Assignment - 1

Z - Transform

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* List the properties of ROC for the Z- Transform.

→ The properties are as follow :-

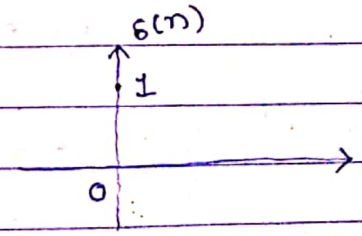
1. The ROC is a ring, whose centre is at origin.
2. ROC cannot certain any poles.
3. If ROC of $x(z)$ include unit circle then only the fourier transform of DT sequnce $x(n)$ converges.
4. The ROC must be a connected region.
5. For a finite Duration sequence, $x(n)$; the ROC is entire Z plane except $z=0$ and $z=\infty$.
6. If $x(n)$ is causal then ROC is exterior part of circle of radius say 'α'.
7. If $x(n)$ is anticausal then ROC is interior part of circle of radius say 'α'.
8. If $x(n)$ is two sided sequence then ROC is intersection of two circle of radii 'α' and 'β'.

* Find Z- Transform of 1) $\delta(n)$
  2) $u(n)$
  3) $n\,a^n u(n)$

→ Z - Transform of $\delta(n)$ (unit impulse)

$\delta(n) = 1$ only at $n=0$

$= 0$ otherwise



unit impulse

According to defination of Z-Transform we have.

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \delta(n) z^{0} = 1$$

ROC : entire z plane.

2. $u(n)$

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$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \frac{1}{1-z^{-1}} \quad ; \text{ROC} : |z^{-1}| < 1$$

$$= \frac{1}{1-1/z} \quad ; \text{ROC} : \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1} \quad ; \text{ROC} : |z| > 1$$

3. $n a^n u(n)$

$$z[n a^n u(n)] = -z \frac{d}{dz} z[a^n u(n)]$$

$$= -z \frac{d}{dz} \left(\frac{z}{z-a} \right)$$

$$= -z \left(\frac{(z-a) - z}{(z-a)^2} \right) = \frac{az}{(z-a)^2} \quad ; \text{ROC} : |z| > |a|$$

* Consider the signal $x(n) = \begin{cases} \frac{1}{3}^n \cos\left(\frac{\pi}{4} n\right), & n \le 0 \\ 0 & , n > 0 \end{cases}$ determine

Poles & ROC for $x[z]$.

→ $$x(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{4} n\right) z^{-n}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n e^{j\pi n/4} z^{-n} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\pi n/4} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{-j\pi n/4} z^{n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{j\pi n/4} z^{n}$$

$$= \frac{1}{2} \frac{1}{1-3e^{-j\pi/4} z} + \frac{1}{2} \frac{1}{1-3e^{j\pi/4} \cdot z} \quad ; |z| < \frac{1}{3}$$

The poles are at $z = \frac{1}{3} e^{j\pi/4}$ and $z = \frac{1}{3} e^{-j\pi/4}$

* Define : The z transform. State & Prove Time shifting & Time reversal properties of Z-Transform.

→ Z-Transform is a mathematical tool use for analysis of discrete time signal and linear time invariant system (LIT).

- Z transform of discrete time signal $x(n)$ is define as $x(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

→ Time shifting :-

State :- $x(n) \xleftrightarrow{z} x(z) \longleftrightarrow x(n-k) \xleftrightarrow{z} z^{-k} x(z)$, ROC of $z^{-k} x(z)$ is same as ROC of $x(z)$ except $z=0$ for $k>0$ and $z=\infty$ for $k<0$.

Proof :- $x(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$x_1(z) = z[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$$

Take $n-k=m \Rightarrow n = k+m$

$n \to -\infty \Rightarrow m \to -\infty$, $n \to \infty \Rightarrow m \to \infty$

$$\therefore z[x(n-k)] = \sum_{m=-\infty}^{\infty} x(m) z^{-(k+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} * z^{-k}$$

$$\therefore z[x(n-k)] = z^{-k} x(z)$$

→ Time reversal :-

State :- If $x(n) \xleftrightarrow{z} x(z)$, ROC: $r_1 < |z| < r_2$ then

$x(-n) \xleftrightarrow{z} x(z^{-1})$, ROC: $1/r_2 < |z| < 1/r_1$

Proof :- $x(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Take $m = -n$

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$$\therefore Z[x(-n)] = \sum_{m=-\infty}^{\infty} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m} = x(z^{-1})$$

ROC of $x(z)$: $r_1 < |z| < r_2$

ROC of $x(z^{-1})$: $r_1 < |z^{-1}| < r_2$

: $r_1 < \left|\frac{1}{z}\right| < r_2$

: $1/r_1 > |z| > 1/r_2$

$\therefore$ $1/r_2 < |z| < 1/r_1$

* Z transform of $\delta(n)$ is $\underline{1}$.

