

Chapter 17. Orthogonal Matrices and Symmetries of Space

Take a random matrix, say

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

and compare the lengths of \mathbf{e}_1 and $A\mathbf{e}_1$. The vector \mathbf{e}_1 has length 1, while $A\mathbf{e}_1 = (1, 4, 7)$ has length $\sqrt{66}$. Most matrices are like this: They change the lengths of vectors. In this chapter we study matrices that do not change lengths. Such matrices are the symmetries of three dimensional space.

Definition of Orthogonal Matrix: *A matrix A is called **orthogonal** if A preserves the lengths of vectors, that is, if*

$$\langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle \quad (1)$$

for all vectors \mathbf{v} in \mathbb{R}^3 .

Remarkably, if a matrix preserves length, then it also preserves angles. This is shown in the following

Proposition: *If A is an orthogonal matrix then $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for any pair of vectors \mathbf{u}, \mathbf{v} in \mathbb{R}^3 . Moreover, the cosine of the angle between $A\mathbf{u}$ and $A\mathbf{v}$ is the same as the cosine of the angle between \mathbf{u} and \mathbf{v} .*

Proof: This is shown by a trick: Since A preserves the lengths of $\mathbf{u} + \mathbf{v}$, and \mathbf{u} and \mathbf{v} , we have

$$\begin{aligned} \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle A(\mathbf{u} + \mathbf{v}), A(\mathbf{u} + \mathbf{v}) \rangle \\ &= \langle A\mathbf{u} + A\mathbf{v}, A\mathbf{u} + A\mathbf{v} \rangle \\ &= \langle A\mathbf{u}, A\mathbf{u} \rangle + 2\langle A\mathbf{u}, A\mathbf{v} \rangle + \langle A\mathbf{v}, A\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle A\mathbf{u}, A\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

Comparing the first and last lines, we get $\langle \mathbf{u}, \mathbf{v} \rangle = \langle A\mathbf{u}, A\mathbf{v} \rangle$, as claimed in the first assertion.

For the second assertion, the geometric form of the dot product shows that the angle between two vectors is completely determined by their dot products with

themselves and each other. We have just shown that all of these dot products are unchanged by A , hence the angle is unchanged. ■

We have defined orthogonal matrices above, in terms of how they act on \mathbb{R}^3 . This definition imparts the geometric meaning of orthogonal matrices. However, given some matrix A , you cannot check the definition for every single vector $\mathbf{v} \in \mathbb{R}^3$, so the definition does not say how to decide if a matrix is orthogonal or not. This problem did not arise in two dimensions, where we defined orthogonal matrices in more practical, yet less meaningful terms. We resolve this as follows.

Three ways to recognize orthogonal matrices: *A matrix A is orthogonal exactly when any one of the following equivalent conditions holds:*

1. *The columns of A form an orthonormal basis of \mathbb{R}^3 ;*

2. *We have*

$$\langle A\mathbf{e}_i, A\mathbf{e}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j; \end{cases}$$

3. *$AA^T = I$. In other words, $A^{-1} = A^T$.*

Proof: Since the columns of A are $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$, condition 2 is just a restatement of condition 1. The entry of AA^T in row i column j is exactly $\langle A\mathbf{e}_i, A\mathbf{e}_j \rangle$, so condition 3 is another way of expressing condition 2. It remains to show that A is orthogonal exactly when condition 1 holds.

Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis, the Proposition above implies that $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$ is also an orthonormal basis. So orthogonal matrices satisfy condition 1. Conversely, suppose that $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$ is an orthonormal basis. Let $\mathbf{u} = (x, y, z)$ be any vector. Then

$$\langle A\mathbf{u}, A\mathbf{u} \rangle = \langle xA\mathbf{e}_1 + yA\mathbf{e}_2 + zA\mathbf{e}_3, xA\mathbf{e}_1 + yA\mathbf{e}_2 + zA\mathbf{e}_3 \rangle.$$

