Differential Calculus

Errors and Approximations

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Definition: Suppose u is a function of x and y i.e. $u = f(x, y)$.

If there is a change in the value of x from x to $(x+\delta x)$ and a change in the value of y from y to $(y+\delta y)$ (where δx and δy are small and may be positive or negative), then there will be a change in the value of u (say) from u to $u+\delta u$.

We may call this change in the value of x i.e. δx as 'increment in x' or 'error in x'. Similarly, δy may be called as 'increment in y' or 'error in y' and so δu is the 'increment in the value of u' or 'error in the value of u'.

Now we have $u = f(x, y)$. (i)

$$\therefore u+\delta u = f(x+\delta x, y+\delta y)$$

$$\Rightarrow \delta u = f(x+\delta x, y+\delta y) - f(x, y). \quad \text{(ii)}$$

Expanding $f(x+\delta x, y+\delta y)$ by Taylor's theorem on two variables

$$f(x+\delta x, y+\delta y) = f(x, y) + \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\delta x^2 + 2\delta x\delta y\frac{\partial^2 f}{\partial x\partial y} + \delta y^2\frac{\partial^2 f}{\partial y^2}\right).....$$

We know that, Taylor's series for a function of one variable is

$$f(x+h)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+..........$$

$$=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+...........$$

Also, Taylor's series for a function of two variables is

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)+\frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}\right)f(x,y)+......$$

Substituting the expansion of $f(x+\delta x,\ y+\delta y)$ in (ii), we obtain

$$\delta u=\left[f(x,y)+\left(\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y\right)+\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\delta x^2+2\delta x\delta y\frac{\partial^2 f}{\partial x\partial y}+\delta y^2\frac{\partial^2 f}{\partial y^2}\right).....\right]-f(x,\ y).$$

As δx and δy are supposed to be very-very small, therefore their squares and higher powers can be neglected.

Thus $\delta u=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y$.

Replacing $\delta x,\ \delta y,\ \delta u$ by $dx,\ dy,\ dz$ respectively, we have

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \quad\Rightarrow \delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y \qquad [\because u=f(x,\ y)].$$

This formula is used in calculating the effect of small errors or increments in measured quantities and is useful in correcting the effect of small errors.

Remarks: If $u=f(x,\ y,\ z,......)$ then, $\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\frac{\partial u}{\partial z}\delta z+..........$

Percentage error:

Definition: $\frac{\delta x}{x}\times 100$ is called percentage error in the value of x, where δx is the change or actual error in the value of x.

Similarly, $\frac{\delta y}{y}\times 100$ is called the percentage error in y, and

$\frac{\delta u}{u}\times 100$ is called the percentage error in u.

where δy and δu are actual errors in y and u respectively.

Relative error: If δx is the error in x, then relative error $= \dfrac{\delta x}{x}$.

Now let us solve some problems related to errors and approximations:

Q.No.1.: Find the percentage error in the area of an ellipse, when an error +1 percent is made in semi-major axis and −1 is made in measuring the semi-minor axis.

Sol.: Since, the area A of an ellipse is given by the relation $A = \pi ab$, where a, b are its semi-major and semi-minor axis.

Here error in a and b are given, therefore we will treat a and b variables. Since, when a and b are treated as variables $\Rightarrow$ A is also a variable.

Taking differentials, we get

$$d(A) = d(\pi ab) = \pi d(ab) = \pi[d(a).b + a.d(b)]$$

$$\Rightarrow dA = \pi b\,.\,da + \pi a\,.\,db$$

$$\Rightarrow \frac{dA}{A} = \frac{\pi b}{A}.da + \frac{\pi a}{A}.db \;= \frac{da}{a} + \frac{db}{b} \qquad \left[\because a = \frac{A}{\pi b},\ \ b = \frac{A}{\pi a}\right]$$

$$\therefore \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 \;= \text{percentage error in a} + \text{percentage error in b}$$

$$= (+1) + (-1) = 0.$$

Hence percentage error in the area of an ellipse is zero.

2nd method:

Since $A = \pi ab$

Taking logarithms on both sides, we get

$$\log A = \log(\pi ab) = \log\pi + \log a + \log b.$$

Now taking differentials, we get

$$\frac{1}{A}.dA = 0 + \frac{1}{a}.da + \frac{1}{b}.db$$

$$\Rightarrow \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 = (+1) + (-1) = 0 \text{ .Ans.}$$

Q.No.2.: If an error committed in measuring the side of a square be 2%. Find the error in calculating the area.

