D’ Alembert’ Ratio Test

D' Alembert' Ratio Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$,then

$\sum u_n$ is convergent if l > 1

$\sum u_n$ is divergent if l < 1

Test fails if l = 1

Q 9 Test the convergence of the series

$$\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots\ldots\ldots\ldots\ \infty$$

The given series can be written as

$$\frac{2.5}{1.5} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots\ldots\ldots\ldots\ \infty$$

Here $u_n = \frac{2.5.8.11\ldots\ldots\ldots(3n-1)}{1.5.9.13\ldots\ldots\ldots(4n-3)}$

=

$$u_{n+1} = \frac{2.5.8.11\ldots\ldots\ldots(3n-1)(3n+2)}{1.5.9.13\ldots\ldots\ldots(4n-3)(4n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n+1}{3n+2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{4+\frac{1}{n}}{3+\frac{2}{n}} = \frac{4}{3} > 1 \quad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty\right)$$

So by D' Alembert' Ratio Test, the given series $\sum u_n$ is also convergent.

Q 10 Test the convergence of the series

$$\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots\ldots\ldots\ldots \infty$$

Here $u_n = \dfrac{x^n}{n+1\sqrt{n+3}}$

$$u_{n+1} = \frac{x^{n+1}}{n+2\sqrt{n+4}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n+1\sqrt{n+3}} \times \frac{n+2\sqrt{n+4}}{x^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \frac{\sqrt{1+\frac{4}{n}}}{\sqrt{1+\frac{3}{n}}}\ \frac{1}{x} = \frac{1}{x}$$

So by D' Alembert' Ratio Test, the given series $\sum u_n$ is convergent if $\frac{1}{x} > 1$ i.e. $x < 1$, and divergent if $\frac{1}{x} < 1$ i.e. $x > 1$.

Ratio Test fails if $\frac{1}{x} = 1$ i.e. $x = 1$

For $x = 1$, $u_n = \frac{1}{(n+1)\sqrt{n+3}}$

$$= \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{3}{n}}}$$

Take $v_n = \frac{1}{n^{\frac{3}{2}}}$

$$\frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{3}{n}}}$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ 1 (finite and non-zero)

So, by the comparison test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge together. But the series $\sum v_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is convergent as $p = \frac{3}{2} > 1$

Hence, the given series $\sum u_n$ is also convergent.

Assignment

Test the convergence and divergence of the following series

(i) $\sum n^{1/2} / (n^2 + 1)$

(ii) $\sum (n^2 - 1)^{1/2} / (n^3 + 1)$