A continuous time LTI system is described by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r; \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; r = \text{unit step}$$

Obtain the exact solution of the state equation and compare it with the solution obtained by discrete-time approximation with T equal to one tenth of the smallest time constant of the system.

7.18 Use the Cayley–Hamilton theorem to find $e^{\mathbf{A}t}$, if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

7.19 Show that for an $n \times n$ diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$e^{\Lambda t}$ is also diagonal with diagonal entries given by

$$e^{\lambda_i t}; i = 1, 2, \ldots n$$

7.20 Obtain a state model for the following.

(a) $\dddot{y} + 2\ddot{y} + 6\dot{y} + 7y = 4r$ (b) $\dfrac{Y(s)}{R(s)} = \dfrac{s^2 + 3s + 2}{s(s+1)(s+3)}$

7.21 A linear time-invariant system is described by the following state model.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_1(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(k)$$

$$y(k) = x_1(k)$$

Transform this state model into a canonical form and therefrom obtain the explicit solution for the state vector and the output when the control force $r(k)$ is a unit step sequence and initial state vector is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Stability Analysis of LTI Systems 8

Introduction

In Chapters 2 and 3 we have discussed system analysis techniques. Here, we focus our attention on the stability of the system behaviour. While rigorous definitions will be addressed in the chapter, in simple terms stability implies that a small change in input or the state of the system itself does not change system output drastically.

In this chapter we introduce the analytical techniques to determine the stability of LTI systems both continuous and discrete-time. Dynamic response of these systems has been studied in details in Chapters 2 and 3 from where the stability determination is just another step forward.

Concepts of stability have already been introduced in Section 1.9 and tests of stability in LTI systems have been addressed in Section 2.19 for continuous-time systems and in Section 3.10 for discrete-time systems. These tests are directly based on system's dynamic response as time.

8.1 GENERAL CONCEPTS OF STABILITY

It has been shown in Section 2.4 that the response of LTI system may be decomposed into zero-input response and zero-state response. We now examine the stability of both these responses.

Stability of zero-input response If a system is excited by only initial conditions (no input is impressed on the system) and the response decays to zero state with time, the system is said to be **asymptotically stable**.

Stability of zero-state response If the response of a system is bounded for bounded input, the zero-state response of the system is called BIBO (bounded-input-bounded-output) stable.

Fortunately for LTI systems BIBO stability implies asymptotic stability and vice-versa, which is not the case with the nonlinear systems. Since, we are concerned with LTI systems, we will discuss only **BIBO** stability.

8.2 STABILITY OF CONTINUOUS-TIME SYSTEMS

For BIBO stability of continuous-time systems, let us consider transfer function of a system relating its zero-state response to its input, which is expressed in the following manner.

$$H(s) = \frac{Y(s)}{R(s)} = \frac{N(s)}{D(s)}; \text{ order of } D(s) > \text{order of } N(s) \tag{8.1a}$$

Then the output is given as

$$Y(s) = H(s)\,R(s) \tag{8.1b}$$

The zero-state response in time-domain can be written by the following convolution integral.

$$y(t) = \int_0^\infty h(\lambda) r(t-\lambda)\mathrm{d}\lambda \tag{8.2}$$

where $h(t) = \mathcal{L}^{-1}H(s)$ = impulse response.

Taking absolute value of Eq. (8.2), we may write the inequality as

$$|y(t)| \le \int_0^\infty |h(\lambda)|\,|r(t-\lambda)|\,\mathrm{d}\lambda \tag{8.3}$$

By the definition of **BIBO** stability for bounded input ($|r(t)| \le \partial_1 \le \infty$), the output should also be bounded ($|y(t)| \le \partial_2 < \infty$) for all t $(0, \infty)$. The inequality in Eq. (8.3) can then be expressed as

$$|y(t)| \le \partial_1 \int_0^\infty |h(\lambda)|\,\mathrm{d}\lambda \le \partial_2 \tag{8.4}$$

It means that, $\int_0^\infty |h(\lambda)|\,\mathrm{d}\lambda$ should be finite. Therefore, BIBO stability depends on nature of impulse response of the system. We know that system impulse response may be determined by the poles of its transfer function or the zeros of

$$\begin{aligned} q(s) &= D(s) \\ D(s) &= a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0 \\ &= \textbf{characteristic equation} \end{aligned} \tag{8.5}$$

For various types of roots of the characteristic equation, the nature of impulse response and BIBO stability are listed in Table 8.1.

Investigating Table 8.1 reveals that stability is determined by the knowledge of roots of the characteristic equations. It is time consuming and cumbersome to determine the roots for answering the stability question. In fact a number of analytical methods are available, which facilitate establishing stability without actually determining roots of $q(s)$.

Table 8.1

Roots of q(s)	*Nature of impulse response*	*BIBO Stability*
1. All the roots have negative real parts.	Impulse response is bounded and decays to zero.	$\int_0^\infty \lvert h(\lambda)\rvert d\lambda$ is finite, BIBO stable.
2. At least one root has a positive real part.	Impulse response is unbounded.	$\int_0^\infty \lvert h(\lambda)\rvert d\lambda$ is infinite, BIBO unstable.
3. Repeated roots on j_-axis or origin.	Impulse response is unbounded.	$\int_0^\infty \lvert h(\lambda)\rvert d\lambda$ is infinite, BIBO unstable.
4. Non-repeated j_-axis roots or at origin.	Impulse response will have either constant amplitude oscillation or constant value ($h(t)$ bounded).	$\int_0^\infty \lvert h(\lambda)\rvert d\lambda$ is infinite. Since $\lvert h(t)\rvert$ is bounded, the system is said to be **marginally or limitedly stable.**

8.3 TEST OF STABILITY BY ANALYTICAL METHODS

It has been shown above that a system is stable provided all the roots of its characteristic equation $q(s) = 0$ have negative real parts. This can be checked without actually finding these roots, i.e., by determining if the characteristic polynomial is Hurwitz. We shall, therefore, present algebraic methods of finding if the polynominal $q(s)$ is Hurwitz. This ensures that all the roots have negative real parts and so the system is stable.

Necessary Condition

The necessary condition for a system to be stable is that the coefficients (a_i's) of its characteristic polynomial are all positive non-zero and no missing terms.

$$q(s) = a_0\, s^n + a_1\, s^{n-1} + a_2\, s^{n-2} + ... + a_n \tag{8.6}$$

While this condition is necessary for system stability, it is not sufficient to ensure the same.

Example 8.1 *Check that the following polynomials satisfy the necessary condition of being Hurwitz.*

(i) $s^3 + 4s^2 + 9s + 10$

(ii) $s^3 + s^2 + 4s + 30$

Solution

(i) This polynomial has all positive coefficients and there are no missing coefficients. It, therefore, satisfies the necessary condition of being Hurwitz.

Let us check the roots, of this polynomial.

$$\begin{aligned} q(s) &= s^3 + 4s^2 + 9s + 10 \\ &= (s + 2)\,(s^2 + 2s + 5) \\ &= (s + 2)\,(s + 1 + 2j)\,(s + 1 - 2j) \end{aligned}$$

Hence the roots are

$$s = -2, -(1 + 2j), (-1 + 2j)$$

We find that the polynomial has all roots with negative real parts. So the polynomial is Hurwitz and it satisfies the necessary condition.

(ii) $$q(s) = s^3 + s^2 + 4s + 30$$

In this polynomial all the coefficients are positive and there are no missing coefficients. Let us check its roots.

$$\begin{aligned} q(s) &= (s + 3)\,(s^2 - 2s + 10) \\ &= (s + 3)\,(s - 1 + 3j)\,(s - 1 - 3j) \end{aligned}$$

We can see the roots are

$$s = -3, +1 - 3j, +1 + 3j$$

The real parts of the complex conjugate roots are positive. Therefore, despite the fact that the polynomial has all positive non-zero coefficients, it is not Hurwitz. From these two examples we conclude that the condition for a Hurwitz polynomial stated above is necessary but not sufficient. However, for a polynomial of order two the necessary condition is sufficient as well.

Necessary and Sufficient Conditions

Routh Stability Criterion This criterion checks both necessary and sufficient condition for a polynomial $q(s)$ to be Hurwitz and therefore, the characteristic equation $q(s) = 0$ pertains to a stable system. Consider the following the n^{th} order characteristic equation.

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

In order to check the stability of the system, we first order its coefficients in the Routh array.

Routh Array

$$\begin{array}{llllllllll}
s^n & a_0 & a_2 & a_4 & a_6 & . & . & . & . \\
s^{n-1} & a_1 & a_3 & a_5 & . & . & . & . & \\
s^{n-2} & b_1 & b_2 & b_3 & . & . & . & & \\
s^{n-3} & c_1 & c_2 & c_3 & . & . & & & \\
s^{n-4} & d_1 & d_2 & & & & & & \\
. & . & . & & & & & & \\
. & . & . & & & & & & \\
s^2 & e_1 & a_n & & & & & & \\
s^1 & f_1 & & & & & & & \\
s^0 & a_n & & & & & & &
\end{array}$$

The coefficients $b_1, b_2, ..,$ are evaluated from the two rows above it in the following manner.

$$b_1 = (a_1a_2 - a_0a_3)/a_1;$$
$$b_2 = (a_1a_4 - a_0a_5)/a_1;...$$

In a similar way, the coefficients of $4^{\text{th}}, 5^{\text{th}}, .., n^{\text{th}}$ and $(n + 1)^{\text{th}}$ rows are evaluated, e.g.

$$c_1 = (b_1a_3 - a_1b_2)/b_1;$$
$$c_2 = (b_1a_5 - a_1b_3)/b_1;...$$

and

$$d_1 = (c_1b_2 - b_1c_2)c_1;$$
$$d_2 = (c_1b_3 - b_1c_3)/c_1;...$$

The process is continued till we reach the row corresponding to s^0.

As the rows are being generated by the steps given above, any row can be divided by a positive constant to avoid dealing with large size numbers. Also missing elements are to be regarded as zeros.

The Routh stability criterion is stated below.

Necessary and sufficient condition, for a system to be stable, is that all the terms of the first column of the Routh array of its characteristic polynomial have the same sign (positive for $a_0 > 0$). Otherwise, the system is unstable with number of roots of the characteristic polynomial with positive real parts being equal to the number of sign changes in the terms of the first column.

Special Cases In generating the Routh array certain special cases arise, which lead to the breakdown of the procedure to complete the array. These cases with remedial measures are described below.

Case 1 First term in a row happens to be zero, while at least one term in the rest of the row is non-zero. Further rows cannot be generated now as division by 0 is encountered.

Remedial Measure Replace first term zero by ε, a small positive number and then complete the array. In the end let $\varepsilon \to \infty$ to determine the signs of the first column terms.

Example 8.2 Consider the characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

The Routh array is

$$\begin{array}{llll}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 5 \\
s^3 & \varepsilon & -2 & \\
s^2 & \dfrac{2\varepsilon - 2}{\varepsilon} & 5 & \\
s^1 & \dfrac{-4\varepsilon - 4 - 5\varepsilon^2}{2\varepsilon + 2} \to -2 & & \\
s_0 & 5 & &
\end{array}$$

Sign of the first term of s^2 row is given by the limit

$$\underset{\varepsilon \to 0}{\mathrm{L}t} \frac{2\varepsilon - 2}{\varepsilon} \quad \Rightarrow \quad \text{negative sign}$$

It is observed from the sign of the first-column terms that there are two changes in sign. Hence the given characteristic equation represents an unstable system, even though all the coefficients of the characteristic polynomial are positive.

Case 2 In forming the Routh array the terms of a row are all zero and so the stability test cannot proceed further.

This situation arises when there are symmetrically located roots in the two half s-planes (pair of real roots with opposite sign or pair of complex conjugate roots on imaginary axis or complex conjugate roots forming quadrates in the s-plane).

Remedial Measure The remedy is to form an auxiliary polynomial $A(s)$ whoes coefficients are the elements of the row above the row of zeros. The order of this polynomial is always even and the row of zeros is replaced by the coefficients of the derivatives of $A(s)$ and then proceed to complete the array.

Example 8.3 *Consider the following sixth-order characteristic polynomial*

$$s^6 + 2s^5 + 4s^4 + 4s^3 + 5s^2 + 2s + 2$$

Check the stability of the system.

Solution We get the Routh array in the following form.

$$\begin{array}{lllll}
s^6 & 1 & 4 & 5 & 2 \\
s^5 & 2 & 4 & 2 & \\
s^4 & 2 & 4 & 2 & \\
s^3 & 0 & 0 & &
\end{array}$$

Form the auxiliary polynomial $A(s)$ from the coefficient of s^4 row.

$$A(s) = 2s^4 + 4s^2 + 2$$

$$\frac{dA(s)}{ds} = 8s^3 + 8s$$

Proceeding furhter

s^4	2	4	2
s^3	8	8	(divide by 8)
s^3	1	1	
s^2	2	2	
s^1	0	0	

The second auxiliary polynomial is

$$A'(s) = s^2 + 1$$

$$\frac{d\,A'(s)}{ds} = 2$$

The later columns of Routh array are

s^1	2	0
s^0	1	

The roots, which cause two consecutive rows of zero to appear are

$$A(s) = (s^4 + 2s^2 + 1) = (s^2 + 1)^2 = 0$$

$$A'(s) = s^2 + 1 = 0$$

The corresponding roots are

$$s = \pm j1,\ s = \pm j1;\ s = \pm j1$$

There are three pairs of complex conjugate roots on the $j\omega$ axis. The system indeed is unstable (refer to Table 8.1).

Example 8.4 *Using the Routh criterion check the stability of the system whose characteristic polynomials is the following.*

$$2s^4 + 5s^3 + 5s^2 + 2s + 1$$

Solution Preparing Routh table for the given polynomial, we get

s^4	2	5	1
s^3	5	2	0
s^2	$\frac{5\times5-2\times2}{5} = 21/5$	$\frac{5\times1-2\times0}{5}$	

s^1	$\dfrac{(21/5)\times 2 - 5\times 1}{21/5}$
	$= 21/17$
s^0	1

Examining the first column of the table, we find that all entries are positive. So the system is stable and the polynomial is Hurwitz.

Example 8.5 *Form the Routh table of the given characteristic polynomial. What conclusion do you draw regarding system stability and characteristic roots?*

$$s^4 + s^3 + 12s^2 + 12s + 36$$

Solution

s^4	1	12	36
s^3	3	12	
s^2	2	9 ; divided by	4
s^1	–3/2		
s^0	9		

It is observed from the table that the elements of the first row changes sign at s^1 row. So the system is unstable. Further there are two changes in sign (plus to minus to plus at s^1 row), so the polynomial has two roots with positive real parts.

Example 8.6 *Find the range of K parameter for the following characteristic polynomial to be stable.*

$$s^4 + 2s^3 + 2s^2 + s + K$$

Solution We can get the Routh table in the following from.

s^4	1	2	K
s^3	2	1	
s^2	1.5	K	
s^0	1– (2/1.5) K		

The system is stable if

$$1 - (2/1.5)\, K > 0$$

or

$$K < 0.75$$

Example 8.7 *Consider a feedback system with block diagram as shown in Fig. 8.1. Find the limit on gain K for the system to be stable. What is the value of K for sustained oscillation of the system and also the frequency of oscillation?*

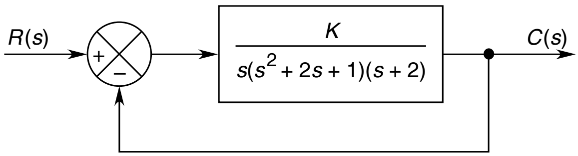


Fig. 8.1

Solution System's characteristic equation is given by the denominator of the following equation.

$$1 + G(s) = 1 + \frac{K}{s(s^2 + 2s + 1)(s + 2)}$$

or

$$\begin{aligned} D(s) &= s(s^2 + 2s + 1)\,(s + 2) + K \\ &= s^4 + 4s^3 + 5s^2 + 2s + K \end{aligned}$$

Routh array is formed below.

s^4	1	5	K
s^3	4	2	
s^2	9/2	K	
s^1	$(4/9K - 1)$		
s^0	1		

For the system to be stable

$$(4/9K - 1) > 0$$
$$K > 9/4$$

For sustained oscillation

$$K = 9/4$$

This makes s^1 row as zeros. This in turn forms the following auxiliary polynomial.

$$(9/2)\,s^2 + K = 0$$

or

$$(9/2)\,s^2 + 9/4 = 0$$

or

$$s^2 = -\tfrac{1}{2}$$

or

$$s = \pm j\,\frac{1}{\sqrt{2}}$$

Frequency of sustained oscillation is

$$\omega_n = \frac{1}{\sqrt{2}}\ \text{rad/s}$$

Example 8.8 *The schematic in block form of the turning control of a tracked vehicle is shown in Fig. 8.2. Turning is achieved by applying different track torque to right and left drive wheels. This creates a differential in the wheel speeds which results in turning the vehicle and changing the direction of the travel. The difference in the two track speeds (proportional to wheel speed) is fedback to the power train and controller which act to align the direction of travel as commanded by the steering.*

Analyse the system for its stability.

Solution The characteristic equation of this signal-loop feedback system is

$$1 + G_c(s)\,G(s) = 0$$

Substituting values, we get

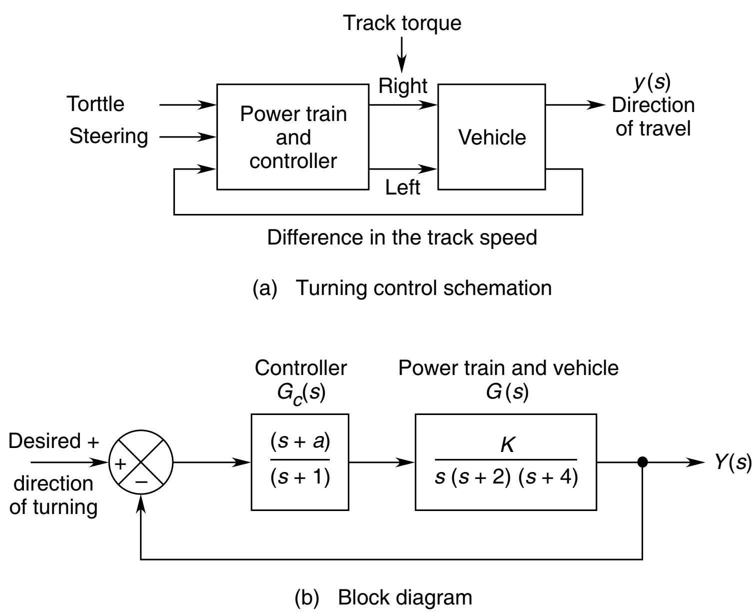


Fig. 8.2 *Turning control of tracked vehicle*

$$1 + \frac{K(s+a)}{s(s+1)(s+2)(s+4)} = 0$$

or

$$s(s + 1)\ (s + 2)\ (s + 4) + K(s + \text{a}) = 0 \tag{i}$$

We can organize this as a polynomial in s, which is written as the following.

$$s^4 + 7s^3 + 14s^2 + (8 + K)\ s + Ka = 0 \tag{ii}$$

The Routh array is

s^4	1	14	Ka
s^3	7	$(8 + K)$	
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

where

$$b_3 = \frac{90 - K}{7}; \quad c_3 = \frac{b_3(8+K) - 7Ka}{b_3}$$

For all the first column terms to have positive sign following conditions should be met

$$K < 90 \tag{iii}$$

$$Ka > 0 \tag{iv}$$

$$(K + 8)\ (90 - K) - 49\ Ka > 0 \tag{v}$$

If K is positive (which is the normal case), then 'a' is positive. The condition of stability of Eq. (v) determines area in the first quadrant of $K - a$ coordinates. The limit of K is the inequality of Eq. (iii). The region of stability is shown in Fig. 8.3 by calculating the curve bounding it by converting Eq. (v) into an equality.

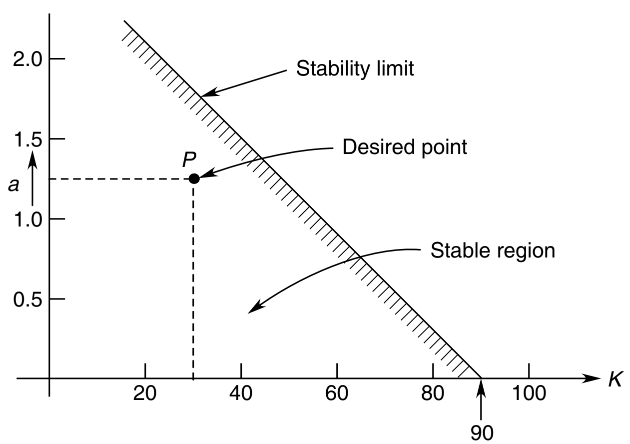


Fig. 8.3 *Stable region for turning control*

The desired point P is fixed from other considerations, which are beyond the scope of this book.

8.4 STABILITY OF LTI DISCRETE TIME SYSTEMS

Most of the general concepts discussed for continuous-time systems are applicable to LTI discrete-time systems as well. Now we shall discuss BIBO stability of discrete-time systems in brief.

Zero-state response of a LTI discrete-time system can be expressed as

$$y(k) = \sum_{i=0}^{k} h(k-i)r(i) \tag{8.7}$$

where y, h and r are respectively the output, impulse response and input to the system.

Consider a bounded input $r(k)$, i.e.,

$$|r(k)| \le \partial_1 < \infty; \text{ for all } k \ge 0 \tag{8.8}$$

For BIBO stability this input (bounded) should produce bounded output sequence. This requires that the impulse response is bounded, i.e.

$$\sum_{i=0}^{\infty} |h(i)| \le \partial_2 < \infty \tag{8.9}$$

Such an impulse response is also known as BIBO stable. The impulse response of a discrete-time system can be expressed as

$$h(k) = Z^{-1}\Sigma H(z)$$

where $H(z)$ is the impulse transfer function of the system, which can be expressed as

$$H(z) = \frac{N(z)}{D(z)} = \frac{b_m z^m + b_1 z^{m-1} + \cdots + b_0}{a_n z^n + a_1 z^{n-1} + \cdots + a_0}; \; m < n \tag{8.10}$$

It has been shown in Section 3.8 and Fig. 3.11 that the impulse response of an LTI discrete-time system decays (exponentially or in an oscillatory manner) as the count $k \to \infty$ provided the poles of $H(z)$ lie within a unit circle in the z-plane. Therefore, as stated in Section 8.1, this is the condition for BIBO stability of a discrete system.

Example 8.9

(a) *Find the condition for an LTI discrete-time system described as*

$$y(k+1) - \alpha y(k) = \alpha r(k)$$

to be BIBO stable.

(b) *Check the BIBO stability of the following.*

$$H(z) = \frac{z-2}{z(z-0.8)}$$

Solution Taking the Z-transform

(a)
$$(z - \alpha)\, Y(z) = \alpha R(z)$$

or

$$H(z) = Y(z)/R(z) = \alpha/(z - \alpha)$$

Pole is located at $z = \alpha$. We know from Chapter 3 that the impulse response decays if $|\alpha| < 1$, which is then the condition for the system to be BIBO stable.

(b)
$$H(z) = \frac{z-2}{(z-0.8)}$$

The systems has a pole at $z = 0.8$. As it lies within the unit circle, therefore, the system is BIBO stable.

As in the case of continuous-time systems, methods are available to determine stability of a discrete-time system without actually finding the location of its poles in the z-plane. We will now advance the tests for this purpose.

Jury Test

This is an analytic test to determine the BIBO stability and is analogous to Routh's test in continuous-time systems.

Let the characteristic polynomial of the system be

$$q(z) = a_n + a_{n-1}z^{n-1} + \dots a_0 = 0;\ a_n > 0 \tag{8.11}$$

Necessary Condition The necessary condition for stability of $D(z)$ of order n is

$$q(1) > 0 \tag{8.12}$$

$$(-1)^n\, q(-1) > 0 \tag{8.13}$$

For Example

$$q(z) = 2z^2 + z + 0.3$$
$$q(1) = 3.3 > 0$$

and

$$(-1)^2\, q(-1) = 2 - 1 + 0.3 = 1.3$$

Thus above polynomial satisfies the necessary condition for stability.

Sufficient Condition For the sufficient condition we prepare JURY TABLE from the coefficients of $q(z)$ as given below.

Row	z^0	z^1	z^2 .... z^{n-2} z^{n-1} z^n
1	a_0	a_1	a_2 .... a_n
2	a_n	a_{n-1}	a_{n-2} .... a_0
3	b_0	b_1	b_2 .... b_{n-1}
4	b_{n-1}	b_{n-2}	b_{n-3}.................. b_0
5	c_0	c_1	c_2 .... c_{n-2}
6	c_{n-2}	c_{n-3}	c_{n-4} .... c_0

where

$$b_j = \begin{vmatrix} a_0 & a_{n-j} \\ a_n & a_j \end{vmatrix}$$

and

$$c_j = \begin{vmatrix} b_0 & b_{n-1-j} \\ b_{n-1} & b_j \end{vmatrix}$$

From the table the sufficient conditions of stability are

$$\left.\begin{aligned} |a_0| &< |a_n| \\ |b_0| &> |b_{n-1}| \\ |c_0| &> |c_{n-2}| \end{aligned}\right\} (n-1) \text{ conditions} \tag{8.14}$$

This is now demonstrated by examples.

Example 8.10 *Check the BIBO stability of the following polynomial.*

$$q(z) = 2z - 0.2z^2 - 0.2z^3$$

Solution

Necessary Condition

$$q(1) = 2 - 0.2 - 0.2 > 0; \text{ satisfied}$$

$$(-1)^3\, q(-1) = -1\,[-2 - 0.2 + 0.24] > 0; \text{ satisfied}$$

Sufficient Conditions

The Jury Table is prepared below.

Row	z^0	z^1	z^2	z^3
1	0	0.24	– 0.2	2
2	2	– 0.2	– 0.24	0
3	– 4	+ 0.4	0.48	
4	0.48	0.4	– 4	
5	15.7	– 1.8		
6	– 8.0	15.7		

Let us check the constraints

$$\begin{array}{lll} |a_0| < a_n & \Rightarrow & |0| < 2;\ \text{satisfied} \\ |b_0| > |b_{n-1}| & \Rightarrow & |-4| > 0.48;\ \text{satisfied} \\ |c_0| > |c_{n-2}| & \Rightarrow & |15.7| > |-1.8|;\ \text{satisfied} \end{array}$$

Thus the polynomial satisfies the sufficients conditions and is BIBO stable.

Bilinear Transformation

This transformation transforms the unit circle in the z-plane, which is the region of stability, into the left-half of the transformed plane, so that the Routh stability criterion could be applied for a discrete-time LTI system also.

Bilinear transformation defined below accomplishes the above requirement.

$$\mathrm{z} = \frac{1+r}{1-r}$$

or

$$r = \frac{z-1}{z+1} \tag{8.14}$$

The unit circle in z-plane, shown in Fig. 8.4 is described by

$z = e^{j\theta}$; θ varying anticlockwise from $-\pi$ to $+\pi$.

Substituting this in Eq (8.14), we get

$$r = \frac{e^{j\theta}-1}{e^{j\theta}+1} = \frac{e^{j\theta/2}-e^{-j\theta/2}}{e^{j\theta/2}+e^{-j\theta/2}}$$

$$= \tanh(j\theta/2) = j\tan(\theta/2) = j\omega_r \tag{8.15}$$

It is seen from this equation that as θ goes through $-\pi \to 0 \to \pi$, ω_r moves along the $j\omega_r$-axis in the r-plane from $-\infty \to 0 \to \infty$. This mapping is indicated in Fig. 8.4. It is observed from both these figures that the region inside the unit

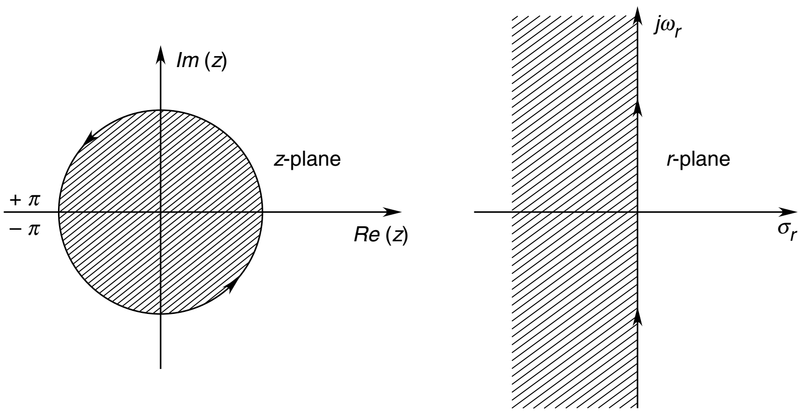


Fig. 8.4

circle in the z-plane maps into the left half of the r-plane. Therefore, the stability criterion, i.e., the roots of the characteristic polynomial in z should lie inside that of the unit circle, modifies to that the roots of the characteristic polynomial in r should lie in the left half of the r-plane, i.e., the roots should have negative real parts. Hence we can use the Routh stability test on the characteristic polynomial in r.

Example 8.11 *Investigate the stability of the discrete-time system with z-domain characteristic polynomial.*

$$z^3 - 0.3\,z^2 + 0.1\,z - 0.1 = 0$$

Use bilinear transformation.

Solution

$$z^3 - 0.3\,z^2 + 0.1z - 0.1 = 0 \tag{i}$$

According to bilinear transformation

$$z = \frac{1+r}{1-r} \tag{ii}$$

Substituting for z of Eq. (ii) in Eq (i), we get

$$\left(\frac{1+r}{1-r}\right)^3 - 0.3\left(\frac{1+r}{1-r}\right)^2 + 0.1\left(\frac{1+r}{1-r}\right) - 0.1 = 0$$

or

$$(1+r)^3 - 0.3(1+r)^2(1-r) + 0.1\,(1+r)\,(1-r)^2 - 0.1\,(1-r)^3 = 0$$

or

$$0.7 + 2.9r + 2.9r^2 + 1.7r^3 = 0$$

Routh table

r^3	1.7	2.9
r^2	2.9	0.7
r	7.22/2.9	
r^0	1	

As there is no sign change in the first column terms, the system is stable.

Problems

8.1 Determine the BIBO stability of a system with impulse response as below.

(a) $h(t) = \begin{cases} \sin t \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$ (b) $h(t) = \dfrac{1}{t+1}$

8.2 Determine the BIBO stability of the transfer functions given below.

(a) $H(s) = \dfrac{s^2 - 1}{s^3 + 2s^2 - 1}$ (b) $H(s) = \dfrac{s^2 - 1}{s^3 + 3s^2 - 4s + 2}$

8.3 Which of the following polynomials pertain to stable systems?

(a) $s^5 + 3s^4 + s^2 + 2s + 10$

(b) $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$

(c) $-s^4 - 2s^3 - 3s^2 - 4s - 1$

8.4 Consider a third-order system with the following characteristic equation.

$$s^3 + 2.5s^2 + 3s + 1 = 0$$

Is this system stable? If so, determine if all its roots have real parts less than – 1.

Hint: Shift origin of s-plane to $s = -1$ by the following substitution and check stability in ν-plane

$$s = \nu - 1$$

8.5 Use Routh criterion to determine if the following characteristic polynomials represent a stable system. If any system is unstable, determine the number of right hand roots, it has.

(a) $s^4 + 5s^3 + 12s^2 + 14s + 8 = 0$

(b) $s^4 + 2s^3 + 3s^2 + 2s + 8 = 0$

(c) $s^5 + 6s^4 + 7s^3 + 6s^2 + 12s + 8 = 0$

(d) $s^5 + 3s^4 + 5s^3 + 5s^2 + 10s + 8 = 0$

8.6 The forward transfer function of unity negative feedback system is

$$G(s) = \frac{K}{(s+1)(s+4)(s^2+2s+2)}$$

Determine the range of K for which the system is stable. What will happen if K is set at the marginal value (just beyond which the system becomes unstable)?

At this value of K can the system be used as an oscillator?

8.7 Consider the following characteristic polynomial of a system.

$$s^3 + 2s^2 + (k + b)\,s + (2k + 1) = 0$$

At what values of k and b the polynomial will have a pair of imaginary roots?

8.8 Consider the following characteristic polynomial of order four in general form.

$$a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$$

Determine the conditions to be met by the coefficients (a's) such that all roots have negative real parts.

8.9 Determine the stability of discrete-time systems described by the z-domain characteristic equations.

(i) $z^3 - 0.5z^2 + 2.49z - 0.486 = 0$ (ii) $z^4 + 2z^3 - 2.24z^2 - 1.24z + 0.24 = 0$

Obtain the answers by

(i) Jury test; and

(ii) Bilinear transformation

8.10 Determine the z-transform of two cascaded systems each described by the following discrete-time equation.

$$y(k) = 0.5y(k-1) + r(k)$$

Check the stability of the system.

9 Analog and Digital Filter Design

Introduction

Filter is a frequency selective network which allows certain range of frequencies of the input signal to pass and to suppress other frequencies which are not desired in the output. Filters are classified according to their frequency domain behaviour, which was specified in terms of their magnitude and phase response. Filters can be classified into the following types.

(i) low-pass (ii) high-pass
(iii) band-pass (iv) band-stop (or reject)

The ideal filter characteristics in frequency domain (amplitude) response are sketched in Figs 9.1(a), (b), (c) and (d). As these are not realizable physically, so various approaches to filter design have been developed which closely approximate the ideal behaviour demanded.

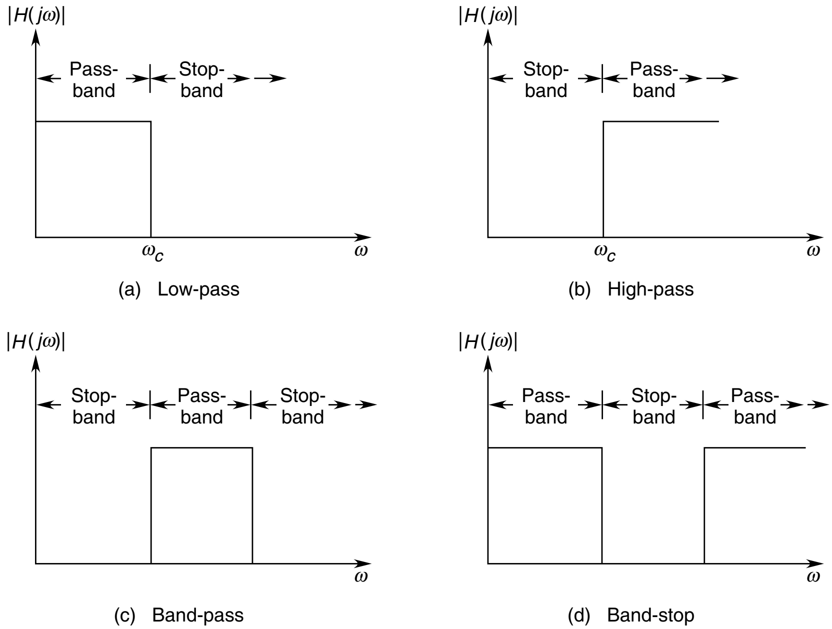


Fig. 9.1

Digital filter design methods depend upon analytical approximations and are related to analog filter design procedures. It is started with analog prototype design, proceeding therefrom to digital filter design, i.e., IIR (infinite inpulse response), and bilinear transformations. However FIR method uses the filter specifications directly in the digital domain itself.

This chapter also deals with parameter quantization effects alongwith filter realization.

9.1 ANALOG FILTERS—ANALYTICAL APPROXIMATIONS AND DESIGN

In this section first we will be examining different analytical approximations used for low-pass filters and then proceed to design other filters using frequency transformations. The term analog filter implies that the filter's magnitude and phase response are continuous function of frequency over a range of frequencies. Following are some of the widely used low-pass filter (LPF) functions approximating the ideal filter characteristics.

(a) Butterworth functions
(b) Chebyshev functions
(c) Elliptical functions

Properties of an Ideal Filter–group Delay and Phase Delay

If an input $r(t)$ is applied to a distortionless filter, its output is

$$y(t) = Gr(t - \tau);\ \text{only change in amplitude and time shift permitted} \tag{9.1}$$

Fourier transforming gives the frequency response of the filter as

$$\overline{H}(\omega) = Ge^{-j\omega\tau};\ \omega = 2\pi f \tag{9.2}$$

whose amplitude and phase response are

$$\left.\begin{aligned} A(\omega) &= G \\ \phi(\omega) &= -\,\omega\tau \end{aligned}\right\} \tag{9.3}$$

Instead of specifying the phase response of a filter, the group delay is specified, which is defined as

$$T_g(\omega) = -\frac{d}{d\omega}\phi(\omega) \tag{9.4}$$

For the ideal filter of Eq. (9.2) and by the use of definition (9.4), we have

$$T_g(\omega) = \tau \tag{9.5}$$

It is then concluded that for a distortionless filter, its gain and group delay are constant over the non-zero range of the input spectrum.

When a single spectral component is applied to a linear filter (LTI), the output is sinusoidal though its amplitude and phase (w.r.t reference) may change. Thus for single frequency input LTI filter is always distortionless.

Let the filter input be

$$r(t) = A \cos \omega t \tag{9.6}$$

The output can be written as

$$y(t) = B \cos (\omega t + \theta) \tag{9.7}$$

where

$$\theta = \text{phase shift}$$

The output can also be expressed as

$$y(t) = B \cos \omega (t - t_0) \tag{9.8}$$

where

$$t_0 = -\theta/\omega \tag{9.9}$$

which is the phase delay (time-wise). In general the phase delay is written as

$$T_p(\omega) = -\frac{\phi(\omega)}{\omega} \tag{9.10}$$

where

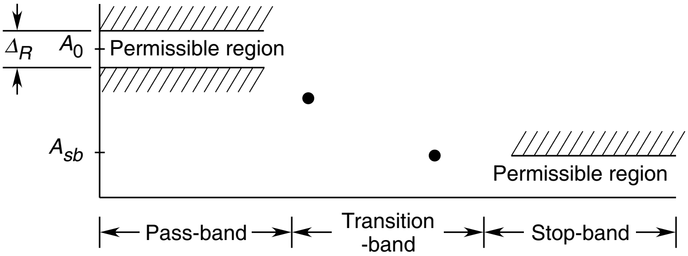
$\phi(\omega)$ = filter phase shift

It is concluded that ideal filters have constant group and phase delays.

However, as explained in Section 2.9, an ideal filter is non-causal and hence its physical realization is not possible. It is therefore, necessary to approximate the transfer function or the characteristics of an ideal filter to that of a realizable filter. Deviation from the ideal amplitude characteristic is called amplitude distortion whereas deviation from the ideal (or linear) phase characteristics is called phase distortion. Filters, applicable in the voice communication are designed such that its amplitude distortion is minimal whereas phase distortion, to some extent, may be tolerated. This is because of the fact that human ear is relatively insensitive to phase distortion. There are some other applications like video or image signal processing where linear phase characteristics is very much desirable, however, amplitude distortion may be allowed to some extent. We, therefore, observe that the characteristics of an analog filter has to be approximated such that on one hand it meets the desired specifications within given tolerance and on the other hand it must be physically realizable.

Approximating Ideal Filters by Practical Filters

Ideal filters are not physically realizable. Therefore, some variation about the nominal amplitude (gain) A_0 has to be permitted in the pass-band to achieve realizabililty. Also the stop-band has to be relaxed to a transition band as shown in Fig. 9.2(a). The variation in the pass-band is limited to a peak-to-peak value of

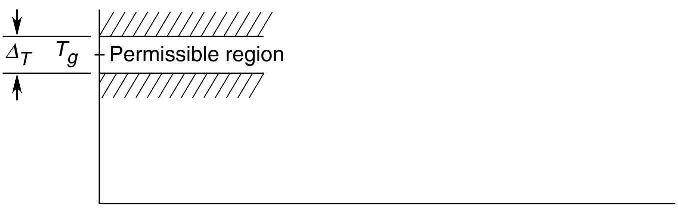


(a) Specification of amplitude response

Fig. 9.2 *Approximation bands in practical filters*

Δ_R. Also in the stop-band the amplitude must be lower than certain value A_{sb}. Just coming out of pass-band and entering the stop-band through a transition-band, the response passes through two points shown by heavy dots. The edge of the pass-band is defined as a frequency where the amplitude attenuates to a value of $1/\sqrt{2}$ (0.707 or 3dB) of the nominal gain. This requirement could be made more stringent.

