Homework 1
Colorado CSCI 5454

RITHIK KUMAR ATHIGANUR SENTHIL

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People I studied with for this homework: Ashwin
Other external resources used: https://www.youtube.com/watch?v=IR$_S 8BC8KI0ab_channel = codebasics$

Problem 1 (8 points)

Let $f(n) = 5n^3 + 2n$.

0.1 Part a (4 points)

To prove: $f(n) \in \Theta(n^3)$

Solution:

$$f(n) = \{\Theta(g(n)) :: \text{ there will be three constants } c_1, c_2 \text{ and } k \text{ such that}$$
$$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \quad \forall \quad n \geq k \text{ and } n \in N\}$$

For all $n \geq k$, the function f(n) lies within a (constant value)*g(n).

which implies, $n^3 \leq 5n^3 + 2n \leq 7n^3$ $\forall$ where $c_1 = 1$, $c_2 = 7$ $\forall$ $n \geq 1$

(1) implies that for a given $k = 1$, we can find $c_1 = 1$ and $c_2 = 7$. This shows that there are three positive constant values c_1, c_2 and k, which satisfies $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ $\forall n \geq k$.

0.2 Part B (2 points)

To prove: $f(n) \in o(n^6)$

Solution:

$$f(n) = \{o(g(n)) : \text{ given a positive constant value } c > 0 \text{ there exists a constant } k > 0 \text{ which satisfies}$$
$$0 \leq f(n) < c \cdot g(n) \ \forall n \geq k \text{ and } n \in N\}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Which implies that, $g(n)$ becomes asymptotically large relative to $f(n)$ as n goes to infinity.
which impplies,

$$\begin{aligned}
\lim_{n \to \infty} \frac{f(n)}{g(n)} &= \lim_{n \to \infty} \frac{5n^3 + 2n}{n^6} \\
&= \lim_{n \to \infty} \frac{5n^3}{n^6} + \lim_{n \to \infty} \frac{2n}{n^6} \\
&= \lim_{n \to \infty} \frac{5}{n^3} + \lim_{n \to \infty} \frac{2}{n^5} \\
&= 0
\end{aligned}$$

0.3 Part C (2 points)

To prove: $f(n) \in \omega(n)$

Solution:

$$f(n) = \{\omega(g(n)) : \text{ given a positive constant } c > 0 \text{ there exists a constant } k > 0 \text{ which satisfies}$$
$$0 \leq c \cdot g(n) < f(n) \ \forall n \geq k \text{ and } n \in N\}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

That is, $f(n)$ becomes asymptotically large relative to $g(n)$ as n goes to infinity. Now,

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{5n^3 + 2n}{n} \\
&= \lim_{n\to\infty} \frac{5n^3}{n} + \lim_{n\to\infty} \frac{2n}{n} \\
&= \lim_{n\to\infty} 5n^2 + \lim_{n\to\infty} 2 \\
&= 2 + \infty \\
&= \infty
\end{aligned}$$

Therefore $f(n) \in \omega(n)$

Problem 2

Question: Let $f(n) = \sum_{j=1}^{n} \sqrt{j}$
Give a simple function g and show that $f(n) = \Theta(g(n))$

Solution:
we need to prove $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$.

According to the definition of $\mathcal{O}(n)$, if there exists two positive numbers C, N such that, $\forall\ \ n \geq$ N $\rightarrow f(n) \leq$ C.g(n) where C $>$ 0 and N $>$ 0.

$$\begin{aligned}\sum_{j=1}^{n} \sqrt{j} &= \sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n-1} + \sqrt{n} \\ &\leq \sqrt{n} + \sqrt{n} + \sqrt{n} + ... + \sqrt{n} + \sqrt{n} \qquad (\forall\ \ n \geq 1) \\ &\leq n.\sqrt{n} \\ &\leq 1.n^{3/2}\end{aligned}$$

$$where\ \ g(n) = n^{3/2}\ \ and\ \ C = 1.$$

This implies that $f(n) \in \mathcal{O}(n^{3/2})$

According to the definition of $\Omega(n)$, if there exists two positive numbers C, N such that, $\forall\ \ n \geq$ N $\rightarrow f(n) \geq$ C.g(n) where C $>$ 0 and N $>$ 0.

