Practical 4

Solution of Differential Equation by Variation of Parameter method

Rule for solving y''+p(x)y'+q(x)y=r(x) by using variation parameter(with constant coefficients)

In[∘]:=
```
(*Given y''[x]+p y'[x]+q y[x]==r[x]...(!)*)
yh = DSolve[y''[x] + p y'[x] + q y[x] == 0, y[x], x]

(*....(2)*)(*solution of associated homogeneous ODE*)
```

Out[∘]= $\left\{\left\{y[x] \to e^{\frac{1}{2}\left(-p-\sqrt{p^2-4q}\right)x} C[1] + e^{\frac{1}{2}\left(-p+\sqrt{p^2-4q}\right)x} C[2]\right\}\right\}$

In[∘]:=
```
rule = {u''[x] + p u'[x] + q u[x] → 0, v''[x] + p v'[x] + q v[x] → 0}

(*u=e^(1/2(-p-sqrt[p^2-4 q])x) and v=e^(1/2(-p+sqrt[p^2-4 q])x)
   are Linearly Independent Solution of associated homogeneous ODE*)
```

Out[∘]= {q u[x] + p u′[x] + u′′[x] → 0, q v[x] + p v′[x] + v′′[x] → 0}

In[∘]:=
```
y[x_] := C1[x] u[x] + C2[x] v[x] (*...(3)*)
(*yh is solution o eq(2) where C[1]=C1[x] and C[2]= C2[x]*)
first = D[y[x], x]
```

Out[∘]= u[x] C1′[x] + v[x] C2′[x] + C1[x] u′[x] + C2[x] v′[x]

In[∘]:=
```
rule1 = {C1'[x] u[x] + C2'[x] v[x] → 0} (*...(4)*)
```

Out[∘]= {u[x] C1′[x] + v[x] C2′[x] → 0}

In[∘]:=
```
output1 = first /. rule1(*...(5)*)(*give y'*)
```

Out[∘]= C1[x] u′[x] + C2[x] v′[x]

In[∘]:=
```
second = D[output1, x] (*...(6)*)(*gives y''*)
```

Out[∘]= C1′[x] u′[x] + C2′[x] v′[x] + C1[x] u′′[x] + C2[x] v′′[x]

```
In[•]:= third = {second + p (output1) + q (y[x]) == r[x]} // Simplify(*using eqn 3,5,and 6 in 1*)
Out[•]= {C1′[x] u′[x] + C2′[x] v′[x] +
            C1[x] (q u[x] + p u′[x] + u″[x]) + C2[x] (q v[x] + p v′[x] + v″[x]) == r[x]}

In[•]:= output2 = third /. rule // Simplify(*...(7)*)
Out[•]= {r[x] == C1′[x] u′[x] + C2′[x] v′[x]}

In[•]:= DSolve[{u[x] C1'[x] + v[x] C2'[x] == 0, r[x] == C1'[x] u'[x] + C2'[x] v'[x]},
          {C1[x], C2[x]}, x]
        (*From eqn(4) and (7)*)(*After using the value of C1[x] and C2[x] in eqn(3),
        we get required solution in the form of y = yc + yp *)
```

$$\text{Out}[\circ]= \{\{C1[x] \to C[1] + \text{Integrate}\left[\frac{r[K[1]]\, v[K[1]]}{v[K[1]]\, u'[K[1]] - u[K[1]]\, v'[K[1]]}, \{K[1], 1, x\}, \text{Assumptions} \to \text{True}\right],$$
$$C2[x] \to C[2] + \text{Integrate}\left[\frac{r[K[2]]\, u[K[2]]}{-v[K[2]]\, u'[K[2]] + u[K[2]]\, v'[K[2]]}, \{K[2], 1, x\}, \text{Assumptions} \to \text{True}\right]\}\}$$

Solve y''+y=cosec(x) by variation of parameters

```
In[•]:= Clear[y, x]

In[•]:= yh = DSolve[y''[x] + y[x] == 0, y[x], x](*Solution of associated Homogeneous ODE*)
Out[•]= {{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

In[•]:= u[x_] := Cos[x]

In[•]:= v[x_] := Sin[x](*we supposed*)

In[•]:= r[x_] := Csc[x]

In[•]:= wronskian = Det[{{u[x], v[x]}, {u'[x], v'[x]}}] // Simplify(*Wronskian*)
Out[•]= 1

In[•]:= C1prime = -v[x] r[x] / wronskian(*=C1'[x]*)
Out[•]= -1

In[•]:= C2prime = u[x] r[x] / wronskian
Out[•]= Cot[x]
```

```
C1[x_] := Integrate[C1prime, x] + c1
C2[x_] := Integrate[C2prime, x] + c2
y = C1[x] u[x] + C2[x] v[x] (*Required General solution*)
```

Out[ ]= (c1 − x) Cos[x] + (c2 + Log[Sin[x]]) Sin[x]

```
In[ ]:= (*General solution of the given ODE is y=yc+yp where yc=c1 Cos[x]+c2 Sin[x],
yp=-x Cos[x]+Sin[x] Log[Sin[x]]*)
```

```
In[ ]:= DSolve[y''[x] + y[x] == Csc[x], y[x], x] // TraditionalForm (*Verified directly*)
```

