Mini Lecture Course: Accelerator Design for a Multi-TeV Muon Collider

Lecture 1: Overview of Muon Collider

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My profile

Katsuya Yonehara

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- Experimental physicist
- Accelerator and Particle Physics
- Currently working on high power targetry for neutrino facilities like NuMI and LBNF

You can contact me through

- During the lecture
- Email to yonehara@fnal.gov
- USMCC Slack chat

Scope of the lecture series

- My goal is to convey our past studies and accumulated knowledges related to muon colliders to the new generation
- Target audience: Lecture materials are being prepared for graduate students, engineers, and researchers who do not have a background in accelerator physics, however, the lecture are open to all interested participants!
- Tentative agenda (1-2 hours per session):
  1. Overview of the muon collider
  2. Emittance evolution and stability conditions in phase space
  3. Review of ionization cooling channels
  4. RF cavities in ionization cooling channels
  5. (TBD) Interactions and kinematics
  6. (TBD) Review of the MICE experiment & demo channel

Recommended paper and textbook

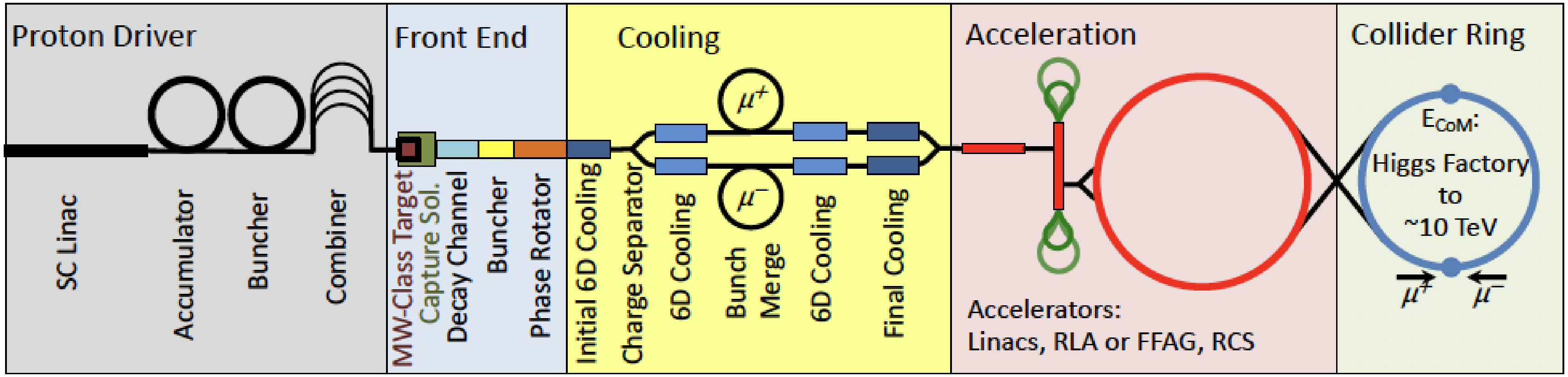
- G. Dugan, PRSTAB 4,104001 (2001) for expert level
- "Theory and Design of Charged Particle Beams", Martin Reiser, Print ISBN:9783527407415 |Online ISBN:9783527622047 |DOI:10.1002/9783527622047 for entry level reader
- "Particle Accelerator Physics", Helmut Wiedemann, https://link.springer.com/book/10.1007/978-3-319-18317-6 for entry and advanced level reader
- "Accelerator Physics", SY Lee, https://library.oapen.org/handle/20.500.12657/50490 for advanced level reader

Overview of Muon Collider Design

Quickly go through the design concept of muon colliders

Overview of Muon Collider

Proton beam-based muon collider



- Our story starts from luminosity…

Number of particles cross at an Interaction Point (IP) per second

$$\mathcal{L} = \frac{N_{\mu^+} \cdot N_{\mu^-} \cdot f \cdot n_b}{4\pi \cdot \sigma_x \cdot \sigma_y} > 10^{34} \quad \text{cm}^{-2}\,\text{s}^{-1}$$

Beam crossing area at IP

N: Num of μ per bunch ($\sim 10^{12}$)
f: Bunch revolution (~ 1000)
n_b: Repetition rate (~ 5 Hz)
σ: RMS Beam spot size at collision ($\sim 10^{-4}$ cm)

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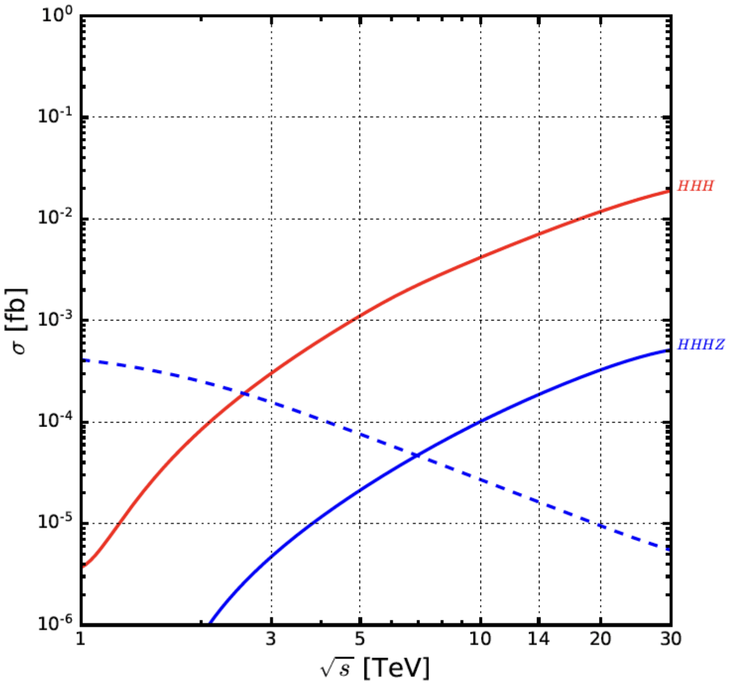
Collider Luminosities

$$\mathcal{L} = \frac{N_{\mu^+} \cdot N_{\mu^-} \cdot f \cdot n_b}{4\pi \cdot \sigma_x \cdot \sigma_y}$$

At the present time

Facility	Ops yr	Beam	COM	σ (μm)	n_b (kHz)	$\mathcal{L}\ (cm^{-2}s^{-1})$
Tevatron	1983-2011	$p \cdot \bar{p}$	1.96 TeV	~ 30	47.7	$4 \cdot 10^{32}$
LHC	2009-	$p \cdot p$	13-14 TeV	~ 17	11.2	$2.1 \cdot 10^{34}$
HL-LHC	2029-	$p \cdot p$	14 TeV	~ 7	11.2	$> 10^{35}$
EIC	2031-	$e^- \cdot p$	20-140 GeV	e: 10~20 p: 5~10	> 10,000	$\sim 10^{34}$
Suer KEKB	2016-	$e^+ \cdot e^-$	10.58 GeV	x: 10 y: 0.06	0.05	$2.4 \cdot 10^{34}$
HERA	1992-2007	$e^{\pm} \cdot p$	318 GeV	omit	0.01	$7 \cdot 10^{31}$
MC	**?**	$\boldsymbol{\mu^+ \cdot \mu^-}$	**3, 10 TeV**	$\boldsymbol{\sim 1}$	**0.005**	$\boldsymbol{> 10^{34}}$
ILC	?	$e^+ \cdot e^-$	1 TeV	x: 0.33 y: 0.005	5 Hz	$4.9 \cdot 10^{34}$
FCC-hh	?	$p \cdot p$	100 TeV	~ 7	?	$3 \cdot 10^{35}$