* Z transform of $x(n-n_0)$ is $\underline{z^{-n_0} x(z)}$.

* for causal signal $x(n)$, Z transform $x(z)$ is

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

* If Z-transform of signal $x(n)$ has ROC $R_1 < |z| < R_2$, find ROC of signal $x(-n)$.

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Take $m = -n$

$$\therefore Z[x(-n)] = \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{m} = x(z^{-1})$$

ROC of $x(z)$: $R_1 < |z| < R_2$

ROC of $x(z^{-1})$: $R_1 < |z^{-1}| < R_2$

: $R_1 < \left|\frac{1}{z}\right| < R_2$

: $1/R_1 > |z| > 1/R_2$

: $1/R_2 < |z| < 1/R_1$

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* Prove differentiation property of z-transform.

→ If $x(n) \xleftrightarrow{z} X(z)$ then $x(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$

ROC of $-z\frac{dX(z)}{dz}$ is same as ROC of $X(z)$.

proof: $X(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

apply differentiation on both the sides.

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n) (-n z^{-n-1})$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} n x(n) z^{-1} z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$\therefore -z\frac{dX(z)}{dz} = z[n x(n)]$$

* Determine the z-transform of following finite duration sequence $x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\}$

→ $X(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\sum_{n=-5}^{0} x(n) z^{-n} = z^5 + 2z^4 + 4z^3 + 5z^2 + 7$$

ROC: entire z plane except $z = \infty$

* Write differentiation in z-domain property of z-transform obtain z-transform of $x(n) = a^n \cos(\Omega_0 n)\, u(n)$

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where a is real and positive.

→ $x_1(n) = a^n \cos(\Omega_0 n) u(n)$

$x(n) \overset{Z}{\longleftrightarrow} \frac{z(z - \cos\Omega_0)}{z^2 - z^2 \cos\Omega_0 + 1}$; ROC : $|z| > 1$

$x_1(n) = a^n x(n) \overset{Z}{\longleftrightarrow} x_1(z) = x(z/a)$

$x_1(z) = \frac{z/a\,(z/a - \cos\Omega_0)}{(z/a)^2 - 2(z/a)\cos\Omega_0 + 1}$; ROC : $|z/a| > 1$

$= \frac{z(z - a\cos\Omega_0)}{z^2 - 2za\cos\Omega_0 + a^2}$; ROC : $|z| > |a|$

* Define z-transform. Explain of convergence.

→ Z transform is a mathematical tool use for analysis of discrete time signal & linear time invarient system.

→ ROC :

- In case of z-transform the limits of summation are from $n = -\infty$ to $n = \infty$. So if we will expand this summation. then we will get an infinite power series. This infinite power series will excist only those values of z for which the series attains a finite value.

→ Defination of ROC :

- The ROC is x(z) is set for all the values of z for which x(z) attains a finite value. Every time when we find the z-transform we must indicates its ROC.

→ Significance of ROC :

- ROC will decide whether a system is stable or unstable.
- ROC also determine the type of sequence that means:

1. Causal or non causal
2. finite or infinite

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* State and proove the initial value theorem.

state: If $x(n)$ is a causal sequence then its initial value is given by. $x(0) = \lim_{z \to \infty} x(z)$

proof:- $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

for causal sequence $n=0$ to $n=\infty$

$\therefore x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$\therefore x(z) = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + \cdots$

Apply limit $z \to \infty$

$\therefore \lim_{z \to \infty} x(z) = \lim_{z \to \infty} x(0) + \lim_{z \to \infty} x(1) z^{-1} + \lim_{z \to \infty} x(2) z^{-2} + \cdots$

$\therefore \lim_{z \to \infty} x(z) = \lim_{z \to \infty} x(0) + \lim_{z \to \infty} \frac{x(1)}{z} + \lim_{x \to \infty} \frac{x(2)}{z^2} + \cdots$

$\therefore \lim_{z \to \infty} x(z) = x(0)$

put $x(0)$ is called as initial value of $x(n)$

Initial value $\Rightarrow x(0) = \lim_{z \to \infty} x(z)$.

* Find z-transform of the signal

$x(n) = \left(\frac{-1}{5}\right)^{n} u(n) + 5\left(\frac{1}{2}\right)^{n} u(-n-1)$

$\rightarrow x(n) = \left(-\frac{1}{5}\right)^{n} u(n) + 5\left(\frac{1}{2}\right)^{-n} u(-n-1)$

