Rabee’s Test
Logarithmic Test
&
Gauss’s Test

Rabee's Test : If $\sum u_n$ is a positive term series, and if

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$,then

$\sum u_n$ is convergent if $l > 1$

$\sum u_n$ is divergent if $l < 1$

Test fails if $l = 1$

Q 11 Test the convergence of the series

$$1 + \frac{1}{2}.\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\frac{x^4}{8} + \frac{1.3.5.7}{2.4.6.8.10}\frac{x^6}{12} + \ldots\ldots\ldots\ldots\ \infty$$

After neglecting first term

$$u_n = \frac{1.3.5.7\ldots\ldots(4n-3)}{2.4.6.8.10\ldots.(4n-2)}.\frac{x^{2n}}{4n}$$

$$u_{n+1} = \frac{1.3.5.7\ldots\ldots(4n-3)(4n-1)(4n+1)}{2.4.6.8.10\ldots.(4n-2)4n(4n+2)}.\frac{x^{2n+2}}{4n+4}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n(4n+2)}{(4n-1)(4n+1)}\cdot\frac{4n+4}{4n}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(4n+2)(4n+4)}{(4n-1)(4n+1)}.\frac{1}{x^2} = \frac{1}{x^2}$$

So by D' Alembert's Ratio Test, the given series $\sum u_n$ is convergent if

$\frac{1}{x^2} > 1$ i.e. $x^2 < 1$, and divergent if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$.

Ratio Test fails if $\frac{1}{x^2} = 1$ i.e. $x^2 = 1$

$$\frac{u_n}{u_{n+1}} = \frac{(4n+2)(4n+4)}{(16n^2-1)}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{16n^2+24n+8-16n^2+1}{(16n^2-1)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n(24n+9)}{(16n^2-1)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{24+\frac{9}{n}}{16-\frac{1}{n^2}} = \frac{24}{16} = \frac{3}{2} > 1$$

So by Raabe's Ratio Test, the given series is convergent.

Hence , the given series is convergent if $x^2 \leq 1$, and divergent if $x^2 > 1$.

Logarithmic Test : If $\sum u_n$ is a positive term series, and if

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l \text{ ,then}$$

$\sum u_n$ is convergent if $l > 1$

$\sum u_n$ is divergent if $l < 1$

Test fails if $l = 1$

Q 12 Test the convergence of the series

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots\ldots\ldots\ldots\ \infty$$

$$u_n = \frac{n^n x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{n+1!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{n+1!}{n+1^{n+1} x^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{n^n}{n+1^{n+1}} \cdot \frac{1}{x} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{x}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{ex}$$

So by Logarithmic Test, the given series $\sum u_n$ is convergent

if $\frac{1}{ex} > 1$ i.e. $x < \frac{1}{e}$, and divergent if $\frac{1}{ex} < 1$ i.e. $x > \frac{1}{e}$

Logarithmic Test fails if $\frac{1}{ex} = 1$ i.e. $x = \frac{1}{e}$

For $x = \frac{1}{e}$

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$n\log\frac{u_n}{u_{n+1}} = \log e - n\log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots\ldots\ldots.\infty\right)$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{3n^3} \ldots\ldots\ldots.\infty$$

$$= \frac{1}{2} < 1$$

By Logarithmic Test, the series is divergent.Hence the given series

$\sum u_n$ is convergent if $x < \frac{1}{e}$, and divergent if $x \geq \frac{1}{e}$.

<u>Gauss's Test :</u> If for the positive term series $\sum u_n$, $\frac{u_n}{u_{n+1}}$ can be expanded in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + O\left(\frac{1}{n^2}\right)$$

Then $\sum u_n$ is convergent if $h > 1$ and divergent if $h \leq 1$.

Q 12 Test the convergence of the series

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}x + \frac{1^2 \cdot 3^2 . 5^2}{2^2 \cdot 4^2 . 6^2}x^2 + \ldots\ldots\ldots\ldots\ldots \infty$$

$$u_n = \frac{1^2 \cdot 3^2 . 5^2 \ldots\ldots (2n-1)^2}{2^2 \cdot 4^2 . 6^2 \ldots\ldots\ldots (2n)^2} x^{n-1}$$

$$u_{n+1} = \frac{1^2 \cdot 3^2 . 5^2 \ldots\ldots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 . 6^2 \ldots\ldots\ldots (2n)^2 (2n+2)^2} x^{n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \left(\frac{2 + \frac{2}{n}}{2 + \frac{1}{n}}\right)^2 \cdot \frac{1}{x} = \frac{1}{x}$$

So by Ratio Test, the given series $\sum u_n$ is convergent if $\frac{1}{x} > 1$ i.e. $x < 1$, and divergent if $\frac{1}{x} < 1$ i.e. $x > 1$

Ratio Test fails if $\frac{1}{x} = 1$ i.e. $x = 1$

For x = 1

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2+8n+4}{4n^2+4n+1}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+8n+4-4n^2-4n-1}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n(4n+3)}{4n^2+4n+1}$$

$$= \lim_{n\to\infty} \frac{n(4n+3)}{4n^2+4n+1}$$

$$= \lim_{n\to\infty} \frac{4+\frac{3}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = 1$$

Raabe's test fail.

Now $$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left[1+(-2)\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= \left[1+\frac{2}{n}-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= \left[1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right]$$

It is of the form

$$\frac{u_n}{u_{n+1}} = 1+\frac{\mu}{n}+O\left(\frac{1}{n^2}\right)$$

$\mu = 1$ By Gauss's test, the series is divergent for x = 1.

Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.