Calculating the area under the curve and approximating it, we can find the lower bound to the curve.

$$\int_0^n f(x)\,dx \leq \sum_{j=1}^{n} \sqrt{j}$$

$$\int_0^n x^{1/2} dx\ \leq\ \sum_{j=1}^{n} j^{1/2}$$

$$\frac{2}{3} x^{3/2} \Big|_0^n\ \ \leq f(n)$$

$$\frac{2}{3} n^{3/2}\ \ \leq f(n)$$

This implies $g(n)\ \in \mathrm{O}f(n)$, where $C = 2/3$, $n \geq 1$
Which also implies that $f(n)\ \in \Omega(g(n))$

Based on the above proofs of $\mathcal{O}(n)$ and $\Omega(n)$,

$$\frac{1}{2^{3/2}}.n^{3/2} \leq \sum_{j=1}^{n} \sqrt{j} \leq 1.n^{3/2}$$

$\therefore$ we can conclude that $f(n) \in \Theta(n^{3/2})$.

Problem 3 (4 points)

Consider the following claim: "On an undirected graph, DFS starting from vertex u will always find the longest simple path in the graph starting at u." Either prove this claim is true, or give a counterexample showing it is false.

Solution:

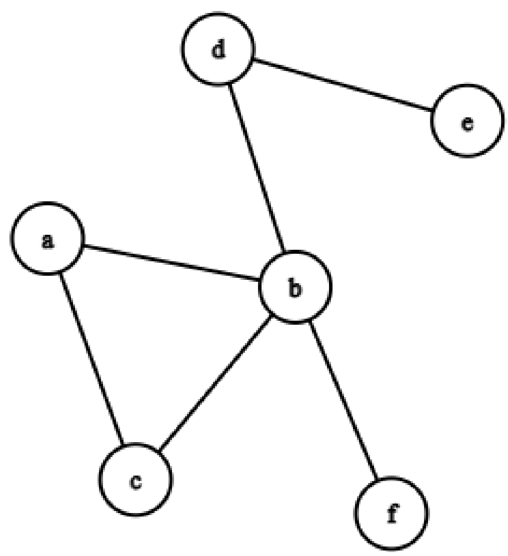

Figure 1: Graph

The above mentioned statement is false, I have taken an undirected graph where if u(starting vertex) is considered to be c, then the longest path must be:
c $\rightarrow$ a $\rightarrow$ b $\rightarrow$ d $\rightarrow$ e , where the length of the path is 4.
Since DFS randomly selects the starting vertex and the subsequent ones, instead of d as the subsequent node after b, if it selects f, then the longest path it can traverse will be:
c $\rightarrow$ a $\rightarrow$ b $\rightarrow$ f, where there the length of the path is 3.

Therefore with this counterexample, we can prove the claim to be false.

Problem 4 (4 points)

Given an undirected graph G = (V;E), an orientation of G is a directed graph that results from assigning a direction to each edge of G.
Prove that any graph has an orientation that is a DAG.

Solution:
Lets take an undirected graph as an example.

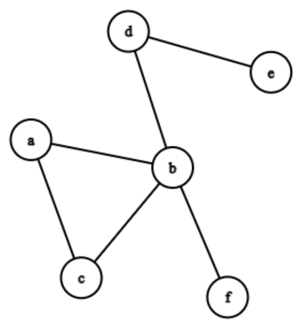

Figure 2: Undirected Graph

Now we can direct the edges of the graph using this algorithm for the orientation.

- Take input of the two values(vertices) i and j which have an edge.
  - direct the edge from i to j if i is lesser than j
  - Mark j as visited.
  - else:
  - direct the edge from j to i if j is lesser than i
  - mark i as visited.
- end
- Once all nodes are marked as visited. The undirected graph would have converted into a directed graph.

- If we take the case of other algorithms like randomly assigning a direction from $i \rightarrow j$ or $j \rightarrow i$, then there is a possibility of forming a cyclic path in itself, that is why we need a set of rules for assigning the direction of the edges.

Now, the graph has turned out to be this:

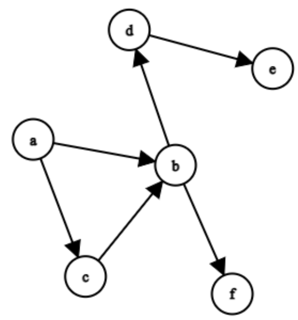

Figure 3: Directed Graph

The topological sort is: a, c, b, d, e, f

The produced directed graph has a topological sort which means it is a Directed Acyclic Graph.
Therefore, after using the above algorithm the resulting orientation of the graph is a DAG.