Sol.: Since the area A of a square is $A = x^2$, where x is the side of a square.

Taking log on both sides, we get

$$\log A = \log x^2 = 2\log x .$$

Taking differentials on both sides, we get

$$\frac{1}{A}dA = 2.\frac{1}{x}dx$$

$$\therefore \frac{dA}{A}\times 100 = 2\left[\frac{dx}{x}\times 100\right]$$

$\Rightarrow$ %age error in $A = 2(\%\text{ age error in } x) = 2\times 2 = 4\%$ .Ans.

Q.No.3.: Find the % error in the area of an ellipse, when an error of +1% is made in measuring the semi-major and semi-minor axis.

Sol.: Since $A = \pi ab$, where a, b are its semi-major and semi-minor axis.

Taking logarithms on both sides, we get

$$\log A = \log(\pi ab) = \log\pi + \log a + \log b .$$

Now taking differentials, we get

$$\frac{1}{A}.dA = 0 + \frac{1}{a}.da + \frac{1}{b}.db$$

$$\Rightarrow \frac{dA}{A}\times 100 = \frac{da}{a}\times 100 + \frac{db}{b}\times 100 = (+1) + (+1) = 2\% \text{ .Ans.}$$

Q.No.4.: The time of swing t, of a pendulum, of length ℓ, under certain conditions is given by $t = 2\pi\sqrt{\frac{\ell}{g'}}$, where $g' = g\left(\frac{r}{r+h}\right)^2$. Find the %age error in t due to the errors of p% in h and q% in ℓ.

Sol.: Given $t = 2\pi\sqrt{\frac{\ell}{g'}}$

Taking log on both sides, we get $\log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g'$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g\left(\frac{r}{r+h}\right)^2$$

$$\Rightarrow \log t = \log 2\pi + \frac{1}{2}\log \ell - \frac{1}{2}\log g - \log r + \log(r+h)$$

Taking differentials, we get

$$\Rightarrow \frac{dt}{t} = 0 + \frac{1}{2}\frac{d\ell}{\ell} - 0 - 0 + \frac{1}{r+h}dh$$

$$\Rightarrow \frac{dt}{t}\times 100 = \frac{1}{2}\left(\frac{d\ell}{\ell}\times 100\right) + \left(\frac{dh}{r+h}\times 100\right)$$

$$\Rightarrow \%\text{age error in } t = \frac{1}{2}q + \frac{1}{r+h}\left(\frac{dh}{h}\times 100\right)h = \left(\frac{1}{2}q + \frac{ph}{r+h}\right)\ \% \text{ .Ans.}$$

Q.No.5.: Using the concept of small errors, find an approximate value of $f(10.02,\ 40.05,\ 29.97)$ where $f(x,\ y,\ z) = x\,y\,z$.

or

Let $f(10.02,\ 40.05,\ 29.97)$ where $f(x,\ y,\ z) = x\,y\,z$.

Using the concept of small errors, find relative error, actual error and approximate value of f.

Sol.: Let $x = 10,\ \delta x = 0.02,\ y = 40,\ \ \delta y = 0.05,\ z = 30,\ \delta z = -0.03$.

Now $f(x, y, z) = xyz$.

Taking log on both sides, we get

$\log f = \log x + \log y + \log z$.

Taking differentials, we get

$$\frac{\delta f}{f} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z} = \frac{0.02}{10} + \frac{0.05}{40} + \frac{-0.03}{30} = 0.002 + .00125 + (-0.001) = 0.00225$$

which is the relative error in f.

$\delta f = 0.00225\,f$, but $f = 10\times 40\times 30 = 12000$. $\delta f = 12000\times 0.00225 = 27$.

$\therefore$ Approximate value of $f = f + \delta f = 12000 + 27 = 12027$.**Ans.**

Actual value = 12026.991

Q.No.6.: If $f(x, y, z) = x^{\ell} y^{m} z^{n}$ and errors of p %, q % and r % are made in measuring x, y, z respectively. Find the error in $f(x, y, z)$.

Sol.: Given $f(x, y, z) = x^{\ell} y^{m} z^{n}$.

Taking log on both sides, we get

$\log f = \ell \log x + m \log y + n \log z$.

