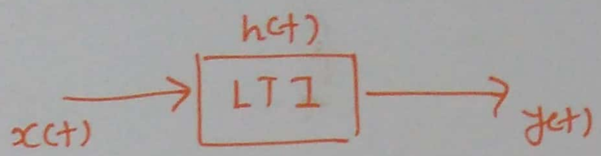
# Laplace Transform

- Laplace transform is also one tool to represent given signal in frequency domain.




- If $x(t) = G e^{st}$
- Hence, $s \rightarrow$ s-plane

$$s = \sigma + j\omega$$

- In fourier transform, we have only imag. plane, while in Laplace x'form we have imag + real plane.
- output

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, G e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \underline{G e^{st}}\, e^{-s\tau}\, d\tau$$

$$= G e^{st} \boxed{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}$$

$$\boxed{y(t) = x(t)\, h(s)}$$

$$\Rightarrow \boxed{h(s) = \frac{y(t)}{x(t)} = \frac{Y(s)}{X(s)}}$$

- $L.T.[x(t)] = \boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$

Scanned by CamScanner

## Relation between Laplace & Fourier transform

→ $L.T.[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ — ①

→ $F.T.[x(t)] = x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ — ②

→ In Laplace transform

$$s = \sigma + j\omega$$

→ from eq.ⁿ ①

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \underline{x(t)\, e^{-\sigma t}}\, e^{-j\omega t}\, dt$$

$$\boxed{L.T.[x(t)] = F.T.[x(t)\, e^{-\sigma t}]}$$

## Inverse Laplace Transform

$$\Rightarrow X(s) = F.T.\left[x(t)e^{-\sigma t}\right]$$

$$\Rightarrow F.T.^{-1}\left[x(s)\right] = x(t)\,e^{-\sigma t}$$

Inver. fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\,e^{j\omega t}\,d\omega.$$

$$\Rightarrow x(t)\,e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)\,e^{j\omega t}\,d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)\,e^{j\omega t}\,e^{\sigma t}\,d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)\,e^{(\sigma+j\omega)t}\,d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)\,e^{st}\,d\omega$$

- If $\sigma + j\omega = s$

$\rightarrow j\,d\omega = ds$

$\Rightarrow d\omega = ds/j$

$$\rightarrow \boxed{X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt}$$

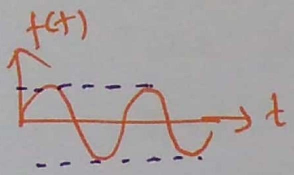
$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)\,e^{st}\,\frac{ds}{j}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} x(s)\,e^{st}\,ds}$$

# Conditions for existence of Laplace transform

- Dirichlet's Conditions are used to define the existence of Laplace transform

**Conditions** ① The function must be absolutely Integrable in the given interval of time.




i.e. $\boxed{\int_{-\infty}^{\infty} |f(t)|\, dt < \infty}$

② The function should have finit number of of maxima & minima

③ there must be finit number of discontinuities in the signal in the given time Interval.

Properties of Laplace transform

1. linearity property $a_1x_1(t) + a_2x_2(t) \xleftrightarrow{LT} a_1X_1(s) + a_2X_2(s)$

2. Time shifting property $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$

3. Freq. shifting property $X(s-s_0) \xleftrightarrow{LT^{-1}} e^{s_0t} x(t)$

4. Time reversal property $x(-t) \xleftrightarrow{LT} - X(-s)$

5. Time scaling $x(at) \xleftrightarrow{LT} \frac{1}{|a|} X(s/a)$

6. Differentiation & Integration property $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s)$

7. Multiplication & convolution property. $\int x(t)\,dt \xleftrightarrow{LT} \frac{X(s)}{s}$

$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s)\,X_2(s)$

$x_1(t)\,x_2(t) \xleftrightarrow{LT} X_1(s) * X_2(s)$

# Linearity property of Laplace transform

- If $x(t) \xleftrightarrow{LT} X(s)$
  $y(t) \xleftrightarrow{LT} Y(s)$
- Then as per Linearity property
  $$a\,x(t) + b\,y(t) \xleftrightarrow{LT} a\,X(s) + b\,Y(s)$$
- $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$
- $Y(s) = \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt$
- Hence, $z(t) = a\,x(t) + b\,y(t)$
- $L.T.[z(t)] = \int_{-\infty}^{\infty} z(t)\, e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} (a\,x(t) + b\,y(t))\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} a\,x(t)\, e^{-st} + b\,y(t)\, e^{-st}\, dt$$

$$= a\,X(s) + b\,Y(s)$$

Time shifting property of Laplace transform

→ If $x(t) \xleftrightarrow{LT} X(s)$

then time shifting of input

$$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s).$$

- $L.T.[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

- $L.T.[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$

- If $t - t_0 = \lambda \Rightarrow t = \lambda + t_0$

$\Rightarrow dt = d\lambda$

$$= \int_{-\infty}^{\infty} x(\lambda) e^{-s(\lambda+t_0)} d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda) e^{-s\lambda} \underline{e^{-st_0}} d\lambda$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(\lambda) e^{-s\lambda} d\lambda$$

$$\Rightarrow \boxed{L.T.[x(t+t_0)] = e^{-st_0} X(s)}$$

- $X(s) = \dfrac{1}{2s+1}$, $x(t) \xleftrightarrow{LT} X(s)$, find LT of $x(t-2)$

$$L.T.[x(t-2)] = e^{-2s} X(s)$$

$$= \frac{e^{-2s}}{2s+1}$$

Frequency shifting property of Laplace transform

- If $x(t) \xleftrightarrow{LT} X(s)$

then freq. shifting means

$$e^{s_0 t} x(t) \xleftrightarrow{LT} X(s-s_0) \checkmark$$

- $L.T[x(t)] = X(s)$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

- $y(t) = e^{s_0 t} x(t)$

- $L.T.[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} e^{s_0 t} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt$$

$$\Rightarrow \boxed{L.T.[e^{s_0 t} x(t)] = X(s-s_0)}$$

- If $X(s) = \dfrac{s}{s+1}$. Find $[L.T.(x(t)\, e^{2t})]$

$$L.T.[x(t)\, e^{2t}] = X(s-2)$$

$$= \frac{(s-2)}{(s-2)+1}$$

$$= \boxed{\frac{s-2}{s-1}}$$

Scanned by CamScanner

Time reversal property of Laplace transform

- If $x(t) \xleftrightarrow{LT} X(s)$

  then time reversal

  $$x(-t) \xleftrightarrow{LT} -X(-s)$$

- $X(s) = L.T\,[x(t)]$

  $$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

- $L.T\,[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-st}\, dt$

  If, $-t = \lambda \Rightarrow t = -\lambda$

  $\Rightarrow dt = -d\lambda$

- $L.T\,[x(-t)] = -\int_{-\infty}^{\infty} x(\lambda)\, e^{-(-s\lambda)}\, d\lambda$

  $$= -X(-s)$$

# Time Scaling property of Laplace transform

- If $x(t) \xleftrightarrow{LT} X(s)$

  then time scaling

  $$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X(s/a)$$

- $LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

- If $y(t) = x(at)$

- $$LT[y(t)] = \int_{-\infty}^{\infty} y(t) e^{-st} dt$$

  $$= \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

  - $at = \lambda \Rightarrow t = \lambda/a$

    $\Rightarrow dt = d\lambda/a$

  $$= \int_{-\infty}^{\infty} x(\lambda) e^{-s\lambda/a} \left(\frac{d\lambda}{a}\right)$$

