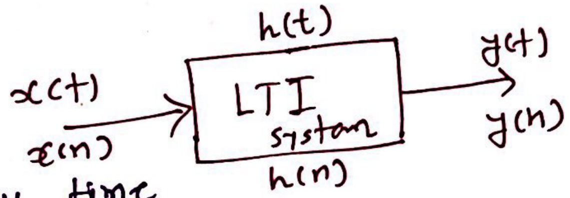
Convolution

- It is a methematical Operation which express Input-Output relation of an $\boxed{\text{LTI System}}$.
- Convolution is widely used in many fields like probability & statistics.
- Convolution of two signals produces a third signal.
- By using Convolution, we can find the zero state response.



- For continuy time

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$\rightarrow Y(\omega) = X(\omega) H(\omega)$

- For discrete signal

$$y(n) = x(n) * h(n)$$

$$= \sum_{K=-\infty}^{\infty} x(n)\, h(n-K)$$

$$= \sum_{K=\infty}^{\infty} x(n-K)\, h(n)$$

Properties of Convolution

① Commutative property

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

② Distributive property

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

③ Associative property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

④ Shifting property

- $x_1(t) * x_2(t) = z(t)$
- $x_1(t) * x_2(t-t_1) = z(t-t_1)$
- $x_1(t-t_1) * x_2(t) = z(t-t_1)$
- $x_1(t-t_1) * x_2(t-t_2) = z(t-t_1-t_2)$

If $x(t) * y(t) = z(t)$

e.g $x(t-3) * y(t+1)$

$= z(t-3+1)$

$= z(t-2)$

⑤ Convolution with Impulse

- $x(t) * \delta(t) = x(t)$
- $x(t) * \delta(t-t_1) = x(t-t_1)$

⊛ $\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$

e.g $x(t) * \delta(t-2)$

$= x(t-2)$

e.g $x(t-2) * \delta(t+1)$

$= x(t-2+1)$

$= x(t-1)$

e.g $x(t) * \delta(2t-6)$

$= x(t) * \delta(2(t-3))$

$= x(t) * \frac{1}{2}\delta(t-3)$

$= \frac{1}{2}[x(t) * \delta(t-3)]$

$= \frac{1}{2}x(t-3)$

Additional properties of Convolution

→ $x(t) * h(t) = y(t)$

→ $\dfrac{d\,y(t)}{dt} = \dfrac{d\,x(t)}{dt} * h(t) = x(t) * \dfrac{d\,h(t)}{dt}$

→ $u(t) * u(t) = r(t)$

→ $u(t-t_1) * u(t-t_2) = r(t-(t_1+t_2))$

→ $y(-t) = x(-t) * h(-t)$

→ $\dfrac{d^n x(t)}{dt^n} * \dfrac{d^m h(t)}{dt^m} = \dfrac{d^{n+m}\, y(t)}{dt^{n+m}}$

Scaling of Convolution

- When Input & Impulse signals are Scaled by α then Output will be

$$\Rightarrow x(t) * h(t) = y(t)$$

$$\Rightarrow x(\alpha t) * h(\alpha t) = \frac{1}{|\alpha|} y(\alpha t)$$

- $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

- $x(\alpha t) * h(\alpha t) = \int_{-\infty}^{\infty} x(\alpha\tau)\, h(\alpha(t-\tau))\, d\tau$

$$= \int_{-\infty}^{\infty} x(\alpha\tau)\, h(\alpha t - \alpha\tau)\, d\tau$$

- If $\alpha\tau = \lambda \Rightarrow \alpha\, d\tau = d\lambda$

$$\Rightarrow d\tau = \frac{d\lambda}{\alpha}$$

$$= \int_{-\infty}^{\infty} \frac{x(\lambda)\, h(\alpha t-\lambda)}{\alpha}\, d\lambda$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} x(\lambda)\, h(\alpha t-\lambda)\, d\lambda$$

