UNIT - 7

UNIFORM PLANE WAVES

' The idea that a changing magnetic field gives rise to an electric field in which the changes in turn give rise to a magnetic field forms the basis of electromagnetic wave propagation.

Wave equations in terms of Electric and Magnetic Fields:

Let us consider the two curl equations of Maxwell,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But, we know that $D = \epsilon E$ and $B = \mu H$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$$

and $$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}.$$

First, let us obtain the wave equation in terms of the electric field,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$\left[\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 E = \nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E},\right]$$

$$\therefore \nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial}{\partial t}\epsilon \vec{E}\right)$$

$$= -\mu \frac{\partial \vec{J}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

or $$\boxed{\nabla^2 E - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\epsilon}\right)}$$

The left side of the above equation is in the characteris-tic form of a wave equation. The solution of such an equation represents a propagating wave. The RHS represents the sources which are responsible for the wave field, i.e the charges and currents. Hence eqn (5) represents the wave equation in $\vec{E}$ for a medium with constant μ and ϵ i.e a homogeneous and isotropic medium.

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Now let us obtain the wave equation in terms of magnetic field,

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} - \epsilon \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla(\nabla \cdot \vec{H}) - \nabla^2 H = \nabla \times \vec{J} - \epsilon\mu \frac{\partial^2 H}{\partial t^2}$$

As per maxwell's equation,

$$\nabla \cdot \vec{B} = 0 \quad \text{i.e } \nabla \cdot \mu\vec{H} = \mu(\nabla \cdot \vec{H}) = 0$$

Since $\mu \neq 0$, $\nabla \cdot \vec{H} = 0$

$$\boxed{\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J}}$$

The above equation represents the wave equation in $\vec{H}$ for a medium with constant μ and ϵ.

If we consider a space where there are no charges, or currents, or even it be a non conducting medium but with no free charges, then $\rho_v = 0$ and $J = 0$

$$\therefore \nabla^2 E - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{——— (1)}$$

$$\nabla^2 H - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{——— (2)}$$

The classical wave equation is represented as,

$$\nabla^2 F - \frac{1}{v^2} \frac{\partial^2 \vec{F}}{\partial t^2}, \quad \text{——— (3)}$$

The above equation represents a wave travelling with a velocity v. comparing eqn (3) with equation (1) and (2), it is clear that

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

with this as reference Maxwell predicted that, the empty space supports the propagation of electromagnetic waves at speed,

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}.$$

Uniform plane waves

Plane waves are waves that possess variation only in the direction of propagation, and their characteristics remain constant across planes normal to the direction of propagation.

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In case of electromagnetic waves propagating along x-axis, they are reffered to as uniform plane waves if the electric field is independent of (which implies that the magnetic field is also independent of) y and z-axes, but is a function of x and t only. Further for such a wave, it is important to note that there will be no field component along the direction of propagation, This can be proved below. [2]

Non-existance of third components along the direction of propagation for a uniform plane wave.
(Transverse nature of electromagnetic waves):-

Consider a uniform plane wave propagating along x-direction in a medium where there are no free charges. From Maxwell's equation, we have at any point.

$$\nabla \cdot \vec{D} = \rho_v$$

Since it is a charge free medium, $\rho_v = 0$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \epsilon \vec{E} = 0$$

$$\epsilon (\nabla \cdot \vec{E}) = 0.$$

$$\therefore \epsilon \neq 0, \quad \nabla \cdot \vec{E} = 0.$$

Expanding, $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$ —— (1)

For a uniform plane wave travelling in x-direction the E_y and E_z components remain constant in y-z plane.

$$\therefore \frac{\partial E_y}{\partial y} = 0 \text{ and } \frac{\partial E_z}{\partial z} = 0$$

Substituting in eqn (1) we get,

$$\frac{\partial E_x}{\partial x} = 0.$$

This eqn is satisfied, both when E_x = Constant and when $E_x = 0$

To understant which of these two holds good for a plane wave, let us consider the wave equation,

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \quad \text{—— (2)}$$

This is in 3-dimensions.

Since E is independent of y and z, eqn (2) is expressed as,

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$$\frac{\partial^2 \vec{E}}{\partial x^2} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

In terms of the 3 components of E, the above equation becomes,

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_z}{\partial t^2}$$

but, we have $\frac{\partial^2 E_x}{\partial x^2} = 0$

$$\therefore \mu\epsilon \left(\frac{\partial^2 E_x}{\partial t^2}\right) = 0$$

Since $\mu \neq 0$, $\epsilon \neq 0$ $\therefore \frac{\partial^2 E_x}{\partial t^2} = 0$.

either $\frac{\partial E_x}{\partial t}$ = constant or $\frac{\partial E_x}{\partial t} = 0$.

$\frac{\partial E_x}{\partial x}$ = Constant means, the value of E_x steadily increases with time indefinitely. But for a wave with E_x are the variable, E_x must undergo periodic variations, i.e, it must have both increasing and decreasing values over a period of time. Hence the condition $\frac{\partial E_x}{\partial x}$ = constant fails to hold good. Now, considering the ~~order~~ other possibility $\frac{\partial E_x}{\partial x} = 0$ to hold good. the solutions are E_x = Constant and $E_x = 0$.

Because of the periodic nature of E_x described above the condition E_x = constant cannot be a solution Hence the only possible solution is $E_x = 0$. The same procedure follows to prove that $H_x = 0$.

$E_x = 0$ means, $\vec{E}$ will take orientations depending on E_y and E_z only, i.e $\vec{E}$ lies only in y-z plane. Thus with variation in values of $\vec{E_y}$ and $\vec{E_z}$ with the passage of time, $\vec{E}$ assumes different orientations in successive y-z planes. Since y-z planes are all at right angles to x-direction, the variation in $\vec{E}$ also occurs only at right angle to x-direction. The same holds good for $\vec{H}$ also.

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Thus it can be said that, a uniform plane electromagnetic wave can have its $\vec{E}$ and $\vec{H}$ components only along normal to the direction of propagation, or in other words, electromagnetic waves are transverse in nature.

Propagation of a Uniform plane wave:-

If we consider an electromagnetic wave propagating in a medium where there are no free charges or currents then we have the wave equation,

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

If we consider a uniform plane wave propagating in x-direction, then the derivatives with respect to y and z vanish. Hence the wave equation becomes,

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\epsilon \frac{\partial^2 \vec{E_x}}{\partial t^2}$$

Since the wave field $\vec{E}$ cannot have a component along the direction of propagation, $E_x = 0$, Hence its other two scalar components only could be considered in above eqn. Let us consider the component E_y. The wave equation for this component can be written as.

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}.$$

For a plane wave moving with a velocity v in the positive x-direction, the solution of the above equation is given by,

$$E_{y_1} = f_1(x - vt) \quad \text{——} (1)$$

where $f_1(x-vt)$ means function of $(x-vt)$

For a plane wave moving with a velocity v in the negative x-direction, the solution is,

$$E_{y_2} = f_2(x + vt) \quad \text{——} (2)$$

where $f_2(x+vt)$ means function of $(x+vt)$

Since the Maxwell's equations are linear, and the linear equations have the property that, the sum of two different solutions is also a solution, we can combine eqn (1) and (2) and write the general solution for the wave equation is,

$$E = f_1(x - vt) + f_2(x + vt)$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$.

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Relation between $\vec{E}$ and $\vec{H}$ for uniform plane wave:

We have the Maxwell's equation for curl of $\vec{H}$ as,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

If we consider a wave to be travelling in space where there are no free charges or current then $\vec{J} = 0$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

Expanding the terms in the above equation we have

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \epsilon \frac{\partial}{\partial t}\left(\vec{E_x}\hat{x} + \vec{E_y}\hat{y} + \vec{E_z}\hat{z}\right)$$

If we consider the wave to be a uniform plane wave travelling in x-direction, then H_x and E_x components are absent, and there is no variation in y and z-directions, i.e $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ operators produce null result.

Thus the surviving factors in the eqn consistitute the following equation.

$$-\left(\frac{\partial H_z}{\partial x}\right)\hat{y} + \left(\frac{\partial H_y}{\partial x}\right)\hat{z} = \epsilon\left(\frac{\partial E_y}{\partial t}\right)\hat{y} + \epsilon\left(\frac{\partial E_z}{\partial t}\right)\hat{z}$$

Equating the components along y-direction of both sides,

$$-\frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t}$$

Similarly for z-direction, we have

$$\frac{\partial H_y}{\partial x} = \epsilon \frac{\partial E_z}{\partial t}$$

The solution for the wave equation for a wave travelling in x-direction is given by,

$$E_y = f_1(x - vt)$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$

& $u = x - vt$

$$E_y = f_1(u)$$

$$\frac{\partial E_y}{\partial t} = \frac{\partial f_1(u)}{\partial t} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial t}$$

But from eqn $u = x - vt$, $\frac{\partial u}{\partial t} = -v$

$$\therefore \frac{\partial f_1(u)}{\partial u} = f_1'$$

$$\therefore \frac{\partial f_1(u)}{\partial t} = \frac{\partial E_y}{\partial t} = -v f_1'$$

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$$-\frac{\partial H_z}{\partial x} = -\epsilon v f_1' = -\sqrt{\frac{\epsilon}{\mu}} f_1'$$

$$\partial H_z = \sqrt{\frac{\epsilon}{\mu}} f_1' \partial x$$

Integrating both sides,

$$H_z = \sqrt{\frac{\epsilon}{\mu}} \int f_1' dx + C$$

Now we can write

$$\frac{\partial E_y}{\partial x} = \frac{\partial f_1(u)}{\partial x} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial x} = f_1' \frac{\partial u}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$

$$f_1' = \frac{\partial E_y}{\partial x}$$

by substituting,

$$H_z = \sqrt{\frac{\epsilon}{\mu}} \int \frac{\partial E_y}{\partial x} dx + C$$

$$H_z = \sqrt{\frac{\epsilon}{\mu}} E_y + C$$

By comparing dimensions on both sides of the equation, it is clear that C must be a magnetic field. Since it is not confined to x-direction, it is dissociated from the wave motion and hence ignored.

$$E_y = \sqrt{\frac{\mu}{\epsilon}} H_z$$

For the same wave, the solution $E_z = f_1(x - vt)$ also holds good.

