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Math 254: Honours Analysis I

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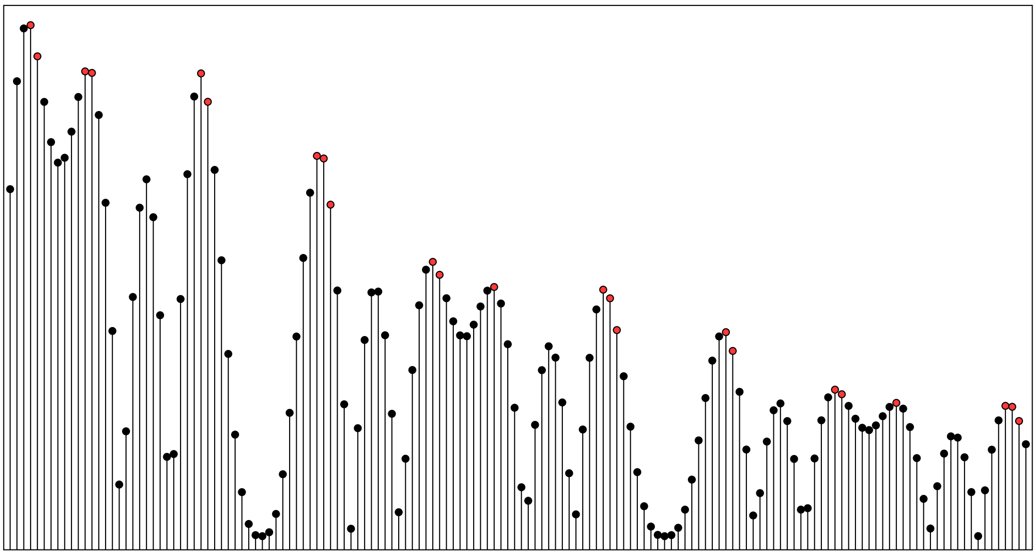

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0 Historical note

In the 5th century B.C., the Greeks discovered that there are numbers beyond the rationals. Consider a right triangle with legs of unit length. By the Pythagorean Theorem, the hypotenuse is $\sqrt{2}$, which is provably not rational.

Proof. Assume that $\sqrt{2}$ is rational. Then there exists $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ such that $\frac{p}{q} = \sqrt{2}$. We may safely assume that $\gcd(p, q) = 1$. From here, we find $p^2 = 2q^2$, so p^2 is even. This means p is also even, so $p = 2r$ for some $r \in \mathbb{Z}$. Now with some rearrangement, we find $q^2 = 2r^2$, implying q^2 is even. Consequently, q is even and is divisible by 2. Therefore p and q have a common divisor greater than 1, and we have arrived at a contradiction. ■

After this discovery of "holes" embedded in the rationals, the stronger system of real numbers was eventually constructed in the 19th century through the efforts of various mathematicians such as Georg Cantor and Richard Dedekind. The rigorous extension of $\mathbb{Q}$ to $\mathbb{R}$ is covered in further analysis courses at McGill. For now, we take $\mathbb{R}$ for granted as the number system underlying this first course in analysis.

1 Preliminaries

1.1 Set theory

Definition 1.1 (Set)**.** A set is a collection of objects.

Definition 1.2 (Set union)**.**

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Union is defined on an infinite collection of sets as

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$$

Definition 1.3 (Set intersection)**.**

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Intersection is defined on an infinite collection of sets as

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}$$

Definition 1.4 (Set complement)**.** Let $A \subseteq X$, where X is some ambient set. Then

$$A^c = \{x : x \in X \text{ and } x \notin A\}$$

Remark. Two useful properties:

(1) $A = B \iff A^c = B^c$

(2) $(A^c)^c = A$

Proposition 1.1. De Morgan's laws for sets A and B can be expanded to the infinite case.

(a) $\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c$

(b) $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$

Furthermore, we have (a) *if and only* if we have (b).

Proof. To prove (a), we have

$$\begin{aligned}
\left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}^c \\
&= \{x : x \notin A_n \text{ for some } n \in \mathbb{N}\} \\
&= \{x : x \in A_n^c \text{ for some } n \in \mathbb{N}\} \\
&= \left(\bigcup_{n=1}^{\infty} A_n^c\right)
\end{aligned}$$

(b) follows in a similar way:

$$\begin{aligned}
\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}^c \\
&= \{x : x \notin A_n \text{ for all } n \in \mathbb{N}\} \\
&= \{x : x \in A_n^c \text{ for all } n \in \mathbb{N}\} \\
&= \left(\bigcap_{n=1}^{\infty} A_n^c\right)
\end{aligned}$$

Now we show $(a) \implies (b)$ (proof of the converse follows a similar structure):

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c \implies \left(\bigcap_{n=1}^{\infty} A_n^c\right)^c = \bigcup_{n=1}^{\infty} (A_n^c)^c = \bigcup_{n=1}^{\infty} A_n$$
$$\implies \left(\left(\bigcap_{n=1}^{\infty} A_n^c\right)^c\right)^c = \left(\bigcup_{n=1}^{\infty} A_n\right)^c$$
$$\implies \bigcap_{n=1}^{\infty} A_n^c = \left(\bigcup_{n=1}^{\infty} A_n\right)^c$$

■

Remark. It would be incorrect to prove Proposition 1.1 with induction, since induction can only prove a statement is true for any *finite* $n \in \mathbb{N}$.

1.2 Functions

Definition 1.5 (Function)**.** Given sets A and B, a function $f : A \to B$ assigns each element $x \in A$ a unique element $f(x) \in B$.

A is the **domain** of f and $f(A) = \{f(x) : x \in A\}$ is the **range** of f. We have $f(A) \subseteq B$. Also, if $C \subseteq A$ then $f(C) = \{f(x) : x \in C\}$ is the **image** of C by the function f.

Example 1.1. Some examples of functions (the last one is called the Dirichlet function).

(a) $f(x) = x^2$, where $A = \mathbb{R}, B = \mathbb{R}, f(A) = [0, \infty)$

(b) $f(x) = e^x$, where $A = \mathbb{R}, B = \mathbb{R}, f(A) = (0, \infty)$

(c) $f(x) = \ln(x)$, where $A = \mathbb{R}, B = \mathbb{R}, f(A) = (-\infty, \infty)$

(d) $f(x) = \sin(x)$, where $A = \mathbb{R}, B = \mathbb{R}, f(A) = [-1, -1]$

(e) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, where $A = \mathbb{R}, B = \mathbb{R}, f(A) = \{0, 1\}$

Proposition 1.2. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

Proof. It suffices to show that $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$ and $f(C_1 \cup C_2) \supseteq f(C_1) \cup f(C_2)$.

First let $y \in f(C_1 \cup C_2)$. Then there exists $x \in C_1 \cup C_2$ such that $y = f(x)$. If $x \in C_1$, then $y \in f(C_1)$, and if $x \in C_2$, then $y \in f(C_2)$. At least one of these cases must be true, so

$y \in f(C_1) \cup f(C_2)$.

Second, let $y \in f(C_1) \cup f(C_2)$. Then $y \in f(C_1)$ or $y \in f(C_2)$, so there exists $x \in C_1$ or $x \in C_2$ such that $y = f(x)$. Hence $y = f(x)$ for some $x \in C_1 \cup C_2$ and we have $y \in f(C_1 \cup C_2)$. ■

Proposition 1.3. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

Proof.

$$\begin{aligned} y \in f(C_1 \cap C_2) &\implies \exists x \in C_1 \cap C_2 \text{ such that } y = f(x) \\ &\implies \exists x \in C_1 \text{ and } \exists x \in C_2 \text{ such that } y = f(x) \\ &\implies y \in f(C_1) \text{ and } y \in f(C_2) \\ &\implies y \in f(C_1) \cap f(C_2) \end{aligned}$$

■

Remark. tly, the reverse implication is false. Certainly, $y \in f(C_1) \cap f(C_2)$ implies there exists $x_1 \in C_1$ and $x_2 \in C_2$ such that $y = f(x_1) = f(x_2)$, **but this does not mean $x_1 = x_2$.**

Example 1.2. Let $f(x) = \sin(x)$, $A = \mathbb{R}$, $B = \mathbb{R}$, $C_1 = [0, 2\pi]$, and $C_2 = [2\pi, 4\pi]$. We have

$$f(C_1) \cap f(C_2) = [-1, 1] \cap [-1, 1] = [-1, 1]$$

However,

$$f(C_1 \cap C_2) = f(\{2\pi\}) = \{0\} \implies f(C_1 \cap C_2) \not\supseteq f(C_1) \cap f(C_2)$$

Definition 1.6 (Inverse image of a set)**.** Let $f : A \to B$ and let $D \subseteq B$. The inverse image of D by f is denoted $f^{-1}(D)$ and is defined by

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

Proposition 1.4. Properties of the inverse image.

(a) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

(b) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$

Remark. Equality holds in both case, which is not true for direct images. In this sense, the inverse image is more well behaved.

Proposition 1.5. Generalization of Proposition 1.4. Let A_n $(n = 1, 2, \dots)$ be an infinite sequence of setes. Then

(a) $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$

(b) $f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$

Proof. We first prove (a):

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

Now we prove (b):

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

■

Proposition 1.6. Let $f : A \to B$ and $D \subseteq B$. Then $f^{-1}(D^c) = [f^{-1}(D)]^c$.

Proof.

$$\begin{aligned} x \in f^{-1}(D^c) &\iff f(x) \in D^c \\ &\iff f(x) \notin D \\ &\iff x \notin f^{-1}(D) \\ &\iff x \in [f^{-1}(D)]^c \end{aligned}$$

■

Remark. The professor notes that this result is important in measure theory.

Example 1.3. In the context of Proposition 1.6, there is generally **no relation** between $f(C^c)$ and $f(C)^c$.

Let $f(x) = \sin(x)$ and take $A = [0, 6\pi]$, $B = [-1, 2]$, and $A \supseteq C = [0, 2\pi]$. Then $f(C) = [-1, 1]$ and $f(C^c) = [-1, 1]$ (taking A as the universe). But $f(C)^c = (1, 2)$ (taking B as the universe). So $f(C^c)$ and $f(C)^c$ are disjoint.

2 The Real Numbers

Axiom of Completeness (AC)
Any non-empty subset of $\mathbb{R}$ that is bounded from above has a least upper bound.

Remark. AC is the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$.

Definition 2.1 (Upper and lower bound)**.** Let $A \subseteq \mathbb{R}$. Then

(1) $b \in \mathbb{R}$ is an upper bound for A if $x \leq b$ for all $x \in A$

(2) $l \in \mathbb{R}$ is a lower bound for A if $x \geq l$ for all $x \in A$

Definition 2.2 (Least upper bound)**.** Let $A \subseteq \mathbb{R}$. Then $s \in \mathbb{R}$ is a least upper bound for A if

(1) s is an upper bound for A

(2) $s \leq b$ for any other upper bound $b \in \mathbb{R}$

If a least upper bound **exists**, then it is **unique**. This is because if s and s^* are both least upper bounds, then $s \leq s^*$ and $s^* \leq s$, so $s = s^*$.

s is called the **supremum** of A, denoted $s = \sup(A)$.

Definition 2.3 (Greatest lower bound)**.** Let $A \subseteq \mathbb{R}$. Then $i \in \mathbb{R}$ is a greatest lower bound for A if

(1) i is a lower bound for A

(2) $i \geq l$ for any other lower bound $l \in \mathbb{R}$

If a greatest lower bound **exists**, then it is **unique**.

i is called the **infimum** of A, denoted $i = \inf(A)$.

Remark. Supremum/infimum should not be confused with maximum/minimum. We have $m \in A \subseteq \mathbb{R}$ is the **maximum** of A if $x \leq m$ for all $x \in A$. But realize that this requires $m \in A$, which is not required of $\sup(A)$. What we can say is that if m exists, then $m = \sup(A)$, as the membership of m in A implies $m \leq b$ for any other upper bound b.

Example 2.1. Let $A = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Clearly, A does not have a maximum, but $\sup(A)$ exists by AC.

Proposition 2.1. Let $A \subseteq \mathbb{R}$ and let s be an upper bound for A. Then $s = \sup(A)$ *if and only if* for any $\varepsilon > 0$, there exists $x \in A$ such that

$$s - \varepsilon < x$$

Proof. Assume $s = \sup(A)$. Then $s - \varepsilon$ cannot be an upper bound for A. To see this, suppose it was. Then we have $s \leq s - \varepsilon$ (since s is the supremum), which implies $\varepsilon \leq 0$, which is a contradiction. Thus $s - \varepsilon$ is not an upper bound for A, so there exists $x \in A$ such that $s - \varepsilon < x$.

For the other direction, assume that for all $\varepsilon > 0$ there exists $x \in A$ such that

$$s - \varepsilon < x$$

We will show $s = \sup(A)$ by contradiction. Assume s is not the supremum. Since A is a nonempty subset of $\mathbb{R}$ that is bounded above, then by AC there exists $b = \sup(A) < s$. Let $\varepsilon = s - b > 0$. Then for some $x \in A$ we have

$$b = s - (s - b) = s - \varepsilon < x$$

We have the contradiction $b < x$, so $s = \sup(A)$.

■

Remark. The idea behind the second part of this proof is that if we assume a smaller supremum b exists, then under the hypothesis we can always wedge an element $x \in A$ between b and s. It is a game that cannot be won – given any $b < s$, we can use the hypothesis to derive an absurdity.

Proposition 2.2. Let $A \subseteq \mathbb{R}$ and let i be an upper bound for A. Then $i = \inf(A)$ *if and only if* for any $\varepsilon > 0$, there exists $x \in A$ such that

$$i + \varepsilon > x$$

2.1 Relation between infimum and supremum

Proposition 2.3. Let $A \subseteq \mathbb{R}$ be nonempty and define $-A = \{-x : x \in A\}$.

(1) If A is bounded from above, then $-\sup(A) = \inf(-A)$

(2) If A is bounded from below, then $-\inf(A) = \sup(-A)$

Proof. We first prove (1). Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Then $\sup(A)$ exists by AC. So we have

$$\begin{aligned} & \sup(A) \geq x && \forall x \in A \\ \iff & -\sup(A) \leq -x && \forall x \in A \\ \iff & -\sup(A) \leq -x && \forall -x \in -A \end{aligned}$$

So $-\sup(A)$ is a lower bound for $-A$. Now suppose a greater lower bound l exists. Then for all $-x \in -A$

$$-\sup(A) < l \leq -x \iff x \leq -l < \sup(A)$$

But this is a contradiction, since $\sup(A)$ is the least upper bound of A. So $-\sup(A)$ is the greatest lower bound of $-A$.

Now we can prove (2) by using (1). Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded from below. Then $-A$ is bounded from above. By (1), we have

$$-\inf(A) = -\inf(-(-A)) = \sup(-A)$$

■

Axiom of Completeness (reformulated)
Any non-empty subset of $\mathbb{R}$ that is bounded from below has a greatest lower bound.

Proposition 2.3 tells us that this is an equivalent formulation of AC. Given the supremum formulation, then for every $A \subseteq \mathbb{R}$ that is nonempty and bounded below, we find that the greatest lower bound exists and is defined by $-\sup(-A)$. So the infimum formulation holds. On the other hand, if we accept the infimum formulation, then for every $A \subseteq \mathbb{R}$ that is nonempty and bounded above, there exists the least upper bound, defined by $-\inf(-A)$. So the supremum formulation holds.

2.2 Consequences of the Axiom of Completeness

2.2.1 Nested Interval Property

Theorem 2.1 (Nested Interval Property)**.** Let $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ ($n \in \mathbb{N}$) be an infinite sequence of bounded intervals such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. We first make two observations.

(1) For $n \in \mathbb{N}$, we have $I_n = [a_n, b_n]$ and $I_{n+1} = [a_{n+1}, b_{n+1}]$ where $I_n \supseteq I_{n+1}$. This means that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall n \geq 1$$

Clearly, the sequence a_n of left endpoints of I_n is increasing, while the sequence b_n of right endpoints is decreasing.

(2) $a_n \leq b_k$ for all $n, k \in \mathbb{N}$.

Case 1: $n \leq k$. Then $a_n \leq a_k$ since the sequence of left endpoints increases, and $a_k \leq b_k$ by definition. Therefore $a_n \leq a_k \leq b_k$.

Case 2: $n > k$. Then $a_n \leq b_n \leq b_k$ by again using the definition of I_n, and the fact that the sequence of right endpoints decreases.

Now let $A = \{a_n : n \in \mathbb{N}\}$. Any b_k, where $k \in \mathbb{N}$, is an upper bound for A by (2). Hence $\sup(A)$ exists by the Axiom of Completeness. As such, for all $k, n \in \mathbb{N}$, we have $\sup(A) \geq a_n$ and we also have $\sup(A) \leq b_k$, since $\sup(A)$ is the *least* upper bound. This gives

$$a_n \leq \sup(A) \leq b_k \quad \forall n, k \in \mathbb{N}$$

Now, let us set $n = k$ to produce

$$\begin{aligned} & a_n \leq \sup(A) \leq b_n \quad \forall n \in \mathbb{N} \\ \implies & \sup(A) \in \bigcap_{n=1}^{\infty} I_n \\ \implies & \bigcap_{n=1}^{\infty} I_n \neq \emptyset \end{aligned}$$

∎

Remark (1). The idea is to construct an increasing sequence of left endpoints, and a decreasing sequence of right endpoints. These two sequences tend towards each other, but there is always something in between them.

Remark (2). Crucially, we need I_n to be closed for this proof to work. If the interval endpoints are open, then everything proceeds until the statement $a_n \leq \sup(A) \leq b_n$. Since $\sup(A)$ may *equal* a_n or b_n, we cannot claim that $\sup(A)$ is in every I_n. In fact, we can show that if $I_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} I_n = \varnothing$, using the Archimedean property.

Remark (3). The Nested Interval Property does not hold in $\mathbb{Q}$. Also, the Nested Interval Property, when combined with the Archimedean property, is an equivalent formulation of the Axiom of Completenes (how so?) – both capture the idea that no holes exist in the real line.

2.2.2 Archimedean Property

Proposition 2.4 (Archimedean Property)**.** This property says that for every real number, there is a larger natural number. Part (b) follows from (a).

(a) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$

(b) $\forall y \in \mathbb{R}, y > 0, \exists n \in \mathbb{N}$ such that $\frac{1}{n} < y$

Proof. To prove (a), assume the negation – namely, there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $n \leq x$.

$$\begin{aligned}
&\implies \sup(\mathbb{N}) \text{ exists, as } \mathbb{N} \text{ is bounded and non-empty} \\
&\implies \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ such that } \sup(\mathbb{N}) - \varepsilon < n \\
&\implies \sup(\mathbb{N}) - 1 < n \text{ for some } n \in \mathbb{N}, \text{by setting } \varepsilon = 1 \\
&\implies \sup(\mathbb{N}) < n + 1 \in \mathbb{N}, \text{since } \mathbb{N} \text{ is closed under addition}
\end{aligned}$$

So $\sup(\mathbb{N})$ is not an upper bound for $\mathbb{N}$, and a contradiction arises.

To prove (b), let $x = \frac{1}{y}$ where $y \in \mathbb{R}$ and $y > 0$. Using (a), we have $n > \frac{1}{y}$ for some $n \in \mathbb{N}$. Therefore $\frac{1}{n} < y$.

