Fourier Series Representation

→ Fourier Series is used to represent time domain signal $x(t)$ into freq. domain signal $x(\omega)$.

→ Fourier Series is used for periodic signals.

→ Standard signals for representation of periodic signal.

$x(t) = \sin \omega_0 t$

$x(t) = e^{j\omega_0 t}$

→ Fourier Series of $x(t)$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 n t}}$$

$$= \cdots + a_{-3} e^{-j3\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + \cdots$$

Fourier Series Co-efficient Calculation

$\rightarrow$ $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\rightarrow$ Multiply $e^{-jn\omega_0 t}$ both the side.

$$\Rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\Rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$\rightarrow$ let, $\int_0^T$ on both sides.

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\, dt$$

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \underline{\int_0^T e^{j(k-n)\omega_0 t}\, dt}$$

$\rightarrow$ From Eulers formula.

$$\Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

$$= \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$$

$$\Rightarrow \boxed{a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt}$$

$$\rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}}$$

Properties of fourier Series

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 nt}}$$

1. Linearity property

If $x(t) \xleftrightarrow{FS} F_x$

$y(t) \xleftrightarrow{FS} F_y$

Then linearity property states that

$$a\,x(t) + b\,y(t) \xleftrightarrow{FS} aF_x + bF_y$$

2. Time Shifting property

If $x(t) \xleftrightarrow{FS} F_n$

Then ~~time~~ time shifting property states that

$$x(t-t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} F_n$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 nt}}$$

3. Frequency shifting property

If $x(t) \xleftrightarrow{FS} F_n$

Then freq shifting property states that

$$e^{jn_0\omega_0 t} x(t) \xleftrightarrow{FS} F(n-n_0) \quad \Big| \quad e^{-jn_0\omega_0 t} x(t) \xleftrightarrow{FS} F(n+n_0).$$

4. Time Scaling Property

If $x(t) \xleftrightarrow{FS} F_n$

Then Time Scaling property states that

$$x(at) \xleftrightarrow{FS} F_n$$

- Time Scaling changes freq components $\frac{1}{T_0}$ to $\frac{a}{T_0}$.

5. multiplication & convolution property

If $x(t) \xleftrightarrow{FS} F_x$

$y(t) \xleftrightarrow{FS} F_y$.

- Then multiplication property states

$$x(t)\,y(t) \xleftrightarrow{FS} F_x * F_y$$

- Convolution property state

$$x(t) * y(t) \xleftrightarrow{FS} F_x F_y.$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 nt}}$$

6. Differentiation property

If $x(t) \xleftrightarrow{FS} F_n$

- Then differentiation property states

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FS} \frac{j2\pi n}{T_0} F_n = j\omega_0 n F_n$$

Symmetry property of fourier transform

Real $f(t) \xleftrightarrow{FT} F_n$

$$f_n^* = f_{-n}$$

- $f_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$
- $f_{-n} = \frac{1}{T}\int_0^T f(t)\, e^{jn\omega_0 t}\, dt$
  - For real signal

$$f(t) = f^*(t).$$

$$= \frac{1}{T}\int_0^T [f^*(t)][e^{-jn\omega_0 t}]^* dt$$

$$= \left[\frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt\right]^*$$

- $\boxed{f_{-n} = f_n^*}$

Imaginary $f(t) \xleftrightarrow{FT} F_n.$

$$f_n^* = -f_{-n}$$

- $f_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$
- $f_n^* = \left[\frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt\right]^*$

$$= \frac{1}{T}\int_0^T f^*(t)\,[e^{-jn\omega_0 t}]^* dt$$

- For Imaginary Signal

$$f(t) = -f^*(t).$$

- $f_n^* = \frac{1}{T}\int_0^T -f(t)\, e^{jn\omega_0 t}\, dt$

$$= -\left[\frac{1}{T}\int_0^T f(t)\, e^{-j(-n)\omega_0 t}\, dt\right]$$

$$\boxed{f_n^* = -f_{-n}}$$

- $x(t) = t$, $x^*(t) = t = x(t).$
- $x(t) = jt$ $x^*(t) = -jt = -x(t)$

Property-1 – Auto Correlation function & power Spectral density are Fourier transform pairs.

$$R(\tau) \xleftrightarrow{FT} S(\omega)$$

where, $S(\omega)$ = Power Spectral density

$R(\tau)$ = Auto Correlation function

$\Rightarrow FT[R(\tau)] = S(\omega)$

$\Rightarrow \boxed{S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau}$

– $R(\tau) = e^{j\omega t}$ $[-T < t < T]$

– $S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega t}\, dt$

$= \int_{-T}^{T} e^{j\omega t}\, e^{-j\omega t}\, dt$

$= \int_{-T}^{T} 1\, dt = 2T$

Property-2 – In a LTI system with $y(t)$ as Output and $H(\omega)$ as transfer function, The relation between Input PSD & Output PSD is

$$S_o(\omega) = |H(\omega)|^2 S_i(\omega)$$

Input $S_i(\omega)$ → [LTI $H(\omega)$] → Output $S_o(\omega) = S_i(\omega)\,|H(\omega)|^2$

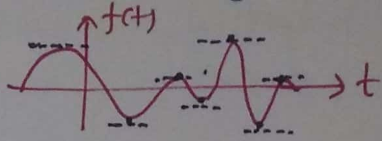
Property-3 - The total area under the density function is equal to the average power.

$$P_{avg} = \int_{0}^{\infty} S(\omega)\, d\omega$$

Dirichlet's Condition for existance of Fourier series

1. The function should be absolutely Integrable.

   i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$



2. There must be finite number of maxima and minima in the function.

3. There must be finite number of discontinuties in the function.

Trigonometric Fourier Series

- In Trigonometric Fourier series we use sine & cosine terms to represent given signal x(t) in freq. domain x(ω).

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \dots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \dots$$

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t}$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt \quad , \quad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt$$

Exponential Fourier Series

- EFS we use $e^{j\omega_0 n}$ as freq. component

- $$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}} = \boxed{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}}$$

$$= a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots\ldots$$
$$+ a_{-1} e^{-j\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + \ldots\ldots$$

- $$\boxed{a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt}$$

Relation between Trigonometric Fourier Series and Exponential Fourier Series.

- Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \quad \text{— ①}$$

- Exponential Fourier Series.

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + F_3 e^{j3\omega_0 t} + \dots$$
$$+ F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + F_{-3} e^{-j3\omega_0 t} + \dots$$

$$[e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t,\ e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t]$$

$$= F_0 + F_1[\cos\omega_0 t + j\sin\omega_0 t] + F_2[\cos 2\omega_0 t + j\sin 2\omega_0 t] + \dots$$
$$+ F_{-1}[\cos\omega_0 t - j\sin\omega_0 t] + F_{-2}[\cos 2\omega_0 t - j\sin 2\omega_0 t] + \dots$$

$$= F_0 + [F_1 + F_{-1}]\cos\omega_0 t + [F_2 + F_{-2}]\cos 2\omega_0 t + \dots$$
$$+ j[F_1 - F_{-1}]\sin\omega_0 t + j[F_2 - F_{-2}]\sin 2\omega_0 t + \dots$$

$$= F_0 + \sum_{n=1}^{\infty} (F_n + F_{-n})\cos n\omega_0 t + j(F_n - F_{-n})\sin n\omega_0 t \quad \text{— ②}$$

- Compare eq^n ① & ②

$$\boxed{a_n = F_n + F_{-n}}$$
$$\boxed{b_n = jF_n - jF_{-n}}$$

$$ja_n = jF_n + \cancel{jF_{-n}}$$
$$b_n = jF_n - \cancel{jF_{-n}}$$

$$\Rightarrow ja_n + b_n = 2jF_n$$

$$\Rightarrow \boxed{F_n = \frac{1}{2}(a_n - jb_n)}$$

$$ja_n = jF_n + jF_{-n}$$
$$b_n = jF_n - jF_{-n}$$

$$\Rightarrow ja_n - b_n = 2jF_{-n}$$

$$\Rightarrow \boxed{F_{-n} = \frac{1}{2}(a_n + jb_n)}$$

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Polar form of Fourier Series.

