Stabilizer Tensor Networks: Universal Quantum Simulator on a Basis of Stabilizer States

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Efficient simulation of quantum computers relies on understanding and exploiting the properties of quantum states. This is the case for methods such as tensor networks, based on entanglement, and the tableau formalism, which represents stabilizer states. In this Letter, we integrate these two approaches to present a generalization of the tableau formalism used for Clifford circuit simulation. We explicitly prove how to update our formalism with Clifford gates, non-Clifford gates, and measurements, enabling universal circuit simulation. We also discuss how the framework allows for efficient simulation of more states, raising some interesting questions on the representation power of tensor networks and the quantum properties of resources such as entanglement and magic, and support our claims with simulations.

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Introduction—Simulation of quantum computing is crucial for two main reasons: driving science in fields like condensed matter physics [1,2] or quantum chemistry [3–5], as long as we do not have large, error-corrected devices, and testing quantum advantage claims [6–8] made by cutting-edge devices [9,10]. To simulate efficiently beyond a few dozen qubits we must find alternative characterizations of the systems as the complexity of brute force approaches grows exponentially. Thus, a great effort is put toward identifying which states are easy and why, given the absence of a universal description of simulability. Resource theories [11] are a useful tool for this task: they characterize the operations that are easy to do (free operations) in a certain framework. We are particularly interested in entanglement [12] and stabilizer rank [13,14], for their relation to tensor networks (TN) and the stabilizer formalism, respectively.

The interest in relating different resources, particularly these two, is not new. Previous research has found some stabilizer states present maximal entanglement, at least in the bipartite sense [15], although most types of entangled states are not achievable with this formalism. Recently, magic in matrix product states (MPS), a type of TN, has also been characterized and looked into [16–18], and it is noteworthy that separable states with a lot of magic are complex in the stabilizer formalism, even though they are trivial to simulate with resource theories of entanglement. This means that these resources are, in some sense, orthogonal, as depicted in Fig. 1(a). In this Letter, we unify simulation strategies for entanglement and magic by using a special basis [19] in conjunction with tensor networks, as shown in Fig. 1(b), focusing on how using both resources can positively enhance simulations based on these strategies individually.

Tensor networks—Entanglement as a resource for simulation is usually characterized by bipartite entanglement between sectors of the system. This applies to methods such as circuit cutting [20], entanglement forging [21], or tensor networks [22]. These simulations rely on limited entanglement, mostly between close neighbors [23], or on a hierarchical structure of entanglement [24,25]. Free operations consist of single-qubit (local) gates and classical communication [26,27]. In the extreme case of no entanglement, the simulation cost of a system grows linearly with its size; in more complex cases, systems can adhere to an area law [28] that allows TN methods like DMRG [29] to perform efficient simulations with great success. Tensor networks encode high-dimensional tensors into the product of smaller, low-dimensional ones, and are in general advantageous whenever long-range correlations are restricted. The tensors can be connected through bonds in various geometries, and networks with more complexity and expressive power entail higher performance costs. We focus on a 1D MPS structure to encode the amplitudes of a quantum state:

$$\mathcal{T}^{i_1 i_2 \ldots i_n} = \sum_{k_1 k_2 \ldots k_{n-1}} (T_1)^{i_1}_{k_1} (T_2)^{i_2 k_1}_{k_2} (T_3)^{i_3 k_2}_{k_3} \ldots (T_n)^{i_n k_{n-1}}. \quad (1)$$

In this structure, the dimension χ of a given bond k corresponds to the entropy between the two subsystems it connects, as measured by the Schmidt rank [30]. We call

