MATH 111 (DAY 3)

REMINDERS:

- QUIZ 1 → CANVAS FRI 09/03
- WEBASSIGN 3 DUE TUES 09/10
- PRECALC MODULES → DUE FRI 09/13
  (2% EC!)

LAST TIME:

The limit, L, exists if and only if
"exactly when"

(i) both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist.

(ii) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ "equal".

$a^2 - b^2 = (a+b)(a-b)$

Day 3- Section 2.5- Continuity

1. Last class, we calculated $\lim_{x\to 3} \frac{x^2-3^2}{x-3}$:

$\frac{x^2-3^2}{x-3}$ and $x+3$ have the same $\lim_{x\to 3}$ but not the same output at $x=3$

$$\lim_{x\to 3} \frac{x^2-3^2}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} \quad \text{by algebra}$$
$$= \lim_{x\to 3}(x+3) \quad \text{Using the "}\lim_{x\to 3}\text{", we don't care what's happening at } x=3.$$
$$= 6 \quad \text{Because } x+3 \text{ is continuous at } x=3.$$

(a) Fill in reasons for the last two lines.

(b) Why couldn't we just plug $x = 3$ in from the start?

B/c $\frac{x^2-3^2}{x-3}$ was undefined at $x=3$. "$\frac{0}{0}$" error.

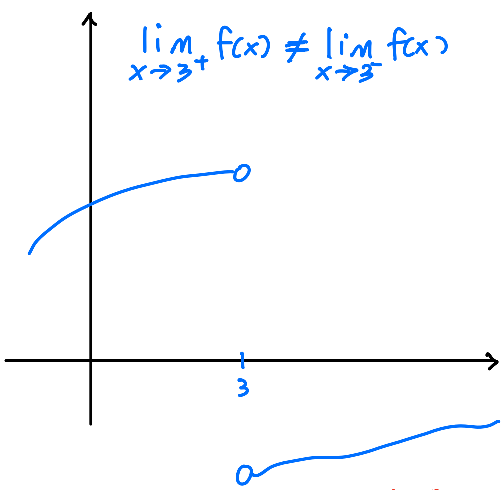
2. *When is* $\lim_{x\to a} f(x) = f(a)$*? That is, when can you just plug in to evaluate a limit?*

Q: "When is $f(x)$ discontinuous"

(a) Sketch some functions $f(x)$ for which $\underbrace{\lim_{x\to 3} f(x)}_{\text{LIMIT}} \neq \underbrace{f(3)}_{\text{OUTPUT}}$. Try to draw a few examples that look really different!

JUMP DISCONTINUITY

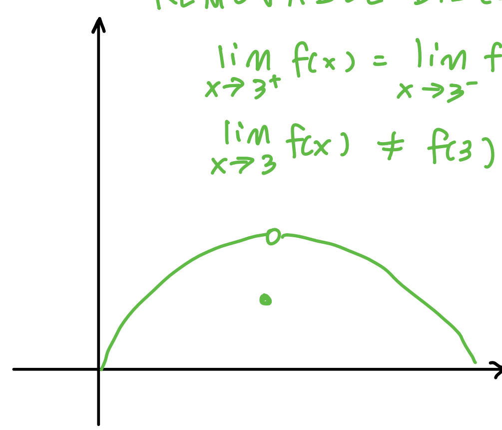
$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$



REMOVABLE DISCONTINUITY

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$\lim_{x\to 3} f(x) \neq f(3)$

