# 2. Set Theory.

A set is any well defined collection of objects, called the elements or members of the set.

Elements may be numbers, points in geometry, letters of alphabets, etc.

capital letters A, B, C ... are ordinarily used to denote the sets and lower case letters a, b, c ... to denote elements of sets.

Well defined means it is possible to decide if a given element belongs to the collection or not.

The statement '$x$ is an element of $A$' or equivalently '$x$ belongs to $A$' (or) $x \in A$.

The statement '$x$ is not an elt of $A$' (or) $x \notin A$.

## Representation of a set:

Two ways
- Roster or Tabular form.
- Rule Method (or) Set Builder form.

### i) Roster or Tabular form:

All the elements of the set are listed, the elements being seperated by commas and enclosed within braces.

eg: 1) Set of binary digits (ie) $A = \{0, 1\}$.

2) Set of vowels in English alphabets : $B = \{a, e, i, o, u\}$

Note :

The order in which the elements of a set are listed is not important.

(ie) $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 3, 1\}$.

(ii) Rule method (or) Set builder form.

A set is defined by specifying a property that elements of the set have in common.

$$\boxed{\begin{array}{l} A = \{x : p(x)\}. \quad (\text{or}) \\ A = \{x / p(x)\}. \end{array}}$$

eg: A set $B = \{1, 4, 9, 16, 25, 36\}$ can be written as

$B = \{x : x = n^2$, where n is a natural number $\leq 6\}$.

Some sets are so important that they are represented by special symbols.

$N = \{x : x$ is a Natural number$\}$

$Z = \{x : x$ is an integer$\}$

$R = \{x : x$ is a real number$\}$

(ie) $N = \{0, 1, 2, 3 \ldots\}$.

$Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$.

$Z^+ = \{1, 2, 3 \ldots\}$ – positive integers

$Q = \{p/q \mid p \in z, q \in z, q \neq 0\}$ – rational nos.

## Finite and Infinite set :

A set with finite number of elements in it, is called a finite set.

An infinite set is a set which contains ~~infite~~ infinite number of elements.

| Finite set | Infinite set |
| --- | --- |
| 1) The set of months in a year | 1) $A$ = set of integers $= \{0, 1, 2 \ldots \ldots\}$ |
| 2) Set of students in a class. | 2) $B = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots\}$ |

## Null set :

A set which contains no elements at all is called the null set (also known as Empty set or Void set). It is denoted by the symbol $\phi$.

eg: $C = \{x : x$ is no. of points in a single throw of a die, $x > 6\}$.

$B = \{x : x$ is a multiple of 4, $x$ is odd$\}$.

$A = \{x : x^2 + 4 = 0,\ x$ is real$\}$.

## Singleton set :

A set which has only one element is called singleton set.

eg: $S = \{a\}$ is a singleton set.

## Universal set :

The set which contains all the objects under consideration is called Universal set and denoted as $U$.

eg: In a study of human population, all people in the world may be assumed to form universal set.

## Subset :

If $A$ and $B$ are sets such that every element of $A$ is also an element of $B$, then $A$ is said to be a subset of $B$. (or $A$ is contained in $B$) and is denoted by $A \subseteq B$.

(ie) $\boxed{A \subseteq B \text{ if } x \in A \text{ and } x \in B.}$

If $A$ is not a subset of $B$.

(ie) atleast one element of $A$ does not belongs to $B$. (ie) $A \not\subseteq B$.

**Note :**

1) Every set $A$ is a subset of itself, $A \subseteq A$.

2) The null set $\phi$ is considered as a subset of any set $A$ (ie) $\phi \subseteq A$.

3) If $A$ is a subset of $B$, & $B$ is subset of $C$, then $A$ is a subset of $C$.

(ie) $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$.

eg: 1) $A = \{1, 3, 4\}$

$B = \{1, 2, 3, 4, 5\}$. $\Rightarrow A \subseteq B$.

2) $A = \{6, 5, 4\}$ & $B = \{4, 5, 6\}$.

then $A \subseteq B$ and $B \subseteq A$.

## Number of subsets of a set.

If a set contains $n$ elements, then the $\boxed{\text{no. of subsets is } 2^n}$.

eg: 1) $A = \{2, 4, 5, 6, 7\}$

Total No. of subsets $= 2^5 = 32$.

2) $A = \{a, b, c\}$.

No. of subsets $= 2^3 = 8$.

Subsets are $\{\ \}$, $\phi$, $\{a\}$. $\{b\}$, $\{c\}$, $\{a, b\}$

$\{b, c\}$, $\{c, a\}$, $\{a, b, c\}$.

## Super set :

If $A$ is a subset of $B$, the $B$ is called the super set of $A$ and is written as $B \supseteq A$.

## Proper subset :

Any subset $A$ is said to be proper subset of another set $B$ if $A$ is the subset of $B$, but there is at least one element of $B$ which does not belong to $A$

i.e) $A \subseteq B$ but $A \neq B$.

i.e) written as $A \subset B$.

eg: $A = \{1, 5\}$
$B = \{1, 5, 6\}$
$C = \{1, 6, 5\}$.

A & B are subsets of C.

But A is proper subset of C.

whereas B is not a proper subset of C

Since $B = C$.

## Equal set :

Two sets A and B are said to be equal iff every element of A is an element of B and consequently every element of B is an element of A.

(ie) $A \subseteq B$ and $B \subseteq A$. (ie) $A = B$.

## Operations on sets :

**UNION:** The union of two sets A and B, denoted by $A \cup B$, is the set of elements that belong to A or to B or to both.

$$\boxed{A \cup B = \{x \mid x \in A \text{ or } x \in B\}}$$

**INTERSECTION :**

The intersection of two sets A and B denoted by $A \cap B$ is the set of elements that belong to both A and B.

$$\boxed{A \cap B = \{x \mid x \in A \text{ and } x \in B\}}$$

## Disjoint :

If $A \cap B$ is the empty set, if A and B do not have any element in common, then the sets A and B are said to be disjoint.

eg: $A = \{1, 3, 5\}$, $B = \{2, 4, 6, 8\}$.

$A \cap B = \phi$.

$\therefore$ A & B are disjoint.

## Complement :

If U is the universal set and A is any set, then the set of elements which belong to U but which do not belong to A is called Complement of A and is denoted by $A'$ (or) $A^c$ (or) $\bar{A}$.

$$\boxed{\bar{A} = \{x \mid x \in U \text{ and } x \notin A\}}$$

eg: $U = \{1, 2, 3, 4, 5\}$. $A = \{1, 3, 5\}$.

then $\bar{A} = \{2, 4\}$.

## Difference / relative component :

If A and B are any two sets, then the set of elements that belong to A but do not belong to B is called difference of A and B or relative component of B w.r.to A and denoted by $A - B$ (or) $A \setminus B$.

$$\boxed{A - B = \{x \mid x \in A \ \& \ x \notin B\}}$$

eg: $A = \{1, 2, 3\}$, $B = \{1, 3, 5, 7\}$ then

$A - B = \{2\}$.

$B - A = \{5, 7\}$

Symmetric difference :

If A and B are any two sets, the set of elements that belong to A or B, but not to both is called the symmetric difference of A and B, is denoted by

$A \oplus B$ (or) $A \Delta B$ (or) $A + B$ :

i.e. $\boxed{A \oplus B = (A - B) \cup (B - A)}$

eg : $A = \{a, b, c, d\}$

$B = \{c, d, e, f\}$

$A \oplus B = \{a, b, e, f\}$

Ordered pairs :

A pair of objects whose components occur in a specific order is called an ordered pair

The ordered pairs $(a, b)$, $(c, d)$ are equal iff $a = c$, $b = d$.

Cartesian product :

If A & B are sets, the set of all ordered pairs whose first component belongs to A and second component belongs to B is called cartesian product of A & B, is denoted by $A \times B$.

$$\boxed{A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.}$$

eg: $A = \{a,b,c\}$, $B = \{1,2\}$.

$A \times B = \{(a,1)(a,2)\ (b,1)(b,2)\ (c,1)(c,2)\}$

$B \times A = \{(1,a)(1,b)\ (1,c)\ ,(2,a)(2,b)(2,c)\}$.

Note :

1) ordered pair $(a,b) \neq (b,a)$ unless $a=b$.

2) cartesian product $A \times B \neq B \times A$ unless $A = \phi$ (or) $B = \phi$ (so that $A \times B = \phi$) (or) $A = B$.

## Algebraic laws of set theory.

Set identities.

| | Identity | Name of the law. |
|---|---|---|
| 1) | $A \cup A = A$ <br> $A \cap A = A$ | Idempotent law |
| 2) | $(A \cup B) \cup C = A \cup (B \cup C)$ <br> $(A \cap B) \cap C = A \cap (B \cap C)$ | Associative laws |
| 3) | $A \cup B = B \cup A$ <br> $A \cap B = B \cap A$ | Commutative laws |
| 4) | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ <br> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributive laws |

## Set identities.

| | | |
|---|---|---|
| 5) | $A \cup \phi = A$<br>$A \cap U = A$ | Identity law |
| 6) | $A \cup U = U$<br>$A \cap \phi = \phi$ | Domination law. |
| 7) | $\overline{\overline{A}} = A$ (or) $(A')' = A$ | Involution (or) Double complement law |
| 8) | $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \phi$<br>$U' = \phi$<br>$\phi' = U$. | complement / Inverse law. |
| 9) | $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption law. |
| 10) | $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$<br>$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ | DeMorgan's law. |

**Problems:**

1) Prove that $(A-C) \cap (C-B) = \phi$ analytically, where $A, B, C$ are sets.

$(A-C) \cap (C-B)$

$= \{x \mid (x \in A \text{ and } x \notin C) \text{ and } (x \in C \text{ and } x \notin B)\}$ Diff.

$= \{x \mid x \in A \text{ and } (x \in C \text{ and } x \in \overline{C}) \text{ and } x \in \overline{B}\}$

$= \{x \mid (x \in A \text{ and } x \in \phi) \text{ and } x \in \overline{B}\}$ commu/: complement

$= \{x \mid x \in A \cap \phi \text{ and } x \in \overline{B}\}$ Domination

$= \{x \mid x \in \phi \text{ and } x \in \overline{B}\}$

$= \{x \mid x \in \phi \cap \overline{B}\} = \{x \mid x \in \phi\} = \phi.$

Domination.

2) If $A, B$ and $C$ are sets, prove analytically that $A-(B\cap C) = (A-B)\cup(A-C)$

$A-(B\cap C) = \{x \mid x\in A \text{ and } x\notin (B\cap C)\}$

$= \{x \mid x\in A \text{ and } (x\notin B \text{ or } x\notin C)\}$ Difference

$= \{x \mid (x\in A \text{ and } x\notin B) \text{ or }$
$(x\in A \text{ and } x\notin C)\}$

$= \{x \mid (x\in(A-B) \text{ or } x\in(A-C)\}$ Diff:

$= \{x \mid x\in (A-B)\cup(A-C)\}$

$= (A-B)\cup(A-C)$

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3) If $A, B$ and $C$ are sets, prove analytically that $A\cap(B-C) = (A\cap B)-(A\cap C)$

$A\cap(B-C) = \{x \mid x\in A \text{ and } x\in (B-C)\}$

$= \{x \mid x\in A \text{ and } (x\in B \text{ and } x\notin C)\}$

$= \{x \mid x\in A \text{ and } (x\in B \text{ and } x\in \bar{C})\}$

$= \{x \mid x\in (A\cap B\cap \bar{C})\}$

$= A\cap B\cap \bar{C}$

$(A\cap B)-(A\cap C) = \{x \mid x\in (A\cap B) \text{ and } x\in \overline{A\cap C}\}$

$= \{x \mid x\in (A\cap B) \text{ and } x\in (\bar{A}\cup\bar{C})\}$

$= \{x \mid x\in (A\cap B) \text{ and } (x\in\bar{A} \text{ or } x\in\bar{C})\}$

$= \{x \mid [x\in (A\cap B) \text{ and } x\in\bar{A}] \text{ or }$
$[x\in(A\cap B) \text{ and } x\in\bar{C}]\}$

$= \{x \mid [x\in A\cap B\cap\bar{A}] \text{ or }$
$[x\in A\cap B\cap\bar{C}]\}$

$= \{x \mid x \in \phi \text{ or } x \in (A \cap B \cap \overline{C})\}$

$= \{x \mid x \in A \cap B \cap \overline{C}\}$

$= A \cap B \cap \overline{C}$.