Now, following the same procedure for E_z as followed earlier for E_y, we say,

$$E_z = f_1(x - vt) = f_1(u)$$

$$\frac{\partial E_z}{\partial t} = \frac{\partial f_1(u)}{\partial t} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial t}$$

$$\frac{\partial E_z}{\partial t} = -v f_1'$$

by substituting.

$$\frac{\partial H_y}{\partial x} = -\epsilon v f_1' = -\sqrt{\frac{\epsilon}{\mu}} f_1'$$

$$H_y = -\sqrt{\frac{\epsilon}{\mu}} \int f_1' dx + C$$

$$\frac{\partial E_z}{\partial x} = \frac{\partial f_1(u)}{\partial x} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial x} = f_1'$$

$$H_y = -\sqrt{\frac{\epsilon}{\mu}} \int \frac{\partial E_z}{\partial x} dx + C$$

$$H_y = -\sqrt{\frac{\epsilon}{\mu}} E_z + C$$

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The constant of integration c is ignored for the wave.

$$E_z = -\sqrt{\frac{\mu}{\epsilon}}\, H_y$$

Squaring and adding,

$$E_y^2 + E_z^2 = \frac{\mu}{\epsilon}(H_y^2 + H_z^2)$$

Since $E_x = 0$ and $H_x = 0$, the total electric and magnetic field strengths E and H are.

$$E = \sqrt{E_y^2 + E_z^2} \quad \text{and} \quad H = \sqrt{H_y^2 + H_z^2}$$

$$\therefore E = \sqrt{\frac{\mu}{\epsilon}}\, H$$

$$\boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}}$$

This is the relation between E and H for a travelling uniform plane wave.

<u>orientation b/w $\vec{E}$ and $\vec{H}$</u> :-

Consider the dot product $\vec{E}\cdot\vec{H}$

$$\vec{E}\cdot\vec{H} = [E_y\cdot\hat{y} + E_z\cdot\hat{z}]\cdot[\vec{H_y}\cdot\hat{y} + \vec{H_z}\cdot\hat{z}]$$

$$= E_yH_y + E_zH_z$$

$$= \sqrt{\frac{\mu}{\epsilon}}\, H_y H_z - \sqrt{\frac{\mu}{\epsilon}}\, H_y H_z$$

$$\vec{E}\cdot\vec{H} = 0$$

$\therefore$ They are right angles to each other.

<u>characteristic or Intrinsic impedence (η)</u>

For free space, i.e for vaccum.

$$\mu = \mu_0 = 4\pi\times10^{-7}\ H/m$$

& $\epsilon = \epsilon_0 = 8.854\times10^{-12}\ F/m$.

by substituting,

$$\frac{E}{H} = \eta = 377 \text{ ohm for free space.}$$

$$\frac{\text{Volt/meter}}{\text{ampere/meter}} = \frac{\text{Volt}}{\text{ampere}} = \text{ohm}.$$

$\therefore$ The ratio $\left(\frac{E}{H}\right) = \eta$ is called characteristic or intrinsic impedence.

<u>Solution of wave equation for Uniform plane wave (wave propagation in free space or lossless medium)</u>

For an electromagnetic wave propagating in a medium where there are no free charges, or currents (i.e lossless

medium), the wave equation for the electric field is given by,

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

let the wave associated with the above equation be a wave for which the field E varies sinusoidally with time. For undamped variation, the variation may be expressed as

$$\vec{E} = \vec{E_0} e^{j\omega t},$$

where $\vec{E}$, is the instantaneous field at time t,

$\vec{E_0}$, is the amplitude or the maximum value of E during the cycle.

ω, is the angular frequency for the wave,

$j = \sqrt{-1}$ and $\omega = 2\pi f$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = j^2\omega^2 (\vec{E_0} e^{j\omega t}) = -\omega^2 \vec{E}$$

we have the wave eqn for lossless medium as.

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0$$

This is called Helmholtz equation.

Wave propagation in good conductor and dielectrics:-

If we consider a uniform plane wave propagating in x-direction, then the derivatives with respect to y and z vanish. The wave equation becomes,

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \omega^2 \mu\epsilon \vec{E} = 0$$

but $E_x = 0$, Among the y and z components, we shall consider the y-component.

$$\frac{\partial^2 E_y}{\partial x^2} + \omega^2 \mu\epsilon E_y = 0$$

let $\omega\sqrt{\mu\epsilon} = \beta$

$$\frac{\partial^2 E_y}{\partial x^2} + \beta^2 E_y = 0$$

The solution for this second order differential eqn is given by

$$E_y = C_1 e^{-j\beta x} + C_2 e^{j\beta x}$$

Here, E_y is the amplitude of y component of $\vec{E}$

$\therefore$ The displacement $E_y(t)$ at any instant t in y-direction is given by,

$$E_y(t) = E_y e^{j\omega t} = \left[C_1 e^{-j\beta x} + C_2 e^{j\beta x}\right] e^{j\omega t}$$

$$= C_1 e^{j(\omega t - \beta x)} + C_2 e^{j(\omega t + \beta x)}$$

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This is the solution of the wave equation, and it represents the sum of two waves travelling in opposite directions, one (i.e the first term) in positive x-direction (i.e away from the source), and the other (i.e the second term) in negative x-direction (i.e towards the source due to reflection).

Velocity of the wave:-

we know that the velocity of a propagation wave is given by,

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

$$v = \omega/\beta.$$

Phase velocity:-

If we consider the wave travelling in positive x-direction, then

$$E_y = C_1 e^{j(\omega t - \beta x)}$$

Considering the real part, we have

$$E_y = C_1 \cos(\omega t - \beta x) = C_1 \cos\omega\left(t - \frac{\beta}{\omega}x\right),$$

$$= C_1 \cos\omega\left(t - x\sqrt{\mu\epsilon}\right)$$

Here the phase term is $\omega(t - x\sqrt{\mu\epsilon})$. Taking its differentials,

$$\omega(dt - dx\sqrt{\mu\epsilon}) = 0$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{\mu\epsilon}}.$$

Since here x refers to a point of particular phase in the wave, (dx/dt) is the velocity with which the constant phase point is transported, and thus termed as phase velocity.

Phase velocity, $v = \frac{1}{\sqrt{\mu\epsilon}}$

which is same as wave velocity.

Phase velocity, $v = \frac{\omega}{\beta}$.

Intrinsic impedence, η :-

The intrinsic impedence or characteristic impedence is given by the ratio (E/H). For a wave travelling in a conducting medium, the ratio is a complex quantity, and in general it is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

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Wave equations in E and H for a conducting medium:-

we have the wave equation in $\vec{E}$ for a homogeonous and isotropic medium given by

$$\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial\vec{J}}{\partial t} + \nabla\left(\frac{\rho}{\epsilon}\right)$$

Let it be conducting medium of conductivity σ (i.e $\vec{J} = \sigma\vec{E}$) but let there be no free charges ($\rho = 0$.

$\therefore$ The above eqn becomes

$$\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0$$

This is the wave eqn in $\vec{E}$ for conducting medium.

Similarly we have wave eqn in $\vec{H}$ as,

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = -\nabla\times\vec{J}.$$

but $\nabla\times\vec{J} = \nabla\times\sigma\vec{E} = \sigma(\nabla\times\vec{E})$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{J} = -\mu\sigma\frac{\partial\vec{H}}{\partial t}$$

$$\therefore \nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = \mu\sigma\frac{\partial\vec{H}}{\partial t}.$$

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = 0$$

This is the wave eqn in $\vec{H}$ for conducting medium,

Uniform plane wave in a conducting medium:

Let us consider the wave equation in $\vec{E}$ for conducting medium.

i.e $\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0$

The solution of the above equation can be written as.

$$\vec{E} = \vec{E_0}e^{j\omega t}$$

$$\frac{\partial\vec{E}}{\partial t} = j\omega(\vec{E_0}e^{j\omega t}) = j\omega\vec{E}$$

$$\frac{\partial^2\vec{E}}{\partial t^2} = j^2\omega^2(\vec{E_0}e^{j\omega t}) = j^2\omega^2\vec{E}.$$

by substituting.

$$\nabla^2\vec{E} - j^2\omega^2\mu\epsilon\vec{E} - j\omega\mu\sigma\vec{E} = 0$$

$$\nabla^2\vec{E} - j\omega\mu(j\omega\epsilon + \sigma)\vec{E} = 0$$

The above eqn can be written as,

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0.$$

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where $\gamma^2 = j\omega\mu(j\omega\epsilon + \sigma)$

γ is called propagation constant.

If we consider a uniform plane wave propagation in x-direction, then considering the y-component, the eqn can be written as,

$$\frac{\partial^2 E_y}{\partial x^2} + \gamma^2 \vec{E_y} = 0$$

The solution for this eqn is

$$E_y = C_1 e^{-\gamma x} + C_2 e^{\gamma x}.$$

where E_y is the amplitude of y-component of $\vec{E}$

The displacement $E_y(t)$ at any instant t in y-direction is given by

$$E_y(t) = E_y e^{j\omega t} = e^{j\omega t}\left[C_1 e^{-\gamma x} + C_2 e^{\gamma x}\right]$$

$$= C_1 e^{(j\omega t - \gamma x)} + C_2 e^{(j\omega t + \gamma x)}$$

If we ignore the reflected wave, then the second term vanishes.

$$E_y(t) = C_1 e^{(j\omega t - \gamma x)}$$

The propagation constant γ is a complex number, and hence can be written,

$$\gamma = \alpha + j\beta.$$

$$E_y(t) = C_1 e^{[j\omega t - (\alpha + j\beta)x]}$$

$$= C_1 e^{-\alpha x} e^{j(\omega t - \beta x)}$$

The real part in this eqn represents a uniform plane wave travelling in a conducting medium in the x-direction. It is a damped wave whose attenuation is given by the factor $e^{-\alpha x}$, α is called the <u>attenuation constant</u>, and β is called <u>phase constant</u>. the values of which are.

$$\alpha = \sqrt{\omega^2\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]\right\}^{1/2} \geq 0.$$

$$\beta = \sqrt{\omega^2\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]\right\}^{1/2} \geq 0.$$

<u>Evaluation of the attenuation constant (α) and phase constant (β)</u>

We have

$$\gamma^2 = j\omega\mu(j\omega\epsilon + \sigma) = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

but $\gamma = \alpha + j\beta$.