- $$L.T[y(t)] = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-(s/a)\lambda} d\lambda$$

- $$\boxed{L.T[x(at)] = \frac{1}{a} X(s/a)}$$

e.g. $x(t) \xleftrightarrow{LT} \frac{1}{s+1}$, find $L.T[x(2t)]$

$$L.T.[x(2t)] = \frac{1}{2} X(s/2)$$

$$= \frac{1}{2}\left[\frac{1}{(s/2)+1}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s+2}\right]$$

$$= \frac{1}{s+2}$$

# Differentiation & Integration property of Laplace X'form.

- If $x(t) \xleftrightarrow{LT} X(s)$
- Then differentiation, $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s)$
- Then Integration, $\int x(t)\,dt \xleftrightarrow{LT} \frac{1}{s}X(s)$.
- For multiple differentiation & Integration.

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{LT} s^n X(s)$$

$$\int\int^n x(t)\,dt \xleftrightarrow{LT} \frac{1}{s^n} X(s)$$

Multiplication & convolution property of Laplace transform

- For $x_1(t)$ and $x_2(t)$

$$x_1(t) \xleftrightarrow{LT} X_1(S)$$

$$x_2(t) \xleftrightarrow{LT} X_2(S)$$

- convolution property

$$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(S)\,X_2(S)$$

- multiplication property

$$x_1(t)\,x_2(t) \xleftrightarrow{LT} X_1(S) * X_2(S).$$

# Initial Value theorm

- Initial value theorm used to Identity initial value of any signal or function $(t = 0^+)$
- If signal in time domain is $f(t)$ and Laplace domain $F(s)$, then Initial value of $f(t) / F(s)$ can be calculated by.

$$F(0^+) = \lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t)$$

- e.g. If $F(s) = \frac{2}{s+2}$. Find Initial value of $F(s)$.

$$F(0^+) = \lim_{s \to \infty} sF(s)$$

$$= \lim_{s \to \infty} s\left[\frac{2}{s+2}\right]$$

$$= \lim_{s \to \infty} \frac{2}{1 + 2/s}$$

$$= \frac{2}{1+0} = \boxed{2}$$

# Final Value theorm

- Final value theorm used to identity final value ($t \to \infty$) of any signal or function
- If signal in time domain is $f(t)$ and in Laplace domain $F(s)$. then final value of $f(t)$ / $f(s)$ can be calculated by

$$F(\infty) = \lim_{s \to 0} s\,F(s) = \lim_{t \to \infty} F(t)$$

- e.g. If $F(s) = \dfrac{1}{s(s+1)(s+2)}$ . Find final value of given function

$$F(\infty) = \lim_{s \to 0} s\,F(s)$$

$$= \lim_{s \to 0} \cancel{s}\left[\frac{1}{\cancel{s}(s+1)(s+2)}\right]$$

$$= \lim_{s \to 0} \frac{1}{(s+1)(s+2)}$$

$$= \frac{1}{(0+1)(0+2)} = \frac{1}{2}$$

Region of Convergence ROC in Laplace transform & it's properties.

S-plane $\longrightarrow s = \underline{\sigma} + j\omega$

- ROC is range of variation of $\sigma$, for which Laplace transform converges is called Region of Convergence.

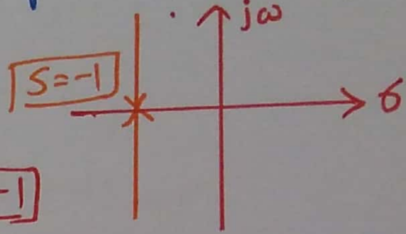
## Properties

1. The ROC of F(s) includes the strip lines parallel to imaginary axis in s-plane.

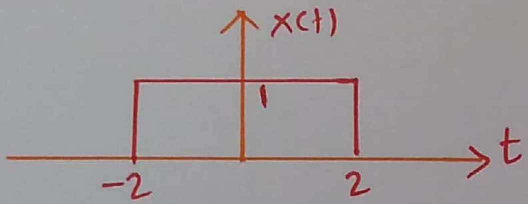
- $f(t) \longrightarrow F(s) = \dfrac{N}{D}$ ← Zeros, ← Poles $= \dfrac{s-1}{s+1}$

Poles defines ROC

Pole $\boxed{s=-1}$



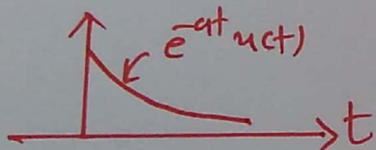

2. If f(t) is absolutely integrable & finite duration then the ROC is entire s-plane.




ROC of x(t) is entire s-plane.

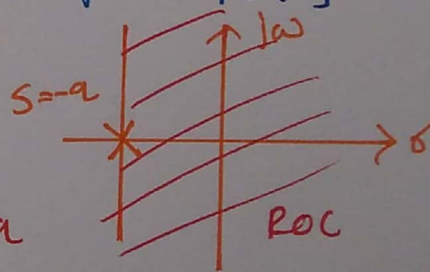
3. If f(t) is right sided signal then ROC: $Re\{s\} > \sigma$

