Cauchy’s Root Test
&
Cauchy’s Integral Test

Cauchy's Root Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

$\sum u_n$ is convergent if $l < 1$

$\sum u_n$ is divergent if $l > 1$

Test fails if $l = 1$

Q 15 Test the convergence of the series

$$\sum \left(\frac{nx}{n+1}\right)^n$$

Here $u_n = \left(\frac{nx}{n+1}\right)^n$

$$(u_n)^{\frac{1}{n}} = \frac{x}{1+\frac{1}{n}} = x$$

So, by Cauchy's Root test, the series is convergent if $x < 1$ and divergent if $x > 1$.

For $x = 1$, $u_n = \left(\frac{n}{n+1}\right)^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0$$

The series is divergent for $x = 1$.

Hence, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

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The series is divergent for $x = 1$.

Hence, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Cauchy's Integral Test : If for $x \geq 1$, $f(x)$ is a non-negative, decreasing function of x such that $f(n) = u_n$ for all positive integral value of n, then the series $\sum u_n$ and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.

Q 16 Show that the series $\sum \frac{1}{n^p}$ converges if p> 1 and diverges if $0 < p \geq 1$

Here $u_n = \frac{1}{n^p} = f(n)$

$\therefore f(x) = \frac{1}{x^p}$

For $x \geq 1$, $f(x)$ is +ve and decreasing function of x.

$\therefore$ Cauchy's Integral test is applicable.

Case I : When $p \neq 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$$

$$\left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

Subcase I : when p > 1 ⇒ p − 1 > 0, so that

$$\int_1^\infty f(x)\, dx =$$

$$\frac{1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^\infty = -\frac{1}{p-1}[0-1]$$

$$= \frac{1}{p-1} = \text{finite value}$$

$$\Rightarrow \quad \int_1^\infty f(x)\, dx \text{ converges.}$$

$\Rightarrow \sum u_n$ is convergent.

Subcase II : when 0 < p < 1, 1 − p > 0, so that

$$\int_1^\infty f(x)\, dx =$$

$$\frac{1}{1-p}\left[x^{1-p}\right]_1^\infty = \frac{1}{1-p}[\infty - 1]$$

$$= \infty$$

$$\Rightarrow \quad \int_1^\infty f(x)\, dx \text{ diverges.}$$

$\Rightarrow \sum u_n$ is divergent.

Case II: when p = 1, $f(x) = \frac{1}{x}$

$$\int_1^\infty f(x)\, dx = \int_1^\infty \frac{1}{x}\, dx = [\log x]_1^\infty = \log \infty - \log 1 = \infty - 0 = \infty$$

$$\Rightarrow \quad \int_1^\infty f(x)\, dx \text{ diverges.}$$

$\Rightarrow \sum u_n$ is divergent.

Hence $\sum u_n = \sum \frac{1}{n^p}$ converges if p > 1 and diverges if $0 < p \leq 1$ VI