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$$\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$$

$$\therefore (\alpha^2 - \beta^2) + j(2\alpha\beta) = -\omega^2\mu\epsilon + j(\omega\mu\sigma)$$

Equating the real and imaginary parts, we have,

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon$$

$$2\alpha\beta = \omega\mu\sigma$$

For the sake of handling the terms, let us make the following substitutions,

$$\omega^2\mu\epsilon = a^2$$

$$\omega\mu\sigma = b^2$$

$$\alpha^2 - \beta^2 = -a^2$$

$$2\alpha\beta = b^2.$$

Squaring the above two eqns and adding

$$(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2 = a^4 + b^4$$

$$\alpha^2 + \beta^2 = \sqrt{a^4 + b^4}$$

adding $\Rightarrow$ $2\alpha^2 = \sqrt{a^4 + b^4} - a^2$

$$\alpha^2 = \frac{a^2}{2}\left[\sqrt{1 + \frac{b^4}{a^4}} - 1\right]$$

$$\frac{b^4}{a^4} = \frac{\omega^2\mu^2\sigma^2}{\omega^4\mu^2\epsilon^2} = \frac{\sigma^2}{\omega^2\epsilon^2}.$$

$$\alpha^2 = \frac{\omega^2\mu\epsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right]$$

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]\right\}^{1/2}$$

Subtracting $\Rightarrow$ $2\beta^2 = \sqrt{a^4 + b^4} + a^2$

$$\beta^2 = \frac{a^2}{2}\left[\sqrt{1 + \frac{b^4}{a^4}} + 1\right]$$

$$\beta^2 = \frac{\omega^2\mu\epsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right]$$

$$\beta = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]\right\}^{1/2}$$

For good conductor, $\frac{\sigma}{\omega\epsilon} >> 1$

For perfect dielectric, $\frac{\sigma}{\omega\epsilon} \to 0$

For good dielectric (low loss dielectric) $\frac{\sigma}{\omega\epsilon} << 1$

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1. Uniform Plane wave in a good conductor:

(a) Attenuation constant α and phase constant β:

wkt for good conductor,

$$\frac{\sigma}{\omega\epsilon} >> 1$$

$$\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 = \sqrt{\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 = \frac{\sigma}{\omega\epsilon} - 1 = \frac{\sigma}{\omega\epsilon}$$

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right\}^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 = \sqrt{\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 = \frac{\sigma}{\omega\epsilon} + 1 = \frac{\sigma}{\omega\epsilon}$$

$$\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

(b) The velocity of the wave (v):-

$$v = \frac{\omega}{\beta} = \omega\sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

(c) Propagation constant (γ):-

$$\gamma = \alpha + j\beta$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}}(1+j)$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}(1+j)^2} = \sqrt{\frac{\omega\mu\sigma}{2}(1-1+2j)}$$

$$= \sqrt{\omega\mu\sigma j} = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

2. Uniform plane wave in a perfect dielectric (Lossless dielectric)

(a) attenuation constant α and phase constant β:

wkt for perfect dielectric $\frac{\sigma}{\omega\epsilon} \to 0$

$$\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 = \sqrt{1+0} - 1 = 0$$

$$\alpha = 0$$

$$\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 = \sqrt{1+0} + 1 = 2$$

$$\beta = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{\cancel{2}}(\cancel{2})\right\}^{1/2}$$

$$\beta = \omega\sqrt{\mu\epsilon}$$

(b) The velocity of the wave:

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 \text{ m/s}.$$

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3. Uniform plane wave in a good dielectric (Low loss dielectric)

a) Attenuation constant α and phase constant β:- [8]

we know that for good dielectric, $\frac{\sigma}{\omega\epsilon} \ll 1$

$$\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} = \left\{1+\left(\frac{\sigma^2}{\omega^2\epsilon^2}\right)\right\}^{1/2}$$

by binomial theorem that

$(1+a)^n \approx 1+na$ i.e $\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} \approx \left(1+\frac{1}{2}\frac{\sigma^2}{\omega^2\epsilon^2}\right)$

$$\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1 \approx \left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}\right) - 1 = \frac{\sigma^2}{2\omega^2\epsilon^2}$$

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left(\frac{\sigma^2}{2\omega^2\epsilon^2}\right)\right\}^{1/2} = \omega\sqrt{\mu\epsilon}\left\{\frac{\sigma}{2\omega\epsilon}\right\}$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} + 1 \approx 1+\frac{\sigma^2}{2\omega^2\epsilon^2} + 1 = 2+\frac{\sigma^2}{2\omega^2\epsilon^2}$$

$$\beta = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[2+\frac{\sigma^2}{2\omega^2\epsilon^2}\right]\right\}^{1/2} = \omega\sqrt{\mu\epsilon}\left\{1+\frac{\sigma^2}{4\omega^2\epsilon^2}\right\}^{1/2}$$

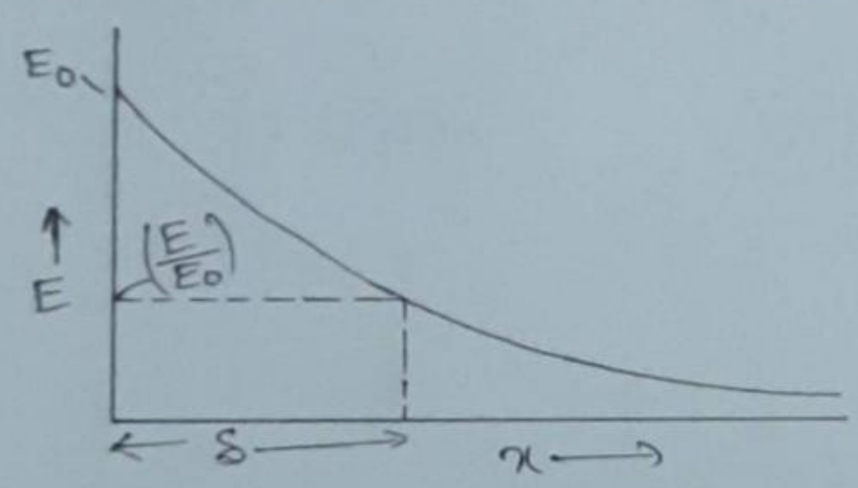
$$\beta = \omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right]$$

The velocity of the wave v:-

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right]} = \frac{1}{\sqrt{\mu\epsilon}}\left[1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right]^{-1}$$

$$v = \frac{1}{\sqrt{\mu\epsilon}}\left(1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$$

Skin depth and Skin effect:-



when an electromagnetic wave enters a conducting medium, its amplitude decreases exponentially and becomes practially zero after penetrating a small distance. As a result, the current induced by the wave exists only near the surface of the conductor. This effect is called Skin effect.

If x is the distance travelled in the medium, and E is the amplitude, then the dependence can be represented as,

$$E = E_0 e^{-\alpha x}$$

where E_0 is the amplitude either at the time of incidence, or at some point in the medium where x is taken as zero.

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α is the attenuation constant.

At the penetrating depth of $x = \delta$, let the value of $x = \frac{1}{\alpha}$ at which time $e^{-\alpha x} = \frac{1}{e}$ $(= 0.368)$.

$$E = \frac{1}{e} E_0$$

δ is called the depth of penetration, or skin depth.

Thus the skin depth, or the depth of penetration, is defined as the depth of a conductor at which the amplitude of an incident wave drops to $(1/e)$ times its value of amplitude at the time of incidence.

But $\delta = \frac{1}{\alpha}$ for good conductor.

$$\alpha = \sqrt{\frac{\omega \mu \sigma}{2}}$$

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}} = \sqrt{\frac{1}{\pi f \mu \sigma}}$$

for copper $\sigma = 5.8 \times 10^7$ S/m

$$\delta = \frac{0.066}{\sqrt{f}}$$

<u>Intrinsic Impedence in different media:</u>

Intrinsic impedence of the medium $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$

(a). <u>for good conductor:</u>

$$\frac{\sigma}{\omega\epsilon} >> 1$$

$$\sigma + j\omega\epsilon \approx \sigma$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

(b) <u>for perfect dielectric:</u>

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j\omega\mu}{\omega\epsilon\left(\frac{\sigma}{\omega\epsilon} + j\right)}}$$

$$\frac{\sigma}{\omega\epsilon} \to 0$$

$$\eta = \sqrt{\frac{j\omega\mu}{\omega\epsilon(0 + j)}}$$

$$= \sqrt{\frac{\mu}{\epsilon}}$$

(c) <u>for good dielectric:</u>

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \neq \sqrt{\frac{\mu}{\frac{1}{j\omega}(\sigma + j\omega\epsilon)}} = \sqrt{\frac{\mu}{\left(\frac{\sigma}{j\omega} + \epsilon\right)}}$$

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$$\eta = \sqrt{\frac{\mu}{\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}} = \sqrt{\frac{\mu}{\epsilon}}\left(1+\frac{\sigma}{j\omega\epsilon}\right)^{-1/2} = \sqrt{\frac{\mu}{\epsilon}}\left(1-\frac{\sigma}{2j\omega\epsilon}\right)$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\left(1+j\frac{\sigma}{2\omega\epsilon}\right)$$

Poynting Vector and Power flow.

Poynting Theorem:

Poynting theorem states that the energy dissipated in a given volume free of any sources is equal to the sum of, the rate at which the decrease in electric and magnetic energies stored in the volume takes place, and the rate at which energy inflow occurs through its surface.

This theorem may be considered as a statement of conservation of energy in electromagnetism.

Poynting vector represents the rate at which energy is transported per unit area at a point on the path of an EMW, and is given by the cross product $\vec{E}\times\vec{H}$ at the same point.

proof: we have maxwell's equation,

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t} = \vec{J}+\epsilon\frac{\partial\vec{E}}{\partial t}$$

Now, let us consider $\nabla\cdot(\vec{E}\times\vec{H})$

$$\nabla\cdot(\vec{A}\times\vec{B}) = B\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$$

$$= \vec{H}\cdot\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) - \vec{E}\cdot\left(\vec{J}+\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\mu\left(\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}\right) - \vec{E}\cdot\vec{J} - \epsilon\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)$$

wkt $\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}$ and $\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}$

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\frac{\mu H^2}{2} - \frac{\partial}{\partial t}\frac{\epsilon E^2}{2} - \vec{E}\cdot\vec{J}$$

Taking volume integral throughout, we have

$$\int_V \nabla\cdot(\vec{E}\times\vec{H})\,dv = -\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}+\frac{\epsilon E^2}{2}\right)dv - \int_V \vec{E}\cdot\vec{J}\,dv.$$

By ~~Stokes~~ gauss' divergence theorem,

$$\int_V \vec{E}\cdot\vec{J}\,dv = -\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}+\frac{\epsilon E^2}{2}\right)dv - \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$

The above expression signifies the Poynting's theorem.

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Let us understand the meaning of each of the terms in the above equation.

(i) meaning of $\int \vec{E}\cdot\vec{J}\,dv$

wkt, for a conductor of any area of cross-section in which a current I is flowing under a potential difference of E, the power dissipated / unit length = EI

∴ power dissipated by a conductor of length l = EIl

∴ power dissipated by a unit volume of the conductor $= \frac{EIl}{v} = \frac{EIl}{l \times A}$

$= \frac{EI}{A} = EJ$

∴ $\int \vec{E}\cdot\vec{J}\,dv$ = power dissipated in a volume v.

(ii) meaning of $-\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right) dv$.

wkt $\frac{\mu H^2}{2}$ and $\frac{\epsilon E^2}{2}$ are respectively the stored magnetic and electric field energies.

