9 B. Pavan Kumar Basu Kiha.

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FRAMES OF REFERENCE

"To know what you are? Look at others."

Inertial frames, Galilean transformation equations, Galilean principle of relativity. Non-inertial frames, fictitious force and concept of Corioli's force. Centre of mass, motion of centre of mass and centre of mass as a frame of reference. —— 7 hours.

Introduction

The state of rest or motion of a body depends on the position of the observer. The object moving with reference to one observer may appear to be at rest with reference to another observer. Thus, a reference frame is essential to represent the state of bodies. To understand the frames of reference, an understanding of some of the terms, which appear frequently in mechanics, like particle, rigid body, event, clock, rest and motion is essential.

1) **Particle:** — 1

A point mass or an object without any extent is called a particle. Generally, a particle is characterized by its mass, energy and charge. A real object of definite shape and considerable size can also be treated as a particle when its dimensions are negligible compared to the distance traveled by it. Even the earth can be treated as a particle in the solar system.

2) **Rigid body:**

A body is said to be rigid if the distance between any two points in it remains constant irrespective of its state of motion. In other words no deformation is observed in the body when it is subjected to an external force.

3) **Event:**

Anything that happens in space is called an event. It is known by its position and time of occurrence and hence specified by space-time co-ordinates (x, y, z, t). For example, explosion of a cracker occurs at a definite position and at a particular time. A moving train crosses a particular place at a definite time.

4) **Clock:**

A clock is a device to measure the time of occurrence of an event in space. It works on the principle of repeated occurrence of something in the universe. For example, the bob of a simple pendulum crosses a point in space after regular interval of time. In relativity, all clocks are synchronized to show the same time and distributed to different regions in the universe. But in Newtonian mechanics time is considered as absolute which flows uniformly for all the bodies in the universe. Therefore there is no need of synchronization of clocks.

5) **Rest & Motion:**

"Rest and uniform motion along a straight line are equally natural"

A particle is said to be at rest if it occupies the same position at all instants of time and it is said to be in motion if it changes its position. But every body in the universe, in one or the other way, is in state of motion. It is a common experience that a stationary tree appears moving and a moving earth appears stationary. This shows that the terms rest and motion are relative terms and should be specified with reference to the origin of a co-ordinate system taken as a reference. Absolute rest and absolute motion are meaningless.

6) **Frames of Reference:**

A frame of reference is a set of Cartesian co-ordinates fixed with respect to an observer used to describe the state of a particle.

A particle is in motion is changing its position relative to something else. A train moves relative to the earth, the earth moves relative to the sun, the sun moves relative to the galaxy and so on. From these examples, it is clear that a reference is needed to define position and motion of particles. At the same time, there is no unique frame of reference in the universe. This leads to the concept of the absence of universal frame of reference and so also the absolute motion. In general, the state o

orbital and escape velocity expression

Millikan's ... Singh R

Malagikumar ... theory of compound pendulum

Stoke's law of viscosity theory

light ...

particle is described relative to a set of Cartesian co-ordinates fixed to the observer called frame of reference. There are two types of frames of reference, inertial and non-inertial frames.

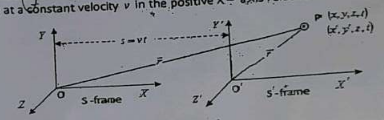
Inertial or un-accelerated frame of reference:

A frame of reference in which Newton's laws of motion and basic principles of physics are valid is called inertial or un-accelerated frame of reference. In such a frame, objects at rest remain at rest and objects in motion continue to move with constant velocities in the absence of external forces. Any frame of reference that moves with constant velocity relative to an inertial frame is also an inertial frame. The force on a particle and the resulting acceleration are the same in all inertial frames. Hence inertial frames are dynamically equivalent. A frame of reference fixed to the earth is an approximation to an inertial frame because of the diurnal rotation of the earth.

Galilean Transformations - *"A bridge between inertial frames"*

An event observed simultaneously in two inertial frames will have two separate sets of coordinates. The transformation of these coordinates from one inertial frame to another is known as Galilean transformation.

Let us consider transformations of position, length, velocity, acceleration and force between two inertial frames. Consider two inertial frames S and S'. Initially the two frames were superposed. The frame S' is moving at a constant velocity v in the positive X – axis relative to the frame S.



1. **Transformation of position**

Consider an event at P in time t whose co-ordinates relative to S and S' be $P(x, y, z, t)$ and $P(x', y', z', t)$ respectively. In time t, S' moves by a distance $s = vt$ in the +X direction. Therefore transformation equations from S to S' are;

$$x' = x - vt,\quad y' = y,\quad z' = z \text{ and } t' = t \quad \text{--- (1)}$$

In general,

$$r' = r - vt \quad \text{---- (2)}$$

2. **Transformation of length or distance:**

Consider a rod kept parallel to x-axis. The x-coordinates of its ends are x_1 and x_2 relative to S-frame and x_1' and x_2' relative to S' respectively.

From the position transformations; $x_1' = x_1 - vt$ and $x_2' = x_2 - vt$

That is, $x_2' - x_1' = x_2 - x_1$

Therefore, $l = l'$ ---- (2)

Thus length or distance is invariant under Galilean transformation

3. **Transformation of velocity:** The velocity relative to S-frame; $u = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{\Delta t}$ ———— (1)

The velocity relative to S'-frame; $u' = \frac{\Delta x'}{\Delta t} = \frac{x_2' - x_1'}{\Delta t}$ ———— (2)

From position transformation, $x' = x - vt$

Substituting for x' in relation (2), $u' = \frac{(x_2 - vt_2) - (x_1 - vt_1)}{\Delta t}$

That is, $u' = \frac{(x_2 - x_1)}{\Delta t} - v\frac{(t_2 - t_1)}{\Delta t}$

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Therefore,

$$u' = u - v$$

This shows that velocity is not invariant (variant) under Galilean transformation

Alternatively we have,

$$r' = r - vt$$

Differentiating with respect to t,

$$\frac{dr'}{dt} = \frac{dr}{dt} - v$$

That is,

$$u' = u - v$$

This is known as Galilean law of addition of velocities.

Transformation of acceleration

From velocity transformation,

$$u' = u - v$$

Differentiating with respect to t,

$$\frac{du'}{dt} = \frac{du}{dt} - 0, \qquad \text{since } v \text{ is a constant.}$$

That is,

$$a' = a$$

Therefore acceleration is invariant under Galilean transformation

5. Transformation of force:

From the transformation of acceleration,

$$a' = a$$

Since m is invariant,

$$ma' = ma$$

That is

$$F' = F$$

Thus force is invariant under Galilean transformations and so also Newton's laws of motion.

The law of conservation of momentum:

Consider an elastic collision between two bodies of masses m_1 and m_2

Relative to S-frame, the principle of conservation of momentum is given by

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \text{ ---------- (1)}$$

From velocity transformation, we have,

$$u = u' + v \text{ ---------- (2)}$$

Substituting (2) in (1) we get,

$$m_1(u_1' + v) + m_2(u_2' + v) = m_1(v_1' + v) + m_2(v_2' + v)$$

$$m_1u_1' + m_2u_2' + (m_1 + m_2)v = m_1v_1' + m_2v_2' + (m_1 + m_2)v$$

That is,

$$m_1u_1' + m_2u_2' = m_1v_1' + m_2v_2'$$

Thus the law of conservation of linear momentum is invariant in all inertial frames of reference.

The law of conservation of energy:

Relative to S-frame, the law of conservation is

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2 \text{ ---------- (1)}$$

From velocity transformation,

$$u = u' + v \text{ ---------- (2)}$$

Substituting (2) in (1) we get,

$$m_1(u_1' + v)^2 + m_2(u_2' + v)^2 = m_1(v_1' + v)^2 + m_2(v_2' + v)^2$$

$$m_1u_1'^2 + m_2u_2'^2 + (m_1 + m_2)v^2 + (m_1u_1' + m_2u_2')2v = m_1v_1'^2 + m_2v_2'^2 + (m_1 + m_2)v^2 + (m_1v_1'^2 + m_2v_2')2v$$

That is,

$$m_1u_1'^2 + m_2u_2'^2 = m_1v_1'^2 + m_2v_2'^2$$

Hence the law of conservation of energy is invariant under Galilean transformations.

Note: Under Galilean transformations, length, acceleration, force and conservation principles are invariant but position and velocity are not invariant

Galilean Principle

According to Newton, the motion of bodies confined to a given space is the same for all frames of reference at rest or moving with a constant velocity relative to each other. This means that, if a closed space ship is moving at a constant velocity relative to the fixed stars then experiments conducted in the ship and all the phenomena observed in it will appear the same to an astronomer in it as if the ship were stationary. This concept of Newtonian or Classical relativity is referred to as Galilean principle of invariance. Thus basic laws of physics are identical in all inertial frames of reference.

Non-inertial frames of reference:

A frame of reference in which Newton's laws of motion are not valid is called non-inertial frame Accelerating systems and rotating frames are non-inertial frames

Fictitious force - *"A pseudo-force with real effect"*

Let S be an inertial frame in which a particle P of mass m moving with a constant acceleration a in the direction positive x-axis and S' be a non-inertial frame moving at a constant acceleration a_1 parallel to a.

The force on P relative to S-frame; $F = ma$ ... comes into action only in accelerated frames and rotating frames. Such forces are called **pseudo, fictitious or frame dependent force.**

Examples:

1. All centrifugal forces.
2. Coriolis force
3. Apparent forces in accelerating lifts - Apparent weight = real weight + Fictitious force

$$R = mg \pm ma = m(g \pm a)$$

The acceleration of the ascending lift a is positive and that of descending lift is negative.

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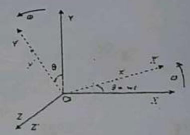
Rotating frame of Reference - *"A non-inertial frame"*

Consider two frames of reference S and S' having a common origin O and a common z- axis. Let S' be rotating about z- axis with a constant angular velocity w. Initially, the two sets of axes the frames were superposed. In time t if θ be the angle covered by S' then $\theta = wt$.
If x' be the sum of components x, y and z along OX', then the position coordinates transform according to the following equations.



$$x' = x\cos XOX' + y\cos YOX' + z\cos ZOX'$$

$$x' = x\cos wt + y\sin wt \text{ ------------ } (1)$$

Angles $XOX^1 = YOX^1 = wt$ & $ZOX^1 = 90^0$

Similarly, $y' = x\cos(90 + wt) + y\cos wt$

That is, $y' = -x\sin wt + y\cos wt$ -------- (2)

And $z' = x\cos 90 + y\cos 90 + z\cos 0$

Therefore, $z = z'$ ------------ (3)

$$\frac{dx'}{dt} = -xw\sin wt + \frac{dx}{dt}\cos wt + yw\cos wt + \frac{dy}{dt}\sin wt$$

Substituting (2) in the above relation, we get,

$$\frac{dx'}{dt} = wy' + \frac{dx}{dt}\cos wt + \frac{dy}{dt}\sin wt$$

Again differentiating, we get,

$$\frac{d^2x'}{dt^2} = w\frac{dy'}{dt} + \frac{d^2x}{dt^2}\cos wt - \frac{dx}{dt}w\sin wt + \frac{d^2y}{dt^2}\sin wt + \frac{dy}{dt}w\cos wt$$

If no force acts in S-frame then, $\frac{d^2x}{dt^2} = 0 = \frac{d^2y}{dt^2}$

$$\frac{d^2x'}{dt^2} = w\frac{dy'}{dt} - \frac{dx}{dt}w\sin wt + \frac{dy}{dt}w\cos wt$$

The first part of fictitious force $Fc = -2m w \times v'$ is called Coriolis force named after G.Coriolis wh discovered it. The Coriolis force comes into existence only when particle is moving in a rotating frame This force is numerically equal to $2mwv' \sin\theta$ and is acting always in a direction perpendicular its pat (v') and it obtained by a rotation through 90 degrees in the opposite direction to that of ω.

The second part $F_{cer} = -mw \times (w \times r)$ is a most familiar centrifugal force. Its magnitude is mw^2r and it is acting always in a direction normal to the axis of rotation and away from it.

Effect of Coriolis force

When a body moves relative to a rotating frame of reference of the earth, the Coriolis force come into action and it deflects the body from its true path. There are two cases of interest.

1. If a body falls freely under the action of gravity, the horizontal component of Coriolis force pushes i towards east in either hemisphere. But the vertical component being small, has almost no effect on the acceleration due to gravity and produces no deviation at all.

2. If a particle is given sufficient horizontal velocity, as in the case of projectile, a small Coriolis force gets sufficient time to act upon it. Therefore the particle moving in the northern hemisphere deflects towards right of its path because latitude θ is positive and rotation of r is clock-wise in northern hemisphere. In southern hemisphere, the particle is deflected towards left because rotation of r is anti-clockwise.