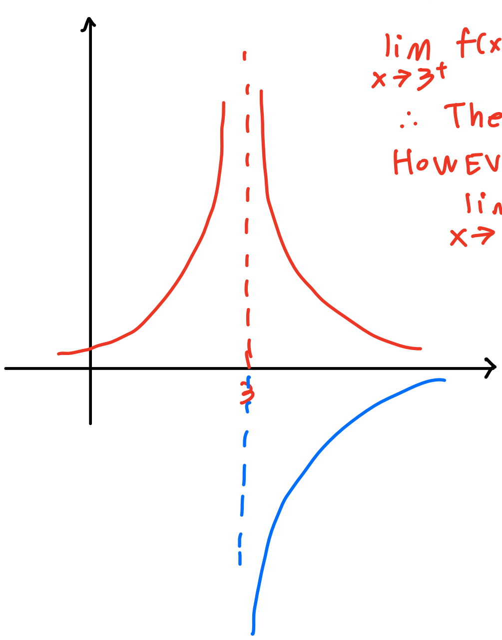


VERTICAL ASYMPTOTE

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = \infty$

$\therefore$ The limit exists

HOWEVER, $\lim_{x\to 3} f(x) \neq f(3)$

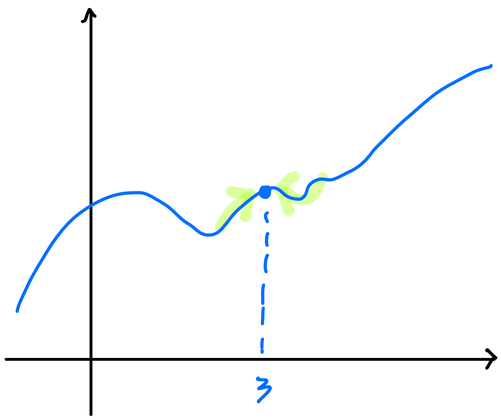


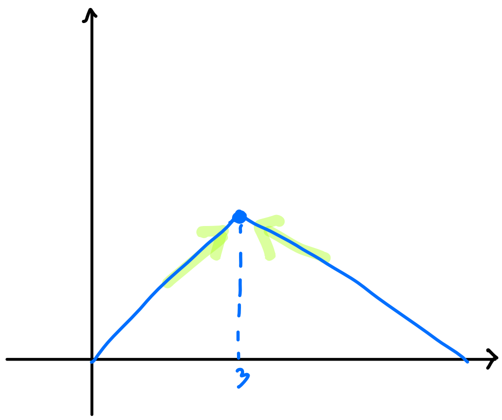
$\lim_{x\to 3^+} f(x) \pm -\infty$

$\lim_{x\to 3^-} f(x) = \infty$

f is continuous

(b) Sketch some functions $f(x)$ for which $\lim_{x\to 3} f(x) = f(3)$. Try to draw a few examples that look really different!





Continuity. We say a function f is **continuous at** a if $\boldsymbol{\lim_{x\to a} f(x) = f(a)}$. In other words, a function is continuous at a point if "The number $f(x)$ approaches as x gets really close to a is, in fact, just $f(a)$."

If a function is continuous at all points in its domain we simply say it is **continuous**.

$f(x) = x+3$

It turns out that most familiar functions from calculus are continuous! $\lim_{x\to 3}(x+3) = 6 \to$ cont. at $x=3$

Theorem[Thm 7, Chapter 2.5] The following types of functions are continuous at every number in their domains:

- polynomials
- trigonometric functions
- exponential functions
- rational functions
- root functions
- logarithmic functions
- inverse trigonometric functions

Also, surprisingly,

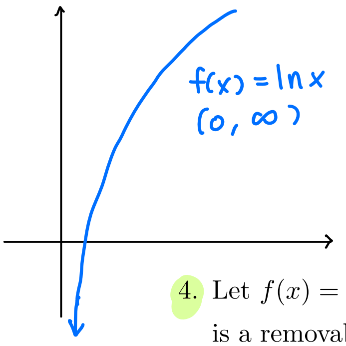
Theorem[Thm 9, Chapter 2.5] The composition of continuous functions is again continuous.

TAKEAWAY: FAMILIAR FCNS FROM ABOVE AND THEIR COMPOSITIONS ARE CONTINUOUS ON THEIR DOMAINS.

what is the domain of the fcn?

