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How to use reduction formulae to integrate even powers of sin and cos functions

We use integration by parts to obtain **reduction formulae**, which express integrals of powers of sin and cos functions in terms of integrals of **lower** powers of the same functions. This is useful because repeated application of the formula can reduce any positive integer power of a sin or cos function to either its first power or its zero'th power (i.e. the constant, 1), both of which are easy to integrate. The derivation of the reduction formula for $\int \sin^n x\,dx$ is given in full detail below, while less detail is given for the derivation of the reduction formula for $\int \cos^n x\,dx$. Also remember that $\int \sin^n(ax+b)\,dx$ and $\int \cos^n(ax+b)\,dx$ can be evaluated by the method for integrating functions of linear expressions.

Reduction formula for $\int \sin^n x\,dx$

$$\begin{aligned}
\int \sin^n x\,dx &= \int \sin x\,\sin^{n-1}x\,dx \\
&= -\cos x\,\sin^{n-1}x - \int (-\cos x)\,.\,(n-1)\sin^{n-2}x\,\cos x\,dx \\
&\qquad\qquad \text{(integrating by parts)} \\
&= -\cos x\,\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,\cos^2 x\,dx \\
&= -\cos x\,\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx \\
&\qquad\qquad \text{(since}\quad \cos^2 x = 1-\sin^2 x\,) \\
&= -\cos x\,\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx\,. \qquad (1)
\end{aligned}$$

There is now a term in $\int \sin^n x\,dx$ on the right-hand side as well as on the left-hand side. Bringing these terms together on the left-hand side, (1) becomes

$$\begin{aligned}
n\int \sin^n x\,dx &= -\cos x\,\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx \\
\therefore \int \sin^n x\,dx &= -\frac{1}{n}\cos x\,\sin^{n-1}x + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx\,. \qquad (2)
\end{aligned}$$

The use of the reduction formula (2) to integrate a power of $\sin x$ is demonstrated in worked example no. 2.

Reduction formula for $\int \cos^n x\, dx$

$$
\begin{aligned}
\int \cos^n x\, dx &= \int \cos x \, \cos^{n-1} x\, dx \\
&= \sin x \, \cos^{n-1} x - \int (\sin x)\,.\,(n-1)\cos^{n-2} x\,(-\sin x)\, dx \\
&\qquad\qquad \text{(integrating by parts)} \\
&= \sin x \, \cos^{n-1} x + (n-1) \int \cos^{n-2} x\,(1-\cos^2 x)\, dx \\
&\qquad\qquad \text{(since} \quad \sin^2 x = 1-\cos^2 x\,\text{)} \\
&= \sin x \, \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx \; - \; (n-1) \int \cos^n x\, dx \\
\therefore\; n \int \cos^n x\, dx &= \sin x \, \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx \\
\therefore\; \int \cos^n x\, dx &= \frac{1}{n} \sin x \, \cos^{n-1} x + \frac{(n-1)}{n} \int \cos^{n-2} x\, dx\,. \qquad (3)
\end{aligned}
$$

The use of the reduction formula (3) to integrate a power of $\cos x$ is demonstrated in worked example no. 1.

Worked examples of using reduction formulae to integrate even powers of sin and cos functions

How to use trigonometric identities to integrate even powers of sin and cos functions

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