Examine Luminosity in Muon Collider



I am, particularly interested in three Higgs productions

- Self coupling
- Possibly coupling with SUSY particles (BSM), or find internal structure of Higgs
- Analogous to a glue-ball (pomeron)

Ex) $\mu\mu \to WW \to HHH$, VBF at 14 TeV

$\sigma = 7.1 \cdot 10^{-3}$ femto barns = $7.1 \cdot 10^{-18}$ barns = $7.1 \cdot 10^{-42}$ cm^{-2}

$1\ barn = 10^{-24}\ cm^{-2}$

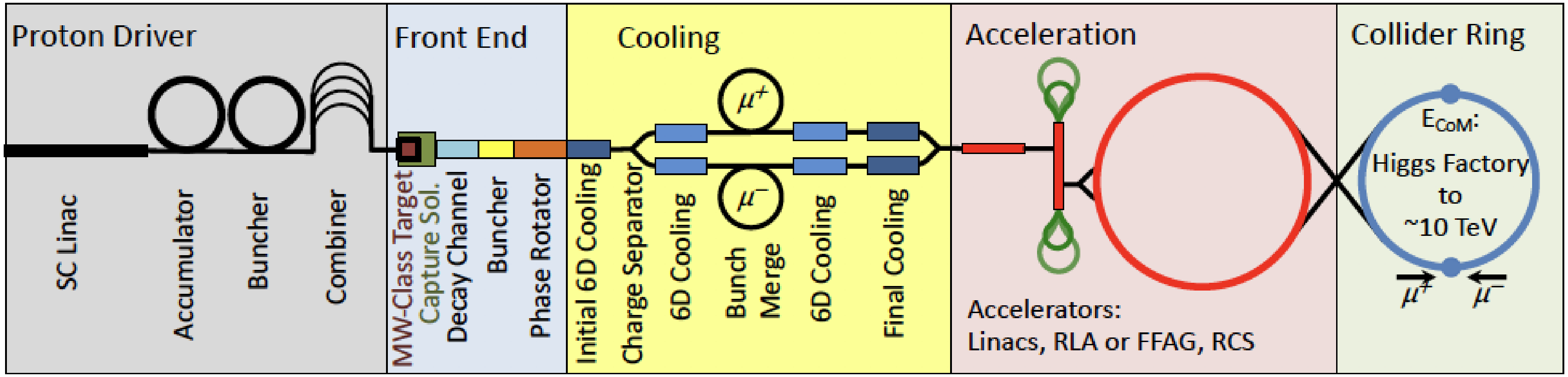
6 months full time beam operation

$\rightarrow t_{operation} \sim 1.56 \cdot 10^{7}$ s

$f_{\mu\mu \to HHH} \sim 7.1 \cdot 10^{-42} \times 1.56 \cdot 10^{7} \times 10^{34} \sim 0.1$ event/year

Overview of Muon Collider

Proton beam based muon collider



- Required muons after cooling, $N{\sim}10^{12}, \varepsilon_{t,n}{\sim}20$ mrad
  - Start from proton beam intensity: $\boldsymbol{\sim 10^{15}}$ **protons/spill**
  - Required $\mu/proton$ yield: $\boldsymbol{\sim 0.1\ \mu/p}$
  - Acceptable μ loss in cooling (and acceleration): $\mathbf{0.01}$
  - We claim that the luminosity gains at higher muon collision energy because muon lifetime longer thus num of rev. (f) also higher
  - Besides, if either transmission or final achievable emittance is improved, the luminosity is increased by square of them!

$$\boxed{\mathcal{L} \propto \frac{N_{\mu^+} \cdot N_{\mu^-} \cdot f}{\sigma_x \cdot \sigma_y}}$$

Low EMittance Muon Accelerator (LEMMA)

- In the LEMMA scheme, muons are produced via **electron-positron annihilation** just above the energy threshold:
  - $e^+ + e^- \rightarrow \mu^+ + \mu^-$
- The size of muon flux after the process is extremely small (possibly a sub micro-meter scale).
- However, the cross section of the process is $\sim 1\ \mu bars = 10^{-30}\ cm^{-2}$ at 45 GeV, the expected muon yield is approximately six orders of magnitude smaller than proton-based scheme
- Currently, we hold this scheme as a backup scenario

Overview of Muon Collider

- Intense proton beam strikes pion production target

$$p + A \rightarrow \pi^{\pm} + A'$$

- Pions are eventually decayed

$$\pi^{+} \rightarrow \mu^{+} + \nu_{\mu}$$
$$\pi^{-} \rightarrow \mu^{-} + \bar{\nu}_{\mu}$$

- Pion lifetime

$\tau = 26\gamma$ ns $\rightarrow \tau \cdot c = 7.8$ m $(\gamma = 1)$

- Muons are eventually decayed as well

$$\mu^{+} \rightarrow e^{+} + \nu_{e} + \bar{\nu}_{\mu}$$
$$\mu^{-} \rightarrow e^{-} + \bar{\nu}_{e} + \nu_{\mu}$$

- Muon lifetime

$\tau = 2.2\gamma$ µs $\rightarrow \tau \cdot c = 659.5$ m $(\gamma = 1)$

Challenge in producing low emittance muon beam

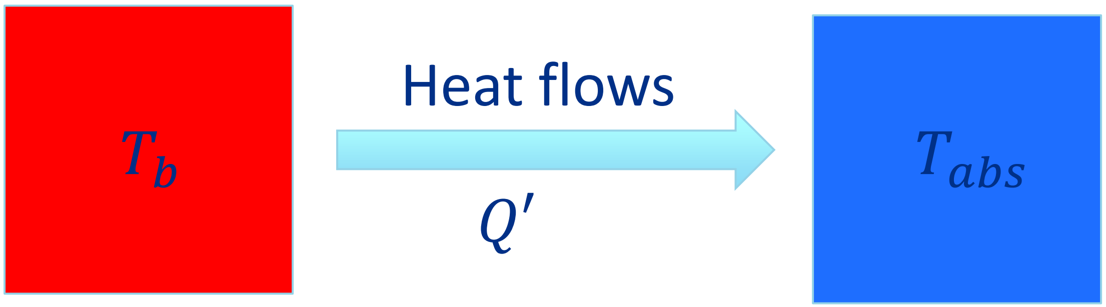
- Initial muon phase space is too large to achieve the goal luminosity
  - Phase space cooling is required
    - Initial muon phase space is similar as basketball
    - Required muon phase space after cooling is sub-millimeter
- Muons have a finite lifetime
  - Need fast cooling scheme
    - Ionization cooling

Intuitive picture of beam cooling

Interpret of the beam cooling:
Similar as refrigerator!

Beam cooling

- Heat flows from warm object to cold object



Heat flows when

$$T_b > T_{abs}$$

Equilibrium condition

$$T_b \sim T_{abs}$$

$$\frac{dQ}{dt} = \dot{Q} = h \cdot A \cdot (T_b - T_{abs})$$

Heat transfer rate

$$\rightarrow \frac{dQ}{ds} = Q' = h \cdot A \cdot (T_b - T_{abs})$$

s is a path length

Since $h \cdot A$ is constant, if ΔT is large, $\dot{Q}$ should be proportionally large

If ΔT is near zero (temperature reaches equilibrium), $\dot{Q}$ is small

Beam cooling

- Beam temperature flows into absorber temperature

Beam

$$\frac{\Delta p_\perp^2}{m} = kT_b$$

q (momentum)-transfer
via coulomb interaction

Absorber

$$\frac{3kT_{abs}}{2}$$

Heat flows when

$$T_b > T_{abs}$$

Equilibrium condition

$$T_b \sim T_{abs}$$

Δp: Momentum in CM frame

How to effectively generate high transverse momentum in a beam?