3. Whirl winds:

The surface of the earth is covered by low pressure air region surrounded by high pressure air region. So air currents are set up from higher pressure region to lower pressure region. When air currents move towards the earth surface they are deflected by the Coriolis forces towards left in southern hemisphere and towards right in northern hemisphere. This leads to whirl winds.

4. Cyclones:

Cyclone is due to low pressure (depression) in the atmosphere in which wind spirals inwards. Because of Coriolis force the wind blows counter clockwise in the northern hemisphere and blows inward in a clockwise direction in southern hemisphere.

5. Greater erosion on the right bank of river:

River water always flows downwards and hence river has a vertical component of velocity. Due to Coriolis force, it gets deviated to its right side and hence erodes the right bank. Thus the right bank is eaten up more rapidly and is steeper than the left bank. This is true both in northern and southern hemispheres.

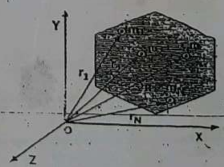
Centre of mass or Centroid:

"A simple concept to solve complex motions"

The concept of centre of mass was introduced to simplify the motion of a system of particles. The centre of mass of a system of particles is defined as a point where the whole mass of the system is assumed to be concentrated. The motion of centre of mass is same as if the external forces acting on the system were applied directly to it.

Consider a system of N particles of masses $m_1, m_2, m_3 ---- m_N$ having their position vectors $r_1, r_2, r_3, ---- r_N$ respectively. If R is the position vector of c.m. then

$$R = \frac{m_1r_1 + m_2r_2 + m_3r_3 + ---- + m_Nr_N}{m_1 + m_2 + m_3 + ---- m_N} = \frac{\sum_{n=1}^{N} m_n r_n}{\sum_{n=1}^{N} m_n}$$



If M is the mass of N particles; $R = \frac{1}{M}\sum_{n=1}^{N} m_n r_n$

That is, $MR = \sum_{n=1}^{N} m_n r_n$

Cartesian components of centre of mass for three dimensional systems:

The position vector R in terms of Cartesian components is given by $R = ix + jy + kz$, where k are unit vectors along X, Y and Z axes respectively and x, y and z are given by

$$x = \frac{m_1x_1 + m_2x_2 + \cdots + m_Nx_N}{m_1 + m_2 + \cdots + m_N} = \frac{\sum_{n=1}^{N} m_n x_n}{M}$$

$$x = \frac{1}{M}\sum_{n=1}^{N} m_n x_n$$

Similarly, $y = \frac{1}{M}\sum_{n=1}^{N} m_n y_n$ and $z = \frac{1}{M}\sum_{n=1}^{N} m_n z_n$

Therefore, $R = \frac{1}{M}\left[i\sum_{n=1}^{N} m_n x_n + j\sum_{n=1}^{N} m_n y_n + k\sum_{n=1}^{N} m_n z_n\right]$

If the particles are closely packed in the system so that the distribution is continuous then summation may be replaced by integration. For such systems the position vector of centre of mass becomes;

$$R = \frac{1}{M}\int r\,dm$$

Motion of centre of mass:

We have, $MR = m_1r_1 + m_2r_2 + \cdots + m_Nr_N$ ---------- (1)

Differentiating equation (1) with respect to t, we get,

$$MV = m_1v_1 + m_2v_2 + \cdots + m_Nv_N \text{ ---------- (2)}$$

In the absence of any external force, the net momentum of the system remains constant.

Therefore, MV = constant. Since mass M of the system is constant, *V=constant.*

This clearly indicates that in the absence of external forces, the velocity of the centre of mass of the system remains constant although the positions of particles relative to each other are changing in a complex manner. As an example, a radioactive nucleus moving at a constant velocity emits α or β particles which move in different directions but velocity of their centre of mass remains constant.

Differentiating equation (2) we get,

$$Ma = m_1a_1 + m_2a_2 + \cdots + m_Na_N$$

That is, $F = F_1 + F_2 + \cdots + F_N$

All the forces $F_1, F_2, \cdots F_N$ must be external because internal forces always exist in pairs of equal and opposite forces which cancel each other and produce no net effect on the system.

Therefore, $Ma = F_{external} = M\frac{dV}{dt} = \frac{d(MV)}{dt} = \frac{dP}{dt}$

Thus the centre of mass of a system moves as if the entire mass of the system is situated at it and the external forces acting on a system at the centre of mass.

Centre of mass frame of reference: Its four characteristics.

A frame of reference fixed rigidly to the centre of mass of an isolated system (a system free from external forces) is called centre of mass frame of reference. In this frame, the position vector of centre of mass, its velocity and momentum are zero.

That is, $R = 0$, $v = 0$ and $P = 0$

Therefore centre of mass frame is called zero momentum frame. In the absence of external force, it is an inertial frame of reference.

Characteristics of centre of mass: Imp

1. Bodies of regular geometrical shapes are symmetrical about their centre of mass. For example, the centers of mass of sphere, disc, ring, metre- stick etc. are their actual-centres.

2. The location of the centre of mass of a body depends on the distribution of mass in it. For dis the centre of mass lies within the material and for ring outside the material.
3. The position of the centre of mass is **independent** of the coordinate system chosen.
4. The centre of mass remains fixed in rotatory motion but it changes in translatory motion.
5. The total moments of particles in an isolated system about the centre of mass are zero.
   That is $MR = \int r\,dm = 0$.
6. If the particles in a system are identical (same mass) then the centre of mass coincides with th geometric centre of the system.
7. **The centre of mass frame is a zero momentum frame.**
8. In the absence of external forces, the centre of mass frame is an inertial frame of reference.
9. When an external force acts on a system of particles, the forces on the individual particles will b parallel to one another and their resultant force always passes through the centre of mass of th system.

Conceptual questions

1. Can a body possess the states, rest and motion simultaneously?
   The terms rest and motion are relative and have no absolute meaning. The state of a body depends on th observer. The body may be rest with respect to one observer and at the same time, in motion with respect t another observer. Hence a body can exhibit the two states at once for different observers in different frame of reference.
2. Why the moon appears to be at rest when you are stationary and appears to be in motion when you ar moving?
   There is a proverb "To know what you are? Look at others." This means that the state of a body lies in the relative state of the observer. If observer is in motion the object (moon) appears to be moving and vice versa.
3. How do you justify the statement that Coriolis force does exist only when a body is moving with respect to rotating frame of reference?
   Coriolis force is given by, $F_c = -2m\omega \times v$, where ω is the angular velocity of the rotating frame and v is the linear velocity of the body. If either ω or v is zero or even ω & v are parallel to each other then the Coriolis force vanishes. Hence Coriolis force comes into comes into play only when a body is moving with respect to rotating frame of reference.
4. Is earth an inertial frame? Justify your answer.
   No, earth has diurnal rotation about its own axis and annual revolution about the sun. Hence earth is a rotating frame or non-inertial frame.
5. Is Newton's II law of motion holds good in a rotating frame of reference? Give explanation.
   No, because rotating frame is a non-inertial frame where in fictitious fore comes into action. Hence Newton's II law fails in rotating frame of reference.
6. Substantiate the statemen: "all inertial frames are dynamically equivalent".
   The acceleration of a body is the same in all inertial frame of reference. Hence the force exerted by the body is the same in all inertial frames. Therefore all inertial frames are said to be dynamically equivalent.
7. **A gun fitted at the centre of a turn table is aiming at the target fixed to the rim of the table.**
   Will the bullet hit the target? Explain.
   No. The moving bullet in the rotating table experiences a Coriolis force $F_c = -2m(\omega \times v)$. If the table is moving in the clockwise direction, the bullet deviates to the right of the radial direction as shown in the fig.

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IM Momentum : It is quantity of motion presc in a body. it is vector quantity given by rel

$$\vec{P} = m\vec{v}$$

Translatory motion : It is motion of body in which the body gets displaced along the direction of force.

Rotatory motion : It is motion of body in which a line of points in the body do not move but other particles move. the line of particles is called axis of rotation

Angular velocity : $\omega = \frac{\theta}{t}$

Angular displacement has definite values $0 \longrightarrow 2\pi$

Angular momentum

$$\vec{P} = mr\omega.$$

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5m ***

Law of Conservation of linear momentum : [the momentum of a body is given by the vectorial summation] ×

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Momentum :

Linear momentum of a body is the moment of a body in linear motion & translatory & along st line.

5m ***

Law of Conservation of linear momentum :

Statement : "when a vector sum of the external forces acting upon a system of particles equals zero, the total linear momentum of the system remains constant."

The law is extended to any No. of particles which are at rest & in motion & free to move

Consider a body having 'n' No. of particles with masses m_1, m_2, m_3 . Consider m_n so that

$$m = m_1 + m_2 + m_3 \ldots m_n$$

The particles will interact with each other & when external force is given. Let the particles will have velocities $v_1, v_2, \ldots v_n$. Now, consider the total momentum of the body which is vectorial sum of individual momenta.

i.e, $\vec{P} = \vec{P_1} + \vec{P_2} + \ldots + \vec{P_n}$

$$\vec{P} = m_1\vec{v_1} + m_2\vec{v_2} + \ldots + m_n\vec{v_n}$$

Consider $\vec{F} = \frac{d\vec{P}}{dt}$ from newtons 2nd law

$$\therefore \vec{F} = \frac{d}{dt}(\vec{P_1} + \vec{P_2} + \ldots + \vec{P_n})$$

Let the net forces acting on a body is zero

i.e, $\vec{F} = 0$

i.e, $\vec{F} = \vec{F_1} + \vec{F_2} + \ldots \vec{F_n} = 0$

$\therefore$ eq v consider

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d\vec{P_1}}{dt} + \frac{d\vec{P_2}}{dt} + \ldots + \frac{d\vec{P_n}}{dt}$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{P_1} + \vec{P_2} + \ldots \vec{P_n})$$

$$\vec{F} = 0 = \frac{d}{dt}(\vec{P_1} + \vec{P_2} \ldots + \vec{P_n})$$

$$\therefore (\vec{P_1} + \vec{P_2} + \ldots + \vec{P_n}) = \text{Constant}$$

The above Eq^n proves the law of Conservation linear momentum.

NOTE : If the body has only 2 particles then

$$\vec{P} = \vec{P_1} + \vec{P_2}$$
$$\vec{P} = m_1\vec{v_1} + m_2\vec{v_2}$$
$$\frac{d\vec{P}}{dt} = \vec{F_1} + \vec{F_2}$$
$$= \vec{F} = 0$$
$$\therefore \frac{d\vec{P}}{dt} = 0 \quad \& \quad \vec{P} = \vec{P_1} + \vec{P_2} + \vec{P_3} = \text{Constant}$$

Collision : 2 bodies are said to be in collision if they interact with each other by their touch & no touch but approach each other.

2 particles collide when large forces act for relatively short interval of time.

