Fermilab U.S. DEPARTMENT OF ENERGY Office of Science

HFOFO study
Solenoid base cooling channel

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Contents

- Basic of solenoid-based beam transport system
- Solenoid-based cooling channel

Betatron tune (from last lecture)

Historically, the betatron tune is defined in a circular beam optics,

$$\nu = \frac{N\sigma}{2\pi} = \frac{1}{2\pi}\int_{s}^{s+C}\frac{ds}{\hat{\beta}}$$

C: circumference

- Tune is an important concept to prevent particle loss in the ring caused by a resonance (see extra slide for a ring design)
  - Also, we often distinguish between "weak" focusing and "strong" focusing by referring tune values
- However, I am always wondering how we define betatron tune in a linear accelerator
  - Sometimes, we extract a tune using Fourier transformation and "revolution"
  - In that sense, we can define the tune in HFOFO
  - According to Yuri, HFOFO has $\nu \geq 1$ in 6 solenoid coils, which is a weak focusing

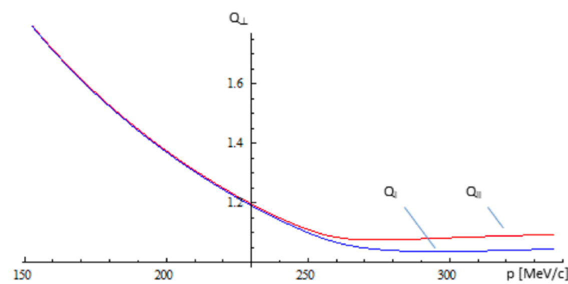


Figure 3. Betatron tunes as function of muon momentum.

Scope of this lecture

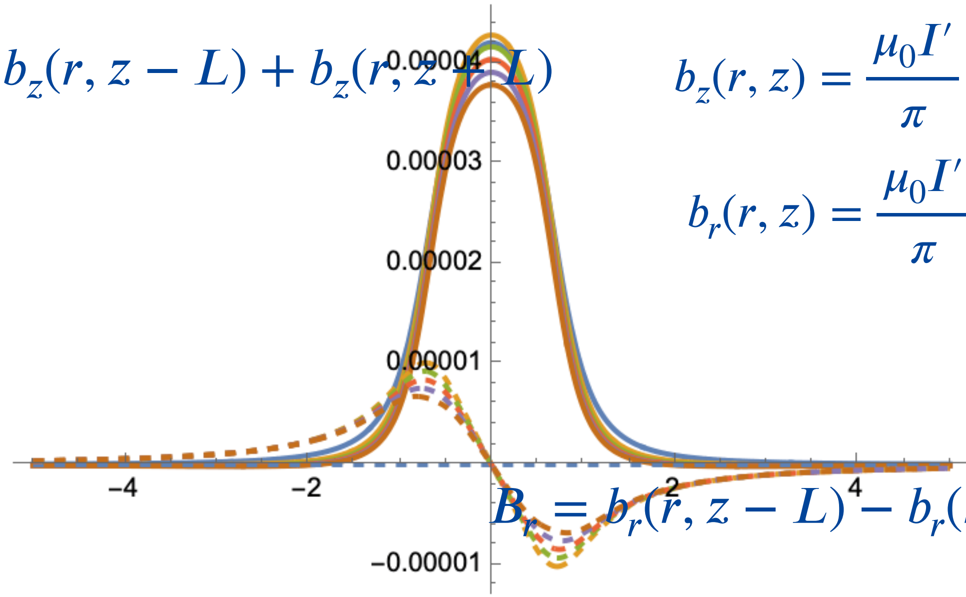
- A comprehensive analysis has been done for solenoid-based cooling channels (so called FOFO channel) and published
  - K.J. Kim and C.Wang, PRL85, 760 (2000)
  - G. Penn and J. Wurtele, PRL85, 764 (2000)
  - G. Dugan, PRAB4, 104001 (2001)
- I will introduce a fundamental part of a solenoid focusing magnet
- Then I will introduce individual cooling channel
  - FOFO using Fernow and Stratakis papers
  - Helical channel using Derbenev paper
  - HFOFO, no theoretical paper published yet

How do we design cooling channel?

- Start from a cooling concept by either forming Hamiltonian or directly deriving equation of motion
- Identify beta functions and dispersion
  - Identify magnetic field of proposed channel by solving Maxwell equations
    - Often start from vector potential (with a current configuration) or scaler potential (solve Laplacian with boundary conditions)
  - Solve Hill's equation with perturbation term if needed
    - It often becomes Mathieu equation, apply Floquet's theorem to solve the equation
  - Find canonical transfer matrix
    - Check stability condition $\left| Tr\left(\tilde{M}\right) \right| \leq 2$
  - Tune $\hat{\beta}$ and D (and a_p: Momentum compaction factor)

Magnetic field in solenoid magnet

$${}_z = -b_z(r, z-L) + b_z(r, z+L)$$



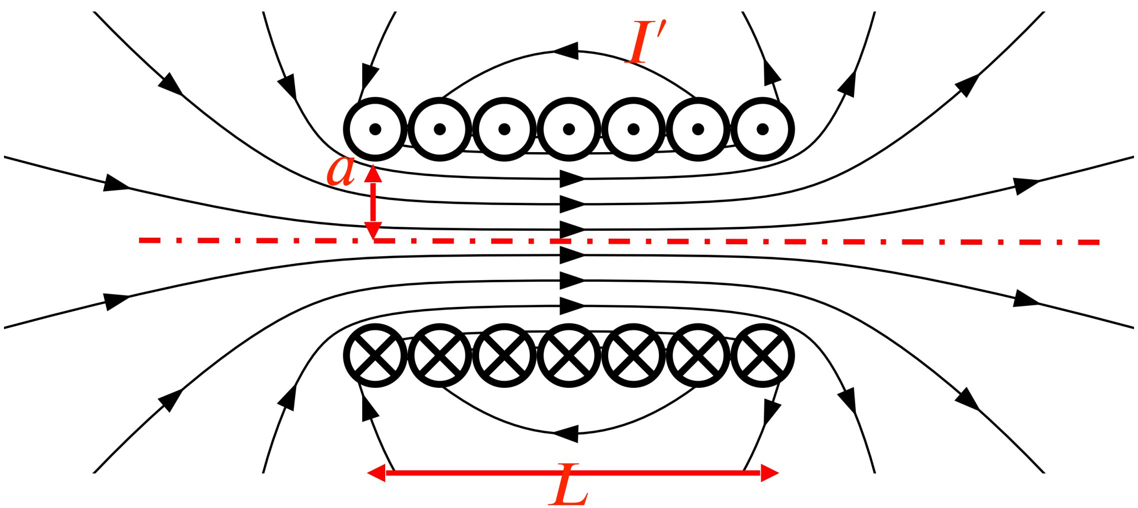
$$b_z(r,z) = \frac{\mu_0 I'}{\pi}\frac{za}{\zeta(a+r)}\left[K(k) + \frac{a-r}{2a}\Pi(k,c) - K(k)\right]$$

$$b_r(r,z) = \frac{\mu_0 I'}{\pi}\frac{\zeta}{4r}\left[2(K(k)-E(k)) + k^2 K(k)\right]$$

$$k = \sqrt{\frac{4ar}{(a+r)^2+z^2}}$$

$$B_r = b_r(r, z-L) - b_r(r, z+L) \quad \zeta = \sqrt{(a+r)^2+z^2}$$

$$c = -\frac{4ar}{(a+r)^2}$$



- B_z is not uniform, even more sinusoidal
- Always B_r generated at the beam off-axis ($r \neq 0$)

Fundamental property of solenoid lens

- Finite long solenoid generates two magnetic components
  - $\vec{B} = \nabla \times \vec{A} \rightarrow \nabla \times \left(\nabla \times \vec{A} \right) = 0$
  - $A_\phi(r,s) = \frac{1}{2} \sum_{n=1,3,5\dots}^{\infty} a_n r^n B^{(n-1)}(s) = \frac{1}{2} r B(s) - \frac{r^3}{16} B''(s) + \dots$

    where $a_1 = 1$, $a_n = -\frac{a_{n-2}}{n^2 - 1}$
  - It extracts

$$B_z(r,z) = B(z) - \frac{r^2}{4} B''(z) + \frac{r^4}{64} B^{(4)}(z) \dots$$

$$B_r(r,z) = -\frac{r}{2} B'(z) + \frac{r^3}{16} B'''(z) + \dots$$

Fundamental property of solenoid lens

- This suggests that when particles enter the solenoid magnet from a field-free region, a Larmor motion is initially induced by B_r, followed by a cyclotron motion induced by B_z
- For a paraxial theory (solenoid lens length ~ 0),

$$A_\phi \sim \frac{r}{2} B(z) \rightarrow \left(\begin{array}{l} B_z = B(Z) \\ B_r = -\frac{r}{2} B'(z) \end{array} \right.$$

