THE LEGENDRE POLYNOMIALS ASSOCIATED WITH BERNOULLI, EULER, HERMITE AND BERNSTEIN POLYNOMIALS

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Abstract. In the present paper, we deal mainly with arithmetic properties of Legendre polynomials by using their orthogonality property. We show that Legendre polynomials are proportional with Bernoulli, Euler, Hermite and Bernstein polynomials.

Keywords and phrases. Legendre polynomials, Bernoulli polynomials, Euler polynomials, Hermite polynomials, Bernstein polynomials, orthogonality.

1. Introduction

Legendre polynomials, which are special cases of Legendre functions, are introduced in 1784 by the French mathematician A. M. Legendre (1752-1833). Legendre functions are a vital and important in problems including spherical coordinates. Due to their orthogonality properties they are also useful in numerical analysis (see [9]). Besides, the Legendre polynomials, $P_n(x)$, are described via the following generating function:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n. \tag{1}$$

Legendre polynomials are the everywhere regular solutions of *Legendre's differential equation* that we can write as follows:

$$\begin{aligned}\left(1-x^2\right)\frac{d}{dx}P_n(x) - 2x\frac{d}{dx}P_n(x) + mP_n(x) &= \\ &= \frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right] + mP_n(x) = 0,\end{aligned}$$

where $m = n(n+1)$ and $n = 0, 1, 2, \cdots$. Taking $x = 1$ in (1) and by using geometric series, we see that $P_n(1) = 1$, so that the Legendre polynomials are *normalized*.

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Legendre polynomials can be generated using *Rodrigue's formula* as follows:

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n . \tag{2}$$

Note that the right hand side of (2) is a polynomial (see [3], [9]).

The Bernoulli polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}, \quad |t| < 2\pi \text{ (see [4], [7])}. \tag{3}$$

By (3), we know that $\frac{dB_n(x)}{dx} = nB_{n-1}(x)$. Taking $x = 0$ in (3), we have $B_n(0) := B_n$ that stands for n-th Bernoulli number.

The Euler polynomials are known to be defined as:

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}. \tag{4}$$

The Euler polynomials can also be expressed by explicit formulas, e.g.

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k},$$

where E_k means the Euler numbers. These numbers are expressed with the Euler polynomials through $E_k = 2^k E_k(1/2)$.

Now also, we give the definition of Hermite polynomials as follows:

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \tag{5}$$

Let $C([0,1])$ be the space of continuous functions on $[0,1]$. For $f \in C([0,1])$, Bernstein operator for f is defined by

$$\mathcal{B}_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k},$$

where $n, k \in \mathbb{N}^* := \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of natural numbers. Here $B_{k,n}(x)$ is called Bernstein polynomials, which are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad x \in [0,1] \text{ (cf. [1], [6].)} \tag{6}$$

In [9], [3], the orthogonality of Legendre polynomials is known as

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{m,n}, \text{ where } \delta_{m,n} \text{ is Kronecker's delta.} \tag{7}$$

In [7], by using orthogonality property of Legendre [7], Kim *et al*. effected interesting identities for them. We also obtain some interesting properties of the Legendre polynomials arising from Bernoulli, Euler, Hermite and Bernstein polynomials.

2. Identities on the Legendre polynomials arising from Bernoulli, Euler, Hermite and Bernstein polynomials

Let $\mathcal{P}_n = \{q(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$. Then we define an inner product on $\mathcal{P}_n$ as follows:

$$\langle q_1(x), q_2(x) \rangle = \int_{-1}^{1} q_1(x) q_2(x) \, dx, \ \left(q_1(x), q_2(x) \in \mathcal{P}_n\right). \tag{8}$$

Note that $P_0(x), P_1(x), ..., P_n(x)$ are the orthogonal basis for $\mathcal{P}_n$. Let us now consider $q(x) \in \mathcal{P}_n$; then we see that

$$q(x) = \sum_{k=0}^{n} C_k P_k(x), \tag{9}$$

where the coefficients C_k are defined over the field of real numbers.

From the above, we readily see that

$$\begin{aligned} C_k &= \frac{2k+1}{2} \langle q(x), P_k(x) \rangle = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) q(x) \, dx \\ &= \frac{2k+1}{k! 2^{k+1}} \int_{-1}^{1} \left(\frac{d^k}{dx^k} \left(x^2-1\right)^k \right) q(x) \, dx. \end{aligned} \tag{10}$$

By (9) and (10), we have the following proposition.

Proposition 2.1. *Let $q(x) \in \mathcal{P}_n$ and $q(x) = \sum_{k=0}^{n} C_k P_k(x)$, then*

$$C_k = \frac{2k+1}{k! 2^{k+1}} \int_{-1}^{1} \left(\frac{d^k}{dx^k} \left(x^2-1\right)^k \right) q(x) \, dx \ \text{(see [7])}.$$

If we take $q(x) = x^n$ in Proposition (2.1), the coefficients C_k can be found as

$$C_k = \frac{(2k+1) 2^{k+1}}{(n+k+2)!} \frac{n! \left(\frac{n+k+2}{2}\right)!}{\left(\frac{n-k}{2}\right)!} \text{ for } n-k \equiv 0 \pmod 2 \ \text{(see [7])}. \tag{11}$$

Let $q(x) = B_n(x)$. Then by using Proposition 2.1 and (11), we have

$$C_k = \frac{2k+1}{k! 2^{k+1}} \int_{-1}^{1} \left(\frac{d^k}{dx^k} \left(x^2-1\right)^k \right) B_n(x) \, dx$$

where $B_n(x)$ are the aforementioned Bernoulli polynomials that can be expressed through Bernoulli numbers B_n as follows:

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j.$$

From this, we have

$$C_k = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} \left[\int_{-1}^{1} \left(\frac{d^k}{dx^k} \left(x^2-1\right)^k \right) x^j dx \right]$$

$$= \left(2^{k+2}k + 2^{k+1}\right) \sum_{j=0}^{n} \frac{j! \binom{n}{j} \left(\frac{j+k+2}{2}\right)!}{\left(\frac{j-k}{2}\right)! \, (j+k+2)!} B_{n-j} \text{ for } j-k \equiv 0 \, (\operatorname{mod} 2) \, .$$

Therefore we have the following theorem.