LHS = RHS.

4) If A, B and C are sets, prove that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$, using set identities

LS: $\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{B \cap C}$ — De Morgans

$= \overline{A} \cap (\overline{B} \cup \overline{C})$ — De Morgans

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$ — Commutative

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$ — commutative.

$=$ RHS

5) Simplify the following using set identities.

(i) $\overline{A} \cup \overline{B} \cup (A \cap B \cap \overline{C})$

$= (\overline{A \cap B}) \cup (A \cap B \cap \overline{C})$ — De Morgans.

$= [\overline{A \cap B} \cup (A \cap B)] \cap [\overline{A \cap B} \cup \overline{C}]$ — Distributive

$= U \cap (\overline{A \cap B} \cup \overline{C})$ — Inverse law.

$= \overline{A \cap B} \cup \overline{C}$ — Identity $U \cap A = A$

$= \overline{A} \cup \overline{B} \cup \overline{C}$ — De Morgans.

(ii) $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))]$

$= (A \cap B) \cup [B \cap \{C \cap (D \cup \overline{D})\}]$ — Distributive law

$= (A \cap B) \cup [B \cap (C \cap U)]$ — Inverse law

$= (A \cap B) \cup [B \cap C]$ Identity

$= (B \cap A) \cup (B \cap C)$ commutative

$= B \cap (A \cup C)$ Distributive

6) Write dual of the following statements :

(i) $A = (\bar{B} \cap A) \cup (A \cap B)$

$\Rightarrow A = (\bar{B} \cup A) \cap (A \cup B)$.

(ii) $(A \cap B) \cup (\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}) = U$

$\Rightarrow (A \cup B) \cap (\bar{A} \cup B) \cap (A \cup \bar{B}) \cap (\bar{A} \cup \bar{B}) = \phi$.

[Replace $\cup$ by $\cap$, $\cap$ by $\cup$, $U$ by $\phi$ and $\phi$ by $U$.]

RELATION :

Relation on sets :

Let $A$ and $B$ be two sets.

A relation from $A$ to $B$ is a subset of the cartesian product $A \times B$

The cartesian product $A \times B$ consists of all ordered pairs whose first element is in $A$ and whose second element is in $B$.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

If $A = \{1, 2, 5\}$, $B = \{2, 4\}$ then

$$A \times B = \{(1,2)\ (1,4)\ (2,2)\ (2,4)\ (5,2)\ (5,4)\}$$

If $x < y$, then the relation is

$R = \{(1,2)\ (1,4)\ (2,4)\}$.

**Note:**

If $R$ is a relation from a set $A$ to itself (ie) $R$ is a subset of $A^2 = A \times A$. then $R$ is a relation on $A$.

## Domain and Range:

Let $A$ and $B$ be two non-empty set. $R$ be binary relation from $A$ to $B$.

The set $\{a \in A : (a, b) \in R$ for some $b \in B\}$ is called **domain of R** and denoted by $Dom(R)$

(ie) It is the set of all first element of the ordered pairs which belong to $R$.

The set $\{b \in B : (a, b) \in R$ for some $a \in A\}$ is called **range of R** and denoted by $Ran(R)$.

(ie) It is the set of all second elements.

**eg:**

1) Let $A = \{2, 3, 4\}$ $B = \{3, 4, 5\}$

$R$ is defined as

a) $a \in A$ is related to $b \in B$, (ie) $aRb$ iff $a < b$.

b) $a \in A$ is related to $b \in B$, (ie) $aRb$ if $a$ & $b$ both are odd nos):

a) $2 \in A$ is less than $3 \in B$ then $2R3$.

$R = \{(2,3)(2,4)(2,5)(3,4)(3,5)(4,5)\}$

$Dom(R) = \{2,3,4\}$.

$Ran(R) = \{3,4,5\}$.

b) $3 \in A$, $3 \in B$ are both odd then $3R3$.

III'y $3R5$.

$R = \{(3,3)(3,5)\}$

$Dom(R) = \{3\}$

$Ran(R) = \{3,5\}$.

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Total No. of distinct relation from set A to set B is $\boxed{2^{mn}}$.

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## Set operations on Relations.

### 1) Intersection:

If R & S denote two relations, the intersection of R and S denoted by $R \cap S$, defined by:

$$\boxed{a(R \cap S)b = (aRb) \wedge (aSb)}$$

### 2) Union:

The union of R and S, denoted by $R \cup S$, is defined by

$$\boxed{a(R \cup S)b = (aRb) \vee (aSb)}$$

### 3) Difference:

The difference R and S, denoted by $R-S$, defined by $\boxed{a(R-S)b = (aRb) \wedge (a \not S b)}$

4. Complement.

The complement of $R$ denoted by $R'$ or $\neg R$ or $\sim R$ is defined by

$$\boxed{a(R')b = a \not R b.}$$

Ex:

Let $A = \{x, y, z\}$ $B = \{1, 2, 3\}$ $C = \{x, y\}$ $D = \{2, 3\}$.

Let $R$ be a relation from $A$ to $B$ defined by

$$R = \{(x,1)\ (x,2),\ (y,3)\}.$$

Let $S$ be a relation from $C$ to $D$ defined by

$$S = \{(x,2),\ (y,3)\}$$

Then
$$R \cap S = \{(x,2)\ (y,3)\}$$
$$R \cup S = R$$
$$R - S = \{(x,1)\}.$$
$$R' = \{(x,3)\ (y,1)(y,2)\ (z,1)(z,2)(z,3)\}$$

Types of relations in a set:

Let $R \subseteq A \times B$ be a relation from $A$ to $B$.

i) Then the inverse relation. $R^{-1} \subseteq B \times A$ defined by

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

It is clear that

(i) $a R b \Leftrightarrow b R^{-1} a$

(ii) $\operatorname{dom} R^{-1} = \operatorname{Ran}(R)$

(iii) $\operatorname{Ran}(R^{-1}) = \operatorname{dom}(R)$

(iv) $(R^{-1})^{-1} = R$.

**Ex : 1**

**Q:** Let $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ and, define a binary relation $R$ from $A$ to $B$ as follows :

For all $(x, y) \in A \times B$, $(x, y) \in R \Leftrightarrow x/y$ write each $R$ and $R^{-1}$ as a (~~x setdivides~~ y) a set of ordered pairs.

(contd) ↓

(ii) **Identity relation:**

A relation $R$ on a set $A$ is called an identity relation, if $R = \{(a, a) \mid a \in A\}$ and denoted by $I_A$.

**Ex-1: soln :**

**Ans:** Here $2 \in A$ divides $6 \in B$.

$R = \{(2, 8)\ (2, 10)\ (2, 6)\ (3, 6)\ (5, 10)\}$

$R^{-1} = \{(8, 2)\ (10, 2)\ (6, 2)\ (6, 3)\ (10, 5)\}$.

$\operatorname{Dom}(R) = \operatorname{Ran}(R^{-1}) = \{2, 3, 5\}$

$\operatorname{Rang}(R) = \operatorname{Dom}(R^{-1}) = \{6, 8, 10\}$

**Ex: 2** $A = \{1, 2, 3\}$

$I_A = \{(1, 1)\ (2, 2)\ (3, 3)\}$ is identity relation in $A$.

## Properties of Relations :

| | Property | Meaning |
|---|---|---|
| 1) | Reflexivity | $(a,a) \in R$<br>(ie) $aRa \ \forall \ a \in A$ |
| 2) | Irreflexivity | $(a,a) \notin R$<br>(ie) $a \not R a \ \forall \ a \in A$ |
| 3) | Symmetry | $(a,b) \in R \Rightarrow (b,a) \in R$<br>(ie) $aRb \Rightarrow bRa \ \forall \ a,b \in A$ |
| 4) | Asymmetry | $(a,b) \in R \Rightarrow (b,a) \notin R$<br>(ie): $aRb \Rightarrow b \not R a \ \forall \ a,b \in A$ |
| 5) | Antisymmetry | $(a,b) \in R \wedge (b,a) \in R \Rightarrow a=b$<br>(ie) $aRb$ and $bRa \Rightarrow a=b$.<br>$\forall \ a,b \in A$ |
| 6) | Transitivity | $(a,b) \in R \wedge (b,c) \in R$<br>$\Rightarrow (a,c) \in R$<br>(ie) $aRb$ and $bRc \Rightarrow aRc$<br>$\forall \ a,b,c \in A$ |

## Composition of Relation :

Let $A, B, C$ be any three sets and $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$, then $R \cdot S$ ($R$ composition $S$) is a relation from $A$ to $C$

eg: 1 $R = \{(1,2)\}$, $S = \{(2,4)\}$. $R \cdot S = \{(1,4)\}$

eg: 2

If $A = \{1, 2, 3, 4\}$.

$R = \{(1,1)(1,2)(2,3)(2,4)(3,4)(4,1)(4,2)\}$

$S = \{(3,1)(4,4)(2,3)(2,4)(1,1)(1,4)\}$

on A. Find 1) $R \cdot R$ 2) $S \cdot R$.

$R \cdot R = \{(1,2)(1,3)(1,4)(2,4)(2,1)(2,2),(3,1)$
$(3,2)(4,1)(4,2)(4,3)(4,4)\}$

$S \cdot R = \{(3,1)(3,2)(4,1)(4,2)(2,\cancel{4})(2,1)(4,1)(1,1)$
$(2,4)\cancel{(2,4)}\cancel{(2,4)},(1,2)(2,2)\}$.

## Equivalence relation :

A relation R on a set A is called Equivalence relation if it is reflexive, symmetric and Transitive.

## Examples :

1) List the ordered pairs in the relation R from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$ where $(a,b) \in R$ iff (i) $a = b$ (ii) $a+b = 4$ (iii) $a > b$ (iv) $a|b$ (a divides b) (v) $\gcd(a,b) = 1$, (vi) $\operatorname{lcm}(a,b) = 2$.

(i) $a = b$.

$a \in A$ & $b \in B$ and $a R b$ when $a = b$.

$R = \{(0,0)(1,1)(2,2)(3,3)\}$.

(ii) $a+b = 4$

$aRb$ iff $a+b = 4$

$R = \{(1,3)(2,2)(3,1)(4,0)\}$.

(iii) $aRb$ iff $a > b$.

$R = \{(1,0)\ (2,0), (3,0)\ (4,0),\ (2,1), (3,1),$
$(3,2)\ (4,1)\ (4,2)\ (4,3)\}$

(iv) $a|b$. ($b$ is divisible by $a$)

$aRb$ iff $a|b$.

$R = \{(1,0)\ (1,1)\ (1,2)\ (1,3), (2,0)\ (2,2),$
$(3,0)\ (3,3)\ (4,0)\}$

$\because \frac{0}{0}$ is indeterminate, (0 does not divide 0).

(v) $\gcd(a,b) = 1$.

$aRb$ iff $\gcd(a,b) = 1$.

$R = \{(0,1)\ (1,0)\ (1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,3),$
$(3,1)\ (3,2)\ (4,1)\ (4,3)\}$.

(vi) $\operatorname{lcm}(a,b) = 2$.

$\therefore aRb$ iff $\operatorname{lcm}(a,b) = 2$.

$R = \{(1,2)\ (2,1)\ (2,2)\}$.

2) The relation $R$ on the set $A = \{1,2,3,4,5\}$ is defined by the rule $(a,b) \in R$ if 3 divides $a-b$

(i) List the elements of $R$ and $R^{-1}$.

(ii) Find the domain and range of $R$

(iii) Find the domain and range of $R^{-1}$.

(iv) List the elements of the complement of $R$.

Given $A = \{1, 2, 3, 4, 5\}$.

The Rule : $(a, b) \in R$.

Cartesian product $A \times A$ consist of

$R_A =$ $A \times A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

(i) List the element of $R, R^{-1}$ if 3 divides $(a-b)$.

$R = \{(1,1), (1,4), (2,2), (2,5), (3,3), (4,1), (4,4), (5,2), (5,5)\}$

$R^{-1}$ (inverse of R)

$= \{(1,1)\ (4,1)\ (2,2)\ (5,2)\ (3,3)\ (1,4)\ (4,4), (2,5)\ (5,5)\}$

(ii) Dom R $=$ Ran R $= \{1, 2, 3, 4, 5\}$

(iii) Dom $R^{-1}$ $=$ Ran $R^{-1}$ $= \{1, 2, 3, 4, 5\}$.

