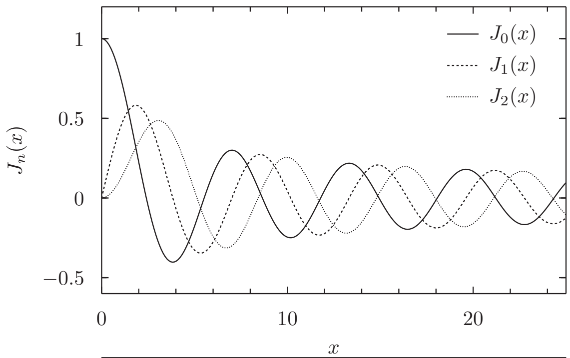
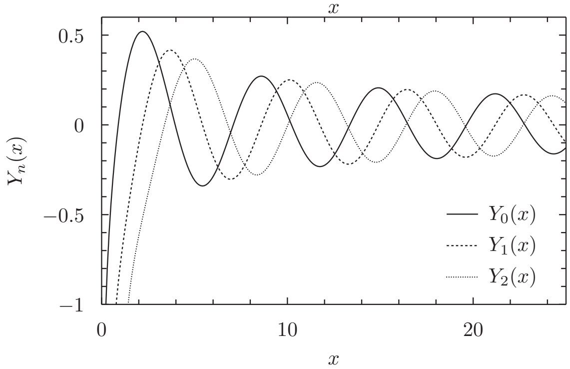
Bessel functions and Legendre polynomials

Bessel's equation of order p

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

Solution: $y(x) = aJ_p(x) + bY_p(x)$
If p is not an integer, $Y_p(x) = J_{-p}(x)$.





Behavior at zero: $J_0(0) = 1$, $J_p(0) = 0$ for $p = 1, 2, 3, \ldots$, $Y_p(0) = -\infty$ for all p

Behavior at infinity: $J_p(x) \to 0$ and $Y_p(x) \to 0$ as $x \to \infty$

Zeros $J_p(\sigma_{pm}) = 0$ for all p and $m = 1, 2, 3, \ldots$. (Each J_p, Y_p has infinite number of zeros labeled σ_{pm}.)

Derivatives

- $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$,
- $\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$
- $\frac{d}{dx}[x^p Y_p(x)] = x^p Y_{p-1}(x)$,
- $\frac{d}{dx}[x^{-p} Y_p(x)] = -x^{-p} Y_{p+1}(x)$

Orthogonality

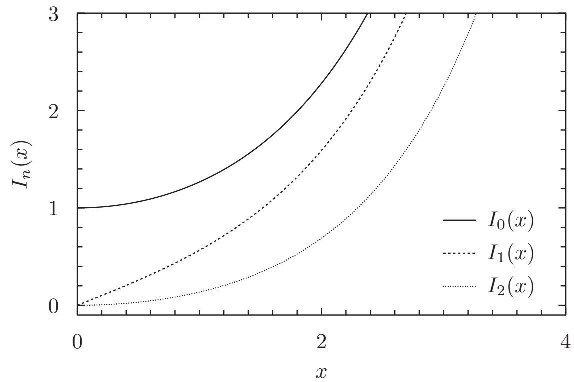
$$\int_o^\ell x J_p(\sigma_{pm} x/\ell) J_p(\sigma_{pn} x/\ell)\, dx = \begin{cases} 0 & n \neq m \\ \frac{\ell^2}{2} J_{p+1}^2(\sigma_{pm}) & n = m \end{cases}$$

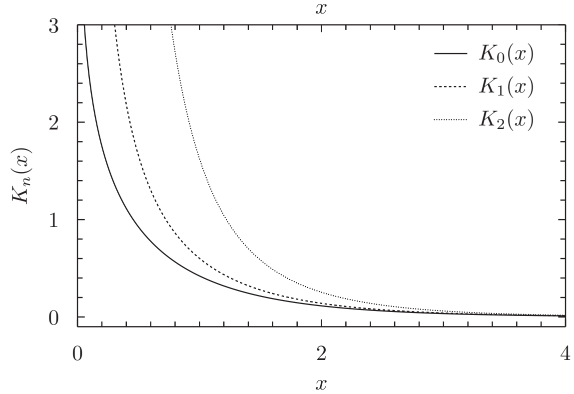
Idea: Bessel functions are like sine and cosine.