3. In each part, for what values of x is the given function continuous?

(a) $x^3 \ln|x-3|$ = $g(x)$ (b) $e^{\tan(x)}$



ln x is defined only for $x > 0$

$|x-3| \neq 0$, $x \neq 3$

$|x-3| \geq 0$ $\boxed{(-\infty, 3) \cup (3, \infty)}$

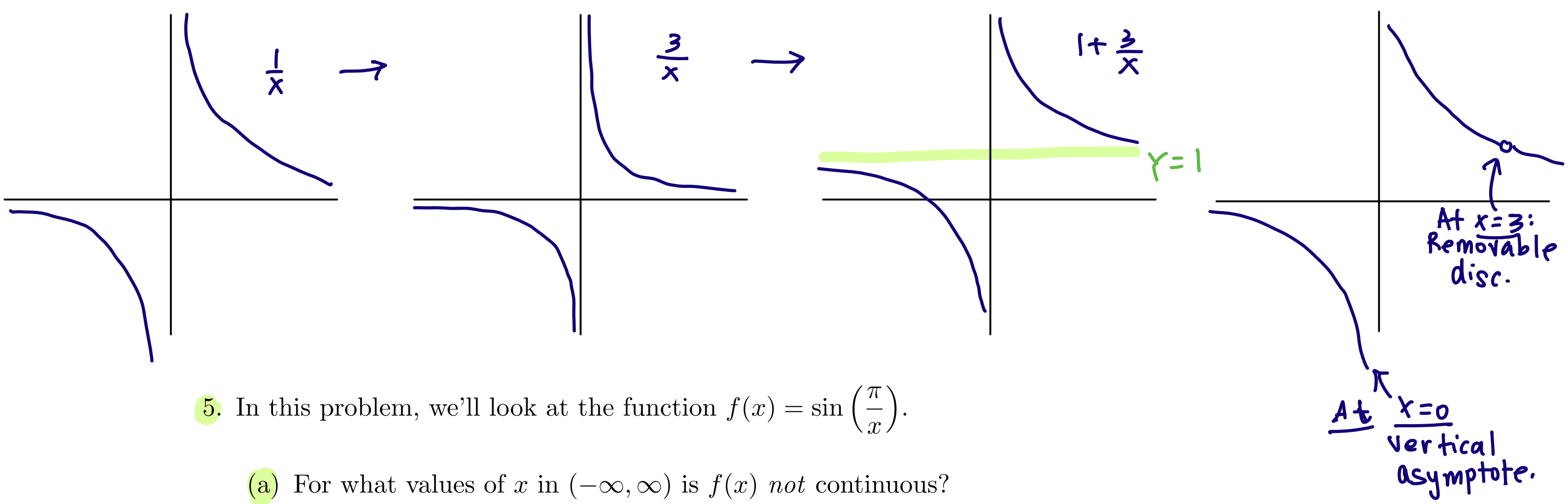
Ans: $g(x)$ is cont. everywhere except when $x = 3$.

4. Let $f(x) = \dfrac{x^2-9}{x^2-3x}$. Find and classify all discontinuities of f (that is, say whether each discontinuity is a removable discontinuity, jump discontinuity, vertical asymptote, or other). Then sketch the graph of $f(x)$.

→ 0 in denom. → f is discont. at 0, 3.

DOMAIN of f(x): $x \neq 0$ and $x \neq 3$

$$f(x) = \frac{x^2-9}{x^2-3x} = \frac{\cancel{(x-3)}(x+3)}{x\cancel{(x-3)}} \overset{?}{=} \frac{x+3}{x} = \frac{x}{x} + \frac{3}{x} = 1 + \frac{3}{x}$$



5. In this problem, we'll look at the function $f(x) = \sin\left(\frac{\pi}{x}\right)$.

(a) For what values of x in $(-\infty, \infty)$ is $f(x)$ *not* continuous?

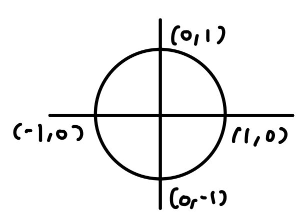
$\sin\left(\frac{\pi}{x}\right)$ undefined at $x = 0$ "$\sin\left(\frac{\pi}{0}\right)$" ??

↳ discontinuity at $x = 0$. Look at the limit: $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$

(b) Let's look numerically at $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$. Fill out the following table of values; what, if anything, does it tell you about $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$?

x	2	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$\sin\left(\frac{\pi}{x}\right)$	$\sin\left(\frac{\pi}{2}\right) = \boxed{1}$	$\sin\left(\frac{\pi}{2/3}\right) = \boxed{-1}$	1.	-1	1	-1

Based on the table: $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ D.N.E



$\sin(k\pi) = 0$ where k is an integer

(c) Harry made the table below and concluded (incorrectly!) that $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$. What was wrong with his reasoning?