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Inelastic Collision :

im it happens

* The motion of the particles after collision is changed
* The total moment of the particle remains Constant

* The essence of collision is redistribution of total momentum & energy of particles.
* There are 2 limits for collision
1) Perfectly Elastic collision
2) " Inelastic collision

Perfectly elastic collision : This is a collision where in total K.E of the particles is fully conserved. In such collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

&

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$

Perfectly Inelastic Collision : This is a collision in which the particles stick permanently together & the loss of K.E is maximum

NOTE :

im Headon collision : Here the direction of colliding particles is exactly opposite

Oblique Collision : It is a collision in which particles collide subtended to an angle

5m

Final velocity of colliding particles

Consider 2 particles of masses m_1 & m_2 undergoing perfectly inelastic collision. Let u_1 & u_2 are the initial velocities after inelastic collision the 2 particles move together with common velocity 'v'. Therefore

∴ The momentum Eq^n is

$$m_1u_1 + m_2u_2 = m_1v + m_2v \rightarrow (1)$$

$$m_1u_1 + m_2u_2 = v(m_1 + m_2)$$

∴ Final velocity $\boxed{v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}} \rightarrow (\mathrm{I})$

Case 1 : Let 2^{nd} particle be at rest then $u_2 = 0$

∴ (1) becomes

$$m_1u_1 = m_1v + m_2v$$

$$m_1u_1 = v(m_1 + m_2)$$

$$\boxed{v = \frac{m_1u_1}{m_1 + m_2}} \rightarrow (\mathrm{II})$$

Consider the K.E of 1^{st} particle

$$KE \text{ of } m_1 = \frac{1}{2}m_1u_1^2 \rightarrow (3)$$

the K.E of 2^{nd} particle is zero because $u_2 = 0$

Final Energy after Collision

$$K.E = \frac{1}{2}(m_1 + m_2)v^2 \rightarrow (4)$$

divide $\frac{(4)}{(3)} = \frac{\cancel{1/2}(m_1 + m_2)v^2}{\cancel{1/2}m_1u_1^2}$

i.e, $(m_1 + m_2)v^2 = m_1u_1^2 \rightarrow (5)$

$$v^2 = \frac{m_1u_1^2}{(m_1 + m_2)}$$

$$v = \sqrt{\frac{m_1u_1^2}{(m_1 + m_2)}}$$

$$\boxed{v = \left(\frac{m_1u_1^2}{m_1 + m_2}\right)^{1/2}} \rightarrow \mathrm{III}$$

also substituting value of v in (5)

Consider

$$(m_1 + m_2)v^2 = m_1u_1^2$$

$$m_1 + m_2\left[\frac{m_1u_1}{m_1 + m_2}\right]^2 = m_1u_1^2$$

i.e, $\cancel{m_1 + m_2}\frac{(m_1^2 \cancel{u_1^2})}{(m_1 + m_2)^2} = \cancel{m_1}\cancel{u_1^2}$

$$\frac{m_1}{(m_1 + m_2)} = 1$$

$$\frac{m_1}{(m_1 + m_2)} < 1$$

∴ Eq^n $\frac{(4)}{(3)} < 1$ that means " The K.E of the 2 bodies in after inelastic collision is **less than**

K.E of the bodies ~~after~~ before collision

5/9/18

10 marks

Law of Conservation of momentum in case of a rocket with variable mass

A rocket is a device which works on the principle of jet propulsion. jet is the ~~ad~~ Emission of hot gases from the rocket through its nossil. when the jet is expelled through nossil with velocity v, the rocket moves in upward direcⁿ with velocity V. the rocket moves only when the fuels are burnt in combustion chamber. In a rocket the fuel that is gunpowder & to burn it the oxidiser that is oxygen, paraffin, hydrogen peroxide are taken in separate chamber & ignited. In combu chamb at certain P. the burnt gases in combustion chamber come out of rocket through nossil of rocket as jet thus the mass of rocket ↓

Let M be the mass of rocket & V be its velocity. The rate of change of mass

$$= \alpha = -\frac{dM}{dt}$$

Let velocity of jet be $(-v+V)$ i.e $(V-v)$ rate of change of moment issuing out of rocket is $\frac{d}{dt}(M(V-v)) = -\alpha(V-v)$

rate of change of moment due to Newton's 2^{nd} law = force acting on rocket

$$\therefore \frac{d}{dt}(MV) = \frac{d}{dt}M(V-v)$$

$$M\frac{dv}{dt} + \cancel{V\frac{dM}{dt}} = \cancel{\frac{dm}{dt}(V)} - \frac{d}{dt}(Mv)$$

$$\text{i.e, } M\frac{dv}{dt} = -\frac{d}{dt}(Mv)$$

$$\therefore \frac{dv}{dt} = -\frac{d}{Mdt}Mv$$

$$\boxed{dV = -\frac{dM}{M}v} \rightarrow (1)$$

The variation in velocity of rocket = the variation of mass of rocket wrt its original mass & the velocity of jet

Integrating above Eqⁿ wrt time we get

$$\int \frac{dv}{dt} = -\frac{1}{M}\frac{dm}{dt}v\,dt$$

$$\boxed{V = \log_e M + c} \rightarrow (2)$$

where c is constant whose value is obtained by considering original condition

Consider

$$\int \frac{dv}{dt} = -\frac{1}{M}\frac{dM}{dt} v\,dt$$

if ~~vo~~ velocity of jet is not constant then

$$\int \frac{dv}{dt} = -\frac{dM}{M} v\,dt$$

$$\boxed{V = \log_e M v + C} \longrightarrow (3)$$

initially let M_0 be mass of rocket, $t = 0$

(3) becomes

$$V = \log_e M_0 v + c$$

$$C = V_0 - v\log_e M_0 \longrightarrow (4)$$

Substitute (4) in (3)

$$V = \log_e M v + V_0 - v\log_e M_0$$

$$V = v\log_e \frac{M}{M_0} + V_0$$

$$\boxed{V = V_0 + v\log_e \frac{M_0}{M}} \longrightarrow (5)$$

The above is Expr^n for velocity of rocket.

Consider Eq (5) & $M = M_0 + \alpha t$.

NOTE : $M = M_0 + \left(1 - \frac{\alpha}{M_0} t\right)$

$$= M_0 + \left(\frac{M_0 + \alpha t}{M_0}\right)$$

12/9/18

then Eq (5) becomes

$$V = V_0 + v\log \frac{\cancel{M_0}}{\cancel{M_0}\left(1 + \frac{\alpha}{M_0} t\right)}$$

$$V = V_0 + v\log_e \frac{1}{1 + \frac{\alpha}{M_0} t}$$

Let $\beta = \left[1 + \frac{\alpha}{M_0} t\right]$

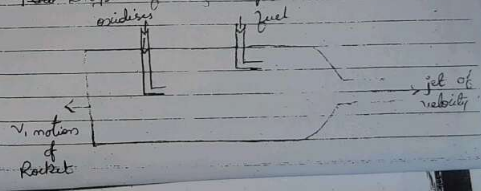
$$V = V_0 + v\log_e \frac{1}{\beta}$$

[or]

$$\boxed{V = V_0 - v\log_e \beta} \longrightarrow (6)$$

Eq (5) & (6) are the Expr^{ns} for velocity of a rocket in terms of 'β'.

~~Ate~~ (4) Diagram for single stage rocket



Numericals

1) A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of fuel is 1.6 Kms^{-1}. Calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is 2 kg per second is

$M_0 = 20 Kg$

$M = M_0 +$ mass of fuel

$= 20 + 180$

$M = 200 \, kg$

$v = 1.6 \, Kms^{-1}$

$= 1.6 \times 10^3 ms^{-1}$ → Next step : Consider $v = v_0 + v \log_e \frac{M_0}{M}$

$= -v \log_e \frac{M_0}{M}$

$V = v \log_e \frac{M_0}{M}$

$V = 1.6 \times 10^3 + \log_e \frac{20}{200}$

$V = 1.6 \times 10^3 - \log_e \frac{200}{20}$

$V = 1.6 \times 10^3 - 2.303 \log_{10} \frac{200}{20} \, 10$

$V = 1.6 \times 10^3 - 2.303 \, [1] \times 10$

$V = 1600 - 2.3$

$V = 1597.6 \, ms^{-1}$

2) A fully fueled rocket of mass 5000 is set to be fired vertically if the rocket ejects it gases at a speed of $3 \times 10^3 m/s$ w.r.t the rocket & burns a fuel at the rate of 50 kg wt is the rockets upward initial acceleration? Also include the effect of gravity

$M_0 = 5000 \, kg$

$v = 3 \times 10^3 m/s$

Let the m be the mass of the fuel burni

∴ rate of burning of fuel

i.e, $\frac{dm}{dt} = 50 \, kg/s$

$F = ?$ [upward force]

$F = v \frac{dm}{dt}$

$= 3 \times 10^3 \times 50$

$F = 150 \times 10^3 \, N$

$F = 1.5 \times 10^6 \, N$

Net thrust $= F - mg$

$= 1.5 \times 10^5 - 5000 \times 9.8$

$= 1.5 \times 10^5 -$

$= 48.998.5$

3) An Empty rocket weighing 5000 Kg & Contains 40,000 kg of fuel. If the exhaust velocity of the fuel is 3 Km/s. calculate the V_{max} gained by the rocket

M_0 = Empty rocket + 40000 = 45,000 Kg

M = 5000 Kg

v = 3 Km/s

= 3×10^3 m/s

V_0 = ?

$$V = V_0 - v \log_e \frac{M_0}{M}$$

$$= 0 - 3\times10^3 \log_e \frac{45000}{5000} 9$$

$$= 0 - 3\times10^3 \times 2.303 \times 9000$$

$$= -3\times10^3 \times$$

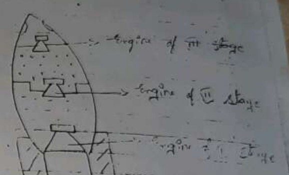
$$V = 0 - 3\times10^3 \log 9$$

$$= 0 - 3\times10^3 \times 2.303 \times 0.9542$$

$$V = -6592.86 \text{ m/s}$$

5m ***

Multi stage rocket :



A multistage rocket is combination of 2 or 3 rockets joined either consecutively or one inside the other

The first stage of the rocket is the largest in dimensions in ~~cot~~ weight & last stage rocket is smallest. when the rocket is launched the 1st stage rocket ignites the fuel is burnt, the rocket shoots with velocity v & then it gets detached or discarded. Now 2nd stage rocket burns the fuel & makes the rocket accl^n i.e, the velocity ↑. usually the fuel consumption in 2nd stage is 100 times more than the 3rd stage

The shape of the rocket is shown in fig so that ~~the~~ when rocket is subjected to intense air pressure lot of heat is generated due to the ~~fr~~ friction of air. here rocket is designed in the shape as shown in fig. acc to aerodynamics A cylindrical shape with tappering & in streamline form

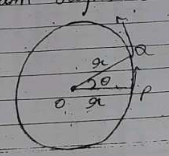
Angular Momentum : It is the measure of motion of a body in rotatary motion

" ~~It is the moment of~~

" Angular momentum is the moment of linear momentum"

Consider a particle of mass 'm' whose position vector is 'r' from origin 'O'

Let the linear momentum of the body ~~is~~ rotating is



$\vec{P} = m\vec{v}$

in circular motion the direct^n of velocity continously changes

→ The angular momentum 'L' of the body w.r.t 'O' is given as vector product of position vector & linear momentum.

i.e, $L = r \times P$

Now L is a vector.

→ If θ is angle b/w r & momentum at P then, L is $\perp^{ar}$ to the plane containing r & P.

Consider ;

$L = r \times P$

but $P = mv$

$\therefore L = r \times mv$

$L = r \times m r \omega$

$\boxed{L = m r^2 \omega}$

angular momentum is vector quantity with unit is kgm^2/s

⇒ The dimension of angluar momentum

$[L] = [ML^2T^{-1}]$

Angular Momentum of System of Particles

Consider a sys of particle having angular momenta $L_1 L_2 L_3$ abt fixed point in sys then

Total angular momentum is

$$\vec{L} = \vec{L_1} + \vec{L_2} + \vec{L_3} + \dots + \vec{L_n}$$

$$\vec{L} = \vec{r_1} \times \vec{P_1} + \vec{r_2} \times \vec{P_2} + \dots$$

$$L = (\vec{r_1} \times P_1) + (r_2 \times P_2) + \dots (r_n \times P_n)$$

$$L = \sum_{i=1}^{n} r_i \times P_i$$

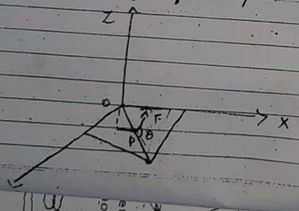
1M

Torque acting ~~of~~ on a particle :

Torque abt a point is defined as rate of change of angular momentum wrt time. It is denoted by $\tau = \frac{dL}{dt}$ it is

$$\tau = \frac{dL}{dt}$$

it is a time derivative so it is a scalar quantity even though "angular momentum (L)" is a vector quantity



Consider a particle P having position vector r given by a force F in an inertial frame of reference the τ acting on particle is defined as

$$\boxed{\tau = r \times \vec{F}}$$

it is vectorial product

UDAYAKUMARA G M
Bsc (Pcm)
V.V.S.C, Ballari

M.Mallikarjun, Lecturer,

3

Rotation of rigid bodies = 3

"A change always encounters hindrance"

Rigid body

A body which undergoes no changes in its shape and size, when external forces are applied on it, is called a rigid body. In other words the distance between any two particles in the rigid body remains same due to the application of forces on the body. Solid bodies are taken as rigid bodies.

Inertia and moment of inertia

According to Newton's first law of motion, a body at rest continues to be at rest and a body in uniform motion along a straight line continues to be in motion. No body tends to change its existing state by itself. This inactive nature of matter is termed as inertia. The inertia of a body increases with the increase of its mass and hence mass in linear motion is referred to as the coefficient of inertia. In other words, mass is the measure of inertia. It is an internal resistance to the change of state.

A body in rotational motion encounters an opposition similar to inertia in rotational motion. The inertia that ~~comes into play~~ during rotational motion is called rotational or moment of inertia. Inertia of a body depends only on the amount of matter in the body but moment of inertia about an axis depends on the distribution of matter as well, about the axis.

Moment of inertia I of a particle of mass m about an axis distant r from the particle is defined as the product of mass of the particle and square of the perpendicular distance between the particle and the axis of rotation.

