HFOFO study
Emittance evolution
and Phase stability conditions

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3/14/2025

Discussion item

- Emittance evolution in Ionization cooling
- Phase space stability
- Thin lens & paraxial approximations

Normalized emittance

$$\varepsilon_n = \beta\gamma \bullet \varepsilon$$

ε: Mechanical emittance

$\beta,\ \gamma$: Lorentz factors

Mechanical emittance (kinetic energy included)

$$\varepsilon_{x,rms} = \sqrt{\begin{vmatrix} \langle x \bullet x \rangle & \langle x \bullet x' \rangle \\ \langle x \bullet x' \rangle & \langle x' \bullet x' \rangle \end{vmatrix}} = \sqrt{Det\left[\sigma^2\right]} = \sqrt{\sigma_x^2 \bullet \sigma_{x'}^2 - \sigma_{xx'}^2}\,.$$

Normalized emittance (no change by kinetic energy)

$$\varepsilon_{x,n} = \sqrt{\begin{vmatrix} \langle x \bullet x \rangle & \left\langle x \bullet \gamma\beta_x \right\rangle \\ \left\langle x \bullet \gamma\beta_x \right\rangle & \left\langle \gamma\beta_x \bullet \gamma\beta_x \right\rangle \end{vmatrix}}$$

We extract $\hat{\beta},\ \hat{\alpha},\ \hat{\gamma}$ (Twiss parameter) from mechanical emittance

Emittance evolution (Transverse)

$$\frac{d\varepsilon_n}{ds} = \frac{d\left(\beta\gamma \bullet \varepsilon\right)}{ds} = \beta\gamma \frac{d\varepsilon}{ds} + \varepsilon \frac{d\beta\gamma}{ds}.$$

First term,

$$\beta\gamma \frac{d\varepsilon_x}{ds} = \frac{\beta\gamma}{2\varepsilon_x} \bullet \frac{d{\varepsilon_x}^2}{ds} = \frac{\beta\gamma}{2\varepsilon_x}\left(\sigma_x^2 \frac{d\sigma_{x'}^2}{ds} + \sigma_{x'}^2 \frac{d\sigma_x^2}{ds} - 2\sigma_{xx'} \frac{d\sigma_{xx'}}{ds}\right)$$

In a toy model beam transport system, there is no beam spot size variation

$$\frac{d\sigma_x^2}{ds} \sim 0.$$

We also assume that there is no coupling between x and x' during ionization cooling in this toy model

$$\frac{d\sigma_{xx'}}{ds} \sim 0.$$

We know that these assumptions are sufficiently accurate for cooling simulations

Emittance evolution (Transverse)

$$\beta\gamma \frac{d\varepsilon_x}{ds} \approx \frac{\beta\gamma}{2\varepsilon_x} \bullet \sigma_x^2 \frac{d\sigma_{x'}^2}{ds}.$$

We involve multiple scattering process,

$$\frac{\beta\gamma}{2\varepsilon_x} \bullet \sigma_x^2 \frac{d\sigma_{x'}^2}{ds} \sim \frac{\beta\gamma}{2\varepsilon_x} \bullet \hat{\beta}_x \varepsilon_x \bullet \sigma_\theta^2.$$

$$\sigma_\theta^2 \sim \left(\frac{13.8\ MeV}{\beta c p}\right)^2 \frac{z^2}{L_R}.$$

We will investigate this later

Thus, we finally gain

$$\beta\gamma \frac{d\varepsilon_x}{ds} \sim \frac{\beta\gamma}{2\varepsilon_x} \bullet \hat{\beta}_x \varepsilon_x \bullet \left(\frac{13.8\ MeV}{\beta c p}\right)^2 \frac{z^2}{L_R}.$$

Emittance evolution (Transverse)

$$\frac{d\varepsilon_n}{ds} = \frac{d\left(\beta\gamma \bullet \varepsilon\right)}{ds} = \beta\gamma\frac{d\varepsilon}{ds} + \varepsilon\frac{d\beta\gamma}{ds}.$$

Second term,

$$\varepsilon_x\frac{d\beta\gamma}{ds} = \frac{\varepsilon_x}{\beta \bullet mc^2}\frac{dE}{ds} = \frac{\beta\gamma \bullet \varepsilon_x}{\beta^2 \bullet \gamma mc^2} \bullet \frac{dE}{ds} = \frac{\varepsilon_{n,x}}{\beta^2 E} \bullet \frac{dE}{ds}.$$

Here, $\frac{dE}{ds}$ is an ionization energy loss rate, thus this amount is negative, or we insert $\frac{dE}{ds}$→- $\left(\frac{dE}{ds}\right)$

Emittance evolution (Transverse)

Final form is

$$\frac{d\varepsilon_{n,x}}{ds} = \frac{\beta\gamma}{2} \bullet \hat{\beta}_x \sigma_\theta^2 - \frac{\varepsilon_{n,x}}{\beta^2 E} \bullet \left(\frac{dE}{ds}\right).$$

The equilibrium emittance is obtained from this,

$$d\varepsilon_{n,x}/ds = 0,$$

$$\varepsilon_{n,\, x,eq} = \frac{\frac{\beta\gamma}{2}\hat{\beta}_x \sigma_\theta^2}{\beta^2 E \bullet \left(\frac{dE}{ds}\right)} = \frac{\hat{\beta}_x \left(13.6\ MeV\right)^2 \bullet z^2}{2\beta m L_R \bullet \left(\frac{dE}{ds}\right)}$$

Emittance evolution (Longitudinal)

$$\frac{d\varepsilon_n}{ds} = \frac{d\left(\beta\gamma \bullet \varepsilon\right)}{ds} = \beta\gamma \frac{d\varepsilon}{ds} + \varepsilon \frac{d\beta\gamma}{ds}.$$

First term for longitudinal,

$$\beta\gamma \frac{d\varepsilon_l}{ds} = \frac{\beta\gamma}{2\varepsilon_l} \bullet \frac{d\varepsilon_l{}^2}{ds} = \frac{\beta\gamma}{2\varepsilon_l}\left(\sigma_{\delta p/p}^2 \frac{d\sigma_{\delta t}^2}{ds} + \sigma_{\delta t}^2 \frac{d\sigma_{\delta p/p}^2}{ds} - 2\sigma_{\delta p/p\bullet\delta t} \frac{d\sigma_{\delta p/p\bullet\delta t}}{ds}\right)$$

