Fermilab

June 9, 2025

Mini Lecture Course: Accelerator Design for a Multi-TeV Muon Collider

Lecture 3: Characterize Solenoid-based cooling channels

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Scope of today's lecture

- Review previous lectures
- Explore solenoid-based cooling channel using simplified toy models
  - Single Solenoid Coil
  - Multiple Solenoid Coils
  - Multiple Solenoid Coils and alternating polarity configuration
- Review key theoretical analysis of FOFO channels
  - K.J. Kim and C.Wang, PRL85, 760 (2000)
  - G. Penn and J. Wurtele, PRL85, 764 (2000)
  - G. Dugan, PRAB4, 104001 (2001)
- Introduction to specific solenoid-based cooling channel designs
  - Rectilinear channel (Stratakis et al.)
  - Helical channel using (Derbenev et al.)
  - HFOFO channel (no formal theoretical paper published)

Quick Review of Previous Lecture (I)

- Beam dynamics is analogous to the dynamics of a simple harmonic motion
  - Solving harmonic oscillator via canonical transformation

$$\begin{pmatrix} H = E = \frac{p^2}{2m} + \frac{1}{2}kq^2 \\ \dot{p} = -\frac{dH}{dq} = -kq \end{pmatrix} \rightarrow \begin{pmatrix} q = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t + \varphi_0) \\ p = \sqrt{2mE}\sin(\omega t + \varphi_0) \end{pmatrix}$$

Or $\mathcal{H}(P,Q) = E = \omega \cdot P, \dot{Q} = \omega = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} q = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t + \phi_0) \\ p = \sqrt{2mE}\sin(\omega t + \phi_0) \end{pmatrix} \rightarrow \begin{pmatrix} q = \sqrt{\frac{2P}{m\dot{Q}}}\cos(\omega t + \phi_0) \\ p = \sqrt{2m\dot{Q}P}\sin(\omega t + \phi_0) \end{pmatrix}$$

Mechanical harmonic oscillation framework

- Harmonic motion → Hill's equation (translate the differentiation from t to s) $\frac{d}{dt} \rightarrow \frac{ds}{dt}\frac{d}{ds}$

$$\dot{p} = -\frac{dH}{dq} = -kq \rightarrow x'' + Kx = 0$$

$$x = \sqrt{2\hat{\beta}_x(s)J_x}\cos[\hat{\mu}_x(s) + \phi_x]$$

Comparing $q = \sqrt{\frac{2P}{m\dot{Q}}}\cos(\omega t + \phi_0)$

Translation from harmonic oscillation to beam dynamics

$$P \rightarrow \frac{J_x}{m}, \dot{Q} \rightarrow \frac{1}{\hat{\beta}_x} = \hat{\mu}'_x \leftarrow (\hat{\mu}_x \rightarrow \omega t), \phi_x \rightarrow \phi_0,$$

Quick Review of Previous Lecture (II)

- Introduce new canonical momentum

$$p_x \equiv \hat{\beta}_x x' + \hat{\alpha}_x x = -\sqrt{2\hat{\beta}_x J_x}\sin\psi_x . \qquad \psi_x = \hat{\mu}_x + \phi_x$$

$$x = \sqrt{2\hat{\beta}_x J_x}\cos\psi_x$$

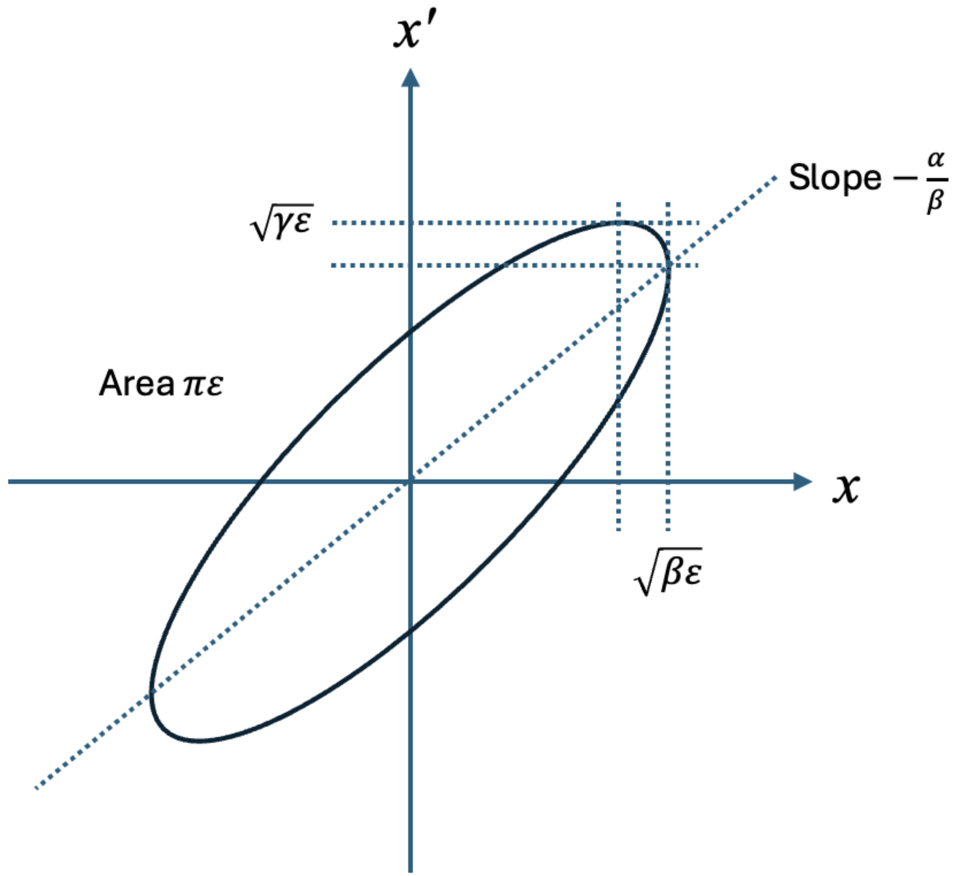
$$\frac{1}{\hat{\beta}_x}(x^2 + p_x^2) = \frac{1}{\hat{\beta}_x}\left\{\left(1+\hat{\alpha}_x{}^2\right)x^2 + 2\hat{\alpha}_x\hat{\beta}_x x x' + \hat{\beta}_x{}^2(x')^2\right\}$$
$$= \hat{\gamma}_x x^2 + 2\hat{\alpha}_x x x' + \hat{\beta}_x (x')^2 = 2J_x \equiv \varepsilon_x.$$

ε_x is called the Courant-Snyder invariant, or emittance

- The beam (gaussian) distribution function is a function of the Courant-Snyder invariant; The σ-matrix is given

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \varepsilon_{rms}\begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}.$$

$$\varepsilon_{x,rms} = \sqrt{Det[\sigma^2]} = \sqrt{\sigma_x^2 \cdot \sigma_{x'}^2 - \sigma_{xx'}^2}.$$



Twiss parameters from phase space

$$\hat{\beta}_x = \frac{\sigma_x^2}{\varepsilon_{rms}} \qquad \hat{\alpha}_x = -\frac{\sigma_{xx'}^2}{\varepsilon_{rms}} \qquad \hat{\gamma}_x = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

$$\hat{\alpha}_x = -\frac{1}{2}\hat{\beta}_x{}', \hat{\gamma}_x = \frac{1+\hat{\alpha}_x{}^2}{\hat{\beta}_x} \rightarrow \hat{\beta}_x\hat{\gamma}_x - \hat{\alpha}_x{}^2 = 1,$$

Quick Review of Previous Lecture (III)

- Solution of Hill's equation $x''(s) + \kappa(s)x(s) = 0$

$$X \rightarrow \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix} \rightarrow \widetilde{M} \cdot X_0$$

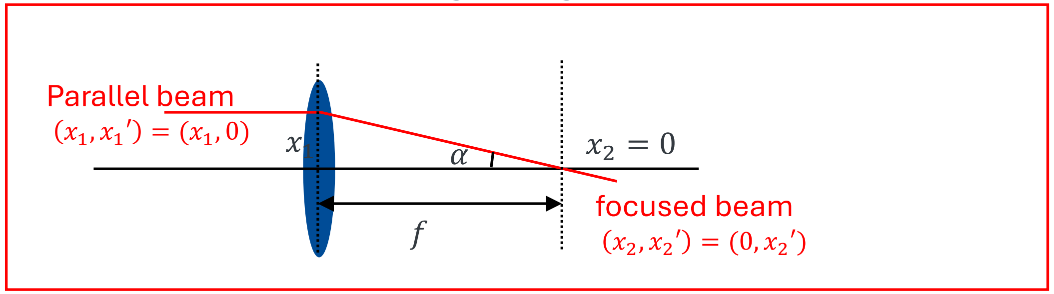
$$det(\widetilde{M} - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \rightarrow \lambda_{1,2} = \cos(\mu) \pm \sqrt{cos^2(\mu) - 1} = \cos(\mu) \pm i\sin(\mu) = e^{\pm i\mu}$$

- $|\mathrm{Tr}(\widetilde{M})| \leq 2, \lambda_{1,2}$ is a periodic solution → <u>Beam motion is stable</u>

$$\widetilde{M} = \begin{bmatrix} \cos\mu + \hat{\alpha}\sin\mu & \hat{\beta}\sin\mu \\ -\hat{\gamma}\sin\mu & \cos\mu - \hat{\alpha}\sin\mu \end{bmatrix} \text{Or } \widetilde{M} = \tilde{I}\cos\mu + \tilde{J}\sin\mu, \text{where } \tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$$

- Thin lens and paraxial approximation

Definition of focusing length

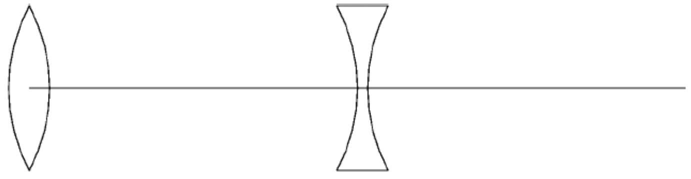


$$\begin{bmatrix} 0 \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{x_1}{f} \end{bmatrix}$$

$$x_2' = -\frac{x_1}{f} = -tan\alpha$$

Quick Review of Previous Lecture (IV)

- Thin lens and paraxial approximation



$$\widetilde{M}_{FODO} = \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{Defocus} \cdot \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{focus}$$

$$= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{bmatrix} = cos(\mu) \cdot \tilde{I} + sin(\mu) \cdot \tilde{J}$$

$$cos\,\mu + \hat{\alpha} \cdot sin\,\mu = 1 - \frac{1}{2}\frac{L}{f} - \frac{1}{4}\frac{L^2}{f^2}$$

$$\hat{\beta} \cdot \sin\mu = L + \frac{L^2}{4f^2}$$

$$-\hat{\gamma} \cdot \sin\mu = -\frac{L}{2f^2}$$

$$cos\,\mu - \hat{\alpha} \cdot sin\,\mu = 1 + \frac{1}{2}\frac{L}{f}$$

- Stability condition, $\left|Tr\left(\widetilde{M}\right)\right| \leq 2$

$$2 - \frac{L^2}{4f^2} = 2cos(\mu) = 2 - 4sin^2\left(\frac{\mu}{2}\right) \rightarrow sin\left(\frac{\mu}{2}\right) = \pm\frac{L}{4f}$$

$$\hat{\gamma}_x x^2 + 2\hat{\alpha}_x x x' + \hat{\beta}_x (x')^2 = \varepsilon_x$$

Demonstrate FODO cell in G4Beamline

Example:

Phase advance per cell $\mu = 70$ degrees

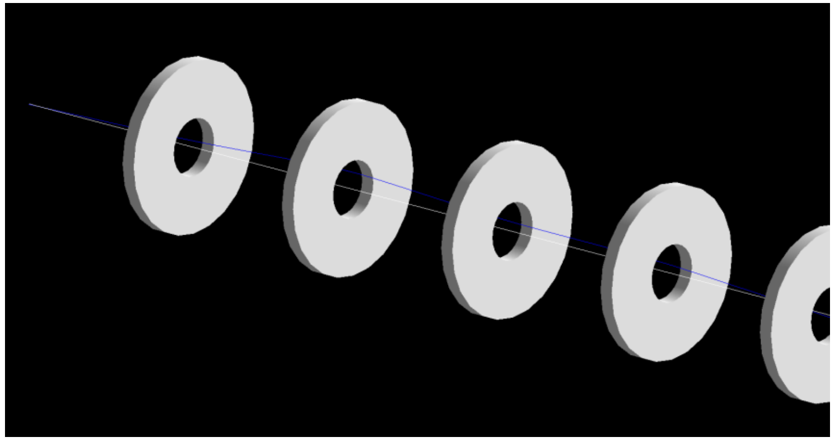
Focusing length $f = 1.23$ m

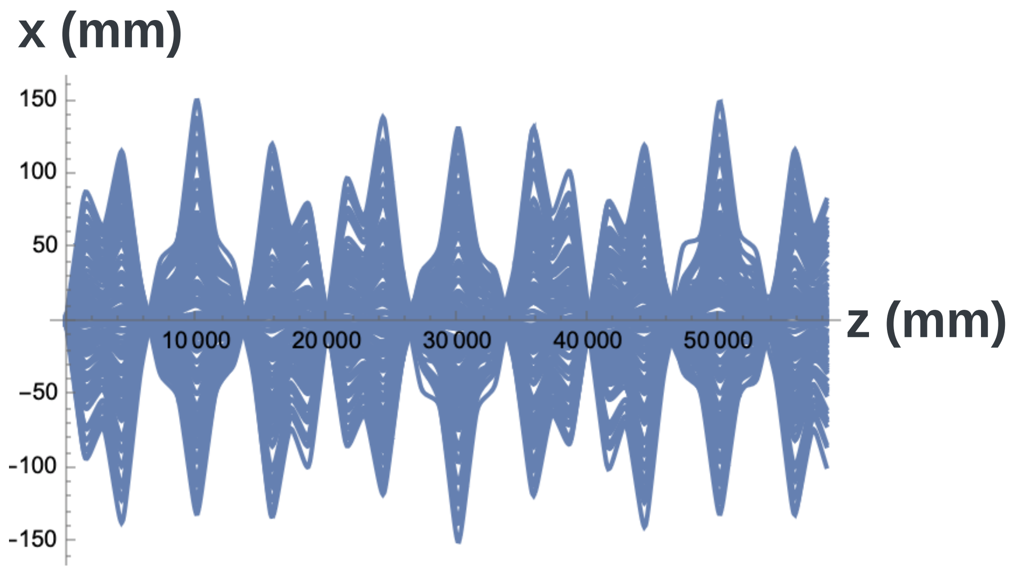
$\rightarrow L = 4f \cdot sin\left(\frac{\mu}{2}\right) = 2.46$ m

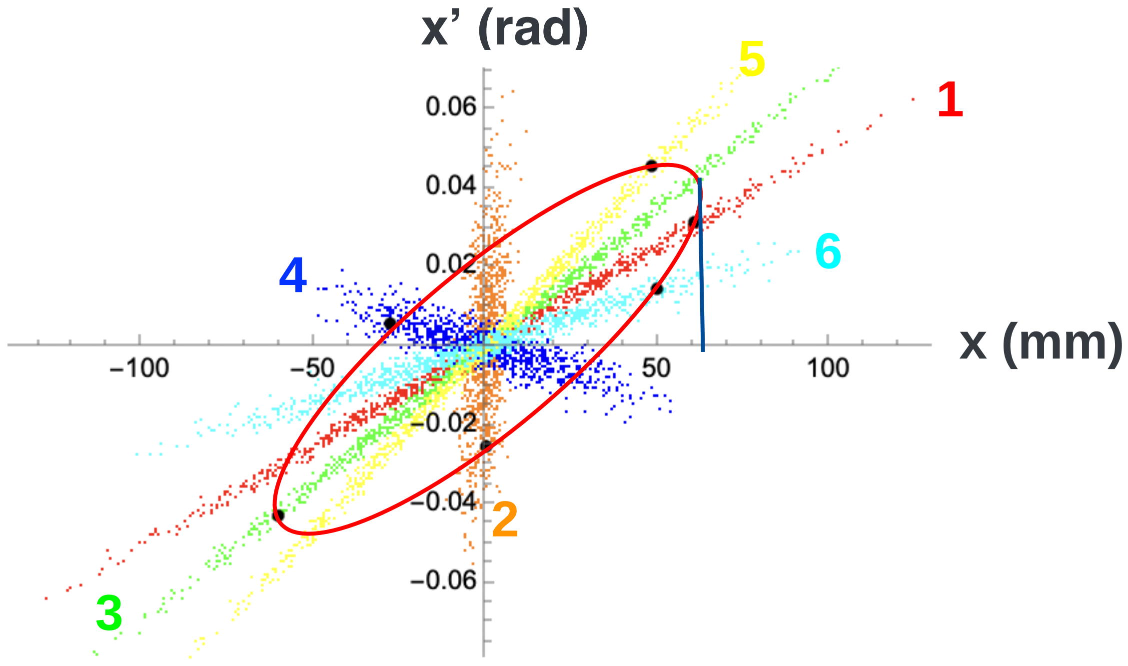
$\hat{\alpha} = -1.26, \hat{\beta} = 2.53, \hat{\gamma} = 1.01$

Demo: FODO cell ($\mu = 70$ deg)

fodocell_phase60_05160030.nb







- Solid black points in x-x' plot is taken with one FOCO-cell period
- Five black points are visible per full revolution, corresponding to ~70 degrees phase advance → 360/70 ~ 5.14 cells/rev
- Observed x ~ 61.6 mm at x'~ 0.0338 rad

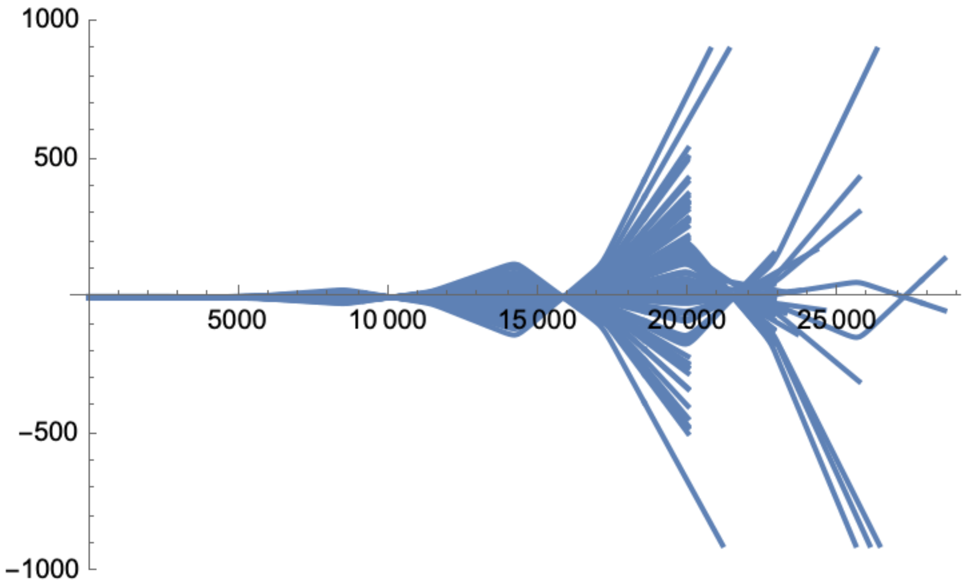
$$\hat{\gamma}_x x^2 + 2\hat{\alpha}_x x x' + \hat{\beta}_x (x')^2 = \varepsilon_x$$

- ε ~ 0.00156 m rad

Demo: FODO cell (Unstable condition)

fodocell_unstable_05160040.nb

- If the FODO lattice set with $\left|Tr\left(\tilde{M}\right)\right| > 2$, the beam optics must be unstable
- With the drift gap increased to 5710 mm, the FODO cell exceeds the stability condition
- Since $sin\left(\frac{\mu}{2}\right) > 1$ has no real solution, the optics become unstable



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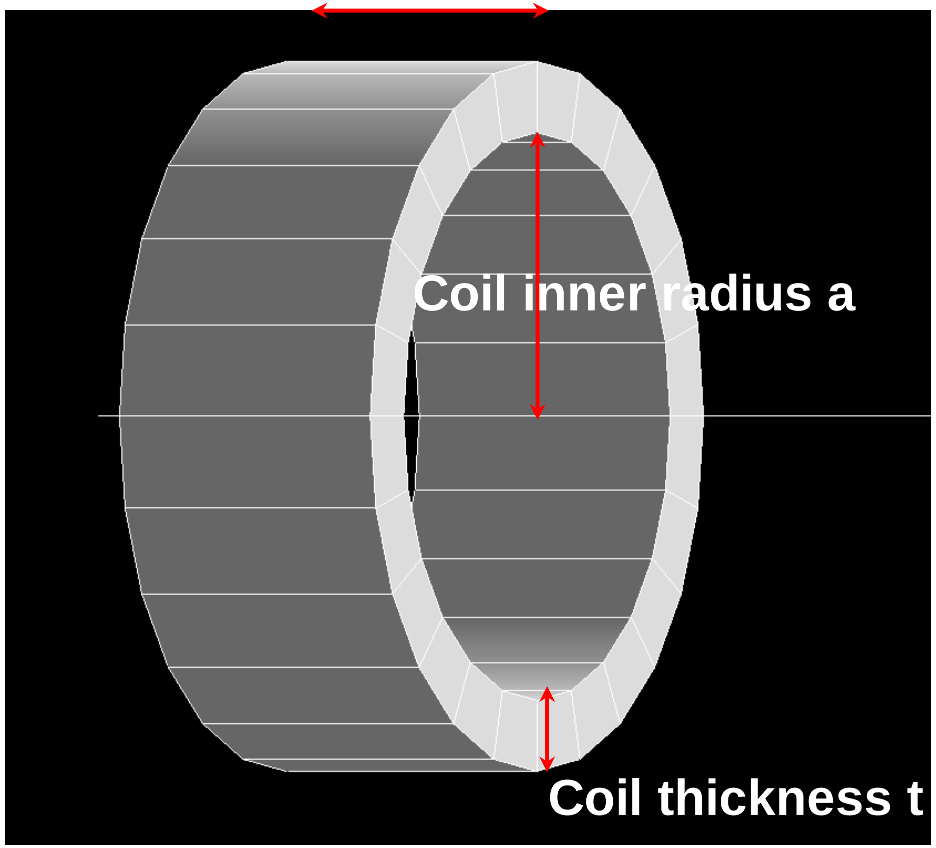
Demo: Single Solenoid Coil

Coil length L



Fernow coil (referred from PRSTAB 10, 064001 (2007)

- Coil length = 400 mm
- Coil inner radius = 400 mm
- Coil thickness = 100 mm
- Current = 100 Amp/mm2

Demo: Single Solenoid Coil

- Conventional Solenoid Focusing Model

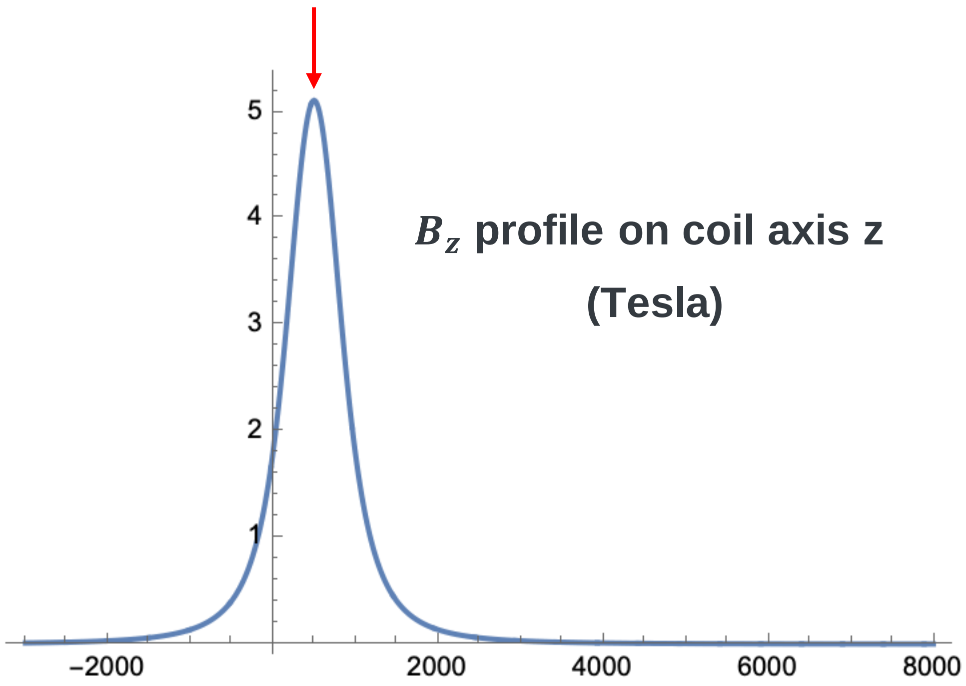
$$\frac{1}{f} = \frac{eB_z}{2m\gamma}$$

- As the formula shows, the focusing length is independent from the coil geometry