That is,

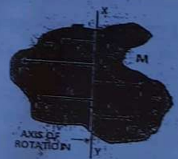
$$i = mr^2$$

rotation product

Moment of inertia of a rigid body about an axis is the sum of the moments of inertia of all the particles in the body about the axis.

$$I = i_1 + i_2 + i_3 + \dots\dots$$

That is, $$I = m_1r_1^2 + m_2r_2^2 + \dots\dots = \sum mr^2$$



Moment of inertia of a rigid body about an axis is also measured as the product of mass of the body and square of the radius of gyration of the body about the axis.

That is, $$I = MK^2$$

Where, K is known as radius of gyration.

Radius of Gyration (K):

Radius of gyration of a body rotating about an axis is defined as the effective distance between the axis and a point at which the entire mass of the body were to be concentrated so that its moment of inertia is same as that with the actual distribution of matter in the body about the axis.

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M.Mallikarjun, Lecturer,

We have, with actual distribution of matter, $I = \sum mr^2$

That is, $I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ---- + = \sum mr^2$

With entire mass concentrated at a single point P we get,

$$I = MK^2$$

For n particles each of mass m, $I = MK^2 = m(r_1^2 + r_2^2 + ----- + r_n^2$

$$MK^2 = (m \times n)\frac{(r_1^2 + r_2^2 + ------ + r_n^2)}{n}$$

Since $M = (m \times n)$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + ------ + r_n^2}{n}}$$ RMS value of the distances of individual particles

Thus, **radius of gyration of a body can also be defined as the square root of the mean of the squares of the distances of the particles of the body from the axis of rotation**.

Note:

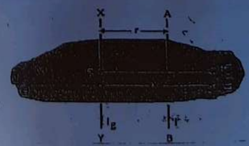
1. M. I. depends on the mass of the body, distribution of matter in the body/shape of the body and the selection of axis of rotation.
2. SI unit of M.I is kg m^2.
3. Dimensions of MI are, one in mass, two in length and zero in time, that is, $[I] = [M^1 L^2 T^0]$
4. The S.I. unit of radius of gyration is metre (m)
5. The value of K varies with axes of rotation. For example, M.I. of a ring about an axis passing through its centre and perpendicular to its plane is MR^2 but M.I. of the same ring about its diameter is (MR^2)/2. Accordingly, $K = R$ in the first case and in the second case, $K = R/\sqrt{2}$

General theorems on M.I

There are two vital theorems used very often in finding M.I. of a body about the required axis

I. Parallel axes theorem:

The theorem states that, **M.I. of a rigid body about an axis is equal to the sum of moment of inertia of the body about a parallel axis passing through its centre of gravity and the product of mass of the body and the square of the separation between the two axes.**



Consider a body of mass M with G as its centre of gravity.

Let AB be any axis about which M.I. of the body be I and XY be the axis parallel to AB and passing through G, about which M.I. of the body be I_g and r be the separation between the two axes.

According to the statement,

$$\boxed{I = I_g + Mr^2}$$

2

$I = I_g + Mr^2$

Proof

Case-1

Let P be a particle of mass m at a distance x from G on the straight line OG produced.

M.I. of the particle about XY is given by, $I_g = mx^2$

M.I. of the entire body about XY, $I_g = \sum mx^2$

Similarly, M.I. of the body about AB, $I = \sum m(r+x)^2 = \sum m(r^2 + x^2 + 2rx)$

That is, $I = \sum mr^2 + \sum mx^2 + \sum 2mrx \longrightarrow (1)$

As the body is balanced about its C.G, the algebraic sum of moments of all the individual particles in it about the C.G is zero.

That is, $\sum mgx = 0 \quad or \quad \sum mx = 0$ ——— (2)

Total mass of the body, $\sum m = M$ ——— (3)

Substituting (2) and (3) in (1) we get, $\boxed{I = I_g + Mr^2}$

Hence, the theorem.

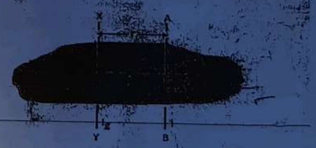
Case-2

Let P be at a distance r from G, any where on the body and PN be the perpendicular drawn from P to OG produced. From right angled triangle OPN, we get,

$$OP^2 = ON^2 + NP^2$$

$$OP^2 = (r+x)^2 + y^2 = (r+x)^2 + r_2^2 - x^2$$

$$r_1^2 = r^2 + r_2^2 + 2rx \text{ --------- } (4)$$



M.I. of the body about AB,

Since, $\sum 2mrx = 0$ and $\sum mr^2 = Mr^2$, we get,

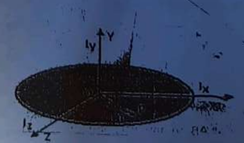
$$I = \sum mr_1^2 = \sum mr^2 + \sum mr_2^2 + \sum 2mrx$$

$$\boxed{I = I_g + Mr^2}$$

~~Hence, the theorem~~

Perpendicular axes theorem

The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of moments of inertia of the lamina about two mutually perpendicular axes on the lamina, so that the three axes meet at a single point on the lamina.



- Let the three mutually perpendicular axes XX', YY' and ZZ' are meeting at a point O on the lamina.

According to the statement,

$$I_z = I_x + I_y$$

Proof

Consider a particle P of mass m at a distance r from the centre O and on the lamina.

M I of the particle P about x, y, z axes are respectively given by;

$$i_x = mx^2, i_y = my^2 \text{ and } i_z = mz^2$$

Therefore,

$$I_x = \sum mx^2, I_y = \sum my^2, I_z = \sum mz^2$$

From the figure,

$$z^2 = x^2 + y^2$$

That is,

$$I_z = \sum mz^2 = \sum m(x^2 + y^2) = \sum mx^2 + \sum my^2$$

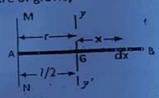
Thus,

$$I_z = I_x + I_y$$

M.I of homogeneous bodies of specific geometrical shapes can be determined by considering a suitable elementary portion of the body. The MI the element about a required axis is obtained and then on integrating it within suitable limits, M I of the entire body about the axis is calculated.

1. M I of a thin uniform rod

Consider a thin uniform bar AB of mass M and length l with G as its centre of gravity.



(i) About an axis yy' passing through C.G and perpendicular to length of rod

Consider an element of length of length dx at a distance x from G.

Mass per unit length of the rod, $m' = \frac{M}{l}$

Mass of the element, $m = m'dx = \frac{M}{l} dx$

Therefore, M.I of the element about yy'. $i_g = mx^2 = \frac{M}{l} x^2 dx$

The MI of the rod about yy',

$$I_g = \sum mx^2 = \sum \frac{M}{l} x^2 dx$$

$$Ig = \frac{M}{l} \int_{-l/2}^{+l/2} x^2 dx = 2\frac{M}{l} \int_0^{l/2} x^2 dx$$

$$Ig = \left[2\frac{M}{l} \times \frac{x^3}{3} \right]_0^{l/2}$$

$$Ig = \frac{Ml^2}{12}$$

$$K_g^2 = l^2/12 \Rightarrow K_g = l/\sqrt{12}$$

$$\boxed{K_g = l/2\sqrt{3}}$$

ii) MI about an axis perpendicular to the length and passing through one end.

According to the parallel axes theorem, $I = I_g + Mr^2$

Here $I_g = Ml^2/12$ and $r = l/2$ we get, $I = Ml^2/12 + Ml^2/4$

Therefore, $I = Ml^2/3$ and $K = L/\sqrt{3}$

2. M.I. of a rectangular lamina:

Consider a rectangular lamina of length l breadth b and negligible thickness. Let M be the mass of the lamina and G be its centre of gravity:

i) About Y-axis passing through G and perpendicular to its length

Consider an element of length dx and breadth b at a distance x from the Y axis

Mass per unit area of the lamina, $m' = M/(l \times b)$

Area of the element, $a = bdx$

Mass of the element, $m = m'a = \frac{M}{l \times b} \times bdx = \frac{M}{l}dx$

M I of the element about Y-axis, $i_y = mx^2 = \frac{M}{l}dx$

M I of the lamina about Y-axis, $I_y = \frac{M}{l}\int_{-l/2}^{l/2} x^2 dx = \frac{2M}{l}\int_{0}^{l/2} x^2 dx = \frac{2M}{l} \times \frac{l^3}{3 \times 8}$

$$I_y = \frac{Ml^2}{12}$$

Similarly, $I_y = \frac{Ml^2}{12}$ and $I_x = \frac{Mb^2}{12}$

Therefore, $I_z = I_x + I_y = \frac{Mb^2}{12} + \frac{Ml^2}{12}$

That is, $\boxed{I_z = M(b^2 + l^2)/12}$

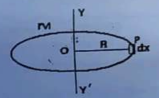
And, $\boxed{K = \sqrt{b^2 + l^2}/2\sqrt{3}}$

3. M I of Circular loop (ring):

Consider a ring of mass M and radius R, with O as its centre as shown in the below figure.

i) About an axis YY' passing through the centre of the ring and perpendicular to its plane

Consider an element of length dx on the ring at a point P as shown in the figure.



Mass per unit length, $m' = M/l = M/2\pi R$

Mass of element, $m = m'dx = (M/2\pi R)dx$

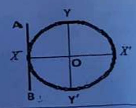
M I of the element about YY', $i_g = mR^2 = \frac{M}{2\pi R}dx \times R^2 = \frac{MR}{2\pi}dx$

M.I of the ring about YY', $Ig = \frac{MR}{2\pi}\int_0^{2\pi R} dx = \frac{MR}{2\pi} \times 2\pi R$

$$I_g = MR^2$$

ii) About the diameter of the ring:

To find the MI of a ring about its diameter, the perpendicular axes theorem is used. From the theorem, M.I of ring about its C.G is equal to MI about one diameter YY' plus MI about perpendicular diameter XX'.



$$I_g = I_y + I_x = I_D + I_D = 2I_D$$

Or, $$I_D = I_g/2$$

$$I_D = MR^2/2$$

iii) About a tangent parallel to the plane of ring (about AB):

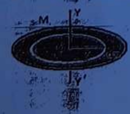
From parallel axes theorem, $I_T = I_D + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$

$\boxed{I_T = \frac{3}{2}mR^2}$

4. MI of a Disc - Circular lamina

Consider a circular lamina of mass M and radius R with O as its centre.

i) About an axis YY' passing through the C.G of the disc and perpendicular to the plane of it.



Consider a ring of radius x and thickness dx as an element.

Mass per unit area of the disc, $m' = M/A = M/\pi R^2$

Area of the element, $a = 2\pi x dx$

Mass of the element, $m = m'a = \frac{M}{\pi R^2} \times 2\pi x\,dx = \frac{2M}{R^2} x\,dx$

M I of element about YY', $i_g = mx^2 = \frac{2M}{R^2} x^3 dx$

$\therefore$ M I of the disc about YY', $I_g = \frac{2M}{R^2} \int_0^R x^3 dx = \frac{2M}{R^2} \times \frac{R^4}{4}$

$$I_g = \frac{MR^2}{2} \qquad K_g = \frac{R}{\sqrt{2}}$$

ii) About the diameter: If I_D is M I of disc about its diameter then;

$$I_g = I_D + I_D$$

$$I_D = I_g / 2$$

$$I_D = MR^2/4 \text{ and } \quad K_D = R/2$$

In case of annular disc of inner radius r and outer radius R we get;

$$i_g = mx^2 = \frac{M}{\pi(R^2 - r^2)} \times 2\pi x\,dx \times x^2$$

$$I_g = \frac{2M}{(R^2 - r^2)} \int_r^R x^3 dx = \frac{2M}{R^2 - r^2} \times \left(\frac{R^4 - r^4}{4}\right)$$

$$I_g = \frac{M(R^2 + r^2)}{2}$$

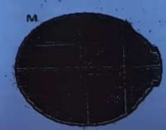
6. M I of a solid sphere

Consider a solid sphere of radius R and mass M. Let 'O' be the centre of the sp here of gravity.

i) About an axis passing through the centre O of the sphere.