Again, in this toy model, there is no mechanism to change time structure and no coupling between δt and $\delta p/p$

$$\frac{d\sigma_{\delta t}^2}{ds} \sim \frac{d\sigma_{\delta p/p\bullet\delta t}}{ds} \sim 0.$$

While the energy straggling effect is included,

$$\frac{\beta\gamma}{2\varepsilon_l} \bullet \sigma_{\delta p/p}^2 \frac{d\sigma_{\delta p/p}^2}{ds} \sim \frac{\beta\gamma}{2} \hat{\beta}_L \bullet \frac{1}{\beta^2 c^2 p^2} \frac{d\left(\Delta E_{rms}^2\right)}{ds}$$

$$\sim \frac{\beta\gamma}{2} \hat{\beta}_L \bullet \frac{1}{\beta^2 c^2 p^2} \bullet \xi W_{max}\left(1 - \frac{\beta^2}{2}\right)$$

$$\xi = 153.4 \frac{z^2}{\beta^2} \frac{Z}{A} \quad \text{keV g/cm}^2$$

$$W_{max} = \frac{2m_e \beta^2 \gamma^2}{1 + 2\gamma m_e/m_\mu + \left(m_e/m_\mu\right)^2}$$

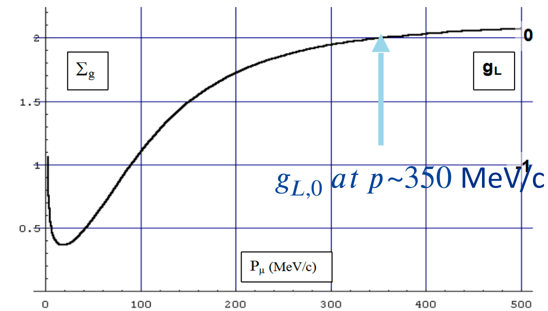
Emittance evolution (Dispersion coupling)

$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y} \bullet \varepsilon_{n,x,y}}{\beta^2 E} \bullet \langle \frac{dE}{ds} \rangle + \frac{(13.6\ MeV)^2 \beta_{x,y}}{2m_\mu E \beta^3 L_R},$$

$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L \bullet \varepsilon_{n,L}}{\beta^2 E} \bullet \langle \frac{dE}{ds} \rangle + \frac{\beta\gamma}{2} \hat{\beta}_L \bullet \frac{1}{\beta^2 c^2 p^2} \bullet \xi W_{max} \left(1 - \frac{\beta^2}{2}\right)$$

Where $g_{x,y}$ and g_L are a partition number

$$\frac{d\left(\frac{dE}{ds}\right)}{ds} = g_{L,0} = -\frac{2}{\gamma^2} + 2\frac{\left(1 - \frac{\beta^2}{\gamma^2}\right)}{\left(ln\left[\frac{2m_e c^2 \beta^2 \gamma^2}{I(Z)}\right] - \beta^2\right)}.$$



To compensate positive $g_{L,0}$, dispersion + wedge absorber

$$g_L \rightarrow g_{L,0} + \frac{D\rho'}{\rho_0}, \quad g_x \rightarrow 1 - \frac{D\rho'}{\rho_0}.$$

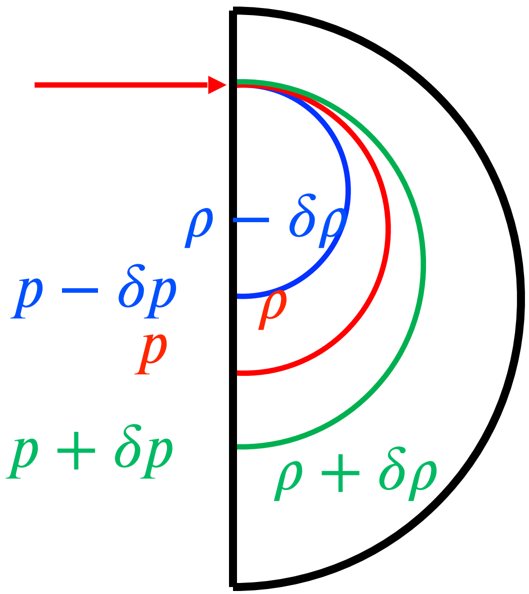
$$\Sigma_g = g_x + g_y + g_L = 2 + g_{L,\,0}\,.$$

Indeed, dispersion induces coupling between transverse and longitudinal

Emittance exchange with Dispersion

Beam position is separated by dipole magnet

Particle position is distributed by $\delta p/p$



$$m\frac{v^2}{\rho} = qv \times B \rightarrow \frac{p}{\rho} = qB$$

for uniform dipole field

$$\rightarrow \frac{p+\delta p}{\rho + \delta\rho} = qB = \mathrm{const}$$

$$\rightarrow \frac{p}{\rho}\left(1 + \frac{\delta p}{p}\right)\left(1 - \frac{\delta\rho}{\rho}\right) = \frac{p}{\rho}$$

$$\rightarrow 1 + \frac{\delta p}{p} - \frac{\delta\rho}{\rho} = 1$$

$$\rightarrow \frac{\delta p}{\delta\rho} = \frac{p}{\rho}$$

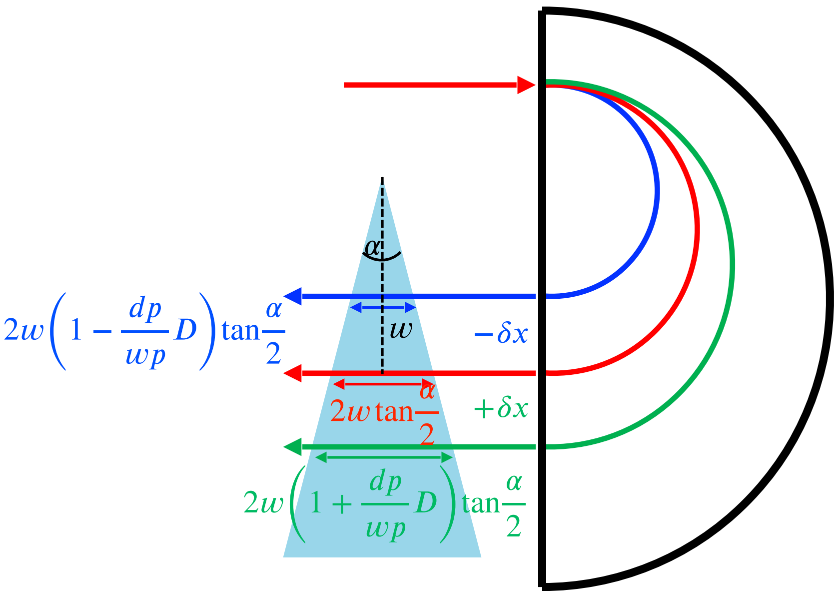
Dispersion (definition)

$$D = p\frac{d\rho}{dp}$$

in a uniform dipole field

$$D = p\frac{d\rho}{dp} = p\frac{\rho}{p} = \rho$$

Longitudinal cooling with emittance exchange



Wedge absorber

Dave's definition: $1 - D\dfrac{\rho'}{\rho}$

$$\rightarrow 1 - \frac{D}{w}\frac{dp}{p}$$

$p - \delta p$ particle traverses with shorter path length than reference in wedge absorber

p particle is a reference particle

$p - \delta p$ particle traverses with longer path length than reference in wedge absorber