∴ $-\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right) dv$ = the rate at which the stored field energy is decreasing in a volume v over which the integration is carried out.

(iii) Meaning of $\oint_S (\vec{E}\times\vec{H})\cdot ds$, and the Poynting vector:

The product $\vec{E}\times\vec{H}$ itself is another vector denoted by $\vec{P}$ directed perpendicular to the plane containing the $\vec{E}$ and $\vec{H}$ vectors, in the sense of a right hand cork screw, $\vec{P}$ is called Poynting vector, named after the mathematician J.H Poynting.

$$\vec{P} = \vec{E}\times\vec{H}.$$

Average Poynting Vector :-

In case where power is changing with time sinusoidally, the time average of the instantaneous power averaged over a cycle shall be considered. The instantaneous power represented by the Poynting vector P(t) is related to the time average of the power over a cycle represented by the Poynting vector $\vec{P_{av}}$, through the relation,

$$\vec{P_{av}} = \frac{1}{T}\int_0^T P(t)\,dt.$$

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We have the poynting vector.

$$\vec{P} = \vec{E} \times \vec{H}$$

Let consider wave travelling in x-direction in a perfect dielectric.

$$E_y = E_{y0} \cos(\omega t - \beta x)$$

$$H_z = \frac{E_{y0}}{\eta} \cos(\omega t - \beta x)$$

where η is real. The power density amplitude is therefore,

$$P_z = \frac{E_{y0}^2}{\eta} \cos^2(\omega t - \beta x)$$

In the case of a lossy dielectric, E_y and H_z are not in time phase. We have,

$$E_y = E_{y0} e^{-\alpha x} \cos(\omega t - \beta x)$$

If we let $\eta = |\eta| \angle \theta_\eta$

$$H_z = \frac{E_{y0}}{|\eta|} e^{-\alpha x} \cos(\omega t - \beta x - \theta_\eta)$$

$$P_z = E_y H_z = \frac{E_{y0}^2}{|\eta|} e^{-2\alpha x} \cos(\omega t - \beta x) \cos(\omega t - \beta x - \theta_\eta)$$

$$P_z = \frac{E_{y0}^2}{|\eta|} e^{-2\alpha x} \left[\cos(2\omega t - 2\beta x - \theta_\eta) + \cos\theta_\eta\right]$$

$$\therefore \langle P_z \rangle = P_{av} = \frac{1}{T}\int_T \frac{E_{y0}^2}{|\eta|} e^{-2\alpha x} \left[\cos(2\omega t - 2\beta x - \theta_\eta) + \cos\theta_\eta\right] dt$$

The second harmonics component of the integrand in the above equation integrates to zero, leaving only the contribution from the dc component. The result is,

$$P_{av} = \frac{1}{2} \frac{E_{y0}^2}{|\eta|} e^{-2\alpha x} \cos\theta_\eta$$

In vector form,

$$P_{av} = \frac{1}{2} \mathrm{Re}(E \times H^*) \ \mathrm{W/m^2}$$

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RMS Value of Electric Field :-

$$E_{rms} = \frac{E_{max}}{\sqrt{2}}$$

But $$P_{av} = \frac{E_{max}^2}{2\eta}$$

$$E_{max} = \sqrt{2\eta P_{av}}$$

$$E_{rms} = \frac{\sqrt{2\eta P_{av}}}{\sqrt{2}}$$

$$E_{rms} = \sqrt{\eta P_{av}}$$

Proof for $\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}$:-

We have $\vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z$

$$\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = (H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z)\cdot\left(\frac{\partial H_x}{\partial t}\hat{a}_x + \frac{\partial H_y}{\partial t}\hat{a}_y + \frac{\partial H_z}{\partial t}\hat{a}_z\right)$$

$$= H_x\frac{\partial H_x}{\partial t} + H_y\frac{\partial H_y}{\partial t} + H_z\frac{\partial H_z}{\partial t}$$

$$= \frac{1}{2}\left\{2H_x\frac{\partial H_x}{\partial t} + 2H_y\frac{\partial H_y}{\partial t} + 2H_z\frac{\partial H_z}{\partial t}\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial H_x^2}{\partial t} + \frac{\partial H_y^2}{\partial t} + \frac{\partial H_z^2}{\partial t}\right\}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(H_x^2 + H_y^2 + H_z^2)$$

$$\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}$$

Intrinsic Impedance in Different media :-

We have the maxwell's equation for curl $\vec{E}$ as,

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t}$$

Expanding the equation, we have,

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\mu\frac{\partial}{\partial t}(H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z)$$

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If we consider a uniform plane wave travelling in x-direction, then E_x and H_x components are absent, and also, there will be no variations in y and z-directions, and hence the operators $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ produce null result. Hence the surviving factors in above equation will then be,

$$+\frac{\partial E_y}{\partial x}\hat{a}_z - \frac{\partial E_z}{\partial x}\hat{a}_y = -\mu\frac{\partial H_y}{\partial t}\hat{a}_y - \mu\frac{\partial H_z}{\partial t}\hat{a}_z$$

$$\frac{\partial E_y}{\partial x} = -\mu\frac{\partial H_z}{\partial t}.$$

Let consider

$$E_y = Ce^{-\gamma x}e^{j\omega t} \quad \& \quad H_z = Ce^{-\gamma x}e^{j\omega t}$$

$$\frac{\partial E_y}{\partial x} = -\gamma Ce^{-\gamma x}e^{j\omega t} \qquad \frac{\partial H_z}{\partial t} = j\omega Ce^{-\gamma x}e^{j\omega t}$$

$$= -\gamma E_y \qquad = j\omega H_z.$$

$$-\gamma E_y = j\omega H_z(-\mu)$$

$$\frac{E_y}{H_z} = \frac{j\omega\mu}{\gamma}$$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma+j\omega\epsilon)}}$$

$$\frac{E_y}{H_z} = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}} = \eta$$

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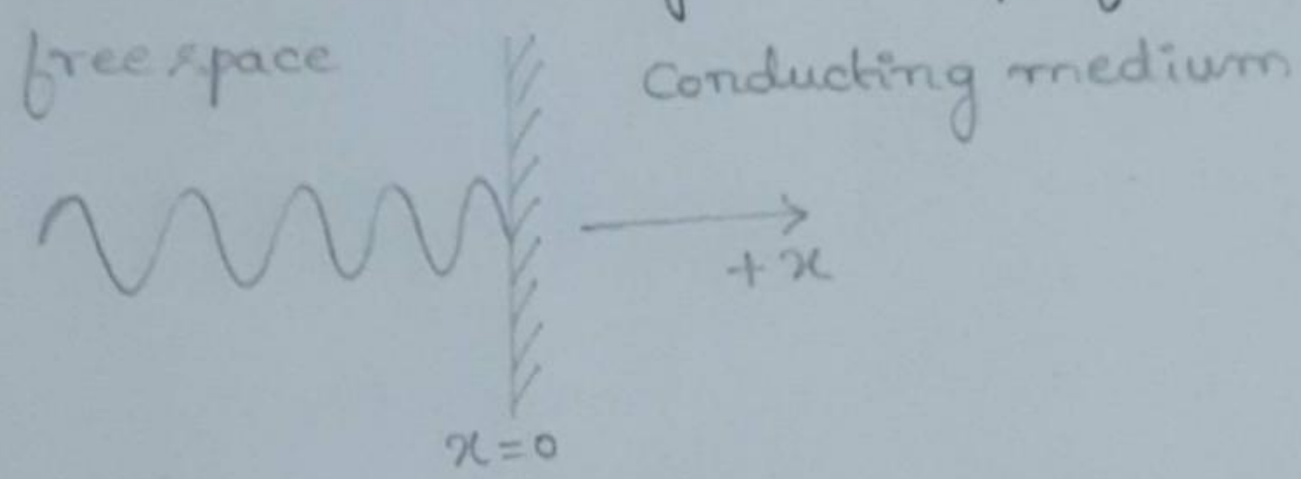
Unit - 8

Reflection of Uniform Plane Waves

Different media are characterized by different values of Conductivity, and dielectric constant for the electric field, and different values of permeability for the magnetic field. If a travelling electromagnetic wave encounters a change in media, then at the boundary of separation of the two media, a part of the wave will be reflected back into the medium in which it was travelling, and the remaining part is transmitted into the adjoining medium. However, the magnitudes of reflection and transmission depend on the nature of the media, i.e whether conducting or non-conducting.

Reflection of plane waves at the surface of a perfect conductor at Normal Incidence :-

Consider a plane wave travelling in positive x-direction to be incident normally on a perfect conductor.



If E_i is the amplitude of the incident wave, then by taking $x=0$ at the surface of the conductor, the electric field of the wave at a distance x from the surface can be written as,

$$E_i e^{-j(\omega t - \beta x)}$$

The part of the wave that is reflected travels in the -ve x-direction. If E_r is the amplitude of the reflected wave, and δ is the phase difference introduced due to reflection then, the electric field of the wave can be written as,

$$E_r e^{-j(\omega t + \beta x + \delta)}$$

The tangential components of $\vec{E}$ must be continuous across the boundary. Hence there cannot be a tangential component of $\vec{E}$ outside the conductor also. This condition is satisfied only when the sum of the electric fields of the

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incident and reflected waves is zero at the boundary i.e

$$E_i + E_r = 0$$

$$E_i = -E_r$$

This means that the reflected wave is out of phase with the incident wave by π.

$$\delta = \pi$$

and the entire energy is returned along the reflected wave.

The resultant wave $E(x,t)$ due to reflection is the sum of the incident and the reflected wave. By adding,

$$E(x,t) = E_i e^{-j(\omega t - \beta x)} + E_r e^{-j(\omega t + \beta x + \delta)}$$

By considering the real part

$$E(x,t) = E_i \left[\cos(\omega t - \beta x) + \cos(\omega t + \beta x + \delta)\right]$$

But $\delta = \pi$

$$E(x,t) = E_i \left[\cos(\omega t - \beta x) - \cos(\omega t + \beta x)\right]$$

$$= 2E_i \sin\omega t \sin\beta x \quad \text{——— (1)}$$

This is the equation for standing wave.

at $x=0$, $E=0$

As per the characteristics of a wave, the same condition, $E=0$ repeat at distances of $(n\lambda/2)$ from the surface, where $n = 0, 1, 2, 3 ----$ and λ is the wavelength.

The magnitude of the electric field will be maximum at distances of $(2n+1)\lambda/4$

If H_i and H_r are the amplitudes of magnetic fields for the wave, then

$H_i = H_r$ holds good.

$$H(x,t) = H_i \left[e^{-j(\omega t - \beta x)} + e^{-j(\omega t + \beta x)}\right]$$

Considering only the real part of RHS, we have

$$H(x,t) = H_i \left[\cos(\omega t - \beta x) + \cos(\omega t + \beta x)\right]$$

$$= 2H_i \cos\omega t \cos\beta x \quad \text{——— (2)}$$

This is also an equation for standing wave.

