Project Report - Mathematical Model

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Introduction

This report shall discuss the mathematical specification of a polyhedral object in two dimensional projections as well as in three dimensions. We shall see how to generate 2D projections from 3D object specification, and vice versa. In the case of conversion of 2D projections to 3D, we shall discuss whether a solution exists, uniqueness of such a solution, and validity of our method in producing one of the valid solutions.

Object specification in 3D and 2D

What data is sufficient to uniquely identify a polyhedral object? By definition, we need points, line segments and faces. A model consisting of only points and line segments is known as a wireframe model. It does not uniquely identify a 3D object as seen in Figure 1.

We exclude the case of objects that are non-contiguous, or those containing faces with bordering line segments not contained in any other face, as these cases do not meet the polyhedral criterion.

Apart from points, lines and planes, no further information can be extracted from a polyhedral object which cannot be derived from these. Hence, if there exists a unique representation, then it shall contain less than or equal information as our model. e.g. Any point in interior of the object is bounded by faces. Putting the coordinates of the point in the equation of the face can tell us which side of a face the point lies. Using this information we can infer that the point lies in the interior, and similarly for exterior points we can determine them to be exterior to our object.

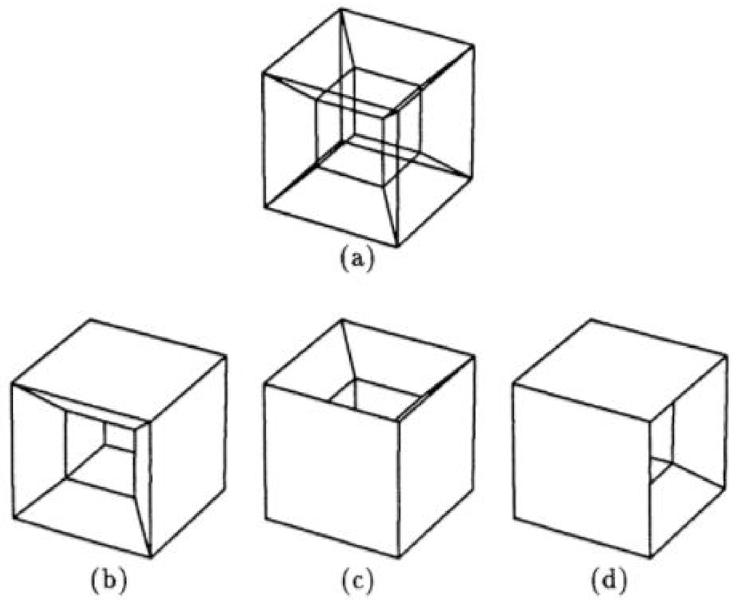

Figure 1: An ambiguous wireframe model

The information of points is stored as a 3-element vector representing x, y and z coordinates of the object with respect to a coordinate system attached to the object. These form nodes in a graph where edges represent line segments between individual points. Faces are represented as sets of line segments.

Computing 2D projections of 3D polyhedral object

Points

Consider a point P with coordinates $r_P = (x, y, z)$ and a plane with normal $n_{plane} = (n_x, n_y, n_z)$. We define an origin on the plane, as well as two orthogonal directions for x and y. The origin is denoted by $r_O = (o_x, o_y, o_z)$ and the two coordinate axes in the plane are denoted by $e_1 = (e_{x1}, e_{y1}, e_{z1}), e_2 = (e_{x2}, e_{y2}, e_{z2})$. We have that $n \cdot e_1 = 0, n_{plane} \cdot e_2 = 0, e_1 \cdot e_2 = 0$. Note that all the direction vectors should be normalized i.e. magnitude should be one.The point P must obey the equation:

$$r_P = r_O + t_1 e_1 + t_2 e_2 + sn$$

where t_1 and t_2 are the 2D coordinates along e_1 and e_2 and s the normal separation (distance) between the plane and the point. These scalars are found by projections:

$$s = n_{plane} \cdot (r_P - r_O)$$

$$t_1 = e_1 \cdot (r_P - r_O)$$

$$t_2 = e_2 \cdot (r_P - r_O)$$

Finally, 2D coordinates of the point P will be (t_1, t_2) in the co-ordinate system of our projection plane where e_1, e_2 are x, y axes respectively.

Line Segments

Consider a line segment L which joins points P_1 and P_2. In the above section on points we have already obtained projections of P_1 and P_2 on the projection plane. Let us denote the projections of these points by p_1 and p_2. The projection of the line segment L shall be the line segment joining p_1 and p_2. We denote this line segment by l.

Proof: The line segment L is a set of points which can be denoted as

$$\{P_i : P_i = xP_1 + (1-x)P_2, 0 \leq x \leq 1\}$$

Since projection involves dot product with a unit vector and dot product of a scalar product is a scalar product of a dot product, projection of $P_i = xp_1 + (1-x)p_2$ which merely denotes a point on the line segment joining p_1 and p_2.

Computing 3D Solid from 2D projections

We assume that we are given the 2D projections in the form of points and line segments on two/three projection planes. The projections of the points are labelled across projections consistently. We shall also assume that the projection planes have linearly independent normals.

Points

We have already observed that

$$r_P = r_O + t_1 e_1 + t_2 e_2 + s n_{plane}$$

where t_1 and t_2 are the 2D coordinates along e_1 and e_2 and s the normal separation (distance) between the plane and the point.

The values e_1, e_2, n_{plane}, r_O remain constant for a single projection plane. Given t_1, t_2 for individual points, we can easily express r_P in terms of s_{plane1}. Similarly using this equation with projection of the same point on another projection plane, we get another equation expressing r_P in terms of s_{plane2}. These are basically lines in 3D geometry along normals of these two planes.

If these lines intersect at a point, then that point is a candidate for r_P. If these lines do not intersect then the projections are inconsistent and no solid is possible. We also need to check the obtained candidate point with the projection on the third plane for a final consistency check (if we are given three projections).

We shall perform above operation for all points in the projections.

Line Segments

Lemma: Each line shall be visible in atleast one projection if we are given ≥ 2 views.

Proof: A line is invisible in a projection if normal to projection plane is parallel to line. However we are given atleast two linearly independent projection planes, thus both their normals can't be parallel to the line.

By the lemma, we can consider a line segment visible in any one projection. Each of its ends shall be labelled in the projection with a set of points. If a line segment has only one label at each of its end points, well simply connect those two points with a line segment in the 3D representation. We will then check the consistency of this line segment with the other views by projecting it onto the planes as described above.

Now, if a line segment has multiple labels at any of its end points, we will iterate from all the possible projections of that line. Using consistency checks we will eliminate the impossible line segments. All the remaining line segments shall be deemed to be part of the 3D figure.

Surfaces

We shall construct faces from coplanar lines. We shall iterate over all the line segments and store the corresponding plane defined by the line, another line intersecting it and their intersection point. After doing this for all line segments, we shall have constructed all the faces.

Each line in a polyhedron is part of a face, which is a subset of a plane. If we iterate over all lines, we shall cover all the faces.

Sufficient Information for 3D Reconstruction

Two views are insufficient to determine the real object (if the views are not labelled). On the other hand, if we have two orthographic projections of an object, in which, every point of the object is labelled in both projections, then we can uniquely determine the object using those two projections.

If we are given three views with no labelling which have linearly independent projection planes then hidden internal structure may not be visible if obscured by external coinciding lines.

Limitations

We have not considered information given in the form of hidden lines i.e. not differentiated between hidden lines and visible lines.