MATH 111 (DAY 2)

REMINDERS:

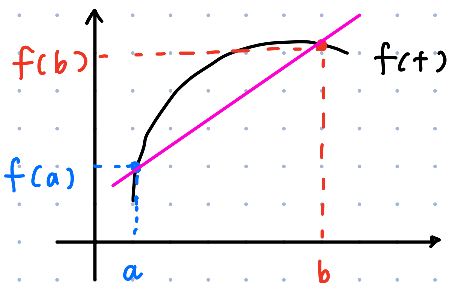
- WEBASSIGN 2 DUE THURS 09/02
- QUIZ 1 → CANVAS FRI 09/03
- PRECALC MODULES → DUE FRI 09/13
  (2% EC!)

LAST TIME:

AVERAGE ROC

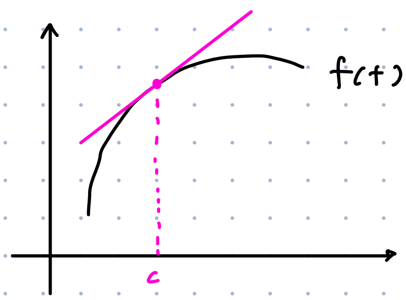
"SLOPE OF SECANT LINE"

↳ CAN COMPUTE ⇒ $\dfrac{f(b) - f(a)}{b - a}$



INSTANTANEOUS ROC

"SLOPE OF TANGENT LINE"



NOTATION: D.N.E = does not exist

Day 2- Section 2.2- Limits

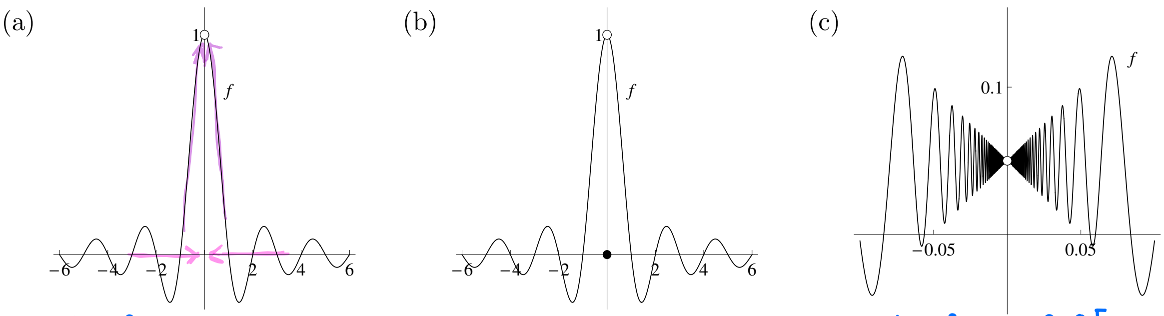
"What # are my outputs getting close to as my inputs get close to a, without equaling a."

Limits. You can think of the notation $\lim\limits_{x \to a} f(x)$ as shorthand for: "What number (if any) does $f(x)$ approach as x gets really close to a, without actually being equal to a?" This number is called the limit of $f(x)$ as x approaches a.

a and L are numbers

More precisely, saying $\lim\limits_{x \to a} f(x) = L$ means that we can make the values of $f(x)$ as close as we like to L by making x sufficiently close to (but not equal to) a.

1. In each of the following examples, what is $\lim\limits_{x \to 0} f(x)$? What is $f(0)$?



(a) $\lim\limits_{x \to 0} f(x) = 1$

$f(0)$ undefined

(b) $\lim\limits_{x \to 0} f(x) = 1$

$f(0) = 0$

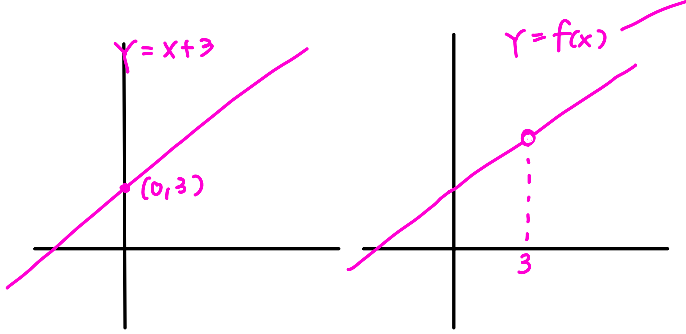
(c) $\lim\limits_{x \to 0} f(x) = 0.05$

$f(0)$ undefined.