All the cross terms are zero, since $\langle A\mathbf{e}_i, A\mathbf{e}_j \rangle = 0$ if $i \neq j$. Therefore

$$\langle A\mathbf{u}, A\mathbf{u} \rangle = x^2 \langle A\mathbf{e}_1, A\mathbf{e}_1 \rangle + y^2 \langle A\mathbf{e}_2, A\mathbf{e}_2 \rangle + z^2 \langle A\mathbf{e}_3, A\mathbf{e}_3 \rangle = x^2 + y^2 + z^2 = \langle \mathbf{u}, \mathbf{u} \rangle.$$

Hence A is orthogonal. ■

Examples of Orthogonal Matrices:

Example 1: A rotation about the z -axis:

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 2: Other rotations

$$\frac{1}{9} \begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - 1 \\ 1 - \frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} - 1 \\ 1 & \sqrt{2} & 1 \end{bmatrix}.$$

Example 3: Let $\mathbf{u} = (a, b, c)$ be a unit vector. Then the matrix $R_{\mathbf{u}}$ of reflection about the plane $ax + by + cz = 0$ is given by

$$R_{\mathbf{u}} = \begin{bmatrix} 1 - 2a^2 & -2ab & -2ac \\ -2ab & 1 - 2b^2 & -2bc \\ -2ac & -2bc & 1 - 2c^2 \end{bmatrix}. \quad (2)$$

A more economical formula for $R_{\mathbf{u}}$ is as follows. For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$R_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{u} \rangle \mathbf{u}. \quad (3)$$

Setting $\mathbf{v} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in the formula (3) gives the matrix formula (2).

Multiplicative properties of Orthogonal matrices

Condition 3 above implies some useful facts about the products of orthogonal matrices:

1. If A and B are orthogonal matrices then AB is also an orthogonal matrix.
2. If A is an orthogonal matrix then A^{-1} is an orthogonal matrix.
3. If A is an orthogonal matrix then $\det(A) = \pm 1$.

Proof: For the first two properties, we use the characterization of orthogonal matrices as those whose inverse equals their transpose. Thus, if we are given that A and B are orthogonal, that means $AA^T = I$ and $BB^T = I$, so we have

$$(AB)(AB)^T = ABB^T A^T = AIA^T = AA^T = I,$$

so AB is orthogonal. Likewise,

$$(A^{-1})(A^{-1})^T = (A^{-1})(A^T)^T = A^{-1}A = I,$$

so A^{-1} is orthogonal.

For the last item, recall that $\det(A^T) = \det(A)$, so that

$$1 = \det(I) = \det(AA^T) = \det(A) \det(A^T) = \det(A) \det(A) = \det(A)^2,$$

so we must have $\det(A) = \pm 1$. ■

We have seen that an orthogonal matrix has determinant ± 1 . In two dimensions, this sign classified the orthogonal matrices into rotations (determinant $+1$) and reflections (determinant -1). There is a similar result in three dimensions, but it is complicated by two facts:

- An orthogonal matrix with determinant $+1$ is a rotation. However, a rotation in three dimensions is not simply determined by its angle of rotation; one must determine the axis of rotation, and specify from which direction you are measuring the angle.
- Orthogonal matrices with determinant -1 are not rotations, but most of them are not reflections either.

The following theorem classifies the rotation matrices and shows how to find the axis and angle of a given rotation.

Rotation Matrix Theorem:

1. *Rotation matrices are precisely those orthogonal matrices A with*

$$\det(A) = +1.$$

2. *If A is a rotation and $A \neq I$, then the eigenvalues of A are $1, e^{i\theta}, e^{-i\theta}$, for some $0 < \theta < 2\pi$. In particular, the 1-eigenspace $E(1)$ is a line.*
3. *A rotation matrix A as in item 2 has axis of rotation $E(1)$ and angle of rotation θ , measured as follows. Take any nonzero vector \mathbf{v} in the plane perpendicular to the axis $E(1)$. Then $\mathbf{v} \times A\mathbf{v}$ is on the axis $E(1)$, and θ is measured counterclockwise when $\mathbf{v} \times A\mathbf{v}$ points at you.*

4. Up to sign, the angle θ can be computed easily from the formula:

$$\cos \theta = \frac{1}{2} [\operatorname{tr}(A) - 1].$$

Proof: For item 1 we must show that any rotation is an orthogonal matrix with determinant $+1$, and conversely, that any orthogonal matrix A with $\det(A) = +1$ is a rotation. We begin with the former.