Out[ ]//TraditionalForm=

$$\{\{y(x) \to c_2 \sin(x) + c_1 \cos(x) - x\cos(x) + \sin(x)\log(\sin(x))\}\}$$

(Alternative)Solve y’’+y=tan(x) by variation of parameter

```
In[ ]:= Clear[y, x]
```

```
In[ ]:= yc = DSolve[y''[x] + y[x] == 0, y[x], x] (*Solution of associated Homogeneous ODE*)
```

Out[ ]= {{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

```
In[ ]:= u[x_] := Cos[x]
```

```
In[ ]:= v[x_] := Sin[x] (*we Supposed*)
```

```
In[ ]:= r[x_] := Tan[x]
```

```
In[ ]:= wronskian = Det[{{u[x], v[x]}, {u'[x], v'[x]}}] // Simplify (*Wronskian*)
```

Out[ ]= 1

```
In[ ]:= C1prime = -v[x] r[x] / wronskian (*=C1'[x]*)
```

Out[ ]= −Sin[x] Tan[x]

```
In[ ]:= C2prime = u[x] r[x] / wronskian
```

Out[ ]= Sin[x]

```
In[ ]:= C1[x_] := Integrate[C1prime, x]
```

```
In[ ]:= C2[x_] := Integrate[C2prime, x]
```

```
In[ ]:= yp = C1[x] u[x] + C2[x] v[x] (*Required General solution*)
```

Out[ ]= $-\mathrm{Cos}[x]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right] - \mathrm{Sin}\left[\frac{x}{2}\right]\right] - \mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right] + \mathrm{Sin}\left[\frac{x}{2}\right]\right] + \mathrm{Sin}[x]\right)$

```
In[ ]:= (*General solution of the given ODE is y=yc+yp where yc=C[1] Cos[x]+C[2]Sin[x],
yp=-x Cos[x]+Sin[x] Log[Sin[x]]*)
```

```
In[ ]:= sol = Evaluate[y[x] /. yc] + yp
```

Out[ ]= $\{C[1]\,\mathrm{Cos}[x] + C[2]\,\mathrm{Sin}[x] - \mathrm{Cos}[x]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right] - \mathrm{Sin}\left[\frac{x}{2}\right]\right] - \mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right] + \mathrm{Sin}\left[\frac{x}{2}\right]\right] + \mathrm{Sin}[x]\right)\}$

In[ ]:=
```
tab = Table[sol /. {C[1] → i, C[2] → j}, {i, -1, 0}, {j, 0, 1}] // Flatten
```

Out[ ]=

$$\Big\{-\mathrm{Cos}[x]-\mathrm{Cos}[x]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right]-\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right]+\mathrm{Sin}[x]\right),$$

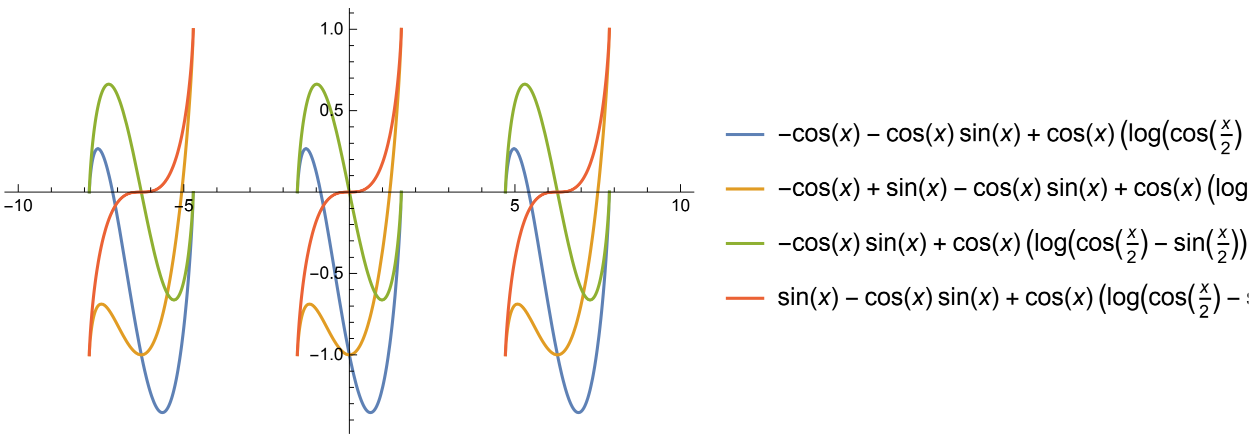
$$-\mathrm{Cos}[x]+\mathrm{Sin}[x]-\mathrm{Cos}[x]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right]-\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right]+\mathrm{Sin}[x]\right),$$

$$-\mathrm{Cos}[x]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right]-\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right]+\mathrm{Sin}[x]\right),$$

$$\mathrm{Sin}[x]-\mathrm{Cos}[x]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\left(\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right]-\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right]+\mathrm{Sin}[x]\right)\Big\}$$

In[ ]:=
```
Plot[Evaluate[tab], {x, -10, 10}, PlotLegends → "Expressions"]
```

Out[ ]=



In[ ]:=
```
Clear[y, x]
```

In[ ]:=
```
DSolve[y''[x] + y[x] == Tan[x], y[x], x] (*Verfied directly*)
```

Out[ ]=

$$\Big\{\Big\{y[x]\to C[1]\,\mathrm{Cos}[x]+\mathrm{Cos}[x]\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right]-\mathrm{Cos}[x]\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right]+C[2]\,\mathrm{Sin}[x]\Big\}\Big\}$$