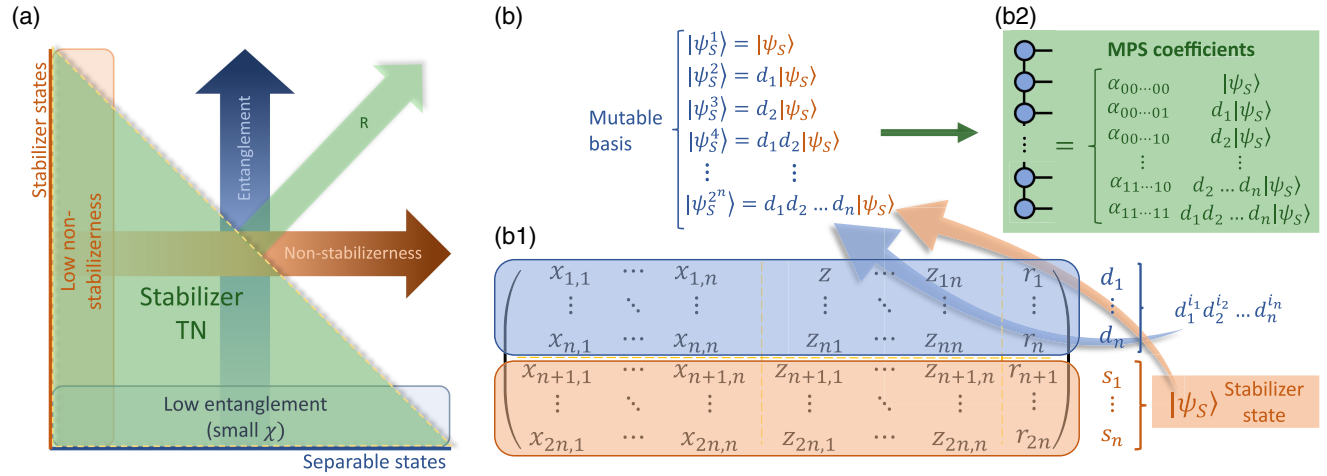

FIG. 1. Showcase of the stabilizer formalism. In (a), we classify states under two different resources. For the resource of entanglement, in blue, (nonstabilizerness, orange), axis y = 0 (x = 0) represents the free states, while its adjacent region represents states with low amounts of entanglement (nonstabilizerness), which are classically simulatable with tensor networks (stabilizer tableaus). They are simultaneously simulatable with stabilizer tensor networks (sTN), in green, and can be characterized with a different resource R. In (b) we show how sTN joins these methods: the tableau (b1) encodes a stabilizer state and a set of destabilizer generators which form a basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ of the Hilbert space. The first n rows of the tableau encode n destabilizer generators, rows $n+1,\ldots,2n$ encode n stabilizer generators, and column r indicates a real phase. The amplitudes of a state $|\psi\rangle$ in this basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ are stored in a TN (green, b2), an MPS in our example.

this χ the bond dimension. A separable state can be encoded into a TN with $\chi = 1$, whereas a state with mostly local entanglement (AKLT state [31]) needs $\chi = 2$, and a maximally entangled state requires up to $\chi = 2^{n/2}$. The bond dimension can also be artificially limited at the cost of precision.

Stabilizer tableaus—The stabilizer tableau formalism [32], on the other hand, can simulate efficiently any circuit composed only by Clifford gates with a classical computer. The states that can be prepared under these constraints are known as stabilizer states. In this context, the set $\mathcal{C}$ of Clifford gates are the free operations, and non-Clifford gates increase the stabilizer rank [33], which constitutes the resource. However, resource theories of stabilizer states are typically studied with other measures such as magic [34], stabilizer Rényi entropy [35], or Wigner positivity [36] due to their interesting properties. This simulation formalism is based on a stabilizer set $\mathcal{S}$ of Pauli operators ($\mathcal{P}_n$). It uniquely defines a state that fulfills $S|\psi_\mathcal{S}\rangle = |\psi_\mathcal{S}\rangle$ for any $S \in \mathcal{S}$. A Pauli operator P can be described with two boolean vectors and a real phase, $P = \alpha \cdot (x_1 x_2 \ldots x_n) \cdot (z_1 z_2 \ldots z_n)$, meaning we only need $2n+1$ boolean values to represent one. Also, a set of n generators of $\mathcal{S}$ are enough to fully define the group. This means a tableau of $n \times (2n+1)$ boolean entries stores all the information about $\mathcal{S}$ [see Fig. 1(b1)]. Since $\mathcal{P}_n$ is closed under the action of any gate $C \in \mathcal{C}$, a Clifford circuit can be simulated by finding the new tableau after applying each gate C in it. An efficient ($O(n^2)$ time) approach to update the tableaus is known [37]; it works by also storing the generators d_i of the destabilizer group $\mathcal{D}$. These operators fulfill $\{d_i, s_i\} = 0$, $[d_i, d_j] = 0$ and $[d_i, s_j] = 0$ for any $i \neq j$.