■

Remark. Notice the use of the Axiom of Completeness. We would not be able to repeat this proof in $\mathbb{Q}$. In fact, we could extend $\mathbb{Q}$ to a larger ordered field containing elements greater than any natural number.

2.2.3 Density of $\mathbb{Q}$ and $\mathbb{J}$ in $\mathbb{R}$

Well-ordering principle
Any non-empty subset of $S \subseteq \mathbb{N}$ has a least element m, such that $m \leq n$ for all $n \in S$.

Remark. The well-ordering principle is equivalent to the Axiom of Induction.

Lemma 2.1. For any $z \in \mathbb{R}$, there exists $m \in \mathbb{Z}$ such that $m - 1 \leq z < m$.

Proof. First, suppose $z \geq 0$. If $0 \leq z < 1$, then we can just let $m = 1$. Now assume $z \geq 1$ and define $S := \{n : 1 \leq z < n \text{ and } n \in \mathbb{N}\}$. By the Archimedean property, S is non-empty. Then by the well-ordering principle, S contains a least element m. Notice that by assuming $z \geq 1$, we constructed $m - 1$ to be a natural number (since $m \geq 2$). This allows us to conclude $m - 1 \notin S$ for the reason that $m - 1 \leq z$ (and not because $m - 1 \notin \mathbb{N}$). Therefore $m - 1 \leq z < m$. The assumption that $z \geq 1$ also guarantees there actually *exists* a natural number bounding it from below.

Now suppose $z < 0$. If $-1 \leq z < 0$, then we are done, simply let $m = 0$ and we have $m - 1 \leq z < m$. Hence let us assume $z < -1$ so that $-z > 1$. Let $S' := \{n : 1 < -z \leq n \text{ and } n \in \mathbb{N}\}$. Then there is a least element $m' \in S'$ with $m' \geq 2$. We have $m' - 1$ is not an element of S', and since $m' - 1 \in \mathbb{N}$, it must be that $m' - 1 < -z$. So there is an interval $m' - 1 < -z \leq m'$, which implies $-m' + 1 > z \geq -m'$. Now we have m because setting $m := -m' + 1 \in \mathbb{Z}$ gives $m - 1 \leq z < m$. ■

Theorem 2.2. Let $a, b \in \mathbb{R}$ such that $a < b$. Then there exists $x \in \mathbb{Q}$ such that

$$a < x < b$$

Proof. First, since $b - a > 0$ then by the Archimedean property there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Hence $na + 1 < nb$, which gives $a < b - \frac{1}{n}$. Let $m \in \mathbb{Z}$ such that $m - 1 \leq na < m$, which exists by Lemma 2.1. Then $na < m$ implies $a < \frac{m}{n}$. We also have $m \leq na + 1$, and the direction of this inequality suggests that the right side should be converted to something involving b. Following this idea gives

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - \frac{1}{n}) + 1 \\ &= nb \end{aligned}$$

So $a < \frac{m}{n}$ and $\frac{m}{n} < b$. ■

Theorem 2.3. Let $a, b \in \mathbb{R}$ such that $a < b$. Then there exists $y \in \mathbb{J}$ such that

$$a < y < b$$

We define $\mathbb{J} = \{y \in \mathbb{R} : y \notin \mathbb{Q}\}$.

Proof. Fix any $y_0 \in \mathbb{J}$. Then

$$\begin{aligned} a - y_0 < b - y_0 &\implies \exists x \in \mathbb{Q} \;\; a - y_0 < x < b - y_0 \\ &\implies a < x + y_0 < b \end{aligned}$$

Now let $y = x + y_0$ and assume it is rational. Then $y - x$ is rational by closure under addition. However, this is a contradiction since y_0 is irrational. ■

Remark. When we fix $y_0 \in \mathbb{J}$, we are implicitly assuming that $\mathbb{J}$ is not empty – that irrational numbers actually exist! The next theorem makes this assumption safe, by showing that $\sqrt{2}$ is a real number. As we know $\sqrt{2}$ is not rational, it follows that $\mathbb{J}$ is not empty.

2.2.4 $\sqrt{2}$ is real

Theorem 2.4. There exists a unique positive real number α such that $\alpha^2 = 2$.

Proof. First we show that α must be unique, if such a number exists. Assume $\alpha^2 = 2$ and $\beta^2 = 2$ for $\alpha, \beta \in \mathbb{R}^+$.

$$\begin{aligned}
&\implies \alpha^2 = \beta^2 \\
&\implies \alpha^2 - \beta^2 = 0 \\
&\implies (\alpha - \beta)(\alpha + \beta) = 0 \\
&\implies \alpha - \beta = 0 \ \ (\text{since } \alpha, \beta > 0) \\
&\implies \alpha = \beta
\end{aligned}$$

Now we show α exists. Define a set $A = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$. A is non-empty since $1 \in A$. Also, A is bounded above. To see this, let $x \in A$ and take $y \geq 2$.

$$\begin{aligned}
&\implies 0 \leq x^2 < 2 < 4 \leq y^2 \\
&\implies x^2 \leq y^2 \\
&\implies x^2 - y^2 \leq 0 \\
&\implies (x - y)(x + y) \leq 0 \\
&\implies x - y \leq 0 \ \ (\text{since } x, y > 0) \\
&\implies x \leq y
\end{aligned}$$

We therefore know $\sup(A) \in \mathbb{R}$ exists by the Axiom of Completeness. Now set $\alpha = \sup(A)$, and the claim is that $\alpha^2 = 2$. We argue this by showing it is impossible to have $\alpha^2 > 2$ or $\alpha^2 < 2$, leaving $\alpha^2 = 2$ as the only remaining option.

Case 1: Assume $\alpha^2 > 2$. Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
(\alpha - \frac{1}{n})^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\
&\geq \alpha^2 - \frac{2\alpha}{n}
\end{aligned}$$

Let $y = \frac{\alpha^2 - 2}{2\alpha}$. We know $y > 0$. By the Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < y$. We thus have $\alpha^2 - 2 > \frac{2\alpha}{n_0}$. It follows that

$$\begin{aligned}
(\alpha - \frac{1}{n_0})^2 &\geq \alpha^2 - \frac{2\alpha}{n_0} \\
&> \alpha^2 - (\alpha^2 - 2) \\
&= 2 \\
&> x^2 \ \ \forall x \in A
\end{aligned}$$

We know $\alpha - \frac{1}{n_0} > 0$ since $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha} \implies \frac{1}{n_0} < \alpha \implies \alpha - \frac{1}{n_0} > 0$. That allows us to conclude $\alpha - \frac{1}{n_0} > x$ for all $x \in A$. But this means we have found an upper bound smaller than $\alpha = \sup(A)$, which is a contradiction.

Case 2: Assume $\alpha^2 < 2$. Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned}(\alpha + \frac{1}{n})^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &\leq \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

Let $y = \frac{2-\alpha^2}{2\alpha+1}$. Again, we know $y > 0$. By the Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < y$. We thus have $2 - \alpha^2 > \frac{2\alpha+1}{n_0}$. It follows that

$$\begin{aligned}(\alpha + \frac{1}{n_0})^2 &\leq \alpha^2 + \frac{2\alpha + 1}{n_0} \\ &< \alpha^2 + (2 - \alpha^2) \\ &= 2\end{aligned}$$

We know $\alpha + \frac{1}{n_0}$ is positive since α is positive (implied by the presence of positive numbers in A). The conclusion is that $\alpha + \frac{1}{n_0} \in A$, which means there is an element in A greater than $\alpha = \sup(A)$, but this is a contradiction.

■

Remark (1). The crux of this proof lies in the argument for existence. The main idea is that by assuming $\sup(A)^2 > 2$ or $\sup(A)^2 < 2$, we can "wedge" a number between $\sup(A)^2$ and 2, making use of the Archimedean property. This lets us derive contradictions of the supremum's two properties. When we assume $\sup(A)^2 > 2$, the wedged number tells us $\sup(A)$ is not the least upper bound – it's not small enough. When we assume $\sup(A)^2 < 2$, the wedged number tells us $\sup(A)$ is not an upper bound – it's not big enough.

Remark (2). Repeat this proof with the set $B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\}$. If we assume AC holds in $\mathbb{Q}$ – namely, that $\alpha = \sup(B) \in \mathbb{Q}$ exists – then we arrive at the conclusion that $\alpha^2 = 2$, which has been impossible for over 2000 years...

2.3 Cardinality

Definition 2.4. Let $f : A \to B$.

(1) f is injective if $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$.

(2) f is surjective if $\forall y \in B, \exists x \in A, f(x) = y$.

(3) f is bijective if it is injective and surjective.

Definition 2.5 (Function composition)**.** Let $f : A \to B$ and $g : B \to C$. We can define $h : A \to C$ as $h(x) = g \circ f(x) = g(f(x))$ for $x \in A$.

Proposition 2.5. Some basic results:

(1) f and g bijective implies $h = g \circ f$ is bijective

(2) Let $E \subseteq C$. Then $h^{-1}(E) = f^{-1}(g^{-1}(E))$.

Definition 2.6 (Inverse function)**.** Let $f : A \to B$ be a **bijective map**. Then the inverse map $f^{-1} : B \to A$ is such that $\forall y \in B, f^{-1}(y) = x$ where x is such that $f(x) = y$.

Remark. Do not confuse inverse function and inverse image of a set. The inverse function only exists for bijective maps, while the inverse image of a set is defined for any function.

Definition 2.7 (Cardinality)**.** Let A, B be two sets. We say A and B have the same cardinality, $A \sim B$, if there exists a bijection $f : A \to B$.

Proposition 2.6. The relation $\sim$ is an equivalence relation. Namely, it satisfies:

(1) Reflexivity: $A \sim A$

(2) Symmetry: If $A \sim B$, then $B \sim A$

(3) Transitivity: If $A \sim B$ and $B \sim C$, then $A \sim C$

Definition 2.8 (Countable set)**.** A set A is countable if $\mathbb{N} \sim A$. In other words, there exists a bijection $f : \mathbb{N} \to A$.

Remark. Strictly speaking, by this definition, a finite set is not countable.

Definition 2.9 (Uncountable set)**.** An infinite set B is uncountable if it is not countable.

Remark. So all uncountable sets are infinite and not in bijective correspondence with $\mathbb{N}$.

2.3.1 Facts about countable sets

We accept the next result without proof.

Theorem 2.5. Let $A \subseteq B$.

(i) If B is finite or countable, then so is A.

(ii) If A is uncountable, then so is B (follows by contraposition of (i))

Theorem 2.6. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

Proof. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(m,n) = 2^m \cdot 3^n$. By the uniqueness of prime factorization, f is an injective function. That is, if $y_1 = y_2$, where $y_1 = 2^{m_1} \cdot 3^{n_1}$ and $y_2 = 2^{m_2} \cdot 3^{n_2}$, then $m_1 = m_2$ and $n_1 = n_2$. This function is not surjective so we cannot claim it is bijective. Luckily however, the next theorem tells us that the existence of an injection $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$ is sufficient to conclude that $\mathbb{N} \times \mathbb{N}$ is finite or countable. Clearly $\mathbb{N} \times \mathbb{N}$ is not finite, so it is countable.

■

Theorem 2.7. Let A be a set. Then the following are equivalent.

(a) A is finite or countable

(b) There exists a surjection from $\mathbb{N}$ onto A

(c) There exists an injection from A into $\mathbb{N}$

Proof. It is sufficient to show $(a) \implies (b) \implies (c) \implies (a)$.

First, let A be finite or countable. If A is finite, then there exists a bijection $h : \{1, 2, \ldots, n\} \to A$. Define $f : \mathbb{N} \to A$ such that for $m \in \mathbb{N}$,

$$f(m) = \begin{cases} h(m) & \text{if } m < n \\ h(n) & \text{if } m \geq n \end{cases}$$

The desired surjection is thus provided by f. On the other hand, if A is countable, then there exists a bijection $f : \mathbb{N} \to A$, which is also a surjection. Hence $(a) \implies (b)$ is proven.

Now assume there exists a surjection $h : \mathbb{N} \to A$. Then for all $a \in A$, there exists $n \in \mathbb{N}$ such that $h(n) = a$. We thus know $h^{-1}(\{a\})$ is non-empty, and by the well-ordering property, $h^{-1}(\{a\})$

contains a least element. Define $f : A \to \mathbb{N}$ such that $f(a)$ gives the least element of $h^{-1}(\{a\})$. It follows that f provides the desired injection. To see this, assume $f(a_1) = f(a_2) = n$. Then n is the least element of both $h^{-1}(\{a_1\})$ and $h^{-1}(\{a_2\})$. That means $h(n) = a_1$ and $h(n) = a_2$, so therefore $a_1 = a_2$. Hence $(b) \implies (c)$ is proven.

Now assume there exists an injection $f : A \to \mathbb{N}$. Since f is injective from A into $\mathbb{N}$, then f also defines a bijection $f : A \to f(A)$. But $f(A) \subseteq \mathbb{N}$ implies $f(A)$ must be finite or countable, since $\mathbb{N}$ is finite or countable. Therefore A must be finite or countable. Hence $(c) \implies (a)$ is proven, and we are done.

■

Remark. Notice that order matters. If we modified (b) as "there exists a surjection from A onto $\mathbb{N}$", then the theorem breaks down. For example, take $A = \mathbb{R}$ and let $f : A \to \mathbb{N}$ such that $f(x) = \lceil |x| \rceil$. This map is surjective, since for any $n \in \mathbb{N}$, taking $x = n$ gives $f(x) = n$. But $\mathbb{R}$ is uncountable.

Theorem 2.8. Let (A_n), $n = 1, 2, \ldots$, be a sequence of sets such that each A_n is finite or countable. Then

$$A = \bigcup_{n=1}^{\infty} A_n$$

is finite or countable.

Proof. We put to good use the equivalence $(a) \iff (b)$ in Theorem 2.7. Since A_n is finite or countable, then by 2.7 there exists a surjection $\phi_n : \mathbb{N} \to A_n$. Now, define $h : \mathbb{N} \times \mathbb{N} \to A$ such that $h(n, m) = \phi_n(m)$. We claim that h is also a surjection. To see this, let $a \in A$. Then $a \in A_n$ for some $n \in \mathbb{N}$. That means $\phi_n(m) = a$ for some $m \in \mathbb{N}$. Hence there exist $n, m \in \mathbb{N}$ such that $h(n, m) = a$, so h is surjective.

Now, by 2.6 there exists a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Since h and f are both surjections, then their composite map $h \circ f : \mathbb{N} \to A$ is a surjection too. Hence, by 2.7 we have that A is finite or countable.

■

Remark. If $A_1, A_2, \ldots, A_N$ are finitely many finite or countable sets, then their union is also finite or countable. The idea is to define $h(n, m) = \phi_n(m)$ if $n < N$ and $h(n, m) = \phi_N(m)$ if $n \geq N$.

2.3.2 Countability of $\mathbb{Q}$, $\mathbb{J}$, and $\mathbb{R}$

Theorem 2.9. The set of rational numbers $\mathbb{Q}$ is countable.

Proof. We write $\mathbb{Q} = A_0 \cup A_1 \cup A_2$ where

$$\begin{aligned} A_0 &= \{0\} \\ A_1 &= \{\frac{m}{n} : m, n \in \mathbb{N}\} \\ A_2 &= \{-\frac{m}{n} : m, n \in \mathbb{N}\} \end{aligned}$$

A_0 is finite. Let us show that A_1 is countable. Define a map $h : \mathbb{N} \times \mathbb{N} \to A_1$ by setting $h(m, n) = \frac{m}{n}$. Clearly, h is a surjection. Since there exists a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, then $h \circ f : \mathbb{N} \to A_1$ is a surjection. It follows that A_1 is countable (A_1 is not finite). In a similar way, define $h(m, n) = -\frac{m}{n}$ to show A_2 is countable.

Since $\mathbb{Q}$ is a union of finite or countable sets, and at least one is countable, then $\mathbb{Q}$ is countable and can be put in bijective correspondence with $\mathbb{N}$.

■

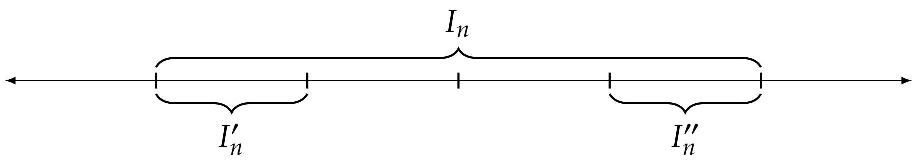
Theorem 2.10. The set of real numbers $\mathbb{R}$ is uncountable.

Proof. Since $\mathbb{Q}$ is countable, our proof will naturally involve AC, the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. The strategy will be to assume the opposite, and derive a contradiction of NIP, which is an equivalent formulation of AC.

Assume $\mathbb{R}$ is countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. This allows us to list all elements of $\mathbb{R}$ as $\{x_1, \ldots, x_n \ldots, \}$. Let us set $f(n) = x_n$ for all $n \in \mathbb{N}$. Now, we construct a sequence (I_n) of closed and bounded intervals. First, define $I_n = [a_n, b_n]$ such that $x_n \notin I_n$. Next, consider two cases to construct I_{n+1}.

Case 1: $x_{n+1} \notin I_n$. Then simply define $I_{n+1} := I_n$.

Case 2: $x_{n+1} \in I_n$. Then divide I_n into four equal parts as shown, where I'_n and I''_n are bounded, closed intervals.



Notice that $x_{n+1} \notin I'_n$ or $x_{n+1} \notin I''_n$ (possibly both are true). We are therefore at liberty to define I_{n+1} as either I'_n or I''_n, such that $x_{n+1} \notin I_{n+1}$.

Continuing this construction for all of $\mathbb{N}$, we obtain an infinite sequence of bounded, closed intervals such that $I_1 \supseteq I_2 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ where $x_m \notin I_m$ for all $m \in \mathbb{N}$. It follows that $x_m \notin \bigcap_{n=1}^{\infty} I_n$ for all $m \in \mathbb{N}$. Thus $\mathbb{R} \cap (\bigcap_{n=1}^{\infty} I_n) = \emptyset$, and since $\bigcap_{n=1}^{\infty} I_n \subseteq \mathbb{R}$, we have

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

But this contradicts NIP, and we are done.

■

Proposition 2.7. The set of irrational numbers $\mathbb{J}$ is uncountable.

Proof. By definition we have that $\mathbb{R} = \mathbb{J} \cup \mathbb{Q}$. If $\mathbb{J}$ were countable, then since $\mathbb{Q}$ is countable we would have $\mathbb{R}$ is countable. But $\mathbb{R}$ is uncountable.

■

Proposition 2.8. Any bounded non-empty open interval (a, b) in $\mathbb{R}$ is uncountable.

Proof. Let $f : (-1, 1) \to \mathbb{R}$ be a map where $f(x) = \frac{x}{x^2-1}$. First, we claim f is surjective. If $y = 0$, then $x = 0$ gives $f(x) = y$ as desired. Suppose then that $y \neq 0$. Then we have

$$y = \frac{x}{x^2 - 1} \iff yx^2 - x - y = 0 \iff x = \frac{1 \pm \sqrt{1 + 4y^2}}{2y}$$

Let $x_1 = \frac{1+\sqrt{1+4y^2}}{2y}$ and $x_2 = \frac{1-\sqrt{1+4y^2}}{2y}$. Observe that $x_1x_2 = -1$, which implies $x_1 \in [-1, 1]$ or $x_2 \in [-1, 1]$. To see this, if this were not true, then $|x_1| > 1$ and $|x_2| > 1$ gives $|x_1||x_2| > 1$, which is a contradiction since $|x_1||x_2| = 1$. We further have that x_1 and x_2 cannot be -1 or 1, since by inspection those points are not possible solutions. So for any y, there exists $x \in (-1, 1)$ that gives $f(x) = y$.

