Mini Lecture Course: Accelerator Design for a Multi-TeV Muon Collider

Lecture 2: Phase stability

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4/28/2025

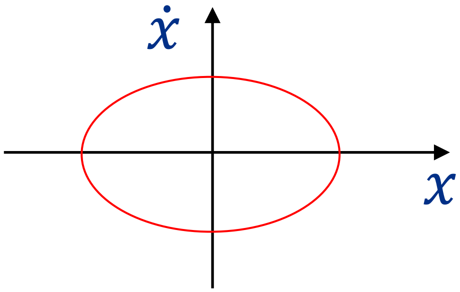
Harmonic Oscillation Refresher

Simple Harmonic Oscillator

- Equation of motion: $m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \omega^2 x = 0 \left(\omega = \sqrt{\frac{k}{m}}\right)$
- General solution:

$$x(t) = A cos(\omega t + \phi)$$
$$\dot{x}(t) = -A\omega \cdot sin\,(\omega t + \phi)$$

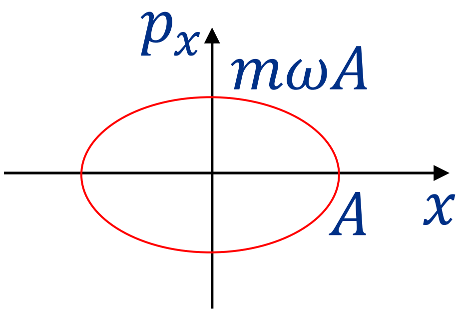
- Energy: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \rightarrow$ conserved quantity
- Phase space trajectory is an **ellipse** in $(x, \dot{x})$: $\frac{kx^2}{2E} + \frac{m\dot{x}^2}{2E} = 1$



Phase Space Area and Action Variable

Area of Phase Space Ellipse

- Let: $x_{max} = A, p_{max} = m\omega A$
- Phase space area: $\mathcal{A} = \pi x_{max} p_{max} = \pi \cdot m\omega A^2 = 2\pi \cdot J$
- **Action variable**: $J = \frac{1}{2\pi}\oint p dq = \frac{E}{\omega}$ (conserved)
- In beam physics: $J \propto \varepsilon$, the emittance



Hamiltonian and Canonical Transformation

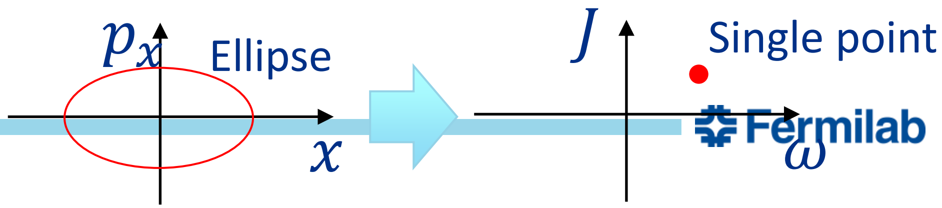
Canonical Variables and Hamiltonian

- Hamiltonian: $H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2$
- Equations of motion: $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \dot{p} = -\frac{\partial H}{\partial q} = -kq$

Canonical Transformation to Action-Angle Coordinates:

- Define: $q = \sqrt{\frac{2J}{m\omega}} cos\psi, p = -\sqrt{2Jm\omega}\, sin\psi$
- Then: $H = \omega J, \dot{\psi} = \omega$ (uniform phase advance)

  - Action angle J and ω are <u>constant</u> in harmonic oscillation
  - Particle trajectory (q,p) is described with <u>action angle and initial condition</u>



Transition to Beam Optics

- Action angle in beam optics is $\varepsilon\ (J)$ and $\hat{\beta}$

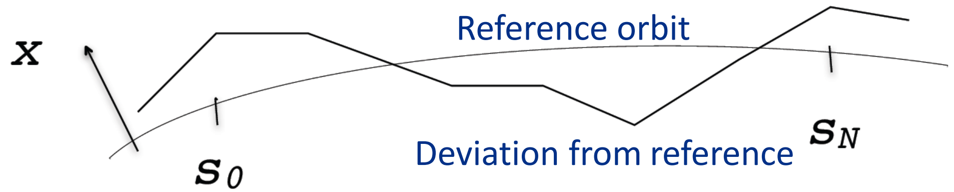
$$p_x \equiv \hat{\beta}_x x' + \hat{\alpha}_x x = -\sqrt{2\hat{\beta}_x J_x} \sin\psi_x\,.$$

$$x = \sqrt{2\hat{\beta}_x J_x} \cos\psi_x\,,$$

$$\hat{\alpha}_x = -\frac{1}{2}\hat{\beta}_x{}',\hat{\gamma}_x = \frac{1+\hat{\alpha}_x{}^2}{\hat{\beta}_x} \rightarrow \hat{\beta}_x\hat{\gamma}_x - \hat{\alpha}_x{}^2 = 1,$$

Basic equation of motion is Hill's equation: $x'' + K(s)x = 0$, which is similar form as $\ddot{x} + \omega^2 x = 0$

- x usually presents the deviation from the reference orbit
- Also note that <u>K is a function of s</u> while <u>ω</u> is a constant value



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Reminder on Symplectic Matrices
What is a Symplectic Matrix?

- A 2×2 matrix is symplectic if: $M^T \Omega M = \Omega$, where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- This preserves phase space area (Liouville's theorem)
- Important property for transport matrices and Twiss formalism

$K = \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}$, with $\hat{\beta}\hat{\gamma} - \hat{\alpha}^2 = 1$

- This matrix is symmetric and symplectic: $K^T \Omega K = \Omega$

Why it matters

- Symplecticity confirms that Twiss parameters are canonical, preserving the Hamiltonian structure of beam motion even after transformation
- Symplecticity also ensures that beam emittance is conserved

Discussion item for lecture 2

- Emittance evolution in Ionization cooling
- Transverse phase space stability
- Thin lens & paraxial approximations
- Longitudinal phase space stability

Normalized emittance

NB: **Normalized emittance** is invariant under acceleration, whereas mechanical emittance shrinks with increasing beam energy

$$\varepsilon_n = \beta\gamma \cdot \varepsilon$$

ε: Mechanical emittance
β, γ: Lorentz factors

Mechanical emittance (kinetic energy included)

$$\varepsilon_{x,rms} = \sqrt{\begin{matrix} \langle x \cdot x \rangle & \langle x \cdot x' \rangle \\ \langle x \cdot x' \rangle & \langle x' \cdot x' \rangle \end{matrix}} = \varepsilon_{rms} \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}$$

$$= \sqrt{Det[\sigma^2]} = \sqrt{\sigma_x^2 \cdot \sigma_{x'}^2 - \sigma_{xx'}^2}.$$

Normalized emittance (no change by kinetic energy)

$$\varepsilon_{x,n} = \sqrt{\begin{matrix} \langle x \cdot x \rangle & \langle x \cdot \gamma\beta_x \rangle \\ \langle x \cdot \gamma\beta_x \rangle & \langle \gamma\beta_x \cdot \gamma\beta_x \rangle \end{matrix}}$$

NB: we extract $\hat{\beta}, \hat{\alpha}, \hat{\gamma}$ (Twiss parameter) from mechanical emittance

Emittance evolution (Transverse)

$$\frac{d\varepsilon_n}{ds} = \frac{d(\beta\gamma \cdot \varepsilon)}{ds} = \beta\gamma\frac{d\varepsilon}{ds} + \varepsilon\frac{d\beta\gamma}{ds}.$$

First term,

$$\beta\gamma\frac{d\varepsilon_x}{ds} = \frac{\beta\gamma}{2\varepsilon_x} \cdot \frac{d{\varepsilon_x}^2}{ds} = \frac{\beta\gamma}{2\varepsilon_x}\left(\sigma_x^2\frac{d\sigma_{x'}^2}{ds} + \sigma_{x'}^2\frac{d\sigma_x^2}{ds} - 2\sigma_{xx'}\frac{d\sigma_{xx'}}{ds}\right)$$

Assume there is no beam spot size variation

$$\frac{d\sigma_x^2}{ds}\sim 0 \left(\rightarrow \frac{d\varepsilon_x\hat{\beta}}{ds}\sim 0\right)$$

NB: We assume that Liouville's theorem is adiabatically preserved during ionization cooling (slide 18)

We further assume that there is no coupling between x and x' during ionization cooling

$$\frac{d\sigma_{xx'}}{ds}\sim 0\left(\rightarrow -\frac{d\varepsilon_x\hat{\alpha}}{ds}\sim 0\right).$$

Emittance evolution (Transverse)

Residual term

$$\beta\gamma \frac{d\varepsilon_x}{ds} \approx \frac{\beta\gamma}{2\varepsilon_x} \cdot \sigma_x^2 \frac{d\sigma_{x'}^2}{ds}$$

We involve multiple scattering process,

$$\frac{\beta\gamma}{2\varepsilon_x} \cdot \sigma_x^2 \frac{d\sigma_{x'}^2}{ds} \sim \frac{\beta\gamma}{2\varepsilon_x} \cdot \hat{\beta}_x \varepsilon_x \cdot \sigma_\theta^2.$$

$$\sigma_\theta^2 \sim \left(\frac{13.8\ MeV}{\beta c p}\right)^2 \frac{z^2}{L_R}.$$

We will investigate multiple scattering in later session

Thus, we finally gain

$$\beta\gamma \frac{d\varepsilon_x}{ds} \sim \frac{\beta\gamma}{2\varepsilon_x} \cdot \hat{\beta}_x \varepsilon_x \cdot \left(\frac{13.8\ MeV}{\beta c p}\right)^2 \frac{z^2}{L_R}.$$

Emittance evolution (Transverse)

$$\frac{d\varepsilon_n}{ds} = \frac{d(\beta\gamma \cdot \varepsilon)}{ds} = \beta\gamma \frac{d\varepsilon}{ds} + \varepsilon \frac{d\beta\gamma}{ds}.$$

Second term,

$$\varepsilon_x \frac{d\beta\gamma}{ds} = \frac{\varepsilon_x}{\beta \cdot mc^2} \frac{dE}{ds} = \frac{\beta\gamma \cdot \varepsilon_x}{\beta^2 \cdot \gamma mc^2} \cdot \frac{dE}{ds} = \frac{\varepsilon_{n,x}}{\beta^2 E} \cdot \frac{dE}{ds}.$$

Since $\frac{dE}{ds}$ is negative for ionization loss, we define it as a positive quantity by adding minus sign explicitly: $\frac{dE}{ds} \rightarrow - \left(\frac{dE}{ds}\right)$ in the emittance evolution equation

Emittance evolution (Transverse)

Final form is

$$\frac{d\varepsilon_{n,x}}{ds} = \frac{\beta\gamma}{2} \cdot \hat{\beta}_x \sigma_\theta^2 - \frac{\varepsilon_{n,x}}{\beta^2 E} \cdot \left(\frac{dE}{ds}\right).$$

The equilibrium emittance is obtained from this,

$$d\varepsilon_{n,x}/ds = 0,$$

$$\varepsilon_{n,x,eq} = \frac{\frac{\beta\gamma}{2} \hat{\beta}_x \sigma_\theta^2}{\beta^2 E \cdot \left(\frac{dE}{ds}\right)} = \frac{\hat{\beta}_x (13.6\ MeV)^2 \cdot z^2}{2\beta m L_R \cdot \left(\frac{dE}{ds}\right)}$$

To reach low emittance: low $\hat{\beta}_x$, large $\left(\frac{dE}{ds}\right)$, and long L_R

Fermilab

Emittance evolution (Longitudinal)

$$\frac{d\varepsilon_n}{ds} = \frac{d(\beta\gamma \cdot \varepsilon)}{ds} = \beta\gamma\frac{d\varepsilon}{ds} + \varepsilon\frac{d\beta\gamma}{ds}.$$

First term for longitudinal,

$$\beta\gamma\frac{d\varepsilon_l}{ds} = \frac{\beta\gamma}{2\varepsilon_l}\cdot\frac{d\varepsilon_l{}^2}{ds} = \frac{\beta\gamma}{2\varepsilon_l}\left(\sigma^2_{\delta p/p}\frac{d\sigma^2_{\delta t}}{ds} + \sigma^2_{\delta t}\frac{d\sigma^2_{\delta p/p}}{ds} - 2\sigma_{\delta p/p\cdot\delta t}\frac{d\sigma_{\delta p/p\cdot\delta t}}{ds}\right)$$

Again, in this toy model, there is no mechanism to change time structure and no coupling between δt and $\delta p/p$

$$\frac{d\sigma^2_{\delta t}}{ds} \sim \frac{d\sigma_{\delta p/p\cdot\delta t}}{ds} \sim 0.$$