$= \frac{z}{z+1/5} + 5 \cdot \frac{z/2}{1-z/2}$; ROC : $|z| > 1/5$ & $|z/2| < 1$

$= \frac{z}{z+1/5} + \frac{5z}{2-z}$; ROC : $|z| > \frac{1}{5}$ & $|z| < 2$

; ROC: $\frac{1}{5} < |z| < 2$

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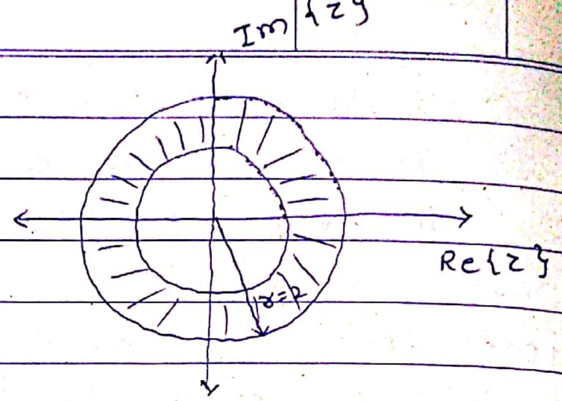
$\left(\because z[u(-n)] = \dfrac{1}{1-z}\right.$

$\left. z[u(-n-1)] = \dfrac{z}{1-z} \right)$



* Determine the z-transform for the following sequence sketch the pole-zero plot & indicate the ROC.

1. $\delta(n+5)$

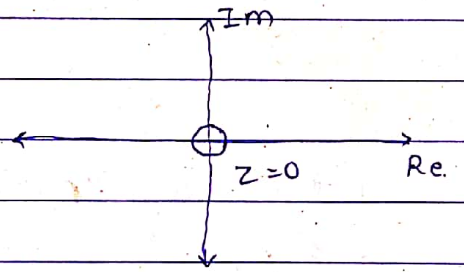
$\delta(n) \xleftrightarrow{z} 1$

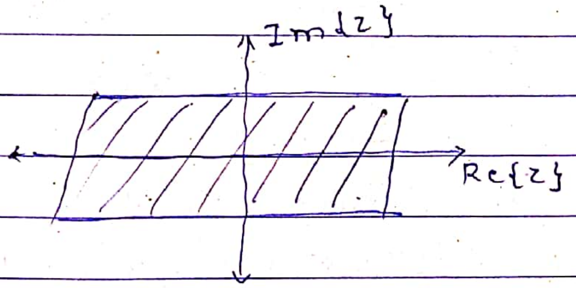
using time shifting property

$x(n+k) \xleftrightarrow{z} z^k x(z)$

$\therefore \delta(n+5) \xleftrightarrow{z} z^5$

entire z plane except $z = \infty$





2. $(1/4)^n u(3-n)$

$u(n) \xleftrightarrow{z} \dfrac{z}{z-1}$; ROC : $|z| > 1$

$u(-n) \xleftrightarrow{z} \dfrac{1}{1-z}$; ROC : $|z| < 1$

$u(3-n) \xleftrightarrow{z} \dfrac{z^3}{1-z}$; ROC : $|z| < 1$

$\left(\dfrac{1}{4}\right)^n u(3-n) \xleftrightarrow{z} \dfrac{(4z)^3}{1-4z}$; ROC : $|z| < 1/4$

* Determine z-transform of following sequences:

1. $x(n) = a^{-|n|}$, $0 < |a| < 1$.

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2. $x(n) = 2^n u(n) + 3^n u(-n-1)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$; ROC : $|z| > 1$

$z^n u(n) \xleftrightarrow{z} \frac{z/2}{z/2 - 1}$; ROC : $\left|\frac{z}{2}\right| > 1$ ($\because$ scalling pro.)

$\therefore z^n u(n) \xleftrightarrow{z} \frac{z}{z-2}$; ROC : $|z| > 2$

$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$; ROC : $|z| < 1$ ($\because$ Reversal pro.)

$u(-n-1) \xleftrightarrow{z} \frac{z}{1-z}$; ROC : $|z| < 1$ ($\because$ shifting pro.)

$3^n u(-n-1) \xleftrightarrow{z} \frac{z}{3-z}$; ROC : $|z| < 3$ ($\because$ scalling)

$\therefore X(z) = \frac{z}{z-2} + \frac{z}{3-z}$; ROC ; $2 < |z| < 3$

* Determine z-transform of following signals.

1. $x(n) = 2^n u(-n)$

$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$; ROC : $|z| < 1$ ($\because$ Reversal pro.)

$2^n u(-n) \xleftrightarrow{z} \frac{1}{1 - z/2}$; ROC : $\left|z/2\right| < 1$ ($\because$ salling pro.)

$2^n u(-n) \xleftrightarrow{z} \frac{z}{2-z}$; ROC : $|z| < 2$

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2. $x(n) = \frac{1}{4^n} u(n-1)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$; ROC : $|z| > 1$

$u(n-1) \xleftrightarrow{z} z^{-1} \times \frac{z}{z-1} = \frac{1}{z-1}$; ROC : $|z|>1$ ($\because$ shifting Pro.)