Main results—In the following equations, we denote the elements of $\mathcal{D}$ as $\delta_{\hat{i}} = d_1^{i_1} \ldots d_n^{i_n}$, with d_i the generators of $\mathcal{D}$ and $\hat{i}$ the binary representation of integer i; the same follows for stabilizers $\sigma_{\hat{i}} \in \mathcal{S}$. These elements $\delta_{\hat{i}}$ and the stabilizer state $|\psi_\mathcal{S}\rangle$ define the set $\{\delta_{\hat{i}}|\psi_\mathcal{S}\rangle\}_{\hat{i}}$, which is a basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ [19] of the Hilbert space $\mathcal{H}^n$:

$$|\psi\rangle = \sum_{i=0}^{2^n} \nu_i \delta_{\hat{i}} |\psi_\mathcal{S}\rangle. \tag{2}$$

This can be seen by checking orthogonality [38]. Note that, as $\mathcal{S} \cup \mathcal{D}$ generates $\mathcal{P}_n$, any operator can be decomposed as $O = \alpha \delta_{\hat{u}} \sigma_{\hat{v}}$ for some integers u, v; this includes unitaries because they can be expressed as a sum of operators. We propose an encoding of the amplitudes of Eq. (2) in an MPS as $|\nu\rangle = \sum_i \nu_i |i\rangle$ [Fig. 1(b2)], and show the necessary *update rules* to change this construction under the action of any unitary gate or measurement: (1) *Clifford gate* G: Update the stabilizer basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ by conjugating with G, following the rules in the tableau formalism (see Ref. [37]) for the update $|\psi_\mathcal{S}\rangle \to G|\psi_\mathcal{S}\rangle = |\psi_{\tilde{\mathcal{S}}}\rangle$. This gives a new basis $\mathcal{B}(\tilde{\mathcal{S}},\tilde{\mathcal{D}})$:

$$G|\psi\rangle = \sum_i \nu_i G \delta_{\hat{i}} |\psi_\mathcal{S}\rangle = \sum_i \nu_i \tilde{\delta}_{\hat{i}} |\psi_{\tilde{\mathcal{S}}}\rangle, \tag{3}$$

and leaves the coefficient state $|\nu\rangle$ unchanged

$$G|\nu\rangle = |\nu\rangle. \tag{4}$$

(2) *Non-Clifford gate* $\mathcal{U}$: Find the decomposition $\mathcal{U} = \sum_i \phi_i \delta_{\hat{u}_i} \sigma_{\hat{v}_i}$, then modify $|\psi\rangle$ as:

$$\mathcal{U}|\psi\rangle = \sum_{i,j} ((-1)^{j\cdot\hat{v}_i}\phi_i\nu_j)\delta_{\hat{j}+\hat{u}_i}|\psi_{\mathcal{S}}\rangle. \tag{5}$$

When the decomposition only has two terms, as in a T gate, this is equivalent to a rotation on $|\nu\rangle$:

$$|\nu'\rangle = \cos(\theta)I - i\sin(\theta)X_{I_x}Y_{I_y}Z_{I_z}|\nu\rangle, \tag{6}$$

for some qubits I_x, I_y, I_z that depend on the non-zero entries of u_i, v_i. The basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ stays unchanged. (3) *Measurement of observable* O: Find the decomposition $O = \alpha\delta_{\hat{u}}\sigma_{\hat{v}}$ and the value of $\langle\mathcal{O}\rangle$ with:

$$\langle\mathcal{O}\rangle = \alpha\langle\nu|X_{\hat{u}}Z_{\hat{v}}|\nu\rangle. \tag{7}$$