While the energy straggling effect is included,

$$\frac{\beta\gamma}{2\varepsilon_l}\cdot\sigma^2_{\delta p/p}\frac{d\sigma^2_{\delta p/p}}{ds} \sim \frac{\beta\gamma}{2}\hat{\beta}_L\cdot\frac{1}{\beta^2c^2p^2}\frac{d\left(\Delta E^2_{rms}\right)}{ds}$$

$$\sim \frac{\beta\gamma}{2}\hat{\beta}_L\cdot\frac{1}{\beta^2c^2p^2}\cdot\xi W_{max}\left(1-\frac{\beta^2}{2}\right)$$

$$\xi = 153.4\frac{z^2}{\beta^2}\frac{Z}{A} \quad \text{keV g/cm}^2$$

$$W_{max} = \frac{2m_e\beta^2\gamma^2}{1+2\gamma\, m_e/m_\mu + \left(m_e/m_\mu\right)^2}$$

Emittance evolution (Dispersion coupling)

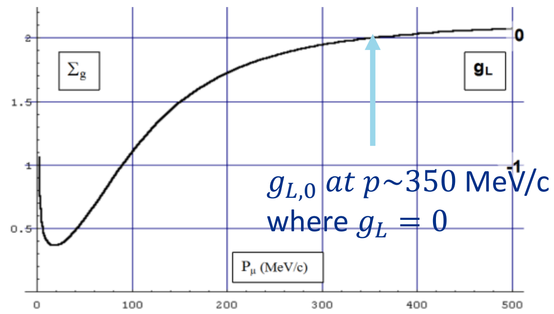
Add coupling parameter

$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y} \cdot \varepsilon_{n,x,y}}{\beta^2 E} \cdot \left\langle \frac{dE}{ds} \right\rangle + \frac{(13.6\ MeV)^2 \beta_{x,y}}{2 m_\mu E \beta^3 L_R},$$

$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L \cdot \varepsilon_{n,L}}{\beta^2 E} \cdot \left\langle \frac{dE}{ds} \right\rangle + \frac{\beta\gamma}{2} \hat{\beta}_L \cdot \frac{1}{\beta^2 c^2 p^2} \cdot \xi W_{max} \left(1 - \frac{\beta^2}{2}\right)$$

Where $g_{x,y}$ and g_L are a partition number

$$\frac{d\left(\frac{dE}{ds}\right)}{ds} = g_L = -\frac{2}{\gamma^2} + 2\frac{\left(1 - \frac{\beta^2}{\gamma^2}\right)}{\left(ln\left[\frac{2 m_e c^2 \beta^2 \gamma^2}{I(Z)}\right] - \beta^2\right)}.$$



To compensate positive $g_{L,0}$, dispersion + wedge absorber

$$g_L \rightarrow g_{L,0} + \frac{D\rho'}{\rho_0}, \qquad g_x \rightarrow 1 - \frac{D\rho'}{\rho_0}.$$

$$\Sigma_g = g_x + g_y + g_L = 2 + g_{L,0}.$$

Indeed, dispersion induces coupling between transverse and longitudinal

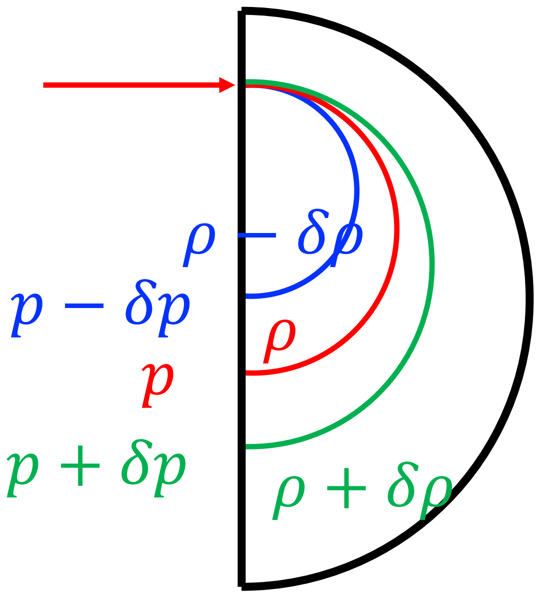
Fermilab

Emittance exchange with Dispersion

Beam momentum is translated into beam position by dipole magnet

NB: Beam emittance is not exchanged without absorber

Particle position is distributed by $\delta p/p$



Top view of dipole magnet

$$m\frac{v^2}{\rho} = qv \times B \rightarrow \frac{p}{\rho} = qB$$

for <u>uniform dipole field</u>

$$\rightarrow \frac{p+\delta p}{\rho+\delta\rho} = qB = \text{const}$$

$$\rightarrow \frac{p}{\rho}\left(1+\frac{\delta p}{p}\right)\left(1-\frac{\delta\rho}{\rho}\right) = \frac{p}{\rho}$$

$$\rightarrow 1 + \frac{\delta p}{p} - \frac{\delta\rho}{\rho} = 1$$

$$\rightarrow \frac{\delta p}{\delta\rho} = \frac{p}{\rho}$$

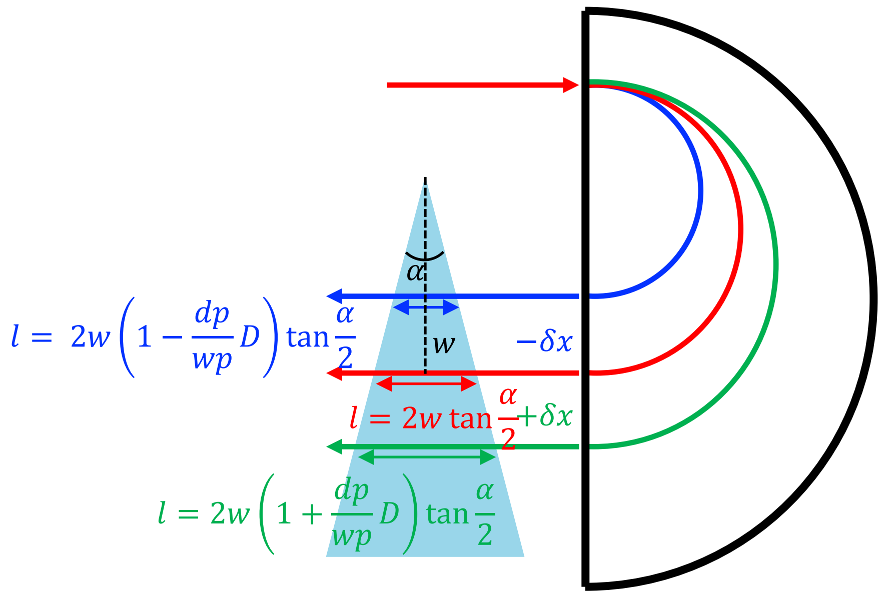
This phenomenon is called dispersion

$$D = p\frac{d\rho}{dp}$$

in a uniform dipole field

$$D = p\frac{d\rho}{dp} = p\frac{\rho}{p} = \rho$$

Longitudinal cooling with emittance exchange



$p - \delta p$ particle traverses with shorter path length than reference in wedge absorber

p particle is a reference particle

$p - \delta p$ particle traverses with longer path length than reference in wedge absorber

δp becomes smaller after passing wedge absorber
→ Longitudinal cooling

Wedge absorber

Dave's definition: $1 - D\frac{\rho'}{\rho}$

$$\rightarrow 1 - \frac{D}{w}\frac{dp}{p}$$

Emittance evolution

Constant term

$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y}\cdot\varepsilon_{n,x,y}}{\beta^2 E}\cdot\left\langle\frac{dE}{ds}\right\rangle + \boxed{\frac{(13.6\ MeV)^2\beta_{x,y}}{2m_\mu E\beta^3 L_R}},$$

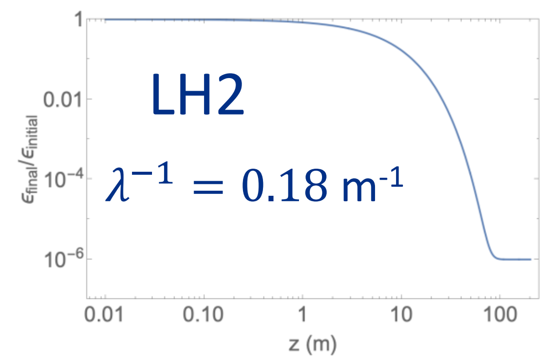
$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L\cdot\varepsilon_{n,L}}{\beta^2 E}\cdot\left\langle\frac{dE}{ds}\right\rangle + \boxed{\frac{\beta\gamma}{2}\hat{\beta}_L\cdot\frac{1}{\beta^2c^2p^2}\cdot\xi W_{max}\left(1-\frac{\beta^2}{2}\right)}$$

$$\frac{d\varepsilon_n}{ds} = -c_1\cdot\varepsilon_n + c_2$$

$$\rightarrow \varepsilon_n(s) = \left(\varepsilon_{n,0}-\varepsilon_{n,eq}\right)\cdot\exp[-\lambda^{-1}_{x,y,z}\cdot s] + \varepsilon_{n,eq}$$

$$\varepsilon_{n,eq} = \frac{c_2}{c_1}$$

Plot shows that the required cooling channel length is < 100 m in LH2 if the equilibrium emittance is 10^{-6} (initial emittance is 1.0) unit is arbitrary in this discussion



How do we treat Liouville's theorem in cooling?

- Regarding Liouville's theorem, the phase space volume remains constant for energy conservative Hamiltonian
  - $\nabla \cdot (n\vec{v}) + \frac{\partial n}{\partial t} = 0$
    - where $n(q,p)$ is a density function and $\vec{v} = \{\dot{q}, p\}$ denotes phase space velocity (continuity equation)
  - Preserving this theorem requires symplectic integration, which is both a necessary and sufficient condition
- Liouville's theorem is not strictly satisfied in beam cooling systems, such as ionization cooling, due to non-Hamiltonian effects
- Nevertheless, if the phase space evolution is close to adiabatic, the theorem remains approximately valid
  - There is a paper to solve the evolution of emittance in cooling by solving Fokker-Planck equation (G. Dugan, PRSTAB4,104001 (2001))

Transverse Phase Space Stability

Beam dynamics and stability condition

Solution of Hill's equation $x''(s) + \kappa(s)x(s) = 0$ is in general

$$x(s) = ax_0 + bx_0'$$
$$x'(s) = c\,x_0 + dx_0'$$

Equation of motion can be solved if initial condition is known

As we usually do for the harmonic oscillation,

$$X \rightarrow \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix} \rightarrow \widetilde{M} \cdot X_0$$

$$Tr\left(\widetilde{M}\right) = a + d$$
$$det\left(\widetilde{M}\right) = ad - bd$$

Extracting eigenvalues from $\widetilde{M}$,

$$det\left(\widetilde{M} - \lambda I\right) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \rightarrow \lambda^2 - Tr\left(\widetilde{M}\right)\lambda + det\left(\widetilde{M}\right) = 0$$

Beam dynamics and stability condition

If there is no energy change, $\frac{dH}{ds} = 0$,

$$det(\widetilde{M}) = ad - bd = 1$$

$\widetilde{M}^T \Omega \widetilde{M} = \Omega$
$\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\rightarrow ad - bc = 1$ (if energy conservation is required, then using symplectic integration is a sufficient condition)

$$\rightarrow \lambda^2 - (a+d) + 1 = 0 \rightarrow \lambda^2 - Tr(\widetilde{M})\lambda + 1 = 0$$

Solution of this equation is

$$\lambda = \frac{Tr(\widetilde{M}) \pm \sqrt{Tr(\widetilde{M})^2 - 4}}{2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2}$$

We notice if $\cos(\sigma) = \frac{1}{2} Tr(\widetilde{M}) = \frac{(a+d)}{2}$,

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2} = \frac{(a+d)}{2} \pm \sqrt{\frac{(a+d)^2}{4} - 1}$$

$$\rightarrow \lambda_{1,2} = \cos(\sigma) \pm \sqrt{cos^2(\sigma) - 1} = \cos(\sigma) \pm i\sin(\sigma) = e^{\pm i\sigma}$$

Beam dynamics and stability condition

If $|a+d| \le 2, \lambda_{1,2}$ is a periodic solution→<u>Beam motion is stable</u>

For our convenience, we use following relations,

$$
\begin{gathered}
a+d = 2\cos(\sigma) \\
a-d = 2\hat{\alpha}\sin(\sigma) \\
b = \hat{\beta}\sin(\sigma) \\
c = -\hat{\gamma}\sin(\sigma)
\end{gathered}
\quad \rightarrow \widetilde{M} = \begin{bmatrix} \cos\sigma + \hat{\alpha}\sin\sigma & \hat{\beta}\sin\sigma \\ -\hat{\gamma}\sin\sigma & \cos\sigma - \hat{\alpha}\sin\sigma \end{bmatrix}
$$

Or $\widetilde{M} = \tilde{I}\cos\sigma + \tilde{J}\sin\sigma$

where $\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$

$$
det\left(\widetilde{M}\right) = ad - bc
$$

$$
= (\cos\sigma + \hat{\alpha}\sin\sigma)\cdot(\cos\sigma - \hat{\alpha}\sin\sigma) - \hat{\beta}\sin\sigma\cdot(-\hat{\gamma}\sin\sigma)
$$

$$
= \hat{\beta}\hat{\gamma} - \hat{\alpha}^2 = 1
$$

These relations are still within a canonical framework!