Next, we claim f is injective. Assume $f(x_1) = f(x_2)$ for $x_1, x_2 \in (-1, 1)$. Then

$$\begin{aligned} \frac{x_1}{x_1^2 - 1} = \frac{x_2}{x_2^2 - 1} &\iff x_1x_2^2 - (x_1^2 - 1)x_2 - x_1 = 0 \\ &\iff x_2 = \frac{x_1^2 - 1 \pm (x_1^2 + 1)}{2x_1} \end{aligned}$$

Thus $x_2 = x_1$ or $x_2 = -\frac{1}{x_1}$. If we take $x_2 = -\frac{1}{x_1}$ as a solution, then

$$-1 < x_1 < 1 \implies -1 < -x_1 < 1 \implies |-x_1| < 1 \implies \left|-\frac{1}{x_1}\right| > 1 \implies |x_2| > 1$$

Which contradicts our assumption that $x_2 \in (-1, 1)$. We are left with $x_2 = x_1$. Therefore f is both injective and surjective, and hence bijective.

Now, let $g : (a, b) \to (-1, 1)$ where $g(x) = \frac{2(x-a)}{b-a} - 1$. The range of g is $(-1, 1)$ since

$$a < x < b \iff 0 < \frac{x - a}{b - a} < 1 \iff -1 < \frac{2(x - a)}{b - a} - 1 < 1$$

Observe that g is bijective. For surjectivity, let $y \in (-1, 1)$. Then

$$y = \frac{2(x - a)}{b - a} - 1 \iff (b - a)(y + 1) = 2(x - a) \iff \frac{1}{2}(b - a)(y + 1) + a = x$$

Notice that $x \in (a, b)$. Now for injectivity, assume $g(x_1) = g(x_2)$ for $x_1, x_2 \in (a, b)$. Then

$$\frac{2(x_1 - a)}{b - a} - 1 = \frac{2(x_2 - a)}{b - a} - 1 \iff x_1 = x_2$$

Thus $f \circ g : (a, b) \to \mathbb{R}$ is bijective. Since $\mathbb{R}$ is uncountable, we conclude (a, b) is also uncountable.

∎

Remark (1). An alternative, shorter proof goes as follows. Write $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. This is true because if $x \in \bigcup_{n=1}^{\infty}(-n, n)$, then clearly $x \in \mathbb{R}$. And if $y \in \mathbb{R}$, then for some $n \in \mathbb{N}$, we have $|y| < n$ by the Archimedean property, so $y \in (-n, n)$. At least one interval $(-m, m)$ is uncountable, since otherwise $\mathbb{R}$ would be countable. Now form a bijection $f : (a, b) \to (-m, m)$ and this provides the conclusion.

Remark (2). The first proof tells us $(-1, 1)$ is not only uncountable, but also a bijection $f : (-1, 1) \to \mathbb{R}$ exists. Uncountability of $(-1, 1)$ follows from the second proof too, but not the existence of a bijective correspondence with $\mathbb{R}$.

2.3.3 Power sets

Definition 2.10 (Power set)**.** $\mathcal{P}(A)$, the power set of A, is the set of all subsets of A. If A is finite with n elements, then $|\mathcal{P}(A)| = 2^n$.

Theorem 2.11. Let A be a set. Then there exists no surjection from A onto $\mathcal{P}(A)$.

Proof. Assume for the sake of contradiction that a surjection $f : A \to \mathcal{P}(A)$ exists. Let $D = \{a \in A : a \notin f(a)\}$. Since D is a subset of A, then $D \in P(A)$. Then there exists $a_0 \in A$ such that $f(a_0) = D$.

Case 1: $a_0 \in D \implies a_0 \notin f(a_0)$, but $f(a_0) = D$. So we cannot assume $a_0 \in D$.

Case 2: $a_0 \notin D \implies a_0 \in f(a_0)$, but $f(a_0) = D$. So we cannot assume $a_0 \notin D$.

Therefore both cases lead to a contradiction.

■

Remark (1). On the other hand, an injection does exist by letting $f(a) = \{a\}$. Since an injection exists, but no bijection, we say $\mathcal{P}(A)$ is *strictly larger* than A. So for example $\mathcal{P}(\mathbb{N}) > \mathbb{N}, \mathcal{P}(\mathcal{P}(\mathbb{N})) > \mathcal{P}(\mathbb{N}), \ldots$.

Remark (2). It happens that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. The continuum hypothesis asks whether there exists a set "between" $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ with cardinality not equal to $|\mathbb{N}|$ and not equal to $|\mathcal{P}(\mathbb{N})|$.

3 Sequences

Definition 3.1 (Sequence)**.** Let A be a set. An A-valued sequence indexed by $\mathbb{N}$ is a map

$$x : \mathbb{N} \to A$$

$x(n) = x_n$ is called the n-th element of the sequence. The sequence itself is abbreviated by $\{x_n\}_{n\in\mathbb{N}}$ or $(x_n)_{n\in\mathbb{N}}$.

Remark. We focus on real sequences, where the index set is $\mathbb{N}$ and $A = \mathbb{R}$.

Definition 3.2 (Convergence)**.** A sequence (x_n) converges to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall n \geq N \quad |x_n - x| < \varepsilon$$

In this case, we write

$$\lim_{n\to\infty} x_n = x$$

Remark. Whenever $\varepsilon > 0$, only finitely many elements of (x_n) are not ε-close to x.

Proposition 3.1 (Triangle inequality)**.** Let $x, y, z \in \mathbb{R}$.

(i) $|x + y| \leq |x| + |y|$

(ii) $|x - y| \leq |x - z| + |z - y|$

Proof.

(i) If $x + y < 0$, then $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$. Or if $x + y \geq 0$, then $|x + y| = x + y \leq |x| + |y|$

(ii) $|x - y| = |(x - z) + (z - y)| \leq |x + z| + |z - y|$.

■

Example 3.1. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 3.2. Show that $\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1$

Definition 3.3 (Divergence)**.** If (x_n) does not converge to any real number x, then (x_n) is divergent.

Example 3.3. Show that $x_n = (-1)^n$ is divergent.

Proof. Suppose that (x_n) is convergent. Let $x = \lim_{n\to\infty} x_n$ and take $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - x| < \varepsilon = 1$. So take $n \geq N$ and we have

$$|x_{n+1} - x_n| = |x_{n+1} - x + x - x_n| \leq |x_{n+1} - x| + |x_n - x| < 1 + 1 = 2$$

But also $|x_{n+1} - x_n| = |(-1)^{N+1} - (-1)^N| = |(-1)^{N+1} + (-1)^{N+1}| = 2$. So $2 < 2$, which is a contradiction.

■

Theorem 3.1. If the limit of a sequence (x_n) exists, then it is unique.

Proof. Assume not. Then there exist $x, y \in \mathbb{R}$, $x \neq y$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$. Let $\varepsilon = \frac{|x-y|}{2}$. Then

$$\begin{aligned} &\exists N_1 \in \mathbb{N}, \forall n \geq N_1 \quad |x_n - x| < \varepsilon \\ &\exists N_2 \in \mathbb{N}, \forall n \geq N_2 \quad |x_n - y| < \varepsilon \end{aligned}$$

Thus

$$\forall n \geq \max(\{N_1, N_2\}) \quad |x_n - x| + |x_n - y| < 2\varepsilon \tag{1}$$
$$\tag{2}$$

But also $|x_n - x| + |x_n - y| = |-x_n + x| + |x_n - y| \geq |x - y|$, so $|x - y| < 2\frac{|x-y|}{2} = |x - y|$. Contradiction arises.

■

Theorem 3.2. If (x_n) is a convergent sequence, then there exists $M > 0$ such that $|x_n| \leq M$ for all $n \geq 1$. In other words, all convergent sequences are bounded.

Proof. Let (x_n) be convergent. Then $\lim\limits_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$. Let $\varepsilon = 1$. Then

$$\exists N \in \mathbb{N}, \forall n \geq N \quad |x_n - x| < 1$$

Now, let us put an upper bound on $|x_n|$ for $n \geq N$. With the triangle inequality,

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$$

We define $M = |x_1| + |x_2| + \cdots + |x_{N-1}| + (1 + |x|)$.

Whenever $n \leq N - 1$, then $|x_n|$ is a summand of M, so $|x_n| \leq M$. On the other hand, whenever $n \geq N$, then $|x_n| \leq 1 + |x|$, which is also a summand of M. We conclude $|x_n| \leq M$ for all $n \geq 1$.

■

Theorem 3.3 (Algebraic properties of limits)**.** Let (x_n) and (y_n) be sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(1) (Constant multiple rule). For any $c \in \mathbb{R}$,

$$\lim_{n\to\infty} c \cdot x_n = c \cdot \lim_{n\to\infty} x_n = c \cdot x$$

(2) (Sum rule).

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

(3) (Product rule).

$$\lim_{n\to\infty} x_n y_n = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} y_n)$$

(4) (Division rule). Suppose $y \neq 0$ and $y_n \neq 0$ for all $n \geq 1$.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

Proof. **These proofs illustrate some generally useful techniques.**

(1) If $c = 0$, then $\lim_{n\to\infty} 0 \cdot x_n = \lim_{n\to\infty} 0 = 0 = 0 \cdot x = 0 \cdot \lim_{n\to\infty} x_n$. Now assume $c \neq 0$ and let $\varepsilon > 0$. Since (x_n) converges, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n - x| < \frac{\varepsilon}{|c|}$. Then

$$\begin{aligned} |x_n - x| < \frac{\varepsilon}{|c|} &\iff |c||x_n - x| < \varepsilon \\ &\iff |c(x_n - x)| < \varepsilon \\ &\iff |cx_n - cx| < \varepsilon \end{aligned}$$

(2) Let $\varepsilon > 0$. Then

$$\begin{aligned} &\exists N_1 \in \mathbb{N}, \forall n \geq N_1 \quad |x_n - x| < \frac{\varepsilon}{2} \\ &\exists N_2 \in \mathbb{N}, \forall n \geq N_2 \quad |y_n - x| < \frac{\varepsilon}{2} \end{aligned}$$

Then for $n \geq \max(\{N_1, N_2\})$, we have

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(3) We know (x_n) is bounded, so there exists $M > 0$ such that $|x_n| \leq M$ for all $n \geq 1$. Now observe that

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &= |x_n||y_n - y| + |y||x_n - x| \\ &\leq M|y_n - y| + |y||x_n - x| \end{aligned}$$

By introducing $|y_n - y|$ and $|x_n - y|$, we can now force $|x_n y_n - xy|$ to be arbitrarily small. Let $\varepsilon > 0$. Let $\varepsilon_1 = \frac{\varepsilon}{2|y|+1}$ and let $\varepsilon_2 = \frac{\varepsilon}{2M}$. We know that

$$\begin{aligned}\exists N_1 \in \mathbb{N}, \forall n \geq N_1 \quad |x_n - x| < \varepsilon_1 \\ \exists N_2 \in \mathbb{N}, \forall n \geq N_2 \quad |y_n - x| < \varepsilon_2\end{aligned}$$

So therefore

$$\begin{aligned}|x_n y_n - xy| &\leq M|y_n - y| + |y||x_n - x| \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2|y|+1} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

The upper bound M was important here since it allowed us to replace $|x_n|$. We added 1 to the denominator of ε_1 to account for the possibility that $y = 0$.

(4) Again, (x_n) is bounded by some $M > 0$. let $\varepsilon > 0$. Let $\varepsilon_1 = \frac{\varepsilon|y|}{2}$ and let $\varepsilon_2 = \frac{\varepsilon|y|^2}{4M}$. By definition,

$$\begin{aligned}\exists N_1 \in \mathbb{N}, \forall n \geq N_1 \quad |x_n - x| < \varepsilon_1 \\ \exists N_2 \in \mathbb{N}, \forall n \geq N_2 \quad |y_n - x| < \varepsilon_2\end{aligned}$$

From here, we have

$$\begin{aligned}\left|\frac{x_n}{y_n} - \frac{x}{y}\right| &= \left|\frac{x_n y - y_n x}{y_n y}\right| \\ &= \left|\frac{x_n y - x_n y_n + x_n y_n - y_n x}{y_n y}\right| \\ &\leq \frac{|x_n y - x_n y_n| + |x_n y_n - y_n x|}{|y_n y|} \\ &= \frac{|x_n||y_n - y| + |y_n||x_n - x|}{|y_n||y|} \\ &= \frac{|x_n|}{|y_n||y|}|y_n - y| + \frac{1}{|y|}|x_n - x|\end{aligned}$$

At this point, $|y_n - y|$ and $|x_n - x|$ have appeared, which is good since we can force them arbitrarily small. $|x_n|$ can be dealt with since it is upper bounded by M. The problem lies in $|y_n|$. Since it is in the denominator, we need a **lower bound** on $|y_n|$ for the correct inequality. The winning idea is that for large enough n, y_n must exceed any fraction of its limit. In particular, $|y_n| > \frac{|y|}{2}$ for large enough n. To see this, we set $\varepsilon_3 = \frac{|y|}{2}$. Then for $n \geq N_3 \in \mathbb{N}$, we have

$$\begin{aligned}|y_n - y| < \frac{|y|}{2} &\implies \frac{-|y|}{2} < y_n - y < \frac{|y|}{2} \\ &\implies \frac{-|y|}{2} + y < y_n < \frac{|y|}{2} + y\end{aligned}$$

If $y < 0$, then

$$\frac{3y}{2} < y_n < \frac{y}{2} \implies \frac{-|3y|}{2} < y_n < \frac{-|y|}{2} \implies \frac{|y|}{2} < |y_n| < \frac{3|y|}{2}$$

If $y > 0$, then

$$\frac{y}{2} < y_n < \frac{3y}{2} \implies \frac{|y|}{2} < |y_n| < \frac{3|y|}{2}$$

In either case, $|y_n| > \frac{|y|}{2}$. With this lower bound in hand, let $n \geq \max(\{N_1, N_2, N_3\})$ and we have

$$\begin{aligned} \left|\frac{x_n}{y_n} - \frac{x}{y}\right| &\leq \frac{|x_n|}{|y_n||y|}|y_n - y| + \frac{1}{|y|}|x_n - x| \\ &< \frac{2M}{|y|^2} \cdot \frac{\varepsilon|y|^2}{4M} + \frac{\varepsilon|y|}{2|y|} \\ &= \varepsilon \end{aligned}$$

■

Remark. Given $X = \{(x_n)\}$ the set of all real-valued sequences, we can define scalar multiplication $c(x_n) = (cx_n)$ and also addition $(x_n) + (y_n) = (x_n + y_n)$. With these operations, X becomes an infinite dimensional vector space.

Theorem 3.4 (Order limit theorem)**.** Let (x_n) and (y_n) be convergent with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(1) If $x_n \geq 0$ for all $n \geq 1$, then $x \geq 0$.

(2) If $y_n \geq x_n$ for all $n \geq 1$, then $y \geq x$.

(3) If $c \in \mathbb{R}$ and $c \leq x_n$ (resp. $c \geq x_n$) then $c \leq x$ (resp. $c \geq x$).

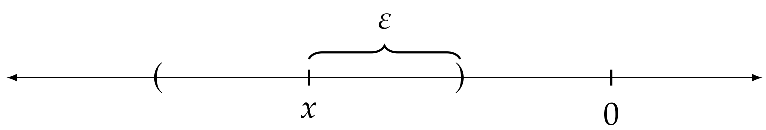
Proof. For (1), assume $x < 0$ towards a contradiction. Let $\varepsilon > 0$ such that $x < -\varepsilon < 0$. That means $x + \varepsilon < 0$. Now, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n - x| < \varepsilon$. It follows that

$$-\varepsilon < x_n - x < \varepsilon \implies -\varepsilon + x < x_n < x + \varepsilon < 0$$

Which is a contradiction since we assumed $x_n \geq 0$ for all $n \geq 1$. We have (2) from (1) by setting $z_n = y_n - x_n \geq 0$, and (3) follows from (2) by setting either $y_n = c$ or $x_n = c$.

■

Remark (1)*.* The intuition here is that by constraining the ε-neighbourhood to never surface above 0, x_n is forced below 0 to reconcile the limit definition.



Remark (2)*.* Concerning (1), if $x_n > 0$ for all $n \geq 1$, we still can only conclude $x \geq 0$. For example, $x_n = \frac{1}{n}$ is a sequence with terms strictly greater than 0, but converges to 0.

Theorem 3.5 (Squeeze theorem)**.** Let (x_n), (y_n), (z_n) be sequences with $x_n \leq y_n \leq z_n$ for all $n \geq 1$. Furthermore, let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$. Then $\lim_{n\to\infty} y_n = l$.

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned}
\exists N_1 \in \mathbb{N}, \forall n \geq N_1 \quad |x_n - l| < \varepsilon \implies -\varepsilon < x_n - l < \varepsilon \\
\exists N_2 \in \mathbb{N}, \forall n \geq N_2 \quad |z_n - l| < \varepsilon \implies -\varepsilon < z_n - l < \varepsilon
\end{aligned}$$

Take $n \geq \max(\{N_1, N_2\})$. Then $y_n - l \leq z_n - l < \varepsilon$ and $y_n - l \geq x_n - l > -\varepsilon$, therefore $|y_n - l| < \varepsilon$. ■

3.1 Monotone convergence theorem

Definition 3.4 (Basic definitions)**.** A sequence (x_n) is

(1) *Increasing* if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

(2) *Decreasing* if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

(3) *Bounded from above* if there exists $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$.

(4) *Bounded from below* if there exists $M \in \mathbb{R}$ such that $x_n \geq M$ for all $n \in \mathbb{N}$.

Theorem 3.6 (Monotone convergence theorem)**.**

(1) Let (x_n) be increasing and bounded from above. Then (x_n) is convergent.

(2) Let (x_n) be decreasing and bounded from below. Then (x_n) is convergent.

Proof. We prove (1), and (2) follows a similar structure (but uses the infimum instead).

Let $A = \{x_n : n \geq 1\}$. Since (x_n) is bounded above, then A is bounded above, and hence $\sup(A)$ exists by AC. Let $\varepsilon > 0$. Then $\sup(A) - \varepsilon$ is not an upper bound of A since $\sup(A)$ is the least upper bound. Thus $\sup(A) - \varepsilon < x_N$ for some $N \in \mathbb{N}$. But since (x_n) is increasing, then for all $n \geq N$ we have $x_N \leq x_n$. It follows that

$$\sup(A) - \varepsilon < x_N \leq x_n \leq \sup(A) < \sup(A) + \varepsilon$$

And therefore $|x_n - \sup(A)| < \varepsilon$, which gives $\lim_{n\to\infty} x_n = \sup(A)$ by definition. ■

Remark (1). (x_n) is called *eventually increasing* if there exists $N_0 \in \mathbb{N}$ such that $x_{n+1} \geq x_n$ for all $n \geq N_0$. MCT works for eventually increasing sequences too (that are bounded above). We just have to set $A = \{x_n : n \geq N_0\}$ to get the supremum we need.