Canonical angular momentum due to the Larmor motion is induced,

$$L_{canonical} = xP_y - yP_x + qrA_\phi = \gamma m r^2 \dot{\theta} + qrA_\phi$$

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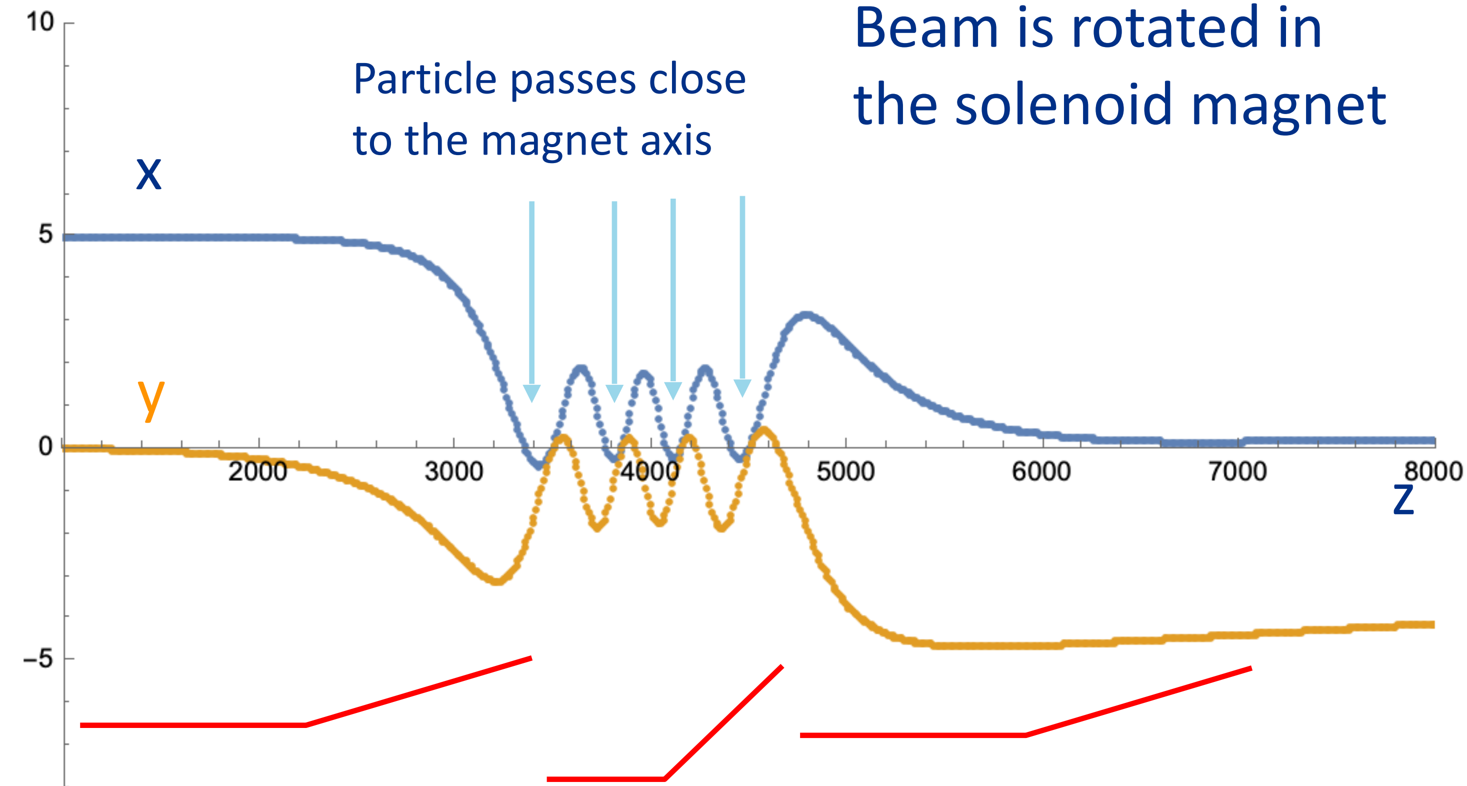
Example particle tracking in a solenoid magnet



Larmor motion

where angular momentum is induced

Cyclotron motion

It appears when a solenoid magnet is long

Larmor motion

Fundamental property of solenoid lens

Set the Larmor rotation frame,

$$X_R = x cos(\varphi) - y sin(\varphi)$$
$$Y_R = x sin(\varphi) + y cos(\varphi)$$
$$\varphi' = \frac{qA_\phi}{P_z r} \sim \frac{qB(z)}{2P} \equiv \kappa = \frac{qB}{2mc\gamma\beta} = \frac{\omega_L}{\beta c}$$

ω_L is a Larmor angular velocity $\quad \omega_L = \frac{qB}{2m\gamma}$

The linearized equations of motion is given,

$$X_R'' + \kappa^2 X_R = 0,\ Y_R'' + \kappa^2 Y_R = 0$$

Fundamental property of solenoid lens

Introduce Twiss parameters,

$$X_R = A_1\sqrt{\hat{\beta}_p}\cos(\Phi - \Phi_1)$$

$$Y_R = A_2\sqrt{\hat{\beta}_p}\cos(\Phi - \Phi_2)$$

$$2\hat{\beta}_p\hat{\beta}_p'' - \left(\hat{\beta}_p'\right)^2 + 4\hat{\beta}_p^{\,2}\kappa^2 - 4 = 0$$

For detail analysis, please find
G. Dugan, PRAB4, 104001 (2001)

$$\kappa = \frac{qB}{2mc\gamma\beta} = \frac{\omega_L}{\beta c}$$

Introduce the normalized canonical angular momentum,

$$L_{canonical} = 2mc\varepsilon_n \bullet \mathscr{L}$$

The transverse covariance matrix includes angular momentum,

$$\frac{M}{mc\epsilon_N} = \begin{pmatrix} \beta_\perp/\langle P_z\rangle & & & \\ -\alpha_\perp & \langle P_z\rangle\gamma_\perp & & \\ 0 & \beta_\perp\kappa - \mathcal{L} & \beta_\perp/\langle P_z\rangle & \\ \mathcal{L} - \beta_\perp\kappa & 0 & -\alpha_\perp & \langle P_z\rangle\gamma_\perp \end{pmatrix}$$

$$det(M) = \left[\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2 - \langle xP_y\rangle\right.$$

Coupling term

Transverse emittance using 4x4 matrix for x-y coupling

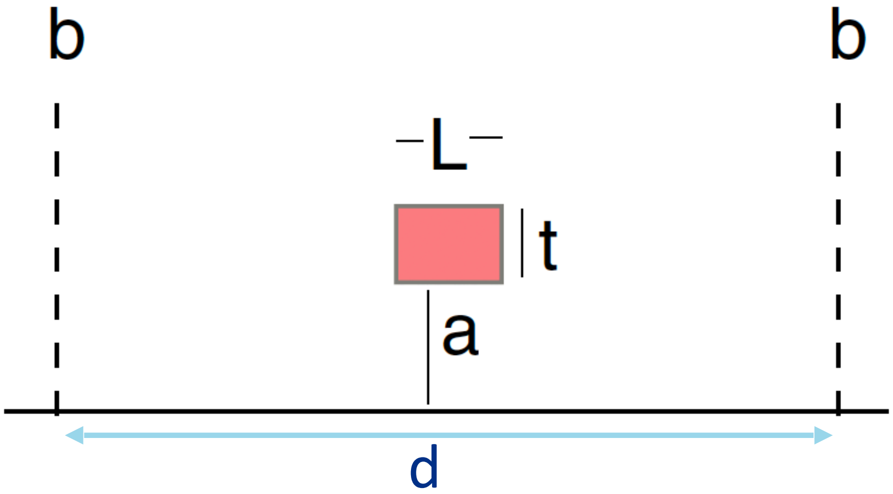
$$\gamma_\perp \equiv \frac{1}{\beta_\perp}[1 + \alpha_\perp^2 + (\beta_\perp\kappa - \mathcal{L})^2]$$