Consider an elementary circular lamina of thickness dx at a distance x from the centre 'O'

Mass per unit volume of the sphere, $m' = \frac{M}{V} = \frac{3M}{4\pi R^3}$



$\therefore$ Mass of the element, $m = m'v = \frac{3M}{4\pi R^3} \pi (R^2 - x^2) dx$

$$m = \frac{3M(R^2 - x^2)dx}{4R^3}$$

Mass per unit volume of the sphere, $\boxed{m' = \frac{M}{V} = \frac{3M}{4\pi R^3}}$

Mass of the element,

$$m = m'v = \frac{3M}{4\pi R^3}\pi(R^2 - x^2)dx$$

That is,

$$m = \frac{3M(R^2 - x^2)dx}{4R^3}$$

MI of the element about its centre C and perpendicular to its plane, (about AB)

$$i_D = \frac{mr^2}{2} = \frac{3M}{2R^3}(R^2 - x^2)\times\frac{(R^2 - x^2)}{4}dx$$

$$i_D = \frac{3M(R^2 - x^2)dx}{8R^3}$$

M.I. of the sphere,

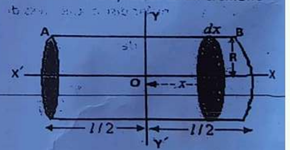
$$I_D = \frac{3M}{8R^3}\int_{-R}^{R}(R^2 - x^2)dx = \frac{3M}{4R^3}\int_{0}^{R}\left[R^4 + x^4 - 2R^2x^2\right]dx$$

$$I_D = \frac{3M}{4R^3}\left[R^5 + \frac{R^5}{5} - 2R^2\frac{R^3}{3}\right] = \frac{3M}{4R^3}\times R^5\left[1 + \frac{1}{5} - \frac{2}{3}\right] = \frac{3MR^2}{4}\times\frac{8}{15}$$

$$I_D = \frac{2}{5}MR^2$$

7. M.I of a solid cylinder

Consider a solid cylinder of mass M, radius R and length l. Let $X'X$ be the axis of the cylinder. A disc of radius R and thickness dx at a distance x from the centre O of the cylinder as an element



Mass per unit volume of the cylinder, $m' = \frac{M}{\pi R^2 l}$

Volume of disc, $v = \pi R^2 dx$

Mass of disc, $m = m'v = \frac{M}{\pi R^2 l}\times \pi R^2 dx = \frac{M}{l}dx$

i) M I about its own axis

The MI of disc about $X'X$ passing through its centre and perpendicular to its plane is;

$$i_g = \frac{mR^2}{2} = \frac{M}{2l}R^2 dx$$

MI of the solid cylinder about $X'X$

$$I_X = \frac{M}{2l}R^2\int_{-l/2}^{l/2}dx = \frac{M}{l}R^2\int_{0}^{l/2}dx = \frac{M}{l}R^2\times\frac{l}{2}$$

$$I_X = MR^2/2$$

ii) About an axis YY' passing through O and perpendicular to its length

M.I of elementary disc about its diameter parallel YY' to is given by,

$$i_0 = \frac{mR^2}{4} = \frac{M}{4l} R^2 dx$$

M.I of disc about YY' (from parallel axes theorem)

$$i_{YY'} = i_D + mx^2 = \frac{M}{4l} R^2 dx + \frac{M}{l} x^2 dx \qquad \text{Since } m = \frac{M}{l} dx$$

M I of whole cylinder about XY,

$$I_{YY'} = \frac{M}{4l} R^2 \int_{-l/2}^{l/2} dx + \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx = \frac{M}{2l} R^2 [x]_0^{l/2} + \frac{2M}{l} \left[\frac{x^3}{3}\right]_0^{l/2}$$

$$I_{YY'} = \frac{MR^2}{2l} \left(\frac{l}{2}\right) + \frac{2M}{3l} \times \frac{l^3}{8} = \frac{MR^2}{4} + \frac{Ml^2}{12}$$

$$\therefore I_{YY'} = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]$$

M.I. of a hollow cylinder:

Consider a hollow cylinder of mass M, length l, inner radius r and outer radius R. Let XX′ be the axis of the cylinder and O be its centre of gravity.

Consider an annular disc of thickness dx as an element at a distance x from the centre O.

Volume of the cylinder, $V = \pi(R^2 - r^2) l$

Mass per unit volume,

$$m' = \frac{M}{V} = \frac{M}{\pi(R^2 - r^2) l}$$

Volume of the element,

$$v = \pi(R^2 - r^2) dx$$

Mass of the element,

$$m = m'v = \frac{M}{\pi(R^2 - r^2) l} \times \pi(R^2 - r^2) dx = \frac{M}{l} dx$$

) About its own axis $X'X$

MI of elementary annular disc about an axis passing through its centre and perpendicular to its plane is given by;

$$i_z = m(R^2 + r^2)/2$$

$$i_z = \frac{M}{l}\left(\frac{R^2 + r^2}{2}\right)dx$$

That is,

M I of the cylinder about XX',

$$I_z = \frac{M}{l}\left(\frac{R^2 + r^2}{2}\right)\int_{-l/2}^{l/2} dx = \frac{2M}{l}\left(\frac{R^2 + r^2}{2}\right)\int_{0}^{l/2} dx$$

$$I_z = \frac{M}{l}(R^2 + r^2)\times l/2 = \frac{M(R^2 + r^2)}{2}$$

$$\boxed{I_z = \frac{M(R^2 + r^2)}{2}}$$

For a solid cylinder, $r = 0$

$$\boxed{I_z = MR^2/2}$$

ii) About an axis YY' passing through O and perpendicular to its length

M I of the element about its diameter parallel to YY' is

$$i_D = \frac{m(R^2 + r^2)}{4} = \frac{M}{l}\left(\frac{R^2 + r^2}{4}\right)dx$$

M I of the element (from parallel axes theorem) about YY' ,

$$i_y = i_D + mx^2 = \frac{M}{4l}(R^2 + r^2)dx + \frac{M}{l}x^2 dx$$

M.I. of the cylinder about YY',

$$I_y = \frac{M}{4l}(R^2 + r^2)\int_{-l/2}^{l/2} dx + \frac{M}{l}\int_{-l/2}^{l/2} x^2 dx$$

$$\boxed{I_y = \frac{M(R^2 + r^2)}{4} + \frac{Ml^2}{12}}$$

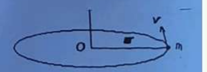
Note:

For a solid cylinder internal radius, $r = 0$ and hence;

$$\boxed{I_Y = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)}$$

Angular momentum and rotational energy

The angular momentum of a particle of mass m rotating about an axis in a circle of radius r with a linear velocity v is given by; $L = I\omega$ (Similar to $p = mv$ in linear motion). Where ω is angular velocity, and I is MI of the particle about YY'



Since, $I = mr^2$ and $v = rw$

We get $L = mr^2\omega = mr^2\frac{v}{r} = mrv$

Imagine a rigid body of mass M rotating about on axis YY^1 naturally the body possesses kinetic energy due to its rotational motion. This energy is called rotational energy

The body is assumed to be made up of a large number of very small particles of masses, $m_1, m_2, m_3, ------ etc.$, situated at distances, $r_1, r_2, r_3, ------ etc.$ respectively from the axis as shown in the figure. All these particles are moving with the same angular velocity w as that of the body but their linear velocities are different.

Let $v_1, v_2, v_3, ---- etc.$ be their respective linear velocities.

The kinetic energies of these particles are given by;

$K.E_1 = \frac{1}{2} m_1 v_1^2, \quad K.E_2 = \frac{1}{2} m_2 v_2^2, \quad K.E_3 = \frac{1}{2} m_3 v_3^2, ------ etc.$

The total K.E. of the rigid body due to rotation is given by,

$$K.E = K.E_1 + K.E_2 + ... = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + ...$$

Since $v = rw$,

$$K.E = \frac{1}{2} m_1 r_1^2 w^2 + \frac{1}{2} m_2 r_2^2 w^2 + ...$$

$$K.E = \frac{1}{2}\left[m_1 r_1^2 + m_2 r_2^2 + ...\right] w^2 = \frac{1}{2}\left[\sum mr^2\right] w^2$$

$$E_K = \frac{1}{2} MK^2 w^2$$

$$E_K = \frac{1}{2} I w^2$$

Where K is radius of gyration of the body about YY^1

<u>Note-1: Kinetic energy,</u> $E_K = \frac{1}{2} I w^2 = \frac{1}{2I}(I^2 w^2) = \frac{1}{2I}(Iw)^2 = \frac{1}{2I} L^2$

Angular momentum, $L = \sqrt{2IE_K}$

The above relation is similar to $P = \sqrt{2mE_K}$ in linear motion which gives the relation between rotational kinetic energy and angular momentum.

Note-2: Kinetic energy, $E_K = \frac{1}{2} I w^2 = \frac{1}{2}(Iw)w = \frac{1}{2} Lw$

K.E of a rolling body

A rolling body has K.E due to both rotational and linear motion. If v and ω are linear and angular velocity of the rolling body of mass M then, K.E due to linear motion, $E_{K_1} = Mv^2/2$

K.E due to rotational motion, $E_{K_2} = \frac{1}{2} I w^2 = \frac{1}{2} MK^2 \left(\frac{v}{R}\right)^2 = \frac{1}{2} M v^2 \frac{K^2}{R^2}$

Total kinetic energy of rolling body; $E_K = E_{K_1} + E_{K_2}$

$$E_K = \frac{1}{2} M v^2 + \frac{1}{2} M v^2 (K/R)^2$$

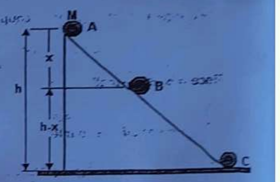
$$\boxed{E_K = \frac{1}{2} M v^2 \left[1 + (K/R)^2\right]}$$

Velocity and acceleration of a body rolling down an inclined plane

Let a body of mass M and radius R be rolling down an inclined plane without slipping

At A, $E_K = 0 \quad \because w_i = 0 \text{ and } v = 0$

$E_p = mgh \quad \therefore T.E = mgh$ ———— (1)



At B, $E_K = \frac{1}{2} m v_B^2 \left[1 + \left(\frac{K}{R}\right)^2\right]$

$E_p = mg(h - x) = \frac{1}{2} m v_B^2$

Since T.E is conserved loss of E_p is equal to gain in E_k

$$\frac{1}{2} m v_B^2 \left[1 + \frac{K^2}{R^2}\right] = mgh$$

$$v_B^2 = \frac{2gh}{\left[1 + K^2/R^2\right]} = \frac{2gl \sin\theta}{\left(1 + K^2/R^2\right)} \quad \because h = l \sin\theta$$

If a is the acceleration then, $v_B^2 = 2al = \frac{2gl \sin\theta}{1 + K^2/R^2}$

Therefore, acceleration of a rolling body, $\boxed{a = \frac{g \sin\theta}{1 + K^2/R^2}}$

Note: 1. Acceleration of rolling body is independent of mass and radius of the body when it is symmetric about the axis of rotation but depends on K and the angle of inclination of the plane (θ)

3. For ring, $K = R \quad \therefore a_{ring} = 0.50 g \sin\theta$, for disc, $K = R/\sqrt{2}$, $\therefore a_{disc} = 0.67 g \sin\theta$ and for sphere,

$$K^2 = \frac{2}{3} R^2$$

$$\therefore a_{sphere} = 0.71 g \sin\theta$$

$$\therefore a_{sphere} > a_{disc} > a_{ring}$$

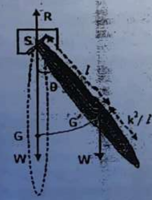
M.Mallikarjun, Lecturer,

Theory of compound pendulum

Any rigid body capable of oscillation in a vertical plane about a horizontal axis is called compound **pendulum**. The oscillations of compound pendulum are simple harmonic and therefore, the pendulum is treated as **harmonic oscillator**. The period of oscillation of the pendulum is given by;

$$T = 2\pi\sqrt{L/g}\ .$$

Consider a compound pendulum of mass m with S as the centre of suspension and G the centre of gravity. The distance between G and S is called the length l of the pendulum. When the pendulum is at equilibrium, G lies vertically bellow S. The weight of the pendulum W acting vertically downward is balanced by the reaction R at S. The reaction R and the weight W are acting along the same vertical line and form no couple. If the pendulum is displaced by a small angle θ, the weight W and its reaction R form a couple called **restoring couple**.



The moment of this restoring couple is given by,

$$C_R = F \times d = W \times AG' = mgl\sin\theta$$

Since θ is very small, $\sin\theta = \theta$

$$C_R = mgl\ \theta \text{ ------------ (1)}$$

The moment of the couple is also given by,

$$C_R = I\frac{dw}{dt} \text{ ------------ (2)}$$

From equations (1) and (2), we get,

$$I\frac{dw}{dt} = mgl\theta$$

That is, acceleration; $a = \mu\theta$ ---------- (3) Where $\mu = \frac{mgl}{I}$ is a constant.

