Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = a$ (i.e. if $f^{(k)}(a)$ exists for all $k = 0, 1, 2, 3, \ldots$) then we can construct the following power series in $(x - a)$:

$$\begin{aligned} & f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \ldots \\ = & \sum_{r=0}^{\infty} \frac{1}{r!}f^{(r)}(a)(x-a)^r. \end{aligned}$$

This series is called the *Taylor series* of f about $x = a$. If $a = 0$ then we usually call this the *Maclaurin series* of f.

We write

$$f(x) = \sum_{r=0}^{\infty} \frac{1}{r!}f^{(r)}(a)(x-a)^r \quad \text{for} \quad |x-a| < R$$

to mean that for all $x = c$, such that $|c - a| < R$, the sum

$$\sum_{r=0}^{\infty} \frac{1}{r!}f^{(r)}(a)(c-a)^r$$

is finite and equals $f(c)$. We call R the *radius of convergence*. If $R = \infty$ then the Taylor series equals $f(x)$ for all $x \in \mathbb{R}$.

1. Standard Maclaurin Series

- For $\alpha \in \mathbb{R}$, $(1+x)^\alpha = 1 + \alpha x + \dfrac{\alpha(\alpha-1)}{2!}x^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots$ for $|x| < 1$
- $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ for $-1 < x \leq 1$
- $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$ for $x \in \mathbb{R}$
- $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$ for $x \in \mathbb{R}$
- $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ for $x \in \mathbb{R}$
- $\sinh(x) = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots$ for $x \in \mathbb{R}$
- $\cosh(x) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots$ for $x \in \mathbb{R}$

Example 1 Using the standard series, find the Maclaurin series for:

(i) $f(x) = \sin(3x)$,

(ii) $f(x) = \cos(x) + \cosh(x)$,

(iii) $f(x) = e^{2x}\ln(1-x)$.

Include all terms up to the fifth power.

(i) Use the series for $\sin(x)$ with x replaced by $3x$:

$$\begin{aligned} \sin(3x) &= 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots \\ &= 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \dots \end{aligned}$$

The above series is valid for $3x \in \mathbb{R}$, i.e. for $x \in \mathbb{R}$.

(ii) The series for $\cos(x)$ and $\cosh(x)$ can be added term by term to obtain the series for their sum:

$$\begin{aligned} \cos(x) + \cosh(x) &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right) \\ &= 2 + \frac{2x^4}{4!} + \dots \\ &= 2 + \frac{x^4}{12} + \dots \end{aligned}$$

The above series is valid for $x \in \mathbb{R}$.

(iii) The series for this product is obtained by multiplying the series for e^{2x} and $\ln(1-x)$, respectively:

$$\begin{aligned} & e^{2x}\ln(1-x) \\ = & \left(1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(3x)^4}{4!} + \dots\right)\left((-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \dots\right) \\ = & \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{27x^4}{8} + \dots\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\right) \\ = & -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - 2x^2 - x^3 - \frac{2x^4}{3} - \frac{x^5}{2} - 2x^3 - x^4 - \frac{2x^5}{3} - \frac{4x^4}{3} - \frac{2x^5}{3} - \frac{27x^5}{8} - \dots \\ = & -x - \frac{5x^2}{2} - \frac{10x^3}{3} - \frac{13x^4}{4} - \frac{649x^5}{120} - \dots \end{aligned}$$

The above series is valid for $-1 < (-x) \le 1$, i.e. for $-1 \le x < 1$. Note that at each stage of the above calculations we kept only enough terms to ensure that we could get all terms with powers up to and including x^5.

