10.3 $p-$series and the ratio test

Let p be a positive number. An infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

is called a $p-$series. If $p = 1$, the series is called the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Classifying $p-$series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a $p-$series with $p = 3$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a $p-$series with $p = 1/2$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is not a $p-$series, it is a geometric series.

Test for convergence of $p-$series:

The $p-$series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

will

- diverge if $0 < p \leq 1$
- converge if $p > 1$.

This is a very simple test, just identify that you have a $p-$series and check if $0 < p \leq 1$(diverge) or if $p > 1$ (converge). Note however, that unlike the geometric series there is no formula for computing the infinite series. The best you can do is to approximate the series using some computational method.

Examples

- $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a $p-$series with $p = 3 > 1$ so this series converges.
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a $p-$series with $p = 1/2 \leq 1$ so this series diverges.
- $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, with $p = 1 \leq 1$ so this series diverges.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with non-zero terms.

- The series converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
- The series diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.
- The test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

The test is particularly useful for factorials and exponentials.

Ratio Test examples

Determine the convergence or divergence of the following series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = \frac{1}{1} + \frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} \ldots$$

Use the ratio test with $a_n = \dfrac{2^n}{n!}$ and you see:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right] \\
&= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right] \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0
\end{aligned}$$

The series converges.

Review of factorials and exponentials

Recall that $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$ so

$$\frac{n!}{(n+1)!} = \frac{n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1}{(n+1)n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1} = \frac{1}{n+1}$$

Also recall that $2^{n+1} = 2 \cdot 2^n$ so

$$\frac{2^{n+1}}{2^n} = \frac{2 \cdot 2^n}{2^n} = 2.$$

Example

Determine the convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^5}$$

We use the ratio test with $a_n = \frac{2^n}{n^5}$ so we have to look at

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)^5} \div \frac{2^n}{n^5} \right] \\
&= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)^5} \cdot \frac{n^5}{2^n} \right] \\
&= \lim_{n\to\infty} 2 \left(\frac{n}{n+1} \right)^5 \\
&= 2
\end{aligned}$$

Since the limit is $2 \geq 1$ the series diverges.

Comparison Test

If you want to test the convergence of a series $\sum_{n=1}^{\infty} a_n$ and you know that $\sum_{n=1}^{\infty} b_n$ converges **and** $a_n \leq b_n$ for all n then you can conclude that $\sum_{n=1}^{\infty} a_n$ converges also.

Also, if you want to test the convergence of a series $\sum_{n=1}^{\infty} a_n$ and you know that $\sum_{n=1}^{\infty} b_n$ diverges **and** $a_n \geq b_n$ for all n then you can conclude that $\sum_{n=1}^{\infty} a_n$ diverges also.

Comparison test example

Consider the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{e^x}{n}$ for $x > 0$.

We know that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

We also know that $e^x > 1$ for $x > 0$ so $\dfrac{e^x}{n} > \dfrac{1}{n}$ and therefore by the comparison test we know that $\displaystyle\sum_{n=1}^{\infty} \frac{e^x}{n}$ for $x > 0$ diverges

Summary of Tests series

Test	Series	Converges	Diverges
n^{th}term	$\sum_{n=1}^{\infty} a_n$	no test	$\lim_{n\to\infty} a_n \neq 0$
Geometric	$\sum_{n=1}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$
$p-$Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| \geq 1$
Comparison	$\sum_{n=1}^{\infty} a_n$	if $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$	if $\sum_{n=1}^{\infty} b_n$ diverges $a_n \geq b_n$

Examples

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + n}$$

Since this is not a geometric series or a $p-$Series (and the n^{th} term test does not apply because the limit of the terms is zero) we have to use the Ratio or Comparison tests. The ratio test might get messy (it usually will if there are sums involved and not just products) we will try the comparison test.

Example: $\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

converges, because it is a geometric series with $r = \frac{1}{3}$, and so we will be able to see this series also converges if we can show that $\frac{1}{3^n+1} \leq \frac{1}{3^n}$ but this is clearly true since $3^n + n \geq 3^n$ for $n \geq 1$. Therefore this series converges by comparison to a geometric series.

Examples

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$$

We know that $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$ so by the n^{th} term test this series diverges.

Examples

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$$

This series is a geometric series with $r = \frac{5}{6} < 1$ so the series converges. In this case we also know the sum of the series is $\dfrac{1}{1-\frac{5}{6}} = 6.$

Examples

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n3^n}{n!}$$

Since this is not a geometric series or a $p-$Series (and the n^{th} term test does not apply because the limit of the terms is zero) we have to use the Ratio or Comparison tests. We will try the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left[\frac{(n+1)3^{n+1}}{(n+1)!} \div \frac{n3^n}{n!} \right] \\
&= \lim_{n\to\infty} \left[\frac{(n+1)3^{n+1}}{(n+1)!} \cdot \frac{n!}{n3^n} \right] \\
&= \lim_{n\to\infty} \frac{3}{n} \\
&= 0
\end{aligned}$$