→ Thin lens approximation

- The estimated focusing length f
  - Coil length L = 400 mm
  - Coil inner radius a = 400 mm
  - Coil thickness t = 100 mm
  - Current I = 100 Amp/mm2
  - Observed peak magnetic field B_z = 5.1 Tesla
  - f = 300 mm@ 200 MeV/c
  - NB: $\underline{f < L}$

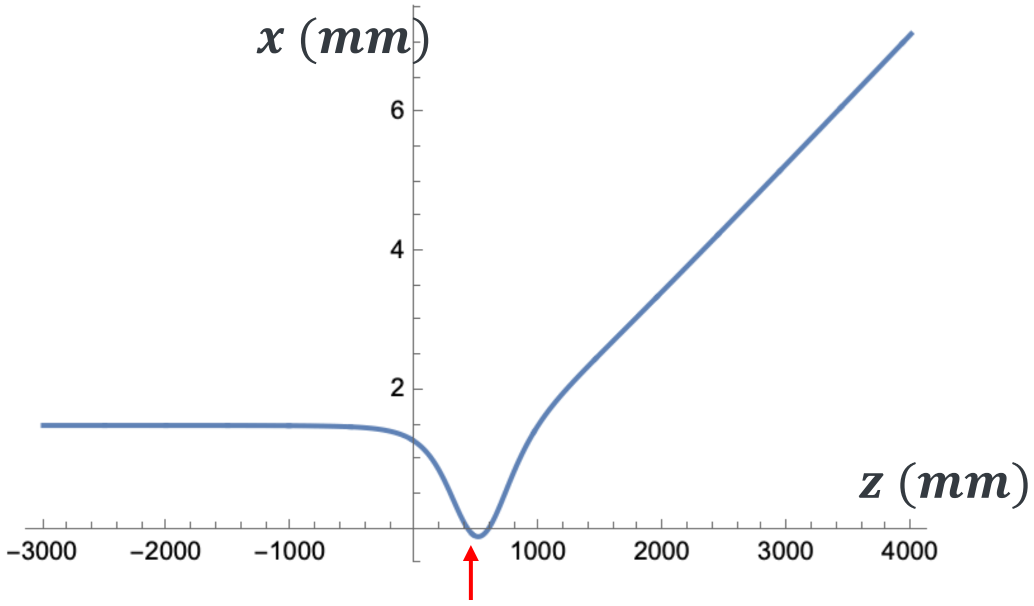
Solenoid center (z = 500 mm)



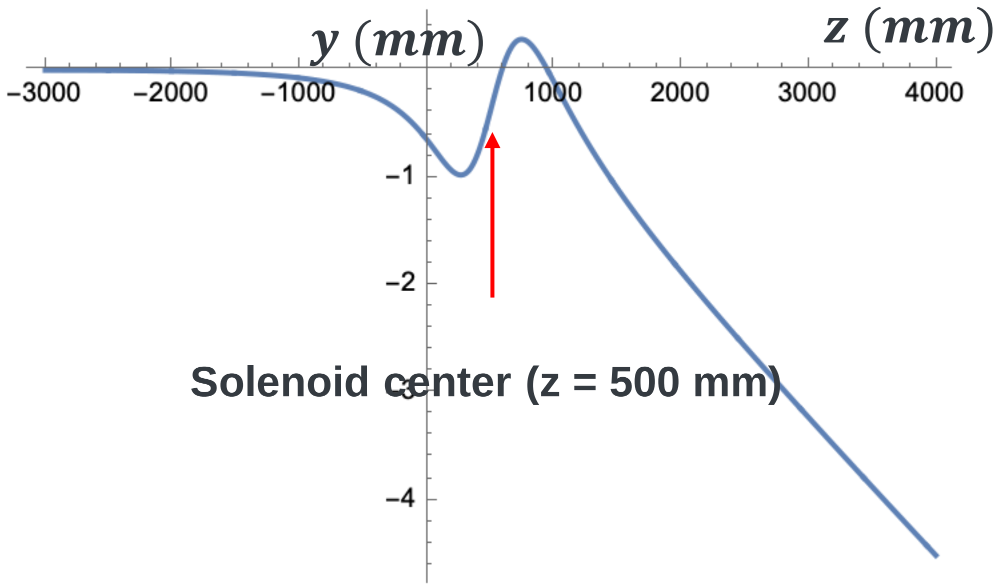
B_z profile on coil axis z (Tesla)

NB: Solenoid field calculation may depend on the simulator; We use G4Beamline which may differ from ICOOL

Demo: Single Solenoid Coil



Solenoid center (z = 500 mm)



NB:

- **Beam is rotated in the coil (Larmor motion)**
- **Focusing strength seems too strong to focus particle**
- **Focusing length should be longer than coil length ($f > L$)**

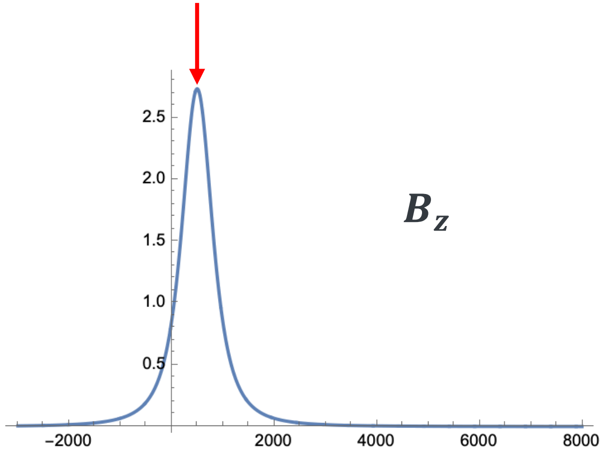
Demo: Single Solenoid Coil

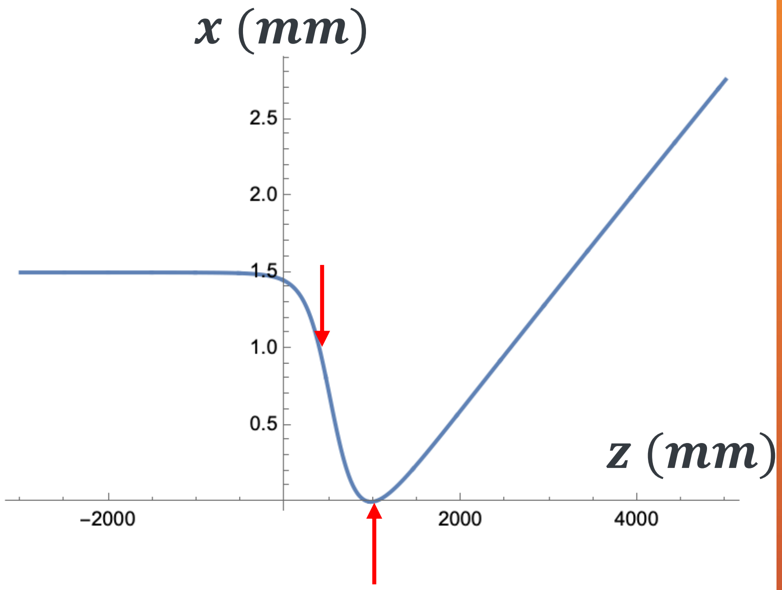
fofocell_fernow_focus_05220070.nb

- Coil length L = 200 mm
- Coil inner radius a = 400 mm
- Coil thickness t = 100 mm
- Current I = 100 Amp/mm2
- Observed peak magnetic field B_z = 2.7 Tesla
- f = 550 mm

Focusing length agrees well with prediction

Solenoid center (z = 500 mm)



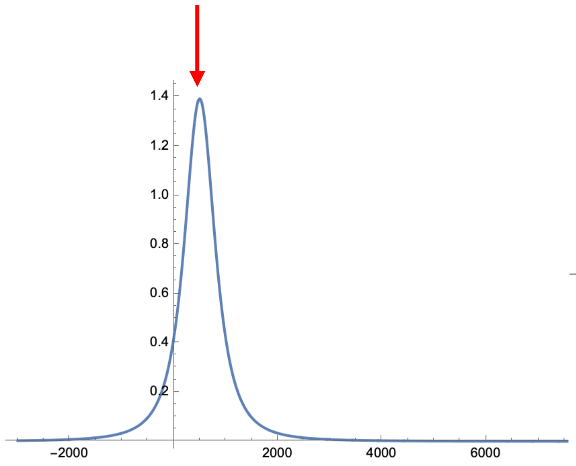


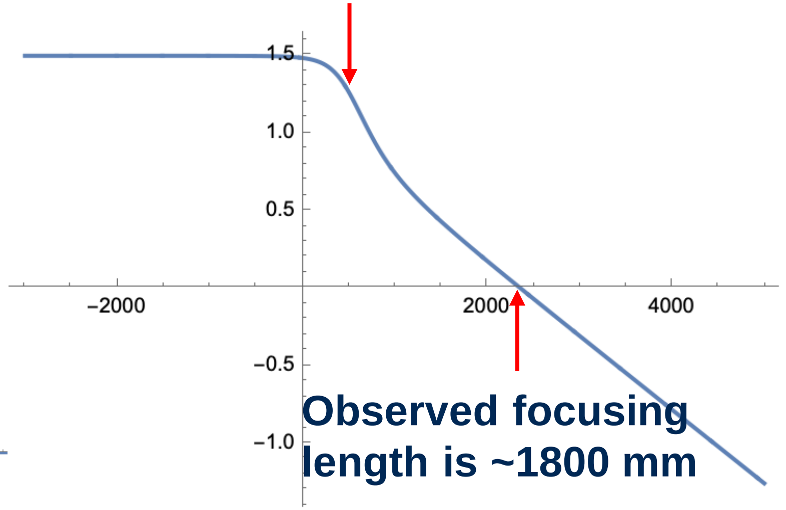
Observed focusing length is ~500 mm

- Coil length L = 100 mm
- Coil inner radius a = 400 mm
- Coil thickness t = 100 mm
- Current I = 100 Amp/mm2
- Observed peak magnetic field B_z = 1.4 Tesla
- f = 1076 mm

Prediction is underestimated

Solenoid center (z = 500 mm)





Observed focusing length is ~1800 mm

Induced Angular Momentum

- Set the Larmor rotation frame

$$X_R = x\cos(\varphi) - y\sin(\varphi)$$
$$Y_R = x\sin(\varphi) + y\cos(\varphi)$$

$$\varphi' = \frac{qA_\phi}{P_z r} \sim \frac{qB(z)}{2P} \equiv \kappa = \frac{qB}{2mc\gamma\beta} = \frac{\omega_L}{\beta c}$$

ω_L is a Larmor angular velocity $\omega_L = \frac{qB}{2m\gamma}$

- The linearized equations of motion is given

$$X_R'' + \kappa^2 X_R = 0, Y_R'' + \kappa^2 Y_R = 0$$

- Twiss parameters

$$X_R = A_1\sqrt{\hat{\beta}_p}\cos(\Phi - \Phi_1)$$

$$Y_R = A_2\sqrt{\hat{\beta}_p}\cos(\Phi - \Phi_2)$$

$$2\hat{\beta}_p\hat{\beta}_p'' - \left(\hat{\beta}_p{}'\right)^2 + 4\hat{\beta}_p{}^2\kappa^2 - 4 = 0$$

- Introduce the normalized canonical angular momentum

$$L_{canonical} = 2mc\varepsilon_n \cdot \mathcal{L}$$

Induced Angular Momentum

- The transverse covariance matrix includes angular momentum

$$\frac{M}{mc\epsilon_N} = \begin{pmatrix} \beta_\perp/\langle P_z\rangle & & & \\ -\alpha_\perp & \langle P_z\rangle\gamma_\perp & & \\ 0 & \beta_\perp\kappa - \mathcal{L} & \beta_\perp/\langle P_z\rangle & \\ \mathcal{L} - \beta_\perp\kappa & 0 & -\alpha_\perp & \langle P_z\rangle\gamma_\perp \end{pmatrix}$$