δp becomes smaller after passing wedge absorber

$\rightarrow$ Longitudinal cooling

Emittance evolution

Constant term

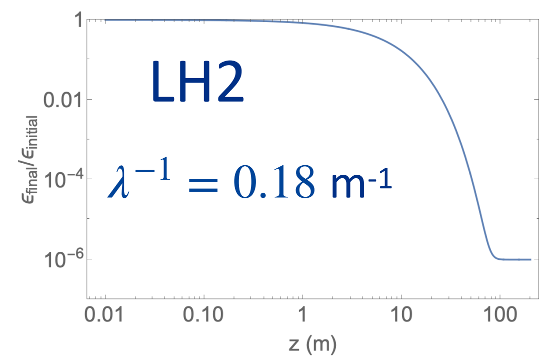
$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y} \bullet \varepsilon_{n,x,y}}{\beta^2 E} \bullet \langle \frac{dE}{ds} \rangle + \boxed{\frac{(13.6\ MeV)^2 \beta_{x,y}}{2 m_\mu E \beta^3 L_R}},$$

$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L \bullet \varepsilon_{n,L}}{\beta^2 E} \bullet \langle \frac{dE}{ds} \rangle \boxed{+ \frac{\beta\gamma}{2} \hat{\beta}_L \bullet \frac{1}{\beta^2 c^2 p^2} \bullet \xi W_{max} \left(1 - \frac{\beta^2}{2} \right)}$$

$$\frac{d\varepsilon_n}{ds} = -c_1 \bullet \varepsilon_n + c_2$$

$$\rightarrow \varepsilon_n(s) = \left(\varepsilon_{n,0} - \varepsilon_{n,eq} \right) \bullet \exp[-\lambda_{x,y,z}^{-1} \bullet s] + \varepsilon_{n,eq}$$

$$\varepsilon_{n,eq} = \frac{c_2}{c_1}$$



Plot shows that the required cooling channel length is < 100 m in LH2 if the equilibrium emittance is 10^{-6} (initial emittance is 1.0) unit is arbitrary in this discussion

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How do we treat Liouville's theorem in cooling?

- Regarding Liouville's theorem, the phase space area remains constant if the theorem holds
  - $\nabla \bullet \left(n\vec{v}\right) + \frac{\partial n}{\partial t} = 0$ where $n\left(q, p\right)$ is a number density and $\vec{v} = \left\{\dot{q}, \dot{p}\right\}$ denotes phase space velocity (continuity equation)
  - The validation of symplectic integration is a necessary and sufficient condition for ensuring phase space conservation
- In this sense, the theorem is not strictly satisfied in the cooling channel
- However, if the phase space evolution is slow, theorem holds approximately
  - This significantly reduces complexity in cooling concepts
- This approximation may breakdown in a specific cooling channel

Beam dynamics and stability condition

Solution of Hill's equation $x''(s) + \kappa(s)x(s) = 0$ is in general

$$x(s) = ax_0 + bx_0'$$
$$x'(s) = c\,x_0 + dx_0'$$

Equation of motion can be solved if initial condition is known

As we usually do for the harmonic oscillation,

$$X \rightarrow \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix} \rightarrow \tilde{M} \bullet X_0$$

$$Tr\left(\tilde{M}\right) = a + d$$
$$det\left(\tilde{M}\right) = ad - bd$$

Extracting eigenvalues from $\tilde{M}$,

$$det\left(\tilde{M} - \lambda I\right) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \rightarrow \lambda^2 - Tr\left(\tilde{M}\right)\lambda + det\left(\tilde{M}\right) = 0$$

Beam dynamics and stability condition

If there is no energy change, $\dfrac{dH}{ds} = 0,$

$$det\left(\tilde{M}\right) = ad - bd = 1$$

$$\rightarrow \lambda^2 - (a+d) + 1 = 0 \rightarrow \lambda^2 - Tr\left(\tilde{M}\right)\lambda + 1 = 0$$

$\tilde{M}^T \Omega \tilde{M} = \Omega$

$\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\rightarrow ad - bc = 1$ (if energy conservation is required, then using symplectic integration is a sufficient condition)

Solution of this equation is

$$\lambda = \frac{Tr\left(\tilde{M}\right) \pm \sqrt{Tr\left(\tilde{M}\right)^2 - 4}}{2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2}$$

We notice if $\cos(\sigma) = \dfrac{1}{2} Tr\left(\tilde{M}\right) = \dfrac{(a+d)}{2},$

$$\frac{\pm\sqrt{(a+d)^2 - 4}}{2} = \frac{(a+d)}{2} \pm \sqrt{\frac{(a+d)^2}{4} - 1} \rightarrow \lambda_{1,2} = \cos(\sigma) \pm \sqrt{cos^2(\sigma) - 1} = \cos(\sigma) \pm$$

Beam dynamics and stability condition

If $\left|a+d\right| \leq 2,\ \lambda_{1,2}$ is a periodic solution→Beam motion is stable

For our convenience, we use following relations,

$$a+d = 2\cos(\sigma)$$
$$a-d = 2\hat{\alpha}\sin(\sigma)$$
$$b = \hat{\beta}\sin(\sigma)$$
$$c = -\hat{\gamma}\sin(\sigma)$$

$$\rightarrow \tilde{M} = \begin{bmatrix} \cos\sigma + \hat{\alpha}\sin\sigma & \hat{\beta}\sin\sigma \\ -\hat{\gamma}\sin\sigma & \cos\sigma - \hat{\alpha}\sin\sigma \end{bmatrix}$$

Or $\tilde{M} = \tilde{I}\cos\sigma + \tilde{J}\sin\sigma$

where $\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$

$$det\left(\tilde{M}\right) = ad - bc$$
$$= \left(\cos\sigma + \hat{\alpha}\sin\sigma\right) \bullet \left(\cos\sigma - \hat{\alpha}\sin\sigma\right) - \hat{\beta}\sin\sigma \bullet \left(-\hat{\gamma}\sin\sigma\right)$$
$$= \hat{\beta}\hat{\gamma} - \hat{\alpha}^2 = 1$$

These relations are still within a canonical framework!

