# LEGENDRE, EULER AND BERNOULLI POLYNOMIALS

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The main aim of this work is to obtain some expansion formulas of Legendre polynomials in either Euler or Bernoulli polynomials. The following formula of finite summation plays an important role in our discussion.

**Lemma 1**. For $\sigma = 0$ or $1$ the following formula is valid

$$\sum_{l=0}^{n} \frac{(2n+2l+2\sigma)!(1-2^{2m+2l+2\sigma+2})B_{2m+2l+2\sigma+2}}{(2l)!(2n-2l)!(2m+2l+2\sigma+2)!} = \frac{(-1)^{\sigma+1}(2n+\sigma)!}{4(4n+2\sigma+1)!}\delta_{mn}, \quad 0 \le m \le n. \tag{1}$$

Here $B_n$ are Bernoulli numbers [3].

**Proof**. Formula (1) appeared in incorrect form in [5], Chapter 5, 5.1.1.7. We shall prove it by using the orthogonality relation of the modified Lommel polynomials [2]. Let $v_n(x) = h_{2n,1/2}\left(\frac{2\sqrt{x}}{\pi}\right)$, where $h_{n,\nu}(x)$ are Lommel polynomials [2]. Then

$$v_n(x) = (-1)^n \sum_{k=0}^{n} \frac{(2n+2k)!}{(2k)!(2n-2k)!}(-\frac{x}{\pi^2})^k, \tag{2}$$

and the sequence $\{v_n(x)\}$ satisfies the following discrete orthogonal relation

$$\sum_{k=0}^{\infty} v_n\left(\frac{1}{(2k+1)^2}\right) v_m\left(\frac{1}{(2k+1)^2}\right) \cdot \frac{1}{(2k+1)^2} = \frac{\pi^2}{8(4n+1)}\delta_{mn}. \tag{3}$$

Putting the expressions for $v_n(x)$ and $v_m(x)$ from (2) into (3), changing the order of summation and expressing the infinite series through Bernoulli numbers ([3], Chapter 1, \$ 1.13) we get formula (1) in case $\sigma = 0$.

For the case $\sigma = 1$ let $t_n(x) = x^{-1}[v_{n+1}(x) + v_n(x)]$. We have

$$t_n(x) = \frac{2(-1)^n(4n+3)}{\pi^2} \sum_{k=0}^{n} \frac{(2n+2k+2)!}{(2k+1)!(2n-2k)!}\left(\frac{-x}{\pi^2}\right)^k. \tag{4}$$

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and the sequence $\{t_n(x)\}$ satisfies the orthogonal relation

$$\sum_{k=0}^{\infty} t_n\left(\frac{1}{(2k+1)^2}\right)\frac{1}{(2k+1)^{2m+4}} = \frac{(2n)!\pi^{2n+2}}{8(4n+1)!}\delta_{mn}, \quad 0 \le m \le n. \tag{5}$$

Putting expression (4) into (5), changing the order of summation and replacing the infinite sum by the formula ([3], Chapter 1) we obtain formula (1) for the case $\sigma = 1$.
Now we shall use the formula of finite summation (1) to prove the following theorem

**Theorem 1**. Let $\{P_n(x)\}$ be Legendre polynomial sequence and $\{E_n(x)\}$ be Euler polynomial sequence [3] . The following equality holds

$$\begin{aligned} P_{2n+\sigma}(2x-1) &= \sum_{k=0}^{n} \frac{(2n+2k+2\sigma)!}{(2k)!(2k+\sigma)!(2n-2k)!} E_{2k+\sigma}(x), \\ & \quad n \ge 0; \quad \sigma = 0 \text{ or } 1. \end{aligned} \tag{6}$$

**Proof**. First, we apply the lemma in case $\sigma = 0$. Applying the formula ([5], chapter 2)

$$\int_0^1 E_m(x)E_n(x)dx = 4(-1)^n(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2}, \tag{7}$$

formula (1) can be replaced by

$$\int_0^1 \frac{E_{2m}(x)}{(2m)!}\sum_{k=0}^{n}\frac{(2n+2k)!E_{2k}(x)}{[(2k)!]^2(2n-2k)!} = \frac{(2n)!\delta_{mn}}{(4n+1)!}, \ 0 \le m \le n. \tag{8}$$

If we set

$$\sum_{k=0}^{n}\frac{(2n+2k)!E_{2k}(x)}{[(2k)!]^2(2n-2k)!} = P^*_{2n}(x), \tag{9}$$

then $P^*_{2n}(x)$ is a polynomial of degree $2n$ and we obtain

$$\int_0^1 E_{2m}(x)P^*_{2n}(x)dx = \frac{\{(2n)!\}^2}{(4n+1)!}\delta_{mn}, \ 0 \le m \le n. \tag{10}$$