- Transverse momentum is maximized at cooling region

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Beam cooling

- Beam temperature flows into absorber temperature

Beam

$$\frac{\Delta p_\perp^2}{m} = kT_b$$

q (momentum)-transfer via coulomb interaction

Absorber

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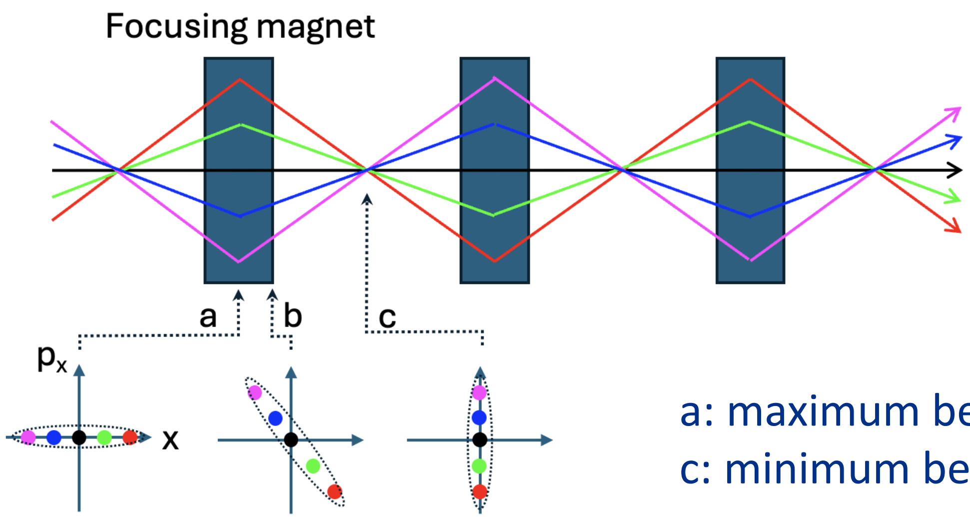
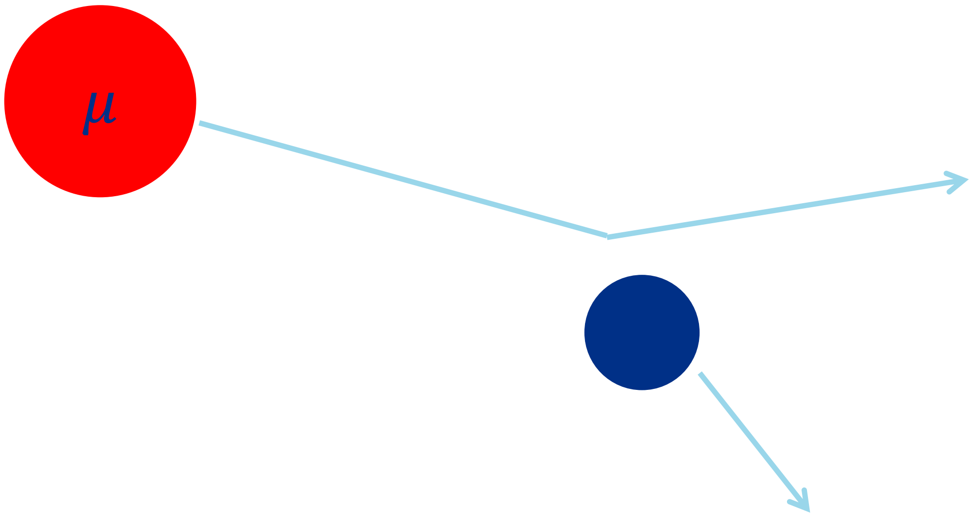


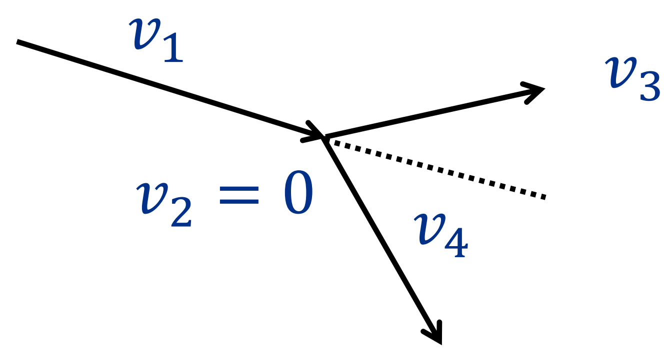
Put absorber at point “c”

a: maximum beta function
c: minimum beta function

Transverse phase space is rotated along beam path

Energy transfer process





Energy transfer in an elastic scattering:

$$\frac{1}{2}m_b v_1^2 = \frac{1}{2}m_b v_3^2 + \frac{1}{2}m_t v_4^2 \rightarrow E_1 - E_3 = \Delta E = E_4$$

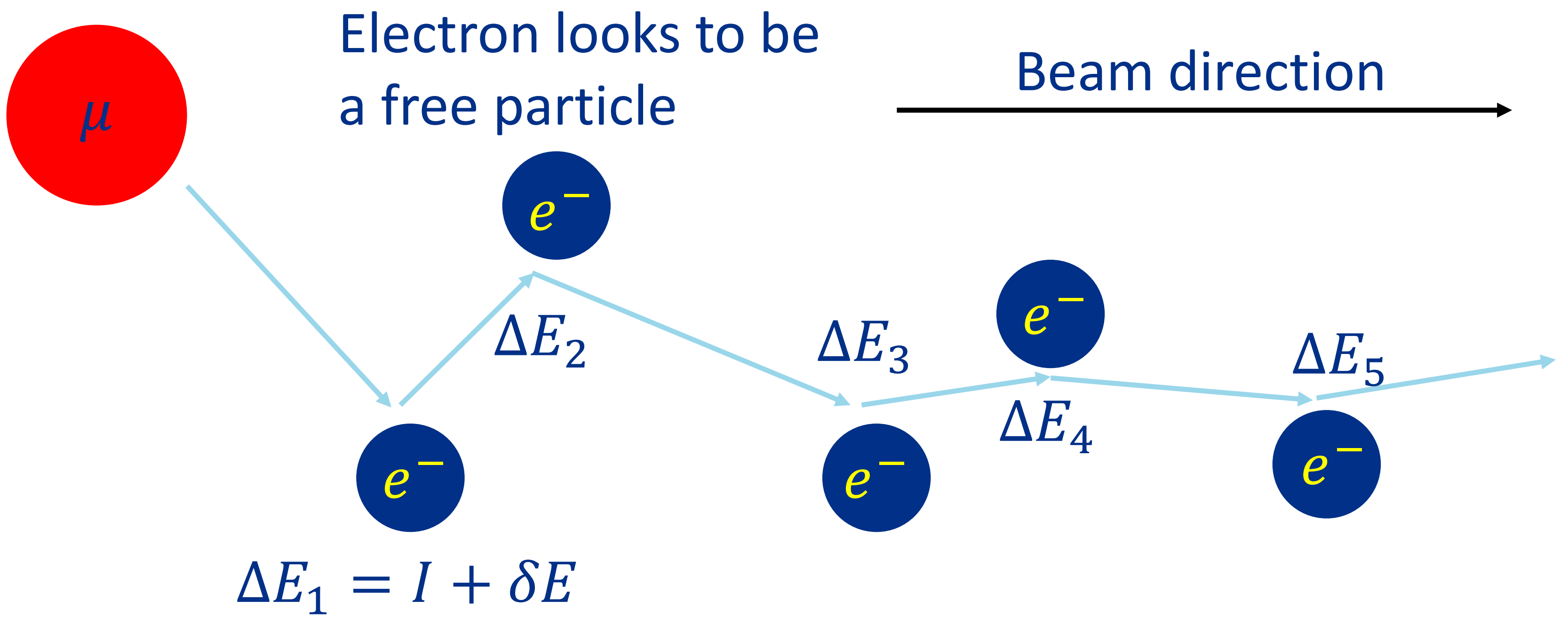
This is a transferred energy

The phase space is cooled by: $\Delta E / E$

Inelastic case: $E_1 - E_3 = \Delta E = E_4 + Q$

Q can be used as excitation of target particle or ionization

Energy transfer in ionization cooling (intuitive approach)



