SIGNALS & SYSTEMS

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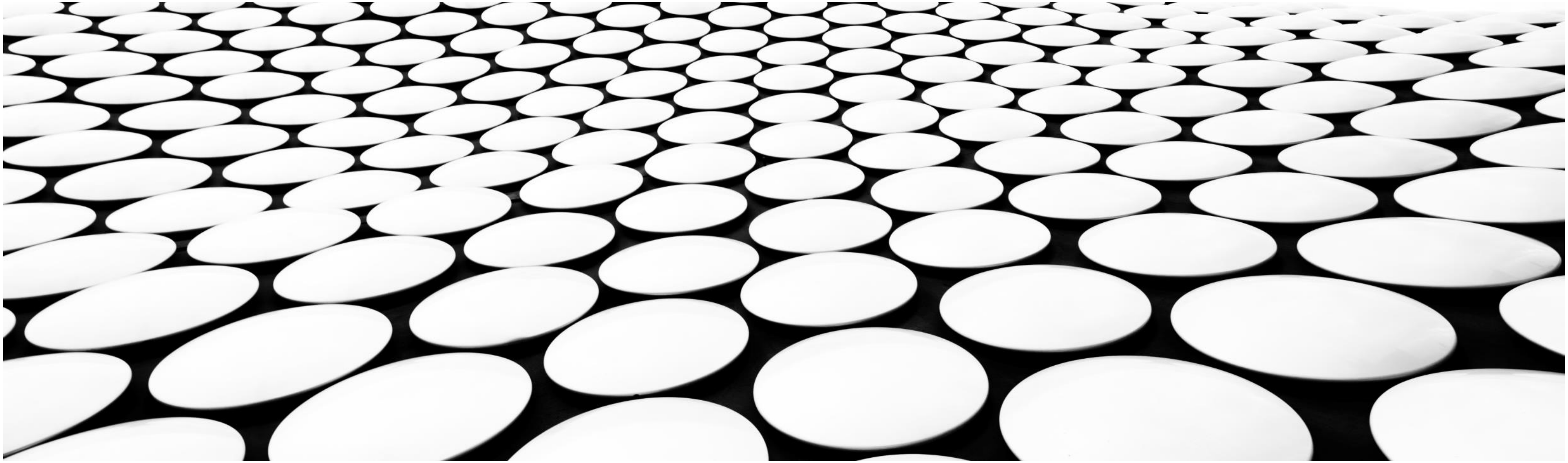

Inverse Laplace Transform

The *Inverse Laplace transform* of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma - j\Omega}^{s=\sigma + j\Omega} X(s)\, e^{st}\, ds}$$

Mainly three cases → Signals with separate poles

→ signals with multiple poles

→ Signals with complex conjugate poles.

We will use partial Fraction method to calculate inverse Laplace transform.

let us consider case 1:- Distinct separate pole

let us consider $X(s) = \dfrac{2}{s(s+1)(s+2)}$

$\Downarrow$

Three distinct poles at $s=0$; $s=-1$ and $s=-2$.

$$X(s) = \boxed{\frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}}$$

Multiply with $s(s+1)(s+2)$ $\rightarrow$ we need to determine the value of A, B and C

$$\therefore \quad 2 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1) \quad \text{—— (i)}$$

put $s=0$; in eq (i)

$$2 = A(0+1)(0+2) + B \cdot 0 \cdot (s+2) + C \cdot 0\,(s+1)$$

$$\Rightarrow 2 = A \cdot 2$$

$$\Rightarrow \boxed{A = 1}$$

Putting $s = -1$, in eq^n I

$2 = B(-1)(-1+2)$

$\Rightarrow 2 = B(-1)$

$\Rightarrow \boxed{B = -2}$

putting $s = -2$ in eq^n 1

$2 = C(-2)(-2+1)$

$\Rightarrow 2 = C(-2)(-1)$

$\Rightarrow \boxed{C = 1.}$

$\therefore$ $X(s)$ can be re-written as $X(s) = \frac{1}{s} + \frac{(-2)}{s+1} + \frac{1}{s+2}$

$\Rightarrow X(s) = \frac{1}{s} - 2\frac{1}{(s+1)} + \frac{1}{(s+2)}$

Inverse Laplace transform convert $X(s)$ into $x(t)$.

From s-domain → Time domain

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - 2 \cdot L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow x(t) = u(t) - 2 \cdot e^{-t} u(t) + e^{-2t} u(t).$$

$$\Rightarrow x(t) = (1 - 2e^{-t} + e^{-2t}) u(t)$$

$$\Rightarrow \boxed{x(t) = (1 - e^{-t})^2 u(t)}$$

Case no: 2 :-

$$X(s) = \frac{2}{s(s+1)(s+2)^2}$$

here multiple pole at $s = -2$

Total no. of poles at $s = 0$, $s = -1$ and two multiple pole at at $s = -2$.

using partial fraction method:-

X(s) can be expressed as

$$X(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \underbrace{\frac{C}{s+2} + \frac{D}{(s+2)^2}}$$

when we have multiple pole this procedure must be adopted.

