10.2 Series and Convergence

- Write sums using sigma notation
- Find the partial sums of series and determine convergence or divergence of infinite series
- Find the N^{th} partial sums of geometric series and determine the convergence or divergence of the series
- Use geometric series to model and solve real-life problems

Sigma Notation

The (finite) sum $a_1 + a_2 + a_3 + ... + a_N$ can be written as

$$\sum_{i=1}^{N} a_i$$

i is called the index of summation and 1 and N are the lower and upper limits of summation, respectively.

Examples

$$1 + 1/2 + 1/3 + 1/4 + 1/5 = \sum_{i=1}^{5} \frac{1}{i}.$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \ldots + (-1)^N \frac{1}{2^N} = \sum_{i=0}^{N} (-1)^i \frac{1}{2^i}.$$

$$4(1) + 4\left(\frac{1}{4}\right) + 4\left(\frac{1}{9}\right) + 4\left(\frac{1}{16}\right) = \sum_{i=1}^{4} 4\left(\frac{1}{i^2}\right)$$

Infinite Series and Partial Sums

The infinite summation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots.$$

is called an infinite series. The sequence of partial sums of the series is denoted by

$$S_1 = a_1,\ S_2 = a_1 + a_2,\ S_3 = a_1 + a_2 + a_3,\ \ldots, S_n = a_1 + a_2 + \ldots + a_n.$$

Convergence and divergence of an infinite series

We say that an infinite series converges to (a single finite value) S if the sequence of partial sums $\{S_n\}$ converges to S. We write

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n = S.$$

Here S is called the sum of the series. If the limit of the sequence of partial sums $\{S_n\}$ does not exist, then the series diverges.

Example: Determining convergence or divergence

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

To see if the series converges or diverges we look at the sequence of partial sums:

$$\begin{aligned}
S_1 &= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}
\end{aligned}$$

Example: cont.

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

We are starting to see a pattern? In general

$$S_n = \frac{2^n - 1}{2^n}.$$

So to determine the sum of the infinite series, now all we have to figure out is what does $\{S_n\}$ converge to (if it does). But we have seen this before and we know that $\lim_{n\to\infty} S_n = 1$, so the infinite series sums to 1.

The n^{th} term test for divergence

One easy way to see if a series diverges is to check if the individual terms of the series are going to zero, if they are not then the series has to diverge.

n^{th}-Term test for divergence

Consider the series $\sum_{n=1}^{\infty} a_n$. If

$$\lim_{n\to\infty} a_n \neq 0$$

then the series diverges.

Example

Determine if the following series converge:

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{n+1}+1}$$

If we look at

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1}+1}$$

we see that as

$$\frac{2^n}{2^{n+1}+1} = \frac{\frac{2^n}{2^{n+1}}}{\frac{2^{n+1}}{2^{n+1}} + \frac{1}{2^{n+1}}} = \frac{1/2}{1 + \frac{1}{2^{n+1}}}$$

So we see that as $n \to \infty$ the terms $a_n \to \frac{1}{2} \neq 0$ so the series must diverge.

Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} r^n$$

where r is any real number is called a geometric series. It has special known convergence properties. In particular if $|r| < 1$ then this series converges:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ if } |r| < 1.$$

Geometric series example

If $r = \frac{1}{2}$, since $|r| < 1$ we have that

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2.$$

How is this different from our previous example

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Note that the second series starts at $n = 1$, but

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + 1 = 2$$

So these two are in agreement.

Geometric Series cont.

If $|r| \geq 1$ then the geometric series will diverge. As an example consider $r = 2$

$$\sum_{n=0}^{\infty} 2^n,$$

has partial sums

$$\begin{aligned} S_0 &= 1 \\ S_1 &= 1 + 2 = 3 \\ S_2 &= 1 + 2 + 4 = 7 \\ S_3 &= 1 + 2 + 4 + 8 = 15 \end{aligned}$$

So we see that $S_n \to \infty$ meaning that the sum diverges.

Example

Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{4}{3^{n+1}}$$

Since this isn't quite in the right form to apply the formula we can factor out both the 4 and one copy of $\frac{1}{3}$ to get

$$\sum_{n=0}^{\infty} \frac{4}{3^{n+1}} = \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{4}{3} \left(\frac{1}{1 - \frac{1}{3}}\right) = \frac{4}{3} \cdot \frac{3}{2} = 2$$

Modeling a bouncing ball

Suppose a ball is dropped from a height of 6 feet and begins to bounce. The height of each bounce is $\frac{3}{4}$ that of the preceding bounce. Find the total distance travelled by the ball.

Solution: When the ball hits the ground the first time it has travelled $D_1 = 6$ feet.

Between the first and second bounces the ball will travel $D_2 = \frac{3}{4}(6) + \frac{3}{4}(6) = \frac{3}{4}(12)$ feet.

Between the second and third bounces the ball will travel $D_3 = \left(\frac{3}{4}\right)^2 (12)$

and so on, so between the $(n-1)^{th}$ and n^{th} bounces the ball will travel $D_n = \left(\frac{3}{4}\right)^{n-1} (12)$.

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bouncing ball cont.

Summing these distances (after D_1) we have

$$\text{distance} = 6 + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n (12)$$

Now this is almost the geometric series, we know

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n (12) = \frac{12}{1 - 3/4} = 48$$

and we can see that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n (12) = 12 + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = 48$$

Therefore the sum we want is 36. Hence

$$\text{distance} = 6 + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n (12) = 6 + 36 = 42 \text{ feet}$$