Taking differentials, we get

$$\frac{\delta f}{f} = \ell \frac{\delta x}{x} + m\frac{\delta y}{y} + n\frac{\delta z}{z}$$

$$\Rightarrow \frac{\delta f}{f}\times 100 = \ell\left(\frac{\delta x}{x}\times 100\right) + m\left(\frac{\delta y}{y}\times 100\right) + n\left(\frac{\delta z}{z}\times 100\right)$$

Hence %age error in $f(x, y, z) = (\ell p + qm + r\,n)\%$ **.** Ans**.**

Q.No.7.: The area S of a triangle is calculated from the length of sides a, b, and c. If a be diminished and b be increased by small amounts x, prove that the consequent change in area is given by $\dfrac{\delta S}{S} = \dfrac{2(a-b)x}{c^2-(a-b)^2}$**.**

Sol.: Hero's Formula: A formula connecting the area of a Δ with its sides

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $s = \dfrac{a+b+c}{2}$ is semi-parameter.

$$\therefore \text{Area } S = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right).\left(\frac{b+c-a}{2}\right).\left(\frac{c+a-b}{2}\right).\left(\frac{a+b-c}{2}\right)}.$$

Taking log on both sides, we get

$$\log S = \frac{1}{2}\left[\left(\log(a+b+c) + \log(b+c-a) + \log(c+a-b) + \log(a+b-c)\right) - 4\log 2\right]$$

Taking differentials, we get

$$\frac{\delta S}{S} = \frac{1}{2}\left[\frac{\delta a + \delta b}{(a+b+c)} + \frac{\delta b - \delta a}{(b+c-a)} + \frac{\delta a - \delta b}{(c+a-b)} + \frac{\delta a + \delta b}{a+b-c}\right]$$

$$= \frac{1}{2}\left[0 + \frac{2x}{(b+c-a)} + \frac{(-2x)}{(c+a-b)} + 0\right]$$

$$= \frac{x}{(b+c-a)} - \frac{x}{(a+c-b)} = x\left[\frac{1}{c-(a-b)} - \frac{1}{c+(a-b)}\right]$$

$$= x\left[\frac{c+a-b-c+a-b}{c^2-(a-b)^2}\right] = \frac{2(a-b)x}{c^2-(a-b)^2}\text{. Ans.}$$

Q.No.8.: The edge of a cube is measured with a positive error of 0.05 cm. Find the relative error in the computed volume, when the edge is found to be 7.5 cm. Also find percentage error in the computed volume.

Sol.: Let x be the edge of the cube.

$\therefore$ volume V of the cube $= x^3$. (i)

Taking log on both sides, we get

$\log V = \log x^3 = 3\log x$. (ii)

Taking differentials, we get

$\frac{1}{V}.dV = 3.\frac{1}{x}dx$. (iii)

$\therefore$ Error in the computed volume $= dV = \frac{3V}{x}dx = \frac{3x^3}{x}dx = 3x^2dx$

$\Rightarrow dV = 3\times(7.5)^2\times(0.05) = 8.44$ cubic cm**.**

Thus, relative error in the computed volume $= \frac{dV}{V} = \frac{3dx}{x} = \frac{3\times 0.05}{7.5} = 0.02$. Ans.

Now, percentage error in the computed volume $= \frac{dV}{V}\times 100 = 0.02\times 100 = 2\%$ **.** Ans**.**

Q.No.9.; The diameter and altitude of a can in the shape of right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Sol.: Let the diameter and altitude of the can be denoted by D and H respectively.

Then radius $= \frac{D}{2}$.

(i) The volume V of the can is given by $V = \pi r^2 h = \frac{\pi}{4}D^2H$ $[= f(D, H)]$

$$\therefore dV = \frac{\partial V}{\partial D}dD + \frac{\partial V}{\partial H}dH = \frac{\pi}{4}\left[2DHdD + D^2dH\right]$$

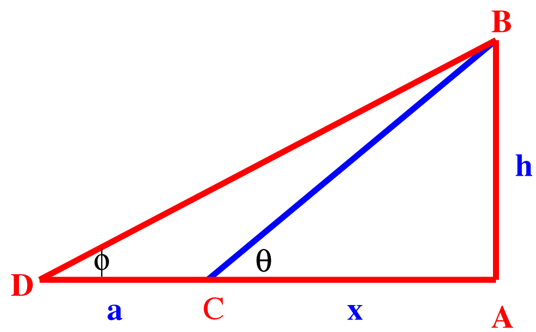
$$= \frac{\pi}{4}\left[2\times 4\times 6\times 0.1 + 4^2\times 0.1\right] = \frac{\pi}{4}(6.4) = 1.6\pi \text{ cubic cm. Ans.}$$