The equation (3) indicates that, **acceleration of the pendulum is directly proportional to its displacement from the mean position and directed always towards it.** Hence motion of pendulum is **simple harmonic.**

The period of SHM is given by; $T = 2\pi\sqrt{1/constant} = 2\pi\sqrt{1/\mu}$

$$T = 2\pi\sqrt{I/mgl} \text{ ------ (4)}$$

If I_g is MI about CG then MI about C.S is given by parallel axes theorem as, $I = I_g + ml^2$

If k is radius of gyration then $I_g = \frac{ml^2}{12} = mk^2$ $\quad k^2 = l^2/12$

That is, $I = m(k^2 + l^2)$ ---------- (5)

Substituting (5) in (4) we get, $T = 2\pi\sqrt{\frac{m(k^2+l^2)}{mgl}} = 2\pi\sqrt{\frac{k^2+l^2}{gl}}$

where $\frac{k^2}{l}+l = L$

$$T = 2\pi\sqrt{\frac{k^2/l+l}{g}} \quad \text{------------ (6)}$$

In case of simple pendulum,

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}} \quad \text{------------ (7)}}$$

The above relation indicates that, the period of oscillation of compound pendulum is same as that of simple pendulum of length $L = (k^2/l)+l$. Therefore the term $(k^2/l)+l$ is called length of an equivalent simple pendulum or the reduced length of the pendulum.

Centre of Oscillation

We have,

$$L = k^2/l + l \text{ and } L = OG + GS$$

The point vertically below the C.G distant k^2/l from G is called centre of oscillation denoted by O.

Note

1. Let OG = $(k^2/l) = l'$ $\Rightarrow$ $k^2 = ll'$ and $(k^2/l') = l$.
2. If $l = 0$ then T=0. No oscillation about C.G.
3. C.O and C.S are the points at unecual distances from C.G and on either side of it, about which the period of pendulum is the same. This principle is used in the experiment to determine g using bar pendulum

Properties of compound pendulum

The compound pendulum possesses three vital properties which are explain ed below.

1. Centre of suspension and centre of oscillation are interchangeable

Period of oscillation about C.S

$$T = 2\pi\sqrt{\frac{k^2/l+l}{g}}$$

If, $k^2/l = l' \Rightarrow k^2/l' = l \Rightarrow k^2 = ll'$

$$T = 2\pi\sqrt{\frac{l'+l}{g}} \longrightarrow 1$$

Period of oscillation about C.O

$$k^2/l = l'$$

$$T' = 2\pi\sqrt{\frac{k^2/l'+l'}{g}} = 2\pi\sqrt{\frac{l+l'}{g}} \quad \text{------------ (2)}$$

From (1) and (2), we have, $T = T'$

In other words, periods about CS and CO are equal and hence CS and CO are interchangeable.

UDAYAKUMARA.G.M
BSC [PCM]

04

ELASTICITY

"Mere elongation is not elasticity"

Review of elastic behavior of solids in general, origin of elastic forces, stress- strain diagram, elastic limit and Hook's Law, Moduli of elasticity and Poisson's ratio for isotropic material, relation among elastic constants: $K=\frac{Y}{3(1-2\sigma)}$, $\eta=\frac{Y}{2(1+\sigma)}$ & $\frac{3}{Y}=\frac{1}{\eta}+\frac{1}{3K}$. Torsion: Expression for couple per unit twist and torsion pendulum (theory and experiment). Bending : Bending moment, Mention of expression for bending moment, theory of light cantilever, uniform bending - beam loaded at the centre (double cantilever) and I-section girder (qualitative).

Review of Elastic behaviour of solids in general:

In mechanics, we study the forces and its impact of material bodies. The material bodies are broadly classified as **rigid and non-rigid bodies.** A material body in which the distance between their atoms or molecules is unaltered under the action of equilibrium forces is called a **rigid body.** The rigid bodies should not undergo any change in size or shape under the action of external forces acting on it.

In practice, a body acted upon by forces in equilibrium undergoes a change in size or shape and hence they are treated as **non-rigid bodies.**

Non-rigid bodies are classified as **elastic** and **plastic bodies**. A body that recovers its original state [size and shape] after the removal of deforming forces acting on it is called **elastic body**. Quartz fibre is the best example for elastic bodies. Generally crystalline solids show elastic nature.

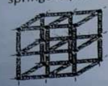
On the other hand, bodies that do not recover its original size and shape after the removal of forces acting on it are called a **plastic body**. Putty ball or mud ball is the best examples for plastic bodies. Generally, amorphous solids exhibit plastic nature.

The bodies observed in reality are neither completely elastic nor completely plastic, but they are **semi-elastic**. For small deformation, the material show elastic behavior and for large force they tend to become semi-elastic. For very large force, the material show plastic behavior.

Origin of elastic forces:

In a solid, atoms or molecules are bonded together by inter atomic or inter-molecular forces and stay in a stable equilibrium position. When a solid is deformed, the atoms or molecules are displaced from their equilibrium positions causing a change in inter- atomic (or intermolecular) distances. When atoms are displaced from their equilibrium position, restoring force come in to picture. Under the action of external deforming force and restoring force, the body is in a state of static equilibrium.

When the deforming force is removed, the restoring force brings the atoms/molecules back to their original positions. Thus the body regains its original shape and size. The restoring mechanism can be visualized by taking a model of spring-ball system shown in the Figure below. Here the balls represent atoms and springs represent inter atomic forces.



If you try to displace any ball from its equilibrium position, the spring system tries to restore the ball back to its original position. Thus elastic behavior of solids can be explained in terms of microscopic nature of the solid

Deforming force:

The force that bring about the change in the shape and size of the body is called deforming force.

Elasticity:

The property a body by virtue of which it regains its original size and shape, after the withdrawal of deforming forces, is called elasticity.

Elasticity is not mere elongation, because, Steel recovers more than rubber even though rubber elongates more than steel for the same force acting on both of them. On the other hand, for the same elongation, the force acting on steel is more than that for rubber. Hence, steel develops more restoring force than rubber, so it has larger elastic property than rubber.

Note:

1. A material for which restoring force is very large compared to deforming force, there is no impact of force on the body and hence it behave like a rigid body.
2. A material for which the restoring force is always equal and opposite to deforming force behaves like a perfectly elastic body.
3. A body for which the restoring force is lesser than the deforming force, the material continuously elongates under the action of constant force and hence it behaves like a perfectly plastic body.

Some definitions:

Load: The term load refers to a set of external forces which tend to deform the body.

Stress:

When deforming forces are applied to a body, the shape and size of the body change. Due to the relative displacement of the molecules, internal restoring forces oppose the change and to try to restore its original condition after the removal of the deforming force.

The internal restoring force per unit area of the body is called stress. In equilibrium, stress is measured by the deforming force applied per unit area,

Stress = F/A

The SI unit of stress is newton/(metre)^2 [Nm^{-2}] and dimensional formula is; $[M^1 L^{-1} T^{-2}]$

Normal stress:

Force acting per unit normal area is called normal stress. It tends to change the size only.

Tangential stress:

The tangential force per unit area is called tangential stress. It tends to change both shape and size.

Strain:

The ratio of change in the dimensions of a body by the application of deforming forces to the original dimensions of the body is called *strain*. It has no unit and dimensions.

Longitudinal or a linear strain:

The ratio of increase in length in the direction of deforming force to the original length is called linear or longitudinal strain. It is equal to elongation per unit length.

$$Linear\ strain = Change\ in\ length / Original\ length = l/L$$

Volume strain:

The ratio of increase in volume to the original volume is called volume strain.

$$Volume\ strain = \frac{Increase\ in\ volume}{Original\ volume} = \frac{v}{V}$$

Shearing strain:

The angle of shear measured in radians is called shearing strain.

Hooke's law:

The law states that, within the elastic limit, stress is directly proportional to strain.

That is, $$Stress \propto Strain$$

Or, $$\frac{Stress}{Strain} = E$$

Where, E is a constant called modulus of elasticity.

Moduli of elasticity

There are three moduli of elasticity.

1. Young's Modulus: $$Y = \frac{Normal\ stress}{Linear\ strain} = \frac{F/A}{l/L} = \frac{F \times L}{A \times l}$$

2. Bulk Modules: $$K = \frac{Normal\ stress}{Volume\ strain} = \frac{F/A}{v/V} = \frac{Pressure \times volume}{Change\ in\ volume} = \frac{PV}{v}$$

Note: Since PV gives a measure of work done, K gives a measure of work done per unit change in volume of the body.

3. Rigidity Modules

$$\eta = \frac{Tangential\ stress}{Shearing\ strain} = \frac{F/A}{\theta} = \frac{F/A}{l/L} = \frac{F \times L}{A \times l}$$

Poisson's Ratio

When a wire is stretched, it becomes longer but thinner. In other words, increase in length is always accompanied by a decrease in its cross section.

Within an elastic limit, ***the ratio of lateral strain to the linear strain is called Poisson's ratio.***

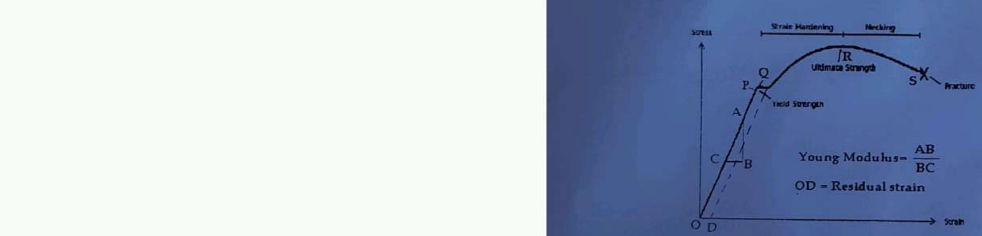
$$\sigma = \frac{Lateral\ strain}{Linear\ strain} = \frac{\beta}{\alpha}$$

Poisson's ratio has no unit and dimensions.

Elastic limit

When the load applied (normal stress) to a wire is gradually increased, initially strain increases linearly in accordance with Hooke's law and on further increase in load leads to random variation and ceases to follow Hooke's law as shown in the graph drawn stress against strain.

The maximum stress up to which the body exhibits elasticity and obeys Hooke's law is called elastic limit. This is shown by the part *OP* of the curve. The stress correspond to point *B* at which a large increase in strain commences is called ***yielding stress***. If the load is removed at this stage, the wire is left with a certain amount of strain called ***residual strain*** represented by OD.



The yielding stops at *R* and further extension become plastic and it can be produced by the gradual increase in the load. The part *RS* represents the ***plastic range***. On further increase in the applied load, the wire finally breaks at *S*. The point *S* is called ***breaking point***. Between QR, the material becomes hard due to strain the material. At R necking starts and the wire finally breaks at S.

Elastic Fatigue:

When an elastic body is subjected to deforming forces time and again (frequently) the body loses its elastic property. This nature of the body is called elastic fatigue. The fatigue may be removed and the body may be recovered to its elastic strength just by allowing the body to take rest for some time.

any 1 in 3 Questions in exam.

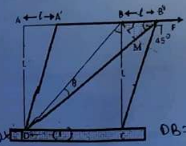
IMP

1. Relation among the elastic constants, η, Y and σ:

Consider a cube of side L fixed at the bottom face and tangential force F is acting on the top face. Under this condition, the cube is sheared through a small angle θ. In the figure, only the front face of the cube is shown. The face $ABCD$ is sheared to $A'B'CD$ by an angle θ.

Let T be a tension per unit length of the diagonal, α be the linear strain per unit tension [stress], β- be the lateral strain per unit tension [stress], BM - be the perpendicular drawn from B to the new diagonal DB' in the deformed position and $AA' = BB' = l$

Since l is very small, the angle θ is very small and hence the angle at B' is 45^0



DB = $L\sqrt{2}$

From the right angle triangle BMB',

$$MB' = BB' \cos 45 = \frac{l}{\sqrt{2}} \text{------------} (1)$$

The extension MB' is due to the combined effect of tensile stress along the diagonal DB and the compressive stress along the diagonal AC.

Extension due to tensile stress $= \alpha \times DB \times T$

$= \alpha L\sqrt{2} \times T$ $= \alpha\, DB \times T$

Extension due to compression stress $= \beta T \times DB \times T$ $= \beta\, DB \times T$

$= \beta L\sqrt{2} \times T$ net ext $=$ $MB' = \alpha L\sqrt{2} + \beta L\sqrt{2}\, T$

Net extension, $MB' = \alpha TL\sqrt{2} + \beta TL\sqrt{2} = \boxed{T(\alpha+\beta)L\sqrt{2}}$ ---- (2) $= T(\alpha+\beta)L\sqrt{2}$

From equations (1) and (2) we get, $T(\alpha+\beta)L\sqrt{2} = \frac{l}{\sqrt{2}}$ $\frac{1}{2} \times \frac{l}{L} = \frac{\theta}{2}$

That is, $$T(\alpha+\beta) = \frac{1}{2} \times \frac{l}{L} = \frac{\theta}{2}$$

That is, $$\frac{T}{\theta} = \frac{1}{2(\alpha+\beta)} = \frac{1}{2\alpha(1+\beta/\alpha)} \text{------------} (3)$$

But, $\frac{T}{\theta} = \frac{\text{tangential stress}}{\text{shearing strain}} = \eta$, $\frac{\beta}{\alpha} = \frac{\text{lateral strain}}{\text{linear strain}} = \sigma$ and $\frac{1}{\alpha} = \frac{\text{normal stress}}{\text{linear strain}} = Y$

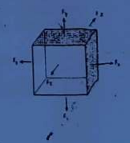
Therefore, $$\eta = \frac{Y}{2(1+\sigma)} \text{----} (4)$$

Equation (4) relates rigidity modulus with Young's modulus and Poisson's ratio.