Remark (2). MCT is an equivalent formulation of AC.

Proposition 3.2. Let (x_n) be a bounded sequence and set $A_n := \{x_k : k \geq n\}$. Define (y_n) as a sequence of supremums such that $y_n = \sup(A_n)$ for all $n \in \mathbb{N}$. Then (y_n) converges and its limit $\lim_{n\to\infty} y_n = y$ exists.

Proof. Observe that every set A_n is bounded above and below. This follows because (x_n) is bounded by some $M \in \mathbb{R}$, so naturally $|x_k| \leq M$ for all $k \geq n$. Thus by AC we can safely say $\sup(A_n)$ and $\inf(A_n)$ exists for all $n \in \mathbb{N}$.

Next, observe that the sets A_n form a nested sequence

$$A_1 \supseteq A_2 \supseteq \dots A_n \supseteq \dots$$

In other words, $A_n \supseteq A_{n+1}$ for every $n \in \mathbb{N}$. This is because if A_{n+1} contains all x_k where $k \geq n+1$, then surely A_n contains them as well, since it contains all x_k where $k \geq n$.

From here, it follows that

$$\sup(A_1) \geq \sup(A_2) \geq \dots \geq \sup(A_n) \geq \dots$$

To see why, assume $\sup(A_n) < \sup(A_{n+1})$ and set $\varepsilon = \sup(A_{n+1}) - \sup(A_n)$. Then there is $k \geq n+1$ such that $x_k > \sup(A_{n+1}) - \varepsilon = \sup(A_n)$. But this is a contradiction since $\sup(A_n)$ is an upper bound of A_n, and the nested property gives that $x_k \in A_n$.

Finally, notice that $\sup(A_n) \geq \inf(A_n) \geq -M$ for all $n \in \mathbb{N}$. What we have now shown then is that the sequence (y_n) is *decreasing* and *bounded below*. Therefore by MCT, (y_n) converges to some limit $y \in \mathbb{R}$.

■

Proposition 3.3. Let (x_n) be a bounded sequence and set $A_n := \{x_k : k \geq n\}$. Define (z_n) as a sequence of infimums such that $z_n = \inf(A_n)$ for all $n \in \mathbb{N}$. Then (z_n) converges and its limit $\lim_{n\to\infty} z_n = z$ exists.

Proof. Uses MCT in a similar fashion as 3.2

■

Definition 3.5 (Limit superior)**.** In context of Proposition 3.2, the *limit superior* of x_n is

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} y_n = y$$

Definition 3.6 (Limit inferior)**.** In context of Proposition 3.3, the *limit inferior* of x_n is

$$\liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} z_n = z$$

Remark. **Importantly,** while a bounded sequence can diverge, the limit inferior and limit superior both exist for **any** bounded sequence (x_n). That (x_n) is bounded was our only assumption in 3.2.

Proposition 3.4 (Ordering of limit superior and limit inferior)**.**

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

Proof. We have $z_n = \inf(A_n) \leq \sup(A_n) = y_n$, so the conclusion follows from the order limit theorem.

■

Theorem 3.7. Let (x_n) be a bounded sequence. Then the following are equivalent:

(i) x_n converges and $\lim_{n\to\infty} x_n = x$

(ii) $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$

Proof. We first show (i) $\implies$ (ii). Assume $\lim_{n\to\infty} x_n = x$ and let $\varepsilon > 0$. Then for sufficiently large $N \in \mathbb{N}$, let $n \geq N$ such that $|x_n - x| < \frac{\varepsilon}{2}$. This means

$$x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}$$

The punchline is that for all $n \geq N$, we have that $x + \frac{\varepsilon}{2}$ is an upper bound of $A_n = \{x_k : k \geq n\}$ and $x - \frac{\varepsilon}{2}$ is a lower bound of A_n. Since $z_n = \inf(A_n)$, then $z_n \geq x - \frac{\varepsilon}{2}$. Since $y_n = \sup(A_n)$, then $y_n \leq x + \frac{\varepsilon}{2}$. Now it follows that

$$x - \frac{\varepsilon}{2} \leq \underbrace{z_n \leq x_n \leq y_n}_{\text{since } x_n \in A_n} \leq x + \frac{\varepsilon}{2}$$

Thus we have the desired conclusion $|z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$ and $|y_n - y| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $n \geq N$.

Next, we show (ii) $\implies$ (i). Assume $\lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n = x$. Let $\varepsilon > 0$. Then there is $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ $|z_n - x| < \varepsilon$. And there is $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ $|y_n - x| < \varepsilon$. Let $n \geq N = \max(\{N_1, N_2\})$. Then

$$x - \varepsilon < z_n < \varepsilon + x$$
$$x - \varepsilon < y_n < \varepsilon + x$$

We have $z_n \leq x_n \leq y_n$ since $x_n \in A_n$. So

$$x - \varepsilon < z_n \leq x_n \leq y_n < x + \varepsilon$$

Which gives $|x_n - x| < \varepsilon$ for all $n \geq N$ as needed.

■

3.2 Subsequences and the Bolzano-Weierstrass Theorem

Definition 3.7 (Subsequence)**.** Let (x_n) be a sequence of real numbers and let

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \dots$$

be a strictly increasing sequence of natural numbers. Then the sequence

$$(x_{n_1}, x_{n_2}, \dots, x_{n_k}, x_{n_{k+1}}, \dots)$$

is called a subsequence of (x_n) denoted as $(x_{n_k})_{k\in\mathbb{N}}$.

Remark (1). **Know that $n_k \geq k$ for all $k \in \mathbb{N}$.** This can be shown by induction. If $k = 1$ then $n_1 \geq 1$ since $n_1 \in \mathbb{N}$ by definition. Assuming $n_k \geq k$ for some $k \in \mathbb{N}$, then $n_{k+1} > n_k \geq k$ so therefore $n_{k+1} \geq k + 1$.

Remark (2). A subsequence converges as $\lim_{n\to\infty} x_{n_k} = x$ if $\forall\varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $|x_{n_k} - x| < \varepsilon$.

Proposition 3.5. Let (x_n) be a sequence that converges to x. Then any subsequence (x_{n_k}) of (x_n) also converges to x.

Proof. Let $\varepsilon > 0$. Since (x_n) converges to x, then there is $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$. Take $K = N$ and let $k \geq K$. Then $n_k \geq k \geq K = N$, so we have that $|x_{n_k} - x| < \varepsilon$.

■

Remark. Taking the contrapositive, if not all subsequences of (x_n) converge to some fixed x, then (x_n) must be divergent.

Lemma 3.1. If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$.

Proof. Let $x_n = b^n$. Clearly $x_n > 0$ for all $n \in \mathbb{N}$ and we also have $x_{n+1} = bx_n < x_n$. Thus x_n is decreasing and bounded below, so it converges to some $x \in \mathbb{R}$ by MCT. It follows that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} bx_n = bx$$

Thus $(1 - b)x = 0$, which implies $x = 0$ since $1 - b > 0$.

■

Theorem 3.8 (Bolzano-Weierstrass Theorem)**.** Every bounded sequence has a convergent subsequence.

Remark. There are three ways to prove this, which all invoke AC.

Proof. We use AC in the form of NIP. Since (x_n) is a bounded sequence, there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Let $I_1 = [-M, M] = [a_1, b_1]$ so that $x_n \in I_1$ for all $n \in \mathbb{N}$. Split I_1 in half, to construct $I_1' = [-M, 0]$ and $I_1'' = [0, M]$. Now set $n_1 = 1$ to define $A_1^{(1)}$ and $A_2^{(1)}$ as

$$A_1^{(1)} = \{n \in \mathbb{N} : n > n_1 \text{ and } x_n \in I_1'\}$$
$$A_2^{(1)} = \{n \in \mathbb{N} : n > n_1 \text{ and } x_n \in I_1''\}$$

Since $A_1^{(1)} \cup A_2^{(1)}$ is infinite, then $A_1^{(1)}$ is infinite or $A_2^{(1)}$ is infinite. If $A_1^{(1)}$ is infinite, set $I_2 = I_1'$ and let $n_2 = \min(A_1^{(1)})$, which is possible due to the well ordering principle. Otherwise set $I_2 = I_1''$ and let $n_2 = \min(A_2^{(1)})$.

Now suppose $I_k = [a_k, b_k]$ is defined along with n_k, and construct I_{k+1} in the same manner. We set $I_k' = [a_k, \frac{b_k - a_k}{2}]$ and $I_k'' = [\frac{b_k - a_k}{2}, b_k]$. Furthermore, we set

$$A_1^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k'\}$$
$$A_2^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k''\}$$

If $A_1^{(k)}$ is infinite, set $I_{k+1} = I_k'$ and let $n_2 = \min(A_1^{(k)})$. Otherwise, we set $I_{k+1} = I_k''$ and $n_{k+1} = \min(A_2^{(k)})$. Now observe that the indices n_k are strictly increasing, since either $n_{k+1} = \min(A_1^{(k)}) > n_k$ or $n_{k+1} = \min(A_2^{(k)}) > n_k$. So we have constructed a subsequence x_{n_k} with the property that $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$. Furthermore, for any $I_k = [a_k, b_k]$ we have that

$$\begin{aligned} b_k - a_k &= \frac{b_{k-1} - a_{k-1}}{2} \\ &= \frac{b_{k-2} - a_{k-1}}{4} \\ &\vdots \\ &= \frac{b_1 - a_1}{2^{k-1}} \\ &= \frac{2M}{2^{k-1}} \\ &= \frac{M}{2^{k-2}} \end{aligned}$$

We have constructed a sequence of infinite sequence of closed and bounded intervals $I_n = [a_n, b_n]$ such that

$$I_1 \supseteq I_2 \supseteq I_3 \supset \dots$$

By NIP, there exists $x \in \bigcap_{k=1}^{\infty} I_k$. The claim then is that $\lim_{k\to\infty} x_{n_k} = x$. To see this, we have

$$\lim_{k\to\infty} \frac{M}{2^{k-2}} = 0$$

Now let $\varepsilon > 0$. Then there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have $b_k - a_k = \frac{M}{2^{k-2}} < \varepsilon$. Since $x \in I_k$ and $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$, then

$$a_k \leq x \leq b_k$$
$$a_k \leq x_{n_k} \leq b_k$$

Hence $a_k - b_k \leq x_{n_k} - x \leq b_k - a_k$ which gives for all $k \geq K$

$$|x_{n_k} - x| \leq |b_k - a_k| < \varepsilon$$

Thus $\lim_{k\to\infty} x_{n_k} = x$, so (x_{n_k}) is a convergent subsequence.

■

Proof. Now we use AC in the form of MCT. First, we claim that every real sequence (x_n) contains a monotone subsequence. We say an element x_m is a **peak** of (x_n) if $x_m \geq x_n$ for all $n \geq m$.

Case 1: (x_n) has infinitely many peaks. Enumerate the peaks as $n_1, n_2, \ldots$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Now $x_{n_k} \geq x_{n_{k+1}}$ for all $k \in \mathbb{N}$, since x_{n_k} is a peak and $n_{k+1} > n_k$. Therefore (x_n) is decreasing.

Case 2: (x_n) has finitely many peaks. Then we can exhaustively enumerate the peaks as $m_1, m_2, \ldots, m_r$ where $r \in \mathbb{N}$. Now let $n_1 = m_r + 1$. Then clearly x_{n_1} is not a peak, so there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Continuing this construction, for every $n_k > m_r$ we have x_{n_k} is not a peak, so there exists $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$. Thus we have a subsequence (x_{n_k}) which is increasing.

By hypothesis (x_n) is a bounded sequence so (x_{n_k}) is also bounded. Therefore by MCT (x_{n_k}) is converging.

■

Proof. There is one more approach we can use. Let (x_n) be a bounded sequence, so there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Let $S = \{x \in \mathbb{R} : x \leq x_n \text{ for infinitely many } n \in \mathbb{N}\}$. Notice that S is nonempty since $-M \in S$. Furthermore, S is bounded from above by M. To see this, if we assumed that $x > M$ for some $x \in S$, then $M < x \leq x_n$ for infinitely many $n \in \mathbb{N}$, which is a contradiction. Therefore by AC, $\sigma = \sup(S)$ exists.

Now let $\varepsilon > 0$ be arbitrary. Then there is $x \in S$ such that $\sigma - \varepsilon < x \leq \sigma$. Since $\sigma + \varepsilon \notin S$, then there are finitely many x_n such that $\sigma + \varepsilon \leq x_n$. But there are infinitely many x_n such that $x \leq x_n$. So there must be infinitely many x_n such that $x_n \in [x, \sigma + \varepsilon)$. Hence there are infinitely many x_n such that $x_n \in (\sigma - \varepsilon, \sigma + \varepsilon)$. Now for the crux of the proof, construct a subsequence (x_{n_k}) such that $x_{n_k} \in (\sigma - \frac{1}{k}, \sigma + \frac{1}{k})$. Particularly, once we have decided x_{n_1} where $n_1 \geq 1$, then for every n_k afterwards where $k \geq 2$, there must exist $m > n_{k-1}$ such that $x_m \in (\sigma - \frac{1}{k}, \sigma + \frac{1}{k})$. This is true since there are infinitely many x_n within $(\sigma - \frac{1}{k}, \sigma + \frac{1}{k})$. So we can inductively set $n_k := m$. We therefore have a converging subsequence (x_{n_k}) with $\lim_{k\to\infty} x_{n_k} = \sigma$, which can be seen by the squeeze theorem.

■

Remark (1). The essential idea of the NIP proof is this. We construct an infinite sequence of closed and bounded intervals $I_k = [a_k, b_k]$. Each interval is half the size of the previous. We can associate x_{n_k} to some element in every interval I_k, thereby constructing a subsequence. By NIP, there is a common element x in every I_k. Furthermore, since the interval lengths are tending to zero, x_{n_k} is brought closer and closer to x. Thus (x_{n_k}) converges. The details early on help ensure that our subsequence goes on infinitely.

Remark (2). The idea of the MCT proof is this. A peak is like a hilltop overlooking the rest of the sequence. We either reach such peaks infinitely many times, or we eventually run out of peaks. If the former, then the peaks themselves define a decreasing subsequence. If the latter, then eventually, no matter how far we continue in the sequence, there will be something greater than our current position. This defines an increasing subsequence. Convergence is then guaranteed by MCT, assuming the sequence is bounded.

Remark (3). The third proof directly invokes AC. An interesting subtle fact is that σ is actually the limit superior of (x_n).

Proof. Let $\sigma = \{x \in \mathbb{R} : x \leq x_n \text{ for infinitely many } n \in \mathbb{N}\}$. Let the limit superior of (x_n) be $l = \limsup_{n\to\infty} x_n$. First, suppose that $l < \sigma$. As we have already shown, there is a subsequence (x_{n_k}) converging to σ. So take an arbitrary $\varepsilon > 0$, for which there exists $K \in \mathbb{N}$ such that $|x_{n_k} - \sigma| < \varepsilon$ for all $k \geq K$. Observe that for a given $k \in \mathbb{N}$, we have $\{x_{n_j} : j \geq k\} \subseteq \{x_n : n \geq k\}$. This follows since $n_j \geq j \geq k$, so $x_{n_j} \in \{x_n : n \geq k\}$. As such, we have $\sup\{x_{n_j} : j \geq k\} \leq \sup\{x_n : n \geq k\}$. This gives the inequality

$$\sigma - \varepsilon < x_{n_k} \leq \sup\{x_{n_j} : j \geq k\} \leq \sup\{x_n : n \geq k\}$$

Hence the order limit theorem gives

$$\sigma - \varepsilon \leq \limsup_{k\to\infty} x_k = l$$

Since we assumed $l < \sigma$, then there is $\varepsilon' > 0$ such that $l < l + \varepsilon' < \sigma$. So $l < \sigma - \varepsilon'$ and $l \geq \sigma - \varepsilon'$, which is a contradiction. Therefore $l \geq \sigma$.

Now suppose $l > \sigma$. Then there exists $\varepsilon > 0$ such that $l > \sigma + 3\varepsilon > \sigma$. Since $l = \limsup_{n\to\infty} x_n$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$l - \varepsilon < \sup\{x_k : k \geq n\} < l + \varepsilon$$

For all $n \geq N$, we know $\sup\{x_k : k \geq n\} - \varepsilon$ is not an upper bound of $\{x_k : k \geq n\}$. So we can find $k_1 \geq 1$ such that

$$x_{k_1} > \sup\{x_k : k \geq 1\} - \varepsilon > l - 2\varepsilon > \sigma + \varepsilon$$

We can then find $k_2 \geq k_1 + 1$ such that

$$x_{k_2} > \sup\{x_k : k \geq k_1\} - \varepsilon > l - 2\varepsilon > \sigma + \varepsilon$$

And so on. Hence there are infinitely many $n \in \mathbb{N}$ such that $x_n > \sigma + \varepsilon$, so $\sigma + \varepsilon \in S$. But this is a contradiction, since σ is the supremum of S and $\sigma < \sigma + \varepsilon$. Therefore $l \leq \sigma$ and $l \geq \sigma$, so $l = \sigma$.

■

3.3 Cauchy sequences

Definition 3.8 (Cauchy sequence)**.** A sequence (x_n) is Cauchy if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$|x_n - x_m| < \varepsilon$$

Lemma 3.2. If (x_n) is convergent, then it is Cauchy.

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then for all $m, n \geq N$ we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \varepsilon \end{aligned}$$

■

Lemma 3.3. Let (x_n) be Cauchy and suppose (x_n) has a convergent subsequence (x_{n_k}). Then (x_n) is also converging.

Proof. Let $\varepsilon > 0$. Since (x_n) is Cauchy, then

$$\exists N \in \mathbb{N}, \forall m, n \geq N \quad |x_n - x_m| < \frac{\varepsilon}{2}$$

Since x_{n_k} is converging to some $x \in \mathbb{R}$, then

$$\exists K \in \mathbb{N}, \forall k \geq K \quad |x_{n_k} - x| < \frac{\varepsilon}{2}$$

Let $K_0 \geq \max(\{K, N\})$. We have $n_{K_0} \geq K_0$ (important). Now take $n \geq N$ and estimate

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_{K_0}} + x_{n_{K_0}} - x| \\ &\leq |x_n - x_{n_{K_0}}| + |x_{n_{K_0}} - x| \\ &< \varepsilon \end{aligned}$$

■

Remark. Intuitively, we use the Cauchy nature of (x_n) to anchor onto, and converge along with, the converging subsequence.

Lemma 3.4. Every Cauchy sequence (x_n) is bounded. Specifically, there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Proof. Let (x_n) be Cauchy, so there is $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$|x_n - x_m| < 1$$

Now fix $m := N$. Then for all $n \geq N$ we have

$$\begin{aligned}
|x_n| &= |x_n - x_N + x_N| \\
&\leq |x_n - x_N| + |x_N| \\
&< 1 + |x_N|
\end{aligned}$$

Set $M := (|x_1| + |x_2| + \cdots + |x_{N-1}|) + (1 + |x_N|)$. By this construction, if $n < N$, then $|x_n| \leq M$, since $|x_n|$ is a summand. And if $n \geq N$, then $|x_n| < 1 + |x_N| \leq M$, since $1 + |x_N|$ is a summand. ∎

Remark. This is almost exactly how we proved that convergent sequences are bounded.

Lemma 3.5. If a sequence (x_n) is Cauchy, then it is convergent.