$\sin\left(\frac{\pi}{1/10}\right) = \sin(10\pi)$

x	1	0.1	0.01	0.001	0.0001	0.00001
$\sin\left(\frac{\pi}{x}\right)$	$\sin(\pi) = \boxed{0}$	$\sin\left(\frac{\pi}{0.1}\right) = \boxed{0}$	0	0	0	0

~~Based on the table: $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$.~~

Comment: we can't use a table to conclude a limit

However, we can use a table to conclude a limit D.N.E by looking at the outputs as $x \to 0$.

(d) Is $x = 0$ a removable discontinuity of $\sin\left(\frac{\pi}{x}\right)$, a jump discontinuity, a vertical asymptote, or none of the above?

Looking at the graph on Desmos, we can see there is chaotic behavior as $x \to 0$.

Q: How can we see this with information above?

From (b) as $x \to 0^+$, we can see the limit is still oscillating b/w $x = -1$ and $x = 1$.

So $\lim_{x\to 0^+} \sin\left(\frac{\pi}{x}\right)$ D.N.E $\Rightarrow \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ D.N.E

Q: What type?

Not removable
jump
vertical asymptote

NEW TYPE!

WILDLY OSCILLATING DISCONTINUITY.

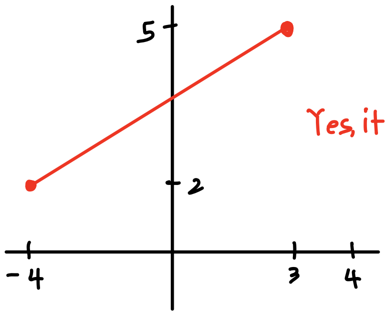
"ZEROS" = "X-INTERCEPTS"

"IS IT POSSIBLE = CAN YOU FIND ONE EXAMPLE"

6. In each part, can you sketch a continuous function $f(x)$ defined on $[-4, 3]$ which has all three of the given properties, or is it impossible to do so?

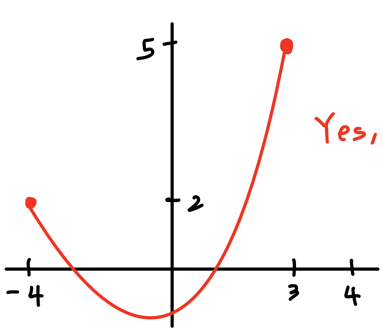
If it is impossible, would it become possible if f did not have to be continuous?

(a)
- $f(-4) = 2$
- $f(3) = 5$
- f has no zeros in $[-4, 3]$



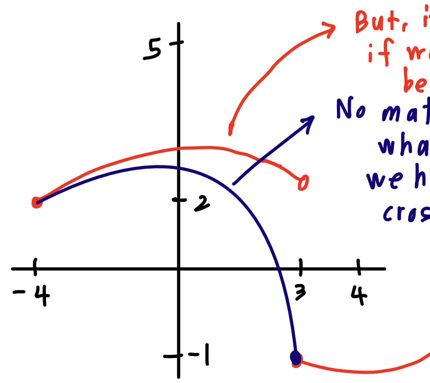
Yes, it is possible!

(b)
- $f(-4) = 2$
- $f(3) = 5$
- f has a zero in $[-4, 3]$



Yes, it is possible!

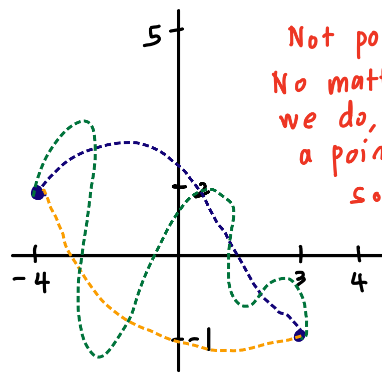
(c)
- $f(-4) = 2$
- $f(3) = -1$
- f has no zeros in $[-4, 3]$



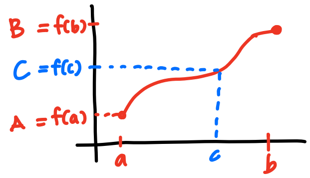
But, it is possible if we allow it to be discontinuous.

