Q. 1

Solution :-

Consider independent and identically distributed data of 'n' sample.

$$X = \{x_1, x_2, x_3, \dots x_n\}$$

Prior distribution can be stated as:-

$$p(\mu) = N(N_0, \sigma_0^2)$$

It is given that sample is drawn from Gaussian distribution.

Mean is unknown and variance is known.

The likelihood of data samples X for a given mean μ

$$p(X|\mu) = p(x_1|\mu) \times p(x_2|\mu) \times \dots \times p(x_n|\mu)$$

$$p(X|\mu) \approx \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) e^{\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right)}$$

$$p(X|\mu) \approx \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)}$$

As variance is constant, first term can be ignored.

$$\therefore p(X|\mu) \approx e^{\left(-\frac{n}{2\sigma^2}(\mu^2 - 2\mu\bar{x})\right)}$$

$$\therefore p(X|\mu) \approx e^{\left(-\frac{1}{2\sigma^2}(\bar{x} - \mu)^2\right)}$$

Now, posterior distribution of mean, $p(\mu|X)$ can be written as:-

$$p(\mu|X) \approx p(X|\mu) \times p(\mu)$$

Substituting for likelihood and prior distribution,

$$p(\mu|X) \approx e^{\left(-\frac{1}{2\sigma^2}(\bar{x} - \mu)^2\right)} \times \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right) e^{\left(-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2\right)}$$

$\therefore p(\mu|x) \approx e^{\left(-\frac{1}{2\sigma^2}(\bar{x}-\mu)^2 - \frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right)}$

$\therefore p(\mu|x) \approx e^{\left(-\frac{1}{2}\left(\frac{1}{\sigma^2}(\bar{x}^2+\mu^2-2\mu\bar{x})+\frac{1}{\sigma_0^2}(\mu^2+\mu_0^2-2\mu\mu_0)\right)\right)}$

$\therefore p(\mu|x) \approx e^{\left(-\frac{1}{2}\left(\mu^2\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)-2\mu\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)+\left(\frac{n\bar{x}^2}{\sigma^2}+\frac{\mu_0^2}{\sigma_0^2}\right)\right)\right)}$

Removing the constants

$p(\mu|x) \approx e^{\left(-\frac{1}{2}\left(\mu^2\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)-2\mu\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)\right)\right)}$ — (1)

Comparing with the form $e^{\left(-\frac{1}{2\sigma_n^2}(\mu-\mu_n)^2\right)}$

we get, $-\frac{1}{2\sigma_n^2} = -\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)$

and $-\frac{1}{2\sigma_n^2}\times\mu_n = -\frac{1}{2}\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)$

$\therefore \frac{1}{\sigma_n^2} = \frac{n}{\sigma^2}+\frac{1}{\sigma_0^2} \Rightarrow \sigma_n^2 = \left(\frac{\sigma^2\sigma_0^2}{\sigma^2+n\sigma_0^2}\right)$

— (2)

and $\mu_n = \frac{2}{2}\sigma_n^2\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)$

$\mu_n = \sigma_n^2\left(\frac{n\bar{x}\sigma_0^2+\mu_0\sigma^2}{\sigma^2\sigma_0^2}\right)$

$\mu_n = \left(\frac{\sigma^2\sigma_0^2}{\sigma^2+n\sigma_0^2}\right)\left(\frac{n\bar{x}\sigma_0^2+\mu_0\sigma^2}{\sigma^2\sigma_0^2}\right)$

$\mu_n = \left(\frac{n\bar{x}\sigma_0^2+\mu_0\sigma^2}{\sigma^2+n\sigma_0^2}\right)$

— (3)

Q.2 From Equation 1, 2, and 3

we have, $p(\mu|X) \approx e^{\left(-\frac{1}{2}\left(\mu^2\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)-2\mu\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)\right)\right)}$

and $\sigma_n^2 = \dfrac{\sigma^2\sigma_0^2}{\sigma^2+n\sigma_0^2}$, $\mu_n = \dfrac{n\bar{X}\sigma_0^2+\mu_0\sigma^2}{\sigma^2+n\sigma_0^2}$

This can be written as:-

$$p(\mu|x) = e^{\left(-\frac{1}{2\sigma_n^2}(\mu-\mu_n)^2\right)}$$

where σ_n^2 & μ_n are given above.