$a^2 - b^2 = (a+b)(a-b)$

2. How about trying to compute limits algebraically: Compute $\lim\limits_{x \to 3} \dfrac{x^2 - 3^2}{x - 3}$.

I need to do more work



$$f(x) = \frac{x^2 - 9}{x - 3}$$

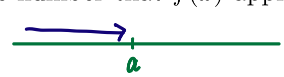
$$f(3) = \frac{3^2 - 9}{3 - 3} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

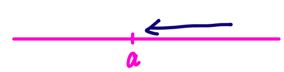
$$= \lim_{x \to 3} x + 3 = \boxed{6}$$

If a is a number, x can get really close to a from two sides, the left and the right. So, we also look at one-sided limits:

- The limit of $f(x)$ as x approaches a *from the left*, $\lim\limits_{x \to a^-} f(x)$, is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly *less* than a.

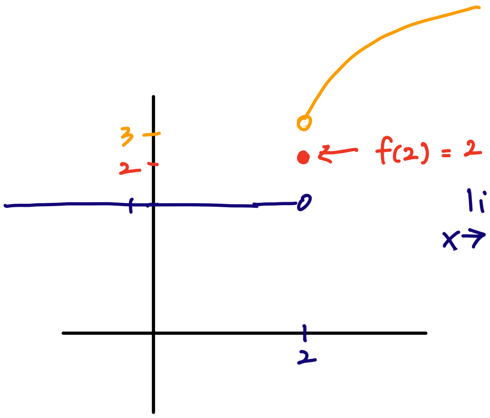


- The limit of $f(x)$ as x approaches a *from the right*, $\lim\limits_{x \to a^+} f(x)$, is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly *greater* than a.



Q: How might these limits connect to $\lim\limits_{x \to a} f(x)$?

3. Sketch an example of a function f for which $\lim_{w\to 2^-} f(w) = 1$ and $\lim_{w\to 2^+} f(w) = 3$. Can you sketch an example where in addition to the two limits above, you also have that $\lim_{w\to 2} f(w)$ exists?



$\leftarrow f(2) = 2$

$\lim_{x\to 2^+} f(x) = 3$

$\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2} f(x)$ D.N.E!

$\lim_{x\to a} f(x) = L$ (if and only if) EXACTLY WHEN $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

(i) Right and left limit exist

(ii) They are equal

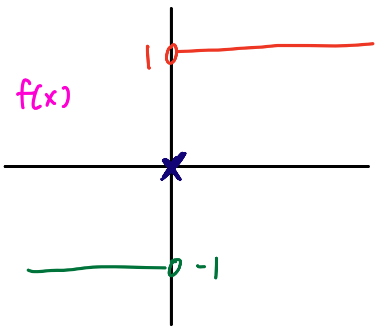
4. Does $\lim_{x\to 0} \frac{x}{|x|}$ exist? Why or why not?

$\left(\frac{0}{0}\right)$ needs more work!

$\frac{5}{|5|} = \frac{5}{5} = 1$

$\frac{-5}{|-5|} = \frac{-5}{5} = -1$

$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$



$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = 1 \\ \lim_{x\to 0^-} f(x) = -1 \end{array}\right\}$ Since $-1 \neq 1$, $\lim_{x\to 0} \frac{x}{|x|}$ D.N.E!

5. Sketch the graph of $f(x) = \frac{1}{x}$, and use it to find the following limits.

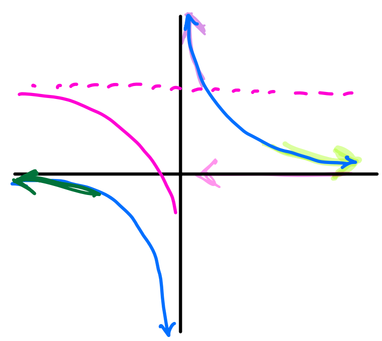
(a) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

(b) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

(c) $\lim_{x\to 0} \frac{1}{x}$ D.N.E!

(d) $\lim_{x\to\infty} \frac{1}{x} = 0$

(e) $\lim_{x\to -\infty} \frac{1}{x} = 0$



TAKEAWAYS:

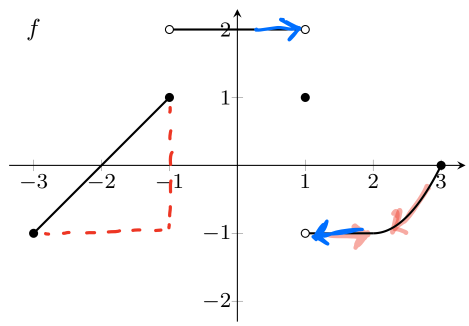
I. ∞ is not a number!

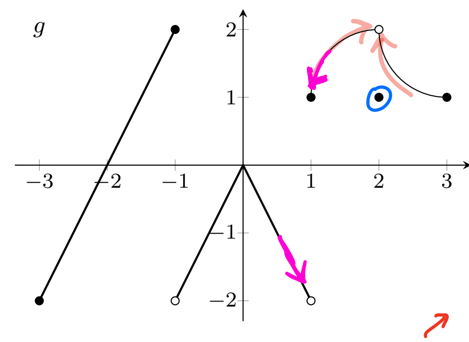
II. When we say $\lim_{x\to 0^+} \frac{1}{x} = \infty$, we really mean $\lim_{x\to 0^+} \frac{1}{x}$ D.N.E, but we can see our outputs are increasing without bound as $x \to 0^+$.