$\frac{1}{4^n} \cdot u(n-1) \xleftrightarrow{z} \frac{1}{z/\frac{1}{4} - 1}$; ROC : $\left|\frac{z}{1/4}\right| > 1$ ($\because$ Scalling pro.)

$= \frac{1/4}{z - 1/4}$; ROC : $|z| > |1/4|$

$= \frac{1}{4z-1}$; ROC : $|z| > 1/4$

* Find z-transform of signal $x(n) = \delta(n+1) - \delta(n-1)$.

$\delta(n) \xleftrightarrow{z} 1$

$\delta(n+1) \xleftrightarrow{z} z$ ($\because$ using shifting

$\delta(n-1) \xleftrightarrow{z} z^{-1}$ theorem)

$x(n) = \delta(n+1) - \delta(n-1)$

$x(z) = z - 1/z$

ROC : entire z-plane except $z=1$ & $z=0$

* Find the z-transform of the following signals :

a. $x(n) = \delta(n - n_0)$

$\delta(n) \xleftrightarrow{z} 1$

using shifting property,

$\delta(n-n_0) \xleftrightarrow{z} z^{-n_0}$

ROC : entire z plane except $z=0$.

b. $x(n) = a^{n+1} u(n+1)$

$x(n) = a \cdot a^n u(n+1)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$; ROC : $|z|>1$

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$u(n+1) \xleftrightarrow{z} \frac{z^2}{z-1}$; ROC : $|z|>1$ (∵ shifting pro.)

$a^n u(n+1) \xleftrightarrow{z} \frac{(z/a)^2}{z/a - 1}$; ROC : $|z/a|>1$ (∵ Salling pro.)

$= \frac{z^2}{a(z-a)}$; ROC : $|z|>|a|$.

c. $x(n) = a^{-n} u(-n) = \left(\frac{1}{a}\right)^n u(-n)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$; ROC : $|z|>1$

$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$; ROC : $|z|<1$ (∵ Reversal pro.)

$\left(\frac{1}{a}\right)^n u(-n) \xleftrightarrow{z} \frac{1}{1-z/\frac{1}{a}}$; ROC : $\left|\frac{z}{1/a}\right|<1$ (∵ Scalling pro.)

$= \frac{1}{1-a^2}$; ROC : $|z|<1/a$

* Find the z-transform of the signal.

$x(n) = \left\{ n\left(\frac{-1}{2}\right)^n u(n) \right\} * \left\{ \left(\frac{1}{4}\right)^{-n} u(-n) \right\}$

$n a^n u(n) \xleftrightarrow{z} \frac{az}{(z-a)^2}$; ROC : $|z|>|a|$

$z\left\{ n\left(\frac{-1}{2}\right)^n u(n) \right\} = \frac{(-1/2)z}{(z+1/2)^2}$; Roc : $|z|>1/2$

$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$; ROC : $|z|<1$

$\left(\frac{1}{4}\right)^{-n} u(-n) \xleftrightarrow{z} \frac{1}{1-z/4}$; ROC : $|z|<4$

$= \frac{4}{4-z}$; ROC : $|z|<4$

$x(z) = \frac{(-1/2)z}{(z+1/2)^2} \times \frac{4}{4-z}$; ROC : $\frac{1}{2}<|z|<4$

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$$= \frac{-2z}{(4-z)(z+\frac{1}{2})^2} \quad ; ROC : \frac{1}{2} < |z| < 4$$

* Find the inverse z transform of $x(z) = \frac{z}{z-1}$; $|z| > 1$

The long division method.

$$\begin{array}{r|l} & 1 + z^{-1} + z^{-2} + z^{-3} + \dots \\ z-1 & z \\ & \underline{z - 1} \\ & 1 \\ & \underline{1 - z^{-1}} \\ & +z^{-1} \\ & \underline{+z^{-1} - z^{-2}} \\ & z^{-2} \\ & \underline{z^{-2} - z^{-3}} \\ & z^{-3} \end{array}$$

$x(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots$

$\therefore x(n) = \{1, 1, 1, 1, \dots\}$

$x(n) = u(n)$

* Determine inverse z transform of following using partial fraction method. Given that ROC of $x(z)$ includes unit circle. $x(z) = \frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$

$$x(z) = \frac{3z^2}{z^2 - \frac{10}{3}z + 1}$$

$$\frac{x(z)}{z} = \frac{3z}{z^2 - \frac{10}{3}z + 1}$$

$$\frac{x(z)}{z} = \frac{3z}{(z-3)(z-\frac{1}{3})}$$

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$$\frac{x(z)}{z} = \frac{A_1}{z-3} + \frac{A_2}{z-1/3}$$

$$A_1 = \left[(z-3) \times \frac{3z}{(z-3)(z-1/3)}\right]_{z=3} = \frac{9}{3-1/3} = \frac{27}{8}$$

$$A_2 = \left[(z-1/3) \times \frac{3z}{(z-3)(z-1/3)}\right]_{z=1/3} = \frac{1}{1/3-3} = \frac{-3}{8}$$

$$\therefore \frac{x(z)}{z} = \frac{27/8}{z-3} - \frac{3/8}{z-1/3}$$

$$\therefore x(z) = \frac{27}{8} \times \frac{z}{z-3} - \frac{3}{8} \times \frac{z}{z-1/3}$$