Noticing that $B_{2m+1} = 0$ for $m > 0$, we have

$$\begin{aligned} &\int_0^1 P_{2n}^*(x) E_{2m+1}(x) dx \\ = &\sum_{k=0}^{n} \frac{4(2m+1)!(2n+2k)!(2^{2m+2k+3}-1)B_{2m+2k+3}}{(2k)!(2n-2k)!(2m+2k+3)!} = 0. \end{aligned} \qquad (11)$$

Hence, the polynomial $P_{2n}^*(x)$ is orthogonal to the system $\{E_0(x), E_1(x), ..., E_{2n-1}(x)\}$ with respect to the weight 1 on the interval $[0, 1]$. For the sequence of Legendre polynomials $\{P_n(2x-1)\}$ is orthogonal on the same interval with the same weight we have

$$P_{2n}^*(x) = \alpha_n P_{2n}(2x-1), \qquad n \geq 0. \qquad (12)$$

Comparing the coefficients of $x^{2n}$ in $P_{2n}^*(x)$ and $P_{2n}(2x-1)$ the scalars $\alpha_n$ can be found to be 1.

Let now $\sigma = 1$. Replacing $(1 - 2^{2m+2l+4})B_{2m+2l+4}/(2m+2l+4)!$ in (1) by integral (7) we get

$$\int_0^1 \frac{E_{2m+1}(x)}{(2m+1)!} \sum_{k=0}^{n} \frac{(2n+2k+2)!E_{2k+1}(x)}{(2k)!(2k+1)!(2n-2k)!} dx = \frac{(2n+1)!}{(4n+3)!} \delta_{mn}, 0 \leq m \leq n. (13)$$

Setting

$$\sum_{k=0}^{n} \frac{(2n+2k+2)!E_{2k+1}(x)}{(2k)!(2k+1)!(2n-2k)!} = P_{2n+1}^*(x), \qquad (14)$$

then $P_{2n+1}^*(x)$ is a polynomial of precise degree $2n+1$. Completely similar to the corresponding step in the proof of the theorem in case $\sigma = 0$ we also obtain

$$\int_0^1 E_{2m}(x) P_{2n+1}^*(x) dx = 0. \qquad (15)$$

Hence, $P_{2n+1}^*(x)$ is orthogonal to the system $\{E_0(x), E_1(x), ..., E_{2n}(x)\}$ with respect to the weight 1 on the interval $[0, 1]$. Hence, there exists a sequence of scalars $\{\beta_n\}$ with

$$P_{2n+1}^*(x) = \beta_n P_{2n+1}(2x-1), \qquad n \geq 0. \qquad (16)$$

By comparing the coefficients of $x^{2n+1}$ in $P_{2n+1}^*(x)$ and $P_{2n+1}(2x-1)$ the scalars $\beta_n$ can be found to be 1, and the proof of the theorem is finished.

Similarly, the following theorem holds

**Theorem 2**. Legendre polynomials are expressed through Bernoulli polynomials $B_n(x)$ [3] by the following formula

$$\begin{aligned} P_{2n+1+\sigma}(2x-1) &= 2\sum_{k=0}^{n} \frac{(2n+2k+2\sigma+1)!B_{2k+\sigma+1}(x)}{(2k+\sigma)!(2k+\sigma+1)!(2n-2k+1)!}, \\ & n \geq 0; \quad \sigma = 0 \text{ or } 1. \end{aligned} \tag{17}$$

The proof of Theorem 2 is based on the Lemma

**Lemma 2**. For $\sigma = 0$ or 1 the formula

$$\begin{aligned} &\sum_{l=0}^{n} \frac{(2n+2l+2\sigma+1)!B_{2m+2l+2\sigma+2}}{(2l+\sigma)!(2n-2l+1)!(2m+2l+2\sigma+2)!} \\ &= (-1)^{\sigma} \frac{(2n+\sigma+1)!}{(4n+2\sigma+3)!2}\delta_{mn}, \quad 0 \leq m \leq n, \end{aligned} \tag{18}$$

is valid.

## References


[1] Chihara T.S. An Introduction to Orthogonal Polynomials. Gordon and Breach, New York - London - Paris, 1978. [2] Dickinson D. Proc. Amer. Math. Soc. 5(1954), p. 946–956. [3] Erdelyi A. et al. Higher Transcendental Functions, Vol.1, McGraw - Hill, New York- Toronto - London, 1952. [4] Erdelyi A. et al. Higher Transcendental Functions, Vol.2, McGraw - Hill, New York- Toronto - London, 1953. [5] Prudnikov A.P., Brychkov Yu. A. and Marichev O.I. Integrals and Series, Vol.3: More Special Functions. Gordon and Breach, Philadelphia - Reading - Paris - Montreux - Tokyo - Melbourne, 1990.



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