$$\Rightarrow x(\alpha t) * h(\alpha t) = \boxed{\frac{1}{\alpha} y(\alpha t)}$$

Area of Convoluted Signal

→ If Input signal $x(t)$ is having area A_i

→ If System $h(t)$ is having area A_h

→ then area of convoluted signal

$$A = A_i A_h$$

→ $y(t) = x(t) * h(t)$

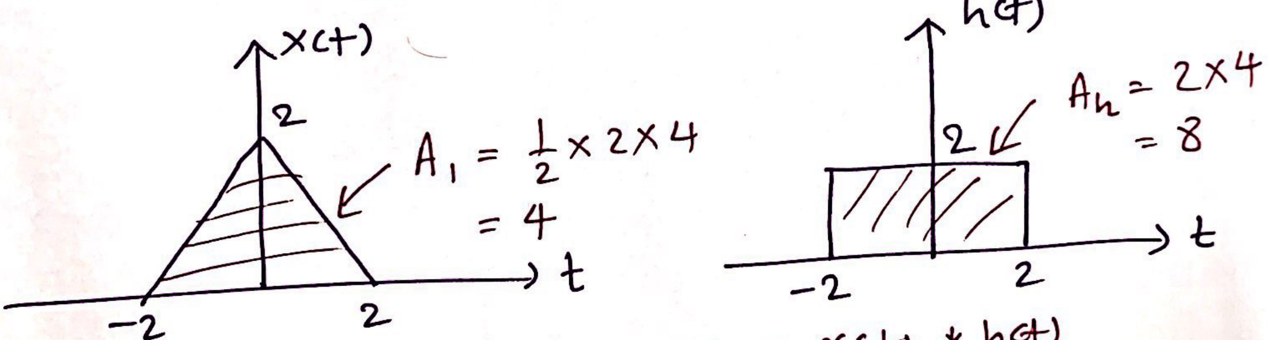
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

→ to get area integrate $y(t)$

$$\int y(t)\, dt = \int\int x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int x(\tau)\, d\tau \int h(t-\tau)\, d\tau$$

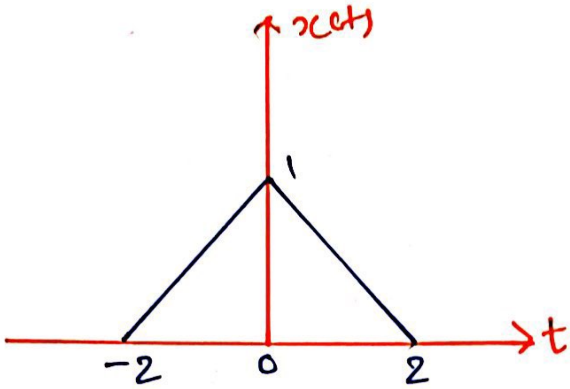
$$\boxed{A = A_i A_h}$$

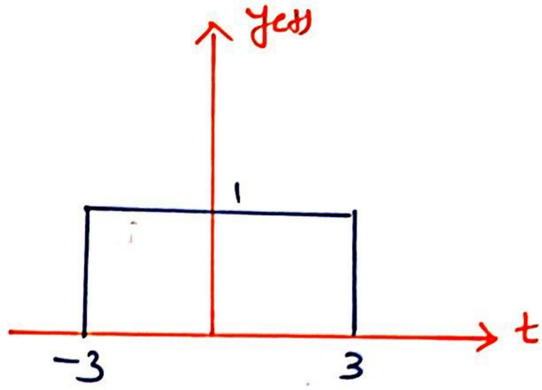


→ convoluted signal $y(t) = x(t) * h(t)$.