New term in phase advance

FOFO channel

If use a thin lens approximation, phase advance

$$\cos(\mu) = 1 - \frac{d}{2f}$$



single solenoid coil configurations

It is known that a focusing length of solenoid lens is

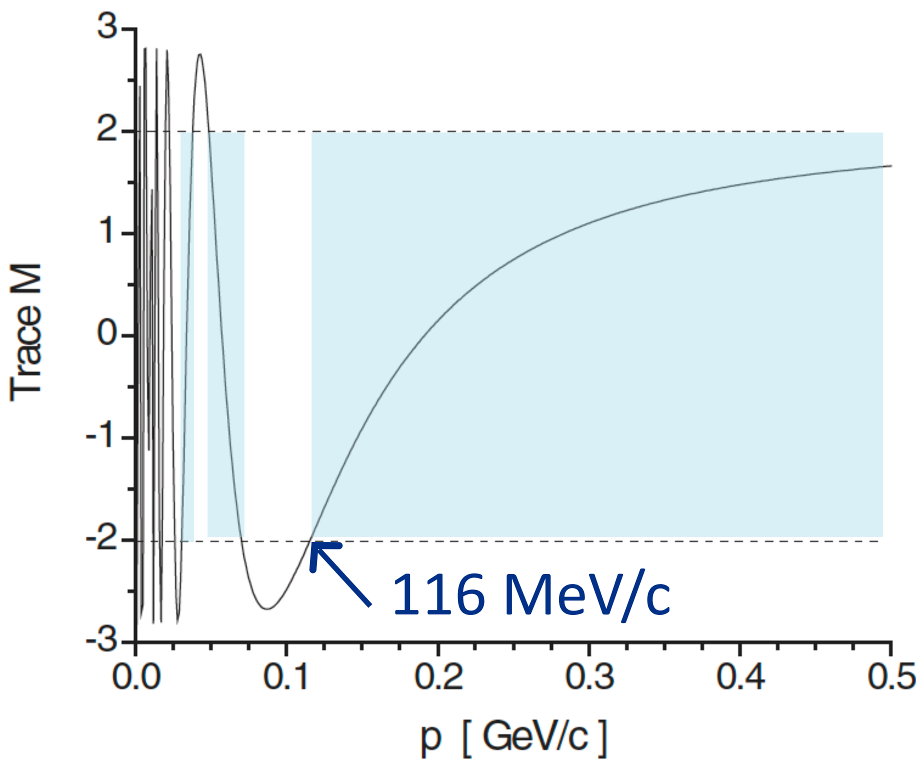
$$f = \frac{4p^2}{e^2 B_0^2 L}$$

Estimate when the phase advance is set π ,

$$\cos\pi = -1 = 1 - \frac{d}{2f} \rightarrow p_\pi^2 = \left(\frac{eB_0}{4}\right)^2 Ld$$

This suggests that the particle momentum should be $p \geq p_\pi$

FOFO channel



Thin lens approximation:

$$p_\pi = \frac{eB_0}{4}\sqrt{Ld} = 173 \text{ MeV/c}$$

While Mathieu equation predicts:

$$p_\pi = \frac{eB_0\lambda}{8\pi\sqrt{q}} = 116 \text{ MeV/c}$$

Fernow claimed that a stable phase space condition from thin lens approximation,

$cos(\mu) = 1 - \frac{d}{2f}$ is not applicable to an FOFO channel

→ B_z has a long tail which breaks an impulse approximation condition

Fernow applied Mathieu equation to find multiple stop band structure as shown in the plot (I guess $Tra(M)$ is from simulation, not analytical)

Introduce Mathieu equation

The radial equation of motion on the Larmor frame (or eliminate canonical momentum in alternate solenoid channel) is

$$\ddot{r} + \frac{r}{4}\left(\frac{eB_z}{mc\gamma\beta}\right)^2 = 0$$

Note: Fernow's paper does not have p, but it should have p to make consistency

Since the solenoid magnet is periodic, we assume that the solenoidal field can be presented with a sinusoidal function

$$B_z \rightarrow B_0 sin(kz)$$

$$\rightarrow \ddot{r} + r\left(\frac{eB_0}{2mc\gamma\beta}\right)^2 sin^2(kz) = 0$$

This equation can be transformed into the canonical form of the Mathieu equation,

$$\frac{d^2y}{dv^2} + \left[a - 2qcos(2v)\right]y = 0$$

Introduce Mathieu equation

$$\ddot{r} = k^{-2} r'' \text{ where } k = \frac{2\pi}{\lambda}$$

$$r'' + k^2 \left[a - 2q\cos(2kz) \right] r = 0$$

$$\rightarrow r'' + k^2 \left[a - 2q\left\{ 1 - 2\sin^2(kz) \right\} \right] r = 0$$

If we set the parameter $a = 2q$

$$\rightarrow r'' + k^2 \left[2q - 2q\left\{ 1 - 2\sin^2(kz) \right\} \right] r = r'' + 4k^2 q \sin^2(kz) \bullet r = 0$$

We obtain

$$\ddot{r} + r\left(\frac{eB_0}{2mc\gamma\beta} \right)^2 \sin^2(kz) = 0$$

$$\rightarrow r'' + r \bullet q\sin^2(kz) = 0 \qquad q \rightarrow \frac{1}{4k^2}\left(\frac{eB_0}{2mc\gamma\beta} \right)^2 = \left(\frac{eB_0\lambda}{8\pi p} \right)^2$$

Introduce Mathieu equation

Stop band is identified as a function of q

TABLE I. Mathieu stop bands.

Stop band	q (low)	q (high)
1	0.3290	0.8898
2	1.8582	3.0391
3	4.6270	6.4259
4	8.6316	11.0480
5	13.8711	16.9047

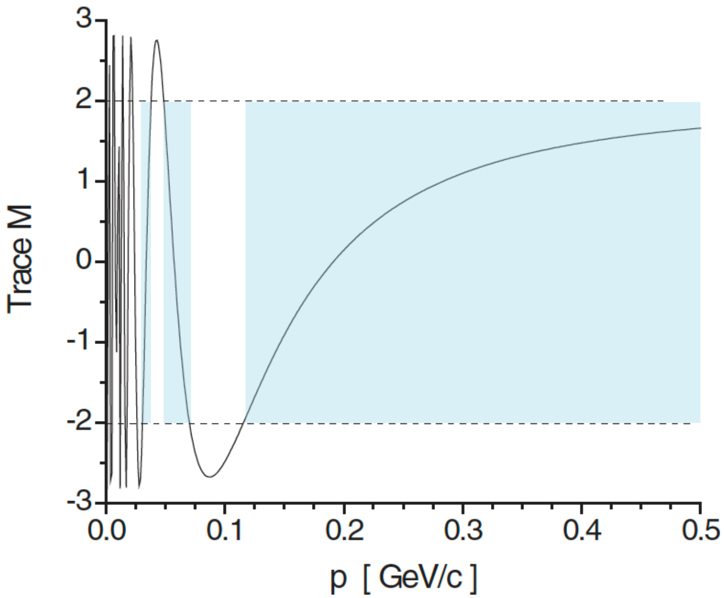
Using $q = \left(\frac{eB_0\lambda}{8\pi p}\right)^2 \rightarrow p = \frac{eB_0\lambda}{8\pi\sqrt{q_M}}$

$B_0 = 2.78$ T, $\lambda = 2$ m, $e = 0.3$ GeV/Tm

TABLE II. Momentum passband locations [MeV/c].

q	One-coil passband	Mathieu theory
0.329–0	116–∞	116–∞
1.86–0.890	49–70	49–70
4.63–3.04	31–37	31–38
8.63–6.42	23–26	23–26
13.87–11.05	18–19	18–20