- If we don't have any Symmetry then to represent given periodic in fourier Series, we use Polar form.

$$g(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)$$

where,

$$d_0 = a_0$$

$$a_n = d_n \cos\theta_n$$

$$b_n = -d_n \sin\theta_n.$$

- Magnitude

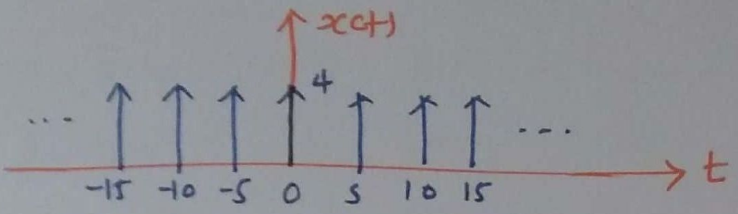
$$|d_n| = \sqrt{a_n^2 + b_n^2}$$

- phase

$$\theta_n = \tan^{-1}(-b_n/a_n) = -\tan^{-1}(b_n/a_n)$$

Example of fourier Series Co-efficient

Find Fourier Series Co-efficient of given Signal



(a) 4/5 ✓ (b) 5/4 (c) $\cos \pi/4$ (d) $\sin \pi/4$

$$- a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} (A\,\delta(t))\, e^{jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[A\, e^{jn\omega_0 t}\right]_{t=0} = \frac{A}{T} = \frac{4}{5}$$

Prediction of Nature of fourier Coefficient

Signal		Fourier Series Coefficient
Real & Even	$\longrightarrow$	Real & Even
Real & odd	$\longrightarrow$	Imag & Odd
Imag & Even	$\longrightarrow$	Imag & Even
Imag & odd	$\longrightarrow$	Real & Odd

S		F
E	$\rightarrow$	E
O	$\rightarrow$	O
E	$\rightarrow$	N ✗
O	$\rightarrow$	N ✓

Example - If $x(n)$ is real & even signal then Fourier series Coefficient will be

(a) Imag & odd

(b) Imag & even

✓ (c) Real & even

(d) Real & odd

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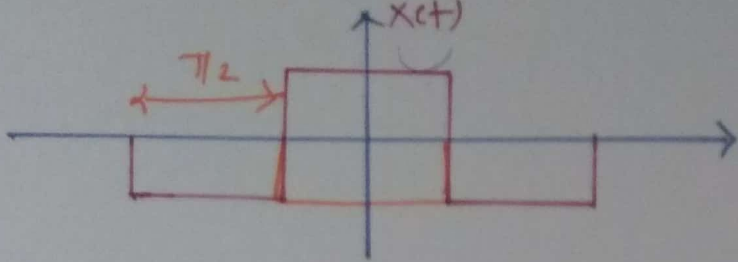
Fourier Series representation using Symmetricity of signal

Even Symmetry with half wave Symmetry



→ $x(t) = x(-t)$ (Even)

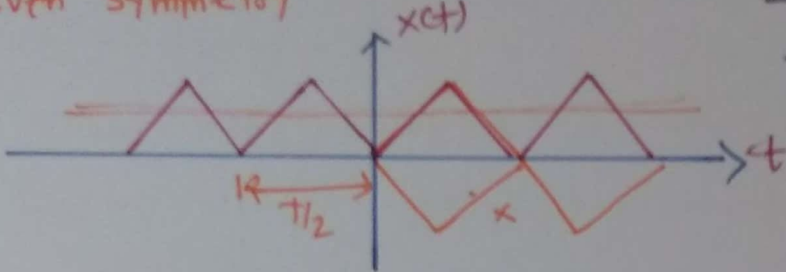
$x(t) = -x(t-T/2)$ (HWS)

→ It has only Cosine terms

→ Odd Harmonics.

$\cos \omega_0 t$, $\cos 3\omega_0 t$, $\cos 5\omega_0 t$, ...

Even Symmetry



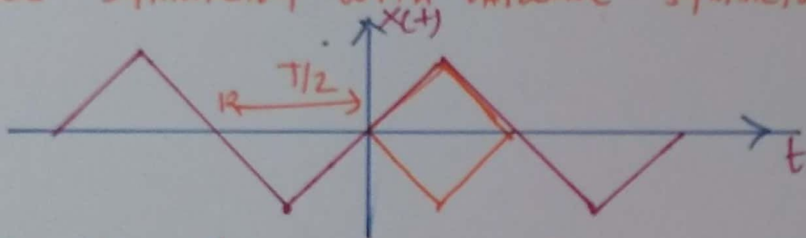
→ $x(t) = x(-t)$ (Even)

→ It has Only cosine terms.

→ all Harmonics.

- $\cos \omega_0 t$, $\cos 2\omega_0 t$, $\cos 3\omega_0 t$, ...
- only Even Symmetry is having DC Components.

Odd Symmetry with halfwave Symmetry



→ $x(t) = -x(-t)$ (Odd)

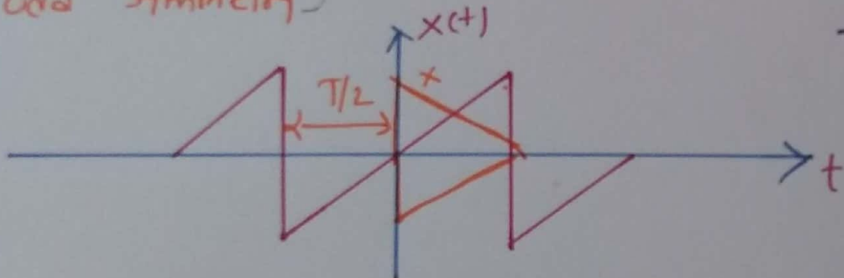
$x(t) = -x(t-T/2)$ (HWS)

→ It has Only Sine terms.

→ Odd harmonics.

$\sin \omega_0 t$, $\sin 3\omega_0 t$, $\sin 5\omega_0 t$, ...

Odd Symmetry



→ $x(t) = -x(-t)$ (Odd).

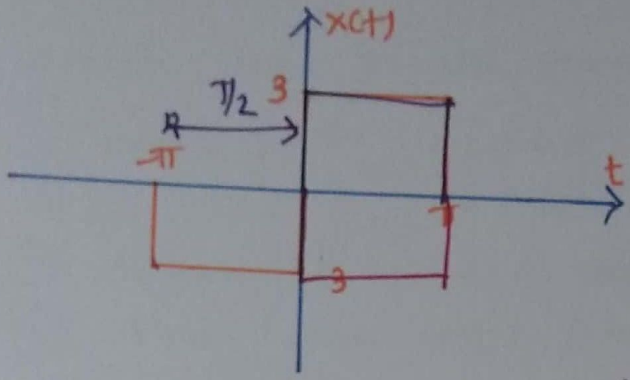
→ It has Only Sine terms.