This equation follows the Gaussian distribution,

$$p(\mu|X) \approx N(\mu_n, \sigma_n^2)$$

Q.3

From equation 2, and 3, we have,

$$\sigma_n^2 = \frac{\sigma^2\sigma_0^2}{\sigma^2+n\sigma_0^2}$$

$$\mu_n = \frac{n\bar{x}\sigma_0^2+\mu_0\sigma^2}{\sigma^2+n\sigma_0^2}$$

Q. 4

we have, $\mu_n = \dfrac{n\bar{X}\sigma_0^2 + \mu_0\sigma^2}{\sigma^2 + n\sigma_0^2}$

$$\therefore \boxed{\mu_n = \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2}\bar{X} + \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}\mu_0}$$

This is the formula for weights.

Q. 5

By looking at the equations, we have weights are inversely proportional to their variance.

This is because, weight of $\bar{X}$ is indirectly proportional to σ^2 and μ_0 is indirectly proportional to σ_0^2.

$$\mu_n = \frac{\dfrac{n\bar{X}\sigma_0^2}{\sigma_0^2\sigma^2} + \dfrac{\mu_0\sigma^2}{\sigma_0^2\sigma^2}}{\dfrac{\sigma^2 + n\sigma_0^2}{\sigma_0^2\sigma^2}}$$

$$\mu_n = \sigma_n^2\left(\frac{\mu_0}{\sigma_0^2} + \frac{\bar{X}}{\frac{\sigma^2}{n}}\right)$$

Q. 6

Sum of weights is :-

$$= \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2} + \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}$$

$$= \frac{\sigma^2 + n\sigma_0^2}{\sigma^2 + n\sigma_0^2}$$

$$\boxed{\text{Sum of weights} = 1}$$

Q. 7

We have weight $= \dfrac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2}$

$$= \frac{1}{\frac{\sigma^2}{n\sigma_0^2} + 1}$$

As the denominator is greater than 1, sample weight is between 0 and 1.

Prior weight $= \dfrac{\sigma^2}{\sigma^2 + n\sigma_0^2} = \dfrac{1}{1 + \frac{n\sigma_0^2}{\sigma^2}}$

Again denominator is greater than 1, Prior weight lies between 0 and 1.

Q.8

From the weights is Q.7, we know that they will lie in the range 0 and 1.

In the worst case, either of them can be 1 and other 0.

$\therefore$ Therefore, value of μ_n must be between $\bar{x}$ and μ_0

$$\min(\bar{x}, \mu_0) \leq \mu_n \leq \max(\bar{x}, \mu_0).$$

Q.9

$$p(x^{new} \mid X) = \int p(x^{new} \mid \mu) \times p(\mu \mid X)\, d\mu$$

It is given that $p(x^i \mid \mu)$ is normal distribution $N(\mu, \sigma^2)$

also, the posterior distribution $p(\mu \mid X)$ is a normal distribution $N(\mu_n, \sigma_n^2)$.

$$\therefore \quad x^{new} = (x^{new} - \mu) + \mu$$

$x^{new} - \mu$ is normal distribution $N(0, \sigma^2)$ and μ is posterior normal distribution. $N(\mu_n, \sigma_n^2)$

$\therefore$ According to a theorem, If $X_1, X_2, \ldots X_n$ are mutually independent normal random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$, then linear combination :-

$Y = \sum_{i=1}^{n} c_i x_i$ follows normal distribution :-

$$N\left(\sum_{i=1}^{n} c_i \mu_i ,\ \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

$$\therefore x_0^{new} = (x^{new} - \mu) + \mu$$

$$\approx N(0, \sigma^2) + N(\mu_n, \sigma_n^2)$$

$$\boxed{x^{new} \approx N(\mu_n, \sigma^2 + \sigma_n^2)}$$

Q.10

Mean of posterior distribution = 5.3079

and variance = 0.0956