SUBJECT CODE: 10EE52 | **SUBJECT: SIGNALS & SYSTEMS** | IA MARKS: 25
EXAM HOURS: 3 | EXAM MARKS: 100
HOURS / WEEK: 4 | TOTAL HOURS: 52

PART – A

UNIT 1:

Introduction: Definitions of a signal and a system, classification of signals, basic Operations on signals, elementary signals, Systems viewed as Interconnections of operations, properties of systems. **07 Hours**

UNIT 2:

Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral. **06 Hours**

UNIT 3:

Time-domain representations for LTI systems – 2: properties of impulse response representation, Differential and difference equation Representations, Block diagram representations. **07 Hours**

UNIT 4:

Fourier representation for signals – 1: Introduction, Discrete time and continuous time Fourier series (derivation of series excluded) and their properties . **06 Hours**

PART – B

UNIT 5:

Fourier representation for signals – 2: Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties. **06 Hours**

UNIT 6:

Applications of Fourier representations: Introduction, Frequency response of LTI systems, Fourier transform representation of periodic signals, Fourier transform representation of discrete time signals. **07 Hours**

UNIT 7:

Z-Transforms – 1: Introduction, Z – transform, properties of ROC, properties of Z – transforms, inversion of Z – transforms. **07 Hours**

UNIT 8:

Z-transforms – 2: Transform analysis of LTI Systems, unilateral Z Transform and its application to solve difference equations. **06 Hours**

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS :

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan**, "Signals and Systems", Scham's outlines, TMH, 2006

3. **B. P. Lathi**, "Linear Systems and Signals", Oxford University Press, 2005

4. **Ganesh Rao and Satish Tunga**, "Signals and Systems", Sanguine Technical Publishers, 2004

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UNIT 1: Introduction **Teaching hours: 7**

1.1	Definitions of a signal	
	Definitions of a system	
1.2	Classification of signals	
1.3	Basic Operations on signals	
1.4	Elementary signals	
1.5	Systems viewed as Interconnections of operations	
1.6	Properties of systems	

1.1.1 Signal definition

A **signal** is a function representing a physical quantity or variable, and typically it contains information about the behaviour or nature of the phenomenon.

For instance, in a RC circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable **'t'**. Usually **'t'** represents time. Thus, a signal is denoted by **x(t)**.

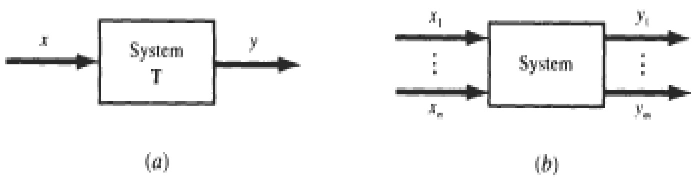
1.1.2 System definition

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of **x** into y. This transformation is represented by the mathematical notation

$$y = Tx \quad \text{--}(1.1)$$

where **T** is the operator representing some well-defined rule by which **x** is transformed into y. Relationship (1.1) is depicted as shown in Fig. 1-1(a). Multiple input and/or output signals are possible as shown in Fig. 1-1(b). We will restrict our attention for the most part in this text to the single-input, single-output case.



1.1 System with single or multiple input and output signals

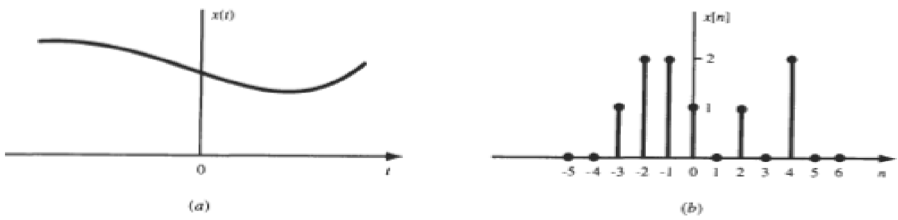
1.2 Classification of signals

Basically seven different classifications are there:

- Continuous-Time and Discrete-Time Signals
- Analog and Digital Signals
- Real and Complex Signals
- Deterministic and Random Signals
- Even and Odd Signals
- Periodic and Nonperiodic Signals
- Energy and Power Signals

Continuous-Time and Discrete-Time Signals

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by {x,) or x[n], where n = integer. Illustrations of a continuous-time signal x(t) and of a discrete-time signal x[n] are shown in Fig. 1-2.



1.2 Graphical representation of (a) continuous-time and (b) discrete-time signals

Analog and Digital Signals

If a continuous-time signal x(t) can take on any value in the continuous interval (a, b), where a may be - ∞ and b may be +∞ then the continuous-time signal x(t) is called an analog signal. If a discrete-time signal x[n] can take on only a finite number of distinct values, then we call this signal a digital signal.

Real and Complex Signals

A signal x(t) is a real signal if its value is a real number, and a signal x(t) is a complex signal if its value is a complex number. A general complex signal x(t) is a function of the form

$$x(t) = x_1(t) + jx_2(t) \text{-------------------------------} 1.2$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$

Note that in Eq. (1.2)**'t'** represents either a continuous or a discrete variable.

Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modelled by a known function of time **'t'.**

Random signals are those signals that take random values at any given time and must be characterized statistically.

Even and Odd Signals

A signal ***x (t)*** or ***x[n]*** is referred to as an ***even*** signal if

$$x(-t) = x(t)$$

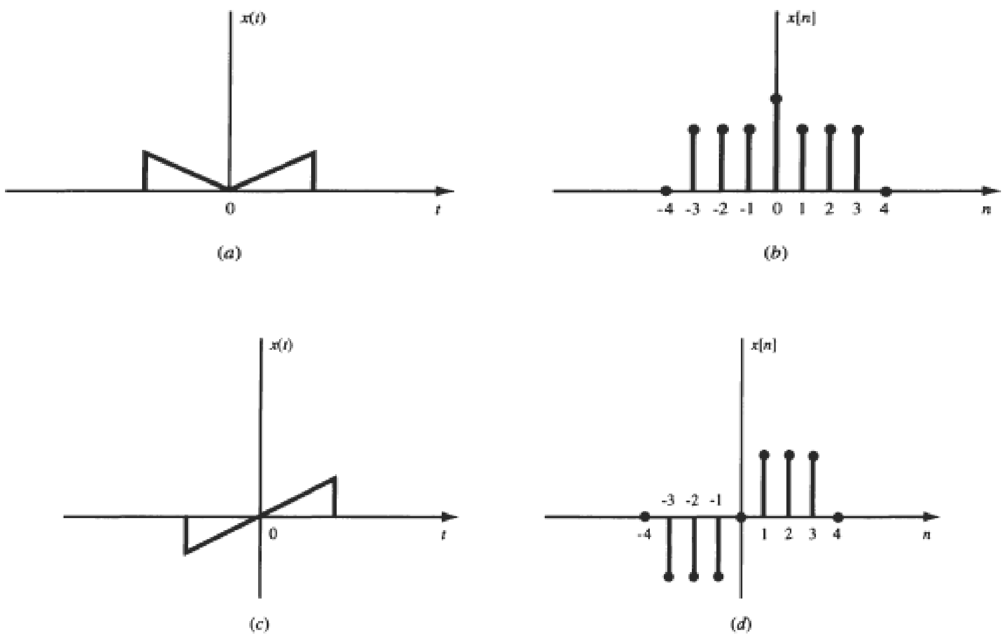
$$x[-n] = x[n] \quad \text{-------------(1.3)}$$

A signal ***x (t)*** or ***x[n]*** is referred to as an ***odd*** signal if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n] \quad \text{--------------(1.4)}$$

Examples of even and odd signals are shown in Fig. 1.3.



1.3 Examples of even signals (a and b) and odd signals (c and d).

Any signal x(t) or x[n] can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_o(t) + x_e(t) \quad \text{-------(1.5)}$$

Where,

$$x_e(t) = \frac{1}{2}(x(t) + x(-t))$$

$$x_o(t) = \frac{1}{2}(x(t) - x(-t)) \quad \text{-----(1.6)}$$

Similarly for x[n],

$$x[n] = x_o[n] + x_e[n] \quad \text{-------(1.7)}$$

Where,

$$x_e[n] = \frac{1}{2}(x[n] + x[-n])$$

$$x_o[n] = \frac{1}{2}(x[n] - x[-n])$$ --------(1.8)

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

Periodic and Nonperiodic Signals

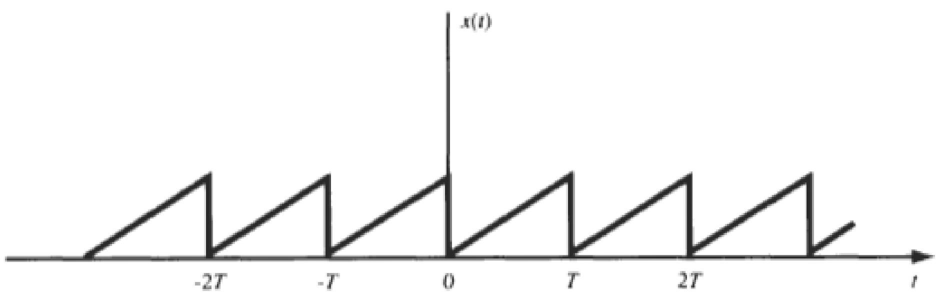
A continuous-time signal x (t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t$$ ………..(1.9)

An example of such a signal is given in Fig. 1-4(a). From Eq. (1.9) or Fig. 1-4(a) it follows that

$$x(t+mT) = x(t)$$ --------------------------(1.10)

for all t and any integer m. The fundamental period T, of x(t) is the smallest positive value of T for which Eq. (1.9) holds. Note that this definition does not work for a constant.



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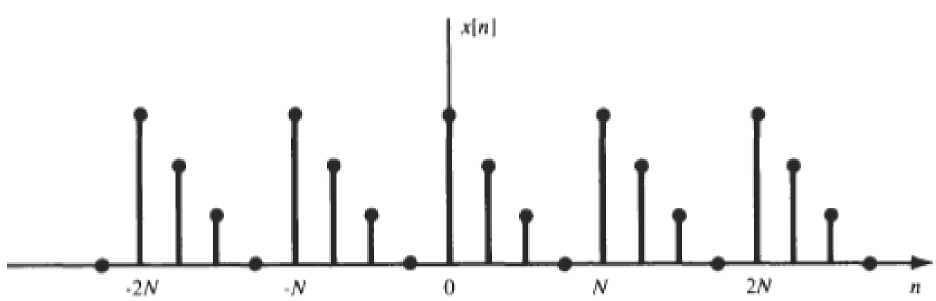


(b)

1.4 Examples of periodic signals.

signal x(t) (known as a dc signal). For a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n \qquad \text{……….(1.11)}$$

An example of such a sequence is given in Fig. 1-4(b). From Eq. (1.11) and Fig. 1-4(b) it follows that

$$x[n+mN] = x[n] \qquad \text{……………………..(1.12)}$$

for all n and any integer m. The fundamental period N_o of x[n] is the smallest positive integer N for which Eq.(1.11) holds. Any sequence which is not periodic is called a nonperiodic (or aperiodic sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic. Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic

Energy and Power Signals

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t) \qquad \text{…………(1.13)}$$

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts} \qquad \text{……(1.14)}$$

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt \qquad \text{…………………(1.15)}$$

The normalized average power P of x(t) is defined as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt \qquad \text{……….. (1.16)}$$

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad \text{………………………….(1.17)}$$

The normalized average power P of x[n] is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \qquad (1.18)$$

Based on definitions (1.15) to (1.18), the following classes of signals are defined:

1. x(t) (or x[n]) is said to be an energy signal (or sequence) if and only if 0 < E < m, and so P = 0.
2. x(t) (or x[n]) is said to be a power signal (or sequence) if and only if 0 < P < m, thus implying that E = m.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period.

1.3 <u>Basic Operations on signals</u>

The operations performed on signals can be broadly classified into two kinds

- Operations on dependent variables
- Operations on independent variables

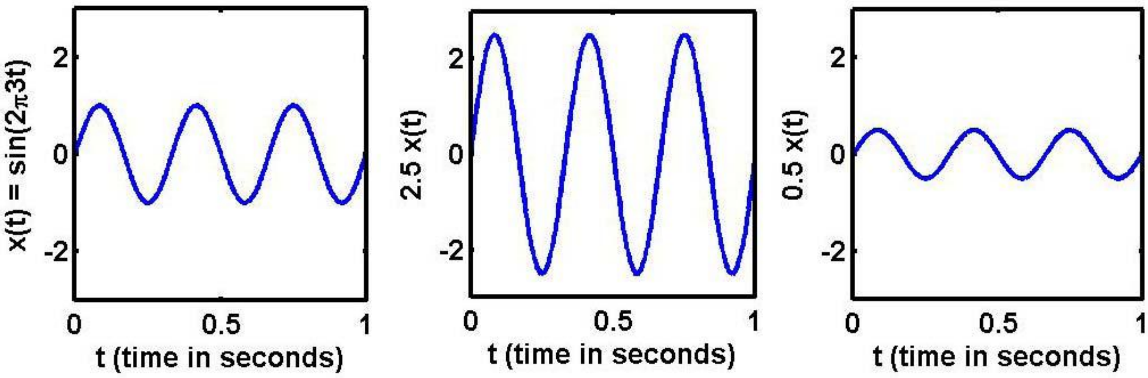
Operations on dependent variables

The operations of the dependent variable can be classified into five types: amplitude scaling, addition, multiplication, integration and differentiation.

Amplitude scaling

Amplitude scaling of a signal $x(t)$ given by equation 1.19, results in amplification of $x(t)$ if $a > 1$, and attenuation if $a < 1$.

$$y(t) = ax(t) \text{……..(1.20)}$$

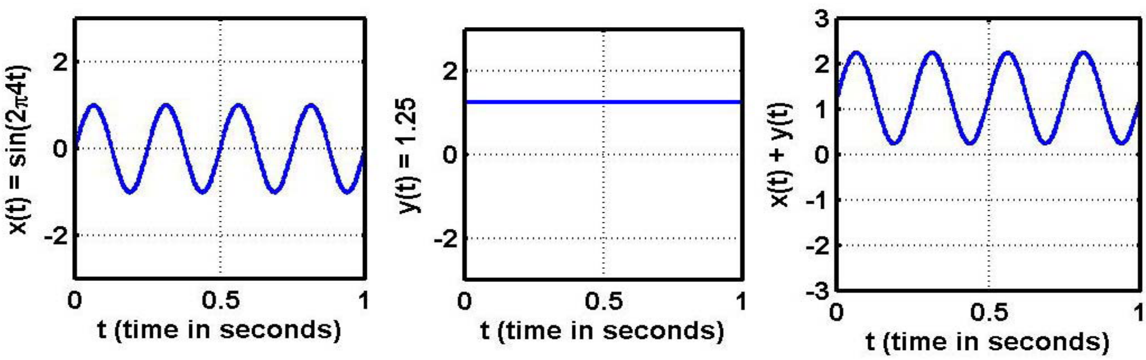


1.5 Amplitude scaling of sinusoidal signal.

Addition

The addition of signals is given by equation of 1.21.

$$y(t) = x_1(t) + x_2(t) \ldots\ldots(1.21)$$



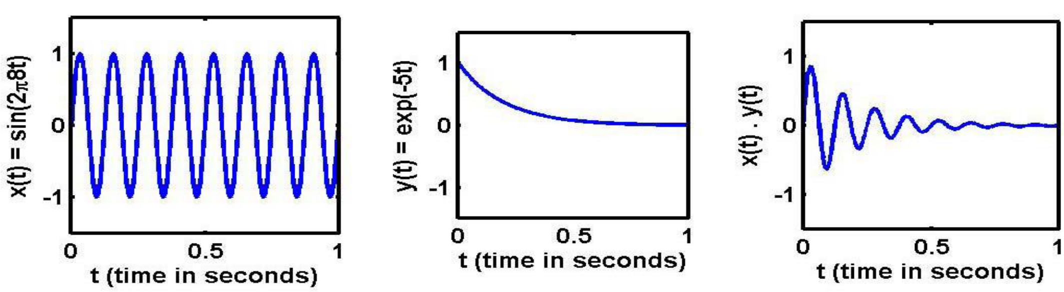
1.6 Example of the addition of a sinusoidal signal with a signal of constant amplitude (positive constant)

Physical significance of this operation is to add two signals like in the addition of the background music along with the human audio. Another example is the undesired addition of noise along with the desired audio signals.

Multiplication

The multiplication of signals is given by the simple equation of 1.22.

$y(t) = x_1(t).x_2(t) \ldots\ldots..(1.22)$



1.7 Example of multiplication of two signals

Differentiation

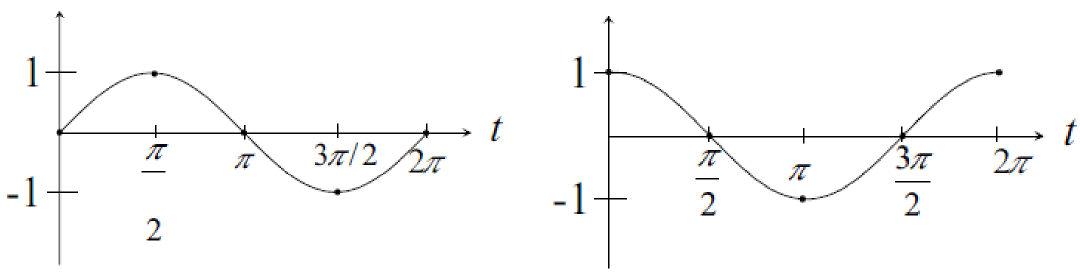
The differentiation of signals is given by the equation of 1.23 for the continuous.

$$y(t) = \frac{d}{dt}x(t) \quad .....1.23$$

The operation of differentiation gives the rate at which the signal changes with respect to time, and can be computed using the following equation, with Δt being a small interval of time.

$$\frac{d}{dt}x(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad ....1.24$$

If a signal doesn't change with time, its derivative is zero, and if it changes at a fixed rate with time, its derivative is constant. This is evident by the example given in figure 1.8.

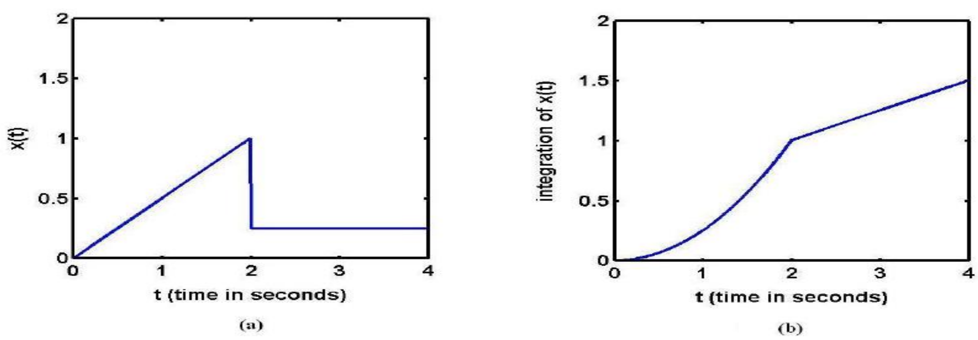


1.8 Differentiation of Sine - Cosine

Integration

The integration of a signal $x(t)$, is given by equation 1.25

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad ......1.25$$



1.9 Integration of x(t)

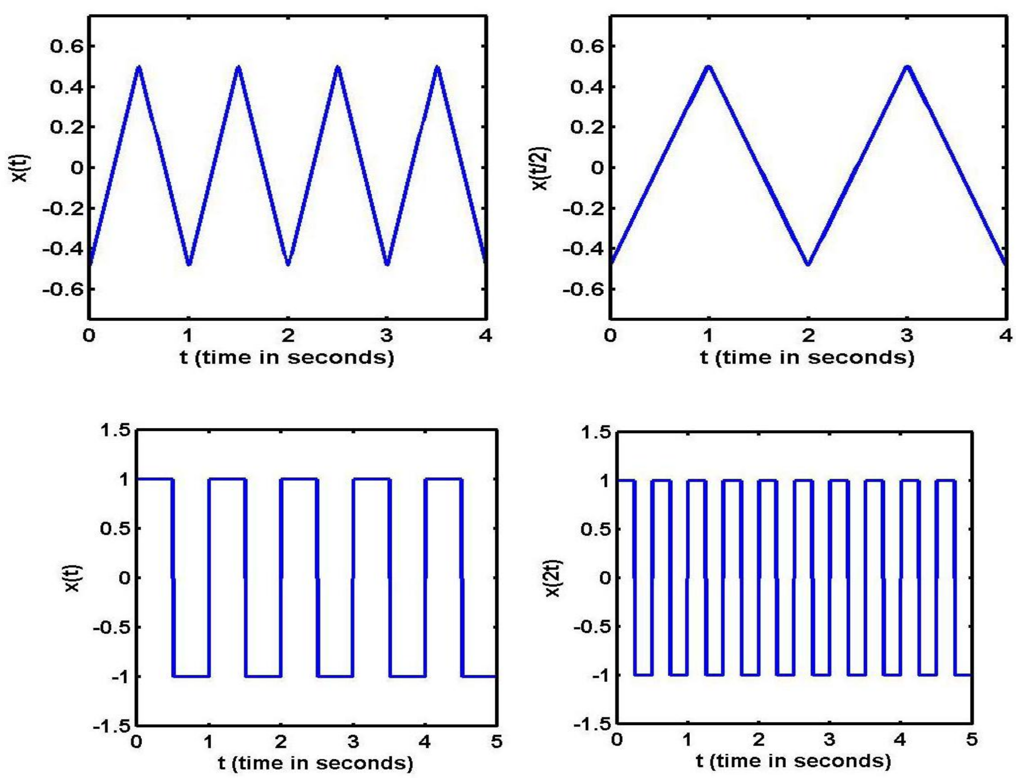
Operations on independent variables

Time scaling

Time scaling operation is given by equation 1.26

$$y(t) = x(at) \ldots\ldots\ldots\ldots\ldots 1.26$$

This operation results in expansion in time for a<1 and compression in time for a>1, as evident from the examples of figure 1.10.



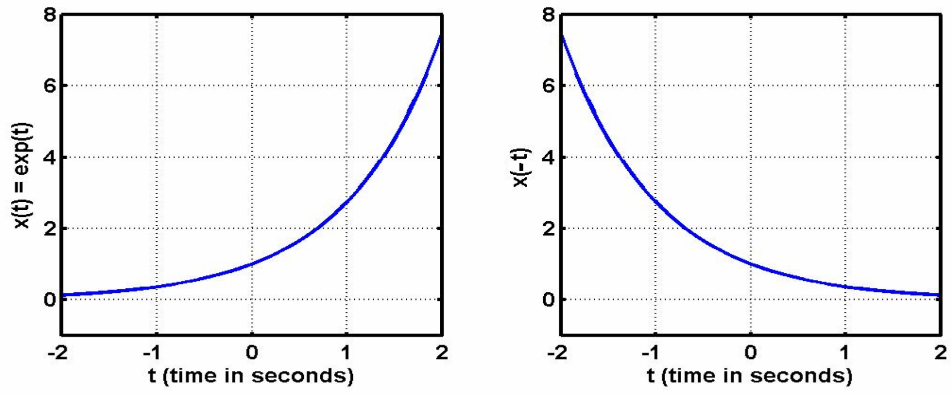
1.10 Examples of time scaling of a continuous time signal

An example of this operation is the compression or expansion of the time scale that results in the '*fast-forward*' or the '*slow motion*' in a video, provided we have the entire video in some stored form.

Time reflection

Time reflection is given by equation (1.27), and some examples are contained in fig1.11.

$$y(t) = x(-t) \ldots\ldots\ldots\ldots 1.27$$



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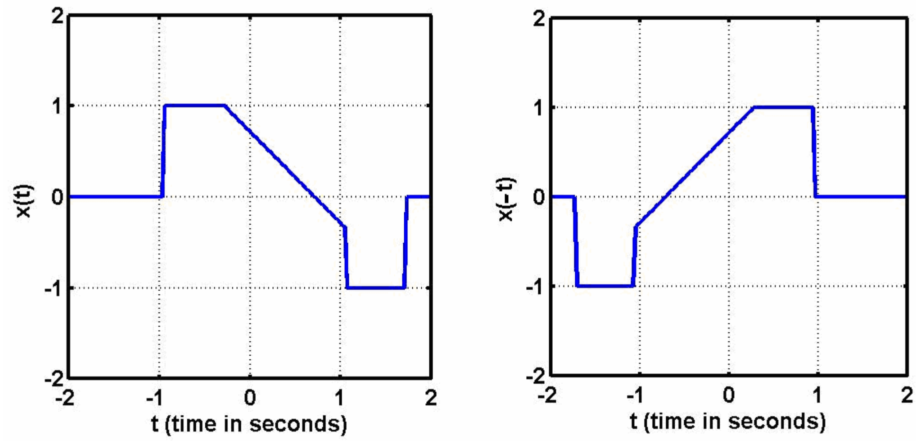

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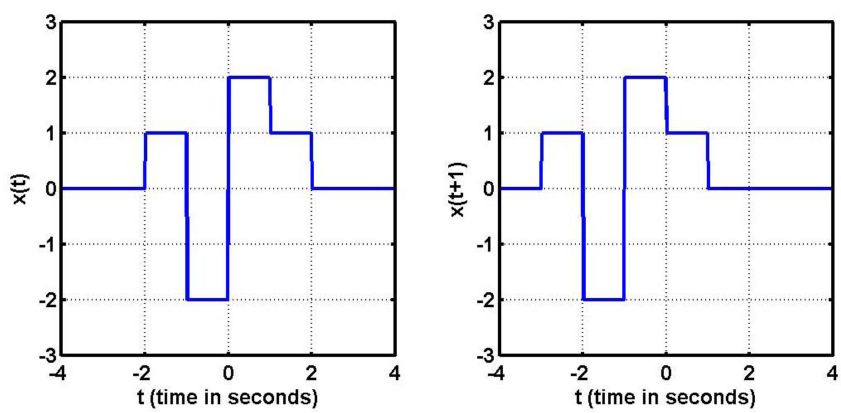
1.11 Examples of time reflection of a continuous time signal

Time shifting

The equation representing time shifting is given by equation (1.28), and examples of this operation are given in figure 1.12.

$$y(t) = x(t - t_0)\ldots\ldots\ldots\ldots..1.28$$



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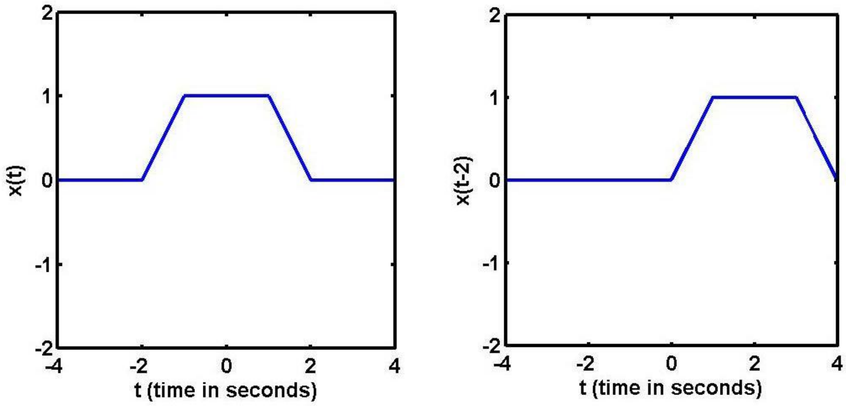

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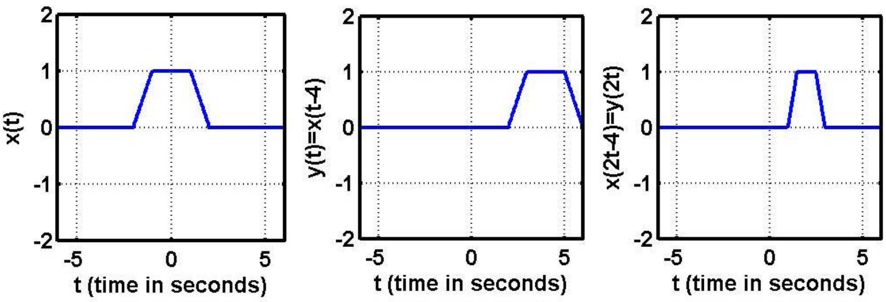
1.12 Examples of time shift of a continuous time signal

Time shifting and scaling

The combined transformation of shifting and scaling is contained in equation (1.29), along with examples in figure 1.13. Here, time shift has a higher precedence than time scale.

$$y(t) = x(at - t_0) \text{ ……………..1.29}$$



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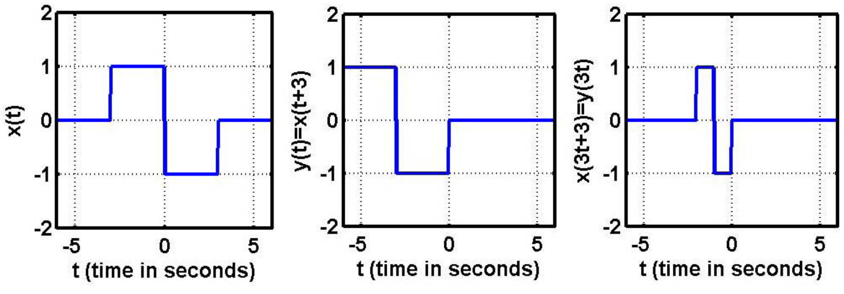


(b)

1.13 Examples of simultaneous time shifting and scaling. The signal has to be shifted first and then time scaled.

1.4 Elementary signals

Exponential signals:

The exponential signal given by equation (1.29), is a monotonically increasing function if $a > 0$, and is a decreasing function if $a < 0$.

$$x(t) = e^{at} \quad \text{.........................(1.29)}$$

It can be seen that, for an exponential signal,

$$x(t + a^{-1}) = e.x(t)$$

$$x(t - a^{-1}) = e^{-1}.x(t) \quad \text{.......................(1.30)}$$

Hence, equation (1.30), shows that change in time by $\pm 1/a$ seconds, results in change in magnitude by $e^{\pm 1}$. The term $1/a$ having units of time, is known as the time-constant. Let us consider a decaying exponential signal

$$x(t) = e^{-at} \quad \text{for } t \geq 0. \quad \text{...............(1.31)}$$

This signal has an initial value $x(0) = 1$, and a final value $x(\infty) = 0$. The magnitude of this signal at five times the time constant is,

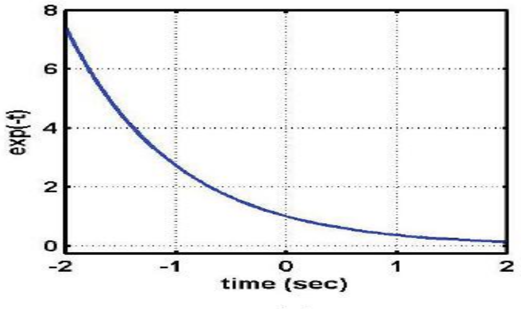
$$x(5/a) = 6.7x10^{-3} \quad \text{......................(1.32)}$$

while at ten times the time constant, it is as low as,

$$x(10/a) = 4.5x10^{-5} \quad \text{...............(1.33)}$$

-

It can be seen that the value at ten times the time constant is almost zero, the final value of the signal. Hence, in most engineering applications, the exponential signal can be said to have reached its final value in about ten times the time constant. If the time constant is 1 second, then final value is achieved in 10 seconds!! We have some examples of the exponential signal in figure 1.14.