→ Area of $y(t) = A_1 A_h = 4 \times 8 = 32$

limits of Convoluted Signal



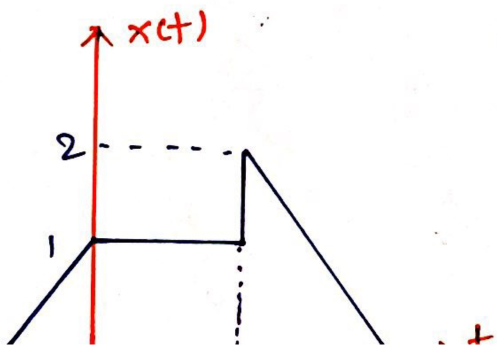


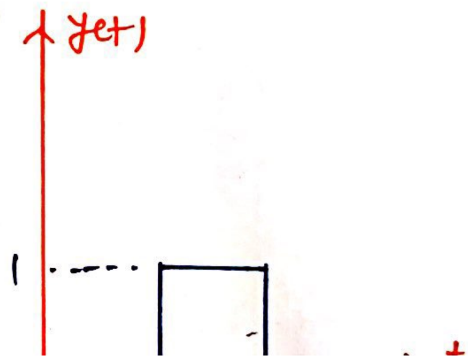
→ $z(t) = x(t) * y(t)$

→ Lower limit of $z(t) = -2 - 3 = -5$

→ Upper limit of $z(t) = 2 + 3 = 5$

limits of $z(t)$ → $\boxed{-5 < t < 5}$





Convolution using Slide & Shift Method

→ $x(t) = t^2 + 2t + 1$

$y(t) = t^2 + 3t + 4$

Find $z(t) = x(t) * y(t)$ using slide & shift method

→ $z(t) = x(t) * y(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, y(\tau)\, d\tau$$

			t^2	$2t$	1	
4	$3t$	t^2				
	4	$3t$	t^2			$= t^4$
		4	$3t$	t^2		$= 3t^3 + 2t^3 = 5t^3$
			4	$3t$	t^2	$= 4t^2 + 6t^2 + t^2 = 11t^2$
				4	$3t$	$= 8t + 3t = 11t$
					4	$= 4$

→ $z(t) = x(t) * y(t)$

$$= t^4 + 5t^3 + 11t^2 + 11t + 4$$

→ $x(t) = 3t^3 + t + 1$

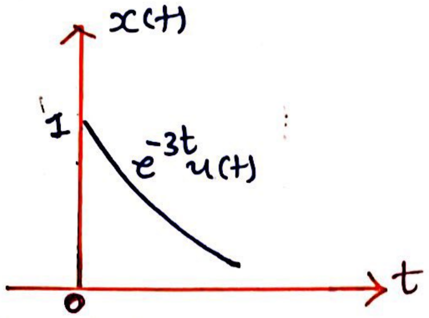
$y(t) = t^2 + t + 2$

$z(t) = x(t) * y(t)$

$$= \boxed{3t^5 + 3t^4 + 6t^3 + 2t^2 + 3t + 2}$$

			$3t^3$	0	t	1	
2	t	t^2					
	2	t	t^2				$= 3t^5$
		2	t	t^2			$= 3t^4$
			2	t	t^2		$= 6t^3 + t^3 = 6t^3$
				2	t	t^2	$= t^2 + t^2 = 2t^2$
					2	t	$= 2t + t = 3t$
						2	$= 2$

Convolution by Image method



Find Convolution using Image method

→ $z(t) = x(t) * h(t)$

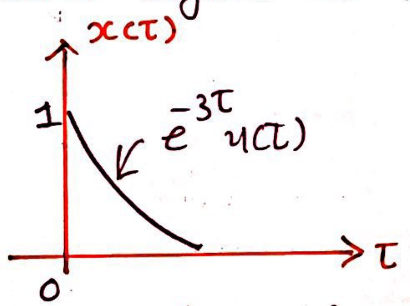
→ limits of $z(t)$

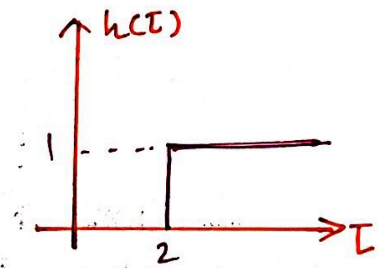
= sum of lower limit < t < sum of upper limit