Fermilab

Beam dynamics and stability condition

Investigate $\tilde{J}$

$$\left(\tilde{J}\right)^2 = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix} \bullet \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\tilde{I}$$

$$\left(\tilde{M}\right)^2 = \left(\tilde{I}\cos\sigma + \tilde{J}\sin\sigma\right)^2$$
$$= \tilde{I}^2 cos^2\sigma + 2\tilde{I} \bullet \tilde{J} \bullet sin\sigma \bullet cos\sigma + \tilde{J}^2 sin^2\sigma$$
$$= \tilde{I}\left(cos^2\sigma - sin^2\sigma\right) + 2\tilde{J} sin\sigma \bullet cos\sigma$$
$$= \tilde{I} cos2\sigma + \tilde{J} sin2\sigma$$

$$\left(\tilde{M}\right)^3 = \left(\tilde{M}\right)^2 \bullet \tilde{M}$$
$$= \left(\tilde{I}cos2\sigma + \tilde{J}sin2\sigma\right) \bullet \left(\tilde{I}cos\sigma + \tilde{J}sin\sigma\right)$$
$$= \tilde{I}^2 cos2\sigma \bullet cos\sigma + \tilde{J}^2 sin2\sigma \bullet sin\sigma$$
$$+ \tilde{I}\tilde{J}cos2\sigma \bullet sin\sigma + \tilde{I}\tilde{J}sin2\sigma \bullet cos\sigma$$
$$cos2\sigma \bullet cos\sigma = \frac{1}{2}(cos3\sigma + cos\sigma)\ldots$$
$$= \tilde{I}cos3\sigma + \tilde{J}sin3\sigma$$

$$\rightarrow \left(\tilde{M}\right)^N = \left(\tilde{I}\cos\sigma + \tilde{J}\sin\sigma\right)^N = \tilde{I}cosN\sigma + \tilde{J}sinN\sigma$$

$$\lambda_{1,2} = \cos(\sigma) \pm i\sin(\sigma) = e^{\pm i\sigma} \rightarrow \lambda_1 = \frac{1}{\lambda_2}$$

This statement is the same general solution from a parametric oscillator, especially it is called Floquet's theorem

Floquet function

Floquet's theorem is applicable to linear differential equations with periodic coefficients

$$\frac{dX}{dt} = A(t)X$$

where $A(t+T) = A(t)$, T is a period

A general solution will be

This is $\lambda_{1,2}$ in our case

$$X(t) = e^{\mu t} p(t)$$

where $\mathrm{p}(t+T) = p(t)$, $p(t)$ is called Floquet function, μ is the Floquet exponent

As we notice, it is the same solution as a forced oscillator

$\rightarrow$ μ is pure imaginary, then $X(t)$ is a stable oscillation

$\rightarrow$ μ contains real, then $X(t)$ is a forced oscillation, growing or decaying exponentially over time

Mathieu equation with Floquet's theorem

As you may have noticed, our process is the reverse of the conventional forced harmonic oscillator problem. The general equation of forced harmonic oscillator in the beam dynamics is so called Mathieu equation,

$$\frac{d^2x}{dt^2} + \left(\delta + \epsilon \bullet cos(t)\right)x = 0$$

δ: stiffness parameter

ϵ: no specific name given

In the case $\epsilon = 0$, it is a condition to obtain Hill's equation

$$\frac{d^2x}{dt^2} + f(t) \bullet x = 0,\ f(t+T) = f(t)$$

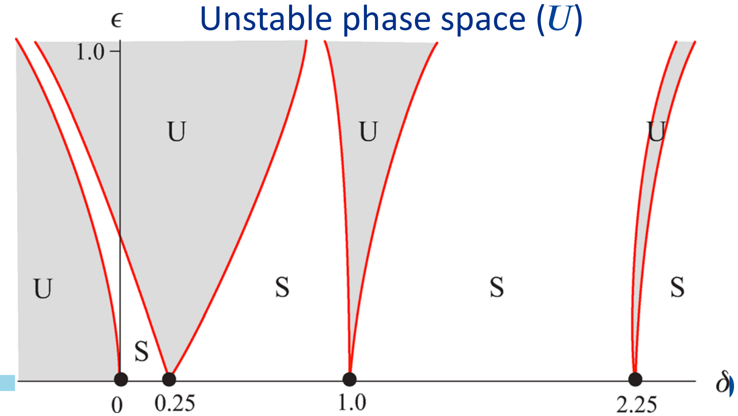
Introduce Floquet function,

$$X(t) = e^{\mu t}p(t),\ \mathrm{X}(t) = \begin{bmatrix} x \\ x' \end{bmatrix}$$

For the stable condition, $\mu \to \pm i\sigma$ as we evaluated

Stable phase space (S)

Unstable phase space (U)



Stable phase space from Mathieu equation

Phase amplitude form

From Floquet's theorem,

$$u = w(s)e^{i\psi(s)}$$
$$v = w(s)e^{-i\psi(s)}$$
$$\rightarrow \begin{cases} u(s+S) = \lambda_1 u(s) \\ v(s+S) = \frac{1}{\lambda_1} v(s) \end{cases}$$

Those are any solutions of $x(s)$ as linear combination of u and v,

$$x(s) = A \bullet w(s) cos\left[\psi(s) + \phi\right]$$

Use Wronskian, $W = uv' - v'u$ as a constant (symplectic condition),

$W = -2iw^2\psi' \equiv -2i$ (as constant)

$$\rightarrow \frac{d\psi}{ds} = \psi' = \frac{1}{w^2}$$

Therefore, Hill's equation yields the differential equation,

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

This is another forms of envelope equation

Envelope equation

Use following,

$$x = A \bullet w cos\left[\psi + \phi\right]$$
$$x' = A\left[w' cos\left[\psi + \phi\right] - w\psi' sin\left[\psi + \phi\right]\right]$$
$$\rightarrow \frac{x^2}{w^2} + (wx' - w'x)^2 = A^2$$

$$\psi' = \frac{1}{w^2}$$

By comparing,

$$\hat{\gamma} x^2 + 2\hat{\alpha} x x' + \hat{\beta} x'^2 = \varepsilon \rightarrow \begin{cases} \hat{\beta} = w^2 \\ \hat{\alpha} = -ww' \\ \hat{\gamma} = \frac{1}{w^2} + w'^2 = \frac{1+\hat{\alpha}^2}{\hat{\beta}} \\ \varepsilon = A^2 \end{cases}$$

