Homework 4
Colorado CSCI 5454

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People I studied with for this homework: Ashwin
Other external resources used: Youtube

Problem 1 (4 points)

In this problem, we will practice the Ford-Fulkerson max-flow framework and recall the associated terminology. We will use the following instance, with capacities in black (all those not shown are zero):

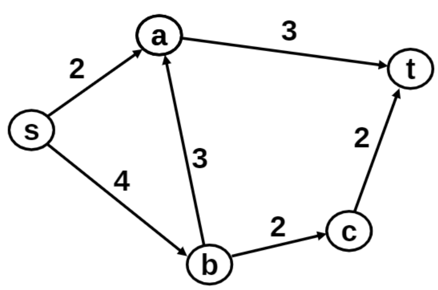

Figure 1: Graph

0.1 Part a(8 Points)

Simulate the Ford-Fulkerson framework on this instance and report a max s-t flow. You may select the path (step 2) in any way you wish. After each round, report all of the following:

- f (draw a fresh graph G and report $f(u, v)$ on all nonzero edges)
- $|f|$
- r_f (draw a fresh graph and report $r_f(u, v)$ on all nonzero edges)

- G_f (draw a fresh graph G_f)

You may omit an edge label if it is zero, but don't omit any others!

Solution:
Let's compute the f, $|f|$, r_f, G_f and compute the max flow from vertices $s \to t$.
I have used the following conventions for all the graphs.

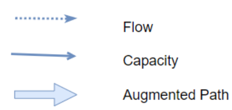

Figure 2: Graph

Cycle 0 **:**
For the Augmented path we will take the path from $s \to a \to t$, the minimum capacity along this path is 2, therefore we take α as 2, i.e $|f| = 2$.

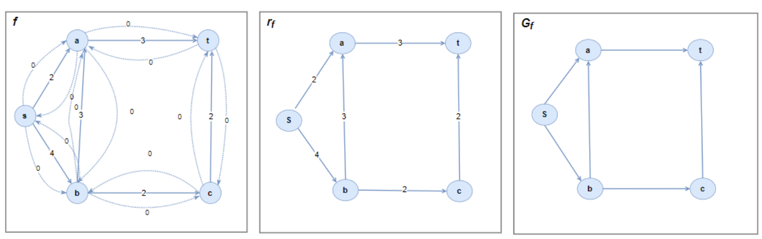

Figure 3: Graph

Cycle 1 **:**
For the Augmented path we will take the path from $s \to b \to c \to t$, the minimum capacity along this path is 2, therefore we take α as 2, i.e $|f| = 4$.

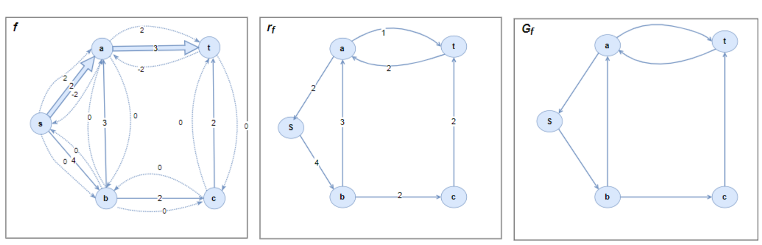

Figure 4: Graph

Cycle 2 **:**
For the Augmented path we will take the path from $s \to b \to a \to t$, the minimum capacity along this path is 1, therefore we take α as 1, i.e $|f| = 5$.

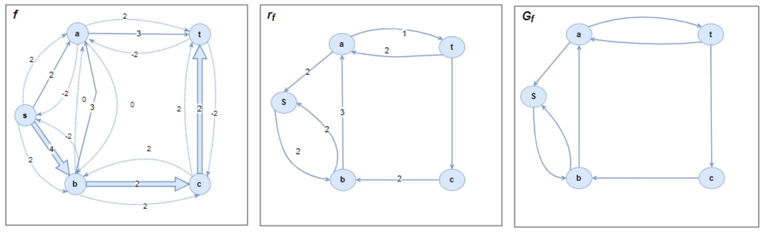

Figure 5: Graph

Cycle 3 **:**
Since, we have the residual capacity of edge s as 1 we do not have nay more augmented paths, hence the max-flow is **5**.

