Convergence and Divergence of Infinite Series

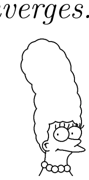
Problem 1 *Determine the convergence or divergence of the series. If possible, find it's sum.*

1. $\sum_{n=1}^{\infty} \frac{(1-2)^{n+1}5}{n}$
   Solution.
   $$= \sum_{n=1}^{\infty} \frac{5(-1)^{n+1}}{n}$$
   This is an alternating series with $a_n = (-1)^n b_n$*, such that* $b_n = \frac{5}{n}$.
   Notice that $b_{n+1} \leq b_n$.

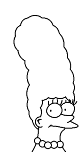
   By the alternating series test, the series converges.

2. $\sum_{n=1}^{\infty} \frac{1}{900n}$
   Solution.
   $$\sum_{n=1}^{\infty} \frac{1}{900n} = \frac{1}{900} \sum_{n=1}^{\infty} \frac{1}{n}$$
   Note that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, so our series diverges.*

   .

3. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$
   Solution.
   $$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3}{n^{3/2}}$$

   The series is a p-series and converges since $p = 3/2 > 1$.

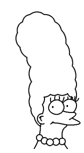
4. $\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$
Solution.

$$\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$$

$$= \frac{1}{1 - \frac{pi}{4}}$$

$$= \frac{1}{\frac{3\pi}{4}}$$

$$= \frac{4}{3\pi}$$



5. $\sum_{n=1}^{100} \frac{n^3+5n+8}{n^4}$
Solution. Note that the sum is over a finite number of terms. Adding a finite number of finite terms certainly yields a finite sum. To see this rigorously (if you're unsatisfied with the two sentences prior to this one), see below.

$$\frac{n^3 + 5n + 8}{n^4} \leq 14 (\textit{for } n = 1)$$

So,

$$\sum_{n=1}^{100} \frac{n^3 + 5n + 8}{n^4} \leq \sum_{n=1}^{100} 14$$

$$= 14(100)$$

$$= 1400$$

Thus, the sum is convergent.

6. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$
Solution.

$$\sum_{n=1}^{\infty} \frac{n}{2n^2 + 1} \geq \sum_{n=1}^{\infty} \frac{n}{2n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the harmonic series diverges, so must ours.

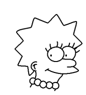
7. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$
Solution.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{9} \left(\frac{3}{2}\right)^n$$

$$= \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{-3}{2}\right)^n$$

This is a geometric series, with $r = -3/2$. The series diverges since



$|r| \geq 1$.

8. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$
Solution.

$$= \frac{10}{3} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series with $p = 3/2$, so the series converges.

9. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$
Solution. Since we see a 2^n, this seems to suggest the ratio test.

$$\lim_{n\to\infty} \frac{\frac{10(n+1)+3}{(n+1)2^{n+1}}}{\frac{10n+3}{n2^n}}$$

$$= \lim_{n\to\infty} \frac{10n+13}{(n+1)2^{n+1}} \cdot \frac{n2^n}{10n+3}$$
$$= \lim_{n\to\infty} \frac{(10n+13)n}{(n+1)\cdot 2 \cdot (10n+3)}$$
$$= \lim_{n\to\infty} \frac{10n^2+13n}{20n^2+23n+6}$$
$$= \frac{1}{2}$$

The series converges since the limit is less than 1.

10. $\sum_{n=1}^{5} 2^{-n+3}$
Solution. *Similar to problem 5, this is over a finite number of terms, so the sum must be finite.*

11. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$
Solution. *Recall that the terms of a series must approach 0 for that series to have a chance at converging. Here, we use the n^{th}-term test for divergence:*

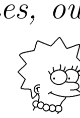
$$\lim_{n\to\infty} 2^n 4n^2 - 1 = \lim_{n\to\infty} \frac{n2^{n-1}}{8n}$$
$$= \lim_{n\to\infty} \frac{n(n-1)2^{n-2}}{8} = \infty$$

So, the series diverges.

12. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$
Solution.

$$\sum_{n=1}^{\infty} \frac{-1}{n^2} \leq \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since our series is bounded above and below by convergent series, our series must also converge.



13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$
Solution. *This is an alternating series, whose terms decrease towards 0. This means the series converges by the alternating series test.*

14. $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$
Solution. *The presence of the 7^n and the $n!$ suggest the ratio test.*

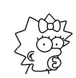
$$\lim_{n\to\infty} \frac{\frac{(n+1)7^{n+1}}{(n+1)!}}{\frac{n7^n}{n!}}$$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \\ &= \lim_{n\to\infty} \frac{7(n+1)}{n(n+1)} \\ &= \lim_{n\to\infty} \frac{7}{n} \\ &= 0 \end{aligned}$$

So,the series converges by the ratio test.

15. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n+1)}$
Solution. *Notice that there is a $(-3)^n$ in the numerator. This suggests the ratio test. Also, the denominator has what looks related to a factorial. Each of these suggest the ratio test.*

Perform the ratio test to find a convergent series.



16. $\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdot \ldots \cdot (2n+1)}{18^n(2n-1)n!}$
Solution. *Notice that there is an 18^n in the denominator. This suggests the ratio test. Also, both the numerator and denominator have factorials or what looks related to a factorial. Each of these suggest the ratio test.*

Perform the ratio test to find a convergent series.