Nominal group delay T_g and allowable variation Δ_T in the pass-band are shown in Fig. 9.2(b).



(b) Specification of group delay

Fig. 9.2 *Approximation bands in practical filters*

Butterworth Low-Pass Filter (LPF)

The magnitude function of the n^{th} order Butterworth filter is

$$|H_{Bn}(j\omega)| = \frac{1}{\sqrt{\left[1 + (\omega/\omega_c)^{2n}\right]}}; \ n = \text{positive integer,}$$

$$\omega_c = \text{cut-off frequency} \tag{9.11}$$

Normalizing ω w.r.t ω_c (or equivalently letting $\omega_c = 1$) Eq. (9.11) takes the following simpler form.

$$|H_{Bn}(j\omega)| = \frac{1}{\sqrt{(1 + \omega^{2n})}} \tag{9.12}$$

where

ω = normalized frequency

The magnitude of this function (Eq.(9.12)) decreases monotonically from $\omega = 0$ onwards with typical values as

$$|H_{Bn}(j\omega)|_{\omega=0} = 1 \tag{9.13}$$

and

$$|H_{Bn}(j\omega)|_{\omega=\omega_c=1} = \frac{1}{\sqrt{(1+1)}} = \frac{1}{\sqrt{2}} \tag{9.14}$$

Equation (9.14) is independent of n (see Fig. 9.3).

The magnitude continues to decrease beyond $\omega = 1$ with no overshoot. This type of response is called **maximally flat**. The response is plotted in Fig. 9.3 for various values of n and it is observed that as n increases the response gets closer to the ideal LPF which is also shown in Fig. 9.3.

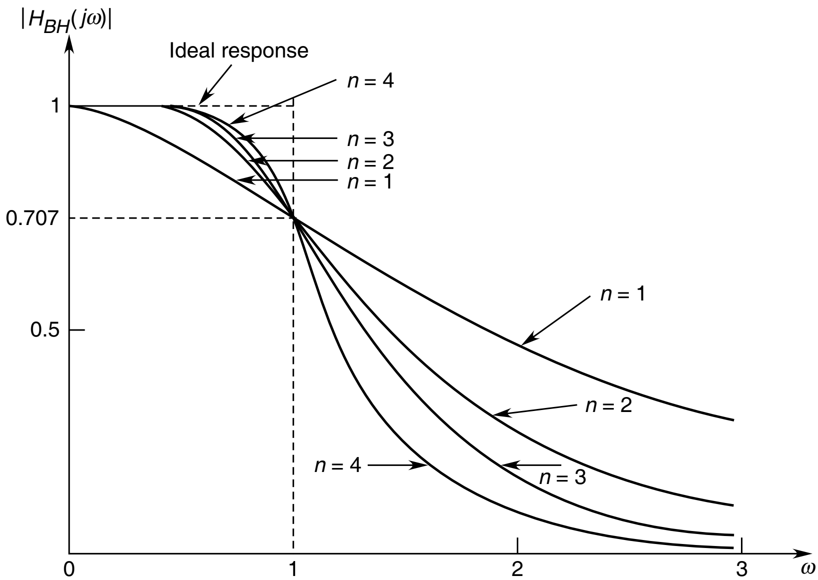


Fig. 9.3 *Butterworth normalized amplitude response*

The s-domain transfer function of BF (Butterworth filter) is obtained from Eq. (9.12) in squared form as

$$|H_{Bn}(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

Letting $\qquad j\omega = s$, we have

$$|H_{Bn}(s)|^2 = \frac{1}{1+(s/j)^{2n}} = \frac{1}{1+(-s^2)^n}$$

or

$$H_{Bn}(s)H_{Bn}(-s) = \frac{1}{1+(-1)^n s^{2n}} \tag{9.15}$$

In order to obtain $H_{Bn}(s)$, we factorize the denominator polynomial $D_n(s)$ (called BF polynomial) of Eq. (9.15) and consider only the roots which lie in the left half of s-plane, so that $H_{Bn}(s)$, whose poles are these, is stable and so realizable.

The procedure shall be illustrated by simple examples.

(i) $n = 1$

$$H_{B1}(s)H_{B1}(-s) = \frac{1}{(1-s^2)} = \frac{1}{(1+s)(1-s)}$$

Therefore

$$H_{B1}(s) = \frac{1}{(1+s)}$$

(ii) $n = 2$

$$H_{B2}(s)H_{B2}(-s) = \frac{1}{1+s^4} = \frac{1}{\left(1+\sqrt{2}s+s^2\right)}\,\frac{1}{\left(1-\sqrt{2}s+s^2\right)}$$

Therefore

$$H_{B2}(s) = \frac{1}{\left(1+\sqrt{2}s+s^2\right)}$$

It similarly follows that

$$H_{B3}(s) = \frac{1}{(1+s)(1+s+s^2)}$$

Table 9.1 BF polynomials of $H_{Bn}(s)$ up to $n = 6$.

Table 9.1 *Butterworth polynomials*

n	*Polynomial*
1	$(s + 1)$
2	$(s^2 + 1.414s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.939s + 1)$

Let us now obtain the roots of BF polynomial. From the denominater of Eq. (9.15), we write

$$D_n(s) = 1 + (-1)^n s^{2n} = 0 \tag{9.17}$$

or

$$(-1)^n s^{2n} = -1$$

or

$$e^{-j\pi n} s^{2n} = e^{j(2m-1)\pi}\;;\; m = 1,2,3...,2n$$

or

$$s^{2n} = e^{j(2m-1)\pi}\, e^{j\pi n} \tag{9.17}$$

The roots will be labelled as s_m, $m = 1,2,...,2n$. From Eq. (9.17), we write

$$s_m = e^{j(2m-1)\pi/2n}\, e^{j\pi/2}$$

or

$$s_m = je^{j(2m-1)\pi/2n} \tag{9.18}$$

Rewriting Eq. (9.18) in sine and cosine form, we get

$$\begin{aligned} s_m &= -\sin\,[(2m-1)\,(\pi/2n)] + j\cos\,[(2m-1)\,(\pi/2n)] \\ &= \sigma_m + j\omega_m \;\;;\; m = 1,2,...,2n \end{aligned} \tag{9.19}$$

It can be seen from Eq. (9.19) that the roots s_m lie on a unit circle and are spaced (π/n) rad apart. Further, no roots can occur on the $j\omega$-axis as $(2m - 1)$ cannot be an even integer. Thus there are n left-half-plane roots and n right-half-plane roots. These are respectively associated with $D_n(s)$ and $D_n(-s)$.

For example, for $n = 4$, the left-half-plane roots are as below. These are plotted in the s-plane in Fig. 9.4.

$$s_1 = e^{j5\pi/8},\; s_2 = e^{j7\pi/8},\; s_3 = e^{j9\pi/8},\; s_4 = e^{j11\pi/8}$$

Substituting these values, we get

$$\begin{aligned} D_4(s) &= (s - s_1)\,(s - s_2)\,(s - s_3)\,(s - s_4) \\ &= 1.2613s + 3.4142s^2 + 2.6131s^3 + s^4 \end{aligned}$$

The BF polynomials given in Table 9.1 are based on normalized cut-off frequency. If cut-off is desired at a particular frequency, denormalization can be carried out as below.

$$H_{Bn}(s) \;\Rightarrow\; H_{Bn}(s/\omega_c);\; \omega_c = \text{cut-off frequency} \tag{9.20}$$

For example for $n = 2$

$$H_{B2}(s) \;\Rightarrow\; H_{B2}(s/\omega_c) = \frac{1}{(s/\omega_c)^2 + \sqrt{2}\,(s/\omega_c) + 1}$$

How the BF filter is designed to meet specifications will now be illustrated through an example.

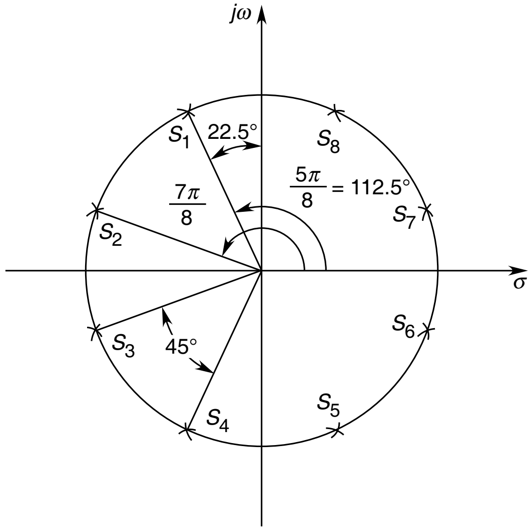


Fig. 9.4 *Location of zeros of BF polynomial, n = 4*

Specifications

1. Attenuation at least 10 dB at $2\omega_c$
2. Cut-off frequency, $f_c = 300$ kHz

Substituting these values in Eq. (9.12) and converting to dB, we get

$$\begin{aligned} 20 \log_{10} |H_{Bn}(j\omega)| &= 20 \log_{10} (1 + \omega^{2n})^{1/2} \\ &= 10 \log_{10} (1 + \omega^{2n}) \qquad \text{(i)} \end{aligned}$$

As per the first specification,

$$10 \log (1 + \omega^{2n}) \geq 10$$

or

$$\log (1 + \omega^{2n}) \geq 1 \qquad \text{(ii)}$$

In normalized frequency

$$\omega = 2\omega_c = 2;\ \omega_c = 1$$

Then

$$\log (1 + 2^{2n}) \geq 1 \qquad \text{(iii)}$$

which gives

$$n = 1.584$$

As n has to be an integer, we choose the higher value of $n = 2$.

From Table 9.1

$$H_{B2}(s) = \frac{1}{(s^2 + 1.4142s + 1)} \qquad \text{(iv)}$$

As per the second specification,

$$f_c = 300 \text{ kHz or } \omega_c = 1.89 \times 10^3 \text{ rad/s}$$

Changing s to (s/ω_c) in Eq. (iv) gives

$$H_{B2}(s/\omega_c) = \frac{1}{(s/\omega_c)^2 + 1.412\,(s/\omega_c) + 1}$$

Substituting the value of ω_c, we get

$$H_{B2}(s/\omega_c) = \frac{1}{2.80 \times 10^{-7} s^2 + 0.7482 \times 10^{-3} s + 1} \tag{v}$$

Normalized band-stop characteristics of a BF for various orders is given in Fig. 9.5 to scale. These are helpful in design work.

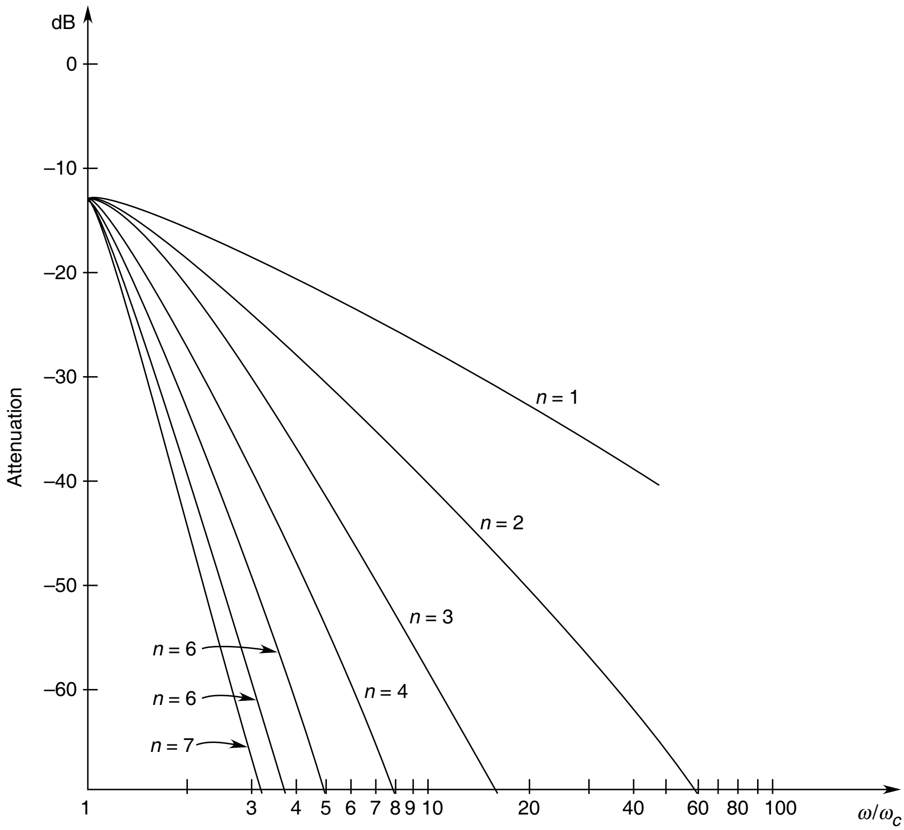


Fig. 9.5 *Normalized BF stop-band*

Chebyshev Filters (CF)

It has been observed from BF that the amplitude response is near ideal for frequencies close to $\omega = 0$, but is a poor approximation in the vicinity of cut-off frequency. Chebyshev filter achieves a sharper cut-off response but amplitude response below cut-off has ripples as sketched in Fig. 9.6.

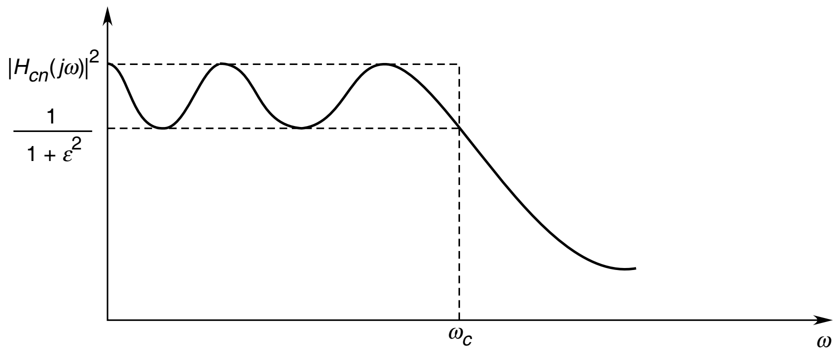


Fig. 9.6 *Amplitude response of CF*

The Chebyshev LPF approximation is given as

$$|H_{Cn}(j\omega)| = \frac{1}{\sqrt{[1+\varepsilon^2 C_n^2(\omega)]}} \tag{9.21}$$

where

$C_n(\omega)$ = Chebyshev polynomial
ε = ripple factor
n = filter order

Chebyshev polynomial is given as

$$C_n(\omega) = \cos(n\cos^{-1}\omega);\ 0 \le \omega \le 1, \text{ i.e. below cut-off frequency} \tag{9.22}$$

and

$$C_n(\omega) = \cosh(n\cosh^{-1}\omega);\ \omega \ge 1, \text{ i.e. above the cut-off frequency} \tag{9.23}$$

In Eqs (9.22) and (9.23) ω is the normalized frequency, i.e., $\omega_c = 1$.

Chebyshev polynomial in these two regions is obtained as below.

(i) $0 \le \omega \le 1$; Frequency Below Cut-off

In Eq. (9.22) let

$$\cos^{-1}\omega = \theta$$

or

$$\omega = \cos\theta$$

Equation (9.22) then is written as

$$C_n(\omega) = \cos n\theta$$

We shall construct the Chebyshev polynomial for values of $n = 0$ onwards.

$$C_0(\omega) = 1;\ n = 0$$

$$C_1(\omega) = \cos\theta = \omega;\ n = 1$$

$$C_2(\omega) = \cos 2\theta = 2\cos^2\theta - 1$$

$$= 2\omega^2 - 1\ ;\ n = 2$$

and so on.

Chebyshev polynomials up to $n = 8$ are listed in Table 9.2.

Table 9.2 *Chebyshev polynomials*

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\ \omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^{s} - 20\omega^3 + 5\omega$
6	$32\omega^7 - 48\omega^4 + 18\omega^2 - 1$
7	$74\omega^7 - 112\omega^5 + 57\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^7 + 170\omega^4 - 32\omega^2 + 1$

Number of Ripples We shall show that in a Chebyshev Filter (CF) the number of ripples (number of maxima plus minima) is equal the filter order, n. For simplicity we choose

Ripple factor, $\varepsilon = 1$

Though the number of ripples are independent of ε, their peak-to-peak value is related to ε. With $\varepsilon = 1$, Eq. (9.21) is written as

$$|H_{Cn}(j\omega)| = \frac{1}{\sqrt{[1 + C_n^2(\omega)]}} \tag{9.24}$$

Consider now the first-order filter, i.e. $n = 1$.

Then

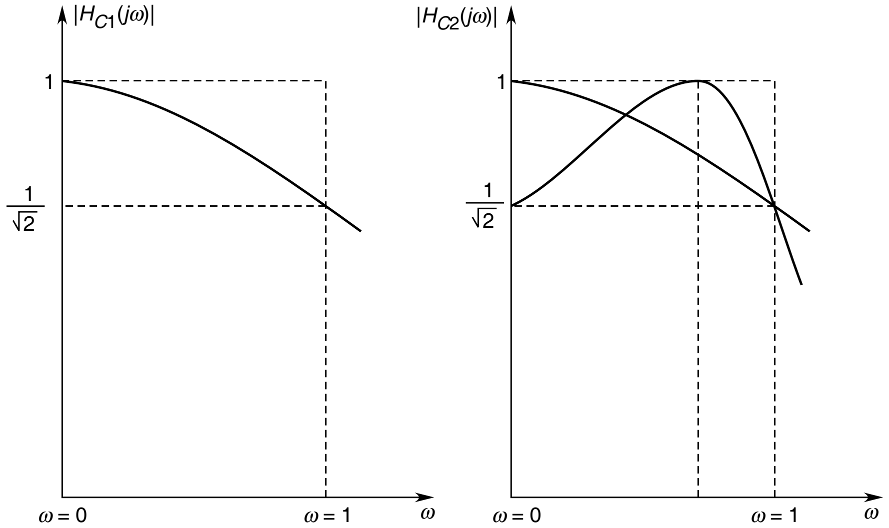
$$|H_{C1}(j\omega)| = \frac{1}{(1 + \omega^2)^{1/2}}$$

Its frequency response reveals the following facts.

$$|H_{C1}(\omega = 0)| = 1\text{, a maxima}$$

$$|H_{C1}(\omega = 1)| = 1/\sqrt{2}\ \text{; a minima ; } \omega = 1 \text{ is cut-off frequency}$$

As shown in Fig. 9.7(a) the response monotonically attenuates from 1 to $1/\sqrt{2}$. For $\omega < \omega_c = 1$, there is only one maxima (or ripple) which means that number of ripples equals $n = 1$.



(a) Magnitude response for $n = 1$

(b) Magnitude response for $n = 2$

Fig. 9.7

Consider now the second-order filter, $n = 2$.

Then

$$C_2 = 2\omega^2 - 1$$

$$|H_{C2}(j\omega)| = \frac{1}{[1 + (2\omega^2 - 1)^2]\sqrt{\ }\ [1 + (2\omega^2 - 1)^2]^{1/2}} \tag{9.25}$$

Its frequency response yields

$$|H_{C2}(\omega = 0)| = 1/\sqrt{2}\text{; a minima}$$
$$|H_{C2}(\omega = 1)| = 1/\sqrt{2}\text{; a minima}$$

By differentiating $|(H_{C2}(j\omega)|$ w.r.t ω and equating to zero, it can be shown that it has maxima of magnitude unity at $\omega = 1/\sqrt{2}$. Its amplitude response is plotted in Fig. 9.7(b).

We easily conclude that for $\omega < \omega_c = 1$, the number of ripples equal $n = 2$

(ii) *ω > Cut-off Frequency*

Let

$$\cosh^{-1} \omega = x \tag{9.26a}$$

or

$$\omega = \cosh x = \frac{e^x + e^{-x}}{2} \tag{9.26b}$$

For $x >> 1$

$$\omega = \cosh x \approx e^x/2 \tag{9.27}$$

or

$$e^x = 2\omega$$

Taking natural log

$$x = \ln 2\omega = \cosh^{-1}\omega;\ \text{as per Eq. (9.26a)} \tag{9.28}$$

Substituting in Eq. (9.23)

$$\begin{aligned} C_n(\omega) &= \cosh(n \ln 2\omega) \\ &= \cosh[\ln(2\omega)^n] \end{aligned}$$

For $\omega >> 1$

$$C_n(\omega) \approx (1/2)\, e^{\ln(2\omega)^n} = (1/2)\,(2\omega)^n \tag{9.29}$$

Chebyshev Filter Design In squared form Eq. (9.21) results into the following equation.

$$|H_{Cn}(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)} \tag{9.30}$$

Letting $j\omega \to s$

$$|H_{Cn}(s)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(s/j)} \tag{9.31}$$

This can be written as

$$H_{Cn}(s)H_{Cn}(-s) = \frac{1}{1+\varepsilon^2 C_n^2(s/j)} \tag{9.32}$$

The roots of the denominator are given as

$$1 + \varepsilon^2 C_n^{\ 2}(s/j) = 0$$

or

$$C_n(s/j) = \pm j/\varepsilon \tag{9.33}$$

For the pass-band ($0 \le \omega \le 1$), from Eqs (9.22) and (9.33), we get

$$\cos[n \cos^{-1}(s/j)] = \pm j/\varepsilon \tag{9.34}$$

Let

$$\cos^{-1}(s/j) = \alpha - j\beta \tag{9.35}$$

Then Eq. (9.34) takes the following form.

$$\cos(n\alpha - jn\beta) = + j/\varepsilon \tag{9.36}$$

or

$$\cos n\alpha \cosh n\beta + j \sin n\alpha \sin n\beta = \pm j/\varepsilon$$

or

$$\cos n\alpha \cosh n\beta + j \sin n\alpha \sin n\beta = \pm j/\varepsilon \tag{9.37}$$

(we have used the identity, $\cos j\theta = \cosh\theta$ and $\sin j\theta = j\sin\theta$)

Equating real and imaginary parts, we get

$$\cos n\alpha \cosh n\beta = 0 \tag{9.38a}$$

and

$$\sin n\alpha \sin n\beta = \pm 1/\varepsilon \tag{9.38b}$$

In Eq. (9.38a) $\cosh n\beta \neq 0$, so we have

$$\cos n\alpha = 0$$

or

$$\alpha = (2m - 1)\ \pi/2n \tag{9.39}$$

Then

$$\begin{aligned}\sin n\alpha &= \sin [(2m-1)\ n\pi/2n] \\ &= \sin [(2m-1)\ \pi/2] \\ &= \pm 1\ ;\ m = 1, 2, 3...\end{aligned}$$

Similarly, Eq. (9.38b) yields

$$\sin n\beta = \pm\ 1/\varepsilon$$

or

$$\beta = \pm(1/n)\ \sin^{-1}(1/\varepsilon) \tag{9.40}$$

Roots s_m of the Chebyshev polynomial are obtained from the relationship of Eq. (9.35). These are given as

$$\cos^{-1}(s/j) = \alpha - j\,\beta$$

or

$$s_m = j\cos(\alpha - j\beta) \tag{9.41}$$

Expanding the cos function, we get

$$s_m = -\sin\alpha \sinh\beta + j\cos\alpha\cosh\beta \tag{9.42}$$

Substituting for α and β from Eqs (9.39) and (9.40), we get

$$\begin{aligned}s_m &= -[\sin(2m-1)(\pi/2n)]\sinh[(1/n)\sinh^{-1}(1/\varepsilon)] \\ &\quad + j\cos[(2m-1)(\pi/2n)]\cosh[(1/n)\sinh^{-1}(1/\varepsilon)] \\ &= \sigma_m + j\omega_m\end{aligned} \tag{9.43}$$

From the above result it follows that

$$\frac{\sigma_m^2}{\left\{\sinh[(1/n)\sinh^{-1}(1/\varepsilon)]\right\}^2} + \frac{\omega_m^2}{\left\{\cosh[(1/n)\sinh^{-1}(1/\varepsilon)]\right\}^2} = 1 \tag{9.44}$$

This equation defines an ellipse on which the roots (s_m) are located.

Transfer function of CF for specified attenuation (dB) and ripple factor can be written as

$$H_{Cn}(s) = \frac{K}{(-1)^n \prod_{m=1}^{n} (s/s_m - 1)} \tag{9.45}$$

in normalized frequency, i.e., $\omega_c = 1$ and K is the specified gain

In denormalized form Eq. (9.45) can be stated in the following manner.

$$H_{Cn}(s) = \frac{K}{(-1)^n \prod_{m=1}^{n}[s/(s_m\omega_c)-1]} \tag{9.46}$$

The Chebyshev polynomial $C_n(\omega)$ (given in Table 9.2) is plotted in the pass-band and beyond for various values of n in Fig. 9.8. It is seen from this figure that within the pass-band, $C_n(\omega)$ has maximum and minimum value of $\pm$ 1. Beyond cut-off frequency (normalized) $C_n(\omega)$ increases monotonically. Therefore $C_n^2(\omega)$ has a maximum value of unity within pass band irrespective of the value of n.

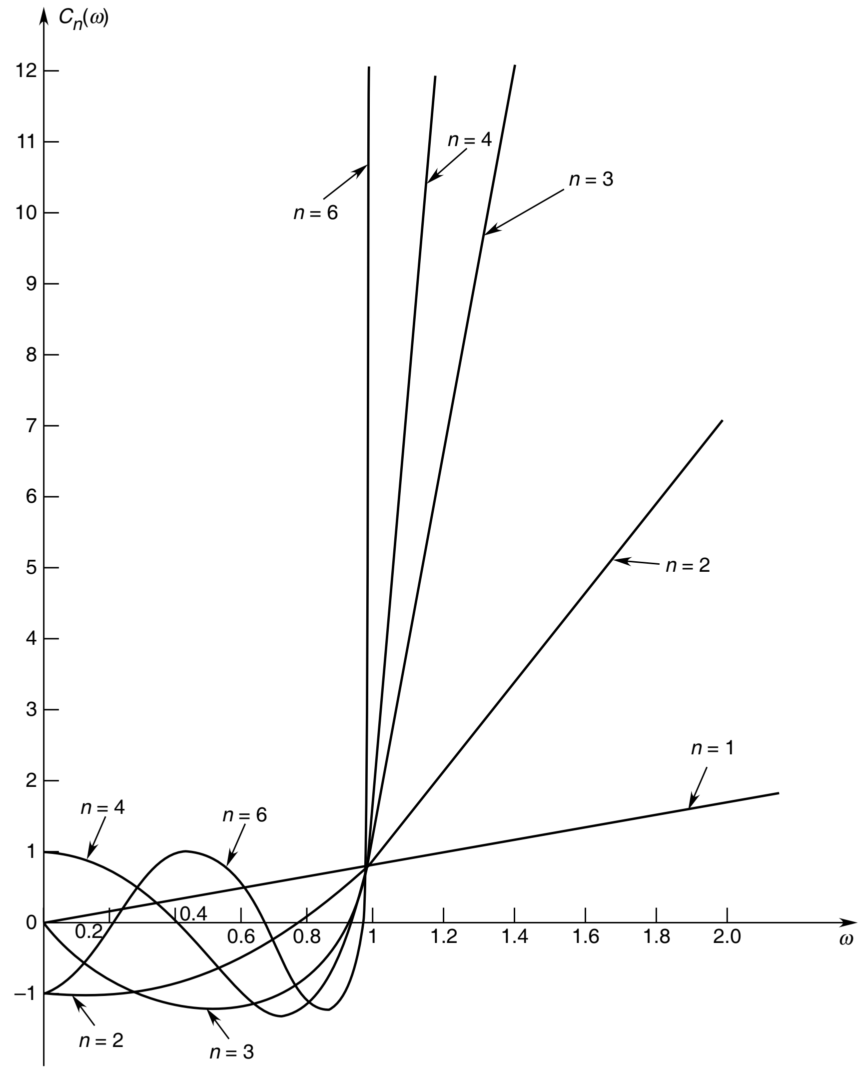


Fig. 9.8 *Chebyshev polynomial for $\omega > 0$*

It then follows from Eq. (9.21) that the minimum value of ripple is

$$\left| H_{Cn}(j\omega) \right|_{\text{min peak}} = \frac{1}{\sqrt{(1+\varepsilon^2)}} \tag{9.47}$$

Therefore, peak-to-peak ripple is

$$\gamma = 1 - \frac{1}{\sqrt{(1+\varepsilon^2)}} \tag{9.48}$$

$$\text{Maximum value of any peak } = 1 \text{ or } 0 \text{ dB}$$

$$\text{Minimum value of a peak is } = -\frac{1}{\sqrt{(1+\varepsilon^2)}}$$

or

$$-\log_{10}(1+\varepsilon^2) \text{ dB}$$

Therefore

$$\text{dB}(\gamma) = 10 \log_{10}(1+\varepsilon^2) \tag{9.49}$$

As $C_n(\omega)$ for $\omega > \omega_c$, increases monotonically, $|H_{cn}(j\omega)|$ decreases monotonically in this region (see Eq.(9.21)). Consider now an example of CF design.

Specifications

(i) Ripple in the pass-band = 1 dB
(ii) Cut-off frequency = 3 kHz
(iii) Amplitude attenuation at least 20 dB at 6 kHz

Design As per Eq. (9.49) and specification (i)

$$10 \log_{10}(1+\varepsilon^2) = 1$$

which gives

$$\varepsilon = 0.51$$

According to specification (ii), cut-off frequency = 3 kHz.

$$6 \text{ kHz} \Rightarrow 6/3 = 2 \text{ (normalized)}$$

Therefore,

$$\text{dB}[|H_{Cn}(\text{j}\omega)|] = 10 \log_{10}\left\lfloor 1 + (0.51)^2 \; C_n^2(2) \right\rfloor; \text{ from Eq. 9.21}$$

As per specification (iii)

$$10 \log_{10}[1 + (0.51)^2 C_n^2(2)] \geq 20 \text{ dB}$$

Solving we get

$$C_n^2(2) \geq 382$$

From Table 9.2, we get

$$C_2^2(2) = 49, \; C_3^2(2) = 676$$

Thus, $n = 3$ meets specification (iii) and of course specifications (i) and (ii).

To find the transfer function $H_{C3}(s)$, we need to determine the roots s_1, s_2, s_3, which are the poles of this function. From Eq. (9.43),

$$\begin{aligned} s_1 &= -\sin(\pi/6)\sinh[(1/3)\sinh^{-1}(1/0.51)] \\ &\quad + j\cos(\pi/6)\cosh[(1/3)\sinh^{-1}(1/0.51)] \\ &= -0.2471 + j0.966 \end{aligned}$$

Similarly,

$$s_2 = -0.4942 \text{ and } s_3 = -0.2471 - j0.966$$

Assuming $K = 1$, normalized trnasfer function as per Eq. (9.45) is

$$H_{C3}(s) = \frac{s_1 s_2 s_3}{(-1)^3 (s - s_1)(s - s_2)(s - s_3)}$$

$$= \frac{0.4913}{s^3 + 0.988s^2 + 1.2384s + 0.4913}$$

$$\omega_c \text{ (given)} = 2\pi \times 3 \times 10^3 = 7\pi \times 10^3 \text{ rad/s}$$

In denormalized form we get

$$H_{C3}(s/\omega_c) = \frac{0.4913}{(s/6\pi \times 10^3) + 0.988(s/6\pi \times 10^3)^2 + 1.2384(s/6\pi \times 10^3) + 0.493}$$

Plot of normalized $H_{C3}(s)$ is drawn in Fig. 9.9.

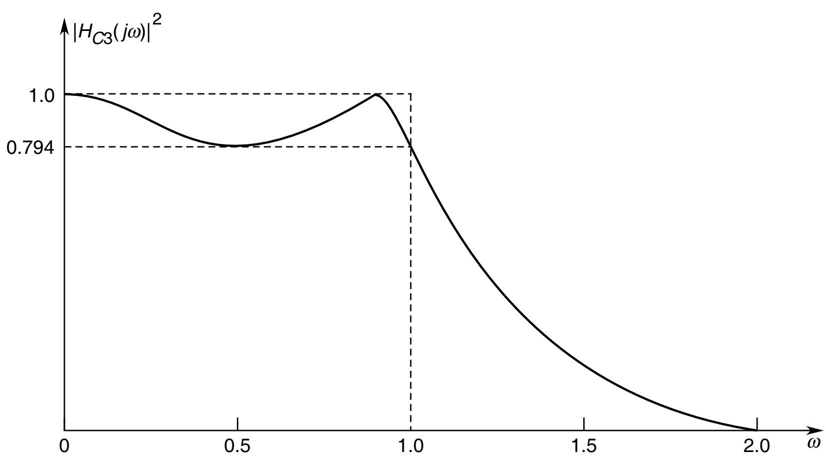


Fig. 9.9 *Plot of normalized magnitude function $H_{C3}(s)$*

Inverse Chebyshev Filter (ICF)

It is often desired to have equal ripples in the stop-band rather than in the pass-band. Inverse Chebyshev filter has the characteristic that permit monotonic passband variation and equal ripples in the stop-band. The inverse Chebyshev filter transfer function is obtained if the function $|H_{Cn}(j\omega)|^2$ is substracted from unity and $1/\omega$ is then substituted for ω. The new transfer function is

$$|H'_{Cn}(j\omega)|^2 = \left|\frac{\varepsilon^2\, C_n^2\,(1/\omega)}{1+\varepsilon^2\, C_n^2\,(1/\omega)}\right| \tag{9.50}$$

We now consider a design example of third order ICF. The normalized Chebyshev filter is given by

$$H_{C3}(s) = \frac{0.4913}{s^3 + 0.988s^2 + 1.2384s + 0.4913}$$

i.e.

$$|H_{C3}(j\omega)|^2 = \frac{1}{1 + 0.2589(4\omega^3 - 3\omega)^2}$$

We now proceed to obtain $|H'_{C3}(j\omega)|^2$ as

$$|H'_{C3}(j\omega)|^2 = 1 - |H_{C3}(j\omega)|^2$$

$$= 1 - \frac{1}{1 + 0.2589\left(\dfrac{4}{\omega^3} - \dfrac{3}{\omega}\right)^2}$$

Thus

$$|H'_{C3}(j\omega)|^2 = \frac{0.2589(4 - 3\omega^2)^2}{\omega^6 + 0.2589(4 - 3\omega^2)^2}$$

The plot of $|H'_{C3}(j\omega)|^2$ is shown in Fig. 9.10. It can be easily seen from this figure that in the pass-band, variation of magnitude is monotonic whereas in the stop-band there are ripples. Thus the characteristics in this case is just inverse of that corresponding Chebyshev filter.

Elliptic Filter

We have seen that

(i) BF characteristic is monotonic both in pass-band and stop-band

(ii) CF characteristic has ripples in pass-band but decreases monotonically in stop-band

An approximation called Elliptic filter has characteristic as shown in Fig. 9.11 with ripples in pass-band as well as stop-band.

EF (Elliptic filter) has transfer function of the following form.

$$|H_{En}(j\omega)| = \frac{1}{\sqrt{\left\lfloor 1 + \varepsilon^2 R_n^2\,(\omega)\right\rfloor}} \tag{9.51}$$

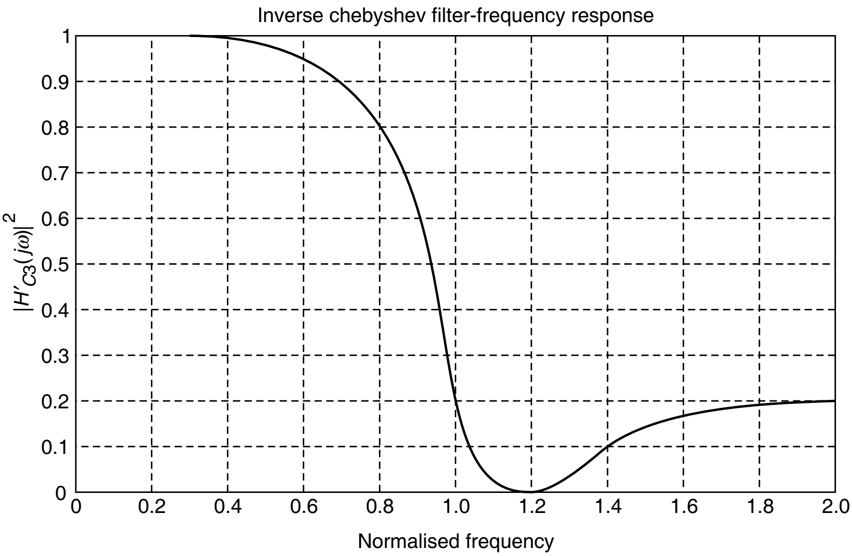


Fig. 9.10 *Characteristics of inverse Chebyshev filter*

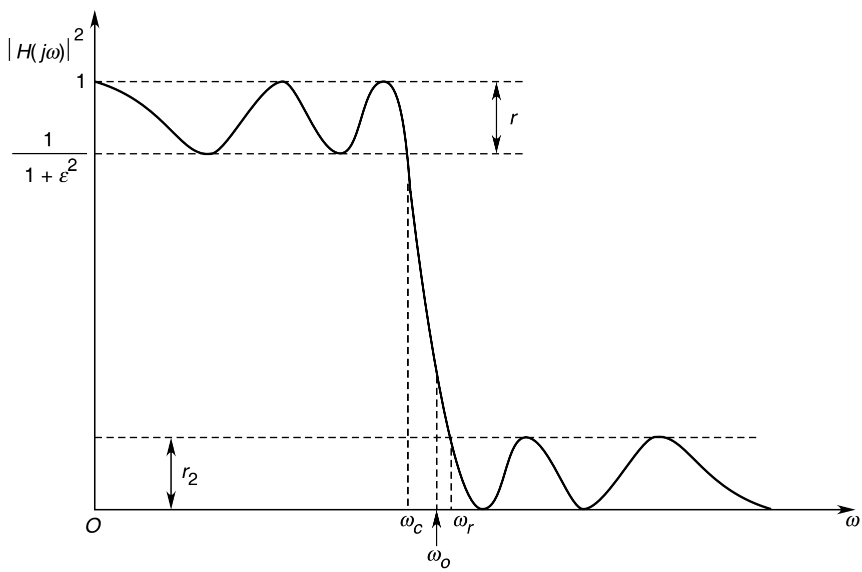


Fig. 9.11 *Characteristics of elliptic filter*

where

$$R_n(\omega) = \text{elliptic rational function}$$

Roots of $R_n(\omega)$ are related to the Jacobian elliptic sine functions.
ε is a paramter related to pass-band ripple. The filter order required to achieve

desired specifications is obtained as

$$n = \frac{K(\omega_c/\omega_r)\, K\left(\sqrt{1+\varepsilon^2/\delta^2)}\right)}{K(\varepsilon/\delta)\, K\left(\sqrt{1-(\omega_c/\omega_r)^2}\right)} \tag{9.52}$$

where $K(x)$ is the complete elliptic integral of the first kind, expressed as

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2 \sin^2\theta}} \tag{9.53}$$

δ is related to the stop-band ripple δ_2 as

$$\delta_2 = 1/\sqrt{1+\delta^2} \tag{9.54}$$

values of this integral have been tabulated in several books. Details of elliptic function is avoided as they are mathematically too complicated. One can use computer programs, already available, for designing elliptic filters from the given specification, as indicated in Eqs 9.51 – 9.54.