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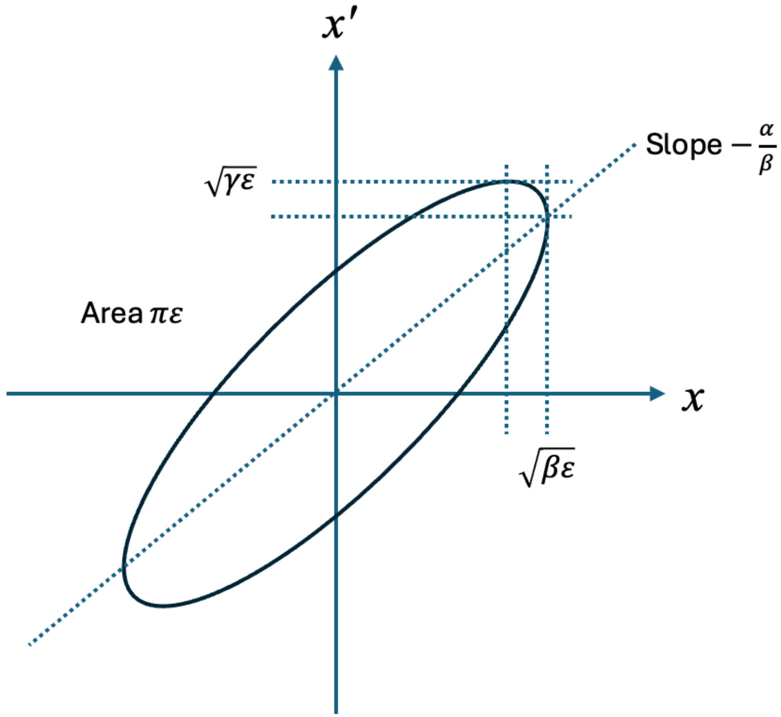
Envelope equation

The maximum point

$$x_m(s) = \sqrt{\varepsilon \bullet \hat{\beta}(s)} = \sqrt{\varepsilon} \bullet w(s)$$

$$\rightarrow x_m'' + \kappa x_m - \frac{\varepsilon^2}{x_m^3} = 0$$

This is another forms of envelope equation



Sometimes the matrix transformation is written by w for envelop calculation

$$\tilde{M} = \begin{pmatrix} cos(\psi) - ww' sin(\psi) & w^2 sin(\psi) \\ -\frac{1 + w^2 w'^2}{w^2} sin(\psi) & cos(\psi) + ww' sin(\psi) \end{pmatrix}$$

$$\sigma = \int_s^{s+S} \frac{ds}{w^2} = \int_s^{s+S} \frac{ds}{\hat{\beta}}$$

Betatron tune (number of betatron oscillations per C)

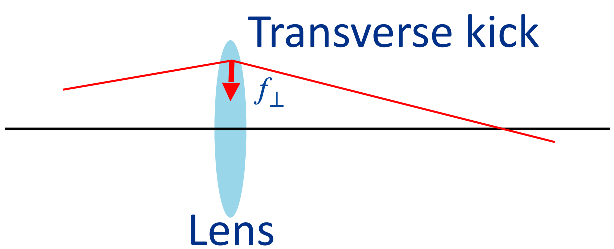
$$\nu = \frac{N\sigma}{2\pi} = \frac{1}{2\pi} \int^{s+C} \frac{ds}{\hat{\beta}}$$

Thin lens & paraxial approximations

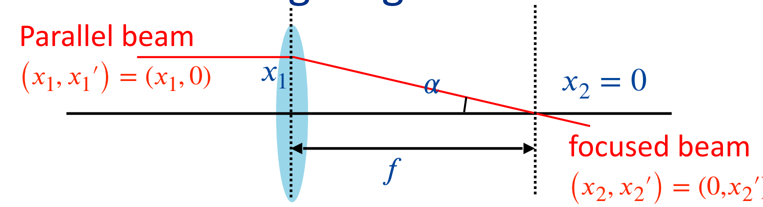
Focusing magnet functions to kick particle into transverse direction

Transverse kick is an impulse force, $f_\perp = p_\perp \bullet \delta(z)$, i.e. thin lens approximation

We also apply a paraxial approximation where p_z is constant



Note: Focusing length is defined below



Kick strength is proportional for most focusing lens

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{after\ kick=2} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ x' \end{bmatrix}_{before\ kick=1}$$

$$\begin{bmatrix} 0 \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{x_1}{f} \end{bmatrix}$$

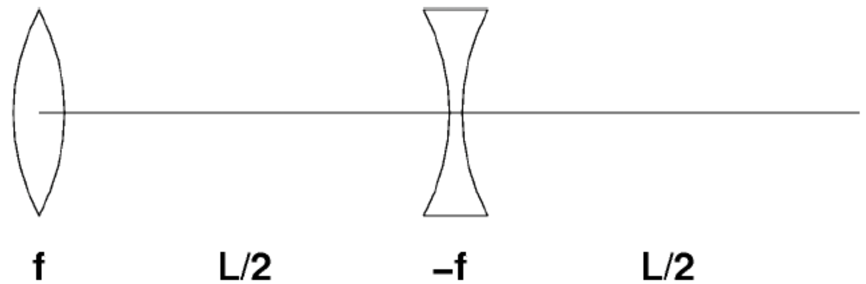
$$x_2' = -\frac{x_1}{f} = -\tan\alpha$$

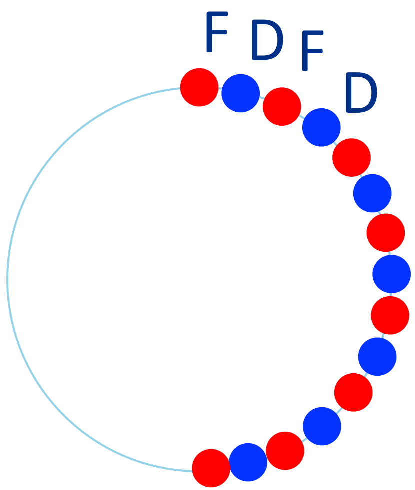
$x_2 = 1 \bullet x_1$ $\quad$ x_2: Transverse beam position does not change right after lens