Suppose A is rotation about some axis through a vector \mathbf{u} , by some angle which appears counterclockwise by θ , as \mathbf{u} points at us. All rotation matrices clearly preserve the lengths of vectors, so the only question is whether $\det A = \pm 1$. For $0 \leq t \leq \theta$, let A_t be rotation about the same axis about the angle t , measured as above. So $A_0 = I$ and $A_\theta = A$. As we vary t , the determinant is either $+1$ or -1 , but $\det A_t$ varies continuously, so it cannot jump between $+1$ and -1 hence must remain constant at either $+1$ or -1 for all t . Since $\det A_0 = \det I = +1$, we must have $\det(A) = \det A_\theta = +1$ as well.

Now suppose A is an orthogonal matrix with $\det A = 1$. Before proving that A is a rotation, we'll first show that the oddness of three (the dimension we're in) forces A to have an eigenvalue equal to 1. Using the properties of transpose and determinant, we get

$$\begin{aligned} \det(I - A) &= \det(A) \det(I - A) \\ &= \det(A^T) \det(I - A) \\ &= \det(A^T - A^T A) \\ &= \det(A^T - I) \\ &= \det(A - I)^T \\ &= \det(A - I). \end{aligned}$$

Since 3 is odd, $\det(A - I) = -\det(I - A)$, so

$$\det(I - A) = -\det(I - A).$$

This means $\det(I - A) = 0$, so the characteristic polynomial $P_A(x) = \det(xI - A)$ has the root $\lambda = 1$.

Now we will see why A is a rotation about the 1-eigenspace. Let \mathbf{u}_1 be a unit vector such that $A\mathbf{u}_1 = \mathbf{u}_1$. Take another unit vector \mathbf{u}_2 which is perpendicular to \mathbf{u}_1 , and let $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is a right-handed orthonormal basis.

Let B be the matrix such that $B\mathbf{e}_i = \mathbf{u}_i$. Then B is orthogonal because its columns are an orthonormal basis. We compute the matrix $B^{-1}AB$.

By the multiplicative properties of orthogonal matrices, proved above, the matrix $B^{-1}AB$ is orthogonal. Moreover,

$$B^{-1}AB\mathbf{e}_1 = B^{-1}A\mathbf{u}_1 = B^{-1}\mathbf{u}_1 = \mathbf{e}_1,$$

so the first column of $B^{-1}AB$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The first entries of the other columns are orthogonal to the first column, so $B^{-1}AB$ looks like

$$B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{bmatrix},$$

and the 2×2 vectors $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix}$ are orthogonal unit vectors. That means $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is orthogonal. Since

$$1 = \det A = \det(B^{-1}AB) = ad - bc,$$

this 2×2 matrix is a rotation and we have

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = \cos \theta$ and $b = \sin \theta$ is the angle of rotation. So we have

$$B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}. \quad (4)$$

We see that $B^{-1}AB$ is rotation in the plane spanned by $\mathbf{e}_2, \mathbf{e}_3$, about the axis through \mathbf{e}_1 , by the angle θ which is counterclockwise when \mathbf{e}_1 points at us. Moreover, a direct computation with the matrix in (4) shows that the eigenvalues of $B^{-1}AB$ are $1, e^{i\theta}, e^{-i\theta}$.