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Note that at $x=0$, the field has a maximum value.

Comparing the eqn (1) and (2), it is clear that, at any instant t, the magnetic and electric fields in a reflected wave are out of phase by $\pi/2$.

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Reflection of Plane waves at the Surface of Dielectric at normal incidence:

Consider a plane wave travelling in positive x-direction in dielectric medium 1. to be incident normally on the surface of another dielectric medium-2. A part of the incident wave is transmitted into medium 2, and the remaining part is reflected back into medium 1.

Let ϵ_1, μ_1 and η_1 be the permittivity, permeability and intrinsic impedance respectively for the first medium, and ϵ_2, μ_2 and η_2 be corresponding constants for the second medium.

With these definition, the following relations holds good

$$E_i = \eta_1 H_i$$

$$E_r = -\eta_1 H_r$$

-ve sign is because, during reflection, the electric field undergoes a phase change of π, where as the magnetic field doesn't)

$$E_t = \eta_2 H_t$$

The ratio of refracted and transmitted fields to the incident field

Reflection Coefficient $\Gamma = \frac{E_r}{E_i}$

Transmission coefficient $\tau = \frac{E_t}{E_i}$

$\frac{H_r}{H_i}$, $\frac{H_t}{H_i}$ in terms of the constants of the medium i.e, η and ϵ.

Total Tangential components in medium 1 = $E_i + E_r$

& Total Tangential components in medium 2 = E_t

But according to boundary conditions

$$E_i + E_r = E_t$$

$$\text{|||}^{ly} \quad H_i + H_r = H_t$$

Now, consider the term $(E_i - E_r)$

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$$E_i - E_r = \eta_1 (H_i + H_r) = \eta_1 H_t$$

$$= \frac{\eta_1}{\eta_2} E_t = \frac{\eta_1}{\eta_2} (E_i + E_r)$$

or $\eta_2 (E_i - E_r) = \eta_1 (E_i + E_r)$

Rearranging,

$$E_i (\eta_2 - \eta_1) = E_r (\eta_2 + \eta_1)$$

$\therefore$ Reflection coefficient $\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Now

$$\frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Transmission coefficient $\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1}$

Considering the ratios of magentic fields we have,

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i}$$

$$\frac{H_r}{H_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

Also,

$$\frac{H_t}{H_i} = \frac{\eta_1 E_t}{\eta_2 E_i} = \frac{\eta_1}{\eta_2} \left(\frac{2\eta_2}{\eta_1 + \eta_2} \right)$$

$$\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2}$$

But wkt $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ & $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$

for free space $\mu_1 = \mu_2 = \mu_0$ [Non magnetic material]

$$\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\frac{\mu_0}{\epsilon_2}} - \sqrt{\frac{\mu_0}{\epsilon_1}}}{\sqrt{\frac{\mu_0}{\epsilon_2}} + \sqrt{\frac{\mu_0}{\epsilon_1}}} = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}}$$

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$$\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1+\eta_2} = \frac{2\sqrt{\frac{\mu_0}{\epsilon_2}}}{\sqrt{\frac{\mu_0}{\epsilon_1}}+\sqrt{\frac{\mu_0}{\epsilon_2}}}$$

$$\frac{E_t}{E_i} = \frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

$$\frac{H_r}{H_i} = \frac{\sqrt{\epsilon_2}-\sqrt{\epsilon_1}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

$$\frac{H_t}{H_i} = \frac{2\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

Relation between Γ & τ

$$\tau - \Gamma = \frac{2\eta_2}{\eta_1+\eta_2} - \frac{\eta_2-\eta_1}{\eta_1+\eta_2} = \frac{\eta_2+\eta_1}{\eta_2+\eta_1} = 1$$

$$\tau - \Gamma = 1$$

$$\boxed{\tau = 1+\Gamma}$$

Standing Wave Ratio (SWR) :-

Standing wave ratio is defined as the ratio of the electric field intensities corresponding to the maximum and minimum amplitudes of the resultant wave pattern in a medium following reflection of waves from another medium.

The electric field for a wave travelling along x-axis and heading for incident can be written as,

$$E_i e^{-j(\omega t - \beta x)}$$

where E_i is the amplitude.

If δ is the phase difference introduced due to reflection then the electric field of the reflected wave can be written as,

$$E_r e^{-j(\omega t + \beta x + \delta)}$$

$\therefore$ The resultant wave $E(x,t)$ due to the interference b/w the waves heading for incidence and the reflected

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waves is,

$$E(x,t) = E_i e^{-j(\omega t - \beta x)} + E_r e^{-j(\omega t + \beta x + \delta)}$$

But $E_r = \Gamma E_i$

$$E(x,t) = E_i \left[e^{-j(\omega t - \beta x)} + \Gamma e^{-j(\omega t + \beta x + \delta)} \right]$$

By adding and subtracting $\Gamma e^{-j(\omega t - \beta x)}$

$$E(x,t) = E_i \left[e^{-j(\omega t - \beta x)} + \Gamma e^{-j(\omega t - \beta x)} - \Gamma e^{-j(\omega t - \beta x)} + \Gamma e^{-j(\omega t + \beta x + \delta)} \right]$$

$$= E_i \left\{ e^{-j(\omega t - \beta x)} (1 + \Gamma) + \Gamma \left[e^{-j(\omega t + \beta x + \delta)} - e^{-j(\omega t - \beta x)} \right] \right\}$$

But $\delta = \pi$

$$\therefore E(x,t) = E_i \left[\tau \cos(\omega t - \beta x) + 2\Gamma \sin \omega t \sin \beta x \right]$$

In the above equation the first part represents a travelling wave of amplitude τE_i and the second part a standing wave of amplitude $2\Gamma E_i$.

At the locations where the phase of E_r and E_i are same, the two fields adds up and the resultant fields becomes maximum.

$$E_{max} = E_i + E_r$$

This is possible for values of $\beta x = n\pi$ where $n = 0, \pm 1, \pm 2 \ldots$

But at the locations where the phase of E_r is in the opposite direction to that of E_i, the resultant field drops to minimum at certain instants, i.e,

$$E_{min} = E_i - E_r$$

This is possible for values of $\beta x = (2n+1)\pi$ where $n = 0, \pm 1, \pm 2, \ldots$

By definition, Standing wave ratio is given by,

$$SWR = \frac{E_{max}}{E_{min}}$$

$$= \frac{E_i + E_r}{E_i - E_r} = \frac{E_i + \Gamma E_r}{E_i - \Gamma E_r} = \frac{E_i(1+\Gamma)}{E_i(1-\Gamma)}$$

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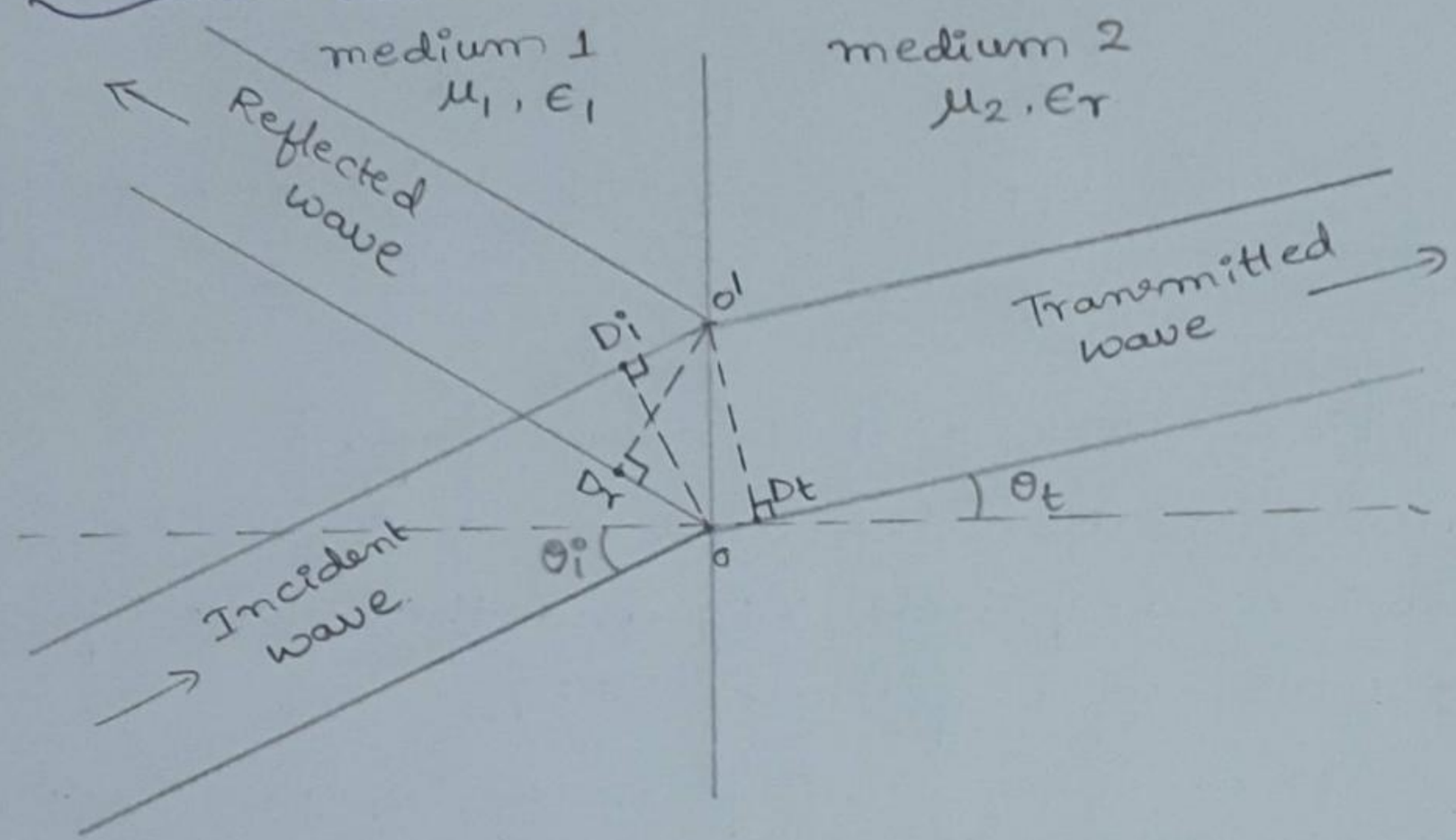
$$SWR = \frac{1+\Gamma}{1-\Gamma}$$