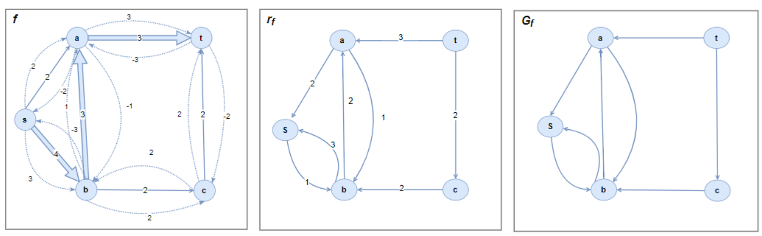

Figure 6: Graph

$f \to$ Flow Graph
$r_f \to$ Residual Flow
$G_f \to$ Residual Graph

0.2 Part b(2 Points)

What is a min s-t cut of the graph? Explain how to use the results of the previous problem to find a min cut.

Solution:
The min s-t cut of a graph is the cut which gives the minimum net flow across the cut.

Once we have the G_f after running the Ford-Fulkerson, we can group the vertices into 2 sets (S, T), where the vertices reachable from s belongs to set S and the rest in set T (Lemma 6).

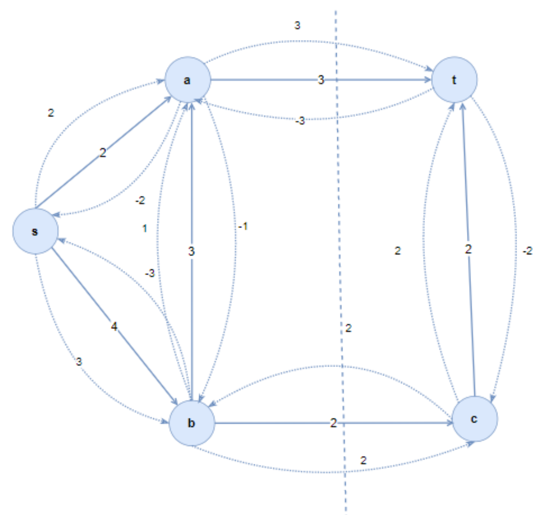

Figure 7: Graph

Now, we have $S = \{$s,a,b$\}$ and $T = \{$c,t$\}$.
The net-flow across the cut is 5.

0.3 Part c(2 Points)

Is that the only min s-t cut in the graph? Explain.
Solution:
For this graph, $S = \{$s,a,b,c$\}$ and $T = \{$t$\}$, $|f| = 5$ and the max-flow is also 5 which satisfies Lemma 3, hence this is also a valid min-cut.

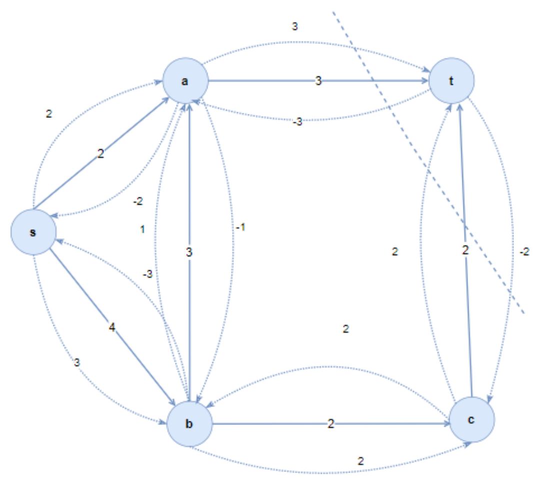

Figure 8: Graph

Problem 2 (4 Points)

Consider the following "Greedy" algorithm for min s-t cut:
Solution:

- Initialize $S = s$, $T = V \quad s$.
- Pick one vertex $u \in T$ and move it from T to S. The chosen vertex is the one that decreases the cut value the most.
- When there is no longer any choice of vertex that decreases the cut value, stop.

Does this algorithm find a min s-t cut? If so, sketch a proof of correctness at a high level. If not, give a proof of incorrectness by counterexample.
Solution:

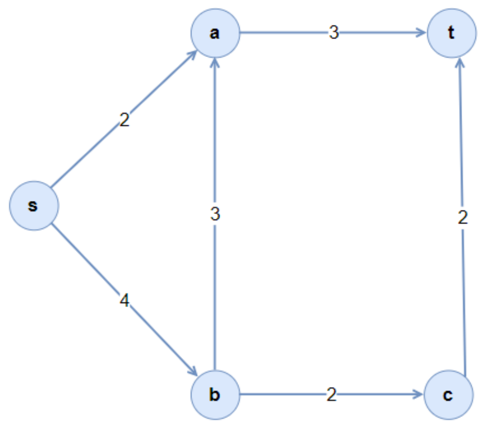

Figure 9: Graph

Lets us take this graph to verify:

Step : 1

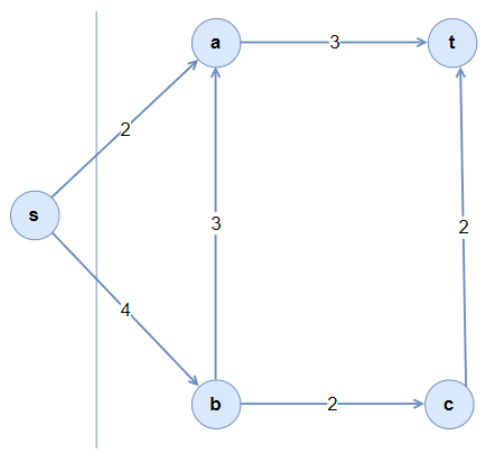

Figure 10: Graph

Here, $S = \{\text{s}\}$ and $T = \{\text{a,b,c,t}\}$ and the cut-value or the capacity is $4 + 2 = 6$.

Step : 2

Choose a vertex from T and move it to S and try the same for all the vertices except t.

i.)

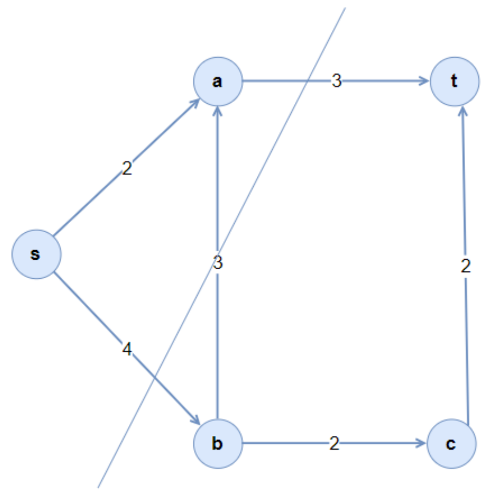

Figure 11: Graph

Here, S = {s,a} and T = {b,c,t} and the cut-value or the capacity is 4 + 3 = 7.

ii.)

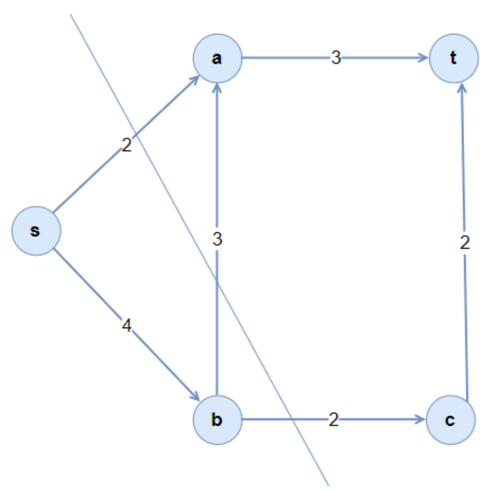

Figure 12: Graph

Here, S = {s,b} and T = {a,c,t} and the cut-value or the capacity is 2 + 3 + 2 = 7.

iii.)

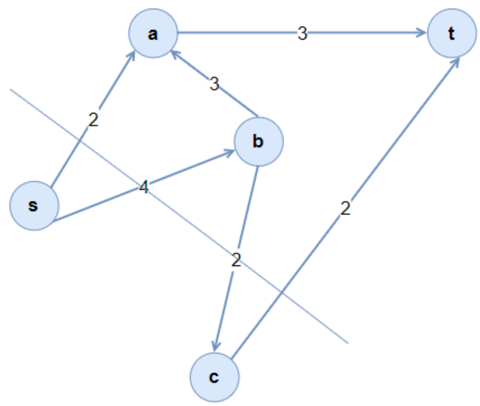

Figure 13: Graph

Here, S = {s,c} and T = {a,b,t} and the cut-value or the capacity is 2 + 4 + 2 = 8.

Since, all the cut-values in step 2 are greater than the cut-value in step 1, i.e (7,7,8) > 6, we can't go any further.

Step : 3
After running the algorithm,
we have :
S = {s} and T = {a,b,c,t}, net-flow = 5 and net-capacity = 6.

But, we have min-cut combinations as:
S = {s,a,b} and T = {c,t}
S = {s,a,b,c} and T = {t}
Holding the min-cut net flow as 5 and min-cut net capacity as 5.
The total capacity after running the algorithm is greater than the min-cut capacity, hence with this prove that the result of the algorithm does not give the min-cut.