- $f(t) = e^{-at}u(t) \longrightarrow F(s) = \dfrac{1}{s+a}$



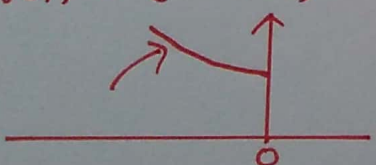

→ Pole is at $s = -a$

→ ROC : $Re\{s\} > -a$



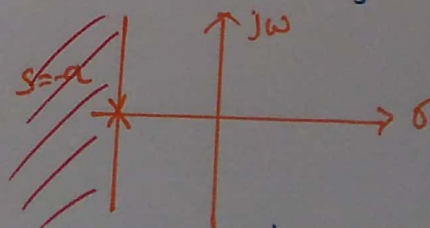

4. If f(t) is left sided signal then ROC : $Re\{s\} < \sigma$

- $f(t) = e^{-at}u(-t) \longrightarrow F(s) = \dfrac{-1}{s+a}$



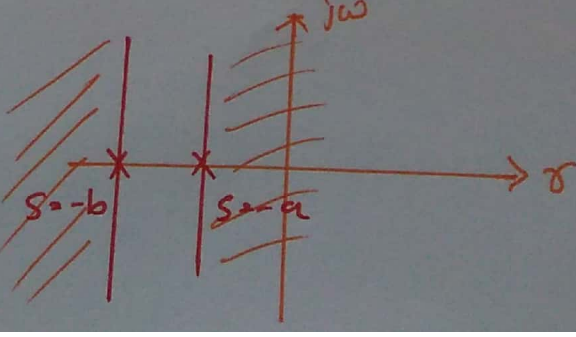

→ Pole is at $s = -a$

→ ROC : $Re\{s\} < -a$



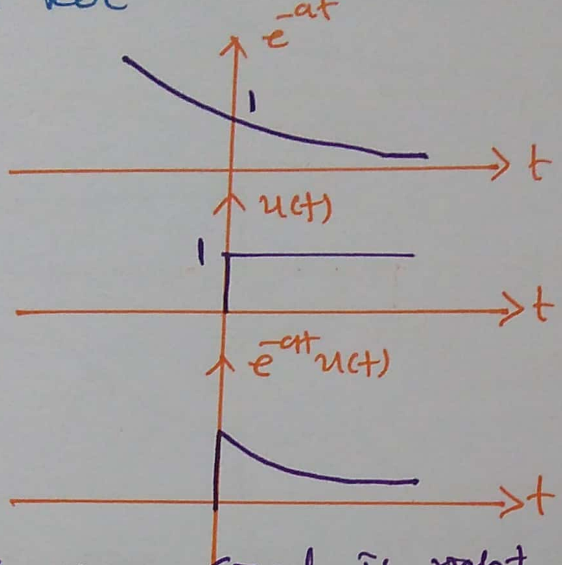

5. If f(t) is two sided then ROC is combination of two region.

- $f(t) = e^{-at}u(t) + e^{-bt}u(-t)$



Example on Laplace transform and Region of Convergence for right Sided Signal

- Find the Laplace transform of $e^{-at}u(t)$ & determine the ROC




$$X(s) = L.T.[x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt$$

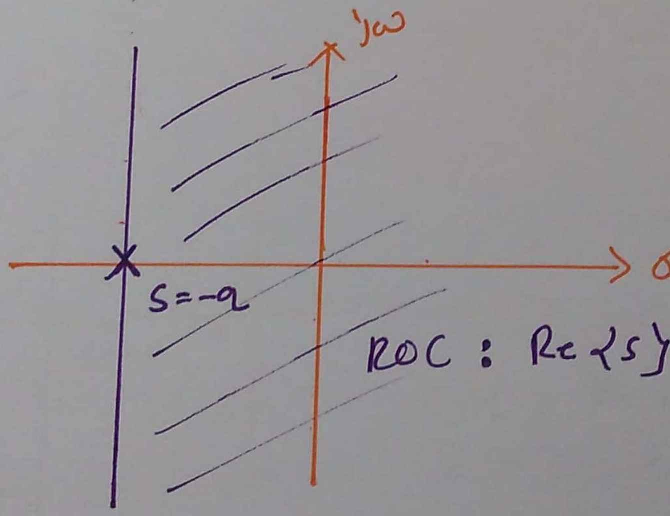
$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= 0 + \frac{1}{+(s+a)} = \frac{1}{s+a}$$

Pole at $s=-a$

- So given signal is right sided signal.

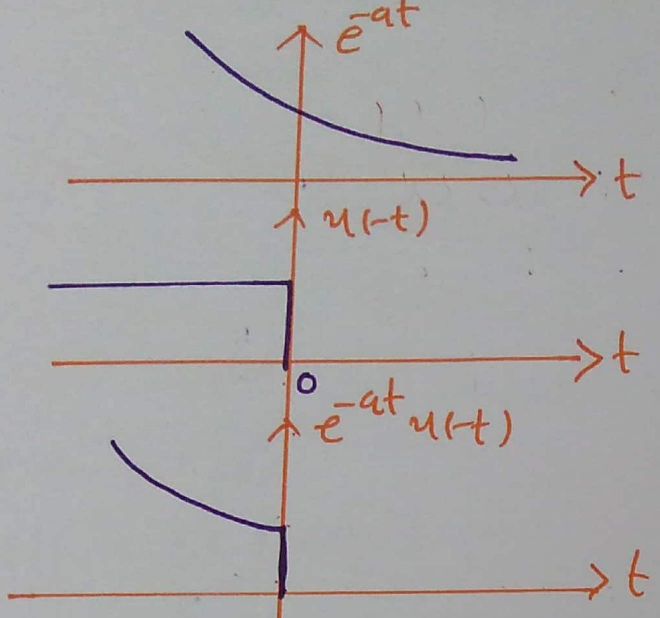



ROC : $Re\{s\} > -a$

Scanned by CamScanner

Example on Laplace transform and Region of Convergence ROC for Left sided Signal

- Find the Laplace transform of $e^{-at}u(-t)$ & determine the ROC.




$$X(s) = L.T[x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(-t)\, e^{-st}\, dt$$

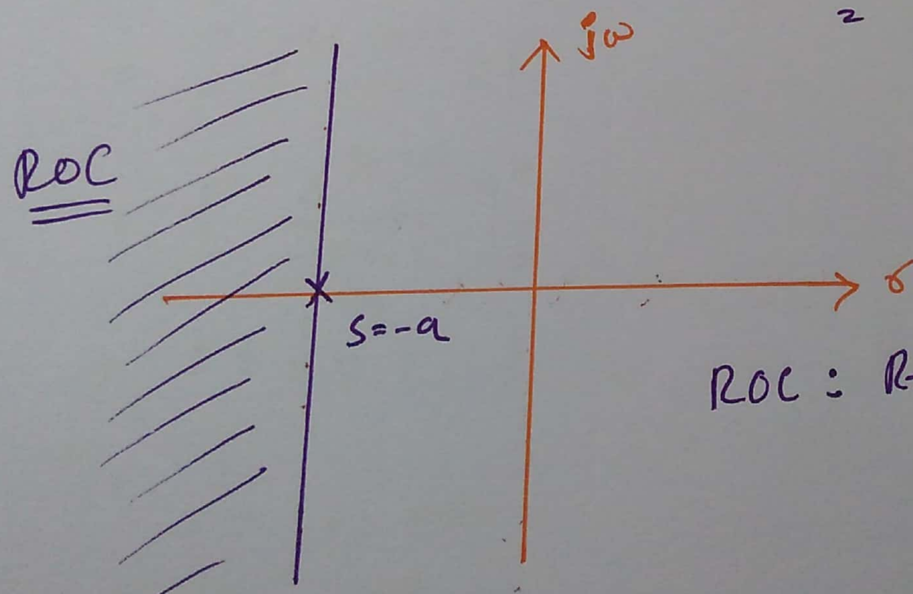
$$= \int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_{-\infty}^{0}$$

$$= \frac{1}{-(s+a)} - 0$$

$$= \frac{-1}{s+a} \leftarrow \text{Pole is at } s = -a$$

- So given signal is left sided signal




ROC : $Re\{s\} < -a$

Example on Laplace transform and Region of Convergence for exponential signal.

- Find Laplace transform & Region of convergence ROC for

① $e^{at}u(t)$ ② $-e^{at}u(-t)$.

**①**

- Signal varies from $t=0$ to $t=\infty$
- So signal is right sided signal.