- ROC : $|z| < 1$

$$x(n) = -\frac{27}{8}(3)^n u(-n-1) + \frac{3}{8}\left(\frac{1}{3}\right)^n u(-n-1)$$

* Using the partial fraction method, determine the sequence that goes with the following z-transform $x(z) = \dfrac{3}{z - 1/4 - 1/8 z^{-1}}$ and $x(n)$ is absolutely summation.

$$x(z) = \frac{3z}{z^2 - 1/4 z - 1/8}$$

$$\frac{x(z)}{z} = \frac{3}{(z-(1/8 + \sqrt{7}/8 i))(z-(1/8 - \sqrt{7}/8 i))}$$

$$\frac{x(z)}{z} = \frac{3}{(z-P_1)(z-P_2)}$$

$$\frac{x(z)}{z} = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2}$$

$$A_1 = \left[(z-P_1) \times \frac{3}{(z-P_1)(z-P_2)}\right]_{z=P_1} = \frac{3}{P_1-P_2}$$

$$A_2 = \left[(z-P_2) \times \frac{3}{(z-P_1)(z-P_2)}\right]_{z=P_2} = \frac{3}{P_2-P_1}$$

$$\therefore \frac{x(z)}{z} = \frac{3}{P_1 - P_2} \times \frac{1}{z - P_1} - \frac{3}{P_1 - P_2} \times \frac{1}{z - P_2}$$

$$\therefore x(z) = \frac{3}{P_1 - P_2} \frac{z}{z - P_1} - \frac{3}{P_1 - P_2} \frac{z}{z - P_2}$$

$$\therefore x(n) = \frac{3}{P_1 - P_2} P_1^n u(n) - \frac{3}{P_1 - P_2} P_2^n u(n)$$

$$\therefore x(n) = \frac{3(P_1^n - P_2^n)u(n)}{P_1 - P_2}$$

where, $P_1 = \frac{1}{8} + \frac{\sqrt{7}}{8} i$ & $P_2 = \frac{1}{8} - \frac{\sqrt{7}}{8} i$

* Using the partial fraction expansion technique find the inverse z transform of $x(z) = \frac{z}{2z^2 - 3z + 1}$; $|z| < \frac{1}{2}$

$$x(z) = \frac{z}{2z^2 - 3z + 1}$$

$$\frac{x(z)}{z} = \frac{1}{2z^2 - 3z + 1}$$

$$\frac{x(z)}{z} = \frac{1}{(z-1)(z - 1/2)}$$

$$\frac{x(z)}{z} = \frac{A_1}{z - 1} + \frac{A_2}{z - 1/2}$$

$$A_1 = \left[(z-1) \times \frac{1}{(z-1)(z-1/2)}\right]_{z=1} = \frac{1}{1 - 1/2} = 2$$

$$A_2 = \left[(z - \tfrac{1}{2}) \times \frac{1}{(z-1)(z-1/2)}\right]_{z=1/2} = \frac{1}{1/2 - 1} = -2$$

$$\frac{x(z)}{z} = \frac{2}{z-1} - \frac{2}{z - 1/2}$$

$$\therefore x(z) = 2\frac{z}{z-1} - 2\frac{z}{z - 1/2}$$

ROC : $|z| < 1/2$

$$\therefore x(n) = -2(1)^n u(-n-1) + 2\left(\frac{1}{2}\right)^n u(-n-1)$$

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* Justify If $x(z)$ is rotional then ROC of $x(z)$ does not contain any poles.

→ For rotential function ROC is always exterior part & the pole. So ROC does not contain pole.

* Consider a causal and stable LTI system S whose input $x(n)$ and output $y(n)$ are related through the second-order difference equation

$y(n) = \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) = x(n)$.

i) Determine the freq. response $H[e^{j\omega}]$ for system S.