Total mean energy loss per unit length:

$$\overline{\frac{dE}{dx}} \sim K \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} ln \frac{2 m_e c^2 \beta^2 \gamma^2 W_{max}}{I^2} - \beta^2 \right] \sim n_c \cdot \overline{\Delta E} \sim n_c \cdot W$$

Bethe equation

W: electron-ion pair production energy

Pure cooling rate

- As an example, in liquid H2 (LH2), $\frac{dE}{dx} \sim 320$ keV/cm at $p_\mu \sim 200$ MeV/c
- It is known from measurements, $W_{LH2} \sim 20$ eV, thus $n_c \sim \frac{320 \cdot 10^3}{20} = 16{,}000$ /cm
- Cooling rate is given,
  - $\lambda_{6D}^{-1} = \left(\frac{dE}{dx}\right) \Big/ _{\beta^2 \cdot E}$ (we will derive this later)
  - In the case of LH2,
    - $\lambda_{6D}^{-1} \sim 320 \cdot 10^3 / 177 \cdot 10^6 \sim 1.8 \cdot 10^{-3}$ /cm
    - $\frac{\varepsilon_{6D,initial}}{\varepsilon_{6D,final}} = exp(-L \cdot \lambda_{6D}^{-1})$, if $L = 100$ meter, $\frac{\varepsilon_{6D,initial}}{\varepsilon_{6D,final}} = 1.5 \cdot 10^{-8}$ (our goal is 10^{-6}!)
  - In reality, there is a heating term due to multiple scattering which will be discussed later

Beam cooling in thermodynamics

- Solving Fokker-Planck Equation (FPE) to describe the cooling with stochastic processes (Expert level! See Dugan's PRSTAB paper to find his beautiful work!)
- Let us assume the probability function $f(\vec{x})$, then the entropy is $S = -\int f(\vec{x}) \cdot ln[f(\vec{x})]d\vec{x}$
- If $f(\vec{x})$ is a multiple gaussian form,
  - $f(\vec{x}) = \frac{1}{(2\pi)^{n/2} \cdot det(\Sigma)^{1/2}} exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right]$
  - Then $S = \frac{1}{2} ln[(2\pi e)^n det(\Sigma)]$
  - Indeed, $det(\Sigma) \propto {\varepsilon_n}^2$
  - $S = ln(\varepsilon_n) + constant$
- If ε_n becomes small, S value is reduced

Our first approach to beam dynamics

Interpret the beam dynamics:
We apply a concept of harmonic oscillation to evaluate the beam dynamics equations

Transverse beam dynamics

- Beam dynamics is analogous to the dynamics of a simple harmonic motion
  - Solving harmonic oscillator via canonical transformation

$$\begin{pmatrix} H = E = \frac{p^2}{2m} + \frac{1}{2}kq^2 \\ \dot{p} = -\frac{dH}{dq} = -kq \end{pmatrix} \rightarrow \begin{pmatrix} q = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t + \varphi_0) \\ p = \sqrt{2mE}\sin(\omega t + \varphi_0) \end{pmatrix}$$

Or $\mathcal{H}(P,Q) = E = \omega \cdot P, \dot{Q} = \omega = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} q = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t + \phi_0) \\ p = \sqrt{2mE}\sin(\omega t + \phi_0) \end{pmatrix} \rightarrow \begin{pmatrix} q = \sqrt{\frac{2P}{m\dot{Q}}}\cos(\omega t + \phi_0) \\ p = \sqrt{2m\dot{Q}P}\sin(\omega t + \phi_0) \end{pmatrix}$$

Transverse beam dynamics

- Harmonic motion → Hill's equation (translate the differentiation from t to s)

$$\dot{p} = -\frac{dH}{dq} = -kq \rightarrow x'' + Kx = 0$$

Non-linear & coupling beam dynamics can be investigated through non-linear harmonic oscillators, which most likely uses a perturbation theory

Let us use following general solution for x

$$x = \sqrt{2\hat{\beta}_x(s)J_x}\cos[\hat{\mu}_x(s) + \phi_x]$$

Comparing $q = \sqrt{\frac{2P}{m\dot{Q}}}\cos(\omega t + \phi_0)$

$$P \rightarrow \frac{J_x}{m}, \dot{Q} \rightarrow \frac{1}{\hat{\beta}_x} = \hat{\mu}_x' \leftarrow (\hat{\mu}_x \rightarrow \omega t), \phi_x \rightarrow \phi_0,$$

Transverse beam dynamics

- Let us confirm
  - Differentiate x and x''

$$x' = \frac{dx}{ds}$$
$$= \sqrt{\frac{J_x}{2\hat{\beta}_x(s)}}\hat{\beta}_x{}'(s)\cos[\hat{\mu}_x(s) + \phi_x] + \sqrt{2\hat{\beta}_x(s)J_x}\sin[\hat{\mu}_x(s) + \phi_x] \cdot \hat{\mu}_x{}'(s),$$

$$x'' = \frac{d^2x}{ds^2} = \sqrt{2\hat{\beta}_x(s)J_x}\left\{\hat{\mu}_x{}''(s) + \frac{\hat{\beta}_x{}'(s)}{\hat{\beta}_x(s)}\hat{\mu}_x{}'(s)\right\}\sin[\hat{\mu}_x(s) + \phi_x]$$
$$+\sqrt{2\hat{\beta}_x(s)J_x}\left\{-\frac{1}{4}\frac{\left(\hat{\beta}_x{}'(s)\right)^2}{\hat{\beta}_x^2} + \frac{1}{2}\frac{\hat{\beta}_x{}''(s)}{\hat{\beta}_x(s)} - (\hat{\mu}_x{}')^2\right\}\cos[\hat{\mu}_x(s) + \phi_x].$$

Transverse beam dynamics

- Substitute into Hill's equation
  - Since right-handed-side of Hill's equation is zero, the coefficient of trigonal functions should be zero

$$\mu_x'' + \frac{\hat{\beta}_x'}{\hat{\beta}_x}\hat{\mu}_x' = 0 \rightarrow \left(\hat{\beta}_x \cdot \hat{\mu}_x'\right)' = 0 \rightarrow \boxed{\hat{\mu}_x' = \frac{1}{\hat{\beta}_x(s)},}$$

This is what we look into!

$$-\frac{1}{4}\frac{\left(\hat{\beta}_x'\right)^2}{\hat{\beta}_x^2} + \frac{1}{2}\frac{\hat{\beta}_x''}{\hat{\beta}_x} - (\hat{\mu}_x')^2 + K_x = 0$$

$$\rightarrow 2\hat{\beta}_x\hat{\beta}_x'' - \left(\hat{\beta}_x'\right)^2 + 4K_x\hat{\beta}_x^2 = 4.$$