Solution from Mathieu equation of FOFO



Introduce Mathieu equation

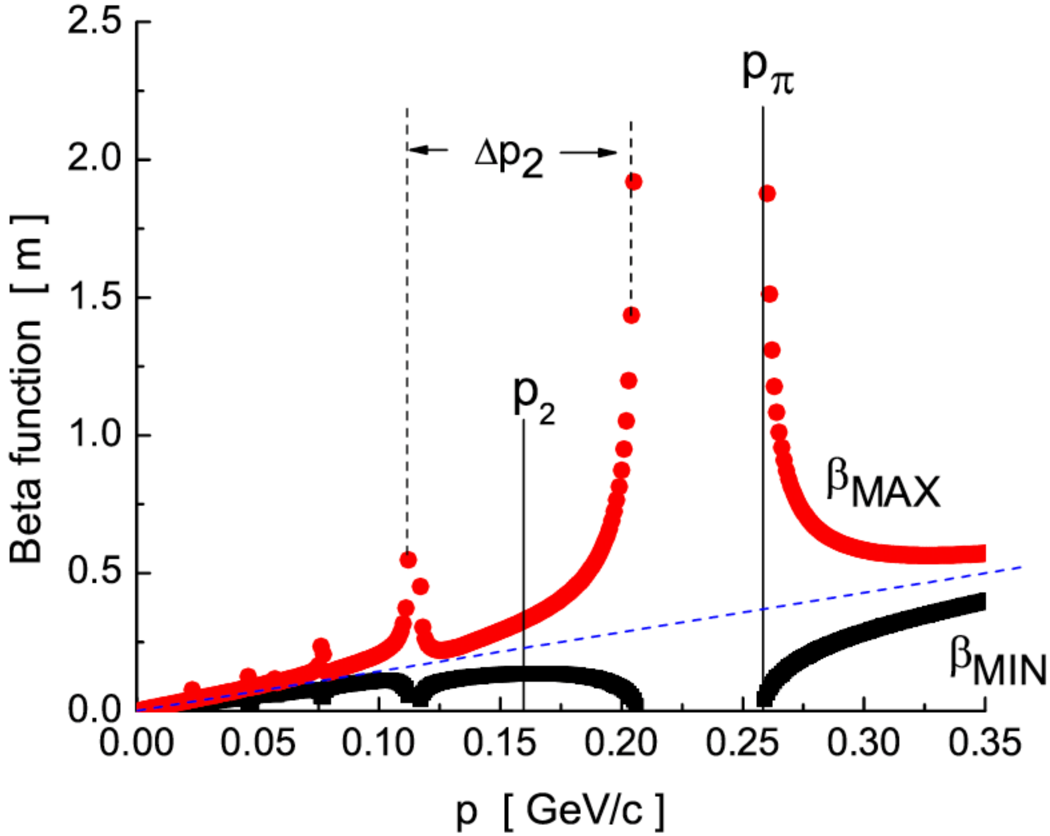


FIG. 4. (Color) Minimum and maximum values of the beta function as a function of p. The parameter values were $d =$ 100 cm, $L = 40$ cm, $a = 40$ cm, $t = 10$ cm, $J = 100\ \mathrm{A/mm}^2$, and constant polarity.

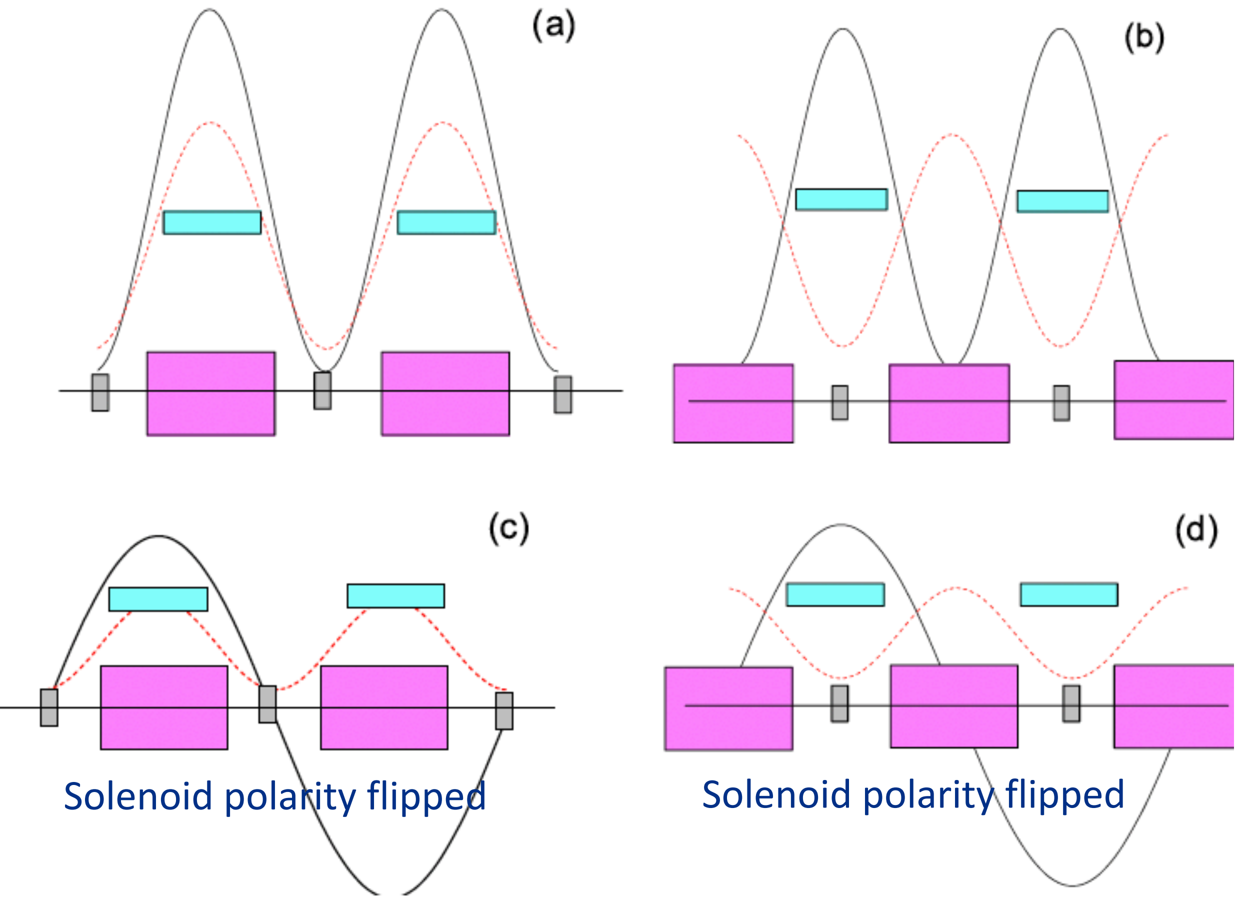
Plot shows simulated $\hat{\beta}_{min}$ and $\hat{\beta}_{max}$ from simualtion

- This stop band structure appears in a periodic channel
- Ferow suggests to design cooling channels at $p > p_\pi$ and $\Delta\, p_2$ range
- This is a basic concept of the rectilinear channel: Initial rectilinear channel is a range $p > p_\pi$ while the later channel is designed in the range $\Delta\, p_2$

Design FOFO channel

FOFO magnet design

(black: B_z, red: $\hat{\beta}$, pink: RF, gray: absorber)

