Harvey Mudd College Math Tutorial:

Eigenvalues and Eigenvectors

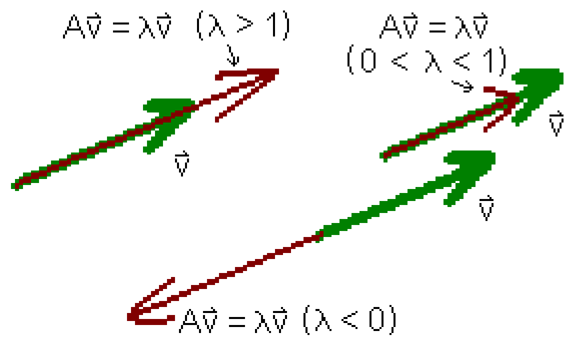
We review here the basics of computing eigenvalues and eigenvectors. Eigenvalues and eigenvectors play a prominent role in the study of ordinary differential equations and in many applications in the physical sciences. Expect to see them come up in a variety of contexts!

Definitions

Let A be an $n \times n$ matrix. The number λ is an **eigenvalue** of A if there exists a non-zero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

In this case, vector $\mathbf{v}$ is called an **eigenvector** of A corresponding to λ.



Computing Eigenvalues and Eigenvectors

We can rewrite the condition $A\mathbf{v} = \lambda\mathbf{v}$ as

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

where I is the $n \times n$ identity matrix. Now, in order for a *non-zero* vector $\mathbf{v}$ to satisfy this equation, $A - \lambda I$ must *not* be invertible.
That is, the determinant of $A - \lambda I$ must equal 0. We call $p(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** of A. The eigenvalues of A are simply the roots of the characteristic polynomial of A.

Otherwise, if $A - \lambda I$ has an inverse,

$$\begin{aligned} (A-\lambda I)^{-1}(A-\lambda I)\mathbf{v} &= (A-\lambda I)^{-1}\mathbf{0} \\ \mathbf{v} &= \mathbf{0}. \end{aligned}$$

But we are looking for a non-zero vector $\mathbf{v}$.

Example

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$. Then
$$\begin{aligned} p(\lambda) &= \det \begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix} \\ &= (2-\lambda)(-1-\lambda) - (-4)(-1) \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2). \end{aligned}$$

Thus, $\lambda_1 = 3$ and $\lambda_2 = -2$ are the eigenvalues of A.

To find eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ corresponding to an eigenvalue λ, we simply solve the system of linear equations given by

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Example

The matrix $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$ of the previous example has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$. Let's find the eigenvectors corresponding to $\lambda_1 = 3$. Let $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$. Then $(A-3I)\mathbf{v} = \mathbf{0}$ gives us

$$\begin{bmatrix} 2-3 & -4 \\ -1 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

from which we obtain the duplicate equations

$$\begin{aligned} -v_1 - 4v_2 &= 0 \\ -v_1 - 4v_2 &= 0. \end{aligned}$$

If we let $v_2 = t$, then $v_1 = -4t$. All eigenvectors corresponding to $\lambda_1 = 3$ are multiples of $\left[\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right]$ and thus the eigenspace corresponding to $\lambda_1 = 3$ is given by the span of $\left[\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right]$. That is, $\left\{\left[\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right]\right\}$ is a **basis** of the eigenspace corresponding to $\lambda_1 = 3$.

Repeating this process with $\lambda_2 = -2$, we find that

$$\begin{aligned} 4v_1 - 4V_2 &= 0 \\ -v_1 + v_2 &= 0 \end{aligned}$$

If we let $v_2 = t$ then $v_1 = t$ as well. Thus, an eigenvector corresponding to $\lambda_2 = -2$ is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and the eigenspace corresponding to $\lambda_2 = -2$ is given by the span of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. $\left\{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right\}$ is a basis for the eigenspace corresponding to $\lambda_2 = -2$.

In the following example, we see a two-dimensional eigenspace.

Example

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Then $p(\lambda) = \det \begin{bmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{bmatrix} = (\lambda - 1)(\lambda + 3)^2$ after some algebra! Thus, $\lambda_1 = 1$ and $\lambda_2 = -3$ are the eigenvalues of A. Eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ corresponding to $\lambda_1 = 1$ must satisfy

$$
\begin{array}{rcrcrcl}
4v_1 & + & 8v_2 & + & 16v_3 & = & 0 \\
4v_1 & & & + & 8v_3 & = & 0 \\
-4v_1 & - & 4v_2 & - & 12v_3 & = & 0.
\end{array}
$$

Letting $v_3 = t$, we find from the second equation that $v_1 = -2t$, and then $v_2 = -t$. All eigenvectors corresponding to $\lambda_1 = 1$ are multiples of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and so the eigenspace corresponding to $\lambda_1 = 1$ is given by the span of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$. $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

Eigenvectors corresponding to $\lambda_2 = -3$ must satisfy

$$
\begin{array}{rcrcrcl}
8v_1 & + & 8v_2 & + & 16v_3 & = & 0 \\
4v_1 & + & 4v_2 & + & 8v_3 & = & 0 \\
-4v_1 & - & 4v_2 & - & 8v_3 & = & 0.
\end{array}
$$

The equations here are just multiples of each other! If we let $v_3 = t$ and $v_2 = s$, then $v_1 = -s - 2t$. Eigenvectors corresponding to $\lambda_2 = -3$ have the form

$$
\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t.
$$

Thus, the eigenspace corresponding to $\lambda_2 = -3$ is two-dimensional and is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_2 = -3$.

Notes

- Eigenvalues and eigenvectors can be complex-valued as well as real-valued.
- The dimension of the eigenspace corresponding to an eigenvalue is less than or equal to the multiplicity of that eigenvalue.
- The techniques used here are practical for 2×2 and 3×3 matrices. Eigenvalues and eigenvectors of larger matrices are often found using other techniques, such as iterative methods.

Key Concepts

Let A be an $n \times n$ matrix. The eigenvalues of A are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

For each eigenvalue λ, we find eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by solving the linear system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

The set of all vectors $\mathbf{v}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$ is called the **eigenspace** of A corresponding to λ.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]