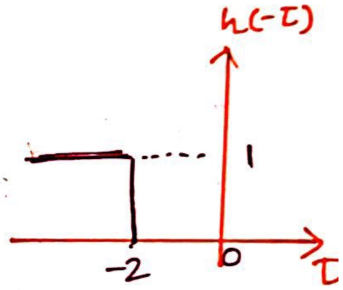
= $0 + 2 < t < \infty + \infty$

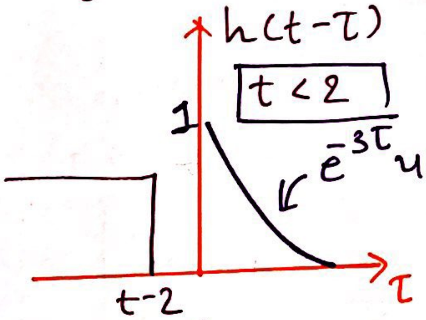
= $\boxed{2 < t < \infty}$

Represent signal in terms of τ

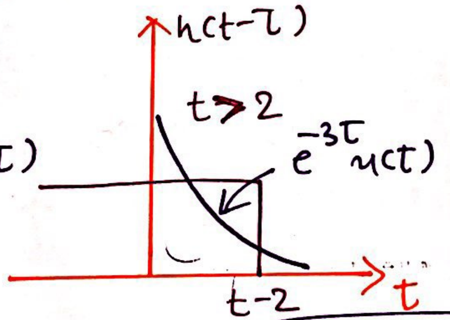




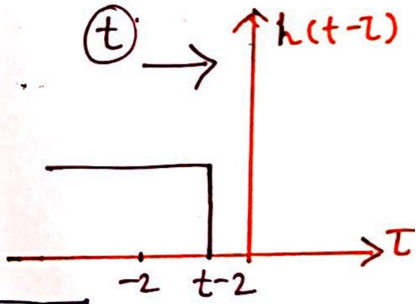


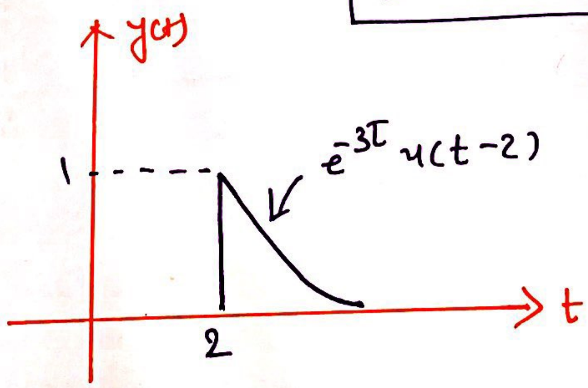


Conv.^{n} = 0



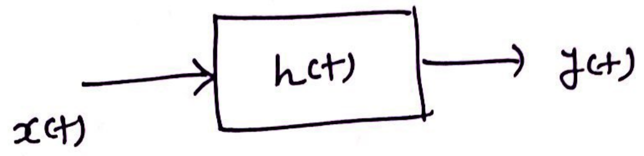
$\boxed{e^{-3\tau}u(t-2)}$





Example of convolution based on distributive property.

The Impulse response of Continuous System is given by $h(t) = \delta(t-1) + \delta(t-3)$. The value of step response at $t = 2$ sec is ___1___