The last step is to translate our information about $B^{-1}AB$ into information about A . We have

$$A\mathbf{u}_2 = B(B^{-1}AB)\mathbf{e}_2 = B(a\mathbf{e}_2 + b\mathbf{e}_3) = a\mathbf{u}_2 + b\mathbf{u}_3.$$

Similarly,

$$A\mathbf{u}_3 = -b\mathbf{u}_2 + a\mathbf{u}_3.$$

So A is the same rotation as $B^{-1}AB$ but now in the plane spanned by \mathbf{u}_2 and \mathbf{u}_3 , namely counterclockwise by θ when \mathbf{u}_1 points at us. The eigenvalues of A are those of $B^{-1}AB$, which are $1, e^{i\theta}, e^{-i\theta}$. Thus we see that A is a rotation, completing the proof of item 1, and we have also verified item 2.

It remains to prove item 3. Since the eigenvalues of A are $1, e^{i\theta}, e^{-i\theta}$, the characteristic polynomial is

$$P_A(x) = (x - 1)(x - e^{i\theta})(x - e^{-i\theta}) = (x - 1)(x^2 - 2\cos\theta x + 1).$$

The coefficient of x^2 is therefore $-(1 + 2\cos\theta)$. But also the coefficient of x^2 is $-\text{tr}(A)$. So $1 + 2\cos\theta = \text{tr}(A)$, and item 3 follows. ■

Calculating Axes and Angles:

Example 4:

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This one is so easy that you don't need the Rotation Matrix Theorem. Here you can see right away that \mathbf{e}_3 is a 1-eigenvector, so the rotation axis is the z -axis, and A rotates by α in the xy -plane.

Example 5:

$$A = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ \sin \alpha & -\cos \alpha & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is easy to see that $AA^T = I$ and that $\det A = 1$, so this is a rotation, but the axis is a bit trickier. It is *not* the z -axis, since the vectors on that line are negated, not fixed. That means that -1 is an eigenvalue, so the angle of rotation, call it θ , must satisfy $e^{i\theta} = -1$, which means $\theta = \pi$. You can confirm this by checking that $A^2 = I$. To find the axis, we compute

$$\ker(I - A) = \mathbb{R}(\cos \alpha + 1, \sin \alpha, 0).$$

So, while an angle α appears in the matrix, it has nothing to do with the angle of rotation (which is π , no matter what α is), but instead α determines the axis of rotation.

Example 6: For

$$B = \frac{1}{9} \begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix},$$

we first check that $BB^T = I$ and compute $\det B = 1$, so that we know B is a rotation. Then using the kernel recipe we find $\ker(I - B) = \mathbb{R}(1, 2, 2)$, which is the axis of rotation. Next, $\text{tr}(B) = -1$, so

$$\cos \theta = \frac{1}{2}(-1 - 1) = -1,$$

so $\theta = \pi$. Thus, B is rotation by π about the line through $(1, 2, 2)$. The reason that $\theta = \pi$ (here, as well as in Example 2) is that $B = B^T$. For that means $B^{-1} = B$, so $B^2 = I$. Any rotation $\neq I$ whose square is I must be rotation by π .

Example 7: For

$$C = \frac{1}{2} \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - 1 \\ 1 - \frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} - 1 \\ 1 & \sqrt{2} & 1 \end{bmatrix},$$

we first check that $CC^T = I$ and compute $\det C = 1$, so that we know C is a rotation. Then using the kernel recipe we find that $\ker(I - C) = \mathbb{R}(1 + \sqrt{2}, -1, 1)$ is the axis, and $\text{tr}(C) = \frac{6 + \sqrt{2}}{4}$, so

$$\cos \theta = \frac{2 + \sqrt{2}}{8}.$$

Then

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \frac{1}{4} \sqrt{\frac{29}{2} - \sqrt{2}},$$

so the eigenvalues (besides 1) are

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta = \frac{2 + \sqrt{2}}{8} \pm \frac{i}{4} \sqrt{\frac{29}{2} - \sqrt{2}}.$$

Note that we found the eigenvalues without ever writing down the characteristic polynomial! But we did have to compute $\det(C)$ to make sure it was a rotation.

Using the Rotation Matrix Theorem we can find the axis and angle of any rotation matrix we may encounter. But suppose we have a particular rotation in mind, and we want to find the matrix that does that. In other words, suppose we have a line L an angle θ . What is the matrix that rotates around L by θ ?