The second equation is often called envelop equation

Transverse beam dynamics

- Proposing further transformations

$$\hat{\alpha}_x = -\frac{1}{2}\hat{\beta}_x{}', \hat{\gamma}_x = \frac{1+\hat{\alpha}_x{}^2}{\hat{\beta}_x} \to \hat{\beta}_x\hat{\gamma}_x - \hat{\alpha}_x{}^2 = 1,$$

- Surprisingly (or expectedly), x and x' becomes much simple

$$x = \sqrt{2\hat{\beta}_x J_x}\cos\psi_x\,,$$

$$x' = \frac{\hat{\beta}_x{}'}{2\beta_x}\cdot\sqrt{2\hat{\beta}_x J_x}\cos\psi_x - \frac{\sqrt{2\hat{\beta}_x J_x}}{\hat{\beta}_x}\sin\psi_x = -\sqrt{\frac{2J_x}{\hat{\beta}_x}}\left(\sin\psi_x + \hat{\alpha}_x\cos\psi_x\right)$$
$$\left(= -\frac{x}{\hat{\beta}_x}\left[\tan\psi_x - \frac{\hat{\beta}_x{}'}{2}\right]\right).$$

- Envelop equation is also simplified

$$2\hat{\beta}_x\hat{\beta}_x{}'' - \left(\hat{\beta}_x{}'\right)^2 + 4K_x\hat{\beta}_x^2 = 4 \to \hat{\alpha}_x{}' = K_x\hat{\beta}_x - \frac{1}{\beta_x}\left[1+\hat{\alpha}_x{}^2\right]$$
$$= K_x\hat{\beta}_x - \hat{\gamma}_x$$

Transverse beam dynamics

- We introduce a (new) canonical momentum

$$p_x \equiv \hat{\beta}_x x' + \hat{\alpha}_x x = -\sqrt{2\hat{\beta}_x J_x} \sin\psi_x \,.$$

$$x = \sqrt{2\hat{\beta}_x J_x} \cos\psi_x \,,$$

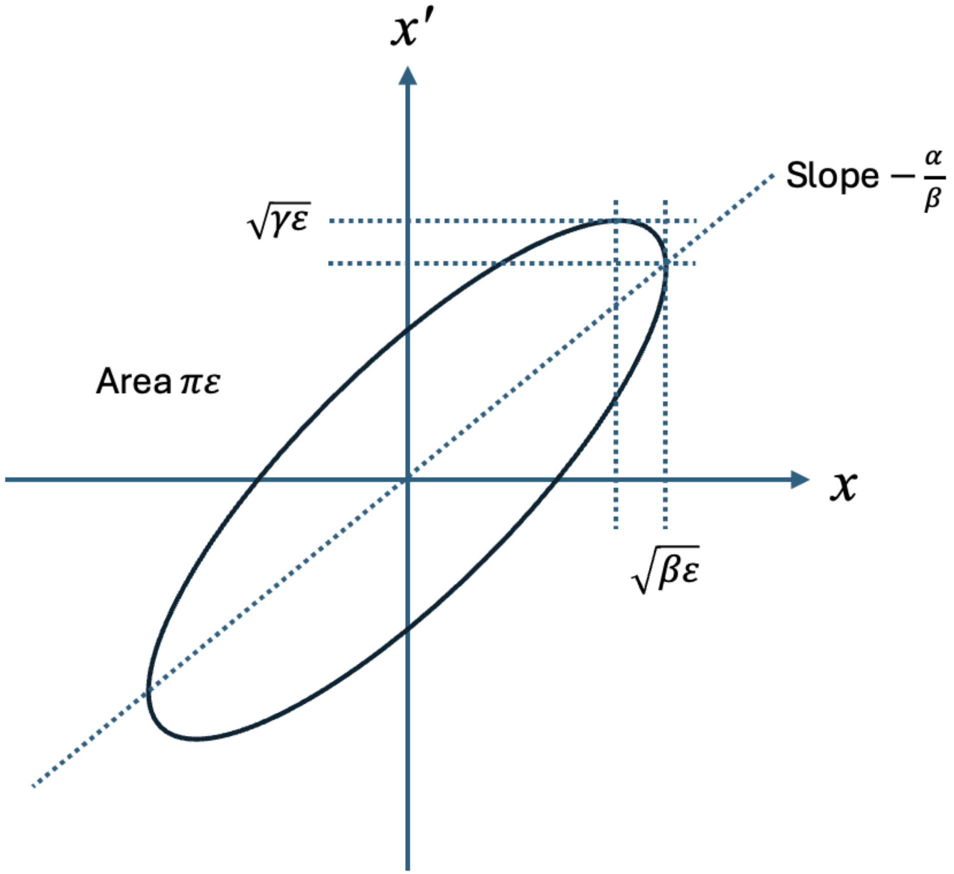
- Then,

$$\hat{\gamma}_x = \frac{1}{\hat{\beta}_x}\left[1 + \hat{\alpha}_x{}^2\right]$$

$$\frac{1}{\hat{\beta}_x}\left(x^2 + p_x^2\right)$$
$$= \frac{1}{\hat{\beta}_x}\left\{\left(1 + \hat{\alpha}_x{}^2\right)x^2 + 2\hat{\alpha}_x\hat{\beta}_x x x' + \hat{\beta}_x{}^2 (x')^2\right\}$$
$$= \hat{\gamma}_x x^2 + 2\hat{\alpha}_x x x' + \hat{\beta}_x (x')^2 = 2J_x \equiv \varepsilon_x.$$

ε_x is called the Courant-Snyder invariant, or emittance

Courant-Snyder invariant (phase ellipse)



x and p_x form a circle with radius $\sqrt{\hat{\beta}_x \cdot \varepsilon_x}$

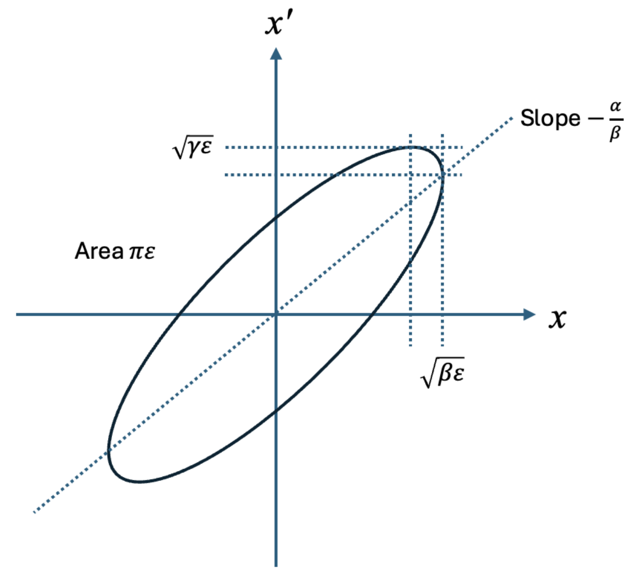
Summary

- We have a solution for particle motion using analogous to the simple harmonic motion

$$x = \sqrt{2\hat{\beta}_x J_x} \cos \psi_x \,,$$

$$x' = -\sqrt{\frac{2J_x}{\hat{\beta}_x}} (\sin \psi_x + \hat{\alpha}_x \cos \psi_x)$$

$$\frac{1}{\hat{\beta}_x} = {\psi_x}'$$



- Smaller β_x makes amplitude of x' larger while amplitude of x smaller
- How do we find β?
  - Can we solve β from envelop equation?