(ii) The lateral surface S of the can is given by $S = 2\pi rh = \pi DH$ $[= f(D, H)]$

$\therefore dS = \frac{\partial S}{\partial D}dD + \frac{\partial S}{\partial H}dH = \pi[HdD + DdH] = \pi[6\times 0.1 + 4\times 0.1] = \pi$ sq. cm. Ans.

Q.No.10.: The height of a tower is determined by observing the elevation θ and ϕ of its summit from two points in a direct line with the foot of the tower and at a distance 'a' apart. Show that the error in the calculated height due to small errors $d\theta$ and $d\phi$ is approximately $a\left(\sin^2\theta d\phi - \sin^2\phi d\theta\right)\mathrm{cosec}^2(\theta-\phi)$.

Sol.: Let h be the height of the tower AB and C and D, the two points of observation so that



$CD = a$, $\angle ACB = \theta$,

$\angle ADB = \phi$. Let $AC = x$

From right angle ΔBAC, $x = h\cot\theta$ (i)

From right angle ΔBAD, $x + a = h\cot\phi$ (ii)

Subtracting (i) from (ii), we get

$$a = h(\cot\phi - \cot\theta) = h\left(\frac{\cos\phi}{\sin\phi} - \frac{\cos\theta}{\sin\theta}\right) = h\left[\frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\sin\theta\sin\phi}\right] = \frac{h\sin(\theta-\phi)}{\sin\theta\sin\phi}$$

$$\Rightarrow h = \frac{a\sin\theta\sin\phi}{\sin(\theta-\phi)}[= f(\theta,\phi)] \qquad \text{(iii)}$$

Taking log on both sides, we get $\log h = \log a + \log\sin\theta + \log\sin\phi - \log\sin(\theta-\phi)$

Taking differentials, we get $\frac{dh}{h} = 0 + \frac{\cos\theta}{\sin\theta}d\theta + \frac{\cos\phi}{\sin\phi}d\phi - \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}(d\theta - d\phi)$

$$\Rightarrow \frac{dh}{h} = \left[\frac{\cos\theta}{\sin\theta} - \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}\right]d\theta + \left[\frac{\cos\phi}{\sin\phi} + \frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}\right]d\phi$$

$$= \frac{\sin(\theta-\phi)\cos\theta - \cos(\theta-\phi)\sin\theta}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin(\theta-\phi)\cos\phi + \cos(\theta-\phi)\sin\phi}{\sin\phi\sin(\theta-\phi)}d\phi$$

$$= \frac{\sin[(\theta-\phi)-\theta]}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin[(\theta-\phi)+\phi]}{\sin\phi\sin(\theta-\phi)}d\phi$$

$$= \frac{\sin(-\phi)}{\sin\theta\sin(\theta-\phi)}d\theta + \frac{\sin\theta}{\sin\phi\sin(\theta-\phi)}d\phi = \frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} \quad [\because \sin(-\phi) = -\sin\phi]$$

$$\therefore dh = h.\frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} = \frac{a\sin\theta\sin\phi}{\sin(\theta-\phi)}.\frac{\sin^2\theta d\phi - \sin^2\phi d\theta}{\sin\theta\sin\phi\sin(\theta-\phi)} \qquad \text{[using (iii)]}$$

$$= a\left(\sin^2\theta d\phi - \sin^2\phi d\theta\right)\operatorname{cosec}^2(\theta - \phi).$$

Hence prove.

Q.No.11.: In some torsion experiment an error of 0.5%, was made in measuring the diameter x. Calculate the corresponding %age error in the stress f, where

$$T + \frac{\pi}{16} f\, x^3 = 0.$$

Sol.: Here $T + \frac{\pi}{16} f\, x^3 = 0 \Rightarrow f = \frac{-16}{11}\frac{T}{x^3}$ $\left[\begin{array}{l}\text{Here only f and x will be treated as} \\ \text{variables as error occurs in these.}\end{array}\right]$

Taking log on both sides, we get

$$\log f = \log\left(\frac{-16}{11}\right) + \log T - 3\log x$$

Taking differentials on both sides, we get

$$\frac{df}{f} = 0 + 0 - 3\frac{dx}{x} \Rightarrow \frac{df}{f} \times 100 = -3\frac{dx}{x} \times 100 = -3 \times (0.5) = -1.5\%\ \textbf{.Ans.}$$

Q.No.12.: In an experiment carried out to find the value of g error of 0.5% and 1% are possible in the value of t and ℓ respectively. Show that the maximum error in the calculated value of g could not be more than 2%.