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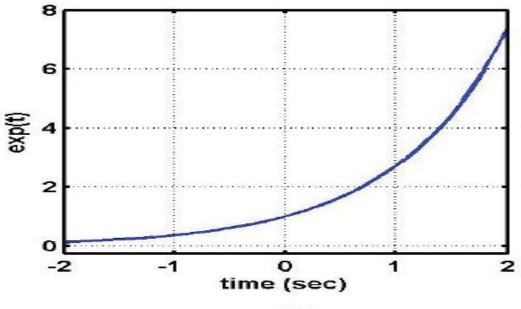

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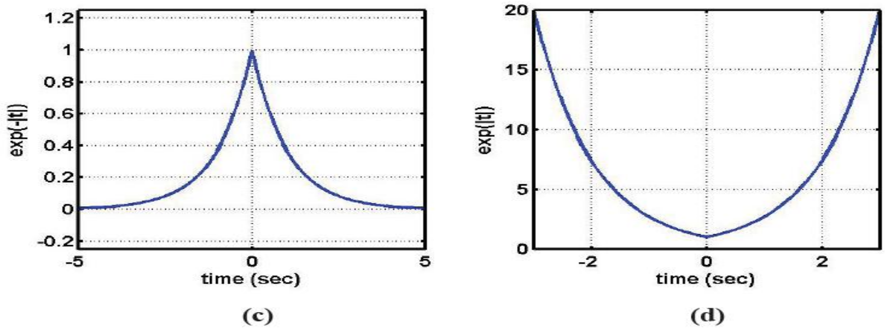


Fig 1.14 The continuous time exponential signal (a) e^{-t}, (b) e^{t}, (c) $e^{-|t|}$, and (d) $e^{|t|}$

The sinusoidal signal:

The sinusoidal continuous time periodic signal is given by equation 1.34, and examples are given in figure 1.15

$$x(t) = A\sin(2\pi ft) \text{ ....(1.34)}$$

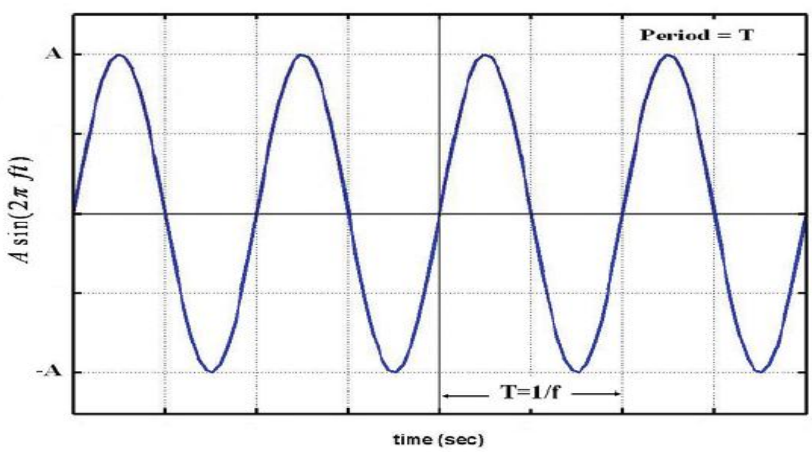
The different parameters are:

Angular frequency $\omega = 2\pi f$ in radians,

Frequency f in Hertz, (cycles per second)

Amplitude A in Volts (or Amperes)

Period T in seconds



The complex exponential:

We now represent the complex exponential using the Euler's identity (equation (1.35)),

$$e^{j\theta} = \left(\cos\theta + j\sin\theta\right) \text{ ....(1.35)}$$

to represent sinusoidal signals. We have the complex exponential signal given by equation (1.36)

$$e^{j\omega t} = (\cos(\omega t) + j\sin(\omega t))$$
$$e^{-j\omega t} = (\cos(\omega t) - j\sin(\omega t)) \quad \text{………(1.36)}$$

Since sine and cosine signals are periodic, the complex exponential is also periodic with the same period as sine or cosine. From equation (1.36), we can see that the real periodic sinusoidal signals can be expressed as:

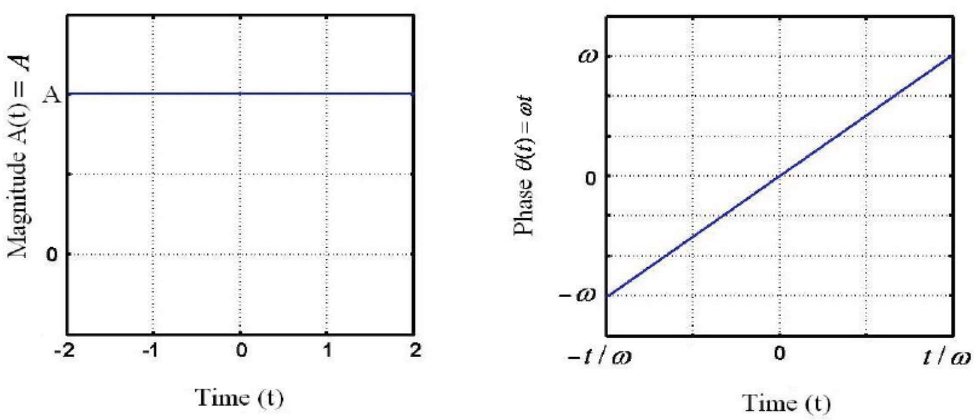
$$\cos(\omega t) = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)$$

$$\sin(\omega t) = \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) \quad \text{………………..(1.37)}$$

Let us consider the signal $x(t)$ given by equation (1.38). The sketch of this is given in fig 1.15

$$x(t) = A(t)e^{j\theta(t)} \quad \text{……………………..(1.38)}$$

$x(t) = Ae^{j\omega t}$

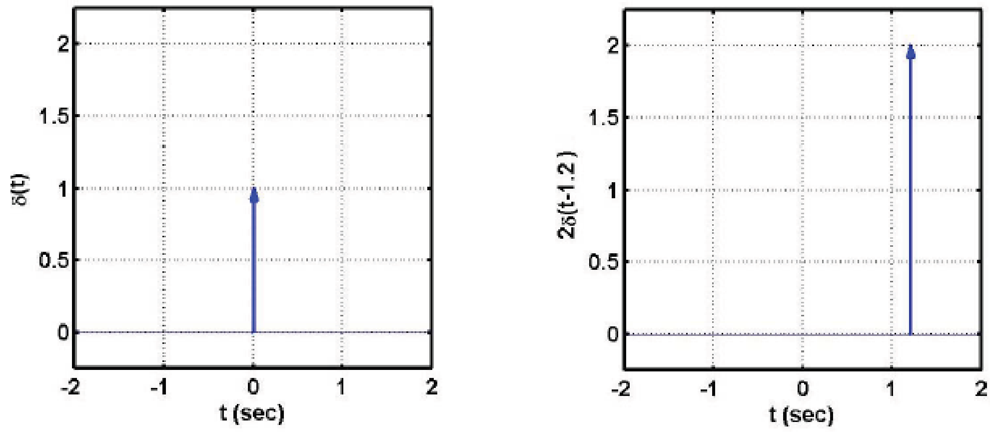


The unit impulse:

The unit impulse usually represented as $\delta(t)$, also known as the dirac delta function, is given by,

$$\delta(t) = 0 \quad for \quad t \neq 0; \quad and \quad \int_{-\infty}^{\infty} \delta(t)dt = 1 \quad \text{…….(1.38)}$$

From equation (1.38), it can be seen that the impulse exists only at $t = 0$, such that its area is 1. This is a function which cannot be practically generated. Figure 1.16, has the plot of the impulse function

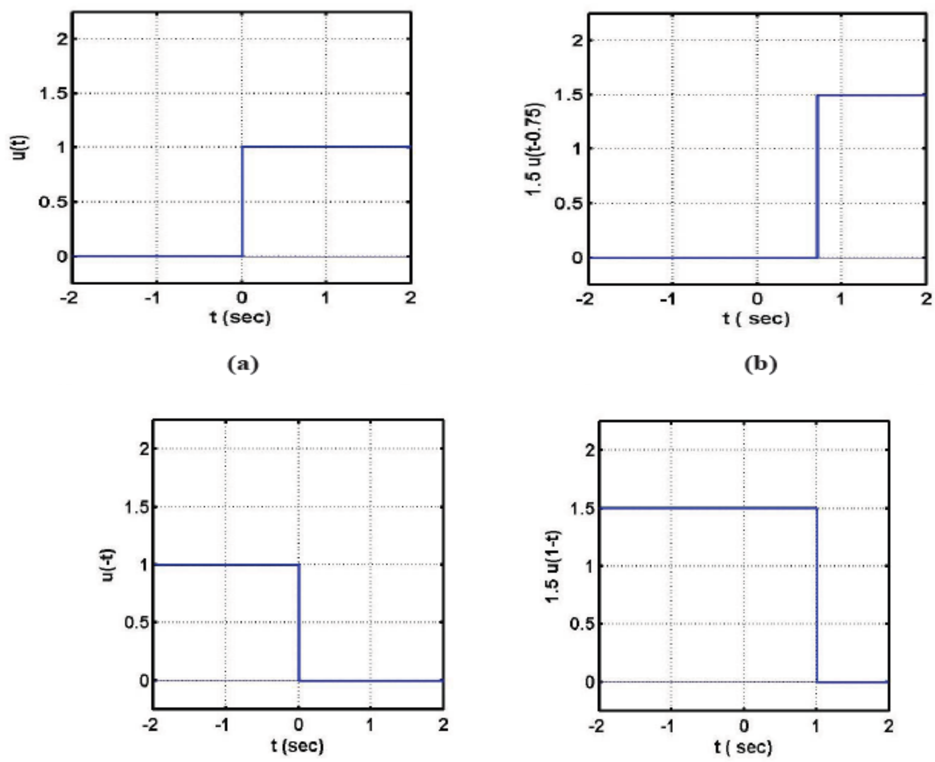


The unit step

The unit step function, usually represented as $u(t)$, is given by,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

……………….(1.39)



(a) (b) (c) (d)

Fig 1.17 Plot of the unit step function along with a few of its transformations

The unit ramp:

The unit ramp function, usually represented as $r(t)$, is given by,

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

…………….(1.40)

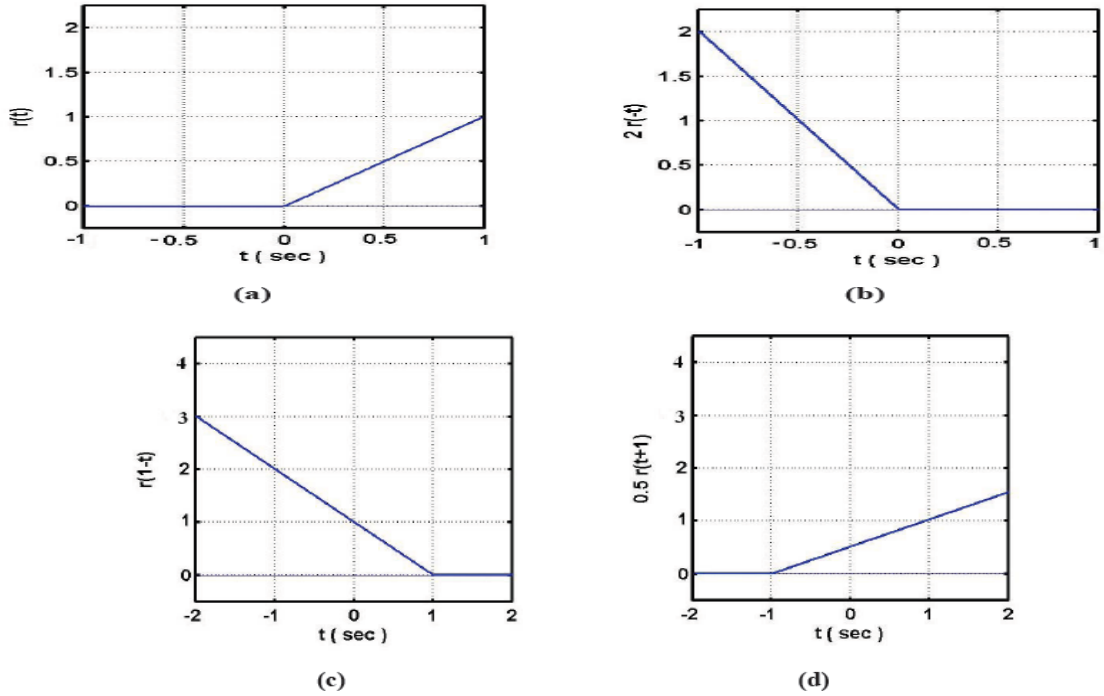


Fig 1.18 Plot of the unit ramp function along with a few of its transformations

The signum function:

The signum function, usually represented as sgn(t) , is given by

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \qquad \textbf{.............................(1.41)}$$

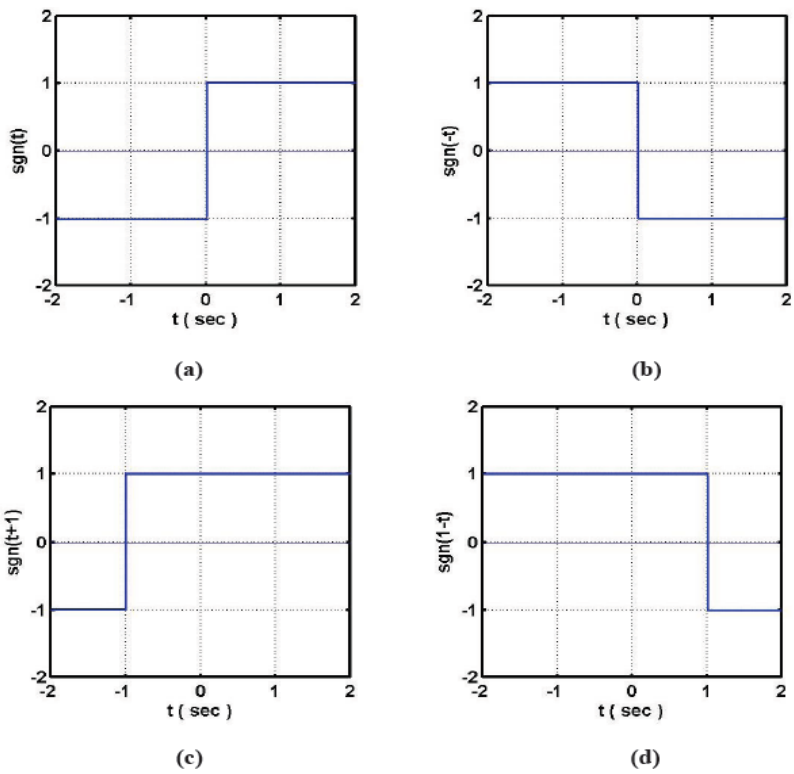


Fig 1.19 Plot of the unit signum function along with a few of its transformations

1.5 System viewed as interconnection of operation:

This article is dealt in detail again in chapter 2/3. This article basically deals with system connected in series or parallel. Further these systems are connected with adders/subtractor, multipliers etc.

1.6 Properties of system:

The discrete time system:

The discrete time system is a device which accepts a discrete time signal as its input, transforms it to another desirable discrete time signal at its output as shown in figure 1.20

Fig 1.2 DT system

Stability

A system is stable if 'bounded input results in a bounded output'. This condition, denoted by BIBO, can be represented by:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad implies \quad \sum_{n=-\infty}^{\infty} |y[n]| < \infty \quad for \quad all \quad n \qquad \text{.......(1.42)}$$

Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

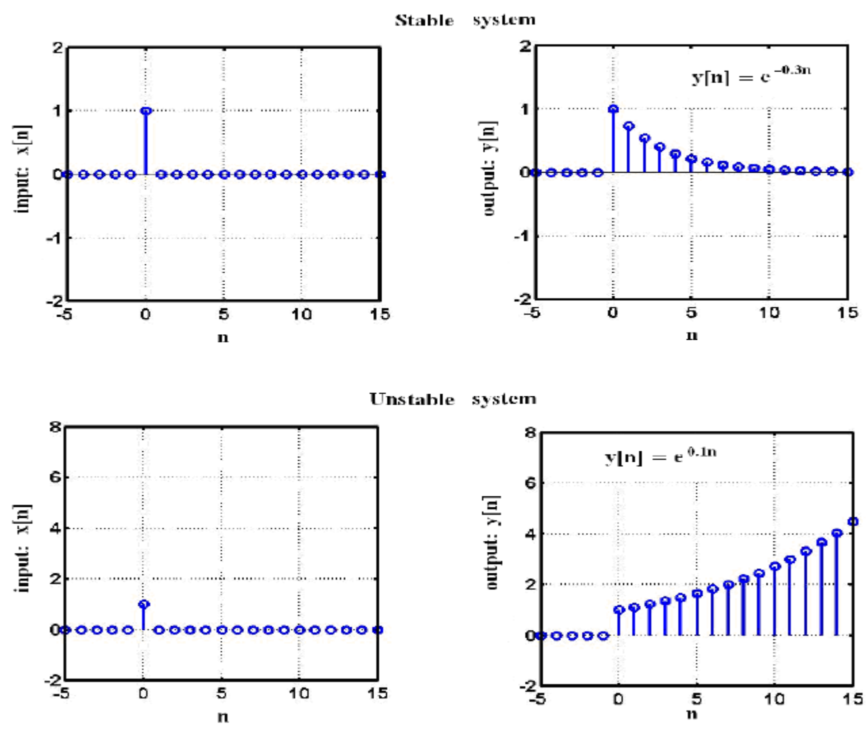


Fig 1.21 Examples for system stability

Memory

The system is memory-less if its instantaneous output depends only on the current input.

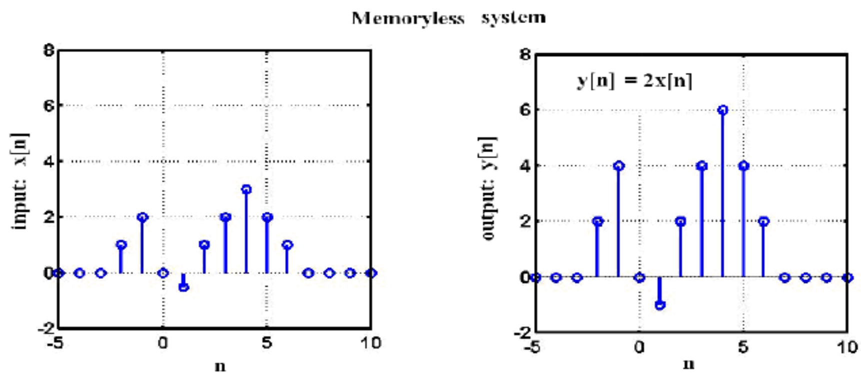
In memory-less systems, the output does not depend on the previous or the future input.

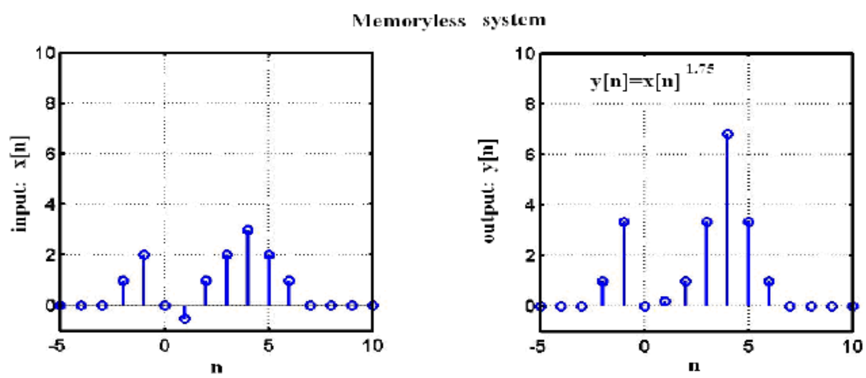
Examples of memory less systems:

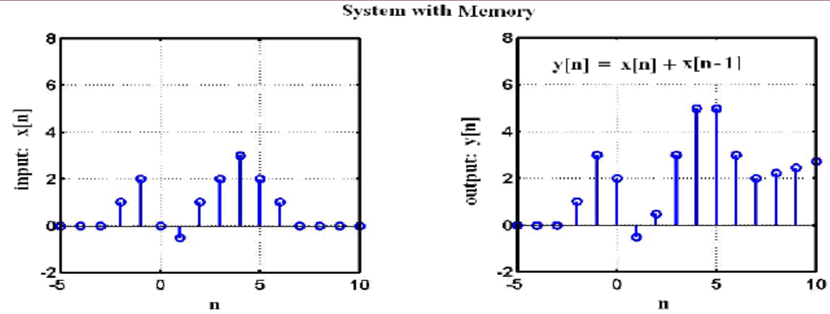
$$y[n] = ax[n]$$

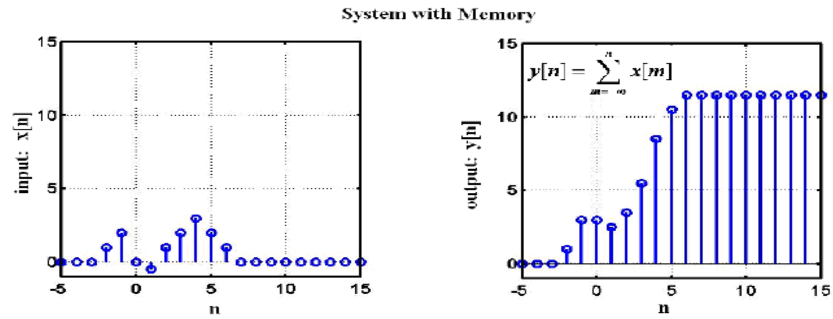
$$y[n] = ax^2[n]$$

$$i[n] = a_0 + a_1 v[n] + a_2 v^2[n] + a_3 v^3[n] + ...$$









Causality:

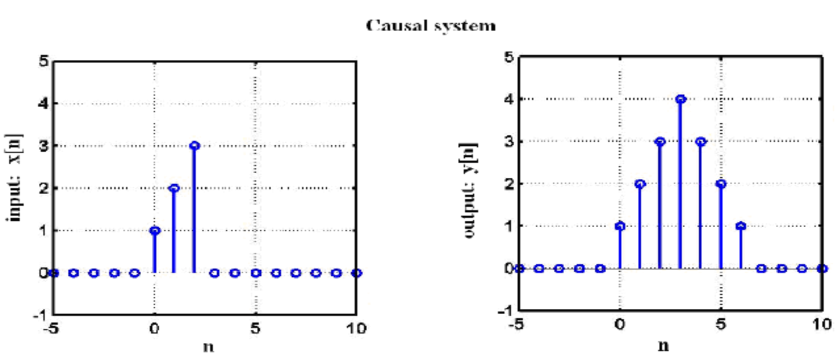
A system is causal, if its output at any instant depends on the current and past values of input. The output of a causal system does not depend on the future values of input. This can be represented as:

$$y[n] = F\left(x[m]\right) \; for \; m \leq n$$

For a causal system, the output should occur only after the input is applied, hence,

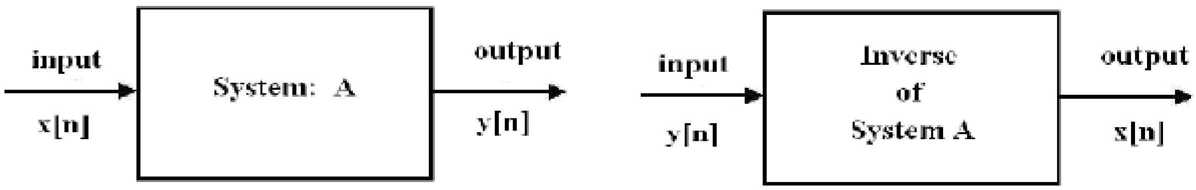
$x[n] = 0 \; for \; n < 0$ implies $y[n] = 0 \; for \; n < 0$

All physical systems are causal (examples in figure 7.5). Non-causal systems do not exist. This classification of a system may seem redundant. But, it is not so. This is because, sometimes, it may be necessary to design systems for given specifications. When a system design problem is attempted, it becomes necessary to test the causality of the system, which if not satisfied, cannot be realized by any means. **Hypothetical examples** of non-causal systems are given in figure below.



Invertibility:

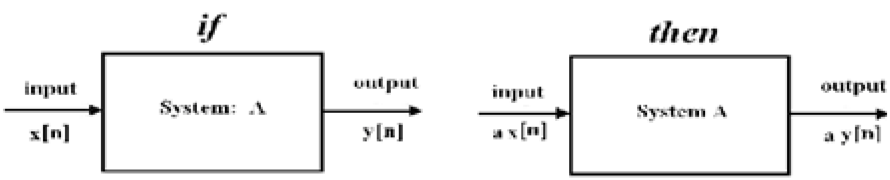
A system is invertible if,



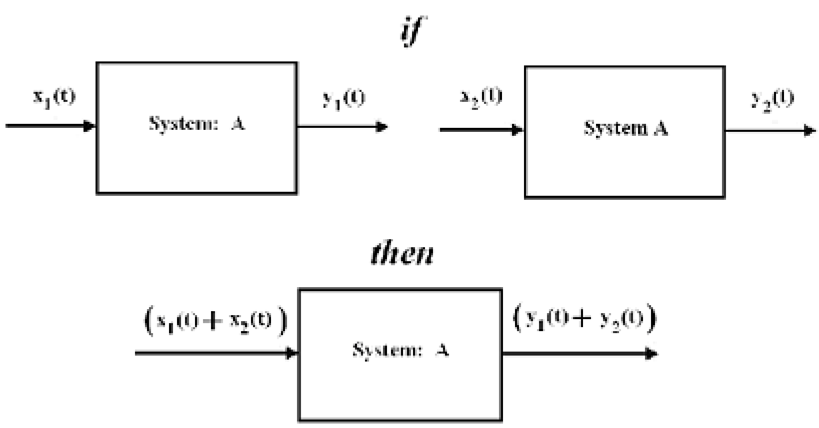
Linearity:

The system is a device which accepts a signal, transforms it to another desirable signal, and is available at its output. We give the signal to the system, because the output is s

Amplitude scaling



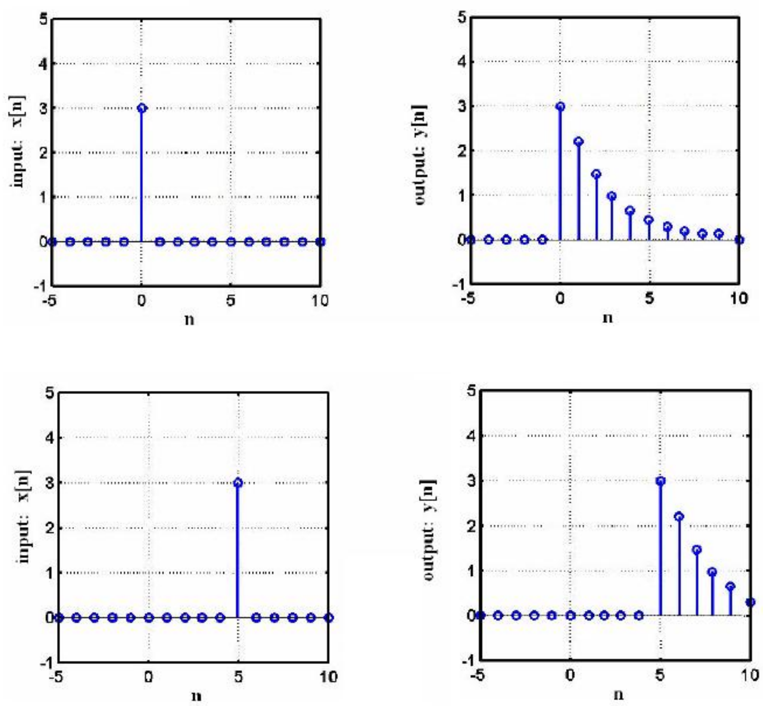
Superposition principle



Time invariance:

A system is time invariant, if its output depends on the input applied, and not on the time of application of the input. Hence, time invariant systems, give delayed outputs for delayed inputs.

Given input-output relation of Time invariant system



UNIT 2: Time-domain representations for LTI systems – 1 **Teaching hours: 6**

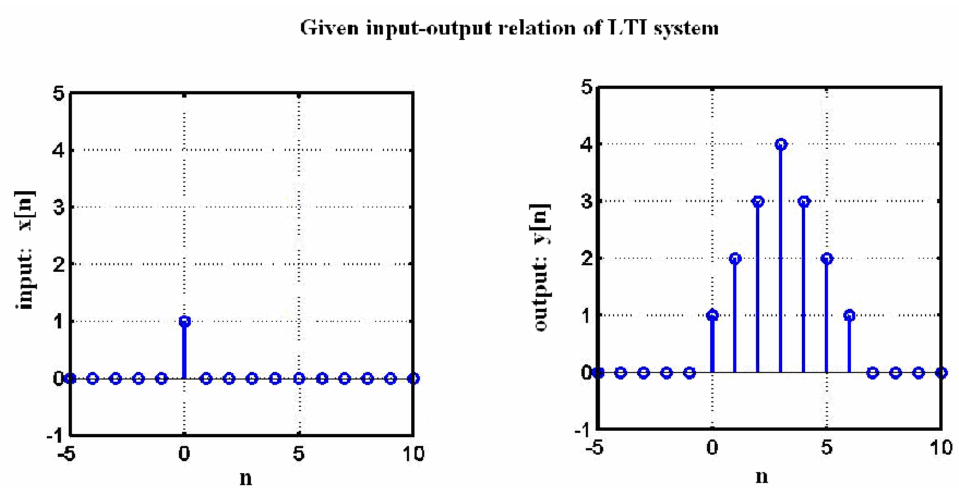
2.1	Convolution	
2.2	Impulse response representation	
2.3	Convolution Sum	
2.4	Convolution Integral	

Introduction:

The Linear time invariant (LTI) system:

Systems which satisfy the condition of linearity as well as time invariance are known as linear time invariant systems. Throughout the rest of the course we shall be dealing with LTI systems. If the output of the system is known for a particular input, it is possible to obtain the output for a number of other inputs. We shall see through examples, the procedure to compute the output from a given input-output relation, for LTI systems.

Example – I:



2.1 Convolution:

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).

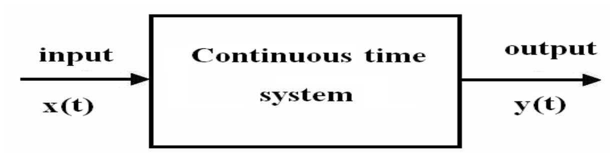


Figure 1: The continuous time system

Some of the different methods of representing the continuous time system are:

i) Differential equation

ii) Block diagram

iii) Impulse response

iv) Frequency response

v) Laplace-transform

vi) Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

2.2 <u>Impulse Response</u>

The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, $\delta(t)$. Usually the impulse response is denoted by $h(t)$.