$$2\hat{\beta}_x {\hat{\beta}_x}'' - \left({\hat{\beta}_x}'\right)^2 + 4K_x\hat{\beta}_x^2 = 4 \rightarrow {\hat{\alpha}_x}' = K_x\hat{\beta}_x - \frac{1}{\hat{\beta}_x}\left[1 + {\hat{\alpha}_x}^2\right]$$

$$= K_x\hat{\beta}_x - \hat{\gamma}_x$$

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Find $\hat{\beta}$,$\hat{\alpha}$,$\hat{\gamma}$(Twiss parameter) from phase space evolution

- Assume the density function is given, $\rho(x, x')$

$$\langle x \rangle = \int x \cdot \rho(x, x') dx dx' , \langle x' \rangle = \int x' \cdot \rho(x, x') dx dx',$$

$$\sigma_x^2 = \int (x - \langle x \rangle)^2 \cdot \rho(x, x') dx dx',$$

$$\sigma_{x'}^2 = \int (x' - \langle x' \rangle)^2 \cdot \rho(x, x') dx dx',$$

$$\sigma_{xx'} = \int (x - \langle x \rangle) (x' - \langle x' \rangle) \cdot \rho(x, x') dx dx'.$$

- The rms beam emittance is then,

$$\varepsilon_{x,rms} = \sqrt{Det[\sigma^2]} = \sqrt{\sigma_x^2 \cdot \sigma_{x'}^2 - \sigma_{xx'}^2}.$$

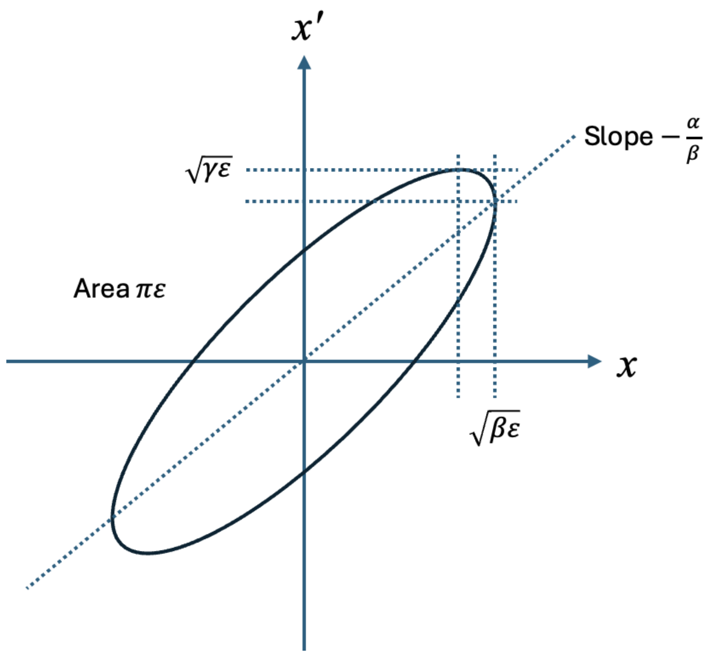
Find $\hat{\beta}$,$\hat{\alpha}$,$\hat{\gamma}$(Twiss parameter) from phase space evolution

- If the beam distribution function is a function of the Courant-Snyder invariant, the σ-matrix is given

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xx\prime} \\ \sigma_{xx\prime} & \sigma_{x\prime}^2 \end{pmatrix} = \varepsilon_{rms} \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}.$$

$$\varepsilon_{x,rms} = \sqrt{Det[\sigma^2]} = \sqrt{\sigma_x^2 \cdot \sigma_{x\prime}^2 - \sigma_{xx\prime}^2}.$$

$$\hat{\beta}_x = \frac{\sigma_x^2}{\varepsilon_{rms}} \quad \hat{\alpha}_x = -\frac{\sigma_{xx\prime}^2}{\varepsilon_{rms}} \quad \hat{\gamma}_x = \frac{\sigma_{x\prime}^2}{\varepsilon_{rms}}$$



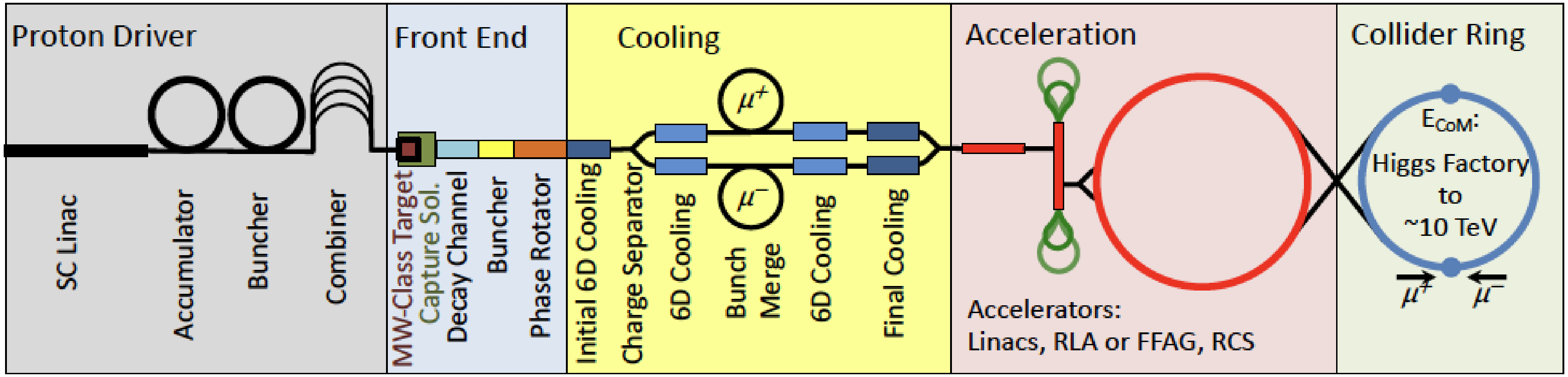
Note: $\widetilde{K} = \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}$ is a symplectic matrix, ie $\widetilde{K}^T \Omega \widetilde{K} = \Omega$