ii) Determine the impulse response $h(n)$ for system S

$$- \quad y(\omega) - \frac{1}{6} e^{-j\omega} y(\omega) - \frac{1}{6} e^{-j2\omega} y(\omega) = x(\omega)$$

$$y(\omega) \left(1 - \frac{1}{6} e^{-j\omega} - \frac{1}{6} e^{-j2\omega}\right) = x(\omega)$$

i) The freq. response is,

$$H(e^{j\omega}) = \frac{Y(\omega)}{x(\omega)}$$

$$\therefore H(e^{j\omega}) = \frac{1}{1 - \frac{1}{6} e^{-j\omega} - \frac{1}{6} j^{-j2\omega}}$$

ii) Converting the eqn. into positive powers,

$$H(e^{j\omega}) = \frac{e^{j2\omega}}{e^{j2\omega} - \frac{1}{6} e^{j\omega} - \frac{1}{6}}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{e^{j2\omega} - \frac{1}{6} e^{j\omega} - \frac{1}{6}}$$

$$= \frac{e^{j\omega}}{(e^{j\omega} - 0.5)(e^{j\omega} + 0.33)}$$

$$= \frac{A_1}{e^{j\omega} - 0.5} + \frac{A_2}{e^{j\omega} + 0.33}$$

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$$A_1 = \frac{0.5}{0.5+0.33} = 0.602$$

$$A_2 = \frac{0.33}{0.33-0.5} = -1.941$$

$$H(e^{j\omega}) = 0.602 \frac{e^{j\omega}}{e^{j\omega}-0.5} - 1.941 \frac{e^{j\omega}}{e^{j\omega}+0.33}$$

$$h(n) = 0.602(0.5)^n u(n) - 1.941(-0.33)^n u(n)$$

* Consider a causal LTI system whose input $x(n)$ and output $y(n)$ are related by the difference equation. $y(n) = \frac{1}{4} y(n-1) + x(n)$. determine $y(n)$ if

$x(n) = \delta(n-1)$

→ $y(n) = \frac{1}{4} y(n-1) + x(n)$

$$y(n) = \frac{1}{4} y(n-1) + \delta(n-1) \qquad (\because x(n) = \delta(n-1))$$

taking z transform both the side.

$$Y(z) = \frac{1}{4} z^{-1} Y(z) + z^{-1}$$

$$Y(z) - \frac{1}{4} z^{-1} Y(z) = z^{-1}$$

$$Y(z)\left(1 - \frac{1}{4} z^{-1}\right) = z^{-1}$$

$$Y(z) = \frac{z^{-1}}{1 - \frac{1}{4} z^{-1}} \qquad Y(z) = z^{-1} \frac{z}{z - \frac{1}{4}}$$

using time shifting property.

$$y(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1)$$

* using power series expontion technique find

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the inverse z-transform of $x(z) = \frac{1}{1-az^{-1}}$; $|z| > |a|$.

$$\begin{array}{r|l} & 1 + a^2z^{-1} + a^2z^{-2} + \dots \\ \hline 1-az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \\ & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\ & \qquad\qquad a^3z^{-3} \end{array}$$

$x(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots$ —— (1)

defination of z transform.

$x(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots$ —— (2)

compare eq^n (1) & (2)

$x(n) = \{1, a, a^2, a^3, \dots\} = a^n u(n)$

* For the differential eq^n. $y(n) - \frac{1}{2}y(n-1) = x(n)$ with input $x(n) = (\frac{1}{3})^n$ and for initially $y(-1) = 1$ find the output $y(n)$.

→ Zero input response is the response of system when input is zero.

$$x(n-k) \xleftrightarrow{z} z^{-k}\left(x(z) + \sum_{n=1}^{k} x(-n)z^n\right)$$

$$y(n) = \frac{1}{2}y(n-1) + x(n)$$

$$\therefore y(z) = \frac{1}{2}z^{-1}[y(z) + y(-1)z^1] + x(z)$$

$$\therefore y(z) = \frac{1}{2}z^{-1}[y(z) + z] + x(z) \quad \text{—— (1)} \quad (\because y(-1) = 1)$$

for $x(n) = 0 \Rightarrow x(z) = 0$

(1) $\Rightarrow y(z) = \frac{1}{2}z^{-1}[y(z) + z]$

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$$y(z) = \frac{1}{2} z^{-1} y(z) + \frac{1}{2}$$

$$y(z) - \frac{1}{2} z^{-1} y(z) = \frac{1}{2}$$

$$y(z)\left(1 - \frac{1}{2} z^{-1}\right) = \frac{1}{2}$$

$$y(z) = \frac{1}{2} \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{2} \frac{z}{z-\frac{1}{2}}$$

$$\therefore y(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

This is zero input response.