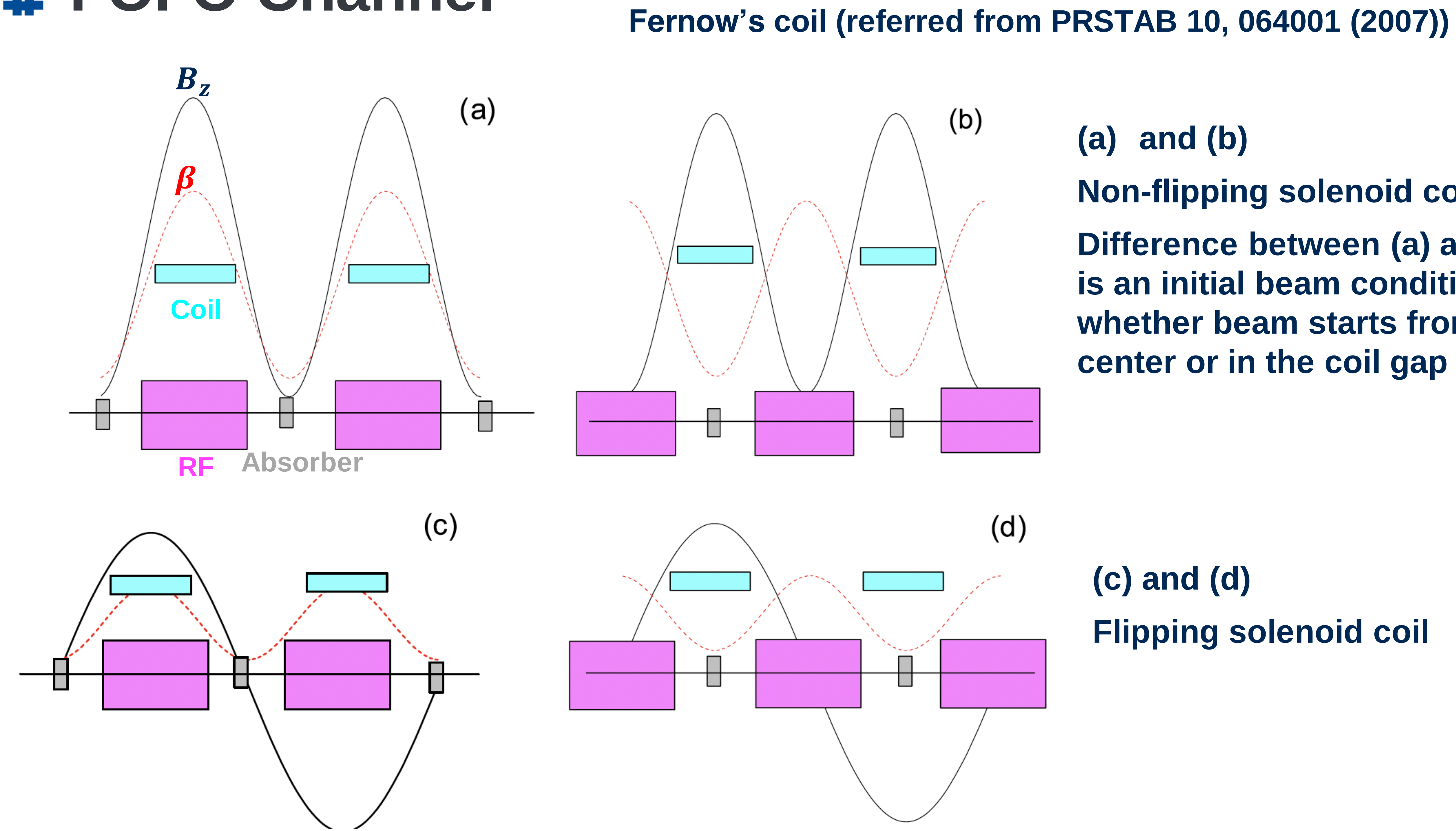
$$det(M) = \left[\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2 - \left\langle xP_y\right\rangle^2\right]$$

$$\gamma_\perp \equiv \frac{1}{\beta_\perp}[1 + \alpha_\perp^2 + (\beta_\perp\kappa - \mathcal{L})^2]$$

Or, taking 4x4 matrix to estimate emittance

$$\boldsymbol{\varepsilon_t = \left(Det\left[\widetilde{M}_{4\times 4}\right]\right)^{1/4}}$$

FOFO Channel



(a) and (b)

Non-flipping solenoid coil

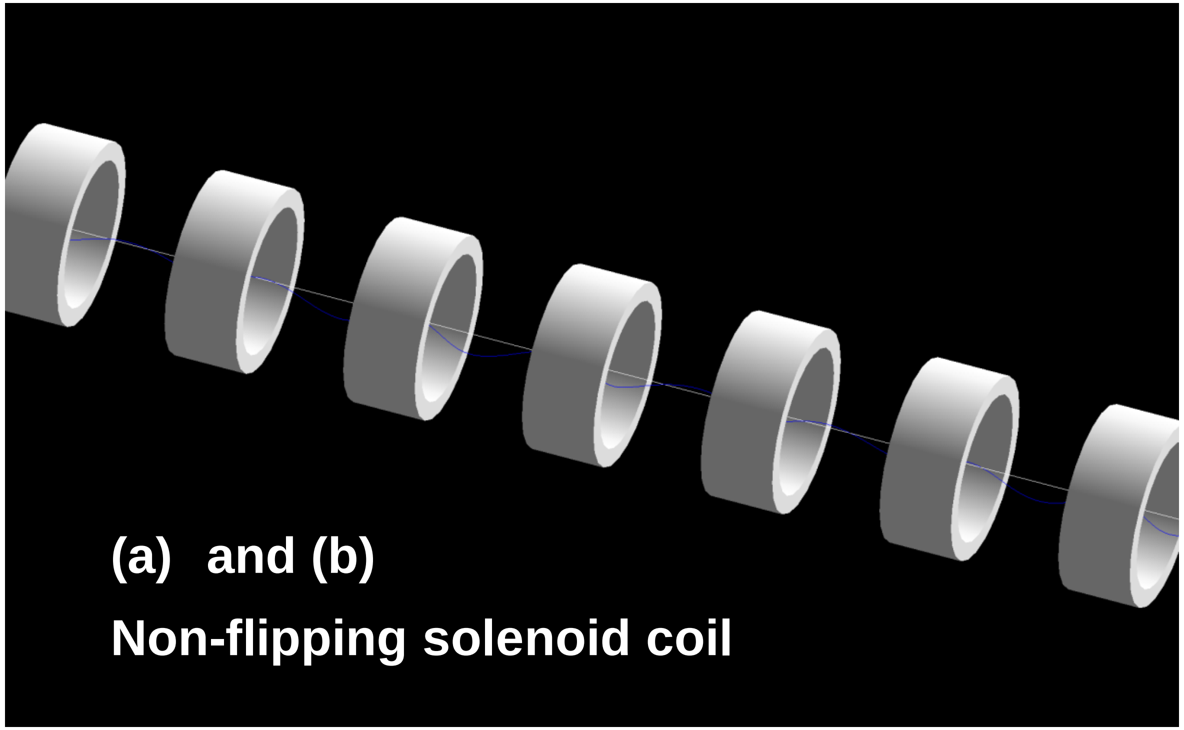
Difference between (a) and (b) is an initial beam condition whether beam starts from coil center or in the coil gap

(c) and (d)

Flipping solenoid coil

Demo: FOFO Channel

fofostudy_fernow_fig1a_05280140.nb

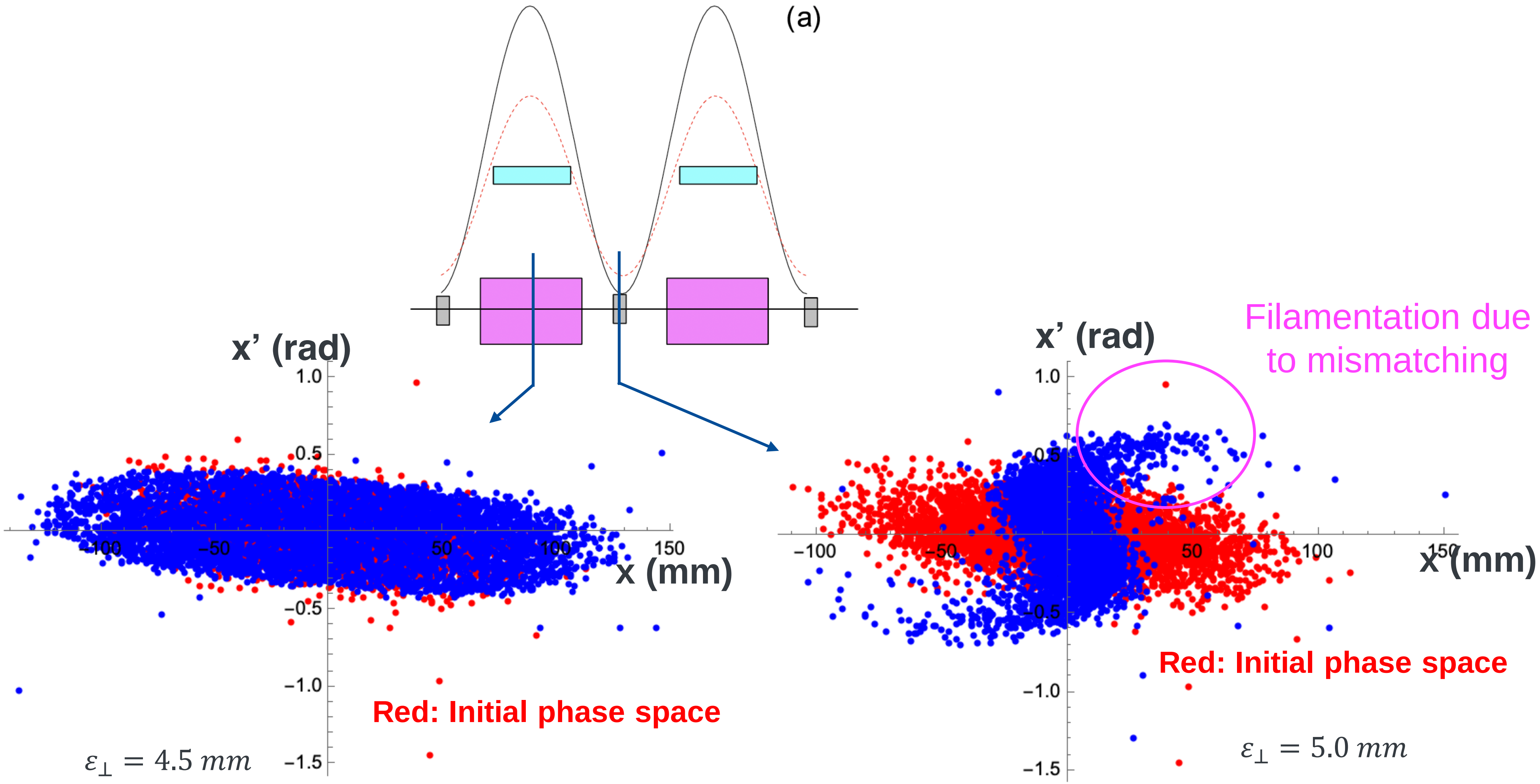


Fernow’s coil (referred from PRSTAB 10, 064001 (2007)

- **Coil length = 400 mm**
- **Coil inner radius = 400 mm**
- **Coil thickness = 100 mm**
- **Current = 100 Amp/mm2**
- **Period = 1000 mm**