Bessel Filters (BEF)

The Bessel filter is a class of filter which gives maximally flat time delay or maximum linear phase response. It is an all-pole filter whose transfer function is defind as follows.

$$H_{BEn}(s) = \frac{1}{BE_n(s)} \tag{9.55}$$

where

$$BE_n(s) = \sum_{k=0}^{n} \frac{(2n-k)!\,s^k}{2^{n-k}k!(n-k)!} = \sum_{k=0}^{n} a_k s^k \tag{9.56}$$

where

$$a_k = \sum_{k=0}^{n} \frac{(2n-k)!}{2^{n-k}k!(n-k)!} \tag{9.57}$$

are the coefficients of the Bessel polynomial, $BE_n(s)$.

The first three polynomials, obtained by using Eq. (9.56) are $BE_1(s) = s + 1$, $BE_2(s) = s^2 + 3s + 3$, and $BE_3(s) = s^3 + 6s^2 + 15s + 15$. It may be noted that $BE_0(s) = 1$. Table 9.3 lists the coefficients a_k up to $n = 6$, used for obtaining Bessel polynomial $BE_n(s)$.

The polynomial $BE_n(s)$, if used as the denominator of the transfer function (Eq. 9.55), one can obtain maximally flat time delay.

For example, we consider first-order and second-order transfer function of the Bessel filter.

Table 9.3 *Coefficients of the polynomials $BE_n(s)$*

n	a_0	a_1	a_2	a_3	a_4	a_5
1	1					
2	3	3				
3	15	15	6			
4	105	105	45	10	0	
5	945	945	420	105	15	
6	10395	10395	4725	1260	210	21

The first-order Bessel filter is

$$H_{BE_1}(s) = \frac{1}{1+s}$$

i.e.

$$H_{BE_1}(j\omega) = \frac{1}{1+j\omega}$$

or

$$H_{BE_1}(j\omega) = \frac{(1-j\omega)}{(1+j\omega)(1-j\omega)} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

Thus the phase angle, $\phi_1(\omega)$, of $H_{BE1}(j\omega)$ is given by

$$\phi_1(\omega) = \tan^{-1}(-\omega) \Rightarrow -\phi_1(\omega) = \tan^{-1}\omega$$

thus the time delay of H_{BE_1} $\phi_1(j\omega)$ is,

$$T_{P1} = \frac{d}{d\omega}\{-\phi_1(\omega)\} = \frac{d}{d\omega}(\tan^{-1}\omega)$$

or

$$T_{P1} = \frac{1}{1+\omega^2}$$

We have

$$H_{BE2}(s) = \frac{1}{s^2+3s+3}$$

or

$$H_{BE2}(j\omega) = \frac{1}{(3-\omega^2)+j\omega} = \frac{(3-\omega^2)}{\omega^4+3\omega^2+9} - j\frac{3\omega}{\omega^4+3\omega+9}$$

Thus the phase angle $\phi_2(\omega)$ is given by

$$-\phi_2(\omega), = \tan^{-1}\left(\frac{3\omega}{3-\omega^2}\right)$$

i.e.

$$T_{P2} = -\frac{d}{d\omega}\{-\phi_2(\omega)\} = \frac{9+3\omega^2}{9+3\omega^2+\omega^4}$$

The plot of T_{P1} (time delay for first order Bessel filter) and T_{P2} (time delay for second order Bessel filter) are shown in Fig. 9.12. It is clear from this figure that as we increase the order of Bessel filter, the time-delay becomes more and more flat within the pass-band, which implies that higher order Bessel filter gives more linear phase response.

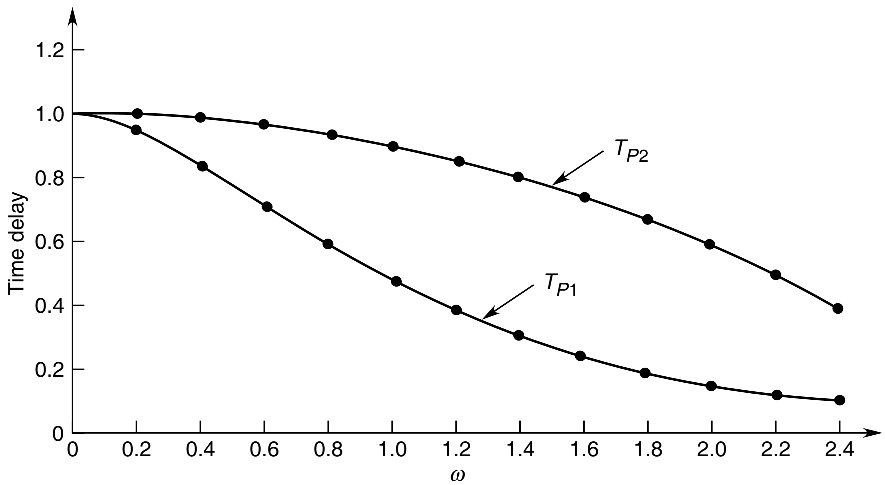


Fig. 9.12 *Plot of time-delay vs ω*

Comparison of Analog Filters The filter functions undertaken for comparison are Butterworth, Chebyshev, Elliptic and Bessel. For a given set of specifications (i.e., attenuation in the pass-band, stop-band and the transition band) the elliptic filter has the lowest order and hence requires minimum cost. In other words, for a given order, the elliptic filter has the lowest transition band compared to all other filters. However, phase characteristic of the elliptic filter is highly non-linear and therefore, offers maximum phase distortion. Bessel filter has linear phase characteristic in the pass-band and therefore, does not offer significant phase distortion. This, however, has the largest transition band compared to other filters, for a given order. Bessel filter, therefore, has the maximum magnitude distortion. It can be easily concluded that elliptic and Bessel filters are the two extreme cases, the former offer the best magnitude characteristics, whereas the latter provides the best phase characteristic.

Butterworth and Chebyshev filters have characteristics lying between the above two. Both have linear phase characteristic almost over three–fourth of the pass-band. The transition band for Butterworth and Chebyshev are more than the elliptic filter. Chebyshev filter, however, has the transition band slightly less than that of Butterworth filter, for a given order of the filter.

Frequency Transformations The discussion will now be focused on how to obtain other normalized filters starting from normalized low-pass prototype filter. This is easily achieved by appropriate frequency transformation.

Low-pass to Low-pass Transformation Condition to be met by this transformation are (ω_T = transformed frequency)

$$\begin{aligned} \omega &= 0 && \Rightarrow \omega_T = \omega_{0T} \\ \omega &= \pm\infty && \Rightarrow \omega_T = \pm\infty \\ \omega &= 1 && \Rightarrow \omega_T = \omega_c \qquad \text{(desired cut-off frequency)} \end{aligned}$$

These conditions are satisfied by the following transformation.

$$s = \frac{s_T}{\omega_c} \quad \text{or} \quad \omega = \frac{\omega_T}{\omega_c}$$

Examples of this kind of transformation have already been discussed in BF and CF designs.

Low-pass to Band-pass Transformation Low-pass to band-pass transformation is depicted in Fig. 9.13. Such a transformation should meet the following requirements.

$$\begin{aligned} \omega &= 0 && \Rightarrow \omega = \pm\,\omega_{0T}, \text{ the centre frequency} \\ \omega &> 0 && \Rightarrow \omega > \omega_{0T} \text{ and } \omega > -\,\omega_{0T} \\ \omega &< 0 && \Rightarrow \omega < \omega_{0T} \text{ and } \omega < -\,\omega_{0T} \end{aligned}$$

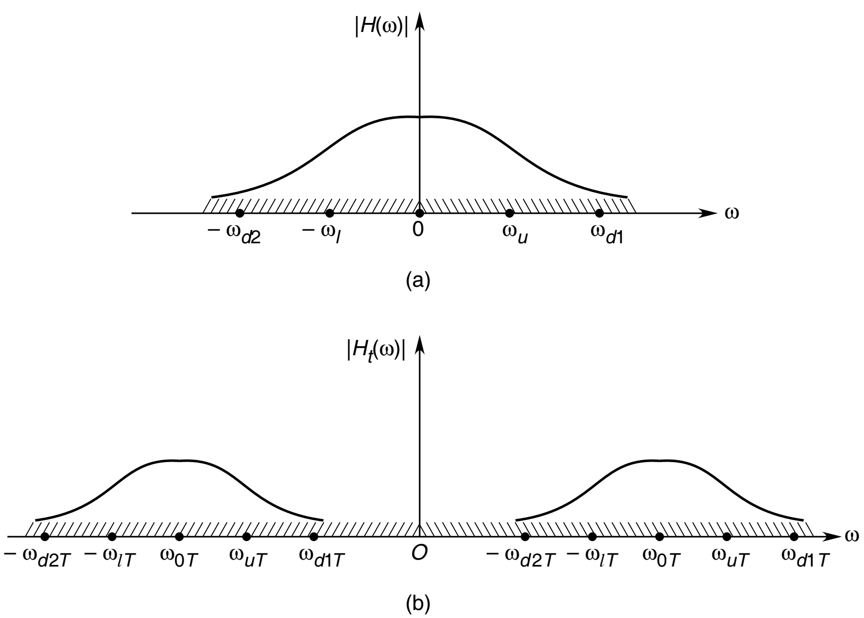


Fig. 9.13 *Transformation from a low-pass to a band-pass filter*

The transformation that accomplishes this requirements is given by

$$s = \frac{\omega_{0T}}{\omega_{bT}}\left(\frac{s_T}{\omega_{0T}} + \frac{\omega_{0T}}{s_T}\right) = \frac{s_T^2 + \omega_{0T}^2}{s_T \omega_{bT}} \tag{9.58}$$

where

ω_{0T} = centre frequency of pass-band
ω_{uT} = upper cut-off frequency of pass-band
$\omega_{\ell T}$ = lower cut-off frequency of pass-band
$\omega_{bT} = \omega_{uT} - \omega_{\ell T}$ = band-width of pass-band

and

$$\omega^2_{0T} = \omega_{uT}\,\omega_{\ell T} \tag{9.59}$$

It is to be observed that in the low-pass (normalized) which is being transformed, $\omega_u = \omega_\ell$; ω_u being positive frequency and $-\omega_\ell$ being negative frequency.

Example 9.1 *Design a band-pass filter to meet the following specifications.*

(a) 3dB attenuation at 10×10^3 rad/s and 15×10^3 rad/s
(b) Attenuation more than 25 dB for frequencies less than 5×10^3 rad/s and more than 20×10^3 rad/s

Solution

$\omega^2_{0T} = 10 \times 10^3 \times 15 \times 10^3 = 150 \times 10^6$
$\omega_{bT} = (15 - 10) \times 10^3 = 5 \times 10^3$ rad/s

In frequency domain the transformation is ($s = j\omega$, $s_T = j\omega_T$ in Eq. (9.58))

$$\omega = \frac{\omega_T^2 - \omega_{0T}^2}{\omega_T\,\omega_{bT}}; \ \omega = \text{normalized frequency of low-pass filter} \tag{i}$$

Substituting values

$$\omega = \frac{\omega_T^2 - 150 \times 10^6}{5 \times 10^3 \omega_T} \tag{ii}$$

With this transformation 3 dB points would be located at $\omega_{uT} = 15 \times 10^3$ rad/s and $\omega_{\ell T} = 10 \times 10^3$ rad/s so that specification (a) is met.

Now let us consider the specification (b) which requires that in pass-band

$$\omega_{d1T} = 20 \times 10^3 \text{ rad/s}, \ \omega_{d2T} = 5 \times 10^3 \text{ rad/s}$$

and

$$\omega_T > \omega_{d1T} \Rightarrow \text{attenuation more than 20 dB}$$
$$\omega_T < \omega_{d2T} \Rightarrow \text{attenuation more than 20 dB}$$

To meet this specification we must find the corresponding ω_{d1} and ω_{d2} in low-pass prototype.

From Eq. (i), we get

$$\omega_{d1T} = \frac{(20\times10^3)^2 - 150\times10^6}{5\times10^3\times5\times10^3} = 10 \text{ rad/s}$$

Letting $s = j\omega$ and $s_T = j\omega_T$ in Eq. (i), we have

$$\omega = \frac{\omega_T^2 - \omega_{0T}^2}{\omega_T\,\omega_{bT}} \tag{7.50}$$

$$\omega_{d2T} = -\frac{(5\times10^3)^2 - 150\times10^3}{5\times10^3\times5\times10^3} = 5 \text{ rad/s}$$

As $\omega_{d2T} < \omega_{d1T}$, we must ensure that $\omega_{d2T} < 5$ has attenuation of more than 20 dB (attenuation at ω_{d1T} will be more than its value at ω_{d2T}).

It must be noted here that the low-pass prototype response is normalized, i.e. $\omega_c = 1$. For a Butterworth prototype, we need to choose the value of n which will meet the above condition. This can be accomplished by means of normalized response curves for Butterworth for increasing value of n given in Fig. 9.5. We will proceed instead by guess and trial.

Let us choose $n = 2$, then

$$H_{B2}(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1} \tag{iv}$$

or

$$H_{B2}(j\omega) = \frac{1}{(1-\omega^2) + j\sqrt{2}\,\omega} \tag{v}$$

At $\quad \omega = \omega_{d2T} = 5$

$$H_{B2}(j5) = \frac{1}{(1-25) + j5\sqrt{2}} = \frac{1}{25}$$

$$\text{dB}|H_{B2}(j5)| = 20\log_{10}\frac{1}{25} = 30 \text{ dB}$$

So it does meet the specification. The transfer function of the high-pass filter is obtained by substituting for ω from Eq. (ii) in Eq. (v), which is given as

$$H_{B2T}(j\omega_T) = \frac{1}{\left[1 - \left(\dfrac{\omega_T^2 - 150\times10^6}{5\times10^3\,\omega_T}\right)^2\right] + j\sqrt{2}\left(\dfrac{\omega_T^2 - 150\times10^6}{5\times10^3\,\omega_T}\right)}$$

$$= \frac{(5\times10^3)\,\omega_T^2}{\omega_T^4 + 275\times10^6\,\omega_T^2 - (150\times10^6)^2 + j\sqrt{2}\times5\times10^3\,(\omega_T^2 - 150\times10^6)} \tag{9.60}$$

Low-pass to Band-stop Transformation This transformation is roughly the inverse of low-pass to band-pass with parameters indicated in Fig. 9.14. The transformation equation is

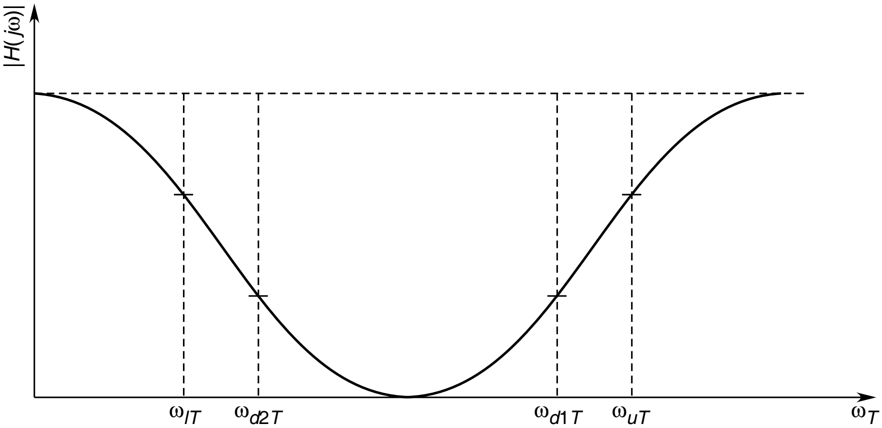


Fig. 9.14 *Band-stop filter parameters*

$$s = \frac{s_T(\omega_{d1T} - \omega_{d2T})\omega_{d1T}}{s_T^2 + \omega_{d1T}\omega_{d2T}} \tag{9.60}$$

or

$$\omega = \frac{\omega_T(\omega_{d1T} - \omega_{d2T})\omega_{d1T}}{-\omega_T^2 + \omega_{d1T}\omega_{d2T}} \tag{9.61}$$

Example 9.2 *Design a band-stop filter to meet the following specifications:*

1. *Attenuation between 1200 rad/s and 2000 rad/s must be at least 20 dB*
2. *Attenuation for less than 400 rad/s and higher than 2500 rad/s must be less than −3dB*

Solution As per transformation Eq.(9.61)

$$\omega = \frac{\omega_T(2000 - 1200)\omega_{d1T}}{-\omega_T^2 + 2000 \times 1200} = f(\omega_T)$$

$$= \frac{800\omega_T\,\omega_{d1T}}{-\omega_T^2 + 2.4 \times 10^6} \tag{i}$$

The values of $f(\omega_T)$ at $\omega_T = 2500$ rad/s and $\omega_T = 400$ rad/s are

$$f(2500) = \frac{2500 \times 800\omega_{d1T}}{-(2500)^2 + 2.4 \times 10^6} = -0.5195\ \omega_{d1T} \tag{ii}$$

$$f(400) = \frac{400 \times 800\omega_{d1T}}{-(400)^2 + 2.4 \times 10^6} = 0.1429\ \omega_{d1T} \tag{iii}$$

We require attenuation to be less than –3dB for $\omega \geq -0.5195\ \omega_{d1T}$ and $\omega \leq 0.1429\ \omega_{d1T}$, so we choose the low-pass cut-off frequency of the prototype as

$$\omega_c = 0.5195\ \omega_{d1T} \tag{iv}$$

For normalized prototype $\omega_c = 1$

$$\omega_{d1T} = 1/0.5195 = 1.925 \text{ rad/s} \tag{v}$$

Additionally, the low-pass prototype should have attenuation of more than 20 dB for $\omega \geq 1.925$ rad/s. From the normalized Butterworth amplitude curve of Fig. 9.5, we find this requires $n = 4$.

The prototype transfer function, therefore, is

$$H(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)} \tag{vi}$$

We have from Eqs (i) and (v)

$$\omega = \frac{800 \times 1.925\omega_T}{-\omega_T^2 + 2.4 \times 10^6} \tag{vii}$$

By substituting (vii) into (vi) and then replacing ω by s/j, the desired band-pass filter transfer function obtained is

$$H_{B4T}(s) = \frac{1}{(5.62 \times 10^{12}\ s_T^4 + 9.55 \times 10^7\ s_T^3 + 3.36 \times 10^9\ s_T^2 + 1464 s_T + 1} \tag{viii}$$

9.2 DIGITAL FILTERS

Digital filter is essentially a process that transforms a set of input sequence into a set of output sequence. This transformation may be carried out either with the help of hardware or with the help of computational algorithm.

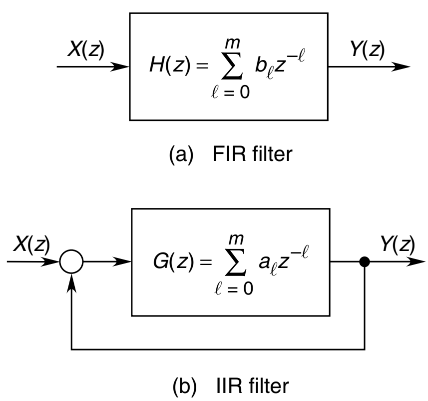


Fig. 9.15

Digital filters, as described later, may be designed in a number of ways depending upon whether the filter is a finite duration impulse response (FIR) or an infinite duration impulse response (IIR). In case of FIR filter, the output is a function of only past and present inputs, as given in Eq. (9.62), so that it is non-recursive in nature. On the other hand, the output of an IIR filter depends not only on present inputs but also on the past outputs, as given in Eq. (9.63), which means that it is recursive in nature. These equations are represented respectively in block diagram form in Figs 9.15(a) and (b). It is easily seen from diagrams that FIR block filter has non-feedback structure, while IIR filter has feedback structure.

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{\ell=0}^{m} b_\ell z^{-\ell} \text{; FIR filter} \tag{9.62}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{m} b_\ell z^{-\ell}}{1+\sum_{\ell=1}^{m} a_\ell z^{-\ell}} \text{; IIR filter} \tag{9.63}$$

9.3 FINITE IMPULSE RESPONSE (FIR) FILTER DESIGN

As stated in the Section 9.2, FIR filters are non-recursive in nature and hence stable. Moreover, these possess linear phase characteristics and hence do not have any phase distortion and also have low coefficient sensitivity. FIR filter design, however, is quite complex, particularly in case of sharp filters.

The FIR filter design can be achieved either by employing discrete Fourier series or by employing discrete Fourier transform. In both these cases, the characteristics function of FIR filter is represented in the form of $H(e^{j\omega T})$ or $H(z)$ to be used in the design process.

Discrete Fourier Series Method

In this method, the given analog filter function is considered to be periodic function so that it could be expressed in terms of a Fourier series, but with the constraint that Fourier series representation is confined to the range of frequencies in the given analog filter function. We, therefore, express the desired periodic digital filter function $H(e^{j\omega T})$ in the following form.

$$H(e^{j\omega T}) = \sum_{\ell=-\infty}^{\infty} h(\ell T) e^{-j\ell\omega T} \tag{9.64}$$

where $h(\ell T)$ is the discrete form of $h(t)$, the unit impulse response sampled at $t = \ell T$ or $\omega_s = 2\pi/T$. The coefficients $h(\ell T)$ is determined using the following expression.

$$h(\ell T) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T})\, e^{j\ell\omega T}\, \mathrm{d}\omega \tag{9.65}$$

where $\omega_s = 2\pi/T$. This is illustrated by an ideal filter with frequency response shown in Fig. 9.16.

Writing $z = e^{j\omega T}$, Eq. (9.64) may be rewritten as

$$H(z) = \sum_{\ell=-\infty}^{\infty} h(\ell T) z^{-\ell} \tag{9.66}$$

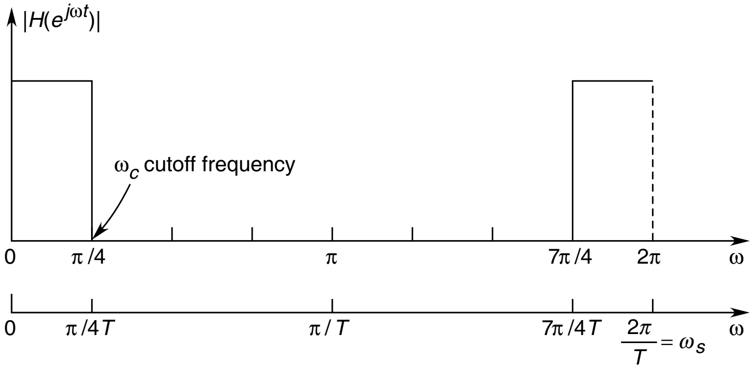


Fig. 9.16 *Desired LP filter characteristic*

The function $H(z)$, in practice, is described by discrete Fourier series with finite number of sample values and Eq. (9.66) may be written in the truncated form as given below:

$$H_l(z) = \sum_{\ell = -\frac{N-1}{2}}^{\frac{N-1}{2}} h(\ell T) z^{-\ell} \tag{9.67}$$

$$= \sum_{\ell = -\frac{N-1}{2}}^{-1} h(\ell T) z^{-\ell} + h(0) + \sum_{\ell = 1}^{\frac{N-1}{2}} h(\ell T) z^{-\ell}$$

$$= \sum_{\ell = 1}^{\frac{N-1}{2}} h(-\ell T) z^{\ell} + h(0) + \sum_{\ell = 1}^{\frac{N-1}{2}} h(\ell T) z^{-\ell}$$

i.e.,

$$H_l(z) = h(0) + \sum_{\ell = 1}^{\frac{N-1}{2}} [h(-\ell T) z^{\ell} + h(\ell T) z^{-\ell}] \tag{9.68}$$

It should be noted that $H_l(z)$ is a non-causal function because it contains positive powers of z. In order to make the function $H_t(z)$ (Eq. 9.68) causal, it is multiplied by a factor $z^{-(N-1)/2}$, so that it takes the following form.

$$H_{tc}(z) = H_t(z)\, z^{-(N-1)/2} \tag{9.69}$$

The factor $z^{-(N-1)/2}$ introduces a phase factor that is proportional to the frequency and thus $H_{tc}(z)$ has linear phase characteristics.

If $h(\ell T)$ is an even function, i.e., $h(\ell T) = h(-\ell T)$, then Eq. (9.68) gets modified as

$$H_l(z) = h(0) + \sum_{\ell = 1}^{\frac{N-1}{2}} h(\ell T)(z^{\ell} + z^{-\ell}) \tag{9.70}$$

We can write

$$z^{\ell} + z^{-\ell} = e^{j\ell\omega \mathrm{T}} + e^{-j\ell\omega \mathrm{T}} = 2\cos\omega T$$

Thus

$$z^{\ell} + z^{-\ell} = 2\cos\frac{2\pi\ell\omega}{\omega_s};\ \omega_s = 2\pi/T \tag{9.71}$$

Substituting Eq. (9.71) in Eq. (9.70), we obtain

$$H_l(z) = h(0) + \sum_{\ell=1}^{\frac{N-1}{2}} h(\ell T)\cos\left(\frac{2\pi\ell\,\omega}{\omega_s}\right) \tag{9.72}$$

Example 9.3 *Design an FIR filter whose frequency characteristics is close to the ideal characteristics shown in Fig. 9.16 using the Fourier series method. Sketch the resulting frequency characteristics for N = 5 and 15.*

Solution The desired filter characteristics, shown in Fig. 9.16, can be mathematically represented for one period, i.e. $\omega_s = 2\pi/T$ as,

$$\begin{aligned}|H(e^{j\omega T})| &= 1\ ;\ 0 \le \omega \le \pi/4T\\ &= 0\ ;\ \pi/4T \le \omega \le 7\pi/4T\\ &= 1\ ;\ 7\pi/4T \le \omega \le 2\pi/T\end{aligned} \tag{i}$$

Its coefficients are given by

$$\begin{aligned}h(\ell T) &= \frac{1}{\omega_s}\int_0^{\omega_S} H(e^{j\omega T})e^{j\ell\omega T}\mathrm{d}\omega\\ &= \frac{T}{2\pi}\int_0^{2\pi T} H(e^{j\ell\omega T})e^{j\ell\omega T}\mathrm{d}\omega\\ &= \frac{T}{2\pi}\left[\int_0^{\pi/4T} e^{j\ell\omega T}\ \mathrm{d}\omega + \int_{7\pi/4T}^{2\pi/T} e^{j\ell\omega T}\ \mathrm{d}\omega\right]\end{aligned} \tag{ii}$$

On carrying out integration, we get

$$h(\ell T) = (1/\ell\pi)\sin(\pi\ell/4) \tag{9.73}$$

Let us now choose $N = 5$ then using Eq. (9.68), the FIR filter function is

$$\begin{aligned}H_{lc}(z) &= h(0) + \sum_{\ell=1}^{(5-1)/2}[h(-\ell T)\,z^{\ell} + h(\ell T)z^{-\ell}]\\ &= h(0) + h(-T)z + h(T)z^{-1} + h(-2T)z^2 + h(-2T)z^{-2}\end{aligned}$$

Then

$$H_{lc}(z) = \mathrm{z}^{-2}[h(0) + h(-T)z + h(T)\,z^{-1} + h(-2T)z^2 + h(2T)z^{-2}]$$

or

$$H_{lc}(z) = h(0)\,z^{-2} + h(-T)z^{-1} + h(T)\,z^{-3} + h(-2T) + h(2T)z^{-4} \tag{iii}$$

From Eq. (ii), we have

$$h(0) = 1,\ h(T) = h(-T) = 0.225 \quad \text{and} \quad h(2T) = h(-2T) = 0.159$$

Thus

$$H_{lc}(z) = z^{-2} + 0.225\, z^{-1} + 0.225\, z^{-3} + 0.159 + 0.159\, z^{-4}$$
$$= 0.159 + 0.225\, z^{-1} + z^{-2} + 0.225\, z^{-3} + 0.159\, z^{-4} \qquad \text{(iv)}$$

Setting $z = e^{j\omega T}$, we get

$$H_{lc}(z)|_{z=e^{j\omega T}} = (0.159 + 0.225 \cos \omega T + \cos 2\omega T + 0.225 \cos 3\omega T + 0.159 \cos 4\omega T)$$
$$-j(0.225 \sin \omega T + \sin 2\omega T + 0.225 \sin 3\omega T + 0.159 \sin 4\omega T) \qquad (9.74)$$

Proceeding in the similar manner, the FIR filter for $N = 15$ is given as,

$$|H_{lc}(z)| = -0.032 - 0.053z^{-1} - 0.045z^{-2} + 0.075z^{-4} + 0.159z^{-5} + 0.225z^{-6} + 0.25z^{-7} + 0.225z^{-8} + 0.159z^{-9} + 0.075z^{-10} - 0.045z^{-12} - 0.053z^{-13} - 0.032^{-14} \qquad (9.75)$$

Plots of $|H_{lc}(z)|_{z=e^{j\omega T}}$ for $N = 5$ and 15 are shown in Fig. 9.17.

It may be observed from Fig. 9.17 that truncation of the Fourier series of the magnitude function leads to ripples in the amplitude response of the filter. Moreover, smaller the number of samples N, wider is the extent of ripples. Truncation of series may be viewed as the multiplication of the original sequence by

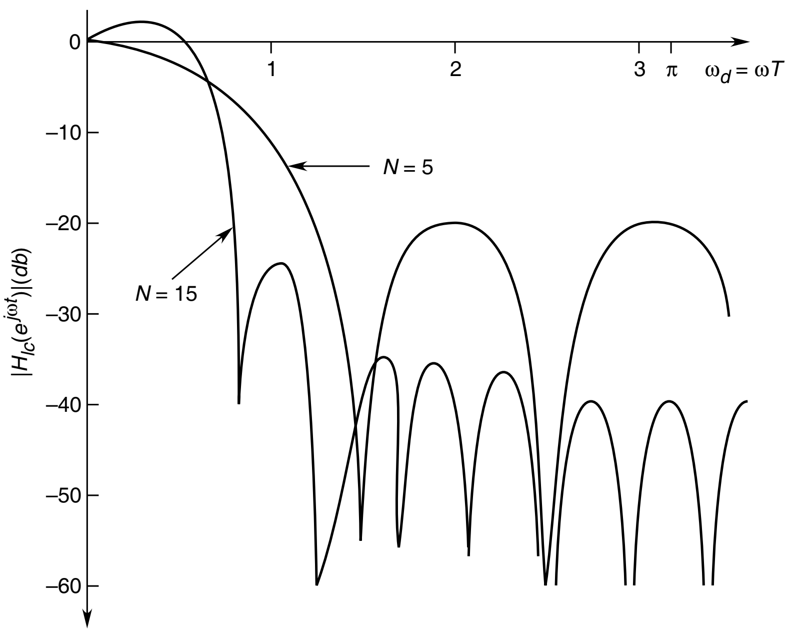


Fig. 9.17 *Plot of amplitude response of FIR filter*

a rectangular window. As a result of this, oscillations arise in the function in the frequency domain due to leakage effect. The leakage effect is reduced by multiplying the original sequence by some appropriate window functions. Important window functions are as follows.

1. **Hamming Window** mathematically described as

$$w(\ell T) = 0.54 + 0.47 \cos(\ell\pi/L)\ ;\ |\ell| \le L = (N-1)/2 \tag{9.76}$$

2. **Blackman Window** given by

$$w(\ell T) = 0.42 + 0.5 \cos(\ell\pi/L) + 0.08 \cos(2\pi\ell/L)$$
$$\text{for } |\ell| \le L = (N-1)/2 \tag{9.77}$$

3. **Hanning Window** given by

$$w(\ell T) = 0.5 + 0.5 \cos(\ell\pi/L);\ |\ell| \le L = (N-1)/2 \tag{9.78}$$

4. **Kaiser Window** Function Kaiser window function provides a great deal of flexibilities and is a near optimal window function. It is formed by zero$^{\text{th}}$ order modified Bessel function of first kind. It is defined as

$$w(\ell T) = \begin{cases} I_0\left[\beta\left\{1-[\ell-\alpha)/\alpha]^2\right\}^{1/2}\right]; 0 \le l \le N \\ 0, \text{ otherwise} \end{cases} \tag{9.79}$$

where $\alpha = \dfrac{N}{2}$ (9.80)

N = number of sample points in the window.

$I_0\{.\}$ = zero$^{\text{th}}$ order modified Bessel function of first kind

β = shape parameter

By varying N and β, the window length and the lobes can be adjusted so as to arrive at desired trade-off between side lobe amplitude and the main lobe width. Characteristics of various window functions are shown in Fig. 9.18.

In order to design an FIR filter using an appropriate window function $w(\ell T)$, the modified truncated sequence of the Fourier coefficients is specified as,

$$h(\ell T) = \begin{cases} \{h_d(\ell T)\, w(\ell T)\ ; 0 \le \ell \le N-1 \\ 0\ ; \text{otherwise} \end{cases} \tag{9.81}$$

Here, $h_d(\ell T)$ is the desired sequence of Fourier coefficient.

FIR Filter Design with Window Function Let us now try to understand the effect of window function on the FIR filter design through some illustrative examples.

We again consider the desired characteristics of the filter as shown in Fig. 9.16. Let us take the example as to how Hamming window affects the FIR approximation. We choose $N = 15$ for obtaining the FIR filter function, approximating the

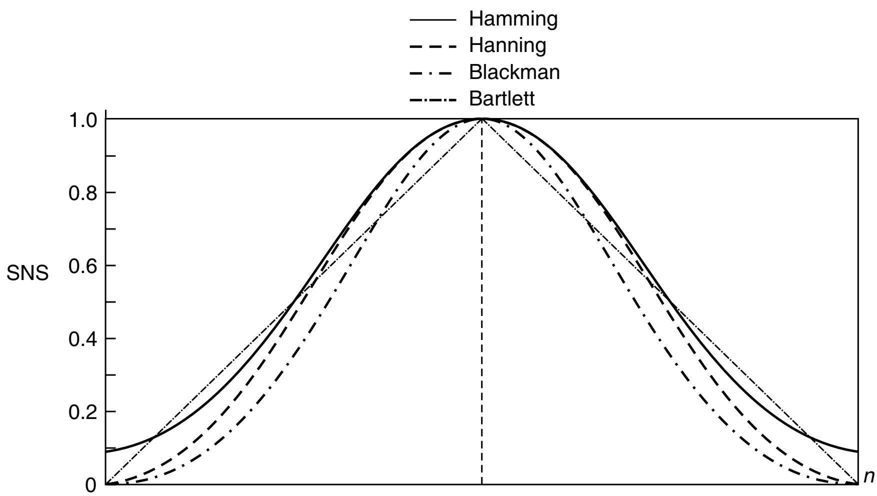


Fig. 9.18 *Commonly used windows for FIR filter design*

desired characteristic. For this characteristic $h(\ell T)$ has already been obtained in closed form in Eq. (9.73), while the Hamming window expression is given in Eq. (9.76). Multiplying these two we get the expression for the approximated impulse response as

$$h(\ell) = \frac{1}{\pi\ell} \sin(\ell\pi/4)\,[0.54 + 0.47 \cos(\ell\pi/7)] \tag{9.82}$$

For $\qquad N = 15 \quad \Rightarrow \quad \ell = 0, \pm 1, \pm 2, \ldots \pm 7$

Specific values of $h(\ell)$ as calculated from Eq. (9.82) are

$$h(0) = 0.25,\ h(\pm 1) = 0.215,\ h(\pm 2) = 0.131$$

$$h(\pm 3) = 0.048,\ h(\pm 4) = 0,\ h(\pm 5) = -0.011$$

$$h(\pm 7) = -0.007, \quad \text{and} \quad h(\pm 7) = 0.003$$

The resulting FIR filter is

$$H_l(z) = \sum_{\ell=-7}^{7} h(\ell T)\, z^{-\ell} \tag{9.83}$$

and corresponding causal filter as per Eq. (9.69) is

$$H_{lc}(z) = z^{-7} H_l(z) \tag{9.84}$$

i.e.

$$\begin{aligned} H_{lc}(z) &= -0.003 - 0.007\, z^{-1} - 0.011\, z^{-2} \\ &\quad + 0.048\, z^{-4} + 131\, z^{-5} + 0.215\, z^{-7} + 0.215\, z^{-8} \\ &\quad + 0.131\, z^{-9} + 0.043\, z^{-10} - 0.011\, z^{-12} - 0.007\, z^{-13} \\ &\quad - 0.003\, z^{-14} \end{aligned} \tag{9.85}$$

The plots of $|H(z)|_{z=e^{j\omega T}}$ in dB vs ω obtained by two different window functions (i) Rectangular function as in Eq. (9.75) and (ii) Hamming window as in Eq. (9.85) are plotted in Fig. 9.19.

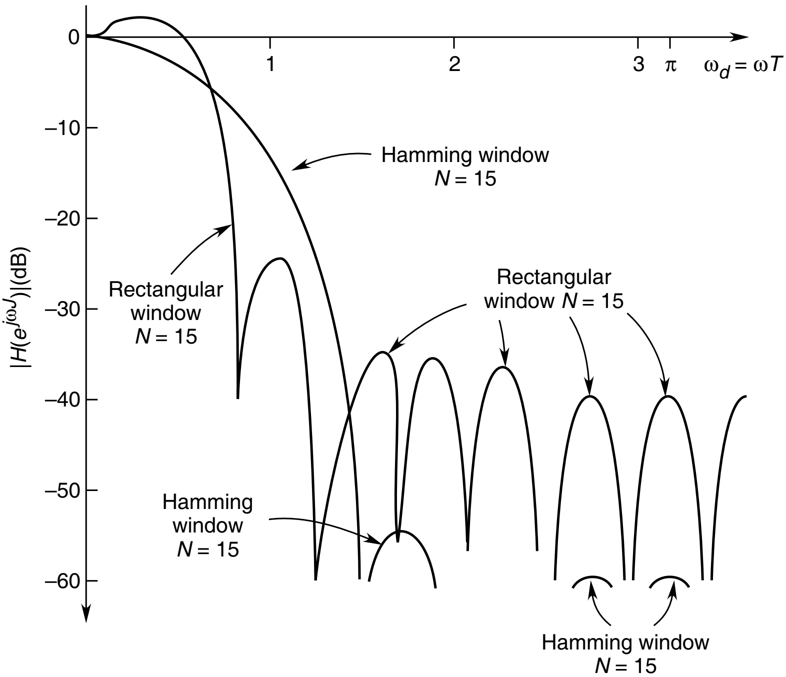


Fig. 9.19

It is evident from Fig. 9.19 that Hamming window removes ripple quite considerably in the stop-band.

If a similar exercise is carried out for other window functions with a view of achieving the desired band-stop characteristic, Blackman window is found to be superior than Hamming and Hanning window. The Kaiser window in fact is the least superset of all the windows mentioned earlier. It will now be illustrated that it has certain desirable features in improving the filter charac-teristics and also has added flexibilities.

We now proceed to design an FIR filter where superior characteristics of a realizable filter can be achieved by the Kaiser window. It has two adjustable parameters.