→ All harmonics.

$\sin \omega_0 t$, $\sin 2\omega_0 t$, $\sin 3\omega_0 t$, ...

$$\boxed{\begin{array}{lll} E & \rightarrow & x(t) = x(-t) \\ O & \rightarrow & x(t) = -x(-t) \\ HW & \rightarrow & x(t) = -x(t-T/2) \end{array}}$$

Example of Fourier series representation using Symmetricity of signal.



(a) $\frac{3}{2} + \frac{2}{\pi}\left[\cos t + \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \ldots\right]$

(b) $\frac{3}{2} + \frac{12}{\pi}\left[\sin t + \frac{1}{2}\sin 2t + \frac{1}{2}\sin 3t + \ldots\right]$ ✗

(c) $\frac{12}{\pi}\left[\cos t + \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \ldots\right]$

(d) $\frac{12}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right]$ ✓

E $\rightarrow$ $x(t) = x(-t)$ ✗

O $\rightarrow$ $x(t) = -x(-t)$ ✓ $\longrightarrow$ Sine terms.

HW $\rightarrow$ $x(t) = -x(t - T/2)$ ✓ $\longrightarrow$ Odd harmonics.

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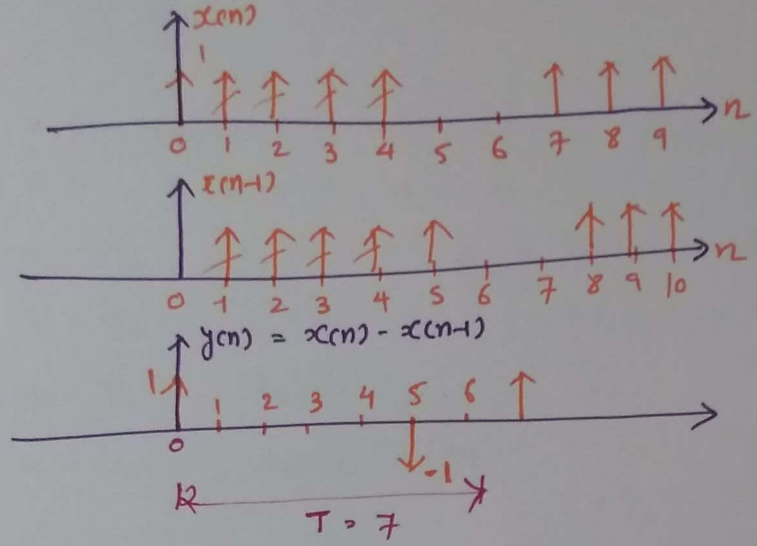
Fourier Series of discrete time Signal.

Given discrete time periodic signal

$$x(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & 5 \le n \le 6 \end{cases}$$

Find (a) Fundamental period of $y(n) = x(n) - x(n-1)$

(b) Find Fourier series coefficient



$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega t}\, dt$$

$$[a_k] = \frac{1}{T}\sum_{k=0}^{\infty} x(n)\, e^{-jk\omega n}$$

$$= \frac{1}{7}\sum_{k=0}^{\infty} x(n)\, e^{-jk\left(\frac{2\pi}{7}\right)n}$$

Relation between Trignometric & Exponential Fourier Series based Coefficient based on phase angle.

$$a_k = 2|F_k| \cos\theta_k$$
$$b_k = 2|F_k| \sin\theta_k.$$

- Avg. power

$$P = F_0^2 + 2[F_1^2 + F_2^2 + \ldots + F_n^2].$$
$$= 0 + 2[0 + {\tfrac{1}{2}}^2 + \ldots + 0]$$
$$= 1/2$$

ex Given F.S. Coefficient $c_2 = 1/2$ & phase Angle is 45°. Then T.F.S. Coefficient will be.

(a) $a_2 = 1/\sqrt{2}$, $b_2 = 1/2$

✓ (b) $a_2 = 1/\sqrt{2}$, $b_2 = 1/\sqrt{2}$

(c) $a_2 = 1/2$, $b_2 = 1/\sqrt{2}$

(d) $a_2 = 1/2$, $b_2 = 1/2$

$$a_2 = 2|C_2| \cos\theta_2$$
$$= 2(1/2) \cos 45$$
$$= 1/\sqrt{2}$$

$$b_2 = 2|C_2| \sin\theta_2$$
$$= 2 \times 1/2 \times 1/\sqrt{2}$$
$$= 1/\sqrt{2}$$

DC Component in Fourier Series

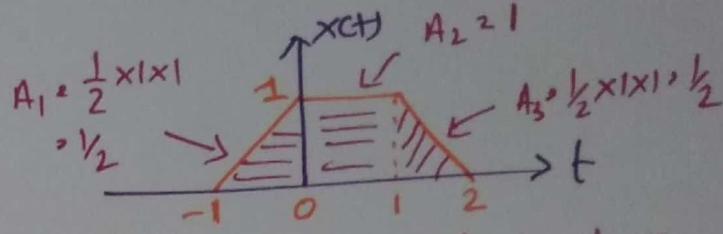
$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$= \cdots + a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + \boxed{a_0} + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \cdots$$

DC Component

- $a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$

$= \dfrac{\text{Area covered by } x(t)}{\text{Time period}}$.



- What is DC component of above signal?

$$a_0 = \frac{\text{Area}}{\text{Time}} = \frac{\frac{1}{2} + 1 + \frac{1}{2}}{3} = \frac{2}{3}$$

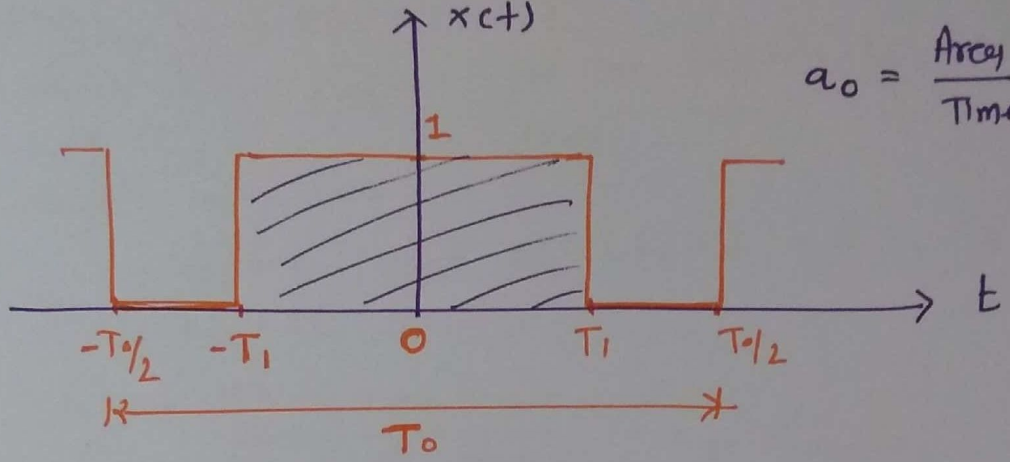
Example of DC Component using Fourier Series.