→ $x(t) = u(t)$

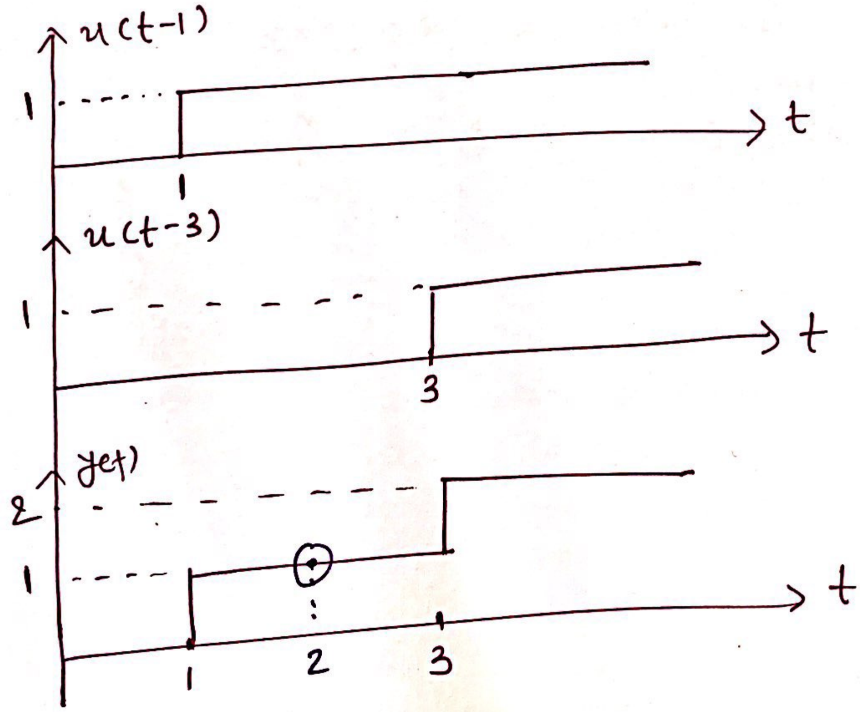
$h(t) = \delta(t-1) + \delta(t-3)$

→ $y(t) = x(t) * h(t)$

$= u(t) * [\delta(t-1) + \delta(t-3)]$

$= u(t) * \delta(t-1) + u(t) * \delta(t-3)$

$= u(t-1) + u(t-3)$



Convolution example

Given $z(t) = \int_{-\infty}^{\infty} x(-\tau+b)\, h(t+\tau)\, d\tau$.

express $z(t)$ Interms of $y(t) = x(t) * h(t)$

$\rightarrow y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\underline{t}-\tau)\, h(\tau)\, d\tau \quad ✓ \quad — ①$$

$$\rightarrow z(t) = \int_{-\infty}^{\infty} x(-\tau+b)\, h(t+\tau)\, d\tau$$

$$t+\tau = \lambda \quad \rightarrow \tau = \lambda - t$$

$$\Rightarrow d\tau = d\lambda$$

$$= \int_{-\infty}^{\infty} x(-(\lambda-t)+b)\, h(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x((\underline{t+b})-\lambda)\, h(\lambda)\, d\lambda \quad — ②$$

$\rightarrow$ compare eq^n ① & ② $\Rightarrow \boxed{z(t) = y(t+b)}$

Example of Convolution based on propeties of Convolution

IF $x(t) = u(t-3) - u(t-5)$

$h(t) = e^{-3t}\,u(t)$

Find $\frac{d\,x(t)}{dt} * h(t)$.

$\rightarrow \frac{d\,x(t)}{dt} = \frac{d\,[u(t-3) - u(t-5)]}{dt}$

$= \delta(t-3) - \delta(t-5)$

$\rightarrow h(t) = e^{-3t}\,u(t)$

$\rightarrow \frac{d(x(t))}{dt} * h(t) = [\delta(t-3) - \delta(t-5)] * e^{-3t}u(t)$

$= \cancel{\delta(t-3) > \delta(t-5}$

$= \underline{\delta(t-3)} * \underline{e^{-3t}u(t)} - \delta(t-5) * e^{-3t}u(t)$

$= e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$

Example based on Area of Convolution and Ramp Convolution

Find area of given Convoluted Signal

$[u(t+1) - u(t-3)] * [u(t+1) - u(t-1)]$

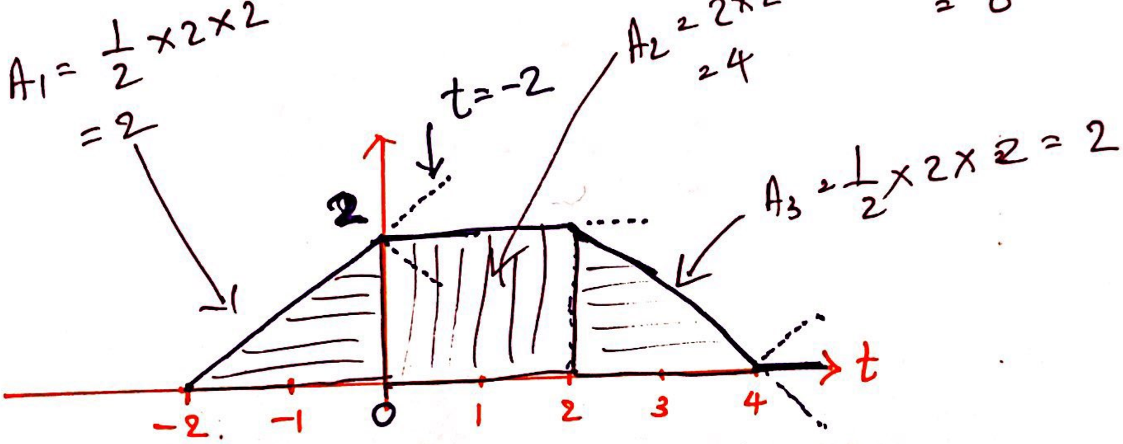
$= u(t+1) * u(t+1) - u(t+1) * u(t-1) - u(t-3) * u(t+1) + u(t-3) * u(t-1)$

