Assignment-1

1) written Questions.

2 - 2D vectors $v_1 = [1, 1, 1]$ & $v_2 = [1, 0, 0]$

Project the following points
$P_1 = [3, 3, 3]$ $P_2 = [1, 2, 3]$ and $P_3 = [0, 0, 1]$

steps :-
1. normalize vector v_1 & v_2
2. Perform inner product

$$\beta^* = \frac{1}{x^T x} * x^T y \text{ - for vector}$$

$$\beta^* = (x^T x)^{-1} x^T y \text{ for matrix}$$

let's consider $X = [v_1 \ v_2]$

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1\times1+1\times1+1\times1 & 1\times1+1\times0+1\times0 \\ 1\times1+0\times1+0\times1 & 1\times1+0\times0+0\times0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

if, $X = \begin{bmatrix} i & j \\ k & h \end{bmatrix}$

$$X^{-1} = \frac{1}{ih - jk} \begin{bmatrix} h & -j \\ -k & i \end{bmatrix}$$

$$\therefore (X^T X)^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(3\times1)-(1\times1)} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

$$(X^T X)^{-1} \cdot X^T = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\times1+(-0.5)\times1 & 0.5+0 & 0.5+0 \\ -0.5+1.5 & -0.5+0 & -0.5+0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix}$$

$$\beta^*_{P_1} = (X^T X)^{-1} X^T P_1 \qquad P_1 = [3, 3, 3]$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0.5\times3+0.5\times3 \\ 3-0.5\times3-0.5\times3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\beta^*_{P_2} = (X^T X)^{-1} X^T P_2 \qquad P_2 = [1, 2, 3]$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0.5\times2+0.5\times3 \\ 1-0.5\times2-0.5\times3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix}$$

$$\beta^*_{P_3} = (X^T X)^{-1} X^T P_3$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+0.5\times1 \\ 0+0-0.5\times1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

projection of P_1 is $\hat{P}_1$

$$\hat{P}_1 = X\beta^*_{P_1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 3+0 \\ 3+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\hat{P}_2 = X\beta^*_{P_2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 2.5-1.5 \\ 2.5+0 \\ 2.5+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \\ 2.5 \end{bmatrix}$$

$$\hat{P}_3 = X\beta^*_{P_3} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5-0.5 \\ 0.5+0 \\ 0.5+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$\therefore$ the projection of the following points on vector v_1 & v_2 are

$P_1 = [3, 3, 3]$ & $\hat{P}_1 = [3, 3, 3]$

$P_2 = [1, 2, 3]$ & $\hat{P}_2 = [1 \ 2.5 \ 2.5]$

$P_3 = [0, 0, 1]$ $\hat{P}_3 = [0 \ 0.5 \ 0.5]$

extra credit Question.

toss a coin 100 times.: Find Probability of Heads 50 or less
Probability of heads $= 2/3$

Two possibilities while flipping a coin → Heads or tails

For $P(\text{Heads}) = \frac{2}{3}$

$p(\text{Tails}) = 1 - \frac{2}{3} = \frac{1}{3}$

$$P(\text{Heads} \leq 50) = P(\text{heads}=50) + P(\text{Heads}=49) + \ldots + P(H=0)$$

use the binomial distribution

here n no. of trials is 100 & $r = 50$; $p = \frac{2}{3}$ & $q = \frac{1}{3}$

$$P(R=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

$$P(R=50) = {}^{100}C_{50} \cdot \left(\frac{2}{3}\right)^{50} \cdot \left(\frac{1}{3}\right)^{100-50}$$

$$P(H=50) = P(R=50) = {}^{100}C_{50} \cdot \left(\frac{2}{3}\right)^{50} \cdot \left(\frac{1}{3}\right)^{50}$$

for $P(H=49) = P(R=49) = {}^{100}C_{49} \cdot \left(\frac{2}{3}\right)^{49} \cdot \left(\frac{1}{3}\right)^{51}$

$$\therefore P(H \leq 50) = {}^{100}C_{50} \cdot \left(\frac{2}{3}\right)^{50} \cdot \left(\frac{1}{3}\right)^{50} + \ldots + {}^{100}C_{0} \cdot \left(\frac{2}{3}\right)^{0} \cdot \left(\frac{1}{3}\right)^{100}$$

which is ~~P(H) =~~

$= 0.00032845$

$= 0.032845\%$

* Let's try with Central Limit theorem

$\mu = np = 100 \cdot \frac{2}{3} = \frac{200}{3}$ & $\sigma = \sqrt{npq}$
$= \sqrt{100 \times \frac{2}{3} \times \frac{1}{2}} = \frac{10\sqrt{2}}{3}$

$$P(0 < x < 50)$$

$$= P\left(\frac{0 - \frac{200}{3}}{\frac{10\sqrt{2}}{3}} < z < \frac{50 - \frac{200}{3}}{\frac{10\sqrt{2}}{3}}\right)$$

$$= P\left(\frac{\frac{-197}{\cancel{3}}}{\frac{10\sqrt{2}}{\cancel{3}}} < z < \frac{\frac{-\cancel{50}^{5}}{\cancel{3}}}{\frac{\cancel{10}\sqrt{2}}{\cancel{3}}}\right)$$

$$= P\left(\frac{-197}{10\sqrt{2}} < z < \frac{-5}{\sqrt{2}}\right) = P\left(\frac{5}{\sqrt{2}} < z < \frac{197}{10\sqrt{2}}\right)$$

$$= P\left(0 < z < \frac{197}{10\sqrt{2}}\right) - P\left(0 < z < \frac{5}{\sqrt{2}}\right)$$