$x(t) \xleftrightarrow{LT} X(s)$

$$-\;X(s) = LT[x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

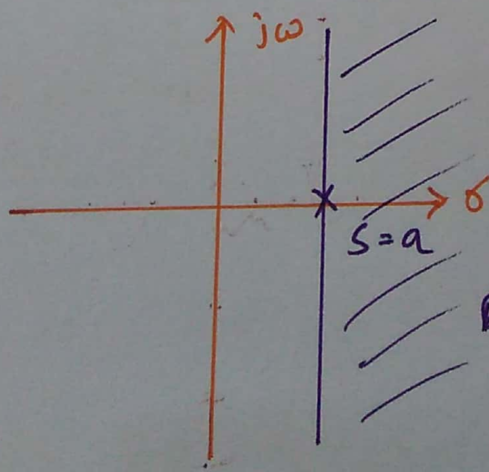
$$= \int_{-\infty}^{\infty} e^{at} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$= \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= 0 + \left[\frac{1}{+(s-a)}\right]$$

$$= \frac{1}{s-a} \leftarrow \text{pole is at } s=a$$




ROC: $Re\{s\} > a$

**②**

- Signal varies from $t=-\infty$ to $t=0$
- So signal is Left sided signal.

$y(t) \xleftrightarrow{LT} Y(s)$

$$-\;Y(s) = LT[y(t)]$$

$$= \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt$$

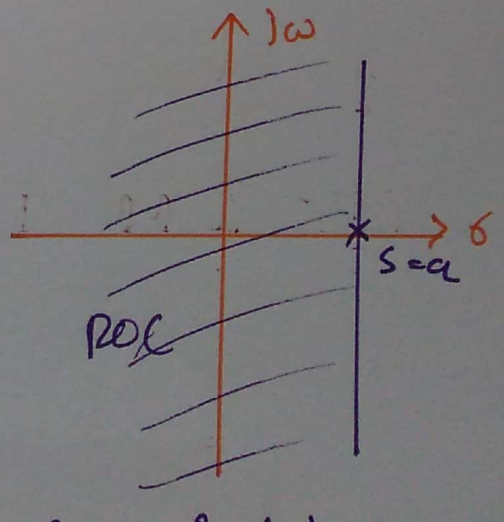
$$= \int_{-\infty}^{\infty} -e^{at} u(-t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} -e^{-(s-a)t}\, dt$$

$$= \left[ \frac{+e^{-(s-a)t}}{+(s-a)} \right]_{-\infty}^{0}$$

$$= \frac{1}{s-a} - 0$$

$$= \frac{1}{s-a} \leftarrow \text{Pole is at } s=a$$




ROC: $Re\{s\} < a$

# Examples of Laplace Transform & Region of Convergence

Find LT & ROC of $e^{-t}u(t) + e^{-2t}u(t)$

- $x(t) = e^{-t}u(t) + e^{-2t}u(t)$
- $X(s) = LT[x(t)]$

$$= LT[e^{-t}u(t) + e^{-2t}u(t)]$$

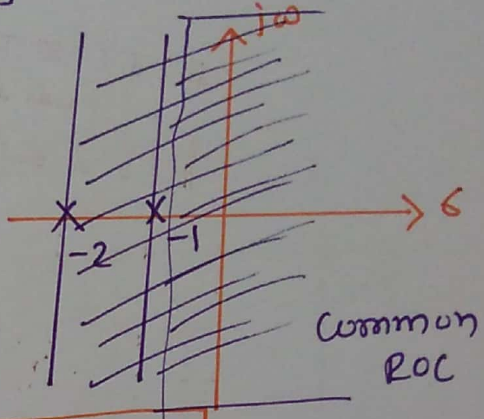
$$= LT[e^{-t}u(t)] + LT[e^{-2t}u(t)]$$

$$LT[e^{-at}u(t)] = \frac{1}{s+a}$$

$$= \frac{1}{s+1} + \frac{1}{s+2}$$

$\frac{1}{s+1}$ ↓
- Right Sided
- Pole - $s=-1$

$\frac{1}{s+2}$ ↓
- Right Sided
- Pole $= -2$




$$\boxed{ROC : Re\{s\} > -1}$$

Find LT & ROC of $e^{-2t}u(t) + e^{-t/2}u(-t)$

- $x(t) = e^{-2t}u(t) + e^{-t/2}u(-t)$
- $X(s) = LT[x(t)]$

$$= LT[e^{-2t}u(t) + e^{-t/2}u(-t)]$$

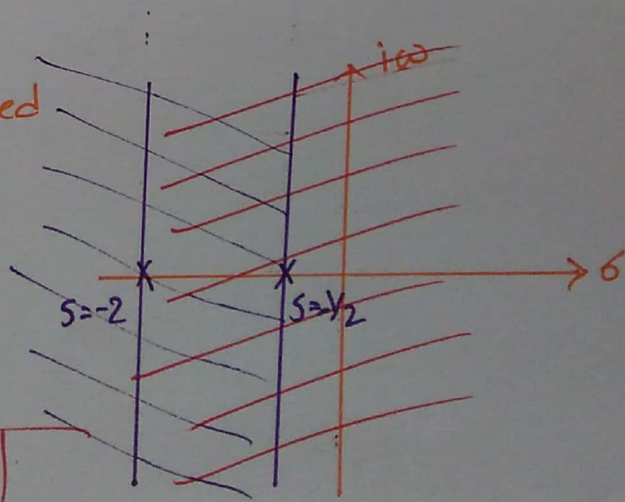
$$= LT[e^{-2t}u(t)] + LT[e^{-t/2}u(-t)]$$