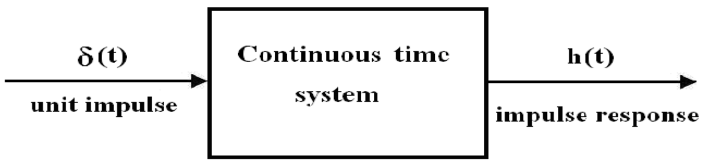
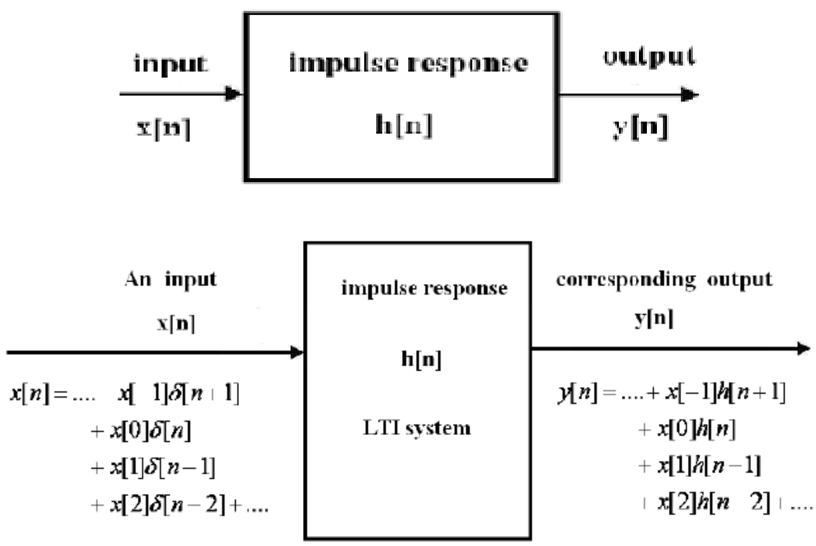
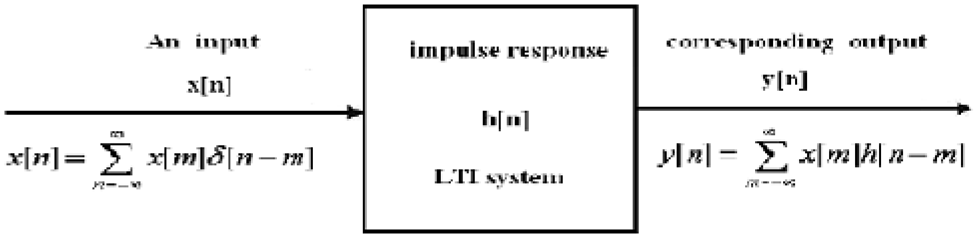


Figure 2: The impulse response of a continuous time system

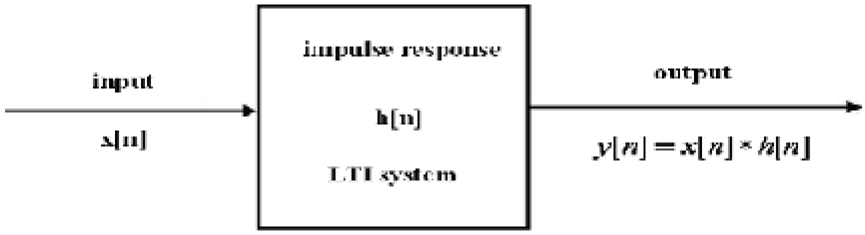
2.3 <u>Convolution Sum:</u>

We now attempt to obtain the output of a digital system for an arbitrary input x[n], from the knowledge of the system impulse response h[n].





time-domain analysis



$$y[n] = x[n] * h[n]$$

Methods of evaluating the convolution sum:

Given the system impulse response h[n], and the input x[n], the system output y[n], is given by the convolution sum:

$$y[n]=\sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Problem:

To obtain the digital system output y[n], given the system impulse response h[n], and the system input x[n] as:

h[n]=[1, -1.5, 3]

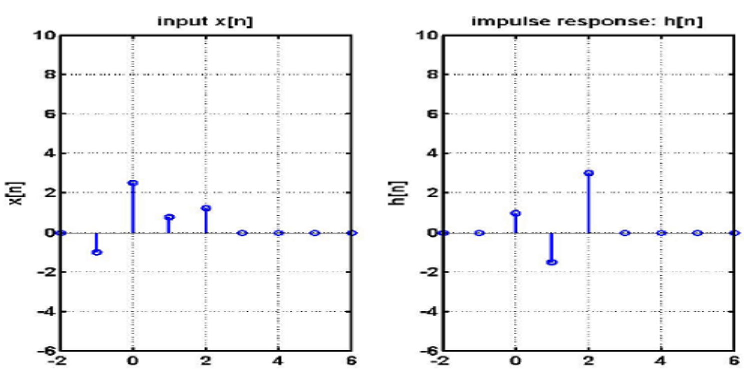
$$x[n]=[-1\ ,\ \underset{\uparrow}{2.5},\ 0.8,\ 1.25]$$

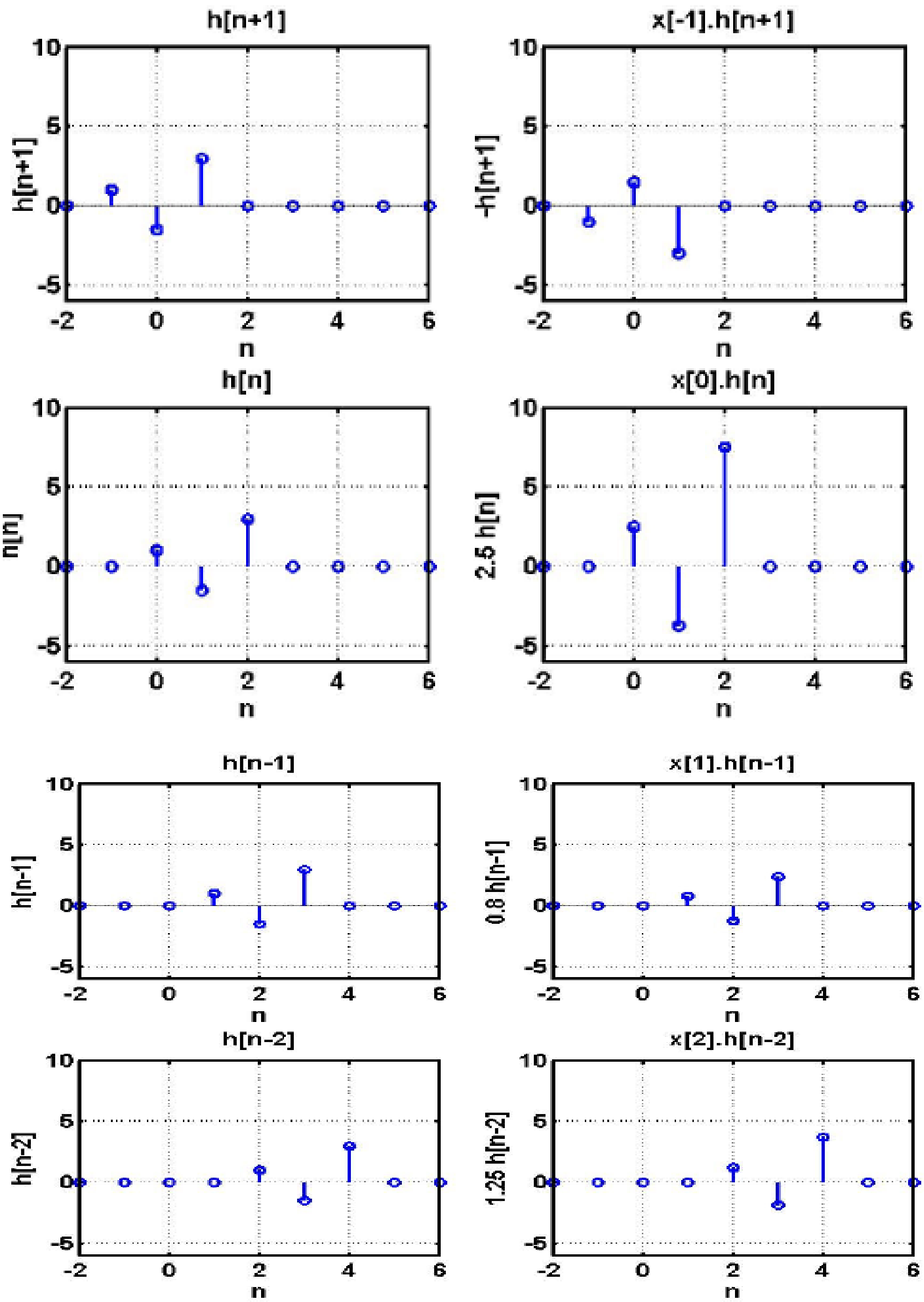
-1 4 -5.95 7.55 0.525 3.75

1. Evaluation as the weighted sum of individual responses

The convolution sum of equation (…), can be equivalently represented as:

$$y[n] = ...+ x[-1]h[n+1]+ x[0]h[n]+ x[1]h[n-1]+.......$$





Convolution as matrix multiplication:

Given

$$x[n] = [x_1 \quad x_2 \quad \dots \quad x_L] \qquad \text{starting from } N_X$$

and

$$h[n] = [h_1 \quad h_2 \quad \dots \quad h_M] \qquad \text{starting from } N_H$$

Step 1: Length of convolved sequence is NUM = (L+M-1)

Step 2: The convolved sequence starts at $i = N_X + N_H$

Step 3: The convolution is given by the following matrix multiplication

$$\begin{bmatrix} y[i] \\ y[i+1] \\ y[i+2] \\ y[i+3] \\ y[i+4] \\ y[i+5] \\ . \\ . \end{bmatrix} = \begin{bmatrix} x_1 & 0 & . & . & 0 \\ x_2 & x_1 & . & . & 0 \\ x_3 & x_2 & . & . & 0 \\ . & x_3 & . & . & 0 \\ . & . & . & . & x_1 \\ x_L & . & . & . & x_2 \\ 0 & x_L & . & . & . \\ 0 & 0 & . & . & x_L \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ h_M \end{bmatrix} = \begin{bmatrix} h_1 & 0 & . & . & 0 \\ h_2 & h_1 & . & . & 0 \\ h_3 & h_2 & . & . & 0 \\ . & h_3 & . & . & 0 \\ . & . & . & . & h_1 \\ h_M & . & . & . & h_2 \\ 0 & h_M & . & . & . \\ 0 & 0 & . & . & h_M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_L \end{bmatrix}$$

The dimensions of the above matrices are:

$$[NUM \quad by \quad 1] = [NUM \quad by \quad M][M \quad by \quad 1] = [NUM \quad by \quad L][L \quad by \quad 1]$$

For the given example:

x[n] is of length L=4, and starts at $N_X = -1$

h[n] is of length M=3 and starts at $N_H = 0$

Step 1: Length of convolved sequence is NUM = (L+M-1)=6

Step 2: The convolved sequence starts at i=(-1+0)=(-1)

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2.5 & -1 & 0 \\ 0.8 & 2.5 & -1 \\ 1.25 & 0.8 & 2.5 \\ 0 & 1.25 & 0.8 \\ 0 & 0 & 1.25 \end{bmatrix} \begin{bmatrix} 1 \\ -1.5 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

or

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 3 & -1.5 & 1 & 0 \\ 0 & 3 & -1.5 & 1 \\ 0 & 0 & 3 & -1.5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2.5 \\ 0.8 \\ 1.25 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

Evaluation using graphical representation:

Another method of computing the convolution is through the direct computation of each value of the output y[n]. This method is based on evaluation of the convolution sum for a single value of *n*, and varying *n* over all possible values.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Step 1: Sketch x[m]

Step 2: Sketch h[-m]

Step 3: Compute y[0] using:

$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[-m]$$

which is the 'sum of the product of the two signals x[m] & h[-m]'

Step 4: Sketch h[1-m], which is right shift of h[-m] by 1.

Step 5: Compute y[1] using:

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$

which is the 'sum of the product of the two signals x[m] & h[1-m]'

Step 6: Sketch h[2-m], which is right shift of h[-m] by 2.

Step 7: Compute y[2] using:

$$y[2] = \sum_{m=-\infty}^{\infty} x[m]h[2-m]$$

which is the 'sum of the product of the two signals x[m] & h[2-m]'

Step 8: Proceed this way until all possible values of y[n], for positive 'n' are computed

Step 9: Sketch h[-1-m], which is left shift of h[-m] by 1.

Step 10: Compute y[-1] using:

$$y[-1] = \sum_{m=-\infty}^{\infty} x[m]h[-1-m]$$

which is the 'sum of the product of the two signals x[m] & h[-1-m]'

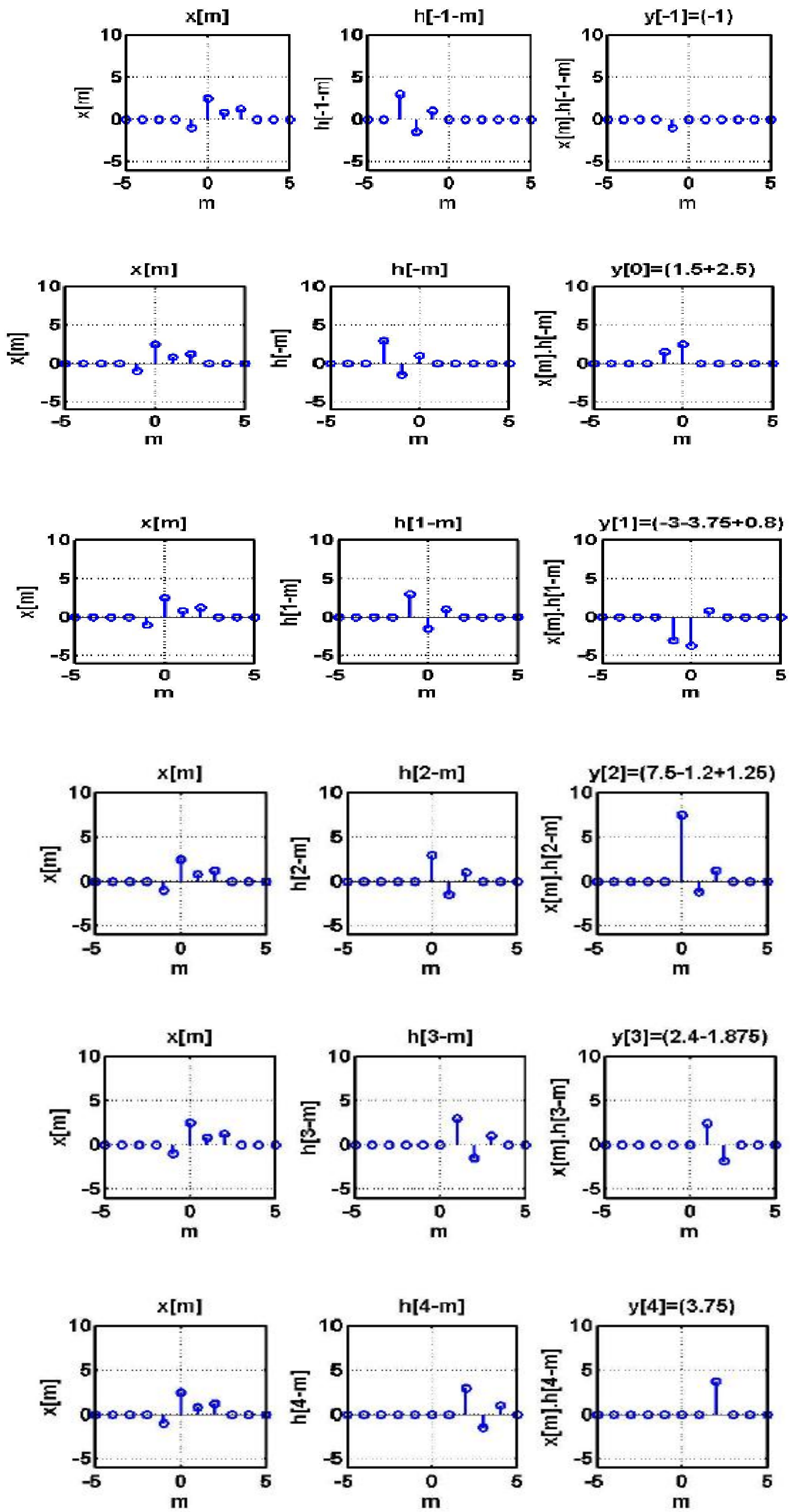
Step 11: Sketch h[-2-m], which is left shift of h[-m] by 2.

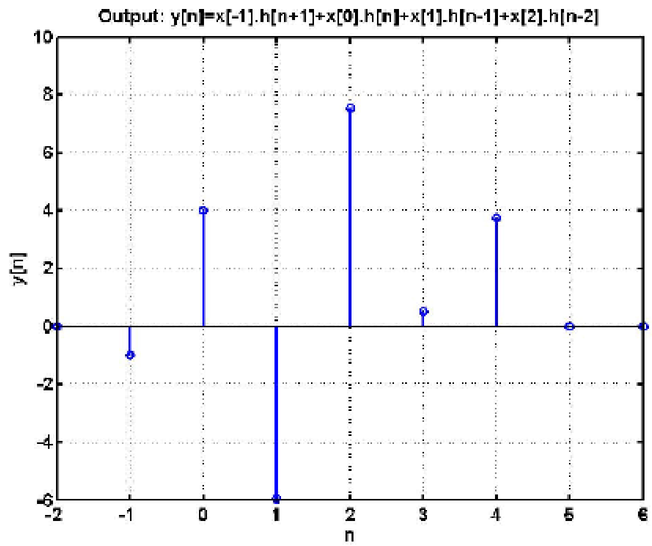
Step 12: Compute y[-2] using:

$$y[-2] = \sum_{m=-\infty}^{\infty} x[m]h[-2-m]$$

which is the 'sum of the product of the two signals x[m] & h[-2-m]'

Step 13: Proceed this way until all possible values of y[n], for negative 'n' are computed


x[m]
h[-1-m]
y[-1]=(-1)
x[m]
h[-m]
y[0]=(1.5+2.5)
x[m]
h[1-m]
y[1]=(-3-3.75+0.8)
x[m]
h[2-m]
y[2]=(7.5-1.2+1.25)
x[m]
h[3-m]
y[3]=(2.4-1.875)
x[m]
h[4-m]
y[4]=(3.75)



Evaluation from direct convolution sum:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the 'convolution sum' of equation (…).

$$since: \quad u[m] = \begin{cases} 0 & for \quad m < 0 \\ 1 & for \quad m \geq 0 \end{cases}$$

$$u[n-m] = \begin{cases} 0 & for \quad (n-m) < 0 \\ 1 & for \quad (n-m) \geq 0 \end{cases}$$

$$= \begin{cases} 0 & for \quad (-m) < n \\ 1 & for \quad (-m) \geq n \end{cases}$$

$$= \begin{cases} 0 & for \quad m > n \\ 1 & for \quad m \leq n \end{cases}$$

Example: A system has impulse response $h[n] = \exp(-0.8n)u[n]$. Obtain the unit step response.

Solution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[m]$$

$$= \sum_{m=-\infty}^{\infty} \{\exp(-0.8(m))\, u[m]\}\{u[n-m]\}$$

$$= \sum_{m=0}^{\infty} \{\exp(-0.8(m))\}\{u[n-m]\}$$

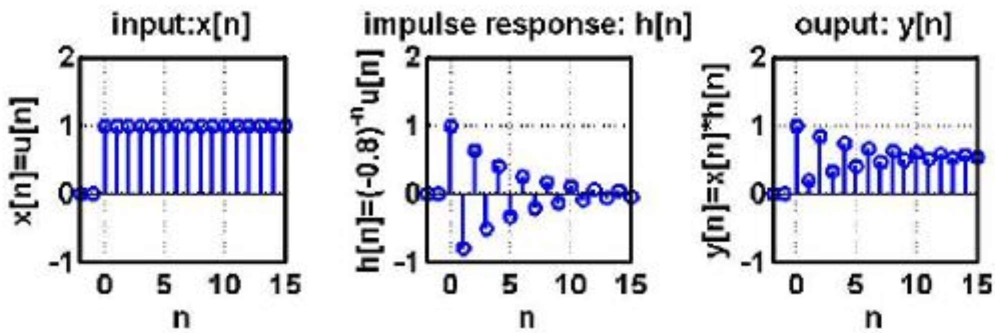
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

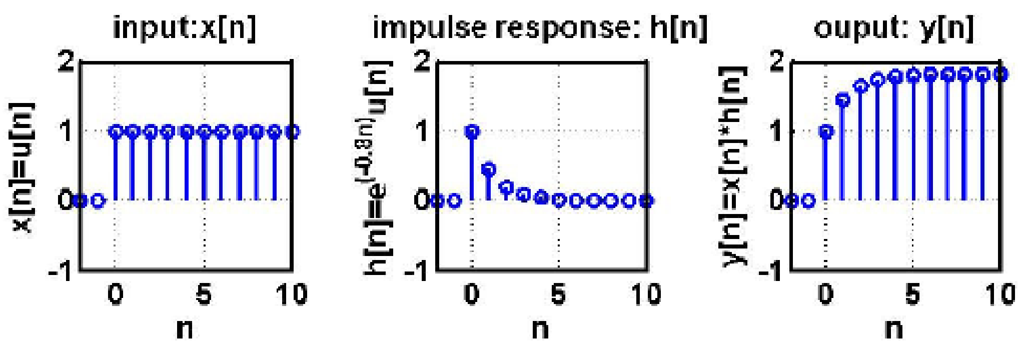
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

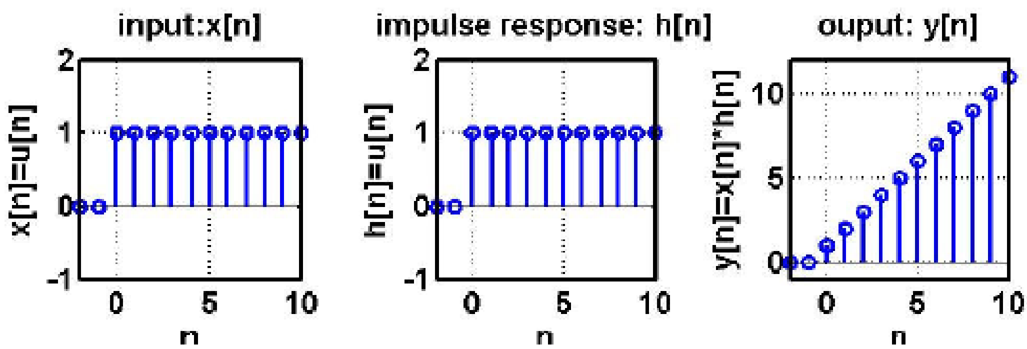
$$= \frac{\left(1-(-0.8)^{n+1}\right)}{\left(1-(-0.8)\right)}$$

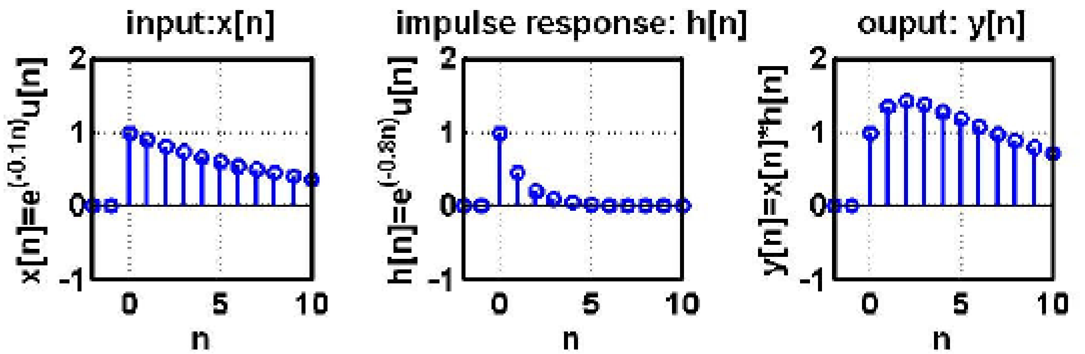
$$y[n] = \sum_{m=-\infty}^{\infty} \left\{(-0.8)^{(n-m)} u[n-m]\right\}$$

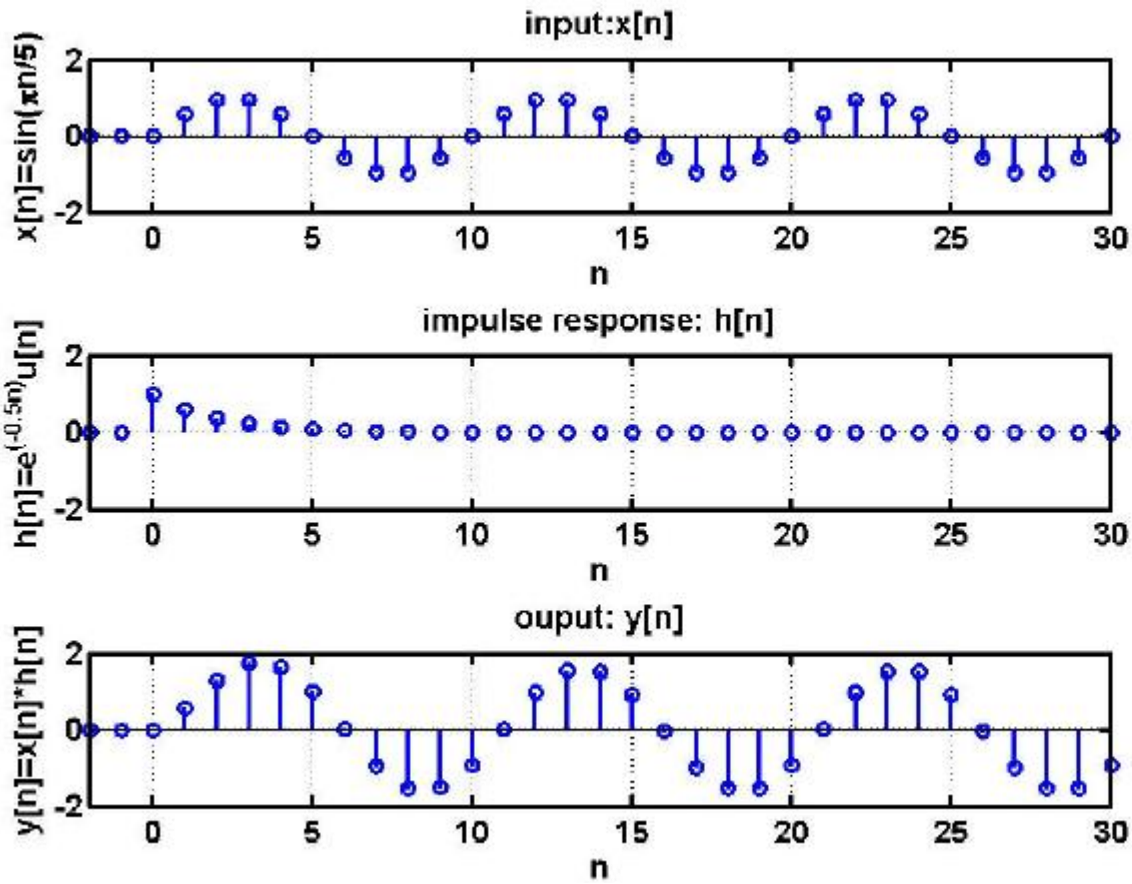
$$= \sum_{m=0}^{\infty} \{\exp(-0.8(n-m))u[n-m]\}$$

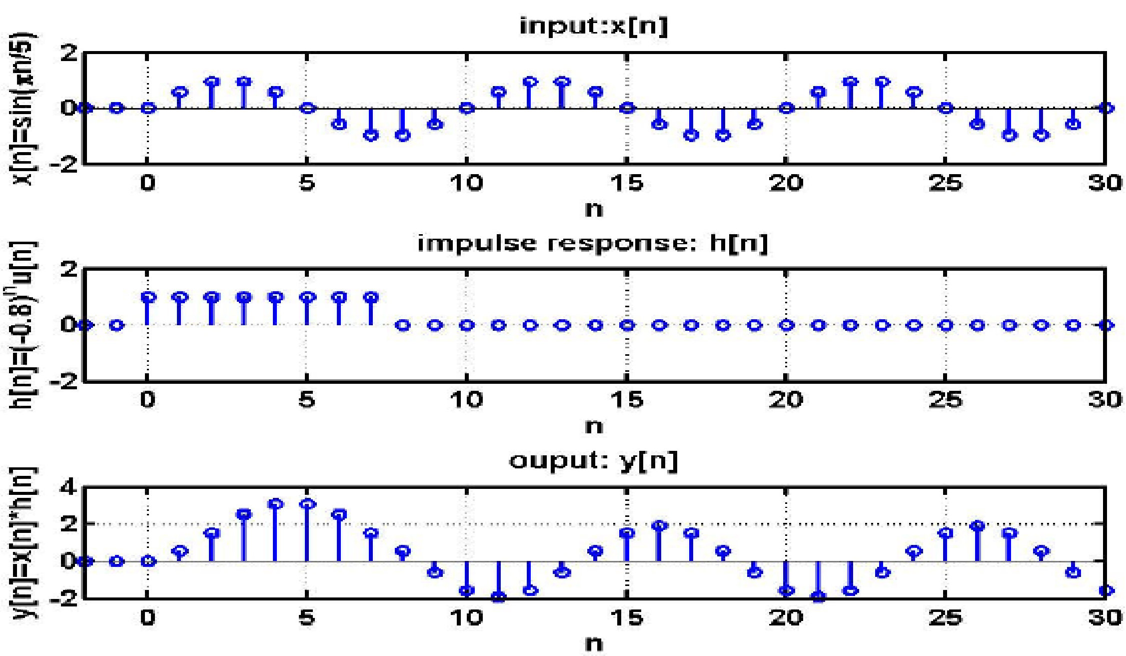






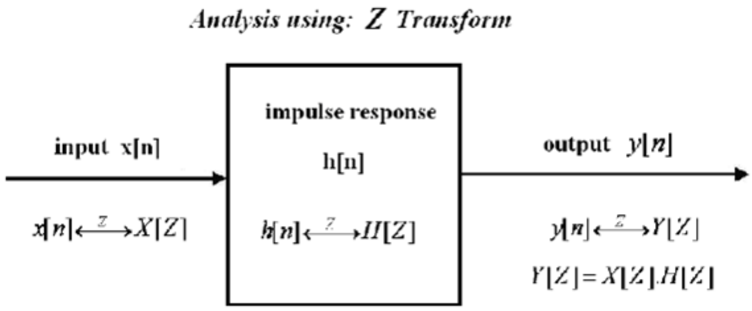

input:x[n]
x[n]=e^(-0.1n)u[n]
impulse response: h[n]
h[n]=e^(-0.8n)u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n


input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=e^(-0.5n)u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n


input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=(-0.8)^n u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n

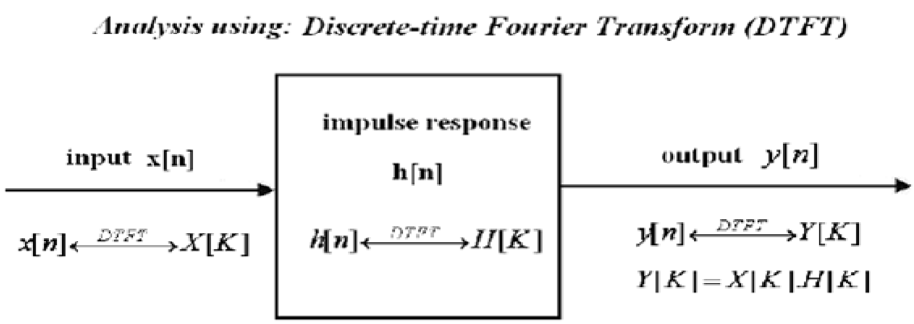
Evaluation from Z-transforms:

Another method of computing the convolution of two sequences is through use of Z-transforms. This method will be discussed later while doing Z-transforms. This approach converts convolution to multiplication in the transformed domain.



Evaluation from Discrete Time Fourier transform (DTFT):

It is possible to compute the convolution of two sequences by transforming them to the frequency domain through application of the Discrete Fourier Transform. This approach also converts the convolution operator to multiplication. Since efficient algorithms for DFT computation exist, this method is often used during software implementation of the convolution operator.



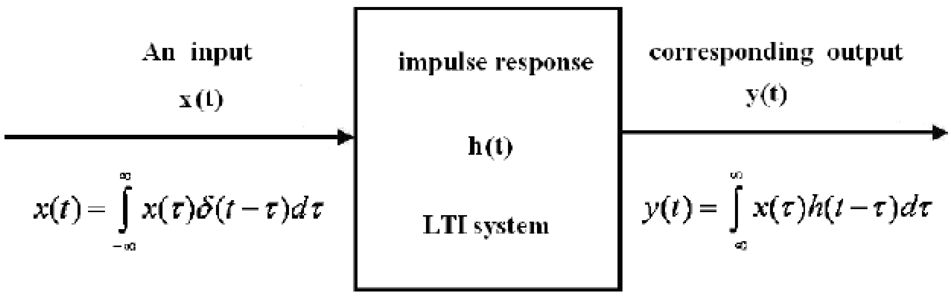
Evaluation from block diagram representation:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the 'convolution sum' .