Demo: FOFO Channel

fofostudy_fernow_fig1a_05280140.nb



$\varepsilon_{\perp} = 4.5\ mm$

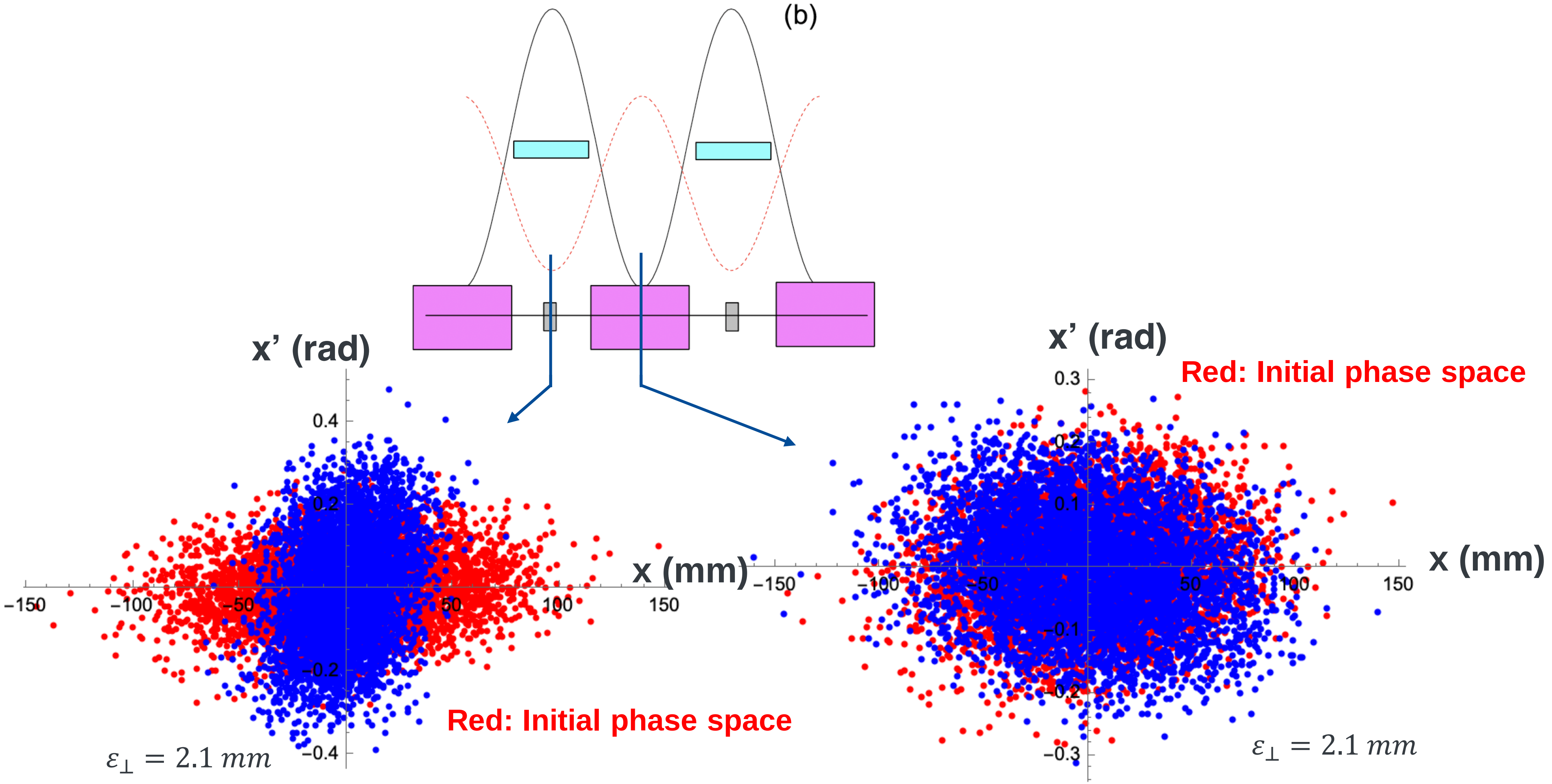
$\beta_{max} = 0.70\ m$

$\varepsilon_{\perp} = 5.0\ mm$

$\beta_{max} = 0.13\ m$

Demo: FOFO Channel

fofostudy_fernow_fig1b_05280150.nb



$\varepsilon_{\perp} = 2.1\ mm$

$\beta_{max} = 0.13\ m$

$\varepsilon_{\perp} = 2.1\ mm$

$\beta_{max} = 0.73\ m$

Beta function and passband (stability condition)

- Stability condition for solenoid-based channel is quite different from FODO channel because the FOFO solenoid magnet does not follow the thin lens approximation

Thin solenoid coil

$$\ddot{r} + \frac{r}{4}\left(\frac{eB_z}{mc\gamma\beta}\right)^2 = 0$$

- Instead, the stability condition can be predicted from Mathieu equation

Thick solenoid coil

$$B_z \rightarrow B_0 sin(kz)$$

$$\rightarrow \ddot{r} + r\left(\frac{eB_0}{2mc\gamma\beta}\right)^2 sin^2(kz) = 0$$

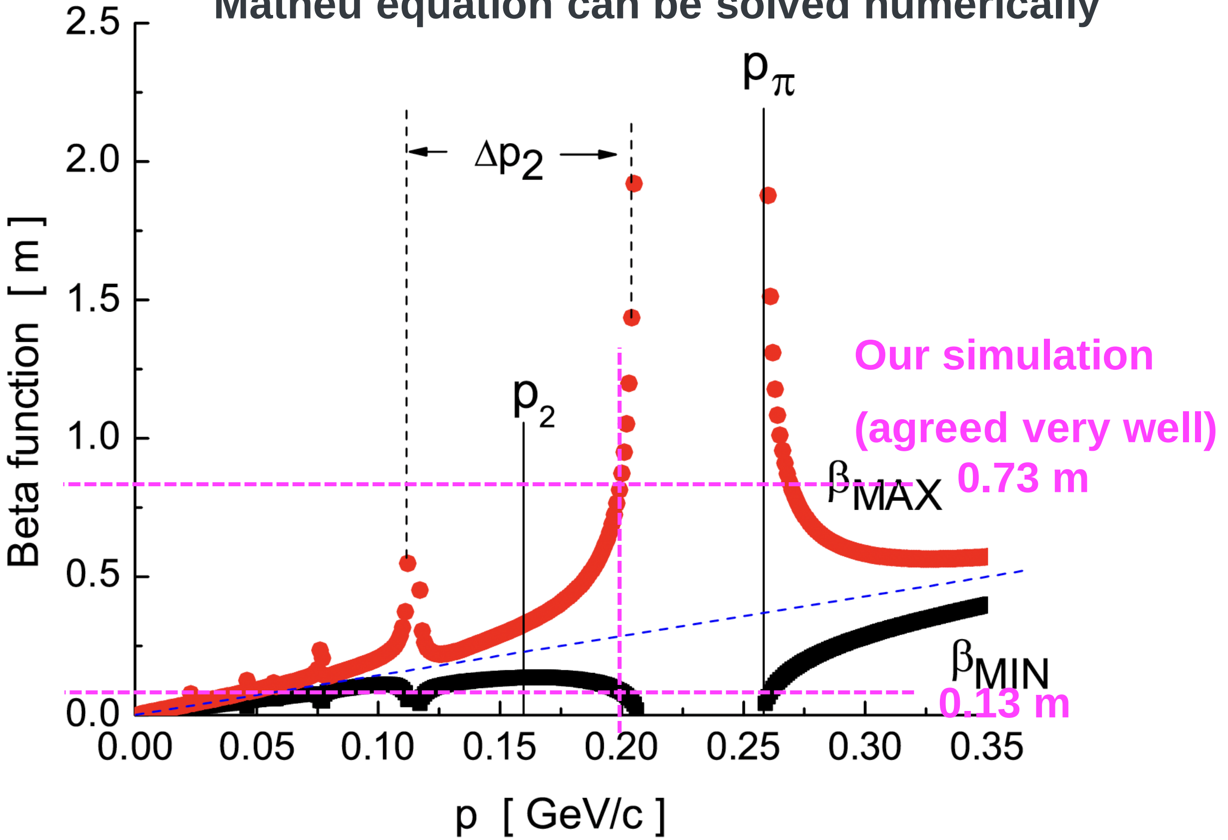
$$\frac{d^2y}{dv^2} + [a - 2q cos(2v)]y = 0$$

TABLE II. Momentum passband locations [MeV/c].

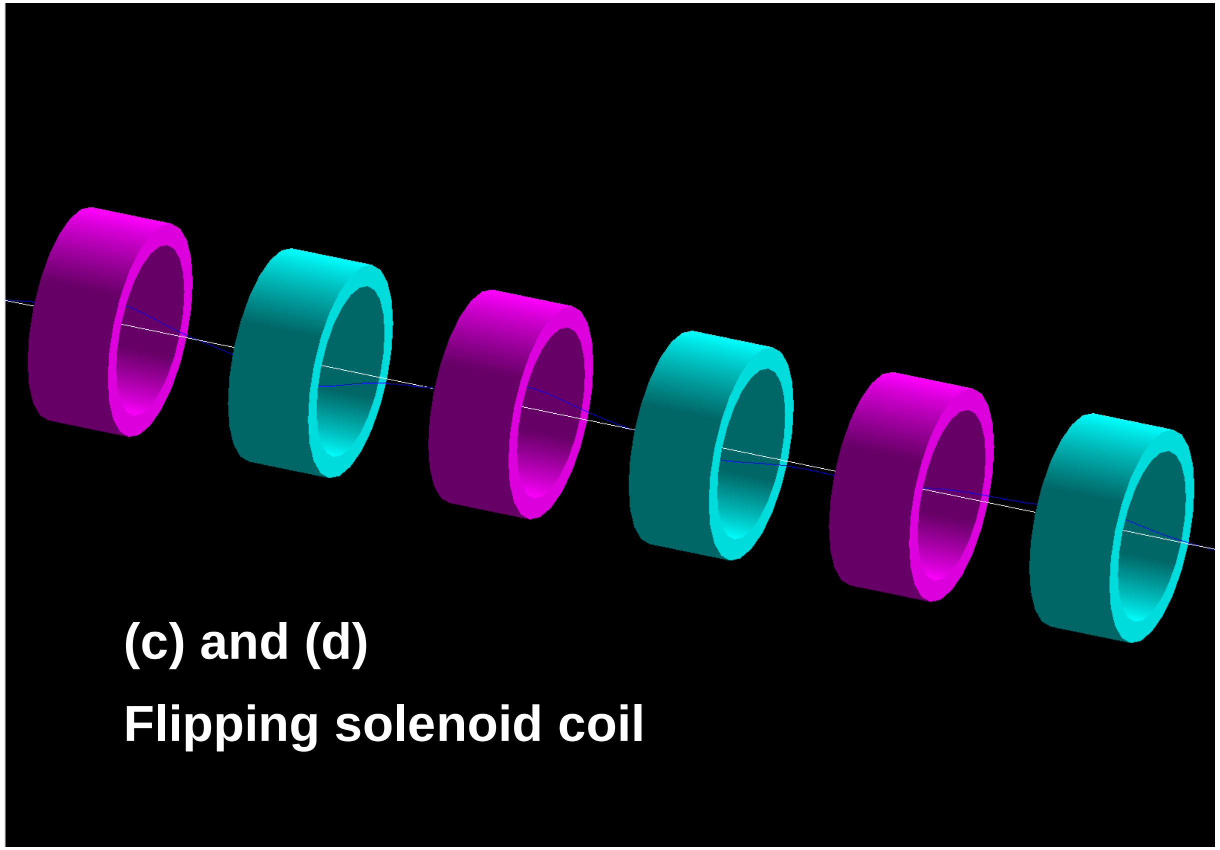
q	One-coil passband	Mathieu theory
0.329–0	116–∞	116–∞
1.86–0.890	49–70	49–70
4.63–3.04	31–37	31–38
8.63–6.42	23–26	23–26
13.87–11.05	18–19	18–20

NB: Table shows passband in different coil configurations from us (e.g. $I = 40\ A/mm2$)