$$LT[e^{-at}u(t)] = \frac{1}{s+a}, \quad LT[e^{-at}u(-t)] = \frac{-1}{s+a}$$

$$= \frac{1}{s+2} + \frac{-1}{s+1/2}$$

$\frac{1}{s+2}$
- Right Sided
- Pole $s=-2$

$\frac{-1}{s+1/2}$
- Left Sided
- $s = -1/2$




ROC : Re{s}

$$\boxed{-2 < Re\{s\} < -1/2}$$

## Example of Laplace Transform & ROC for no ROC.

Find LT & ROC of $e^{-2t}u(-t) + e^{5t}u(-t)$

- $x(t) = e^{-2t}u(-t) + e^{5t}u(-t)$

- $X(s) = LT[x(t)]$

$x(t) = e^{-2t}u(-t) + \underline{e^{5t}u(t)}$

$$= LT[e^{-2t}u(-t) + e^{5t}u(-t)]$$

$$= LT[e^{-2t}u(-t)] + LT[e^{5t}u(-t)]$$

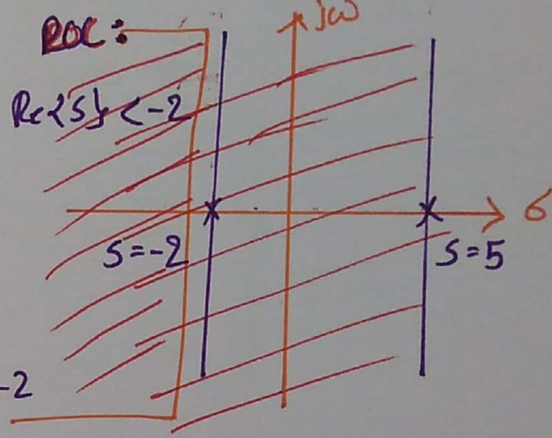
$$LT[e^{-at}u(-t)] = \frac{-1}{s+a}$$

$$= \frac{-1}{s+2} + \frac{-1}{s-5}$$

- Left Sided
- Pole $s=-2$

- Left Soded
- pole, $s=5$

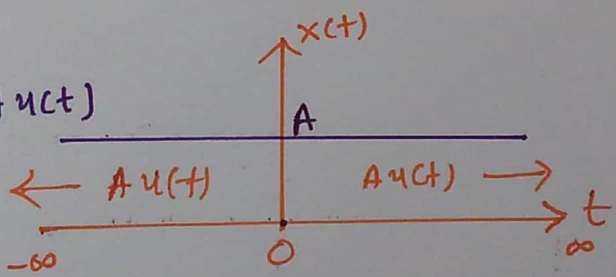
ROC : $Re\{s\} < -2$




## Laplace transform & ROC of DC signal

$$x(t) = A$$

$$= Au(-t) + Au(t)$$




- $X(s) = LT[x(t)]$

$$= LT[Au(-t) + Au(t)]$$

$$= LT[Au(-t)] + LT[Au(t)]$$

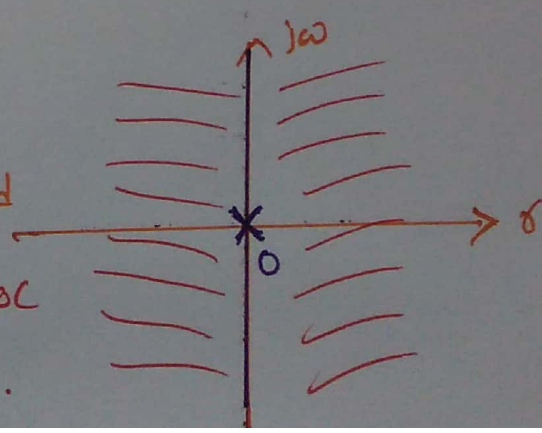
$$LT[u(t)] = 1/s \quad , \quad LT[u(-t)] = -1/s$$

$$= \underline{A[-1/s]} + \underline{A[1/s]}$$

$= 0$

left Sided

Right Sided



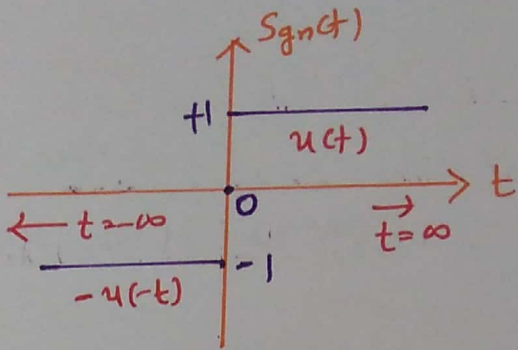

- There is no common ROC
- So Here, ROC is empty.

# Example of Laplace transform & ROC of Signum function

- $x(t) = \mathrm{Sgn}(t)$

$$= \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$= u(t) - u(-t)$$




- $X(s) = LT[x(t)]$

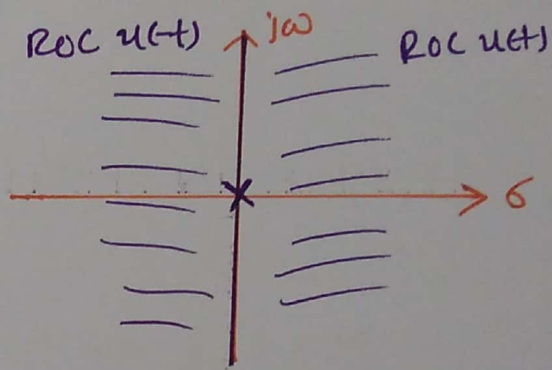
$$= LT[u(t) - u(-t)]$$

$$= LT[u(t)] - LT[u(-t)]$$

$$= 1/s - (-1/s)$$

$$= 2/s$$

$\boxed{\text{- No ROC}}$



L Unilateral Laplace Transform

$$ULT[x(t)] = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

- $x(t) = u(t+1) + u(t-4)$. find ULT & LT

- $LT(x(t)) = LT[u(t+1) + u(t-4)]$

$= LT[\underline{u(t+1)}] + LT[\underline{u(t-4)}]$

$t = -1$ $\quad$ $t = 4$

$x(t) \xleftrightarrow{LT} X(s)$

$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$

$u(t) \longleftrightarrow 1/s$

$= e^{s} \cdot (1/s) + e^{-4s} \cdot (1/s)$

$= e^{s}/s + e^{-4s}/s$

- For ULT, we need consider time $t = 0$ to $t = \infty$

- $ULT(x(t)) = \dfrac{e^{-4s}}{s}$

Laplace Transform & Unilateral Laplace transform of Impulse Signal using time shifting property

Find LT & ULT of $\delta(t+1) + \delta(t-1) + \delta(t-2)$

$- \; x(t) = \delta(t+1) + \delta(t-1) + \delta(t-2)$

$- \; LT[x(t)] = LT[\delta(t+1) + \delta(t-1) + \delta(t-2)]$

$= LT[\underbrace{\delta(t+1)}_{t=-1}] + LT[\underbrace{\delta(t-1)}_{t=1}] + LT[\underbrace{\delta(t-2)}_{t=2}]$

$x(t) \xleftrightarrow{LT} X(s)$

$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$

$\delta(t) \xleftrightarrow{LT} 1$

$= e^{s}.1 + e^{-s}.1 + e^{-2s}.1$

$= e^{s} + e^{-s} + e^{-2s}$

×

→ ULT is in the range of time $t=0$ to $t=\infty$

$- \; ULT[x(t)] = e^{-s} + e^{-2s}$

## Standard Signals with their Laplace transform & ROC

| | $f(t)$ | $F(s)$ | ROC |
|---|---|---|---|
| 1 | $u(t)$ | $1/s$ | $Re\{s\} > 0$ |
| 2 | $t\,u(t)$ | $1/s^2$ | $Re\{s\} > 0$ |
| 3 | $t^n\,u(t)$ | $n!/s^{n+1}$ | $Re\{s\} > 0$ |
| 4 | $e^{at}\,u(t)$ | $\frac{1}{s-a}$ | $Re\{s\} > a$ |
| 5 | $e^{-at}\,u(t)$ | $\frac{1}{s+a}$ | $Re\{s\} > -a$ |
| 6 | $e^{-at}\,u(-t)$ | $\frac{-1}{s+a}$ | $Re\{s\} < -a$ |
| 7 | $e^{at}\,u(-t)$ | $\frac{-1}{s-a}$ | $Re\{s\} < a$ |
| 8 | $t e^{-at}\,u(t)$ | $\frac{1}{(s+a)^2}$ | $Re\{s\} > -a$ |
| 9 | $t^n e^{-at}\,u(t)$ | $\frac{n!}{(s+a)^{n+1}}$ | $Re\{s\} > -a$ |
| 10 | $t\,e^{at}\,u(t)$ | $\frac{1}{(s-a)^2}$ | $Re\{s\} > a$ |
| 11 | $t^n e^{at}\,u(t)$ | $\frac{n!}{(s-a)^{n+1}}$ | $Re\{s\} > a$ |
| 12 | $t^n e^{-at}\,u(-t)$ | $-\frac{n!}{(s+a)^{n+1}}$ | $Re\{s\} < -a$ |
| 13 | $t^n e^{+at}\,u(-t)$ | $-\frac{n!}{(s-a)^{n+1}}$ | $Re\{s\} < a$ |
| 14 | $e^{-at}\cos bt\;u(t)$ | $\frac{s+a}{(s+a)^2+b^2}$ | $Re\{s\} > -a$ |
| 15 | $e^{-at}\sin bt\;u(-t)$ | $\frac{b}{(s+a)^2+b^2}$ | $Re\{s\} < -a$ |
| | $\cos bt$ | $\frac{s}{s^2+b^2}$ | |
| | $\sin bt$ | $\frac{b}{s^2+b^2}$ | |