No matter what we do we have to cross the x-axis.

(d)
- $f(-4) = 2$
- $f(3) = -1$
- f does not attain the value 1 on $[-4, 3]$



Not possible! No matter what we do, we can find a point s.t $f = 1$ for some x.



Intermediate Value Theorem. If f is continuous on the closed interval $[a, b]$ and $f(a) = A$, $f(b) = B$, then somewhere in the interval $[a, b]$, f attains every value between A and B.

7. Is the following argument correct or incorrect? Why?

> Imagine our basketball team beats Georgia Tech's 98-60. Since we had 0 points at the beginning of the game and 98 at the end, the Intermediate Value Theorem says that we must have had exactly 25 points at some moment in the game

FALSE! Points fcn is <u>not</u> continuous! So IVT, does not apply.

8. In December 2004, Facebook announced that it had 1 million users. For the next 5 years, analysts observed that the number of Facebook users appeared to be growing exponentially; since then, the number of Facebook users has appeared to grow linearly. Mark would like to model the number of users U, measured in millions, as a continuous function of time t, where t is measured in years since December 2004. Based on the given information, he models $U(t)$ as a piecewise function:

$$U(t) = \begin{cases} Ce^{kt} & \text{for } 0 \le t < 5 \\ at + b & \text{for } 5 \le t \end{cases}$$

(a) What does the fact that Facebook had 1 million users at time $t = 0$ tell Mark about C, k, a, and b?

$$U(0) = 1 \Rightarrow Ce^{k\cdot 0} = 1 \Rightarrow C\cancel{e^{0}}^{\,1} = 1$$
$$\Rightarrow C = 1.$$

(b) By fitting recent data, Mark determines that $a = 220$ and $b = 380$ are reasonable values. What should k be so that Mark's model is continuous?

Problem area: point at which two piecewise fcns connect.
↳ In this case $t = 5$.

Need $\lim_{t \to 5} U(t) = U(5)$

"more than 5" ← (1) $\underbrace{\lim_{t \to 5^+} U(t) = U(5)}$

automatically true, b/c for $t \ge 5$, $U(t) = 220t + 380$. which is continuous

For $0 < t < 5$, $U(t) = Ce^{kt}$ since $C = 1$ by (a), $U(t) = e^{kt}$

(2) $\lim_{t \to 5^-} U(t) = U(5)$

$\Rightarrow \underbrace{\lim_{t \to 5^-} e^{kt}} = 220(5) + 380$

cont. at $t = 5$ so we can plug in.

$e^{k(5)} = 1480$

$\ln(e^{k \cdot 5}) = \ln(1480)$

$5k = \ln(1480)$

$$\boxed{k = \frac{\ln(1480)}{5}}$$

More Practice

1. *Summarize*

   (a) What does it mean for a function $f(x)$ to be continuous at a point $x = a$ in its domain?

   (b) What are the different kinds of discontinuities and how do we describe them?

2. Are the following functions continuous? If not, would it be reasonable to model them continuously?

   (a) The temperature in this classroom as a function of time.

   (b) The number of registered instagram users as a function of time.

   (c) The amount of federal income tax a person with taxable income of M dollars in 2014 must pay, as a function of M.

   (d) The number of points our basketball team had t minutes into the game, as a function of t.

3. Let a be a constant and f be the function defined by

$$f(x) = \begin{cases} \frac{ax}{6} & \text{for } x < 1 \\ \frac{1}{x+a} & \text{for } x \geq 1 \end{cases}.$$

   Find all values of a for which f is continuous on $(-\infty, \infty)$. (Do this without using a calculator!)

4. *Reflect Back.*

   Find and classify all discontinuities of $f(x) = e^{1/x}$. (When we ask you to "classify" a discontinuity, we mean "determine what type of discontinuity this is.")