Math 421/521, Fall 2016.

Homework Assignment 3: Due Monday, September 26th.

MTH 421 students: Submit problems 1 and 2 as Section A, and problems 3, 4, and 5 as Section B.
MTH 521 students: Do problems 3, 4, 5, 6, and 7.

1. Prove the Latin square property: if G is a finite group, then each element of G appears exactly once in each column and exactly once in each row of its Cayley table.

2. Suppose H is a nonempty *finite* subset of a group G with the property that if $a, b \in H$, then $ab \in H$. Prove that H is a subgroup of G.

3. Let $n \geq 1$ be an integer; recall

$$\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$$

is a group with respect to addition modulo n.
(a) Prove that for any k, $0 \leq k \leq n-1$, $< [k] >=< [\gcd(n, k)] >$.
(b) Find all generators for the group $\mathbb{Z}_{30}$.
(c) How many generators does $\mathbb{Z}_{8400}$ have?

4. Show that for every element a in a group G, $|a| = |a^{-1}|$. Make sure you treat all cases.

5. Recall that $S_{\mathbb{Z}}$ is the group of permutations of $\mathbb{Z}$. Let $F \subseteq S_{\mathbb{Z}}$ consist of those permutations which fix all but finitely many elements of $\mathbb{Z}$. Prove that F is a subgroup of $S_{\mathbb{Z}}$.

6. Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

7. A *quasigroup* is a groupoid G such that for all $a \in G$, the left multiplication map $L_a : G \to G$ and the right multiplication map $R_a : G \to G$ defined (respectively) by $L_a(x) = ax$ and $R_a(x) = xa$ are bijective. (This is equivalent to requiring that the Cayley table of G be a Latin square.)
Prove that an associative quasigroup is a group.