$= \boxed{r(t+2) - r(t) - r(t-2) + r(t-4)}$

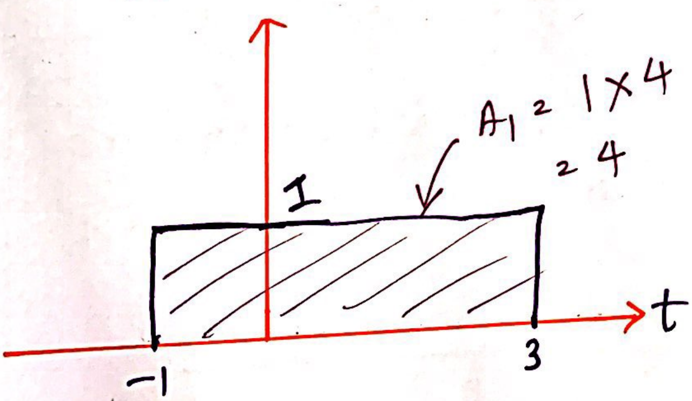
total Area $= A_1 + A_2 + A_3$

$= 2 + 4 + 2$

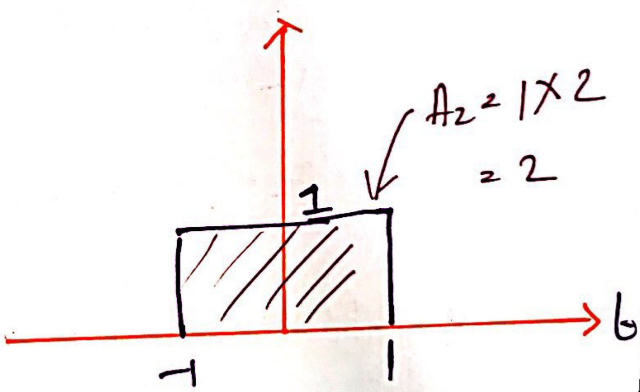
$= 8$



$u(t+1) - u(t-3)$



$u(t+1) - u(t-1)$



→ Area of convoluted signal

$A = A_1 A_2$

$= 4 \times 2 = 8$

- linear convolution for discrete signal

$x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$ and $h(n) = \{1, 1, \underset{\uparrow}{-1}, 1\}$

Find linear convolution of above signal

- $y(n) = x(n) * h(n)$
- total samples $= m + n - 1 = 4 + 4 - 1 = 7$

	1	2	3	4
1	1	2	3	4
1	1	2	3	4
-1	-1	-2	-3	-4
1	1	2	3	4

$y(n) = x(n) * h(n)$

$y = \{1, 3, 4, \underset{\uparrow}{6}, 3, -1, 4\}$

Find linear convolution of given signals

$x(n) = \{1, 3, \underset{\uparrow}{2}, 1\}$

$y(n) = \{1, \underset{\uparrow}{2}\}$

- $z(n) = x(n) * y(n)$
- total samples $= m + n - 1 = 4 + 2 - 1 = 5$

	1	3	2	1
1	1	3	2	1
2	2	6	4	2

$z(n) = x(n) * y(n)$

$= \{1, 5, 8, \underset{\uparrow}{5}, 2\}$

* Circular Convolution or Periodic Convolution

- $x(n) = \{1, 2, 1, 1\}$ & $y(n) = \{1, 1, -1, -1\}$
Find Periodic Convolution of above Signal

	1	2	1	1
1	1	2	1	1
1	1	2	1	1
-1	-1	-2	-1	-1
-1	-1	-2	-1	-1

Simples   linear $\boxed{n+m-1}$ Circular. $\boxed{\max(n,m)}$

- $z(n) = x(n) * y(n)$

$= \{1, 3, 2, -1, -2, -2, -1\}$

$-2, -2, -1$

$= \{-1, 1, 1, -1\}$

- $x(n) = \{1, 2, 1, 2\}$ & $y(n) = \{2, 1, 1\}$
Find Circular Convolution of above Signal.