where using $\hat{\beta}\hat{\gamma} = \hat{\alpha}^2 + 1$

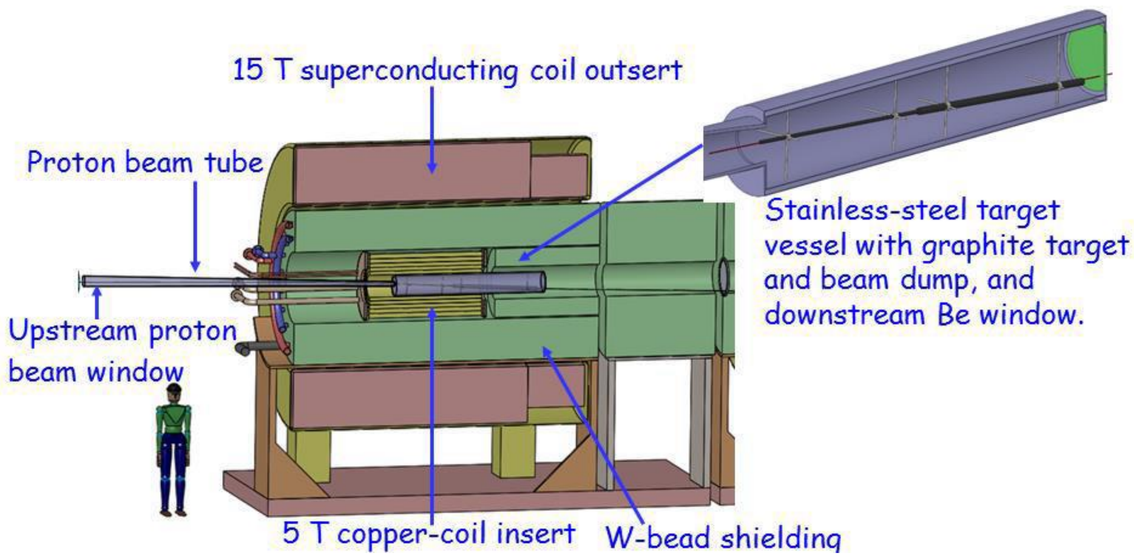
Extra slides

Overview of Muon Collider

Proton beam based muon collider



- $\pi^{\pm}/\mu^{\pm}$ capture magnets



Solenoid base

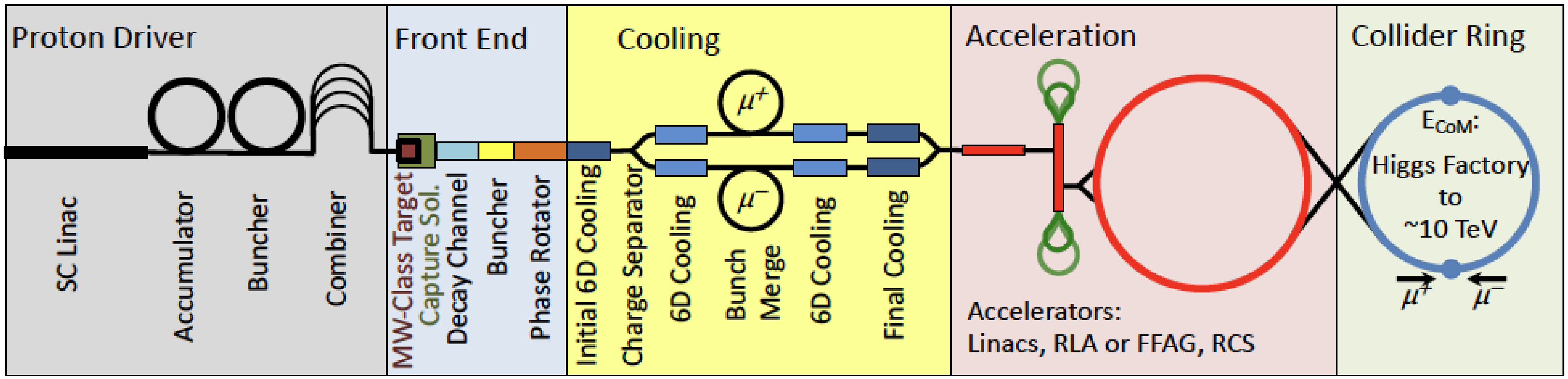
- Massive & expensive
- Most efficient

Horn base

- Cheap
- Charge dependence

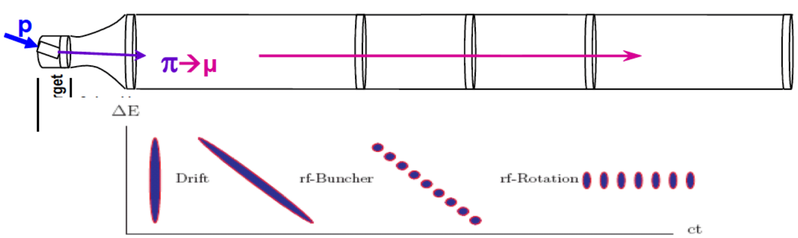
Overview of Muon Collider

Proton beam based muon collider



- Buncher and Phase rotator

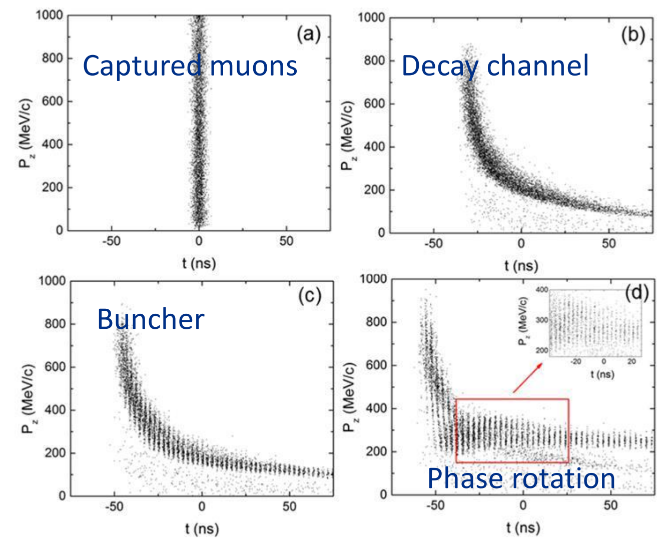
Neuffer



Concept

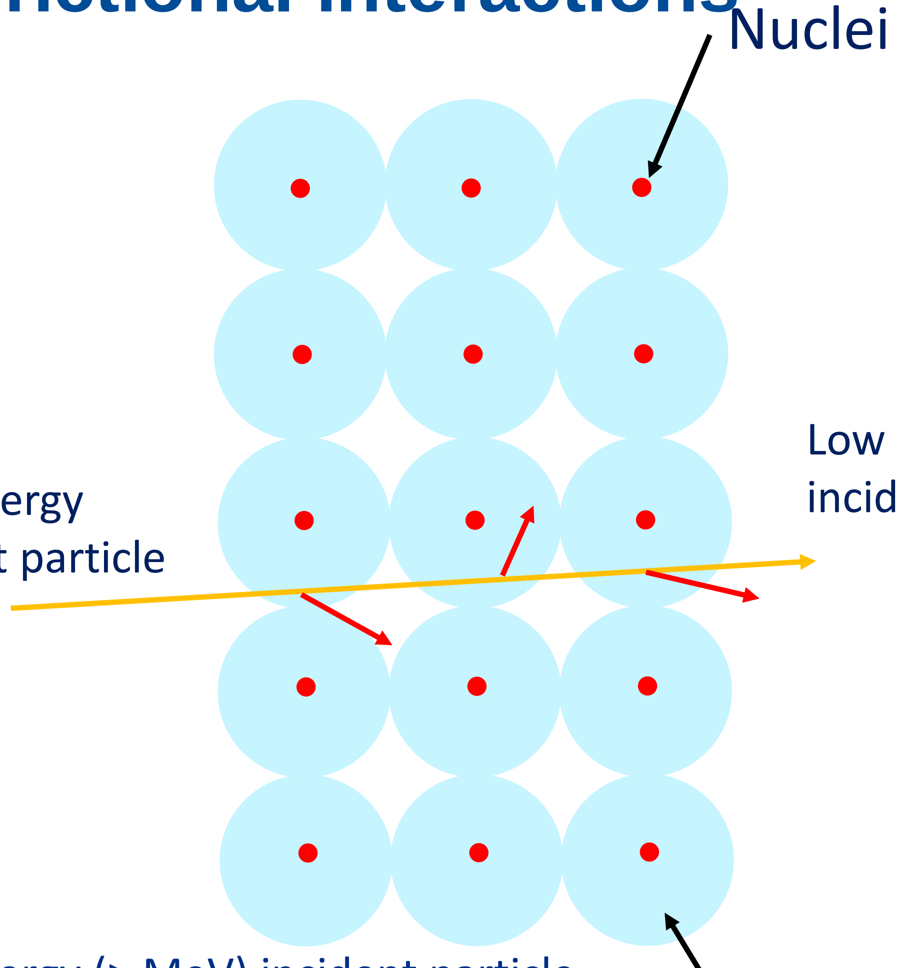
Simulation

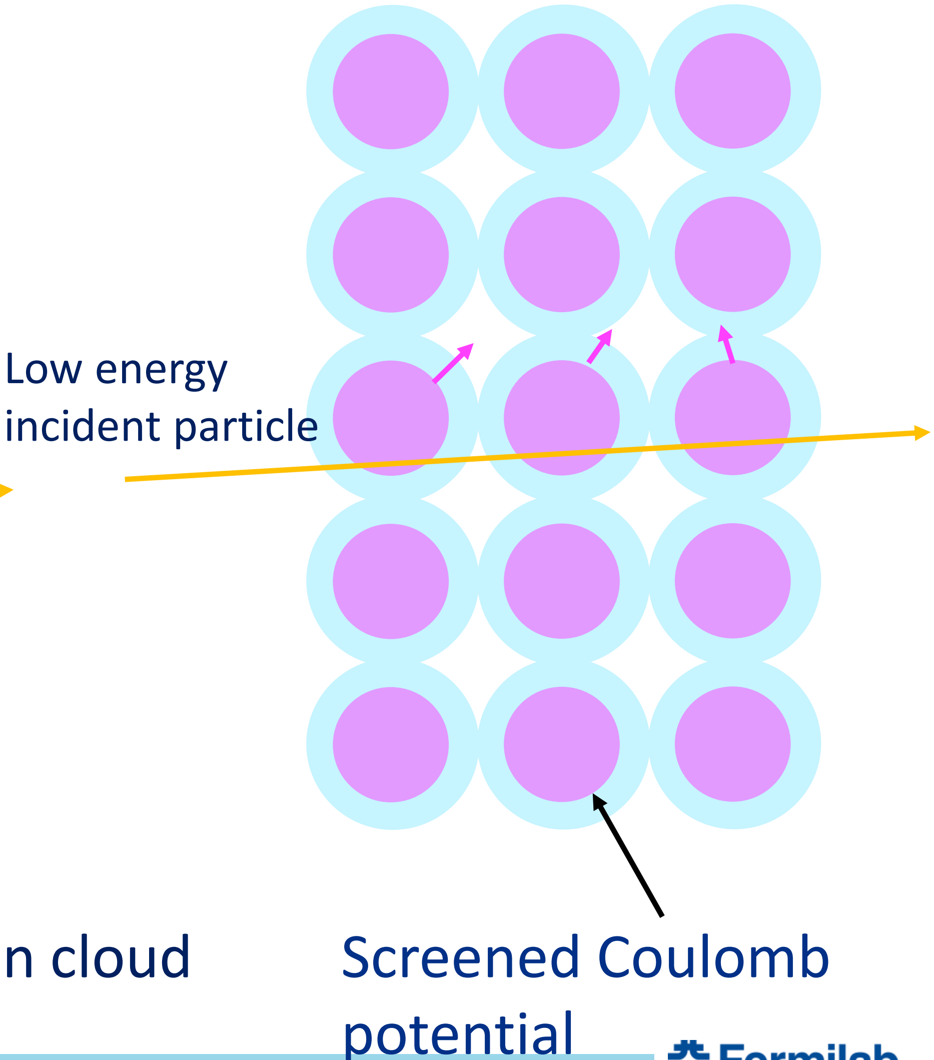