- for $x(n) = \left(\frac{1}{3}\right)^n u(n) \Rightarrow x(z) = \frac{z}{z-1/3}$

$$(1) \Rightarrow y(z) = \frac{1}{2} z^{-1} y(z) + \frac{z}{z-1/3}$$

$$y(z) - \frac{1}{2} z^{-1} y(z) = \frac{z}{z-1/3}$$

$$y(z)\left(1 - \frac{1}{2} z^{-1}\right) = \frac{z}{z-1/3}$$

$$y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{z}{z-1/3}$$

$$y(z) = \frac{z}{z-1/2} \times \frac{z}{z-1/3}$$

$$\frac{y(z)}{z} = \frac{z}{(z-1/2)(z-1/3)}$$

$$\frac{y(z)}{z} = \frac{A_1}{z-1/2} + \frac{A_2}{z-1/3}$$

$$A_1 = \left(z-\frac{1}{2}\right)\frac{y(z)}{z}\Bigg|_{z=\frac{1}{2}} = \frac{1/2}{1/2 - 1/3} = \frac{1/2}{1/6} = 3$$

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$$A_2 = \left(z-\frac{1}{3}\right)\frac{y(z)}{z}\Bigg|_{z=1/3} = \frac{1/3}{1/3-1/2} = -2$$

$$\therefore \frac{y(z)}{z} = \frac{3}{z-1/2} - \frac{2}{z-1/3}$$

$$\therefore y(z) = \frac{3z}{z-1/2} - \frac{2z}{z-1/3}$$

$$\therefore y(n) = 3\left(1/2\right)^n u(n) - 2\left(1/3\right)^n u(n)$$

This is zero state response.

$$y(n) = ZIR + ZSR$$

$$y(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

* Find the inverse z-transform of $x(z) = \dfrac{z^3-10z^2-4z+4}{2z^2-2z-4}$ with ROC $|z|<1$.

$$\begin{array}{r|l} & \tfrac{1}{2}z \\ 2z^2-2z-4 & z^3-10z^2-4z+4 \\ & z^3 - z^2 - 2z \\ & -\quad +\quad + \\ \hline & -9z^2-2z+4 \end{array}$$

$$\therefore x(z) = \frac{1}{2}z + \frac{-9z^2-2z+4}{2z^2-2z-4}$$

$$\therefore x_1(z) = \frac{-9z^2-2z+4}{2z^2-2z-4}$$

$$x_1(z) = \frac{-4.5z^2-z+2}{z^2-z-2}$$

$$\frac{x_1(z)}{z} = \frac{-4.5z^2-z+2}{z^2-z-2}$$

$$\frac{x_1(z)}{z} = \frac{-4.5z^2-z+2}{z(z+1)(z-2)}$$

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$$\frac{x_1(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z-2} + \frac{A_3}{z+1}$$

$$A_1 = \left.\frac{-4.5z^2 - z + 2}{(z+1)(z-2)}\right|_{z=0} = \frac{2}{(1)(-2)} = -1$$

$$A_2 = \left.\frac{-4.5z^2 - z + 2}{z(z+1)}\right|_{z=2} = -\frac{9}{3} = -3$$

$$A_3 = \left.\frac{-4.5z^2 - z + 2}{z(z-2)}\right|_{z=-1} = \frac{-4.5+1+2}{(-3)(-1)} = \frac{-1.5}{3} = -0.5$$

$$\frac{x_1(z)}{z} = \frac{-1}{z} - \frac{3}{z-2} - \frac{0.5}{z+1}$$

$$x_1(z) = -1 - \frac{3z}{z-2} - 0.5\frac{z}{z+1}$$

$$x_1(n) = -\delta(n) - 3(2)^n\, u(-n-1) - 0.5(-1)^n\, u(-n-1)$$

$$IZT\left\{\frac{1}{2}z\right\} = \frac{1}{2}\delta(n+1)$$

$$x(n) = \frac{1}{2}\delta(n+1) - \delta(n) - 3(2)^n u(n-1) - (0.5)(-1)^n u(-n-1)$$

* Determine the inverse z-transform of the following $x(z)$ by the partial fraction expansion method.

$$x(z) = \frac{z+1}{2z^2 - 7z + 3}$$

$$x(z) = \frac{0.5z + 0.5}{z^2 - 3.5z + 1.5}$$

$$\frac{x(z)}{z} = \frac{0.5z + 0.5}{z(z-3)(z-0.5)}$$

$$\frac{x(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z-3} + \frac{A_3}{z-0.5}$$

$$A_1 = \left.\frac{0.5z + 0.5}{(z-3)(z-0.5)}\right|_{z=0} = \frac{0.5}{(-3)(-0.5)} = \frac{1}{3} = 0.33$$

$$A_2 = \left.\frac{0.5z + 0.5}{z(z-0.5)}\right|_{z=3} = \frac{1.5+0.5}{3(2.5)} = 0.27$$