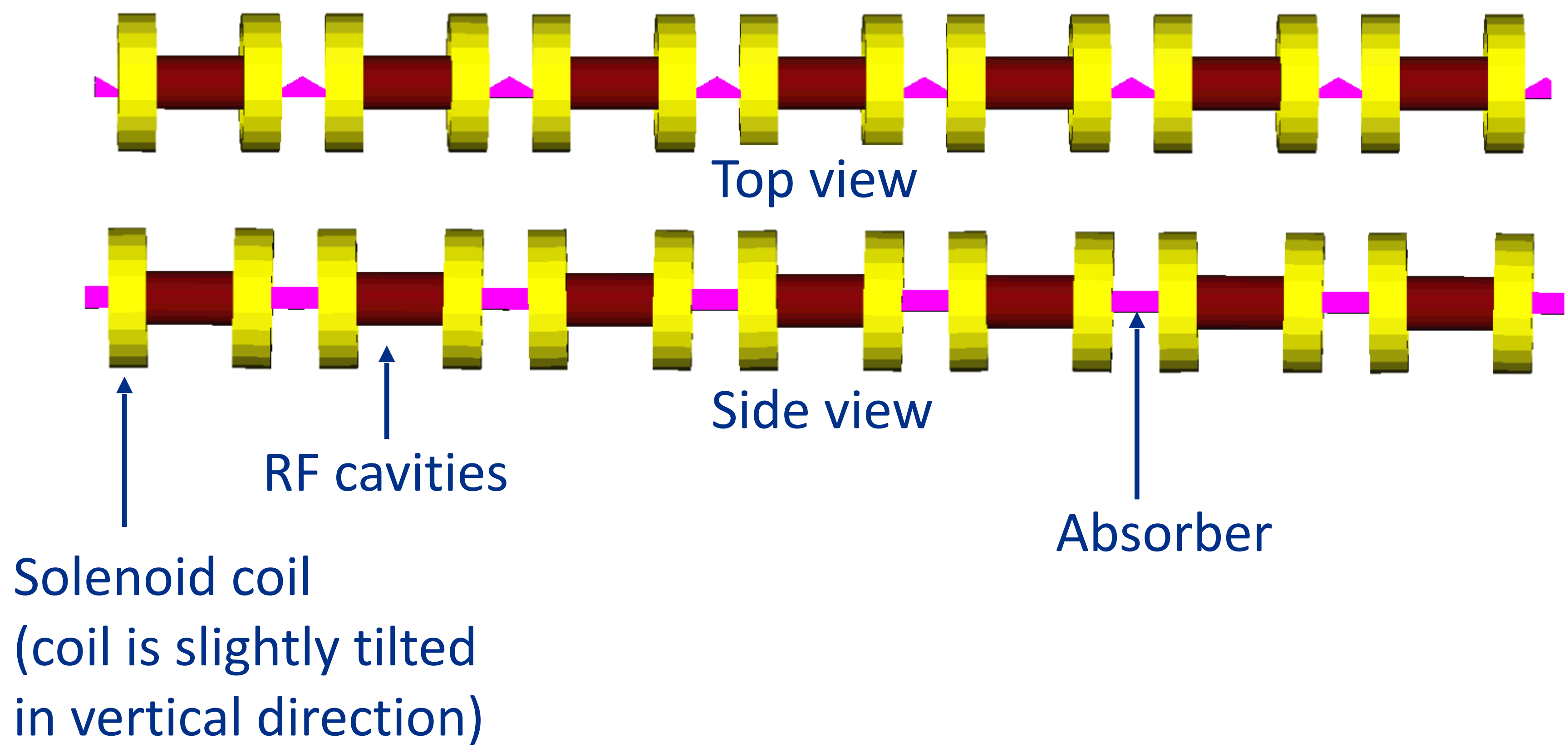


Fernow evaluated the stop band with 4 different magnet configurations with numerical simulation

A cooling performance was evaluated→ They suggests to use the polarity flipping which can eliminate accumulation of a canonical angular momentum

Rectilinear channel

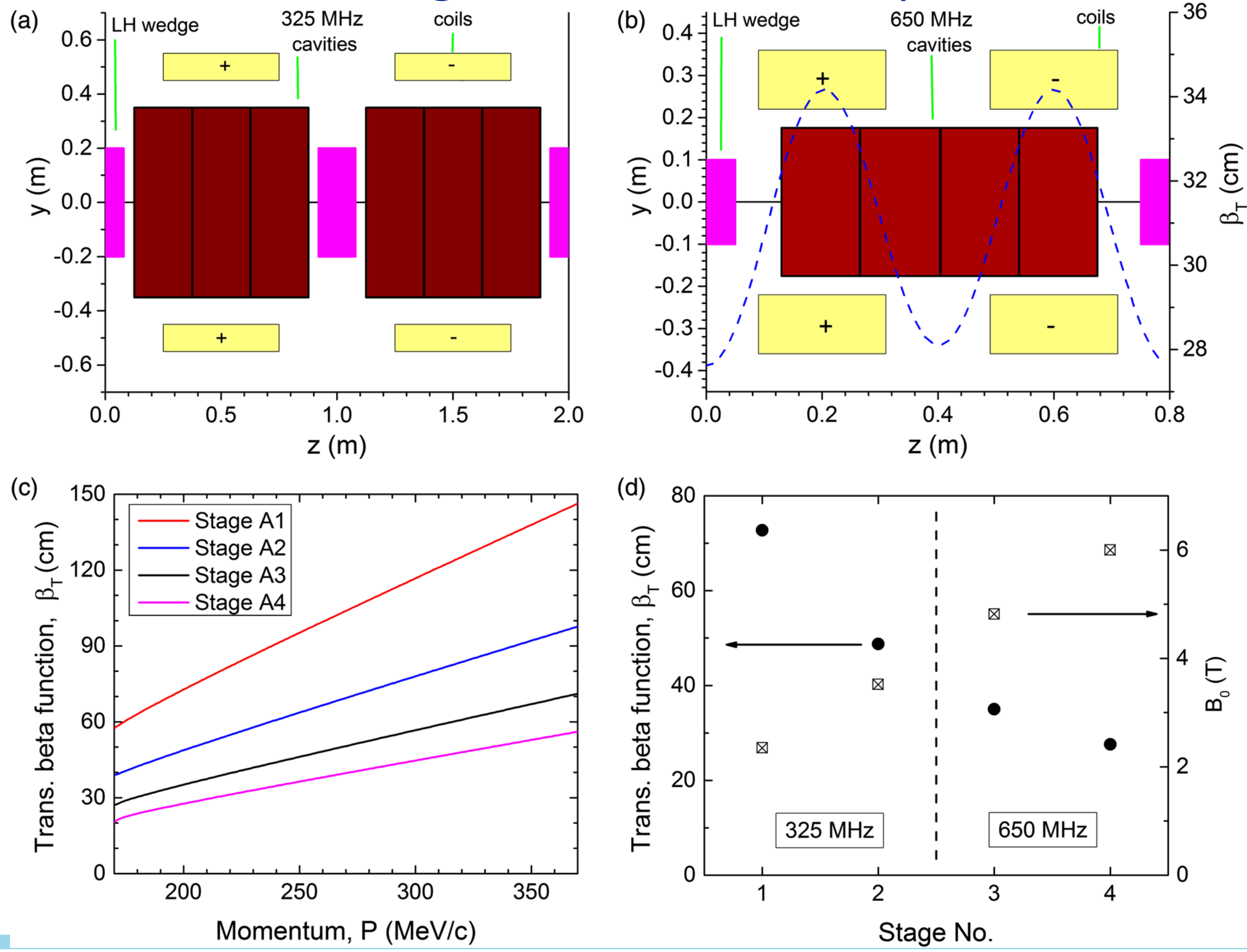
Stratakis demonstrated the advanced FOFO channel
→ So called Rectilinear channel
→ Solenoid coil is tilted in vertical direction to induce dispersion



