MATH 111 (DAY 4)

REMINDERS:

- QUIZ 2 → CANVAS FRI 09/13
- WEBASSIGN 4 DUE THURS 09/12 ← LAST ONE W/ NO EXTENSION PENALTY!
- PRECALC MODULES → DUE FRI 09/13 (2% EC!)

LAST TIME:

CONTINUOUS AT $x = a$ IF

(1) $\lim_{x \to a} f(x)$ exists

(2) $f(a)$ is defined

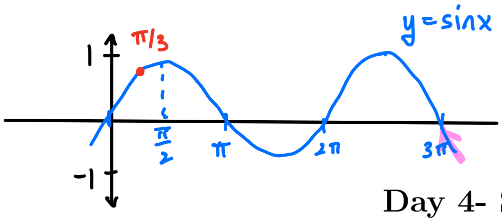
(3) $\lim_{x \to a} f(x) = f(a)$

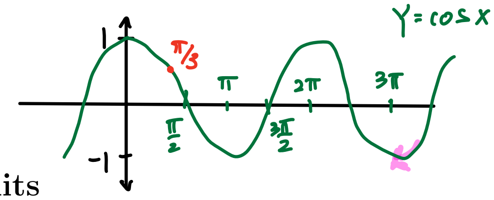
THM: INTERMEDIATE VALUE THM

(1) If $f(x)$ is continuous on $[a, b]$

(2) $f(a) < w < f(b)$

$\Rightarrow$ There is atleast one value $x = c$ such that $f(c) = w$





Day 4- Section 2.3- Computing Limits

1. *Working algebraically with limits.* Can you evaluate the following limits?

(a) $\lim_{x\to\frac{\pi}{3}} \frac{\sin x}{\cos x} = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$

$f(\pi/3)$

(b) $\lim_{x\to 3\pi^+} \frac{\sin x}{\cos x}$

$\sin(3\pi) = 0$

$\cos(3\pi) = -1$

$= \frac{0}{-1} = \boxed{0}$

(c) $\lim_{x\to 3\pi^+} \frac{\cos x}{\sin x}$

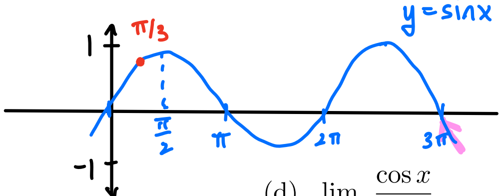
$\cos(3\pi) = -1$

$\sin(3\pi) = 0$

(I.) NON-ZERO NUM. / 0 — lim DNE

(II.) CAN WE SAY MORE? → ∞, → $-\infty$

STRATEGY: Look at SIGN of NUM & DENOM As $x\to 3\pi^+$.



As $x\to 3\pi^+$, $\sin x \to$ ~~-0.000000001~~ small neg. #

$\lim_{x\to 3\pi^+} \frac{\cos x}{\sin x} = \frac{-1}{\text{small neg \#}} = \boxed{+\infty}$ (**)

(d) $\lim_{x\to 3\pi} \frac{\cos x}{\sin x}$

(1) From (c) we've checked $\lim_{x\to 3\pi^+} \frac{\cos x}{\sin x}$ DNE but $+\infty$.

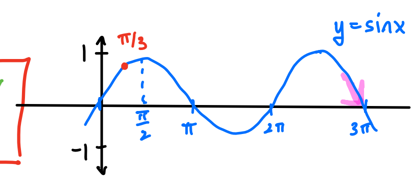
"non-zero / 0" D.N.E.

(2) $\lim_{x\to 3\pi^-} \frac{\cos x}{\sin x} = \frac{-1}{\text{small pos \#}} = \boxed{-\infty}$ (*)

As $x\to 3\pi^-$ $\cos x \to -1$

$x\to 3\pi^-$ $\sin x \to$ small pos. #.

From (*) and (**), $\lim_{x\to 3\pi} \frac{\cos x}{\sin x}$ D.N.E.