2.4 Convolution Integral:

The output y(t) is given by, using the notation, y(t)=R{x(t)}.

$$
\begin{aligned}
y(t) &= R\{x(t)\} \\
&= R\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\
&= \int_{-\infty}^{\infty} x(\tau)R\{\delta(t-\tau)\}\,d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= x(t)*h(t)
\end{aligned}
$$



Methods of evaluating the convolution integral: (Same as Convolution sum)

Given the system impulse response h(t), and the input x(t), the system output y(t), is given by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Some of the different methods of evaluating the convolution integral are: Graphical representation, Mathematical equation, Laplace-transforms, Fourier Transform, Differential equation, Block diagram representation, and finally by going to the digital domain.

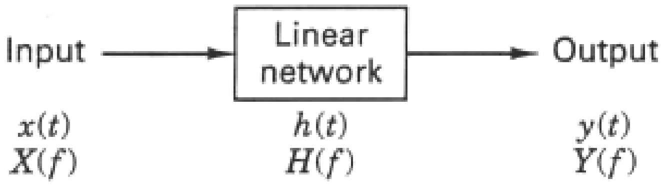
UNIT 3: Time-domain representations for LTI systems – 2 **Teaching hours: 7**

3.1	Properties of impulse response representation	
3.2	Differential equation representation	
3.3	Difference equation representation	
3.4	Block diagram representation	

3.1 Properties of impulse response representation:

Impulse Response

Def. Linear system: system that satisfies superposition theorem.



For any system, we can define its impulse response as:

$$h(t) = y(t) \quad \text{when } x(t) = \delta(t)$$

For linear time invariant system, the output can be modeled as the convolution of the impulse response of the system with the input.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

For casual system, it can be modeled as convolution integral.

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

3.2 Differential equation representation:

General form of differential equation is

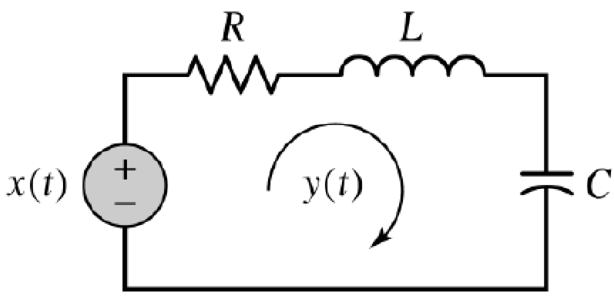
$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

where *ak* and *bk* are coefficients, *x*(.) is input and *y*(.) is output and order of differential or difference equation is (*M*,*N*).

Example of Differential equation

Consider the RLC circuit as shown in figure below. Let $x(t)$ be the input voltage source and $y(t)$ be the output current. Then summing up the voltage drops around the loop gives

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$



Solving differential equation:

A wide variety of continuous time systems are described the linear differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

- Just as before, in order to solve the equation for $y(t)$, we need the ICs. In this case, the ICs are given by specifying the value of y and its derivatives 1 through $N-1$ at $t=0^-$ (time "just before" $t=0$): $y(0^-), y^{(1)}(0^-), \ldots, y^{(N-1)}(0^-)$. where $y^{(i)}(t)$ denotes the i^{th} derivative of $y(t)$, and $y^{(0)}(t) = y(t)$.
- Note: the ICs are given at $t=0^-$ to allow for impulses and other discontinuities at $t=0$.
- Systems described in this way are
- linear time-invariant (LTI): easy to verify by inspection
- causal: the value of the output at time t depends only on the output and the input at times $0 \le \tau \le t$
- As in the case of discrete-time system, the solution $y(t)$ can be decomposed into $y(t) = y_h(t) + y_p(t)$
  where homogeneous solution or zero-input response (ZIR), $y_h(t)$ satisfies the equation

$$y_h^N(t) + \sum_{i=0}^{N-1} a_i y_h^{(i)}(t) = 0, \qquad t \ge 0$$

  with the ICs $y^{(1)}(0^-), \ldots,$
- The zero-state response (ZSR) or particular solution $y_p(t)$ satisfies the equation

$$y_h^N(t) + \sum_{i=0}^{N-1} a_i y_h^{(i)}(t) = \sum_{i=0}^{m} b_i x^{(M-i)}(t), \qquad t \ge 0$$

  with ICs $y_p(0^-) = y_p^{(1)}(0^-) = \ldots = y_p^{(N-1)}(0^-) = 0$.

Homogeneous solution (ZIR) for CT

- A standard method for obtaining the homogeneous solution or (ZIR) is by setting all terms involving the input to zero.

$$\sum_{i=0}^{N} a_i y_h^{(i)}(t) = 0, \qquad t \geq 0$$

and homogeneous solution is of the form

$$y_h(t) = \sum_{i=1}^{N} C_i e^{r_i t}$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Homogeneous solution (ZIR) for DT

- The solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

is

$$y_h[n] = \sum_{i=1}^{N} c_i r_i^n$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Example 1 (ZIR)

- Solution of

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

is

$$y_h(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

- Solution of $y[n] - 9/16y[n-2] = x[n-1]$ is $y_h[n] = c_1(3/4)^n + c_2(-3/4)^{n}$

Example 2 (ZIR)

- Consider the first order recursive system described by the difference equation $y[n]-\rho y[n-1]=x[n]$, find the homogeneous solution.
- The homogeneous equation (by setting input to zero) is $y[n]-\rho y[n-1]=0$.
- The homogeneous solution for $N=1$ is $y_h[n]=c_1 r_1^n$.
- r_1 is obtained from the characteristics equation $r_1-\rho=0$, hence $r_1=\rho$
- The homogeneous solution is $y_h[n]=c_1\rho^n$

Example 3 (ZIR)

- Consider the RC circuit described by $y(t)+RC\frac{d}{dt}y(t)=x(t)$
- The homogeneous equation is $y(t)+RC\frac{d}{dt}y(t)=0$
- Then the homogeneous solution is

$$y_h(t)=c_1 e^{r_1 t}$$

  where r_1 is the root of characteristic equation $1+RCr_1=0$

- This gives $r_1=-\frac{1}{RC}$
- The homogeneous solution is

$$y_h(t)=c_1 e^{\frac{-t}{RC}}$$

Particular solution (ZSR)

- Particular solution or ZSR represents solution of the differential or difference equation for the given input.
- To obtain the particular solution or ZSR, one would have to use the method of integrating factors.
- y_p is not unique.
- Usually it is obtained by assuming an output of the same general form as the input.
- If $x[n] = \alpha_n$ then assume $y_p[n] = c\alpha^n$ and find the constant c so that $y_p[n]$ is the solution of given equation

1.1.3 Examples

Example 1 (ZSR)

- Consider the first order recursive system described by the difference equation $y[n] - \rho y[n-1] = x[n]$, find the particular solution when $x[n] = (1/2)^n$.
- Assume a particular solution of the form $y_p[n] = c_p(1/2)^n$.
- Put the values of $y_p[n]$ and $x[n]$ in the equation then we get $c_p(\frac{1}{2})^n - \rho c_p(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$
- Multiply both the sides of the equation by $(1/2)^n$ we get $c_p = 1/(1 - 2\rho)$.
- Then the particular solution is

$$y_p[n] = \frac{1}{1-2\rho}(\frac{1}{2})^n$$

- For $\rho = (1/2)$ particular solution has the same form as the homogeneous solution
- However no coefficient c_p satisfies this condition and we must assume a particular solution of the form $y_p[n] = c_p n(1/2)^n$.
- Substituting this in the difference equation gives $c_p n(1-2\rho)+2\rho c_p = 1$
- Using $\rho = (1/2)$ we find that $c_p = 1$.

Example 2 (ZSR)

- Consider the RC circuit described by $y(t)+RC\frac{d}{dt}y(t) = x(t)$
- Assume a particular solution of the form $y_p(t) = c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)$.
- Replacing $y(t)$ by $y_p(t)$ and $x(t)$ by $\cos(\omega_0 t)$ gives

$$c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)-RC\omega_0 c_1\sin(\omega_0 t)+RC\omega_0 c_2\cos(\omega_0 t) = \cos(\omega_0 t)$$

- The coefficients c_1 and c_2 are obtained by separately equating the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, gives

$$c_1 = \frac{1}{1+(RC\omega_0)^2} \qquad \textit{and} \qquad c_2 = \frac{RC\omega_0}{1+(RC\omega_0)^2}$$

- Then the particular solution is

$$y_p(t) = \frac{1}{1+(RC\omega_0)^2}\cos(\omega_0 t)+\frac{RC\omega_0}{1+(RC\omega_0)^2}\sin(\omega_0 t)$$

Complete solution

- Find the form of the homogeneous solution y_h from the roots of the characteristic equation
- Find a particular solution y_p by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution $y = y_h + y_p$ satisfies the initial conditions

3.3 Difference equation representation:

- A wide variety of discrete-time systems are described by linear difference equations:

$$y[n]+\sum_{k=1}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k], \qquad n=0,1,2,\ldots$$

  where the coefficients $a_1,\ldots,a_N$ and $b_0,\ldots,b_M$ do not depend on n. In order to be able to compute the system output, we also need to specify the initial conditions (ICs) $y[-1],y[-2]\ldots y[-N]$

- Systems of this kind are
  - linear time-invariant (LTI): easy to verify by inspection
  - causal: the output at time n depends only on past outputs $y[n-1],\ldots,y[n-N]$ and on current and past inputs $x[n],x[n-1],\ldots,x[n-M]$

- Systems of this kind are also called Auto Regressive Moving-Average (ARMA) filters. The name comes from considering two special cases.

- auto regressive (AR) filter of order N, $AR(N)$: $b_0=\ldots=b_M=0$

$$y[n]+\sum_{k=1}^{N} a_k y[n-k]=0 \qquad n=0,1,2,\ldots$$

  In the AR case, the system output at time n is a linear combination of N past outputs; need to specify the ICs $y[-1],\ldots,y[-N]$.

- moving-average (MA) filter of order $N, AR(N)$: $a_0=\ldots=a_N=0$

$$y[n]=\sum_{k=0}^{M} b_k x[n-k] \qquad n=0,1,2,\ldots$$

  In the MA case, the system output at time n is a linear combination of the current input and M past inputs; no need to specify ICs.

- An ARMA(N, M) filter is a combination of both.

- Let us first rearrange the system equation:

$$y[n]=-\sum_{k=1}^{N} a_k y[n-k]+\sum_{k=0}^{M} b_k x[n-k] \qquad n=0,1,2,\ldots$$

- at $n = 0$

$$y[0] = \underbrace{-\sum_{k=1}^{N} a_k y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[-k]}_{depends\ on\ input\ x[0] \to x[-M]}$$

- at $n = 1$

$$y[1] = \sum_{k=1}^{N} a_k y[1-k] + \sum_{k=0}^{M} b_k x[1-k]$$

After rearranging

$$y[1] = -a_1 y[0] \underbrace{- \sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input x[1] \ldots x[1-M]}$$

- at $n = 2$

$$y[2] = \sum_{k=1}^{N} a_k y[2-k] + \sum_{k=0}^{M} b_k x[2-k]$$

After rearranging

$$y[2] = -a_1 y[1] - a_2 y[0] \underbrace{- \sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input\ x[2] \ldots x[2-M]}$$

Example of Difference equation

- An example of II order difference equation is

$$y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n] + 2x[n-1]$$

- Memory in discrete system is analogous to energy storage in continuous system
- Number of initial conditions required to determine output is equal to maximum memory of the system

- In discrete case, for an N^{th} order system the N initial value are

$$y[-N], y[-N+1], \ldots, y[-1]$$

- The initial conditions for N^{th}-order differential equation are the values of the first N derivatives of the output

$$y(t)|_{t=0}, \frac{d}{dt}y(t)|_{t=0}, \frac{d^2}{dt^2}y(t)|_{t=0}, \cdots \frac{d^{N-1}}{dt^{N-1}}y(t)|_{t=0}$$

Solving difference equation

- Consider an example of difference equation $y[n]+ay[n-1]=x[n], \quad n=0,1,2\ldots$ with $y[-1]=0$ Then

$$\begin{aligned}
y[0] &= -ay[-1]+x[0] \\
y[1] &= -ay[0]+x[1] \\
&= -a(-ay[-1]+x[0])+x[1] \\
&= a^2y[-1]-ax[0])+x[1] \\
y[2] &= -ay[1]+x[2] \\
&= -a(-a^2y[-1]-ax[0]+x[1])+x[2] \\
&= a^3y[-1]+a^2x[0]-ax[1]+x[2]
\end{aligned}$$

and so on

- We get $y[n]$ as a sum of two terms:
$y[n]=(-a)^{n+1}y[-1]+\Sigma_{i=0}^{n}(-a)^{n-i}x[i], \qquad n=0,1,2,\ldots$
- First term $(-a)^{n+1}y[-1]$ depends on IC's but not on input

- Second term $\sum_{i=0}^{n}(-a)^{n-i}x[i]$ depends only on the input, but not on the IC's
- This is true for any ARMA (auto regressive moving average) system: the system output at time n is a sum of the AR-only and the MA-only outputs at time n.
- Consider an ARMA (N,M) system $y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i]$, $\quad n = 0, 1, 2, \ldots$ with the initial conditions $y[-1], \ldots, y[-N]$.
- Output at time n is:

$$y[n] = y_h[n] + y_p[n]$$

  where $y_h[n]$ and $y_p[n]$ are homogeneous and particular solutions
- First term depends on IC's but not on input
- Second term depends only on the input, but not on the IC's
- Note that $y_h[n]$ is the output of the system determined by the ICs only (setting the input to zero), while $y_p[n]$ is the output of the system determined by the input only (setting the ICs to zero).
- $y_h[n]$ is often called the zero-input response (ZIR) usually referred as homogeneous solution of the filter (referring to the fact that it is determined by the ICs only)
- $y_p[n]$ is called the zero-state response (ZSR) usually referred as particular solution of the filter (referring to the fact that it is determined by the input only, with the ICs set to zero).

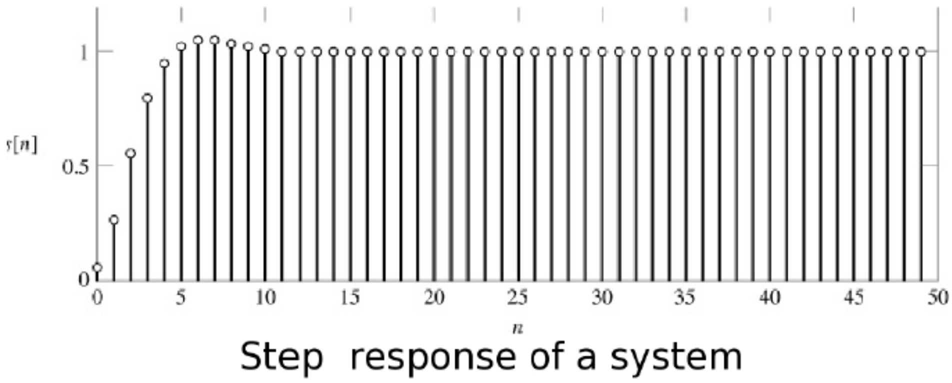


Figure 1.2: Step response

- Consider the output decomposition $y[n] = y_h[n] + y_p[n]$ of an ARMA (N, M) filter

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0, 1, 2, \ldots$$

with the ICs $y[-1], \ldots, y[-N]$.

- The output of an ARMA filter at time n is the sum of the ZIR and the ZSR at time n.

Example of difference equation

- example: A system is described by $y[n] - 1.143y[n-1] + 0.4128y[n-2] = 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$
- Rewrite the equation as $y[n] = 1.143y[n-1] - 0.4128y[n-2] + 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$

3.4 Block Diagram representation:

- A block diagram is an interconnection of elementary operations that act on the input signal
- This method is more detailed representation of the system than impulse response or differential/difference equation representations

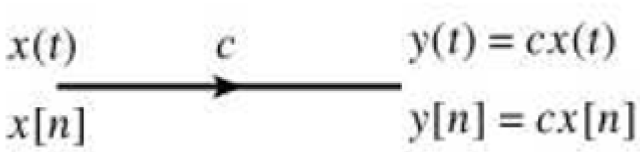
- The impulse response and differential/difference equation descriptions represent only the input-output behavior of a system, block diagram representation describes how the operations are ordered

- Each block diagram representation describes a different set of internal computations used to determine the system output

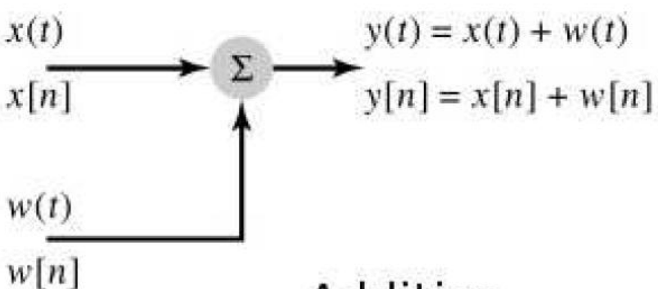
- Block diagram consists of three elementary operations on the signals:
  - Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar
  - Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$.

- Block diagram consists of three elementary operations on the signals:
  - Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
    Time shift for discrete time LTI system: $y[n] = x[n-1]$

- Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar

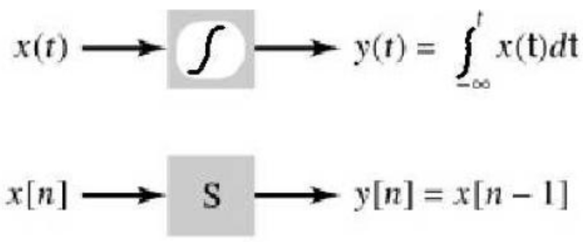


Scalar Multiplication



Addition

- Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$
- Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
  Time shift for discrete time LTI system: $y[n] = x[n-1]$



Integration and timeshifting

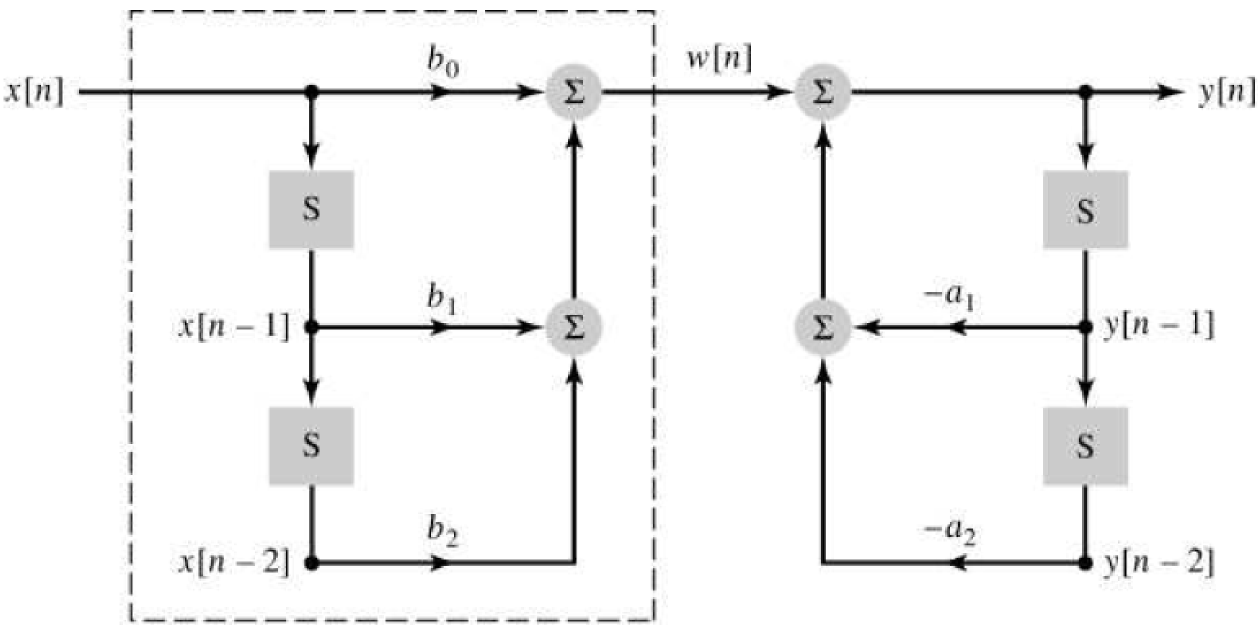


Figure 1.10: Example 1: Direct form I

Example 1

- Consider the system described by the block diagram as in Figure 1.10
- Consider the part within the dashed box
- The input $x[n]$ is time shifted by 1 to get $x[n-1]$ and again time shifted by one to get $x[n-2]$. The scalar multiplications are carried out and

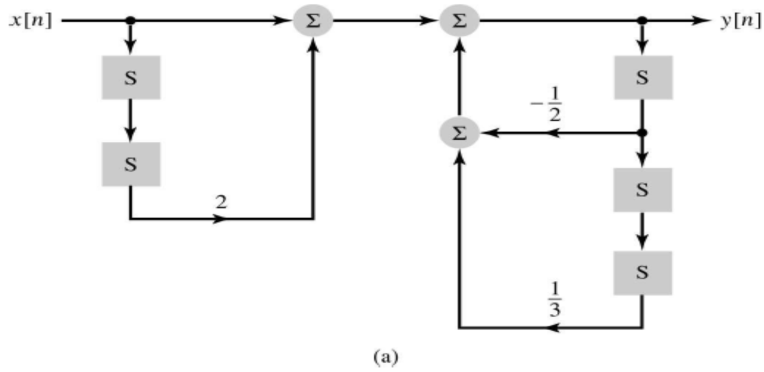


Figure 1.11: Example 2: Direct form I

they are added to get $w[n]$ and is given by

$$w[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2].$$

- Write $y[n]$ in terms of $w[n]$ as input $y[n] = w[n] - a_1y[n-1] - a_2y[n-2]$
- Put the value of $w[n]$ and we get $y[n] = -a_1y[n-1] - a_2y[n-2] + b_0x[n]$ $+ b_1x[n-1] + b_2x[n-2]$
  and $y[n] + a_1y[n-1] + a_2y[n-2] = \quad b_0x[n] + b_1x[n-1] + b_2x[n-2]$
- The block diagram represents an LTI system

Example 2

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] - (1/3)y[n-3] = x[n] + 2x[n-2]$

Example 3

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] + (1/4)y[n-2] = x[n-1]$

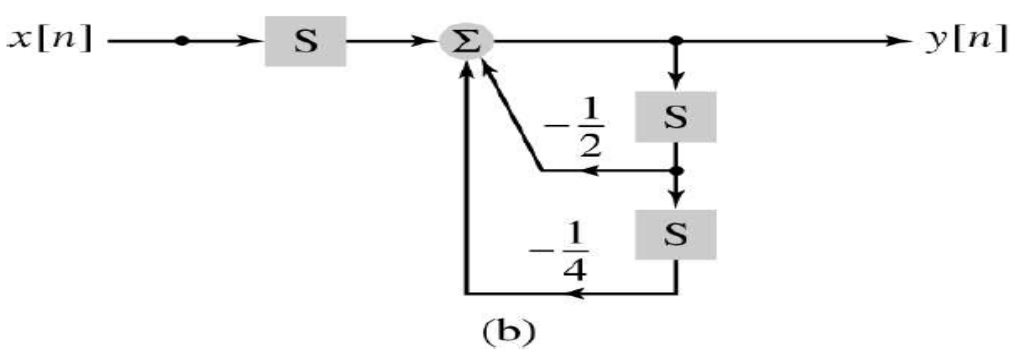


Figure 1.12: Example 3: Direct form I

- Block diagram representation is not unique, direct form II structure of Example 1
- We can change the order without changing the input output behavior Let the output of a new system be $f[n]$ and given input $x[n]$ are related by

$$f[n] = -a_1 f[n-1] - a_2 f[n-2] + x[n]$$

- The signal $f[n]$ acts as an input to the second system and output of second system is

$$y[n] = b_0 f[n] + b_1 f[n-1] + b_2 f[n-2].$$

- The block diagram representation of an LTI system is not unique

Continuous time

- Rewrite the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

as an integral equation. Let $v^{(0)}(t) = v(t)$ be an arbitrary signal, and set

$$v^{(n)}(t) = \int_{-\infty}^{t} v^{(n-1)}(\tau)\, d\tau, \qquad n = 1, 2, 3, \ldots$$

where $v^{(n)}(t)$ is the n-fold integral of $v(t)$ with respect to time

- Rewrite in terms of an initial condition on the integrator as

$$v^{(n)}(t) = \int_{0}^{t} v^{(n-1)}(\tau)\, d\tau + v^{(n)}(0), \quad n = 1, 2, 3, \ldots$$

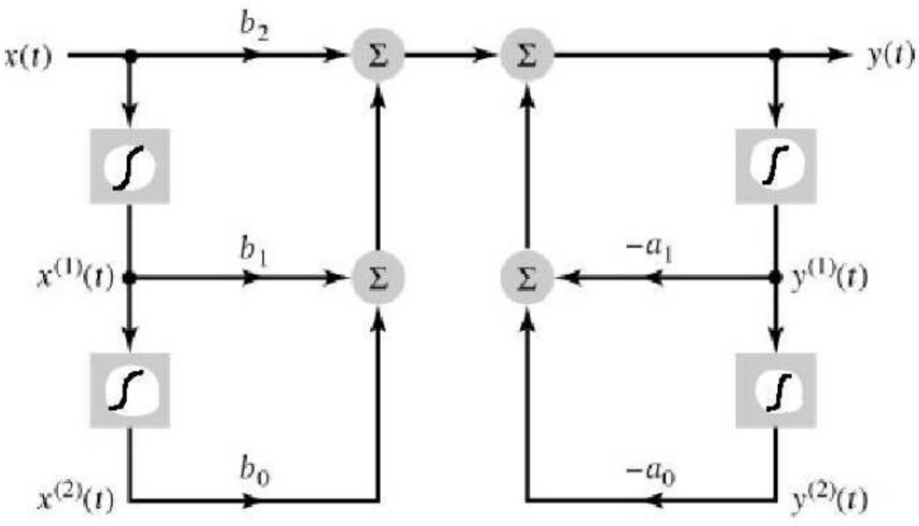
- If we assume zero ICs, then differentiation and integration are inverse operations, ie.

$$\frac{d}{dt} v^{(n)}(t) = v^{(n-1)}(t), \quad t > 0 \text{ and } n = 1, 2, 3, \ldots$$

- Thus, if $N \geq M$ and integrate N times, we get the integral description of the system

$$\sum k = 0^N a_k y^{(N-k)}(t) = \sum k = 0^M b_k x^{(N-k)}(t)$$

- For second order system with $a_0 = 1$, the differential equation can be

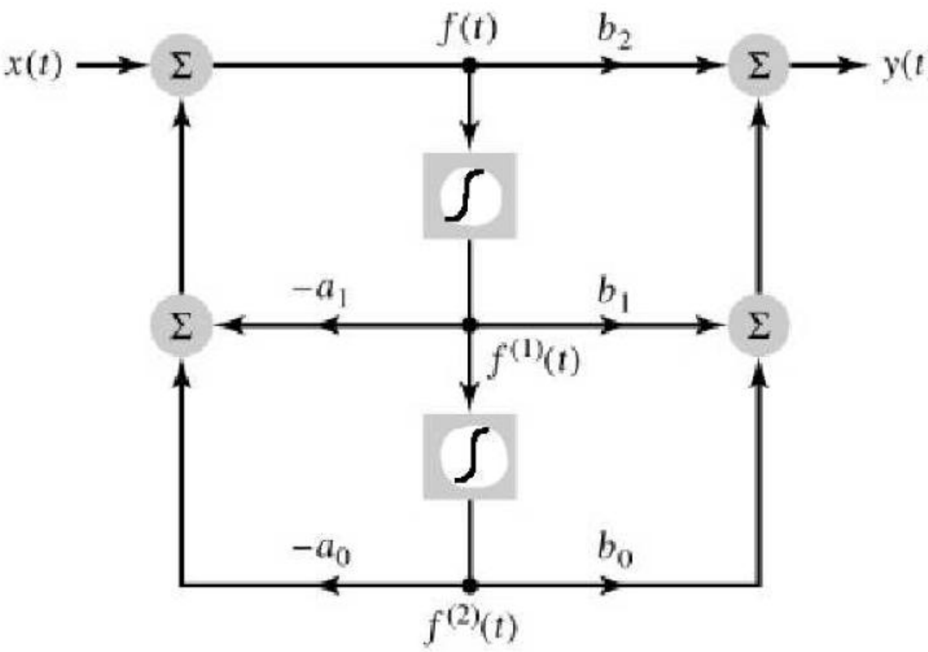


Direct form I structure

Figure 1.25: Direct form I

written as

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + a_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$



Direct form II structure

UNIT 4: Fourier representation for signals – 1 **Teaching hours: 6**

4.1	Introduction	
4.2	Discrete time Fourier series	
4.3	Continuous time Fourier series	
4.4	Properties	

4.1 Introduction:

Fourier series has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

• In 1807, Jean Baptiste Joseph Fourier Submitted a paper of using trigonometric series to represent "any" periodic signal.

• But Lagrange rejected it!

• In 1822, Fourier published a book "The Analytical Theory of Heat" Fourier's main contributions: Studied vibration, heat diffusion, etc. and found that a series of harmonically related sinusoids is useful in representing the temperature distribution through a body.

• He also claimed that "any" periodic signal could be represented by Fourier series. These arguments were still imprecise and it remained for P.L.Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a FS.

• He however obtained a representation for aperiodic signals i.e., Fourier integral or transform

• Fourier did not actually contribute to the mathematical theory of Fourier series.

• Hence out of this long history what emerged is a powerful and cohesive framework for the analysis of continuous- time and discrete-time signals and systems and an extraordinarily broad array of existing and potential application.

<u>The Response of LTI Systems to Complex Exponentials</u>:

We have seen in previous chapters how advantageous it is in LTI systems to represent signals as a linear combinations of basic signals having the following properties.

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.
2. Set of basic signals used to construct a broad class of signals.
3. The response of an LTI system to each signal should be simple enough in structure.
4. It then provides us with a convenient representation for the response of the system.
5. Response is then a linear combination of basic signal.

Eigenfunctions and Values :

• One of the reasons the Fourier series is so important is that it represents a signal in terms of eigenfunctions of LTI systems.

• When I put a complex exponential function like x(t) = ejωt through a linear time-invariant system, the output is y(t) = H(s)x(t) = H(s) ejωt where H(s) is a complex constant (it does not depend on time).

• The LTI system scales the complex exponential ejωt .

Historical background

There are antecedents to the notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the century. In [5], Fourier deals with the problem of describing the evolution of the temperature of a thin wire of length X. He proposed that the initial temperature could be expanded in a series of sine functions:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \qquad (2)$$

The Fourier sine series, defined in Eq.s (1) and (2), is a special case of a more general concept: the Fourier series for a *periodic function.* Periodic functions arise in the study of wave motion, when a basic waveform repeats itself periodically. Such periodic waveforms occur in musical tones, in the plane waves of electromagnetic vibrations, and in the vibration of strings. These are just a few examples. Periodic effects also arise in the motion of the planets, in ac-electricity, and (to a degree) in animal heartbeats.

A function f is said to have period P if $f(x+P) = f(x)$ for all x. For notational simplicity, we shall restrict our discussion to functions of period 2π. There is no loss of generality in doing so, since we can always use a simple change of scale $x = (P/2\pi)t$ to convert a function of period P into one of period 2π.

