Solving Systems of Linear Equations Using Matrices

I. Gauss–Jordan Method

A very systematic method of solving linear systems of equations is called the Gauss-Jordan Method. This method involves the use of matrices, the plural of the word matrix. A **matrix** is a rectangular array of numbers arranged in rows and columns. The numbers in the array are called the elements of the matrix.

To solve linear systems of equations, we will first form the **augmented matrix** by writing the numerical coefficients and constants of each equation in a matrix. Before forming the matrix, be sure to line-up the variables so that they are in the same column; any missing terms will require a coefficient of 0. We separate the coefficients from the constants with a vertical line. Two examples are:

System of Linear Equations	**Augmented Matrix**
$\begin{cases} 5x+8y = 2 \\ 7x-3y = 88 \end{cases}$	$\left[\begin{array}{cc\|c} 5 & 8 & 2 \\ 7 & -3 & 88 \end{array}\right]$
$\begin{cases} x + 2y + 4z = 24 \\ 2x - y - 3z = -22 \\ 3x \quad\quad + 5z = 19 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & 2 & 4 & 24 \\ 2 & -1 & -3 & -22 \\ 3 & 0 & 5 & 19 \end{array}\right]$

If two systems of linear equations have the same solution sets, then the systems are said to be equivalent. Similarly, the augmented matrices of equivalent systems are also equivalent.

Example: Verify that the augmented matrices A and B are equivalent:

$$A = \left[\begin{array}{ccc|c} 1 & 6 & -2 & -14 \\ 3 & 0 & 2 & 4 \\ 5 & -3 & 3 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

By definition, the two augmented matrices are equivalent if the associated system of linear equations have the same solution sets. The solution set of the system associated with matrix B can be readily identified by converting back to equations: $\begin{cases} x = -4 \\ y = 1 \\ z = 8 \end{cases}$

We must show that this is also a solution to the system associated with matrix A:

$$\begin{cases} x + 6y - 2z = -14 \\ 3x \qquad + 2z = 4 \\ 5x - 3y + 3z = 1 \end{cases}$$

Substituting in -4 for x, 1 for y, and 8 for z produces a true statement in each case (verify this!) so the matrices are indeed equivalent.

In the **Gauss–Jordan Method**, our goal is to transform our original augmented matrix into one like matrix B in the above example, with ones along the main diagonal from upper left to lower right and zeros everywhere else. Then the solution will be readily identified. To do this, we use **elementary row operations**, which produce equivalent matrices. Each row operation corresponds to an operation which can be performed on a system of equations to produce an equivalent system.

Elementary Row Operations

1. Any two rows of the matrix may be interchanged. (This corresponds to interchanging the position of any two equations in a system.)

2. The elements of any row of the matrix may be multiplied or divided by a nonzero constant. (This corresponds to multiplying or dividing both sides of an equation by the same nonzero constant.)

3. A nonzero multiple of the elements of any row may be added to the corresponding elements in any other row of the matrix. (This is what we do when we use the **elimination/addition method**!)

A definite strategy must be used to transform the augmented matrix into the form required. The following procedure will always work.

1. Work column by column, left to right.
2. Within a given column, get the "one" in the correct position first. This can be done by dividing the row by the entry or interchanging two rows. As you progress through this procedure, you will only interchange a given row with one **below** it.
3. Use the row with the "one" in it to get the "zeros". This is done using row operation 3. You will multiply the row with the "one" in it by the opposite of the entry that you wish to make zero. Then, when you add the rows together, you will get the zero in that position. Note that you do not actually **change** the row with the "one" in it in the matrix; it is only **used** to get the zero.