(iv) $R'$ (complement of R).

$= \{(1,2)\ (1,3)\ (1,5)\ (2,1)\ (2,3)\ (2,4), (3,1)\ (3,2)\ (3,4)\ (3,5), (4,2)\ (4,3)\ (4,5)\ (5,1)\ (5,3)\ (5,4)\}$

---

3) If $R = \{(1,2)\ (2,4)\ (3,3)\}$ . and

$S = \{(1,3)\ (2,4)\ (4,2)\}$

Find (i) $R \cup S$ (ii) $R \cap S$ (iii) $R - S$.

(iv) $S - R$ (v) $R \oplus S$.

Verify vi) $dom(R \cup S) = dom(R) \cup dom(S)$

vii) $Ran(R \cap S) \subseteq Ran(R) \cap Ran(S)$.

1) Given $R = \{(1,2)(2,4)(3,3)\}$

$S = \{(1,3)(2,4)(4,2)\}$.

(i) $R \cup S = \{(1,2)(1,3)(2,4)(3,3),(4,2)\}$

(ii) $R \cap S = \{(2,4)\}$

(iii) $R - S = \{(1,2)(3,3)\}$

(iv) $S - R = \{(1,3)(4,2)\}$

(v) $R \oplus S = (R-S) \cup (S-R)$ (or) $(R \cup S) - (R \cap S)$

$= \{(1,2)(1,3),(3,3),(4,2)\}$.

(vi) dom $R = \{1,2,3\}$

dom $S = \{1,2,4\}$.

$dom(R) \cup dom(S) = \{1,2,3,4\}$.

$dom(R \cup S) = \{1,2,3,4\}$.

LHS = RHS.

(vii) Range $R = \{2,3,4\}$

Range $S = \{2,3,4\}$

$Ran(R \cap S) = \{4\}$

$Ran(R) \cap Ran(S) = \{2,3,4\}$.

clearly $\{4\} \subseteq \{2,3,4\}$.

Hence, $Ran(R \cap S) \subseteq Ran(R) \cap Ran(S)$

4) (i) If $R$ is the relation on the set of ordered pairs of positive integers such that $(a,b), (c,d) \in R$ whenever $ad = bc$, show that $R$ is an equivalence relation.

(ii) If $R$ is the relation on the set of positive integers such that $(a,b) \in R$ iff $ab$ is perfect square. Show that $R$ is an equivalence relation.

Soln:

(i) Given $(a,b), (c,d) \in R$ whenever $ad = bc$.

To prove : $R$ is an equivalence relation.

→ $(a,b)\ R\ (a,b)$ $\because ab = ba$. eg: $\{(1,2)\}$

$\therefore$ $R$ is reflexive

→ when $(a,b)\ R\ (c,d)$, $ad = bc$. eg: $\{(1,2)(2,4)\}$

ie) $cb = da$

⇒ $(c,d)\ R\ (a,b)$

$\therefore$ $R$ is symmetric.

→ when $(a,b)\ R\ (c,d)$, $ad = bc$. eg: $\{(1,2)(3,6),(4,8)\}$

when $(c,d)\ R\ (e,f)$, $cf = de$

⇒ $af = be$ ie) $(a,b)\ R\ (e,f)$.

$(\because c \ \& \ d > 0)$.

$\therefore$ $R$ is transitive.

Hence, $R$ is an equivalence relation.

(ii) Given $(a,b) \in R$. iff $ab$ is perfect square.

To prove : $R$ is an equivalence relation.

→ $(a,a) \in R$, $\because a^2$ is perfect square

$\therefore$ $R$ is reflexive.

→ $(a,b) \in R$ if $ab$ is perfect square.

Then $ba$ is also perfect square.

$a R b \Rightarrow b R a$.

$\therefore$ R is symmetric

→ If $aRb$, $ab = x^2$.

If $bRc$, $bc = y^2$.

product of above

$$a(b^2)c = x^2y^2.$$

$$ac = \frac{x^2y^2}{b^2} = \left(\frac{xy}{b}\right)^2 \text{ is a perfect square.}$$

$\therefore a R c$.

(i.e) R is transitive.

Hence R is an equivalence relation.

---

5) Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transistive where $aRb$ iff (i) $a \neq b$ (ii) $ab \geq 0$ (iii) $ab \geq 1$.

(iv) a is a multiple of b (v) $a \equiv b \pmod 7$

(vi) $|a-b| = 1$ (vii) $a = b^2$ (viii) $a \geq b^2$

(i) $a \neq b$.

→ $a \neq a$ is not true. $\therefore$ R is not reflexive

→ $a \neq b \Rightarrow b \neq a$. $\therefore$ R is symmetric.

→ Here $a \neq b$. $\therefore$ R is not antisymmetric.

→ $a \neq b$ & $b \neq c \not\Rightarrow a \neq c$ R is not transitive.

(ii) $ab \geq 0$.

$\rightarrow a^2 \geq 0$ $\therefore$ R is reflexive

$\rightarrow ab \geq 0 \Rightarrow ba \geq 0$ $\therefore$ R is symmetric.

$\rightarrow ab \geq 0$ and $bc \geq 0 \not\Rightarrow ac \geq 0$. if $a < 0$ (or) $c < 0$. $\therefore$ R is not transitive

eg: $(2, 0)$ and $(0, -3) \in R$
($a$, $b$ and $b$, $c$)

$\Rightarrow (2, -3) \notin R$ as $2(-3) < 0$. (Gn condition $ab \geq 0$ not satisfied).

(iii) $ab \geq 1$.

$\rightarrow a^2 \geq 1$ is not true. since a may be zero.

$\therefore$ R is not reflexive

$\rightarrow ab \geq 1 \Rightarrow ba \geq 1$, then R is symmetric.

$\rightarrow$ If all of $a, b, c > 0$,

Least $a$ = least $b$ = least $c$ = 1.

$\therefore ac \geq 1$. (or)

greatest $a$ = greatest $b$ = greatest $c$ = $-1$.

if $a, b, c < 0$.

$\therefore ac \geq 1$.

$\rightarrow \therefore$ R is transitive.

(iv) a is multiple of b.

$\rightarrow$ a is multiple of a R is reflexive

$\rightarrow$ a is multiple of b, b is not a multiple of a R is not symmetric

$\rightarrow$ (ie) a mul. of b & b mul. of a only if $a = b$. R is antisymmetric.

$\rightarrow$ a mul. of b, b mul. of c $\Rightarrow$ a is mul. of c. R is transitive.

(v) $(a-b)$ is a multiple of 7.

$\rightarrow$ $(a-a)$ is a mul. of 7. R is reflexive.

$\rightarrow$ $(a-b)$ is mul. of 7. R is symmetric.

$\Rightarrow (b-a)$ is mul. of 7

$\rightarrow$ $(a-b)$ and $(b-c)$ are mul. of 7.

$(a-b)+(b-c) = (a-c)$ R is transitive

$\therefore (a-c)$ is a mul. of 7

(vi) $|a-b| = 1$.

$\rightarrow$ $|a-a| \neq 1$ R is reflexive

$\rightarrow$ $|a-b| = 1 \Rightarrow |b-a| = 1$ R is symmetric.

$\rightarrow$ $|a-b| = 1 \Rightarrow a-b = 1$ (or) $-1$

$|b-c| = 1 \Rightarrow b-c = 1$ (or) $-1$

Adding, $a-c = \pm 2$ or $0$. R is not transitive.

$\Rightarrow |a-c| = 2$ (or) $0$

(ie) $|a-c| \neq 1$

(vii) $a = b^2$.

$\rightarrow$ $a = a^2$ is not true for all integers R is not reflexive

$\rightarrow$ $a = b^2$ and $b = a^2$, R is antisymmetric

when $a = b = 0$ (or)

$a = b = 1$.

$\rightarrow$ $a = b^2$ and $b = c^2$ R is not transitive.

$\not\Rightarrow a = c^2$

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(viii) $a \geq b^2$

$\rightarrow a \geq a^2$ Not true $\forall$ integers — R is not reflexive

$\rightarrow a \geq b^2$ & $b \geq a^2 \Rightarrow$ if $a = b = 0$, $a = b = 1$ — R is antisymmetric

$\rightarrow a \geq b^2$ and $b \geq c^2 \Rightarrow a \geq c^2$ — R is transitive.

---

6) (i) If R is the relation on the set of positive integers such that $(a,b) \in R$ iff $a^2+b$ is even, Prove R is an equivalence relation.

(ii) If R is the relation on the set of integers such that $(a,b) \in R$ iff $3a + 4b = 7n$ for some integer $n$, P.T R is an equivalence relation.

i) Given $(a,b) \in R$

$\rightarrow a^2 + a = a(a+1) =$ even $\rightarrow a$ & $a+1$ are consecutive +ve integers.

$\therefore$ R is reflexive.

$\rightarrow a^2+b$ is even, $(a,b) \in R$.

$a$ & $b$ must be both even (or) both odd.

$\Rightarrow b^2 + a$ is even

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

$\therefore$ R is symmetric.

$\rightarrow$ If $a, b, c$ are even

$a^2+b$ & $b^2+c$ are even $\Rightarrow a^2+c$ is even.

If $a, b, c$ are odd

$a^2+b$ & $b^2+c$ are even $\Rightarrow a^2+c$ is even

$(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$. $\therefore$ R is transitive.

$\therefore$ R is an equi. relation.

(ii) $3a+4b=7n$ , $n$ – for some integer.

$(a,b)\in R$.

$\to$ $3a+4a=7a$ , $a$ is an integer.

$(a,a)\in R$.

$\therefore$ R is reflexive

$\to$ Gi $3a+4b=7n$.

$$3b+4a = (7a-3a)+(7b-4b)$$

$$= 7a+7b-(3a+4b)$$

$$= 7a+7b-7n$$

$$= 7(a+b-n).$$

Here $(a+b-n)$ is an integer.

$(b,a)\in R$ when $(a,b)\in R$.

$\therefore$ R is symmetric.

$\to$ Let $(a,b)\in R$ & $(b,c)\in R$.

i.e. $3a+4b=7n$

$3b+4c=7m$.

$$\Rightarrow\quad 3a+4c = 7n+7m-7b.$$

$$= 7(m+n-b).$$

where $(m+n-b)$ is an integer.

$\therefore$ $(a,c)\in R$.

i.e. R is transitive

$\therefore$ R is an equivalence relation.

## The Matrix of a Relation:

Let $A = \{a_1, a_2, a_3 \ldots a_m\}$ and $B = \{b_1, b_2, b_3 \ldots b_n\}$ are finite sets containing $m$ and $n$ elements respectively and let $R$ be a relation from $A$ to $B$. Then $R$ can be represented by $m\,n$ matrix.

$$\boxed{\begin{aligned} M_R &= [m_{ij}] \\ m_{ij} &= \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases} \end{aligned}}$$

where

The matrix $M_R$ is called the matrix of $R$.

The matrix $M_R$ is a binary matrix.

## Operations on Matrices:

(i) UNION : $M_{R \cup S} = M_R \cup M_S$

(ii) INTERSECTION : $M_{R \cap S} = M_R \cap M_S$

INVERSE : $R$ $M_R^{-1} = M_R^T$.

composition of a Matrix : $M_{R \circ S} = M_R \cdot M_S$.

Symmetric difference : $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$.

Note : $R^2 = R \cdot R = R$, Hence $M_{R^2} = M_R$.

## Problems:

1) If $R$ is the relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$, iff $a + b =$ even. Find the relational matrix $M_R$. Also find relational Matrices $R^{-1}$, $\bar{R}$ & $R^2$.

Given $A = \{1,2,3\}$. $(a,b) \in R$ iff $a+b = \text{even}$

$\therefore R = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(i) $$M_{R^{-1}} = M_R^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(ii) $M_{\bar{R}} = ?$

$$\bar{R} = \{(1,2),(2,1),(2,3),(3,2)\}$$

$$M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ In R matrix, (change 0's to 1's, 1's to 0's)

(iii) $M_{R^2} = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \\ 0\vee 0\vee 0 & 0\vee 1\vee 0 & 0\vee 0\vee 0 \\ 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2) If R and S be relations on a set A represented by matrices.