If the function f has period 2π, then its *Fourier series* is

$$c_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \tag{4}$$

with *Fourier coefficients* c_0, a_n, and b_n defined by the integrals

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{5}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{6}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{7}$$

The following relationships can be readily established, and will be used in subsequent sections for derivation of useful formulas for the unknown Fourier coefficients, in both time and frequency domains.

$$\int_0^T \sin(kw_0t)dt = \int_0^T \cos(kw_0t)dt \tag{1}$$

$$= 0$$

$$\int_0^T \sin^2(kw_0t)dt = \int_0^T \cos^2(kw_0t)dt \tag{2}$$

$$= \frac{T}{2}$$

$$\int_0^T \cos(kw_0t)\sin(gw_0t)dt = 0 \tag{3}$$

$$\int_0^T \sin(kw_0t)\sin(gw_0t)dt = 0 \tag{4}$$

$$\int_0^T \cos(kw_0t)\cos(gw_0t)dt = 0 \tag{5}$$

where

$$w_0 = 2\pi f \tag{6}$$

$$f = \frac{1}{T} \tag{7}$$

where f and T represents the frequency (in cycles/time) and period (in seconds) respectively. Also, k and g are integers.

A periodic function $f(t)$ with a period T should satisfy the following equation

$$f(t+T) = f(t) \tag{8}$$

Example 1

Prove that

$$\int_0^{\pi} \sin(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$A = \int_0^{T} \sin(kw_0 t)dt \tag{9}$$

$$= -\left(\frac{1}{kw_0}\right)\left[\cos(kw_0 t)\right]_0^T$$

$$A = \left(\frac{-1}{kw_0}\right)\left[\cos(kw_0 T) - \cos(0)\right] \tag{10}$$

$$= \left(\frac{-1}{kw_0}\right)\left[\cos(k2\pi) - 1\right]$$

$$= 0$$

Example 2

Prove that

$$\int_0^{\pi} \sin^2(kw_0 t) = \frac{T}{2}$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$B = \int_0^T \sin^2(kw_0 t)dt \tag{11}$$

Recall

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{12}$$

Thus,

$$B = \int_o^T \left[\frac{1}{2} - \frac{1}{2}\cos(2kw_0 t)\right]dt \tag{13}$$

$$= \left[\left(\frac{1}{2}\right)t - \left(\frac{1}{2}\right)\left(\frac{1}{2kw_0}\right)\sin(2kw_0 t)\right]_0^T$$

$$B = \left[\frac{T}{2} - \frac{1}{4kw_0}\sin(2kw_0 T)\right] - [0] \tag{14}$$

$$= \frac{T}{2} - \left(\frac{1}{4kw_0}\right)\sin(2k * 2\pi)$$

$$= \frac{T}{2}$$

Example 3

Prove that

$$\int_0^\pi \sin(gw_0 t)\cos(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k and g are integers.

Solution

Let

$$C = \int_0^T \sin(gw_0 t)\cos(kw_0 t)dt \tag{15}$$

Recall that

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha) \tag{16}$$

Hence,

$$C=\int_0^T \left[\sin\left((g+k)w_0 t\right)-\sin(kw_0t)\cos(gw_0t)\right]dt \tag{17}$$

$$=\int_0^T \sin\left[(g+k)w_0 t\right]dt-\int_0^T \sin(kw_0t)\cos(gw_0t)dt \tag{18}$$

From Equation (1),

$$\int_0^T [\sin(g+k)w_0t]dt=0$$

then

$$C=0-\int_0^T \sin(kw_0t)\cos(gw_0t)dt \tag{19}$$

Adding Equations (15), (19),

$$2C=\int_0^T \sin(gw_0t)\cos(kw_0t)dt-\int_0^T \sin(kw_0t)\cos(gw_0t)dt$$

$$=\int_0^T \sin\left[(gw_0t)-(kw_0t)\right]dt=\int_0^T \sin\left[(g-k)w_0t\right]dt \tag{20}$$

$2C=0$, since the right side of the above equation is zero (see Equation 1). Thus,

$$C=\int_o^T \sin(gw_0t)\cos(kw_0t)dt=0 \tag{21}$$

$$=0$$

Example 4

Prove that

$$\int_0^T \sin(kw_0t)\sin(gw_0t)dt=0$$

for

$$w_0=2\pi f$$

$$f=\frac{1}{T}$$

$$k, g=\text{integers}$$

Solution

$$\text{Let } D = \int_0^T \sin(kw_0 t)\sin(gw_0 t)dt \tag{22}$$

Since

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

or

$$\sin(\alpha)\sin(\beta) = \cos(\alpha)\cos(\beta) - \cos(\alpha+\beta)$$

Thus,

$$D = \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt - \int_0^T \cos[(k+g)w_0 t]dt \tag{23}$$

From Equation (1)

then

$$D = \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt - 0 \tag{24}$$

Adding Equations (23), (26)

$$2D = \int_0^T \sin(kw_0 t)\sin(gw_0 t) + \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt$$

$$= \int_0^T \cos[kw_0 t - gw_0 t]dt \tag{25}$$

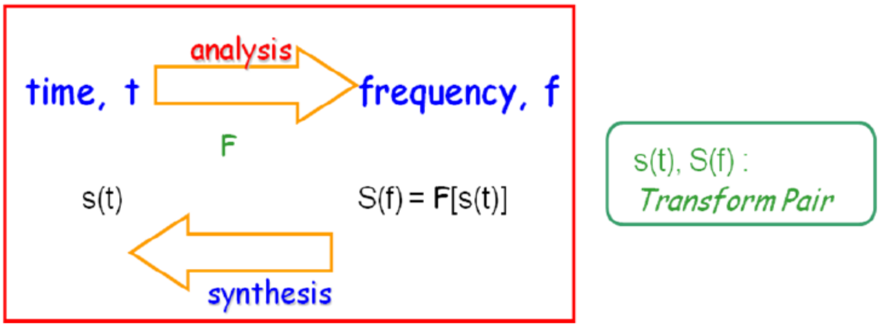
$$= \int_0^T \cos[(k-g)w_0 t]dt$$

2D = 0, since the right side of the above equation is zero (see Equation 1). Thus,

$$D \equiv \int_0^T \sin(kw_0 t)\sin(gw_0 t)dt = 0 \tag{26}$$

Need for Frequency Analysis

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns.
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.



Orthogonality of the Complex exponentials

• For continuous-time signals with period T, the complex exponentials $e^{jn\omega_o t}$ must satisfy the orthogonality condition

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t *}\, dt \;-\begin{cases} 0 \text{ if } n\neq m \\ 1 \text{ if } n=m \end{cases}$$

Where $T = \frac{2\pi}{\omega_o}$

and f*(t) denotes the complex conjugate of f(t)

Theorem:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t *}\, dt \qquad \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; \bar{e}^{jm\omega_o t} dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{j(n-m)\omega_o t}$$

When n≠m, let n-m=p. Then

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t *}\, dt \qquad \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jp\omega_o t}\, dt$$

$$= \frac{e^{jp\omega_o t}}{Tjp\omega_o}\Bigg|_{-\frac{T}{2}}^{+\frac{T}{2}} \qquad \frac{e^{jp\pi} - e^{-jp\pi}}{Tjp\omega_o} = 0$$

On the other hand when n=m, then equation becomes

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\ e^{jm\omega_o t*}\ \mathrm{dt} = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{0}\ \mathrm{dt} = 1$$

Hence the theorem may be stated in general

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\ e^{jm\omega_o t*}\ \mathrm{dt} \ \begin{cases} 0 \text{ if } n\neq m \\ 1 \text{ if } n=m \end{cases}$$

Harmonically Related Complex Exponentials

$\emptyset(t) = e^{jk\omega_o t} = e^{jk\left(\frac{2\pi}{T}\right)t}, k = 0, \pm1, \pm2 \ldots\ldots$ where $\omega_{o=}\left(\frac{2\pi}{T}\right)$

Fourier Series Representation

$$x(t) = \sum_{k=-\infty}^{+\infty} ak\ \emptyset(t) = \sum_{k=-\infty}^{+\infty} ak\ e^{jk\omega_o t}$$

$$= \sum_{k=-\infty}^{+\infty} ak\, e^{jk\left(\frac{2\pi}{T}\right)t}$$ Fourier Series representation

Where, k=+1,-1; the first harmonic components or the fundamental Component and k=+2,-2; the second harmonic components or the fundamental Component ….. etc.

Fourier Series Representation of CT Periodic Signals

Example 1

$$\mathbf{x(t)} = \sum_{k=-3}^{+3} a_k\, e^{jk(2\pi)t}$$

$a_0 = 1$; $a_{-1}, a_{+1} = 1/4$;
$a_{-2}, a_{+2} = 1/2$;
$a_{-3}, a_{+3} = 1/3$

Substituting for all vaues of k

$$\Rightarrow \mathbf{x(t)} = 1+\frac{1}{4}(e^{j2\pi t}+e^{-j2\pi t})+\frac{1}{2}(e^{j4\pi t}+e^{-j4\pi t})$$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$+\frac{1}{3}(e^{j6\pi t}+e^{-j6\pi t})$$

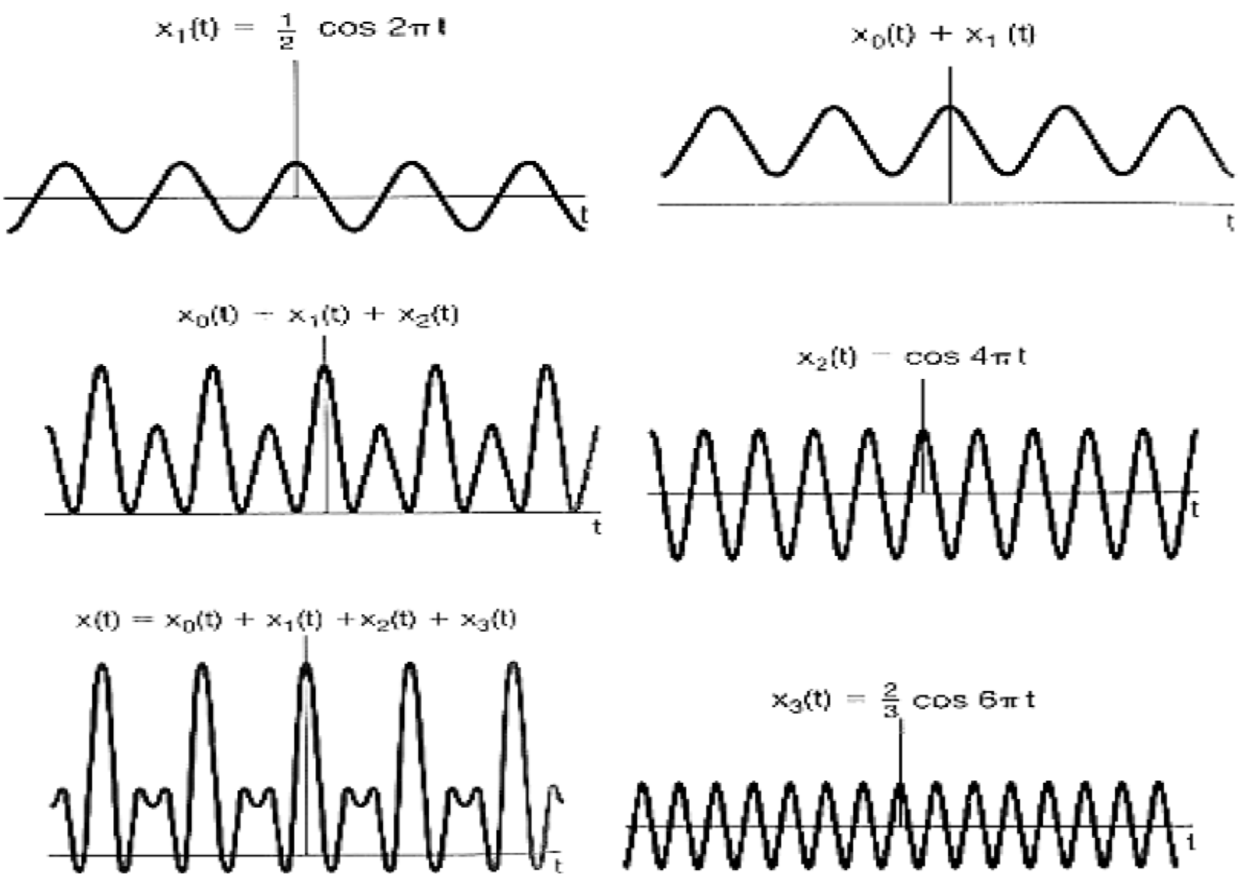
Equivalently, using Euler's relation (an alternate form of Fourier series)

$$\cos(\theta)=\frac{1}{2}(e^{j\theta}+e^{-j\theta})$$

$$\sin(\theta)=\frac{1}{2j}(e^{j\theta}-e^{-j\theta})$$

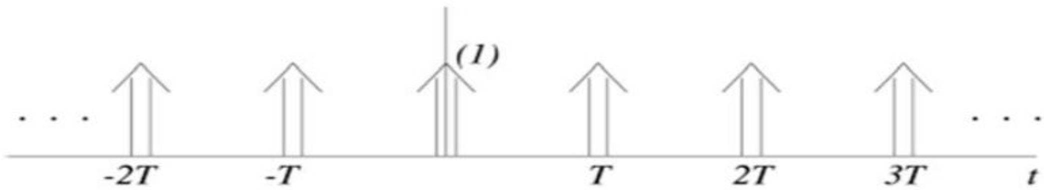
$$\Rightarrow \quad \mathbf{x(t)} = 1+\frac{1}{2}\cos 2\pi t+\cos 4\pi t+\frac{2}{3}\cos 6\pi t$$

Example 1 Graphical Representation



Another (important!) example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$ — Sampling function, important for sampling later

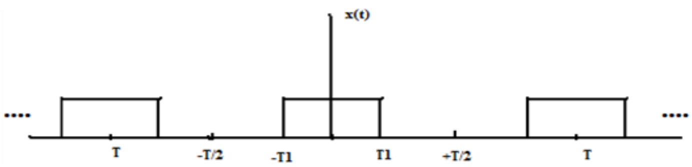


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_o t} dt$$

$$\Downarrow$$

$$= \frac{1}{T} \quad \text{for all } k\ ! \qquad x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{jk\omega_o t}$$

Example 2



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

To determine the FS Coefficient for $x(t)$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t} dt = -\frac{1}{jkw_0 T} e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{kw_0 T}\left[\frac{e^{jkw_0 T_1} - e^{-jkw_0 T_1}}{2j}\right]$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

Dirichlets Condition

Condition 1:
Krupa Over any period, x(t) must be absolutely integrable, i.e each coefficient is to be finite.
Condition 2:
In any finite interval, x(t) is of bounded variation; i.e., – There are no more than a finite number of maxima and minima during any single period of the signal
Condition 3:
In any finite interval, x(t) has only finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Properties of Fourier Representation

1.Linear Properties

2. Translation or Time Shift Properties

3. Frequency shift properties

4. Time Differentiation

5. Time Domain Convolution

6. Modulation or Multiplication Theorem

7. parsevals Relationship

1. Linear Properties

- Linearity:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$$\Rightarrow \quad z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

The Fourier series coefficient c_k are given by the same linear combination of FS coefficients for x(t) and y(t)

2. Frequency shift properties

$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ **Let** $z(t) = e^{-jk_o w_0 t}\, x(t)$

Then $z(t) \overset{\mathcal{FS}}{\longleftrightarrow} z_k = a_{k-k_o}$

Proof:

$$z_k = \frac{1}{T}\int_T z(t)\, e^{-jkw_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{-jk_o w_0 t}\, x(t)\; e^{-jkw_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\; e^{-j(k-k_o)w_0 t}\, dt$$

$$z_k = a_{k-k_o}$$

3) Scaling Properties

- $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$ and fundamental frequency αw_0
- $x(t)$: periodic signals with period T and fundamental frequency $w_0 = \frac{2\pi}{T}$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \quad \Big| \quad x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0(\alpha t)}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\alpha\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha w_0)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{\left(\frac{T}{\alpha}\right)}\right)t} \quad \Big| \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

4) Time Differentiation

- $x(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\frac{d}{dt}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0\, a_k$$

Proof: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$

then $\frac{d}{dt}x(t) = \sum_{k=-\infty}^{+\infty} a_k\, jkw_0 e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} (jkw_0\, a_k)\, e^{jkw_0 t}$

$= jkw_0\, a_k$ ------- **Proved**

5) Modulation or Multiplication theorem

- $x(t), y(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T $\qquad z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$

$$z(t) = x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- $x(t)$: periodic signal with period T

$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$ **Then** $z(t) = x(t)\ y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k * b_k = c_k$

$$= \frac{1}{T}\int_T x(t)\ y(t)\ e^{-jkw_0 t} dt$$

$$x(t) = \sum_{l=-\infty}^{+\infty} a_l\ e^{jlw_0 t}$$

$$= \frac{1}{T}\int_T \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}\ y(t)\ e^{-jkw_0 t} dt$$

$$= \frac{1}{T}\sum_{l=-\infty}^{+\infty} a_l \int_T y(t)\ e^{-j(k-l)w_0 t} dt = \frac{1}{T}\sum_{l=-\infty}^{+\infty} a_l\ b_{(k-l)}$$

$c_k = a_k * b_k$ **Then** $z(t) = x(t)\ y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k * b_k = c_k$

$$c_k = \frac{1}{T}\int_T z(t)\ e^{-jkw_0 t} dt$$

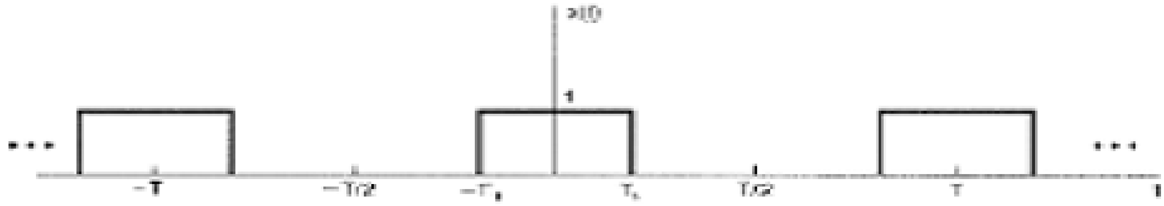
$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

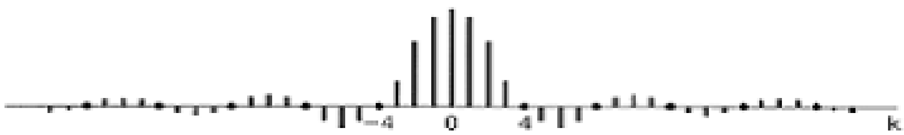
$$z(t) = x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

6) Parsevals Relationships

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$





(b)

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- Parseval's relation states that
  the total average power in a periodic signal
  equals
  the sum of the average powers
  in all of its harmonic components

Summary

Property	$\boldsymbol{x(t), y(t)}$	$\boldsymbol{X(j\omega), Y(j\omega)}$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{j\omega}{a})$
Convolution	$x(t) ** y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$X(j\omega) ** Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

UNIT 5: Fourier representation for signals – 2 **Teaching hours: 6**

5.1	Continuous Fourier transforms	
5.1.1	Continuous Fourier transforms properties.	
5.2	Discrete Fourier transforms	
5.2.2	Discrete Fourier transforms properties.	

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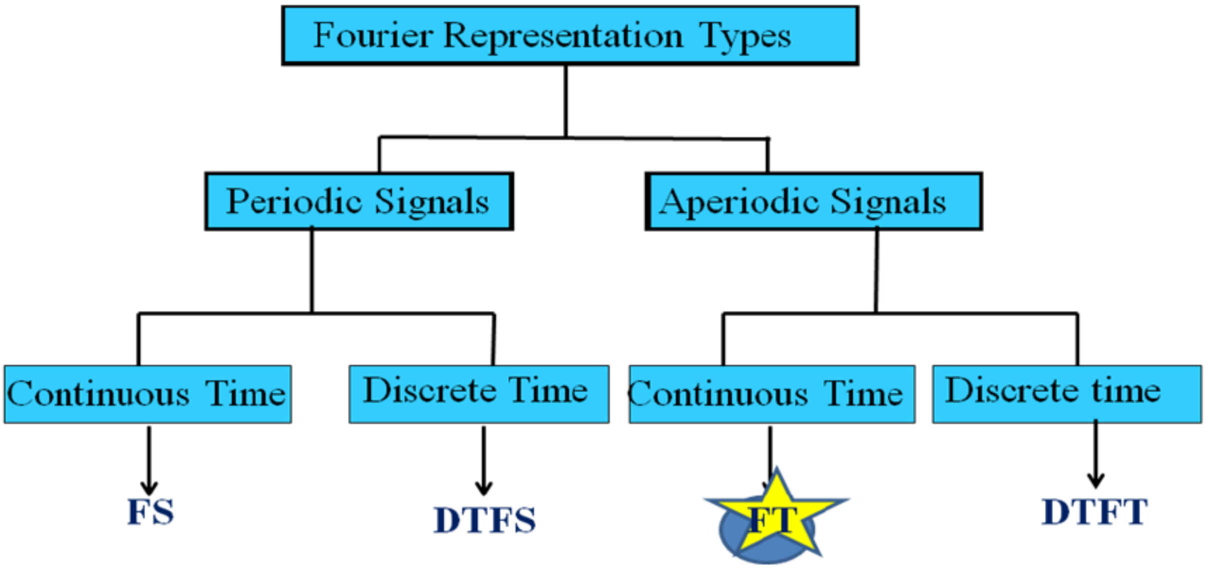
5.1 Introduction:

Fourier Representation for four Signal Classes



5.1.1Mathematical Development of Fourier Transform

For a periodic signal x(t) with period T and its exponential FS coefficients a_k are given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_o t} dt$$

If the period is stretched without limit, the periodic signal no longer remains periodic but remains a single pulse x(t), corresponding to one period of the periodic pulse, x(t). It also represents a transition from a power signal to an energy signal. As,

$T \rightarrow \infty,\ \omega o = \frac{2\pi}{T} \rightarrow 0,$

and the line spectrum becomes a continuous spectrum.

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_o t} dt$$

Also if we replace ωo by an infinitesimally small quantity dω ⇨ 0, the discrete frequency kωo may be replaced by a continuous frequency ω. The factor1/T in the equation means that

$$Ta_k = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_o t}\, dt$$

$$\lim_{T\to\infty} Ta_k = X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt$$

Continuous Time Fourier Transform

Hence, we can extend the formula for continuous-time Fourier series coefficients for a periodic signal

$$a_k = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jk\omega_o t} dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_o t} dt$$

to periodic signals as well. The continuous-time Fourier series is not defined for aperiodic signals, but we call the formula

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

the (continuous time) **Fourier transform**

5.2.2 Inverse Transforms

Replacing a_k with Ta_k and multiplying and dividing the RHS of FS Synthesis equation *by $2\pi/\omega o$ i.e*

$$x(t) = \sum_{k=-\infty}^{+\infty} \frac{T a_k e^{jk\omega_o t}}{T} = \sum_{k=-\infty}^{+\infty} T a_k e^{jk\omega_o t}\ \omega o/2\pi$$

As $T \rightarrow \infty$, Ta_k *becomes* $X(j\omega)$. *Also, as*

and $k\omega_o \rightarrow \omega$, *the summation becomes an integration from*

$-\infty$ *to* $+\infty$. *With* ω_o $d\omega$ o, *we get,*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

Inverse Transforms

If we have the full sequence of Fourier coefficients for a periodic signal, we can reconstruct it by multiplying the complex sinusoids of frequency $\omega_0 k$ by the weights X_k and summing:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

We can perform a similar reconstruction for aperiodic signals:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

These are called the inverse transforms

Comparison of Fourier Series and Fourier Transform

	Synthesis	Analysis
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$

FS coefficients *are a complex-valued function of integer k*, **FT** *is a complex-valued function of the variable* $-\infty < \omega < \infty$

Synthesis	
$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$x(t) \xleftrightarrow{FS} a_k$
shows how much there is of the signal at frequency kωo	
shows how much phase shift is needed at frequency kωo	
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$	$x(t) \xleftrightarrow{FT} X(j\omega)$
shows how much there is in the signal at frequency ω	
shows how much phase shift is needed at frequency ω	

5.1.3 Convergence Issues

Dirichlets Conditions

Condition 1:
Over any period, x(t) must be absolutely integrable, i.e each coefficient to be finite
Condition 2:
In any finite interval, x(t) is of bounded variation; i.e., – There are no more than a finite number of maxima and minima during any single period of the signal
Condition 3:
- In any finite interval, x(t) has only finite number of discontinuities.
- Furthermore, each of these discontinuities is finite

Convergence Issues

- Note that the above are sufficient conditions and not necessary conditions.
- Fourier transform for the analysis of many useful signals would be impossible if these were necessary conditions.
- FT Examples for typical signals

5.1.4FT Examples for typical signals

- Examples1
- Find the Fourier transform of the following functions:

(a) The unit Impulse
(b) The rect function
(c) The decaying exponential

- Ex 1(a) : The Unit Impulse

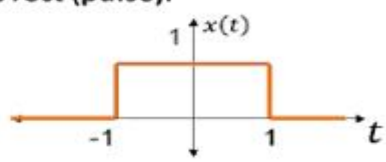
$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t} dt \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$= e^{-j\omega t} \Big|_{t=0} = 1$$

All the coefficients will have the same value i.e '1'

Example 1(b) : Find the Fourier transform of the rect (pulse).

$$x(t) = \begin{cases} 1 \; for -1 < t < 1 \\ 0 \; otherwise \end{cases}$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt = \int_{-1}^{+1} (1)\, e^{-j\omega t} dt$$

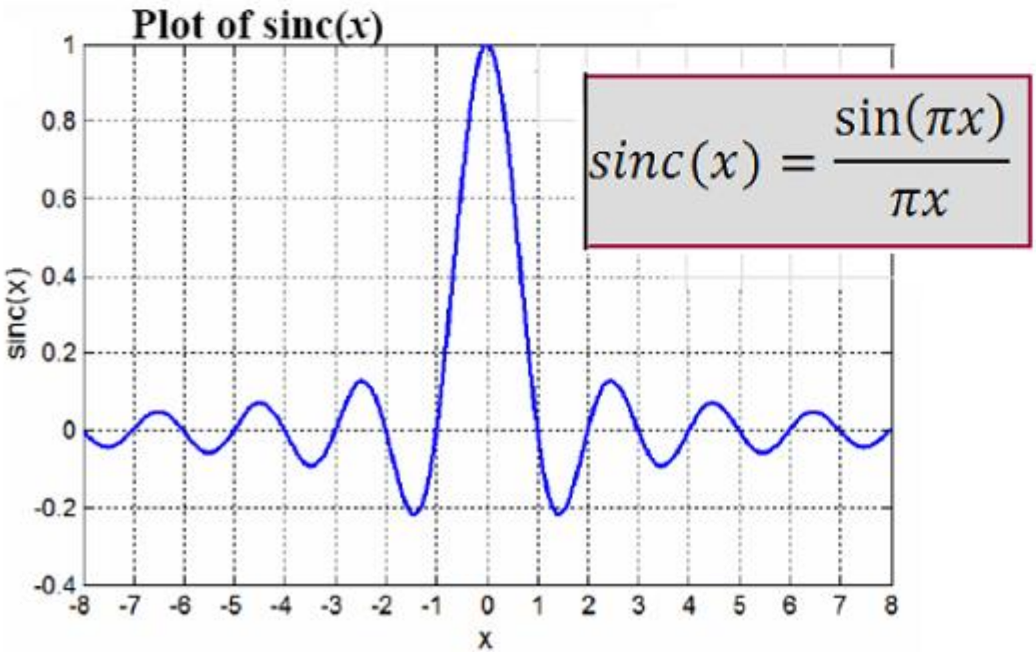
$$\frac{1}{-j\omega} e^{-j\omega t} \Big|_{t=-1}^{1} = \frac{1}{-j\omega}\left(e^{-j\omega} - e^{j\omega}\right)$$

$$\frac{1}{-j\omega}\left(-2j\sin(\omega)\right) = 2\sin\frac{(\omega)}{\omega} = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right) \quad \text{... a sinc function}$$

5.1.5 Definition of Sinc Function

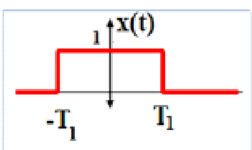
Definition of "Sinc" Function

The result we just found had this mathematical form:



$$sinc(x)=\frac{\sin(\pi x)}{\pi x}$$

Sync function



$x(t) \xleftrightarrow{FT} X(j\omega)$ $\qquad X(j\omega)=\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}dt$

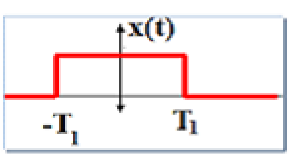
$$X(j\omega)=\int_{-T1}^{+T1} 1.e^{-j\omega t}dt = -\frac{e^{-j\omega t}}{(-j\omega)}\Big|_{-T1}^{+T1} \qquad X(j\omega)=\frac{2}{(\omega)}\frac{[e^{j\omega T1}-e^{-j\omega T1}]}{2j}$$

$$=\frac{2Sin(\omega T1)}{\omega}=\frac{2Sin\left(\frac{\omega T1\pi}{\pi}\right)}{\omega}=\frac{\frac{\pi T1}{\pi}2Sin\left(\left(\frac{\omega T1)\pi}{\pi}\right)\right)}{\pi\frac{\omega T1}{\pi}}=2T1sinc\left(\frac{\omega T1}{\pi}\right)$$

- $X(j\omega)==\frac{2Sin(\omega T1)}{\omega}=2\,T1sinc\left(\frac{\omega T1}{\pi}\right)$, *we know,* $\quad sinc(x)=\frac{sin(\pi x)}{\pi x}$

5.1.6 Band Limited and Time Concept

To Summarize for sync Function



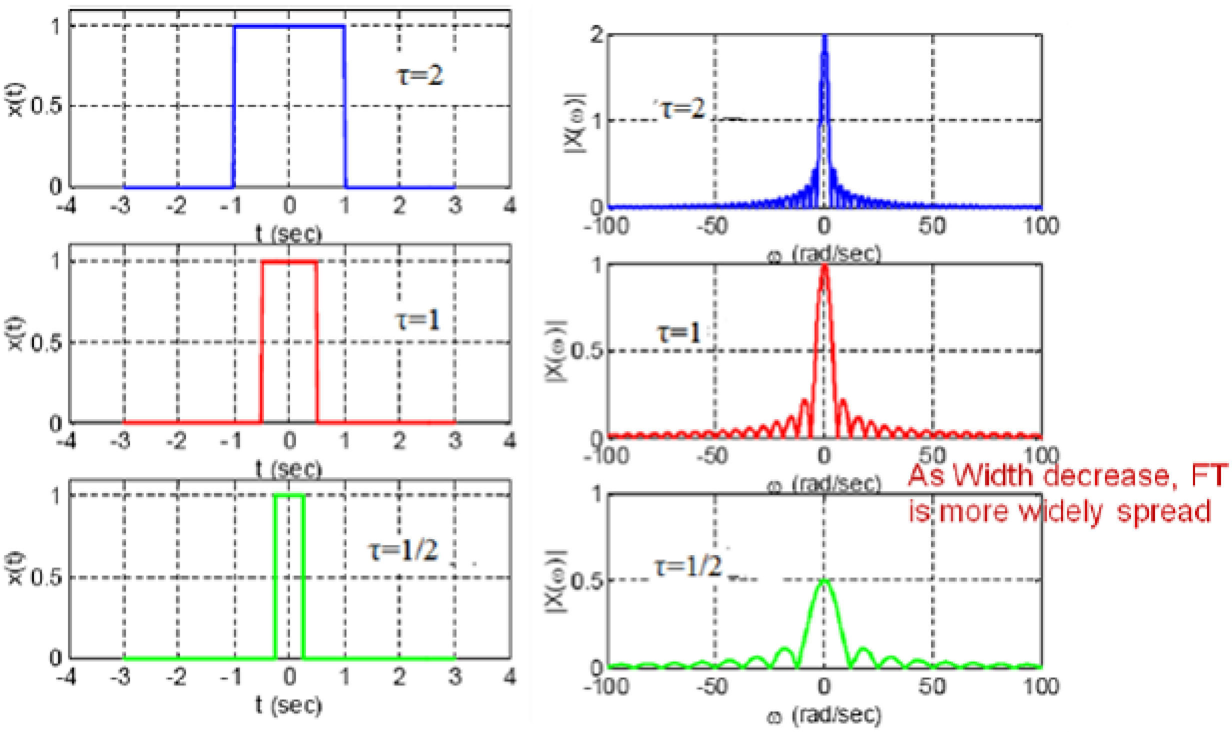
$x(t) \xleftrightarrow{FT} X(j\omega)$ $\qquad X(j\omega)=\int_{-T1}^{+T1} x(t)\, e^{-j\omega t}dt$

$$X(j\omega) = \frac{2Sin(\omega T1)}{\omega} = 2\,T1 sinc\left(\frac{\omega T1}{\pi}\right)$$

Example:

$$X(j\omega) = \int_{T1=-1}^{+1} 1.e^{-j\omega t}dt = \frac{2Sin(\omega)}{\omega} = 2\ sinc\left(\frac{\omega}{\pi}\right)$$

$$X(j\omega) = \int_{T1=-2}^{+2} 1.e^{-j\omega t}dt = \frac{2Sin(2\omega)}{\omega} = 4sinc\left(\frac{2\omega}{\pi}\right)$$



5.1.8 Properties of Fourier Transform

- Linearity
- Time Shift
- Frequency Shift
- Scaling
- Frequency Differentiation
- Time differentiation
- Convolution
- Integration or Accumulation
- Modulation
- Parsevals theorem or Rayleigh's theorem
- Duality or similarity theorem
- Symmetry

1. Linearity

$$If\ x(t) \xleftrightarrow{FT} X(\omega)$$

$$y(t) \xleftrightarrow{FT} Y(\omega)$$

$$\boxed{then\ \ [ax(t)+by(t)] \longleftrightarrow [aX(\omega)+bY(\omega)]}$$

Proof: $Z(j\omega) = \int_{-\infty}^{+\infty} [ax(t)+by(t)]\, e^{-j\omega t} dt$

$$= \int_{-\infty}^{+\infty} ax(t)\, e^{-j\omega t} dt + \int_{-\infty}^{+\infty} by(t)\, e^{-j\omega t} dt$$

$$= a\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt + b\int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t} dt$$

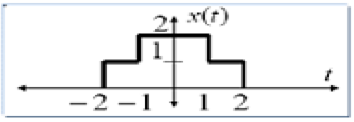
$Z(j\omega) = aX(j\omega) + bY(j\omega)$ ....Proved

$$Let\quad z(t) = ax(t) + by(t)$$

$$Let\quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt Let\quad Y(j\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t} dt$$

$$and\ \ Z(j\omega) = \int_{-\infty}^{+\infty} z(t)\, e^{-j\omega t} dt$$

- Find the FT of the following signal



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$= \int_{-2}^{-1} x(t)\, e^{-j\omega t} dt += \int_{-1}^{+1} x(t)\, e^{-j\omega t} dt + \int_{1}^{2} x(t)\, e^{-j\omega t} dt$$

$$= \int_{-2}^{-1} e^{-j\omega t} dt + 2\int_{-1}^{1} e^{-j\omega t} dt + \int_{1}^{2} e^{-j\omega t} dt$$

Finding this using straight forward approach is using definition is not difficult but tedious.