(e) $\lim_{x\to 2} \frac{(\sin x) - 2}{(x-2)^2}$

→ "non-zero / 0" D.N.E

As $x\to 2$, $(x-2)^2 \to 0^+$ small but pos #

$\boxed{\sin x - 2}$

$-1 \le \sin x \le 1$

$-1-2 \le \sin x - 2 \le 1-2$

$-3 \le \sin x - 2 \le -1$

$\lim_{x\to 2} \frac{\sin x - 2}{(x-2)^2} = \frac{\text{neg num}}{0^+} = \boxed{-\infty}$

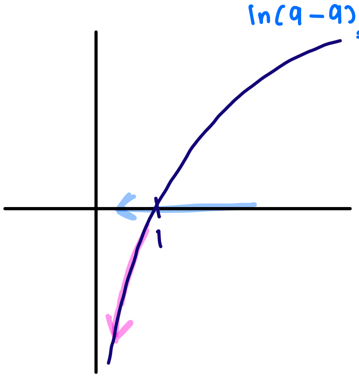
(f) $\lim_{x\to 3^+} \ln(x^2-9)$

$\ln(9-9) = \ln(0)$ = und.

As $x\to 3^+$ (values bigger than 3)

e.g 3.11111

$x^2-9 \to 0^+$ (small pos #)



As $x\to 0^+$, $\ln x \to -\infty$

$\lim_{x\to 3^+} \ln(x^2-9) = \boxed{-\infty}$

2. Summarize: if you're trying to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$, and you first find that $\lim_{x\to a} g(x) = 0$, what else do you need to look at?

① $\lim_{x\to a} f(x) = 0$, then this is a $\left(\frac{0}{0}\right)$ limit → Algebraically "Needs more work"

② $\lim_{x\to a} f(x) \neq 0$, then this is a $\left(\frac{\text{non-zero}}{0}\right)$ limit. The final answer in this case is:

(a) DNE
(b) DNE and $+\infty$
(c) DNE and $-\infty$

} STRAT: Check signs of num & denom.

3. Last class we saw that $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist. In this problem, we'll look at $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$.

(a) Amani is thinking about this limit and says, "As $x \to 0$, x^2 approaches 0. 0 times anything is 0, so $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$ must be 0." What do you think of Amani's reasoning?

E.g. $\lim_{x\to 0} \cancel{x^2} \cdot \frac{1}{\cancel{x^2}} = \lim_{x\to 0} 1 = \boxed{1} \neq 0$

No AMANI IS NOT RIGHT!

$\lim_{x\to 0} \cancel{x^2} \cdot \frac{1}{x^{\cancel{4}\,2}} = \lim_{x\to 0} \frac{1}{x^2} = \boxed{\text{DNE but } +\infty} \neq 0$

(b) Find $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$. Explain your reasoning carefully.

can't plug in b/c $\sin\left(\frac{\pi}{x}\right)$ is und. at $x=0$. "not continuous"

$$-1 \cdot x^2 \leq \sin\left(\frac{\pi}{x}\right) \cdot x^2 \leq 1 \cdot x^2$$

$$\underbrace{-x^2}_{\text{nice!}} \leq \underbrace{x^2 \sin\left(\frac{\pi}{x}\right)}_{\text{messy!}} \leq \underbrace{x^2}_{\text{nice!}}$$

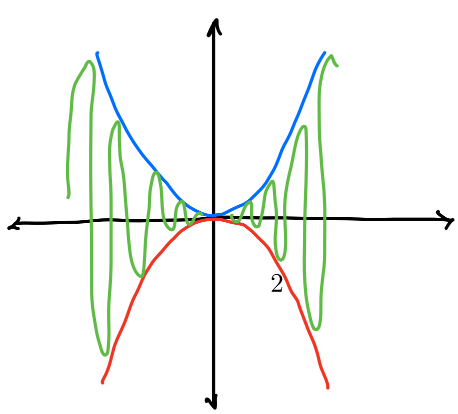
$$\lim_{x\to 0} -x^2 \leq \lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) \leq \lim_{x\to 0} x^2$$

$$0 \leq \lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) \leq 0$$

↳ $= 0$

(c) Sketch a rough graph of $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$ for x near 0.

