# MTH 331 – Statement 54

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**Statement 54.** *Let $x$ be a non-zero real number. If $x + 1/x$ is an integer, then for all integers $n$ with $n \geq 0$, $x^n + 1/x^n$ is an integer.*

*Proof.* *(i)* $n = 0$, $x^0 + 1/x^0 = 2$ is an integer

*(ii)* $n = 1$, $x^1 + 1/x^1 = x + 1/x$

*(iii)* Let $x + 1/x$ be an integer. For all $n \geq 2$, if, for all $0 \leq k \leq n - 1$, $x^k + 1/x^k$ is an integer then $x^n + 1/x^n$ is an integer. Suppose for all k, $x^k + 1/x^k$ is an integer.

$$\begin{aligned}(x + 1/x)(x^{k-1} + 1/x^{k-1}) &= x^k + x^{2-k} + x^{k-2} + x^{-k} \\ &= x^k + x^{-k} + x^{k-2} + x^{-(k-2)} \\ &= x^k + 1/x^k + x^{k-2} + 1/x^{k-2} \\ \to x^{k-2} + 1/x^{k-2} &= (x + 1/x)(x^{k-1} + 1/x^{k-1}) - (x^k + 1/x^k)\end{aligned}$$

We know that $x + 1/x$,$x^k + 1/x^k$ and $x^{k-1} + 1/x^{k-1}$ are integers and they imply that $x^{k-2} + 1/x^{k-2}$ is an integer thus $x^n + 1/x^n$ is an integer.

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