We have seen from our earlier analysis a FT of a pulse

$$X(j\omega) = 4sinc\left(\frac{2\omega}{\pi}\right) + 2sinc\left(\frac{\omega}{\pi}\right)$$

$$X(j\omega) = \int_{-2}^{+2} x1(t)e^{-j\omega t}dt + \int_{-1}^{+1} x2(t)e^{-j\omega t}dt$$

2. Time Shift Property

Shift of Time Signal gives"Linear" Phase Shift of Frequency Components.

$$If\ x(t) \xleftrightarrow{FT} X(j\omega)$$

$$then\ x(t-To) \xleftrightarrow{FT} X(j\omega)e^{-j\omega To}$$

No Change in Magnitude observed,

$$but\ produces\ \ phase\ shift\ of\ +\angle\ e^{-j\omega To}$$

$$i.e\ Phase\ Shift = \angle X(j\omega) + \angle\ e^{-j\omega To} \quad = \angle X(j\omega) + \omega To$$

$$If x(t) \xleftrightarrow{FT} X(j\omega) \qquad y(t) = x(at) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

If the time signal is Time Scaled by '*a*'. *Then…The FT is Freq. Scaled by '1/a'. And Amplitude scaling by in the frequency domain.That means Compression of x(t) to x(at) in time domain leads to Expansion of X(jω) by a An interesting "**duality**"!!!*

$$If x(t) \xleftrightarrow{FT} X(j\omega) \quad \text{then,} \qquad y(t) = x(at) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Proof: $$Y(j\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}dt$$

$$i.e\quad Y(j\omega) = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t}dt \qquad Let\ a > 0;\ \ put\ at = \lambda$$

$$Then, \qquad dt = \frac{d\lambda}{a}$$

$$accordingly, \qquad Y(j\omega) = \int_{-\infty}^{+\infty} x(\lambda)\, e^{-j(\lambda/a)\omega}\frac{d\lambda}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\lambda)\, e^{-j\omega(\lambda/a)}d\lambda \qquad = \frac{1}{a}X\left(\frac{\omega}{a}\right) \qquad \text{...... proved}$$

3) Frequency Shift Property

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

Then, $$y(t) = e^{j\beta t}x(t) \xleftrightarrow{FT} Y(j\omega) = X(j(\omega - \beta))$$

Proof : We know $$Y(j\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} e^{j\beta t}x(t)\, e^{-j\omega t}dt \quad = \int_{-\infty}^{+\infty} x(t)\, e^{-j(\omega-\beta)t}dt$$

$$= Xj(\omega - \beta)$$ ....proved

therefore $$y(t) = e^{j\beta t}x(t) \xleftrightarrow{FT} Y(j\omega) = X(j(\omega - \beta))$$

4) Time Differentiation

If

$$x(t) \xleftrightarrow{FT} X(j\omega)$$ Then, $$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Proof : we know, $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}d\omega$$

Differentiating both sides $$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)(j\omega\, e^{j\omega t})d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,(j\omega)e^{j\omega t}d\omega$$

$$= j\omega X(j\omega)$$ ....proved i.e $$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

5) Frequency Differentiation

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$then \quad -jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$$

Proof :

$$we\ know, \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

Differentiating both sides with respect to ω

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{+\infty} x(t)[-jt\, e^{-j\omega t}]dt \quad = \int_{-\infty}^{+\infty} [-jtx(t)\, e^{-j\omega t}]dt$$

$$Hence, -jtx(t) = \frac{d}{d\omega} X(j\omega) \ldots\ldots proved$$

6) Convolution

$$If \quad x(t) \xleftrightarrow{FT} X(j\omega) \quad and \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$then \ \ z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(j\omega) = X(j\omega)Y(j\omega)$$

Proof :

$$we\ know, \quad Z(j\omega) = \int_{t=-\infty}^{+\infty} z(t)\, e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{+\infty} x(t) * y(t)\, e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{+\infty} \left[\int_{T=-\infty}^{+\infty} x(T)y(t-T)dT \right] e^{-j\omega t} dt$$

rearranging the above, we get

$$Z(j\omega) = \int_{T=-\infty}^{+\infty} x(T) \left[\int_{t=-\infty}^{+\infty} y(t-T) e^{-j\omega t} dt \right] dT$$

$$put\ t-T=\lambda, then\ dt=d\lambda,$$

Accordingly we get,
$$=\int_{T=-\infty}^{+\infty} x(T)e^{-j\omega(T)}dT\left[\int_{\lambda=-\infty}^{+\infty} y(\lambda)e^{-j\omega\lambda}d\lambda\right]$$

$$Z(j\omega)=X(j\omega)Y(j\omega)\ldots\ldots Proved$$

7) Integration and Accumulation

$$If \quad x(t)\xleftrightarrow{FT}X(j\omega)$$

$$then, \quad \int_{-\infty}^{t} x(T)\xleftrightarrow{FT}\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

Proof : we can write, $\int_{-\infty}^{t} x(T)=x(t)*u(t)$

$$Hence, \quad \int_{-\infty}^{t} x(T)dT\xleftrightarrow{FT}X(j\omega)U(j\omega)\ldots.(1)$$

It has been proved that

$$u(t)\xleftrightarrow{FT}U(j\omega)=\pi\delta(\omega)+\frac{1}{j\omega}\ldots..(2)$$

Substituting (2) in (1)we get

$$\int_{-\infty}^{t} x(T)dT\xleftrightarrow{FT}X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$$

please note that

$$X(j\omega)\delta(\omega)=X(0)\delta(\omega)$$

$$\int_{-\infty}^{t} x(T)dT\xleftrightarrow{FT}X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$$

$$\xleftrightarrow{FT}\pi X(j\omega)\delta(\omega)+X(j\omega)\frac{1}{j\omega}$$

$$\xleftrightarrow{FT}\pi X(0)\delta(\omega)+X(j\omega)\frac{1}{j\omega}\ldots\ldots proved$$

5.2 Discrete Fourier transforms:

5.2.1 Fourier Transform for Discrete Time Signal Definition

Development of the Discrete-Time Fourier Transform

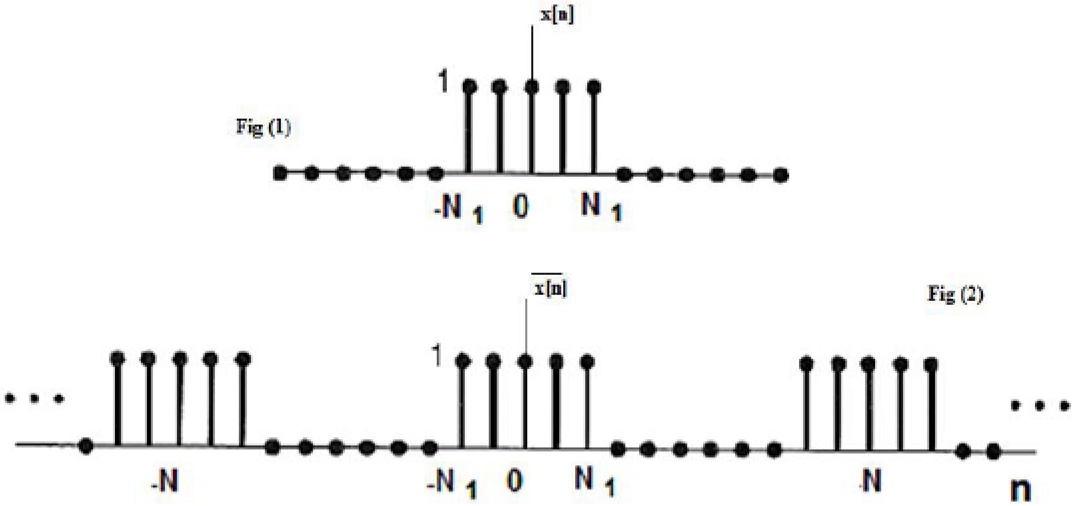


Fig 1: is a Finite duration signal x[n]; Fig 2 periodic signal

as we choose the period N to be larger, $\ddot{x}[n]$
is identical to $x[n]$ over a longer interval,
and as $N \rightarrow\gg \infty$, $\ddot{x}[n] = x[n]$

$As\ N\ increases\ \omega o\ decreses, and\ as\ N \rightarrow\gg \infty,$
$the\ synthsis\ equation\ passes\ to\ integral.$
$The\ coefficient\ is\ seen\ to\ be\ periodic\ in\ \Omega$
$with\ a\ period\ 2\pi\ \ and\ so\ is\ e^{j\Omega n}.$
$Hence, the\ X(j\Omega)\ will\ also\ be\ periodic,$
$\omega o = \dfrac{2\pi}{N}, the\ total\ integration\ will\ always$
$have\ a\ width\ of\ 2\pi.$

FT of an Aperiodic Signal

- **Define**

$$X(j\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}$$

$$over\ a\ period - N1 \le n \le +N1$$

$$a_k = \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk\omega_o n} \qquad = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} X(j\Omega) \qquad where, \omega o = \frac{2\pi}{N} \qquad \text{also } \frac{1}{N} = \frac{\omega o}{2\pi}$$

$$\ddot{x}[n] = \sum_{k=<N>} \frac{1}{N} X(j\Omega) e^{jk(2\pi/N)n} \qquad x[n] = \frac{1}{2\pi} \int_{2} X(j\Omega) e^{j\Omega n} d\Omega$$

Synthesis Equation

$$Therefore\ \ we\ can\ write\ as\ for\ DTFT$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(j\Omega) e^{j\Omega n}\, d\Omega \text{ .... Synthesis Equation}$$

$$X(j\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n} \text{ .... Analysis Equation}$$

$$a_k = \frac{1}{N} X(j\Omega) \qquad where, \omega o = \frac{2\pi}{N}$$

$$x[n] \xleftrightarrow{DTFT} X(j\Omega)$$

DTFT

It is convenient to think of a_k as being defined for all integers K. So:

1) a_k+N = a_k – Special property of DT Fourier coefficients
2) We only use N consecutive values of a_k in the synthesis equation
3) Since N is periodic, it is specified by a sequence of N numbers, either in the
   Time or in the Frequency domain.
4) It is convenient to think of a_k as being defined for all integers K. So:
   a) a_k+N = a_k – Special property of DT Fourier coefficients

   b) We only use N consecutive values of a_k in the synthesis equation
   c) Since N is periodic, it is specified by a sequence of N numbers, either in the Time or in the Frequency domain

	Time Domain	Frequency Domain	Time Domain	Frequency domain
Fourier Series FS	$x(t) \xleftrightarrow{FS} a_k$ $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ Continuous/Periodic	$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$ Aperiodic/Discrete	$x[n] \xleftrightarrow{DTFS} a_k$ Duality $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$ Discrete/Periodic	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk\omega_0 n}$ Periodic/Discrete
Fourier Transform FT	$x(t) \xleftrightarrow{FT} X(j\omega)$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ Continuous/aperiodic	Duality $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$ Aperiodic/Continuous	$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ $x[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ Discrete/aperiodic	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$ periodic/Continuous

5.2.1 Properties of Discrete-Time Fourier Transform

- **Linearity**
- **Time Shifting and Frequency Shifting**
- **Conjugation and Conjugate Symmetry**
- **Differencing and Accumulation**
- **Time Reversal**
- **Time Expansion**
- **Differentiation in frequency**
- **Parseval's Relation**

Discrete-Time Fourier Transform Pair

- **Notation**

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

- **Synthesis Equation**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

- **Analysis Equation**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\,e^{-j\omega n}$$

- **Also if**

$$x[n] = a^n u[n]$$

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

- **Periodicity of DTFT**

$$X(e^{j\omega+2\pi}) = \sum_{n=-\infty}^{+\infty} x[n]\,e^{-j\omega n}$$

$$= X(e^{j\omega})$$

Linearity Property

$$If, x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$y[n] \xleftrightarrow{DTFT} Y(e^{j\omega})$$

$$ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$$

Time Shifting

$$If, x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x[n - n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

Frequency Shifting

$$e^{-j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(\omega - \omega_0)$$

Conjugation and Conjugate Symmetry

$$if, x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$then\ x^*[n] \xleftrightarrow{DTFT} X^*(e^{j\omega})$$

- Also, if x[n] is real valued, its transform X(jΩ) is a Conjugate Symmetric. That is,
- From this it follows that Re{X(jΩ)} is an even function of ω and Im{X(jΩ)} is an odd function of ω
- Similarly, the magnitude is an even function and the phase angle is an odd function.

Conjugation and Conjugate Symmetry

- Futhermore,

$$Ev\{x[n]\} \xleftrightarrow{DTFT} Re\{X(e^{j\omega})\}$$

- And

$$Od\{x[n]\} \xleftrightarrow{DTFT} jIm\{X(e^{j\omega})\}$$

- Where 'Ev' and 'Od' denote the even and odd parts respectively of x[n]
- Eg.

$$Re/Ev\{x[n] = a^{|n|}\} \xleftrightarrow{DTFT} Re/Ev\{X(e^{j\omega})\}$$

Differencing and Accumulation

- Here we consider the Discrete time counter part of integration—i.e Accumulation--- and its inverse.

$$If, x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

- Then from Linearity and time Shifting properties, the FT of the first-difference signal

$$x[n] - x[n-1] \xleftrightarrow{DTFT} (1 - e^{-j\omega})X(e^{j\omega})$$

$$x[n] - x[n-1] \xleftrightarrow{DTFT} (1 - e^{-j\omega})X(e^{j\omega})$$

- Next consider the signal

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]$$

- Next, $y[n] - y[n-1] = x[n]$

$$\Rightarrow (1 - e^{-j\omega})Y(e^{j\omega}) = X(e^{j\omega}) \Rightarrow Y(e^{j\omega}) = \frac{1}{(1 - e^{-j\omega})}X(e^{j\omega})$$

Differencing and Accumulation

- This is partly right, but as with integration in CT, the relationship is

$$\sum_{m=-\infty}^{+\infty} x[m] \xleftrightarrow{DTFT} \frac{1}{(1 - e^{-j\omega})}X(e^{j\omega}) + \pi X(j0) + \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

- The Impulse train on the right-hand side reflects the dc or average value that results due to summation.

Example

- Derive the FT X(jΩ) of the unit step x[n]=u[n] using the accumulation property.
- We know $g[n] = \delta[n] \xleftrightarrow{DTFT} G(e^{j\omega}) = 1$
- Also,a unit step is the running sum of unit impulse. $x[n] = \sum_{m=-\infty}^{n} g[m]$
- Taking FT of both sides and using accumulation yields

Differentiation in Frequency

- ***Proof :***
- *Let,* $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$
- Differentiating both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega}\sum_{n=-\infty}^{+\infty} x[n]\,e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} -jnx[n]\,e^{-j\omega n}$$

- The RHS is the FT of .Multiplying both sides by j, we see that

$$j\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} nx[n]\,e^{-j\omega n} \qquad nx[n] \xleftrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$$

Time Reversal

Let, $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ *and* $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\,e^{-j\omega n}$

Let, $y[n] = x[-n]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]\,e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]\,e^{-j\omega n}$$

- Substituting m=-n, we obtain

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]\,e^{-j(-\omega)m} = X(e^{-j\omega})$$

- That is $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$

Time Expansion

- The relation between time and frequency scaling in DT takes on a different form as that of CT.
- In CT, $y(t) = x(at) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$
- However to try to take x[an], we run into difficulties if a is not an integer.
- Even for integers, x[2n]—we do not merely speed up the original signal. Since x[2n] consists of the even samples of x[n] alone

Let,

$$x_{(k)}[n]=\begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

- For, k=3, $x_{(k)}[n]$ is obtained from x[n] by placing k-1 zeroes between successive values of the original signal. It's a slowed down version of x[r], where r=n/k,

- The FT is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]\, e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]\, e^{-j\omega rk}$$

$$\text{Further,} \quad x_{(k)}[rk] = x[r] \quad = \sum_{r=-\infty}^{+\infty} x[r]\, e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$\text{That is, } x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$$

Time Expansion

- The relation between time and frequency scaling in DT takes on a different form as that of CT.
- In CT, $y(t) = x(at) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$
- However to try to take x[an], we run into difficulties if a is not an integer.
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Let,

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- For, k=3, $x_{(k)}[n]$ is obtained from x[n] by placing k-1 zeroes between successive values of the original signal. It's a slowed down version of x[r], where r=n/k,
- The FT is then given by

$$\boxed{x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})}$$

- **Proof:** The FT is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]\, e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]\, e^{-j\omega rk}$$

$$further, \quad x_{(k)}[rk] = x[r] = \sum_{r=-\infty}^{+\infty} x[r]\, e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$That\ is, x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$$

Parsewal's Relation

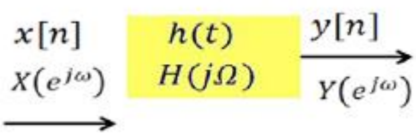
- The quantity in the LHS is the total energy x[n], and Parsewals relation states that this energy can also be determined by integrating the energy per unit frequency

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$\sum_{n=-\infty}^{+\infty} |\, x[n]\, |^2 = \frac{1}{2\pi}\int_{2\pi} |\, X(e^{j\omega})\, |^2\, d\omega$$

Convolution Property

$$y[n] = x[n] * h[n] = \sum_{n=-\infty}^{+\infty} x[k]h[n-k]$$



$$y[n] \xleftrightarrow{DTFT} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Duality Property

- In CTFT, we observed a symmetry or duality between the analysis equation and the synthesis equation.
- Such Duality is not seen in DTFT.

- However, such Duality exists between the analysis and synthesis equation of DTFS
- In addition, there is a duality relationship between the DTFT and the CTFS

	Time Domain	Frequency Domain	Time Domain	Frequency domain
Fourier Series FS	$x(t) \xleftrightarrow{FS} a_k$ $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ Continuous/Periodic	$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$ Aperiodic/Discrete	Duality $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ Discrete/Periodic	$a_k = \frac{1}{N}\sum_{k=<N>} x[n] e^{-jk\omega_0 n}$ Periodic/Discrete
Fourier Transform FT	$x(t) \xleftrightarrow{FT} X(j\omega)$ Duality $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$ Continuous/aperiodic	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$ Aperiodic/Continuous	Duality $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ $x[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$ Discrete/aperiodic	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$ periodic/Continuous

UNIT 6: Applications of Fourier representations **Teaching hours: 7**

6.1	Introduction	
6.2	Frequency response of LTI systems	
6.3	Fourier transform representation of periodic signals	
6.4	Fourier transform representation of discrete time signals	

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6.1 Introduction:

The Response of LTI Systems to Complex Exponentials

- The set of complex exponential signals

  Signals of the form e^{st} in CT

  Signals of the form z^n *in DT*

- The Response of an LTI System:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

CT : $e^{st} \rightarrow H(s)e^{st}$ ⟶ **Eiigenfunction**

DT : $z^n \rightarrow H(z)z^n$ ⟶ **Eiigenvalue**

As an illustration, consider an LTI system for which the input x(t) and output y(t) are related by a time shift of 3, i.e y(t)=x(t-3). If the input to this system is the complex exponential signal $x(t) = e^{j2t}$

- Soln:

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$$
$$= H(s)x(t)$$
$$where, H(s) = e^{-j6} (an\ eigen\ value)$$
$$and\ x(t) = e^{j2t}\ (an\ eigen\ function)$$

The impulse response is

$$and\ so\ H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - 3)\, e^{-s\tau} d\tau$$
$$= e^{-3s}$$
$$so\ that\ H(j2) = e^{-j6} \qquad h(t) = \delta(\tau - 3)d\tau$$

Fourier Series and LTI System

- Fourier series representation can be used to construct any periodic signals in discrete as well as continuous-time signals of practical importance.
- We have also seen the response of an LTI system to a linear combination of complex exponentials taking a simple form.
- Now, let us see how Fourier representation is used to analyze the response of LTI System.

Consider the CTFS synthesis equation for x(t) given by
Suppose we apply this signal as an input to an LTI System with impulse respose h(t). Then, since each of the complex exponentials in the expression is an eigen function of the system. Then, with $sk = jk\omega o$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega o}) e^{jk\omega ot}$$

Thus y(t) is periodic with frequency as x(t). Further, if ak is the set of Fourier series coefficients for the input x(t), then $\{a_k H(e^{jk\omega o})\}$ is the set of coefficient for the y(t). Hence in LTI, modify each of the Fourier coefficient of the input by multiplying by the frequency response at the corresponding frequency.

Example:

Consider a periodic signal x(t), with fundamental frequency 2π, that is expressed in the form

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \quad \text{.......(1)}$$

$$where, \quad a_0=1, \quad a_1=a_{-1}=1/4\ , \quad a_2=a_{-2}=1/2, \quad a_3=a_{-3}=1/3,$$

Suppose that the this periodic signal is input to an LTI system with impulse response To calculate the FS Coeff. Of o/p y(t), lets compute the frequency response.The impulse response is therefore,

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau \qquad = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau} \Big|_0^{\infty}$$

and

$$H(j\omega) = \frac{1}{1+j\omega}$$

Y(t) at ωo = 2π . We obtain,

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t} \qquad with\ b_k = a_k H(jk2\pi), so\ that$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right) b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right) b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right) b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right) \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

$$b_o = 1$$

The above o/p coefficients. Could be substituted in

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have H(ω).

To find the frequency response H(ω) for a system, we can:

1. Put the input $x(t) = e^{i\omega t}$ into the system definition
2. Put in the corresponding output $y(t) = H(\omega)\, e^{i\omega t}$
3. Solve for the frequency response H(ω). (The terms depending on t will cancel.)

Example:

Consider a system with impulse response

$$h(t) = \begin{cases} \frac{1}{5} & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output corresponding to the input x(t) = cos(10 t).

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{\tau=0}^{5} \frac{1}{5} \cos(10(t-\tau))\, d\tau$$

$$y(t) = \frac{1}{5}\left(-\frac{1}{10}\sin(10(t-\tau))\right)\Big|_{\tau=0}^{5} = \frac{1}{50}\left(\sin(10t) - \sin(10(t-5))\right)$$

Differential and Difference Equation Descriptions

Frequency Response is the system's steady state response to a sinusoid. In contrast to differential and difference-equation descriptions for a system, the frequency response description cannot represent initial conditions, it can only describe a system in a steady state condition. The differential-equation representation for a continuous-time system is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

$$since, \frac{d}{dt} g(t) \xleftrightarrow{FT} j\omega G(j\omega)$$

Rearranging the equation we get

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The frequency of the response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Hence, the equation implies the frequency response of a system described by a linear constant-coefficient differential equation is a ratio of polynomials in jω.

The difference equation representation for a discrete-time system is of the form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take the DTFT of both sides of this equation, using the time-shift property.

$$g[n-k] \xleftrightarrow{DTFT} e^{-jk\omega} G(e^{j\omega})$$

To obtain

$$\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k Y(e^{j\omega}) = \sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k X(e^{j\omega})$$

- Rewrite this equation as the ratio

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k (e^{j\omega})^k}$$

- The frequency response is the polynomial in $e^{j\omega}$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k (e^{j\omega})^k}$$

Differential Equation Descriptions

Ex: Solve the following differential Eqn using FT.

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

For all t where, $x(t) = (1 + e^{-t})u(t)$

Soln:we have

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

FT gives,

$$[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$$

$$and\, x(t) = (1 + e^{-t})u(t) \qquad x(t) = u(t) + (e^{-t})u(t)$$

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}\; since\; u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

$$and (e^{-t})u(t) \xleftrightarrow{FT} \frac{1}{j\omega + 1}$$

Hence we have

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}$$

And $[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$

i.e

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega)^2 + 4(j\omega) + 5]}X(j\omega)$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1]}\left[\frac{1}{j\omega} + \pi\delta(\omega) + \frac{1}{(j\omega + 1)}\right]$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega)^2 + 4(j\omega) + 5]}\left[\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}\right]$$

$$\boxed{Y(j\omega) = Y(1) + Y(2) + Y(3)}$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1]j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1](j\omega + 1)}$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1]j\omega} + \frac{(3j(\omega = 0) + 1)\pi[\delta(0) = 1]}{[(j(\omega = 0) + 2)^2 + 1]j(\omega = 0)} + \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1](j\omega + 1)}$$

$$Y(1) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1]j\omega} \quad Y(1) = \frac{A}{j\omega} + \frac{Bj\omega + C}{[(j\omega + 2)^2 + 1]}$$

$$Performing\; partial\; fraction\; we\; get\; A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{11}{5}$$

$$\boxed{Y(1) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega + 2)^2 + 1]}}$$

Similarly

$$Y(3) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1](j\omega + 1)}$$

$$Y(3) = \frac{R}{(j\omega + 1)} + \frac{Pj\omega + Q}{[(j\omega + 2)^2 + 1]}$$

$$Performing\; partial\; fraction\; we\; get\; R = -1, P = 1, Q = 6$$

$$\boxed{Y(3) = \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]}}$$

$$Y(3) = \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]} Y(j\omega) = Y(1) + Y(2) + Y(3)$$

$Hence, we\ have$

$$Y(1) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega + 2)^2 + 1]}$$

$$Y(2) = \frac{\pi}{5}\delta(\omega)$$

$Readjusting$

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega + 2)^2 + 1]} + \frac{\pi}{5}\delta(\omega) + \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]}$$

$$\boxed{Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega + 1)} + \frac{1}{5}\left[\frac{4j\omega + 41}{[(j\omega + 2)^2 + 1]}\right]}$$

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{11/5 - 1/5j\omega}{[(j\omega + 2)^2 + 1]} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]} - \frac{1}{(j\omega + 1)}$$

$we\ know\ that,$

$$e^{-\beta t}\cos\omega_o t u(t) \xleftrightarrow{FT} \frac{\beta + j\omega}{[(\beta + j\omega)^2 + \omega_o^2]}$$

$$e^{-\beta t}\sin\omega_o t u(t) \xleftrightarrow{FT} \frac{\omega_o}{[(\beta + j\omega)^2 + \omega_o^2]}$$

$Readjusting\ the\ last\ term, we\ get$

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega + 1)} + \frac{4}{5}\left[\frac{j\omega + 2}{[(j\omega + 2)^2 + 1]}\right] + \frac{33}{5}\left[\frac{1}{[(j\omega + 2)^2 + 1]}\right]$$

$Now, taking\ the\ inverse\ Fourier\ Transform, we\ get$

$$\boxed{y(t) = \frac{1}{5}u(t) - e^{-t}u(t) + \frac{4}{5}e^{-2t}\cos t\, u(t) + \frac{33}{5}e^{-2t}\sin t\, u(t)}$$

<u>Differential Equation Descriptions</u>

- Ex: Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

Here we have N=2, M=1. Substituting the coefficients of this differential equation in

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Differential Equation Descriptions

We obtain

$$H(j\omega)=\frac{2j\omega+1}{(j\omega)^2+3j\omega+2}$$

The impulse response is given by the inverse FT of H(jω). Rewrite H(jω) using the partial fraction expansion.

$$H(j\omega)=\frac{A}{j\omega+1}+\frac{B}{j\omega+2}$$

Solving for A and B we get, A=-1 and B=3. Hence

$$H(j\omega)=\frac{-1}{j\omega+1}+\frac{3}{j\omega+2}$$

The inverse FT gives the impulse response

$$\boxed{h(t)=3e^{-2t}u(t)-e^{-t}u(t)}$$

Difference Equation

Ex: Consider an LTI system characterized by the following second order linear constant coefficient difference equation.

$$y[n]=1.3433y[n-1]-0.9025y[n-2]+x[n]$$
$$-1.4142x[n-1]+x[n-2]$$

Find the frequency response of the system.