Proof. Since (x_n) is Cauchy, then (x_n) is bounded by Lemma 3.4. By BW, then (x_n) has a convergent subsequence (x_{n_k}). Then by Lemma 3.3, we have that (x_n) must converge. ∎

Remark. This is finally where the Axiom of Completeness becomes involved.

Theorem 3.9. A sequence (x_n) converges if and only if it is Cauchy.

Proof. Given by Lemma 3.2, and Lemma 3.5. ∎

Definition 3.9 (Contractive sequence)**.** A sequence (x_n) is called contractive with contractive constant K, where $0 < K < 1$ if for all $n \geq 1$

$$|x_{n+2} - x_{n+1}| \leq K|x_{n+1} - x_n|$$

The distances between successive elements are contracted by at least a factor of K.

Remark. If (x_n) is contractive, then for $n \geq 3$ we have $|x_n - x_{n-1}| \leq K^{n-2}|x_2 - x_1|$. This is obtained by recursively applying the definition

$$\begin{aligned}
|x_n - x_{n-1}| &\leq K|x_{n-1} - x_{n-2}| \\
&\leq K^2|x_{n-2} - x_{n-3}| \\
&\vdots \\
&\leq K^{n-2}|x_2 - x_1|
\end{aligned}$$

Theorem 3.10. Let (x_n) be contractive with contractive constant K. Then (x_n) is Cauchy (and hence converging).

Proof. Let $\varepsilon > 0$. Since $0 < K < 1$, then by Lemma 3.1 we have $K^{m-1} \to 0$ as $m \to \infty$. So there exists $M \in \mathbb{N}$ such that for all $m \geq M$

$$K^{m-1} < \frac{\varepsilon(1-K)}{|x_2 - x_1| + 1}$$

Let $m, n \in \mathbb{N}$ with $n > m \geq \max(\{M, 2\})$. Then we have

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq K^{n-2}|x_2 - x_1| + K^{n-3}|x_2 - x_1| + \cdots + K^{m-1}|x_2 - x_1| \\
&= K^{m-1}|x_2 - x_1|(1 + K + K^2 + \cdots + K^{n-m-1}) \\
&= K^{m-1}|x_2 - x_1|\frac{1 - K^{n-m}}{1-K} \\
&< \frac{K^{m-1}|x_2 - x_1|}{1-K} \\
&< \frac{\varepsilon(1-K)|x_2 - x_1|}{(1-K)(|x_2 - x_1| + 1)} \\
&< \varepsilon
\end{aligned}$$

Note that we needed the geometric sum formula.

■

Remark. This result will become important in Analysis 2 when we see the Banach fixed-point theorem.

3.4 Limits of recursively defined sequences

Let (x_n) be a sequence such that $x_1 = 1, x_2 = 2$, and $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$ for all $n > 1$. One approach to solving x_n goes as follows. First, we establish various properties of (x_n).

(i) Claim: $1 \leq x_n \leq 2$ for all $n \geq 1$.

Proof. Let $S \subseteq \mathbb{N}$ be the set of natural numbers such that $1 \leq x_n \leq 2$. Then $x_1 = 1 \in S$. Suppose that $\{1, \ldots, n\} \subseteq S$. If $n = 1$, then $x_2 = 2 \in S$ so $n + 1 \in S$. If $n > 1$, then $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$. But $1 \leq x_n \leq 2$ and $1 \leq x_{n-1} \leq 2$ by the inductive assumption. Hence $1 \leq x_{n+1} \leq 2$ so $n + 1 \in S$. Therefore $S = \mathbb{N}$.

■

(ii) Claim: $|x_n - x_{n+1}| = \frac{1}{2^{n-1}}$ for all $n \geq 1$.

Proof. Let $S \subseteq \mathbb{N}$ such that $|x_n - x_{n+1}| = \frac{1}{2^{n-1}}$ for all $n \in S$. Then $|x_1 - x_2| = 1 = \frac{1}{2^0}$ so $1 \in S$. Now suppose $n \in S$. Then

$$\begin{aligned}
|x_{n+1} - x_{n+2}| &= |x_{n+1} - \frac{1}{2}(x_{n+1} + x_n)| \\
&= \frac{1}{2}|x_{n+1} - x_n| \\
&= \frac{1}{2} \cdot \frac{1}{2^{n-1}} \\
&= \frac{1}{2^n}
\end{aligned}$$

■

(iii) Claim: (x_n) is a Cauchy sequence, so $\lim_{n\to\infty} x_n$ exists.

Proof. Let $\varepsilon > 0$. There exists N such that $\frac{1}{2^{N-2}} = \frac{4}{2^N} < \varepsilon$. Such N exists because $\lim_{k\to\infty} \frac{1}{2^{k-2}} = 0$. Now let $n, m \geq N$ and without loss of generality suppose $n > m$. When $n = m$, there is nothing to do since then $|x_n - x_m| = 0 < \varepsilon$. With a telescoping sum and the generalized triangle inequality,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} + x_{n-2}| + \cdots + |x_{m+1} - x_m|
\end{aligned}$$

By property (ii) and the geometric sum formula we have

$$\begin{aligned}
|x_n - x_m| &\leq \frac{1}{2^{m-1}} + \frac{1}{2^m} + \cdots + \frac{1}{2^{n-2}} \\
&= \frac{1}{2^{m-1}}(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}) \\
&= \frac{1}{2^{m-1}}\left(\frac{1 - (\frac{1}{2})^{n-m}}{\frac{1}{2}}\right) \\
&= \frac{1}{2^{m-2}} - \frac{1}{2^{n-2}} \\
&\leq \frac{1}{2^{m-2}} \\
&\leq \frac{1}{2^{N-2}} \\
&< \varepsilon
\end{aligned}$$

■

(iv) Claim: The subsequence $(x_{2k+1})_{k\in\mathbb{N}}$ of (x_n) satisfies for all $k \geq 1$

$$x_{2k+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2k-1}}$$

Proof. We first show that $x_{2n} \geq x_{2n-1}$ and $x_{2n} \geq x_{2n+1}$ for all $n \geq 1$. Let $S \subseteq \mathbb{N}$ be the set of natural numbers for which this statement holds. Then $x_1 = 1, x_2 = 2, x_3 = \frac{3}{2}$ so clearly $1 \in S$. Suppose $n \in S$ so that $x_{2n} \geq x_{2n-1}$ and $x_{2n} \geq x_{2n+1}$. Then

$$x_{2n+2} = \frac{1}{2}(x_{2n+1} + x_{2n}) \geq \frac{1}{2}(x_{2n+1} + x_{2n+1}) = x_{2n+1}$$

And

$$x_{2n+3} = \frac{1}{2}(x_{2n+2+}x_{2n+1}) \leq \frac{1}{2}(x_{n+2} + x_{n+2}) = x_{2n+2}$$

So $x_{2n+2} \geq x_{2n+1}$ and $x_{2n+2} \geq x_{2n+3}$. Therefore $S = \mathbb{N}$.

$\blacksquare$

3.5 Euler Number e

The Euler number e is important in power series, complex analysis, and trigonometric/hyperbolic functions. Up to a constant coefficient, the function e^x is the only function satisfying the differential equation $y'(x) = y(x)$.

How is e defined? Let $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$. Then we set

$$e := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

To show that x_n and y_n are both converging and converge to the same limit, we need Bernoulli's inequality.

Lemma 3.6 (Bernoulli's Inequality)**.** If $x > -1$, then for all $n \in \mathbb{N}$

$$(1 + x)^n \geq 1 + nx$$

Proof. We prove this by induction. Let $x > -1$. Then for the base case $n = 1$, we have $(1 + x)^1 = 1 + x \geq 1 + x$. For the inductive step, assume $(1 + x)^n \geq 1 + nx$ for some $n \geq 1$. Then

$$\begin{aligned}
(1 + x)^{n+1} &= (1 + x)(1 + x)^n \\
&\geq (1 + x)(1 + nx) && \text{(using I.H., given that } 1 + x > 0) \\
&= nx + nx^2 + 1 + x \\
&= 1 + (n + 1)x + nx^2 \\
&\geq 1 + (n + 1)x
\end{aligned}$$

∎

Remark. We could have proved this using the binomial theorem too, since

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1} x + \ldots \geq 1 + nx$$

Theorem 3.11. Let $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$. Then x_n and y_n are both converging and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Proof. The proof requires several steps. On a high level, we use MCT to show convergence and then the algebraic limit theorem to show the two limits are equal.

(1) First we show that x_n is strictly increasing, namely $\frac{x_{n+1}}{x_n} > 1$ for all $n \geq N$.

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} \\
&= \frac{(n+2)^{n+1} \cdot n^n}{(n+1)^{n+1} \cdot (n+1)^n} \\
&= \frac{n+2}{n+1} \cdot \left(\frac{(n+2)n}{(n+1)^2}\right)^n \\
&= \frac{n+2}{n+1} \cdot \left(\frac{n^2+2n+1-1}{n^2+2n+1}\right)^n \\
&= \frac{n+2}{n+1} \cdot \left(1 - \frac{1}{n^2+2n+1}\right)^n \\
&\geq \frac{n+2}{n+1} \cdot \left(1 + n\left(\frac{-1}{(n+1)^2}\right)\right) \\
&= \frac{n+2(n^2+n+1)}{(n+1)^3} \\
&= \frac{n^3+3n^2+3n+2}{n^2+3n^2+3n+1} \\
&> 1
\end{aligned}$$

(2) Next we show that y_n is strictly decreasing, namely $\frac{y_n}{y_{n+1}} > 1$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
\frac{y_n}{y_{n+1}} &= \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n+1})^{n+2}} \\
&= \frac{n+1}{n+2} \cdot \frac{(n+1)^{2(n+1)}}{n^{n+1} \cdot (n+2)^{n+1}} \\
&= \frac{n+1}{n+2} \cdot \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+1} \\
&= \frac{n+1}{n+2} \cdot \left(1 + \frac{1}{n^2+2n}\right)^{n+1} \\
&\geq \frac{n+1}{n+2} \cdot \left(1 + \frac{n+1}{n^2+2n}\right) \\
&= \frac{n+1}{n+2} \cdot \left(\frac{n^2+3n+1}{n^2+2n}\right) \\
&= \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} \\
&> 1
\end{aligned}$$

(3) Now, we show $x_n < y_k$ for all $n, k \in \mathbb{N}$.

Case 1: $n = k$. Then $x_n = (1 + \frac{1}{n})^n < (1 + \frac{1}{n})(1 + \frac{1}{n})^n = (1 + \frac{1}{n})^{n+1} = y_n$.

Case 2: $n < k$. Then $x_n < x_k$ since (x_n) is strictly increasing, and $x_k < y_k$ by Case 1.

Case 3: $n > k$. Then $x_n < y_n$ by Case 1 and $y_n < y_k$ since (y_n) is strictly decreasing.

(4) Furthermore, (x_n) is bounded above and (y_n) is bounded below. This easily follows from (3) since $x_n < y_1$ for all $n \in \mathbb{N}$ and $y_n > x_1$ for all $n \in \mathbb{N}$.

(5) By MCT, (x_n) and (y_n) both converge. Now we know $y_n = (1 + \frac{1}{n})^{n+1} = (1 + \frac{1}{n})x_n$. Since $1 + \frac{1}{n}$ converges to 1, then using the algebraic limit theorem we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (1 + \frac{1}{n})x_n = \lim_{n\to\infty} (1 + \frac{1}{n}) \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

■

Remark (1). From the proof of MCT we further know $e = \inf(\{y_n : n \in \mathbb{N}\}) = \sup(\{x_n : n \in \mathbb{N}\})$. This means that for all $n \in \mathbb{N}$

$$(1 + \frac{1}{n})^n \leq e \leq (1 + \frac{1}{n})^{n+1}$$

Furthermore, because x_n and y_n are *strictly* increasing/decreasing, there does not exist $n_0 \in \mathbb{N}$ such that $x_{n_0} = e$ or $y_{n_0} = e$. Otherwise, we would have $x_{n_0+1} > x_{n_0} = e$ or $y_{n_0+1} < y_{n_0} = e$, both of which are contradictions. This lets us conclude that for all $n \in \mathbb{N}$

$$(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}$$

Which is useful to state as it allows for estimating the value of e.

Remark (2). We can also introduce e as the limit $\lim_{n\to\infty} S_n$ where the sequence (S_n) is defined as

$$S_n = \sum_{k=0}^{n} \frac{1}{k!}$$

Particularly, S_n is a Cauchy sequence. To see this, let $\varepsilon > 0$ and set $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Let $m, n \geq N$ and assume $n > m$ without loss of generality (the case $n = m$ is trivial). We then have

$$\begin{aligned} |S_n - S_m| &= \Big|\sum_{k=0}^{n} \frac{1}{k!} - \sum_{k=0}^{m} \frac{1}{k!}\Big| \\ &= \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \cdots + \frac{1}{n!} \\ &\leq \frac{1}{(m+1)m} + \frac{1}{(m+2)(m+1)} + \cdots + \frac{1}{n(n-1)} \\ &= (\frac{1}{m} - \frac{1}{m+1}) + (\frac{1}{m+1} - \frac{1}{m+2}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \leq \frac{1}{N} < \varepsilon \end{aligned}$$

It is noted that the irrationality of e can be shown using S_n and the formula $\lim_{n\to\infty} S_n = e$.

3.6 Returning to the Limit Superior and Limit Inferior

Recall a few basic facts we established earlier:

(1) $\liminf x_n$ and $\limsup x_n$ exist for any bounded sequence (x_n) (Prop. 3.2, 3.3).

(2) $\liminf x_n \leq \limsup x_n$ (Prop. 3.4)

(3) A bounded sequence (x_n) converges to $x \iff \liminf x_n = \limsup x_n = x$ (Thm. 3.7).

Furthermore, $\liminf x_n$ is increasing and bounded above, while $\limsup x_n$ is decreasing and bounded below.

Proposition 3.6. For any bounded sequence (x_n), $\limsup(-x_n) = -\liminf(x_n)$.

Proof. Let $A_n = \{x_k : k \geq n\}$. Recall from homework that we then have $\inf(A_n) = -\sup(-A_n)$. Now let $t_n = \inf(A_n)$ and let $s_n = \sup(-A_n)$, meaning that $t_n = -s_n$. So then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} t_n = -\lim_{n\to\infty} s_n = -\limsup_{n\to\infty} x_n$$

■

Important note
Algebraic operations that involve lim inf and lim sup require care. Take for example a bounded sequence (x_n) and $\alpha \geq 0$. Then

$$\limsup_{n\to\infty}(\alpha x_n) = \alpha \limsup_{n\to\infty} x_n$$
$$\liminf_{n\to\infty}(\alpha x_n) = \alpha \liminf_{n\to\infty} x_n$$

Both of these statements follow from the fact that $\sup(\alpha A) = \alpha \sup(A)$ and $\inf(\alpha A) = \alpha \inf(A)$, as long as $A \subseteq \mathbb{R}$ is bounded. **<u>However</u>**, if say $\alpha < 0$, then

$$\limsup_{n\to\infty}(\alpha x_n) = \alpha \liminf_{n\to\infty} x_n$$
$$\liminf_{n\to\infty}(\alpha x_n) = \alpha \limsup_{n\to\infty} x_n$$

This is a consequence of Proposition 3.6, since

$$\begin{aligned}\limsup_{n\to\infty}(\alpha x_n) &= \limsup_{n\to\infty} -|\alpha| x_n \\ &= -\liminf_{n\to\infty} |\alpha| x_n \\ &= -|\alpha| \liminf_{n\to\infty} x_n \\ &= \alpha \liminf_{n\to\infty} x_n\end{aligned}$$

An analogous proof can be formulated to show $\liminf_{n\to\infty}(\alpha x_n) = \alpha \limsup_{n\to\infty} x_n$.

Proposition 3.7. Let (x_n) and (y_n) be bounded sequences. Then

(1) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$

(2) $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$

Proof.

(1) Let $A_n = \{x_k + y_k : k \geq n\}$, $B_n = \{x_k : k \geq n\}$, and $C_n = \{y_k : k \geq n\}$. Then

$$B_n + C_n = \{x_k + y_j : k \geq n, j \geq n\}$$

We know that $\sup(B_n + C_n)$ exists since all elements in $B_n + C_n$ are less than some fixed constant. Furthermore, since $A_n \subseteq B_n + C_n$, then $\sup(A_n) \leq \sup(B_n + C_n)$. Also, one property we established in homework 2 is that

$$\sup(B_n + C_n) = \sup(B_n) + \sup(C_n)$$

Now, let $t_n = \sup(A_n), r_n = \sup(B_n)$, and $s_n = \sup(C_n)$. We have $t_n \leq r_n + s_n$ because $A_n \subseteq B_n + C_n$. Since $(t_n), (r_n)$, and (s_n) are all decreasing and bounded below, their limits exist. The order limit theorem gives

$$\begin{aligned} \lim_{n\to\infty} t_n &\leq \lim_{n\to\infty} (r_n + s_n) \\ &= \lim_{n\to\infty} r_n + \lim_{n\to\infty} s_n \end{aligned}$$

(2) This can be derived from (1), since

$$\begin{aligned} -\liminf(x_n + y_n) &= \limsup((-x_n) + (-y_n)) \\ &\leq \limsup(-x_n) + \limsup(-y_n) \\ &= -\liminf(x_n) - \liminf(y_n) \end{aligned}$$

Which gives

$$\liminf(x_n + y_n) \geq \liminf(x_n) + \liminf(y_n)$$

■

Proposition 3.8. Let (x_n) be a bounded sequence. Then

(1) $\limsup(x_n) = \inf \Big\{t : \{n : x_n > t\} \text{ is either empty or finite}\Big\}$

(2) $\liminf(x_n) = \sup \Big\{t : \{n : x_n < t\} \text{ is either empty or finite}\Big\}$

Proof. We first prove (1). Let $A = \Big\{t : \{n : x_n > t\} \text{ is empty or finite}\Big\}$. Since (x_n) is bounded, then there exists $M', M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$

$$M' \leq x_n \leq M$$

It follows that $\{n : x_n > M\}$ is empty, so $M \in A$, which shows that A is non-empty. Furthermore, we have that $t \geq M'$ for any $t \in A$. To see this, suppose $t < M'$ for some $t \in A$, and set $\varepsilon = M' - t > 0$. But then we have the contradiction that the set $\{n : x_n > t = M' - \varepsilon\}$ is infinite since $x_n \geq M' > M' - \varepsilon$ for all $n \in \mathbb{N}$. So A is bounded below by M'. Therefore $\inf(A)$ exists.

Next, we show that $\limsup(x_n) = \inf(A)$. First we show $\limsup(x_n) \leq \inf(A)$. Take any eventual upper bound $t \in A$. Then $\{n : x_n > t\}$ is empty or finite. If empty, then let $n_t := 1$. Otherwise if finite, then let $n_t := \max(\{n : x_n > t\})$. It follows that $x_n \leq t$ for all $n > n_t$, meaning that t is an upper bound of $\{x_n : n > n_t\}$. Now define a sequence (Y_m) such that $Y_m := \sup(\{x_k : k \geq m\})$. We can define Y_m in this way since $\{x_k : k \geq m\}$ is bounded above and non-empty for any choice of m. Then for all $m > n_t$ we have $Y_m \leq t$ since Y_m is the least upper bound. Therefore Y_m is a lower bound of A, as long as $m > n_t$.