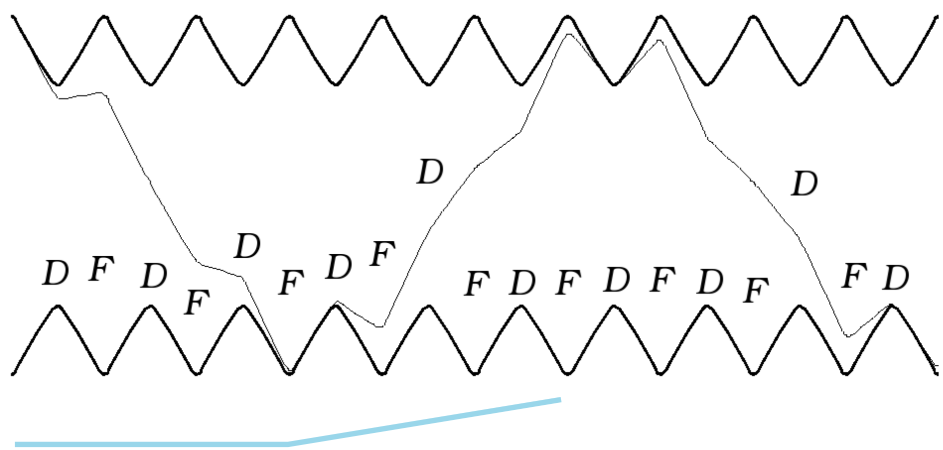
$x_2' = -\frac{1}{f} \bullet x_1 + 1 \bullet x_1'$ $\quad$ x_2': Transverse beam angle is changed by transverse kick plus initial angle

FODO cell in collider ring

FODO cell consists of Focusing-Drift-Defocusing-Drift component







6.5 cell to form one betatron period

FODO cell in collider ring

FODO cell consists of Focusing-Drift-Defocusing-Drift component

$$\tilde{M}_{FODO} = \tilde{M}_{half\,drift} \bullet \tilde{M}_{Defocus} \bullet \tilde{M}_{half\,drift} \bullet \tilde{M}_{focus}$$

$$= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{bmatrix} = cos(\mu) \bullet \tilde{I} + sin(\mu) \bullet \tilde{J}$$

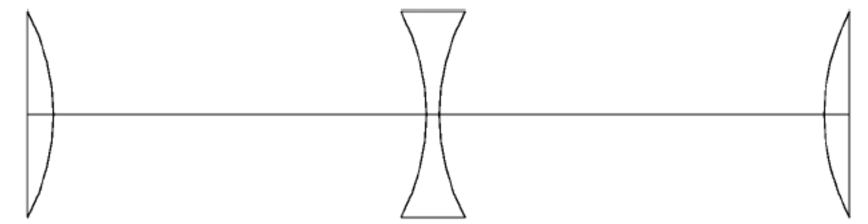
$$\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \tilde{J} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\gamma} & -\hat{\alpha} \end{bmatrix}$$

Stability condition, $\left| Tr\left(\tilde{M}\right) \right| \leq 2$

$$2 - \frac{L^2}{4f^2} = 2cos(\mu) = 2 - 4sin^2\left(\frac{\mu}{2}\right) \rightarrow sin\left(\frac{\mu}{2}\right) = \pm\frac{L}{4f}$$

FODO cell in collider ring

In the FODO cell, maximum $\hat{\beta}$ appears in the center of focusing magnet to the next center of focusing magnet,



$$\tilde{M}_{f\ to\ f} = \tilde{M}_{half\ focus} \bullet \tilde{M}_{half\ drift} \bullet \tilde{M}_{Defocus} \bullet \tilde{M}_{half\ drift} \bullet \tilde{M}_{half\ focus}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ \frac{L}{4f^2}\left(\frac{L}{4f} - 1\right) & 1 - \frac{L^2}{8f^2} \end{bmatrix}$$

FODO cell in collider ring

Because maximum $\hat{\beta}$, $\hat{\alpha} = \frac{\hat{\beta}'}{2} = 0$

$$\tilde{M}_{f\ to\ f} = \begin{bmatrix} cos(\mu) & \hat{\beta}_{max} sin(\mu) \\ -\frac{sin(\mu)}{\hat{\beta}_{max}} & cos(\mu) \end{bmatrix} = \begin{bmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ \frac{L}{4f^2}\left(\frac{L}{4f} - 1\right) & 1 - \frac{L^2}{8f^2} \end{bmatrix}$$

$$cos(\mu) = 1 - \frac{L^2}{8f^2} \rightarrow sin\left(\frac{\mu}{2}\right) = \frac{L}{4f}$$

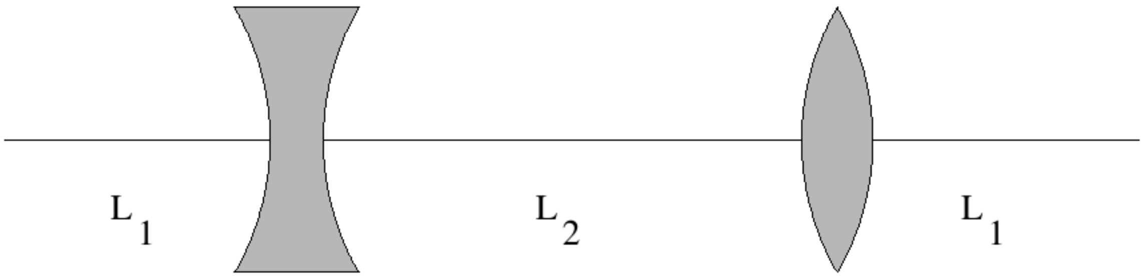
$$\hat{\beta}_{max} = \frac{L\left(1 + \frac{L}{4f}\right)}{sin(\mu)} = L\left(\frac{1 + sin\left(\frac{\mu}{2}\right)}{sin(\mu)}\right)$$

Minimum $\hat{\beta}$ occurs in defocusing-to-defocusing magnets,

$$\hat{\beta}_{min} = L\left(\frac{1 - sin\left(\frac{\mu}{2}\right)}{sin(\mu)}\right)$$

