

Assignment-2

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Download all python codes from here

<https://github.com/rithvikreddy6300/Assignment-2/tree/main/code>

and latex-tikz codes from

<https://github.com/rithvikreddy6300/Assignment-2/blob/main/Assignment-2.tex>

Gate Problem-77

If a random variable X assumes only positive integral values, with the probability

$$P(X = x) = \frac{2}{3} \left(\frac{1}{3}\right)^{x-1}, x = 1, 2, 3, \dots$$

(A) $\frac{2}{9}$

(C) 1

(B) $\frac{2}{3}$

(D) $\frac{3}{2}$

Then E(X) is ?

SOLUTION

Let $Y = \{0, 1\}$ be a set of random variables of a Bernoulli's distribution with 0 representing a loss and 1 a win and let $Y_i \in Y$ for $i=0, 1, 2, 3, \dots$, Y_i is the outcome of i^{th} try. So

$$\begin{aligned} X = x &\implies Y_i = 0 \text{ if } i < x, Y_i = 1 \text{ if } i = x \\ &\implies X = \{Y_1 = 0, Y_2 = 0, Y_3 = 0, \dots, Y_x = 1\} \end{aligned}$$

For given bernoulli's trial $p = \frac{2}{3}$ and $q = 1 - p = \frac{1}{3}$.
The given probability distribution is

$$\begin{aligned} P(X = x) &= \frac{2}{3} \left(\frac{1}{3}\right)^{x-1} \\ \implies P(X = x) &= p(1-p)^{x-1} \\ \implies P(X = x) &= P(Y_{i=x} = 1)P(Y_{i < x} = 0)^{x-1} \end{aligned}$$

The expectation value of X represented by E(X) is given by

$$E(X) = \sum_{i=1}^{\infty} Pr(x=i) \times i$$

Let $S = E(X)$,

$$\implies E(X) = S = \sum_{i=1}^{\infty} Pr(x=i) \times i \quad (1)$$

$$\implies S = \sum_{i=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \times i \quad (2)$$

$$\implies S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \times i \quad (3)$$

Multiplying (2) with $\frac{1}{3}$ on both sides gives

$$\frac{1}{3}S = \sum_{i=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^i \times i \quad (4)$$

In (3) $\sum_{i=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^i \times i$ can be written as $\sum_{i=2}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \times (i-1)$

$$\implies \frac{1}{3}S = \sum_{i=2}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \times (i-1) \quad (5)$$

$$(3)-(5) \text{ gives : } \frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \times (i - (i-1)) \quad (6)$$

$$\implies \frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \quad (7)$$

$$\implies S = 1 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i \quad (8)$$

$$\implies S = 1 + \frac{1/3}{1 - \frac{1}{3}} \quad (9)$$

$$\implies S = \frac{3}{2} \quad (10)$$

The Variance Var(X) is given by $\sum x^2 P(x) - E(X)$ for the given distribution,

$$Var(X) = \sum_{i=1}^{\infty} i^2 P(x=i) - E(X) \quad (11)$$

$$\text{let } S = \sum_{i=1}^{\infty} i^2 P(x=i) = \sum_{i=1}^{\infty} i^2 \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \quad (12)$$

$$S/3 = \sum_{i=1}^{\infty} i^2 \frac{2}{3} \left(\frac{1}{3}\right)^i = \sum_{i=0}^{\infty} i^2 \frac{2}{3} \left(\frac{1}{3}\right)^i \quad (13)$$

$$= \sum_{i=1}^{\infty} (i-1)^2 \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \quad (14)$$

(12)-(14) gives us

$$\frac{2S}{3} = \sum_{i=1}^{\infty} (i^2 - (i-1)^2) \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} \quad (15)$$

$$S = \sum_{i=1}^{\infty} (2i-1) \left(\frac{1}{3}\right)^{i-1} \quad (16)$$

$$\Rightarrow S = 3 \sum_{i=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{i-1} i - \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} \quad (17)$$

$$\Rightarrow S = 3E(X) - \frac{1}{1-1/3} \quad (18)$$

$$\Rightarrow S = \frac{9}{2} - \frac{3}{2} = 3 \quad (19)$$

From (19) and (11) we can write

$$Var(X) = 3 - \frac{3}{2} = \frac{3}{2}$$

From (10) we can say that the expectation value of X given by $E(X)=S=\frac{3}{2}$ and $Var(X) = \frac{3}{2}$

(Option D).

The theoretical vs simulated probabilities are as follows,