* Shape factor (β)
* Length (N)

These offer greater flexibility for the Kaiser window. It has been numerically established by Kaiser that to determine the values of β and N to meet the filter specification, the following emprical formulae could be used.

$$\Delta\omega = \omega_s - \omega_p \tag{9.86}$$

and

$$A = -20 \log_{10} \delta \tag{9.87}$$

where $\Delta\omega$ is the transition band for a low-pass filter approximation, ω_p the upper edge of the pass-band and ω_s the lower edge of the stop-band and δ the ripple width of the magnitude response both in the pass-band as well as stop-band. The shape factor β is related to A according to the expression given below.

$$\beta = \begin{cases} \{0.1102(A - 8.7), A > 50 \\ 0.5842\,(A - 21)^{0.4} + 0.07886\,(A - 21), 21 \leq A \leq 50 \\ 0, A < 21 \end{cases} \tag{9.88}$$

N is calculated as

$$N = \frac{A - 8}{2.285\,\Delta\omega} \tag{9.89}$$

Let us now design a low-pass filter which is required to meet following specifications

$$\omega_p = 0.4\pi, \omega_s = 0.7\pi, \delta = 0.001$$

and

$$H(j\omega T) = \begin{cases} e^{-j\omega N/2}\,; |\omega| < \omega_c \\ 0\,; \qquad \omega_c < |\omega| \leq \pi \end{cases}$$

* From Eq. (9.65)

$$h(\ell T) = (1/\omega_s) \int_{-\omega_{s/2}}^{-\omega_{s/2}} H(e^{j\omega T})e^{j\ell\omega T}\, \mathrm{d}\omega; \omega_s = 2\pi/T$$

Using $H_d(e^{j\omega T})$ as given in the specifications above, the impulse response $h(\ell T)$ is obtained as:

$$h(\ell T) = (T/2\pi) \int_{-\omega_c}^{\omega_c} H_d(e^{j\omega T})e^{j\ell\omega T}\, \mathrm{d}\omega$$

$H_d(j\omega T)$ = 0 outside the range ($-\omega_c$ to ω_c) as given. Then

$$h(\ell T) = (T/2\pi) \int_{-\omega_c}^{\omega_c} e^{jN\omega T/2}\, e^{j\ell\omega T}\, \mathrm{d}\omega$$

$$= \frac{T}{2\pi}\left[\frac{e^{j\omega T(\ell - N/2)}}{jT(\ell - N/2)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{2j \sin\{\omega_c T(\ell - N/2)\}}{2\pi j\,(\ell - N/2)}$$

$$= \frac{\sin \omega_c T(\ell - \alpha)}{\pi(\ell - \alpha)}; \alpha = N/2$$

The cut-off frequency ω_c is taken to be at mid-point between ω_p and ω_s thus

$$\omega_c = \frac{0.4\pi + 0.6\pi}{2} = 0.5\,\pi \tag{9.90}$$

$$\Delta\omega = \omega_s - \omega_p = 0.2\,\pi$$

$$A = -20 \log_{10} (0.001) = 60$$

Using Eq. (9.88) β can be determined as,

$$\beta = 0.1102\,(A - 8.7) = 5.653$$

Now using Eq. (9.89), N is found to be 37. According to the above specifications the desired impulse response $h_d(\ell)$ is obtained*, using Eq. (9.65) as

$$h_d(\ell)\ \frac{\sin[\omega_c T(\ell - \alpha)]}{\pi(\ell - \alpha)} \tag{9.91}$$

Thus

$$h(\ell) = h_d(\ell)\, w(\ell)$$

Using $w(\ell)$ from Eq. (9.79) for Kaiser window

$$h(\ell) = \left[\frac{\sin\omega_c(\ell - \alpha)}{\pi(\ell - \alpha)} \cdot \frac{[\beta\,\{1 - (\ell - \alpha)/\alpha)\}^{½}]}{\mathrm{I}_0\beta}\right] \tag{9.92}$$

Here, $\alpha = N/2 = 37/2 = 17.5$ is used to evaluate $h(\ell)$, from which ultimately FIR filter function is found. The complete calculation is not given here, as the approach has already been discussed in earlier examples. However, the magni-tude response of FIR filter function with Kaiser window is shown in Fig. 9.20.

Symmetric and anti-symmetric properties of FIR filters and their significance in realizing the FIR filter are discussed in Section 9.7 (Linear Phase FIR systems).

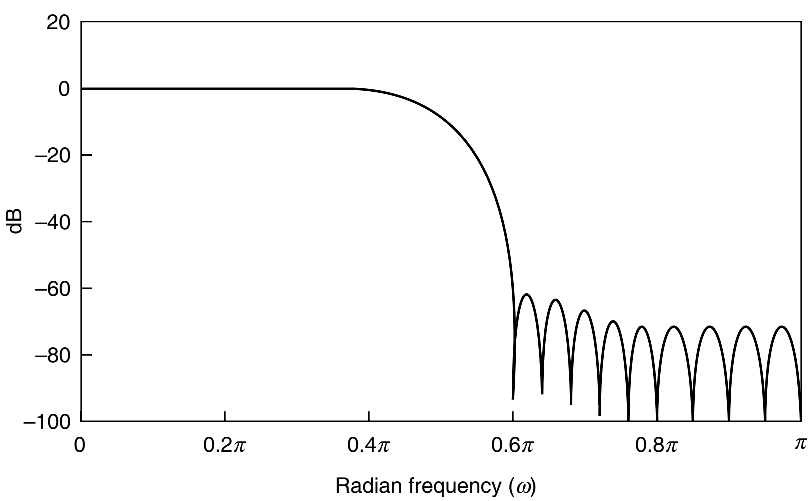


Fig. 9.20 *Plot of magnitude response*

9.4 INFINITE IMPULSE RESPONSE(IIR) FILTER DESIGN

Infinite impulse response filters are realized using feedback (recursive) structures. Designing of IIR filter usually begins with analog transfer function, $H(s)$ and ends up with obtaining the discrete-domain transfer function, $H(z)$. It is necessary to emphasize that satisfactory correspondence between the analog and digital filters in the time domain does not ensure a satisfactory correspondence even in the frequency domain.

In this design technique, a digital filter is found to be equivalent to an analog filter in the time domain invariance sense, that is same inputs yield same outputs. The most often used invariant design is the impulse-invariant digital filter.

In this technique it is assumed that impulse response of a digital filter $h(nT)$ is equivalent to the impulse response of the analog filter in the sampled form $h_a(nT)$, as shown in Fig. 9.21.

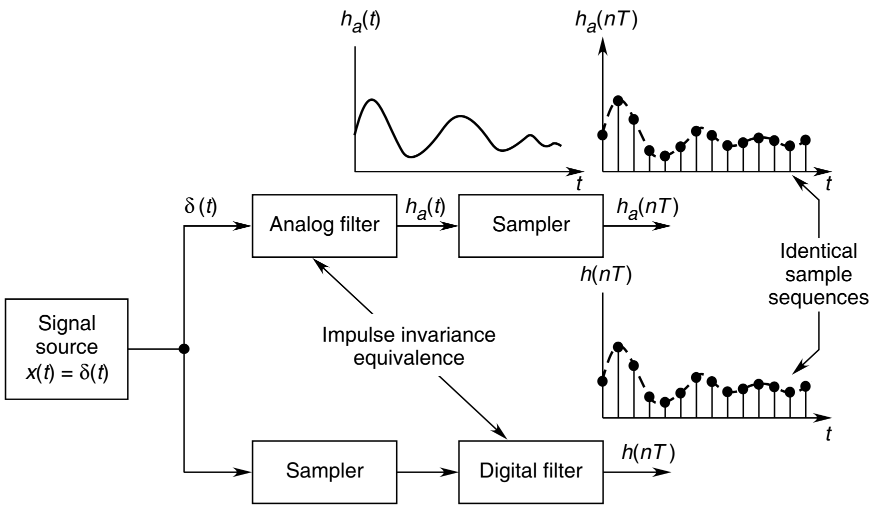


Fig. 9.21 *Schematic of IIR filter*

The design of IIR digital filter employs following steps.

1. Obtain an appropriate analog transfer function in s-domain that meets the given filter specifications.
2. Determine the impulse response $h(t)$ of the analog filter by applying inverse Laplace transform.
3. Sample $h(t)$ at the interval of T-seconds so as to obtain the sequence $\{h_a(nT)\}$.
4. Finally, derive the transfer function of the digital filter $H(z)$ by taking Z-transform of $h_a(nT)$.

Suppose that the transfer function of an analog filter with certain desired characteristics is given as

$$H(s) = \sum_{i=1}^{p} \frac{A_i}{(s + s_i)} \tag{9.93}$$

If it is assumed that all the poles of $H(s)$, i.e. $s = -s_i$ are real and distinct, then it's impulse response function is obtained as,

$$h(t) = \pounds^{-1}\{H(s)\} = \sum_{i=1}^{p} A_i\, e^{-s_i t} \tag{9.94}$$

Let $h_a(nT)$ be the sampled version of analog $h(t)$. Then,

$$h_a(nT) = \sum_{i=1}^{p} A_i\, e^{-s_i nT} \tag{9.95}$$

Taking z-transform of $h(nT)$ yields

$$H(z) = \sum_{n=0}^{\infty} h_a(nT) z^{-n} \tag{9.96}$$

Substituting for $h_a(nT)$ in Eq. (9.96) from Eq. (9.95) one obtain

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} z^{-n} \sum_{i=1}^{p} A_i e^{-s_i nT} \\ &= \sum_{i=1}^{p} A_i \sum_{n=0}^{\infty} z^{-n}\, e^{-s_i nT} \end{aligned} \tag{9.97}$$

Using geometric sum-series formula, $H(z)$ may be rewritten as,

$$H(z) = \sum_{i=1}^{p} \frac{A_i}{1 - e^{-s_i nT} z^{-1}} \tag{9.98}$$

By comparing Eq. (9.93) and Eq. (9.98) it may be observed that one can arrive at $H(z)$ from $H(s)$ by setting

$$s + s_i = 1 - e^{-s_i T} z^{-1} \tag{9.99}$$

Thus $H(z)$ can be obtained from $H(s)$ without determining $h_a(nT)$, which is not the case in FIR filter designing method.

It is to be noted that if the analog filter is stable, i.e. if the real part of s_i lie in the left half of s-plane, then the poles of $H(z)$, which are at $z_i = e^{s_i T}$, will lie inside the unit circle. This is because, the magnitude of $e^{s_i T}$ for $s_i < 0$ will be less than unity and hence $H(z)$ will also be stable. The frequency response of digital filter, however, will not be identical to the frequency response of analog filter, in general. This is because of the fact that practical analog filters are not band-limited and hence aliasing will occur during the sampling.

The sampled version $h_s(t)$ of the impulse response $h(t)$ is expressed as

$$h_s(t) = h(nT) = \sum_{n=-\infty}^{\infty} h(t)\,\delta\,(t - nT) \tag{9.100}$$

Taking its Laplace transform, we get

$$h_s(s) = \text{£}\left\{\sum_{n=-\infty}^{\infty} h(t)\,\delta\,(t - nT)\right\} \tag{9.101}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} H(s - n\omega_s) \tag{9.102}$$

where

$$\omega_s = 2\pi/T = \text{sampling frequency}$$

we shall now prove a useful result.

As per definition

$$F(z) = z\{f(n)\}$$

Setting

$$z = e^{sT} = e^{j\omega T} \tag{9.103}$$

We can write

$$s = (1/T)\ln z \tag{9.104}$$

Using the transformation of Eq. (9.104) in $h_s(s)$ of Eq. (9.102), we can obtain $H(z)$ in the form given below.

$$H(z) = \sum_{i=1}^{p} \frac{A_i}{1 - e^{s_i T} z^{-1}}$$

$$= H_s(s)\big|_{s=(1/T)\ln z} \tag{9.105}$$

Using this result we write Eq. (9.102) as

$$H(z) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H(s - n\omega_s)\big|_{s=(1/T)\ln z} \tag{9.106}$$

From Eq. (9.95) it is clear that for $n = 0$ in the base-band, we can see the following.

$$-\frac{\omega_s}{2} \le \omega T \le \frac{\omega_s}{2}$$

The frequency response of $H(z)$ differs from that of $H(s)$ and the magnitude of this difference is manifested in the form of aliasing error. It is to be noted that this aliasing error will not exist, if we have $|H(s)| = 0$ for $|\omega| > \omega_s/2$ and the frequency response characteristics of the digital and analog filters become identical, when

$$H(z)\big|_{z=e^{j\omega T}} = \frac{1}{T} H_s(s)\Big|_{s=\frac{1}{T}\ln z} \;;\ \text{for } \omega T \le \pi \tag{9.107}$$

We now undertake an example to design a digital filter starting from a given analog filter. Suppose we are provided with an analog filter function as

$$H(s) = \frac{s+4}{2(s+1)(s+2)} \tag{9.108}$$

By using partial fraction expansion we can write Eq .(9.108) in the following form.

$$H(s) = \frac{1.5}{(s+1)} - \frac{1}{(s+2)}$$

Taking the inverse Laplace transform, the unit impulse response of the analog filter is obtained as,

$$h(t) = 1.5e^{-t} - e^{-2t}\text{; for } t \geq 0 \tag{9.109}$$

The unit impulse response of the digital filter, equivalent to the given analog filter, is then given as,

$$h(nT) = 1.5\, e^{-nT} - e^{-2nT}\text{; for } n \geq 0 \tag{9.110}$$

Taking Z-transform of $h(n\mathrm{T})$, yields the following result.

$$H(z) = \frac{1.5T}{1-e^{-T}z^{-1}} - \frac{T}{1-e^{-2T}\,z^{-1}} \tag{9.111}$$

Let us now compare the frequency response of analog filter and the digital filter.

Letting $s = j\omega$ in Eq. (9.108), we get

$$H(j\omega) = \frac{0.5\,(j\omega+4)}{2(j\omega+1)\,(j\omega+2)}\text{; frequency response of analog filter} \tag{9.112}$$

Similarly letting $z = e^{j\omega T}$, in Eq. (9.111), we get

$$H(e^{j\omega T}) = \frac{1.5T}{1-e^{-T}e^{-j\omega T}} - \frac{T}{1-e^{-2T}e^{-j\omega T}} \tag{9.113}$$

$$= \text{frequency response of digital filter}$$

At $\omega = 0$, we have

$$H(j\omega)|_{\omega=0} = \frac{0.5\times 4}{1\times 2} = 1 \tag{9.114}$$

and

$$H(e^{j\omega T})|_{\omega=0} = \frac{1.5T}{1-e^{-T}} - \frac{T}{1-e^{-2T}} \tag{9.115}$$

which indicates that dc frequency response of $H(s)$ and $H(z)$ are different. However, if we make T small such that

$$e^{T} \approx 1 - T \quad \text{and} \quad e^{-2T} \approx 1 - 2T$$

Then

$$H(e^{j\omega T})|_{\omega=0} \approx \frac{1.5T}{1-(1-T)} - \frac{T}{1-(1-2T)} = 1 \tag{9.116}$$

Thus it may easily be interpreted that as the sampling frequency is made large, i.e. $T \to 0$, the difference between the frequency response of analog filter and digital filter can be considerably reduced. In other words, by making the sampling period very small, the aliasing error can be made negligible. These facts are evident from Fig. 9.22 (a) and (b) also. It can be observed from these figures that the characteristics of amplitude response in case of both analog as well as digital filters remain more or less the same, however, the phase characteristic for the two cases differ appreciably as T is increased.

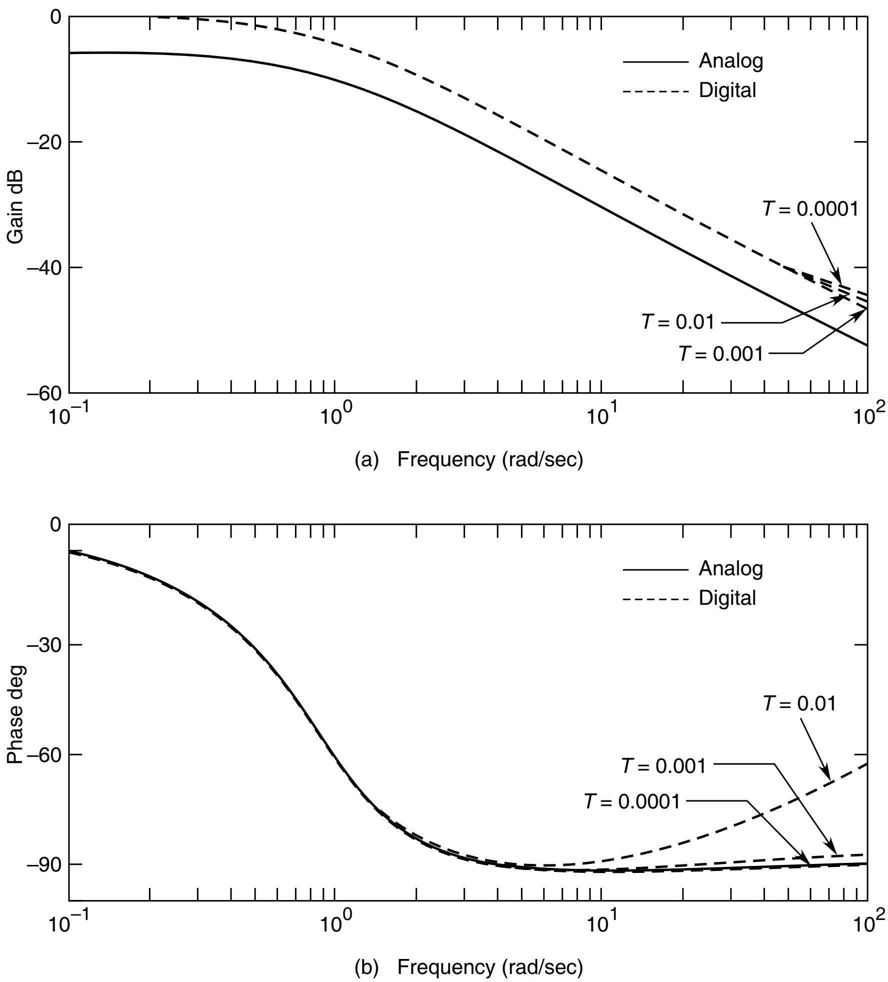


Fig. 9.22 *Plot of amplitude and phase response*

9.5 DESIGNING DIGITAL FILTER USING BILINEAR Z-TRANSFORM

As we have seen that impulse-invariant technique poses certain problems like the one where digital filter contains aliasing error. Moreover, there is a need to obtain the inverse Laplace transform of analog filter function, which gives rise to computational requirements. The method called Bilinear Z-transform overcomes these problems.

The Bilinear Z-transform method is in fact, equivalent to trapezoidal integration. The Bilinear transformation, as such, is a mapping of rational Laplace transform into a rational Z-transform . This mapping has the following two important properties.

(i) If $H(s)$ is the Laplace transform of a causal and stable LTI system, then $H(z)$ is also a causal and stable.
(ii) The important characteristics of $H(s)$ are preserved in $H(z)$.

There is, however, an inherent problem associated with Bilinear Z-transformation, that is, frequency in the analog filter is related non-linearly with the frequencies in the digital filter.

The Bilinear Z-transformation is achieved in the following manner.

If $H(s)$ is the transfer function of an analog filter, then the transfer function of its equivalent digital filter is obtained by the following transformation.

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \tag{9.117}$$

The inverse transformation is

$$z = \frac{1+\left(\frac{T}{2}\right)s}{1-\left(\frac{T}{2}\right)s} \tag{9.118}$$

Substituting $s = \sigma + j\omega$, Eq. (9.118) is written as

$$z = \frac{1+\frac{T}{2}\sigma + j\frac{T}{2}\omega}{1-\frac{T}{2}\sigma - j\frac{T}{2}\omega}$$

$$|z| = \frac{\sqrt{\left(1+\frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}}{\sqrt{\left(1-\frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}} \tag{9.119}$$

For $\sigma < 0$, the numerator of Eq. (9.119) will be less than its denominator, which implies $|z| < 1$, i.e. $|z|$ will lie within the unit circle, indicating that the system is stable. If $\sigma = 0$, then $|z| = 1$, i.e. $|z|$ will lie on the circumference of the unit circle, indicating a stable system. If $\sigma > 0$, then $|z| > 1$ and hence $|z|$ will lie outside the unit circle indicating that the system is unstable.

Let us now investigate the properties of the transformation, given in Eq. (9.117) For $z = e^{j\omega_d}$; $\omega_d = \omega T$.

i.e.
$$s = \frac{2}{T}\left[\frac{1 - e^{-j\omega_d}}{1 + e^{-j\omega_d}}\right] \tag{9.120}$$

We can write $1 = e^{-j\omega_d/2}\, e^{j\omega_d/2}$

Equation (9.120) then gets modified as

$$s = \frac{2}{T}\left[\frac{e^{-j\omega_d/2}.e^{j\omega_d/2} - e^{-j\omega_d}}{e^{-j\omega_d/2}.e^{j\omega_d/2} + e^{-j\omega_d}}\right]$$

Simplification yields the following result.

$$s = \frac{2}{T}\left[\frac{j\sin(\omega_d/2)}{\cos(\omega_d/2)}\right]$$

$$= j\frac{2}{T}\tan(\omega_d/2) \tag{9.121a}$$

As $s = j\omega$, the above equation takes the following form.

$$\omega = \frac{2}{T}\tan(\omega_d/2) \tag{9.121b}$$

From Eq. (9.121a) it is obvious that s has only imaginary part and the real part is zero. This means that poles of $H(z)$ will lie on the circumference of the unit circle and hence will be stable.

It is easily seen from Eq. (9.117) that there is a non-linear relationship between the frequency in the analog domain and frequency in the digital domain. This is called **frequency warping**. Therefore, this technique is applicable only in those cases where non-linearities in frequency may be tolerated or compensated.

If, however, one wishes to design a digital low-pass filter without any change in the critical frequency, then one can employ **frequency prewarping** technique. Design using prewarping technique is achieved in following steps.

(i) Begin with given specifications of digital filter.

(ii) Transform the critical frequencies in the digital domain into corresponding frequencies in the analog domain using the following relation.

$$\omega_{ac} = (2/T)\tan(\omega_{dc}T/2) \tag{9.122}$$

where ω_{dc} is the critical frequency (rad/s) in the digital domain, say cut-off frequency of a given low-pass digital filter and ω_{ac} is the corresponding cut-off frequency in the analog domain, obtained using Eq. (9.122).

(iii) Deduce the analog filter function using the critical frequencies, say ω_{ac}, as obtained in Step (ii).

(iv) Finally apply transformation $s = \dfrac{2}{T}\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]$ to obtain $H(z)$ from $H(s)$ obtained in step (iii).

The above design process is illustrated through an example below.

Let us assume that we are required to design first-order high-pass digital filter with the following specifications.

Cut-off frequency $\omega_{dc} = 2 \times 10^3$ rad/s and sampling frequency = 10^4 samples/s.

Let us first design a high-pass filter in the s-domain with its cut-off frequency, obtained in the following way.

$$\omega_{ac} = \frac{2}{T}\tan\left[\frac{\omega_{dc}T}{2}\right] \quad \text{(i)}$$

$$= \frac{2}{10^{-4}}\tan\left[\frac{2\pi \times 10^{-3} \times 10^{-4}}{2}\right]$$

$$= 0.65 \times 10^{-4} \text{ rad/s}$$

The first order high-pass filter in the s-domain, therefore, is

$$H(s) = \frac{(s/\omega_{ac})}{(s/\omega_{ac})+1} = \frac{(s/0.65 \times 10^{-4})}{1+(s/0.65 \times 10^{-4})}$$

$$= \frac{s}{s + 0.65 \times 10^{-4}} \quad \text{(ii)}$$

The first-order high-pass filter in the digital domain is now obtained by applying bilinear Z-transformation to Eq. (ii), i.e.

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}}$$

i.e.

$$H(z) = \frac{2 \times 10^{-4}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)}{2 \times 10^{-4}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 2 \times 0.325 \times 10^{-4}}$$

$$= \frac{z-1}{1.325z - 0.625} \quad \text{(iii)}$$

It may be noted that the sampling period T gets cancelled in the process of designing the filter using Bilinear Z-transform.

9.6 STRUCTURES OF DIGITAL SYSTEMS

The problem of synthesizing a digital system will be discussed here. Synthesization or realization of digital network is carried out so that it meets the specifications of a given digital system. This may be achieved in number of ways. However, main considerations in the choice of a particular method or form of synthesization is the computational complexities. It is often desirable to have the structure of the system which requires minimum number of constant multipliers and delay units or registers. This is largely because of the fact that multiplication operation takes longer time and more delay units mean requirement of more number of memory elements. Another important consideration is the effects of finite word-length, also called register length. According to this consideration, the most appropriate structure or form is the one which is less sensitive to finite register length effects. We now examine several techniques for realization of a digital network.

Realization of IIR Filters

Some of the most commonly used network forms for IIR filters are the following.

(i) Direct form
(ii) Cascade form
(iii) Parallel form

Methods of realizing IIR filters using above mentioned forms are discussed below.

Direct-Form Direct-form realization can be classified as (i) Direct-form I and (ii) Direct-form II. In Direct-form I, a digital network is realized without bringing any modifications in the given transfer function $H(z)$.

Direct-Form I Realization Let us consider a transfer-function $H(z)$, given as

$$H(z) = \frac{\left[\sum_{i=0}^{m} a_i z^{-i}\right]}{\left[1 + \sum_{i=1}^{m} b_i z^{-i}\right]} \tag{9.123}$$

It may be recalled from Section 9.2 that $H(z)$ expressed in Eq. (9.63) is a recursive or IIR filter function. This is because in this case output is dependent not only on the past and present inputs, but also on the past outputs.

Let us now modify Eq. (9.123) in the form of inputs and outputs as a function of discrete time in the following manner.

$$H(\mathrm{z}) = \frac{N(z)}{D(z)} = \frac{Y(z)}{X(z)} = \frac{\left[a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_m z^{-m}\right]}{\left[1 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m}\right]} \tag{9.124}$$

i.e.

$$a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + \cdots + a_m z^{-m} X(z)$$
$$= Y(z) + b_1 z^{-1} Y(z) + b_2 z^{-2} Y(z) + \cdots + b_m z^{-m} Y(z)$$

i.e.

$$\begin{aligned} Y(z) &= a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + \cdots + a_m z^{-m} X(z) \\ &\quad - b_1 z^{-1} Y(z) - b_2 z^{-2} Y(z) \cdots b_m z^{-m} Y(z) \end{aligned} \tag{9.125}$$

Taking inverse Z-transform (IZT) on both sides of Eq. (9.125), one obtains

$$\begin{aligned} y(n) &= a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + \cdots + a_m x(n-m) \\ &\quad - b_1 y(n-1) - b_2 y(n-2) \cdots b_m y(n-m) \end{aligned} \tag{9.126}$$

It may be noted that $x(n)$ and $y(n)$ indicate inputs and outputs of the given system.

Realization of Direct-form I based on Eq. (9.126) is shown in Fig. 9.23.

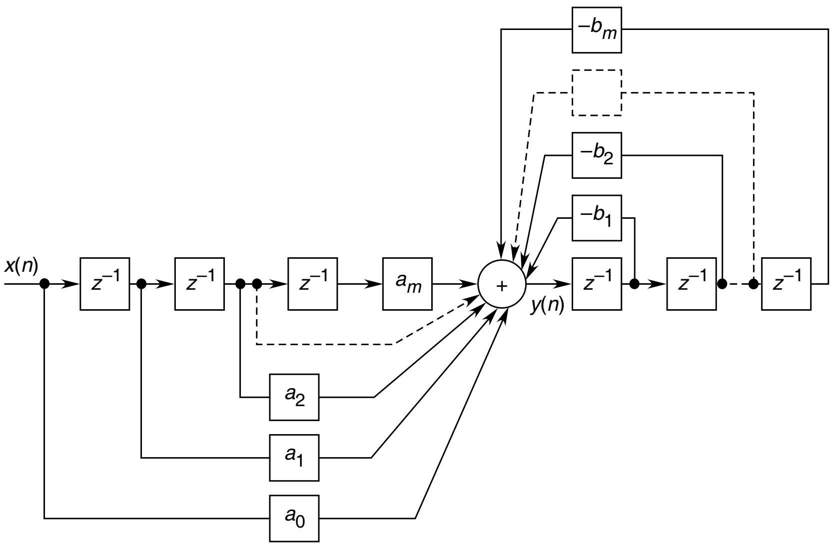


Fig. 9.23 *Direct-form I realization*

It may be observed from Fig. 9.23 that m^{th} order IIR filter requires 2m registers as indicated by delay units z^{-1} and equal number of constant multipliers,

$$a_0, a_1, \ldots a_m, b_1, b_2 \ldots, b_m$$

Requirement of registers, however, can be reduced to only m if an m^{th} order IIR filter is realized using Direct-form II, which is discussed below.

Direct-Form II Realization In Direct-form II, the given transfer function being realized is,

$$H(z) = \frac{N(z)}{D(z)} = \frac{Y(z)}{X(z)}$$

i.e.

$$Y(z) = \frac{N(z)X(z)}{D(z)} \tag{9.127}$$

We now define $W(z) = \dfrac{X(z)}{D(z)}$ (9.128)

so that

$$Y(z) = N(z)\, W(z) \tag{9.129a}$$

and

$$X(z) = W(z)\, D(z) \tag{9.129b}$$

Let us again consider the realization of the earlier example of IIR filter function, whose transfer-function $H(z)$ is expressed in Eq. (9.124),

Thus

$$D(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m} \tag{9.130}$$

$$N(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_m z^{-m} \tag{9.131}$$

$$X(z) = W(z)\,[1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}] \tag{9.132}$$

and

$$Y(z) = W(z)\,[a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_m z^{-m}] \tag{9.133}$$

Taking IZT on both sides of Eqs. (9.132) and (9.133) yields,

$$x(n) = w(n) + b_1 w(n-1) + b_2 w(n-2) + \cdots + b_m w(n-m)$$

or

$$w(n) = x(n) - b_1 w(n-1) - b_2\, w(n-2) - \cdots - b_m w(n-m) \tag{9.134}$$

and

$$y(n) = a_0 w(n) + a_1 w(n-1) + a_2 w(n-2) + \cdots + a_m w(n-m) \tag{9.135}$$

Direct-form II is finally, achieved by combining Eqs (9.134) and (9.135), as shown in Fig. 9.24.

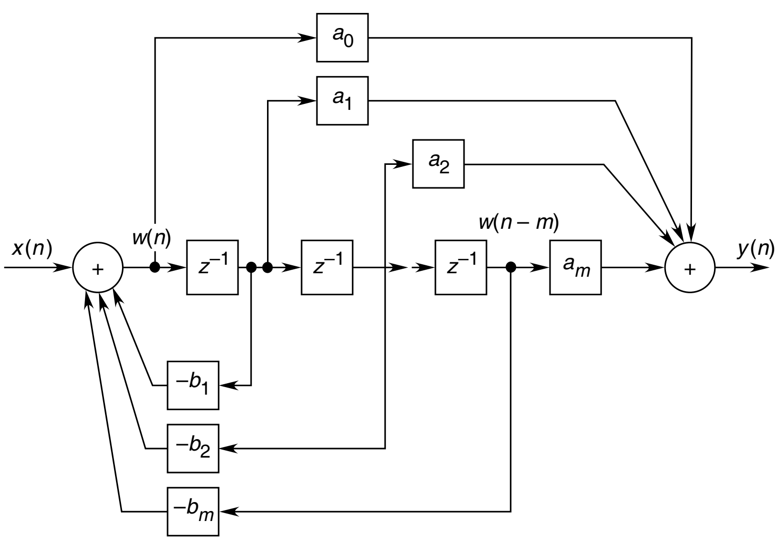


Fig. 9.24 *Direct-form II realization*

It is clear from the structure shown in Fig. 9.24, that Direct-form II requires only m number of registers, which is just half the number, required in case of Direct-form I. However, the number of constant multipliers in this case also is same as the number required in Direct-form I.

Cascade Form In the cascade form, a given transfer function H(z), is obtained in the factored form as below.

$$H(z) = K\,H_1(z)\,H_2(z)\,H_3(z)\,...\,H_m(z) \tag{9.136}$$

m is a positive integer and each $H_m(z)$ is a first order or second order transfer function. Since, poles and zeros of the transfer function are either real or exist in complex conjugate pair, $H(z)$ may also be written in the following form.

$$H(z) = K\,\frac{\prod_{i=1}^{p}(1-a_i\,z^{-1})\prod_{i=1}^{q}(1-b_i\,z^{-1})(1-b_i^*\,z^{-1})}{\prod_{i=1}^{r}(1-c_i\,z^{-1})\prod_{i=1}^{\ell}(1-d_i\,z^{-1})(1-d_i^*\,z^{-1})} \tag{9.137}$$

It is clear from the expression of $H(z)$ in Eq. (9.137) that first order factors represent real zeros at a_i and real poles at c_i, whereas second order factors represent complex-conjugate zeros at b_i and b_i^* and complex-conjugate poles at d_i and d_i^*. The transfer function involving complex conjugate pairs of poles and zeros are realized as second order section because the multiplier constants whether in the form of $(b_i + b_i^*)$ or $(b_i\,b_i^*)$ will always be real.

The factors of $H(z)$ may have any composition and may be cascaded in any order but it is important that cascade form is achieved with minimum of storage elements. A cascade structure with minimum requirement of memory elements may be possible, if each second order section is realized using Direct-form II. The cascade form may, however, poses problem, i.e. one section may cause loading to next section, if their impedances are not properly matched.

We now discuss some examples, which will illustrate as to how a digital network can be obtained in the cascade form.

Let

$$H(z) = \frac{0.7(z^2 - 0.36)}{z^2 + 0.1z - 0.72} \tag{i}$$

It may be rewritten in factored form as

$$H(z) = \frac{0.7\,(z+0.6)\,(z-0.6)}{(z+0.9z)\,(z-0.8)}$$

$$= \frac{0.7\,(1+0.6z^{-1})\,(1-0.6z^{-1})}{(1+0.9z^{-1})\,(1-0.8z^{-1})} \tag{ii}$$

Comparing with Eq. (9.136), we identify

$$K = 0.7,\ H_1(z) = \frac{(1+0.6z^{-1})}{(1+0.9z^{-1})} \tag{iii}$$

and

$$H_2(z) = \frac{(1-0.6z^{-1})}{(1-0.8z^{-1})} \tag{iv}$$

We now write Eq. (9.136) in the following manner.

$$Y(z) = KH_1(z)\,H_2(z)\,...\,H_m(z)\,X(z) \tag{9.138}$$

Since the example under consideration has factors corresponding only to $m = 1$ and 2, so Eq. (9.138) is rewritten as

$$Y(z) = K_0\,H_1(z)\,H_2(z)\,X(z) \tag{v}$$

We define

$$W(z) = H_2(z)\,X(z)$$

or

$$H_2(z) = \frac{W(z)}{X(z)} = \frac{(1-0.6z^{-1})}{(1-0.8z^{-1})} \tag{vi}$$

From Eq. (vi) we can write the following equation.

$$W(z) = 0.8\,W(z-1) + X(z) - 0.7X(z-1) \tag{vii}$$

Taking IZT on both sides of Eq. (vii) results in the following equation.

$$w(n) = 0.8\,w(n-1) + x(n) - 0.7X(n-1) \tag{viii}$$

One can write Eq. (v) as

$$H(z) = KH_1(z)W(z) \tag{ix}$$

i.e.

$$H_1(z) = \frac{Y(z)}{KW(z)} = \frac{(1+0.6z^{-1})}{(1+0.9z^{-1})} \tag{x}$$

i.e. $$Y(z)\,\{1 + 0.9z^{-1}\} = 0.7\,W(z)\,\{1 + 0.7z^{-1}\}$$

Taking IZT on both sides, one gets the following equation.

$$y(n) = 0.7\,w(n) + 0.42\,w(n-1) - 0.9\,y(n-1) \tag{xi}$$

The cascade-form of given $H(z)$ using Eqs (vii) and (xi) is shown in Fig. 9.25.

Parallel Form The parallel-form of digital network is obtained by using partial fraction expansion of the system function $H(z)$. The general form of partial fraction is given by

$$H(z) = \sum_{i=0}^{p} a_i z^{-i} + \sum_{j=1}^{q} \frac{b_j}{(1-c_j z^{-1})} + \sum_{\ell=1}^{r} \frac{d_\ell\,(1-c_\ell z^{-1})}{(1-f_\ell z^{-1})\,(1-f_\ell^* z^{-1})} \tag{9.139}$$

Let us illustrate parallel-form realization through the following example.

Given

$$H(z) = \frac{0.7(z+0.6)\,(z-0.6)}{(z+0.9)\,(z-0.8)}; \text{ (same as in earlier example)} \tag{i}$$

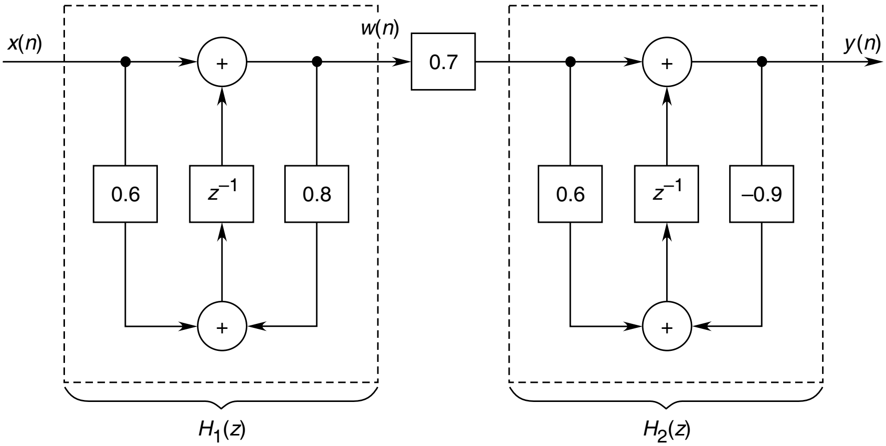


Fig. 9.25 *Cascade form realization*

Using partial fraction expansion, the above transfer function is rewritten as,

$$H(z) = 0.35 + \left\{\frac{0.206}{1+0.9z^{-1}}\right\} + \left\{\frac{0.144}{1-0.8z^{-1}}\right\} \quad \text{(ii)}$$

i.e.

$$\mathrm{Y}(z) = 0.35X(z) + \left\{\frac{0.206X(z)}{1+0.9z^{-1}}\right\} + \left\{\frac{0.144X(z)}{1-0.8z^{-1}}\right\} \quad \text{(iii)}$$

Let

$$0.35X(z) = Y_1(z)$$

$$\frac{0.206X(z)}{1+0.9z^{-1}} = Y_2(z)$$

i.e.,

$$Y_2(z) = -0.9z^{-1}Y_2(z) + 0.207X(z)$$

Again, assume

$$\frac{0.144X(z)}{1-0.8z^{-1}} = Y_3(z)$$

So that

$$Y_3(z) = 0.8z^{-1}\, Y_3(z) + 0.144X(z)$$

Then

$$Y(z) = Y_1(z) + Y_2(z) + Y_3(z) \quad \text{(iv)}$$

Taking IZT of Eq. (iv), we get

$$y(n) = y_1(n) + y_2(n) + y_3(n) \quad \text{(v)}$$

$y_1(n), y_2(n)$ and $y_3(n)$ as obtained by taking IZT of $Y_1(z), Y_2(z)$ and $Y_3(z)$ are

and
$$\left.\begin{aligned} y_1(n) &= 0.35\, x(n) \\ y_2(n) &= -\,0.9\, y_2(n-1) + 0.207\, X(n) \\ y_2(n) &= 0.8\, y_3(n-1) + 0.144\, X(n) \end{aligned}\right\} \qquad \text{(vi)}$$

The parallel-form realization of this $H(z)$ using Eq. (vi) is shown in Fig. 9.26.