Beam dynamics and stability condition

Investigate $\tilde{J}$

$$(\tilde{J})^2 = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix} \cdot \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\tilde{I}$$

$$\left(\widetilde{M}\right)^2 = (\tilde{I}\cos\sigma + \tilde{J}\sin\sigma)^2$$
$$= \tilde{I}^2 cos^2\sigma + 2\tilde{I}\cdot\tilde{J}\cdot sin\sigma\cdot cos\sigma + \tilde{J}^2 sin^2\sigma$$
$$= \tilde{I}(cos^2\sigma - sin^2\sigma) + 2\tilde{J} sin\sigma\cdot cos\sigma$$
$$= \tilde{I}cos2\sigma + \tilde{J}sin2\sigma$$

$$\left(\widetilde{M}\right)^3 = \left(\widetilde{M}\right)^2\cdot\widetilde{M}$$
$$= (\tilde{I}cos2\sigma + \tilde{J}sin2\sigma)\cdot(\tilde{I}cos\sigma + \tilde{J}sin\sigma)$$
$$= \tilde{I}^2 cos2\sigma\cdot cos\sigma + \tilde{J}^2 sin2\sigma\cdot sin\sigma$$
$$+\tilde{I}\tilde{J}cos2\sigma\cdot sin\sigma + \tilde{I}\tilde{J}sin2\sigma\cdot cos\sigma$$

$$cos2\sigma\cdot cos\sigma = \frac{1}{2}(cos3\sigma + cos\sigma)\ \ldots$$

$$= \tilde{I}cos3\sigma + \tilde{J}sin3\sigma$$

$$\rightarrow \left(\widetilde{M}\right)^N = (\tilde{I}\cos\sigma + \tilde{J}\sin\sigma)^N = \tilde{I}cosN\sigma + \tilde{J}sinN\sigma$$

$$\lambda_{1,2} = \cos(\sigma) \pm i\sin(\sigma) = e^{\pm i\sigma}$$

$$\rightarrow \lambda_1 = \frac{1}{\lambda_2}$$

This statement is the same general solution from a parametric oscillator, especially it is called Floquet's theorem

Stability of Periodic Phase Space from Mathieu equation

Floquet function

Floquet's theorem is applicable to linear differential equations with periodic coefficients

$$\frac{dX}{dt} = A(t)X$$

where $A(t+T) = A(t)$, T is a period

A general solution will be

This is $\lambda_{1,2}$ in our case

$$X(t) = e^{\mu t} p(t)$$

where $\mathrm{p}(t+T) = p(t)$, $p(t)$ is called Floquet function, μ is the Floquet exponent

As we notice, it is similar as a forced oscillator

$\rightarrow$ μ is pure imaginary, then $X(t)$ is a stable oscillation
$\rightarrow$ μ contains real, then $X(t)$ is a forced oscillation, growing or decaying exponentially over time

Mathieu equation with Floquet's theorem

As you may have noticed, our process is the reverse of the conventional forced harmonic oscillator problem. The general equation of forced harmonic oscillator in the beam dynamics is so called Mathieu equation,

$$\frac{d^2x}{dt^2} + \big(\delta + \epsilon \cdot cos(t)\big)x = 0$$

δ: stiffness parameter
ϵ: no specific name given

In the case $\epsilon = 0$, it is a condition to obtain Hill's equation

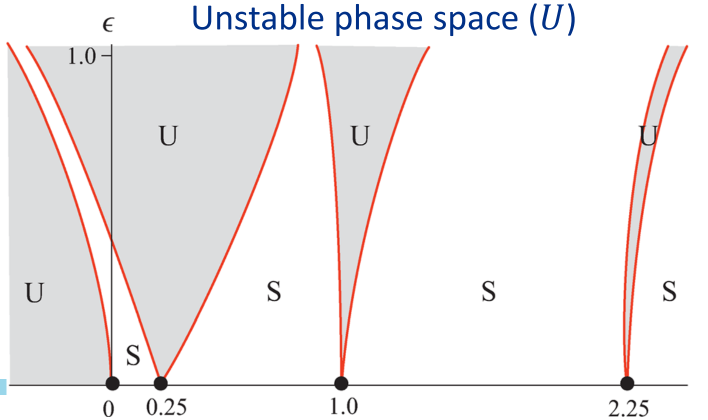
$$\frac{d^2x}{dt^2} + f(t) \cdot x = 0, f(t+T) = f(t)$$

Introduce Floquet function,

$$X(t) = e^{\mu t}p(t), \mathrm{X}(t) = \begin{bmatrix} x \\ x' \end{bmatrix}$$

For the stable condition, $\mu \to \pm i\sigma$ as we evaluated

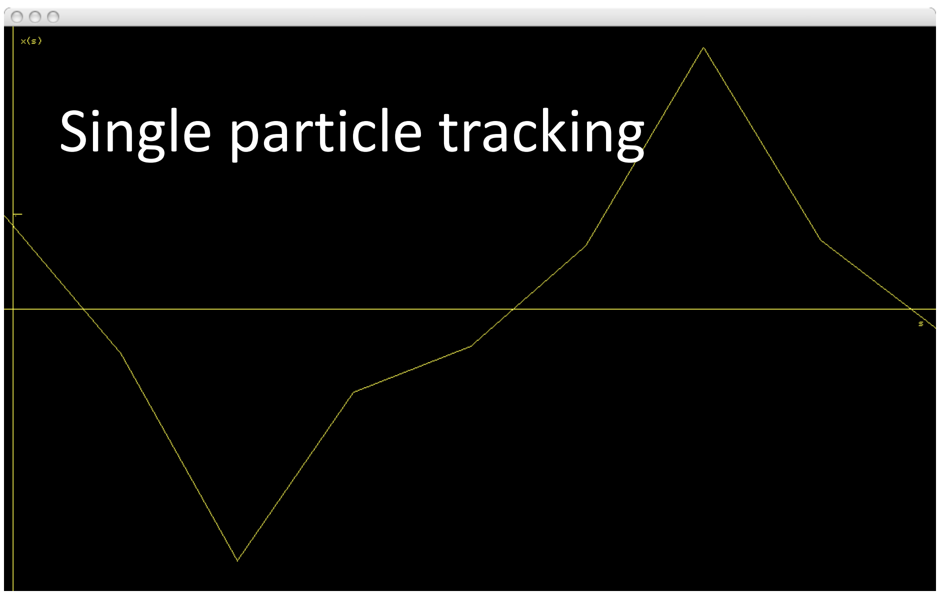
Stable phase space (S)
Unstable phase space (U)

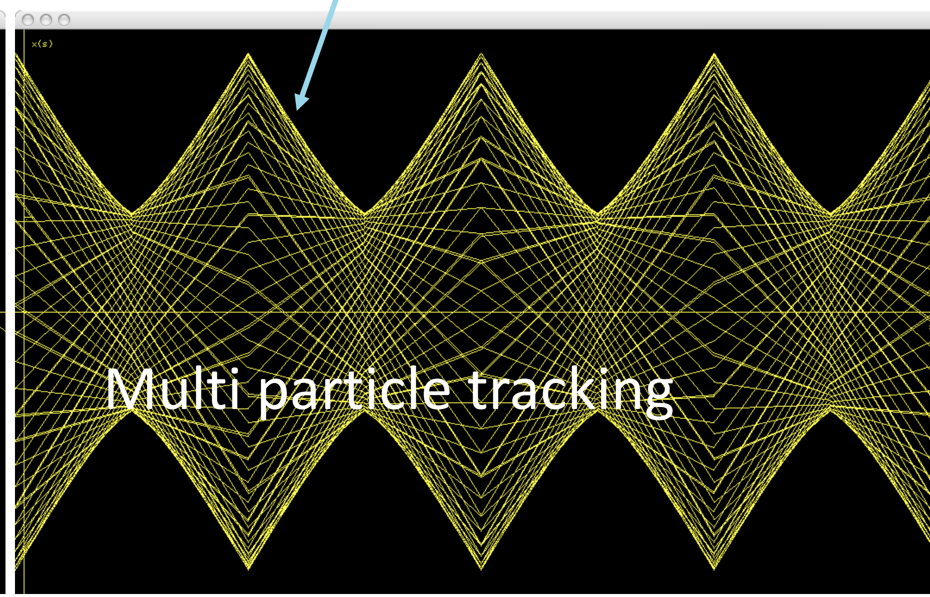


Stable phase space from Mathieu equation

Envelope equation

Here is an envelope of beam





Phase amplitude form

From Floquet's theorem,

$$u = w(s)e^{i\psi(s)}$$
$$v = w(s)e^{-i\psi(s)}$$

$$\rightarrow \left(\begin{array}{l} u(s+S) = \lambda_1 u(s) \\ v(s+S) = \dfrac{1}{\lambda_1} v(s) \end{array} \right.$$

Those are any solutions of $x(s)$ as linear combination of u and v,

$$x(s) = A \cdot w(s) cos[\psi(s) + \phi]$$

Use Wronskian, $W = uv' - vu'$ as a constant (symplectic condition),

$W = -2iw^2\psi' \equiv -2i$ (as constant)

$$\rightarrow \frac{d\psi}{ds} = \psi' = \frac{1}{w^2}$$

Therefore, Hill's equation yields the differential equation,

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

This is another forms of envelope equation

Envelope equation

Use following,

$$x = A \cdot w cos[\psi + \phi]$$

$$x' = A\left[w' cos[\psi + \phi] - w\psi' sin[\psi + \phi]\right]$$

$$\psi' = \frac{1}{w^2}$$

$$\rightarrow \frac{x^2}{w^2} + (wx' - w'x)^2 = A^2$$

By comparing,

$$\hat{\gamma}x^2 + 2\hat{\alpha}xx' + \hat{\beta}x'^2 = \varepsilon \rightarrow \left(\begin{array}{c} \hat{\beta} = w^2 \\ \hat{\alpha} = -ww' \\ \hat{\gamma} = \dfrac{1}{w^2} + w'^2 = \dfrac{1 + \hat{\alpha}^2}{\hat{\beta}} \\ \varepsilon = A^2 \end{array} \right.$$

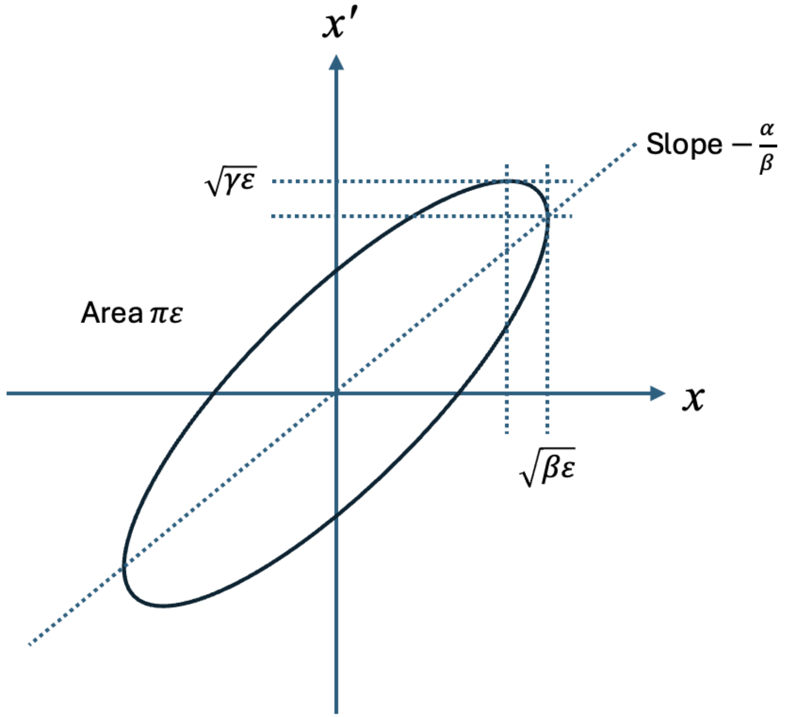
Fermilab

Envelope equation

The maximum point

$$x_m(s) = \sqrt{\varepsilon \cdot \hat{\beta}(s)} = \sqrt{\varepsilon} \cdot w(s)$$

$$\rightarrow x_m'' + \kappa x_m - \frac{\varepsilon^2}{x_m^3} = 0$$



Sometimes the matrix transformation is written by w for envelop calculation

$$\widetilde{M} = \begin{pmatrix} cos(\psi) - ww' sin(\psi) & w^2 sin(\psi) \\ -\frac{1 + w^2 w'^2}{w^2} sin(\psi) & cos(\psi) + ww' sin(\psi) \end{pmatrix}$$

$$\sigma = \int_s^{s+S} \frac{ds}{w^2} = \int_s^{s+S} \frac{ds}{\hat{\beta}}$$

Betatron tune (number of betatron oscillations per C)

$$\nu = \frac{N\sigma}{2\pi} = \frac{1}{2\pi} \int_s^{s+C} \frac{ds}{\hat{\beta}}$$