Matheu equation can be solved numerically



Demo: FOFO Channel

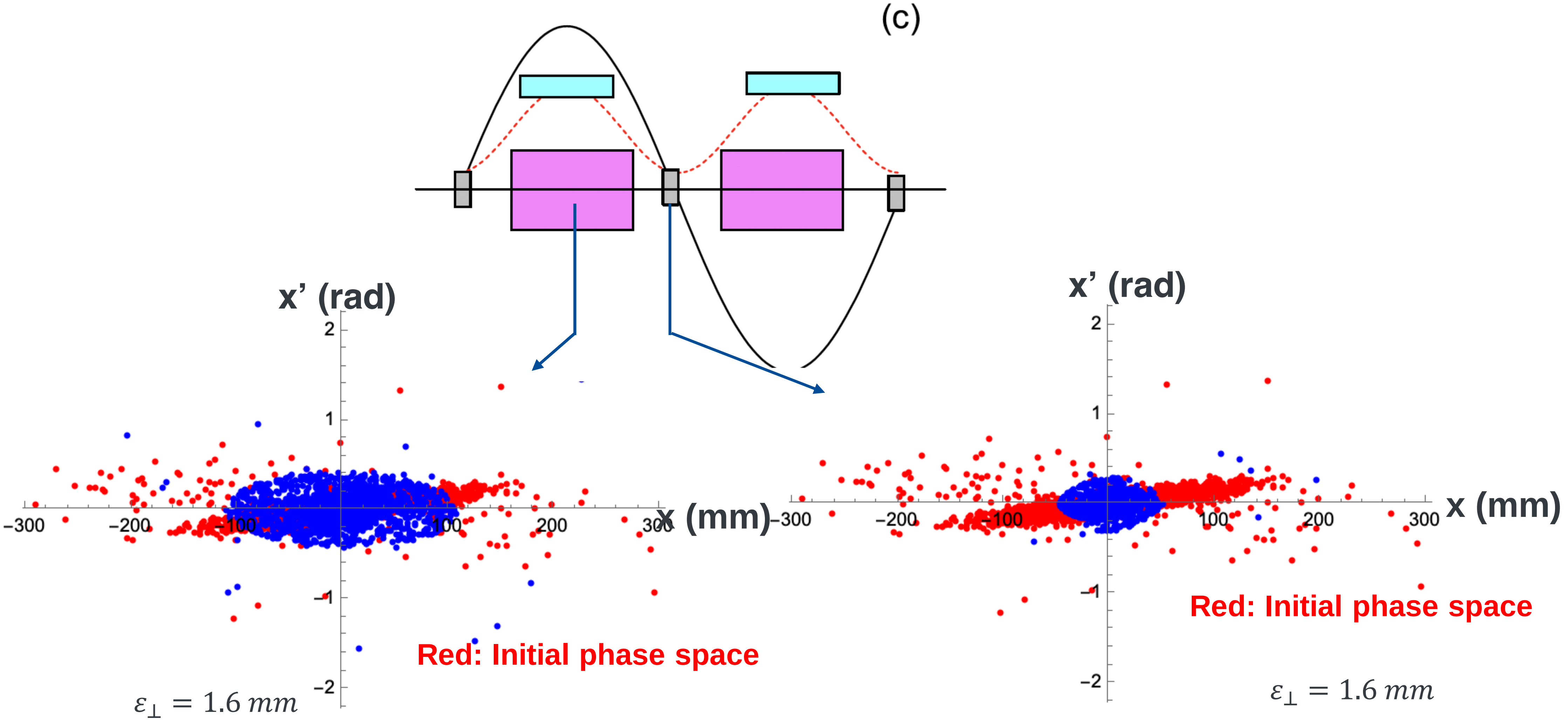


Fernow's coil (referred from PRSTAB 10, 064001 (2007)

- **Coil length = 400 mm**
- **Coil inner radius = 400 mm**
- **Coil thickness = 100 mm**
- **Current = $\pm$100 Amp/mm2**
- **Period = 1000 mm**