Scanned by CamScanner

Example of Laplace tranform using time shifting property

Find Laplace tranform of the signal $e^{-2t}u(t-1)$

- $x(t) = e^{-2t}u(t-1)$

  - $e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2}$

  - $x(t) \xleftrightarrow{LT} X(s)$

    $x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$

- $x(t) = e^{-2t}u(t-1)$

$$= e^{-2(t-1+1)}u(t-1)$$

$$= e^{-2(t-1)}e^{-2}u(t-1)$$

$$= e^{-2}\left[e^{-2(t-1)}u(t-1)\right]$$

$$X(s) = LT[x(t)]$$

$$= LT\left[e^{-2}e^{-2(t-1)}u(t-1)\right]$$

$$= e^{-2}\left[LT\left(e^{-2(t-1)}u(t-1)\right)\right]$$

$$= e^{-2}e^{-s} \times \frac{1}{s+2}$$

$$= \frac{e^{-(s+2)}}{s+2}$$

## Example of Inverse Laplace transform using time shifting Property

Find Inverse Laplace transform of $Z(s) = \frac{e^{-2s}}{(s+1)(s+2)}$

- $Z(s) = e^{-2s} X(s)$

- $X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$\Rightarrow 1 = A(s+2) + B(s+1)$

- If $s = -1$

$\Rightarrow 1 = A(-1+2)$

$\Rightarrow A = 1$

- If $s = -2$

$\Rightarrow 1 = B(-2+1)$

$\Rightarrow B = -1$

- $X(s) = \frac{1}{s+1} - \frac{1}{s+2}$

- $Z(s) = \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+2}$

- $x(t) \xleftrightarrow{LT} X(s)$

$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$

$\frac{1}{s+a} \xleftrightarrow{ILT} e^{-at}$

$z(t) = ILT[Z(s)]$

$= e^{-(t-2)} u(t-2) - e^{-2(t-2)} u(t-2)$

Laplace transform using freq. shifting property

- Find Laplace transform of $\underbrace{\cos \omega t \; u(t)}_{\frac{s}{s^2+\omega^2}}$,

$- \; x(t) = \cos \omega t \; u(t)$

$$= \left[ \frac{e^{j\omega t} + e^{-j\omega t}}{2} \right] u(t)$$

$$= \frac{1}{2} e^{j\omega t} u(t) + \frac{1}{2} e^{-j\omega t} u(t)$$

$- \; X(s) = LT[x(t)]$

$$= LT\left[ \frac{1}{2} e^{j\omega t} u(t) + \frac{1}{2} e^{-j\omega t} u(t) \right]$$

$$= \frac{1}{2} LT[e^{j\omega t} u(t)] + \frac{1}{2} LT[e^{-j\omega t} u(t)]$$

$x(t) \xleftrightarrow{LT} X(s)$ | $u(t) \xleftrightarrow{LT} 1/s$

$e^{s_0 t} x(t) \xleftrightarrow{LT} X(s-s_0)$

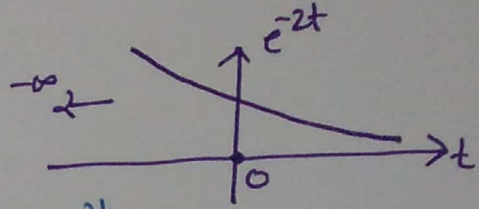
$$= \frac{1}{2}\left[ \frac{1}{s-j\omega} \right] + \frac{1}{2}\left[ \frac{1}{s+j\omega} \right]$$

$$= \frac{1}{2}\left[ \frac{s+\cancel{j\omega}+s-\cancel{j\omega}}{(s-j\omega)(s+j\omega)} \right]$$

$$= \frac{1}{\cancel{2}}\left[ \frac{\cancel{2}s}{s^2-j^2\omega^2} \right] = \boxed{\frac{s}{s^2+\omega^2}}$$

Example of Inverse Laplace transform.

Determine the signal $x(t)$ for which the Laplace transform $X(s)$ is given by




$$X(s) = \frac{s+1}{s^2+3s+2}$$

(a) $e^{2t}u(t)$ (b) $e^{-2t}u(t)$ ✓ (c) $e^{2t}$ (d) $e^{-2t}$

– $X(s) = \dfrac{s+1}{s^2+3s+2} = \dfrac{\cancel{s+1}}{(\cancel{s+1})(s+2)} = \dfrac{1}{s+2}$

– $x(t) = \text{ILT}[X(s)] = \text{ILT}\left[\dfrac{1}{s+2}\right] = e^{-2t}u(t)$

# Examples of Inverse Laplace Transform

- Find ILT of $X(s) = \dfrac{s+2}{s^2+2s+10}$

- $$X(s) = \frac{s+2}{s^2+2s+10}$$

$$= \frac{s+2}{(s^2+2s+1)+9}$$

$$= \frac{s+2}{(s+1)^2+3^2}$$

$$= \frac{s+1+1}{(s+1)^2+3^2}$$

$$= \frac{s+1}{(s+1)^2+3^2} + \frac{1}{3}\,\frac{3}{(s+1)^2+3^2}$$

$$e^{-at}\cos bt \xleftrightarrow{LT} \frac{s+a}{(s+a)^2+b^2}$$

$$e^{-at}\sin bt \xleftrightarrow{LT} \frac{b}{(s+a)^2+b^2}$$

$$x(t) = ILT\,[x(s)]$$

$$= e^{-t}\cos 3t\; u(t) + \frac{1}{3}e^{-t}\sin 3t\; u(t)$$

- Find ILT of $X(s) = \dfrac{2s+4}{s^2+4s+3}$

- $$X(s) = \frac{2s+4}{s^2+4s+3}$$

$$= \frac{2s+4}{(s+3)(s+1)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$\Rightarrow 2s+4 = A(s+3) + B(s+1)$$

- $s = -1$

$$\Rightarrow -2+4 = A(-1+3)$$

$$\Rightarrow A = 1$$

- $s = -3$

$$\Rightarrow -6+4 = B(-3+1)$$

$$\Rightarrow B = 1$$

- $$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

- $$x(t) = ILT\,[x(s)]$$

$$= ILT\left[\frac{1}{s+1} + \frac{1}{s+3}\right]$$

$$= ILT\left[\frac{1}{s+1}\right] + ILT\left[\frac{1}{s+3}\right]$$

$$= e^{-t}u(t) + e^{-3t}u(t)$$

Impulse response from transfer function by Inverse Laplace transform

Transfer function of stable system is $H(s) = \frac{s}{(s+1)^2}$.

The impulse response is ______

- $h(t) \xleftrightarrow{LT} H(s)$

Impulse response     Transfer function.