Rectilinear channel

Initial cooling channel use $p > p_\pi$ region

→ Benefit a large momentum acceptance



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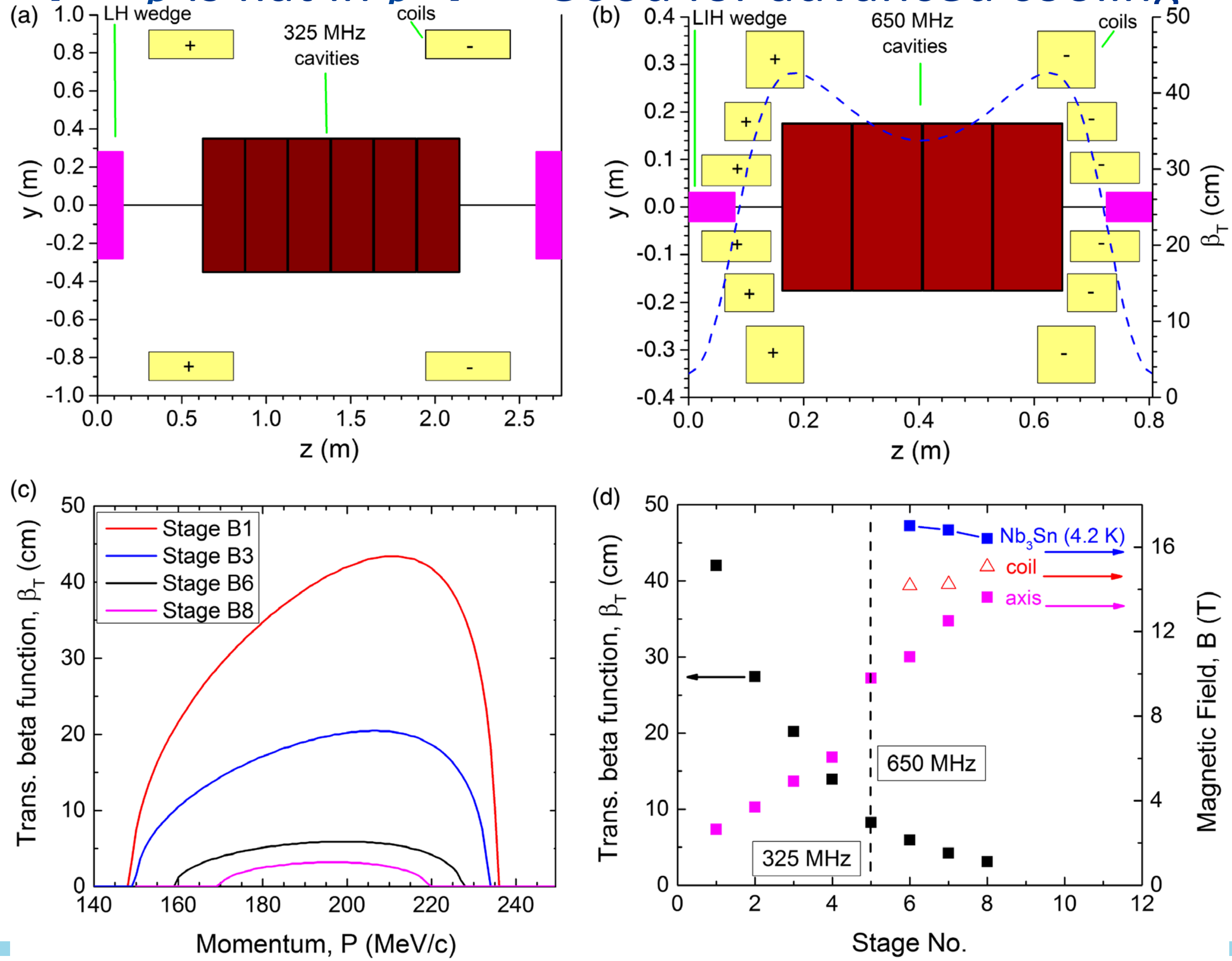
Rectilinear channel

Later cooling channel use $\Delta\, p_2$ region

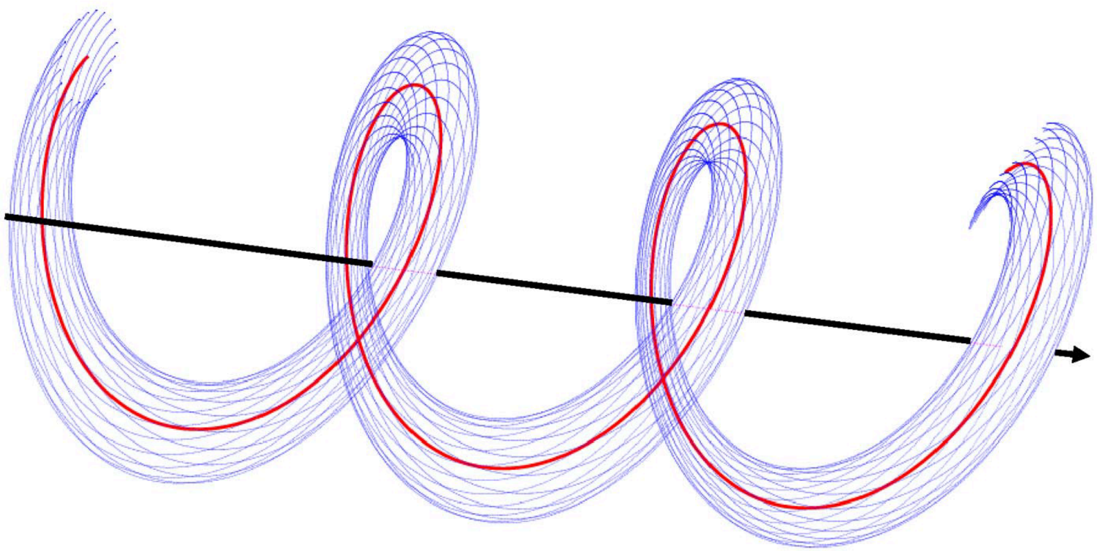
→ $\hat{\beta}$ is flat in p → Good for advanced cooling



Helical cooling channel

Use continuous combined magnet (straight solenoid + helical dipole & quadrupole magnets)

→ No stop band due to a periodic focusing channel



Rotation of reference particle (red line), which is based on cyclotron motion

Equations of motion on the reference frame

$$u_1'' + \frac{q-1}{1+\kappa^2} k u_2' + k^2 \hat{D}^{-1} u_1 = 0$$

$$u_2'' - (q-1) k u_1' + k^2 (q-g) u_2 = 0$$

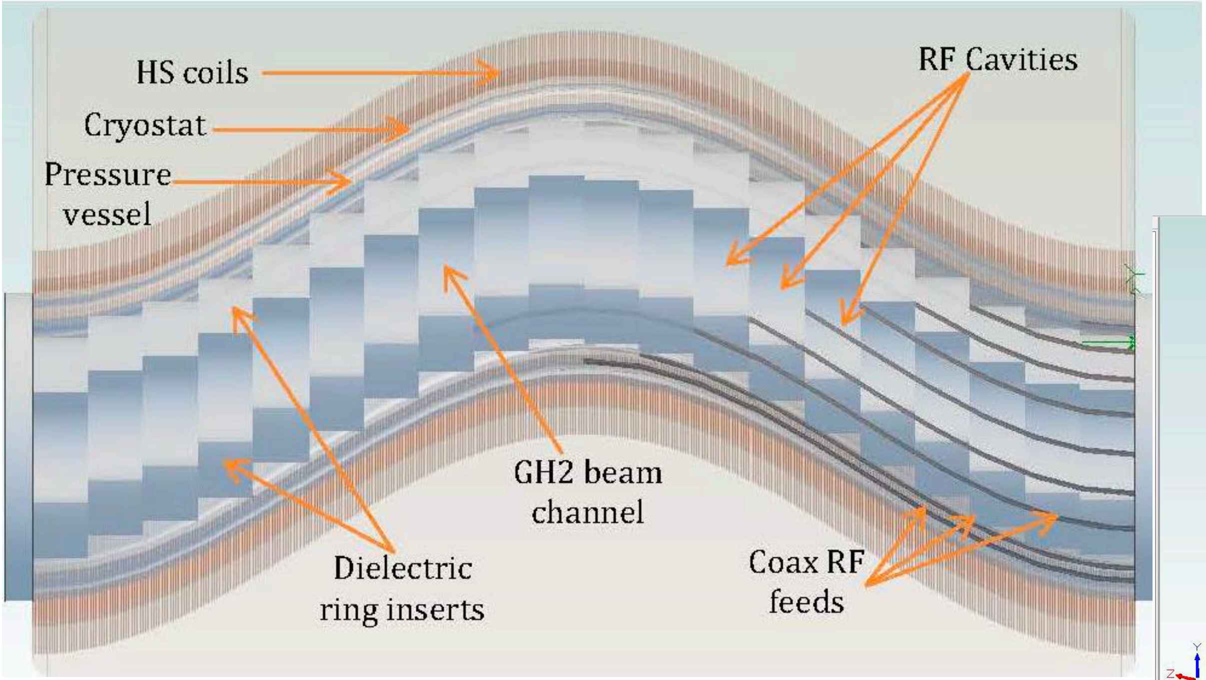
$$\hat{D}^{-1} = \frac{\kappa^2 + (1-\kappa^2) q}{1+\kappa^2} + g$$