Soln:

$$y[n]=1.3433y[n-1]-0.9025y[n-2]+x[n]$$
$$-1.4142x[n-1]+x[n-2]$$

$$Y(e^{j\omega})=1.3433(e^{-j\omega})Y(e^{j\omega})$$
$$-0.9025(e^{-j2\omega})Y(e^{j\omega})+X(e^{j\omega})$$
$$-1.4142(e^{-j\omega})X(e^{j\omega})+(e^{-j2\omega})X(e^{j\omega})$$

$$we\ know, y[n-k]\xleftrightarrow{DTFT}e^{-jk\omega}Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$
$$= \frac{1 - 1.4142e^{-j\omega} + e^{-j2\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-j2\omega}}$$

Ex: If the unit impulse response of an LTI System is $h(n)=\alpha^n u[n]$, find the response of the system to an input defined by $x[n] = \beta^n u[n]$, where $\beta, \alpha < 1$ and $\alpha \neq \beta$

Soln:

$$y[n] = h[n] * x[n]$$

Taking DTFT on both sides of the equation, we get

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}} = \frac{A}{1 - \alpha e^{-j\omega}} \text{x} \frac{B}{1 - \beta e^{-j\omega}}$$

where A and B are constants to be found by using partial fractions

$$Let, \ e^{-j\omega} = v \qquad Then, Y(e^{j\omega}) = \frac{A}{1 - \alpha v} \text{x} \frac{B}{1 - \beta v}$$

By performing partial fractions, we get $A = \frac{\alpha}{\alpha - \beta}$, $B = \frac{-\beta}{\alpha - \beta}$

$$Therefore, Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha - \beta}}{1 - \alpha\, e^{-j\omega}} \text{x} \frac{\frac{-\beta}{\alpha - \beta}}{1 - \beta\, e^{-j\omega}}$$

Taking inverse DTFT, we get

$$\boxed{y[n] = \left[\frac{\alpha}{\alpha - \beta}\alpha^n - \frac{\beta}{\alpha - \beta}\alpha^n\right] u[n]}$$

Sampling

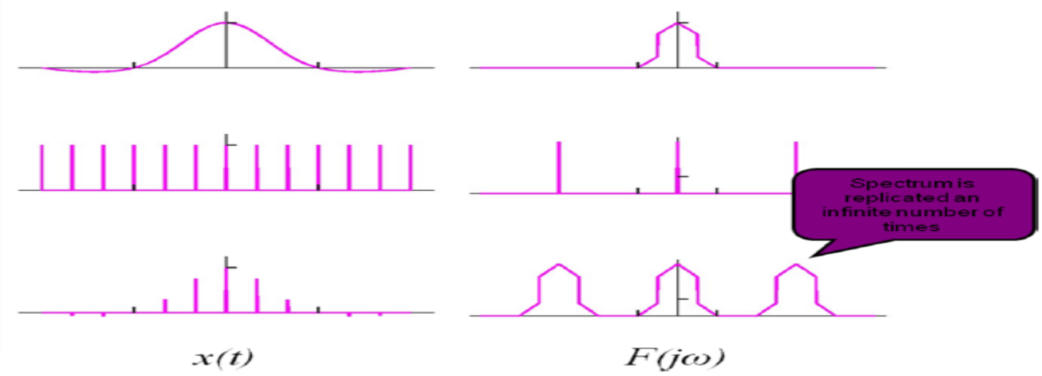
In this chapter let us understand the meaning of sampling and which are the different methods of sampling. There are the two types. Sampling Continuous-time signals and **Sub-sampling**. In this again we have *Sampling Discrete-time signals*.

Sampling Continuous-time signals

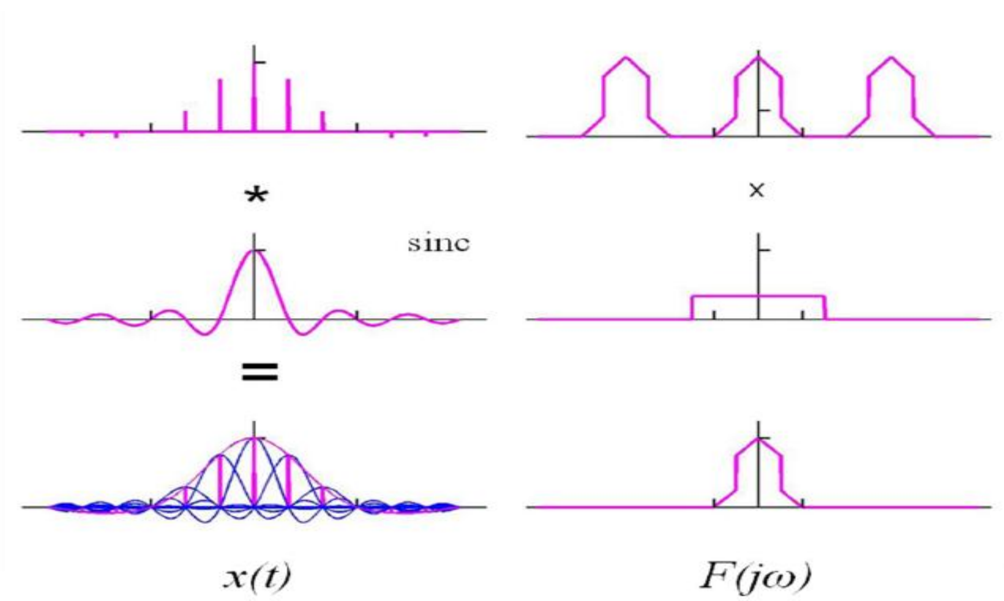
Sampling of continuous-time signals is performed to process the signal using digital processors. The sampling operation generates a discrete-time signal from a continuous-time signal.DTFT is used to analyze the effects of uniformly sampling a signal.Let us see, how a DTFT of a sampled signal is related to FT of the continuous-time signal.

- Sampling: Spatial Domain: A continuous signal x(t) is measured at fixed instances spaced apart by an interval 'T'. The data points so obtained form a discrete signal x[n]=x[nT]. Here, ΔT is the sampling period and 1/ ΔT is the sampling frequency.Hence, sampling is the multiplication of the signal with an impulse signal.

- **Sampling theory**



- **Reconstruction theory**



Sampling: Spatial Domain

From the Figure we can see

Where x[n] is equal to the samples of x(t) at integer multiples of a sampling interval T

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

Now substitute $x(nT)$ for $x[n]$ to obtain

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n\tau)\, \delta(t - n\tau)$$

$$since\ x(t)\delta(t - n\tau) = x(n\tau)\delta(t - n\tau)$$

we may rewrite $x_\delta(t)$ as a product of time functions

$$x_\delta(t) = x(t)p(t) \qquad where, \qquad p(t) = \delta(t - n\tau)$$

Hence, Sampling is the multiplication of the signal with an impulse train.

The effect of sampling is determined by relating the FT of $x_\delta(t)$ to the FT of $x(t)$. Since Multiplication in the time domain corresponds to convolution in the frequency domain, we have

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

Substituting the value of $P(j\omega)$ as the FT of the pulse train i.e

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

We get,

$$P(j\omega) = \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$where, \omega_s = \frac{2\pi}{\tau}, is\ the\ sampling\ frequency. Now$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

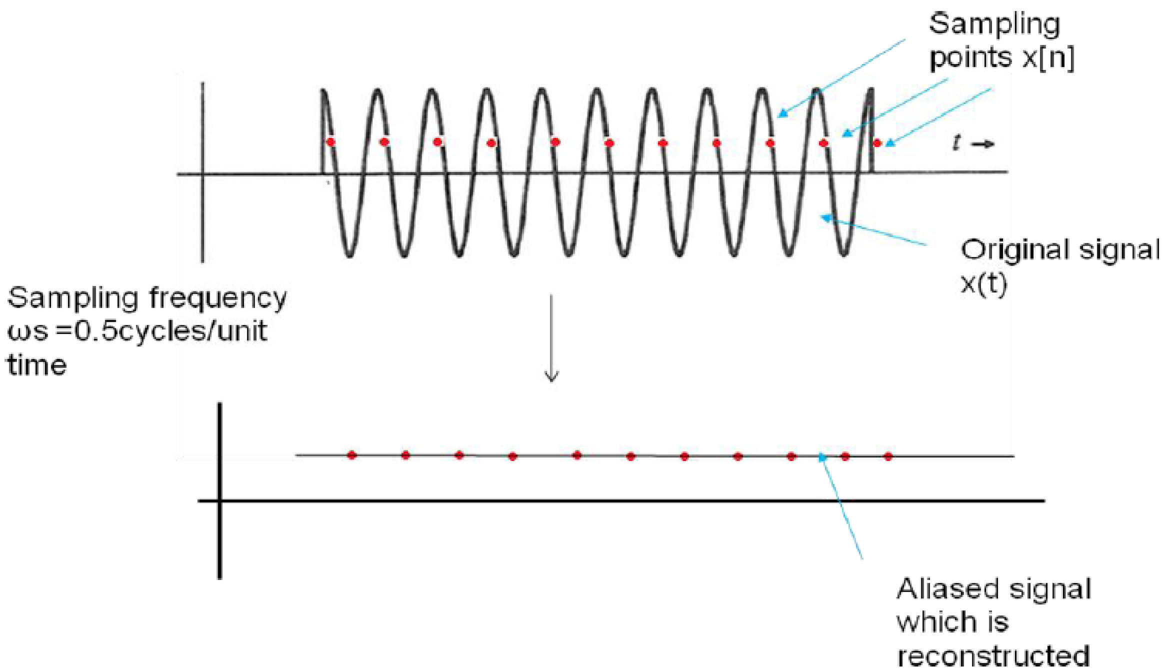
$$\boxed{X_\delta(j\omega) = \frac{1}{\tau} \sum_{n=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

The FT of the sampled signal is given by an infinite sum of shifted version of the original signals FT and the offsets are integer multiples of ω_s.

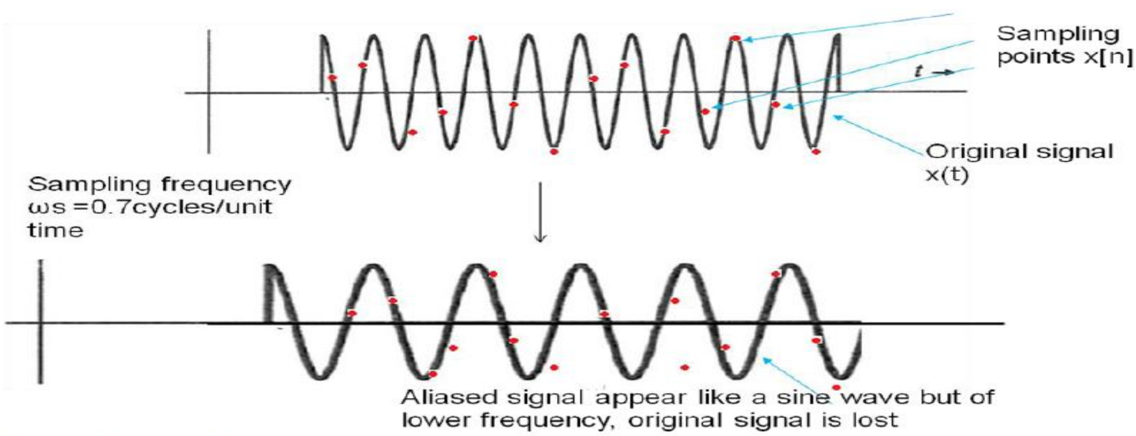
Aliasing : an example

Frequency of original signal is 0.5 oscillations per time unit). Sampling frequency is also 0.5 oscillations per time unit). Original signal cannot be recovered.

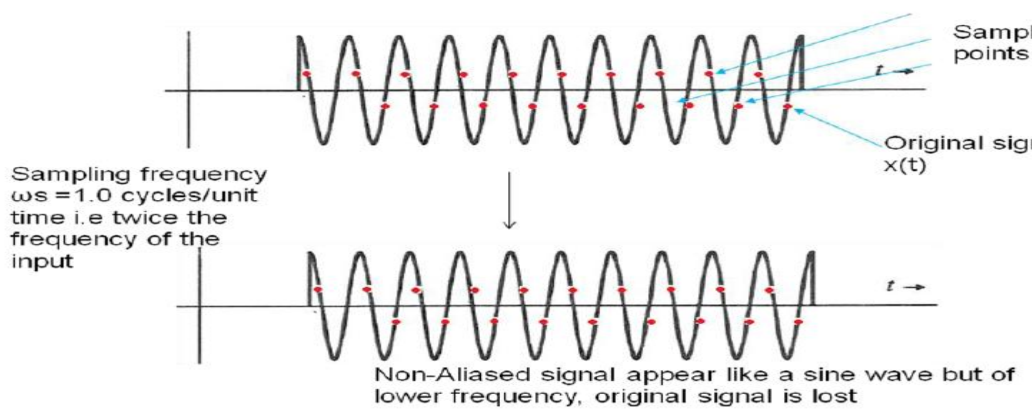
Aliasing Ex:1



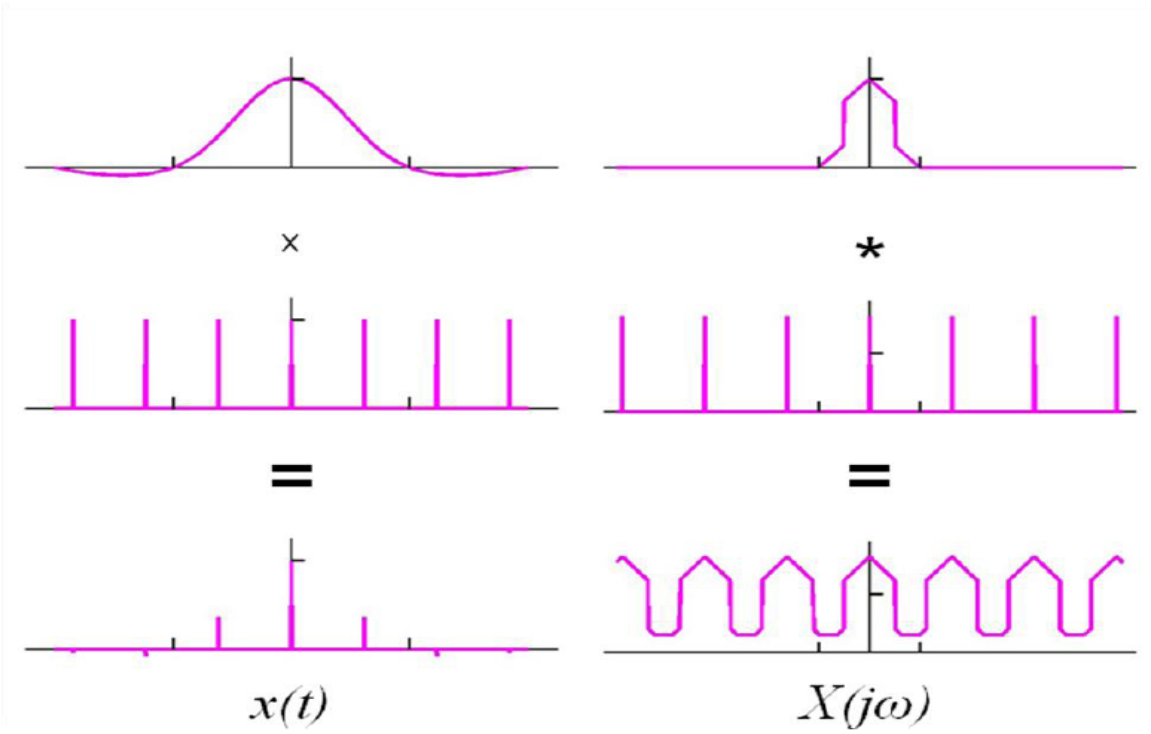
Aliasing Ex:2



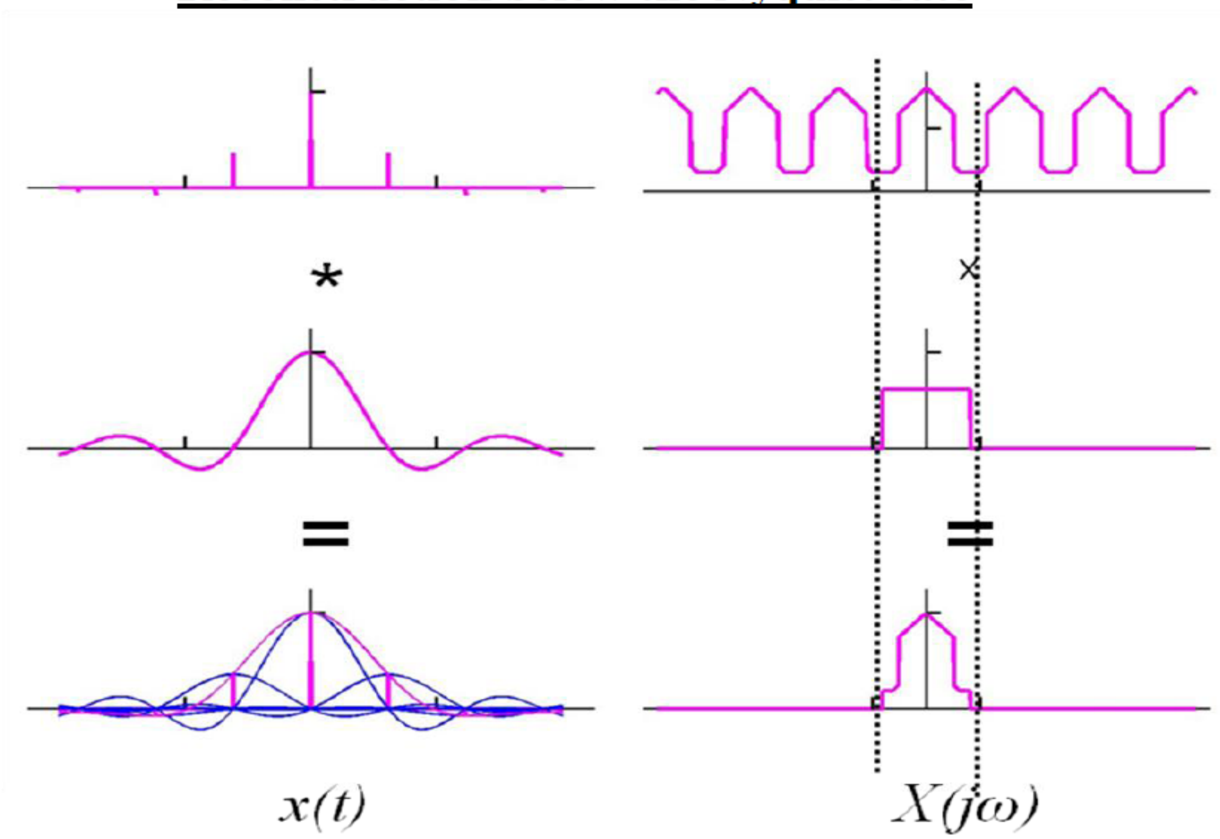
Non-Aliasing: Ex 3



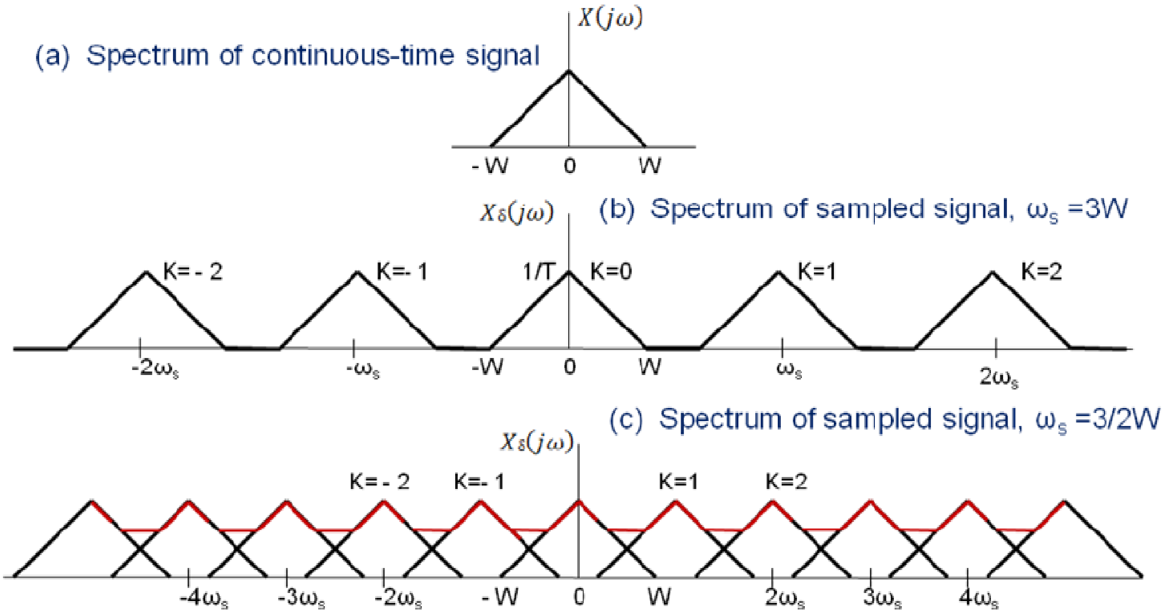
Sampling below the Nyquist rate



Reconstruction below the Nyquist rate



FT of sampled signal for different sampling frequency



- Reconstruction problem is addressed as follows.
- Aliasing is prevented by choosing the sampling interval T so that $\omega_s > 2W$, where W is the highest frequency component in the signal.
- This implies we must satisfy $T < \pi/W$.
- Also, DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ using the relationship $\Omega = \omega T$, that is

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X_\delta(j\omega)\big|_{\omega = \Omega/T}$$

- This scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$

Subsampling: Sampling discrete-time signal

- FT is also used in discrete sampling signal.
- Let $y[n] = x[qn]$ be a subsampled version x[n], where q is a positive integer.
- Relating DTFT of y[n] to the DTFT of x[n], by using FT to represent x[n] as a sampled versioned of a continuous time signal x(t).
- Expressing now y[n] as a sampled version of the sampled version of the same underlying CT x(t) obtained using a sampling interval q that associated with x[n]
- We know to represent the sampling version of x[n] as the impulse sampled CT signal with sampling interval T.

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

- Suppose, x[n] are the samples of a CT signal x(t), obtained at integer multiples of T. That is, x[n]=x[nT]. Let $x(t) \xleftrightarrow{FT} X(j\omega)$ and applying it to obtain

$$\boxed{X_\delta(j\omega) = \frac{1}{\tau} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

- Since y[n] is formed using every qth sample of x[n], we may also express y[n] as a sampled version of x(t).we have $y[n] = x[qn] = x(nq\tau)$
- Hence, active sampling rate for y]n] is T'=qT. Hence

$$y_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau') \xleftrightarrow{FT} Y_\delta(j\omega) = \frac{1}{\tau'} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s'))$$

- Hence substituting T'=q T, and $\omega_s' = \omega_s/q$

$$Y_\delta(j\omega) = \frac{1}{q\tau} \sum_{k=-\infty}^{+\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

- We have expressed both $Y_\delta(j\omega)$ and $X_\delta(j\omega)$ as a function of .
- Expressing $X(j\omega)$ as a function of $X_\delta(j\omega)$. Let us write k/q as a proper function, we get

$$\frac{k}{q} = l + \frac{m}{q},$$

where l is the integer portion of $\frac{k}{q}$, *and m is the remainder*

allowing k to range from $-\infty$ *to* $+\infty$ *corresponds*

to having l range from $-\infty$ *to* $+\infty$ *and m from 0 to* $q-1$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} \left\{ \frac{1}{\tau} \sum_{l=-\infty}^{+\infty} X_\delta\left(j\left(\omega - l\omega_s - \frac{m}{q}\omega_s\right)\right)\right\}$$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} X_\delta\left(j\left(\omega - \frac{m}{q}\omega_s\right)\right)$$

which represents a sum of shifted versions of

$X_{\delta}(j\omega)$ *normalized by q.*

Converting from the FT representation back to DTFT and substituting $\Omega = \omega\tau'$ *above*

and also $X(e^{j\Omega}) = X_{\delta}(j\Omega/\tau)$ *, we write this result as*

$$\boxed{Y_{\delta}(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-i} X_{q}\left(e^{j(\Omega - m2\pi)}\right)}$$

where, $X_{q}(e^{j\Omega}) = X(e^{j\Omega/q})$ *– a scaled DTFT version*

UNIT 7: Z-Transforms – 1 **Teaching hours: 7**

7.1	Introduction	
7.2	Z – transform	
7.3	Properties of ROC	
7.4	Properties of Z – transforms	
7.5	Inversion of Z – transforms	

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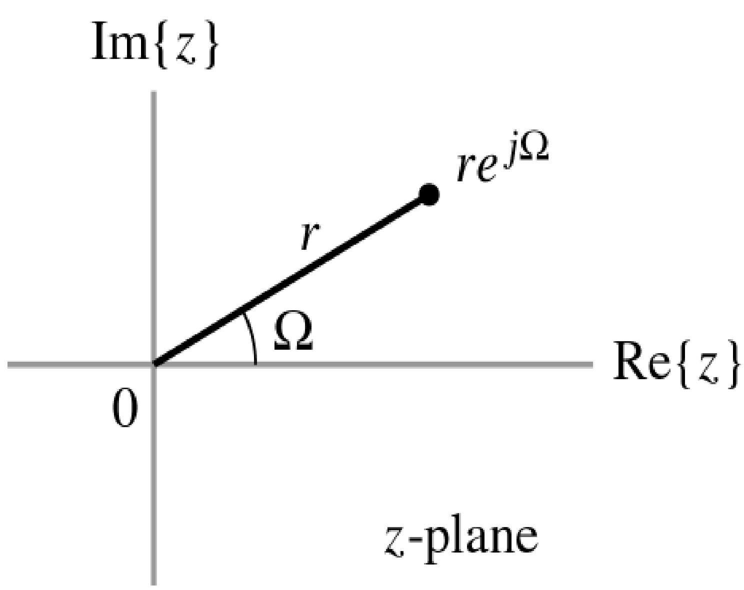
7.1 Introduction to *z*-transform:

The *z*-transform is a transform for sequences. Just like the Laplace transform takes a function of *t* and replaces it with another function of an auxiliary variable *s*. The *z*-transform takes a sequence and replaces it with a function of an auxiliary variable, *z*. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace transform, where taking the Laplace transform makes it easier to solve differential equations. A difference equation is an equation which tells you what the *k*+2th term in a sequence is in terms of the *k*+1th and *k*th terms, for example. Difference equations arise in numerical treatments of differential equations, in discrete time sampling and when studying systems that are intrinsically discrete, such as population models in ecology and epidemiology and mathematical modelling of mylinated nerves. Generalizes the complex sinusoidal representations of DTFT to more generalized representation using complex exponential signals



• It is the discrete time counterpart of Laplace transform

The z-Plane

• Complex number $z = re^{j\Omega}$ is represented as a location in a complex plane (z-*plane*)

7.2 The z-transform:

• Let $z = re^{j\Omega}$ be a complex number with magnitude r and angle Ω.

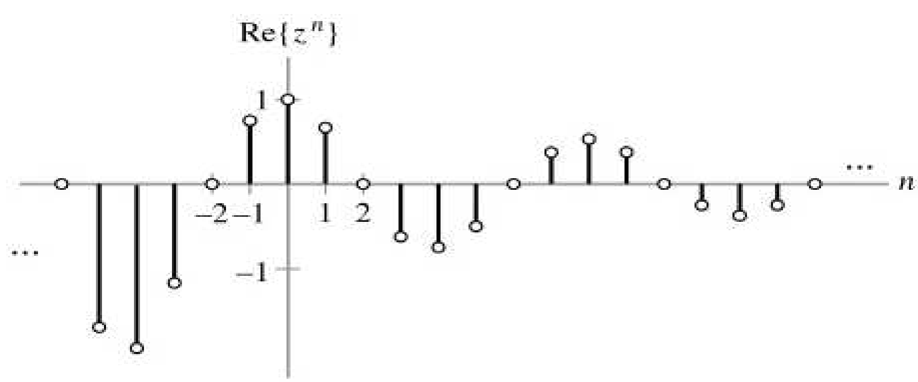
• The signal $x[n] = z^n$ is a complex exponential and $x[n] = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$

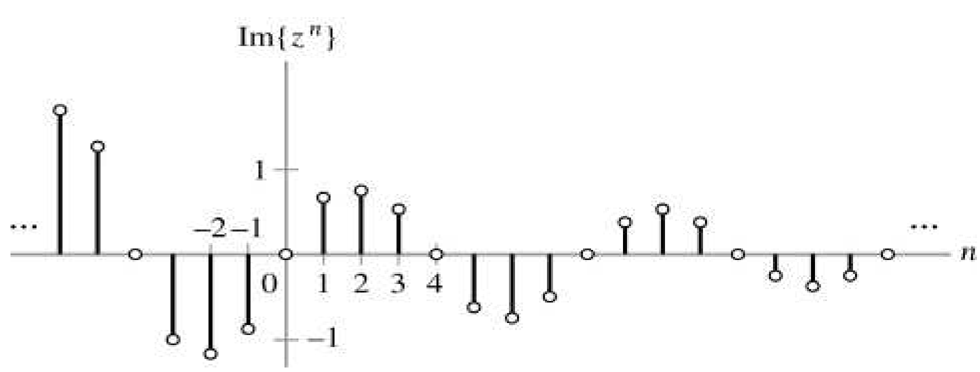
• The real part of $x[n]$ is exponentially damped cosine

• The imaginary part of $x[n]$ is exponentially damped sine

• Apply $x[n]$ to an LTI system with impulse response $h[n]$, Then

$$y[n] = H\{x[n]\} = h[n] * x[n]$$





$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- If

$$x[n] = z^n$$

we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- The z-transform is defined as

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

we may write as

$$H(z^n) = H(z)z^n$$

You can see that when you do the *z*-transformit sums up all the sequence, and so the individual terms affect the dependence on *z*, but the resulting function is just a function of *z*, it has no *k* in it. It will become clearer later why we might do this.

• This has the form of an eigen relation, where z_n is the eigen function and $H(z)$ is the eigenvalue.

• The action of an LTI system is equivalent to multiplication of the input by the complex number H(z).

- If $H(z) = |H(z)|e^{j\phi(z)}$ then the system output is

$$y[n] = |H(z)|e^{j\phi(z)}z^n$$

- Using $z = re^{j\Omega}$ we get

$$y[n] = |H(re^{j\Omega})|r^n \cos(\Omega n + \phi(re^{j\Omega}) +$$
$$j|H(re^{j\Omega})|r^n \sin(\Omega n + \phi(re^{j\Omega})$$

- Rewriting $x[n]$

$$x[n] = z^n = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$$

- If we compare $x[n]$ and $y[n]$, we see that the system modifies
  - the amplitude of the input by $|H(re^{j\Omega})|$ and
  - shifts the phase by $\phi(re^{j\Omega})$

DTFT and the z-transform

- Put the value of z in the transform then we get

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n](re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (h[n]r^{-n})e^{-j\Omega n}$$

- We see that $H(re^{j\Omega})$ corresponds to DTFT of $h[n]r^{-n}$.
- The inverse DTFT of $H(re^{j\Omega})$ must be $h[n]r^{-n}$.

- We can write

$$h[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

The z-transform contd..