Summary

- Emittance evolution with ionization cooling
  - Longitudinal cooling is induced by dispersion, which is so called emittance exchange

$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y} \cdot \varepsilon_{n,x,y}}{\beta^2 E} \cdot \langle \frac{dE}{ds} \rangle + \frac{(13.6\ MeV)^2 \beta_{x,y}}{2 m_\mu E \beta^3 L_R},$$

$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L \cdot \varepsilon_{n,L}}{\beta^2 E} \cdot \langle \frac{dE}{ds} \rangle + \frac{\beta\gamma}{2} \hat{\beta}_L \cdot \frac{1}{\beta^2 c^2 p^2} \cdot \xi W_{max} \left(1 - \frac{\beta^2}{2}\right)$$

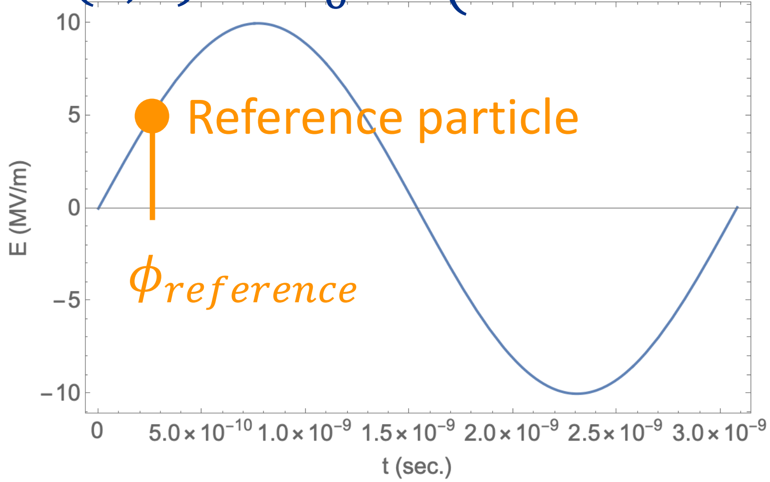
- Phase space stability
  - Stability condition, $\left|Tr\left(\widetilde{M}\right)\right| \leq 2$, where $\widetilde{M}$ is a transfer matrix
  - Stability condition is identified from Mathieu equation
- Thin lens and paraxial approximations to examine a simple transfer matrix

Longitudinal beam dynamics

- Define beam parameters related to longitudinal motion
- Find stable phase space

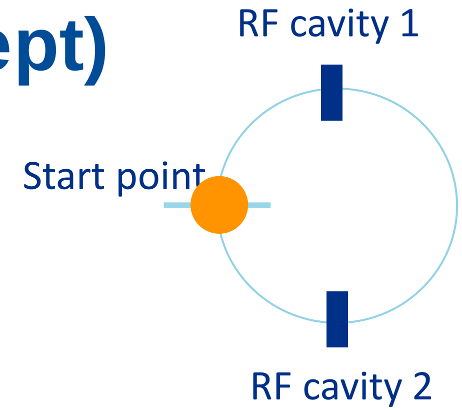
RF acceleration (I) (General concept)

$$E(s,t) = E_0 sin\left(ks + \omega t + \phi_{reference}\right)$$



RF cavity 1

s = s1

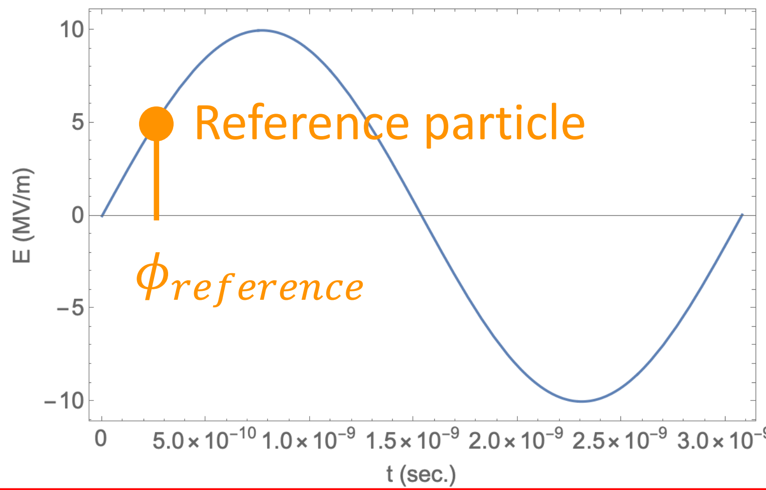


Most channel designs, the RF phase of reference particle is the same for each RF cavity.

Standing wave electric field generated in a pill-box (cylinder) metal wall resonator (detail will be discussed in later session)

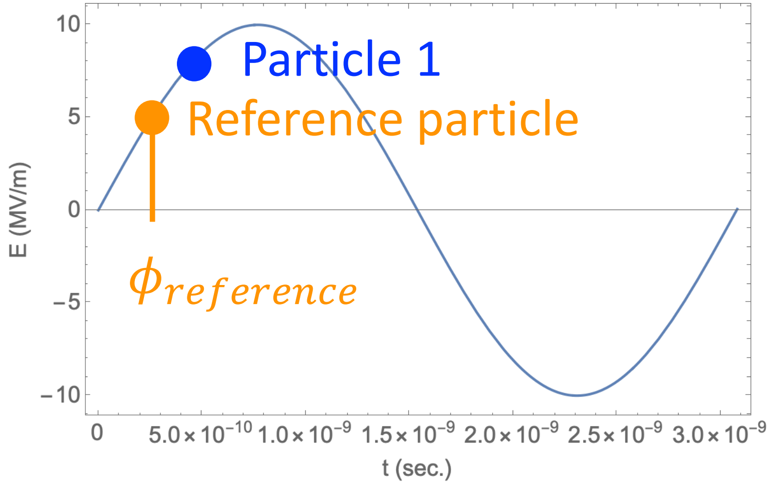
RF cavity 2

s = s1+s2



NB: In a cooling channel, the RF cavity is used to compensate Energy loss in absorber, thus the particle's kinetic energy is constant

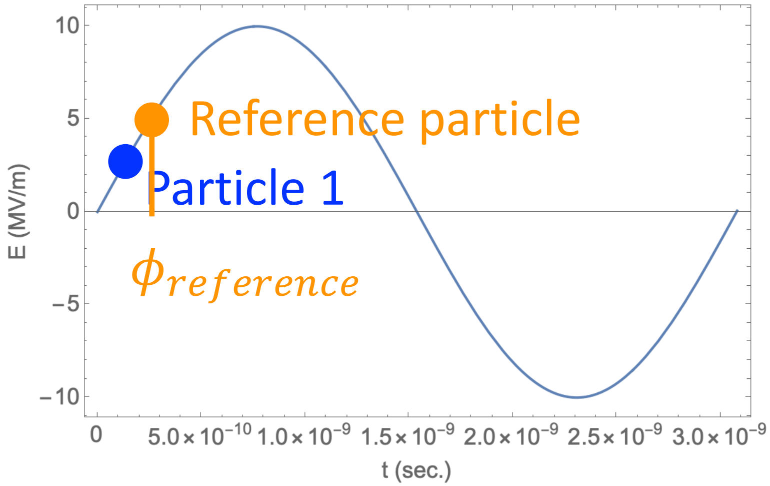
RF acceleration (II) (General concept)



RF cavity 1
s = s1

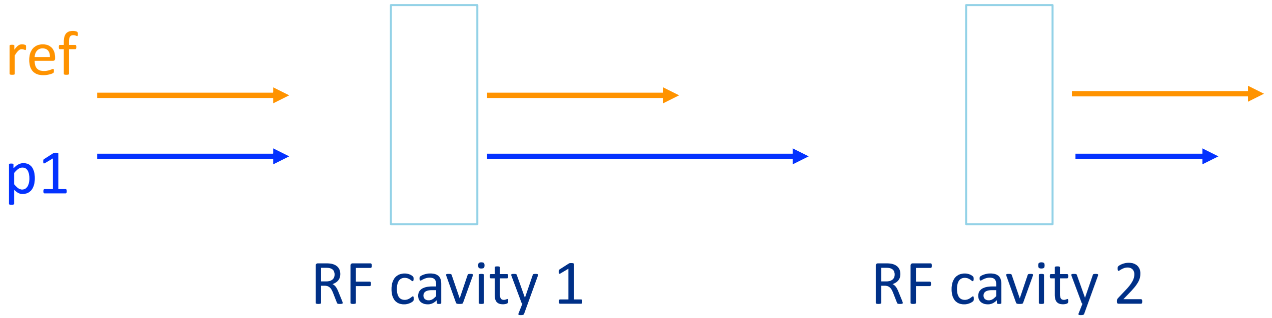
On the other hand, non-reference particles have a different RF phase at each RF cavity.

RF cavity 2
s = s1+s2



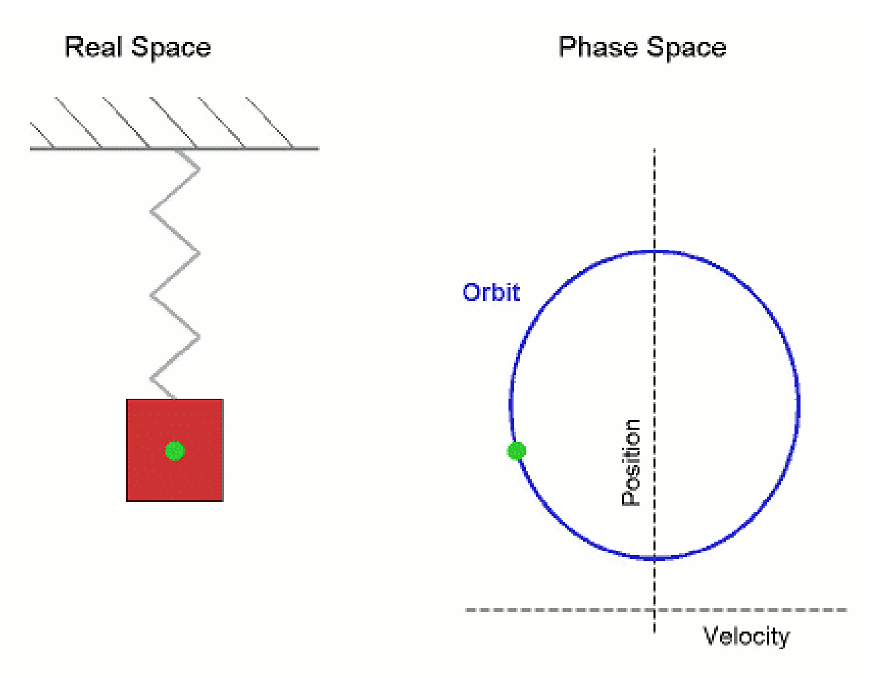
Synchrotron motion (General concept)