Theorem 2.2. *Let* $B_n(x) = \sum_{k=0}^{n} C_k P_k(x) \in \mathcal{P}_n$. *Then we have*

$$B_n(x) = 2 \sum_{k=0}^{n} \Bigg(\left(2^{k+2}k + 2^{k+1}\right) \times \times \sum_{j-k \equiv 0 (\operatorname{mod} 2)}^{n} \frac{j! \binom{n}{j} \left(\frac{j+k+2}{2}\right)!}{\left(\frac{j-k}{2}\right)! \, (j+k+2)!} B_{n-j} \Bigg) P_k(x) \, .$$

Let $H_n(x) \in \mathcal{P}_n$. By Proposition 2.1 and (11), we have the following theorem.

Theorem 2.3. *Let* $H_n(x) = \sum_{k=0}^{n} C_k P_k(x) \in \mathcal{P}_n$. *Then we have*

$$H_n(x) = \sum_{k=0}^{n} \Bigg(\left(2^{k+2}k + 2^{k+1}\right) \times \times \sum_{j-k \equiv 0 (\operatorname{mod} 2)}^{n} \frac{2^j \binom{n}{j} j! \left(\frac{j+k+2}{2}\right)!}{(j+k+2)! \left(\frac{j-k}{2}\right)!} H_{n-j} \Bigg) P_k(x) \, .$$

Let the Bernstein polynomials $B_{j,n}(x) \in \mathcal{P}_n$. By Proposition 2.1 and (11), we have the following theorem.

Theorem 2.4. *Let* $B_{j,n}(x) = \sum_{k=0}^{n} C_k P_k(x) \in \mathcal{P}_n$. *We have*

$$B_{j,n}(x) = \sum_{k=0}^{n} \Bigg(\left(2^{k+2}k + 2^{k+1}\right) \times \times \sum_{l+j-k \equiv 0 (\operatorname{mod} 2)}^{n-j} \frac{\binom{n-j}{l} (-1)^l (l+j)! \left(\frac{l+j+k+2}{2}\right)!}{(l+j+k+2)! \left(\frac{l+j-k}{2}\right)!} \Bigg) P_k(x) \, .$$

The following equality is defined by Kim et al. in [7]:

$$\sum_{k=0}^{n} B_k(x) B_{n-k}(x) = \frac{2}{n+2} \sum_{l=0}^{n-2} \binom{n+2}{l} B_{n-l} B_l(x) + (n+1) B_n(x) \, . \tag{12}$$

Let $\sum_{k=0}^{n} B_k(x) B_{n-k}(x) \in \mathcal{P}_n$. By Proposition 2.1 and (11), we get the following theorem.

Theorem 2.5. *Let* $\sum_{k=0}^{n} B_k(x) B_{n-k}(x) \in \mathcal{P}_n$. *Then we have*

$$\begin{aligned}\sum_{k=0}^{n} B_k(x) B_{n-k}(x) = &\sum_{k=0}^{n} \left(2^{k+2}k + 2^{k+1}\right) \times \\ &\times \left[\frac{2}{n+2} \sum_{l=0}^{n-2} \sum_{j-k \equiv 0 (\mathrm{mod}\, 2)}^{l} B_{n-l} B_{l-j} \frac{\binom{n+2}{l}\binom{l}{j} j! \left(\frac{j+k+2}{2}\right)!}{\left(\frac{j-k}{2}\right)! (j+k+2)!} \right. \\ &\left. + (n+1) \sum_{l-k \equiv 0 (\mathrm{mod}\, 2)}^{n} \binom{n}{l} B_{n-l} \frac{l! \left(\frac{l+k+2}{2}\right)!}{\left(\frac{l-k}{2}\right)! (l+k+2)!} \right] P_k(x).\end{aligned}$$

Let $q(x) = \sum_{k=0}^{n} E_k(x) E_{n-k}(x) \in \mathcal{P}_n$. In [8], Kim et al derived convolution formula for the Euler polynomials as

$$\sum_{k=0}^{n} E_k(x) E_{n-k}(x) = -\frac{4}{n+2} \sum_{l=0}^{n} \binom{n+2}{l} E_{n-l+1} B_l(x).$$

By Proposition 2.1 and (11), we get the following theorem.

Theorem 2.6. *The following equality holds true:*

$$\begin{aligned}\sum_{k=0}^{n} E_k(x) E_{n-k}(x) &= -\frac{8}{n+2} \sum_{k=0}^{n} \left(2^{k+1}k + 2^k\right) \\ &\quad \times \{\sum_{l=0}^{n} \sum_{j-k \equiv 0 (\mathrm{mod}\, 2)}^{l} \binom{n+2}{l} \binom{l}{j} E_{n-l+1} \\ &\quad \times B_{l-j} \frac{j! \left(\frac{j+k+2}{2}\right)!}{\left(\frac{j-k}{2}\right)! (j+k+2)!} \} P_k(x).\end{aligned}$$

Remark 2.7. By using Theorem 2.1, we can find many interesting identities for the special polynomials in connection with Legendre polynomials.

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