Now choose an outcome $m \in \{+,-\}$ with probability $p_+ = [(I + \langle O\rangle)/2]$, $p_- = 1 - p_+$, and let k be the position of the first 1 in $\hat{u}$. Then update the stabilizer basis to $\mathcal{B}'(\mathcal{S}',\mathcal{D}')$ following the rules for a measurement in the original formalism, and find $|\psi'\rangle = (I + m\mathcal{O})/2|\psi\rangle$ as

$$|\psi'\rangle = \sum_i \delta_{i_k,0}\left(\frac{1}{\sqrt{2}}\nu_{\hat{i}} + m\frac{\alpha(-1)^{\hat{i}\cdot\hat{v}}}{\sqrt{2}}\nu_{\hat{i}+\hat{u}}\right)\delta_{\hat{i}}|\psi_{\mathcal{S}'}\rangle, \tag{8}$$

which equals to a projection and rotation on $|\nu\rangle$:

$$|\nu'\rangle = |0\rangle\langle 0|_k\left(\frac{1}{\sqrt{2}}I + m\frac{\alpha(-i)^{|I_y|}}{\sqrt{2}}X_{I_x}Y_{I_y}Z_{I_z}\right)|\nu\rangle. \tag{9}$$

When simulating a circuit, we use a decomposition into CNOT and single qubit rotations, as is usually done in real devices. This ensures that all non-Clifford gates conform to the particular case of Eq. (6), although we have also implemented the generalization for $\mathcal{U}$ with more terms [46]. In [38] we prove the rules using two Lemmas and present clarifying examples. To compute the change of basis for Clifford gates, we employ already known [37] efficient methods to update the tableau. Since the amplitudes do not change, they preserve χ. Non-Clifford gates and measurements, on the other hand, can introduce correlations between amplitudes that increase χ and make calculations more expensive. Consequently, χ constitutes our resource and the free operations include all Clifford gates *and also* non-Clifford gates $\mathcal{U}$ such that Eq. (6) is a local rotation on $|\nu\rangle$.

The key ingredient to stabilizer tensor networks (sTN) is allowing the basis to change beyond local rotations. The tableau algorithm replaces the computational basis with a basis of stabilizer states, which can have some entanglement, and forgoes the correspondence between qubits and tensors. In a way, entanglement is transferred from the tensor network $|\nu\rangle$ representation into the basis, at the cost of single qubit gates potentially becoming entangling on the amplitudes of $|\nu\rangle$. In general, this only happens if the circuit already contained entangling gates and thus that part of the circuit was entangling to begin with. Therefore, we argue that the formalism does not generate fictitious entanglement. Instead, we say we store *potential entanglement* in the basis.

We mentioned several resources linked to nonstabilizerness. Among those, stabilizer rank has a direct link to our formalism. For an arbitrary state $|\psi\rangle$, its stabilizer rank is the smallest ξ that allows a decomposition into stabilizer states $|\psi_S\rangle$:

$$|\psi\rangle = \sum_{i=1}^{\xi}\alpha_i|\psi_S^i\rangle. \tag{10}$$

Stabilizer states have $\xi = 1$. The structure on the basis states we use does not mean that a low stabilizer rank translates into a simple $|\nu\rangle$, as the necessary stabilizer states might not be simultaneously in $\mathcal{B}(\mathcal{S},\mathcal{D})$. We can define a pseudo-stabilizer rank $\tilde{\xi}$ as the amount of nonzero coefficients in $|\nu\rangle$. This is obviously an upper bound to ξ, but a thorough characterization of how these quantities relate is left for future work.