Modified Bessel's equation of order p

$$x^2 y'' + xy' - (x^2 + p^2)y = 0$$

Solution: $y(x) = aI_p(x) + bK_p(x)$
If p is not an integer, $K_p(x) = I_{-p}(x)$.





Behavior at zero: $I_0(0) = 1$, $I_p(0) = 0$, $p = 1, 2, 3, \ldots$ $K_p(0) = \infty$ for all p

Behavior at infinity: $I_p(x) \to \infty$, $K_p(x) \to 0$ as $x \to \infty$

Derivatives

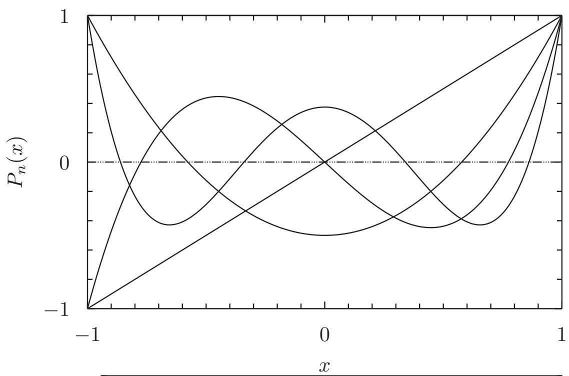
- $\frac{d}{dx}[x^p I_p(x)] = x^p I_{p-1}(x)$,
- $\frac{d}{dx}[x^{-p} I_p(x)] = x^{-p} I_{p+1}(x)$
- $\frac{d}{dx}[x^p K_p(x)] = -x^p K_{p-1}(x)$,
- $\frac{d}{dx}[x^{-p} K_p(x)] = -x^{-p} K_{p+1}(x)$

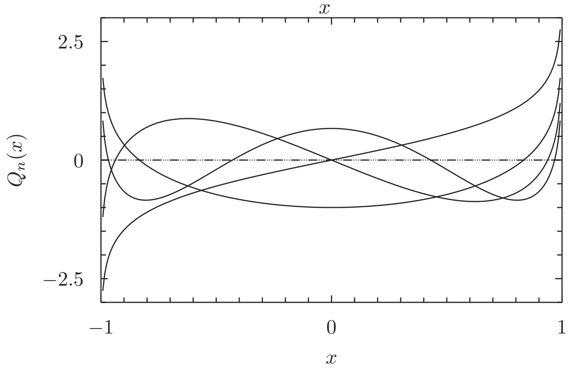
Idea: Modified Bessel functions are like e^{px} and e^{-px}.

Legendre's equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

where n is an integer.
Solution: $y(x) = aP_n(x) + bQ_n(x)$





- $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
- $P_0 = 1,\ P_1 = x,\ P_2 = \frac{1}{2}(3x^2-1),\ \ldots$
- $Q_n(x) = \frac{1}{2}P_n(x)\ln\left(\frac{1+x}{1-x}\right) - \sum_{m=1}^{n}\frac{1}{n}P_{m-1}P_{n-m}$ for $n = 1, 2, 3, \ldots$
- $Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Note: $Q_n(x)$ only finite on $-1 < x < 1$

Properties on $-1 < x < 1$

- n even: $P_n(-1) = 1, P_n(1) = 1$
- n odd: $P_n(-1) = -1, P_n(1) = 1$
- $Q_n(-1) = \pm\infty$, $Q_n(1) = \pm\infty$

Zeros:

- $P_n(x)$ has n zeros.
- $P_n(0) = 0$ if n odd.
- $Q_n(x)$ has $n+1$ zeros.

Orthogonality:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}$$

$$\int_0^{\pi} P_n(\cos\phi)P_m(\cos\phi)\sin\phi\,d\phi$$
$$= \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}$$

Idea: $P_n(x)$ is a specific polynomial in x of order n. $Q_n(x)$ is a polynomial times a log.