We know Y_m is decreasing for all $m \geq 1$ since $\{x_k : k \geq m\} \supseteq \{x_k : k \geq m+1\}$. Furthermore $\{Y_m : m \geq 1\}$ is bounded below by any lower bound l of (x_n), since

$$Y_m = \sup(\{x_k : k \geq m\}) \geq x_m \geq \inf(\{x_n : k \geq m\}) \geq l$$

As such $\lim_{n\to\infty} Y_m$ exists by MCT. Particularly, (Y_m) converges to $\inf\{Y_m : m \geq 1\}$. Fixing any $m_0 > n_t$, we have

$$\begin{aligned}\limsup_{n\to\infty} x_n &= \lim_{m\to\infty} Y_m \\ &= \inf\{Y_m : m \geq 1\} \\ &\leq Y_{m_0} \\ &\leq t\end{aligned}$$

Therefore $\limsup_{n\to\infty} x_n \leq \inf(A)$. Now we need to show that $\limsup_{n\to\infty} x_n \geq \inf(A)$. Let $\alpha = \inf(A)$ and set $\varepsilon > 0$. Then $\alpha - \varepsilon \notin A$, so $\alpha - \varepsilon$ is not an eventual upper bound.

Consequently, it follows that for all $m \in \mathbb{N}$ we have $Y_m = \sup(\{x_n : n \geq m\}) > \alpha - \varepsilon$. To see this, assume there exists $m \in \mathbb{N}$ such that $\sup\{x_n : n \geq m\} \leq \alpha - \varepsilon$. But since $\{n : x_n > \alpha - \varepsilon\}$ is infinite, there exists $n \geq m$ such that $x_n > \alpha - \varepsilon$. This leads to the absurdity

$$x_n > \alpha - \varepsilon \geq \sup(\{x_n : n \geq m\})$$

Now, using the order limit theorem

$$Y_m > \alpha - \varepsilon \implies \lim_{n\to\infty} Y_m \geq \alpha - \varepsilon$$

The final step is to notice that since ε was *chosen arbitrarily*, then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} Y_m \geq \alpha - \varepsilon \implies \limsup_{n\to\infty} x_n = \lim_{n\to\infty} Y_m \geq \alpha$$

Therefore $\limsup_{n\to\infty} x_n = \inf(A)$, completing the proof.

∎

Remark (1). **Important:** To see why $\lim Y_m \geq \alpha - \varepsilon \implies \lim Y_m \geq \alpha$ when $\varepsilon > 0$ is arbitrary, assume for the sake of contradiction that $\lim Y_m < \alpha$. Then there exists $\varepsilon > 0$ such that $\lim Y_m + \varepsilon < \alpha$, for example take $\varepsilon := \frac{1}{2}(\alpha - \lim Y_m)$. But then $\lim Y_m < \alpha - \varepsilon$, which contradicts the hypothesis $\lim Y_m \geq \alpha - \varepsilon$.

Remark (2). (2) may be proved by repeating the argument of (1), or derived from (1) using the fact that $\liminf x_n = -\limsup(-x_n)$.

Remark (3). We used the idea that $\{n : x_n > t\}$ is empty or finite if and only if t is an eventual upper bound – namely there exists $n_t \in \mathbb{N}$ such that $x_n \leq t$ for all $n \geq n_t$. We could restate (1) as

$$\limsup x_n = \inf \left\{t : \text{t is an eventual upper bound for } (x_n)\right\}$$

Remark (4). Try drawing eventual lower bounds for some simple bounded sequence. They will approach the largest limit point $\limsup x_n$, but will never cross it.

3.7 Limit points

Definition 3.10. Let (x_n) be a sequence. Then $x \in \mathbb{R}$ is a limit or accumulation point of (x_n) if there exists a subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} x_{n_k} = x$$

We denote $\mathcal{L}$ as the set of all limit points of (x_n).

Remark. In general, $\mathcal{L}$ could be the empty set. However, if (x_n) happens to be bounded, then there exists a convergent subsequence by BW so $\mathcal{L} \neq \emptyset$.

Proposition 3.9. Let (x_n) be a sequence. Then

$$x \in \mathcal{L} \iff \forall \varepsilon > 0 \ \ \{n : |x_n - x| < \varepsilon\} \text{ is infinite}$$

Proof. Let $x \in \mathcal{L}$ so that there exists a subsequence (x_{n_k}) such that $\lim x_{n_k} = x$. Let $\varepsilon > 0$. Then there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have $|x_k - x| < \varepsilon$. It follows that

$$\{n_k : k \geq K\} \subseteq \{n : |x_n - x| < \varepsilon\}$$

But since $\{n_k : k \geq K\}$ is infinite, then $\{n : |x_n - x| < \varepsilon\}$ is also infinite.

Now suppose that for all $\varepsilon > 0$ the set $\{n : |x_n - x| < \varepsilon\}$ is infinite. Let $\varepsilon = 1$. By the well-ordering principle, we can define $n_1 := \min(\{n : |x_n - x| < 1\})$. Now for all $k \geq 1$, recursively define

$$n_{k+1} := \min(\{n : n > n_k \text{ and } |x_n - x| < \frac{1}{k+1}\})$$

Observe that the set $\{n : n > n_k \text{ and } |x_n - x| < \frac{1}{k+1}\}$ is infinite, since $\{n : n > n_k\}$ is infinite and $\{n : |x_n - x| < \frac{1}{k+1}\}$ is infinite. So we are always dealing with non-empty subsets of $\mathbb{N}$, hence we can always obtain the next term in the sequence with the well-ordering principle.

By construction, we have defined a strictly increasing sequence $(n_k)_{k\geq 1}$ such that $|x_{n_k} - x| < \frac{1}{k}$ for all $k \in \mathbb{N}$. Since $0 \leq |x_{n_k} - x| < \frac{1}{k}$, then by the squeeze theorem $\lim_{k\to\infty} |x_{n_k} - x| = 0$. This gives $\lim_{k\to\infty} x_{n_k} = x$.

∎

Theorem 3.12. Let (x_n) be a bounded sequence. Then

(1) $\limsup x_n = \sup(\mathcal{L})$

(2) $\liminf x_n = \inf(\mathcal{L})$

Proof. We will prove (1), and (2) is left as an exercise. First, we show that $\limsup x_n \geq \sup(\mathcal{L})$. Let $x \in \mathcal{L}$ and let (x_{n_k}) be a subsequence such that $\lim_{k\to\infty} x_{n_k} = x$. Then let $y_n = \sup(\{x_m : m \geq n\})$. We have that $y_n \geq x_n$ and $\limsup x_n = \lim y_n$ by definition. Particularly, this gives that $y_{n_k} \geq x_{n_k}$ for all $k \in \mathbb{N}$, and that $\limsup x_n = \lim_{k\to\infty} y_{n_k}$. The latter fact follows because y_n is convergent so any of its subsequences converge to the same limit. Therefore by the order limit theorem

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} y_{n_k} \geq \lim_{k\to\infty} x_{n_k} = x$$

Since $x \in \mathcal{L}$ was chosen arbitrarily, then $\limsup x_n$ is an upper bound of $\mathcal{L}$, and so $\limsup x_n \geq \sup(\mathcal{L})$.

Next, we show that $\limsup x_n \leq \sup(\mathcal{L})$. We will prove a stronger statement, namely that $\limsup x_n \in \mathcal{L}$. Set $n_1 = 1$ and suppose n_k is defined. We need to define n_{k+1}. We have that $y_{n_k+1} = \sup(\{x_m : m \geq n_k + 1\})$. Then $y_{n_k+1} - \frac{1}{k+1}$ is not an upper bound for $\{x_m : m \geq n_k + 1\}$. So there exists some choice of n_{k+1}, where $n_{k+1} \geq n_k + 1 > n_k$ such that

$$y_{n_k+1} - \frac{1}{k+1} < x_{n_{k+1}} \leq y_{n_{k+1}}$$

We have hence defined a strictly increasing sequence $(n_k)_{k\geq 1}$ such that for all $k > 2$

$$y_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq y_{n_k}$$

Since all subsequences of (y_n) converge to the same limit, then

$$\lim_{k\to\infty} (y_{n_{k-1}+1} - \frac{1}{k}) = \lim_{k\to\infty} y_{n_k} = \limsup_{n\to\infty} x_n$$

Then by the squeeze theorem

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$$

So $\limsup x_n \in \mathcal{L}$, which implies $\limsup x_n \leq \sup(\mathcal{L})$ as needed.

■

Remark. The fact that the proof gives $\limsup x_n \in \mathcal{L}$ and $\liminf x_n \in \mathcal{L}$ is useful for showing the following corollary. Also, notice that since $\limsup x_n \in \mathcal{L}$ and $\limsup x_n = \sup(\mathcal{L})$, then we can say $\limsup x_n$ is the largest limit point of a bounded sequence. Analogously, $\liminf x_n$ is the smallest limit point.

Remark. Focussing on y_{n_k+1} was vital to establish the inequality $n_{k+1} \geq n_k + 1 > n_k$.

Corollary 3.12.1. Let (x_n) be a bounded sequence and set

$$\alpha = \liminf_{n\to\infty} x_n$$
$$\beta = \limsup_{n\to\infty} x_n$$

Then $\alpha, \beta \in \mathcal{L}$ and for any $x \in \mathcal{L}$

$$\alpha \leq x \leq \beta$$

Proof. The proof of Theorem 3.12 showed that $\alpha \in \mathcal{L}$ and $\beta \in \mathcal{L}$. By 3.12 we know that $\alpha = \inf(\mathcal{L})$ and $\beta = \sup(\mathcal{L})$, so then $\alpha \leq x \leq \beta$ for any $x \in \mathcal{L}$.

■

3.8 Properly divergent sequences

Definition 3.11. Let (x_n) be a sequence. We say (x_n) properly diverges to ∞ if

$$\forall \alpha \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geq N \quad x_n \geq \alpha$$

We abbreviate this notion as

$$\lim_{n\to\infty} x_n = \infty$$

Definition 3.12. Let (x_n) be a sequence. We say (x_n) properly diverges to $-\infty$ if

$$\forall \alpha \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geq N \quad x_n \leq \alpha$$

We abbreviate this notion as

$$\lim_{n\to\infty} x_n = -\infty$$

Proposition 3.10. Let (x_n) be an increasing sequence. Then $\lim x_n = \infty \iff (x_n)$ is not bounded from above.

Proposition 3.11. Let (x_n) be a decreasing sequence. Then $\lim x_n = -\infty \iff (x_n)$ is not bounded from below.

Proposition 3.12. Let (x_n) and (y_n) be sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then

(1) $\lim x_n = \infty \implies \lim y_n = \infty$

(2) $\lim y_n = -\infty \implies \lim x_n = -\infty$

Proposition 3.13. Let (x_n) be a sequence and let $c > 0$. Then $\lim x_n = \infty \iff \lim c x_n = \infty$.

Proposition 3.14. Let (x_n) and (y_n) be positive sequences. Suppose $\lim \frac{x_n}{y_n} = L$ for some $L > 0$. Then

$$\lim x_n = \infty \iff \lim y_n = \infty$$

Proposition 3.15. Let (x_n) and (y_n) be two sequences such that (x_n) is properly diverging and (y_n) is bounded. Then $(x_n + y_n)$ is also properly diverging.

4 Functional Limits and Continuity

4.1 Cluster points

Definition 4.1 (Cluster point of a set)**.** Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point (or limit point) of A if for all $\varepsilon > 0$ there is $x \in A$, $x \neq c$, such that

$$0 < |x - c| < \varepsilon$$

Remark. This definition requires that $c \in A$. We will define the notion of a limit only at cluster points.

Proposition 4.1 (Sequential characterization of cluster points)**.** Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. The following are equivalent.

(1) c is a cluster point of A.

(2) There exists a sequence (x_n) in A such that we have $x_n \in A$, $x_n \neq c$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} x_n = c$$

Proof. Let c be a cluster point in A and take $\varepsilon = \frac{1}{n}$. Then there exists $x_n \in A$ with $x_n \neq c$ such that $0 < |x_n - c| < \varepsilon = \frac{1}{n}$. This defines a sequence with $(x_n) \in A$ and $x_n \neq c$ for all $n \in \mathbb{N}$, such that

$$|x_n - c| < \frac{1}{n}$$

Hence $\lim |x_n - c| = 0$ so $\lim x_n = c$.

Now let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - c| < \varepsilon$. Take $x = x_n$. Then $|x - c| < \varepsilon$ and by hypothesis $x \in A$, $x \neq c$. Hence c is a cluster point of A.

■

4.2 Functional limits

Definition 4.2 (Functional limit)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Let c be a cluster point of A. Then we say the limit of f at c is L, abbreviated as

$$\lim_{x \to c} f(x) = L$$

If for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ satisfying $0 < |x - c| < \delta$ we have $|f(x) - L| < \varepsilon$.

Remark. The cluster point c itself may or may not be an element of A. For example, $\lim_{x \to 0} \frac{\sin x}{x} = 1$, but $0 \notin A$. However, it is crucial that c be a cluster point of A. Intuitively, we cannot get arbitrarily close to c if c is not a cluster point.

Theorem 4.1 (Sequential characterization of functional limits)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.

(1) $\lim_{x \to c} f(x) = L$

(2) For any sequence (x_n) in A, with $x_n \neq c$ such that $\lim x_n = c$, we have that the sequence $\big(f(x_n)\big)$ converges to L by
$$\lim_{n \to \infty} f(x_n) = L$$

Proof. First we show that (1) $\implies$ (2). Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in A$ satisfying $0 < |x - c| < \delta$ we have $|f(x) - L| < \varepsilon$. Since c is a cluster point of A, there exists (x_n) in A with $x_n \neq c$ such that $\lim x_n = c$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|x_n - c| < \delta$$

Since $x_n \in A$ for all $n \in \mathbb{N}$, then we have

$$|f(x_n) - c| < \varepsilon$$

Thus $\lim f(x_n) = L$.

Now we show (2) $\implies$ (1) by contradiction. Assume (2) and assume (1) is false. Then negating the functional limit definition gives

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in A)(0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon)$$

Fix this ε. Let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$. Then we have a sequence (x_n) in A such that $x_n \neq c$ and $0 < |x_n - c| < \frac{1}{n}$. Thus $\lim x_n = c$. But also $|f(x_n) - L| \geq \varepsilon$ for all $n \in \mathbb{N}$, so $f(x_n)$ does not converge to L. This contradicts (2).

■

Proposition 4.2. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of A. Then

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} = M \implies L = M$$

In other words, if the functional limit exists, then it is unique.

Proof. We first prove this using the sequential characterization of functional limits. Since c is a cluster point, let (x_n) be a sequence in A such that $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. Then we have

$$\lim_{x\to c} f(x) = L \implies \lim_{n\to\infty} f(x_n) = L$$
$$\lim_{x\to c} f(x) = M \implies \lim_{n\to\infty} f(x_n) = M$$

Limits of sequences are unique, so $L = M$.

Now we can also prove this using the $\varepsilon - \delta$ definition. Assume $L \neq M$ and let $\varepsilon = \frac{|L-M|}{2}$. Then

$$\lim_{x\to c} f(x) = L \implies \exists \delta_1 > 0, \forall x \in A, x \neq c, 0 < |x-c| < \delta_1 \implies |f(x) - L| < \varepsilon$$
$$\lim_{x\to c} f(x) = M \implies \exists \delta_2 > 0, \forall x \in A, x \neq c, 0 < |x-c| < \delta_2 \implies |f(x) - M| < \varepsilon$$

Let $\delta = \min(\{\delta_1, \delta_2\})$. Take $x \in A$ such that $x \neq c$. Then

$$|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M| < 2\varepsilon = |L - M|$$

Which is a contradiction.

■

Remark. We used an identical idea to prove uniqueness for limits of convergent sequences.

Theorem 4.2 (Algebraic properties of functional limits)**.** Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and let c be a cluster point of A. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

(1) $\lim_{x\to c} k \cdot f(x) = k \cdot L \quad \forall k \in \mathbb{R}$

(2) $\lim_{x\to c}(f(x) + g(x)) = L + M$

(3) $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$

(4) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$ and $g(x) \neq 0 \; \forall x \in A$

Proof. We will prove (3) for now, again in two ways. First, we use the sequential characterization of functional limits. Since c is a cluster point, there exists (x_n) in A such that $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. Then we have that

$$\lim_{n\to\infty} f(x_n) = L$$
$$\lim_{n\to\infty} g(x_n) = M$$

By the product rule for converging sequences

$$\lim_{n\to\infty} f(x_n)g(x_n) = L \cdot M$$

Now, invoking the other direction of the sequential characterization

$$\lim_{x\to c} f(x)g(x) = L \cdot M$$

Let us prove this again using the $\varepsilon - \delta$ definition. Let $\varepsilon = 1$. Then there is $\delta_1 > 0$ such that for all $x \in A$, $x \neq c$, satisfying $0 < |x - c| < \delta_1$, we have $|f(x) - L| < 1$. For such x we have

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L|$$

Now let $\varepsilon > 0$. We can find $\delta_2 > 0$ such that for all $x \in A$, $x \neq c$, satisfying $0 < |x - c| < \delta_2$, we have

$$|f(x) - L| < \frac{\varepsilon}{2(|M| + 1)}$$
$$|g(x) - M| < \frac{\varepsilon}{2(|L| + 1)}$$

Take now $\delta = \min(\{\delta_1, \delta_2\})$ and let $x \in A$, $x \neq c$, such that $0 < |x - c| < \delta$. Then

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\
&= |f(x)||g(x) - M| + |M||f(x) - L| \\
&< (1 + |L|)|g(x) - M| + |M||f(x) - L| \\
&< (1 + |L|) \cdot \frac{\varepsilon}{2(1 + |L|)} + |M| \cdot \frac{\varepsilon}{2(1 + |M|)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

■

Remark. The $\varepsilon - \delta$ proof of the product rule bears many similarities to how we proved the product rule for sequences. Here, we controlled $|f(x)|$ by using $\lim_{x\to c} f(x) = L$. For sequences, we controlled $|x_n|$ by using the fact that (x_n) is bounded.

Theorem 4.3 (Squeeze theorem for functional limits)**.** Let $A \subseteq \mathbb{R}$ and $f, g, h : A \to \mathbb{R}$. Let c be a cluster point of A. Suppose that for all $x \in A$ we have

$$f(x) \leq g(x) \leq h(x)$$

With $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$. Then $\lim_{x\to c} h(x) = L$.

Proof. As usual, first we give the proof by sequential characterization. Let (x_n) be a sequence in A such that $x_n \neq c$ and $\lim_{n\to\infty} = c$. Remember such a sequence exists by the sequential characterization of cluster points. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = L$$
$$\lim_{n\to\infty} h(x_n) = \lim_{x\to c} h(x) = L$$

Now for all $n \in \mathbb{N}$ we have $f(x_n) \leq g(x_n) \leq h(x_n)$. By the squeeze theorem proved for sequences we have

$$\lim_{n\to\infty} g(x_n) = L$$

So for any (x_n) in A, $x_n \neq c$, and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} g(x_n) = L$. Invoking the other direction of the sequential characterization, we get

$$\lim_{x\to c} g(x) = L$$

Now for the $\varepsilon - \delta$ proof. Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists $\delta_1 > 0$ such that for all $x \in A$, $x \neq c$, satisfying $0 < |x - c| < \delta_1$, we have

$$|f(x) - L| < \varepsilon$$

Since $\lim_{x\to c} h(x) = L$, there exists $\delta_2 > 0$ such that for all $x \in A$, $x \neq c$, satisfying $0 < |x-c| < \delta_2$, we have

$$|h(x) - L| < \varepsilon$$

Let now $\delta = \min(\{\delta_1, \delta_2\})$ and let $x \in A$, $x \neq c$, such that $0 < |x - c| < \delta$. Then

$$-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$$

So $|g(x) - L| < \varepsilon$ and therefore $\lim_{x\to c} g(x) = L$.