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices that represent.

(i) $R \cup S$ (ii) $R \cap S$ (iii) $R \cdot S$ (iv) $S \cdot R$ (v) $R \oplus S$.

(i) $M_{R \cup S} = M_R \cup M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} \quad \left( \begin{matrix} 0 \vee 0 = 0 \\ 0 \vee 1 = 1 \\ 1 \vee 1 = 1 \end{matrix} \right)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(ii) $M_{R \cap S} = M_R \cap M_S$.

$$= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} \quad \left( \begin{matrix} 0 \wedge 0 = 0 \\ 0 \wedge 1 = 0 \\ 1 \wedge 1 = 1 \end{matrix} \right)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(iii) $M_{R \cdot S} = M_R \cdot M_S$

$$= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 1 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(iv) $M_{S \cdot R} = M_S \cdot M_R$

$$= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(v) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

---

## Partition of a set :

A partition of a set A is a set of non empty subsets of A denoted by $\{A_1, A_2 - - - A_k\}$ such that (union of $A_i$'s) (i) $\bigcup_{i \in I} A_i = A$.

(ii) $A_i \cap A_j = \phi$, $i \neq j$
(Subsets are pairwise disjoint)

(iii) $A_i \neq \phi$, $i \in I$

The subsets in a partition are also called blocks of the partition.

eg: $S = \{1, 2, 3, 4\}$.

→ Partition of the set $= \{\{1, 2\}, \{3\}, \{4\}\}$

→ If the partition is $\{\{1, 2\} \{2, 3\}, \{4\}\}$ is not so because 2 appears in 2 different blocks. Hence subsets are not pairwise disj/.

→ The set $\{\{1, 2\} \{4\}\}$ is also not a partition either $3 \in S$ is not in block and $\bigcup A_i \neq S$.

## Partial order relation :

A relation $R$ on a set $S$ is called a Partial order if it is .

* Reflexive : $aRa \ \forall a \in S$
* Antisymmetric : $aRb$ and $bRa \Rightarrow a=b$
* Transitive : $aRb$ and $bRc \Rightarrow aRc$.

A set $S$ together with partial order $R$ is called partial order set (or) a poset and is denoted by $(S,R)$ .

---

Problem :

1) (i) Prove that the relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets

(ii) If $R$ is the relation on the set of integers such that $(a,b) \in R$ iff $b = a^m$ for some positive integer $m$, S.T $R$ is a partial ordering.

Soln :

(i) $(A,B) \in R$ iff $A \subseteq B$ .

$\rightarrow$ $A \subseteq A$ .

$\therefore (A, A) \in R$ .

i.e., $R$ is reflexive .

$\rightarrow$ If $A \subseteq B$ & $B \subseteq A$ then $A = B$ .

$\therefore$ $R$ is antisymmetric .

$\rightarrow$ If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$ .

i.e., $(A,B) \in R$ & $(B,C) \in R \Rightarrow (A,C) \in R$ .

$\therefore$ $R$ is transitive . $\Rightarrow$ $R$ is a partial ordering.

(ii) Given $b = a^m$, $m$ - +ve integer $(a,b) \in R$.

→ $a = a^1$, ($m = 1$ is true)

$\therefore (a,a) \in R$, $R$ is reflexive.

→ Let $(a,b) \in R$ & $(b,a) \in R$.

(ie) $b = a^m$ & $a = b^n$ $m, n$ - +ve integer.

$a = (a^m)^n = a^{mn}$.

means that $mn = 1$ (or) $a = 1$ (or) $a = -1$.

(i) If $mn = 1$, $m = 1$, $n = 1$.

$\therefore a = b$.

(ii) If $a = 1$,

$b = 1^m = 1 = a \Rightarrow a = b$.

(iii) If $a = -1$,

then $b = -1$. $\Rightarrow a = b$.

$\therefore$ R is antisymmetric

→ Let $(a,b) \in R$ & $(b,c) \in R$.

(ie) $b = a^m$ & $c = b^n$.

$c = (a^m)^n = a^{mn}$.

$(a,c) \in R$.

(ie) R is transitive.

$\therefore$ R is partial ordering.

---

(2) If R is the equivalence relation on set $A = \{1,2,3,4,5,6\}$ given below, find partition of A induced by R.

$R = \{(1,1)(1,2)(2,1)(2,2)(3,3)(4,4)(4,5)(5,4)(5,5)(6,6)\}$

Soln:

The element,

$[1]_R = \{1,2\}$

$[2]_R = \{1,2\}$

$[3]_R = \{3\}$

$[4]_R = \{4,5\}$

$[5]_R = \{4,5\}$

$[6]_R = \{6\}$

Hence $\{1,2\}, \{3\}, \{4,5\}, \{6\}$ is the partition induced by R.

---

3) If R is the equivalence relation on the set

$A = \{(-4,-20)\ (-3,-9)\ (-2,-4)\ (-1,-11)\ (-1,-3), (1,2)\ (1,5)\ (2,10)\ (2,14)\ (3,6)\ (4,8), (4,12)\}$

$(a,b)\ R\ (c,d)$ if $ad = bc$. Find equivalence classes of R.

Soln: The elements.

$[(-4,-20)] = \{(-4,-20)\}, (1,5)\ (2,10)\}$

$[(-3,-9)] = \{(-3,-9)\ (-1,-3), (4,12)\}$

$[(-2,-4)] = \{(-2,-4)\ (1,2)\ (3,6)\ (4,8)\}$

$[(-1,-11)] = \{(-1,-11)\}$

$[(2,14)] = \{(2,14)\}$

The partition induced by R ~~is~~

$\{-4,-20\}\ \{-3,-9\}\ \{-2,-4\}\ \{-1,-11\}, \{2,14\}$.

is the equivalence classes.

---

4) If $A = \{1,2,3,4\} \times \{1,2,3,4\}$ and the relation $R$ is defined on $A$ by $(a,b) \, R \, (c,d)$ iff $a+b = c+d$. Verify $A$ is an equivalence relation on $A$ and also find the quotient set of $A$ by $R$.

**Quotient set of A by R**

The collection of all equivalence classes of elements of $A$ under an equivalence relation $R$ is called quotient set denoted by $A/R$.

$$A/R = \{[a] \mid a \in A\}.$$

The quotient set $A/R$ is a partition of A.

Soln:

$$A = \{(1,1) (1,2) (1,3) (1,4),$$
$$(2,1) (2,2) (2,3) (2,4),$$
$$(3,1) (3,2) (3,3) (3,4),$$
$$(4,1) (4,2) (4,3) (4,4)\}$$

$[(1,1)] = \{(1,1)\}$

$[(1,2)] = \{(1,2) (2,1)\}$

$[(1,3)] = \{(1,3) (3,1) (2,2)\}$

$[(1,4)] = \{(1,4) (3,2) (2,3) (4,1)\}$

$[(2,4)] = \{(2,4) (3,3) (4,2)\}$

$[(3,4)] = \{(3,4) (4,3)\}$

$[(4,4)] = \{(4,4)\}$

Equivalence classes of $A$ by $R$ is

$[(1,1)]$ $[(1,2)]$ $[(1,3)]$ $[(1,4)]$ $[(2,4)]$ $[(3,4)]$ $[(4,4)]$.

5) If the relation $R$ on the set of integers $Z$ is defined by $aRb$ if $a \equiv b \pmod 4$. Find the partition induced by $R$.

$a \equiv b \pmod m$ if $m$ divides $(a-b)$.

$(a-b) \bmod m = 0$.

$(a-b) = km$. (or) $a = b + km$.

$10 \equiv 4 \pmod 6 \Rightarrow (10-4)$ is divisible by 6.

Soln:

Gn $a \equiv b \pmod 4$

$(a-b)$ is divisible by 4, (Rem $\rightarrow < 4$)

$[0]_R = \{\cdots -8, -4, 0, 4, 8, 12 \cdots\}$

$[1]_R = \{\cdots -7, -3, 1, 5, 9, 13 \cdots\}$

$[2]_R = \{\cdots -6, -2, 2, 6, 10, 14 \cdots\}$

$[3]_R = \{\cdots -5, -1, 3, 7, 11, 15 \cdots\}$

Equivalence classes are $[0], [1], [2], [3]$.

## Interpretation of Posets and Relations:

| Poset | Relation. |
|---|---|
| $(Z^+, \mid)$ | divides. |
| $(R, \leq)$ | Less than (or) equal to |
| $(R, \geq)$ | greater than (or) equal to |
| $(P(A), \subseteq)$ | Set inclusion. |

## Representation of Relation by Graphs:

### Hasse diagram :

This is the systematic diagram of representing partial ordering set.

i.e) 1) Elements of the poset are represented by means of dots

2) Self loops need not be shown.

3) If $a \leq b$, then there is a line segment joining the dots representing A and B.

4) If $a \leq b$ and $b \leq c$, there is no line segment from a to c, but there is line segment from a to b & from b to c.

5) Assuming that all the edges are directed upward, directions of the edges are not shown.

### Digraph :

A directed graph / digraph, is a graph that is a set of vertices connected by edges, where the edges have a direction associated with them.

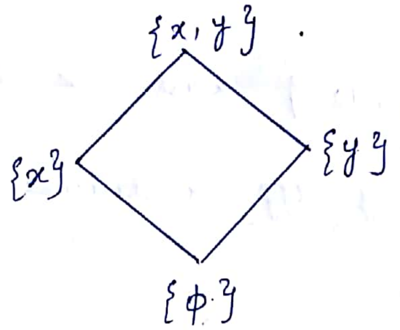
---

1) Problems:

Draw the Hasse diagram for the following $(P(A), \subseteq)$. where $A = \{x, y\}$.

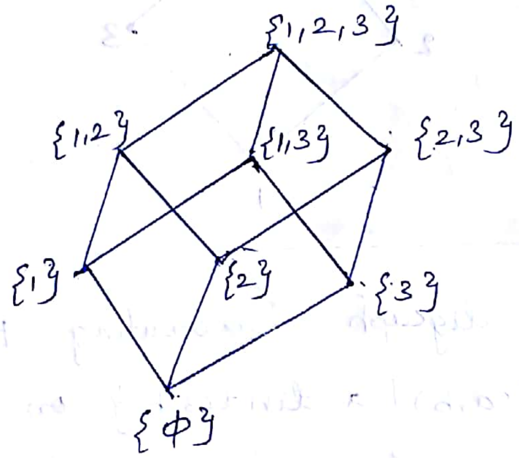
Soln :

Subset of $A = \{\{\phi\}, \{x\}, \{y\}, \{x, y\}\}$



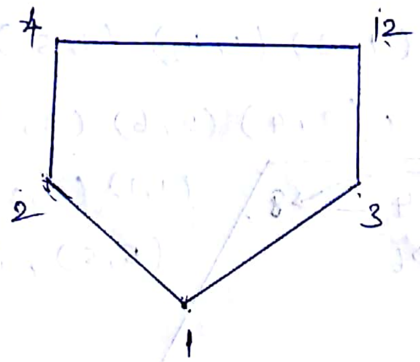

2) Draw the Hasse diagram for $(P(A), \subseteq)$ where $A = \{1, 2, 3\}$.

Subset of A = $\{\{\phi\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$




3) Let $A = \{1, 2, 3, 4, 12\}$. Consider the partial order set divisibility on A. Draw the Hasse diagram for the poset $(A, |)$

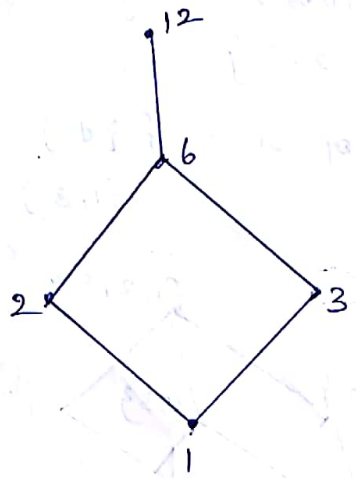
Poset of A are $\{(1,2)\ (1,3)\ (2,4)\ (3,12)\ (4,12)\}$



4) Let $X = \{1, 2, 3, 6, 12\}$ and $\leq$ be the divides relation on $X$ : $x \leq y$ iff $x$ divides $y$.

