MTH 331 – Lemma 41

Robert Ritchie

November 5, 2015

Let U and V be finite dimensional vector spaces (over the same field F) and let $T : U \to V$ be a linear transformation. If $B = \{u_1, ..., u_k\}$ is a basis for U, then for all $u \in U$, there exists unique $a_1, ..., a_k \in F$ such that $T(u) = a_1T(u_1) + ... + a_kT(u_k)$.

Proof. Suppose $B = \{u_1, ..., u_k\}$ is a basis for U. Let $x \in U$ then there exist scalars $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k \in F$ such that $x = \alpha_1 u_1 + ... + \alpha_k u_k = \beta_1 u_1 + ... + \beta_k u_k$. $\to (\alpha_1 - \beta_1)u_1 + ... + (\alpha_k - \beta_k)u_k = 0$
Since the basis is independant, the only linear combination of its vectors that makes zero is the one where all of the scalars are zero so $\alpha_1 - \beta_1 = ... = \alpha_k - \beta_k = 0 \to \alpha_1 = \beta_1, ..., \alpha_k = \beta_k$
Since $B = \{u_1, ..., u_k\}$ is a basis for U, $\forall u \in U, \exists! \alpha_1, ..., \alpha_k \in F$ such that $u = \alpha_1 u_1 + ... + \alpha_k u_k$.

$$\begin{aligned} T(u) &= T(\alpha_1 u_1 + ... + \alpha_k u_k) \\ &= \alpha_1 T(u_1) + .. + \alpha_k T(u_k) \end{aligned}$$

$\square$