Demo: FOFO Channel

fofostudy_fernow_fig1c_05290160.nb



$\varepsilon_{\perp} = 1.6\, mm$

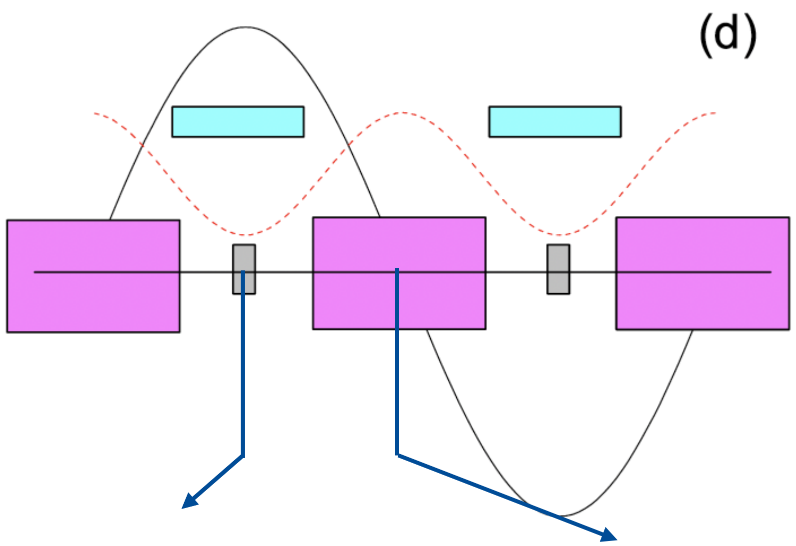
$\beta_{max} = 1.08\, m$

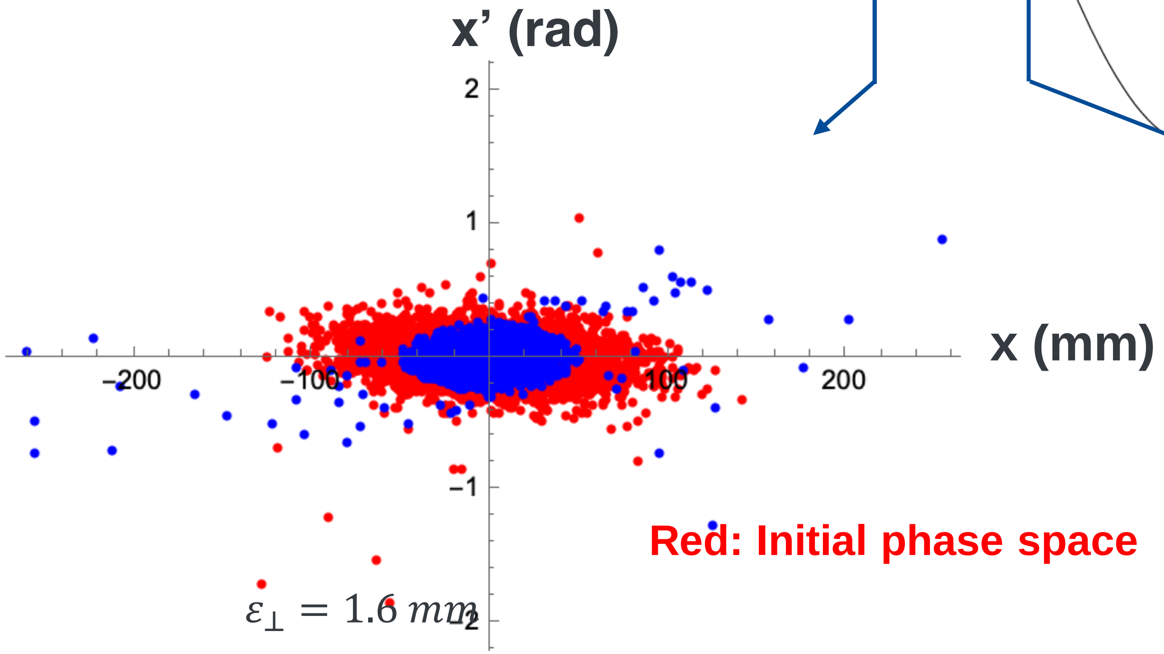
$\varepsilon_{\perp} = 1.6\, mm$

$\beta_{max} = 0.18\, m$

Demo: FOFO Channel

fofostudy_fernow_fig1d_05280170.nb

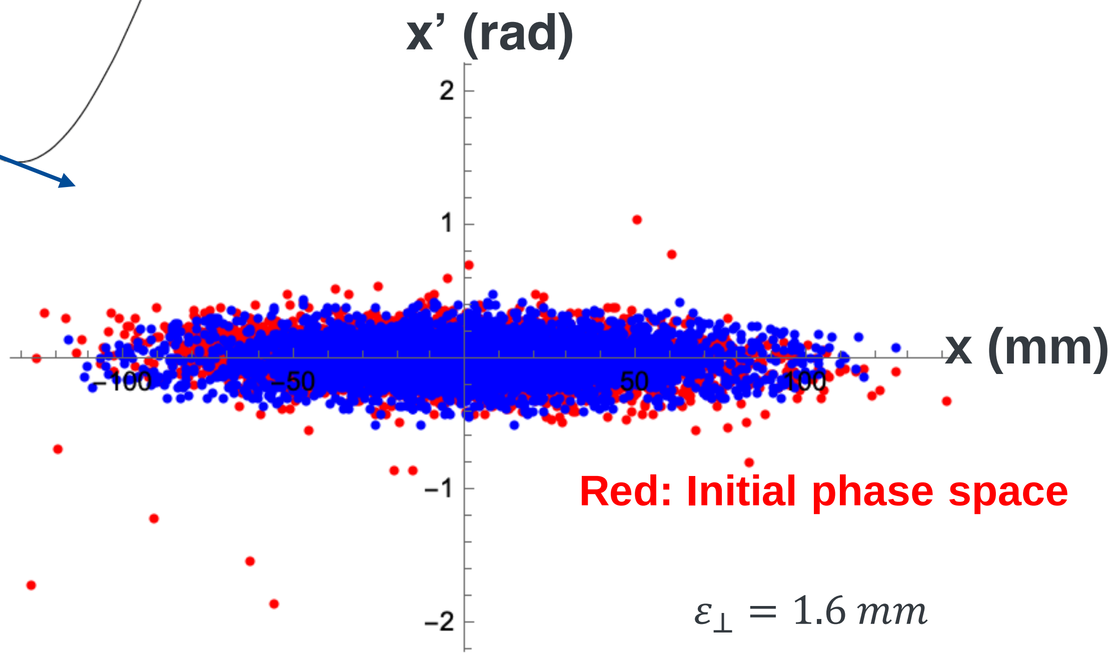




Red: Initial phase space

$\varepsilon_{\perp} = 1.6\ mm$

$\beta_{max} = 0.97\ m$

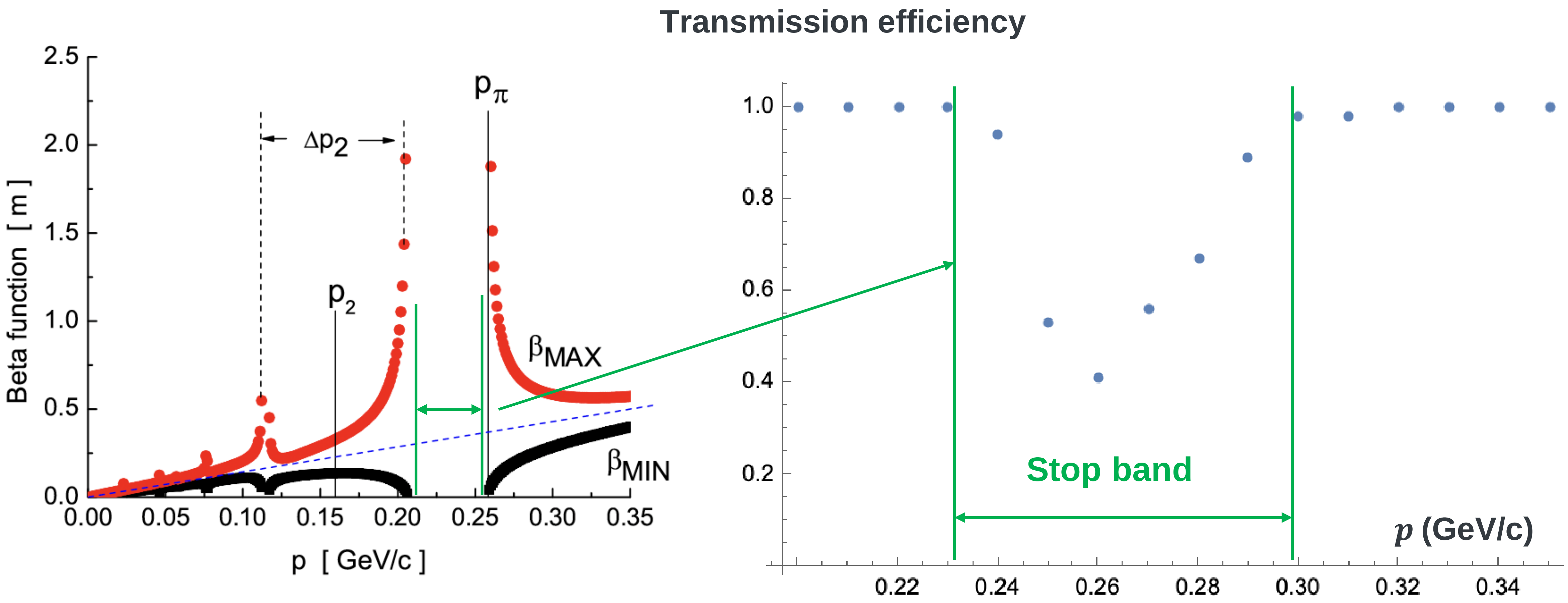


Red: Initial phase space

$\varepsilon_{\perp} = 1.6\ mm$

$\beta_{max} = 0.18\ m$

FOFO Bandpass



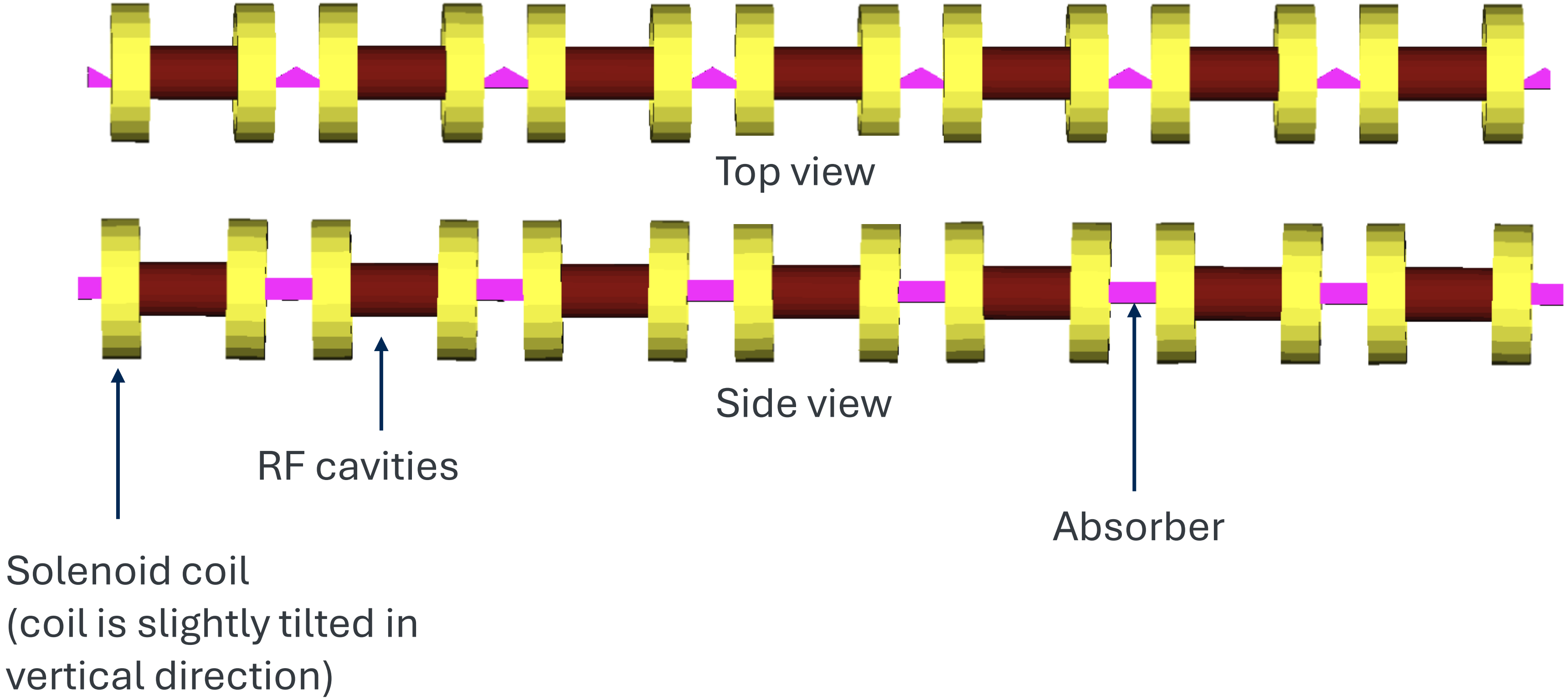
It seems that the stop band momentum range is different from Fernow’s paper; This may be because g4beamline uses different field calculation formula from ICOOL (?)

Rectilinear Channel

Stratakis demonstrated the advanced FOFO channel

→ So called Rectilinear channel

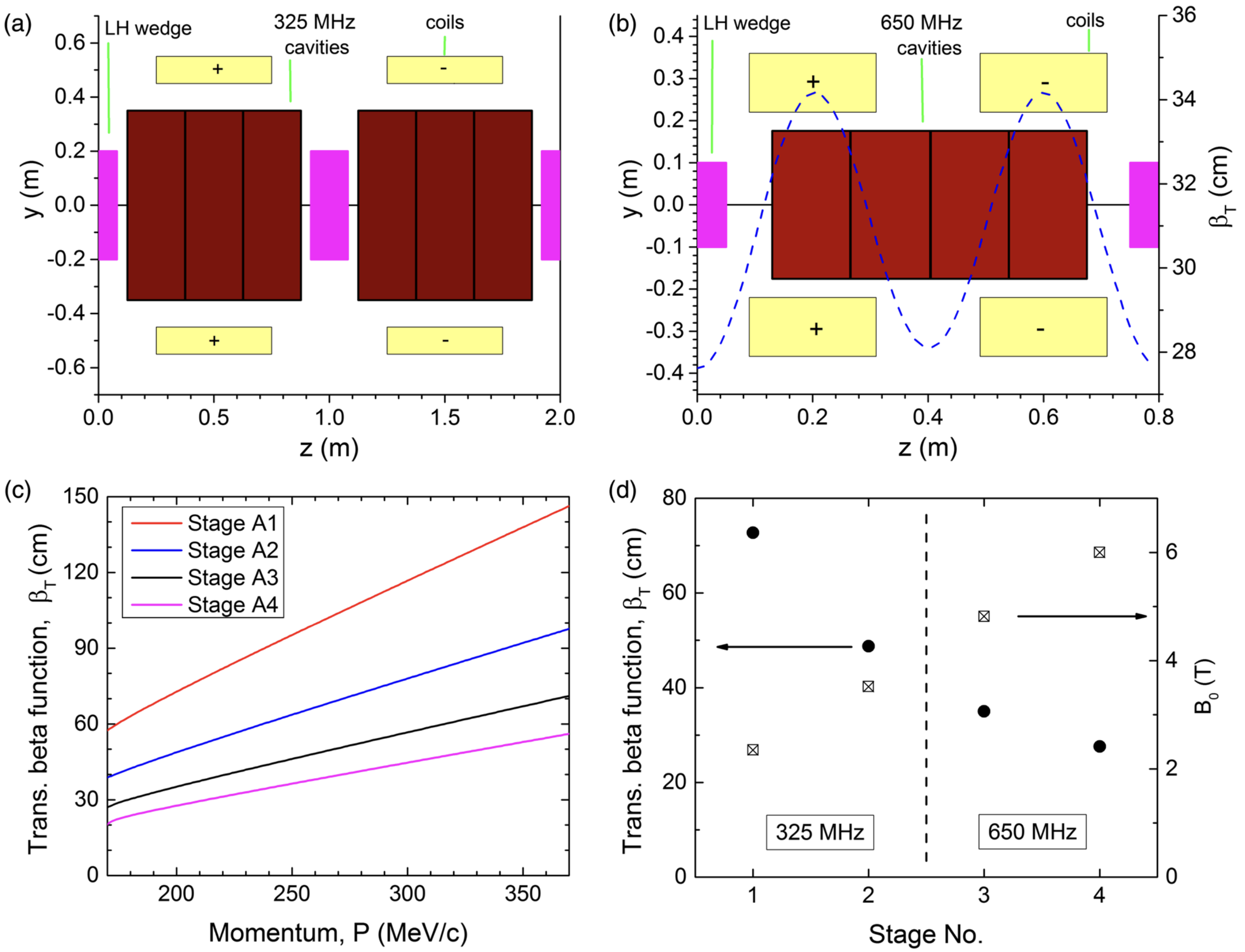
→ Solenoid coil is tilted in vertical direction to induce dispersion



Rectilinear Channel

Initial cooling channel use $p > p_{\pi}$ region

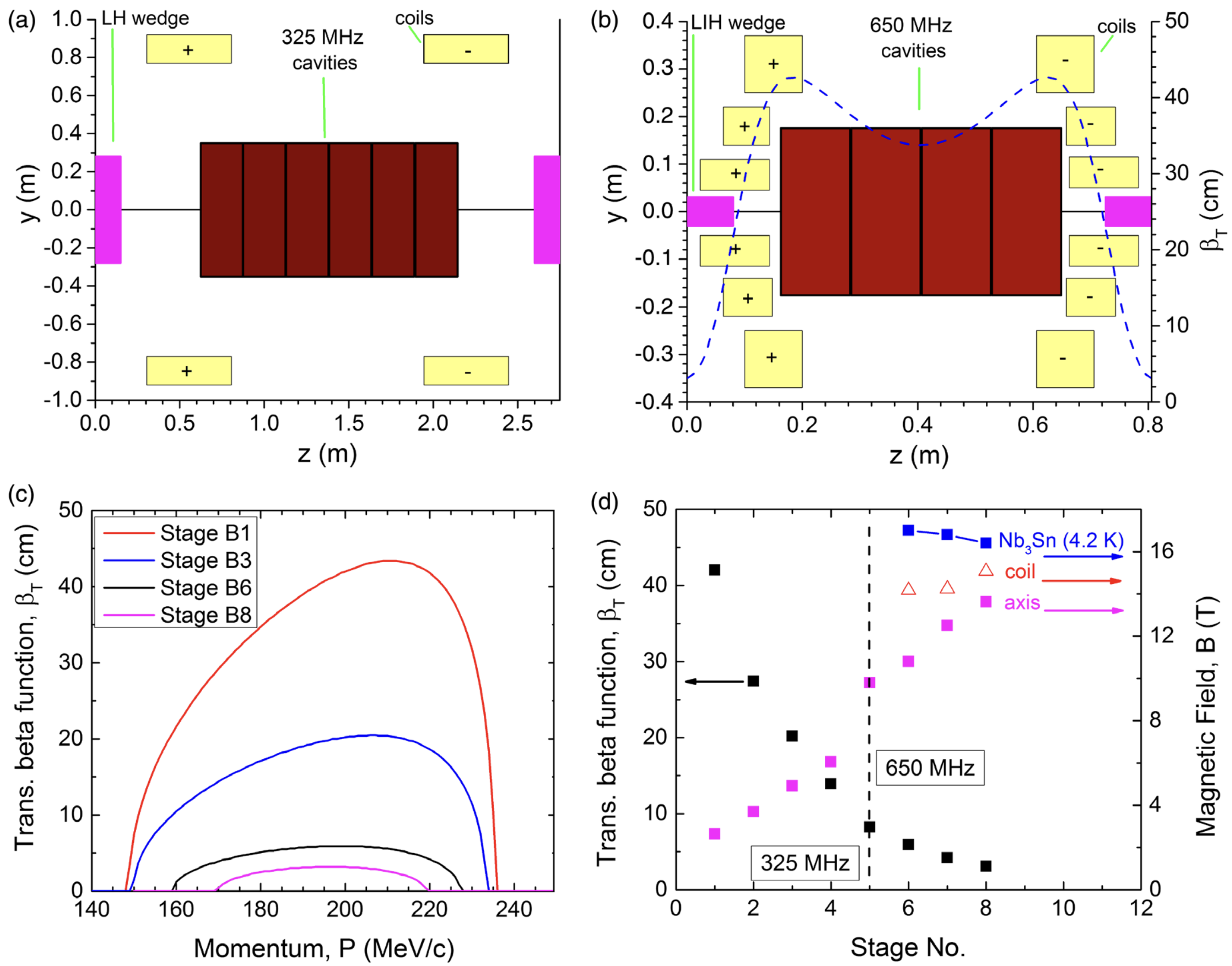
→ Benefit a large momentum acceptance



Rectilinear Channel

Later cooling channel use Δp_2 region

→ $\hat{\beta}$ is flat in p → Good for advanced cooling



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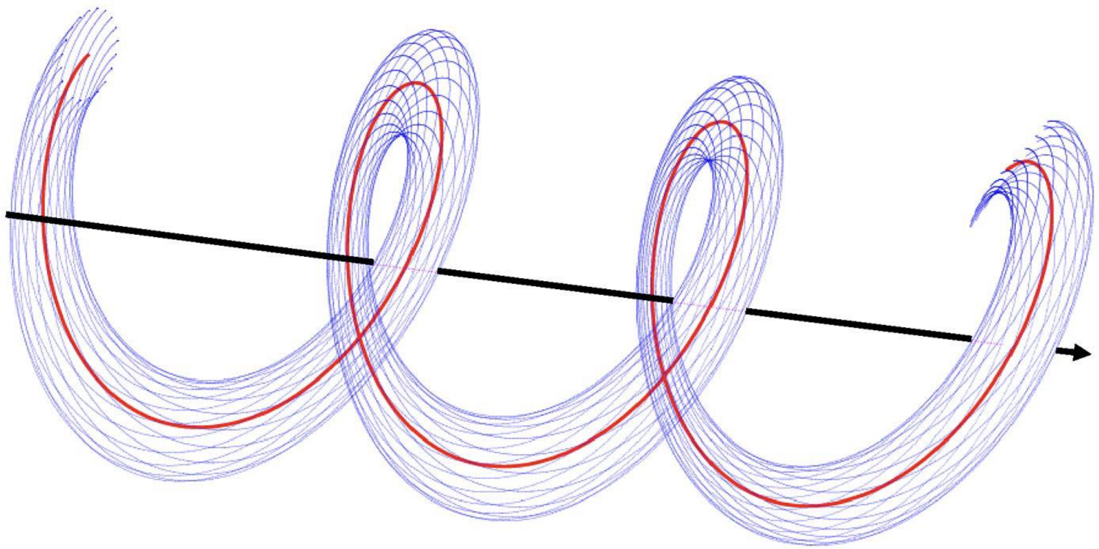
Helical Cooling Channel