IMP

2. Relation among K, Y and σ:

Consider a cube of side L subjected to three pairs of forces $F_x, -F_x, F_y, -F_y$, and $F_z, -F_z$ acting along X, Y and Z axes respectively as shown in the figure. Let α and β be the linear and lateral strain per unit tension.



The side of the cube parallel to X-axis is increased by αLF_x due to a pair of forces F_x and $-F_x$. At the same time it is decreased by βLF_y due to a pair of forces F_y and $-F_y$ and also decreased by βLF_z due to F_z and $-F_z$

$$L_x = L + \alpha LF_x - \beta LF_y - \beta LF_z$$

That is, $$L_x = L\left[1+\alpha F_x - \beta F_y - \beta F_z\right]$$

Similarly the other two sides become,

$$L_y = L\left[1+\alpha F_y - \beta F_z - \beta F_x\right] \text{ and } L_z = L\left[1+\alpha F_z - \beta F_x - \beta F_y\right]$$

Therefore the new volume of the cube is given by;

$$V = L_x L_y L_z = L^3\left[1+\alpha F_x - \beta F_y - \beta F_z\right]\left[1+\alpha F_y - \beta F_z - \beta F_x\right]\left[1+\alpha F_z - \beta F_x - \beta F_y\right]$$

$$V = V_0\left[1+\alpha F_y - \beta F_z - \beta F_x + \alpha F_x - \beta F_y - \beta F_z\right]\left[1+\alpha F_z - \beta F_x - \beta F_y\right]$$

Since α and β are small the terms containing α^2, β^2 and $\alpha\beta$ are neglected. = 0

$$V = V_0\left[1+\alpha F_y - \beta F_z - \beta F_x + \alpha F_x - \beta F_y - \beta F_z + \alpha F_z - \beta F_x - \beta F_y\right]$$

$$V = V_0\left[1+\alpha\left(F_x + F_y + F_z\right) - 2\beta\left(F_x + F_y + F_z\right)\right]$$

If $$F_x = F_y = F_z = F$$

Then, $$V = V_0\left[1+3\alpha F - 6\beta F\right]$$

Increase in volume, $$v = V - V_0 = V_0\left[1+3\alpha F - 6\beta F\right] - V_0$$

$$v = V_0\left[\alpha - 2\beta\right]\times 3F$$

Therefore, $$v/V_0 = 3F(\alpha - 2\beta)$$

Therefore, $$\frac{F}{v/V_0} = \frac{1}{3(\alpha - 2\beta)} = \frac{1/\alpha}{3(1-2\beta/\alpha)} = \frac{1/\alpha}{\left(1 - \frac{2\beta}{\alpha}\right)} = \frac{Y}{3(1-2\sigma)}$$

That is, $$K = \frac{Y}{3(1-2\sigma)}$$

IMP

3. Relation among η, Y and K:

We have, Rigidity modulus, $\eta = \dfrac{Y}{2(1+\sigma)} \Rightarrow 2(1+\sigma) = \dfrac{Y}{\eta}$ ------------ (a)

And, Bulk modulus, $K = \dfrac{Y}{3(1-2\sigma)} \Rightarrow (1-2\sigma) = \dfrac{Y}{3K}$ ------------ (b)

Adding equations (a) and (b) we get, $3 = \dfrac{Y}{\eta} + \dfrac{Y}{3K}$

That is, $\dfrac{3}{Y} = \dfrac{1}{\eta} + \dfrac{1}{3K}$ ----(c)

This is also written as,

$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K} \text{ ----}(d)$$

Limiting values of Poisson's ratio:

From equations (a) and (b) we get, $2\eta(1+\sigma) = 3K(1-2\sigma)$ ------------ (A)

Here, the values of η and K are positive.

1. For σ to be **positive**, both sides of equation (A) must be positive.

That is, $(1-2\sigma) > 0$ or $\sigma < 1/2$

2. For σ to be **negative** RHS of equation (A) becomes positive and hence LHS must also be positive

That is, $(1+\sigma) > 0$ i.e. $\sigma > -1$

Thus the limiting values of Poisson's ratio are given by;

$$\left[-1 < \sigma < 1/2\right]$$

Work done in deforming a body

[Energy stored per unit volume in a deformed body]

1. Energy stored inTwisting a wire –[in producing shear strain]

Let a wire be initially twisted by an angle θ by applying a couple C. The work done to twist it further by a small angle dθ is given by; $dW = C\,d\theta$

If 'c' is couple per unit twist, then the total couple for producing a twist is $c \times \theta$

Total work done to produce a twist θ; $W = \int_0^\theta C d\theta = \int_0^\theta c\theta\, d\theta = c \times \frac{\theta^2}{2} = \frac{1}{2} c\theta \times \theta$

Therefore,

$$W = C\theta/2$$

This work done is stored up in the wire in the form of potential energy.

Potential energy per unit volume, $E = \frac{W}{V} = \frac{1}{2}\frac{C\theta}{V} = \frac{1}{2}\frac{FL}{AL} \times \theta = \frac{1}{2} \times \frac{F}{A} \times \theta$

That is,

$$E = \frac{1}{2} \times \text{Tangential Stress} \times \text{Sheer Strain}$$

2. Work done in stretching a wire - in producing extension strain:

A wire of length L is subjected to a deforming force F to produce in it an extension l. The work done to produce further extension of dl is given by;

$$dW = F\,dl = \left(\frac{YA}{L}\right) l\, dl$$

Since the Young's modulus, $Y = \frac{F/A}{l/L} \Rightarrow F = \left(\frac{YA}{L}\right) l$

Therefore work done to produce extension l is given by;

$$W = \int_0^l F dl = \int_0^l \left(\frac{YA}{L}\right) l\, dl = \frac{1}{2} \times \left(\frac{YA}{L}\right) \times l^2 = \frac{1}{2} \times \left(\frac{YA}{L}\right) l \times l$$

That is,

$$W = \frac{1}{2} F \times l$$

Work done per unit volume of the wire; $E = \frac{W}{V} = \frac{1}{2} \times \frac{F \times l}{A \times L} = \frac{1}{2}\left(\frac{F}{A}\right) \times \left(\frac{l}{L}\right)$

That is,

$$E = \frac{1}{2} \times \text{Normal Stress} \times \text{Longitudinal Strain}$$

3. Work done in producing volume strain:

Let V be the volume of a gas at a pressure P. The work done to change the volume further by dv is given by;

$$dW = P\,dv$$

Since bulk modulus, $K = P\left(\frac{V}{v}\right) \Rightarrow P = K\frac{v}{V}$

Total work done to change the volume by 'v' is given by;

$$W = \int_0^v P dv = \int_0^v \frac{Kv}{V} dv = \frac{1}{2}\left(\frac{K}{V}\right) v^2 = \frac{1}{2} \times P \times v$$

Energy stored per unit volume [= work done per unit volume] in the gas is given by,

$$E = \frac{W}{V} = \frac{1}{2} \times P \times \left(\frac{v}{V}\right)$$

$$E = \frac{1}{2} \times \text{Normal Stress} \times \text{Volume Strain}$$

Note: In all the cases we get the same expression for the stored energy.

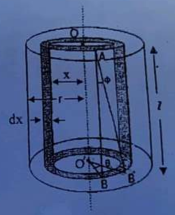
IMP

Twisting couple on a cylindrical wire

Consider a cylindrical wire of radius r and length l. The wire is clamped at one end and twisted at the other end by applying twisting couple. Due to the elastic nature, a restoring couple is set into action in the material of the wire to resist the twisting couple. At equilibrium, these two couples balance each other.

The cylinder is assumed to be made up of a large number of co-axial hollow cylinders. Let us consider one such hallow cylinder of radius x thickness dx and length l. Imagine a line ~~AB lying~~ on the cylindrical shell and parallel to its axis. When the wire is twisted at the bottom end by an angle θ, the line AB sheers to the position AB' by an angle ϕ.

At the bottom, the angle of twist θ is same for all shells but angle of sheer ϕ is zero on the axis and increases towards the rim and becomes highest at the rim.
From triangle ABB' and $O'BB'$ we get, $BB' = l\phi = x\theta$



Angle of sheer, $\phi = \frac{x}{l}\theta$ ---------- (1)

$$\text{Tangential stress} = \frac{\text{force}}{\text{cross section of shell}} = \frac{F}{2\pi x dx} \text{ ---------- (2)}$$

By the definition we get,

$$\eta = \frac{\text{tangential stress}}{\text{shearing strain}} \text{ ------------ (3)}$$

Substituting (1) & (2) in (3) we get,

$$\eta = \frac{F}{2\pi x dx} \times \frac{l}{x\theta}$$

Or,

$$F = \frac{2\pi\eta\theta}{l} x^2 dx$$

Moment of force on the shell about the axis of the cylinder;

$$C' = F \times x = \frac{2\pi\eta\theta}{l} x^3 dx$$

Therefore moment of the force on the entire wire;

$$C = \frac{\cancel{2}\pi\eta\theta}{l}\int_0^r x^3 dx = \frac{\cancel{2}\pi\eta\theta}{l} \times \frac{r^4}{\cancel{4}\,2}$$

$$C = \frac{\pi\eta r^4}{2l} \times \theta = c\theta \Rightarrow \boxed{C = \frac{\pi\eta r^4\theta}{2l}}$$

Where $c = \frac{\pi\eta r^4}{2l}$ is called **twisting couple per unit twist** or **torsional rigidity.**

Note:

If a hollow cylinder and a solid cylinder are of the same mass and dimensions then to twist the two cylinders through the same angle θ, a larger couple is required for the hallow cylinder

BASAVA

BASAVARAJ G M
BSc I SEM [PCM]
VVSC. Ballari

$\boxed{05}$ -- GRAVITATION

"The weakest force that controls the evolution and motion of the universe"

Newton's law of gravitation. Kepler's laws of planetary motion – explanation with out derivation. Elements of satellite motion and geo-stationary satellite. ------------ 3hours.

Introduction

Motion of planets has been a fascinating subject since ancient days. Greeks and Indians have made systematic study of the celestial objects. Ptolemy, a Greek astronomer suggested the "*geocentric theory*". In the 16th century, Copernicus proposed the "*heliocentric theory*". Tycho Brahe collected the data of positions of planets and their motions. In the 17th century, *Kepler* formulated three laws regarding the planetary motion. Isaac Newton reduced these three laws into a single law called *Newton's law of gravitation*. Galileo was the first to demonstrate, from the *leaning tower* of Pisa, that all bodies falling with the *same acceleration* towards the centre of the earth.

Newton's law of gravitation

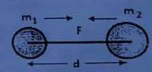
The force of attraction between any two bodies in the universe is called *gravitation* and the force of attraction between the earth and any other body is called *gravity*. This secret of nature was first disclosed by Newton inspired by the apple falling from the tree. He argued himself that, apples are falling from tree and stones thrown upwards are falling but why not the stars? This idea led him to discover the most basic force of nature in the year 1687.

The law states that **"every body in the universe attracts every other body with a force which is directly proportional to the product of masses of the two bodies and inversely proportional to the square of their separation"**

If two bodies of masses m_1 and m_2 are separated by a distance d then the gravitational force between them is given by,

$$\vec{F} \propto \frac{m_1 m_2}{d^2} \quad OR \quad \vec{F} = G\frac{m_1 m_2}{d^2} \quad \text{------- (1)}$$

$\vec{F} \propto \frac{m_1 m_2}{d}$



In equation (1) G is a constant called universal gravitational constant.

Definition of G:

In equation (1), if $m_1 = m_2 = 1kg$ and $d = 1m$, then $G = F$

Therefore, **the universal gravitational constant G is defined as the gravitational force of attraction between two bodies each of mass 1 kg separated by a distance of 1m**

Note:

1. **Dimensions of G**

   We have, $$[G] = \left[\frac{F \times d^2}{m_1 \times m_2}\right] = \left[\frac{MLT^{-2} \times L^2}{M \times M}\right] = \left[M^{-1}L^3T^{-2}\right]$$

   The dimensions of G are (-1, 3, -2) i.e. -1 in mass, 3 in length and -2 in time.
2. The value of G is $6.67 \times 10^{-11}\,Nm^2kg^{-2}$.
3. The numerical value of G depends on the system of units used.
4. Unlike electrostatic force, gravitation is independent of intervening medium.
5. Newton's law holds good for small & large masses, any separation.