Let us demonstrate that our formalism can efficiently simulate two different scenarios: low entanglement and low stabilizer rank. In the original tableau formalism [37], it was already shown how we can simulate any circuit and encode any state in the n qubit Hilbert space with a superposition of tableaus. However, this is akin to a brute-force simulation with the state vector approach and grows exponentially with the number t of non-Clifford gates. Instead, our formalism can take advantage of all the tools that have been developed for tensor networks simulations. Consider the state $|T\rangle^n$, which can be prepared with

$$|T\rangle^n = \prod_{i=1}^{n} T_i \prod_{i=1}^{n} H_i|0\rangle^{\otimes n}. \tag{11}$$

The first layer of Hadamards, which are Clifford gates and only updates the tableau, sets the stabilizer basis to $s_i = X_i$, $d_i = Z_i$. In this basis, each T-gate on qubit i,

$$T_i = \cos\left(\frac{\pi}{8}\right)I - i\sin\left(\frac{\pi}{8}\right)Z_i = \cos\left(\frac{\pi}{8}\right)I - i\sin\left(\frac{\pi}{8}\right)d_i, \tag{12}$$

fulfills the criteria for a free operation, so the resulting state is represented by a trivial MPS with $\chi = 1$. Notice that, in this case, the pseudo-stabilizer rank is maximal $\tilde{\xi} = 2^n$. With the conventional generalization of tableaus, we would

need 2^n copies, as each T gate duplicates the number of necessary tableaus. This is not the best that can be achieved: the stabilizer rank of $|T\rangle^n$ for small n has been shown to be low [47], meaning an optimal decomposition requires fewer tableaus (and also $\xi \ll \tilde{\xi}$). However, a general method to find these decompositions is not known. Most importantly, the growth of ξ with n is expected to be exponential [48] unless quantum computing is completely simulatable (even though a supralinear lower bound has not been found [33]), whereas sTN can represent these states efficiently for any n. On the other hand, some stabilizer states have been shown to have maximum bipartite entanglement [15], as mentioned earlier. These states can be prepared with a Clifford circuit, so in a simulation with sTN they are an element of the basis $\mathcal{B}(\mathcal{S},\mathcal{D})$, and therefore trivial to represent with $\xi = \tilde{\xi} = 1$, despite being expensive with a regular MPS.

A good understanding of how operators decompose for a given basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ is useful to study other advantageous scenarios. A 1D cluster state [49], for example, has $s_i = Z_{i-1}X_iZ_{i+1}, d_i = Z_i$, meaning that a k-local $\mathcal{U}$ becomes effectively $(k+2)$ local on the sTN. For states that are not 1D local, the mapping of the qubits is crucial to finding when k locality is somewhat maintained or improved, but other 1-local operators can become highly nonlocal. In the Toric code [50] on $t \times t$ plaquettes, for example, a X rotation on the qubits of any one plaquette $X_{i_1}X_{i_2}X_{i_3}X_{i_4}$ is a 4-local operator that becomes 1 local; the operator $X_1 \ldots X_n$ becomes $[(t-1)^2+1]$ local, and some simple X_i operators can become $O(t)$ local. These effects are further detailed in [38] with some examples, including explicit derivations.

In the context of all the examples above, there likely exists a different resource R that captures the power of the approach, defining whether a state can be efficiently represented or not with a single metric as illustrated in Fig. 1(a). This resource R must be related to the two discussed resources, in the sense that both low entanglement or low stabilizer rank imply low R, because we have seen that we can simulate either case. However, since stabilizer states can be very entangled, and separable states can have high nonstabilizerness, it follows that high entanglement does not imply high R, nor does low stabilizer rank imply high R, indicating that it isn't trivially connected to these resources.

Beyond the cases of complete stabilizerness or no entanglement, the efficiency should persist when these resources are present in a low amount for the formalism to be useful. Our discussion so far highlights two advantages in that regard. First, notice that we can always process Clifford gates directly on $|\nu\rangle$ instead of changing the basis so that the TN behaves traditionally. This means any state simulatable with tensor networks is also feasible in our approach. Additionally, we have seen that Clifford gates don't change $|\nu\rangle$, independently of its χ, so we can always move freely in the space of states with fixed stabilizer rank.