Use continuous combined magnet (straight solenoid + helical dipole & quadrupole magnets)
→ No stop band due to a periodic focusing channel



Rotation of reference particle (red line), which is based on cyclotron motion

Equations of motion on the reference frame

$$u_1'' + \frac{q-1}{1+\kappa^2} k u_2' + k^2 \hat{D}^{-1} u_1 = 0$$

$$u_2'' - (q-1) k u_1' + k^2 (q-g) u_2 = 0$$

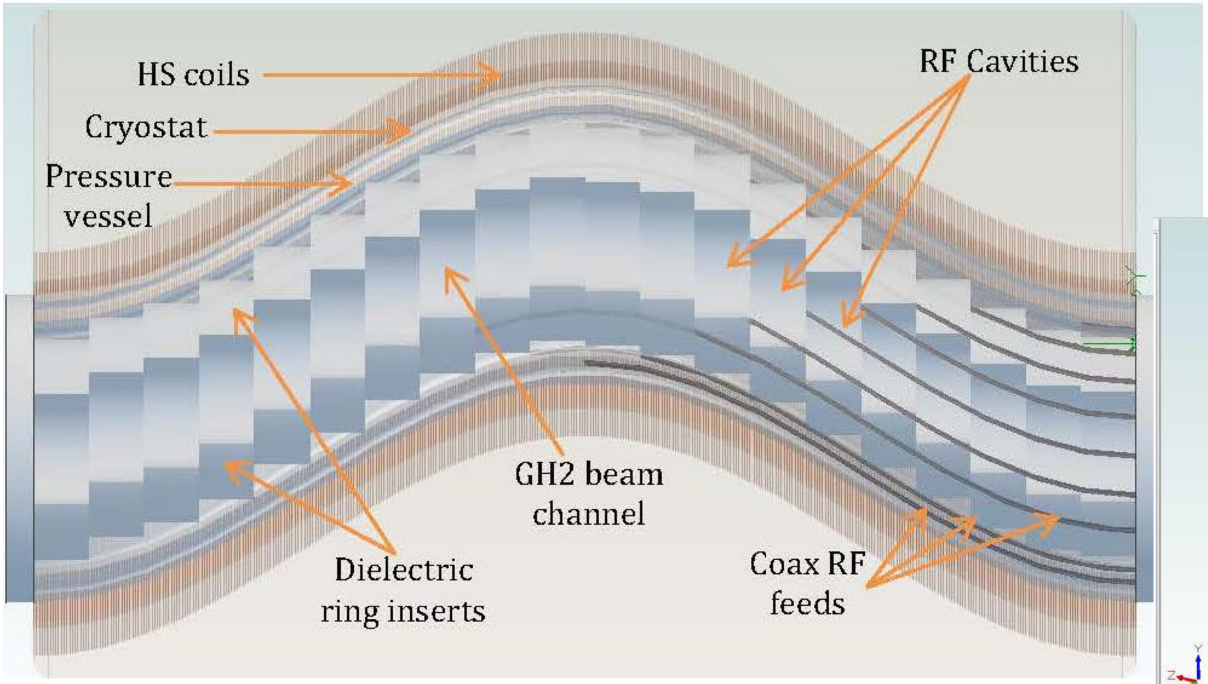
$$\hat{D}^{-1} = \frac{\kappa^2 + (1-\kappa^2) q}{1+\kappa^2} + g$$

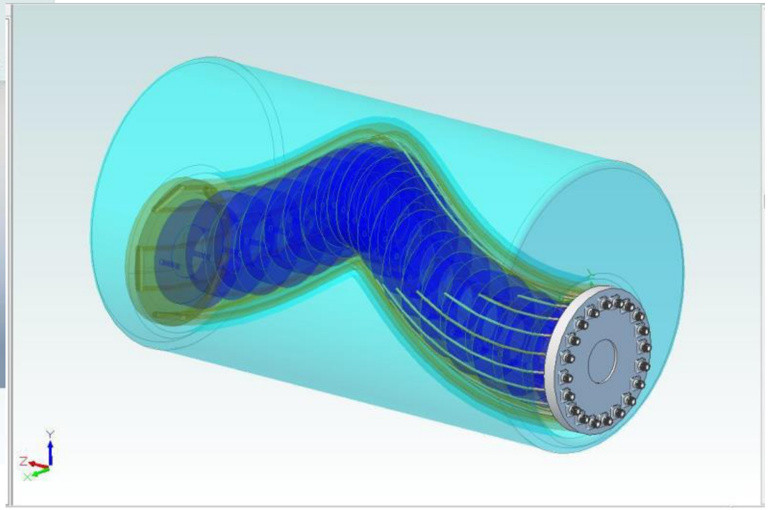
$$g = \frac{-(1+\kappa^2)^{3/2}}{pk^2} \frac{\partial b}{\partial a}$$

Helical Cooling Channel

Challenge of designing helical cooling channel is following

- Helical magnet is complicated
- RF cavities must be inserted into a helical magnet
- If helical channel is used, the matching has to flip sign of the momentum compaction factor from positive to negative





Helical FOFO

Yuri claims that FOFO has an issue to produce low $\hat{\beta}$ and high D, (which is ideal for ionization cooling) because

$$D_{x,RFOFO} \sim \frac{R}{Q_x^2} \sim \frac{\bar{\beta}_x^2}{R}$$

R: circumference of RFOFO, or the length of one cell period of rectilinear channel

His concept starts from a periodic orbit with near pi resonance

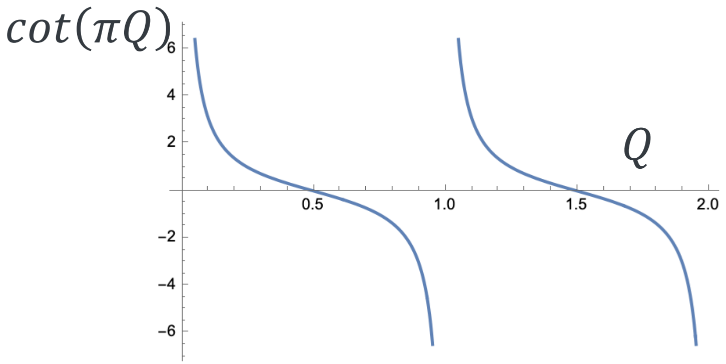
$$x_{p.o.} = \frac{1}{sin(\pi Q_x)}$$

where $x_{p.o.}$ is a periodic orbit, Q_x is a tune, then

$$D_x = \frac{\delta x_{p.o.}}{\delta p} \sim -\pi Q_x' \frac{cos(\pi Q_x)}{sin^2(\pi Q_x)} = -\pi Q_x' \cdot x_{p.o.} cot(\pi Q_x)$$

HFOFO

$$D_x = \frac{\delta x_{p.o.}}{\delta p} \sim -\pi Q_x' \cdot x_{p.o.} cot(\pi Q_x)$$

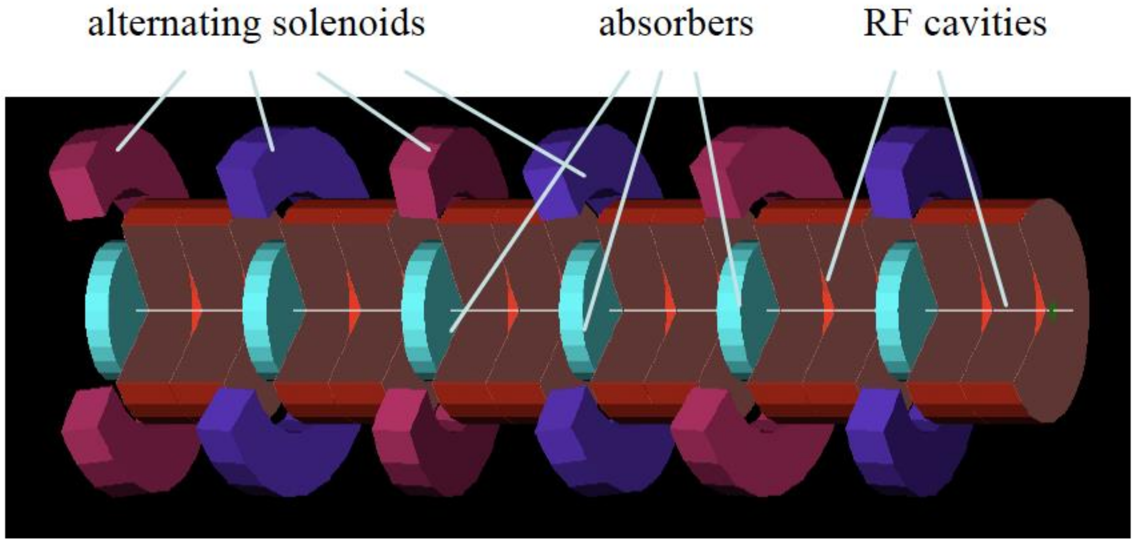


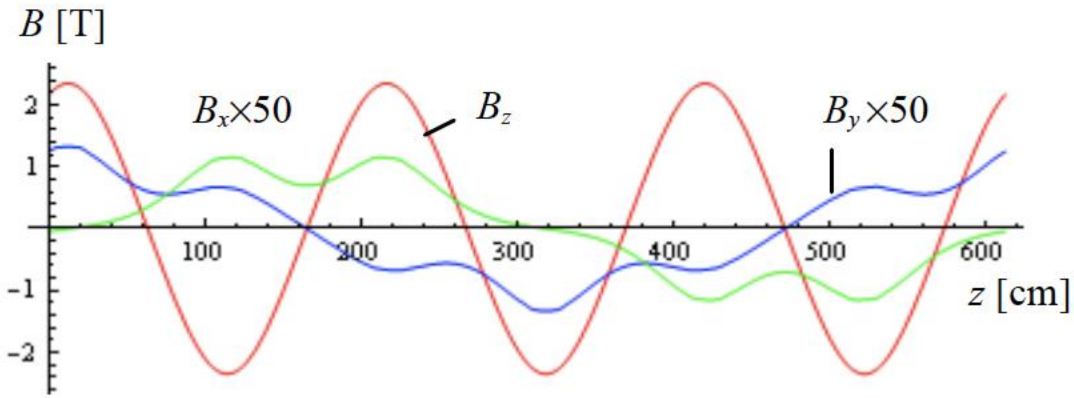
This is good since D can be extremely large when Q_x is integer
($Q_x{}'$ is called a chromaticity)
Making integer Q channel means making π phase advanced FOFO channel → Helical FOFO channel

Helical FOFO channel could have a large momentum acceptance by tuning higher order chromaticity

- Helical orbit permits a large acceptance
- Alternative solenoid eliminates angular momentum
- Forming one period with multiple solenoid cells induces a weak focusing (6 solenoid coils induce one period)

HFOFO





- Initial HFOFO channel has 6 solenoid coils to induce one beta oscillation
- Reference particle has a position offset from solenoid coil center
- Absorber is located low $\hat{\beta}$ and high D region while RF cavities are located other regions
- Tilted solenoid induces dispersion

Feasibility of Cooling Channel

- Channel requires strong magnetic fields
- Channel requires high gradient RF fields
- Channel should be compact, so that integration is challenging
- Channel should be accessible and replaceable
- Channel should have a large tolerance
- Channel should be radiation hard
- Channel requires high resolution beam instrumentation

Summary

M. Palmer, Muon Collider Workshop 2019 in CERN

- VCC→ Rectilinear Cooling Channel
- HCC→ Helical Cooling Channel
- HFOFO is not shown this plot but it has similar as Initial (x)
- Final Cooling Channel expected to be High Field Solenoid Channel
- Bunch merge transforms multi-bunched beam into single bunched beam