A Periodic signal $x(t)$ of period T_0 is given by

$$x(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , T_1 < |t| < T_0/2 \end{cases}$$

The DC component of $x(t)$ is

ⓐ $\frac{T_1}{T_0}$ ⓑ $\frac{T_1}{2T_0}$ ⓒ $\frac{2T_1}{T_0}$ ⓓ $\frac{T_0}{T_1}$



$$a_0 = \frac{\text{Area}}{\text{Time}} = \boxed{\frac{2T_1}{T_0}}$$

Amplitude of Fundumental Components in Fourier Series

Find the amplitude of fundamental Components of the signal

$$x(t) = \underbrace{3 \sin(\overset{\downarrow \omega_1}{6t} + 30^\circ)}_{1^{st}\ \text{Harmonic}} + \underbrace{4 \cos(\overset{\downarrow \omega_2 = 2\omega_0}{12t} - 60^\circ)}_{2^{nd}\ \text{Harmonic.}}$$

1^{st} Harmonics $\longrightarrow \omega_0 \longleftarrow$ Fundamental Freq.

2^{nd} Harmonics $\longrightarrow 2\omega_0$

3^{rd} Harmonic $\longrightarrow 3\omega_0$

we Can Identify by GCD of all freq.

$\vdots$

n^{th} Harmonic $\longrightarrow n\omega_0$

$\omega_1 = 6$ $\longrightarrow$ GCD $[6, 12] = 6$, $\omega_0 = \omega_1 = 6$ rad/sec

$\omega_2 = 12$

$$x(t) = \underbrace{2 \sin(3t + 15^\circ)}_{1^{st}\ \text{Harmonic}} + \underbrace{6 \sin(9t + 30^\circ)}_{3^{rd}\ \text{Harmonic}} + \underbrace{8 \cos(12t + 15^\circ)}_{4^{th}\ \text{Harmonic}}$$

$\omega_1 = 3$ $\longrightarrow$ GCD $[3, 9, 12] = 3$

$\omega_2 = 9$

$\omega_3 = 12$

Fourier Transforms.

$x(t) \xrightarrow{FT} X(\omega)$

- It is a tool to represent time domain signal into freq. domain.
- Fourier Series is mainly used for periodic signals whereas Fourier transform is used for aperiodic signals.

Conditions for existence of Fourier transform.

1. f(t) should be absolutely Integrable. $\boxed{\int_{-\infty}^{\infty} f(t)\, dt < \infty}$
2. The function must have finite number of maxima & minima
3. The function must have finite number of discontinuities.

Fourier Transform equation from Fourier Series

→ Fourier Series in discrete time.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

- $\omega = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

- Let $\frac{1}{T} = \Delta f$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi\Delta f t}$$

→ Fourier Co-efficient a_k

- $a_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega t} dt$
- $a_k = \Delta f \int_{-T/2}^{T/2} x(t) e^{-jk2\pi\Delta f t} dt$

$$x(t) = \sum_{k=-\infty}^{\infty} \Delta f \left(\int_{-T/2}^{T/2} x(t) e^{-jk(2\pi)\Delta f t} dt \right) e^{jk(2\pi)\Delta f t}$$

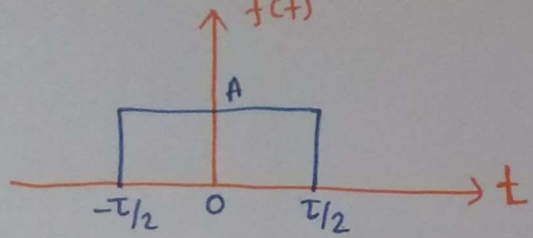
- $T \to \infty$, $\Sigma \to \int$, $\Delta f \to df$, $k\Delta f \to f$

$$x(t) = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{\text{Fourier Transform}} \, e^{j\omega t} df$$

- $\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$ ← Fourier Transform.

- $\boxed{x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega}$ ← Inverse Fourier transform.

Fourier Transform of Rectangular Pulse or Gate function



$$f(t) = \begin{cases} A & -\tau/2 < t < \tau/2 \\ 0 & \text{else.} \end{cases}$$

$$F(\omega) = F.T.[f(t)]$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\tau/2}^{\tau/2} A\, e^{-j\omega t}\, dt$$

$$= A \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{j\omega} \left[-e^{-j\omega\tau/2} + e^{j\omega\tau/2} \right]$$

$$= \frac{2A}{\omega} \left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \right] \qquad \left[\sin t = \frac{e^{jt} - e^{-jt}}{2j} \right]$$

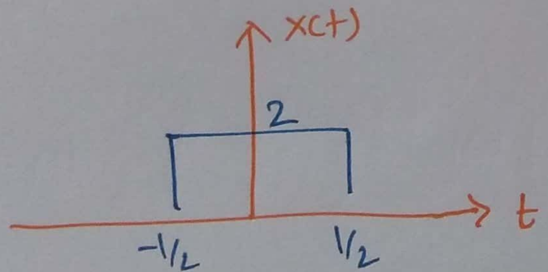
$$= \boxed{\frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)}$$

$$\left[Sa(t) = \frac{\sin t}{t} \right]$$

$$= A\tau \left[\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \right]$$

$$= \boxed{A\tau\, Sa\left(\frac{\omega\tau}{2}\right)}$$

example



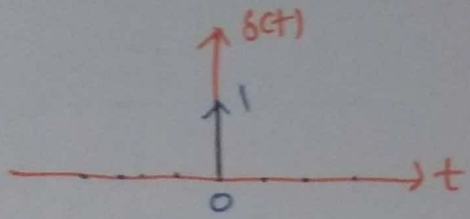
$\rightarrow A = 2$

$\rightarrow \tau = \cancel{1/2}\ 1$

$$X(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right) = \overset{2\times 1}{\cancel{2\times\frac{1}{2}}}\, Sa\left(\frac{\omega}{2}\right) = 2\, Sa\left(\frac{\omega}{2}\right)$$

$$= \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right) = \frac{2\times 2}{\omega} \sin\left(\frac{\omega}{2}\right) = \frac{4}{\omega} \sin\left(\frac{\omega}{2}\right)$$

Fourier Transform of Impulse function



$$\delta(t) = 1 \quad t = 0$$
$$= 0 \quad t \neq 0$$

$\rightarrow X(\omega) = \text{F.T.}\,[\delta(t)]$

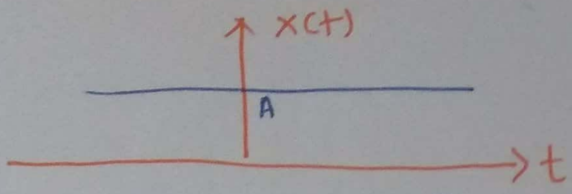
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \left[\delta(t)\, e^{-j\omega t}\right]_{t=0}$$

$$= \delta(0)\, e^{-0}$$

$$= 1$$

Fourier Transform of DC signal.