By transposing,

$$\Gamma = \frac{SWR-1}{SWR+1}$$

Snell's Law:-



* Figure shows the boundary between 2 lossless media with constants μ_1, ϵ_1 and μ_2, ϵ_2
* θ_i is the angle of Incidence, θ_r angle of reflection and θ_t angle of transmission.
  All these angles are measured with respect to the lines which are $\perp^r$ to the boundary.
* A point on the incident wave strikes the boundary at 'O' Later a point on the wavefront strikes the boundary at 'O'.
* The reflected and transmitted waves are produced from these points.
* The wave front of the incident wave is OD_i, that of the transmitted wave is OD_t and that of reflected wave is OD_r.
* The time taken by the incident wave to travel a distance $O'D_i$, the time taken by the transmitted wave to travel OD_t and the time taken by a reflected wave to travel a

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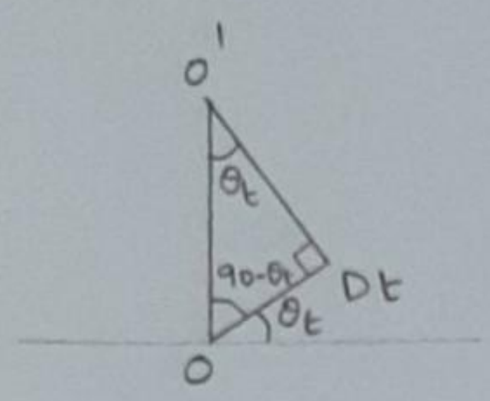
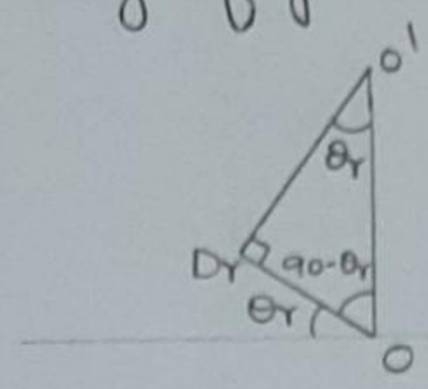
distance OD_r is the same.

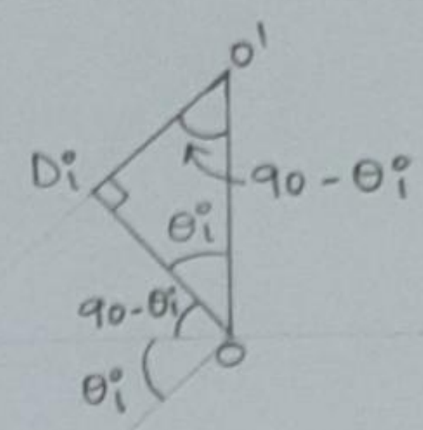
$$\frac{OD_t}{v_2} = \frac{OD_r}{v_1} = \frac{O'D_i}{v_1} \quad \text{—— (1)}$$

v_1 = velocity in the medium 1
v_2 = velocity in the medium 2

Consider the following figures,







$$\sin\theta_t = \frac{OD_t}{OO'}$$

$$OD_t = OO' \sin\theta_t \quad \text{—— (2)}$$

$$\sin\theta_r = \frac{OD_r}{OO'}$$

$$OD_r = OO' \sin\theta_r \quad \text{—— (3)}$$

$$\sin\theta_i = \frac{O'D_i}{OO'}$$

$$O'D_i = OO' \sin\theta_i \quad \text{—— (4)}$$

By substituting eqn (2), (3) and (4) in eqn (1)

$$\frac{OO'\sin\theta_t}{v_2} = \frac{OO'\sin\theta_r}{v_1} = \frac{OO'\sin\theta_i}{v_1}$$

Let consider,

$$\frac{\sin\theta_r}{v_1} = \frac{\sin\theta_i}{v_1}$$

$$\Rightarrow \theta_r = \theta_i$$

The angle of reflection is equal to the angle of incidence.
This is called Snell's law of reflection.

$$\frac{\sin\theta_t}{v_2} = \frac{\sin\theta_i}{v_1}$$

$$v_1 = \frac{c}{\sqrt{\mu_{r1}\epsilon_{r1}}} \qquad v_2 = \frac{c}{\sqrt{\mu_{r2}\epsilon_{r2}}}$$

$$\sin\theta_t \sqrt{\mu_{r2}\epsilon_{r2}} = \sin\theta_i \sqrt{\mu_{r1}\epsilon_{r1}}$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_{r1}\epsilon_{r1}}{\mu_{r2}\epsilon_{r2}}} = \frac{v_2}{v_1}$$

This is called Snell's law of refraction.

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The refractive index is given by,

$$n = \frac{c}{v}$$

$$= \frac{\text{velocity of wave in free space}}{\text{velocity of wave in the medium under consideration}}$$

$$= \frac{c}{c/\sqrt{\mu_r \epsilon_r}} = \sqrt{\mu_r \epsilon_r}$$

for medium 1, $n_1 = \sqrt{\mu_{r1} \epsilon_{r1}}$

medium 2, $n_2 = \sqrt{\mu_{r2} \epsilon_{r2}}$

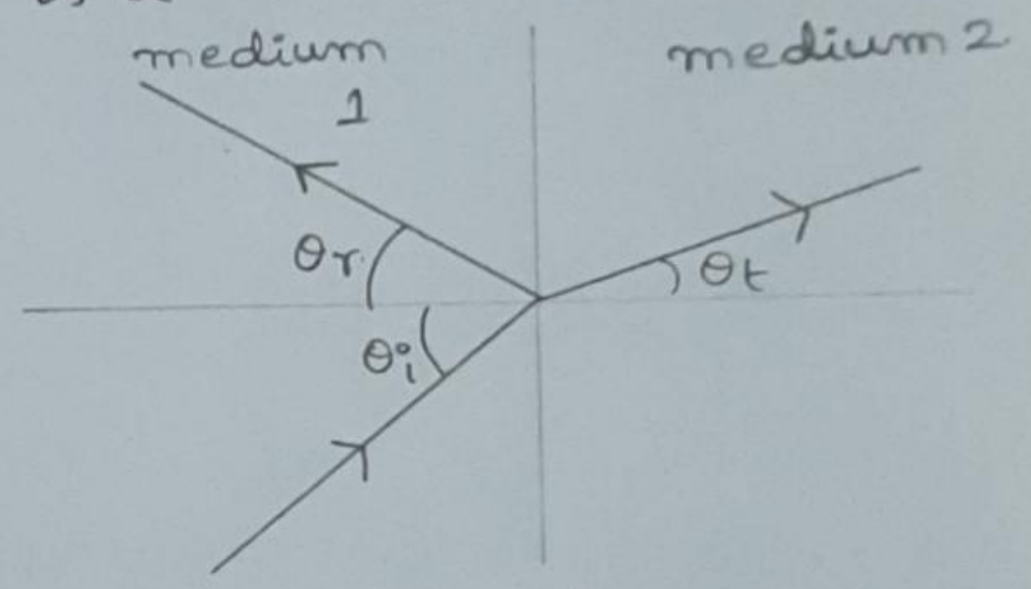
So $\frac{n_1}{n_2} = \sqrt{\frac{\mu_{r1} \epsilon_{r1}}{\mu_{r2} \epsilon_{r2}}}$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\mu_{r1} \epsilon_{r1}}{\mu_{r2} \epsilon_{r2}}}$$

Consider the following cases:-

Case 1: $n_2 > n_1$

In this case, medium 2 is a denser medium



$\frac{n_1}{n_2} < 1$

$\frac{\sin\theta_t}{\sin\theta_i} < 1$

$\sin\theta_t < \sin\theta_i$

$\Rightarrow \theta_t < \theta_i$

In this case the angle of transmission is less than the angle of incidence.

Case 2: $n_2 < n_1$

medium 1 is denser medium

$\frac{n_1}{n_2} > 1$

$\sin\theta_t > \sin\theta_i$

$\theta_t > \theta_i$

In this case the angle of transmission is greater than the angle of incidence.

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Case 3 : Critical angle of incidence

When the angle of incidence is such that the when angle of transmittion is 90°, then the transmitted wave will go along the boundary.

The angle of incidence corresponding to $\theta_t = 90°$ is called critical angle of incidence denoted by θ_c

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}$$

$$\frac{\sin 90°}{\sin\theta_c} = \frac{n_1}{n_2}$$

$$\sin\theta_c = \frac{n_2}{n_1} = \sqrt{\frac{\mu_{r2}\epsilon_{r2}}{\mu_{r1}\epsilon_{r1}}}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{\mu_{r2}\epsilon_{r2}}{\mu_{r1}\epsilon_{r1}}}$$

For $\theta_i > \theta_c$, the entire incident wave is returned along the reflected wave. This kind of reflection is called Total internal reflection.

Case 4 : Brewster angle :-

$$\frac{E_r}{E_i} = 0$$

$E_r = 0$ i.e there will be no reflected wave.

The angle of incidence θ_i which obeys this condition is called Brewster angle θ_B.

Thus, Brewster angle is the angle of incidence at which when a vertically polarized wave is incident at the boundary of two perfect dielectric media there will be no reflected wave.