6. Gravitational force is a **mutual force** of attraction that acts always along the straight line joining the centres of the two bodies and towards each other - **central force.**
7. The gravitation constitutes **action and reaction pair** - force on B exerted by A is always equal and opposite to the force on A by B.
8. Among the basic forces, the gravitational force is the weakest.
9. The gravitational force is a **conservative force** – work done by this force does not depend on the path followed but it depends only on the initial and final points.
10. Gravitation is independent of nature of masses, unaffected by any change in temperature, pressure and any other physical condition - totally the law is universal.
11. **The weight of a body is measured by the reaction which supports the body.**
12. **There is no shield against gravitation.**

Therefore, $F = W = mg$ ------------ (2)

From equations (1) and (2), $mg = GmM/R^2$

That is $g = GM/R^2$ ------------ (3)

Where m is the mass of the body and M is the mass of the earth. The above relation shows that g is independent of mass, material, shape and size of the body. This was experimentally verified by Galileo in 1590 by dropping bodies of different masses, sizes, shapes and materials from the **leaning tower of Pisa.**

13. The **mean density** of earth, $\rho = \frac{M}{V} = \frac{M}{4/3\pi R^3}$

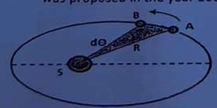
$M = 4/3\pi\rho R^3$ ------------ (4)

From equations (3) & (4), $g = 4/3\pi R\rho G$ ------------ (5)

6 **Kepler's laws of planetary motion;**

Kepler studied the data collected by Tyco Brahe for 22 years to evolve the three laws of planetary motion.

1. **The law of elliptic orbit:** Planets are revolving around the sun in different elliptical orbits with the sun being at one of the two foci. This was proposed n 1602.
2. The law of areas: The radius vector drawn from the sun to the planet sweeps equal areas in equal intervals of time. In other words, the areal velocity of a planet is constant. This law was proposed in the year 1605.



$$dA/dt = constant$$

$$dA = 1/2 \times SA \times AB = 1/2 \times R \times R\,d\theta = 1/2R^2 d\theta$$

Therefore, $\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt} = constant$

3. **The law of harmonics:** The **square** of the **planet's year** is proportional to the *cube* of the average distance between the planet and the sun. This law was proposed in the year 1618. If T is the planets year and R is the average distance of the planet and the sun, then according to the law, $T^2 \;\alpha\; R^3$

A

Elements of satellite motion

Satellite:

A satellite is an artificial object which has been intentionally placed into orbit. Such objects are sometimes called **artificial satellites** to distinguish them from natural satellites such as the Moon.

The Goddard Space Flight Center's lists 2,271 satellites currently in orbit. Russia has the most satellites currently in orbit, with 1,324 satellites, followed by the U.S. with 658. There are 71 space satellites launched by India so far, the first one in the year 1975.

We have never heard of instances of one satellite colliding with another in space in spite of so many satellites revolving around the earth. This is due to the reason that these satellites have fixed orbital elements that define the orbit of a satellite. These elements are generally called Keplerian elements. The study of orbital elements is important because these elements decide the position and its direction of motion at any moment.

Keplerian orbital elements:

1. **Epoch time** : It is the time at which the Keplerian elements were defined. It represents a snapshot of where and how fast the satellite was going.
2. **Orbital Inclination:** It is the angle between the equator and the orbit of the satellite when observed from the centre of the earth. It ranges from 0 to 180 degrees.
3. **Right Ascension of Ascending Node:** [RAAN]: Ascending node is a place where the satellite crosses the equator while going from Southern Hemisphere to Northern Hemisphere. Since earth rotates, a fixed object in space Aries is used as reference to define this angle. The angle from the centre of the earth between Aries [a small constellation fixed in space – first sign of zodiac] and the ascending node is called the right ascension of ascending node.
4. **Eccentricity:** It gives a measure of the shape of the orbit. Orbit is circular if eccentricity is zero. When eccentricity is close to 1 the orbit is very flat.
5. **Argument of Perigee**: Since the orbit is elliptical, the satellite is close to earth at one point than the other. The point where the earth is closest to the earth is called the perigee. The point where the satellite is the furthest from the earth is called apogee. The argument of perigee is the angle formed between the perigee and the ascending node. If the perigee would occur at the ascending node, the argument of perigee would be zero.
6. **Mean motion:** The mean motion tells about the velocity of the satellite. According to Kepler's law, the velocity of satellite is given by the formula, $v = \frac{GM}{r}$, where v is the velocity of the satellite, $M = 5.98 \times 10^{24}$ kg is the mass of the earth, $G = 6.662 \times 10^{-11}$ N m^2 kg^{-2} is gravitational constant and r is the distance of satellite from the centre of the earth. Thus by knowing the velocity of the satellite the distance of satellite from the earth can be calculated.
7. **Mean Anomaly:** It gives an idea of the exact position of satellite in its orbital path. It is defined with reference to perigee of the orbit. It ranges from 0 to 360 degrees. If the satellite were at the perigee, the mean anomaly would be 0.
8. **Drag [optional]:** Several factors affect the velocity of a satellite. If the satellite were in a low orbit, then the atmosphere would produce drag. This would cause the satellite to come closer to the earth thus speeding up the satellite. Another factor that can affect satellite orbits is the gravitational pull from stellar bodies such as the sun or the moon. These bodies could pull the satellite away from the earth causing it to slow down.

Geo-Stationary Satellite

An artificial satellite whose revolution around the earth is synchronized perfectly with the rotation of the earth is called geo-stationary satellite. The period of geo-centric satellite is same as that of diurnal rotation of the earth (24hours).This satellite is moving from west to east with its orbital plane passing through the centre of the earth. The reason is that the orbit will be stable only when the centripetal force is directed towards the centre of the earth. Though the satellite is revolving around the earth, it appears to be at rest for the observer on the earth and hence it is called stationary satellite.

If r is the radius of the orbit of the satellite then its period of revolution is given by,

$$T = 2\pi\sqrt{\frac{r}{g'}} \Rightarrow r = \frac{g'T^2}{4\pi^2} \text{------} (1)$$

The acceleration due to gravity g' at a distance r from the centre of the earth is given by,

$$g' = g\left(\frac{R}{r}\right)^2 \text{------} (2)$$

From equations (1) and (2) we get, $r = \left(gR^2/r^2\right)\times\left(T^2/4\pi^2\right)$

$$r^3 = \frac{gT^2R^2}{4\pi^2} \Rightarrow \boxed{r = \left[\frac{gT^2R^2}{4\pi^2}\right]^{1/3}}$$

$$r = \left[\frac{9.8\times(24\times3600)^2\times(6.4\times10^6)^2}{4\times9.87}\right]^{1/3}$$

$$r = 42.34\times10^6\,m$$

We have,

$$r = R + h$$

$$h = r - R = 42.34\times10^6 - 6.4\times10^6 = 35.94\times10^6\,m.$$

$$h = 35940\,km$$

This is the required height for the geo-stationary satellite from the earth's surface.

The orbital speed of synchronous satellite is given by;

$$v_0 = 2\pi r/T = \frac{2\pi\times42.34\times10^6}{24}$$

$$v_0 = 11086\,km\,h^{-1} = \frac{11086}{36000} = 3.079\,km\,s^{-1}$$

###

UDAYAKUMAR. G.M.

VISCOSITY - A

(1)

Flow of fluid is divided into two types namely 1) Stream line flow & 2) turbulant flow.

(2) **Stream line flow** :- If a fluid flows such that its velocity at a point is always the same in magnitude & direction, the fluid is said to have stream line flow. Stream line flow is also known as steady flow (or) orderely flow.

A stream line is a straight or curved path such that tangent to it at a point gives the direction of flow of liquid at that point.

Equation Of Continuity :-

Let a_1 & a_2 be the ~~above~~ areas of cross sections at A & B respectively. v_1 & v_2 be the velocities at A & B. Then, flow of fluid will be stream line if $\boxed{a_1 v_1 = a_2 v_2}$ This is the eqn of continuity for steady flow. From the above eqn, it is clear that

$$v \propto \frac{1}{a}$$

It has been observed that a liquid can possess streamlined motion only its velocity is less than a limiting velocity called the ~~initial~~ critical velocity

Turbulant Flow:- A fluid is said to be at turbulant if its velocities not same at a given point w.r.t. time.

When the velocity of the flow exceeds a certain limiting values for the liquid, the orderly motion in streamline motion of the fluid breaks down

B.SHARAN

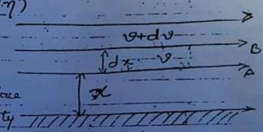
B.SHAR

In turbulant flow, paths of the particles changes continuously. Portion of the fluid acquire a rotation motion forming eddies or vortices.

Viscosity :- The property of a liquid by virtue of which it tends to resist relative motion b/w different layers of it is called viscosity.

Co-efficient Of Viscosity :- (η)

Let v & $v+dv$ be the velocities of two layers at distances x & $x+dx$ from two layer in contact with the surface



$dv \rightarrow$ is the change in velocity

$dx \rightarrow$ is the separation b/w the two levels.

$\frac{dv}{dx} \rightarrow$ is the velocity $\boxed{\text{gradient}}$

Experimentally it has been found that the tangential dragging force F acting on any layer of liquid in motion is i) direction proportional to A, ii) directly proportional to the velocity gradient dv/dx

Then $F \propto A \frac{dv}{dx}$

$$F = -\eta \cdot A \cdot \frac{dv}{dx}$$

$\eta \rightarrow$ is called co-efficient of viscosity.

If $A = 1$, $\frac{dv}{dx} = 1$ then

$$F = \eta$$

Thus co-efficient of viscosity is defined as the tangential force per unit area required to maintain unit velocity gradient.

Unit of η :- S.I unit of η is $Nm^{-2} \cdot s$.

CGS unit of η is Poise

$1 \text{ poise} = 10^{-1} Nm^{-2}$

SHARAN

Dimension of η :- Consider $\eta = \dfrac{F}{A \cdot \frac{dv}{dx}}$

$$\eta = \frac{F \cdot dx}{A \cdot dv}$$

$$= \frac{MLT^{-2}\,L}{L^2\,LT^{-1}}$$

$$[\eta] = [M^1 L^{-1} T^{-1}]$$

Variation of viscosity with temparature :-

The viscosity of liquid decreases with increase in temparature. No definite relation has been found to exist b/w viscosity & temp, but approximate ~~the viscosity & temperature~~ emperical relations are suggested.

One such relation is : $\boxed{\log \eta = a + b/T}$

Where 'a' & 'b' are constants 'η' is the co-efficient of viscosity at absolute temparature T.

In case of gases, the viscosity of increases with increase in temparature. The variation of 'η' with temp. T is given by : $\boxed{\eta = a\eta_0 T^{1/2}}$

where 'a' is a constant, η_0 is the co-efficient of viscosity at 0°C & η is the co-efficient of viscosity at the absolute temp. T.

(1) Poiseulle's Equation For Flow Of Liquid Through A Tube :-

Rate of flow of liquid through a capillary tube depends upon :-

a) length of the capillary tube (l)

b) radius of the capillary tube (r)

c) pressure difference b/w the ends of the capillary tube (p) &

d) co-efficient of viscosity of the liquid (η)

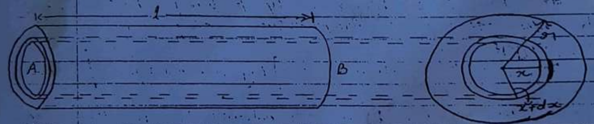
It is also assumed that:

i) The flow of liquid is stream line.

ii) The liquid in contact with the walls of the tube is at rest.

iii) The tube is kept horizontal.

iv) The pressure at any given cross section is constant & there is no radial flow of the liquid.



Consider a liquid flowing in a capillary tube AB of length 'l' & radius 'r'. The layer at the centre of the tube has maximum velocity & the velocity decreases from the axis to the walls of the tube.

→ Let a layer be at a distance 'x' from the axis. The other layer is at a distance $x+dx$ from the axis. The tangential force acting in the opposite direction to the direction of flow is given by:-

$$F = -\eta A \frac{du}{dx}$$

Here $A = 2\pi x l$

$$\therefore F = -\eta (2\pi x) l \frac{du}{dx} \longrightarrow (1)$$

The force at the end of the tube is

$$F = P \times \pi x^2 \longrightarrow (2)$$

where 'P' is the difference in pressure b/w the two ends of the capillary tube; For steady