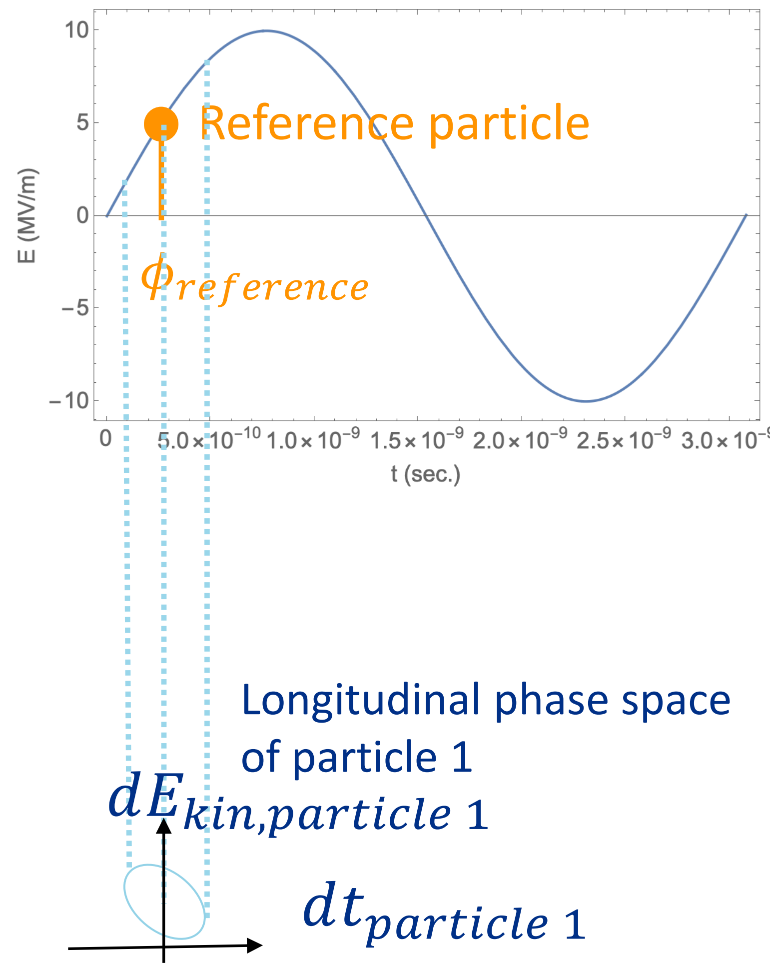
- Particle 1 gains higher energy than the reference particle at RF cavity 1 since the RF gradient of Particle 1 is higher than the reference particle
- At RF cavity 2, Particle 1 arrives earlier than the reference particle since Particle 1 has more kinetic energy at RF cavity 1, then Particle 1 gain less energy than the reference particle
- As a result, Particle 1 is oscillated around the reference particle
- This is referred as <u>synchrotron motion</u>



Analogy of Pedram (General concept)

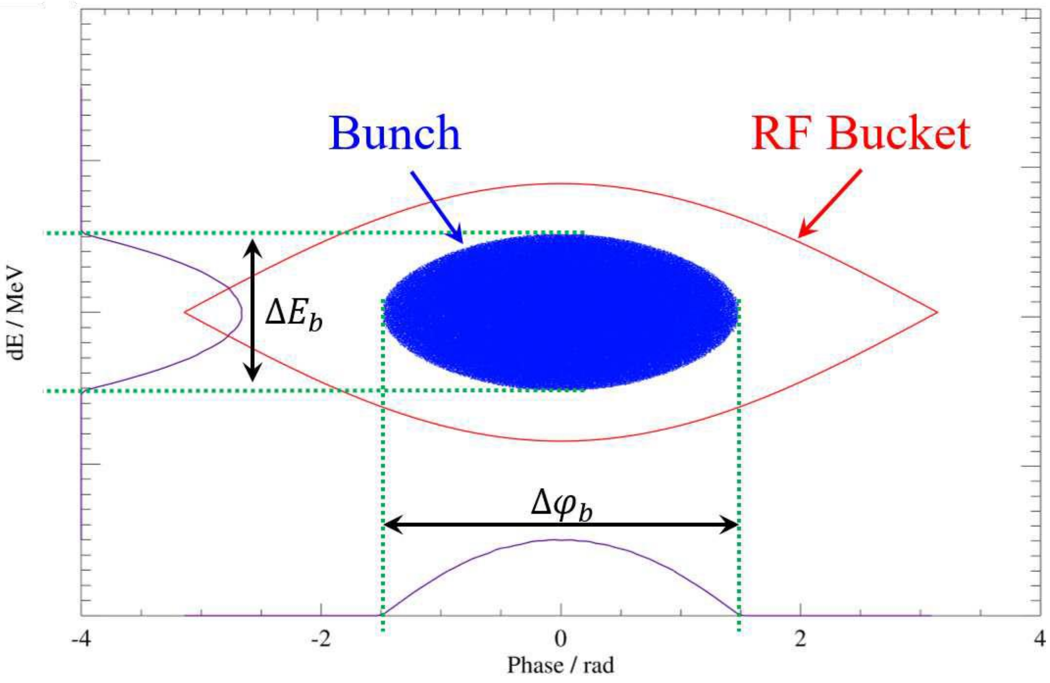
From wikipedia



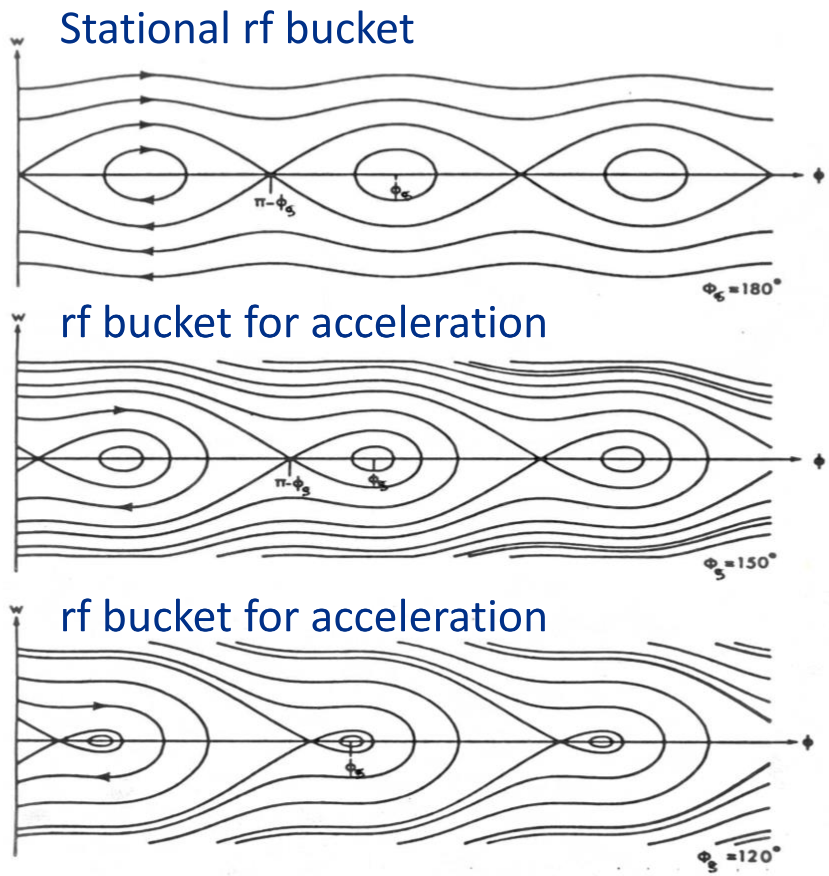


NB: Early arrival particle gains low energy while late arrival particle gains high energy

Separatrix

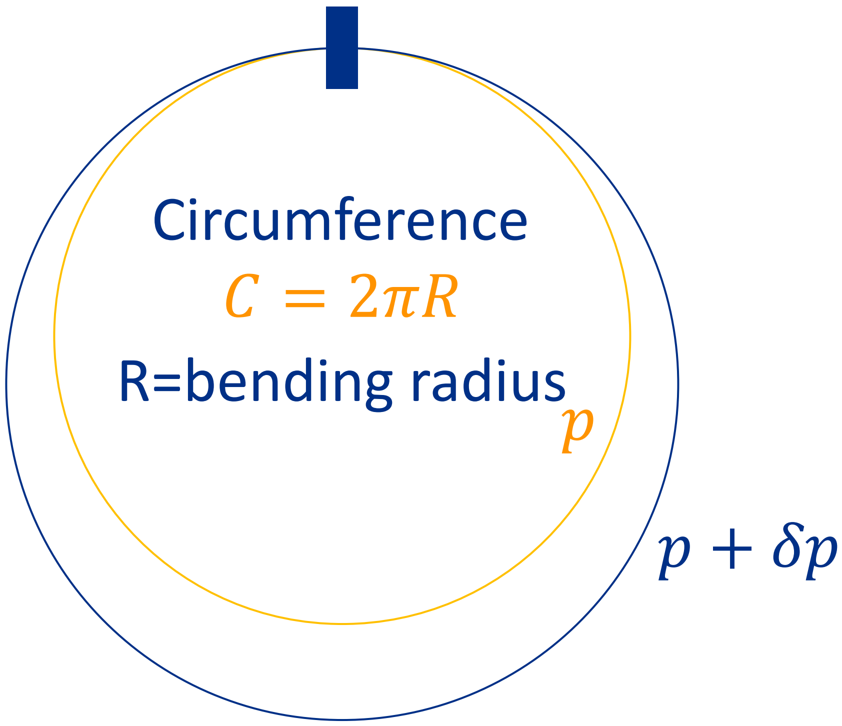


Bunch
RF Bucket
ΔE_b
$\Delta\varphi_b$
dE / MeV
Phase / rad
-4
-2
0
2
4



Stational rf bucket
rf bucket for acceleration
rf bucket for acceleration
$\phi_s = 180°$
$\phi_s = 150°$
$\phi_s = 120°$

Traveling path length variation with momentum



- Momentum compaction factor

(path length variation by p)

$$\alpha_c = \frac{p}{2\pi R}\frac{2\pi dR}{dp} = \frac{p}{R}\frac{dR}{dp} = \frac{p}{C}\frac{dC}{dp}$$

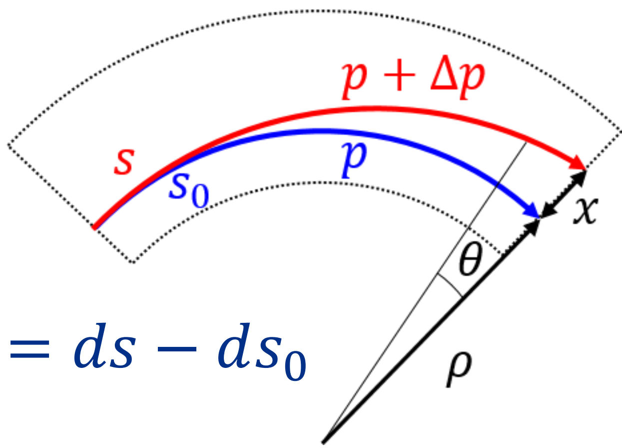
- (Phase) Slip factor

(particle revolution variation by p)

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

$$p = mv = \beta\gamma mc \rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d(1-\beta^2)^{-1/2}}{(1-\beta^2)^{-1/2}} = (1-\beta^2)^{-1}\frac{d\beta}{\beta} = \gamma^2\frac{d\beta}{\beta}$$

Dispersion, Momentum compaction and Slip factor



$$dl = ds - ds_0$$

For individual dipole, a particle position is varied by its momentum

$$ds_0 = \rho d\theta \rightarrow ds = (\rho + x)d\theta$$

Dispersion is given

$$\frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D}{\rho}\frac{dp}{p} \rightarrow D = x\frac{p}{dp}$$

(see slide 15, where $x = d\rho$)

In circular periodic motion,

$$\Delta C = \oint dl = \oint x \cdot d\theta = \oint x \cdot \frac{ds_0}{\rho} = \oint x \cdot \frac{p}{dp} \cdot \frac{dp}{p}\frac{ds_0}{\rho} = \oint D \cdot \frac{dp}{p}\frac{ds_0}{\rho}$$

$$\alpha_c = \frac{p}{C}\frac{dC}{dp} = \frac{1}{C}\oint D \cdot \frac{ds_0}{\rho} \rightarrow \frac{1}{C}\sum_i \overline{D}_i \cdot \theta_i$$

$\overline{D}_i$: Average dispersion per beam element

Dispersion, Momentum compaction and Slip factor

Number of revolutions

$$f_r = \frac{v}{2\pi R} = \frac{\beta\gamma}{2\pi R} \rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

$$p = \beta\gamma mc \rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\gamma}{\gamma} = \gamma^2 \frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p} = \frac{1}{\gamma^2}\frac{dp}{p} - \alpha_c \frac{dp}{p} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p}$$