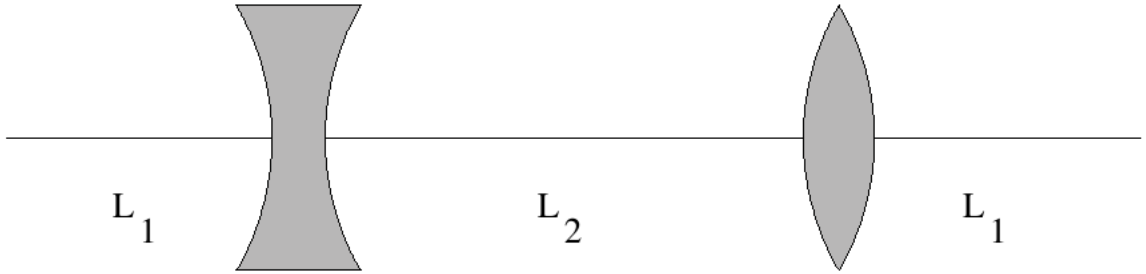
Low $\hat{\beta}$ insertion section

- Low beta insertion is designed for making a long straight section, which is needed to locate RF cavities, beam injection/extraction, and collider detectors



We would like to make a long L_2 line as a long straight section

Low $\hat{\beta}$ insertion section



$$\tilde{M}_{f\ to\ f} = \tilde{M}_{L_1} \bullet \tilde{M}_F \bullet \tilde{M}_{L_2} \bullet \tilde{M}_D \bullet \tilde{M}_{L_1}$$

$$= \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L_1 L_2}{f^2} + \frac{L_2}{f} & 2L_1 + L_2 - \frac{L_1^2 L_2}{f^2} \\ -\frac{L_2}{f^2} & 1 - \frac{L_1 L_2}{f^2} - \frac{L_2}{f} \end{bmatrix}$$

Low $\hat{\beta}$ insertion section

$$\begin{bmatrix} 1-\frac{L_1L_2}{f^2}+\frac{L_2}{f} & 2L_1+L_2-\frac{L_1^2L_2}{f^2} \\ -\frac{L_2}{f^2} & 1-\frac{L_1L_2}{f^2}-\frac{L_2}{f} \end{bmatrix}$$

$$= \begin{bmatrix} cos(\mu)+\hat{\alpha}sin(\mu) & \hat{\beta}sin(\mu) \\ -\hat{\gamma}sin(\mu) & cos(\mu)-\hat{\alpha}sin(\mu) \end{bmatrix}$$

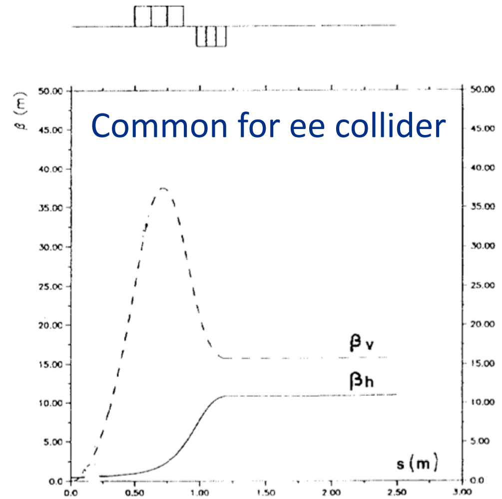
$$\frac{L_2}{f} = \hat{\alpha}sin(\mu) \rightarrow L_2 = \hat{\alpha}fsin(\mu)$$

To maximize L_2, $\mu = \frac{\pi}{2}$

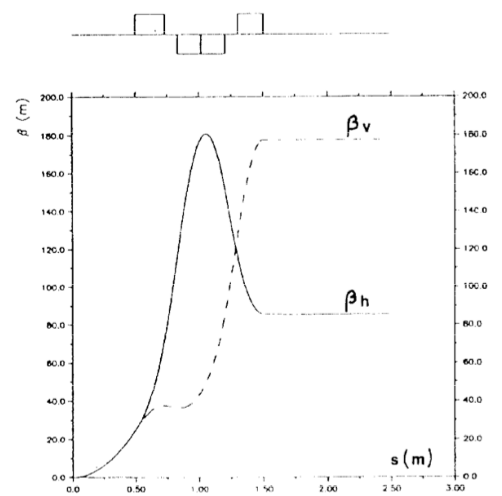
$$cos\left(\frac{\pi}{2}\right) = 0 \rightarrow 1 - \frac{L_1L_2}{f^2} = 0,$$

$$f^2 = L_1L_2,\ \hat{\alpha} = \frac{L_2}{f},\ \hat{\gamma} = \frac{L_2}{f^2},\ \hat{\beta} = L_1 + L_2$$

It is worth to note that the transfer matrix at a $\pi/2$ insertion is $\tilde{M} = \tilde{J}$



igure 1. Flat-beam low-β insertion with $\beta_h^* = 50$ cm, $\beta_v^* = 1$ cm and k = 8 m-2. The distance to the first quadrupole is d = 0.5 m, the inter-quadrupole space is 0.1 m.



igure 2. Round-beam low-β insertion with $\beta_h^* = \beta_v^* = 1$ cm and k = 8 m-2. The distance to the first quadrupole is d = 0.5 m, the inter-quadrupole space is 0.1 m.

Summary

- Emittance evolution with ionization cooling
  - Longitudinal cooling is induced by dispersion, which is so called emittance exchange

$$\frac{d\varepsilon_{n,x,y}}{ds} = -\frac{g_{x,y} \bullet \varepsilon_{n,x,y}}{\beta^2 E} \bullet \langle\frac{dE}{ds}\rangle + \frac{\left(13.6\ MeV\right)^2 \beta_{x,y}}{2m_\mu E\beta^3 L_R},$$

$$\frac{d\varepsilon_{n,L}}{ds} = -\frac{g_L \bullet \varepsilon_{n,L}}{\beta^2 E} \bullet \langle\frac{dE}{ds}\rangle + \frac{\beta\gamma}{2}\hat{\beta}_L \bullet \frac{1}{\beta^2 c^2 p^2} \bullet \xi W_{max}\left(1 - \frac{\beta^2}{2}\right)$$

- Phase space stability
  - Stability condition, $\left|Tr\left(\tilde{M}\right)\right| \leq 2$, where $\tilde{M}$ is a transfer matrix
  - Stability condition is identified from Mathieu equation
- Thin lens and paraxial approximations to examine a simple transfer matrix

Fermilab

Extra slide

Unstable condition

If $\left|a+d\right|>2$, we could have different condition

We will use following conditions

$$a+d=2\cosh(\sigma)$$
$$a-d=2\hat{\alpha}\sinh(\sigma)$$
$$b=\hat{\beta}\sinh(\sigma)$$
$$c=-\hat{\gamma}\sinh(\sigma)$$

