A Reduction Formula

Problem: Integrate $I = \int (\sin x)^n \, dx$

Try integration by parts with

$$\begin{aligned} u &= (\sin x)^{n-1} & v &= -\cos x \\ du &= (n-1)(\sin x)^{n-2} \cos x \, dx & dv &= \sin x \, dx \end{aligned}$$

We get

$$\begin{aligned} I = \int (\sin x)^n \, dx = \int u \, dv = uv - \int v \, du & \\ &= -\cos x (\sin x)^{n-1} - \int (-\cos x)(n-1)(\sin x)^{n-2} \cos x \, dx \\ &= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \cos^2 x \, dx \\ &= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} (1 - \sin^2 x) \, dx \\ &= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} - (\sin x)^n \, dx \\ &= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \, dx - (n-1) \int (\sin x)^n \, dx \\ &= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \, dx - (n-1) \cdot I \end{aligned}$$

Solving for I:

$$(n-1)I + I = -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \, dx$$

or

$$nI = -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \, dx$$

Dividing by n gives the

Reduction Formula: $\displaystyle \int (\sin x)^n \, dx = -\frac{1}{n} \cos x (\sin x)^{n-1} + \frac{n-1}{n} \int (\sin x)^{n-2} \, dx$

Example: Using this formula three times, with $n = 6$, $n = 4$, and $n = 2$ allow us to integrate $\sin^6 x$, as follows:

$$\begin{aligned}
\int (\sin x)^6\,dx &= -\frac{1}{6}\cos x(\sin x)^5 + \frac{5}{6}\int (\sin x)^4\,dx \\
&= -\frac{1}{6}\cos x(\sin x)^5 + \frac{5}{6}\Big[-\frac{1}{4}\cos x(\sin x)^3 + \frac{3}{4}\int (\sin x)^2\,dx\Big] \\
&= -\frac{1}{6}\cos x(\sin x)^5 - \frac{5}{6}\frac{1}{4}\cos x(\sin x)^3 + \frac{5}{6}\frac{3}{4}\Big[-\frac{1}{2}\cos x(\sin x)^1 + \frac{1}{2}\int (\sin x)^0\,dx\Big] \\
&= -\frac{1}{6}\cos x(\sin x)^5 - \frac{5}{6}\frac{1}{4}\cos x(\sin x)^3 - \frac{5}{6}\frac{3}{4}\frac{1}{2}\cos x\sin x + \frac{5}{6}\frac{3}{4}\frac{1}{2}x + C
\end{aligned}$$

Remark. Note that when $n = 2$ it is often more convenient to use the double angle formula:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Substituting $\cos^2 x = 1 - \sin^2 x$ gives us

$$\cos(2x) = 1 - 2\sin^2 x$$

while substituting $\sin^2 x = 1 - \cos^2 x$ gives us

$$\cos(2x) = 2\cos^2 x - 1.$$

Solving these two equations for $\sin^2 x$ and $\cos^2 c$ gives

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

Thus,

$$\begin{aligned}
\int \sin^2 x\,dx &= \int \frac{1-\cos(2x)}{2}\,dx \\
&= \int \frac{1}{2}\,dx - \int \frac{1}{2}\cos(2x)\,dx \\
&= \frac{1}{2}x - \frac{1}{2}\frac{1}{2}\sin(2x) + C.
\end{aligned}$$

Problem 2: Integrate $I = \displaystyle\int (\sec x)^n \, dx$

Try integration by parts with

$$\begin{aligned} u &= (\sec x)^{n-2} & v &= \tan x \\ du &= (n-2)(\sec x)^{n-3} \sec x \tan x \, dx & dv &= \sec^2 x \, dx \end{aligned}$$

We get

$$\begin{aligned} I = \int (\sec x)^n \, dx = \int u \, dv = uv - \int v \, du \\ &= \tan x (\sec x)^{n-2} - \int (\tan x)(n-2)(\sec x)^{n-3} \sec x \tan x \, dx \\ &= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} \tan^2 x \, dx \\ &= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} (\sec^2 x - 1) \, dx \\ &= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^n - (\sec x)^{n-2} \, dx \\ &= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^n \, dx + (n-2) \int (\sec x)^{n-2} \, dx \\ &= \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx - (n-2) \cdot I \end{aligned}$$

Solving for I:

$$(n-2)I + I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

or

$$(n-1)I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

Dividing by n gives the

Reduction Formula: $\displaystyle\int (\sec x)^n \, dx = \frac{1}{n-1} \tan x (\sec x)^{n-2} + \frac{n-2}{n-1} \int (\sec x)^{n-2} \, dx$

Remark. Note that you cannot use this formula when $n = 1$ (Why not?) Instead you must use the formula

$$\int \sec(x)\,dx = \ln(\sec x + \tan x) + C$$

Example: Using this formula three times, with $n = 5$ and $n = 3$ allow us to integrate $\sec^5 x$, as follows:

$$\begin{aligned}\int (\sin x)^5\,dx &= \frac{1}{4}\tan x(\sec x)^3 + \frac{3}{4}\int (\sec x)^3\,dx \\ &= \frac{1}{4}\tan x(\sec x)^4 + \frac{3}{4}\left[\frac{1}{2}\tan x(\sec x)^1 + \frac{1}{2}\int (\sec x)^1\,dx\right] \\ &= \frac{1}{4}\tan x(\sec x)^4 + \frac{3}{4}\frac{1}{2}\tan x \sec x + \frac{3}{4}\frac{1}{2}\ln(\sec x + \tan x) + C\end{aligned}$$

using the formula above for $\int \sec x\,dx$.

If n is even, it is simpler to write $\sec^n(x)$ as $\sec^{n-2}(x) \cdot \sec^2 x$ and then use the substitution $u = \tan x$ and the trig identity $\sec^2 x = \tan^2 x + 1$.

Example: Integrate $\int \sec^4 x\,dx$

First write

$$\begin{aligned}\int \sec^4 x\,dx &= \int (\sec^2 x)\,\sec^2 x\,dx \\ &= \int (\tan^2 x + 1)\,\sec^2 x\,dx\end{aligned}$$

Now substitute $u = \tan x$, $du = \sec^2 x\,dx$ so that

$$\begin{aligned}\int \sec^4 x\,dx &= \int (u^2 + 1)\,du \\ &= \frac{1}{3}u^3 + u + c \\ &= \frac{1}{3}\tan^3 x + \tan x + c\end{aligned}$$