- Multiplying $h[n]r^{-n}$ with r^n gives

$$h[n] = \frac{r^n}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})(re^{j\Omega})^n d\Omega$$

- We can convert this equation into an integral over z by putting $re^{j\Omega} = z$
- Integration is over Ω, we may consider r as a constant

- We have

$$dz = jre^{j\Omega}d\Omega = jzd\Omega$$

$$d\Omega = \frac{1}{j}z^{-1}dz$$

- Consider limits on integral
  - Ω varies from $-\pi$ to π
  - z traverses a circle of radius r in a counterclockwise direction
- We can write $h[n]$ as $h[n] = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz$ where $\oint$ is integration around the circle of radius $|z| = r$ in a counter clockwise direction
- The *z-transform* of any signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The *inverse z-transform* of is

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

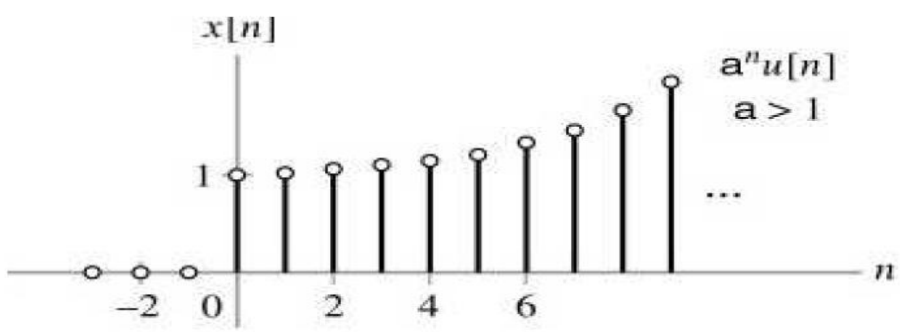
- *Inverse z-transform* expresses $x[n]$ as a weighted superposition of complex exponentials z^n
- The weights are $(\frac{1}{2\pi j}) X(z) z^{-1} dz$
- This requires the knowledge of complex variable theory

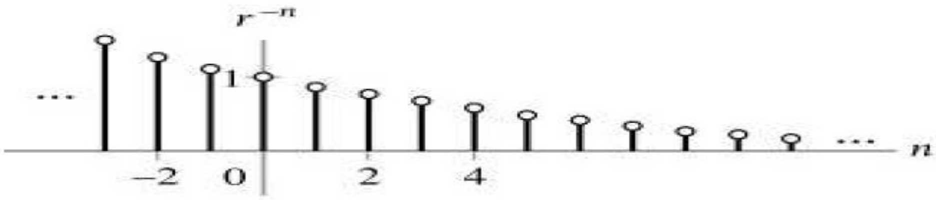
Convergence

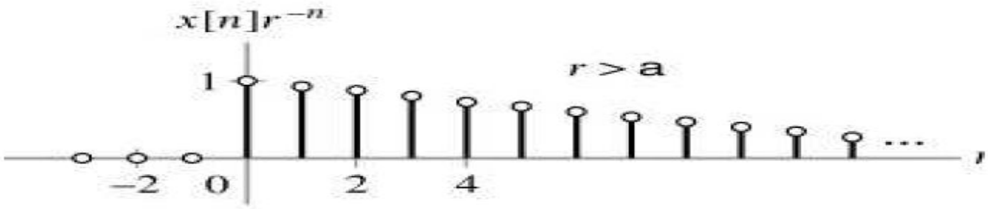
- Existence of *z-transform*: exists only if $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges
- Necessary condition: absolute summability of $x[n]z^{-n}$, since $|x[n]z^{-n}| = |x[n]r^{-n}|$, the condition is

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- The range r for which the condition is satisfied is called the *range of convergence* (ROC) of the *z-transform*
- ROC is very important in analyzing the system stability and behavior
- We may get identical *z-transform* for two different signals and only ROC differentiates the two signals
- The *z-transform* exists for signals that do not have DTFT.
- existence of DTFT: absolute summability of $x[n]$
- by limiting restricted values for r we can ensure that $x[n]r^{-n}$ is absolutely summable even though $x[n]$ is not
- Consider an example: the DTFT of $x[n] = \alpha^n u[n]$ does not exists for $|\alpha| > 1$
- If $r > \alpha$, then r^{-n} decays faster than $x[n]$ grows
- Signal $x[n]r^{-n}$ is absolutely summable and *z-transform* exists







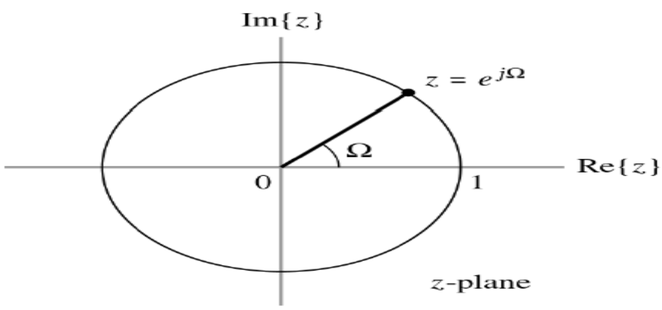


Figure 1.31: DTFT and z-transform

The z-Plane and DTFT

- If $x[n]$ is absolutely summable, then DTFT is obtained from the z-*transform* by setting $r = 1$ ($z = e^{j\Omega}$), ie. $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}$ as shown in Figure **??**

Poles and Zeros

- Commonly encountered form of the z-transform is the ratio of two polynomials in z^{-1}

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + b_N z^{-N}}$$

- It is useful to rewrite $X(z)$ as product of terms involving roots of the numerator and denominator polynomials

$$X(z) = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where $\tilde{b} = b_0 / a_0$

Poles and Zeros contd..

- Zeros: The c_k are the roots of numerator polynomials
- Poles: The d_k are the roots of denominator polynomials
- Locations of zeros and poles are denoted by "○" and "×" respectively

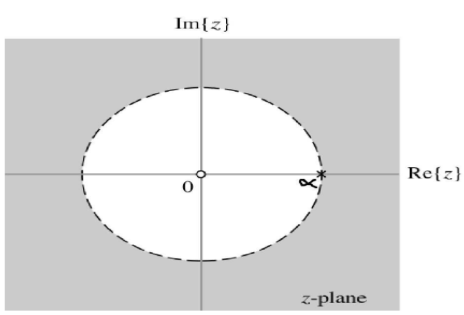
Example 1:

- The z-transform and DTFT of $x[n] = \{1, 2, -1, 1\}$ starting at $n = -1$
- $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-1}^{2} x[n]z^{-n} = z + 2 - z^{-1} + z^{-2}$
- $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}} = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$

- The z-transform and DTFT of $x[n] = \{1, 2, -1, 1\}$ starting at $n = -1$
- $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-1}^{2} x[n]z^{-n} = z + 2 - z^{-1} + z^{-2}$
- $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}} = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$

Example 2

- Find the z-transform of $x[n] = \alpha^n u[n]$, Depict the ROC and the poles and zeros
- Solution: $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} (\frac{\alpha}{z})^n$
  The series converges if $|z| > |\alpha|$
  $X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}, \qquad |z| > |\alpha|.$
  Hence pole at $z = \alpha$ and a zero at $z = 0$
- The ROC is



Properties of Region of Convergence:

- ROC is related to characteristics of $x[n]$
- ROC can be identified from $X(z)$ and limited knowledge of $x[n]$
- The relationship between ROC and characteristics of the $x[n]$ is used to find inverse z-transform

Property 1

ROC can not contain any poles

- ROC is the set of all z for which z-transform converges
- $X(z)$ must be finite for all z
- If p is a pole, then $|H(p)| = \infty$ and z-transform does not converge at the pole
- Pole can not lie in the ROC

Property 2

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- Let $x[n]$ be nonzero on the interval $n_1 \leq n \leq n_2$. The z-transform is

$$X(z) = \sum_{n=n_1}^{n_2} x[n] z^{-n}$$

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- If a signal is causal ($n_2 > 0$) then $X(z)$ will have a term containing z^{-1}, hence ROC can not include $z = 0$
- If a signal is non-causal ($n_1 < 0$) then $X(z)$ will have a term containing powers of z, hence ROC can not include $z = \infty$

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- If $n_2 \leq 0$ then the ROC will include $z = 0$
- If $n_1 \geq 0$ then the ROC will include $z = \infty$
- This shows the only signal whose ROC is entire z-plane is $x[n] = c\delta[n]$, where c is a constant

Finite duration signals

- The condition for convergence is $|X(z)| < \infty$

$$|X(z)| = |\sum_{n=-\infty}^{\infty} x[n]z^{-n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}|$$

magnitude of sum of complex numbers $\leq$ sum of individual magnitudes

- Magnitude of the product is equal to product of the magnitudes

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}|$$

- split the sum into negative and positive time parts
- Let

$$I_{-}(z) = \sum_{n=-\infty}^{-1} |x[n]||z|^{-n}$$

$$I_{+}(z) = \sum_{n=0}^{\infty} |x[n]||z|^{-n}$$

- Note that $X(z) = I_-(z) + I_+(z)$. If both $I_-(z)$ and $I_+(z)$ are finite, then $X(z)$ if finite

- If $x[n]$ is bounded for smallest $+ve$ constants A_-, A_+, r_- and r_+ such that

$$|x[n]| \leq A_-(r_-)^n, \quad n < 0$$

$$|x[n]| \leq A_+(r_+)^n, \quad n \geq 0$$

- The signal that satisfies above two bounds grows no faster than $(r_+)^n$ for $+ve\ n$ and $(r_-)^n$ for $-ve\ n$

- If the $n < 0$ bound is satisfied then

$$I_-(z) \leq A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n}$$

$$= A_- \sum_{n=-\infty}^{-1} (\frac{r_-}{|z|})^n = A_- \sum_{k=1}^{\infty} (\frac{|z|}{r_-})^k$$

- Sum converges if $|z| \leq r_-$

- If the $n \geq 0$ bound is satisfied then

$$I_+(z) = A_+ \sum_{n=0}^{\infty} (r_+)^n |z|^{-n}$$

$$= A_+ \sum_{n=0}^{\infty} (\frac{r_+}{|z|})^n$$

- Sum converges if $|z| > r_+$

- If $r_+ < |z| < r_-$, then both $I_+(z)$ and $I_-(z)$ converge and $X(z)$ converges

Properties of Z – transform:

- Linearity
- Time reversal
- Time shift
- Multiplication by α^n
- Convolution
- Differentiation in the z-domain

The z-transform

- The *z-transform* of any signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The *inverse z-transform* of $X(z)$ is

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- We assume that

$$x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$y[n] \overset{z}{\longleftrightarrow} Y(z), \qquad \text{with ROC} \quad R_y$$

- General form of the ROC is a ring in the z-plane, so the effect of an operation on the ROC is described by the a change in the radii of ROC

P1: Linearity

- The z-transform of a sum of signals is the sum of individual z-transforms

$$ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z),$$

$$\text{with ROC at least} \quad R_x \cap R_y$$

- The ROC is the intersection of the individual ROCs, since the z-transform of the sum is valid only when both converge

P1: Linearity

- The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.
- Consider an example: $x[n] = (\frac{1}{2})^n u[n] - (\frac{3}{2})^n u[-n-1]$
- We have $x[n] \xleftrightarrow{z} X(z)$

P2: Time reversal

- Time reversal or reflection corresponds to replacing z by z^{-1}. Hence, if R_x is of the form $a < |z| < b$ then the ROC of the reflected signal is $a < 1/|z| < b$ or $1/b < |z| < 1/a$

$$\text{If} \quad x[n] \xleftrightarrow{z} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad x[-n] \xleftrightarrow{z} X(\frac{1}{z}), \qquad \text{with ROC} \quad \frac{1}{R_x}$$

Proof: Time reversal

- Let $y[n] = x[-n]$
  $Y(z) = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$
  Let $l = -n$, then
  $Y(z) = \sum_{l=-\infty}^{\infty} x[l] z^{l}$

$Y(z) = \sum_{l=-\infty}^{\infty} x[l](\frac{1}{z})^{-l}$
$Y(z) = X(\frac{1}{z})$

P3: Time shift

- Time shift of n_o in the time domain corresponds to multiplication of z^{-n_o} in the z-domain

$$\text{If} \quad x[n] \xleftrightarrow{z} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad x[n-n_o] \xleftrightarrow{z} z^{-n_o}X(z),$$

$$\text{with ROC} \;\; R_x \;\; \text{except} \;\; z=0 \text{ or } |z|=\infty$$

P3: Time shift, $n_o > 0$

- Multiplication by z^{-n_o} introduces a pole of order n_o at $z=0$
- The ROC can not include $z=0$, even if R_x does include $z=0$
- If $X(z)$ has a zero of at least order n_o at $z=0$ that cancels all of the new poles then ROC can include $z=0$

P3: Time shift, $n_o < 0$

- Multiplication by z^{-n_o} introduces n_o poles at infinity
- If these poles are not canceled by zeros at infinity in $X(z)$ then the ROC of $z^{-n_o}X(z)$ can not include $|z|=\infty$

Proof: Time shift

- Let $y[n] = x[n-n_o]$
  $Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_o]z^{-n}$
  Let $l = n-n_o$, then
  $Y(z) = \sum_{l=-\infty}^{\infty} x[l]z^{-(l+n_o)}$
  $Y(z) = z^{-n_o}\sum_{l=-\infty}^{\infty} x[l]z^{-l}$
  $Y(z) = z^{-n_o}X(z)$

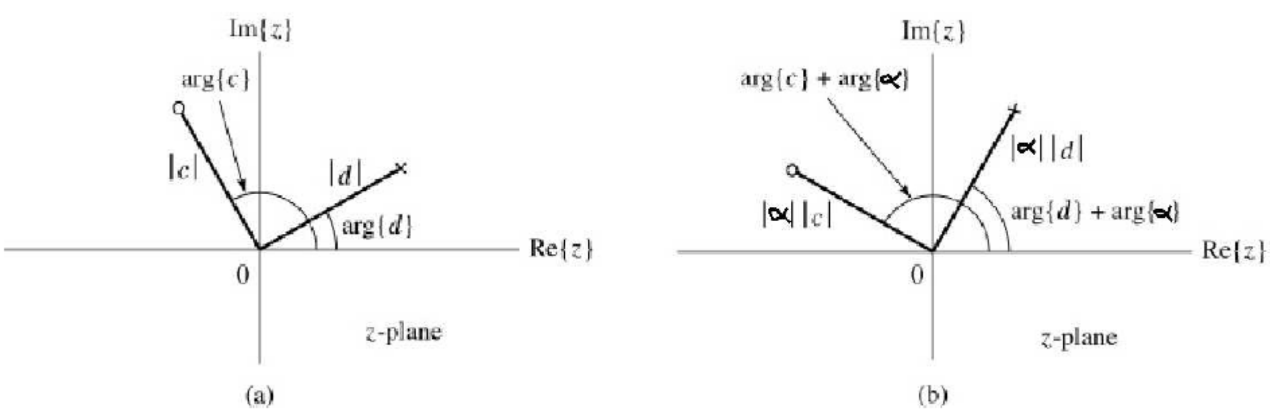
P4: Multiplication by α^n

- Let α be a complex number

$$\text{If} \quad x[n] \xleftrightarrow{z} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad \alpha^n x[n] \xleftrightarrow{z} X(\frac{z}{\alpha}), \qquad \text{with ROC} \quad |\alpha| R_x$$

- $|\alpha|R_x$ indicates that the ROC boundaries are multiplied by $|\alpha|$.
- If R_x is $a < |z| < b$ then the new ROC is $|\alpha|a < |z| < |\alpha|b$
- If $X(z)$ contains a pole d, ie. the factor $(z-d)$ is in the denominator then $X(\frac{z}{\alpha})$ has a factor $(z-\alpha d)$ in the denominator and thus a pole at αd.
- If $X(z)$ contains a zero c, then $X(\frac{z}{\alpha})$ has a zero at αc
- This indicates that the poles and zeros of $X(z)$ have their radii changed by $|\alpha|$
- Their angles are changed by $arg\{\alpha\}$



- If $|\alpha| = 1$ then the radius is unchanged and if α is +ve real number then the angle is unchanged

Proof: Multiplication by α^n

- Let $y[n] = \alpha^n x[n]$

$$Y(z) = \sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n}$$

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] (\frac{z}{\alpha})^{-n}$$

$$Y(z) = X(\frac{z}{\alpha})$$

P5: Convolution

- Convolution in time domain corresponds to multiplication in the z-domain If $x[n] \xleftrightarrow{z} X(z)$, with ROC R_x If $y[n] \xleftrightarrow{z} Y(z)$, with ROC R_y Then $x[n] * y[n] \xleftrightarrow{z} X(z)Y(z)$,

  with ROC at least $R_x \cap R_y$

- Similar to linearity the ROC may be larger than the intersection of R_x and R_y

Proof: Convolution

- Let $c[n] = x[n] * y[n]$

$$C(z) = \sum_{n=-\infty}^{\infty} (x[n] * y[n]) z^{-n}$$

$$C(z) = \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x[k] * y[n-k]) z^{-n}$$

$$C(z) = \sum_{k=-\infty}^{\infty} x[k] (\underbrace{(\sum_{n=-\infty}^{\infty} y[n-k]) z^{-(n-k)})}_{Y(z)} z^{-k}$$

$$C(z) = (\underbrace{\sum_{k=-\infty}^{\infty} x[k] z^{-k}}_{X(z)}) Y(z)$$

$$C(z) = X(z)Y(z)$$

P6: Differentiation in the z domain

- Multiplication by n in the time domain corresponds to differentiation with respect to z and multiplication of the result by $-z$ in the z-domain
If $x[n] \xleftrightarrow{z} X(z)$, with ROC R_x Then $nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z)$ with ROC R_x
- ROC remains unchanged

Proof: Differentiation in the z domain

- We know

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Differentiate with respect to z

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n}z^{-1}$$

- Multiply with $-z$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} -(-n)x[n]z^{-n}z^{-1}z$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Then $nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z)$ with ROC R_x

Example 1

Use the z-transform properties to determine the z-transform

- $x[n] = n((\frac{-1}{2})^n u[n]) * (\frac{1}{4})^{-n}u[-n]$
- Solution is:
$a[n] = (\frac{-1}{2})^n u[n] \xleftrightarrow{z} A(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$
$b[n] = na[n] \xleftrightarrow{z} B(z) = -z\frac{d}{dz}A(z) = -z\frac{d}{dz}(\frac{1}{1+\frac{1}{2}z^{-1}}), \quad |z| > \frac{1}{2}$
$b[n] = na[n] \xleftrightarrow{z} B(z) = \frac{\frac{-1}{2}z}{(1+\frac{1}{2}z)^2}, \quad |z| > \frac{1}{2}$
$c[n] = (\frac{1}{4})^n u[n] \xleftrightarrow{z} C(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$

Use the z-transform properties to determine the z-transform

- $x[n] = n((\frac{-1}{2})^n u[n]) * (\frac{1}{4})^{-n} u[-n]$

$$d[n] = c[-n] = (\tfrac{1}{4})^{-n} u[-n] \xleftrightarrow{z} D(z) = C(\tfrac{1}{z}) = \frac{1}{1-\frac{1}{4}z}, \quad |z| < 4$$
$$x[n] = (b[n] * d[n]) \xleftrightarrow{z} X(z) = B(z)D(z), \quad \tfrac{1}{2} < |z| < 4$$
$$x[n] = (b[n] * d[n]) \xleftrightarrow{z} \frac{\frac{-1}{2}z}{(1+\frac{1}{2}z)^2} \frac{1}{(1-\frac{1}{4}z)}, \quad \tfrac{1}{2} < |z| < 4$$
$$x[n] = (b[n] * d[n]) \xleftrightarrow{z} \frac{2z}{(1+\frac{1}{2}z)^2 (z-4)}, \quad \tfrac{1}{2} < |z| < 4$$

Example 2

Use the z-transform properties to determine the z-transform

- $x[n] = a^n \cos(\Omega_o n) u[n]$, where a is real and +ve

- Solution is:

  $b[n] = a^n u[n] \xleftrightarrow{z} B(z) = \frac{1}{1-az^{-1}}, \quad |z| > a$

  Put $\cos(\Omega_o n) = \frac{1}{2} e^{j\Omega_o n} + \frac{1}{2} e^{-j\Omega_o n}$, so we get

  $x[n] = \frac{1}{2} e^{j\Omega_o n} b[n] + \frac{1}{2} e^{-j\Omega_o n} b[n]$

Use the z-transform properties to determine the z-transform

- $x[n] = a^n \cos(\Omega_o n) u[n]$, where a is real and +ve

- Solution continued

$$x[n] \xleftrightarrow{z} X(z) = \tfrac{1}{2} B(e^{j\Omega_o} z) + \tfrac{1}{2} B(e^{-j\Omega_o} z), \quad |z| > a$$
$$x[n] \xleftrightarrow{z} X(z) = \frac{1}{2} \frac{1}{1-ae^{j\Omega_o} z^{-1}} + \frac{1}{2} \frac{1}{1-ae^{-j\Omega_o} z^{-1}}, \quad |z| > a$$
$$x[n] \xleftrightarrow{z} X(z) = \frac{1}{2} \left(\frac{1-ae^{j\Omega_o} z^{-1} + 1 - ae^{-j\Omega_o}}{(1-ae^{j\Omega_o} z^{-1})(1-ae^{-j\Omega_o} z^{-1})} \right)$$
$$x[n] \xleftrightarrow{z} X(z) = \frac{1-a\cos(\Omega_o) z^{-1}}{1-2a\cos(\Omega_o) z^{-1} + a^2 z^{-2}}, \quad |z| > a$$

Inverse Z transform:

Three different methods are:

1. Partial fraction method
2. Power series method
3. Long division method

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where $\tilde{B}(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \stackrel{z}{\longleftrightarrow} \delta[n]$$

$$z^{-n_o} \stackrel{z}{\longleftrightarrow} \delta[n - n_o]$$

$$\text{with ROC} \quad z > d_k \qquad \text{OR}$$

$$-A_k(d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1-d_k z^{-1}},$$

$$\text{with ROC} \quad z < d_k$$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \xleftrightarrow{z} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC}|z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse z-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \xleftrightarrow{z} \frac{A}{(1-d_i z^{-1})^m}, \quad \text{with ROC}|z| < d_i$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.

- In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

- Find the inverse z-transform of

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})},$$

with ROC $1 < |z| < 2$

- Solution: Use partial fraction and rewrite the expression

$$X(z) = \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - 2z^{-1})} + \frac{A_3}{(1 - z^{-1})}$$

- Solving for A_1, A_2 and A_3 gives the values as $A_1 = 1$, $A_2 = 2$ and $A_3 = -2$,

$$X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})} + \frac{2}{(1 - 2z^{-1})} - \frac{2}{(1 - z^{-1})}$$

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued
- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen

- If the signal is stable, then t is absolutely summable and has DTFT
- Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. $|z| = 1$
- The inverse z-transform is determined by comparing the poles and the unit circle
- If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
- If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

<u>Power series expansion method</u>

- Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation
- The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals
- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal

Long division method:

- Find the z-transform of

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}}, \text{with ROC} \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence

- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \dots$$

- Compare with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \dots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in z using long division method

- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \dots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2] - 32\delta[n+3] + \dots$$

- Find the z-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for e^a and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$

$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

UNIT 8: Z-Transforms – 2 **Teaching hours: 6**

8.1	Transform analysis of LTI Systems	
8.2	Unilateral Z- Transform	
8.3	Application to solve difference equations	

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8.1 Transform analysis of LTI systems:

- We have defined the transfer function as the z-transform of the impulse response of an LTI system

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- Then we have $y[n] = x[n] * h[n]$ and $Y(z) = X(z)H(z)$
- This is another method of representing the system
- The transfer function can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

- This is true for all z in the ROCs of $X(z)$ and $Y(z)$ for which $X(z)$ in nonzero
- The impulse response is the z-transform of the transfer function
- We need to know ROC in order to uniquely find the impulse response

- If ROC is unknown, then we must know other characteristics such as stability or causality in order to uniquely find the impulse response

System identification

- Finding a system description by using input and output is known as system identification
- Ex1: find the system, if the input is $x[n] = (-1//3)^n u[n]$ and the out is $y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$

- Solution: Find the z-transform of input and output. Use $X(z)$ and $Y(z)$ to find $H(z)$, then find $h(n)$ using the inverse z-transform

$$X(z) = \frac{1}{(1+(\frac{1}{3})z^{-1})}, \qquad \text{with ROC } |z| > \frac{1}{3}$$

$$Y(z) = \frac{3}{(1+z^{-1})} + \frac{1}{(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We can write $Y(z)$ as

$$Y(z) = \frac{4}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We know $H(z) = Y(z)/X(z)$, so we get

$$H(z) = \frac{4(1+(\frac{1}{3})z^{-1})}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})} \quad \text{with ROC } |z| > 1$$

- We need to find inverse z-transform to find $x[n]$, so use partial fraction and write $H(z)$ as

$$H(z) = \frac{2}{1+z^{-1}} + \frac{2}{1-(\frac{1}{3})z^{-1}} \quad \text{with ROC } |z| > 1$$

- Impulse response $x[n]$ is given by

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n]$$

Relation between transfer function and difference equation

- The transfer can be obtained directly from the difference-equation description of an LTI system
- We know that
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
- We know that the transfer function $H(z)$ is an eigen value of the system associated with the eigen function z^n, ie. if $x[n] = z^n$ then the output of an LTI system $y[n] = z^n H(z)$
- Put $x[n-k] = z^{n-k}$ and $y[n-k] = z^{n-k}H(z)$ in the difference equation,

  we get
$$z^n \sum_{k=0}^{N} a_k z^{-k} H(z) = z^n \sum_{k=0}^{M} b_k z^{-k}$$
- We can solve for $H(z)$
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$
- The transfer function described by a difference equation is a ratio of polynomials in z^{-1} and is termed as a rational transfer function.
- The coefficient of z^{-k} in the numerator polynomial is the coefficient associated with $x[n-k]$ in the difference equation
- The coefficient of z^{-k} in the denominator polynomial is the coefficient associated with $y[n-k]$ in the difference equation
- This relation allows us to find the transfer function and also find the difference equation description for a system, given a rational function

Transfer function:

- The poles and zeros of a rational function offer much insight into LTI system characteristics
- The transfer function can be expressed in pole-zero form by factoring the numerator and denominator polynomial
- If c_k and d_k are zeros and poles of the system respectively and $\tilde{b} = b_0/a_0$ is the gain factor, then

$$H(z) = \frac{\tilde{b}\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- This form assumes there are no poles and zeros at $z = 0$
- The p^{th} order pole at $z = 0$ occurs when $b_0 = b_1 = \ldots = b_{p-1} = 0$
- The l^{th} order zero at $z = 0$ occurs when $a_0 = a_1 = \ldots = a_{l-1} = 0$
- Then we can write $H(z)$ as

$$H(z) = \frac{\tilde{b}z^{-p}\prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l}\prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

  where $\tilde{b} = b_p/a_l$

- In the example we had first order pole at $z = 0$
- The poles, zeros and gain factor $\tilde{b}$ uniquely determine the transfer function
- This is another description for input-output behavior of the system
- The poles are the roots of characteristic equation

8.2 Unilateral Z- transforms:

- Useful in case of causal signals and LTI systems
- The choice of time origin is arbitrary, so we may choose $n = 0$ as the time at which the input is applied and then study the response for times $n \geq 0$

Advantages

- We do not need to use ROCs
- It allows the study of LTI systems described by the difference equation with initial conditions

Unilateral z-transform

- The unilateral z-transform of a signal $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

which depends only on $x[n]$ for $n \geq 0$

- The unilateral and bilateral z-transforms are equivalent for causal signals

$$\alpha^n u[n] \xleftrightarrow{z_u} \frac{1}{1 - \alpha z^{-1}}$$

$$a^n \cos(\Omega_o n) u[n] \xleftrightarrow{z_u} \frac{1 - a\cos(\Omega_o) z^{-1}}{1 - 2a\cos(\Omega_o) z^{-1} + a^2 z^{-2}}$$

Properties of unilateral Z transform:

- The same properties are satisfied by both unilateral and bilateral z-transforms with one exception: the time shift property
- The time shift property for unilateral z-transform: Let $w[n] = x[n-1]$

- The unilateral z-transform of $w[n]$ is

$$W(z) = \sum_{n=0}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{m=0}^{\infty} x[m]z^{-(m+1)}$$

- The unilateral z-transform of $w[n]$ is

$$W(z) = x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m}$$

$$W(z) = x[-1] + z^{-1}X(z)$$

- A one-unit time shift results in multiplication by z^{-1} and addition of the constant $x[-1]$
- In a similar way, the time-shift property for delays greater than unity is

$$x[n-k] \xleftrightarrow{z_u} x[-k] + x[-k+1]z^{-1} +$$
$$\ldots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for} \quad k > 0$$

- In the case of time advance, the time-shift property changes to

$$x[n+k] \xleftrightarrow{z_u} -x[0]z^{k} - x[-1]z^{k-1} +$$
$$\ldots - x[k-1]z + z^{k}X(z) \quad \text{for} \quad k > 0$$

8.3 Application to solve difference equations

Solving Differential equations using initial conditions:

- Consider the difference equation description of an LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- We may write the z-transform as

$$A(z)Y(z) + C(z) = B(z)X(z)$$

where

$$A(z) = \sum_{k=0}^{N} a_k z^{-k} \quad \text{and} \quad B(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- We get

$$C(z) = \sum_{m=0}^{N-1} \sum_{k=m+1}^{N} a_k y[-k+m] z^{-m}$$

- We have assumed that $x[n]$ is causal and

$$x[n-k] \xleftrightarrow{z_u} z^{-k} X(z)$$

- The term $C(z)$ depends on the N initial conditions $y[-1], y[-2], \ldots, y[-N]$ and the a_k
- $C(z)$ is zero if all the initial conditions are zero

- Solving for $Y(z)$, gives

$$Y(z) = \frac{B(z)}{A(z)}X(z) - \frac{C(z)}{A(z)}$$

- The output is the sum of the forced response due to the input and the natural response induced by the initial conditions

- The forced response due to the input

$$\frac{B(z)}{A(z)}X(z)$$

- The natural response induced by the initial conditions

$$\frac{C(z)}{A(z)}$$

- $C(z)$ is the polynomial, the poles of the natural response are the roots of $A(z)$, which are also the poles of the transfer function

- The form of natural response depends only on the poles of the system, which are the roots of the characteristic equation

First order recursive system

- Consider the first order system described by a difference equation

$$y[n] - \rho y[n-1] = x[n]$$

where $\rho = 1 + r/100$, and r is the interest rate per period in percent and $y[n]$ is the balance after the deposit or withdrawal of $x[n]$

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- Assume bank account has an initial balance of \$10,000/- and earns 6% interest compounded monthly. Starting in the first month of the second year, the owner withdraws \$100 per month from the account at the beginning of each month. Determine the balance at the start of each month.

- Solution: Take unilateral z-transform and use time-shift property we get

$$Y(z) - \rho(y[-1] + z^{-1}Y(z)) = X(z)$$

- Rearrange the terms to find $Y(z)$, we get

$$(1 - \rho z^{-1})Y(z) = X(z) + \rho y[-1]$$

$$Y(z) = \frac{X(z)}{1 - \rho z^{-1}} + \frac{\rho y[-1]}{1 - \rho z^{-1}}$$

- $Y(z)$ consists of two terms
  - one that depends on the input: the forced response of the system
  - another that depends on the initial conditions: the natural response of the system
- The initial balance of \$10,000 at the start of the first month is the initial condition $y[-1]$, and there is an offset of two between the time index n and the month index
- $y[n]$ represents the balance in the account at the start of the $n+2^{nd}$ month.
- We have $\rho = 1 + \frac{\frac{6}{12}}{100} = 1.005$
- Since the owner withdraws \$100 per month at the start of month 13 $(n = 11)$
- We may express the input to the system as $x[n] = -100u[n-11]$, we get

$$X(z) = \frac{-100z^{-11}}{1 - z^{-1}}$$

- We get

$$Y(z) = \frac{-100z^{-11}}{(1 - z^{-1})(1 - 1.005z^{-1})} + \frac{1.005(10,000)}{1 - 1.005z^{-1}}$$

- After a partial fraction expansion we get

$$Y(z) = \frac{20,000z^{-11}}{1-z^{-1}} + \frac{20,000z^{-11}}{1-1.005z^{-1}} + \frac{10,050}{1-1.005z^{-1}}$$

- Monthly account balance is obtained by inverse z-transforming $Y(z)$ We get

$$y[n] = 20,000u[n-11] - 20,000(1.005)^{n-11}u[n-11] + 10,050(1.005)^n u[n]$$

- The last term $10,050(1.005)^n u[n]$ is the natural response with the initial balance
- The account balance
- The natural balance
- The forced response