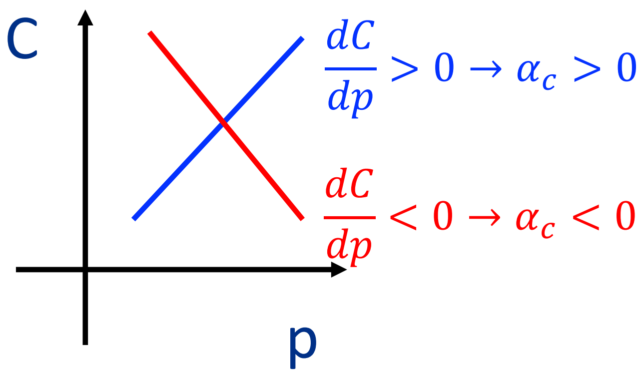
(Phase) Slip factor

$$\eta = \left(\frac{1}{\gamma^2} - \alpha_c\right)$$

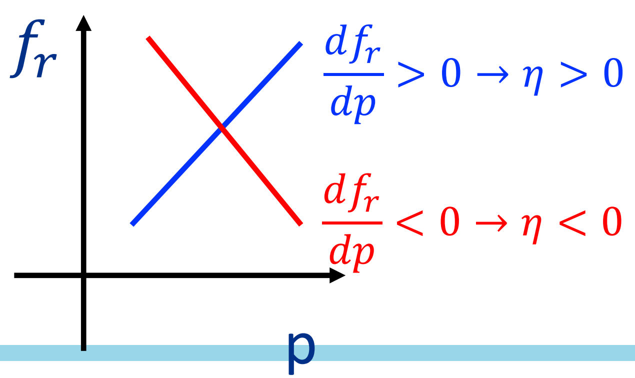
NB: We use a circulator motion to introduce the momentum compaction factor and slip factor, but those can be defined for any periodic channel, e.g. R and f_r are a Larmor radius and Larmor frequency, respectively

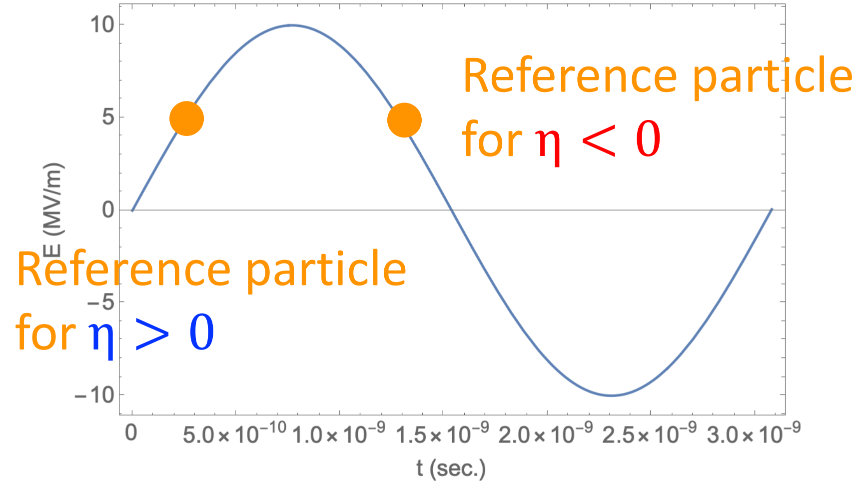
Interpret momentum compaction and slip factor

$$\alpha_c = \frac{p}{C}\frac{dC}{dp}$$



$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p} = \eta\frac{dp}{p}$$



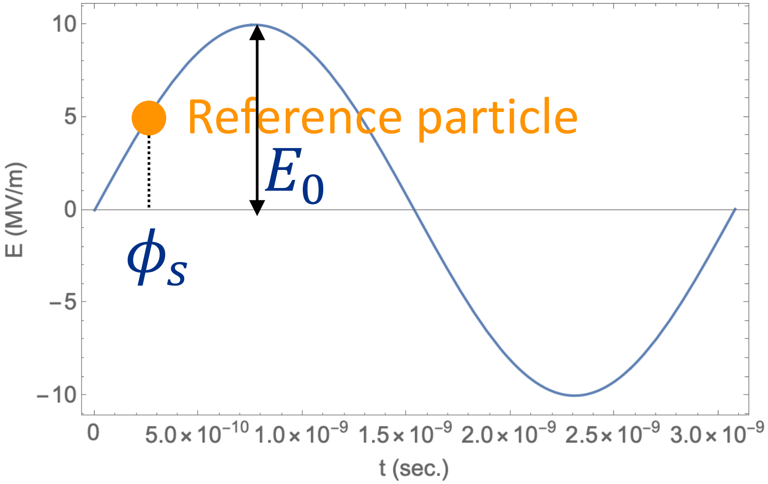


- $\eta = 0$: Transition, isochronous
- $\eta > 0$: Below transition, higher momentum particle revolute faster
- $\eta < 0$: Above transition, higher momentum particle revolve slower

Longitudinal phase space

- Energy gain for synchrotron particle (reference particle) is
  - $\frac{dE_s}{dz} = \frac{dp_s}{dt} = e \cdot E_0 sin(\phi_s)$
  - For non-reference particle
  - $\frac{dE_{kin}}{dz} = eE_0[sin(\phi_s - \phi) - sin\phi_s]$

    $\sim eE_0 cos(\phi_s) \cdot \phi$



- The rate of change of the phase is
  - $\frac{d\phi}{dz} = \omega_{rf}\left[\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right] = \omega_{rf}\left(\frac{1}{v} - \frac{1}{v_s}\right) \sim -\frac{\omega_{rf}}{v_s^2}(v - v_s)$
- Using $d\gamma = \gamma^3\beta \cdot d\beta$,
  - $\Delta E_{kin} = E - E_s = m_0c^2(\gamma - \gamma_s) = m_0c^2\gamma_s^3\beta_s \cdot d\beta = m_0\gamma_s^3 v_s(v - v_s)$
  - $\frac{d\phi}{dz} = -\frac{\omega_{rf}}{m_0 v_s^3 \gamma_s^3} \cdot \Delta E_{kin}$

Longitudinal phase space

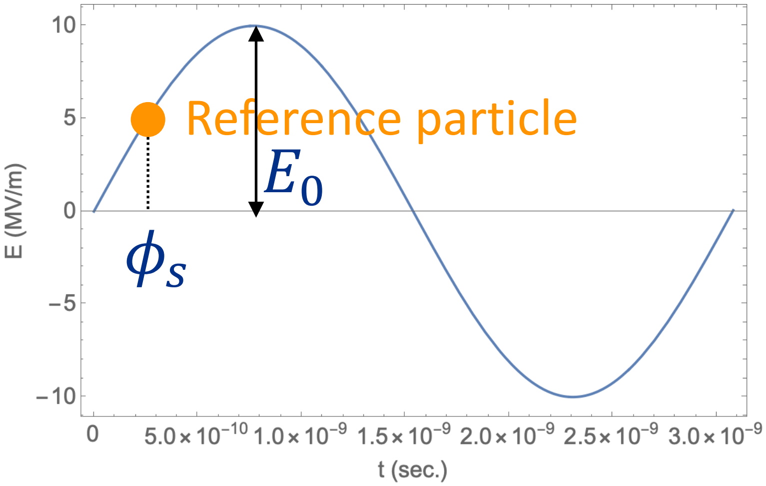
- Apply differentiation of z on both terms

$$-\frac{d^2\phi}{dz^2} = -\frac{\omega_{rf}}{m_0 v_s^3 \gamma_s^3}\frac{dE_{kin}}{dz}$$

$$= -\frac{\omega_{rf}}{m_0 v_s^3 \gamma_s^3} eE_0 cos(\phi_s) \cdot \phi$$

$$-\frac{d^2\phi}{dz^2} + \Omega_s^2 \phi = 0$$

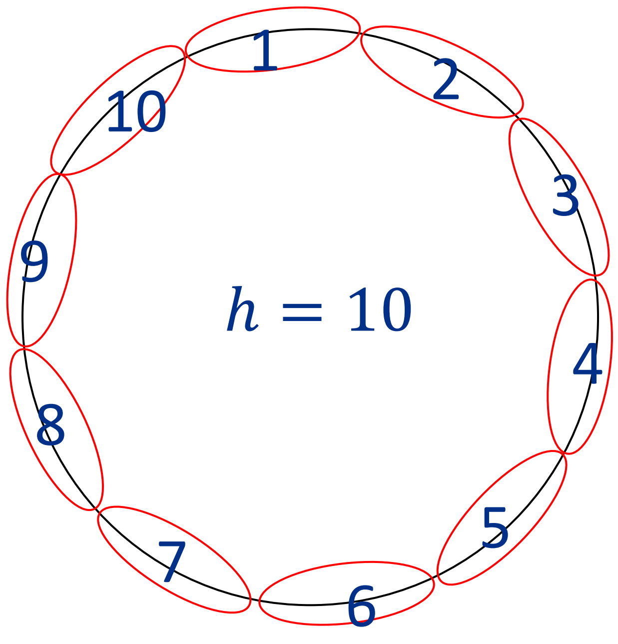
$$-\ \Omega_s^2 = \frac{\omega_{rf}}{m_0 v_s^3 \gamma_s^3} eE_0 cos(\phi_s)$$



- These show the synchrotron motion for linac where dispersion is assumed to be zero

Longitudinal phase space

- If the beam magnet and rf cavity are positioned with period, following relation must be satisfied
  - $\omega_{rf} = h \cdot \omega_r$,
    - $\omega_{rf}$is rf frequency, ω_r is a revolution frequency, and h is a harmonic number (h must be <u>integer</u>)



Accelerator ring

Red ellipse is rf bucket

Longitudinal phase space

Published by CERN in the Proceedings of the CAS-CERN Accelerator School: Advanced Accelerator Physics, Trondheim, Norway, 19–29 August 2013, edited by W. Herr, CERN–2014–009 (CERN, Geneva, 2014)

Longitudinal Beam Dynamics

F. Tecker
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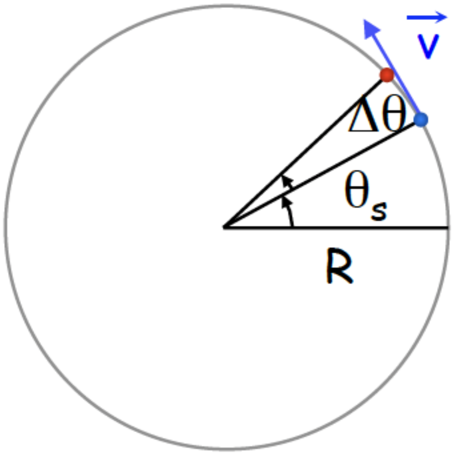
particle RF phase:	$\Delta\phi = \phi - \phi_s$,
particle momentum:	$\Delta p = p - p_s$,
particle energy:	$\Delta E = E - E_s$,
azimuth angle:	$\Delta\theta = \theta - \theta_s$.