- $h(t) = ILT[H(s)]$

- $$H(s) = \frac{s}{(s+1)^2}$$
$$= \frac{s+1-1}{(s+1)^2}$$
$$= \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\frac{1}{s+a} \xleftrightarrow{ILT} e^{-at}u(t)$$
$$\frac{1}{(s+a)^2} \xleftrightarrow{ILT} t e^{-at}u(t).$$

- $$h(t) = ILT[H(s)]$$
$$= ILT\left[\frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$
$$= ILT\left[\frac{1}{s+1}\right] - ILT\left[\frac{1}{(s+1)^2}\right]$$
$$= \boxed{e^{-t}u(t) - t e^{-t}u(t)}$$

Impulse response from transfer function by Inverse Laplace transform

Transfer function of stable system is $H(s) = \frac{s^2+2s-4}{s^2-4}$

Find Impulse response $h(t)$.

- $h(t) \xleftrightarrow{LT} h(s)$

Impulse response — Transfer function

$e^{-at}\cos bt \xleftrightarrow{LT} \frac{s+a}{(s+a)^2+b^2}$

$e^{-at}\sin bt \xleftrightarrow{LT} \frac{b}{(s+a)^2+b^2}$

- $h(t) = ILT[H(s)]$

- $$H(s) = \frac{s^2+2s-4}{s^2-4}$$

$$= \frac{s^2-4+2s}{s^2-4}$$

$$= 1 + \frac{2s}{s^2-4}$$

$$= 1 + \frac{2s}{(s-2)(s+2)}$$

$\Rightarrow \frac{2s}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}$

$\Rightarrow 2s = A(s+2) + B(s-2)$

If $s=2$

$\Rightarrow 4 = A(2+2)$

$\Rightarrow A = 1$

$\Rightarrow -4 = B(-2-2)$

$\Rightarrow B = 1$

$$= 1 + \frac{1}{s-2} + \frac{1}{s+2}$$

$\delta(t) \xleftrightarrow{LT} 1$

$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$

- $$H(t) = ILT[H(s)]$$

$$= ILT\left[1 + \frac{1}{s-2} + \frac{1}{s+2}\right]$$

$$= ILT[1] + ILT\left[\frac{1}{s-2}\right] + ILT\left[\frac{1}{s+2}\right]$$

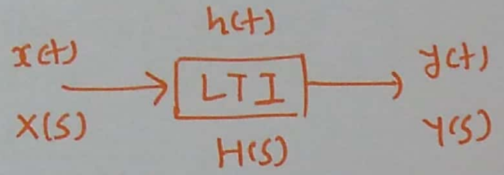
$$= \boxed{\delta(t) + e^{+2t}u(t) + e^{-2t}u(t)}$$

Impulse response from differential eq$^n$ using Inverse Laplace transform

The relation between the Input $x(t)$ and the Output $y(t)$ of a causal system is given as

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 20\,y(t) = -2x(t) + 5\frac{dx(t)}{dt}$$

Find Impulse response of system.




- $H(s) = \dfrac{Y(s)}{X(s)}$

- $h(t) = ILT\,[H(s)]$

$\Rightarrow s^2 Y(s) + s\,Y(s) - 20\,Y(s) = -2X(s) + 5s\,X(s)$

$\Rightarrow Y(s)\,[s^2 + s - 20] = X(s)\,[5s - 2]$

$\Rightarrow H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{5s-2}{s^2+s-20}$

$= \dfrac{5s-2}{s^2+5s-4s-20}$

$= \dfrac{5s-2}{s(s+5)-4(s+5)}$

$= \dfrac{5s-2}{(s+5)(s-4)}$

$\Rightarrow \dfrac{5s-2}{(s+5)(s-4)} = \dfrac{A}{s+5} + \dfrac{B}{s-4}$

$\Rightarrow 5s - 2 = A(s-4) + B(s+5)$

If $s = -5$

$\Rightarrow -25 - 2 = A(-5-4)$

$\Rightarrow A = 3$

If $s = 4$

$\Rightarrow 20 - 2 = B(4+5)$

$\Rightarrow B = 2$

$\rightarrow H(s) = \dfrac{3}{s+5} + \dfrac{2}{s-4}$

$H(t) = ILT\,[H(s)]$

$= ILT\left[\dfrac{3}{s+5}\right] + ILT\left[\dfrac{2}{s-4}\right]$

$e^{-at}u(t) \xleftrightarrow{LT} \dfrac{1}{s+a}$

$= \boxed{3e^{-5t}u(t) + 2e^{4t}u(t)}$

# Examples of Final Value theorm

From Final Value theorm, Find the value of $x(t)$ with Laplace transform $X(s) = \dfrac{s+2}{s(s^2+3s+1)}$

(a) 1 (b) 2 ✓ (c) 0 (d) ∞

$$X(\infty) = \lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t)$$

$$= \lim_{s\to 0} \cancel{s}\left[\frac{s+2}{\cancel{s}(s^2+3s+1)}\right]$$

$$= \frac{0+2}{0+0+1} = 2$$

$$X(0^+) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \cancel{s}\left[\frac{s+2}{\cancel{s}(s^2+3s+1)}\right]$$

$$= \lim_{s\to\infty} \frac{\cancel{s}(1+2/s)}{s^{\cancel{2}}(1+3/s+1/s^2)}$$

$$= 0$$

The Laplace transform of $I(t)$ is given by $I(s) = \dfrac{2}{s(1+s)}$ at $t\to\infty$, the value of $I(t)$ is ____ [GATE]

(a) 0 (b) ∞ (c) 2 ✓ (d) None

$$I(\infty) = \lim_{s\to 0} sI(s) = \lim_{t\to\infty} I(t)$$

$$= \lim_{s\to 0} \cancel{s}\left[\frac{2}{\cancel{s}(1+s)}\right]$$

$$= \frac{2}{1+0} = 2$$

$$I(0^+) = \lim_{s\to\infty} sI(s) = \lim_{t\to 0} I(t)$$

$$= \lim_{s\to\infty} \cancel{s}\left[\frac{2}{\cancel{s}(1+s)}\right]$$

$$= \lim_{s\to\infty} \frac{2}{1+s} = \frac{2}{\infty} = 0$$

If $F(s) = \dfrac{\omega}{s^2+\omega^2}$. the value of $\lim_{t\to\infty} f(t)$ is

(a) can not determined (b) sine terms
(c) ∞ (d) zero ✓ [GATE]

$$F(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s\left[\frac{\omega}{s^2+\omega^2}\right]$$

$$= 0\left[\frac{\omega}{0+\omega^2}\right]$$

$$= 0$$

$$- F(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s\left[\frac{\omega}{s^2+\omega^2}\right]$$

$$= \lim_{s\to\infty} \frac{s}{s^2}\left[\frac{\omega}{1+\omega^2/s^2}\right]$$

$$= \lim_{s\to\infty} \frac{1}{s}\left[\frac{\omega}{1+\omega^2/s^2}\right] = 0\left[\frac{\omega}{1+0}\right]$$

$$= 0$$

## Unilateral Laplace transform of Shifted Impulse.

The Unilateral Laplace transform of $\delta(t+k)$ is

(a) 0 (b) 1 (c) 2 (d) not defined

- $LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$
- $ULT[x(t)] = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$
- In ULT, range of time varies from 0 to $\infty$
- $x(t) = \delta(t+k)$

  $\hookrightarrow$ time $t = -k$
- $ULT[x(t)] = 0$ as $t = -k$ is not there in range of 0 to $\infty$
- $LT[x(t)] = LT[\delta(t+k)] = e^{ks} \cdot 1 = e^{ks}$

  $\delta(t) \xleftrightarrow{LT} 1$

  $\delta(t-t_0) \longleftrightarrow e^{-t_0 s} \cdot 1$

# Causality & Stability using Laplace transform

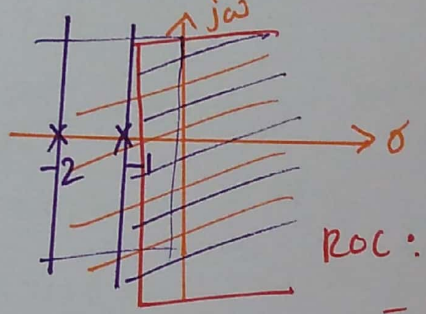
- The system with transfer function is said to be both causal & stable when all the poles lies in the left half of s plane.