■

4.3 Divergence criteria

Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. The sequential characterization of functional limits can be used to prove that a limit $\lim_{x\to c} f(x)$ does not exist.

Proposition 4.3. Suppose there exists (x_n) in A, $x_n \neq c$, such that $\lim_{n\to\infty} f(x_n)$ is not convergent. Then $\lim_{x\to c} f(x)$ does not exist.

Proof. If $\lim_{x\to c} f(x)$ did exist, then $\lim_{n\to\infty} f(x_n)$ would converge by the sequential characterization.

■

Proposition 4.4. Suppose there exists (x_n), (y_n) in A with $x_n \neq c, y_n \neq c$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$$

If we have

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

Then $\lim_{x\to c} f(x)$ does not exist.

Proof. If $\lim_{x\to c} f(x)$ did exist, then (x_n) and (y_n) would converge, and converge to the same limit.

■

Example 4.1. Let $A = (0, \infty)$, $f(x) = \sin(\frac{1}{x})$, and $c = 0$. Then $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Proof. The first divergence criterion can be used. Let $x_n = \frac{1}{(2n+1)\frac{\pi}{2}}$, which satisfies $x_n \neq 0$ and $\lim_{n\to\infty} x_n = 0$. We have $f(x_n) = \sin(\frac{1}{x_n}) = \sin((2n+1)\frac{\pi}{2}) = (-1)^n$. The sequence $(f(x_n))$ does not converge, so $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

■

Proof. The second divergence criterion can be used too. Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi+\frac{\pi}{2}}$, which satisfies $x_n \neq 0$, $y_n \neq 0$, and $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} y_n = 0$. Now

$$f(x_n) = \sin(\frac{1}{x_n}) = \sin(2n\pi) = 0 \quad \forall n \in \mathbb{N}$$
$$f(y_n) = \sin(\frac{1}{y_n}) = \sin(2n\pi + \frac{\pi}{2}) = 1 \quad \forall n \in \mathbb{N}$$

Since $0 \neq 1$, then $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

■

4.4 Extensions of the functional limit concept

Definition 4.3 (Right and left limits)**.**

(1) Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and suppose c is a cluster point of $A \cap (c, \infty) = \{x \in A : x > c\}$. Suppose that $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in A$ satisfying $0 < x - c < \delta$ we have

$$|f(x) - L| < \varepsilon$$

Then we say $L \in \mathbb{R}$ is the right limit of f at c

$$\lim_{x\to c+} f(x) = L$$

(2) Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and suppose c is a cluster point of $A \cap (-\infty, c) = \{x \in A : x < c\}$. Suppose that $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in A$ satisfying $-\delta < x - c < 0$ we have

$$|f(x) - L| < \varepsilon$$

Then we say $L \in \mathbb{R}$ is the left limit of f at c

$$\lim_{x\to c-} f(x) = L$$

Remark. Sometimes, the left/right hand limits reduce to the usual limits. If $A = (a, b)$ (or $[a, b)$, $(a, b]$, $[a, b]$), then a cannot possibly be a cluster point of $A \cap (-\infty, a)$, since it is empty. So the left hand limit at a does not exist. But the right hand limit and usual functional limit can both exist. We would have

$$\lim_{x\to a+} f(x) = \lim_{x\to a} f(x)$$

Proposition 4.5. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then the following are equivalent.

(1) $\lim_{x \to c} f(x) = L$

(2) $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = L$

Proof. First we show (1) $\implies$ (2). Let $\varepsilon > 0$ and let $\delta > 0$ such that for all $x \in A$ satisfying $0 < |x - c| < \delta$ we have $|f(x) - L| < \varepsilon$. But then $|f(x) - L| < \varepsilon$ whenever $0 < x - c < \delta$ and $|f(x) - L| < \varepsilon$ whenever $-\delta < x - c < 0$. So $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = L$.

Next we show (2) $\implies$ (1). Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < x - c < \delta_1$, then $|f(x) - L| < \varepsilon$. There exists $\delta_2 > 0$ such that if $-\delta_2 < x - c < 0$, then $|f(x) - L| < \varepsilon$. Let $\delta = \min(\{\delta_1, \delta_2\})$. Then if $0 < |x - c| < \delta$, we have that either

$$0 < x - c < \delta \leq \delta_1$$

Or

$$-\delta_2 \leq \delta < x - c < 0$$

In either case $|f(x) - L| < \varepsilon$. So $\lim_{x \to c} f(x) = L$.

■

Theorem 4.4 (Sequential characterization of the right hand limit)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of the set $A \cap (c, \infty)$. Then the following are equivalent.

(1) $\lim_{x \to c+} f(x) = L$

(2) For any (x_n) in A such that $x_n > c$ and $\lim_{n \to \infty} x_n = c$, we have $\lim_{n \to \infty} f(x_n) = L$.

Theorem 4.5 (Sequential characterization of the left hand limit)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of the set $A \cap (-\infty, c)$. Then the following are equivalent.

(1) $\lim_{x \to c-} f(x) = L$

(2) For any (x_n) in A such that $x_n < c$ and $\lim_{n \to \infty} x_n = c$, we have $\lim_{n \to \infty} f(x_n) = L$.

Definition 4.4 (Infinite limits)**.** Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and c a cluster point of A. Then

(1) We say $\lim_{x \to c} f(x) = \infty$ if for any $M \in \mathbb{R}$ there exists $\delta > 0$ such that for all $x \in A$, $x \neq c$ satisfying $0 < |x - c| < \delta$ we have $f(x) \geq M$.

(2) We say $\lim_{x \to c} f(x) = -\infty$ if for any $M \in \mathbb{R}$ there exists $\delta > 0$ such that for all $x \in A$, $x \neq c$ satisfying $0 < |x - c| < \delta$ we have $f(x) \leq M$.

Proposition 4.6 (Order properties for infinite limits)**.** Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and suppose $f(x) \leq g(x)$ for all $x \in A$. Let c be a cluster point of A. Then

(1) If $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} g(x) = \infty$.

(2) If $\lim_{x \to c} g(x) = -\infty$, then $\lim_{x \to c} f(x) = -\infty$.

Definition 4.5 (Limit at ∞)**.** Let $A \subseteq \mathbb{R}$ and suppose $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Let $f : A \to \mathbb{R}$. We say $L \in \mathbb{R}$ is the limit of f at ∞, if for all $\varepsilon > 0$ there exists $K > a$ such that for all $x \geq K$ we have

$$|f(x) - L| < \varepsilon$$

Definition 4.6 (Limit at $-\infty$)**.** Let $A \subseteq \mathbb{R}$ and suppose $(-\infty, a) \subseteq A$ for some $a \in \mathbb{R}$. Let $f : A \to \mathbb{R}$. We say $L \in \mathbb{R}$ is the limit of f at $-\infty$, if for all $\varepsilon > 0$ there exists $K > a$ such that for all $x \leq K$ we have

$$|f(x) - L| < \varepsilon$$

4.5 Continuity

Definition 4.7. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$ not necessarily a cluster point. We say f is continuous at c if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$. If further f is continuous at all $c \in A$, then we say f is continuous. Otherwise, f is discontinuous.

Remark. Now we have finally characterized the notion that a continuous function is one where we do not need to lift our pen to draw its graph. The definition encodes the idea that as we approach c, we also approach $f(c)$, and no abrupt changes ever occur.

Definition 4.8 (Isolated point)**.** If $c \in A$ is not a cluster point of A, then we say c is an isolated point of A.

Proposition 4.7. Let $A \subseteq \mathbb{R}$. Then $c \in A$ is an isolated point of A if and only if there exists $\delta > 0$ such that $\{x \in A : |x - c| < \delta\} = \{c\}$.

Proof. Follows by negating the definition of cluster point. ∎

Proposition 4.8. If $c \in A$ is an isolated point of A, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Let c be an isolated point of A. Then there is $\delta > 0$ such that $\{x \in A : |x - c| < \delta\} = \{c\}$. Now let $f : A \to \mathbb{R}$ and let $\varepsilon > 0$. It follows that if $x \in A$ and $|x - c| < \delta$, then $x = c$. As such $|f(c) - f(c)| = 0 < \varepsilon$.

■

Proposition 4.9. Let $c \in A$ be a cluster point of A. Then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Proof. Suppose f is continuous at c. Then for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$. Since c is a cluster point, there is $x \neq c$ satisfying $0 < |x - c| < \delta$. Hence $\lim_{x\to c} f(x) = f(c)$.

Now suppose $\lim_{x\to c} f(x) = f(c)$. Then for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in A$, $x \neq c$, $0 < |x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Furthermore, if $x = c$, then $|f(c) - f(c)| = 0 < \varepsilon$. So whenever $x \in A$ satisfies $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. Hence f is continuous at c.

■

Remark. If c is not a cluster point of A, then we cannot define the functional limit so this theorem does not apply. Notice we also need c to be in the domain of f.

Theorem 4.6 (Sequential characterization of continuity)**.** Let $f : A \to \mathbb{R}$ and $c \in A$. Then the following are equivalent.

(1) f is continuous at c.

(2) For any (x_n) in A such that $\lim_{n\to\infty} x_n = c$ we have that

$$\lim_{n\to\infty} f(x_n) = f(c)$$

Proof. We first show (1) $\implies$ (2). Let f be continuous at c and let (x_n) be a sequence in A with $\lim_{n\to\infty} x_n = c$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. Since $\lim_{n\to\infty} x_n = c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - c| < \delta$. But then $|f(x_n) - f(c)| < \varepsilon$ for all $n \geq N$. So $\lim_{n\to\infty} f(x_n) = f(c)$.

Next we show (2) $\implies$ (1). Assume the hypothesis of (2) and further assume that f is not continuous at c, for the sake of contradiction. This implies

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in A, |x - c| < \delta \text{ and } |f(x) - f(c)| \geq \varepsilon$$

Fix this ε. By the above statement, we can safely define (x_n) in A such that $|x_n - c| < \frac{1}{n}$ for all $n \in \mathbb{N}$, by setting $\delta = \frac{1}{n}$. Then $|f(x_n) - f(c)| \geq \varepsilon$ for all $n \in \mathbb{N}$. But we also have $\lim_{n\to\infty} x_n = c$, and by (2) the sequence $f(x_n)$ converges to $f(c)$. So there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $|f(x_n) - f(c) < \varepsilon$. Therefore a contradiction arises.

■

Remark. These sequential characterization proofs all follow an essentially similar template. The implication (1) $\implies$ (2) follows directly, while (2) $\implies$ (1) requires contradiction.

Remark. We can also deduce this theorem from the sequential characterization of functional limits. In the first case, if c is a cluster point of A, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$. But $\lim_{x\to c} f(x) = f(c)$ if and only if there is (x_n) in A, $x_n \neq c$, $\lim_{n\to\infty} x_n = c$, such that $\lim_{n\to\infty} f(x_n) = f(c)$. In the second case, if c is an isolated point of A, then f is immediately continuous at c. Any (x_n) in A satisfying $\lim_{n\to\infty} x_n = c$ must have $x_n = c$ for large enough n.

Proposition 4.10 (Algebraic properties of continuity)**.** Let $f, g : A \to \mathbb{R}$ and let $c \in A$. Suppose f and g are continuous at c. Then

(1) kf is continuous at c for any $k \in \mathbb{R}$.

(2) $f + g$ is continuous at c.

(3) $h = f \cdot g$ is continuous at c.

(4) $h = \frac{f}{g}$ is continuous at c, if $g(x) \neq 0$ for all $x \in A$.

Proof. We prove (3) by the sequential characterization of continuity. Let (x_n) be a sequence in A such that $\lim_{n\to\infty} x_n = c$. Since f and g are continuous at c, we have $\lim_{n\to\infty} f(x_n) = f(c)$ and $\lim_{n\to\infty} g(x_n) = g(c)$. Then by the algebraic limit theorem

$$\lim_{n\to\infty} f(x_n) \cdot g(x_n) = \lim_{n\to\infty} f(x_n) \cdot \lim_{n\to\infty} g(x_n) = f(c) \cdot g(c)$$

So for any sequence (x_n) in A such that $\lim_{n\to\infty} x_n = c$, we have that $\lim_{n\to\infty} h(x_n) = h(c)$ where $h = f \cdot g$. Therefore h is continuous at c, by the reverse implication in the sequential characterization.

■

Theorem 4.7 (Function composition and continuity)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ such that $f(A) = \{f(x) : x \in A\} \subseteq B$. This ensures $h(x) = g \circ f(x)$ is well defined on A. Now suppose $c \in A$, f is continuous at c, and g is continuous at $f(c)$. Then h is continuous at c.

Proof. We first prove this by the sequential characterization of continuity. Let (x_n) be a sequence in A such that $\lim_{n\to\infty} x_n = c$. Remember this is possible independent of whether c is a cluster point or an isolated point. Since f is continuous at c, we have $\lim_{n\to\infty} f(x_n) = f(c)$. Now $(f(x_n))$ is a sequence in B such that

$$\lim_{n\to\infty} f(x_n) = f(c)$$

We are given that g is continuous at $f(c)$. So it follows that $\lim_{n\to\infty} g(f(x_n)) = g(f(c))$. Thus for any (x_n) in A such that $\lim_{n\to\infty} x_n = c$, we have that $\lim_{n\to\infty} h(x_n) = h(c)$ where $h = g \circ f$. Therefore h is continuous at c.

Now we give an ε-δ proof. Let $\varepsilon > 0$. Since g is continuous at $f(c)$, there is $\delta_1 > 0$ such that for all $y \in B$

$$|y - f(c)| < \delta_1 \implies |g(y) - g(f(c))| < \varepsilon$$

Since f is continuous at c, there is $\delta_2 > 0$ such that for all $x \in A$

$$|x - c| < \delta_2 \implies |f(x) - f(c)| < \delta_1$$

So for any $x \in A$

$$|x - c| < \delta_2 \implies |f(x) - f(c)| < \delta_1 \implies |g(f(x)) - g(f(c))| < \varepsilon$$

Therefore $h = g \circ f$ is continuous at $x = c$.

■

Example 4.2. We can show $\sin(x)$ is continuous at any $c \in \mathbb{R}$, assuming a few basic properties. Namely,

$$|\sin x| \leq |x| \text{ and } |\cos x| \leq 1$$
$$\sin x - \sin y = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$$

Fix $c \in \mathbb{R}$ and let $\varepsilon > 0$. Take $\delta = \varepsilon$ and let $x \in \mathbb{R}$ such that $|x - c| < \delta$. Then

$$\begin{aligned} |\sin x - \sin y| &= |2\sin(\frac{x-c}{2})\cos(\frac{x+y}{2})| \\ &= |2\sin(\frac{x-c}{2})||\cos(\frac{x+y}{2})| \\ &\leq 2\left|\frac{x-c}{2}\right| \\ &= |x - c| < \delta = \varepsilon \end{aligned}$$

So $\sin x$ is continuous at any $c \in \mathbb{R}$.

Remark. We write

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To understand why $\sin x, \cos x, e^x$ are characterized as such requires a detour through complex analysis. In complex analysis, these three functions become "the same" since

$$e^{ix} = \cos x + i \sin x \quad i^2 = -1$$

For now we just rely on our intuition about these functions.

4.6 Extensions by continuity

Let $f : A \to \mathbb{R}$ and let c be a cluster point of A such that $c \notin A$. Since $c \notin A$, then **we cannot determine the continuity** of f at c, as f is undefined at c. To deal with this, we can extend the domain of f to $\{c\} \cup A$ by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ L & \text{if } x = c \end{cases}$$

Since c is a cluster point of $A \cup \{c\}$, then it follows that

$$F \text{ continuous at } c \iff \lim_{x \to c} F(x) = L \overset{(*)}{\iff} \lim_{x \to c} f(x) = L$$

The equivalence $(*)$ holds by observing $|F(x) - L| = |f(x) - L|$ when $x \in A$ and $x \neq c$. This can then be substituted in the functional limit definition.

The point then is that if $\lim_{x \to c} f(x) = L$, then the extended function $F(x)$ is continuous at c. Importantly, if $\lim_{x \to c} f(x) = L$ does not exist, then f **cannot** be continuously extended at c. Nothing prevents us from extending it, but the resulting function would not be continuous.

Example 4.3. Let $f(x) = x \sin \frac{1}{x}$, which is defined on $A = (-\infty, 0) \cup (0, \infty)$. We have that 0 is a cluster point of A. Furthermore, for all $x \in A$

$$0 \leq |f(x)| \leq |x|$$

Hence by the squeeze theorem $\lim_{x \to 0} f(x) = 0$. Therefore we can continuously extend f to 0 by setting $f(0) := 0$. So the extension by continuity of f is continuous at $c = 0$. Furthermore, f is continuous everywhere else on $\mathbb{R}$. To see this, observe x is continuous on A, and $\frac{1}{x}$ is continuous on A by algebraic properties. Furthermore, $\sin(\frac{1}{x})$ is continuous on A by the composition theorem. So therefore $x \sin(\frac{1}{x})$ is continuous on A by algebraic properties.

4.7 Continuity on bounded closed intervals

Theorem 4.8. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded, namely there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Proof. Suppose there exists a continuous function $f : [a, b] \to \mathbb{R}$ that is not bounded. Then for every $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $f(x_n) > n$. Since $a \leq x_n \leq b$, then (x_n) is a bounded sequence and so a convergent subsequence (x_{n_k}) exists by the Bolzano-Weierstrass theorem. By the order limit theorem

$$\lim_{k \to \infty} x_{n_k} = x \in [a, b]$$

Since f is continuous everywhere in $[a, b]$ including x, then the sequential characterization of continuity gives

$$\lim_{k \to \infty} f(x_{n_k}) = f(x)$$

By construction of (x_n), we have $|f(x_{n_k})| > n_k \geq k$. Thus $(f(x_{n_k}))$ is a convergent sequence of real numbers that is not bounded. But this contradicts Theorem 3.2.

■

Remark. This proof relied on two things. First, we needed a bounded interval, since otherwise we could not invoke the Bolzano-Weierstrass theorem. Second, we needed the interval to be closed, since otherwise we could not invoke the sequential characterization of continuity. Particularly, after defining $a < x_n < b$, we may have that (x_{n_k}) converges to a or b by order limit theorem. If so, then $x \notin (a, b)$ and so we cannot apply the sequential characterization of continuity. Unfortunately then we have no say in the convergence of $\lim_{k\to\infty} f(x_{n_k})$. Consequently, a contradiction cannot be derived.

Example 4.4. Let $f : (0, 1] \to \mathbb{R}$ where $f(x) = \frac{1}{x}$ is continuous. The interval $(0, 1]$ is bounded but not closed. Now f is not bounded since for any $M > 0$, if we take $0 < x < \frac{1}{M+1}$ we get $f(x) > M + 1 > M$.

Definition 4.9 (Absolute maximum)**.** Let $f : A \to \mathbb{R}$. Then f has absolute maximum at $\bar{x} \in A$ if $f(\bar{x}) \geq f(x)$ for all $x \in A$.