Now, $2 = A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1)(s+2) + D \cdot s(s+1)$ — (1)

Putting $s=0$ in eqn (1) $\Rightarrow 2 = A(0+1)(0+2)^2$

$\Rightarrow 2 = A \times 4 \Rightarrow \boxed{A = 1/2}$

putting $s=-1$ in eqn ① $\Rightarrow$

$$2 = B(-1)(-1+2)^2$$

$$\Rightarrow 2 = B(-1)(1)$$

$$\Rightarrow \boxed{B=-2}$$

putting $s=-2$ in eqn ① $\Rightarrow$

$$2 = D(-2)(-2+1)$$

$$\Rightarrow 2 = D(-2)(-1)$$

$$\Rightarrow \boxed{D=1}$$

In this case, C is left and to find C reconsider eqn ①; i.e;

$$2 = A(s+1)(s+2)^2 + B\,s(s+2)^2 + C\,s(s+1)(s+2) + D\,s(s+1)$$

$$\Rightarrow 2 = A\left[(s+1)(s^2+4+4s)\right] + B\,s(s^2+4+2s) + C\,s(s^2+2s+s+2) + D(s^2+s)$$

$$\Rightarrow 2 = A(s^3+4s+4s^2+s^2+4+4s) + B(s^3+4s+2s^2) + C(s^3+3s^2+2s) + D(s^2+s)$$

$$\Rightarrow 2 = A(s^3+5s^2+8s+4) + B(s^3+2s^2+4s) + C(s^3+3s^2+2s) + D(s^2+s)$$

$$\Rightarrow 2 = (A+B+C)s^3 + (5A+2B+3C+D)s^2 + (8A+4B+2C+D)s + (4A)s^0 \quad \text{—— ②}$$

From eqⁿ ②, comparing the co-efficients of s^3 on both sides we get –

$$A+B+C=0$$

since we already found the value of $A = 1/2$ and $B=-2$

$$\Rightarrow \ 1/2 + (-2) + C = 0$$

$$\Rightarrow \ C = 2 - 1/2 = \frac{4-1}{2}$$

$$\Rightarrow \ \boxed{C = \frac{3}{2}}$$

$$X(s) = \frac{1/2}{s} + \frac{-2}{s+1} + \frac{3/2}{s+2} + \frac{1}{(s+2)^2}$$

$$\Rightarrow X(s) = \frac{1}{2}\cdot\frac{1}{s} - 2\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+2} + \frac{1}{(s+2)^2}$$

$$\therefore L^{-1}\{X(s)\} = \frac{1}{2}L^{-1}\{1/s\} - 2L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$\Rightarrow x(t) = \frac{1}{2}u(t) - 2e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) + te^{-2t}u(t)$$

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Case 3:-

$$X(s) = \frac{1}{(s+2)(s^2+s+1)}$$

quadratic eqn of s, which gives complex conjugate poles.

$$\Rightarrow X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

This pattern must be followed for quad. eqn

$$\therefore \quad 1 = A(s^2+s+1) + (Bs+C)(s+2) \quad —(1)$$

Now, in this case eqn (1) must be expanded to calculate the value of A, B and C.

$$1 = A(s^2+s+1) + (Bs^2 + 2Bs + Cs + 2C)$$

$$\Rightarrow 1 = (A+B)s^2 + (A+2B+C)s + (A+2C)s^0 \quad —(2)$$

Comparing the coefficients of s^2, s and s^0 for eqn (2), we get

$A+B=0 \Rightarrow A=-B$

$A+2B+C=0 \Rightarrow -B+2B+C=0 \Rightarrow \boxed{B+C=0} \longrightarrow (x)$

$A+2C=1$

$\Rightarrow \boxed{-B+2C=1} \longrightarrow (y)$

Adding (x) and (y) $\Rightarrow 3C=1$

$\Rightarrow \boxed{C=1/3}$

From (x) $\Rightarrow B=-C$

$\Rightarrow \boxed{B=-1/3}$

Also $A=-B$

$\Rightarrow \boxed{A=1/3}$

$$\therefore X(s) = \frac{1/3}{s+2} + \frac{-1/3\, s + 1/3}{s^2+s+1}$$

$$= \frac{1}{3}\cdot\frac{1}{s+2} + \frac{-\frac{1}{3}(s-1)}{s^2+s+1}$$

Try to get into $(x+y)^2 = x^2+2xy+y^2$ form

$$= \frac{1}{3}\cdot\frac{1}{s+2} - \frac{1}{3}\,\frac{s-1}{s^2+2\times s\times 0.5+(0.5)^2+1-(0.5)^2}$$

$$= \frac{1}{3}\cdot\frac{1}{s+2} - \frac{1}{3}\,\frac{s-1}{(s+0.5)^2+0.75}$$

$$= \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5-1-0.5}{(s+0.5)^2+\sqrt{0.75}^2}$$

$$= \frac{1}{3}\cdot\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5-1.5}{(s+0.5)^2+(0.866)^2}$$

$$\boxed{\sqrt{0.75} = 0.866}$$

$$X(s) = \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5}{(s+0.5)^2+(0.866)^2} - \frac{1}{3}\frac{1.5}{(s+0.5)^2+(0.866)^2}$$

$$\Rightarrow X(s) = \frac{1}{3}\cdot\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5}{(s+0.5)^2+(0.866)^2} - \frac{1}{3}\times\frac{1.5}{0.866}\times\frac{0.866}{(s+0.5)^2+(0.866)^2}$$

$\therefore$ Now $x(t) = L^{-1}\{X(s)\} = \frac{1}{3}e^{-2t}u(t) - \frac{1}{3}e^{-0.5t}\cos 0.866t\, u(t) + 0.577\, e^{-0.5t}\sin 0.866t\, u(t)$ #

Formula used $\rightarrow$ $L\{e^{-at}\sin\omega_0 t\, u(t)\} = \dfrac{\omega_0}{(s+a)^2+\omega_0^2}$

$$L\{e^{-at}\cos\omega_0 t\, u(t)\} = \frac{s+a}{(s+a)^2+\omega_0^2}$$