Problem 3 (10 points)

In **Max Flow with Vertex Capacities**, we are given an instance of max flow (G, c, s, t) along with a vertex capacity function k : V $\rightarrow$ **R**$\geq$ 0. We consider the *net flow through a vertex u*:

$$\frac{1}{2}\sum_{v}|f(u,v)|$$

Here |f(u,v)| is the absolute value of the flow from u to v. This adds up every unit of flow through u twice: once for negative flow (i.e. incoming flow), and once for positive flow (i.e. outgoing flow). Therefore, we divide by 2 to get the net flow through the vertex. (If this is confusing, try drawing an example, calculate the net flow through u, and see why it's correct!)
In Max Flow with Vertex Capacities, the problem is to find the maximum s-t flow, with the added requirement that any flow must also satisfy the *vertex capacity constraint*:

$$\frac{1}{2}\sum_{v}|f(u,v)| \leq k(u) \quad \forall \quad u$$

The algorithm should return the value of the flow and the flow itself.

Question: Give an algorithm for Max Flow with Vertex Capacities. Briefly justify correctness (one to two paragraphs). Do so by reducing this problem to standard max flow.

Solution: For a given graph, $G = \{\text{V,E}\}$, we can split each vertex $u \in$ V into u_{in} and u_{out} with their capacity, thus we have made a reduction of the original graph.
Now, we can find a max-flow on the new graph.

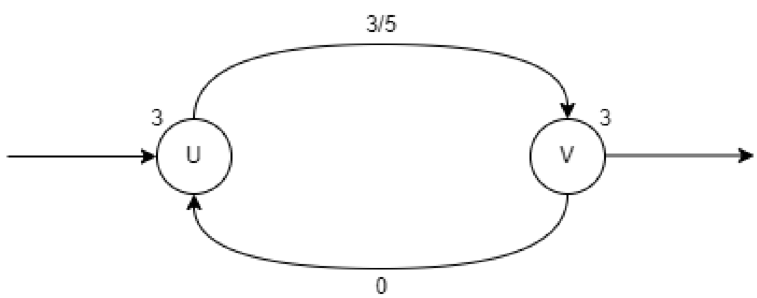

Figure 14: Graph

Now, we can try to find the max-flow with vertex capacities graph by reducing it to max-flow.
For the above graph, the capacity of u is 3 and the capacity of v is also 3, we have

$$\frac{1}{2}\sum_{v}|f(u,v)| \leq k(u)$$

Now, we will try the reduction:

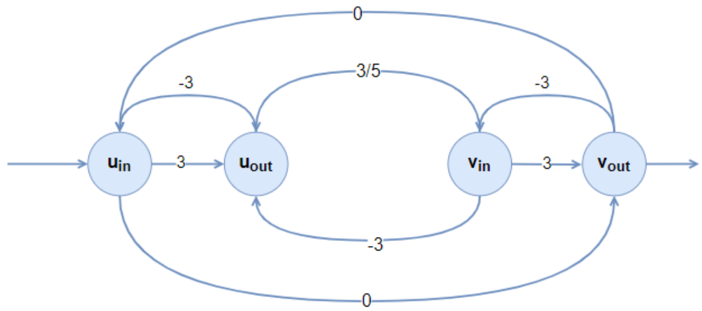

Figure 15: Graph

On the reduced graph, we have split u and v into u_{in}, v_{in} and u_{out}, v_{out} with their capacities.
Incoming flow at a vertex u $= f(u_{in}, u_{out})$ which satisfies flow conservation.
This also satisfies the capacity conservation at a vertex u as:
$\sum_{u \in V} max\{f(u,v), 0\} \leq$ k(u)
$f(u_{in}, u_{out}) \leq k(u)$ which satisfies the edge capacity constraint.
We can noe check the inequality of the graph as follows and using $Using\ |a+b| \leq |a|+|b|\ rule:$

$$\frac{1}{2}\sum_v |f(u,v)| \leq k(u) \rightarrow \frac{1}{2}|f'(u_{out}, v_{in}) + f'(v_{out}, u_{in})| \leq k(u).$$
$$\frac{1}{2}|f'(u_{out}, v_{in}) + f'(v_{out}, u_{in})| \leq \frac{1}{2}(|f'(u_{out}, v_{in})| + |f'(v_{out}, u_{in})|) \leq k(u)$$
$$Since\ f'(u_{out}, v_{in}) \leq k(u)\ and\ f'(v_{out}, u_{in}) \leq k(u).$$
$$\frac{1}{2}(|f'(u_{out}, v_{in})| + |f'(v_{out}, u_{in})|) \rightarrow \frac{1}{2}(k(u) + k(u)) \leq k(u)$$
$$\text{Hence, } k(u) \leq k(u).$$

Hence the inequality is satisfied and we have proved the statement and the correctness.