$x(t) = A \quad -\infty < t < \infty$

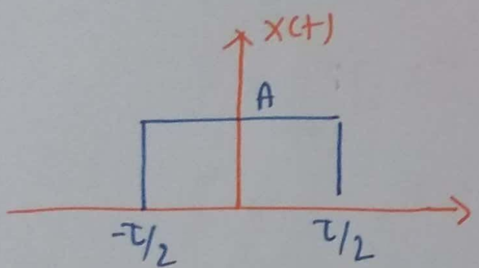
- As Dirichlet Cond^n (function should be Absolutely Integrable).

$$\int_{-\infty}^{\infty} x(t)\,dt < \infty \quad \Big| \quad \int_{-\infty}^{\infty} A\,dt = A[t]_{-\infty}^{\infty}$$

$$= A[\infty + \infty]$$

$$= \infty$$

- So we can not Calculate F.T of DC signal.



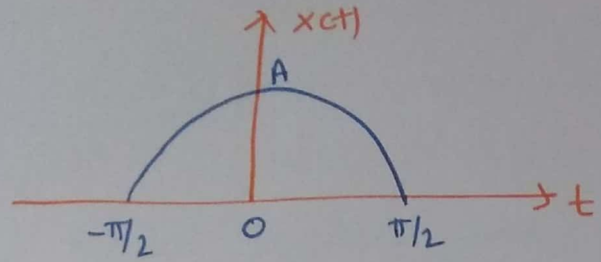
$X(\omega) = FT[x(t)]$

$= A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$

- If $\tau \to \infty$ then Rectangular pulse will be DC signal.
- So Fourier Transform of DC signal.

$$\boxed{X(\omega) = \lim_{\tau \to \infty} A\tau\, Sa\left(\frac{\omega\tau}{2}\right)}$$

Fourier Transform of Cosine Signal.



→ $T/2 = \pi \Rightarrow T = 2\pi$

→ $\omega = 2\pi f = \frac{2\pi}{T} = 1$ rad/sec

→ $x(t) = A\cos\omega t$

$= A\cos t$

→ $X(\omega) = FT[x(t)]$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\pi/2}^{\pi/2} A\cos t\, e^{-j\omega t}\, dt$$

$$= A\int_{-\pi/2}^{\pi/2} \left[\frac{e^{jt} + e^{-jt}}{2}\right] e^{-j\omega t}\, dt$$

$$= \frac{A}{2}\int_{-\pi/2}^{\pi/2} e^{jt(1-\omega)} + e^{-jt(1+\omega)}\, dt$$

$$= \frac{A}{2}\left[\frac{e^{jt(1-\omega)}}{j(1-\omega)} + \frac{e^{-jt(1+\omega)}}{-j(1+\omega)}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{A}{2}\left[\frac{e^{j(1-\omega)\pi/2} - e^{-j(1-\omega)\pi/2}}{j(1-\omega)} + \frac{e^{-j(1+\omega)\pi/2} - e^{j(1+\omega)\pi/2}}{-j(1+\omega)}\right]$$

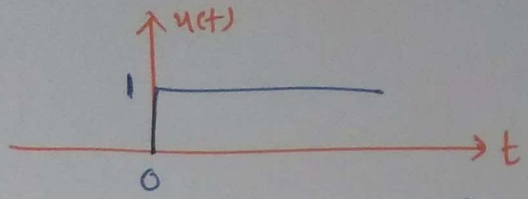
$$= \frac{A}{\quad}\left[\frac{1}{(1-\omega)}\left[\frac{e^{j(1-\omega)\pi/2} - e^{-j(1-\omega)\pi/2}}{2j}\right] + \frac{1}{1+\omega}\left[\frac{e^{j(1+\omega)\pi/2} - e^{-j(1+\omega)\pi/2}}{2j}\right]\right]$$

$$= \frac{A\pi}{2}\left[\frac{\sin(1-\omega)\pi/2}{(1-\omega)\pi/2} + \frac{\sin(1+\omega)\pi/2}{(1+\omega)\pi/2}\right]$$

$Sa(t) = \frac{\sin t}{t}$

$$= \boxed{\frac{A\pi}{2}\left[Sa(1-\omega)\pi/2 + Sa(1+\omega)\pi/2\right]}$$

Fourier Transform of unit Step function



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

1) As per Dirichlet cond.^{n} function should absolutely Integrable.

$$\int_{-\infty}^{\infty} f(t)\, dt < \infty \quad \Bigg| \quad \int_{0}^{\infty} 1\, dt = [t]_0^{\infty} = \infty$$

- This function is not absolutely Integrable so directly we can not have fourier transform of unit step function.

$$x(t) = \lim_{\omega_0 \to 0} \cos \omega_0 t \; u(t)$$

- If $\omega_0 \to 0$ then function $\cos \omega_0 t \cdot u(t)$ will be unit step function and then we can Calculate FT of unit step.

$$X(\omega) = FT[\underline{\cos \omega_0 t \; u(t)}]$$

$$= \frac{\pi}{2}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$X(\omega) = FT[\sin \omega_0 t \; u(t)]$$

$$= \frac{\pi}{2}[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

- $X(\omega) = FT[x(t)]$

$$= FT[\cos \omega_0 t \; u(t)]$$

$$= \lim_{\omega_0 \to 0} \frac{\pi}{2}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{\pi}{2}[\delta(\omega) + \delta(\omega)] + \frac{j\omega}{-\omega^2}$$

$$= \pi\,\delta(\omega) + \frac{j}{-\omega}$$

$$= \boxed{\pi\,\delta(\omega) + \frac{1}{j\omega}}$$

Fourier Transform of Some Standard basic Signals.

No	$x(t)$		$X(\omega)$
1	$\delta(t)$	$\xleftrightarrow{FT}$	1
2	$\text{Rect}(t)$	$\xleftrightarrow{FT}$	$\text{Sinc}\left(\frac{\omega}{2\pi}\right)$
3	$\text{Tri}(t)$	$\xleftrightarrow{FT}$	$\text{sinc}^2\left(\frac{\omega}{2\pi}\right)$
4	$\text{Sinc}(t)$	$\xleftrightarrow{FT}$	$\text{Rect}\left[\frac{\omega}{2\pi}\right]$
5	$\cos(2\pi a t)$	$\xleftrightarrow{FT}$	$\pi[\delta(\omega+2\pi a)+\delta(\omega-2\pi a)]$
6	$\sin(2\pi a t)$	$\xleftrightarrow{FT}$	$j\pi[\delta(\omega+2\pi a)-\delta(\omega-2\pi a)]$
7	$e^{-at}u(t)$	$\xleftrightarrow{FT}$	$\frac{1}{a+j\omega}$
8	$t^n e^{-at}u(t)$	$\xleftrightarrow{FT}$	$\frac{1}{(a+j\omega)^{n+1}}$
9	$e^{-a\|t\|}$	$\xleftrightarrow{FT}$	$\frac{2a}{a^2+\omega^2}$
10	$e^{-\pi t^2}$	$\xleftrightarrow{FT}$	$e^{-\omega^2/4\pi}$
11	$\text{Sgn}(t)$	$\xleftrightarrow{FT}$	$\frac{2}{j\omega}$
12	$u(t)$	$\xleftrightarrow{FT}$	$\pi\delta(\omega)+\frac{1}{j\omega}$
13	$e^{-at}\cos(2\pi b t)\,u(t)$	$\xleftrightarrow{FT}$	$\frac{a+j\omega}{(a+j\omega)^2+(2\pi b)^2}$

Properties of Fourier Transform

① linearity property

$$\left.\begin{array}{l} x_1(t) \xleftrightarrow{FT} X_1(\omega) \\ x_2(t) \xleftrightarrow{FT} X_2(\omega) \end{array}\right\} \rightarrow a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{FT} a_1 X_1(\omega) + a_2 X_2(\omega).$$