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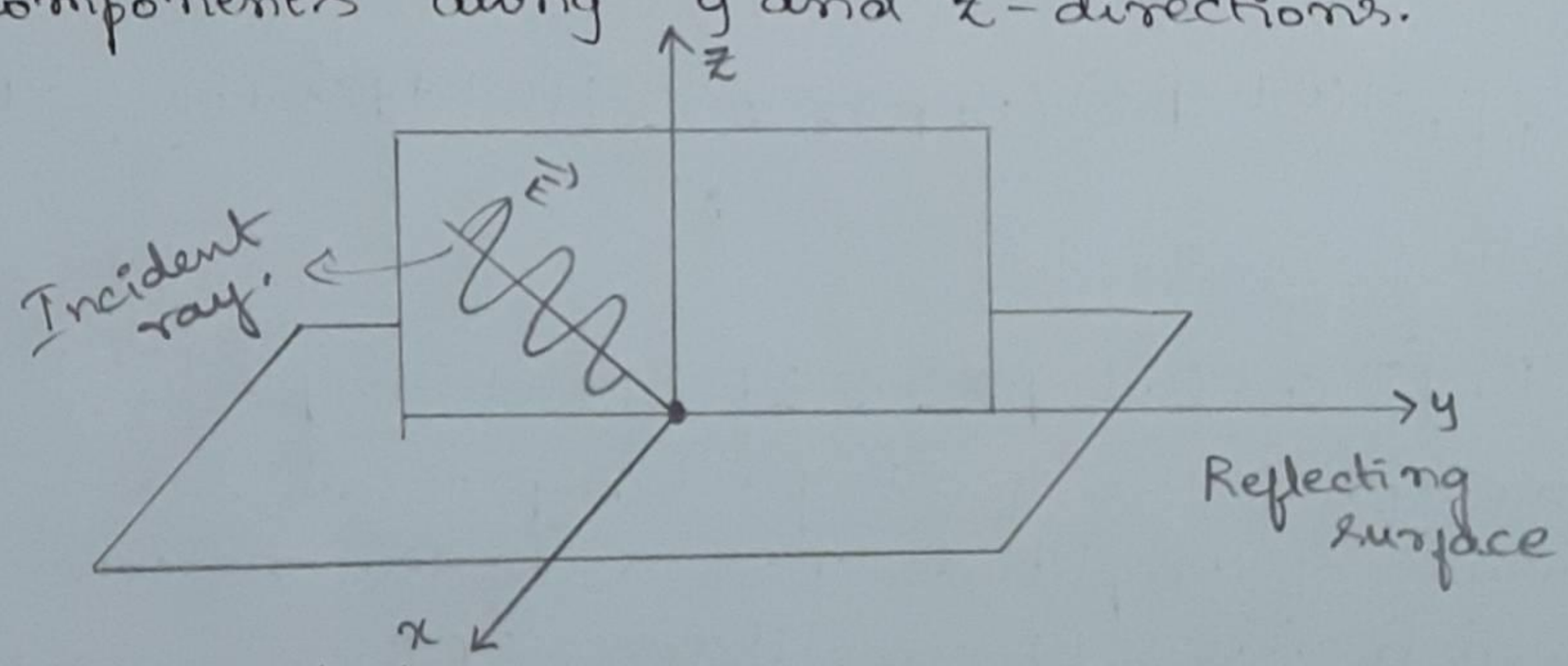
Meaning of the terms, Plane of Incidence, Horizontal Polarisation and Vertical Polarization.

Plane of Incidence: An imaginary plane in which both the incident ray and the normal to the reflecting surface could be accommodated is called plane of incidence.

Polarization: On a reflecting surface, when a wave is incident obliquely, it gives rise to two possibilities.

a) The electric vector is perpendicular to the plane of incidence. For this configuration, the electric vector is aligned parallel to the conducting surface, hence it is said to be horizontally polarized. It will not have any components along y and z-directions.



(b) The electric field vector is parallel to the plane of incidence - For this configuration, the plane having the electric field vector will always be perpendicular to the reflecting surface. Hence it is said to be vertically polarized. It will not have x- components.

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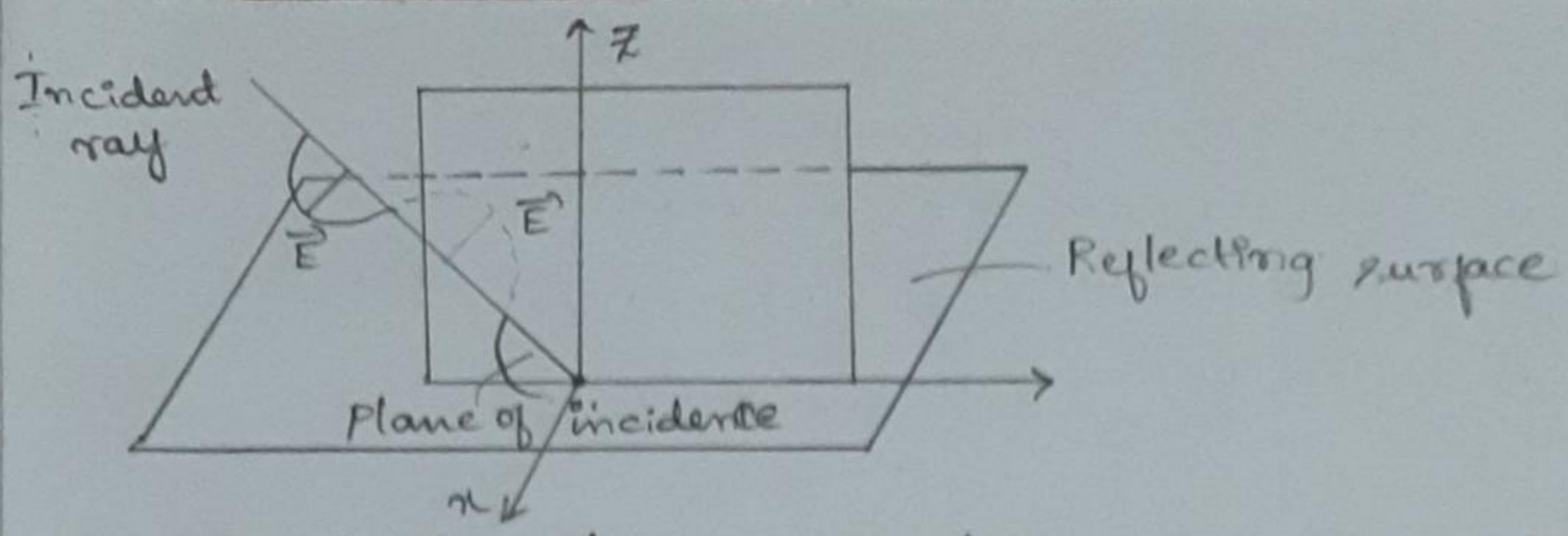
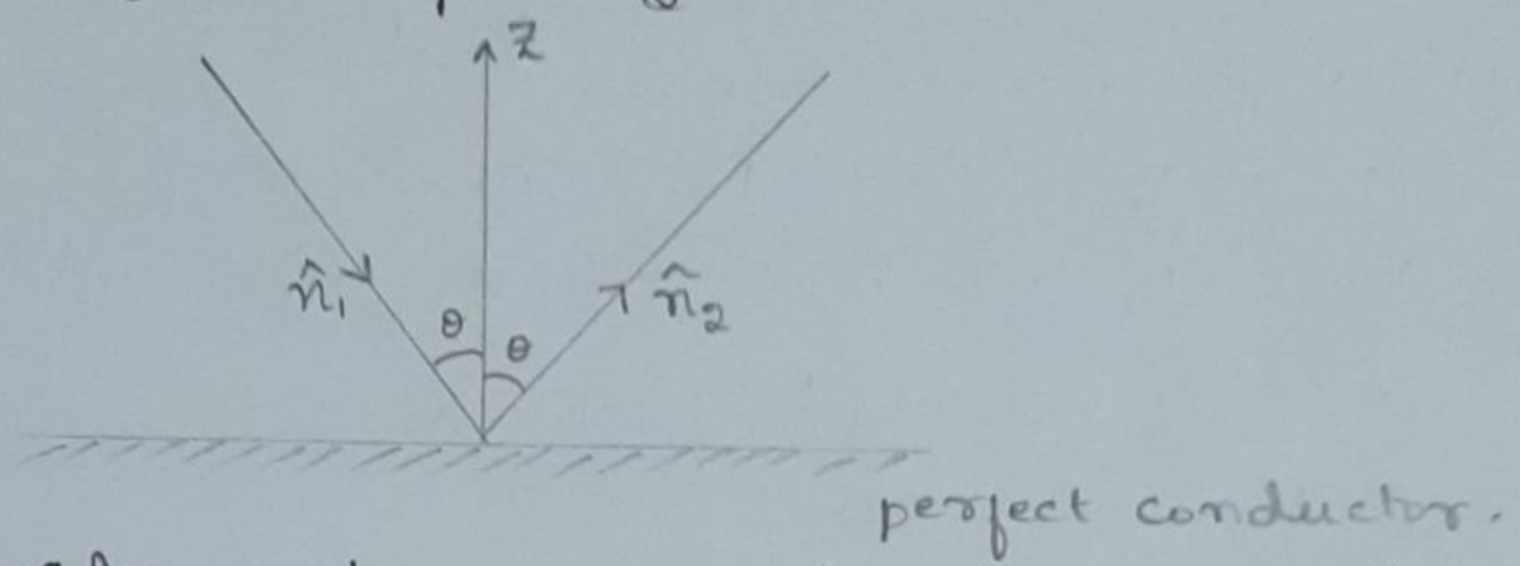


The oblique incidence of a plane wave on a reflecting surface can be considered under two categories, namely
(a) Horizontal polarization and
(b) Vertical polarization.

(a) Horizontal Polarization:

Let us consider oblique incidence, for which we know that $\vec{E}$ is perpendicular to the plane of incidence i.e horizontal polarization.



Consider a plane wave incidence ~~let~~ its. ~~do~~ on the surface of perfect conductor with $\vec{E}$ perpendicular to the plane of incidence. Let its direction of propagation make an angle θ with the normal along which we have the z-direction. The direction of propagation of the reflected wave also makes the same angle θ with z-axis. Because of same angles of orientation, the wave compone-nts along z-axis will have same wavelength, but will be set in opposite direction. These cdns lead to setting a standing wave along z-axis.

Let $\hat{n}_1$ be the unit vector along the direction of incidence.

The eqn for the incident wave can be written as,

$$E_{incident} = E_i e^{-j\beta \hat{n}_1 \cdot \vec{r}}$$

where E_i is the amplitude of the wave.

If θ_1, θ_2 and θ_3 are the angles between the incident directions, and x, y and z-axes respectively, then.

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$\hat{n}_1 \cdot \vec{r} = \hat{x}\cos\theta_1 + \hat{y}\cos\theta_2 + \hat{z}\cos\theta_3.$

$\therefore$ The direction of propagation is,

1) at right angles $= \theta_1$ to x-axis.

2) at an angle $\theta_2 = \left(\frac{\pi}{2} - \theta\right)$ to y axis.

3) at an angle $\theta_3 = (\pi - \theta)$ to z-axis.

$$\hat{n}_1 \cdot \vec{r} = x\cos\frac{\pi}{2} + y\cos\left(\frac{\pi}{2} - \theta\right) + z\cos(\pi - \theta)$$

$$= y\sin\theta - z\cos\theta.$$

$$\therefore E_{incident} = E_i e^{-j\beta(y\sin\theta - z\cos\theta)}$$

Now, let us consider the reflected wave.

$\therefore$ The equation for the wave can be written as,

$$E_{reflected} = E_r e^{-j\beta\hat{n}_2 \cdot \vec{r}}$$

where E_r is the amplitude of the wave.

For the wave, the direction of propagation is,

1) at right angles $= \theta_1$ to x-axis.

2) at an angle $\theta_2 = \left(\frac{\pi}{2} - \theta\right)$ to the y-axis and

3) at an angle $\theta_3 = \theta$ with the z-axis.

$$\hat{n}_2 \cdot \vec{r} = x\cos\frac{\pi}{2} + y\cos\left(\frac{\pi}{2} - \theta\right) + z\cos\theta.$$

$$= y\sin\theta + z\cos\theta.$$

$$E_{reflected} = E_r e^{-j\beta(y\sin\theta + z\cos\theta)}$$

Applying the property that no electric field can exist within a perfect conductor, $E_i = -E_r$.