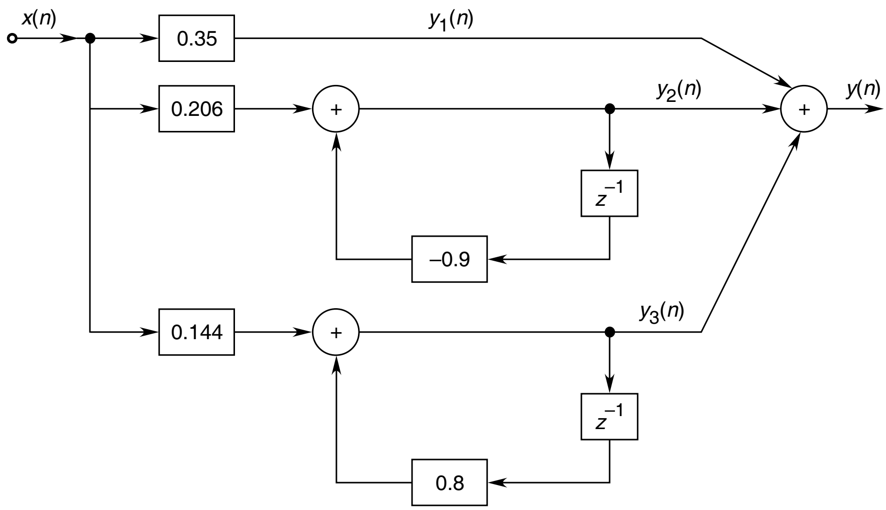


Fig. 9.26 *Parallel-form realization*

Effects of Finite Word-length and Structures of Digital Systems

It is known, that for a given system having a transfer function $H(z)$, the coefficients of the numerator and denominator polynomials can only be approximated, if encoded in the binary form. This is because of the availability of registers of finite word-length. It is, therefore, important that we realize such digital systems in the form, which is less sensitive to the changes in the roots of the transfer function. If the forms are found to be highly sensitive to the changes in the roots, then it may so happen that poles closer to unity may move out of the unit circle, thereby, making stable system unstable.

The digital system realized in Direct-form I is found to have the roots more sensitive to changes as compared to those realized either in cascaded-form or parallel-form. This is illustrated in the following example.

Let us consider a second order recursive function $H(z)$, given as

$$H(z) = \frac{1}{(1-0.5z^{-1})\,(1-0.4z^{-1})} \qquad \text{(i)}$$

If we wish to realize this $H(z)$, in Direct-form I, then it is modified as

$$H(z) = \frac{1}{1 - 0.9z^{-1} + 0.2z^{-2}} \tag{ii}$$

Again it is assumed that coefficients are encoded in the binary form using registers having word-length of four bits. If the dynamic range is two, then number of quantized level will be equal to $2^4 = 16$ and the step-size will be the following.

$$q = 2/16 = 0.125$$

Thus

$$q/2 = 0.0625$$

It is evident that for encoding the coefficients in the binary form, we choose the quantized value of coefficients such that the magnitude of the difference of original value is less than 0.0625. In such conditions we may encode the coefficients 0.9 as 0.875 and 0.2 and 0.250. Thus modified transfer function $H_b(z)$, obtained from given $H(z)$, is given as

$$H_b(z) = \frac{1}{1 - 0.875z^{-1} + 0.250z^{-2}} \tag{iii}$$

Let us now realize $H(z)$, given in Eq. (i), in the cascaded-form. For this, $H(z)$ is rewritten as

$$H(z) = H_1(z)\, H_2(z) \tag{iv}$$

With either

$$H_1(z) = \frac{1}{1 - 0.5z^{-1}},\ H_2(z)\ \frac{1}{(1 - 0.4z^{-1})}$$

or

$$H_1(z) = \frac{1}{(1 - 0.4z^{-1})},\ H_2(z) = \frac{1}{(1 - 0.5z^{-1})} \tag{v}$$

In both cases, the poles remain at 0.4 and 0.5 and they may be approximated as 0.375 and 0.5 respectively, if encoded in the binary form.

This modified transfer function, suitable to cascade-form, is given either as,

$$H'_b(z) = \frac{1}{(1 - 0.5z^{-1})} \cdot \frac{1}{(1 - 0.375z^{-1})}$$

or

$$H'_b(z) = \frac{1}{(1 - 0.375z^{-1})} \cdot \frac{1}{(1 - 0.5z^{-1})} \tag{vi}$$

It is quite clear from Eqs (iii) and (vi) that roots of $H'_b(z)$ are closer to the original roots (i.e., roots of $H(z)$). Hence, if the digital system having finite register length is realized in cascaded form then the roots will be found to be less sensitive to the changes due to encoding.

In the similar fashion it can be illustrated that if the system is realized in parallel-form then again roots will be less sensitive to changes as compared to the Direct-form I.

Realization of FIR Filters

The transfer function of an FIR filter may be obtained from Eq. (9.123) by making the coefficients in denominator of the polynomial (b_i ; for i = 1, 2, 3…, n) zero. $H(z)$ thus obtained is represented in the following form.

$$H(z) = \sum_{i=0}^{m} a_i z^{-i} \tag{9.140}$$

It is obvious from Eq. (9.140) that the transfer function of an FIR filter contain only zeros and no poles.

Some of the most commonly used network forms for FIR filters are the following.

(i) Direct-form
(ii) Cascade-form
(iii) Network for linear phase FIR filters
(iv) Frequency sampling structure

Only first three will be discussed here.

Direct Form In the case of FIR filters, both direct-form I and direct-form II reduce to the direct form as shown in Fig. 9.29.

Figure 9.27 has been directly obtained from Fig. 9.23 making $b_i = 0$ for i = 1, 2, 3... m, etc.

Cascade Form The cascade form for FIR systems is obtained in factored form as given below.

$$H(z) = \sum_{\ell=0}^{N} h(\ell) z^{-\ell} = K\, H_1(z)\, H_2(z)\, ...\, H_p(z) \tag{9.141}$$

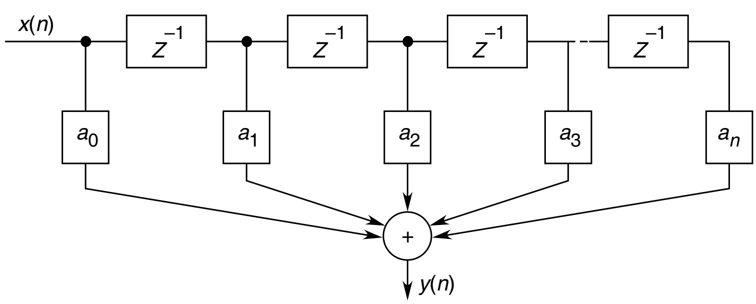


Fig. 9.27 *Direct-form FIR filter*

Here, p is a positive integer and each $H_i(z)$ is a first-order or second-order transfer function.

The cascade form of realization is illustrated through an example below.

It is given that

$$H(z) = 0.7 - 0.35z^{-1} + 0.042z^{-2} \qquad \text{(i)}$$

This can be written in factored form as

$$H(z) = 0.7(1 - 0.2z^{-1})\,(1 - 0.3z^{-1})$$

i.e.

$$Y(z) = 0.7(1 - 0.2z^{-1})\,(1 - 0.3z^{-2})\,X(z) \qquad \text{(ii)}$$

Let

$$(1 - 0.2z^{-1})\,X(z) = W(z) \qquad \text{(iii)}$$

Taking IZT of Eq. (iii) one gets

$$w(n) = x(n) - 0.2\,x(n-1) \qquad \text{(iv)}$$

Using Eqs (ii) and (iii), we get

$$Y(z) = 0.7\,W(z)\,(1 - 0.3\,z^{-1}) \qquad \text{(v)}$$

$$y(n) = 0.7\,[w(n) - 0.3\,w(n-1)] \qquad \text{(vi)}$$

Using Eqs (iv) and (vi) the Cascade-form of given transfer function $H(z)$ is obtained as shown in Fig. 9.28.

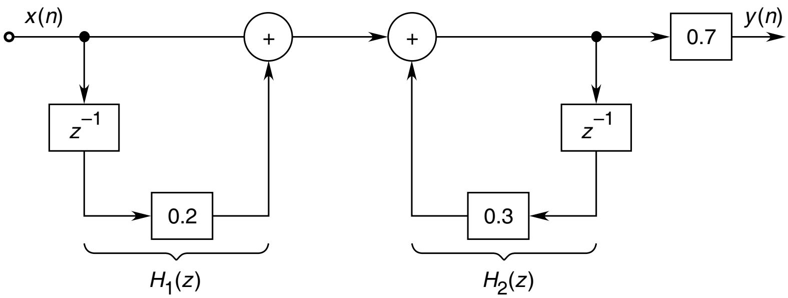


Fig. 9.28

Linear Phase FIR Systems The unit-impulse response of a linear phase causal FIR system can be either symmetric or anti-symmetric.

The symmetric impulse response is defined as

$$h(n) = h(N - n)\;;\; 0 \le n \le N;\ \text{even or odd} \qquad (9.142)$$

The anti-symmetric impulse response is defined as

$$h(n) = -h(N - n)\;;\; 0 \le n \le N;\ \text{even or odd} \qquad (9.143)$$

As a result of these properties the number of multiplier units in FIR filter realization can be reduced to half for the even case and nearly half for the odd case. This procedure is now illustrated through an example.

Consider a low-pass filter with desired characteristics given as

$$H_d(e^{j\omega T}) = e^{-j\omega NT/2}\,;\ |\omega| \le \omega_c \qquad \text{(i)}$$
$$= 0;\ \text{otherwise}$$

The impulse response corresponding to Eq. (i) is same as obtained in Eq. 9.91, i.e.

$$h(nT) = \frac{\sin\{\omega_c T\,(n - N/2)\}}{\pi\,(n - N/2)} \qquad \text{(ii)}$$

The approximated filter characteristics, using FIR design technique, therefore, is

$$H(z) = \sum_{n=0}^{N-1} h(nT) z^{-n} \qquad \text{(iii)}$$

On replacing n by $(N - n)$ in Eq. (ii), we write

$$h\{(N-n)T\} = [\sin\{\omega_c T\,(N/2 - n)\}]/[\pi(N/2 - n)]$$
$$= -[\sin\{\omega_c T\,(N/2 - n)\}]/-[\pi(N/2 - n)]$$
$$= h(nT) \qquad \text{(iv)}$$

This example has illustrated that the impulse response of a linear FIR filter has the even symmetric property. This property will now be explained.

Let us assume $N = 5$ and write Eq. (iii) in the expanded form, i.e.

$$H(z) = h(0) + h(T)z^{-1} + h(2T)z^{-2} + h(3T)z^{-3} + h(4T)z^{-4} \qquad \text{(v)}$$

which implies that

$$Y(z) = [h(0) + h(T)z^{-1} + h(2T)z^{-2} + h(3T)z^{-3} + h(4T)z^{-4}]\,X(z) \qquad \text{(vi)}$$

Taking IZT of Eq. (vi) we get

$$y(n) = h(0)\,x(n) + h(T)\,x(n-1) + h(2T)\,x(n-2) + h(3T)\,x(n-3)$$
$$+ h(4T)\,x(n-4) \qquad \text{(vii)}$$

From Eq. (vii) the direct form realization of the FIR filter designed as above is drawn in Fig. 9.29(a).

Using the symmetric property, we can write ($N = 5$)

$$h(T) = h(4T); \quad \text{and} \quad h(2T) = h(3T)$$

This allows us to rewrite Eq. (vii) as

$$y(n) = h(0)x(n) + h(T)x(n-1) + h(2T)x(n-2) + h(2T)$$
$$x(n-3) + h(T)x(n-4) \qquad \text{(viii)}$$

This modified equation leads to the realization of Fig. 9.29(b). It is easily observed that the realization of Fig. 9.29(b) requires $\left(\frac{N-1}{2}\right)$ multipliers instead of N multipliers as required in Fig. 9.29(a).

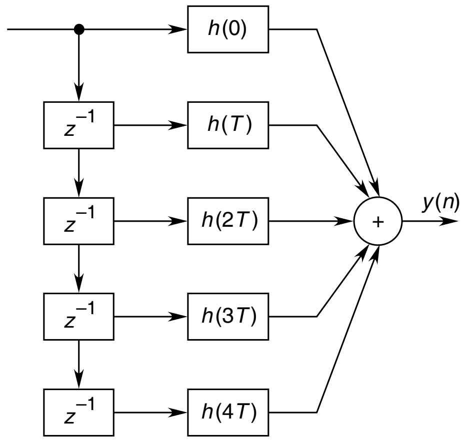


Fig. 9.29(a)

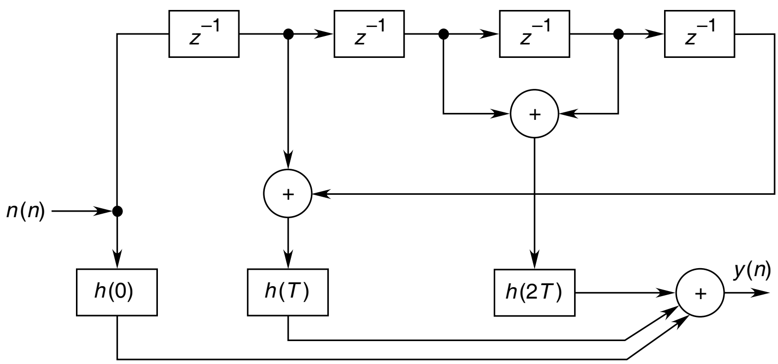


Fig. 9.29(b)

If we take N to be even then it will be found that realization using symmetric property requires only $N/2$ multiplier units.

Additional Examples

Example 9.8 *An analog filter is characterized as $H(s) = 4/(s + 2)^2$.*

Design the corresponding IIR filter. It is given that sampling frequency is 10 Hz.

Solution Taking the inverse Laplace transform gives the following result.

$$h(t) = \mathcal{L}^{-1}[4/(s+2)^2]$$
$$= 4t\,e^{-2t}\,u(t) \tag{i}$$

Then

$$h(nT) = 4nT\,e^{-2nT};\ n \ge 0$$
$$= 0.4n\,e^{-0.2n};\ T = 1/0.1\text{s} \qquad \text{(ii)}$$

Taking the z-transform of Eq. (ii) we get

$$H(z) = \sum_{n=0}^{\infty} 0.4n\,e^{-0.2n}\,z^{-n} \qquad \text{(iii)}$$

or

$$H(z) = \frac{\lfloor 0.3275z^{-1} \rfloor}{[1 - 0.818z^{-1}]} \qquad \text{(iv)}$$

Example 9.5 *Design a low-pass filter that approximates*

$$|H(e^{j\omega T})| = 1;\ -1000 \le f \le 1000\ Hz$$
$$= 0;\quad otherwise$$

f_m is the maximum frequency component of the filter function. Take sampling frequency as 8000 samples per second (sps). The impulse response duration is to be limited to 2.5ms. Plot the magnitude response of the filter.

Solution According to sampling theorem,

$$T \text{ (critical sampling period)} = 1/2f_m$$

or

$$f_m = 1/2T \qquad \text{(i)}$$

Sampling frequency is given as

$$f_s = 1/T = 8000 \text{ Hz},\ \omega_s = 2\,\pi \times 8000 \text{ rad/s} \qquad \text{(ii)}$$
$$f_m = f_s/2 = 4000 \text{ Hz} \qquad \text{(iii)}$$

It is also specified that

$$t_m = 2.5 \times 10^{-3}\text{s}$$

Therefore, number of samples in the impulse response is

$$N = \frac{2.5 \times 10^{-3}}{T} = 8000 \times 2.5 \times 10^{-3} = 20 \qquad \text{(iv)}$$

With initial point included, sample points are

$$N = 21 \quad \text{or} \quad \frac{N-1}{2} = 10 \qquad \text{(v)}$$

Then ℓ takes on values 0, ± 1, ± 2, ., ± 10.

Frequency range of analog filter is –1000 to 1000 Hz or $-\omega_s/8$ to $\omega_s/8$.

Sampled impulse response is obtained below.

$$h(\ell T) = \frac{1}{\omega_s} \int_{-\omega_s/8}^{\omega_s/8} e^{j\omega T}\, d\omega$$

$$= \frac{1}{\omega_s}\frac{1}{j\ell T}[e^{j\ell\omega T}]_{-\omega_s/8}^{\omega_s/8}$$

$$= \frac{1}{\pi\ell}\sin\left[\ell\frac{\omega_s}{8}T\right] = \frac{1}{\pi\ell}\sin\left(\frac{\pi\ell}{4}\right) \quad \text{(vi)}$$

($h(\ell T)$), therefore, has even symmetry for 2ℓ samples

$$H_{tc}(z) = z^{-10}\left[\sum_{\ell=-10}^{10} h(\ell T)z^{-\ell}\right] \quad \text{(vii)}$$

Computing $h(\ell T)$ from Eq. (vii) and utilizing symmetry property, we arrive at the following result for the transfer function of the approximated causal low-pass filter.

$$\begin{aligned} H_{lc}(z) &= 0.0318 + 0.025z^{-1} + (-0.032)z^{-3} + (-0.053)z^{-4} \\ &+ (-0.045)z^{-5} + 0.075z^{-7} + 0.1591z^{-8} + 0.225z^{-9} + z^{-10} \\ &+ 0.225z^{-11} + 0.1591z^{-12} + 0.075z^{-13} + (-0.045)z^{-15} \\ &+ (-0.053)z^{-16} + (-0.032)z^{-17} + 0.025z^{-19} + 0.03182z^{-20} \end{aligned} \quad \text{(viii)}$$

The plot of amplitude response of the filter function, as obtained in Eq. (viii), is shown in Fig. 9.30(a), (b), (c) and (d).

Example 9.6 *Design a band-pass filter that approximates the following transfer function.*

$$\begin{aligned} H(e^{j\omega T}) &= 1\,; \qquad 160 \le f \le 200 \text{ Hz} \\ &= 0\,; \qquad 200 \text{ Hz} \le f \le f_m \text{ Hz} \end{aligned}$$

f_m is the maximum frequency component in the above transfer function. The sampling period is (1/800)s and duration of the impulse response is 50 ms.

Solution We have $t_m = NT = 50 \times 10^{-3}$s (i)

or

$$N = \frac{50\times10^{-3}}{1/800} = 40 \quad \text{(ii)}$$

Including initial point $N = 41$, so that $\ell = 0, \pm 1, \pm 2, \ldots \pm 20$.

Now,

$$f_s = 1/T = 800 \text{ sps} \quad \text{(iii)}$$

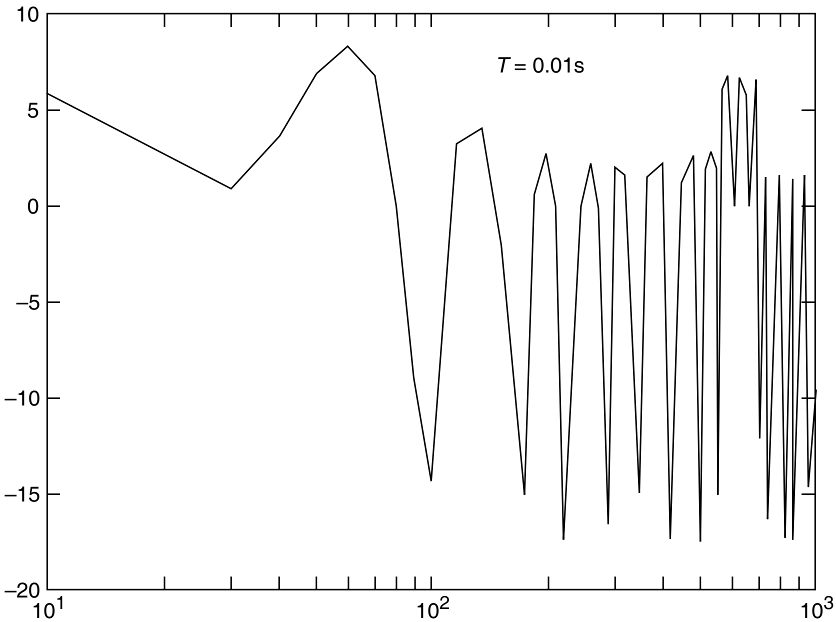
$$\omega_s = 2\omega_m = 2\pi \times 800 \text{ rad/s} \quad \text{(iv)}$$

Also,

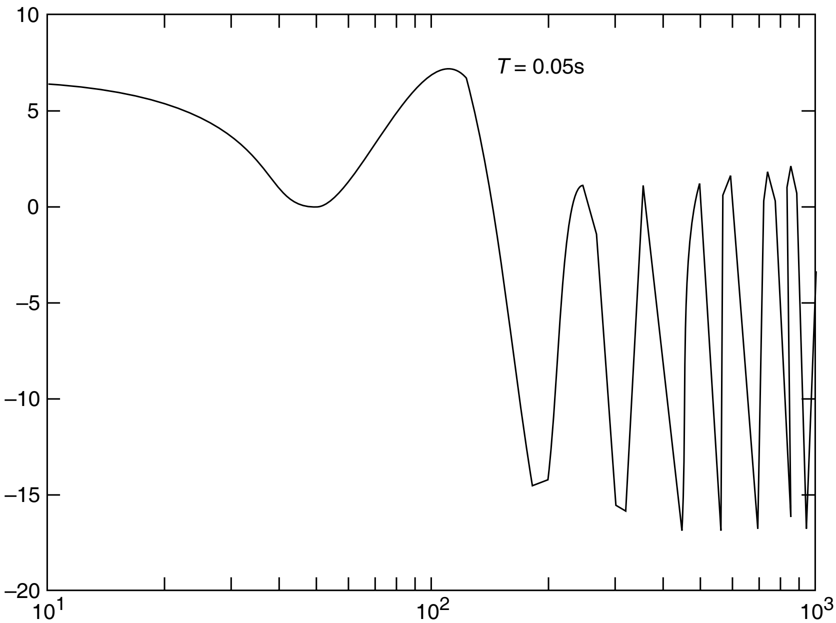
$$f = 160 \text{ Hz} \Rightarrow 2\pi \times 160 = \omega_s/5 \quad \text{(v)}$$

and

$$f = 200 \text{ Hz} \Rightarrow 2\pi \times 200 = \omega_s/4 \quad \text{(vi)}$$

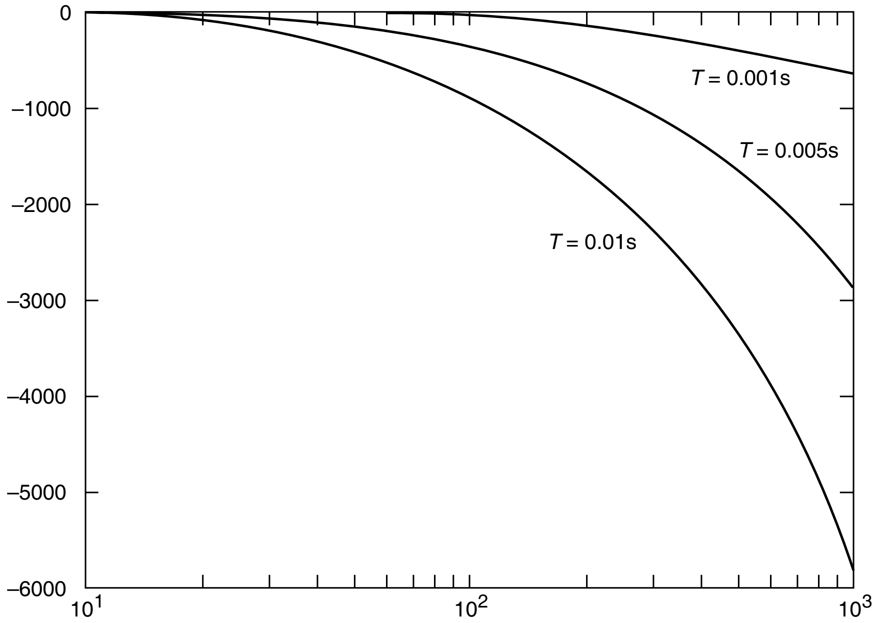


(a)

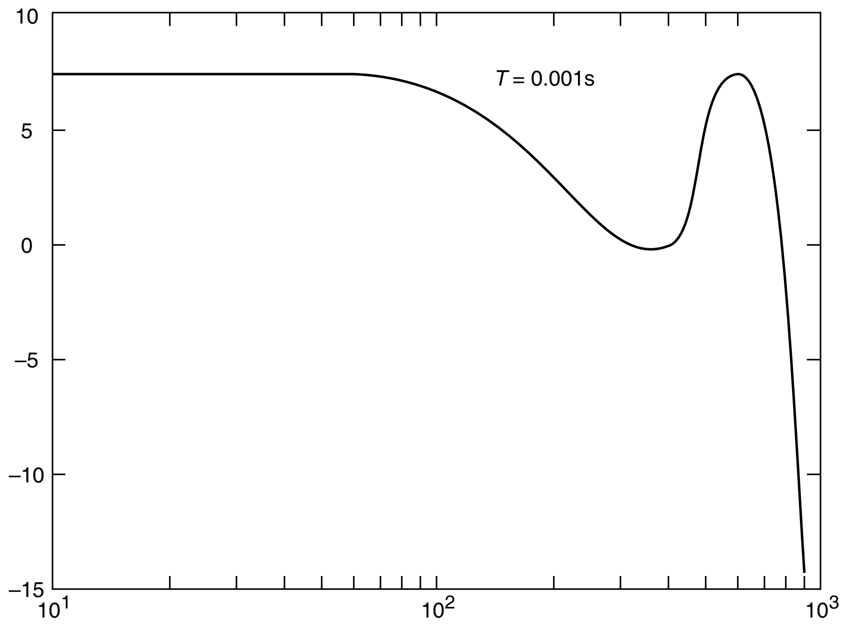


(b)

Fig. 9.30 *Contd.*



(c)



(d)

Fig. 9.30 *Plot of amplitude response of low-pass filter* sampled impulse response is

$$h(\ell T) = \frac{1}{\omega_s}\left[\int_{-\omega_s/4}^{-\omega_s/5} e^{j\ell\omega T}\,d\omega + \int_{-\omega_s/5}^{\omega_s/4} e^{j\ell\omega T}\,d\omega\right]$$

$$= \frac{1}{2\pi j \times 800\ell T}\left[e^{\frac{-jl\omega_s T}{5}} - e^{\frac{-jl\omega_s T}{4}} + e^{\frac{-jl\omega_s T}{4}} - e^{\frac{-jl\omega_s T}{5}}\right]$$

$$= \frac{-2j\sin[(\omega_s/5)\mathrm{T}\ell]}{2\pi j\ell} = -\frac{\sin(2\pi\ell/5)}{\pi\ell} \quad \text{(vii)}$$

As per Eq. (vii), $h(\ell T)$ satisfies the symmetry property.

For 41 samples

$$H_{tc}(z) = z^{-20}\left[\sum_{t=-20}^{20} h(\ell T) z^{-\ell}\right] \quad \text{(viii)}$$

Computation and design can now proceed routinety and is left to the reader.

Example 9.7 *Design a second-order Digital Butterworth filter with cut-off frequency 1 kHz at a sampling rate of 10^4 sps. Plot the corresponding ampli-tude and phase response.*

Solution It is given that $f_s = 1/T = 10^4$ sps

or

$$T = 10^{-4} \text{ s/sample} \quad \text{(i)}$$

Now the critical frequency,

$$\omega_{dc} = 2\pi \times 10^3 \text{ rad/s} \quad \text{(ii)}$$

The corresponding prewarped cut-off frequency in the analog domain is

$$\omega_{ac} = \frac{2}{T}\tan\left(\frac{\omega_{dc}T}{2}\right)$$

Substituting for value, we get

$$\omega_{ac} = 2 \times 10^4 \tan(\pi/10) = 0.65 \times 10^4 \quad \text{(iii)}$$

The second-order BF is given by

$$H(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1}; \text{ normalized} \quad \text{(iv)}$$

Using the value of ω_{ac} from Eq. (iii)

$$H(s/\omega_{ac}) = \frac{1}{(s/0.65 \times 10^4)^2 + \sqrt{2}\,(s/0.65 \times 10^4) + 1}$$

On simplification, we get

$$H(s/\omega_{ac}) = \frac{(0.65 \times 10^4)^2}{s^2 + \sqrt{2}\,s \times 0.65 \times 10^4 + (0.65 \times 10^4)^2} \quad \text{(v)}$$

Using Bilinear Transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{(0.65\times 10^4)^2}{(2\times 10^4)^2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\times 0.65\times 10^4 + (0.65\times 10^4)^2}$$

On simplification, we get

$$H(z) = \frac{0.0676\,(z^2+2z+1)}{z^2-1.14z+0.412} \qquad \text{(vi)}$$

Eq. (vi) gives the transfer function of second-order BF in the digital domain. The plot of amplitude and phase response of $H(z)$ are shown in Figs 9.31(a) and (b) respectively.

Example 9.8 *Consider designing a digital filter with system function H(z) from a continuous time filter with rational function H(s) by the following transformation.*

$$H(z) = H(s)\Big|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]}$$

where α is a non-zero integer and β is real.

(a) *If $\alpha > 0$, determine the range of β for which a stable, causal continuous-time filter H(s) will always lead to a stable, causal digital filter with rational H(z).*

(b) *Repeat part (a) for $\alpha < 0$.*

(c) *Given $H(s)|_{s=j\omega}$, as shown in Fig. 9.32, sketch H(z) for $\beta = 1$ and $\alpha = 1$ with phase angle.*

Solution We have

(a)
$$s = \beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right] = \beta\left[\frac{z^{\alpha}-1}{z^{\alpha}+1}\right] \qquad \text{(i)}$$

or

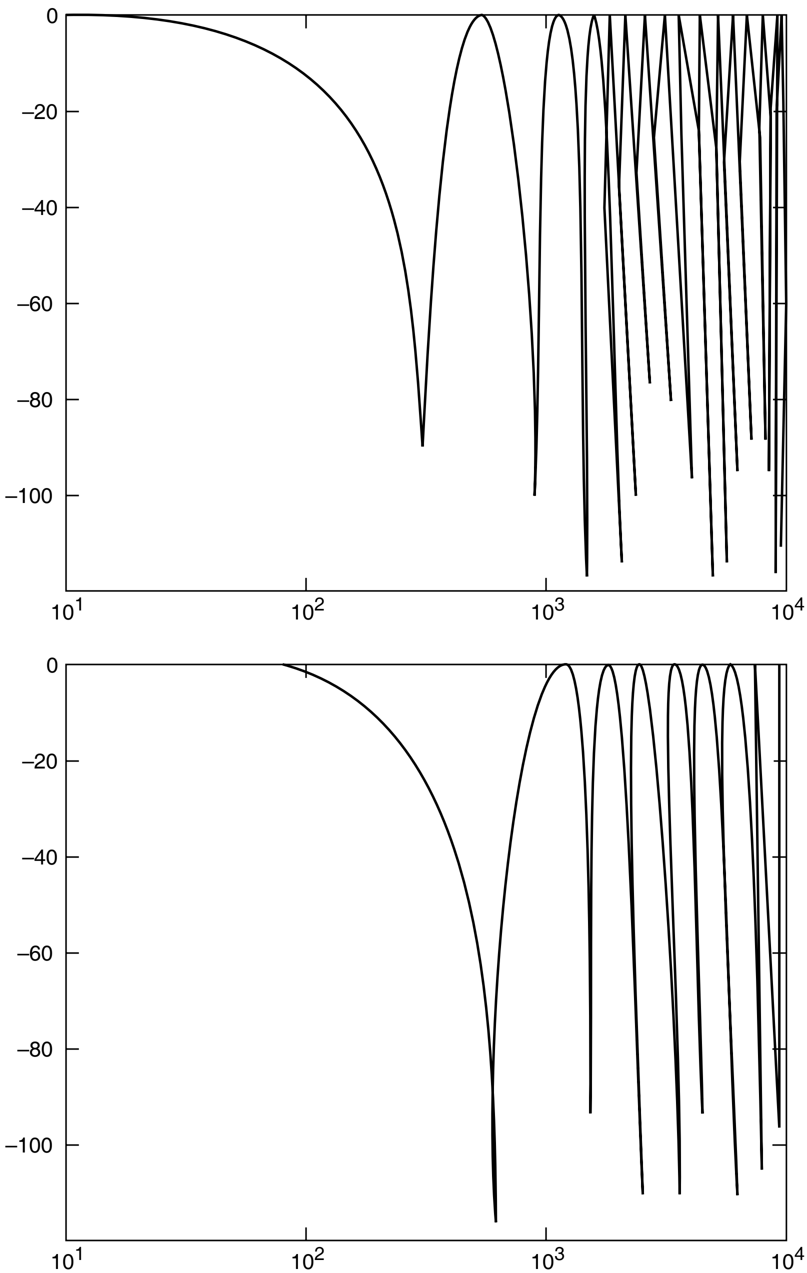
$$s(z^{\alpha}+1) = \beta(z^{\alpha}-1)$$

Solving for z we get

$$z^{\alpha} = \frac{s+\beta}{\beta-s} = \left[\frac{1+s/\beta}{1-s/\beta}\right] \qquad \text{(ii)}$$

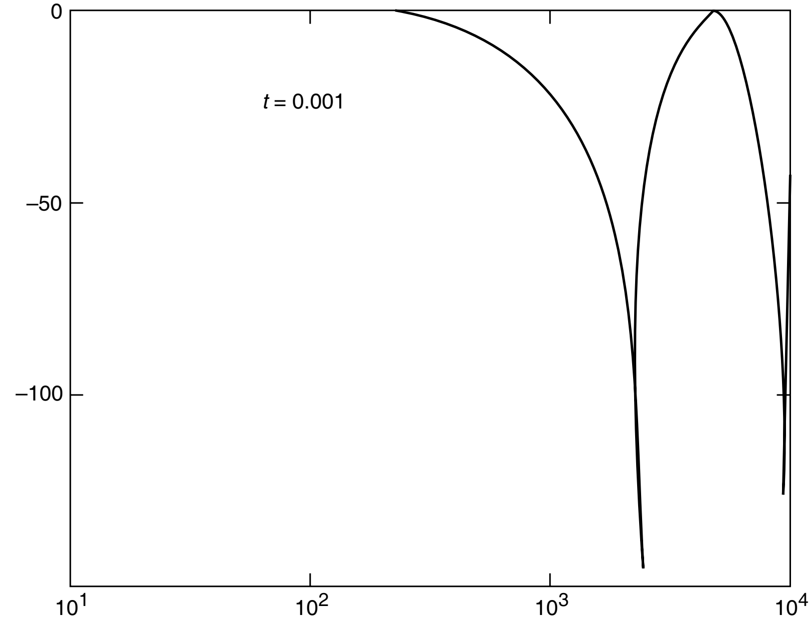
But

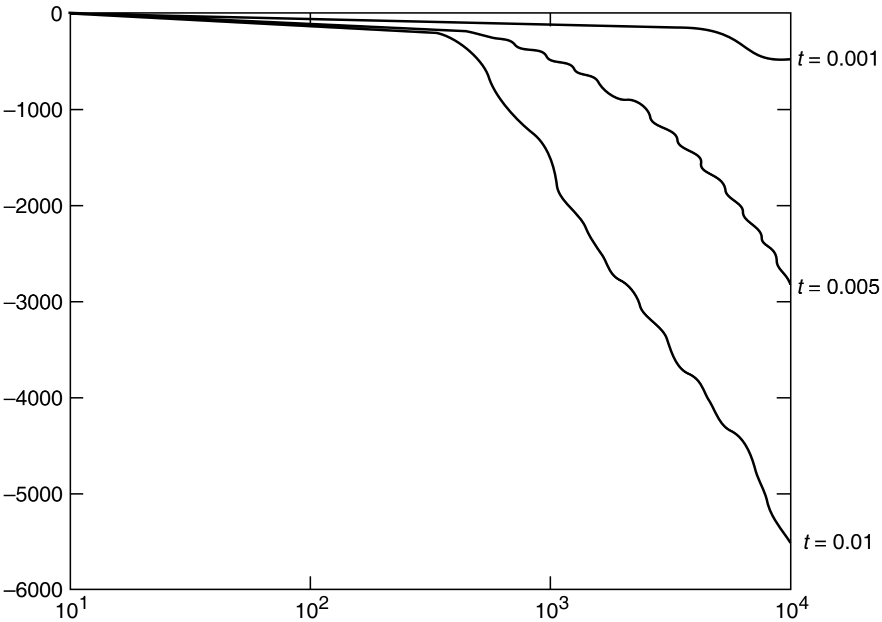
$$s = \sigma + j\omega$$



(a) Analog version

Fig. 9.31 *Contd.*





(b) Digital version

Fig. 9.31 *Butterworth filter*

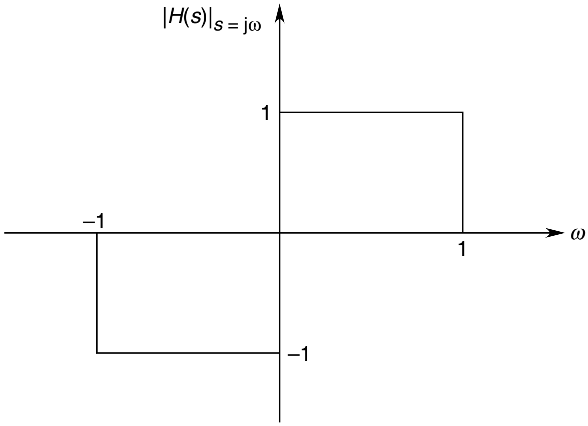


Fig. 9.32 *Sketch of H(s) with phase angle*

Then

$$z^{\alpha} = \frac{(1+\sigma/\beta + j\omega/\beta)}{1-\sigma/\beta - j\omega/\beta} \tag{iii}$$

We can then write

$$|z^{\alpha}| = \left[\frac{(1+\sigma/\beta)^2 + (\omega/\beta)^2}{(1-\sigma/\beta)^2 + (\omega/\beta)^2}\right]^{1/2} \tag{iv}$$

For system in z-domain to be causal and stable we must have $\sigma < 0$. So that $|z^{\alpha}|$ lies inside unit circle. Thus for α a non-zero integer, $|z^{\alpha}|$ will lie inside the unit circle only when $\beta \geq 0$.

(b) One can similarly show that for $\alpha < 0$, $|z^{\alpha}|$ will lie inside unit circle only when $\beta < 0$.

(c) According to the sketch given in Fig. 9.32, we have

$$H(s)|_{s=j\omega} = \omega; \quad \text{for} \quad -1 \leq \omega \leq 1 \tag{v}$$

so that

$$H(s) = s/j \tag{vi}$$

Thus

$$H(z) = H(s)\Bigg|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]} \tag{vii}$$

However, given that $\beta = 1$ and $\alpha = 1$, then

$$s = \left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \left[\frac{1-e^{-j\omega_d T}}{1+e^{-j\omega_d T}}\right] = j\tan\left(\frac{\omega_d T}{2}\right) \tag{viii}$$

From Eqs (vi) and (viii), we have

$$H(s) = \frac{j \tan(\omega_d T/2)}{j} = \tan\left(\frac{\omega_d T}{2}\right) \qquad \text{(ix)}$$

Plot of $H(s)$ with phase angle is shown in Fig. 9.33.

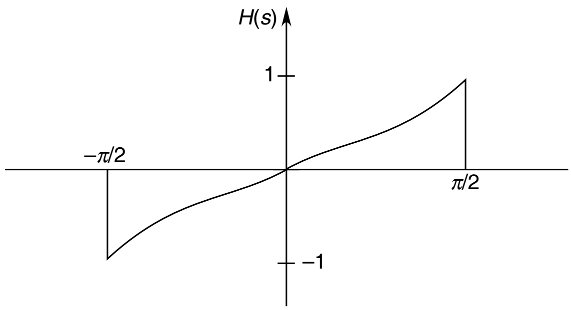


Fig. 9.33 *Plot of H(s) with $\omega_d T$*

Example 9.9 *A discrete-time LPF with frequency response $H(e^{j\omega_d T})$ is to be designed to meet the following specifications*

$$0.8 < |H(e^{j\omega_d T})| < 1.2; \quad \text{for } 0 \le |\omega_d| \le 0.2\pi$$

$$|H(e^{j\omega_d T})| < 1.2; \quad \text{for } 0.8\pi \le |\omega_d| \le \pi$$

The design procedure consists of applying the bilinear transformation to an appropriate continuous-time Butterworth LPF.