Sol.: Time period of pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$ (i)

Where T = time period, ℓ = length of pendulum, g = acceleration due to gravity.

On squaring (i), we get

$$T^2 = 4\pi^2\frac{\ell}{g} \Rightarrow g = 4\pi^2\frac{\ell}{T^2}. \qquad \text{(ii)}$$

On differentiating, we get

$$dg = 4\pi^2\left[\frac{1}{T^2}d\ell - \frac{2}{T^2}\ell dT\right] = \frac{4\pi^2}{T^2}\ell\left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right]$$

$$dg = g\left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right]$$

$$\Rightarrow \frac{dg}{g} = \left[\frac{d\ell}{\ell} - 2\frac{dT}{T}\right] \qquad \text{(iii)}$$

Given, %age error in length, $\frac{d\ell}{\ell}\times 100 = 1\%$ (iv)

%age error in Time period, $\frac{dT}{T}\times 100 = .5\%$ (v)

Putting the values from (iv) and (v) in (iii), we get

$$\frac{dg}{g}\times 100 = 1 - 2(-.5) = (1+1)\% = 2\%.$$

Therefore maximum error in the value of g = 2%. Ans.

Q.No.13.: In measuring the value of angle θ, an error of 0.1^0 was made. Find the corresponding error in the value of the sine of the angle**.**

Sol.: Given $d\theta = 0.1^0 = \frac{0.1\times\pi}{180}\text{Radian} = 0.00175\text{Radian}$

$dy = ?$ where $y = \sin\theta$

$\therefore dy = \cos\theta d\theta = 0.00175\cos\theta$ **.**Ans**.**

Q.No.14.: If H. .P. required to propel a steamer is proportional to the cube of its velocity and square of its length, prove that a 2% increase in velocity and 3% increase in length will require approximately a 12% increase in H. P**.**

Sol.: Given $P \propto v^3\ell^2 \Rightarrow P = kv^3\ell^2$, where k is the constant of proportionality.

Taking log on both sides, we get

$\log P = \log k + 3\log v + 2\log \ell$.

Taking differentials, we get

$$\frac{dP}{P} = 0 + 3\frac{dv}{v} + 2\frac{d\ell}{\ell}$$

$$\Rightarrow \frac{dP}{P}\times 100 = 3\left(\frac{dv}{v}\times 100\right) + 2\left(\frac{d\ell}{\ell}\times 100\right) = (3\times 2) + (2\times 3) = 6 + 6 = 12\ \%$$ **.**Ans**.**

Q.No.15.: The indicated horse power I of an engine is calculated from the formula $I = \frac{PLAN}{33000}$, where $A = \frac{\pi}{4}d^2$. Assuming that error of r % may have been made in measuring P, L, N and d. Find the greatest possible error in I**.**

Sol.: Given $I = \frac{PLAN}{33000}$.

Taking log on both sides, we get

$\log I = \log P + \log L + \log A + \log N - \log 33000$

Taking differentials, we get

$$\Rightarrow \frac{dI}{I} = \frac{dP}{P} + \frac{dL}{L} + 2.\frac{d(d)}{dN} + \frac{dN}{N} - 0 \qquad \left[\because \log A = \log\left(\frac{\pi}{4}d^2\right) = \log\frac{\pi}{4} + \log d^2 = \log\frac{\pi}{4} + 2\log d\right]$$

$$\Rightarrow \frac{dI}{I}\times 100 = \frac{dP}{P}\times 100 + \frac{dL}{L}\times 100 + 2\frac{d(d)}{d}\times 100 + \frac{dN}{N}\times 100$$

$$= r + r + 2r + r = 5r\% \text{. Ans.}$$

Q.No.16.: The time period of a simple pendulum is given by $t = 2\pi.\sqrt{\frac{\ell}{g}}$.