Finding the matrix of a given rotation

Let $\mathbf{u} = (a, b, c)$ be a unit vector and let $\theta \in (0, \pi]$ be an angle. I claim that the matrix that rotates about the line through \mathbf{u} by the angle θ , measured as \mathbf{u} points at you, is given by

$$A = I + (\sin \theta)U + (1 - \cos \theta)U^2, \quad \text{where} \quad U = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}. \quad (5)$$

We will use the Rotation Matrix Theorem to show that A performs as advertised. The matrix U , whose entries come from the unit vector \mathbf{u} , satisfies

$$U^T = -U, \quad U^3 = -U, \quad U\mathbf{u} = \mathbf{0}, \quad \text{tr}(U) = 0, \quad \text{tr}(U^2) = -2. \quad (6)$$

Using the first two facts, one can check that $AA^T = I$, so that A is an orthogonal matrix. To show that $\det(A) = 1$ we use again the continuity argument from the Rotation Matrix Theorem: Replace θ by a variable $t \in [0, \theta]$ so that we have a path of orthogonal matrices

$$A_t = I + (\sin t)U + (1 - \cos t)U^2$$

with $A_0 = I$ and $A_\theta = A$. Again, since $\det A_t = \pm 1$ and $\det A_0 = \det I = 1$ and $\det A_t$ varies continuously, we must have $\det A_t = 1$ for all t , so $\det A = \det A_\theta = 1$. Hence A is indeed a rotation matrix. Since $U\mathbf{u} = \mathbf{0}$, it is clear that $A\mathbf{u} = \mathbf{u}$. So the axis of rotation is the line through \mathbf{u} . The trace of A is

$$\text{tr}(A) = \text{tr}(I) + (\sin \theta) \text{tr}(U) + (1 - \cos \theta) \text{tr}(U^2) = 3 + 0 - 2(1 - \cos \theta) = 1 + 2 \cos \theta,$$

so A rotates by θ , in one direction or the other. To pin that down, first recall that $a^2 + b^2 + c^2 = 1$, so $c^2 \leq 1$. Suppose $c^2 < 1$. Then $a^2 + b^2 > 0$, so the vector $\mathbf{v} = (-b, a, 0)$ is nonzero and is orthogonal to \mathbf{u} . We have

$$\mathbf{v} \times A\mathbf{v} = (-b, a, 0) \times (-ac, -abc, 1 - c^2) = (1 - c^2)\mathbf{u},$$

which is a positive multiple of \mathbf{u} . By the Rotation Matrix Theorem, this means A is indeed rotation by θ , as \mathbf{u} points at you. We leave the cases $c = \pm 1$ to the exercises.

Where does the matrix U comes from? Note that

$$\mathbf{u} \times \mathbf{e}_1 = (0, b, -c), \quad \mathbf{u} \times \mathbf{e}_2 = (-b, 0, a), \quad \mathbf{u} \times \mathbf{e}_3 = (c, -a, 0).$$

So in fact U is the matrix which sends an arbitrary vector \mathbf{v} to $U\mathbf{v} = \mathbf{u} \times \mathbf{v}$. We may rewrite equation (5) as

$$A\mathbf{v} = \mathbf{v} + (\sin \theta)\mathbf{u} \times \mathbf{v} + (1 - \cos \theta)\mathbf{u} \times (\mathbf{u} \times \mathbf{v}). \quad (7)$$

This version is less mysterious and easier to remember, but probably equation (5) is easier for computations. The matrix equation $U^3 = -U$ expresses the cross-product equation

$$\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v})) = -\mathbf{u} \times \mathbf{v},$$

which you can see by drawing a picture (see also exercise 17.11).