Stratakis et al.



Ionization interactions vs frictional interactions

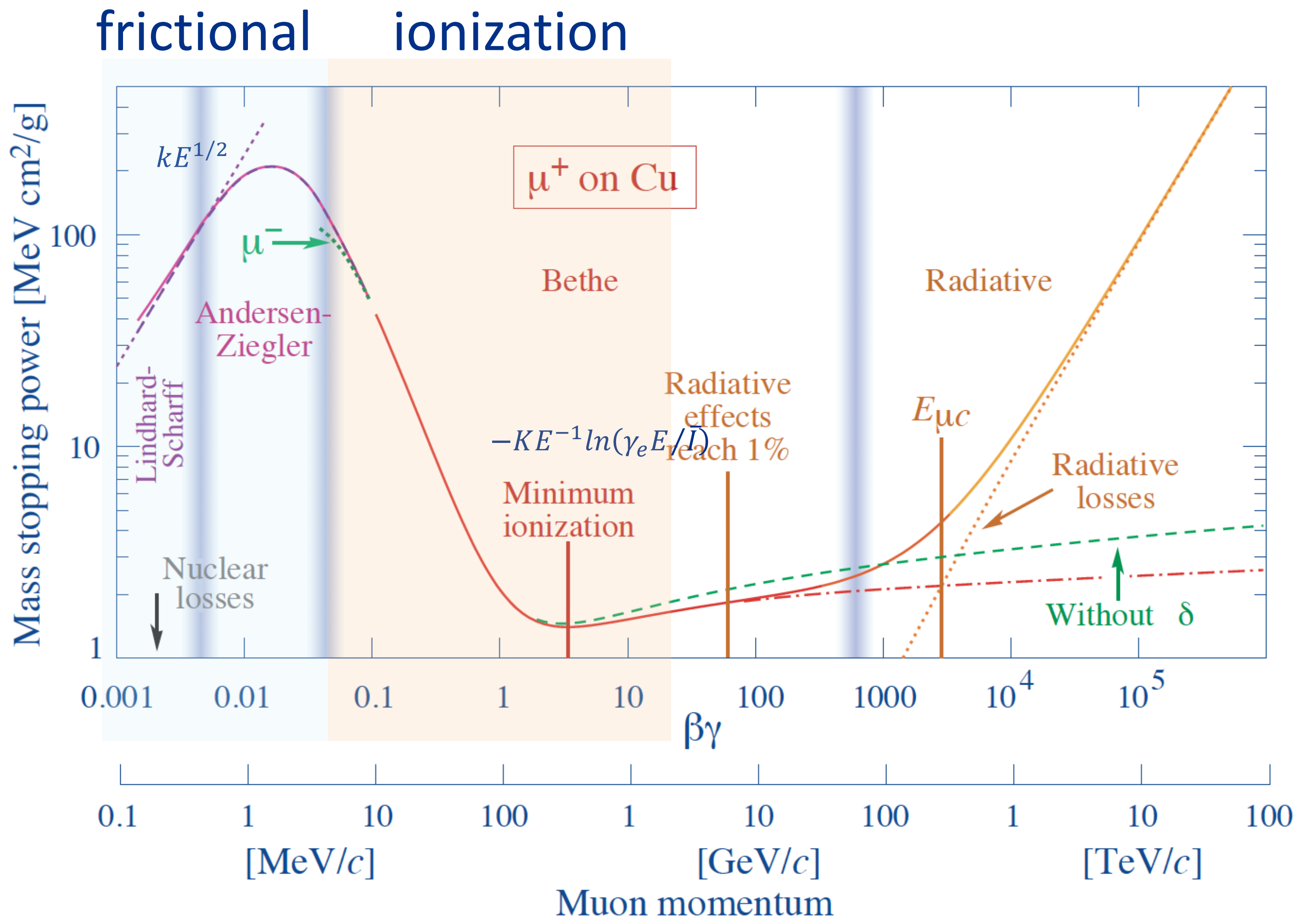
Low energy (< 100 eV) incident particle feels a screened Coulomb potential
$S_e(E)_{Lindhard-Scharff} = kE^{1/2}$





High energy (> MeV) incident particle interact individual atom (electrons)
$$S_e(E)_{Bethe} = KE^{-1}ln(\gamma_e E/\bar{I})$$

Wide range of energy loss value in PDG



Muon momentum range is typically 100 ~ 300 MeV/c for ionization cooling (HF solenoid final channel uses p ~ 50 MeV/c)

Other beam cooling techniques

- Stochastic cooling
- Electron cooling
- Laser cooling