(a) with T = 2, determine necessary specifications on the continuous-time Butterworth LPF, which when mapped through the bilinear transformation will result in the desired discrete-time filter.

(b) determine the lowest order continuous-time Butterworth filter that meets the specifications as determined in (a).

Solution

(a) We have

$$\omega = \frac{2}{T} \tan\left(\frac{\omega_d T}{2}\right)$$

For $T = 2$

$$\omega = 2 \tan(\omega_d) \qquad \text{(i)}$$

At $\omega = 0$

$\omega_d = 0$

$\omega = 0.2\pi;\ \omega_d = 0.325$

and

$\omega = 0.8\pi;\ \omega_d = 3.078$

Thus specifications of the analog filter is

$$0.8 < |H(s)| < 1.2; \quad \text{for } 0 \le \omega \le 0.325$$

and

$$|H(s)| < 0.2; \quad \text{for } \omega \ge 3.078 \tag{ii}$$

According to the specification obtained in Eq. (ii), we get

$$\frac{1}{\left[1+\left(\frac{0.325}{\omega_c}\right)^{2n}\right]} = (0.8)^2 \tag{iii}$$

and

$$\frac{1}{\left[1+\left(\frac{3.078}{\omega_c}\right)^{2n}\right]} = (0.2)^2 \tag{iv}$$

By solving Eqs (iii) and (iv), we get

$$\left(\frac{0.325}{3.078}\right)^{2n} = \frac{0.5625}{24}$$

or

$$n = 0.8 \approx 1$$

ω_c will take values either 0.433 or 0.641.

The first-order BF, therefore, is

$$H_B = \frac{1}{(s/\omega_c)+1} \tag{v}$$

Example 9.10 *A continuous-time high-pass filter can be obtained from a continuous-time LPF by replacing s by (1/s) in a transfer function, i.e., if G(s) is the transfer function of LPF and H(s) is the transfer function of corresponding high-pass filter then*

$$H(s) = G(s)\,|_{s=(1/s)}$$

Asume that a discrete-time LPF G(z) and a discrete-time HPF H(z) are obtained from G(s) and H(s) respectively, using bilinear transformation. Show that H(z) can be obtained from G(z) by replacing z by some function in z denoted by m(z). Determine m(z).

Solution We have

$$G(z) = G(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \tag{i}$$

and

$$H(z) = G(1/s)\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \tag{ii}$$

or

$$H(z) = G(s')\Bigg|_{s'=\frac{2}{T}\left[\frac{1\times z^{-1}}{1-z^{-1}}\right]} \tag{iii}$$

In Eq. (iii), it is assumed that $s' = 1/s$.

It is, therefore, evident that for obtaining $H(z)$ from $G(z)$, the bilinear transformation to be applied is $s' = m(z)$, where s' is some function of z. Let us use x as dummy variable for z for transforming LPF from analog domain to digital domain, i.e.

$$s = \frac{2}{T}\left[\frac{1-x^{-1}}{1+x^{-1}}\right] \tag{iv}$$

Let us choose x as a function $m(z)$ such that

$$\frac{2}{T}\left[\frac{1-x^{-1}}{1+x^{-1}}\right] = \frac{T}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right] \tag{v}$$

or

$$x = m(z) = \frac{T^2\,(z+1)+4(z-1)}{4(z-1)-T^2\,(z+1)} \tag{vi}$$

Problems

9.1 Determine the frequencies at which the amplitude response of the following low-pass filters with cut-off frequency ω_c = 1(rad/s) will become 0.1 times of its maximum amplitude, which is unity at $\omega = 0$.

(i) First-order Butterworth (ii) Second-order Butterworth

9.2 Determine the frequency of the first-order low-pass Butterworth filter at which its amplitude response will drop by 6dB as compared to that at zero frequency. The cut-off frequency of this filter is ω_c = 1 rad/s and amplitude response at zero frequency is 0 dB.

9.3 Design a low-pass Butterworth filter with following specifications.

(i) attenuation of at least 40 dB at three times the normalized cut-off frequency (rad/s).

9.4 Design a LP Butterworth filter whose maximum amplitude is unity at ω = 0 (rad/s) and at 100 (rad/s) the amplitude response becomes 0.5 times the maximum amplitude response. The cut-off frequency ω_c = 62.8 (rad/s) where the amplitude response is 0.707 times that of the maximum amplitude.

9.5 Deduce the transfer function of second-order Butterworth LP filter with the help of the roots of Butterworth polynomial.

9.6 An active network shown in Fig. P-9.6 consists of an op-amp with gain K. Determine the values of K, C_1, G_1 and G_2 so that the circuit behaves like a second-order LP Butterworth filter with cut-off frequency $\omega_c = 2.5 \times 10^3$ rad/s. Assume $(G_1/C_1) = 1500\ \Omega/F$ and $C_2 = 0.05$ F.

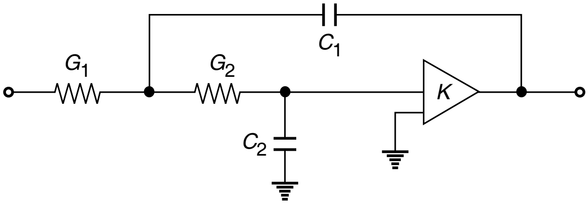


Fig. P-9.6

9.7 In the passive network shown in Fig. P-9.7 determine the value of L so that the circuit behaves like a first-order LP Butterworth filter that achieves an attenuation of at least 5 dB at twice the cut-off frequency. The cut-off frequency is required to be $\omega_c = 10^5$ rad/s.

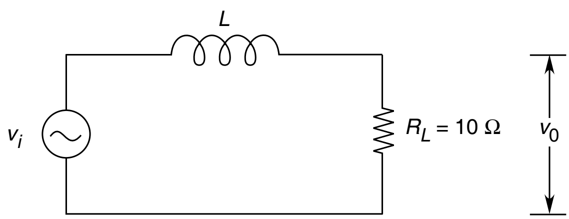


Fig. P-9.7

9.8 Write the following Chebyshev functions, if $\omega_c = 2$ rad/s and $\varepsilon = 0.6$.

(i) $H_{c1}(j\omega)$ (ii) $H_{c3}(j\omega)$ (iii) $H_{c4}(j\omega)$

9.9 Determine the poles of a LP Chebyshev filter for $n = 4$ and $\varepsilon = 1.25$. Plot the poles on complex plane.

9.10 Deduce the transfer function of a Chebyshev filter with following specifications.

(i) 0.5 dB ripple in the pass-band

(ii) Cut-off frequency 1 kHz

(iii) Attenuation of at least 10 dB at $\omega = 3$ kHz

9.11 Deduce the transfer function of a second-order Chebyshev LP filter with cut-off frequency of 100 Hz, pass-band ripple of 0.5 dB and amplitude response of unity at zero Hz.

9.12 Compare the attenuation at 10 kHz achieved by Butterworth and Chebyshev LP filters of same order, when 3 dB cut-off frequencies are assumed to be same for both the filters. The order of both the filters should be choosen to be one. For the purpose of comparison choose $\varepsilon = 0.51$, 0.98 and 1.25 for Chebyshev filter.

9.13 Show that the half power point of Butterworth LP filter is independent of the order of the filter.

9.14 Deduce half power point for Chebyshev LP filter.

9.15 Design a high-pass Chebyshev filter with cut-off frequency at 3 rad/s using low-pass to high-pass frequency transformation. Following are the specifications of low-pass Chebyshev filter.

(i) 1.0 dB ripple in the pass-band
(ii) Attenuation of at least 20 dB at twice the cut-off frequency ω_c = 4 rad/s

9.16 Design a band-pass filter using frequency transformation with following specifications.

(i) 3 dB attenuation at $2\pi \times 5 \times 10^2$ rad/s and $2\pi \times 7 \times 10^2$ rad/s
(ii) 40 dB attenuation at frequencies less than $2\pi \times 4 \times 10^2$ rad/s and higher than $2\pi \times 10^3$ rad/s

9.17 Design a band-stop filter using frequency transformation that meets the following specifications.

(i) 20 dB attenuation between 5 kHz and 5.02 kHz
(ii) 3 dB attenuation for less than 3.5 kHz and more than 6.5 kHz

9.18 Deduce the transfer function of a second order notch filter with a notch centre frequency of 1.5 kHz and a notch band-width of 500 Hz. Plot accurately the amplitude and phase response of the filter.

9.19 Deduce transfer function for an FIR filter which approximates the function shown in Fig. P-9.19. Plot the resulting FIR filter in terms of both amplitude as well as phase response and compare with the given function. For deducing FIR filter, take the following set of values of sampling period (T) and number of samples (N).

$$T = 0.175\ ;\ N = 5$$

9.20 Obtain frequency response of the following window functions.

(i) Rectangular (ii) Hamming (iii) Hanning

9.21 Find $H(z)$ using the impulse-invariant method for a second-order normalized Chebyshev filter having 0.5 dB ripple factor.

9.22 Determine the following for the second-order Chebyshev filter with a pass-band ripple of 0.5 dB.

(a) Corresponding impulse-invariant digital filter for T = 0.00001s, 0.0001s and 0.01s.
(b) Compare the amplitude and phase response of resulting IIR filter for three values of T given in part (a).

9.23 Determine the impulse-invariant digital filter corresponding to the following analog transfer functions. The sampling period T, is to be taken as unity

(a) $H(s) = 4/\{(s + 1)(s + 4)\}$ (b) $H(s) = (s + 8)/\{(s + 1)(s + 2)(s + 4)\}$
(c) $H(s) = 4/\{(s + 1)(s + 2)^2\}$

9.24 Analog filter is given as $H(s) = 9/(s + 3)^2$.

(a) Find the impulse-invariant filter using a sampling frequency of 10 Hz.
(b) Plot the amplitude function for sampling frequencies 20 Hz and 100 Hz and compare the two.

9.25 Determine $H(z)$, using the bilinear z-transform technique corresponding to the following analog transfer function.

$$H(s) = 8/(s^2 + 6s + 8)$$

Both the analog and digital filters are to have the same magnitude and phase response at 10 Hz. Assume sampling frequency of 100 Hz.

9.26 Determine $H(z)$, using the bilinear z-transform technique, for the following analog transfer function.

$$H(s) = 4/\{(s + 1)\,(s + 4)\}$$

Assume a sampling frequency of 10 Hz and express your result as a ratio of polynomials in z^{-1}. Consider the following two cases.

(a) The digital filter is to have a response closely approximating the analog filter for very low frequencies.

(b) The digital filter and the analog filter are to have the same response at a frequency of 2 Hz.

9.27 Show that a notch (band reject) filter can be developed from a low-pass prototype by using the following correspondence.

$$s \to \left[\frac{a - z^{-2}}{1 - bz^{-1} + z^{-2}}\right]$$

Determine the relationship between 'a' and the bandwidth 'b' and the frequency of the notch.

9.28 Determine and sketch the Direct-Form I and Direct Form-II realization for each of the following transfer functions.

(a) $H(z) = \dfrac{2 - 0.1z^{-1} + 0.2z^{-2} + 0.3z^{-3}}{1 - 0.8z^{-1} + 0.5z^{-2} + 0.7z^{-3}}$ (b) $H(z) = \dfrac{1 + 0.2z^{-1}}{1 - 0.4z^{-1} + 0.3z^{-2}}$

(c) $H(z) = \dfrac{(1 - z^{-1})^2}{(1 + z^{-1})^2}$ (d) $H(z) = \dfrac{(1 - z^{-1})(1 - 0.2z^{-1})}{(1 - 0.4z^{-1})(1 - 0.2z^{-2})}$

9.29 Determine the transfer function, $H(z)$, of the system shown in Fig. P-9.29.

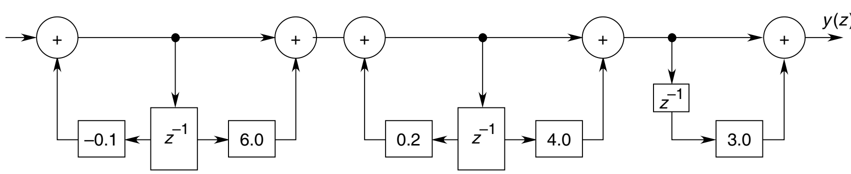


Fig. P-9.29

9.30 Determine the Direct-Form I and Direct-Form II realizations for the following transfer functions.

(a) $H(z) = 1 - 0.5z^{-1} + 0.2z^{-2}$ (b) $H(z) = 0.5 + z^{-1} + 0.5z^{-2}$

9.31 Determine a cascade and parallel realization, using only first-order structure, for each of the following transfer functions.

(a) $H(z) = \dfrac{z^{-1} - z^{-2}}{(1 - z^{-1})^2\,(1 - 0.1z^{-1})^2}$ (b) $H(z) = \dfrac{1}{(1 + 0.1z^{-1} - 0.02z^{-1})}$

(c) $H(z) = \dfrac{1}{(1 - 0.2z^{-1})(1 + 0.4z^{-1} - 0.21z^{2})}$

9.32 Determine parallel realization for each of the following transfer functions.

(a) $H(z) = z^{-3}/(1 - 0.2z^{-1})^2$ (b) $H(z) = \dfrac{1 - 0.25z^{-3}}{(1 - z^{-1})(1 + 0.1z^{-1})^3}$

9.33 For the system, shown in Fig. P-9.33, determine the transfer function and determine the Direct form II realization.

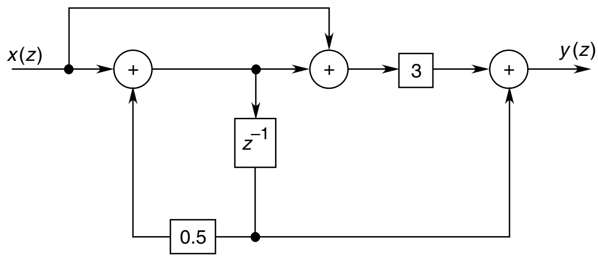


Fig. P-9.33

9.34 Realize the system, shown in Fig. P-9.34. using only one delay unit (z^{-1}).

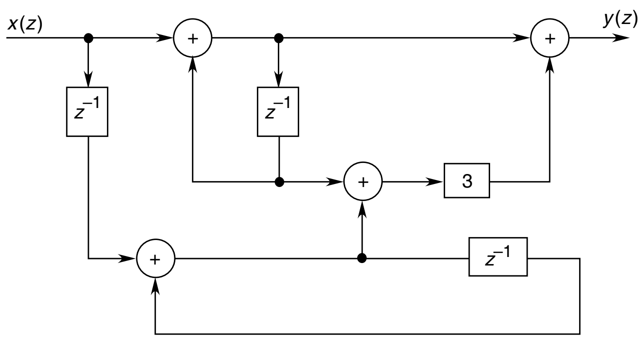


Fig. P-9.34

9.35 Determine the transfer function of the system, shown in Fig. P-9.35, as a ratio of polynomials z^{-1}. The parameters *a*, *b*, *c*, and *d* are arbitrary.

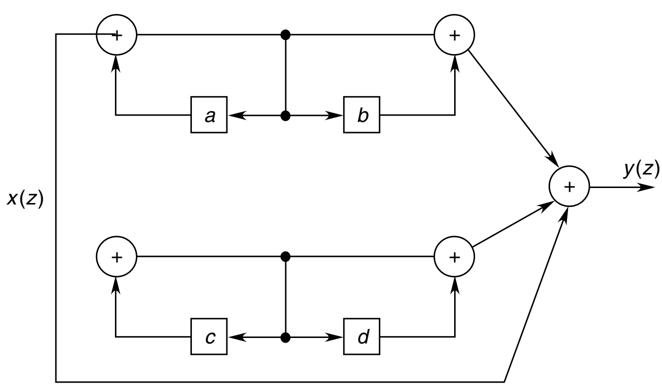


Fig. P-9.35

MATLAB Tools for Analysis of Signals and Systems

10

Introduction

MATLAB is a software package with complex mathematics and simulation environment for high-performance numerical computations. It has several built-in-functions for numerical computation, design graphics and animation. It is, as a matter of fact, capable of handling linear, non-linear, continuous and discrete systems. The main advantage of this package is that it provides easy extensibility with its own high-level programming language. It also provides an external interface to run programs, written in Fortran or C codes, from within MATLAB. Moreover, in addition to built-in-functions, user can create his own functions in MATLAB language. Once created, these functions can be used just like the built-in-functions. Matrix is the main building block of MATLAB. Complex matrix is the only data-type and one need not declare it. In fact, vectors, scalars, real matrices and integer matrices are automatically dealt with as special cases of the basic data type.

MATLAB offers several optional tool boxes, such as control systems, signal processing, optimization, Robust-control, Neural Networks and System Identification. Simulink is a graphical environment, offered by MATLAB for modelling and simulating block diagrams and general non-linear systems. The basic MATLAB and toolboxes are command driven with ON-LINE help facilities. The syntax and structure of most commands are similar and hence there is no need to memorize the commands. In this chapter, only important and commonly used commands for computations and design have been discussed. For knowing other built-in functions one can look at the index of built-in function given in the MATLAB package.

10.1 BASIC STRUCTURE OF MATLAB

MATLAB works through three basic windows on all Unix systems, PCs and Macintosh computers, which are discussed below.

(i) Command Window It is the main window and is characterized by the prompt '>>'. All commands including those for running user-written programs can be typed in this window.

(ii) Graphics Window The output of all graphic commands, typed in the command window are flushed to the graphics window. The user can create as many graphic windows as the system memory permits.

(iii) Edit Window In this window, a user can write, edit, create and save his own programs in files, called M-files.

It is noted that MATLAB is case sensitive, thus '*A*' and '*a*' are two different file names.

10.2 FILE TYPES

MATLAB offers three types of files for storing informations which are (i) M-files (ii) Mat-files and (iii) Mex-files.

M-files are standard ASCII text-files, with a .m extension to the file name. There are two types of M-files. These are *script files* and *function files*. Script file consists of a set of valid commands in it. Script file can be executed by typing the name of the file without the .m extension on the command line. Script-files work on global variables, i.e. variables currently available in the workspace. Results obtained on the execution of script files are left in the workspace. This kind of file is useful when one needs to use a set of commands time and again. This is illustrated through following examples.

Example 10.1 *Write a script file to solve the following set of linear equations.*

$$\begin{bmatrix} 2 & a \\ 1 & 2a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let us name the script file to be created as lineq.m. Set of commands under this file name are as follows.

```
A = [2 a; 1 2*a];  % create matrix A
B = [1 ; 0];      % create vector B
x = A/B           % find x
```

It may be noted that last command does not contain semicolon at the end. This means that result of these commands will be displayed on the screen when this script file is executed.

In order to execute the script file we type the following.

```
>> a = 1    % specify a value of a
>> lineaq   % execute the script file lineq.m
x = -0.33
     0.66
```

Function file as such is also an M-file and is different from script file only in the sense that the variables in a function file are all local. A function file begins with a function definition line with well defined list of inputs and outputs. In fact, without this line, the file is nothing but a script file. The syntax of the function definition line is the following.

```
Function [output variables] = function - name (input variable)
```

It is important to note that function name is same as file name. The function definition line may look different, depending upon number of outputs or no output. For example;

```
Function definition line              file name
Function [a, b, c] = square (x)       square.m
Function [ ] = angles (p, q)          angles.m
Function area(x)                      area.m
```

The first word in the definition line must be typed in lower case. Single-output variable need not be enclosed within the square bracket, but multiple output variables must be enclosed within the square bracket. In the event of the absence of any output variable, the bracket as well as equal sign need not be typed.

In order to understand the operation of function file we proceed with the same example discussed in Example 10.1. Let us give some other name to function file as lineqn.m. The contents of this file are as below.

```
Function [x]= lineqn(a)
          A = [2 a; 1 2*a]     ; create matrix A
          B = [1 ; 0]          ; create vector B
          x = A/B              ; find x
```

Here, *a* and *x* are local variables, which means any other variable name may be used in their places. The execution of this file in MATLAB is discussed below.

```
>> [y] = lineqn(1)
>> y
ans = - 0.33
         0.66
```

These are the same values of x_1 and x_2 we get in Example 9.1.

Matlab-files are generated by MATLAB when data is saved using the save command. It is a binary data-file with a .mat extension to the file name. Mat-files can again be loaded into MATLAB using load command.

Mex-files are used to call Fortran and C programs. Mex-files are MATLAB with a mex extension to the filename.

10.3 SOME BASIC SYSTEM COMMANDS IN MATLAB

Some of the key commands, often used while working with MATLAB package are explained in Table. 10.1.

Table 10.1

`>> help`	; It is used to get on-screen information and it lists all the topics on which help is available.
`>> help command name`	; It provides informations about particular command name, as mentioned.
`>> expo`	; Executes the demonstration programs.
`>> who`	; Displays variables currently in the workspace.
`>> whos`	; Displays variables currently in the workspace with their size.
`>> what`	; It shows the MATLAB files and MATLAB data files in the current directory.
`>> pwd`	; Shows the current working directory.
`>> cd`	; Changes the current working directory.
`>> dir`	; Displays contents of the current directory.
`>> print`	; To print the contents of the current active window, e.g. command, figure, edit window, etc.
`>> save`	; To save the variables in the file in binary or ASCII format.
`>> load filename`	; It loads the variables saved in the file filename.mat into the MATLAB workspace.
`>> quit`	; Quits MATLAB.
`>> exit`	; Does the same job as the quit command.

10.4 SOME BASIC LOOPING AND CONDITIONING COMMANDS AND OPERATORS

MATLAB supports some basic programming structures that allow looping and conditioning commands along with relational and logic operators. The syntax and use of these commands and operators are very similar to those found in other high level languages such as C, BASIC, FORTRAN, etc. Certain commonly used commands for looping and conditioning of programs in MATLAB are described in Table 10.2.

Some important relational and logic operators often used are as discussed in Table 10.3.

It is to be noted that operators = = (equal) and ~ = (not equal) can compare both real as well as imaginary parts, whereas all other operators are capable of comparing only the real parts.

10.5 MATLAB TOOLBOXES

Following are the toolboxes available in MATLAB.

(i) Control System (ii) Signal processing

Table 10.2

`>> for`	;	This is used to execute series of statements iteratively. The syntax of this command is as given below. >> for variable = expression statement 1 statement 2 : : end;
`>> while`	;	It allows conditional looping in which statements within the loop are executed as long as the condition is true. The syntax is; >> while expression statement 1 statement 2 : : end;
`>> if`	;	These commands are also used for conditional execution of a set of commands. The syntax is
`>> else`	;	
		>> if expression, statement 1, statement 2, ….
`else if`	;	end;

Table 10.3

<	;	implies less than
>	;	greater than
< =	;	less than or equal
> =	;	greater than or equal
= =	;	equal
~ =	;	not equal
&	;	logical AND
|	;	logical OR
~	;	logical NOT

(iii) Optimization
(iv) Robust control
(v) Splive
(vi) System identification
(vii) Neural networks

If one desires to use a particular toolbox for his programs then he has to enter the desired toolbox by typing following commands.

```
>> C :  MATLAB\name of the toolbox
```

For example, if one wants to use signal processing toolbox then he should type as follows;

```
>> C : MATLAB\signal processing
```

Thus the user enters into the signal processing toolbox environment in which following built-in-functions with their syntax and structure are given in Table 10.4.

Table 10.4

`>> filter (B, A, X)`	;	The syntax of this function is Y = filter (B, A, X). It filters the data in vector X with filter described by vectors A and B and create the filtered data Y.
`>> fft (X)`	;	It is the discrete Fourier transform of vector X.
`>> fft (Y, N)`	;	It gives N-point FFT, padded with zero if Y has less than N-points and truncated if it has more.
`>> ifft (X)`	;	It obtains inverse Fourier transform of vector X.
`>> ifft (X, N)`	;	It obtains N-point inverse Fourier transform.
`>> conv (X, Y)`	;	The syntax of this function is Z = conv (X, Y). It convolves vectors X and Y and results into Z vector of length = length of X + length of Y – 1.
`>> deconv (P, Q)`	;	The syntax of this built-in function is [A, B] = deconv (P, Q). It deconvolves vector Q out of vector P and the resultant is returned in vector A with remainder in vector B such that P = conv (A, Q) + B.
`>> impulse (A, B, C, D, IU)`	;	It plots the time response of linear system described by $\dot{X} = AX + Bu$ $Y = CX + Du$ To an impulse applied to the signal input IU. The time vector is automatically determined.
`>> impulse (N, D)`	;	It plots the impulse response of the transform function H(s) = N(s) / D(s) where N(s) and D(s) contain the polynomial coefficients in descending powers of s.
`>>impulse(A,B,C,D,T)` or `>>impulse(N, D, T)`	;	These two make use of user-supplied time vector T which must be regularly spaced.

Contd.

Table 10.4 (Contd.)

`>> deimpulse (A,B,C, D,IU)`	; Plots time-response of a discrete-time linear system given by $x(n+1) = \mathrm{A}\,x(n) + \mathrm{Bu}(n)$, $y(n) = \mathrm{C}x(n) + \mathrm{Du}(n)$ to an unit sample input IU. The number of points are determined automatically.
`>> deimpulse (N, D)`	; Plots the impulse response of the transfer function H(z) = N(z) / D(z) where N(z) and D(z) contains the polynomial coefficients in descending powers of z.
`>> abs (X)`	; It gives the absolute value of the elements of X. If X is complex then abs(X) is the magnitude of the elements of Y.
`>> angle (X)`	; It returns phase angle of a matrix with complex elements in radians.
`>> plot (X, Y)`	; Plots vector X versus vector Y. If X or Y is a matrix then the vector is plotted versus rows or columns of the matrix, whichever line is up.
`>> x label ('text')`	; It adds text beside the X-axis on the current axis
`>> y label ('text')`	; It adds text beside the Y-axis on the current axis

10.6 SOME BASIC IDEAS OF COMPUTATIONS IN MATLAB

In this section, some basic ideas about how to invoke MATLAB, do some trivial computations, work with arrays of numbers, create row and column vectors, execute script file, etc. have been touched upon.

To work with MATLAB an user should first of all, log on to the system and enter into MATLAB command window. The screen, as discussed in introduction, will show the following prompt.

\>>

Now, user can perform desired operation with the help of MATLAB. The example below will give clear idea as to how one can perform simple calculations in MATLAB.

```
>> 1000 + 9900
 ans =
   10,900
```

; note that result of an unassigned mathematical expression is saved in the default variable "ans"

```
>> A = 1000 + 9900
   A =
   10,900
```

; User can assign the value of an expression to his own defined variable, say A in this case

```
>> X = 2 ^ 2 + log(pi) * sin (pi);;   It should be noted that semicolon at the end
                                      suppresses the output at the screen. However,
>> X                                  one can recall the value by typing X
    X =
          4.0
```

We now discuss an example which illustrates how to create rows, vectors, how to do array operations and how to do arithmetic operations on vectors.

```
>> A = [1 2 3]                 ;  Here A is a row vector with elements 1, 3
   A =                            and 2
     1 2 3
>> B = [4; 1; 3]               ;  B is a column vector with elements 4, 1 and
   B = 4                          3
       1
       3
>> C = [4 3 7]
>> X = A + C                   ;  X is obtained by adding vectors A and C
   X =
    5 5 10
>> TH = linespace (0, 90, 3);  ;  It will create a vector TH with 3 elements
   TH =                           spaced between 0 and 90
     0  45  90
>> Y = sin (TH)
      Y = 0
        0  0.707  1
```

10.7 MATLAB COMMANDS FOR SIGNALS AND SYSTEMS PROBLEMS

MATLAB usages are better understood by working with it. A mastery can only be achieved by actually working and solving problems in MATLAB. However, important commands are described below that can be useful to start with those students who are working first time.

Transfer-Function Related Commands Two important commands are discussed taking help of Example 10.2. These commands are

tf2zp $\rightarrow$ Transfer function to zeros and poe
Zp2tf $\rightarrow$ zeros and poles to transfer function

Example 10.2 *Explain commands tf2zp and zp2tf considering transfer function.*

$$T(s) = \frac{s+2}{s^2+2s+3}$$

In MATLAB T(s) can be defined as

```
>>      num = [1 2];
>>      den = [1 2 3];
```

To find zeros and poles, the following commands are typed

```
>>    [z, p] = tf2zp(num den)
```

Zeros are stored in z and poles are stored in P.

If zeros and poles are known then transfer function $T(s)$ can be obtained by typing the following commands

```
>>[num, den] = zp2tf(z, p)
```

Step and Impulse Response

Step response and impulse response can be obtained using step and impulse commands. These commands are explained using Example 10.3.

Example 10.3 *Find step and impulse response of the system described by its transfer function*

$$T(s) = \frac{s}{s+2}$$

Step response of Example 10.3 can be calculated as

```
>>   num = 1;
>>   den = [1 2];
>>   t = 0.0: 0.001: 3; % where 0.0 is start time,
     0.001 is time step and 3 is the end time.
>>   y = Step (num, den, t);
>>   Plot (t, y) % Plot of step response.
```

Impulse response of Example 10.3 can be calculated as

```
>> num = 1;
>> den = [1 2];
>> t = 0.0; 0.001: 3;
>> y = sim (num, den , s, t) % Impulse response.
>> Plot (t, y)
```

State Space Commands Two important commands (ss2tf & tf2ss), are illustrated below:

Let state space representation be

$$\dot{x} = \text{Ax} + \text{Bu}$$
$$\text{y} = \text{Cx} + \text{Du}$$

In MATLAB, the above state space representation can be written as

```
sys = ss(A, B, C, D) % convenient way to represent state space
```

The command ss2tf is used to find transfer function from state space representation as

```
                  [num, den] = ss2tf (A, B, C, D);
or                [num, den] = ss2tf (sys);
```

The commands **tf2ss** is used to get state space from the given transfer function as

```
              [A, B, C, D] = tfss (num, den);
```

Discrete Time System Analysis

Four commands are introduced under discrete time systems

```
Conv %        Convolving two sequences
deConv %      deConvolution
dstep %       discrete-time step response
dimpulse %    discrete time impulse response.
```

Example 10.4

```
            >>     X = [1 1 1 1 1];
            >>     h = [1 2 3];
            >>     y = Conv(X, h); will give output
sequence           y = [1 3 6 6 6 5 3].
```

Detailed MATLAB code is discussed as Example 10.12.

Example 10.5 *If a transfer function of discrete time sytem is*

$$T(z) = \frac{z^2 + 2z + 3}{z^2 + z + 4}, \text{ then find}$$

The output sequence $y(n)$ for unit step input

$y(n)$ can be obtained using MATLAB commands follows

```
>>       n = 0: 10;
>>     num = [1 2 3];
>>     den = [1 1 4];
>>       y = dstep (num, den, length (n));
```

Impulse response of the same system may be obtained as

```
>>       y = dim pulse (num, den, length (n));
```

Example 10.6 *For the system given in Example 10.5; state space model with sampling time Ts = 0.01 can be obtained as*

```
>>   num = [1 2 3];
>>   den = [1 1 4];
>>   Sys = tf (num, den, Ts);
>>   Sys
```

Additional Examples

Example 10.7 *Obtain phase and magnitude plot of a discrete system whose transfer function is* $H(z) = (z + 3) / (z^2 + 3z + 2)$.

Solution

```
num=[1,3];
den=[1,3,2];
y=dimpulse(num,den);
f=fft(y);
z=abs(f);
zz=20*log(z);
subplot(2,1,1),plot(zz);
ylabel('magnitude in dB');
xlabel('frequency');
ff=f';
p=phase(ff);
subplot(2,1,2),plot(p);
xlabel('frequency');
ylabel('phase in radians');
end;
```

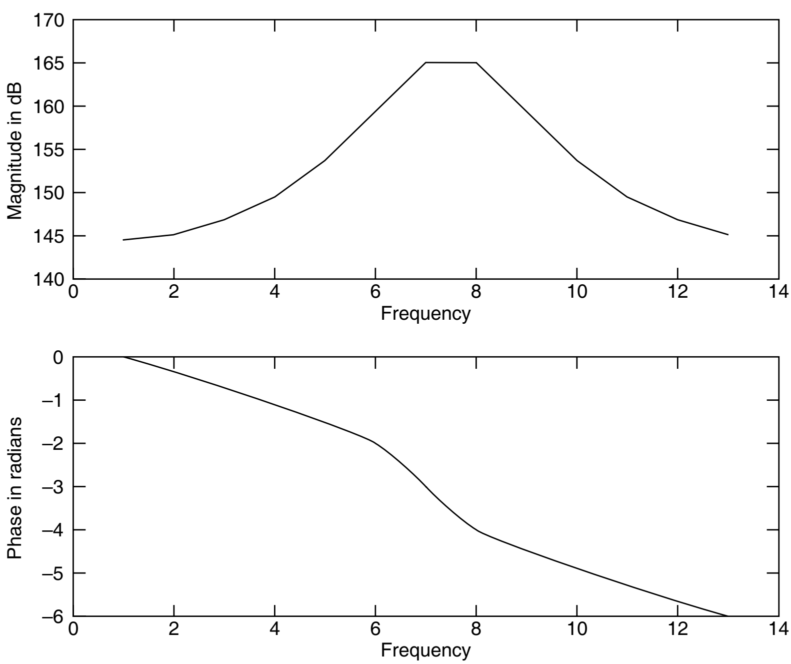


Fig. 10.1

Example 10.8 *Given transfer function $H(s) = (s + 3) / (s^3 + 5s^2 + 8s + 4)$; find frequency response (magnitude and phase response of the system).*

Solution

```
num=[1,3];
den=[1,5,8,4];
y=impulse(num,den);
x=fft(y);
z=abs(x);
zz=20*log(z);
xx=x';
p=phase(xx);
disp(z);
subplot(2,1,1),plot(zz);
xlabel('frequency');
ylabel('magnitude in dB');
disp(p);
subplot(2,1,2),plot(p);
xlabel('frequency');
ylabel('phase in radians');
end;
```

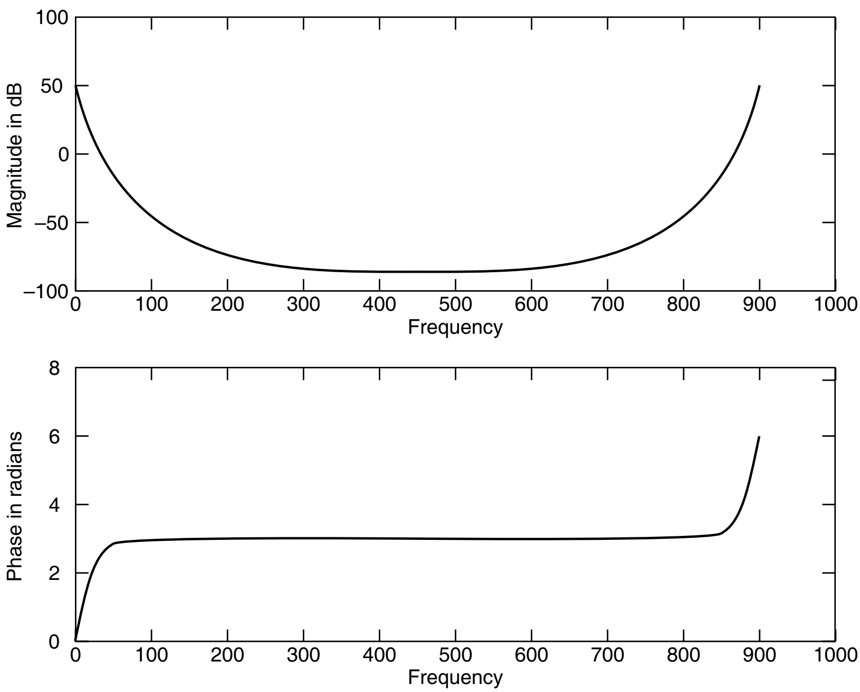


Fig. 10.2

Example 10.9 *Given transfer function $H(z) = (z + 3)/(z^2 + 3z + 2)$; find impulse response.*

Solution

```
num=[1,3];
den=[1,3,2];
y=dimpulse(num,den);
disp(y);
plot(y);
xlabel('time');
ylabel('impulse response');
end;
```

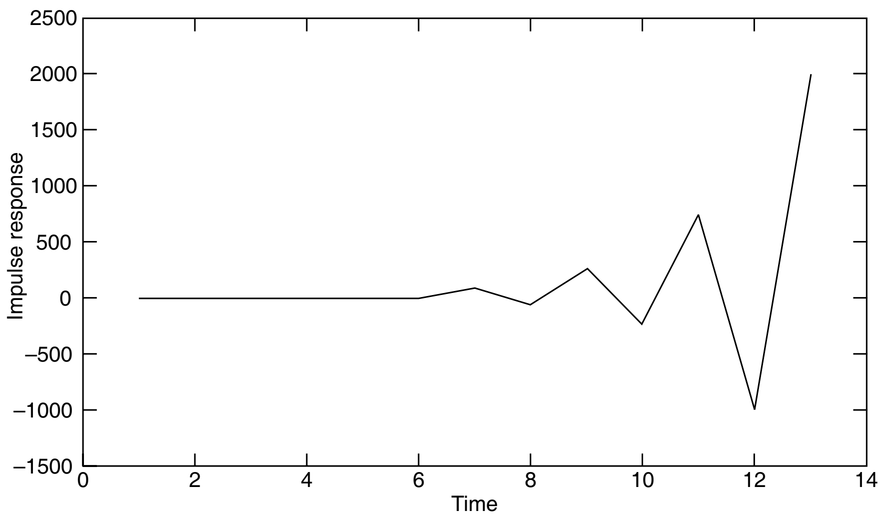


Fig. 10.3

Example 10.10 *Given transfer function $H(s) = (s + 3)/(s^3 + 5s^2 + 8s + 4)$; find the step response.*

Solution

```
num=[1,3];
den=[1,5,8,4];
y=step(num,den);
disp(y);
plot(y);
xlabel('time');
ylabel('impulse response');
end;
```

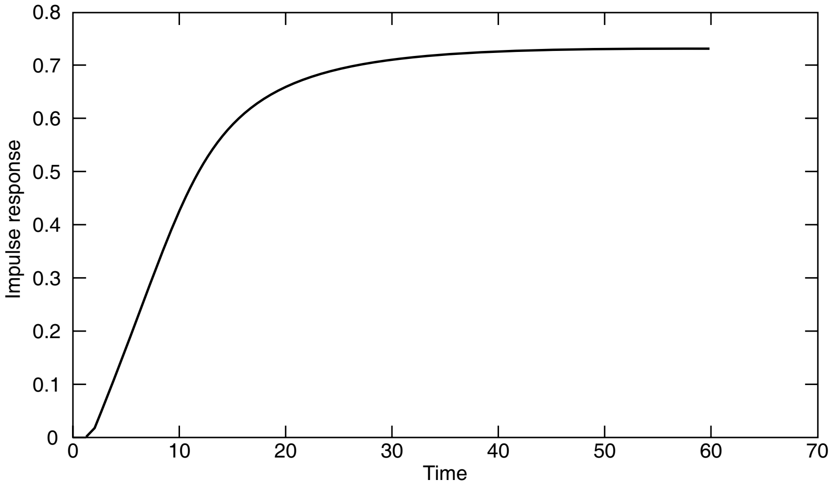


Fig. 10.4

Example 10.11 *Given transfer function $H(s) = (s + 3)/(s^3 + 5s^2 + 8s + 4)$, find the impulse response.*

```
num=[1,3];
den=[1,5,8,4];
y=impulse(num,den);
disp(y);
plot(y);
xlabel('time');
ylabel('impulse response');
end;
```

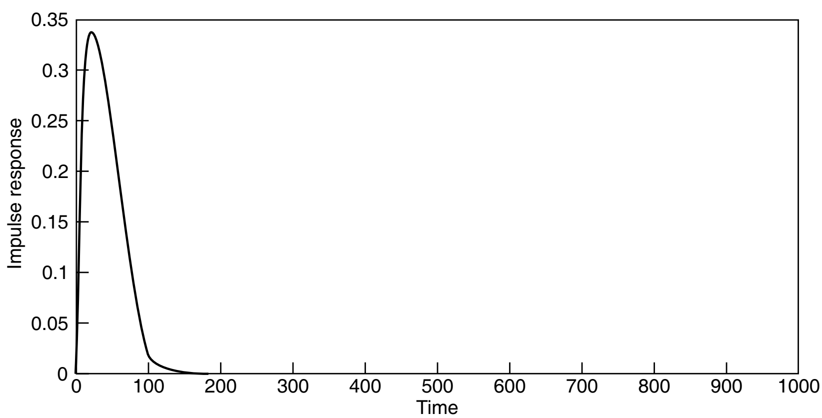


Fig. 10.5

Example 10.12 *Find the convolution of sequences x and h using MATLAB. Also plot the sequences x, h and y.*

$$x = [1\ 1\ 1\ 1\ 1]$$
$$h = [1\ 2\ 3]$$

Matlab Code:

```
clc;
clear all;
close all;
x=input('enter the 1st sequence');
h=input('enter the 2nd sequence');
y=conv(x, h);
figure;
subplot(3, 1, 1);
stem(x);
ylabel('Amplitude --->');
xlabel('(a) n --->');
subplot(3, 1, 2)
stem(h); % Figure 10.6
ylabel('Amplitude --->');
xlabel('(b) n --->');
subplot(3, 1, 3);
stem(y); % Figure 10.6
ylabel('Amplitude --->');
xlabel('(c) n --->');
disp ('The resultant signal is');
y
%enter the 1st sequence [1 1 1 1 1]
%enter the 2nd sequence [1 2 3]
%The resultant signal is
% y = 1 3 6 6 6 5 3
```

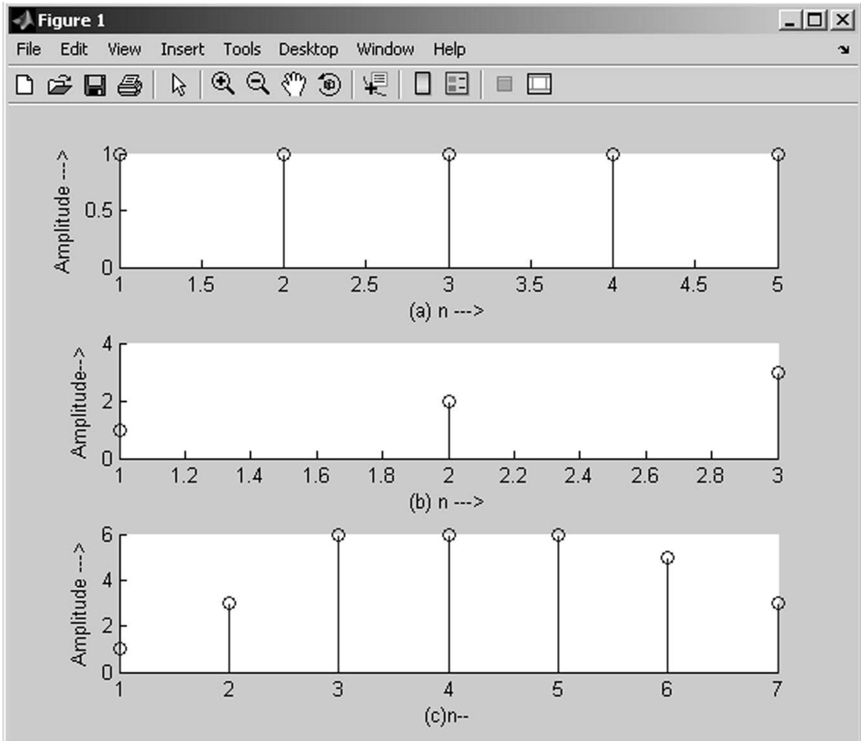


Fig. 10.6

Example 10.13 *$h_1(k)$ and $h_2(k)$ are impulse responses of two systems.*

$$h_1(k) = \begin{cases} \frac{1}{4} & 0 \le u \le 3 \\ 0 & \text{oham} \end{cases}$$

$$h_2(k) = \begin{cases} \frac{1}{4}, & n = 0, 2 \\ -\frac{1}{4}, & n = 1, 3 \\ 0 & \text{oham} \end{cases}$$

Use convolution command to plot the first 15 values of the step response for both the systems.