We still have

$$Tr\left(\tilde{M}\right)=1\rightarrow\left(1-\hat{\alpha}^2\right)+\hat{\beta}\hat{\gamma}=0$$

$$\rightarrow\tilde{M}=\begin{bmatrix}\cosh\sigma+\hat{\alpha}\sinh(\sigma) & \hat{\beta}\sinh(\sigma)\\ -\hat{\gamma}\sinh(\sigma) & \cosh\sigma-\hat{\alpha}\sinh(\sigma)\end{bmatrix}$$

Or $\tilde{M}=\tilde{I}\cosh\sigma+\tilde{J}\sinh\sigma$

where $\tilde{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tilde{J}=\begin{bmatrix}\hat{\alpha} & \hat{\beta}\\ -\hat{\gamma} & -\hat{\alpha}\end{bmatrix}$

Note: $\tilde{J}$ is still a symplectic matrix, i.e. $\tilde{J}^T\Omega\tilde{J}=\Omega$

Hyperbolic functions represent that particle motions are divergence along s

Harmonic oscillator

- Demonstrate analogy of harmonic oscillators
  - How the canonical transformation is useful by using an example of simple harmonic oscillator

Simple Harmonic Oscillator

Let us begin from Hamiltonian for a simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \rightarrow \begin{pmatrix} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \end{pmatrix}$$

Of course, we can solve these from $\ddot{x} + {\omega_0}^2 x = 0$,

but x and p are coupled

We approach different way using different canonical variables

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix} \bullet \begin{bmatrix} x \\ p \end{bmatrix} \rightarrow \frac{d\vec{\eta}}{dt} = \tilde{M} \bullet \vec{\eta}$$

This shows that the matrix $\tilde{M}$ is transforming $\vec{\dot{\eta}} \rightarrow \vec{\eta}$

Simple Harmonic Oscillator

Obtain eigenvalue of $\tilde{M}$ via solving $det\left(\tilde{M} - \lambda \bullet I\right) = 0$

$$\lambda^2 - \frac{1}{m} \bullet \left(-m\omega^2\right) = 0 \rightarrow \lambda = \pm i\omega$$

Corresponding eigenvector is

$$\vec{v}_{\pm} = \alpha \begin{bmatrix} 1 \\ \pm im\omega \end{bmatrix}$$

We would like to convert the original canonical basis from $\left\{\vec{e}_x, \vec{e}_p\right\}$ to eigenvector basis $\left\{\vec{e}_{v_+}, \vec{e}_{v_-}\right\}$, We introduce

$$T = \left\{\vec{v}_+, \vec{v}_-\right\} = \alpha \begin{bmatrix} 1 & 1 \\ +im\omega & -im\omega \end{bmatrix}, T^{-1} = \frac{1}{2\alpha} \begin{bmatrix} 1 & -\frac{1}{im\omega} \\ 1 & \frac{1}{im\omega} \end{bmatrix}$$

Simple Harmonic Oscillator

Then the new basis is

$$\vec{a} = \begin{bmatrix} \vec{a}_- \\ \vec{a}_+ \end{bmatrix} = T^{-1} \bullet \vec{\eta} = \frac{1}{2\alpha} \begin{bmatrix} x - \frac{ip}{m\omega} \\ x + \frac{ip}{m\omega} \end{bmatrix}$$

Note $\vec{a}_- = \vec{a}_+^{\,*}$, complex conjugate

So, let us redefine $a = a_+,\ a^* = a_-$

$$2\alpha^2 a^* a = 2\alpha^2 \left(\frac{1}{2\alpha}\left(x - \frac{ip}{m\omega}\right) \bullet \frac{1}{2\alpha}\left(x + \frac{ip}{m\omega}\right) \right) = \frac{p^2}{2m^2\omega^2} + \frac{x^2}{2} = \frac{H}{m\omega^2}$$

Let us put $\alpha = \frac{1}{\sqrt{2m\omega}}$, then $H = \omega J,$ new basis is

$$\begin{cases} a = \sqrt{\frac{m\omega}{2}}\left(x + \frac{ip}{m\omega}\right) \\ a^* = \sqrt{\frac{m\omega}{2}}\left(x - \frac{ip}{m\omega}\right) \end{cases}$$

Simple Harmonic Oscillator

The time evolution of the new basis are

$$\dot{a} = \{a, H\} = -\,i\omega a$$
$$\dot{a}^* = \left\{a^*, H\right\} = +\,i\omega a^*$$

where we use $H = \omega a a^*$ and $\left\{a, a^*\right\} = -\,i$

$$\rightarrow \frac{d}{dt}\begin{bmatrix} a^* \\ a \end{bmatrix} = \begin{bmatrix} +i\omega & 0 \\ 0 & -i\omega \end{bmatrix}\begin{bmatrix} a^* \\ a \end{bmatrix}$$

These can be easily solved

$$\begin{cases} a(t) = a\left(t_0\right)e^{-i\omega\left(t\,-\,t_0\right)} \\ a^*(t) = a^*\left(t_0\right)e^{-i\omega\left(t\,-\,t_0\right)} \end{cases}$$

Let us put back x and p

$$\begin{cases} x = \sqrt{\frac{1}{2m\omega}}\left(a^* + a\right) = \sqrt{\frac{2}{m\omega}}Re(a) \\ p = i\sqrt{\frac{1}{2m\omega}}\left(a^* - a\right) = \sqrt{2m\omega}Im(a) \end{cases}$$

Simple Harmonic Oscillator

Implement an initial condition,

$$a(t_0) = \sqrt{\frac{m\omega}{2}}\left(x_0 + i\frac{p_0}{m\omega}\right)$$

$$\rightarrow \begin{pmatrix} x(t) = x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t) \\ p(t) = p_0\cos(\omega t) - m\omega x_0\sin(\omega t) \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \frac{1}{m\omega}\sin(\omega t) \\ -m\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \bullet \begin{bmatrix} x_0 \\ p_0 \end{bmatrix}$$

For 2nd order quantization (quantize field) in Quantum mechanics,

$$\{a, a^*\} = -i \rightarrow \left[\hat{a}, \hat{a}^\dagger\right] = \frac{h}{2\pi}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$ Annihilation operator

$$\hat{N} = \hat{a}^\dagger \hat{a}$$

$$\hat{N}|n\rangle = n|n\rangle$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$ Creation operator