III. When we say $\lim_{x\to\infty} \frac{1}{x} = 0$

$\hookrightarrow$ END BEHAVIOUR

6. The graphs of f and g are given below.





$\lim_{x\to 1} f(x)$ $\boxed{DNE}$, $\lim_{x\to 1} g(x)$ $\boxed{DNE}$ → DOES NOT MEAN THAT $\lim_{x\to 1} \frac{f(x)}{g(x)}$ $\boxed{DNE}$

Evaluate each of the following quantities.

From (a), (b)

$\lim_{x\to 2} g(x)$ does not care about $g(2)$!

(a) $\lim_{x\to 2} g(x)$ $= 2$

(b) $g(2)$ $= 1$

(c) $\lim_{x\to 2} \frac{\boxed{f(x)} \to -1}{\boxed{g(x)} \to 2}$ $= \boxed{-\frac{1}{2}}$

(d) $\lim_{x\to 1} \frac{f(x)}{g(x)}$

(i) $\lim_{x\to 1^-} \frac{f(x) \to 2}{g(x) \to -2} = \frac{2}{-2} = \boxed{-1}$

(ii) $\lim_{x\to 1^+} \frac{f(x) \to -1}{g(x) \to 1} = \frac{-1}{1} = \boxed{-1}$

$\lim_{x\to 1} \frac{f(x)}{g(x)} = \boxed{-1}$

(e) $\lim_{x\to -1} \frac{f(x)}{g(x)}$

(i) $\lim_{x\to -1^-} \frac{f(x) \to 1}{g(x) \to 2} = \boxed{\frac{1}{2}}$

(ii) $\lim_{x\to -1^+} \frac{f(x) \to 2}{g(x) \to -2} = \frac{2}{-2} = \boxed{-1}$

$\lim_{x\to -1} \frac{f(x)}{g(x)}$ D.N.E

(f) $\lim_{x\to -2} \frac{f(x)}{g(x)}$ $=$ "$\frac{0}{0}$" limit → need to do more work.

We know for linear lines $y = mx + b$

m → slope, b → y-intercept

$$\begin{cases} f(x) = x + 2 \\ g(x) = 2x + 4 = 2(x+2) \end{cases}$$

$$\lim_{x\to -2} \frac{f(x)}{g(x)} = \lim_{x\to -2} \frac{x + \cancel{2}}{2(x+\cancel{2})} = \boxed{\frac{1}{2}}$$

More Practice

1. Find each limit. (It will help to sketch the relevant functions.)

   (a) $\lim_{x\to 0} \frac{1}{x^2}$

   (b) $\lim_{x\to \infty} \frac{1}{x^2}$

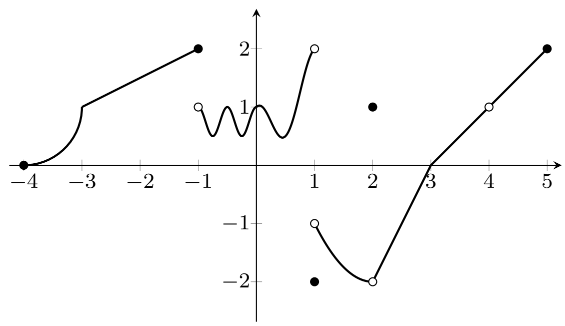
   (c) $\lim_{x\to \infty} \sqrt{x}$

   (d) $\lim_{x\to -\infty} 3^x$

   (e) $\lim_{x\to \infty} \sin x$

   (f) $\lim_{x\to 1^-} \arcsin x$

2. The graph of a function g is given below.



   Evaluate the following, or explain why they do not exist.

   (a) $\lim_{x\to 0} g(x)$

   (b) $\lim_{x\to 1^+} g(x)$

   (c) $\lim_{x\to 1^-} g(x)$

   (d) $\lim_{x\to 1} g(x)$

   (e) $g(1)$

   (f) $\lim_{x\to 2^-} g(x)$

   (g) $\lim_{x\to 2^+} g(x)$

   (h) $\lim_{x\to 2} g(x)$

   (i) $g(2)$

   (j) $\lim_{x\to -4^+} g(x)$

   (k) $\lim_{x\to -3} g(x)$

   (l) $g(-3)$

   (m) $\lim_{x\to -1} g(x)$

   (n) $g(-1)$

   (o) $\lim_{x\to 4} g(x)$

   (p) $g(4)$