Definition 4.10 (Absolute minimum)**.** Let $f : A \to \mathbb{R}$. Then f has absolute minimum at $\underline{x} \in A$ if $f(\underline{x}) \leq f(x)$ for all $x \in A$.

Theorem 4.9. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f has absolute maximum and absolute minimum on $[a, b]$.

Proof. We prove the absolute maximum case. The absolute minimum case can be shown by repeating the proof, or by applying the absolute maximum case on $g := -f$.

Since f is continuous over a closed and bounded interval $[a, b]$, then $f([a, b]) = \{f(x) : x \in [a, b]\}$ is bounded. That is to say there exists $M > 0$ such that $f([a, b]) \subseteq [-M, M]$. By the Axiom of Completeness, $s = \sup(f([a, b]))$ exists. Now for every $n \in \mathbb{N}$, we have $s - \frac{1}{n}$ is not an upper bound. Thus there exists $x_n \in [a, b]$ such that

$$s - \frac{1}{n} < f(x_n) < s$$

Since (x_n) is bounded, then by the Bolzano-Weierstrass theorem there exists a convergent subsequence (x_{n_k}). Let $\bar{x} = \lim_{k\to\infty} x_{n_k}$. By the order limit theorem $\bar{x} \in [a, b]$. Now since $\bar{x} \in [a, b]$ and f is continuous, then the sequential characterization of continuity give

$$f(\bar{x}) = \lim_{k\to\infty} f(x_{n_k})$$

Now notice that since $n_k \geq k$ for all $k \in \mathbb{N}$, then $-\frac{1}{k} \leq -\frac{1}{n_k}$ for all $k \in \mathbb{N}$. This gives

$$s - \frac{1}{k} \leq s - \frac{1}{n_k} < f(x_{n_k}) < s$$

So $f(\bar{x}) = \lim_{k\to\infty} f(x_{n_k}) = s$. Alternatively, this also follows since $f(x_n) \to s$ so any subsequence $f(x_{n_k})$ must also converge to s. We have shown that $s \in f([a,b])$ and thus f has absolute maximum on $[a,b]$.

■

Remark (1). From our initially rough construction of $f(x_n)$, we were able to establish that it has a convergent subsequence $f(x_{n_k})$ by the Bolzano-Weierstrass theorem. We could then establish that the limit of this subsequence is actually an element of $f([a,b])$. The punchline was then that this subsequence converges to the supremum – thus revealing that the supremum belongs to $f([a,b])$.

Remark (2). This result will hold over any "compact metric space".

Theorem 4.10 (Location of roots theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that

$$f(a) < 0 < f(b)$$

Then there exists c such that $a < c < b$ and $f(c) = 0$.

Proof. Let $S = \{x \in [a,b] : f(x) < 0\}$. Then S is non-empty since $a \in S$, and S is bounded. Hence the supremum $c = \sup(S)$ exists. The strategy will be to first show that $a < c < b$. Then we will show that $f(c) \leq 0$ and $f(c) \geq 0$, which will give $f(c) = 0$.

We now show that $a < c < b$. Let $\varepsilon = \min(\{\frac{|f(a)|}{2}, \frac{f(b)}{2}\}) > 0$. Since f is continuous at a, there exists $\delta_1 > 0$ such that for all $x \in [a, a+\delta_1)$ we have

$$|f(x) - f(a)| < \varepsilon \leq \frac{|f(a)|}{2}$$

This gives $f(x) < \frac{f(a)}{2} < 0$. Since f is also continuous at b, there exists $\delta_2 > 0$ such that for all $x \in (b - \delta_2, b]$ we have

$$|f(x) - f(b)| < \varepsilon \leq \frac{f(b)}{2}$$

This gives $f(x) - f(b) > -\frac{f(b)}{2}$, so $f(x) > \frac{f(b)}{2} > 0$. Now let $\delta = \min(\{\delta_1, \delta_2, \frac{b-a}{2}\})$. Then for all $x \in [a, a+\delta]$ we have $f(x) < 0$ and hence $[a, a+\delta) \subseteq S$. Therefore $c \geq a + \delta$. On the other hand, we have $S \subseteq [a,b]$, but since $f(x) > 0$ for all $x \in (b-\delta, b]$ we can more sharply say $S \subseteq [a, b-\delta]$. Thus $b - \delta$ is an upper bound of S, so $c \leq b - \delta$. This gives

$$a < a + \delta \leq c \leq b - \delta < b$$

So $a < c < b$ as desired.

Now we show that $f(c) \leq 0$. Since $c = \sup S$, then for all $n \in \mathbb{N}$ we have $c - \frac{1}{n}$ is not an upper bound for S. Hence there exists $x_n \in S$ such that

$$c - \frac{1}{n} < x_n < c$$

So we have $\lim_{n\to\infty} x_n = c$. Since $c \in [a,b]$, then by the sequential characterization of continuity

$$\lim_{n\to\infty} f(x_n) = f(c)$$

But also $f(x_n) \leq 0$ since $x_n \in S$. By the order limit theorem we have $f(c) \leq 0$. Now we show that $f(c) \geq 0$. We have shown that $c < b$, so there exists a sequence (x_n) with $x_n > c$ such that $\lim_{n\to\infty} x_n = c$. For example, some variant of $x_n = c + \frac{1}{n}$ would work, as long as $x_n \in [a, b]$ for all $n \in \mathbb{N}$. Now $c \in S$ and we have $\lim_{n\to\infty} x_n = c$. By the sequential characterization of f, we have

$$\lim_{n\to\infty} f(x_n) = f(c)$$

But $f(x_n) > 0$ for all $n \in \mathbb{N}$ since $x_n \notin S$, so we must have $f(c) \geq 0$.

■

Remark. This is an equivalent formulation of the Axiom of Completness.

4.8 Review of intervals in $\mathbb{R}$

We have four types of bounded intervals in $\mathbb{R}$.

$$\begin{aligned}
[a, b] &= \{x : a \leq x \leq b\} \\
[a, b) &= \{x : a \leq x < b\} \\
(a, b] &= \{x : a < x \leq b\} \\
(a, b) &= \{x : a < x < b\}
\end{aligned}$$

We have four types of unbounded intervals in $\mathbb{R}$.

$$\begin{aligned}
[a, \infty) &= \{x : x \geq a\} \\
(a, \infty) &= \{x : x > b\} \\
(-\infty, a] &= \{x : x \leq a\} \\
(-\infty, a) &= \{x : x < a\}
\end{aligned}$$

The ninth interval is $\mathbb{R} = (-\infty, \infty)$. Given any of the nine intervals, and any elements x, y inside with $x < y$, then $[x, y]$ is completely contained within the given interval.

Theorem 4.11 (Interval characterization theorem)**.** Let $S \subseteq \mathbb{R}$ that contains more than two points. Suppose S has the property that for all $x, y \in S$, if $x < y$ then we have $[x, y] \subseteq S$. Then S is one of the nine intervals.

Proof. We show the case where S is bounded. Let $\alpha = \inf S$ and $\beta = \sup S$. For any $x \in S$ then we have $\alpha \leq x \leq \beta$, so $S \subseteq [\alpha, \beta]$. Let now $\alpha < z < \beta$. Since $z < \beta$ then z is not an upper bound for S. Hence there exists $y \in S$ such that $z < y$. Since $\alpha < z$, then z is not a lower bound for S. Hence there exists $x \in S$ such that $x < z$. But then $x < z < y$ and by our assumption, we have $[x, y] \subseteq S$, and so $z \in S$. It follows that

$$(\alpha, \beta) \subseteq S \subseteq [\alpha, \beta]$$

S is one of the four bounded intervals in $\mathbb{R}$, depending on whether $\alpha \in S$ or $\beta \in S$.

■

Theorem 4.12 (Bolzano's Intermediate Value Theorem)**.** Let I be an interval and $f : I \longrightarrow \mathbb{R}$ be a continuous function on I. If $a, b \in I$ and $k \in \mathbb{R}$ such that

$$f(a) < k < f(b)$$

Then there exists $c \in I$, with $a < c < b$, such that $f(c) = k$.

Proof. Assume first that $a < b$ and let $h(x) = f(x) - k$. Then h is continuous on $[a, b]$ and $h(a) = f(a) - k < 0$, while $h(b) = f(b - k > 0)$. By the location of roots theorem, there exists $c \in I$ with $a < c < b$ such that $h(c) = 0$, or equivalently $f(c) = k$.

Now assume $b < a$ and let $h(x) = k - f(x)$. Then $h(b) = k - f(b) < 0$ and $h(a) = k - f(a) > 0$. By the location of roots theorem, there exists $c \in I$ with $b < c < a$ such that $h(c) = 0$, or equivalently $f(c) = k$.

■

Theorem 4.13. Let $I = [a, b]$ and let $f : I \longrightarrow \mathbb{R}$ be a continuous function on I. If $k \in \mathbb{R}$ satisfies

$$\inf f(I) \leq k \leq \sup f(I)$$

Then there exists $c \in I$ such that $f(c) = k$.

Proof. By the minimum-maximum theorem, there exists $\bar{x}$ and $\underline{x}$ in I such that $f(\bar{x}) = \sup f(I)$ and $f(\underline{x}) = \inf f(I)$. Thus is $\inf f(I) \leq k \leq \sup f(I)$ is satisfied, then we have $f(\underline{x}) \leq k \leq f(\bar{x})$. If $f(\underline{x}) = k$ or $f(\bar{x}) = k$, then we are done. Otherwise, there exists $c \in I$ such that $f(c) = k$ by Bolzano's intermediate value theorem, as $f(\underline{x}) < k < f(\bar{x})$.

■

Theorem 4.14. Let $I = [a, b]$ and let $f : I \longrightarrow \mathbb{R}$ be a continuous function on I. Then $f(I) = \{f(x) : x \in I\}$ is a closed and bounded interval.

Proof. Let $m = \inf f(I)$ and let $M = \sup f(I)$, which both exist as $f(I)$ is bounded and non-empty. Then immediately $f(I) \subseteq [m, M]$. Now let $m \leq k \leq M$. Then there exists $c \in I$ such that $f(c) = k$, by the previous theorem. Therefore $f(I) = [m, M] = [\inf f(I), \sup f(I)]$, which is closed and bounded.

■

Theorem 4.15. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \longrightarrow \mathbb{R}$ be a continuous function on I. Assume f is not a constant function so that $f(I)$ contains at least two points. Then $f(I)$ is an interval.

Proof. Let $\alpha, \beta \in f(I)$ such that $\alpha < \beta$. Then there exists $\alpha, \beta \in I$ such that $f(a) = \alpha$ and $f(b) = \beta$. By the Bolzano intermediate value theorem, if $\alpha < k < \beta$, then there exists $c \in I$ such that $f(c) = k$. Thus $[\alpha, \beta] \subseteq f(I)$. By the interval characterization theorem, $f(I)$ is an interval.

■

Remark. In this case, we do not know if I is closed and bounded. Hence $f(I)$ is not necessarily closed and bounded, like it was in the previous theorem. For example, take $I = (0,1]$ and let $f(x) = \frac{1}{x}$. Then $f(I) = [1,\infty)$.

4.9 Uniform continuity

Recall the definition of continuity over $A \subseteq \mathbb{R}$ for a function $f : A \to \mathbb{R}$. In terms of quantifiers,

$$\forall c \in A, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in A,\ |x-c| < \delta \implies |f(x)-f(c)| < \varepsilon$$

Crucially, the particular value of δ depends on the choice of c and ε. Uniform continuity modifies this notion so that δ depends only on ε.

Definition 4.11 (Uniform continuity)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then f is uniformly continuous on A if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $c \in A$ and all $x \in A$ that satisfy $|x-c| < \delta$, we have $|f(x)-f(c)| < \varepsilon$.

Remark. In terms of quantifiers, this definition says

$$\forall \varepsilon > 0, \exists \delta > 0, \forall c \in A, \forall x \in A,\ |x-c| < \delta \implies |f(x)-f(c)| < \varepsilon$$

The game now is that given any $\varepsilon > 0$, we must find a $\delta > 0$ such that no matter where we slide a δ-wide vertical bar along the x-axis, the function does not change more than ε within the bar.

Example 4.5. $f(x) = x$ is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then for any $c, x \in \mathbb{R}$, $|x-c| < \delta \implies |f(x)-f(c)| = |x-c| < \delta = \varepsilon$.

Example 4.6. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Intuitively, if we fix any ε, we can move right (or left) far enough such that the function grows too quickly to be contained by any δ.

Suppose $f(x)$ is uniformly continuous and let $\varepsilon = 1$. Let $\delta > 0$ such that for all $c, x \in \mathbb{R}$ we have $|x-c| < \delta \implies |f(x)-f(c)| < \varepsilon$. Now take $x = \frac{1}{\delta} + \delta$ and $c = \frac{1}{\delta} + \frac{\delta}{2}$. Then $|x-c| = \frac{\delta}{2} < \delta$ but $|x^2 - c^2| = 1 + \frac{3}{4}\delta^2 > 1$. Contradiction arises.

Example 4.7.

Example 4.8.

Example 4.9.

Theorem 4.16 (Sequential characterization of non-uniform continuity)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous. Then the following are equivalent.

(1) f is **not** uniformly continuous.

(2) There exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A such that $\lim_{n\to\infty} |x_n - y_n| = 0$ and

$$|f(x_n) - f(y_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$$

Proof. First we show (1) $\implies$ (2). If f is uniformly continuous, then

$$\forall \varepsilon > 0, \exists \delta > 0, \forall c \in A, \forall x \in A, |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Negating this gives

$$\exists \varepsilon_0 > 0, \forall \delta > 0, \exists c \in A, \exists x \in A, |x - c| < \delta \text{ and } |f(x) - f(c)| \geq \varepsilon_0$$

Take this ε_0 and let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$. Then there exists $x_n, y_n \in A$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Now this defines $(x_n), (y_n)$ in A such that $\lim_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

We show now that (2) $\implies$ (1) by contradiction. Suppose there exists a continuous function $f : A \to \mathbb{R}$ such that (2) holds but (1) does not. Then f is uniformly continuous, and there also exist $(x_n), (y_n)$ in A such that $\lim_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. By the definition of uniform continuity, there exists $\delta > 0$ such that for all $x, y \in A$ satisfying $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon_0$. Since $\lim_{n\to\infty} |x_n - y_n| = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - y_n| < \delta$. So for all $n \geq N$ we have $|f(x_n) - f(y_n)| < \varepsilon_0$. But this is a contradiction.

■

Remark. As the two sequences get arbitrarily close, there is a gap between their functional values that can never be crossed.

Example 4.10. We can now show $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ using the sequential characterization. Let $x_n = \frac{1}{n} + n$ and $y_n = n$. Then $x_n - y_n = \frac{1}{n}$ and $\lim_{n\to\infty} |x_n - y_n| = 0$. On the other hand, we have $f(x_n) - f(y_n) = n^2 + 2 + \frac{1}{n^2} - n^2 = 2 + \frac{1}{n^2} \geq 2$. Hence not uniformly continuous.

Example 4.11. We can do the same for $f(x) = \sin(\frac{1}{x})$ on $(0, 1]$. Take $x_n = \frac{1}{n\pi + \frac{\pi}{2}}$ and $y_n = \frac{1}{n\pi}$. Then $\lim_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| = 1$.

Theorem 4.17 (Uniform continuity theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous on $[a,b]$.

Proof. We show this by contradiction. Suppose there exists a continuous function $f : [a,b] \to \mathbb{R}$ that is not uniformly continuous on $[a,b]$. Then by the sequential characterization, there is $\varepsilon_0 > 0$ and $(x_n), (y_n)$ in A such that $\lim_{n\to\infty} |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}) with limit $z \in [a,b]$. We explicitly require that $[a,b]$ is closed to claim that $z \in [a,b]$, since z may equal a or b by the order limit theorem. Now we have the estimate

$$\begin{aligned} 0 \leq |y_{n_k} - z| &= |y_{n_k} - x_{n_k} + x_{n_k} - z| \\ &\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z| \end{aligned}$$

Since the entire right side converges to 0, then the squeeze theorem gives $\lim_{k\to\infty} |y_{n_k} - z| = 0$. So we have that $\lim_{k\to\infty} y_{n_k} = z$ and $\lim_{k\to\infty} x_{n_k} = z$. Since f is continuous on $[a,b]$ we have that

$$\lim_{k\to\infty} f(x_{n_k}) = f(z) \text{ and } \lim_{k\to\infty} f(y_{n_k}) = f(z)$$

But then $\lim_{k\to\infty}(f(x_{n_k}) - f(y_{n_k})) = 0$ which implies the existence of $K \in \mathbb{N}$ such that for all $k \geq K$

$$|f(x_{n_k}) - f(y_{n_k})| < \varepsilon_0$$

This contradicts the statement that $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

∎

Theorem 4.18 (Preservation of Cauchy Criterion by uniformly continuous functions)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a uniformly continuous function. Let (x_n) be a sequence in A and assume (x_n) is Cauchy. Then $(f(x_n))$ is also a Cauchy sequence.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on A, there exists $\delta > 0$ such that for all $x, y \in A$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \delta$. But then $|f(x_n) - f(x_m)| < \varepsilon$. So $(f(x_n))$ is Cauchy.

∎

Theorem 4.19 (Continuous extension theorem)**.** Let (a,b) be a bounded **open** interval and let $f : (a,b) \to \mathbb{R}$ be a continuous function. The following are equivalent

(1) f is uniformly continuous on (a,b).

(2) f can be defined at the end points a and b such that it is continuous on $[a,b]$.

Proof. We start with the easier direction (2) $\implies$ (1). If f can be extended to $[a,b]$ so that it is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$ by the uniform continuity theorem. But then f is uniformly continuous on any subset of $[a,b]$, including (a,b).

Now we show that (1) $\implies$ (2). Let (x_n) be any sequence in (a, b) that converges to the end point a, so that $\lim_{n\to\infty} x_n = a$. Then (x_n) is a Cauchy sequence and by the previous theorem, $(f(x_n))$ is also Cauchy. Hence set $L = \lim_{n\to\infty} f(x_n)$, and we define $f(a) = L$.

We would like to now show that f is continuous at a. Let (u_n) be any sequence in (a, b) that converges to a, so that $\lim_{n\to\infty} u_n = a$. Let $\varepsilon > 0$. Since f is uniformly continuous on (a, b) there exists $\delta > 0$ such that for all $x, y \in (a, b)$

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Notice that we have $\lim_{n\to\infty}(u_n - x_n) = 0$. So there exists $N_1 \in \mathbb{N}$ such that $|u_n - x_n| < \delta$ for all $n \geq N_1$. The definition of uniform continuity gives that $|f(x_n) - f(u_n)| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. We further know that $\lim_{n\to\infty} f(x_n) = L$, so there is $N_2 \in \mathbb{N}$ such that $|f(x_n) - L| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. If $N = \max\{N_1, N_2\}$, then for $n \geq N$ we have

$$\begin{aligned} |f(u_n) - L| &\leq |f(u_n) - f(x_n)| + |f(x_n) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Hence $\lim_{n\to\infty} u_n = L = f(a)$. Since (u_n) was arbitrary in A and $a \in A$, then f is continuous at a by the sequential characterization of continuity. We can repeat this argument with b to show that the extended function is continuous at b.

■

Remark. Extension by uniform continuity is a commonly used tool in analysis.