Exercise 17.1 Find the axis and angle of rotation of the following matrices.

$$\text{(a)} \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \text{(b)} \ B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{(c)} \ C = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Exercise 17.2 Find the axis and the cosine of the angle of rotation for

$$A = \frac{1}{4} \begin{bmatrix} 2\sqrt{3} & -\sqrt{3} & 1 \\ 2 & 3 & -\sqrt{3} \\ 0 & 2 & 2\sqrt{3} \end{bmatrix}.$$

Exercise 17.3 Show that for any number t , the matrix

$$A_t = \frac{1}{1+t+t^2} \begin{bmatrix} -t & t+t^2 & 1+t \\ 1+t & -t & t+t^2 \\ t+t^2 & 1+t & -t \end{bmatrix}$$

is a rotation, find its axis of rotation, and the cosine of the angle of rotation.

Comments: (i) You can use the continuity argument to easily compute $\det A_t$.

(ii) The formula for A_t shows how to write down lots of rotations with rational entries: just take t to be an integer.

Exercise 17.4 Find the matrix that rotates about the axis through $\mathbf{u} = \frac{1}{3}(-1, 2, 2)$ by the angle $\pi/4$, measured as \mathbf{u} points toward you.

Exercise 17.5 What rotation matrices do you get from the formula (5) if $c = \pm 1$?

Exercise 17.6 Find a 2×2 matrix U such that

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = I + (\sin \theta)U + (1 - \cos \theta)U^2.$$

Exercise 17.7 Let $R_{\mathbf{u}}$ be the reflection matrix in formula (2), about the plane $ax + by + cz = 0$, where $\mathbf{u} = (a, b, c)$ is a unit vector. Show that the matrix $A = -R_{\mathbf{u}}$ is a rotation and determine its axis and angle of rotation.

Hint: this can be done without writing down any matrices, and very little calculation.

Exercise 17.8 In two dimensions, an orthogonal matrix is either a rotation or a reflection. This is no longer true in three dimensions.

(a) Show that the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

is an orthogonal matrix that is neither a reflection nor a rotation.

(b) In three dimensions, we have $\det(-I) = -1$, so any orthogonal matrix with determinant -1 is of the form $-A$, where A is a rotation. For what kinds of rotations A is $-A$ a reflection?

Exercise 17.9 How many orthogonal matrices have all their entries integers? How many of these are rotations? What are the axes of these rotations? The connection between this problem and the octahedron will be discussed in class.

Exercise 17.10 Show that if A is a rotation then for any two vectors \mathbf{u}, \mathbf{v} we have

$$(A\mathbf{u}) \times (A\mathbf{v}) = A(\mathbf{u} \times \mathbf{v}).$$

What is the analogous result if A is orthogonal but not a rotation?

Hint: Use the formula $\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \det[\mathbf{u} \ \mathbf{v} \ \mathbf{w}]$.

Exercise 17.11 ¹ Let \mathbf{u} be a unit vector in \mathbb{R}^3 .

(a) Suppose \mathbf{w} in \mathbb{R}^3 is perpendicular to \mathbf{u} . Show that

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{w}) = -\mathbf{w}.$$

(b) Suppose \mathbf{v} is any vector in \mathbb{R}^3 . Show that

$$\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v})) = -\mathbf{u} \times \mathbf{v}.$$

Exercise 17.12 Show directly that equation (7) gives the desired rotation, without matrix computations. You can do this in the following steps. Prove:

(a) $A\mathbf{u} = \mathbf{u}$.

(b) For any $\mathbf{v} \in \mathbb{R}^3$ we have $\langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$.

Hint: Use the fact that $\langle \mathbf{v}, \mathbf{u} \times \mathbf{w} \rangle = -\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle$ for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

(c) A is a rotation about \mathbf{u} . (Use the continuity argument.)

(d) If \mathbf{v} is perpendicular to \mathbf{u} then

$$\langle \mathbf{v}, A\mathbf{v} \rangle = |\mathbf{v}|^2 \cos \theta, \quad |\mathbf{v} \times A\mathbf{v}| = |\mathbf{v}|^2 \sin \theta.$$

(e) From (c) we know that A is rotation by some angle α . Use (d) to show that $\alpha = \theta$.

¹This problem was not on the assigned homework. It replaces the old 17.11 which duplicated 17.5.