Relation be

$R = \{(1,2)\ (1,3)\ (2,6)\ (3,6)\ (6,12)\}$

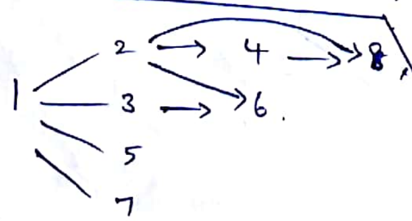



5) Draw the digraph representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on the set $\{1,2,3,4,5,6,7,8\}$. Reduce it to the Hasse diagram representing the given partial ordering.

Given $A = \{1,2,3,4,5,6,7,8\}$.

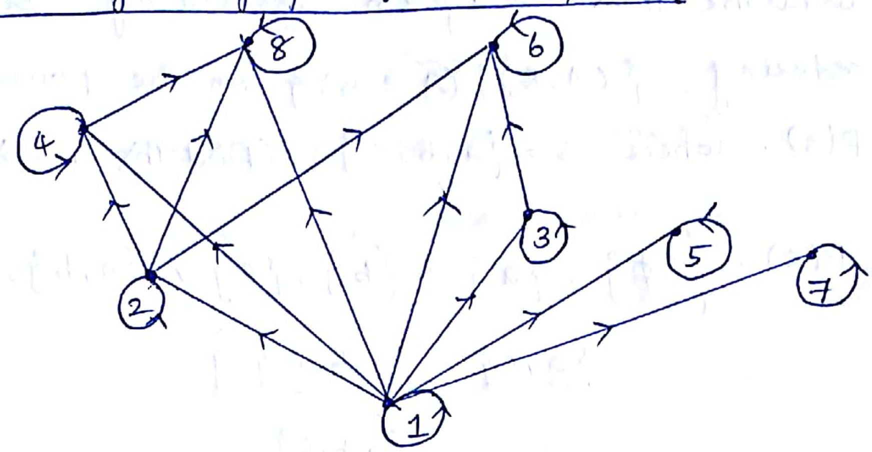
$R = \{(a,b) \mid a \text{ divides } b\}$.

$R = \{(1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)\ (1,7),$
$(1,8),\ (2,4)\ (2,6)\ (2,8)\ (3,6)\ (4,8)\}$



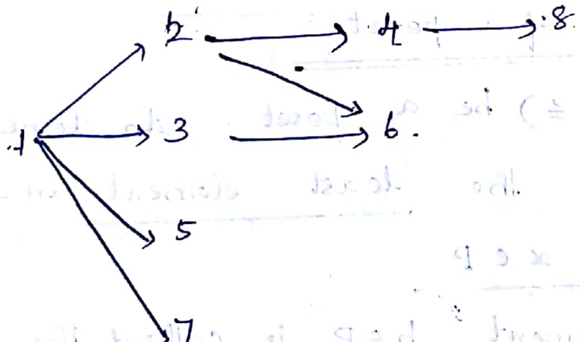

$(1,1)\ (2,2)\ (3,3), (4,4),$
$(5,5), (6,6)\ (7,7) (8,8)\}$

(i) Representing diagraph for the poset.



(ii) Representing Hasse diagram for the poset.

* Delete all the loops at the vertices.
* Delete all the edges occuring due to transitivity.
* Arranging all the edges to point upward.
* Delete all arrows.



Transitive

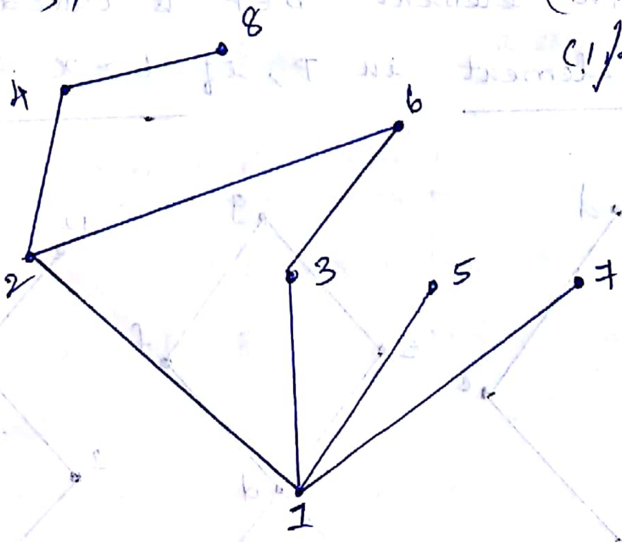
(1,2) (2,4) → (1/4)

(1,3) (3,6) → (1/6)

(1,2) (2,6) → (1/6)

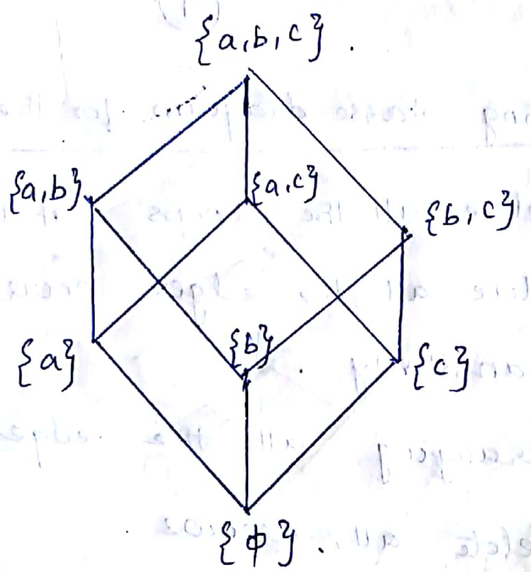
(2,4) (4,8) → (2/8)

(1/4) (4,8) → (1/8)



6) Draw the Hasse diagram representing the partial ordering $\{(A,B) \mid (A \subseteq B)\}$ on the power set $P(S)$, where $S = \{a,b,c\}$. (Find the maximal,

$P(S) = \{\{\phi\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

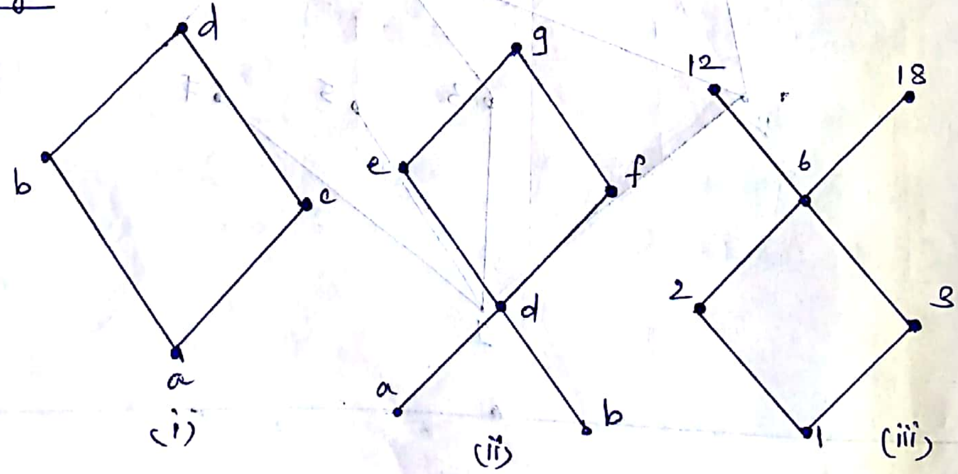



**Special elements of a poset:**

Let $(P, \leq)$ be a poset. An element $a \in P$ is called the least element in $P$, if $a \leq x \ \forall x \in P$.

An element $b \in P$ is called the greatest element in $P$, if $b \geq x \ \forall x \in P$.

eg:



In (i), a is the least element.
d is the greatest element.

In (ii) g is the greatest element.
There is no least element.

In (iii) 1 is the least element.
There is no greatest element.

## upper bound / lower bound:

Let $(P, \leq)$ be an poset and A be any non empty subset of P.

An element $a \in P$ is an upper bound of A if $a \geq x \ \forall x \in A$.

An element $b \in P$ is said to be lower bound of A if $b \leq x \ \forall x \in A$.

## Least upper bound (LUB):

Let $(P, \leq)$ be a poset and $A \subseteq P$.
An element $a \in P$ is said to be least upper bound (LUB) (or) supremium (sup) of A if

(1) a is upper bound of A.

(2) $a \leq c$, where c is any other upper bound of A.

## Greatest lower bound (GLB):

Let $(P, \leq)$ be a poset and $A \subseteq P$.
An element $b \in P$ is said to be greatest lower bound (GLB) (or) infimum of A if

(1) b is lower bound of A.

(2) $b \geq d$ where d is any other lower bound of A.

Example:

Consider $X = \{1, 2, 3, 4, 6, 12\}$

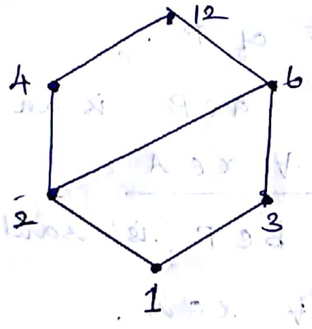
$R = \{(a,b) \mid a|b\}$ find LUB and GLB.

for the poset $(X, R)$

Soln:

$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6)$
$(2,12)\ (3,6)\ (3,12)\ (4,12), (6,12)\}$

Hasse diagram for $(X, R)$ is




$(1,2)(2,4) = (1/4)$
$(1,2)\ (2,6) = (1/6)$
$(2,4)(4,12) = (2/12)$
$(1,3)\ (3,6) = (1/6)$
$(1,6)\ (6,12) = (1/12)$

| LUB | GLB |
|---|---|
| 1) $UB = \{1,3\} = \{3,6,12\}$<br>$LUB\ \{1,3\} = 3$. | 1) $LB\ \{1,3\} = \{1\}$<br>$GLB\ \{1,3\} = 1$ |
| 2) $UB = \{1,2,3\} = \{6,12\}$<br>$LUB\ \{1,2,3\} = 6$. | 2) $LB\ \{1,2,3\} = \{1\}$<br>$GLB\ \{1,2,3\} = 1$ |
| 3) $UB = \{2,3\},\ \{6,12\}$<br>$LUB\ \{2,3\} = 6$ | 3) $LB\ \{2,3\} = \{1\}$<br>$GLB\ \{2,3\} = 1$. |
| 4) $UB\ \{2,3,6\} = \{6,12\}$<br>$LUB\ \{2,3,6\} = 6$ | 4) $LB\ \{2,3,6\} = \{1\}$<br>$GLB\ \{2,3,6\} = 1$ |
| 5) $UB\ \{4,6\} = \{12\}$<br>$LUB\ \{4,6\} = 12$ | 5) $LB\ \{4,6\} = \{1,2\}$<br>$GLB\ \{4,6\} = 1$ |

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## Representation of Relation by Graphs (finite set)

**Vertex / Node** :- Represented by dots (or) small circles.

**Edge** – Joining the nodes $x_i$ & $y_i$

**Equivalence relation :**

1) **Reflexive** – There is a loop at every vertex

2) **Symmetric** – If every edge between distint vertices in the digraph there is an opposite direction.

3) **Transitive** – If there is an edge between a to b and b to c, then there is an edge from a to c.

**Problems :**

List the ordered pairs in the relation on $\{1,2,3,4\}$ corresponding to the following matrix

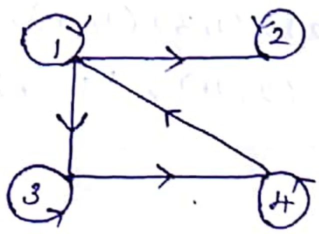
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also, draw the graph representing this is reflexive, symmetric & transitive.

**Soln :**

**Ordered pair :** $\{(1,1),(1,2),(1,3),(2,2),(3,3),(3,4),(4,1),(4,4)\}$

**Graph :**



## To prove equivalence relation :

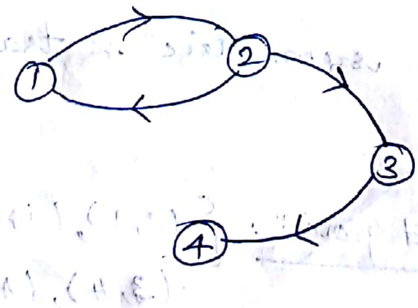
* It is reflexive, since there is a loop at every vertex of the diagram
* It is not symmetric, since there is an edge from 1 to 2, there is no edge in the opposite direction
* It is not transitive, since though there are edges from 1 to 3 and 3 to 4 there is no edge from 1 to 4

## Transitive closure :

The Transitive closure of a binary relation R on a set X is the smallest relation on X that contains R and is transitive, Denoted by $R^T$.

eg: Given $R = \{(1,2),(2,3),(3,4),(2,1)\}$ defined on $A = \{1,2,3,4\}$. Find Transitive closure of R.

