Written Question 1

Vector $V_1 = [1\ 1\ 1]^T$

Vector $V_2 = [1\ 0\ 0]^T$

Point $P_1 = [3\ 3\ 3]^T$

$P_2 = [1\ 2\ 3]^T$

$P_3 = [0\ 0\ 1]^T$

$\beta^* = [\beta_1^*, \beta_2^*]^T$ $\quad \beta^* = \dfrac{X^T y}{X^T X}$

Vector matrix $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\left\{ X = [V_1\ V_2] \right\}$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\times1 + 1\times1 + 1\times1 & 1\times1 + 0\times1 + 0\times1 \\ 1\times1 + 0\times1 + 0\times1 & 1\times1 + 0\times0 + 0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(X^T X)^{-1} = \frac{1}{(3\times1) - (1\times1)} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$\left\{ \text{if } X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right.$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

$$(X^TX)^{-1}X^T = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\times1-0.5\times1 & 0.5\times1-0.5\times0 & 0.5\times1-0.5\times0 \\ -0.5\times1+1.5\times1 & -0.5\times0+1.5\times1 & -0.5\times1+1.5\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix}$$

Projection of point P1 on the subspace is $\hat{P}_1$.

$\beta^*_{P_1}$ is $(X^TX)^{-1}X^TP_1$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times0+3\times0.5+3\times0.5 \\ 3\times1-3\times0.5-3\times0.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Projection of point P_2 on the subspace is $\hat{P}_2$.

$\beta^*_{P_2}$ is $(X^TX)^{-1}XP_2$

$$= \begin{bmatrix} 0 & 0.3 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0+0.5\times2+3\times0.5 \\ 1\times1-0.5\times2-3\times0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix}$$

Projection of point P_3 on the subspace $\hat{P_3}$.

$\beta_{P_3}^*$ is $(X^TX)^{-1}X^TP_3$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times 0 + 0.5\times 0 + 0.5\times 1 \\ 1\times 0 \cancel{+} - 0.5\times 0 - 0.5\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$$\hat{P_1} = X\beta P_1^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times 3 + 1\times 0 \\ 1\times 3 + 0\times 0 \\ 1\times 3 + 0\times 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\hat{P_2} = X\beta_{P_2}^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 2.5\times 1 - 1.5\times 1 \\ 2.5\times 1 - 0\times 1.5 \\ 2.5\times 1 - 1.5\times 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \\ 2.5 \end{bmatrix}$$

$$\hat{P_3} = X\beta_{P_3}^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1\times 0.5 - 0.5\times 1 \\ 1\times 0.5 - 0.5\times 0 \\ 1\times 0.5 - 0\times 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Written Question 2

Given probability of heads is 2/3

$P(H) = \frac{2}{3} \Rightarrow$ Probability of tails $P(T) = 1/3$

we need to find probability of heads less than or equal to 50.

$P(H \leq 50) = P(H=50) + P(H=49) + \dots + P(H=1) + P(H=0)$

$$P(X=x) = {}^{n}C_x \, p^x q^{n-x}$$
$${}^{n}C_r = \frac{n!}{(n-r)!\,r!} \; ; \; q = 1-p$$

Now,

$$P(H=50) = {}^{100}C_{50} \left(\frac{2}{3}\right)^{50} \left(\frac{1}{3}\right)^{50}$$

$$P(H=49) = {}^{100}C_{49} \left(\frac{2}{3}\right)^{49} \left(\frac{1}{3}\right)^{51}$$

$$\vdots$$

$$P(H=1) = {}^{100}C_{1} \left(\frac{2}{3}\right)^{1} \left(\frac{1}{3}\right)^{99}$$

$$P(H=0) = {}^{100}C_{0} \left(\frac{2}{3}\right)^{0} \left(\frac{1}{3}\right)^{100}$$

$$\therefore P(H \leq 50) = {}^{100}C_{50} \left(\frac{2}{3}\right)^{50} \left(\frac{1}{3}\right)^{50} + {}^{100}C_{49} \left(\frac{2}{3}\right)^{49} \left(\frac{1}{3}\right)^{51} + \dots + {}^{100}C_{1} \left(\frac{2}{3}\right)^{1} \left(\frac{1}{3}\right)^{99} + {}^{100}C_{0} \left(\frac{2}{3}\right)^{0} \left(\frac{1}{3}\right)^{100}$$

The computation is performed using python code.

$\therefore$ The probability for 50 or fewer heads is $P(H)$ is 0.00032845 or 0.032845%.