	1	2	1	2
2	2	4	2	4
1	1	2	1	2
1	1	2	1	2

Total samples

linear $\boxed{n+m-1}$ Circular $\boxed{\max(n,m)}$

$z(n) = x(n) * y(n)$

$= \{2, 5, 5, 7, 3, 2\}$

$3, 2$

$= \{5, 7, 5, 7\}$

Circular Convolution by matrix method

- Find Circular Convolution of given Signals

$x(n) = \{1, 1, -1, 2\}$

$h(n) = \{1, 2, 3, 4\}$

$$= \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & 2 & -1 \\ -1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+4-3+4 \\ 1+2+6-4 \\ -1+2+3+8 \\ 2-2+3+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 12 \\ 7 \end{bmatrix}$$

$\rightarrow z(n) = x(n) \circledast h(n)$

$= \{6, 5, 12, 7\}.$

- Find Circular convolution by matrix method for given signals

$x(n) = \{1, 2, 1, 1\}$

$y(n) = \{2, 2, 1, 3\}$

$$= \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+2+1+6 \\ 4+2+1+3 \\ 2+4+1+3 \\ 2+2+2+3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ 10 \\ 9 \end{bmatrix}$$

$\Rightarrow z(n) = x(n) \circledast y(n) = \{11, 10, 10, 9\}.$

Deconvolution

IF a signal $x(n) = \{1, 1, -1\}$ is convoluted with an unknown signal & the convolution result is noted as $\{1, 3, 1, -2\}$. Find $h(n)$.

→ $y(n) = x(n) * h(n)$

→ In linear convolution total samples

$= m+n-1$

$\Rightarrow 4 = 3+n-1$

$\Rightarrow n = 2$

	1	1	-1
a	a	a	-a
b	b	b	-b

→ $h(n) = \{a, b\}$

→ $y(n) = x(n) * h(n) = \{a, b+a, b-a, -b\}$.

$= \{1, 3, 1, -2\}$

→ $a = 1, b = 2$

→ $h(n) = \{1, 2\}$

- Hence, $y(n) = x(n) * h(n)$.

IF $y(n) = \{2, 1, 3, 4, 1\}$
$x(n) = \{1, 1, 2\}$, then find $h(n)$

→ total samples $= m+n-1$

$\Rightarrow 5 = 3+n-1$

$\Rightarrow n = 3$

	1	1	2
a	a	a	2a
b	b	b	2b
c	c	c	2c

→ $h(n) = \{a, b, c\}$.

→ If $y(n) = x(n) * h(n)$

$= \{a, b+a, c+b+2a, c+2b, 2c\}$.

$= \{2, 1, 3, 4, 1\}$

→ $a = 2$, $b = -1$, $c+b+2a = 3$

$\Rightarrow c = 3-b-2a$

$= 3+1-4 = 0$

→ $h(n)$ can not be calculated as it is incorrect.

Division method for deconvolution

- If $y(n) = x(n) * h(n)$

where $x(n) = \{1, 1, -1\}$

$y(n) = \{1, 3, 1, -2\}$

Find $h(n)$.

$$
\begin{array}{r|l}
 & 1 + 2x \\
\hline
1 + x - x^2 & 1 + 3x + x^2 - 2x^3 \\
 & \underline{-\ 1 \pm x \mp x^2} \\
 & 0 + 2x + 2x^2 - 2x^3 \\
 & \underline{-\ 2x \pm 2x^2 \mp 2x^3} \\
 & 0
\end{array}
$$

→ As reminder is 0

→ $h(n) = \{1, 2\}$.