Digraph:




$R^T = \{(1,1)\ (1,2),\ (1,3)\ (1,4),\ (2,1),\ (2,2),\ (2,3)\ (2,4),\ (3,4)\}$

# Warshall's Algorithm. (Matrix generation)

Recurrence relating elements $R^k$ to elements of $R^{(k-1)}$ is

$$R^k(i,j) = R^{(k-1)}[i,j] \text{ (or) } [R^{(k-1)}(i,k) \ \& \ R^{(k-1)}(k,j)]$$

## Rules for generating $R^k$ from $R^{k-1}$

→ If an element in row i and column j is 1 in $R^{(k-1)}$, it remains 1 in $R^k$.

→ If an element in row i and column j is 0 in $R^{k-1}$, it has to be changed to 1 in $R^k$ iff the element in its row i and column k and the element in its column j and row k are both 1's in $R^{k-1}$.

→ Put 1's in position of $(P_i, q_j)$ (ie) (col, row)

## Problem:

1) Find the transitive closure of the relation whose matrix is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using warshall's Algorithm.

Soln:

Let $W_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

| K | In $W_{k-1}$ Position of 1's in col. K ($p_j$) | Position of 1's in row K ($q_j$) | $W_k$ has 1's in ($p_j, q_j$) | $W_k$ |
|---|---|---|---|---|
| 1 | 1 | 1,4 | (1,1)(1,4) | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 2 | 2 | 2,4 | (2,2)(2,4) | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 3 | – | 4 | – | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 4 | 1,2,3,4 | 4 | (1,4) (2,4) (3,4) (4,4) | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |

$R^T = \{(1,1), (1,4), (2,2)(2,4), (3,4), (4,4)\}$

2) using warshall Algorithm, find Transitive closure of relation

$R = \{(1,1) (1,3) (1,5) (2,3) (2,4) (3,3) (3,5),$
$(4,2) (4,4) (5,4)\}$

defined on $A = \{1,2,3,4,5\}$. Let $W_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

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| K | In $W_{K-1}$: Position of 1's in coln k $(P_j)$ | In $W_{K-1}$: Position of 1's in Row k $(q_i)$ | $W_K$ has 1's in $(P_j, q_i)$ | $W_K$ |
|---|---|---|---|---|
| 1 | 1 | 1, 3, 5 | (1,1) (1,3) (1,5) | $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ |
| 2 | 4 | 3, 4 | (4,3) (4,4) | $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ |
| 3 | 1, 2, 3, 4 | 3, 5 | (1,3) (1,5) (2,3) (2,5) (3,3) (3,5) (4,3) (4,5) | $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ |
| 4 | 2, 4, 5 | 2, 3, 4, 5 | (2,2) (2,3) (2,4) (2,5) (4,2) (4,3) (4,4) (4,5) (5,2) (5,3) (5,4) (5,5) | $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ |
| 5 | 1, 2, 3, 4, 5 | 2, 3, 4, 5 | (1,2) (1,3) (1,4) (1,5) (2,2) (2,3) (2,4) (2,5) (3,2) (3,3) (3,4) (3,5) (4,2) (4,3) (4,4) (4,5) (5,2) (5,3) (5,4) (5,5) | $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ |

Transitive closure of R is

$R^T = \{(1,1) (1,2) (1,3) (1,4) (1,5) (2,2) (2,3) (2,4), (2,5) (3,2) (3,3) (3,4) (3,5) (4,2) (4,3) (4,4), (4,5) (5,2) (5,3) (5,4) (5,5)\}$

Find Transitive closure of R using warshall Algorithm

1) $R = \{(1,4),(2,1),(2,2),(2,3),(3,2)(4,3)(4,5)(5,1)\}$

2) $R = \{(1,4)(2,1)(2,3)(3,1)(3,4)(4,3)\}$

3) $R = \{(1,1)(1,2)(2,1)(3,3)(3,4)(4,3)(4,4),$
$(4,5)(5,4)(5,5)\}$

4) $R = \{(1,1)(1,3)(1,4)(2,2)(3,4)(4,1)\}$

---

Partition of a set :

Let $A$ be any non-empty set. The collection of subsets of $A$ i.e. $A_1, A_2 \ldots A_n$ is called partition of $A$ if

1) $\cup A_i = A$

2) $A_i \cap A_j = \phi$

3) $A_i \neq \phi \ \forall i$

---

Min sets or Min terms :

Let $A$ be any non-empty set and $B_1, B_2$ be subsets of $A$. Then

$\boxed{B_1 \cap B_2, \ B_1^c \cap B_2, \ B_1 \cap B_2^c \text{ and } B_1^c \cap B_2^c}$

are called Min sets or Minterms generated by $B_1$ and $B_2$.

If $B_1, B_2, B_3$ are subsets of $A$ then minterms are

$$\boxed{\begin{array}{l} B_1 \cap B_2 \cap B_3, \ B_1 \cap B_2 \cap B_3^c, \ B_1 \cap B_2^c \cap B_3, \ B_1^c \cap B_2 \cap B_3, \\ B_1^c \cap B_2^c \cap B_3, \ B_1 \cap B_2^c \cap B_3^c, \ B_1^c \cap B_2 \cap B_3^c, \ B_1^c \cap B_2^c \cap B_3^c \end{array}}$$

The No. of Minterms is $2^n$.
The minterms of A will form a partition of A.

1) Let $A = \{1,2,3,4,5,6\}$. Find the minsets generated by $B_1 = \{1,3,5\}$, $B_2 = \{1,2,3\}$. Give the partition of A using Minsets.

Gn
$B_1 = \{1,3,5\}$ $B_2 = \{1,2,3\}$
$B_1^c = \{2,4,6\}, B_2^c = \{4,5,6\}$

Minterms are.

$B_1 \cap B_2 = \{1,3\} = A_1$ (say)
$B_1^c \cap B_2 = \{2\} = A_2$ (say)
$B_1 \cap B_2^c = \{5\} = A_3$ (say)
$B_1^c \cap B_2^c = \{4,6\} = A_4$ (say)

Partition of A are $\{\{1,3\},\{2\},\{5\},\{4,6\}\}$.
$\Rightarrow$ each subset is Non empty.
$\Rightarrow A_i \cap A_j = \phi \ \forall \ i \neq j$.
$\Rightarrow \cup A_i = A_1 \cup A_2 \cup A_3 \cup A_4 = A$.

2) Let $A = \{1,2,3,4,5,6,7,8,9\}$.
$B_1 = \{1,5,6,7\}$, $B_2 = \{2,4,5,9\}$, $B_3 = \{3,4,5,6,8,9\}$
Find the minsets generated by $B_1, B_2, B_3$ & give the partition of A.

$B_1 = \{1,5,6,7\}$ $B_2 = \{2,4,5,9\}$ $B_3 = \{3,4,5,6,8,9\}$
$B_1^c = \{2,3,4,8,9\}$, $B_2^c = \{1,3,6,7,8\}$, $B_3^c = \{1,2,7\}$

Minsets :

$B_1 \cap B_2 \cap B_3 = \{5\}$
$B_1 \cap B_2 \cap B_3^c = \{\phi\}$
$B_1 \cap B_2^c \cap B_3 = \{6\}$
$B_1^c \cap B_2 \cap B_3 = \{4,9\}$
$B_1 \cap B_2^c \cap B_3^c = \{1,7\}$
$B_1^c \cap B_2 \cap B_3^c = \{2\}$
$B_1^c \cap B_2^c \cap B_3 = \{3,8\}$
$B_1^c \cap B_2^c \cap B_3^c = \{\phi\}$

Minsets form a partition of A are

$\{\{5\}, \{6\}, \{4, 9\}, \{1, 7\}, \{2\}, \{3, 8\}\}$.

## Maxsets :

The dual of Minsets is called Max sets.

Let A be any set and $B_1, B_2$ be the subsets of A.

Maxsets are $\boxed{B_1 \cup B_2,\ B_1^c \cup B_2,\ B_1 \cup B_2^c,\ B_1^c \cup B_2^c}$

Note : Maxsets need not form a partition.

1) Find Maxsets for $A = \{1, 2, 3, 4, 5, 6\}$ generated by $\{1, 3, 5\}$; $\{1, 2, 3\}$.

Let $B_1 = \{1, 3, 5\}$, $B_1^c = \{2, 4, 6\}$

$B_2 = \{1, 2, 3\}$. $B_2^c = \{4, 5, 6\}$.

Max sets are
$$B_1 \cup B_2 = \{1, 2, 3, 5\}$$
$$B_1^c \cup B_2 = \{1, 2, 3, 4, 6\}$$
$$B_1 \cup B_2^c = \{1, 3, 4, 5, 6\}$$
$$B_1^c \cup B_2^c = \{2, 4, 5, 6\}$$

## Graphs of Relation

1) Let $A = \{1, 2, 3, 4\}$ and $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Draw the directed graph. Use the graph to find if R is reflexive, symmetric & Transitive. Also find indegree & outdegree of each vertex.

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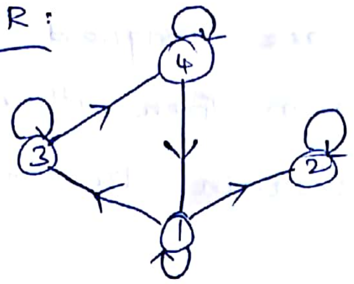
The relation $R = \{(1,1)(1,2), (1,3) (2,2) (3,3) (3,4), (4,1) (4,4)\}$

Graph of R :



From the graph,

→ R is reflexive.
since each vertex has a loop.

→ R is not symmetric.
Since there is an edge from 1 to 2
and there is no edge from 2 to 1.

→ R is not transitive.
Since there are edges from 1 to 3, 3 to 4
but there is no edge from 1 to 4.

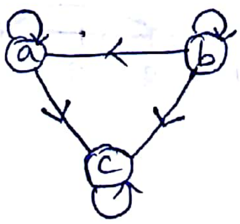
∴ R is not an equivalence relation.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Indegree | 2 | 2 | 2 | 2 |
| outdegree | 3 | 1 | 2 | 2 |

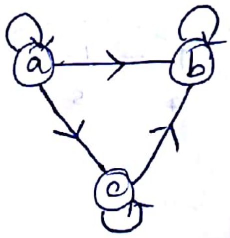
2) Let $A = \{a,b,c\}$ & $R = \{(a,a)(a,c)(b,a)(b,b)(b,c)(c,c)\}$
Draw the digraph of $R, R^{-1}$ & $\bar{R}$.

Diagraph :

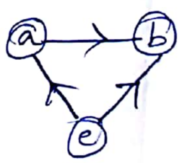
R



$R^{-1}$



$\bar{R}$
$= \{(a,b), (c,a) (c,b)\}$



## Pigeonhole Principle :

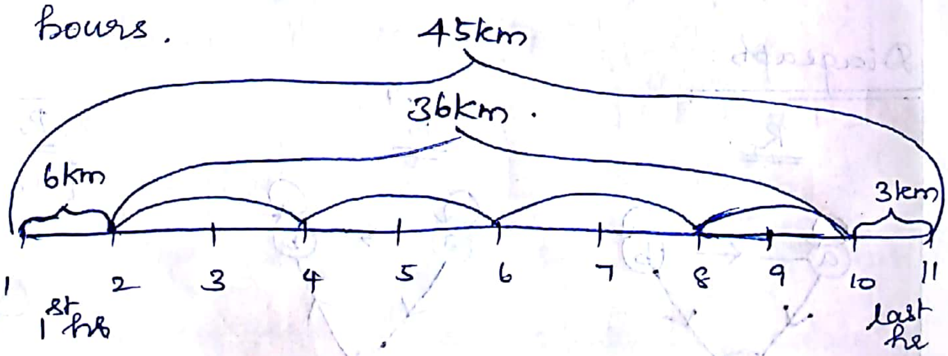
If $n$ Pigeons are assigned to $m$ pigeonholes and $n \geq m$ then atleast one pigeonhole will contain two or more pigeons.

