MTH 331 – Theorem 39

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Let T be a linear transformation on a finite vector space V. Then $dim(V) = dim(kerT) + dim(ranT)$.

Proof. Let $B' = \{k_1, ..., k_m\}$ be a basis for kerT. Since B' is independant, we can extend it to a basis for V say $B = \{k_1, ..., k_m, u_1, ..., u_n\}$. Let $U = \{Tu_1, ..., Tu_n\}$ Let $w \in ranT$, this implies that $\exists v \in V$ such that $Tv = w \rightarrow \exists \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n \in F$ such that

$$\begin{aligned} v &= \alpha_1 k_1 + ... + \alpha_m k_m + \beta_1 u_1 + ... + \beta_n u_n \\ Tv &= T(\alpha_1 k_1 + ... + \alpha_m k_m + \beta_1 u_1 + ... + \beta_n u_n) \\ w &= \alpha_1 Tk_1 + ... + \alpha_m Tk_m + \beta_1 Tu_1 + ... + \beta_n Tu_n \\ &= \beta_1 Tu_1 + ... + \beta_n Tu_n \\ &\rightarrow \bigvee \{Tu_1, ..., Tu_n\} = ranT \end{aligned}$$

Let $\gamma_1, ..., \gamma_n \in F$. Suppose $T\gamma_1 u_1 + ... + T\gamma_n u_n = 0$
$\rightarrow T(\gamma_1 u_1 + ... + \gamma_n u_n) = 0$
$\rightarrow \gamma_1 u_1 + ... + \gamma_n u_n \in kerT$
$\rightarrow \gamma_1 u_1 + ... + \gamma_n u_n = \beta_1 k_1 + ... + \beta_m k_m$ and since $\{k_1, ..., k_m, u_1, ..., u_n\}$ is independant, $\gamma_1 = ... = \gamma_n = \beta_1 = ... = \beta_n = 0$ so $\{Tu_1, ..., Tu_n\}$ is independant. $\square$