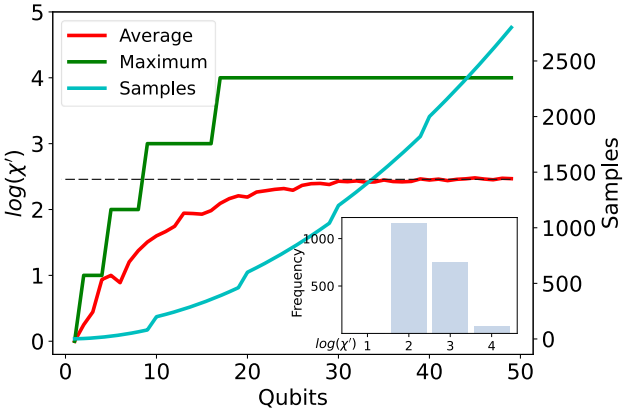

FIG. 2. Average and maximum increase of entanglement after applying a single T gate on a random Clifford tableau, measured with $\log(\chi')$, where χ' is the maximum bond dimension of the MPS in the stabilizer TN. The average is done over $\sim n^2$ uniformly sampled Clifford circuits. In the inset, the distribution of $\log(\chi')$ for $n = 40$.

On the other hand, existing simulations approaches in the low T-count regime that rely on approximations [51] can now be systematically studied by controlling the TN structure of $|\nu\rangle$ and the amount of correlations allowed by χ_ν.

Nonetheless, a gate that entangles $|\psi\rangle$ by a certain amount could in principle become a more entangling gate on $|\nu\rangle$; alternatively, a gate that does not increase stabilizer rank by much could also add a lot of entanglement to $|\nu\rangle$. To show this is not the case, it suffices to look at single-qubit rotations $\mathcal{R}$ for both cases, due to the decomposition we employ. When θ is not a multiple of $\pi/4$, this rotation is also a good example of an operation that only increases stabilizer rank slightly (Eq. (12)). We can bind the growth of χ in our MPS after applying $\mathcal{R}$ by using Eq. (6) (which is equivalent to a CNOT cascade [52]). The worst-case scenario for an MPS is $\chi' = 2^4\chi$ when swap gates are needed (see Ref. [38]), although our simulations (Fig. 2) show that, on average, it is only $\sim 2^{2.46}$, and that it does not grow as $n \to \infty$. For other TN structures with more connectivity, this bound can decrease. Regardless, since it is bounded, we can ensure efficient simulation with a low amount of non-Clifford single-qubit rotations. Notice that the worst-case scenario only happens if the circuit applies CNOT gates before $\mathcal{R}$, which were free in the stabilizer TN and stored *potential entanglement* in the basis.

Conclusions and outlook—We have presented a new technique for circuit simulation, unifying two different frameworks each with its characterizing resource, and discussed how it can impact simulation approaches in general. We have also shown in which instances it offers an advantage and identified its free operations for the characterization of a resource theoretic description. In addition, developing the formalism has identified several interesting research directions. First, one could decide to

not always use a Clifford gate to update the stabilizer basis of the TN and apply it directly to $|\nu\rangle$ instead. The criteria used for this decision would directly affect the growth of the resource in the simulation. Also, the storage and retrieval of the *potential* entanglement in $\mathcal{B}(\mathcal{S},\mathcal{D})$ is possible with entangling Clifford gates, but changing the basis also changes $|\nu\rangle$. This means that, during a simulation, we cannot trivially decrease the cost of the next nonstabilizer gate without potentially increasing χ of the current TN. A thorough study of how to optimally allocate this resource to decrease the cost of the simulation is left for future work.

Bounding χ of the MPS and checking the accuracy of results on states with different amounts of entanglement, magic, or other resources is a strong candidate to characterize the resource R, which relates nontrivially to both entanglement and stabilizer rank. In general, being able to relate χ to a magnitude other than bipartite entanglement also opens up the field of tensor networks to the use of other resources. The evident locality of χ is an obstacle for which resources can be used in this way, making a link between magic and χ of our vector $|\nu\rangle$ a specially interesting objective. We can also use stabilizer TNs as a practical tool to bound the simulation hardness of a circuit based on the amount of T gates it contains, a research that is usually restricted to theoretical approaches [53]. Thus, one can look into improving the efficiency of the implementation. An obvious candidate is integration with the handling of tableaus as done by STIM [54], the most performant Python library for stabilizer simulation.

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Data availability—The method described in this Letter is available in a Python implementation [55] that can simulate any circuit.

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