## Generalised Pigeonhole Principle :

If $n$ Pigeons are assigned to $m$ pigeon holes then one of the pigeonhole will contain atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$, which is a real number.

## Problems :

1) A man hiked for 10 hours and covered a total distance of 45km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked atleast 9 km within a certain period of 2 consecutive hours.



$\because$ the man has covered 9km in the first hr & last hour,

He has to cover 36 km during the remaining hours.

The consecutive hrs between 2 & 10 are

{2nd, 3rd}, {4th, 5th}, {6th, 7th}, {8th, 9th}

Let Pigeons = 36 (n), Pigeon holes = 4 (m), $n > m$.

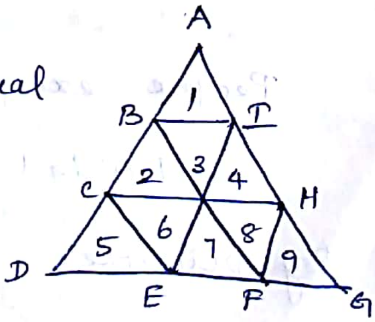
By the principle,

$$\left[\frac{n-1}{m}\right]+1 = \left[\frac{35}{4}\right]+1$$
$$= [8.75]+1$$
$$= 9$$

Hence, the man must have hiked atleast 9km in one time period of 2 consecutive hours.

---

2) If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be atleast two points whose distance apart is less than $1/3$.




Let ADG be given equilateral triangle.

Divide the $\Delta^{le}$ into 9, equilateral $\Delta^{le}$ of side $1/3$.

Let Interior points $= 10$ $(n)$ , subtriangles $= 9$ $(m)$

('n'- Pigeons) ('m' - pigeonholes).

By the principle,

$$\left[\frac{n-1}{m}\right]+1 = \left[\frac{10-1}{9}\right]+1$$
$$= 2$$

Hence, atleast one sub triangle must contain 2 interior points

$\therefore$ The distance between any 2 interior points of any sub triangle cannot exceed the length of side $\frac{1}{3}$.

---

3) Prove that in any group of 6 people, atleast 3 must be mutual friends or at least 3 must be mutual strangers.

Let A be one of the 6 people, other 5 people are divided in to 2 sets that can be taken as

{Friends of A} , {strangers of A}.

People except A = 5 (n- pigeons)

Divided sets = 2 (m - pigeonholes).

By principle, $\left[\frac{n-1}{m}\right]+1 = \left[\frac{5-1}{2}\right]+1 = 3$.

Hence, $\exists$ atleast 3 mutual friends (or) atleast 3 mutual strangers.

---

## Functions :

A function from a set $A$ into a set $B$ is a relation from $A$ to $B$ ∋ each element of $A$ is related to exactly one element of the set $B$.

The set $A$ is called domain of the fn/:

The set $B$ is called co-domain of the fn/:

**Note :** If $f : A \to B$ ; $a \in A$, then the element in $B$ which is assigned to 'a' is called the image of a and is denoted by $f(a)$.

**eg :** Let $f$ assign to each country in the world its capital city.

Domain of $f$ : set of all countries in the world

Co-domain of $f$ : capital cities in the world

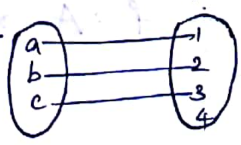
ie, $f$( France) = Paris.

## Range :

The set of all images of elements of $X$ is called range of $f$. ie, $f(x) \subseteq Y$

eg: Let the function $f : R \to R$ be defined by the formula $f(x) = x^2$. Then the range of $f$ consists of the positive real nos/: and zero.

eg:




Domain → {a, b, c}

Co-Domain → {1, 2, 3, 4}

Range → {1, 2, 3}

## One-One function (Injective function)

Let $f : A \to B$.

Then $f$ is called one-one fn if different element in B are assigned to different elements in A ie) $f(a) = f(a')$ $\Rightarrow a = a'$.

(or) equivalently $a \neq a' \Rightarrow f(a) \neq f(a')$

eg:- Let $f : R \to R$ be defined by the fn. $f(x) = x^2$.

Then $f(-2) = 4$

$f(2) = 4$

but $2 \neq -2 \Rightarrow f$ is not injective.

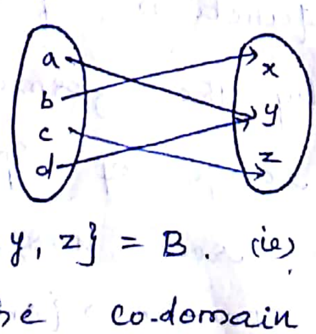
## Onto function (surjective fn/:)

Let $f$ be a function of A into B

If $f(A) = B$ ie) if every member of B appears as the image of atleast one elements of A, then we say "$f$ is a function of A onto B".

eg: Let $A = \{a, b, c, d\}$, $B = \{x, y, z\}$

Let $f : A \to B$




$f(A) = \{x, y, z\} = B$. ie) the range of $f$ is equal to the co-domain B. $\therefore f : A \to B$.

$\therefore f$ is onto mapping.

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## Identity function :

Let A be any set. Let the function $f : A \to A$ be defined by $f(x) = x$.

(ie) f assign to each element in A the element itself.

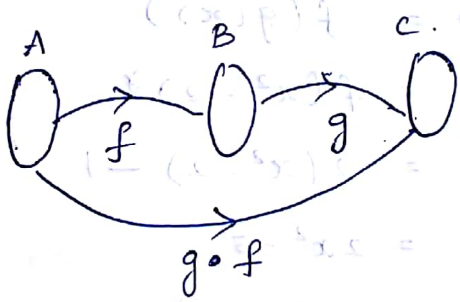
Then f is called identity function.

denoted by 1 (or) $1_A$.

## Product or Composite of functions :

If $f : A \to B$ and $g : B \to C$, then the composition of f and g is a new function from A to C denoted by $g \circ f$ is given by

$$(g \circ f)x = g\{f(x)\}, \ \forall x \in A.$$




## Bijection :

A function $f : x \to y$ is called bijection if it is both 1-1 and onto.

## Inverse function :

If $f : A \to B$ and $g : B \to A$, then the fn: g is called inverse of f if $f \circ g = I_B$ & $g \circ f = I_A$, denoted by $f^{-1}$.

eg : Let $g : R \to R$, $g(x) = x^2 \to$ $g(-2) = 4$
$g(2) = 4$.

$\Rightarrow g^{-1}(4) = 2, -2$.

$g^{-1}(0) = 0$ & $g^{-1}(-1) = \phi$.

Problems :

1) If $f(x) = 4x-1$, $g(x) = \cos x$

Find $f\circ g$ and $g\circ f$.

$$(f\circ g)(x) = f(g(x))$$
$$= f(\cos x)$$
$$= 4(\cos x) - 1.$$
$$(g\circ f)(x) = g(f(x))$$
$$= g(4x-1)$$
$$= \cos(4x-1)$$

2) $f(x) = 2x-1$, $g(x) = x^2 - 2$.

Find $f\circ g$ & $g\circ f$

$$(f\circ g)(x) = f(g(x))$$
$$= f(x^2-2)$$
$$= 2(x^2-2) - 1$$
$$= 2x^2 - 5.$$
$$(g\circ f)(x) = g(f(x))$$
$$= g(2x-1)$$
$$= (2x-1)^2 - 2$$
$$= 4x^2 - 4x - 1.$$

3) $f(x) = x^3 - 4x$, $g(x) = \frac{1}{x^2+1}$, $h(x) = x^4$

Show the composition of fns is associative

To prove : $f\circ(g\circ h) = (f\circ g)\circ h$.

$$f\circ(g\circ h) = f[g(h(x))]$$
$$= f[g(x^4)]$$
$$= f\left[\frac{1}{(x^4)^2+1}\right]$$
$$= \left[\frac{1}{x^8+1}\right]^3 - 4\left[\frac{1}{x^8+1}\right]$$

$$(f\circ g)\circ h = [f(g(x))]\circ h$$
$$= \left[f\left(\frac{1}{x^2+1}\right)\right]\circ h$$
$$= \left[\left(\frac{1}{(x^2+1)}\right)^3 - 4\left(\frac{1}{x^2+1}\right)\right]\circ h$$
$$= [f\circ g]\, h(x).$$
$$= (f\circ g).\, x^4$$
$$= \left(\frac{1}{x^8+1}\right)^3 - 4\left(\frac{1}{x^8+1}\right)$$

LHS = RHS, Hence proved.

---

Note:

1) If $f^{-1}$ exists, then $f$ is invertible.

2) If ~~is~~ $f$ is bijective, then $f^{-1}$ is bijective.

---

4) If $f : Z \to N$ is defined by $f(x) = \begin{cases} 2x-1, & x>0 \\ -2x, & x\le 0 \end{cases}$

then prove $f$ is bijective & determine $f^{-1}$.

Proof:

To prove : $f$ is bijective $\begin{cases} 1-1 \\ \text{onto} \end{cases}$.

case -(i) : To prove $f$ is $1-1$.

→ If $x>0$, let $x_1, x_2 \in Z$.

$\therefore f(x_1) = f(x_2)$.

$2x_1 - 1 = 2x_2 - 1$.

$x_1 = x_2$.

$\rightarrow$ If $x \le 0$,

$f(x_1) = f(x_2)$

$-2x_1 = -2x_2$

$x_1 = x_2$.

$\therefore f$ is 1-1 fn.

Case (ii):- $f$ is onto (to prove)

$\rightarrow$ If $x > 0$.

Let $y = 2x - 1$.

$\Rightarrow \frac{y+1}{2} = x$.

$\rightarrow$ If $x \le 0$.

Let $y = -2x$.

$x = -y/2$.

$\therefore$ For any $y \in N$, the preimage is

$\frac{y+1}{2} \in Z$ (or) $\frac{-y}{2} \in Z$.

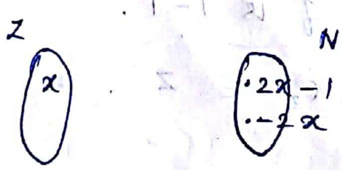
$\therefore f$ is onto.

Hence $f$ is bijective.

To find $f^{-1}$:

Gn: $f : Z \rightarrow N$.

$\Rightarrow f^{-1} : N \rightarrow Z$.



for $f(x) = y$

$\therefore \ f^{-1}(y) = x.$

$$= \begin{cases} \frac{y+1}{2}, & y = 1, 3, 5 \\ -y/2, & y = 0, 2, 4, 6 \ldots \end{cases}$$

---

5) If $A = \{x \in R \mid x \neq 1/2\}$ and $f: A \to R$ is defined by $f(x) = \frac{4x}{2x-1}$, find.

(i) Range of $f$

(ii) $f$ is invertible (prove)

(iii) Domain of $f^{-1}$.

(iv) Range $f^{-1}$

(v) formula for $f^{-1}$.

Soln:

To prove: $f$ is 1-1.

Let $x_1, x_2 \in A$.

Then $f(x_1) = f(x_2)$

$$\frac{4x_1}{2x_1 - 1} = \frac{4x_2}{2x_2 - 1}.$$

$$4x_1(2x_2 - 1) = 4x_2(2x_1 - 1)$$

$$\cancel{8x_1x_2} - 4x_1 = \cancel{8x_1x_2} - 4x_2$$

$$x_1 = x_2.$$

$\therefore f$ is 1-1 fn.

To prove: $f$ is onto.

Let $y \in R$, $x \in A$.

$\ni$, $f(x) = y \Rightarrow f^{-1}(y) = x$

Let $y = \frac{4x}{2x - 1}$.

$$y(2x - 1) = 4x$$

$$2xy - 4x = y$$

$$2x(y-2) = y$$

$$x = \frac{y}{2(y-2)}$$

$\therefore$ For every $y \in R$, the preimage is $\frac{y}{2(y-2)} \in A$.

$\therefore$ $f$ is onto

Hence, $f$ is bijection.

$\Rightarrow$ $f^{-1}$ exists.

$\Rightarrow$ $f$ is invertible.