Find the error in t due to error $\delta\ell$ and δg in ℓ and g. What is the max. %age error in t if there is an error of 1% in ℓ and g**.**

Sol.: Given $t = 2\pi.\sqrt{\frac{\ell}{g}}$.

Taking log on both sides, we get

$$\log t = \log 2\pi + \frac{1}{2}\log\ell - \frac{1}{2}\log g .$$

Taking differentials, we get

$$\frac{dt}{t} = 0 + \frac{1}{2}\frac{d\ell}{\ell} - \frac{1}{2}\frac{dg}{g}$$

$$\Rightarrow \left(\frac{dt}{t}\times 100\right) = \frac{1}{2}\left(\frac{d\ell}{\ell}\times 100\right) - \frac{1}{2}\left(\frac{dg}{g}\times 100\right)$$

(i) $= \frac{1}{2}(+1) - \frac{1}{2}(+1) = 0\%$ (Not max)**.** Ans**.**

(ii) $= \frac{1}{2}(+1) - \frac{1}{2}(-1) = +1\%$ (Max)

(iii) $= \frac{1}{2}(-1) - \frac{1}{2}(+1) = -1\%$ (Max)**.** Ans**.**

(iv) $= \frac{1}{2}(-1) - \frac{1}{2}(-1) = 0\%$ (Not Max)

$\therefore$ Max. %age error in t $= \pm 1$ % . Ans**.**

Q.No.17.: The slope of a hanging rod of uniform strength is given by $y = A\exp.\left(\frac{w}{f}x\right)$, where y is the radius at any height x above a fixed point at A is constant. Find the change in y produced by small changes δw in w and δf in f. Show that the %age error in y is $\frac{wx}{f}$ times the difference in the %age errors in w and f.

Sol.: Given $y = Ae^{wx/f}$.

Taking log on both sides, we get $\log y = \log A + \frac{wx}{f}\log e$.

Taking differentials on both sides, we get

$$\frac{\delta y}{y} = 0 + x\left(\frac{f\delta w - w\delta f}{f^2}\right)$$

$$\Rightarrow \delta y = \frac{xy}{f}\left(\frac{f\delta w - w\delta f}{f}\right) = \frac{wxy}{f}\left(\frac{f\delta w - w\delta f}{wf}\right).$$ **Ans.**

Also $\frac{\delta y}{y}\times 100 = x\left[\frac{f\delta y - w\delta f}{f^2}\right]\times 100 = \frac{wx}{f}\left[\frac{f\delta w - w\delta f}{wf}\right]\times 100$

$$= \frac{wx}{f}\left[\frac{\delta w}{w}\times 100 - \frac{\delta f}{f}\times 100\right].$$ **Ans.**

Q.No.18.: If $R = \frac{E}{C}$, find the max. error and the %age error in R if $C = 20$ with a possible error of ± 0.1 and $E = 120$ with a possible error of ± 0.05.

Sol.: Given $R = \frac{E}{C}$.

Taking log on both sides, we get

$\log R = \log E - \log C$.

Taking differentials on both sides, we get

$$\frac{\delta R}{R} = \frac{\delta E}{E} - \frac{\delta C}{C} \qquad \Rightarrow \delta R = R\left(\frac{\delta E}{E} - \frac{\delta C}{C}\right) = 6\left[\frac{0.05}{120} - \left(\frac{-0.1}{20}\right)\right] = 0.0324.(\max)$$

which is the required max. error in R.

Now $\frac{\delta R}{R}\times 100 = \frac{\delta E}{E}\times 100 - \frac{\delta C}{C}\times 100$

$$\Rightarrow \frac{\delta R}{R}\times 100 = \frac{+0.05}{120}\times 100 - \frac{-0.1}{20}\times 100 = \frac{5}{120}+\frac{1}{2} = 0.54\% \text{ (max)}. \text{ Ans.}$$

which is the required max. % error in R.

Q.No.19.: In calculating the volume of a right circular cone, errors of +2% and minus one percent are made in the height and radius of the base respectively. Find the %age error in the volume. What is the percentage error in calculating value of the surface area of the cone ?

Sol.: Given %age error in height = 2% and %age error in radius $= -1\%$.

Since we know that the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$.

$$\log V = \log\frac{1}{3} + \log\pi + 2\log r + \log h$$

Taking differentials, we get

$$\frac{dV}{V} = \frac{2dr}{r} + \frac{dh}{h} \quad \Rightarrow \frac{dV}{V}\times 100 = \frac{2dr}{r}\times 100 + \frac{dh}{h}\times 100.$$

$\therefore$ %age error in volume $= 2(-1) + 2 = 0\%$. Ans.