Example: Solve the following system using the **Gauss-Jordan Method**:

$$\begin{cases} 4x + y - 3z = 11 & (1) \\ 2x - 3y + 2z = 9 & (2) \\ x + y + z = -3 & (3) \end{cases}$$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 4 & 1 & -3 & 11 \\ 2 & -3 & 2 & 9 \\ 1 & 1 & 1 & -3 \end{array}\right]$$

Start with column 1. To get a one in the top position, we may interchange rows 1 and 3 to get the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 2 & -3 & 2 & 9 \\ 4 & 1 & -3 & 11 \end{array}\right]$$

Now we will use row 1 to get the zeros in rows 2 and 3. To get the zero in row 2, we multiply row 1 by – 2 and add it to row 2:

	– 2 (row 1):	– 2	– 2	– 2	6
+	old row 2:	2	– 3	2	9
	new row 2:	0	– 5	0	15

To get the zero in row 3, we multiply row 1 by – 4 and add it to row 3:

	– 4 (row 1):	– 4	– 4	– 4	12
+	old row 3:	4	1	– 3	11
	new row 3:	0	– 3	– 7	23

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -5 & 0 & 15 \\ 0 & -3 & -7 & 23 \end{array}\right]$$

The next step is to move to column 2. We first get the one in the proper position by dividing row 2 by – 5: New row 2: 0 1 0 – 3

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & -3 & -7 & 23 \end{array}\right]$$

We will now use row 2 to get the required zeros. To get the zero in row 1, we multiply row 2 by – 1 and add it to row 1:

	– 1 (row 2):	0	– 1	0	3
+	old row 1:	1	1	1	– 3
	new row 1:	1	0	1	0

To get the zero in row 3, we multiply row 2 by 3 and add it to row 3:

	3 (row 1):	0	3	0	– 9
+	old row 3:	0	– 3	– 7	23
	new row 3:	0	0	– 7	14

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -7 & 14 \end{array}\right]$$

To finish, we look at column 3 and get the one by dividing the entries by -7:
New row 3: 0 0 1 -2

Our augmented matrix now looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We will now use row 3 to get the required zeros. To get the zero in row 1, we multiply row 3 by -1 and add it to row 1:

	-1 (row 3):	0	0	-1	2
+	old row 1:	1	0	1	0
	new row 1:	1	0	0	2

Row 2 already has a zero in the third position so we are finished! Our final augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This means that the solution to the system is $(x,y,z) = (2,-3,-2)$.

The **Gauss–Jordan Method** is a very systematic method for solving systems and it works especially well with large systems. Sometimes you end up with fractional entries which makes computations more difficult, but overall it is a very organized method.

In the process of solving a system of linear equations using the **Gauss–Jordan Method**, two results can happen that indicate that the system is inconsistent or dependent:

1. If the elements of any row are all zeros to the left of the vertical bar and a nonzero constant to the right side, then the associated system is inconsistent. (Note that this would correspond to an equation like $0 = 7$)
2. If the elements of any row are all zeros (to the left and right of the vertical bar), then the associated system is dependent.
(Note that this would correspond to the equation $0 = 0$)

HOMEWORK ASSIGNMENT
GAUSS–JORDAN METHOD

Solve each of the following systems using the Gauss-Jordan Method. Clearly show your steps!

1. $\begin{cases} x - y = 3 \\ -x - y = -3 \end{cases}$

2. $\begin{cases} x - 2y = -5 \\ 2x + 3y = 11 \end{cases}$

3. $\begin{cases} x - 3y = 15 \\ y = \frac{1}{3}x - 5 \end{cases}$

4. $\begin{cases} 3x - 2y = 1 \\ 6x - 4y = 5 \end{cases}$

5. $\begin{cases} x + y = 4 \\ y + z = -8 \\ x + z = 2 \end{cases}$

6. $\begin{cases} x + y + 3z = 1 \\ 2x + 5y + 2z = 0 \\ 3x - 2y - z = 3 \end{cases}$

7. $\begin{cases} 6x + 3y + 2z = 1 \\ 5x + 4y + 3z = 0 \\ x + y + z = 0 \end{cases}$

8. $\begin{cases} x + y - z = -10 \\ 2x + y + z = 2 \\ 3x + 5y - 8z = -66 \end{cases}$

9. $\begin{cases} x + 3y - 2z = 14 \\ 3x - 2y + z = -8 \\ -2x - 6y + 4z = -30 \end{cases}$

10. $\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ -x + 2y = 1 \end{cases}$

11. $\begin{cases} x + 3w = 4 \\ 2y - z - w = 0 \\ 3y - 2w = 1 \\ 2x - y + 4z = 5 \end{cases}$