RF phase $\Delta\phi$ changes as

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$$f_{RF} = h \cdot f_r \rightarrow \Delta\phi = -h \cdot \Delta\theta \text{ with } \theta = \int \omega_r dt$$

For a given particle with respect to the reference one, the change in angular revolution frequency is

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Longitudinal phase space

Since $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$, $E^2 = E_0^2 + p^2c^2$ and $\Delta E = v_s\Delta p = \omega_{rs}R_s\Delta p$, one gets the first order equation

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

The second first-order equation follows from the energy gain of a particle:

$$\frac{dE}{dt} = \frac{\omega_r}{2\pi} e\hat{V} sin(\phi),$$

$$\rightarrow 2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\big(sin(\phi) - sin(\phi_s)\big)$$

Fermilab

Longitudinal phase space

Therefore, we obtain following 2nd order differential equation:

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}\left(sin(\phi) - sin(\phi_s)\right) = 0$$

$$\ddot{\phi} + \frac{\Omega_s^2}{cos(\phi_s)}\left(sin(\phi) - sin(\phi_s)\right) = 0$$

$\rightarrow \ddot{\phi} + \Omega_s^2\phi = 0$ (if (ϕ) is small)

$$\Omega_s^2 = \frac{h\eta\omega_{rs}e\hat{V}cos(\phi_s)}{2\pi R_s p_s}$$

Stability condition when Ω_s is real→$\eta \cdot cos(\phi_s) > 0$

Since a cooling channel is periodic structure, we often use this synchrotron frequency and set $h = 1$

Fermilab

Longitudinal phase space

Let us extract the Hamiltonian of the longitudinal motion

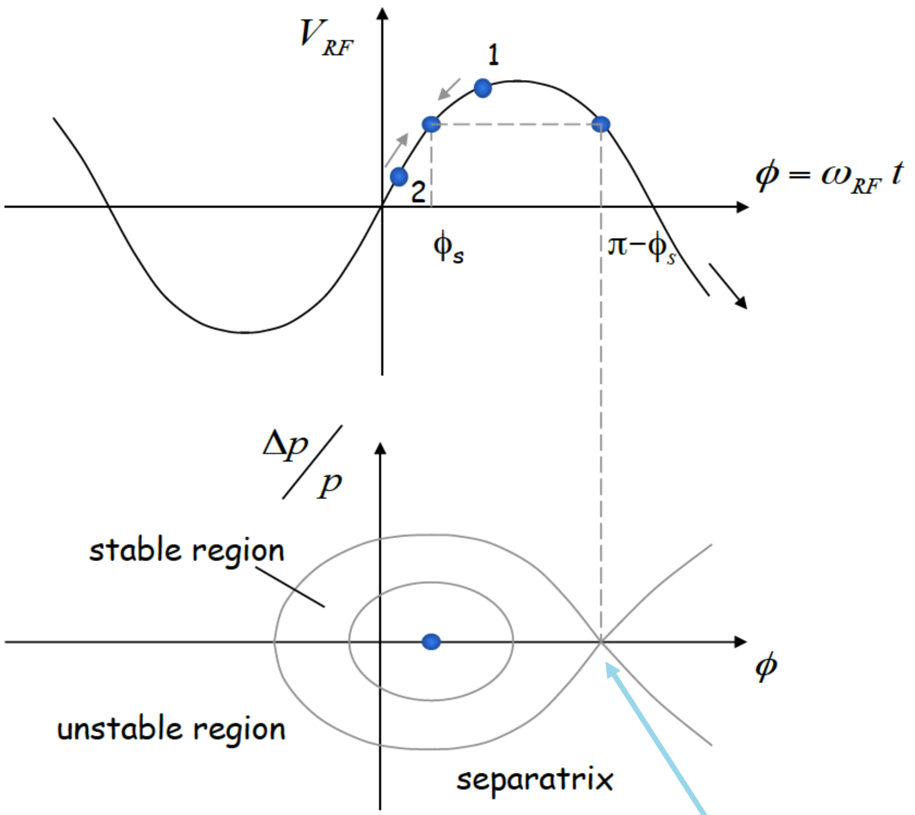
$$\ddot{\phi} \cdot \dot{\phi} + \frac{\Omega_s^2}{cos(\phi_s)}\left(\dot{\phi} \cdot sin(\phi) - \dot{\phi} \cdot sin(\phi_s)\right) = 0$$

$$\rightarrow \frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{cos(\phi_s)}\left(cos(\phi) + \phi \cdot sin(\phi_s)\right) = \mathrm{I}$$

$$\rightarrow \frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta\phi)^2}{2} = \mathrm{I}'$$

Longitudinal phase space

Hamiltonian shows the separatrix



ϕ_m

Transition point of the separatrix

Longitudinal phase space

Hamiltonian is constant at ϕ_m

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{cos(\phi_s)}\left(cos(\phi) + \phi \cdot sin(\phi_s)\right)$$
$$= -\frac{\Omega_s^2}{cos(\phi_s)}\left(cos(\pi - \phi_s) + (\pi - \phi_s) \cdot sin(\phi_s)\right)$$

Maximum phase $\ddot{\phi} = 0$ happens with $\phi = \phi_s$,

$$\frac{\dot{\phi}_{max}^2}{2} - \frac{\Omega_s^2}{cos(\phi_s)}\left(cos(\phi_s) + \phi_s \cdot sin(\phi_s)\right)$$
$$= -\frac{\Omega_s^2}{cos(\phi_s)}\left(cos(\pi - \phi_s) + (\pi - \phi_s) \cdot sin(\phi_s)\right)$$

Longitudinal phase space

Maximum phase

$$\dot{\phi}_{max}^2 = 2\Omega_s^2[2 + (2\phi_s - \pi)tan(\phi_s)]$$

Which translates into the acceptance in energy

$$\left(\frac{\Delta E}{E_s}\right)_{max} = \pm\beta\sqrt{-\frac{e\hat{V}}{\pi h\eta E_s}[2\,cos(\phi_s) + (2\phi_s - \pi)sin(\phi_s)]}$$

For a cooling channel, we set $h = 1$

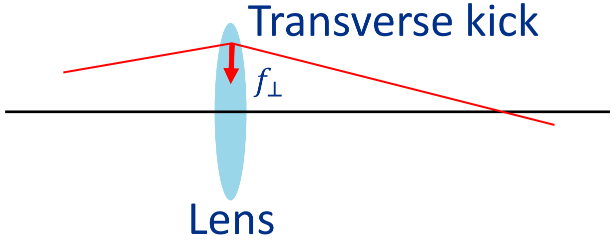
Extra slide

Thin lens & paraxial approximations

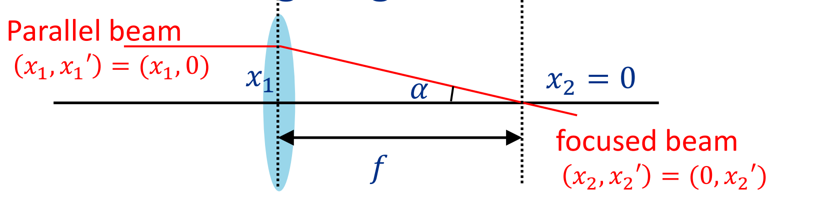
Focusing magnet functions to kick particle into transverse direction

Transverse kick is an impulse force, $f_\perp = p_\perp \cdot \delta(z)$, i.e. thin lens approximation

We also apply a paraxial approximation where p_z is constant



Note: Focusing length is defined below



Kick strength is proportional for most focusing lens

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{after\ kick=2} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{before\ kick=1}$$

$$\begin{bmatrix} 0 \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{x_1}{f} \end{bmatrix}$$

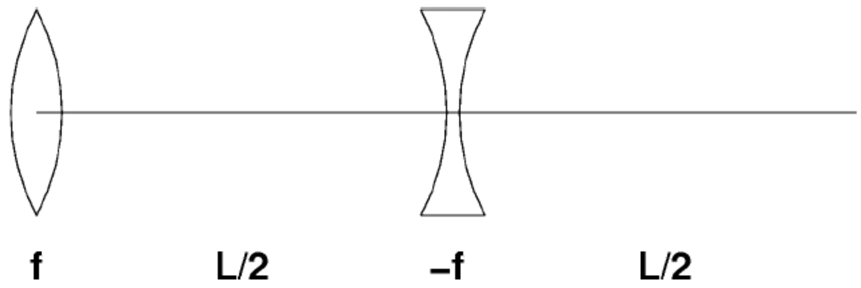
$$x_2' = -\frac{x_1}{f} = -tan\alpha$$

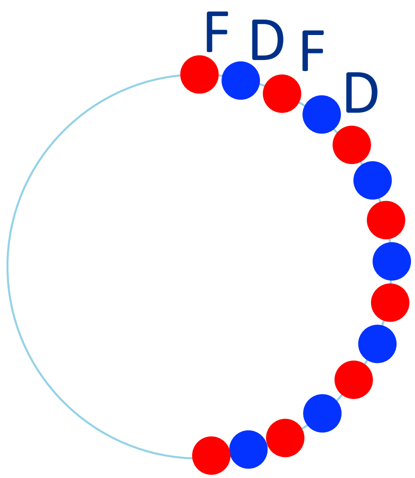
$x_2 = 1 \cdot x_1$ $\quad$ x_2: Transverse beam position does not change right after lens

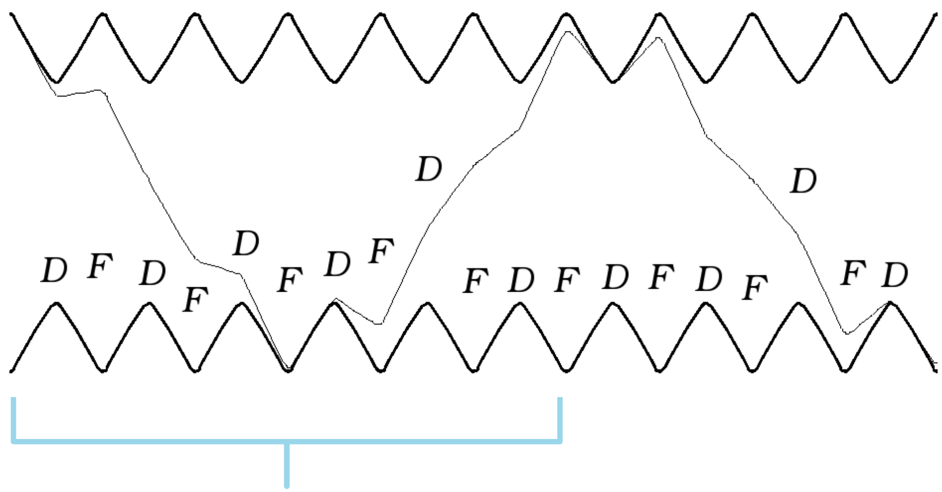
$x_2' = -\frac{1}{f} \cdot x_1 + 1 \cdot x_1'$ $\quad$ x_2': Transverse beam angle is changed by transverse kick plus initial angle

FODO cell in collider ring

FODO cell consists of Focusing-Drift-Defocusing-Drift component







6.5 cell to form one betatron period

FODO cell in collider ring

FODO cell consists of Focusing-Drift-Defocusing-Drift component

$$\widetilde{M}_{FODO} = \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{Defocus} \cdot \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{focus}$$

$$= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{bmatrix} = cos(\mu) \cdot \tilde{I} + sin(\mu) \cdot \tilde{J}$$

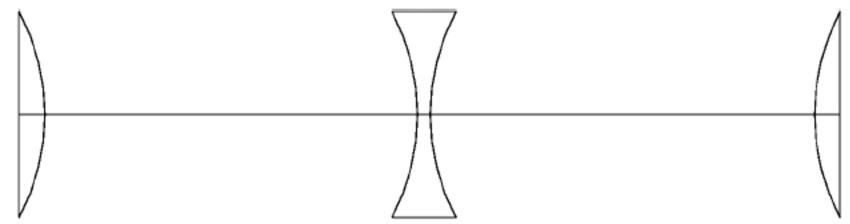
$$\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$$

Stability condition, $\left|Tr\left(\widetilde{M}\right)\right| \leq 2$

$$2 - \frac{L^2}{4f^2} = 2cos(\mu) = 2 - 4sin^2\left(\frac{\mu}{2}\right) \rightarrow sin\left(\frac{\mu}{2}\right) = \pm\frac{L}{4f}$$

FODO cell in collider ring

In the FODO cell, maximum $\hat{\beta}$ appears in the center of focusing magnet to the next center of focusing magnet,



$$\widetilde{M}_{f\ to\ f} = \widetilde{M}_{half\ focus} \cdot \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{Defocus} \cdot \widetilde{M}_{half\ drift} \cdot \widetilde{M}_{half\ focus}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ \frac{L}{4f^2}\left(\frac{L}{4f} - 1\right) & 1 - \frac{L^2}{8f^2} \end{bmatrix}$$

FODO cell in collider ring

Because maximum $\hat{\beta}$, $\hat{\alpha} = \frac{\hat{\beta}'}{2} = 0$

$$\widetilde{M}_{f\ to\ f} = \begin{bmatrix} cos(\mu) & \hat{\beta}_{max} sin(\mu) \\ -\frac{sin(\mu)}{\hat{\beta}_{max}} & cos(\mu) \end{bmatrix} = \begin{bmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ \frac{L}{4f^2}\left(\frac{L}{4f} - 1\right) & 1 - \frac{L^2}{8f^2} \end{bmatrix}$$

$$cos(\mu) = 1 - \frac{L^2}{8f^2} \rightarrow sin\left(\frac{\mu}{2}\right) = \frac{L}{4f}$$