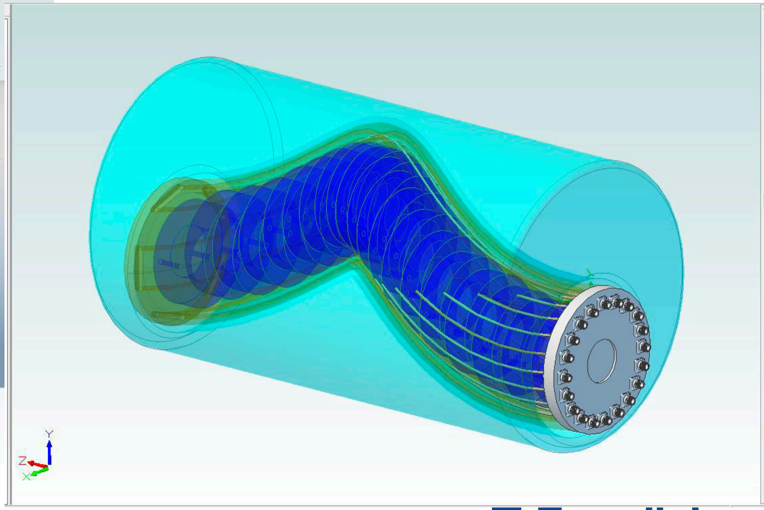
$$g = \frac{-(1+\kappa^2)^{3/2}}{pk^2} \frac{\partial b}{\partial}$$

Helical cooling channel

Challenge of designing helical cooling channel is following

- Helical magnet is complicated
- RF cavities must be inserted into a helical magnet
- If helical channel is used, the matching has to flip sign of the momentum compaction factor from positive to negative





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Concept of HFOFO

Yuri claims that FOFO has an issue to produce low $\hat{\beta}$ and high D, (which is ideal for ionization cooling) because

$$D_{x,RFOFO} \sim \frac{R}{Q_x^2} \sim \frac{\bar{\beta}_x^2}{R}$$

R: circumference of RFOFO, or the length of one cell period of rectilinear channel

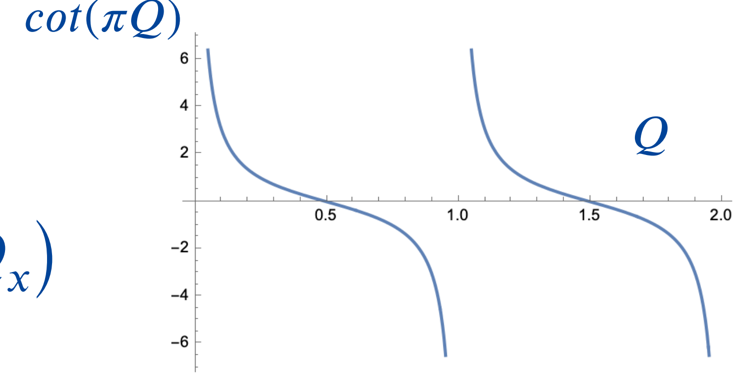
His concept starts from a periodic orbit with near pi resonance

$$x_{p.o.} = \frac{1}{sin\left(\pi Q_x\right)}$$

where $x_{p.o.}$ is a periodic orbit, Q_x is a tune, then

$$D_x = \frac{\delta x_{p.o.}}{\delta p} \sim -\pi Q_x' \frac{cos\left(\pi Q_x\right)}{sin^2\left(\pi Q_x\right)} = -\pi Q_x' \bullet x_{p.o.} cot\left(\pi Q_x\right)$$

Concept of HFOFO



$$D_x = \frac{\delta x_{p.o.}}{\delta p} \sim -\pi Q_x' \bullet x_{p.o.} cot\left(\pi Q_x\right)$$

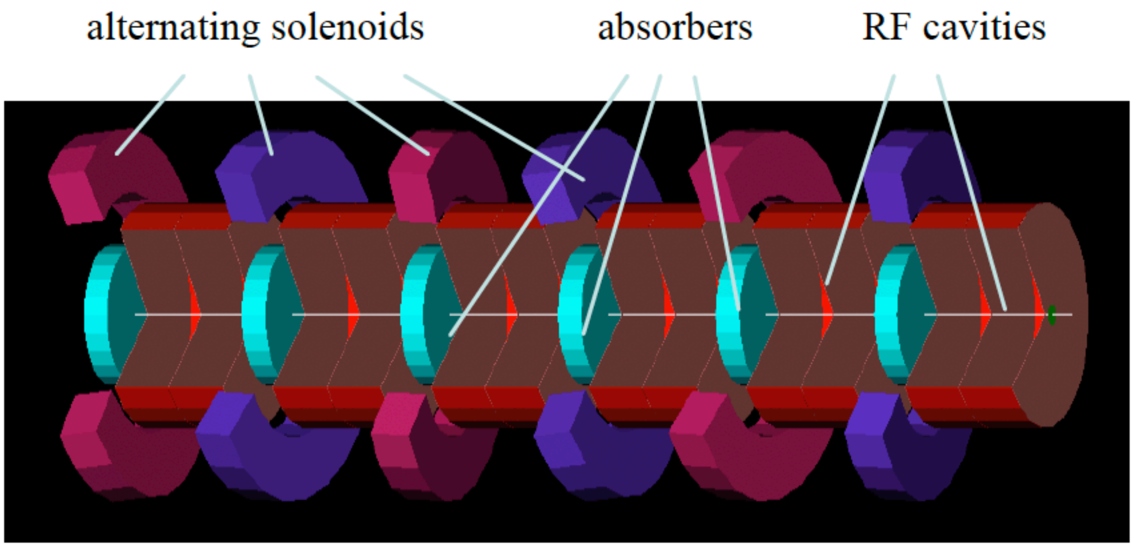
This is good since D can be extremely large when Q_x is integer (Q_x' is called a chromaticity)

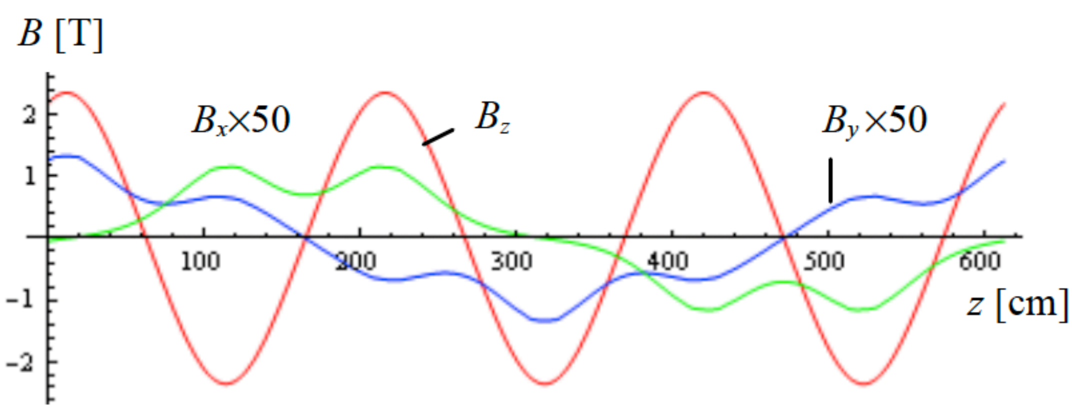
Making integer Q channel means making π phase advanced FOFO channel → Helical FOFO channel

Helical FOFO channel could have a large momentum acceptance by tuning higher order chromaticity

- Helical orbit permits a large acceptance
- Alternative solenoid eliminates angular momentum
- Forming one period with multiple solenoid cells induces a weak focusing (6 solenoid coils induce one period)