Solution:

```
clc;
clear all;
close all;
x=input('enter the 1st sequence');
```

```
h=input('enter the 2ndsequence');
y=conv(x,h);
figure;
subplot(3,1,1);
stem(x);
ylabel('Amplitude --->');
xlabel('(a) n --->');
subplot(3,1,2);
stem(h);
ylabel('Amplitude -->');
xlabel('(b) n --->');
subplot(3,1,3);
stem(y); % Figure 10.7
ylabel('Amplitude --->');
xlabel('(c)n--');
disp('The resultant signal is');
y
%enter the 1st sequence [0 0 0.25 0.25 0.25 0.25 0 0]
%enter the 2nd sequence [0 0 0.25 -0.25 0.25 -0.25 0 0]
%The resultant signal is
%y= [0 0 0 0 0.0625 0 0.0625 0 -0.0625 0 -0.0625 0 0 0 0]
```

Example 10.14 *Design using MATLAB a second-order digital Butterworth filter with a cut-off frequency of 1200 Hz at a sampling rate of 10^4 sps. Also plot the corresponding amplitude and phase response.*

Solution

Algorithm

1. Get the passband and stopband ripples.
2. Get the passband and stopband edge frequencies.
3. Get the sampling frequency.
4. Calculate the order of the filter.
5. Find the filter coefficients.
6. Draw the magnitude and phase responses.

```
%Program for the design of Butterworth
Low-pass digital filter
    clc;
    close all; clear all;
    format long
    rp=input('enter the passband ripple');
```

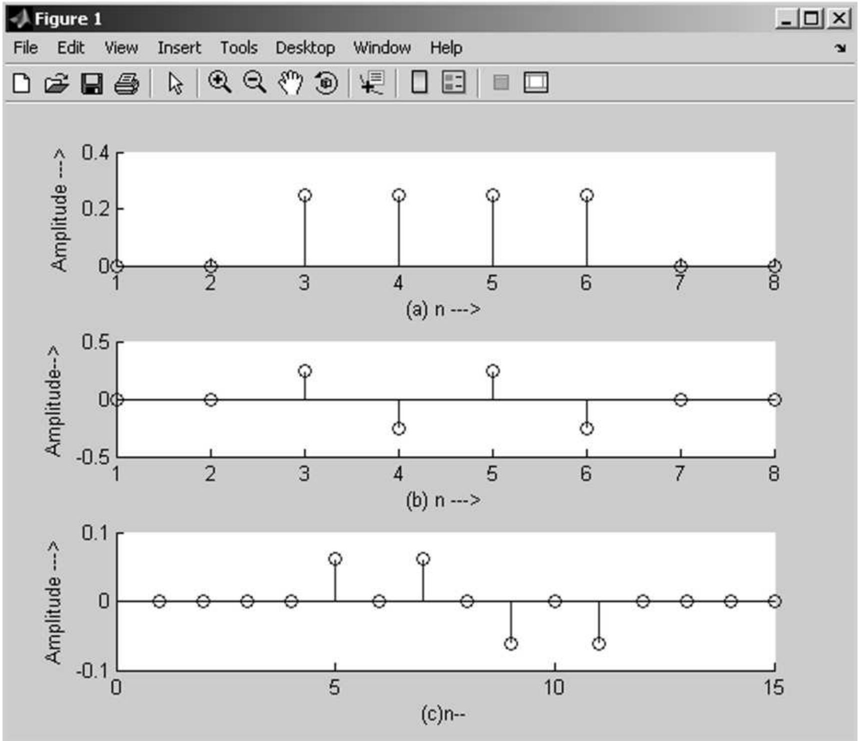


Fig. 10.7

```
rs=input('enter the stopband ripple');
wp=input('enter the passband freq');
ws=input('enter the stopband freq');
fs=input('enter the sampling freq');
wl=2*wp/fs; w2=2*ws/fs;
[n,wn]=buttord(wl,w2,rp,rs);
[b,a]=butter(n,wn);
w=0:0.0l:pi;
[h,om]=freqz(b,a,w);
m=20*loglO(abs(h))
an=angle(h);
subplot(2,l,l);plot(om/pi,m);
ylabel('Gain in dB-->');xlabel('(a) Normalised
frequency-->'); subplot(2,l,2); plot(om/pi,an);
xlabel('(b) Normalised frequency -->');
ylabel('Phase in radians -->');
```

```
As an example,
enter the passband ripple   0.5
enter the stopband ripple  50
enter the passband freq  1200
enter the stopband freq  2400
enter the sampling freq 10000
```

The amplitude and phase responses of Butterworth low-pass digital filter are shown in Fig. 10.8.

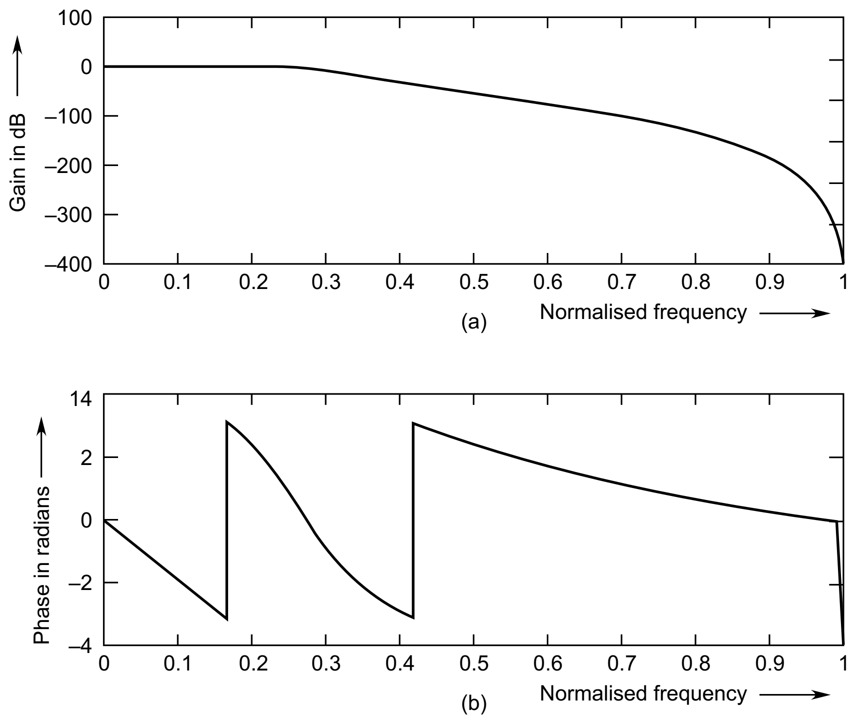


Fig. 10.8

Example 10.15 *Using an analog filter with* $H(s) = \dfrac{0.0343}{s^2 + 0.3249\,s + 0.1056}$

Design an IIR low-pass filter having 3-dB cutoff frequency $\Omega_c = 0.2\pi$. *Plot impulse response of digital IIR low-pass filter.*

Solution:

As $\qquad \Omega_c = 0.2\,\pi \qquad ; \qquad -\pi \le \omega_r \le \pi$

Let us take $\omega_r = \dfrac{\pi}{2}$ and substitute it into

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2} = \frac{2}{T}\tan\frac{\pi}{4} = 0.2\,\pi$$

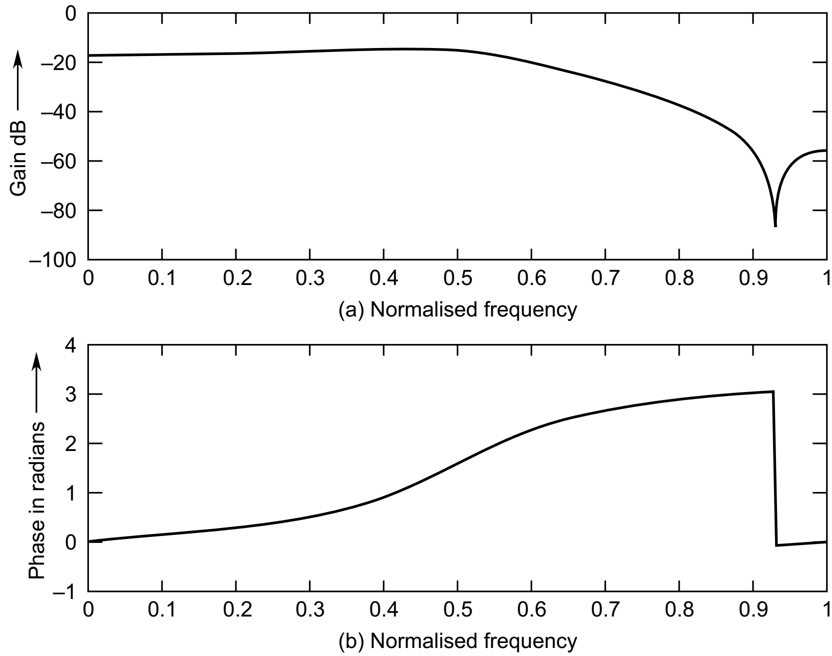


Fig. 10.9

$$\Rightarrow \qquad T = \frac{2}{0.2\,\pi} \times 1 = \frac{10}{\pi}$$

The system response of the digital filter is given by

$$H(z) = H(s)\Big|_s = \frac{2(z-1)}{T(z+1)}$$

$$= \frac{0.318 + 0.62\,z^{-1} + 0.318\,z^{-2}}{7 + 0.z^{-1} + 3\,z^{-2}}$$

MATLAB Code

```
b=[0.318 .62 .318];
a=[3 0 7];
w=0:0.01:pi;
[h,om]=freqz(b,a,w);
m=20*log10(abs(h))
an=angle(h);
subplot(2,1,1);
plot(om/pi,m);
ylabel('gaindb--->');
xlabel('(a)normalized freq--->');
subplot(2,1,2);
```

```
plot(om/pi,an);
xlabel('(b) normalized frequency--->');
ylabel('phase in radian--->');
```

Problems

10.1 A discrete system is described by the following difference equation $y(n) + y(n-1) + y(n-2) = \delta(n)$ where y's correspond to outputs and $\delta(n)$ to the initial conditions of the system are

$$y(0) = 0 \text{ and } y(1) = 1$$

Determine the response of the system by

(i) varying n from 1 to 20; and (ii) varying n from 1 to 100

Also obtain the plot of the response.

10.2 State variable model of a dynamic system given as

$$\begin{aligned} x_1(t) &= -x_1 + r \\ x_2(t) &= -2x_2 + r \\ y(t) &= 4x_1 - 4x_2 \end{aligned}$$

Find the value step response of the system and obtain plot of the response.

10.3 Obtain the unit impulse response of the dynamic system, whose state-variable model is given in Problem 10.2. Also obtain the time-response plot.

10.4 Determine the linear convolution of the following two sequence $f_1(n)$ and $f_2(n)$.

$$\begin{aligned} f_1(n) &= (2\ 1\ 2\ 1) \\ f_2(n) &= (1\ 2\ 3\ 4) \end{aligned}$$

10.5 Obtain the include convolution of sequence $f_1(n)$ and $f_2(n)$, defined in Problem 10.4.

10.6 Write a MATLAB program to plot a continuous-time sinusoidel signal and its sampled version.

10.7 Using MATLAB verify general properties of DFT.

10.8 Write a MATLAB probram to compute the circular convolution of the following two sequences.

$$\begin{aligned} f_1(n) &= (3 + j2 \quad -2 + j \quad 0 + j4 \quad 1 + j4 \quad -3 + j3) \\ f_2(n) &= (1 - j3 \quad 4 + j2 \quad 2 - j2 \quad -3 + j4 \quad 2 + j) \end{aligned}$$

10.9 With the help of MATLAB obtain the factored form of following function.

$$H(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

10.10 Given $H(s) = (s + 0.1) / (s + 0.1)^2 + r^2$

For $r = 3$, convert the given analog filter into IIR Digital filter. Plot pole–zero plot for analog filter as well as the frequency response of both digital and analog filter using MATLAB.

10.11 Using the bilinear transform method, obtain digital filter as specified below.

Prototype: Chebyshev second order low-pass filter pass band ripple $\leq$ 0.25 dB

$$f_s = 8 \text{ kHz};\ f_c = 2 \text{ kHz}$$

Using MATLAB.

(i) determine the transfer function
(ii) plot the amplitude and phase response of the filter
(iii) plot the poles and zeros

Appendix I
Partial Fraction Expansion

We have seen in Chapters 2 and 3 respectively that it is possible to obtain inverse z-transform or inverse Laplace-transform of a rational algebraic functions if the response $Y(x)$ can be expressed as

$$Y(x) = ay_1(x) + by_2(x) + cy_3(x) + \dots \tag{I.1}$$

For each of which the transform pair table gives the corresponding time function as $ay_1(t)$, $by_2(t)$, $cy_3(t)$ …, etc. It may be noted that the variables x in case of the z-transform symbolizes the variables z whereas in case of the Laplace transform, it symbolizes the variables s.

A rational algebraic function is a ratio of two polynomials in x such that

$$F(x) = \frac{N(x)}{D(x)} = \frac{\sum_{i=0}^{m} b_i x^{m-i}}{\sum_{j=0}^{n} a_j x^{n-j}} \tag{I.2}$$

This function can be classified in terms of the highest powers m (for numerator polynomial) and n (for denominator polynomial), as

(i) proper functions if $m < n$
(ii) improper function if $m \geq n$

There are well developed methods to expand a proper function into partial fractions.

In order to expand a proper function into the partial fraction, the denominator polynomial $D(x)$ is factorized into n first order factor and then depending upon the type of roots, the function is expanded into partial fractions.

A proper function can have the following three kind of poles.

(i) distinct, real poles.
(ii) distinct, complex conjugate poles.
(iii) repeated pole of the above two kinds.

In case of an improper function, the improper component is first separated out by long division such that it is written in the following form.

$$F'(x) = d(x) + F(x) \tag{I.3}$$

where

$$F(x) = \frac{N(x)}{D(x)};\ m < n \tag{I.4}$$

Now $D(x)$ is factored out to find its roots, which are the poles of $F(x)$. We can then split $F(x)$ into its partial portion. The method is illustrated for various kinds of poles.

F(x) with Distinct Real Poles $F(x)$ can the written with denominator in factored form as

$$F(x) = \frac{N(x)}{D(x)} = \frac{N(x)}{(x+p_1)(x+p_2)\ldots(x+p_e)\ldots(x+p_n)} \tag{I.5}$$

The poles of $F(x)$ are located at $x = -p_k$; $k = 1, 2, \ldots, n$. We can write $F(x)$ in form of partial fractions as

$$F(x) = \frac{A_1}{(x+p_1)} + \frac{A_2}{(x+p_2)} + \ldots.\frac{A_e}{(x+p_e)} + \ldots. + \frac{A_n}{(x+p_n)} \tag{I.6}$$

where coefficients A_k's are the *residues* at the respective poles.

To find the coefficient A_k, multiply $F(x)$ by $(x + p_k)$ and evaluate at $x = -p_k$. Thus

$$A_k = (x + p_k)\, F(x)\,|_{x=-p_k}$$

Example I.1 *Consider the following improper function.*

$$F'(x) = \frac{3x^2 + x + 1}{2x^2 + 3x}\quad m = n$$

Find its partial fractions.

Solution Dividing the numerator by denominator, we get

$$F'(x) = 1 + \frac{1-2x}{2x(x+3/2)} = 1 + F(x)$$

In partial fraction form

$$F(x) = \frac{A_1}{x} + \frac{A_2}{(x+3/2)}$$

We now find A_1 and A_2.

$$A_1 = xF(x) = \left.\frac{1-2x}{x(x+3/2)}\right|_{x=0} = 1/3$$

and

$$A_2 = (x+3/2)\cdot F(x)|_{x=-3/2}$$

$$= \left.\frac{(1-2x)}{2x}\right|_{x=-3/2} = -4/3$$

Thus

$$F'(x) = 1 + \frac{1/3}{x} - \frac{4/3}{(x + 3/2)}$$

Depending upon whether $F(x)$ is the function in the Z-domain or Laplace-domain one can apply suitable transform-pair to obtain $f(t)$.

F(x) with Complex-conjugate and Distinct Poles Suppose there is a function of the following form.

$$F(x) = \frac{N(x)}{(x + a + jb)(x + a - jb)(x + p_3)(x + p_4) + \ldots + (x + p_n)} \tag{I.7}$$

where $x = -a - jb$ and $x = -a + jb$ form the distinct pair of complex-conjugate roots and $p_3, p_4 \ldots p_n$, etc form the distinct real roots of the denominator polynomial. Then $F(x)$ can be expanded into partial fraction as

$$F(x) = \frac{A_1}{(x + a + jb)} + \frac{A_1^*}{(x + a - jb)} + \frac{A_3}{(x + p_3)} + \frac{A_4}{(x + p_4)} + \ldots + \frac{A_n}{(x + p_n)} \tag{I.8}$$

A_1 and A_1^* are residues at the poles $x = -a - jb$ and $x = -a + jb$ respectively.

The inverse Laplace transform of Eq. (I.8) is

$$f(t) = \mathcal{L}^{-1}[F(s)] = 2\,\mathrm{Re}\left\lfloor A_1 e^{-(a+jl)t} \right\rfloor + A_3\, e^{-p_3 t} + \ldots + A_n e^{-p_n A} \tag{I.9}$$

It is to be noted that in obtaining inverse Laplace transform of $F(x)$ in Eq. (I.8) it is assumed that $x \to s$.

Similarly, if we obtain inverse Z transform of Eq. (I.8) with the assumption that $x \to z$ we obtain

$$f(t) = 2\ \mathrm{Re}\left\lfloor A_1\{-(a + jl)^k\} \right\rfloor + A_3(-p_3)^k + \ldots + A_n(-p_n)^k \tag{I.10}$$

We now consider an example

$$\begin{aligned} F(x) &= \frac{x^2 + 2x}{(x+2)(x^2 + 2x + 5)} = \frac{N(x)}{D(x)} \\ &= \frac{x^2 + 2x}{(x+2)(x + 2 + j1)(x + 2 - j1)} \\ &= \frac{A_1}{(x + 2 + j1)} + \frac{A_1}{(x + 2 - j1)} + \frac{A_3}{(x+2)} \end{aligned}$$

The residues A_1, A_1^* and A_3 are determined as given below.

$$A_1 = (x + 2 + j1)\ F(x)\big|_{x=-(2+j1)} = \frac{1}{2} - j$$

$$A_1^* = (x + 2 - j1)\ F(x)\big|_{x=-(2-j1)} = \frac{1}{2} + j$$

and

$$A_3 = (x+2)\ F(x)\big|_{x=-2} = 0$$

F(x) with Repeated Poles If $F(x)$, the proper function, has repeated poles, i.e. $F(x)$ has a pole of multiplicity r at $x = p_i$ and all other poles are distinct, then the partial fraction expansion of $F(x)$ have terms of the following form.

$$F(x) = \sum_{k=1}^{q} \frac{A_k}{(x+p_k)} + \sum_{k=1}^{r} \frac{B_k}{(x+p_i)^k} \tag{I.11}$$

Here q is the number of distinct poles in $F(x)$. Coefficients A_k are determined in the manner, discussed before, depending upon whether poles are distinct and real or distinct and complex-conjugate. The coefficients B_k, corresponding to repeated poles, are determined as described below.

$$B_k = \frac{1}{(r-k)!}\left[\frac{d^{r-k}}{d\,x^{r-k}}\{(x+p_i)^r\ F(x)\}\right]_{x=-p_i} \tag{I.12}$$

Let us consider an example $F(x) = \dfrac{x^2-9}{(x-1)(x-2)^3}$

Using the Equation I.11, the above expression can be expanded as

$$F(x) = \frac{A_1}{(x-1)} + \sum_{k=1}^{3} \frac{B_k}{(x-2)^3}$$

$$= \frac{A_1}{(x-1)} + \frac{B_1}{(x-1)} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}$$

$$A_1 = (x-1)\ F(x)\big|_{x=1} = 8$$

$$B_3 = \frac{1}{0!}\left[(x-2)^3\ F(x)\right]\Big|_{x=2} = -5$$

$$B_2 = \frac{1}{1!}\left[\frac{d}{\mathrm{d}x}\{(x-2)^3\ F(x)\}\right]_{x=2}$$

$$= \frac{x^2-2x+9}{(x-1)^2}\Bigg|_{x=2} = 9$$

$$B_1 = \frac{1}{2!}\left[\frac{d^2}{\mathrm{d}x^2}\{(x-2)^3 F(x)\}\right]_{x=2}$$

$$= \frac{1}{2}\left[\frac{16}{(x-1)^3}\right]_{x=2} = -8$$

Thus

$$F(x) = \frac{8}{(x-1)} - \frac{8}{(x-2)} + \frac{9}{(x-2)^2} + \frac{5}{(x-2)^3}$$

Appendix II
Matrix Analysis and Matrix Operations

II.1 DEFINITION

Matrix A matrix is an ordered array of elements which can be real numbers, complex numbers, functions or operators. Thus a matrix A can be represented as the following

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{31} & a_{32} \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \tag{II.1}$$

This is a rectangular array of $m \times n$ elements. The elements a_{ij} correspond to i^{th} row and j^{th} column. The matrix given above has m rows and n columns. This matrix is said to be a rectangular matrix of order $m \times n$.

An $(m \times 1)$ matrix is a column vector, whereas $(1 \times n)$ matrix is a row vector.

When $m = n$, then number of rows are equal to number of columns and such matrix is said to be a square matrix. The following matrix is the example of a square matrix because in this case $m = n = 3$.

$$\mathbf{B} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 2 & 7 \\ 1 & 3 & 2 \end{bmatrix} \tag{II.2}$$

Diagonal Matrix A diagonal matrix is a square matrix whose elements leaving the main diagonal (from upper left to lower right) are zero. For example the following matrix is a diagonal matrix.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{II.3}$$

If all the diagonal elements of a diagonal matrix are equal to unity then such matrix is called unit or identity (I) matrix. The example of an unit matrix is given below.

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{II.4}$$

A matrix with all its elements equal to zero is called null matrix (Φ), i.e.

$$\Phi = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{II.5}$$

Transpose of a Matrix The transpose of matrix **A** is formed by interchanging the rows and columns and is denoted as $\mathbf{A}^T$. For example if

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 7 \\ -1 & 3 & 8 \\ 2 & 4 & -6 \\ 1 & 5 & 2 \end{bmatrix}$$

Then

$$\mathbf{A}^T = \begin{bmatrix} 4 & -1 & 2 & 1 \\ -2 & 3 & 4 & 5 \\ 7 & 8 & -6 & 2 \end{bmatrix}$$

A square matrix is symmetric if it is equal to its transpose, i.e.

$$A = A^T \tag{II.6}$$

where A is assumed to be square matrix. It may be noted that $(A^T)^T = A$. (II.7)

If a square matrix is equal to the negative of its transpose then such matrix is called skew-symmetric matrix, i.e.

$$A = -A^T \tag{II.8}$$

The example of an skew-symmetric matrix is given below.

$$A = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}$$

Determinant of a Matrix The determinant of a square matrix is a scalar, obtained by calculating the parallelepiped hyper volume specified by considering each row as an m vector in m-dimensional space. The determinant of a matrix A is denoted as $|\mathbf{A}|$. Let us consider an example.

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & -1 \\ 4 & 2 & 6 \\ 3 & -2 & 1 \end{bmatrix}$$

then

$$|A| = 7\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix}$$

$$= 7\{2-(-12)\} - 4\{3-2\} + 3\{18-(-2)\}$$

i.e $\quad |\mathbf{A}| = 7(14) - 4(1) + 3(20) = 154$

Minor If the i^{th} row and j^{th} column of determinant A are deleted then remaining $(n-1)$ rows and $(n-1)$ columns form a determinant, called minor of the element a_{ij} and is denoted as M_{ij}.

If a minor of a $|A|$ has diagonal elements of $|A|$, then such a minor is known as principal minor.

Determinant of a Matrix The value of determinant of a matrix is obtained by Laplace expansion formula, which is expressed as

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any integer } i\,;\ 1 \le i \le n \tag{II.9}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any integer } j\,;\ 1 \le j \le n$$

where M_{ij} is the minor of the element a_{ij}.

Co-factors The co-factor of the element a_{ij} of the square matrix **A** is $(-1)^{i+j}$ times the determinant of the matrix, obtained by deleting i^{th} row and j^{th} column of the given matrix **A**. For example, if we have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the co-factor is

$$a_{13} = (-1)^{1+3} \left\| \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & {}_{32} \end{bmatrix} \right\| = a_{21}a_{32} - a_{31}a_{22}$$

Inverse of the Matrix The inverse of a square matrix **A**, denoted by $\mathbf{A}^{-1}$, is defined as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{II.10}$$

I is the identity or unit matrix. The inverse matrix is calculated as

$$\mathbf{A}^{-1} = \left[\frac{\text{co-factor } a_{ij}}{|\mathbf{A}|}\right]^T \tag{II.11}$$

That is, the inverse of a square of a square matrix A is determined by substituting each element of **A** by its corresponding co-factor, transposing and then dividing by the determinant | **A** |.

Singular and Non-Singular Matrices A square matrix whose determinant is zero is called singular and if the determinant is non-zero then it is called non-singular matrix.

Rank of a Matrix A matrix is said to have rank r if there exists an $r \times r$ submatrix of **A** which is non-singular and all other $q \times q$ submatrices ($q \geq r + 1$) are singular. For example a 4×4 matrix **A**, given as

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

has its determinant $|\mathbf{A}| = 0$ and the sub-matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \neq 0$

then the above matrix **A** has rank 3.

Conjugate Matrix The conjugate of a matrix A, denoted as A^*, is the matrix in which each element is the complex-conjugate of the corresponding elements in matrix A.

Real Matrix If $A = A^*$ then the matrix **A** has real elements and **A** is said to be real matrix.

II.2 ELEMENTARY OPERATIONS IN A MATRIX

Equality Two matrices **A** and **B** are said to be equal if both possess same number of rows and columns and also the elements of the corresponding orientations are equal.

Multiplication by Scalar A matrix is said to have been multiplied by a factor k if all its elements are multiplied by k. For example if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$\mathbf{kA} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Moreover $$[kA]^T = kA^T \tag{II.12}$$

Addition and Subtraction Addition of two matrices A and B give rise to a new matrix C whose element C_{ij} is equal to the sum of the corresponding elements a_{ij} and b_{ij}. It is denoted as

$$C = A + B \tag{II.13}$$

For example if we have

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 2 \\ 3 & 7 & 1 \\ 2 & 4 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 1 & 3 \\ 5 & 4 & 7 \\ 3 & 2 & 8 \end{bmatrix}$$

then

$$C = \begin{bmatrix} 10 & 5 & 5 \\ 8 & 11 & 8 \\ 5 & 6 & 13 \end{bmatrix}$$

It may be noted that

$$(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}} \tag{II.14}$$

If **A** is a square matrix then **A** may be obtained as the sum of a symmetric matrix A_s and skew-matrix A_{sk}, i.e.

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_{sk} \tag{II.15}$$

Eq. (II.15) can be easily proved as given below. We take transpose on both sides of Eq. (II.15) so that

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}_s^{\mathbf{T}} + \mathbf{A}_{sk}^{\mathbf{T}} = \mathbf{A}_s - \mathbf{A}_{sk} \tag{II.16}$$

by simultaneously solving Eqs (II.15) and (II.16), we obtain

$$\mathbf{A}_{\mathbf{s}} = \frac{\mathbf{A} + \mathbf{A}^{\mathbf{T}}}{2} \quad \text{and} \quad \mathbf{A}_{sk} = \frac{\mathbf{A} - \mathbf{A}^{\mathbf{T}}}{2}$$

Multiplication of Matrices If a matrix **A** is of order $m \times n$ and **B** of order $n \times q$. The element c_{ij} of the product $C = AB$ are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} \tag{II.17}$$

For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where,

$$c_{11} = a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31}$$

$$c_{12} = a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32}$$
$$c_{13} = a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33}$$
$$c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31}$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is to be noted that, in general, matrix multiplication is not commutative, i.e.

$$\mathbf{AB} \neq \mathbf{BA} \tag{II.18}$$

Multiplication is associative, i.e.

$$\mathbf{(AB)\, C} = \mathbf{A(BC)} \tag{II.19}$$

Multiplication is distributive with respect to addition, i.e.

$$\mathbf{A(B + C)} = \mathbf{AB} + \mathbf{AC} \tag{II.20}$$

Multiplication of any matrix **A** by an unit matrix results in matrix **A** itself, i.e.

$$\mathbf{AI} = \mathbf{A} \tag{II.21}$$

The transpose of the product of two matrices is the product of their transpose in reverse order, i.e.

$$\mathbf{(AB)^T} = \mathbf{B^T\, A^T} \tag{II.22}$$

Cramer's Rule Let us have a set of following equations.

$$7V_1 - 3V_2 - 4V_3 = -11 \quad \ldots \quad (1)$$
$$-3V_1 + 6V_2 - 2V_3 = 3 \quad \ldots \quad (2)$$
$$-4V_1 - 2V_2 + 11V_3 = 25 \quad \ldots \quad (3)$$

Matrix

$$[A] = \begin{bmatrix} 7 & -3 & -4 \\ -3 & 6 & -2 \\ -1 & -2 & 11 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 25 \end{bmatrix} \tag{4}$$

It should be noted that matrix itself does not have any value. It's determinant has the value.

The value of any determinant as explained before is obtained by expanding it in terms of its minors. To do this, we select any row (j) or any column (k), Multiply each element in that row or column by its minor and by $(-1)^{j+k}$ and then add the products. The minor of the elements appearing in both row j and column k is the determinant which is obtained by removing row j and column k.

Let us take an example as

$$\Delta_A = \det \cdot A = \begin{vmatrix} 7 & -3 & -4 \\ -3 & 6 & -2 \\ -4 & -2 & 11 \end{vmatrix}$$

Take $k = 3$

Then

$$-4\,(-1)^{1+3}\begin{vmatrix} -3 & 6 \\ -4 & -2 \end{vmatrix} + (-2)\,(-1)^{2+3}\begin{vmatrix} 7 & -3 \\ -4 & -2 \end{vmatrix}$$

$$+\,11(-1)^{3+3}\begin{vmatrix} 7 & -3 \\ -3 & 6 \end{vmatrix}$$

$$(-4)[(-3\times -2)-(6\times -4)] + (+2)[(7\times -2)-(-3)\times(-4)]$$

$$+\,(-11)[(7\times 6)-(-3)\times(-3)]$$

$$= -4[+6+24] + 2[-14-12] + 11\,[42-9]$$

$$= -120 + 2\times(-26) + 11\times 33 = -120 - 52 + 363$$

$$= 191$$

Let us find same determinant using $j = 1$

$$\Delta_A = 7\,(-1)^{1+1}\begin{vmatrix} 6 & -2 \\ -2 & 11 \end{vmatrix} + (-3)\,(-1)^{1+2}\begin{vmatrix} -3 & -2 \\ -4 & 11 \end{vmatrix}$$

$$+\,(-4)\,(-1)^{1+3}\begin{vmatrix} -3 & 6 \\ -4 & -2 \end{vmatrix}$$

$$= 7[66-4] + 3\,[-33-8] + (-4)\,[6+24]$$

$$= 7\times 62 + 3\times(-41) + (-4)\times 30$$

$$= 434 - 123 - 120 = 191$$

We now try to apply Cramers rule to determine unknowns.

In order to determine say unknown V_1 using Crammer's rule, we define v_1 as the determinant which is obtained by replacing first column of L.H.S matrix in Eq. (4) by constant in R.H.S, i.e.

$$\Delta v_1 = \begin{vmatrix} -11 & -3 & -4 \\ 3 & 6 & -2 \\ 25 & -2 & 11 \end{vmatrix}$$

$$\Delta v_1 = (-11)\,(-1)^{1+1}\begin{vmatrix} 6 & -2 \\ -2 & 11 \end{vmatrix} + 3(-1)^{2+1}\begin{vmatrix} -3 & -4 \\ -2 & 11 \end{vmatrix}$$

$$+\,25\,(-1)^{3+1}\begin{vmatrix} -3 & -4 \\ 6 & -2 \end{vmatrix}$$

$$= (-11)\,\{66-4\} - 3\,\{-33-8\} + 25\,\{6+24\}$$

$$= (-11)\times 62 + 3\times 41 + 25\times 30 = 191$$

So

$$V_1 = \frac{\Delta v_1}{\Delta A} = \frac{191}{191} = 1$$

$$\Delta V_2 = \begin{vmatrix} 7 & -11 & -4 \\ -3 & 3 & -2 \\ -4 & 25 & 11 \end{vmatrix}$$

$$= 7^{\,(-1)\,1+1} \begin{vmatrix} 3 & -2 \\ 25 & 11 \end{vmatrix} + (-3)\,(-1)^{2+1} \begin{vmatrix} -11 & -4 \\ 25 & 11 \end{vmatrix}$$

$$+ (-4)\,(-1)^{3+1} \begin{vmatrix} -11 & -4 \\ 3 & -2 \end{vmatrix}$$

$$= 7\{33 + 50\} + 3\,\{-121 + 100\} - 4\{22 + 12\}$$

$$= 7 \times 83 - 3 \times 21 - 4 \times 34 = 382$$

So,

$$V_2 = \frac{\Delta V_2}{\Delta A} = \frac{382}{191} = 2\text{V}$$

Similarly,

$$v_3 = \frac{\begin{vmatrix} 7 & -3 & -11 \\ -3 & 6 & 3 \\ -4 & -2 & 25 \end{vmatrix}}{111} = \frac{\Delta_{v_3}}{\Delta_A} = 3\text{V}$$

Thus Cramer's rule is applicable to a system of N simulatenous equations (linear) in N unknown; for the i^{th} variable ; $\dfrac{\Delta_{v_i}}{\Delta}$.

Appendix III
Answers to Problems

CHAPTER 1

1.1 (i) Non-linear, Causal, time-invarient and static
(ii) Linear, Causal, time-invarient and dynamic
(iii) Non-linear, causal, time-invarient and static
(iv) Linear, causal, time-varient and static
(v) Linear, causal, time-invarient and dynamic
(vi) Non-linear, causal, time-varient and static
(vii) Linear, causal, time-invarient and static
(viii) Non-linear, causal, time-invarient and dynamic
(ix) Linear, Non-causal, time-invarient and dynamic
(x) Linear, causal, time-varient and static
(xi) Linear, Non-causal, time-invarient and static
(xii) Linear, causal, time-invarient and static
(xiii) Linear, causal, time-invarient and dynamic
(xiv) Non-linear, causal, time-varient and dynamic

1.2 Linear, causal and time-invarient system

1.3 $y(t) = h_1(t) * h_2(t) * \ldots * h_n(t)$

* Symbolizes convolution operation

1.5 Diode circuit, in general, is a non-linear system

1.6 $$R_2 C_1 C_2 \frac{d^2 e_1}{dt^2} + \left[\left(1 + \frac{R_2}{R_1}\right) C_1 + C_2\right] \frac{de_1}{\mathrm{d}t} + \frac{1}{R_1} e_1 = \frac{1}{R_1} e$$

where input $\dfrac{F(t)}{M_2} = r(t)$

1.7 At the end of rth month amount to be repaid by David is

$$R(n) = P(1 + i)^n - I[(1 + i)^n + (1 + i)^{n-1} + \ldots (1 + i) + 1]$$

where P = Rs. 40000/– is the initial investment made to purchase car and $i = 16\%$ is the interest rate compound monthly.