Range $(f) = \{y \in R \mid y \neq 2\}$

Domain $f^{-1} = \{y \in R \mid y \neq 2\}$

Range $f^{-1} = \{x \in A \mid x \neq 1/2\}$

Formula for $f^{-1}$.

$f : A \to R$,

$\Rightarrow f^{-1} : R \to A$.

$$f^{-1}(y) = \frac{y}{2(y-2)}, \quad y \neq 2.$$

---

Theorem: 1

If $f : A \to B$, $g : B \to C$ are bijective fn.

then $g \circ f : A \to C$, $g \circ f$ is bijective.

Proof:

To prove: $g \circ f$ is 1-1.

Let $a_1, a_2 \in A$.

$\Rightarrow (g \circ f) a_1 = (g \circ f) a_2$.

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$f(a_1) = f(a_2)$ $\therefore$ $g$ is 1-1.

$\Rightarrow a_1 = a_2 \quad \because f$ is 1-1.

$\therefore g \circ f$ is 1-1.

<u>To prove :</u> $g \circ f$ is onto.

Let $c \in C$.

$\because g$ is onto, $\exists$ an elt $b \in B$.

$\ni g(b) = c$

$\because f$ is onto, $\exists$ an elt $a \in A \ni f(a) = b$.

Now $(g \circ f)a = g(f(a))$

$= g(b)$

$= c$.

For every $c \in C$, $\exists$ an elt $a \in A \ni$

$(g \circ f)a = c$.

$\therefore g \circ f$ is onto

Hence, $g \circ f$ is bijective.

---

<u>Theorem - 2 :</u>

composition of function is associative

If $f : A \to B$, $g : B \to C$ & $h : C \to D$ are fns.

then $h \circ (g \circ f) = (h \circ g) \circ f$.

<u>Proof :</u> $g \circ f : A \to C$.

$h \circ (g \circ f) : A \to D$.

Also, $h \circ g : B \to D$.

& $(h \circ g) \circ f : A \to D$

Both $(h \circ g) \circ f$ & ~~$(h \circ g)$~~ $h \circ (g \circ f)$

are fns/. from $A$ to $D$.

Let $x \in A$, then $\exists\ y \in B \ni f(x) = y$

Also for every $y \in B$, $\exists\ z \in C \ni g(y) = z$.

Now $[(h \circ g) \circ f]x = (h \circ g)[f(x)]$

$= (h \circ g)\, y$

$= h[g(y)]$

$= h(z) \longrightarrow (1)$

Now $[h \circ (g \circ f)]x = h \circ [(g \circ f)(x)]$

$= h(z) \longrightarrow (2)$.

$\because [(g \circ f)x = g(f(x)) = g(y) = z]$

From (1) & (2),

$(h \circ g) \circ f = h \circ (g \circ f)$.

---

Theorem - 3 :

If $f : A \to B$ and $g : B \to C$ are invertible fn/: then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof :

$(g \circ f)^{-1} : C \to A$.

$f^{-1} : B \to A$, $g^{-1} : C \to B$.

Hence, $f^{-1} \circ g^{-1} : C \to A$.

Both $f^{-1} \circ g^{-1}$ & $(g \circ f)^{-1}$ are fns from $C \to A$.

Let $x \in A$, $\exists\ y \in B \ni f(x) = y$.

ie) $x = f^{-1}(y)$.

Also $\forall\ y \in B\ \exists\ z \in C \ni g(y) = z$.

$y = g^{-1}(z)$

Now $(g \circ f)x = g(f(x))$
$= g(y) = z$.

$\therefore (g \circ f)x = z$.

$(g \circ f)^{-1} z = x \longrightarrow (1)$

Now $(f^{-1} \circ g^{-1})z = f^{-1}(g^{-1}(z))$
$= f^{-1}(y)$
$= x \longrightarrow (2)$.

From (1), & (2), $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = x$.

---

Theorem : 4 :

A function $f : A \to B$ is invertible iff $f$ is 1-1 & onto.

Proof :

Let $f : A \to B$ be invertible.

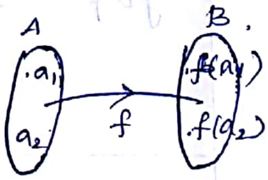
To prove : $f$ is 1-1 & onto.

Since $f$ is invertible, $\exists$ a unique fn/: $g : B \to A$ $\ni$ $g \circ f = I_A$ & $f \circ g = I_B \longrightarrow (1)$.

(i) Let $a_1, a_2 \in A$.

$f(a_1) = f(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$

$\because g : B \to A$ is a fn.




$\Rightarrow (g \circ f)a_1 = (g \circ f)a_2$.

$\Rightarrow I_A(a_1) = I_A(a_2)$

$a_1 = a_2$.

$\therefore f$ is 1-1 fn.

(ii) Let $b \in B$, then $g(b) \in A$.

Now, $b = I_B \cdot b$.

$= (f \circ g)\, b$

$= f(g(b))$

$\therefore$ For every $b \in B$ $\exists$ $g(b) \in A \ni$

$f(g(b)) = b$

$\therefore$ $f$ is onto.

Conversely, if $f$ is 1-1 & onto.

To prove, $f$ is invertible.

Since, $f$ is onto $\forall$ $b \in B$, $\exists$ $a \in A \ni$

$f(a) = b$.

Hence, $g: B \to A \ni g(b) = a$ where

$f(a) = b$.

Let $g(b_1) = a_1$ & $g(b_2) = a_2$ where

$a_1 \neq a_2$

then $f(a_1) = b$, $f(a_2) = b$. which is impossible, since $f$ is 1-1.

$\therefore$ $g: B \to A$ is a unique fn. $\ni$

$g \circ f = I_B$ & $f \circ g = I_A$.

$\therefore$ $f$ is invertible.

---

1) The inverse of function $f$ is unique if it exists.

Proof:

Let $g$ & $h$ be inverse of fn $f$.

Let $f: A \to B$.

Then $g\circ f = I_A$ & $f\circ g = I_B$.

$h\circ f = I_A$ & $f\circ h = I_B$.

$h = h\circ I_B = h\circ(f\circ g) = (h\circ f)\circ g = I_A\circ g = g$

$\Rightarrow h = g$.

$\therefore$ The inverse of $f$ is unique.

---

1) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 8, 9\}$ and the functions $f: A \to B$ and $g: A \to A$ are defined by $f = \{(1,8)\ (3,9)\ (4,3)\ (2,1)\ (5,2)\}$ and $g = \{(1,2)\ (3,1)\ (2,2)\ (4,3)\ (5,2)\}$. Find $f\circ g$, $g\circ f$, $f\circ f$ & $g\circ g$. if they exists.

(i) $f\circ g \to$ $\text{Ran } g = \{1,2,3\} \subseteq \text{Dom } f = \{1,2,3,4,5\}$.
$\therefore f\circ g$ is defined.

$(f\circ g)(1) = f[g(1)] = f(2) = 1$.

$(f\circ g)(2) = f[g(2)] = f(2) = 1$

$(f\circ g)(3) = f[g(3)] = f(1) = 8$

$(f\circ g)(4) = f[g(4)] = f(3) = 9$

$(f\circ g)(5) = f[g(5)] = f(2) = 1$

$f\circ g = \{(1,1)\ (2,1)\ (3,8)\ (4,9)\ (5,1)\}$.

(ii) Now Range $f = \{1,2,3,8,9\}$

Domain $g = \{1, 2, 3, 4, 5\}$.

Range $f \not\subseteq$ Dom $g$.

$\therefore (g\circ f)a = g(f(a))$ is not defined.

(ie) $g\circ f$ is not defined.

(iii) Range $f = \{1, 2, 3, 8, 9\}$

Dom $f = \{1, 2, 3, 4, 5\}$

$\therefore$ Ran $f \not\subseteq$ Dom $f$

$\therefore$ $f \circ f$ is not defined.

(iv) Range $g = \{1, 2, 3\}$

Dom $g = \{1, 2, 3, 4, 5\}$.

Ran $g \subseteq$ Dom $g$

$\therefore$ $g \circ g$ is defined.

$(g \circ g)(1) = g(g(1)) = g(2) = 2$

$(g \circ g)(2) = g(g(2)) = g(2) = 2$

$(g \circ g)(3) = g(g(3)) = g(1) = 2$

$(g \circ g)(4) = g(g(4)) = g(3) = 1$

$(g \circ g)(5) = g(g(5)) = g(2) = 2$.

$g \circ g = \{(1,2), (2,2), (3,2), (4,1), (5,2)\}$

---

2) If $S = \{1, 2, 3, 4, 5\}$ and if the functions $f, g, h : S \to S$ are given by

$f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$

$g = \{(1,3)\ (2,5)\ (3,1)\ (4,2)\ (5,4)\}$

$h = \{(1,2)\ (2,2)\ (3,4)\ (4,3)\ (5,1)\}$.

(i) Verify $f \circ g = g \circ f$.

(ii) Explain why $f$ & $g$ have inverses but $h$ does not.

(iii) Find $f^{-1}$ and $g^{-1}$

(iv) S.T $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$.

(i) Solution:

$$(f \circ g)(1) = f(g(1)) = f(3) = 4$$
$$(f \circ g)(2) = f(g(2)) = f(5) = 3$$
$$(f \circ g)(3) = f(g(3)) = f(1) = 2$$
$$(f \circ g)(4) = f(g(4)) = f(2) = 1$$
$$(f \circ g)(5) = f(g(5)) = f(4) = 5$$

$\therefore (f \circ g) = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$

$$(g \circ f)(1) = g(f(1)) = g(2) = 5$$
$$(g \circ f)(2) = g(f(2)) = g(1) = 3$$
$$(g \circ f)(3) = g(f(3)) = g(4) = 2$$
$$(g \circ f)(4) = g(f(4)) = g(5) = 4$$
$$(g \circ f)(5) = g(f(5)) = g(3) = 1$$

$\therefore g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$

Hence $f \circ g \neq g \circ f$.

(iii) Gn $f = \{(1,2) (2,1) (3,4) (4,5) (5,3)\}$.

$f^{-1} = \{(2,1) (1,2) (4,3) (5,4) (3,5)\}$

Verified.

$f \circ f^{-1} = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

$f^{-1} \circ f = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

$$f \circ f^{-1} = f^{-1} \circ f = I$$

Gn, $g = \{(1,3)(2,5)(3,1)(4,2)(5,4)\}$.

$g^{-1} = \{(3,1)(5,2)(1,3)(2,4)(4,5)\}$.

(IV) $(f \circ g) = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$

$\rightarrow (f \circ g)^{-1} = \{(4,1)(3,2)(2,3)(1,4)(5,5)\}$

$g^{-1} = \{(3,1)(5,2)(1,3)(2,4)(4,5)\}$

$f^{-1} = \{(2,1)(1,2)(4,3)(5,4)(3,5)\}$

$\rightarrow g^{-1} \circ f^{-1} = \{(2,3),(1,4)(4,1),(5,5),(3,2)\}$

$\rightarrow f^{-1} \circ g^{-1} = \{(3,2)(5,1)(1,5)(2,3)(4,4)\}$

Hence $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

(ii) Gn $f = \{(1,2)(2,1)(3,4)(4,5),(5,3)\}$.

$f(1) = 2$; $f(2) = 1$, $f(3) = 4$, $f(4) = 5$,

$f(5) = 3$.

$\therefore$ $f$ is 1-1!

$g = \{(1,3),(2,5)(3,1)(4,2)(5,4)\}$.

$g(1) = 3$, $g(2) = 5$, $g(3) = 1$, $g(4) = 2$, $g(5) = 4$

$\therefore$ $g$ is 1-1

Now,

Range of $f = \{1,2,3,4,5\} = S$ (co-domain)

Range of $g = \{1,2,3,4,5\} =$ (co-domain)

Hence $f$ & $g$ are onto.

$\therefore$ f&g are bijective.

$\Rightarrow$ $f^{-1}$ exists. $\therefore$ f&g are invertible.

~~If~~ for given,

$f = \{(1,2)(2,2)(3,4)(4,3),(5,1)\}$

$f(1) = f(2) = 2$ , but $1 \neq 2$.

$\therefore$ $f$ is not 1-1.

Range of $f = \{1,2,3,4\} \neq$ (co-domain)

$\therefore$ $f$ is not onto.

Hence $f^{-1}$ does not exists.