The resultant wave satisfies the equation,

$$E = E_{incident} + E_{reflected}.$$

$$E = E_i\left[e^{-j\beta(y\sin\theta - z\cos\theta)} - e^{-j\beta(y\sin\theta + z\cos\theta)}\right]$$

$$= E_i\left[e^{-j\beta z\cos\theta} - e^{-j\beta z\cos\theta}\right]e^{-j\beta y\sin\theta}.$$

$$= E_i\left[\cos(\beta z\cos\theta) + j\sin(\beta z\cos\theta) - \cos(\beta z\cos\theta) + j\sin(\beta z\cos\theta)\right]e^{-j\beta y\sin\theta}$$

$$E = 2jE_i\sin(\beta z\cos\theta)e^{-j\beta y\sin\theta}.$$

The above equation represents a standing wave set up along z-axis.

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(b) Vertical Polarization:-

Now, we shall consider oblique incidence, for which we know that $\vec{E}$ lies in the plane of incidence, i.e vertical polarization.

Consider a plane wave incident on the surface of a perfect conductor, with $\vec{E}$ in the plane of incidence. let its direction of propagation make an angle θ with the normal i.e the z-axis.

let $\hat{n}_1$ be the unit vector along the direction of incidence

∴ The equation for the incident wave is,

$$E_{incident} = E_i e^{-j\beta \hat{n}_1 \cdot \vec{r}}$$

$$= E_i e^{-j\beta(y\sin\theta - z\cos\theta)}$$

The same treatment applies to magnetic field also, for which the incident wave can be written as

$$H_{incident} = H_i e^{-j\beta(y\sin\theta - z\cos\theta)}$$

where H_i is the amplitude of the incident wave.

But $\frac{E_i}{H_i} = \eta$ the intrinsic impedence

$$E_i = \eta H_i$$

The component of E_i along z-direction is

$$E_{zi} = E_i \cos(90-\theta) = E_i \sin\theta$$

$$= \eta H_i \sin\theta$$

$$E_{zi} e^{-j\beta(y\sin\theta - z\cos\theta)} = \eta H_i \sin\theta \, e^{-j\beta(y\sin\theta - z\cos\theta)}$$

The component of E_i along y-direction is

$$E_{yi} = E_i \cos\theta = \eta H_i \cos\theta$$

The wave equation for component along y-direction due to the incident wave is,

$$E_{yi} e^{-j\beta(y\sin\theta - z\cos\theta)} = \eta H_i \cos\theta \, e^{-j\beta(y\sin\theta - z\cos\theta)}$$

The equation for the reflected wave is,

$$E_{reflected} = \vec{E_r} e^{-j\beta \hat{n}_2 \cdot \vec{r}}$$

$$= E_r e^{-j\beta(y\sin\theta + z\cos\theta)}$$

$$\vec{H}_{reflected} = H_r e^{-j\beta(y\sin\theta + z\cos\theta)}$$

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$$\frac{E_r}{H_r} = \eta$$

$$E_r = \eta H_r$$

The component of E_r along z-direction

$$E_{zr} = E_r \cos(90-\theta) = E_r \sin\theta$$

$$= \eta H_r \sin\theta.$$

$$E_{zr}\, e^{-j\beta(y\sin\theta + z\cos\theta)} = \eta H_r \sin\theta\, e^{-j\beta(y\sin\theta + z\cos\theta)}$$

The component of E_r along y-direction is

$$E_{yr} = -E_r\cos\theta = -\eta H_r \cos\theta.$$

$$E_{yr}\, e^{-j\beta(y\sin\theta + z\cos\theta)} = -\eta H_r \cos\theta\, e^{-j\beta(y\sin\theta + z\cos\theta)}$$

The total z-components of electric fields along z-direction due to both the incident and reflected waves is the sum of the right side of eqns

$$E_z = \eta H_i \sin\theta\, e^{-j\beta(y\sin\theta - z\cos\theta)} + \eta H_r \sin\theta\, e^{-j\beta(y\sin\theta + z\cos\theta)}$$

But during reflection $H_i = H_r$.

$$E_z = \eta H_i \sin\theta \cdot e^{-j\beta y \sin\theta}\left[e^{+j\beta z\cos\theta} + e^{-j\beta z\cos\theta}\right]$$

$$= 2\eta H_i \sin\theta \cos(\beta z\cos\theta)\, e^{-j\beta y\sin\theta}.$$

Similarly the total y-components of electric fields along y-direction due to both the waves is the sum of right side of eqns

$$E_y = \eta H_i \cos\theta\, e^{-j\beta(y\sin\theta - z\cos\theta)} - \eta H_r \cos\theta\, e^{-j\beta(y\sin\theta + z\cos\theta)}$$

$$= \eta H_i \cos\theta\, e^{-j\beta y\sin\theta}\left[e^{+j\beta z\cos\theta} - e^{-j\beta z\cos\theta}\right]$$

$$E_y = 2j\eta H_i \cos\theta \sin(\beta z\cos\theta)\, e^{-j\beta y\sin\theta}.$$

Oblique Incidence :-

Let a plane wave be incident at the interface of two perfect dielectric media, at an angle to the normal drawn to the interface. Because of the change of media, a part of the incident wave energy is transmitted and the rest is reflected. Let E_i and E_r be the amplitudes of the electric fields of the incident and the transmitted waves

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respectively, and ϵ_1 and ϵ_2 be the permittivities of the 6 media in which the wave is incident, and transmitted respectively.

If θ_1, θ_2 and θ_3 are the angle of incidence, angle of refraction, and angle of reflection, it can be shown that during reflection,

$$\theta_1 = \theta_3$$

and for refraction,

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

This is well known law in optics by the name Snell's law. Here two cases arise - namely horizontal polarization and vertical polarization.

For horizontal polarization i.e when the electric field vector is perpendicular to the plane of incidence, the following equation holds good.

$$\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{(\epsilon_2/\epsilon_1) - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{(\epsilon_2/\epsilon_1) - \sin^2\theta_1}}$$

For vertical polarization, i.e when the electric field vector is parallel to the plane of incidence, corresponding equation is,

$$\frac{E_r}{E_i} = \frac{(\epsilon_2/\epsilon_1)\cos\theta_1 - \sqrt{(\epsilon_2/\epsilon_1) - \sin^2\theta_1}}{(\epsilon_2/\epsilon_1)\cos\theta_1 + \sqrt{(\epsilon_2/\epsilon_1) - \sin^2\theta_1}}$$

Brewster Angle:-

Let consider

$$(\epsilon_2/\epsilon_1)\cos\theta_1 = \sqrt{(\epsilon_2/\epsilon_1) - \sin^2\theta_1}$$

which gives

$$\tan\theta_1 = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

In other words, if the angle of incident θ_1 is such that it obeys,

$$\frac{E_r}{E_t} = 0,$$

or $E_r = 0$; i.e there will be no reflected wave.

The angle of incidence θ_1 which obeys in above eqn is called the Brewster angle θ_B.

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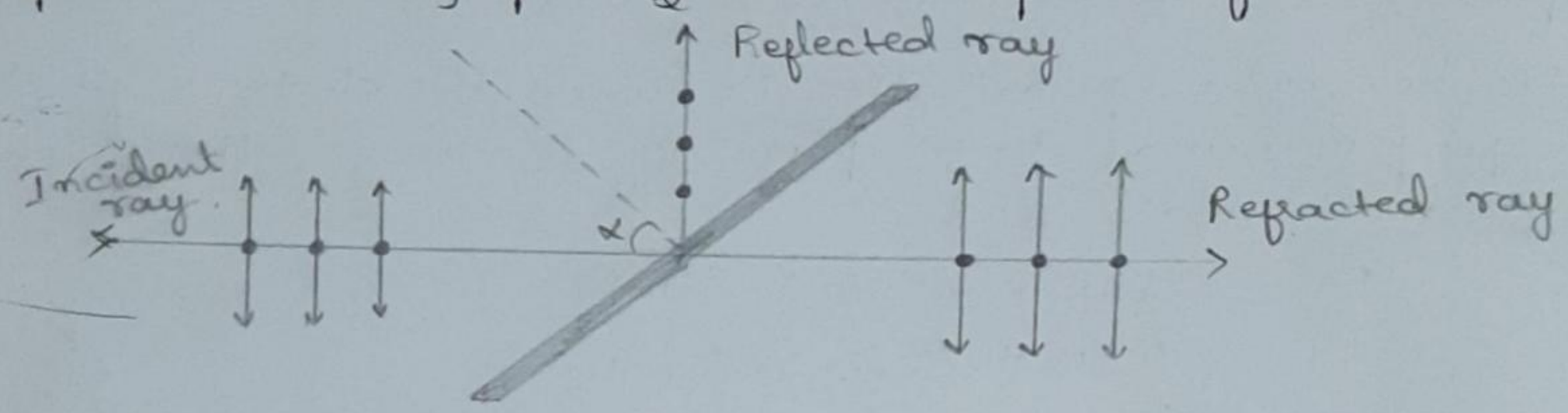
Thus, Brewster angle is the angle of incidence at which when a vertically (or parallel) polarized wave is incident at the boundary of two perfect dielectric media there will be no reflected wave.

Brewster Window:

A Brewster window is designed on the principle of Brewster angle. Brewster angle is the angle of incidence for a ray such that is satisfies the condition, $\tan\alpha = \mu$ where μ is the refractive index of the material on which the light is incident. For such an incidence, the part of the ray reflected from the surface is polarized normally to the plane of incidence, and the part of the ray refracted is predominantly polarized in the plane of incidence.



The absence of reflection for the incidence of a vertically polarized wave at Brewster angle, holds good whether $\epsilon_1 > \epsilon_2$ or $\epsilon_2 > \epsilon_1$. It is interesting to note that, there is a condition under which there will be no refraction for the incident ray that is either polarized or un-polarized. This is well known as Total Internal Reflection.

Critical angle and Total Internal Reflection:

we have $\dfrac{\sin\theta_1}{\sin\theta_2} = \sqrt{\dfrac{\epsilon_2}{\epsilon_1}}$

For given ϵ_1 and ϵ_2, θ_2 depends on θ_1. Consider the case $\theta_2 > \theta_1$ (As per the above eqn, this is possible only when $\epsilon_1 > \epsilon_2$) Let $\epsilon_1 > \epsilon_2$. The maximum value of θ_2 can have is 90° at which time. let the corresponding value of θ_1 be θ_c

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

The angle θ_c is called the critical angle.

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