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$$A_3 = \frac{0.5z + 0.5}{z(z-3)}\Bigg|_{z=0.5} = \frac{0.5(0.5)+0.5}{0.5(0.5-3)} = -0.6$$

$$\frac{x(z)}{z} = \frac{0.33}{z} + \frac{0.27}{z-3} - \frac{0.6}{z-0.5}$$

$$x(z) = 0.33 + 0.27\frac{z}{z-3} - 0.6\frac{z}{z-0.5}$$

$$x(n) = 0.33\delta(n) + 0.27(3)^n u(n) - 0.6(0.5)^n u(n).$$

* Find the inverse z transform of $x(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)$

$$x(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)$$

$$x(z) = z^2 - \frac{1}{2}z$$

$\delta(n) \xleftrightarrow{z} 1$, $\delta(n+1) \longleftrightarrow z$ & $\delta(n+2) \longleftrightarrow z^2$

$$\therefore x(n) = \delta(n+2) - \frac{1}{2}\delta(n+1)$$

$$\therefore x(n) = \{1, -1/2, 0\}$$

* If signal $x(n)$ has z transform $x(z) = \frac{z^2}{z^2-16}$; $|z|<4$ using properties find z transform of signal $z^{-n}x(n)$.

$$x(z) = \frac{z^2}{z^2-16} \;;\; |z|<4$$

Scalling property, $a^n x(n) \xleftrightarrow{z} x(z/a)$

$$z\left[z^{-n}x(n)\right] = z\left\{\left(\frac{1}{2}\right)^n x(n)\right\}$$

$$= \frac{(2z)^2}{(2z)^2-16}$$

$$= \frac{4z^2}{4z^2-16} \;;\; |z|<8$$

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* $x(n) = a^{-|n|}$, $0 < \alpha < 1$

The given signal is

$x(n) = \alpha |n|$, $0 < |\alpha| < 1$

nature of $x(n)$, let $\alpha = 1/2$

$n = 0 \Rightarrow x(0) = (1/2)^0 = 1$

$n = 1 \Rightarrow x(1) = (1/2)^1 = 1/2$

$n = 2 \Rightarrow x(2) = (1/2)^2 = 1/4$

$n = 3 \Rightarrow x(3) = (1/2)^3 = 1/8$

$x(n)$ for negative values of n.

$n = -1 \rightarrow x(-1) = \alpha^{|-1|} = 1/2$

$n = -2 \rightarrow x(-2) = \alpha^{|-2|} = 1/4$

$n = -3 \rightarrow x(-3) = \alpha^{|-3|} = 1/8$

$x(n) = \alpha^{|n|} = \alpha^n u(n) + \alpha^{-n} u(-n-1)$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(z) = \sum_{n=-\infty}^{\infty} [\alpha^n u(n)] + \alpha^n u(-n-1)] z^{-n}$$

$$x(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{\infty} \alpha^{-n} z^{-n}$$

$x(z) = x_1(z) + x_2(z)$

$$\therefore x_1(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

z transform.

$x_1(z) = \frac{z}{z-\alpha}$ ROC: $|z| > |\alpha|$

$$z_2(z) = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n}$$

$$\therefore x_2(n) = \sum_{n=1}^{\infty} \alpha^1 z^1 - \sum_{z=1}^{\infty} (az)^1$$

$\therefore x_2(z) = (\alpha z) + (\alpha z)^2 + (\alpha z)^3 + (\alpha z)^4 + \dots$

Taking $\alpha(z)$ common.

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$x_2(z) = \alpha z\,[1 + \alpha z + (\alpha z)^2 + \dots]$

$x_2(z) = \alpha z \left(\frac{1}{-\alpha z + 1}\right)$

$\therefore x(z) = \frac{z}{z-a} + \frac{\alpha z}{1-\alpha z}$, ROC : $|z| > |\alpha|$
$|z| < 1/|\alpha|$

* $\left(\frac{1}{4}\right)^n u(3-n)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$

$u(n-3) \xleftrightarrow{z} \frac{z^{-3} \times z}{z-1} = \frac{z^{-2}}{z-1}$

$u[-(n-3)] \xleftrightarrow{z} \frac{(z^{-1})^{-2}}{(z^{-1})-1}$

$= \frac{z^2}{1/z - 1} = z^3/1-z$

$\begin{cases} \frac{1}{3}^n \cos\left(\pi/4\, n\right) & , n \leq 0 \\ 0 & , n > 0 \end{cases}$

* $z[a^n \cos\omega_0 n\, u(n)] = \frac{z(z - a\cos\omega_0)}{z^2 - 2az\cos\omega_0 + a^2}$, $|z| > |a|$

$z\left[\left(\frac{1}{3}\right)^n \cos\left(\frac{n\pi}{4}\right)\right] = \frac{z\left[z - \left(1/3\right)\cos\left(\pi/4\right)\right]}{z^2 - 2\left(\frac{1}{3}\right) z \cos\left(\pi/4\right) + \left(\frac{1}{3}\right)^2}$

$= \frac{z\left[z - 1/3\sqrt{2}\right]}{z^2 - 2/3\left(1/\sqrt{2}\right)z + 1/9}$, ROC : $|z| > 1/3$ ROC , $|z| > 1/3$

$= \frac{z\left(z - 1/3\sqrt{2}\right)}{z^2 - \sqrt{2}/3\, z + 1/9}$

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