② Time shifting property

$$x(t) \xleftrightarrow{FT} X(\omega) \Big\} \rightarrow \begin{array}{l} x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega) \\ x(t+t_0) \xleftrightarrow{FT} e^{j\omega t_0} X(\omega). \end{array}$$

③ Frequency shifting property

$$x(t) \xleftrightarrow{FT} X(\omega) \Big\} \rightarrow \begin{array}{l} x(t)\, e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega-\omega_0) \\ x(t)\, e^{-j\omega_0 t} \xleftrightarrow{FT} X(\omega+\omega_0). \end{array}$$

④ Time Differentiation & Integration properties

$$x(t) \xleftrightarrow{FT} X(\omega) \Big\} \begin{array}{l} \rightarrow \dfrac{dx(t)}{dt} \xleftrightarrow{FT} (j\omega)\, X(\omega) \\ \rightarrow \int x(t)\, dt \xleftrightarrow{FT} \dfrac{1}{j\omega} X(\omega) \end{array}$$

⑤ Time Scaling property

$$x(t) \xleftrightarrow{FT} X(\omega) \Big\} \longrightarrow x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

⑥ Multiplication & Convolution property.

$$\left.\begin{array}{l} x_1(t) \xleftrightarrow{FT} X_1(\omega) \\ x_2(t) \xleftrightarrow{FT} X_2(\omega) \end{array}\right\} \begin{array}{l} \rightarrow x_1(t)\, x_2(t) \xleftrightarrow{FT} X_1(\omega) * X_2(\omega) \\ \rightarrow x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)\, X_2(\omega). \end{array}$$

⑦ Duality ~~or~~ Similarity property

$$x(t) \xleftrightarrow{FT} X(\omega) \Big\} \longrightarrow x(t) \xleftrightarrow{FT} 2\pi X(-\omega).$$

Linearity property of Fourier Transform [Statement, Proof & Example]

Statement

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$

$x_2(t) \xleftrightarrow{FT} X_2(\omega)$

Then linearity property states that

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{FT} a_1 X_1(\omega) + a_2 X_2(\omega)$$

Proof

→ $x(t) = a_1 x_1(t) + a_2 x_2(t)$

→ $X(\omega) = FT[x(t)]$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-j\omega t}\, dt$$

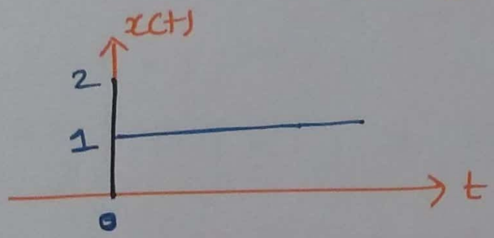
$$= \int_{-\infty}^{\infty} a_1 x_1(t) e^{-j\omega t} + a_2 x_2(t) e^{-j\omega t}\, dt$$

$$= a_1 \underline{\int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt} + a_2 \underline{\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt}$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega).$$

Example Find Fourier Transform of given signal

$$x(t) = \begin{cases} 2 & t = 0 \\ 1 & t > 0 \end{cases}$$



$x(t) = \delta(t) + u(t)$

$$FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= FT[\delta(t)] + FT[u(t)].$$

$$= 1 + \pi\,\delta(\omega) + \frac{1}{j\omega}.$$

Example Find FT of

$$x(t) = e^{-at}u(t) + u(t) + \delta(t).$$

$$X(\omega) = FT[e^{-at}u(t)] + FT[u(t)] + FT[\delta(t)]$$

$$= \frac{1}{a+j\omega} + \pi\,\delta(\omega) + \frac{1}{j\omega} + 1$$

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Time shifting property of Fourier Transform [Statement, Proof & Example]

statement If $x(t) \xleftrightarrow{FT} X(\omega)$

Then as per time shifting property.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

Proof

$$- X(\omega) = F.T.[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$- F.T.[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

$\Rightarrow t - t_0 = a \Rightarrow t = a + t_0$ | - limits will be $-\infty$ to ∞

$\Rightarrow dt = da$

$$= \int_{-\infty}^{\infty} x(a) e^{-j\omega(a+t_0)} da$$

$$= \int_{-\infty}^{\infty} x(a) e^{-j\omega a} e^{-j\omega t_0} da$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} \underline{x(a) e^{-j\omega a} da}$$

$$\boxed{- FT[x(t-t_0)] = e^{-j\omega t_0} X(\omega)}$$

example Find Fourier Transform of Rect $(t+1)$

$$- FT[\text{rect}(t)] = A\tau \, Sa\left(\frac{\omega\tau}{2}\right)$$

$$A = 1, \tau = 1$$

$$- FT[\text{Rect}(t)] = Sa(\omega/2)$$

$$\text{Rect}(t) \xleftrightarrow{FT} Sa(\omega/2)$$

$$\text{Rect}(t+1) \xleftrightarrow{FT} \boxed{e^{+j\omega} Sa(\omega/2)}$$

example Find Fourier Transform of $u(t-1)$.

$$- FT[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$- FT[u(t-1)] = e^{-j\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right].$$

Frequency shifting property of Fourier Transform [Statement, Proof, Example]

Statement

$$f(t) \xleftrightarrow{FT} F(\omega)$$

Then as per frequency shifting property

$$f(t)\, e^{j\omega_0 t} \xleftrightarrow{FT} F(\omega - \omega_0) \quad \Big| \quad f(t)\, e^{-j\omega_0 t} \xleftrightarrow{FT} F(\omega + \omega_0).$$

Proof

- $F(\omega) = FT[f(t)]$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

- $$FT[\underline{f(t)\, e^{j\omega_0 t}}] = \int_{-\infty}^{\infty} \underline{f(t)\, e^{j\omega_0 t}}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-jt(\underline{\omega - \omega_0})}\, dt$$

$$FT[f(t)\, e^{j\omega_0 t}] = \boxed{F(\omega - \omega_0)}$$

Example Find Fourier Transform of $\cos \omega_0 t$

- $x(t) = \cos \omega_0 t$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

- $1 \xleftrightarrow{FT} 2\pi\, \delta(\omega)$
- $1 \cdot e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\, \delta(\omega - \omega_0)$
- $1 \cdot e^{-j\omega_0 t} \xleftrightarrow{FT} 2\pi\, \delta(\omega + \omega_0)$

$F.T[x(t)]$

$$= \frac{1}{2}(2\pi\, \delta(\omega - \omega_0)) + \frac{1}{2}(2\pi\, \delta(\omega + \omega_0))$$

$$= \boxed{\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}$$

Differentiation & Integration properties of Fourier Transform

[Statement , Proof & Example].