1.8 Balance amount at the beginning of nth month

where $P1 = P \times 1.03$ with P being the principal amount

1.9 Population in kth year is

$$P(k) = P(k-1)\,\{1 + B - D\} - m(k)$$

where $B = \dfrac{b\%}{100}$ and $D = \dfrac{b\%}{100}$

1.13 (a) memoryless (b) causal (c) time-invarient (d) linear

1.14 (a) not invertable (b) $(x-2)$ (c) $x(x/3)$ (d) $x(n)$ for add n

1.15 (a) no (b) A (c) not invertable

1.16 (a) period 32 (b) aperiodic (c) period 24

1.18 (a) $A \cos 3t$ (b) $A \cos 3\left(t - \dfrac{1}{3}\right)$

CHAPTER 2

2.1 $f(t) = \displaystyle\sum_{k=-\infty}^{\infty} \frac{A}{(\pi k)^2} e^{jk\omega_0 t}$; for k odd

$= 0$ for k even

2.2 $F_k = \dfrac{2}{(\pi k)^2}\,(1 + j)$; for k even

$= \dfrac{2}{(\pi k)^2}\,(j - 1)$; for k odd

2.3 (a) Average value of $f(t) = 2$

(b) Effective value of $f(t) = 4.122$

(c) Period T of $f(t) = 0.2$ secs

(d) Value of $f(t)$ at 0.05s = – 0.5

2.5 (i) $f(t) = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\omega_0 t$

(ii) $f(t) = 1 + 3 \cos 20\pi t + 2 \cos 40\pi t$

(iii) $f(t)$ at 0.04 s $= -0.81$

2.6 (a) odd symmetry (b) Half-wave symmetry (c) Even symmetry

2.7 $8 \cos\left(\dfrac{3\pi t}{8}\right) - 4 \sin\left(\dfrac{\pi t}{8}\right)$

2.8 $2 + \dfrac{1}{2}\, e^{j2\pi t/3} - 2j\, e^{j7\pi/3}$

2.9 $F_k = \dfrac{4}{k\pi} \sin \dfrac{k\pi}{2}\, e^{-j\frac{\pi k}{2}}$

2.10 (a) $F_1 = F_{-1}^* = 2.23$; $F_2 = F_{-2}^* = 0.35$

$F_3 = F_{-3}^* = 0.13$; $F_4 = F_{-4}^* = 0.07$

$F_5 = F_{-5}^* = 0.04$; $F_6 = F_{-6}^* = 0.03$

$F_7 = F_{-7}^* = 0.02$; $F_8 = F_{-8}^* = 0.01$

(b) Average power delivered to $5\,\Omega$ resistor = 56.5 watts (approx.)

2.12 (a) $F(\omega) = \dfrac{1}{\omega^2}[(e^{-j\omega} - 1) + \{1 - j(3 - 3\,\omega)\}e^{-j3\omega} - \{1 + j(4 - 4\,\omega)\}e^{-j4\omega}]$

(b) $F(\omega) = 0.25\left[\dfrac{1}{(1+\omega)}\sin\{(1+\omega)\,\pi\} + \dfrac{1}{(1-\omega)}\sin\{(1-\omega)\pi\}\right]$

$$- 0.25\,j\left[\frac{1}{(1+\omega)} - \frac{\cos\{(1+\omega)\pi\}}{(1+\omega)} - \frac{1}{(1-\omega)} + \frac{\cos\{(1-\omega)\pi}{(1-\omega)}\right]$$

(c) $F(\omega) = \dfrac{4}{\omega}\ [\sin 4\,\omega + j\ \{\cos 4\,\omega - \text{sinc } 4\,\omega\}]$

2.13 (a) $F(\omega) = \dfrac{1}{(1 + j\omega)}$ (b) same as part (a)

2.14 $\lim\limits_{\omega \to \infty} |F(\omega)| \to \dfrac{\omega^4}{\omega^2} = \infty$

2.15 This proof can be carried out by taking Fourier transform of

$$f(t) = \begin{cases} 1\,; -1 \le t \le 0^- \\ \dfrac{1}{2}\,; 0^- \le t \le 0^+ \\ 1\,; 0^+ \le t \le 1 \end{cases}$$

which comes out to be $\dfrac{2\sin\omega}{\omega}$

Now by taking inverse Fourier transform of $F(\omega) = \dfrac{2\sin\omega}{\omega}$ one arrives at the function $f(t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \dfrac{\sin\omega}{\omega}\ e^{j\omega t}\ d\omega$

2.16 (a) Energy $E = \displaystyle\int_{t_1}^{t_2} [f_1(\tau) + f_2(\tau)]^2\ d\tau$

(b) $E = \displaystyle\int_{t_1}^{t_2} [f_1^2(\tau) + f_2^2(\tau)]\,d\tau$

because $\displaystyle\int_{t_1}^{t_2} f_1^{(2)}(\tau)\ f_2(\tau)\ d\tau = 0$

2.17 This proof can be easily arrived at by evaluating the integral

$$P = \frac{1}{T}\int_{T/2}^{T/2} v(t)\ i(t)\ \mathrm{d}t$$

2.18 (a) $g(t) = 0.25\left[\text{sinc}\left\{(1+t)\dfrac{\pi}{2}\right\} + \text{sinc}\left\{(1-t)\dfrac{\pi}{2}\right\}\right]$

(b) $f(t) = 2g(t)\cos\omega_0 t$

$$= \frac{\cos\omega_0 t}{2}\left[\text{sinc}\left\{(1+t)\frac{\pi}{2}\right\} + \text{sinc}\left\{(1-t)\frac{\pi}{2}\right\}\right]$$

(c) $f(t) = \cos \{\omega_0 (t^1 - 1)\} \operatorname{sinc} (\pi t^1/2)$
where $t^1 = (1 + t)$

(d) $Y(j\omega) = AG(j\omega); -\omega_1 \le \omega \le \omega_1$

2.19 (a) $g(t) = \dfrac{2A}{\pi t} \sin at \cos \omega_0 t$

2.20 $Y(j\omega) = \cos \omega\,; -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2}$

if $\omega_0\,|\omega_1| = \left|\dfrac{\pi}{2}\right|$ and $A = 1$

2.21 (a) Energy $= \int_{-\infty}^{\infty} \operatorname{sinc}^2 (t)\, dt = \pi$

sinc function is defined as

$$\frac{\sin(\pi f_0 t)}{\pi f_0 t} = \operatorname{sinc}(f_0 t)$$

then Energy $= \int_{-\infty}^{\infty} \operatorname{sinc}^2 (t)\, dt = 1$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(Hx^2)^2} = \frac{\pi}{2}$

2.22 $F_s(\omega) = \displaystyle\sum_{k=-\infty}^{\infty} g_{\omega_0}(\omega - k\omega_0)$

2.24 For the specified bandwidth of the Band Pass filter the signal cannot be recovered.

2.25 Energy contained in the output signal $y(t)$ is

$$E_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega = \frac{1}{2\pi}\left[\frac{W}{4 + W^2} + \tan^{-1}\left(\frac{W}{4}\right)\right]$$

since $E_0 = \dfrac{1}{2} E_{in} = 0.125$

so $W = 0.0136$ rad/s

2.26 (a) $h(t) = \dfrac{R}{L} e^{-\frac{R}{L}t}\, u(t)$

(b) same as part (a)

2.27 $h(t) = (1 - t)\, e^{-t}\, u(t)$

2.28 $h(t) = e^{-\frac{R}{L}t}\, \delta(t) + \left(-\dfrac{R}{L}\right) e^{-\frac{R}{L}t}\, u(t)$

and step response for the same system

$y(t) = 2e^{-\frac{R}{L}t} - 1;\ t \ge 0$

2.29 $y(t) = Re^{-t}\,[u(t) - 2.72u\,(t - 1)]$

2.30 (a) $r(t) = \left(\dfrac{R_1 + R_2}{R_2}\right) y(t) + R_1 C \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}$

where both $r(t)$ and $y(t)$ are assumed to be voltage signals

(b) $h(t) = \dfrac{1}{R_1 C} e^{-\left(\frac{R_1 + R_2}{R_1 R_2 C}\right) t} u(t)$

(c) $y(t) = \left(\dfrac{R_2}{R_1 + R_2}\right)\left[1 - e^{-\left(\frac{R_1 + R_2}{R_1 R_2 C}\right)}\right]; t \geq 0$

(d) $y(t) = \dfrac{2}{R_1 C} \{e^{-at} - 1\}$

where $a = \left(\dfrac{R_1 + R_2}{R_1 R_2 C}\right)$

2.31 (*a*) $F(s) = \left[\dfrac{1}{s} + \dfrac{1}{(3+s)}\right]$; RoC = Re($s$) = $\sigma > 0$

(b) $F(s) = [1 - e^{5s}]$; RoC = Re(s) = $\sigma > 0$

(c) $F(s) = \dfrac{1}{\left(1 + s^2\right)} \{e^{-s\pi} + 1\}$ RoC = $\sigma > 0$

(d) $F(s) = \dfrac{1}{\left(s^2 + a^2\right)^2}$

2.32 $Y(s) = \left[\dfrac{1}{s+5} - \dfrac{e^4}{(s+4)}\right]$; RoC = $\sigma > -4$

2.33 (a) $F(s) = \dfrac{s}{s^2 + (100\pi)^2}$; RoC = $\sigma > 0$

(b) $F(s) = \dfrac{100\pi}{s^2 + (100\pi)^2}$

(c) $F(s) = \dfrac{s}{s^2 + (100\pi)^2} + \dfrac{100\pi}{s^2 + (100\pi)^2}$

2.34 $Y(s) = \dfrac{(1 - e^{-s})^2}{s^2}$

2.35 $i(t) = 1.2\, e^{-0.01t} u(t)$

2.36 $i(t) = 0.536\, (e^{-0.76t} e^{-5.24t})\, u(t)$

2.37 (a) $e^{-4t} (\sin 2t + 1.5 \cos 2t)\, u(t)$

(b) $e^{-3t}\left(\sin 3t + \dfrac{1}{3}\cos 3t\right) u(t)$

2.38 (a) $\lim_{s \to \infty} sF(s) = f(0^+) = 1$

$\lim_{s \to 0} sF(s) = f(\infty) = 1.5$

(b) $f(0^+) = 1$ and $f(\infty) = 0$

(c) $f(0^+) = \infty$ and $f(\infty) = 3.5$

2.39 $f(t) = [7e^{-t} - 78e^{-2t} - 4te^{-2t} + 12t^2 e^{-2t}]\, u(t)$

2.41 (a) £ $[y(t/a) = \int_0^\infty y\,(t/a)\, e^{-st} - \mathrm{d}t$

if $\frac{t}{a} = \lambda$ then above integral is written at

$\int_0^\infty y(\lambda)\, e^{-sa\lambda}\,(a\,\mathrm{d}\lambda) = aY(as)$

(b) $Y(s) = \int_0^\infty y(t)\, e^{-st}\,\mathrm{d}t$

or $\frac{\mathrm{d}y(s)}{\mathrm{d}s} = -t \int_0^\infty y(t)\, e^{-st}\,\mathrm{d}t = -ty(t)$

2.42 (a) $f_1(t) = e^{-4t}\,u(t) - e^{-4(t-2)}\,u(t-2)$

(b) $f_2(t) = te^{-2t}\,u(t)$

(c) $f_3(t) = (e^{-6t} - 1)\,u(t)$

2.43 (a) $f_1(0^+) = 1$; and $f_1(\infty) = 1.5$

$f_1(t) = [1.5 - 0.5\,e^{-2t}]\,u(t)$

(b) $f_2(0^+) = 1$ and $f_2(\infty) = 0$

$f_2(t) = [9e^{-t} - 8e^{-2t}]\,u(t)$

2.44 $V(s) = \frac{A}{s}[1 - 2e^{-sT} + e^{-2sT}]$

2.45 $i(t) = [5 - 2.5\,e^{-10t}]\,u(t)$

2.46 $v(t) = 9e^{-t}\cos(56.31 + 2t);\ t > 0$

2.47 $V_0(s) = \dfrac{8s}{(s^2+1)(6s^2+3s+2)}$

2.48 $f(t) = \left[\frac{T^2}{2}e^{-(2T+t)} + Tt\,e^{-(2T+t)} + \frac{t^2}{2}\,e^{-(2T+t)}\right]u(t)$

2.49 $y(t) = [1 - 3.83\,e^{-0.382t} + 2.392\,e^{-2.62t}]\,u(t)$

2.50 $R(s) = \frac{1}{s}\,[1 + e^{-st} + e^{-2st} + e^{-2st} + 4e^{-4st}]$

CHAPTER 3

3.1 (i) $\{y(k)\} = r(k-n)$

(ii) $\{y(k)\} = \{r(k)\}$

(iii) $\{y(k)\} = \{\delta(k - m - n)\}$

(iv) $\{y(k)\} = (k + 1)\ \{u(k)\}$

3.2 $y(k) = h_2(k) * h_1(k) = h_1(k) * h_2(k)$

3.3 $y(k) = \cos 2k$

3.4 $\{r_1(k)\} = \alpha^k [u(k) - u(k-1)]$

$\{r_1(k)\}$ has value 1 at $k = 0$ and zero event other instances

3.5 (i) $\{y(k)\} = \{(0.5)^k + 3^k\}\ \{(k + 1)\, u(k)\}$

(ii) $\{y(k)\} = 0.5\ \{3^{k+1} - 1\}$

(iii) $\{y(k)\} = \{6\ 13\ 10\ 10\ -11\ 2\}$

3.6 (a) causal, stable (b) non-causal, unstable

(c) causal, unstable (d) causal, stable

3.7 $(-3)^n\, u(n)$

3.9 (a) $N = 10,\ A_k = \dfrac{\sin\left(\dfrac{7\pi}{10}k\right)}{\sin\left(\dfrac{\pi}{10}k\right)}$

(b) $\dfrac{1}{10}\left[\dfrac{\left(1 - e^{-jk\left(\frac{2\pi}{10}\right)}\right)\sin\left(\dfrac{7\pi}{10}k\right)}{\sin\left(\dfrac{\pi}{10}k\right)}\right]$

3.10 (b) $A_0 = \dfrac{1}{2},\ A_1 = -\dfrac{1+j}{2},\ A_2 = -1,\ A_3 = -\dfrac{1-j}{4}$

(c) $A_k = \dfrac{1}{4}\left[1 + 2\, e^{-jk\frac{\pi}{2}} - e^{-jk\frac{3\pi}{2}}\right]$

3.11 $A_k = \dfrac{1}{4} + 2\left(1 - \dfrac{1}{\sqrt{2}}\right)(-1)^k \cos\left(\dfrac{\pi}{2}k\right)$

3.12 $A_0 = 0.125,\ A_1 = 0.004 - j\,0.5,\ A_2 = 0$

$A_3 = -0.0366 + j\,0.052,\ A_4 = 0.6768$

$A_5 = -j\,0.026,\ A_6 = 0.125 + j\,0.0366$

$A_7 = -j\,0.516$

3.15 $2 + \sin\left(\dfrac{4\pi}{7}n + \dfrac{5\pi}{6}\right) + \sin\left(\dfrac{8\pi}{7}n + \dfrac{7\pi}{10}\right)$

3.17 (i) $\dfrac{z - 3z^2}{(z-1)^3}$ (ii) $\dfrac{z}{(z-\alpha)^2}$

(iii) $\dfrac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$ (iv) $\dfrac{z^2 + 3\alpha}{z^2 + 2z\alpha + \alpha^2}$

3.18 $T^2\left[\dfrac{z-3z^2}{(z-1)^3}\right]$

3.19 $1 + 4.7\,z^{-1} + 0.75\,z^{-4} + 1.414\,z^{-5}$

3.20 (a) $x(z) = 5z^{-4}$

(b) (i) $x(z) = 5z^{-4}$

(ii) $\dfrac{zT}{z^2 - 2e^{-\alpha T}z - e^{-2\alpha T}}$

(iii) $\dfrac{e^{\alpha T}\alpha^T(\sin\omega T)z}{e^{2\alpha T}z^2 - 2\alpha^T e^{\alpha T}z(\cos\omega T) + \alpha^{2I}}$

3.21 (a) LTI and causal system

(b) LTI and causal system

(c) LTI and non-causal system

3.22 $y(nT) = \{0\ 0\ 0\ 0\ 30\ 38\ 34\ 44\ 42\ 58\ 42\ 20\ 40\}$

3.23 $y(nT) = \{56\ 68\ 66\ 99\ 120\ 121\ 113\ 115\ 82\ 59\ 42\ 27\ 16\ 8\ 4\ 0\}$

3.24 (i) $h(k) = \left(\dfrac{1}{3}\right)^{k+1} u(k)$

(ii) $h(k) = (3)^{k-1}\,u(k)$

(iii) $h(k) = 2(-0.5)^k\,u(k)$

(iv) $h(k) = (-0.56)\,(-0.438)^{k-1} + 3.56\,(-4.561)^{k-1}$

3.25 $h(k) = 1.45\,(1.25)^k - 1.28\,(1.11)^k$

3.26 $h(2) = \alpha_1(\alpha_3 - \alpha_1\alpha_2)$ and so on

3.27 (i) $y(k) = 5.3\cos(40.9 - 60°\,k)$

(ii) $y(k) = \dfrac{1}{9}\,u(k-1) - (6.6)\,(-0.7)^{k-1} - 0.22\,(-4.303)^{k-1}$

3.28 Monthly installment $R = \dfrac{210085.7(0.14)^k}{1-(0.14)^k}$

3.29 (i) $\dfrac{\alpha}{z(z-\alpha)}$ (ii) $\dfrac{z^2 - z - 3}{z^2(z-3)}$

(iv) $\dfrac{z}{z-e^{\alpha}}$ (v) $F(z) = 0$ for $k \geq 0$

3.30 (i) $F(z) = \dfrac{0.125z^4 + 0.1875z^2 + 0.375z^2 + 0.75z}{(z-0.5)}$

RoC $|z| < |10.5|$

(ii) $F(z) = \dfrac{z}{(z-\alpha_1)} - \dfrac{z}{(z-\alpha_2)} - 1$

RoC $\alpha_1 \leq |z| \leq \alpha_2$

(iii) $F(z) = \dfrac{z^2 + 0.5 - 0.5z}{z(z - 0.5)}$

RoC $|z| > 0.5$

(iv) $F(z) = z^3$

3.31 (i) $f(k)$ is causal

(ii) $f(k)$ is non-causal

(iii) $f(k) = 0.5\,[(-1)^k + (1)^k];\ k \geq 0$

(iv) $f(k) = -0.25\,(-1)^k + 0.125\,(1)^k + 0.1\left(\dfrac{-1}{3}\right)^k$

(v) $f(k)$ is non-causal

3.32 (i) $h(k) = -1;\ k \geq 0$

(ii) $h(k)\ 3.625\,(-3.96)^k - 2.625\,(-1.94)^k$

(iii) $h(k) = (k + 1) + (k - 1) = 2k$

3.33 (i) $y(k) = A_1\ \alpha_1^{k-1} + A_2\alpha_2^{k-1}$

$A_1 = \dfrac{\alpha_1^2 + \alpha_2}{\alpha_1 - \alpha_2},\ A_2 = \dfrac{\alpha_1\alpha_2 + \alpha_2}{\alpha_2 - \alpha_1}$

(ii) $y(k) = (k + 1)\,(2)^k - (2)^k + 1^k = k(2)^k - 1^k$

(iii) $y(k) = 1^{k-1}$

3.34 $y(k) = 2.5\,u(k) + 7.9\cos(26.6°\,k - 108.4°)$

3.35 $y(k) = \delta(k) + \delta(k - 1) + 18.03\,(-1)^k + 35.61\,(-0.73)^k$
$-50.05\,(-0.86)^k + 9.8 \times 10^{-4}\,(4 - 84)^k$

3.36 Causal, unstable

3.38 Non-causal

3.39 LTI stable

3.40 Non-causal

CHAPTER 4

4.1 (a) $e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}$ (b) $\dfrac{e^{j\omega}}{2}\,\dfrac{1}{(1 - \frac{1}{2}e^{j\omega})}$

(c) $\dfrac{e^{j2\omega}}{4} \cdot \dfrac{1}{\left(1 - \frac{1}{4}e^{j\omega}\right)}$

4.2 (a) $-\dfrac{1}{2j}\left[\dfrac{1}{1 - \frac{1}{2}e^{j\frac{\pi}{3}}e^{j\omega}} - \dfrac{1}{1 - \frac{1}{2}e^{-j\frac{\pi}{3}}e^{j\omega}}\right]$

(b) $\dfrac{\pi}{j}\left[\delta\left(\omega-\dfrac{\pi}{2}\right)-\delta\left(\omega+\dfrac{\pi}{2}\right)\right]+\pi\left[\delta(\omega-1)+\delta(\omega+1)\right];\ 0\le\omega\le\pi$

(c) $\dfrac{4}{5-3\cos\omega}$

4.3 (a) $\delta(n)+3\,\delta(n-1)+2\,\delta(n-2)-4\,\delta(n-5)$

(b) $\dfrac{j(-1)^{n+1}\sin\dfrac{\pi}{3}}{\pi\left(n-\dfrac{1}{3}\right)}$

4.4 $\delta(n)+\dfrac{1}{2}\delta(n-2)+\dfrac{1}{2}\delta(n+2)+\dfrac{1}{2}\delta(n-6)+\dfrac{1}{2}\delta(n+6)$

4.5 (a) $\left(1+\dfrac{1}{N}\right)F(k)\,W_N^k;\ W_N=e^{-j2\pi/N}$

(b) Re $[F(e^{j\omega})]$

4.6 $2\,\delta(n)-\delta(n-1)+\delta(n-3)$

4.7 $\dfrac{3}{4}$

4.8 (a) $y(n)=-2\left(\dfrac{1}{3}\right)^n u(n)+\dfrac{1}{3}\left(\dfrac{1}{6}\right)^n u(n)-(n+1)\left(\dfrac{1}{6}\right)^n u(n)$

(b) $y(n)=\dfrac{8\pi}{3}\displaystyle\sum_{k=-\infty}^{\infty}\delta(\omega-(2k+1)\pi)$

4.9 $3\,\delta(n+3)-\delta(n+1)+\delta(n)-3\,\delta(n-2)+2\,\delta(n-3)$
$-\,\delta(n-4)+3\,\delta(n-5)\ +2\,\delta(n-7)$

4.10 (a) $y(n)=(n+1)\left(-\dfrac{1}{3}\right)^n u(n)-\dfrac{1}{6}n\left(-\dfrac{1}{3}\right)^n u(n)$

(b) $y(n)=\dfrac{1}{3}\left(-\dfrac{1}{3}\right)^n u(n)+\dfrac{2}{3}\left(\dfrac{1}{6}\right)^n u(n)$

(c) $y(n)=\left(-\dfrac{1}{3}\right)^n u(n)+3\left(-\dfrac{1}{3}\right)^{n-2} u(n-2)$

4.11 (a) $y(n)=\left(-\dfrac{1}{3}\right)^n(n+1)\,u(n)$

(b) $-\dfrac{1}{4}\left(-\dfrac{1}{3}\right)^n u(n)+\dfrac{1}{4}\left(\dfrac{1}{3}\right)^n u(n)+\dfrac{1}{6}\left(\dfrac{1}{9}\right)^{n-1} u(n)$

(c) $\left[\dfrac{1}{9}\left(\dfrac{1}{6}\right)^n+\dfrac{8}{9}\left(-\dfrac{1}{3}\right)^n+\dfrac{2}{3}n\left(-\dfrac{1}{3}\right)^n\right]u(n)$

4.12 $y(n) = x(n) + \frac{3}{4}\,y(n-1) - \frac{1}{8}\,y(n-2)$

4.13 (a) $F(k) = \frac{1}{2}\left[\frac{1-\left[3\,e^{j\pi\left(\frac{1}{3}-\frac{2k}{N}\right)}\right]^N}{1-3\,e^{j\pi\left(\frac{1}{3}-\frac{2k}{N}\right)}} - \frac{1-\left[3\,e^{j\pi\left(\frac{1}{3}+\frac{2k}{N}\right)}\right]^N}{1-3\,e^{-j\pi\left(\frac{1}{3}+\frac{2k}{N}\right)}}\right]$

(b) $\frac{1}{2j}\left[\frac{1-e^{-j2\pi(k-k_0)}}{1-e^{-j\frac{2\pi}{N}(k-k_0)}}\right] + \frac{1}{2j}\left[\frac{1-e^{-j2\pi(k+k_0)}}{1-e^{-j\frac{2\pi}{N}(k+k_0)}}\right];\ k_0 = \frac{N}{3}$

4.14 $h_2(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$

4.15 (a) $F(n) = \frac{4j}{N}\left[\sin\left(n\frac{\pi}{3}\right) + \sin\left(2\pi\frac{n}{3}\right)\right]$

4.16 (a) $$H(e^{j\omega}) = \frac{\left(-\frac{1}{8}+j\frac{5\sqrt{3}}{24}\right)}{1-\left(\frac{1}{6}+j\frac{1}{2\sqrt{3}}\right)e^{-j\omega}} + \frac{\left(-\frac{1}{8}-\frac{5\sqrt{3}}{24}\right)}{1-\left(\frac{1}{6}-j\frac{1}{2\sqrt{3}}\right)e^{-j\omega}} + \frac{\frac{9}{4}}{\left(1+\frac{1}{3}e^{-j\omega}\right)}$$

(b) $$h(n) = \left[\left(-\frac{1}{8}+j\frac{5\sqrt{3}}{24}\right)\left(\frac{1}{6}+j\frac{1}{2\sqrt{3}}\right)^n + \left(-\frac{1}{8}-j\frac{5\sqrt{3}}{24}\right)\left(\frac{1}{6}\,j\frac{1}{2\sqrt{3}}\right)^n + \frac{9}{4}\left(-\frac{1}{3}\right)^n\right]u(n)$$

4.17 $W = e^{-2\pi/16} = e^{-\delta\pi/8}$

so W's are spread on the circum of unit circle in complex plane. Spacing between any two consecutive W is $\pi/8$

4.18 $F_1(0) = 28;\ F_1(1) = -4 + j9.6$

$F_1(2) = -4 + 4j;\ F_1(3) = -4 + j1.66$

$F_1(4) = -4;\ F_1(5) = -4 - j17.9;$

$F_1(6) = -4 - j1.66;\ F_1(7) = -4 - j9.6$

4.19 (a) $F(k) = u(k)$

(b) $F(k) = e^{2\pi km/N};\ 0 \le m \le N$

(c) $F(k) = \sum_{u=0}^{N-1} a^n \; e^{-j2\pi km/N}$

4.20 $F(0) = 8$

$F(1) = -4$

$F(2) = -4 - 4j$

$F(3) = -4$

4.21 (i) $F_0 = 32$

$F_k = 0, k = 1, 2, \ldots, 15$

4.22 For $N = 4$

$F_k = 1.552, 0.96 \angle -25.7°, 0.718 \angle 0°, 0.92, \angle 20°$

4.25 Frequency spacing

$4f = 9.76\,\text{Hz}$

4.27 $\sum_{n=0}^{2} |f_n|^2 = \frac{1}{3} \sum_{k=0}^{2} |f_k|^2 = 18$

so Parseval's theorem is valid

4.32 $f_3(n) =$

$n = 0$	0
$n = 1$	0
$n = 2$	1
$n = 3$	3
$n = 4$	2
$n = 5$	0
$n = 6$	0

4.33 $f_3(n) = 2, 5, 6, 5, 1, 2, 0$

CHAPTER 5

5.1 Beyond $(\omega - 50\,\pi) < |\,\omega_m\,| < \omega + 50\,\pi$

5.2 (i) $4000\,\pi$ (ii) $40\,\pi$

(iii) $4000\,\pi$ (iv) $4000\,\pi$

5.3 (i) $2\,\omega_s$ (ii) $4\,\omega_s$

(iii) $\omega_{s/2}$ (iv) ω_s

5.4 (i) ω_s (ii) $\omega_s + 2\,\omega_0$

5.5 (i) ∞ (ii) 2 Hz

5.6 $T \le \frac{1}{4}$ sec (ii) $T \le \frac{1}{2}$ sec

5.7 $2\,(\omega_{m_1} + \omega_{m_2})$

5.9 $\omega_s = 2\pi$ rad/s

5.13 $y(n) = x(n)$

CHAPTER 6

6.1 $$\frac{1 + 2R_1Cs + R_1R_2C^2s^2}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2}$$

6.2 $$\frac{1}{1 + \frac{3}{2}RCs}$$

6.3 $$\frac{3.98}{(1 + j\omega/20)(1 + j\omega/800)}, -46.4°, -133.6°$$

6.4 Break frequencies

(a) 0.25,50 (b) 7.9

6.5 $$\frac{2(1 + j\omega/0.2)}{(1 + j\omega/10)(1 + j\omega/50)}$$

6.6 $$H_2(j\omega) = \frac{(1 + j\omega/40)}{(1 + j\omega)(1 + j\omega/8)}$$

6.7 $$\frac{G_1G_2}{H_1G_1 + H_2G_2}$$

6.10 $$\frac{KG_1(s)G_2(s)/s}{1 + G_1(s)H_3(s) + G_1(s)G_2(s)H_2(s) + KG_1(s)G_2(s)H_1(s)/s}$$

6.11 $$\frac{G_5 + G_1G_6G_4}{1 + G_1G_2H_1 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2}$$

6.12 143.7 (1 + 0.167 s)

6.13 $$\left(1 + \frac{R_f}{R_1}\right)$$

CHAPTER 7

7.1 (a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} & -\frac{1}{L_1} \\ 0 & \frac{-(R_1 + R_2)}{L_2} & 0 & \frac{1}{L_2} \\ \frac{1}{c_1} & 0 & 0 & 0 \\ \frac{1}{c_2} & -\frac{1}{c_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r(t)$$

(b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-(k_1+k_2)}{M_1} & 0 & \dfrac{k_2}{M_2} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{M_2} & \dfrac{-k_2}{M_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} + r(t)$$

7.2 (a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} r(t)$$

(b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1}{M_1} & \dfrac{-B_1}{M_1} & \dfrac{k_1}{M_1} & \dfrac{B_1}{M_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_1}{M_2} & \dfrac{-(k_1+k_2)}{M_2} & \dfrac{B_1}{M_2} & \dfrac{-B_1}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} r(t)$$

7.3 $$y = [(-4b_0 + b_3)\ (-b_0 + b_2)\ (-b_0 + b_1)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 4b_0$$

7.4 (a) (i) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(b)$$

$$y = [20\ 5\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(ii) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t)$$

$$y = [10\ {-15}\ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(iii) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} r(t)$$

(b) state transition matrix—Part (a) (ii)

$$e^{At} = \begin{bmatrix} e^{0t} & 1 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

7.5 $e^{At} = 3x$ matrix

Its elements are

(1, 1) $4e^{-t} - 4e^{-2t} - e^{-3t}$

(1, 2) $4e^{-t} - 5.5e^{-2t} - 1.5e^{-3t}$

(1, 3) $e^{-t} - 1.5e^{-2t} - 4.5e^{-3t}$

$(\overline{2,\ 1})$ $-6e^{-t} + 9e^{-2t} - 27e^{-3t}$

(2, 2) $-7e^{-t} + 12.5e^{-2t} + 4.85e^{-3t}$

(2, 3) $-2e^{-t} + 3.5e^{-2t} + 25.5e^{-3t}$

$(\overline{3,\ 1})$ $12e^{-t} - 21e^{-2t} - 153e^{-3t}$

(3, 2) $16e^{-t} - 29.5e^{-2t} - 253.5e^{-3t}$

(3, 3) $5e^{-t} - 8.5e^{-2t} - 104.5e^{-3t}$

7.6 (i) $A^k = \begin{bmatrix} 2(-1)^k & -(-2)^k & 2(-1)^k & -2(-2)^k \\ -(-1)^k & +t(-2)^k & -(-1) & +2(-2)^k \end{bmatrix}$

(ii) can be solved on same lines

7.7 $\phi(t) = e^{-4t}\begin{bmatrix} \sqrt{2}\cos(t - 45°) & 5.1\cos(t + 78.7°) \\ 2.237\cos(t + 63.4°) & \sqrt{2}\cos(t + 45°) \end{bmatrix}$

7.8 $H(s) = \dfrac{3s^2 + 6s + 4}{s^3 + 3s^2 + 2s + 1}$

7.9 (a) $\lambda^2 + 5\lambda + 6 = 0$

(b) Eigen values –2, –3

Eigen values

(–2) $[-3\ 1]^T$

(–3) $[-2\ 1]^T$

7.10 $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 - te^{-t} \\ 1 - e^{-t} \end{bmatrix} u(t)$

7.11 Zero input response

$$x(t) = \begin{bmatrix} 3e^{-t} - 12e^{-2t} + 9e^{-3t} \\ 1.5e^{-t} - 16e^{-2t} + 13.5e^{-3t} \\ 0.5e^{-t} - 14e^{-2t} + 4.5e^{-3t} \end{bmatrix}$$

7.12 $x(k) = \begin{bmatrix} (-1)^k - (-2)^k \\ -(-1)^k + 2(-2)^k \end{bmatrix}$

7.13 Zero input response

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t} - 2e^{-2t} + e^{-3t} \\ -e^{-t} + 4e^{-2t} + 3e^{-3t} \\ e^{-t} - 8e^{-2t} + 4e^{-3t} \end{bmatrix}$$

7.14 (a) $y(k) = 2(2)^k + k$

(b) $y(k) = 2(2)^k + k$

7.15 (i) $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ k_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [r - terms]$$

(ii) Jordan cannonical form

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$[2r(k+3) - 12r(k+2) + 13(k+1) - r(k)]$

7.16 (i) Cannonical form

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

(ii) $$A^k = \begin{bmatrix} 3(-2)^k - 2(-3)^k & (-2)^k - (-3)^k \\ -3(-2)^k + 6(-3)^k & -(-2)^k + 3(-2)^k \end{bmatrix}$$

(iii) $y(k) = 4(-2)^k - 3(-3)^k$

7.17 Solve using MATLAB tools

7.18 $$e^{At} = \begin{bmatrix} \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{4}{3}e^{-4t}\,\frac{1}{3}e^{-4t} \\ -\frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t} & -\frac{19}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

7.19 $$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & 0 \\ 0 & & \ddots & \\ & & & e^{-\lambda_n t} \end{bmatrix}$$

7.20 (a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r;\ y = [1\ \ 0\ \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & s \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

7.21 $$x(k) = \begin{bmatrix} 3(-2)^k + 2(-2)^k = 5(-2)^k \\ -3(-2)^k - 3(3)^k \end{bmatrix}$$

$$+(-2)^{k-1}\begin{bmatrix} 2\left[1-\left(-\frac{1}{2}\right)^k\right]+\frac{4}{3}\left[1-\left(-\frac{1}{2}\right)^k\right]=\frac{10}{3}\left[1-\left(-\frac{1}{2}\right)^k\right] \\ -2\left[1-\left(-\frac{1}{2}\right)^k\right]-\frac{9}{2}\left[1-\left(\frac{1}{3}\right)^k\right] \end{bmatrix}$$

$y = x_1(k)$ is obvious

CHAPTER 8 (ANSWERS)

8.1 (a) unstable in BIBO sense
(b) unstable in BIBO sense

8.2 (a) Unstable system
(b) Unstable system

8.3 (a) Unstable
(b) limitedly stable
(c) stable

8.4 Stable system. It's two roots have values greater than–1.

8.5 (a) Stable
(b) Unstable system. It has two roots with positive real part
(c) Unstable system. Its two roots have positive real part
(d) Unstable. Its two roots have positive real part

8.6 $-8 < k < 26.6$

8.7 $b = \frac{1}{2}$ and $k = \frac{1}{2}$

8.8 a_0, a_1, a_2, a_3 and a_4, should be positive non zero. Also

$$\frac{a_1a_2 - a_0a_3}{a_1} > 0 \quad \text{and} \quad \frac{a_1a_2a_3 - a_0a_3^2 - a_1^2a_4}{a_1^2} > 0$$

8.9 (i) According to both Jury test and Billinear transformation test the given system is unstable.
(ii) Unstable according to both the tests.

8.10 Unstable system.

CHAPTER 9

9.1 $\omega = 3.15$ rad/s

9.2 $\omega = 1.732$ rad/s

9.3 $H(s) = \dfrac{1}{(s+1)(s^2+0.347s+1)(s^2+s+1)(s^2+1.53s+1)(s^2+1.9s+1)}$

9.4 $H_{B2}(s) = \dfrac{1}{\left(\dfrac{s}{62.8}\right)^2 + \dfrac{\sqrt{2}}{62.8}s + 1}$

9.5 $H_{B2}(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

9.6 $k = 1$, $G_1 = 152.2\ \mho$, G2 = 208.33 $\mho$ and $c_1 = 0.1015F$

9.7 $L = 10^{-4}$ Henery

9.8 (i) $H_{C1}(j\omega) = 0.64$

(ii) $H_{C2}(j\omega) = 0.204$

(iii) $H_{C3}(j\omega) = 0.052$

9.9 $s_1 = -0.07 + j0.98;\ s_2 = -0.17 + j0.4$

$s_3 = -0.17 - j0.4;\ s_4 = -0.07 - j0.98$

$s_5 = 0.07 - j0.98;\ s_6 = 0.17 - j0.4$

$s_7 = 0.17 + j04;\ s_8 = 0.07 + j0.98$

9.10 $H(s) = \dfrac{1}{\prod_{k=1}^{2}\left\{\dfrac{s/2\pi \times 10^3}{s_k} - 1\right\}}$

9.11 It is again second order Chabyshev low-pass filter.

9.12 Chebyshev LPF achieves less attenuation than Butterworth LPF for $\varepsilon < 1$. However, for $\varepsilon > 1$ behaviour of two filters is reversed.

9.13 Butterworth LPF achieves half power point at the cut-off frequency.

9.14 In the case of Chebyshev LPF, the half-power point is obtained at frequency $\omega_i = \cos h \left\{\dfrac{1}{n} \cos h^{-1}\left(\dfrac{1}{\varepsilon}\right)\right\}$

9.15 $Hc_{3(s)} = \dfrac{0.4913}{(s/4)^3 + 0.98\,(s/4)^2 + 1.24\,(s/4) + 0.4913}$

when $Hc_3(s)$ is transformed to high pass filter than transfer function of high pass filter is

$$Hc_3Hp(s) = \frac{0.4913s^3}{0.4913s^3 + 0.93s^2 + 0.553s + 0.422}$$

9.16 $H_{B3}(s) = \dfrac{1}{(s+1)\,(s^2 + s + 1)}$

After applying LP-to-BP transformation the transfer function is

$$H_{BT3}(s) = \frac{1}{\left[\dfrac{s_T^2 - 1382.8\times10^4}{12.57\times10^2 s_T} + 1\right]\left[\left\{\dfrac{s_T^2 - 1382.8\times10^4}{12.57\times10^2 s}\right\}^2 + \left\{\dfrac{s_T^2 - 1382.8\times10^4}{12.57\times10^2 s_T} + 1\right\}\right]}$$

CHAPTER 10

10.1 (i)
```
y(1) = 0;
y(2) = 1;
for i = 3:20,
y(i) = -y (i - 2) - y (i - 1) + 1;
end ;
disp (y);
plot (y);
x label ('time');
y label ('response');
end;
```
(ii)
```
y(1) = 0;
y(2) = 1;
for i = 3:100
y(i) = -y (i - 2) -y(i - 1) + 1;
end;
disp (y);
plot (y);
x label ('time');
y label ('response');
end;
```

10.5
```
f1 = [2 1 2 1];
f2 = [1 2 3 4];
% step 1 – take 4-point dft of f1 (n)
x1 = fft (f1, 4);
% step 2 – take 4-point dft of f2 (n)
x2 = fft (f2, 4);
% step 3—multiply x1 and x2
x = x1 * x2;
% take ifft of x
y = ifft (x);
disp (y);
end;
```

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