Hodge Decomposition Theorem: Introduction

Rithwik S V

Let us start with two finite dimensional vector spaces U and V and a linear map T between them. The adjoint of this map is denoted by T^*. When can we find solutions to the equation $Tu = v$, for a fixed $v \in V$. One necessary condition is that $v \perp Ker(T^*)$, since had a solution u_0 existed and $x \in ker(T^*)$, then $\langle v, x\rangle = \langle Tu_0, x\rangle = \langle u_0, T^*x\rangle = 0$. In fact, for the finite dimensional case this is a sufficient condition as it can be shown without a lot of trouble that $Im(T) = (Ker(T^*))^{\perp}$. The Hodge Decomposition Theorem is a generalization of this simple theorem.

Consider the the differential operator $L := \frac{d}{dx^2}$ acting on smooth function on the real line. This is just a linear map that sends a function f to its second derivative f''. In order to define an inner product on this space, we will restrict our functions to have a period of 2π. Define, $\langle f, g\rangle := \int_0^{2\pi} fg dx$. Then using integration by parts, we can show that L is in fact self-adjoint. What is the kernel of L? $u \in Ker(L) \implies u'' = 0$. This forces u to be linear of the form $u = ax + b$, but it has to be 2π periodic, this constrains $a = 0$. Hence $Ker(L) = constants$.

Suppose we are trying to look for a solution of $Lu = f$, i.e, $u'' = f$. Integrating both sides from 0 to x, we see $u'(x) = u'(0) + \int_0^x f$. Now suppose we have a solution, then it has to be 2π periodic, and $u'(0) = u'(2\pi) \implies \int_0^{2\pi} f = 0$. Recall that L is a self adjoint operator, hence $Ker(L^*) = Ker(L) = constants$. Clearly $\int_0^{2\pi} f = 0 \implies f \perp Ker(L)$. We have a necessary condition, which in fact turns out to be sufficient. Hence, even in this infinite dimensional case, $Lu = f$ has a solution iff $f \perp Ker(L)$.

This is precisely the Hodge Decomposition Theorem where your manifold is $\mathbb{R}$. For a general manifold, L turns out to be what is called the Hodge-Laplacian, Δ which is a self adjoint linear operator on smooth forms. The Hodge-Decomposition Theorem states that $\Delta u = f$ has a solution precisely when $f \in (Ker(\Delta))^{\perp}$. However there are a lot of subtle things that need to be done since we don't have a well defined notion of a PDE on a manifold to start with, or an inner product on the space of smooth forms.

The goal of this project is to give a self contained proof of the Hodge Decomposition Theorem. The result has far reaching consequences in getting information about the cohomology groups of the manifold which is an algebraic quantity and as such not an easy quantity to calculate. Below is a brief outline of the chapters.

The proof involves a study of PDE on manifolds. But what does it mean to have a PDE on a manifold? Is it independent of the co-ordinate system? Does it match with our usual definitions when we consider $\mathbb{R}^n$ as a smooth manifold. All these questions are dealt with in the first chapter. The analysis of PDE involves an inevitable study of Sobolev Spaces. This is a vast theory and studying it in it's generality will sufficiently take us off course. Hence we limit our study to the case of periodic functions which is far easier to deal with. The second chapter involves a brief section on Fourier Analysis, as does any theory on periodic functions, then we define Sobolev spaces and see generalisations of some things from calculus.

The third chapter deals with defining PDE and looking at some properties of the solutions. Most of these are for a special class of PDE, called elliptic PDE's. We proceed to develop the theory of Sobolev Spaces for periodic functions using techniques from Fourier Analysis. We will see this is sufficient for our purpose on compact manifolds. The final chapter gives us the proof the main analytical theorems that will be used to prove the Hodge-Decomposition Theorem. It is a self contained read, with the reader assumed to have a working understanding of functional analysis and differential geometry along with the elementary notions of a Riemannian metric and the concept of partitions of unity.