Statement $x(t) \xleftrightarrow{FT} X(\omega)$

- Then as per differentiation property

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega) \Rightarrow \frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(\omega).$$

- Then as per Integration property

$$\int x(t)\,dt \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega) \Rightarrow \underbrace{\iiint}_{n} x(t)\,dt \xleftrightarrow{FT} \frac{1}{(j\omega)^n} X(\omega).$$

Proof

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\Rightarrow \frac{dx(t)}{dt} = \int_{-\infty}^{\infty} X(\omega) \frac{d e^{j\omega t}}{dt} d\omega.$$

$$\Rightarrow \frac{dx(t)}{dt} = \int_{-\infty}^{\infty} X(\omega) (j\omega) e^{j\omega t} d\omega.$$

$$\Rightarrow \frac{dx(t)}{dt} = (j\omega) \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

$$\Rightarrow \frac{dX(t)}{dt} = (j\omega) \; IFT[X(\omega)]$$

→ take Fourier Transform

$$\Rightarrow FT\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega)$$

$$\Rightarrow \int x(t)\,dt = \int_{-\infty}^{\infty} X(\omega) \int e^{j\omega t} dt \, d\omega.$$

$$\Rightarrow \int x(t)\,dt = \int_{-\infty}^{\infty} X(\omega) \frac{e^{j\omega}}{j\omega} d\omega.$$

$$\Rightarrow \int x(t)\,dt = \frac{1}{j\omega} \int_{-\infty}^{\infty} X(\omega) e^{j\omega} d\omega.$$

$$\Rightarrow \int x(t)\,dt = \frac{1}{j\omega} \; IFT[X(\omega)]$$

→ take Fourier Transform

$$\Rightarrow FT\left[\int x(t)\,dt\right] = \frac{1}{j\omega} X(\omega)$$

example Find Fourier Transform of Signum function.

$$Sgn = \begin{cases} -1 & t<0 \\ 0 & t=0 \\ 1 & t>0 \end{cases} \qquad Sgn(t) = 2u(t) - 1$$

$$- \frac{d\,Sgn(t)}{dt} = \frac{d}{dt}(2u(t) - 1)$$

$$- \frac{d\,Sgn(t)}{dt} = 2\delta(t)$$

- take Fourier Transform

$$\Rightarrow j\omega X(\omega) = 2$$

$$\Rightarrow \boxed{X(\omega) = \frac{2}{j\omega}}$$

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Time Scaling Property of Fourier Transform

[Statement, Proof & Example]

Statement

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then as per time scaling property.

$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Proof

- $X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

- $FT[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$

$\Rightarrow at = y \Rightarrow t = y/a$

$\Rightarrow dt = \frac{dy}{a}$

- limits will be $-\infty$ to ∞

- $FT[x(at)] = \int_{-\infty}^{\infty} x(y)\, e^{-j\omega y/a}\, \frac{dy}{|a|}$

$= \frac{1}{|a|} \int_{-\infty}^{\infty} x(y)\, e^{-j\left(\frac{\omega}{a}\right) y}\, dy.$

- $\boxed{FT[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$

Example - Find Fourier transform of $\delta(2t)$

$\delta(t) \xleftrightarrow{FT} 1$

$\delta(2t) \xleftrightarrow{FT} \frac{1}{|2|} \times 1 = \frac{1}{2}$

Example - Find Fourier transform of $Sgn(2t)$

$Sgn(t) \xleftrightarrow{FT} \frac{2}{j\omega}.$

$Sgn(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$= \frac{1}{2} \times \left[\frac{2}{j(\omega/2)}\right]$

$= \frac{2}{j\omega}.$

Multiplication & Convolution property of Fourier Transform

[Statement & Examples]

Statement

If $f_1(t) \xleftrightarrow{FT} F_1(\omega)$

$f_2(t) \xleftrightarrow{FT} F_2(\omega)$

- Then multiplication property state that

$$f_1(t)\, f_2(t) \xleftrightarrow{FT} F_1(\omega) * F_2(\omega).$$

- Then Convolution property state that

$$f_1(t) * f_2(t) \xleftrightarrow{FT} F_1(\omega)\, F_2(\omega)$$

example Find Fourier transform of

$$e^{-at}u(t) * e^{-bt}u(t).$$

- $f_1(t) = e^{-at}u(t) \xleftrightarrow{FT} F_1(\omega) = \dfrac{1}{a+j\omega}.$

- $f_2(t) = e^{-bt}u(t) \xleftrightarrow{FT} F_2(\omega) = \dfrac{1}{b+j\omega}.$

- $F.T\,[f_1(t) * f_2(t)] = F_1(\omega)\, F_2(\omega)$

$$= \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right).$$

example Find Inverse Fourier Transform of

$$\left[\frac{1}{a+j\omega} * \frac{2}{j\omega}\right]$$

- $F_1(\omega) = \dfrac{1}{a+j\omega}. \xleftrightarrow{IFT} F_1(t) = e^{-at}u(t)$

- $F_2(\omega) = \dfrac{2}{j\omega} \xleftrightarrow{IFT} F_2(t) = Sgn(t).$

- $IFT\,[F_1(\omega) * F_2(\omega)] = F_1(t)\, F_2(t)$

$$= \left(e^{-at}u(t)\right)\left(Sgn(t)\right).$$

Duality or Similarity property of Fourier Transform

[Statement & Examples]

Statement

If $x(t) \xleftrightarrow{FT} X(\omega)$

then

$x(t) \xleftrightarrow{FT} 2\pi X(-\omega)$.

Example Find Fourier Transform of $\frac{6}{t^2+9}$.

- $e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{\omega^2+a^2}$

- $e^{-3|t|} \xleftrightarrow{FT} \frac{6}{\omega^2+9}$

If $e^{-3|t|} \xleftrightarrow{FT} \frac{6}{\omega^2+9}$

$\frac{6}{t^2+9} \xleftrightarrow{FT} 2\pi X(-\omega)$

$= 2\pi e^{-3|-\omega|}$

$= 2\pi e^{-3|\omega|}$

Example - Find Fourier Transform of $\frac{1}{jt}$

$\mathrm{Sgn}(t) \xleftrightarrow{FT} \frac{2}{j\omega}$.

$\Rightarrow \frac{1}{2}\mathrm{Sgn}(t) \xleftrightarrow{FT} \frac{1}{j\omega}$.

If $\frac{1}{2}\mathrm{Sgn}(t) \xleftrightarrow{FT} \frac{1}{j\omega}$.

$\frac{1}{jt} \xleftrightarrow{FT} 2\pi X(-\omega)$.

$= 2\pi\left[\frac{1}{2}\mathrm{Sgn}(-\omega)\right]$

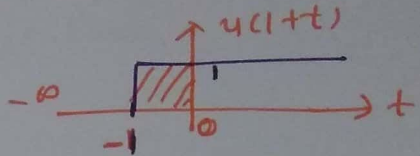
$= \pi\,\mathrm{Sgn}(-\omega)$

Fourier Transform of Shifted Unit Step Signal

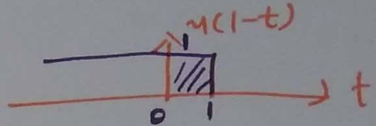
Find the Fourier transform of $x(t) = u(1-|t|)$

$$X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$x(t) = u(1-t) \quad t>0$



$= u(1+t) \quad t<0$



$$X(\omega) = \int_{-\infty}^{\infty} u(1-|t|)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} u(1+t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} u(1-t)\, e^{-j\omega t}\, dt$$

$\Rightarrow 1+t=0$ $\qquad$ $\Rightarrow 1-t=0$

$\Rightarrow t=-1$ $\qquad$ $\Rightarrow t=1$

$$= \int_{-1}^{0} e^{-j\omega t}\, dt + \int_{0}^{1} e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{0} + \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{1}$$

$$= \cancel{\frac{1}{-j\omega}} + \frac{e^{j\omega}}{j\omega} + \frac{e^{-j\omega}}{-j\omega} + \cancel{\frac{1}{j\omega}}$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega} - e^{-j\omega}}{j2}\right]$$

$$= \boxed{\frac{2}{\omega}\sin\omega} = 2\left(\frac{\sin\omega}{\omega}\right) = \boxed{2\, Sa(\omega)}$$

Fourier Transform of two Sided Exponential Signal.