Ⓘ. SET UP INEQUALITY
Ⓘⓘ. GRAPH OR SOLVE ALGEBRAIC.



The Squeeze Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\underbrace{\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L}_{\text{easier to compute}}$$

Then $\lim_{x\to c} f(x) = L$

4. There is one special limit that we want you to know. We'll make sense of these eventually. Assume that θ is in radians. Then:

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1} \rightarrow \text{COOL PROPERTY MEMORIZE!}$$

Use this to compute the following:

(a) $\lim_{x\to 0} \dfrac{\sin x}{\pi x} = \dfrac{1}{\pi} \cdot \lim_{x\to 0} \dfrac{\sin x}{x} = \dfrac{1}{\pi} \cdot \overbrace{1}^{\text{from identity}} = \boxed{\dfrac{1}{\pi}}$

look at graph "linear" $y=x$ near $x=0$.

Plug in $x=0.0001$ $\sin(x) =$

(b) $\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2}$

As $x \to 0^+$, $\sin x \approx x$

$\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2} \approx \lim_{x\to 0} \dfrac{x^2}{x^2} = \boxed{1}$

Let $u = x^2$

$\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2} = \lim_{x\to 0} \dfrac{\sin(u)}{u} = \boxed{1}$

(c) $\lim_{x\to 0} \dfrac{\sin(x^2)}{x}$

ss

$\lim_{x\to 0} \dfrac{x^2}{x} = \lim_{x\to 0} x = 0$

$\underbrace{\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2}}_{(*)} \cdot x$ from (b) $(*) \to 1$

$\lim_{x\to 0} 1 \cdot x = 0$

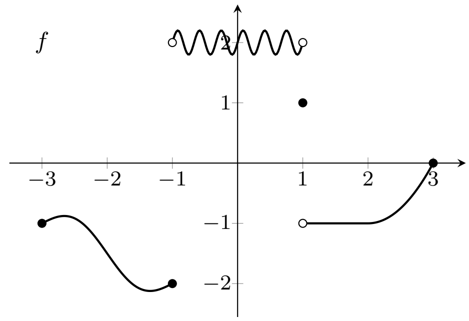
More Practice

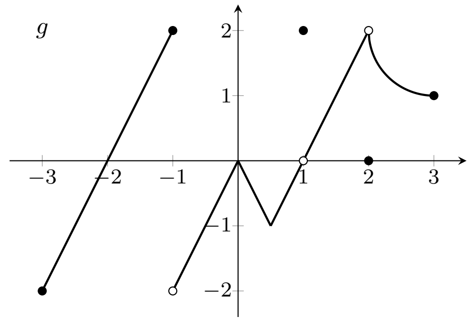
1. (a) Evaluate $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$.

   (b) Sketch $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ together to explain why your answer to (a) makes sense.

2. Compute the following limit: $\lim_{t\to 0} f(t)$, where $1 - t^2 \leq f(t) \leq 1 + t^2$. Can you draw a picture to show why your answer makes sense?

3. The graphs of f and g are given below.





Evaluate each of the following limits.

As $x \to 2$, $g(x) \to 2$

(a) $\lim_{x\to 2}(g(x) - f(x))$, $f(x) \to -1$

$\lim_{x\to 2}(g(x)-f(x)) = 2-(-1)$
$= \boxed{3}$

(b) $\lim_{x\to 0} \frac{f(x)}{g(x)}$

(c) $\lim_{x\to -2} \frac{f(x)}{g(x)}$

(d) $\lim_{x\to 1} \frac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} \frac{f(x)}{g(x)}$

As $x \to 0$, $g(x) \to 0$
As $x \to 0$, $f(x) \to 2$ $\Big]$ $\frac{\text{non-zero}}{0}$

As $x \to 0^+, 0^-$, $g(x)$ is neg.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\text{positive \#}}{\text{small neg \#}} = \boxed{-\infty}$$