- If the ROC contains $j\omega$ axis then system is stable.

- $H(s) = \dfrac{1}{(s+1)(s+2)}$

- Position poles $-1$ & $-2$

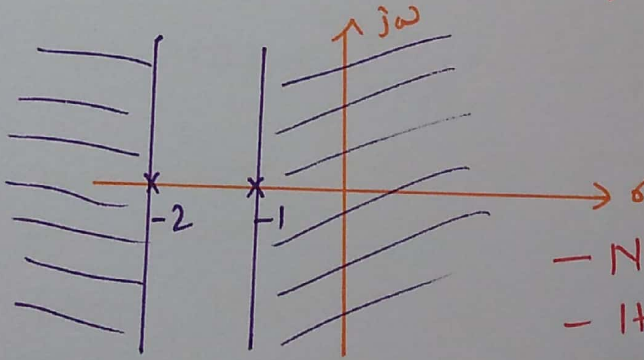
- System is stable & Causal.




① ROC : $Re\{s\} > -1$ &
$Re\{s\} > -2$

ROC : $Re\{s\} > -1$

- As combine ROC includes $j\omega$ axis for above case system is stable.



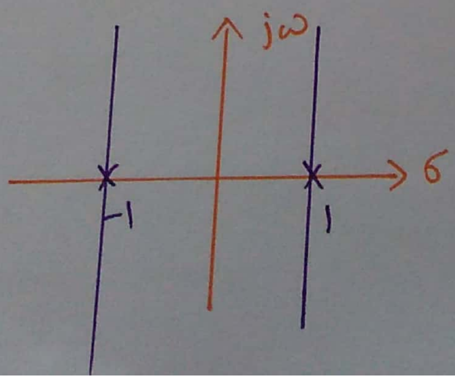

② ROC : $Re\{s\} > -1$
$Re\{s\} < -2$

- No combine ROC for poles.
- It does not includes $j\omega$ axis.
- So system is unstable.
- but position of poles in left half plane so system is causal.

$H(s) = \dfrac{1}{(s+1)(s-1)}$

③

-> Poles at $s = -1, 1$



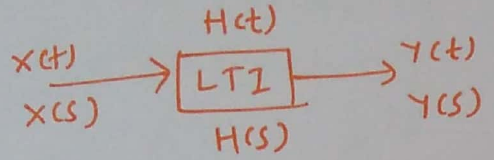

- One pole is on RHS plane
- System is unstable and Non causal.

Causality & stability of system from differential eq.$^n$

Find the transfer function & determine the causality and stability of a system, described by eq.$^n$

$$\frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} - 2x(t), \quad \cancel{ROC: Re\{s\} > 2}$$

ROC : $Re\{s\} > 1$




- $H(s) = \frac{Y(s)}{X(s)}$

$$\Rightarrow \frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} - 2x(t)$$

- take Laplace x'form

$$\Rightarrow sY(s) + 2Y(s) = s^2X(s) + sX(s) - 2X(s)$$

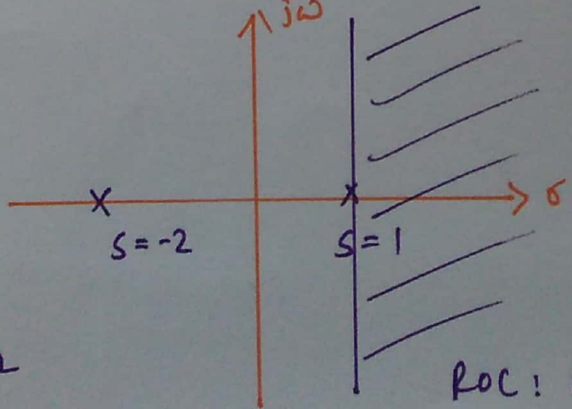
$$\Rightarrow Y(s)(s+2) = X(s)(s^2+s-2)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+s-2}{s+2}$$

- Here no of zeros are greater than no of poles, so this transfer function is improper transfer function.
- take $H_{in}(s) = \frac{1}{H(s)}$
- check stability and causality for $H_{in}(s)$

$$H_{in}(s) = \frac{1}{H(s)} = \frac{s+2}{s^2+s-2} = \frac{s+2}{(s+2)(s-1)}$$

- Position of poles for $H_{in}(s)$ $s = -2$, $s = 1$




- Here one pole is there is RHP at $s = 1$
- So system is not causal system

- Here ROC does not includes jω axis.
- System is not stable.

ROC : $Re\{s\} > 1$

# Laplace tranform of Ramp and Unit step combination

Find the Laplace transform of $r(t-1) - r(t-3) - u(t-3)$

$$x(t) = r(t-1) - r(t-3) - u(t-3)$$

$$r(t) \xleftrightarrow{LT} 1/s^2 \qquad x(t) \xleftrightarrow{LT} X(s)$$

$$u(t) \xleftrightarrow{LT} 1/s \qquad x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$$

$$X(s) = LT\left[ r(t-1) - r(t-3) - u(t-3) \right]$$

$$= LT[r(t-1)] - LT[r(t-3)] - LT[u(t-3)]$$

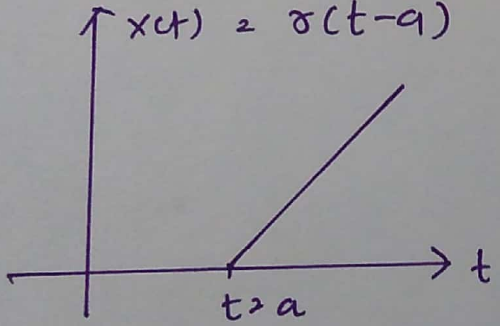
$$= \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$$

$$= \frac{1}{s^2}\left[ e^{-s} - e^{-3s} - s\,e^{-3s} \right]$$

The Laplace transform of unit ramp function starting at $t = a$ is

(a) $\frac{1}{(s+a)^2}$ (b) $\frac{e^{as}}{(s+a)^2}$ (c) ✓ $\frac{e^{-as}}{s^2}$ (d) $\frac{a}{s^2}$

$x(t) = r(t-a)$




- $r(t) \xleftrightarrow{LT} 1/s^2$
- $x(t) \xleftrightarrow{LT} X(s)$

  $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$
- $LT[r(t-a)] = \frac{e^{-as}}{s^2}$