Example 2 Find the first three nonzero terms in the Taylor Series of f about $x = a$ if

(i) $f(x) = \cos(2x)$, $a = \dfrac{\pi}{8}$

(ii) $f(x) = (x+1)^3$, $a = 2$

(iii) $f(x) = x\ln(x)$, $a = 1$

(i) $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$

$$\begin{aligned} & f\left(\frac{\pi}{8}\right) + f'\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right) + \frac{1}{2}f''\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right)^2 + \dots \\ = \; & \cos\left(\frac{\pi}{4}\right) + \left(-2\sin\left(\frac{\pi}{4}\right)\right)\left(x - \frac{\pi}{8}\right) + \frac{1}{2}\left(-4\cos\left(\frac{\pi}{4}\right)\right)\left(x - \frac{\pi}{8}\right)^2 + \dots \\ = \; & \frac{1}{\sqrt{2}} - \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 + \dots \end{aligned}$$

(ii) $f'(x) = 3(x+1)^2$, $f''(x) = 6(x+1)$

$$\begin{aligned} & f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \dots \\ = \; & 3^3 + 3 \times 3^2(x-2) + \frac{1}{2} \times 6 \times 3(x-2)^2 \\ = \; & 27 + 27(x-2) + 9(x-2)^2 + \dots \end{aligned}$$

(iii) $f'(x) = \ln(x) + 1$, $f''(x) = \dfrac{1}{x}$, $f'''(x) = -\dfrac{1}{x^2}$

$$\begin{aligned} & f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3 + \dots \\ = \; & 1\ln(1) + (\ln(1)+1)(x-1) + \frac{1}{2} \times \frac{1}{1}(x-1)^2 + \frac{1}{6} \times \frac{-1}{1}(x-1)^3 + \dots \\ = \; & (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \dots \end{aligned}$$

2. Error Estimation

The Taylor series of f about the point $x = a$ can be truncated in order to provide an approximating polynomial for the function. We also obtain an expression describing the error between this polynomial and f.

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n \\ &= P_n(x) + R_n \end{aligned}$$

- $P_n(x)$ is the *approximating (or Taylor) polynomial* of degree n
- R_n is the *truncating error*
- The truncating error can be written in various ways; we shall use Lagrange's form

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where c is some value that lies between a and x. We don't know the value of c, but we can obtain upper and/or lower bounds for the error.

Example 3 Find the Taylor polynomial of degree 3 for $\sin(x)$ about $x = \dfrac{\pi}{3}$. Hence estimate $\sin 64^\circ$ and find the maximum error in this approximation.

First note that $64^\circ = \left(\dfrac{60+4}{180}\right)\pi$ radians $= \dfrac{\pi}{3} + \dfrac{\pi}{45}$ radians.

$$f(x) = \sin(x) \Rightarrow f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos(x) \Rightarrow f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f^{(2)}(x) = -\sin(x) \Rightarrow f^{(2)}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f^{(3)}(x) = -\cos(x) \Rightarrow f^{(3)}\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(c) = \sin(c)$$

Therefore

$$\sin(x) \simeq \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3$$

and so

$$\begin{aligned}
\sin 64^\circ &= \sin\left(\frac{\pi}{3} + \frac{\pi}{45}\right) \\
&= \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\pi}{45} - \frac{\sqrt{3}}{4}\left(\frac{\pi}{45}\right)^2 - \frac{1}{12}\left(\frac{\pi}{45}\right)^3 \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{90} + \frac{\sqrt{3}\pi^2}{8100} + \frac{\pi^3}{1093500}
\end{aligned}$$

The truncating error is

$$R_3 = \frac{f^{(4)}(c)}{4!}\left(x - \frac{\pi}{3}\right)^4 = \frac{\sin(c)}{4!}\left(\frac{\pi}{45}\right)^4,$$

where $\frac{\pi}{3} < c < \frac{64\pi}{180}$. In order to estimate the maximum error we are required to find the maximum value of $\sin(c)$ (as all other terms are known). The best we can say is that $\sin(c) \leq 1$ and so

$$R_3 \leq \frac{1}{4!}\left(\frac{\pi}{45}\right)^4 = \frac{\pi^4}{24 \times 45^4}.$$