Find the Fourier Transform of the signal $x(t) = e^{-3|t|}$

(a) $\frac{6}{\omega^2+9}$ (b) $\frac{3}{\omega^2+9}$ (c) $\frac{2}{\omega^2+9}$ (d) None.

$$x(t) = \boxed{e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}}$$

$$x(t) = e^{-3|t|} \xleftrightarrow{FT} \frac{6}{\omega^2+9.}$$

$\rightarrow x(t) = e^{-a|t|}$

$\rightarrow X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$x(t) = e^{-at} \quad t>0$$
$$= e^{at} \quad t<0$$

$\rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-3|t|} e^{-j\omega t} dt$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty}$$

$$= \left[\frac{1}{a-j\omega} - 0 \right] + \left[0 - \frac{1}{-(a+j\omega)} \right]$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{(a-j\omega)(a+j\omega)}$$

$$= \frac{2a}{(a^2-j^2\omega^2)} = \boxed{\frac{2a}{a^2+\omega^2}} \xleftrightarrow{IFT} e^{-a|t|}$$

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Examples of Inverse Fourier Transform

- Find Inverse Fourier Transform of $X(\omega) = \frac{3+j\omega}{(1+j\omega)^2}$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}.$$

$$t^n e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^{n+1}}$$

- $X(\omega) = \frac{3+j\omega}{(1+j\omega)^2} = \frac{1+j\omega+2}{(1+j\omega)^2} = \frac{1}{1+j\omega} + 2\left[\frac{1}{(1+j\omega)^2}\right]$

- take Inverse Fourier transform.

$$x(t) = e^{-t}u(t) + 2te^{-t}u(t)$$

$$= \boxed{(1+2t)e^{-t}u(t).}$$

Find Inverse Fourier Transform of $X(\omega) = \frac{\omega^2+21}{\omega^2+9}$

$$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{\omega^2+a^2}$$

- $X(\omega) = \frac{\omega^2+21}{\omega^2+9} = \frac{\omega^2+9+12}{\omega^2+9} = 1 + \frac{12}{\omega^2+9} = 1+\left(\frac{2\times 3}{\omega^2+3^2}\right)\times 2$

- take Inverse Fourier transform.

$$\boxed{x(t) = \delta(t) + 2e^{-3|t|}}$$

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Find the Inverse Fourier Transform of

$$x(\omega) = \frac{5+2j\omega}{(j\omega)^2+5j\omega+6}.$$

(a) $(e^{3t}+e^{2t})u(t)$

(b) $(e^{-3t}+e^{-2t})u(t)$

(c) $(3e^{-3t}+2e^{2t})u(t)$

(d) None

$$- X(\omega) = \frac{5+2j\omega}{(2+j\omega)(3+j\omega)} = \frac{A}{2+j\omega} + \frac{B}{3+j\omega}.$$

$$\Rightarrow 5+2j\omega = A(3+j\omega) + B(2+j\omega).$$

If, $j\omega = -2$

$\Rightarrow 5+2(-2) = A(3-2)$

$\Rightarrow \boxed{A=1}$

If $j\omega = -3$

$\Rightarrow 5+2(-3) = B(2-3)$

$\Rightarrow \boxed{B=1}$

$$- X(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}.$$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}.$$

- take Inverse Fourier transform

$$- x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

$$= \boxed{(e^{-2t}+e^{-3t})u(t).}$$

Example of Fourier Transform of Spectral Property.

The power spectral density of a deterministic signal is given by $\left[\frac{\sin f}{f}\right]^2$. where f is freq., The auto correlation function of this signal in the time domain is,

$$S(\omega) = \left[\frac{\sin f}{f}\right]^2 = (Sa(\omega))^2$$

(a) Rectangular pulse

(b) A delta function

(c) A sine pulse

(d) Triangular pulse ✓

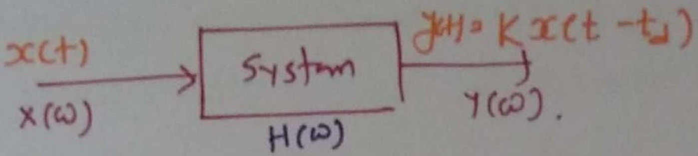
$$\Rightarrow R(\tau) \xleftrightarrow{FT} S(\omega)$$

$$\Rightarrow \underline{R(\tau)} \xleftrightarrow{FT} (Sa(\omega))^2$$

$\uparrow$

Triangular function.

Distortion Less Transmission of Signal.



$-\ y(t) = k\, x(t - t_d)$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(t-t_d) \xleftrightarrow{FT} e^{-j\omega t_d} X(\omega)$$

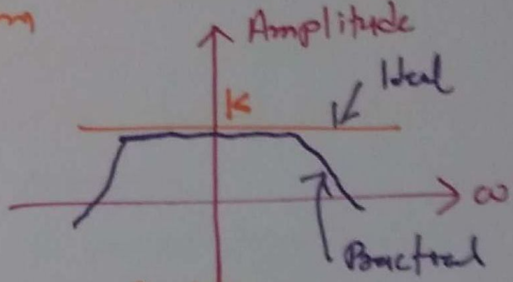
$$\Rightarrow Y(\omega) = K\ FT[x(t-t_d)]$$

$$\Rightarrow Y(\omega) = K e^{-j\omega t_d} X(\omega).$$

– Transfer function of System

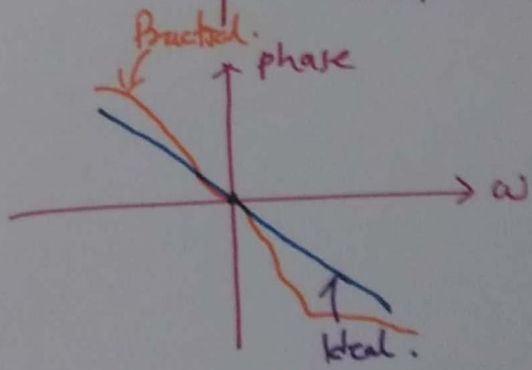
$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \underline{\underline{K e^{-j\omega t_d}}}$$



magnitude Spectrum $= K$

phase Spectrum $\phi = -\omega t_d$



Hilbert Transform

- Hilbert transform of a signal $x(t)$ is defined as the transform in which phase angle of all components of signal is shifted by $\pm 90^\circ$.

- Hilbert Transform of signal $x(t)$ represented with $\hat{x}(t)$ & given by

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\omega)}{t-\omega}\, d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(s)}{t-s}\, ds$$

Properties of Hilbert Transform

- a signal $x(t)$ & its hilbert transform follows
  - same amplitude spectrum
  - same Auto correlation function
  - same ESD [Energy spectral density]
- $x(t)$ & $\hat{x}(t)$ are Orthogonal to each other.
- IF Fourier transform of for energy & power signal exist then Hilbert Transform also exist. $\boxed{\int_{-\infty}^{\infty} x(t)\, dt < \infty}$