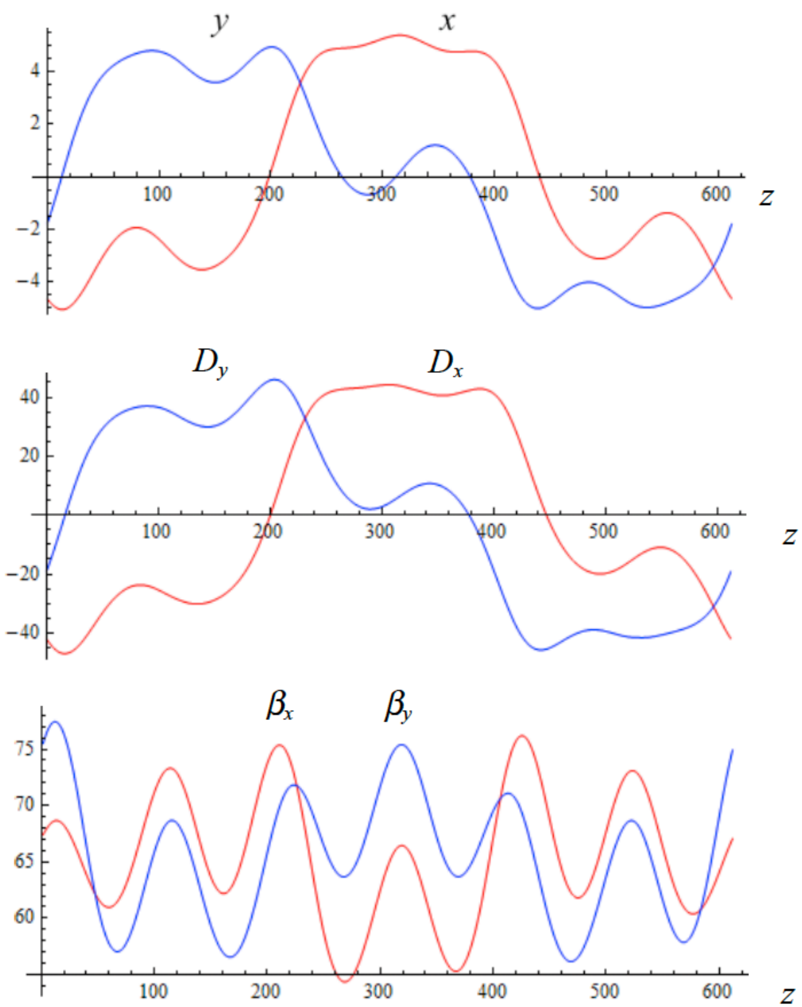
Design HFOFO





- Initial HFOFO channel has 6 solenoid coils to induce one beta oscillation
- Reference particle has a position offset from solenoid coil center
- Absorber is located low $\hat{\beta}$ and high D region while RF cavities are located other regions
- Tilted solenoid induces dispersion

Design HFOFO



- Reference particle position has offset from the solenoid coil center

- Large dispersion is created with a tilted solenoid coil
- Wedge absorber is also rotated with respect to the dispersion

- Betatron functions are low at the absorber (Absorber is located six positions in this example)

Estimated cooling rate

Estimated beta tune has imaginary part

Table 1. Normal mode tunes and normalized equilibrium emittances.

Parameter	Mode I	Mode II	Mode III
Tune	$1.2271 + 0.0100i$	$1.2375 + 0.0036i$	$0.1886 + 0.0049i$
Emittance (mm)	2.28	6.13	1.93

The imaginary part is a growth rate of the oscillation amplitude so we should suppress this growth through ionization cooling

The damping rate is given

$$\frac{d}{ds} ln\left(\varepsilon_j\right) = -2 \times \frac{2\pi}{L_{period}} Im\left(Q_j\right) \qquad j = 1,2,3$$

Increase imaginary part makes more cooling but the beam phase stability is lower

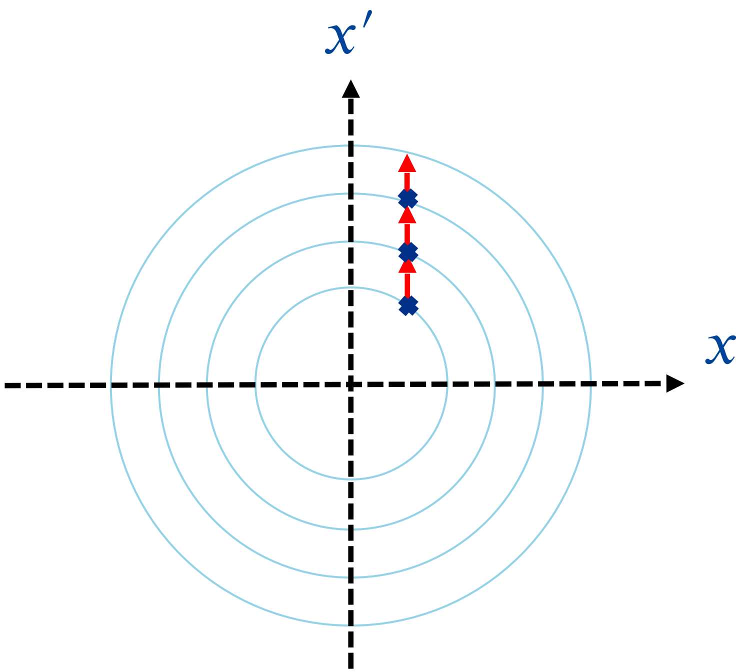
Potential parameters to improve HFOFO

- Once the constant momentum HFOFO is set, we can study HFOFO with following interests
  – Investigate beta tune change and find the cooling rate
    - Adjust size of imaginary part of beta tune
  – Yuri has already demonstrated adding quadrupole component, G_q
    - Getting lower beta function on one plane but other plane has larger beta functions
    - If the heating is not significant, this scheme works for cooling on both planes
- Is it possible to form Hamiltonian or equation of motion in HFOFO?
  – It sounds hard to find a formula for tilted solenoid fields
    - Is it possible to use a pure solenoid field + dipole field?

Extra slide

Tune resonance

- Designing a channel with an integer tune induces a risk because it can drive resonant particle motion, potentially leading to beam instability



Integer resonance

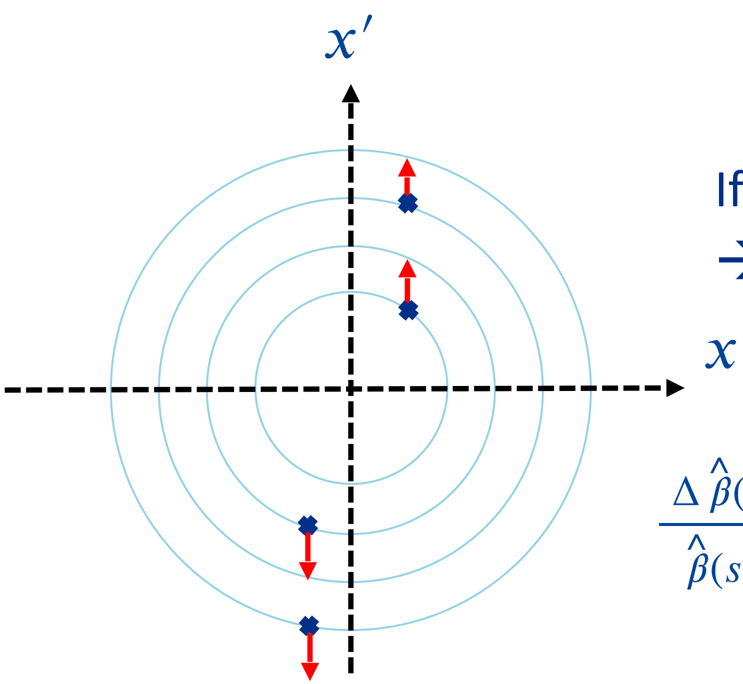
- In the integer channel, a particle passes the same location at specific longitudinal position (s)
- If there is an irregular magnetic field at a specific position, the transverse kick due to the field enhances during the particle revolutions
- A particle amplitude is growing → this process is called tune resonance

Tune resonance

- Resonance happens even a half integer tune since the kick occurs at pi phase after the first kick

For integer resonance, after one-turn period,

$$u_{co}(s) = \frac{\sqrt{\hat{\beta}(s)}}{2sin(\pi\nu)} \sum \sqrt{\hat{\beta}_i}\theta_i cos\left(\pi\nu - \varphi(s) + \varphi_i\right)$$



If ν is integer, $sin(\nu\pi) = 0$, then $u_{co}(s)$ is infinite

→ Particle motion unstable

For a half integer resonance, after one-turn period,

$$\frac{\Delta\hat{\beta}(s)}{\hat{\beta}(s)} = -\frac{1}{2sin(2\pi\nu)}\int \hat{\beta}(s_1)K(s_1)cos\left(2\pi\nu_0 - 2\varphi(s) + 2\varphi(s_1)\right)ds_1$$

Half integer resonance

If ν is integer, $sin(\nu\pi) = 0$, $\frac{\Delta\hat{\beta}(s)}{\hat{\beta}(s)}$ is infinite