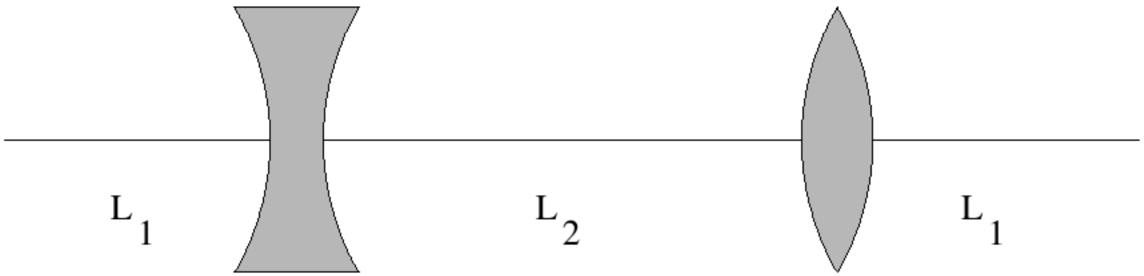
$$\hat{\beta}_{max} = \frac{L\left(1 + \frac{L}{4f}\right)}{sin(\mu)} = L\left(\frac{1 + sin\left(\frac{\mu}{2}\right)}{sin(\mu)}\right)$$

Minimum $\hat{\beta}$ occurs in defocusing-to-defocusing magnets,

$$\hat{\beta}_{min} = L\left(\frac{1 - sin\left(\frac{\mu}{2}\right)}{sin(\mu)}\right)$$

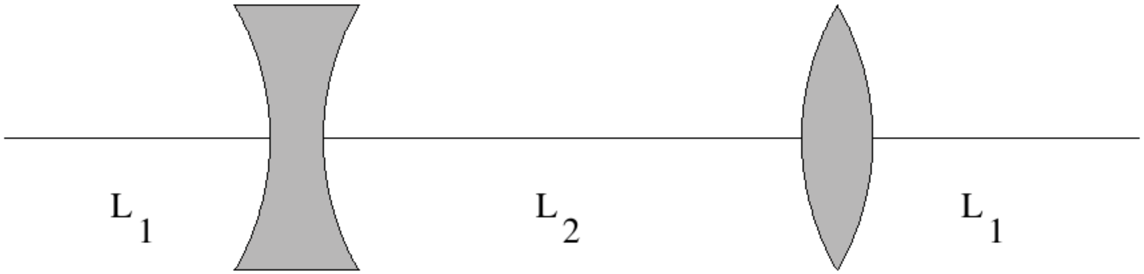
Low $\hat{\beta}$ insertion section

- Low beta insertion is designed for making a long straight section, which is needed to locate RF cavities, beam injection/extraction, and collider detectors



We would like to make a long L_2 line as a long straight section

Low $\widehat{\beta}$ insertion section



$$\widetilde{M}_{f\ to\ f} = \widetilde{M}_{L_1} \cdot \widetilde{M}_F \cdot \widetilde{M}_{L_2} \cdot \widetilde{M}_D \cdot \widetilde{M}_{L_1}$$

$$= \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L_1 L_2}{f^2} + \frac{L_2}{f} & 2L_1 + L_2 - \frac{L_1^2 L_2}{f^2} \\ -\frac{L_2}{f^2} & 1 - \frac{L_1 L_2}{f^2} - \frac{L_2}{f} \end{bmatrix}$$

Fermilab

Low $\widehat{\beta}$ insertion section

$$\begin{bmatrix} 1-\frac{L_1L_2}{f^2}+\frac{L_2}{f} & 2L_1+L_2-\frac{L_1^2L_2}{f^2} \\ -\frac{L_2}{f^2} & 1-\frac{L_1L_2}{f^2}-\frac{L_2}{f} \end{bmatrix}$$
$$= \begin{bmatrix} cos(\mu)+\hat{\alpha}sin(\mu) & \hat{\beta}sin(\mu) \\ -\hat{\gamma}sin(\mu) & cos(\mu)-\hat{\alpha}sin(\mu) \end{bmatrix}$$

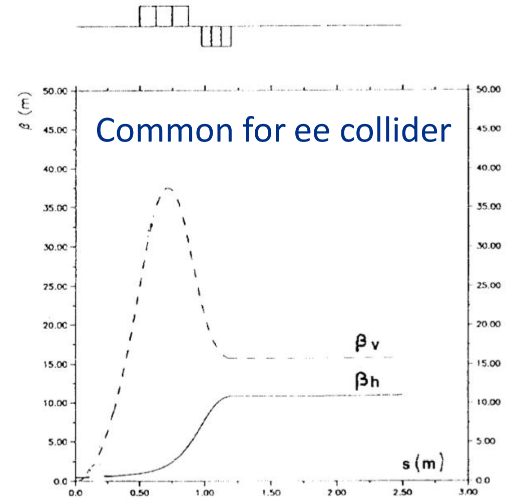
$$\frac{L_2}{f} = \hat{\alpha}sin(\mu) \rightarrow L_2 = \hat{\alpha}f sin(\mu)$$

To maximize L_2, $\mu = \frac{\pi}{2}$

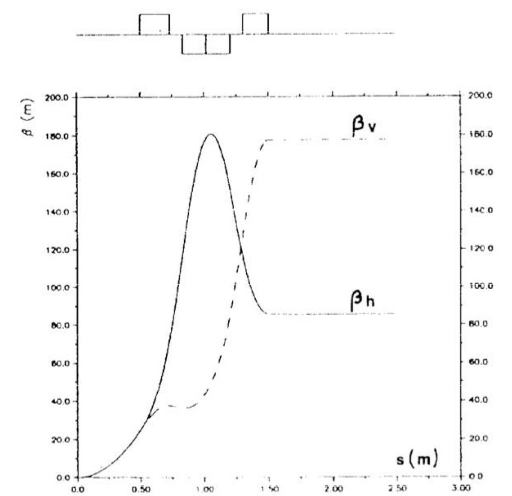
$$cos\left(\frac{\pi}{2}\right) = 0 \rightarrow 1-\frac{L_1L_2}{f^2} = 0,$$

$$f^2 = L_1L_2, \hat{\alpha} = \frac{L_2}{f}, \hat{\gamma} = \frac{L_2}{f^2}, \hat{\beta} = L_1+L_2$$

It is worth to note that the transfer matrix at a $\pi/2$ insertion is $\widetilde{M} = \tilde{J}$



igure 1. Flat-beam low-β insertion with $\beta_h^* = 50$ cm, $\beta_v^* = 1$ cm and k = 8 m-2. The distance to the first quadrupole is d = 0.5 m, the inter-quadrupole space is 0.1 m.



igure 2. Round-beam low-β insertion with $\beta_h^* = \beta_v^* = 1$ cm and k = 8 m-2. The distance to the first quadrupole is d = 0.5 m, the inter-quadrupole space is 0.1 m.

Unstable condition

If $|a + d| > 2$, we could have different condition
We will use following conditions

$$a + d = 2\cosh(\sigma)$$
$$a - d = 2\hat{\alpha}\sinh(\sigma)$$
$$b = \hat{\beta}\sinh(\sigma)$$
$$c = -\hat{\gamma}\sinh(\sigma)$$

We still have
$$Tr(\widetilde{M}) = 1 \rightarrow (1 - \hat{\alpha}^2) + \hat{\beta}\hat{\gamma} = 0$$

$$\rightarrow \widetilde{M} = \begin{bmatrix} \cosh\sigma + \hat{\alpha}\sinh(\sigma) & \hat{\beta}\sinh(\sigma) \\ -\hat{\gamma}\sinh(\sigma) & \cosh\sigma - \hat{\alpha}\sinh(\sigma) \end{bmatrix}$$

Or $\widetilde{M} = \tilde{I}\cosh\sigma + \tilde{J}\sinh\sigma$

where $\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$

Note: $\tilde{J}$ is still a symplectic matrix, i.e. $\tilde{J}^T \Omega \tilde{J} = \Omega$

Hyperbolic functions represent that particle motions are divergence along s

Harmonic oscillator

- Demonstrate analogy of harmonic oscillators
  - How the canonical transformation is useful by using an example of simple harmonic oscillator

Simple Harmonic Oscillator

Let us begin from Hamiltonian for a simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \rightarrow \left(\begin{array}{c} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \end{array} \right.$$

Of course, we can solve these from $\ddot{x} + {\omega_0}^2 x = 0$,
but x and p are coupled
We approach different way using different canonical variables

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} \rightarrow \frac{d\vec{\eta}}{dt} = \widetilde{M} \cdot \vec{\eta}$$

This shows that the matrix $\widetilde{M}$ is transforming $\dot{\vec{\eta}} \rightarrow \vec{\eta}$

Simple Harmonic Oscillator

Obtain eigenvalue of $\widetilde{M}$ via solving $det\left(\widetilde{M} - \lambda \cdot I\right) = 0$

$$\lambda^2 - \frac{1}{m} \cdot (-m\omega^2) = 0 \rightarrow \lambda = \pm i\omega$$

Corresponding eigenvector is

$$\vec{v}_\pm = \alpha \begin{bmatrix} 1 \\ \pm i m\omega \end{bmatrix}$$

We would like to convert the original canonical basis from $\left\{\vec{e}_x, \vec{e}_p\right\}$ to eigenvector basis $\left\{\vec{e}_{v_+}, \vec{e}_{v_-}\right\}$, We introduce

$$T = \{\vec{v}_+, \vec{v}_-\} = \alpha \begin{bmatrix} 1 & 1 \\ +im\omega & -im\omega \end{bmatrix}, T^{-1} = \frac{1}{2\alpha} \begin{bmatrix} 1 & -\frac{1}{im\omega} \\ 1 & \frac{1}{im\omega} \end{bmatrix}$$

Simple Harmonic Oscillator

Then the new basis is

$$\vec{a} = \begin{bmatrix} \vec{a}_- \\ \vec{a}_+ \end{bmatrix} = T^{-1} \cdot \vec{\eta} = \frac{1}{2\alpha} \begin{bmatrix} x - \frac{ip}{m\omega} \\ x + \frac{ip}{m\omega} \end{bmatrix}$$

Note $\vec{a}_- = \vec{a}_+{}^*$, complex conjugate

So, let us redefine $a = a_+, a^* = a_-$

$$2\alpha^2 a^* a = 2\alpha^2 \left(\frac{1}{2\alpha} \left(x - \frac{ip}{m\omega} \right) \cdot \frac{1}{2\alpha} \left(x + \frac{ip}{m\omega} \right) \right) = \frac{p^2}{2m^2\omega^2} + \frac{x^2}{2} = \frac{H}{m\omega^2}$$

Let us put $\alpha = \frac{1}{\sqrt{2m\omega}}$, then $H = \omega J$, new basis is

$$\begin{pmatrix} a = \sqrt{\frac{m\omega}{2}} \left(x + \frac{ip}{m\omega} \right) \\ a^* = \sqrt{\frac{m\omega}{2}} \left(x - \frac{ip}{m\omega} \right) \end{pmatrix}$$

Simple Harmonic Oscillator

The time evolution of the new basis are

$$\dot{a} = \{a, H\} = -i\omega a$$
$$\dot{a}^* = \{a^*, H\} = +i\omega a^*$$

where we use $H = \omega a a^*$ and $\{a, a^*\} = -i$

$$\rightarrow \frac{d}{dt}\begin{bmatrix} a^* \\ a \end{bmatrix} = \begin{bmatrix} +i\omega & 0 \\ 0 & -i\omega \end{bmatrix}\begin{bmatrix} a^* \\ a \end{bmatrix}$$

These can be easily solved

$$\begin{cases} a(t) = a(t_0)e^{-i\omega(t-t_0)} \\ a^*(t) = a^*(t_0)e^{-i\omega(t-t_0)} \end{cases}$$

Let us put back x and p

$$\begin{cases} x = \sqrt{\dfrac{1}{2m\omega}}(a^* + a) = \sqrt{\dfrac{2}{m\omega}}Re(a) \\ p = i\sqrt{\dfrac{1}{2m\omega}}(a^* - a) = \sqrt{2m\omega}Im(a) \end{cases}$$

Simple Harmonic Oscillator

Implement an initial condition,

$$a(t_0) = \sqrt{\frac{m\omega}{2}}\left(x_0 + i\,\frac{p_0}{m\omega}\right)$$

$$\rightarrow \left(\begin{array}{l} x(t) = x_0\cos(\omega t) + \dfrac{p_0}{m\omega}\sin(\omega t) \\ p(t) = p_0\cos(\omega t) - m\omega x_0\sin(\omega t) \end{array} \right.$$

$$\rightarrow \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \dfrac{1}{m\omega}\sin(\omega t) \\ -m\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ p_0 \end{bmatrix}$$

For 2nd order quantization (quantize field) in Quantum mechanics,

$$\{a, a^*\} = -i \rightarrow \left[\hat{a}, \hat{a}^\dagger\right] = \frac{h}{2\pi}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$ Annihilation operator

$$\widehat{N} = \hat{a}^\dagger\,\hat{a}$$

$$\hat{N}|n\rangle = n|n\rangle$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$ Creation operator