Q.No.20.: In estimating the cost of a pile of bricks measured as 6 by 50 by 4 feet, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to 1foot^3 and bricks cost Rs. 100 per 1000, find the approximate error in the cost**.**

Sol.: Let ℓ, b and h feet be the length, breadth and height of the pile so that its volume $V = \ell \times b \times h$.

Taking log on both sides, we get

$$\log V = \log\ell + \log b + \log h.$$

Taking differentials, we get

$$\frac{\delta V}{V} = \frac{\delta\ell}{\ell} + \frac{\delta b}{b} + \frac{\delta h}{h}.$$

Since $V = 6\times 50\times 4 = 1200\,\text{ft}^3$ and $\frac{\delta\ell}{\ell}\times 100 = \frac{\delta b}{b}\times 100 = \frac{\delta h}{h}\times 100 = 1\%$.

$$\therefore \delta V = 1200\left(\frac{3}{100}\right) = 36\ \text{ft}^3.$$

Number of bricks in $\delta V = 36\times 12 = 432$.

Thus error in the cost $= 432 \times \dfrac{100}{1000} = \text{Rs. } 43.20$,

which is less to the brick seller.

Q.No.21.: Two quantities x_1 and x_2 are related to each other by the formula,

$x_2 = a(x_1)^n$, where a and n are constant quantities. Small errors of p % and q% are made in measuring a and n, show that the calculated value of x_2 for a given value of X' of x_1 will have a percentage error of $p + nq\log_e X'$.

Sol.: Given that %age error in a = p%. %age error in n = q%

Since given $x_2 = a(x_1)^n$.

Taking log on both sides, we get $\log x_2 = \log a + n \log x_1$.

Differentiating on both sides, we get $\dfrac{dx_2}{x_2} = \dfrac{da}{a} + dn \log x_1 + \dfrac{ndx_1}{x_1}$

$$\Rightarrow \frac{dx_2}{x_2} \times 100 = \frac{da}{a} \times 100 + \left(\frac{dn}{n} \times 100\right) n \log x_1 + \frac{ndx_1}{x_1} \times 100$$

$$\Rightarrow \frac{dx_2}{x_2} \times 100 = p + qn\log_e X' + 0 = p + nq\log_e X'.$$

Thus %age error in $x_2 = (p + nq\log_e X')\%$.

Q.No.22.: The acceleration of a piston is equal to $rw^2 \cos\theta + \dfrac{r^2w^2}{\ell}\cos 2\theta$. In measuring $\theta\ (= 30°)$ and w small error minus 1 percent each was detected. Prove that calculated value of acceleration is minus 1.5%. Take $4r = \ell$.

Sol.: Given acceleration of a piston $a = rw^2 \cos 2\theta + \dfrac{r^2w^2}{\ell}\cos 2\theta$. (i)

Putting $4r = \ell$ in (i), we get

$$a = rw^2 \cos\theta + \frac{rw^2}{4}\cos 2\theta$$

$$\delta a = 2rw\cos\theta\delta w + rw^2(-\sin\theta)\delta\theta + \frac{r\cos 2\theta}{4}2w\delta w + \frac{rw^2}{4}2(-\sin 2\theta)\delta\theta$$

$$\delta a = rw^2\left[2\cos\theta\left(\frac{\delta w}{w}\right) + (-\sin\theta)\left(\frac{\delta\theta}{\theta}\right)\theta + \frac{\cos 2\theta}{2}\left(\frac{\delta w}{w}\right) - \frac{\sin 2\theta}{2}\left(\frac{\delta\theta}{\theta}\right)\theta\right]$$

$$\frac{\delta a}{a}=\frac{rw^2\left[2\cos\theta\left(\frac{\delta w}{w}\right)-\sin\theta\left(\frac{\delta\theta}{\theta}\right)\theta+\frac{\cos 2\theta}{2}\left(\frac{\delta w}{w}\right)-\frac{\sin 2\theta}{2}\left(\frac{\delta\theta}{\theta}\right)\theta\right]}{rw^2\left[\cos\theta+\frac{\cos 2\theta}{4}\right]}$$

Dividing and multiplying by 100 and putting $\frac{\partial w}{w}=-1\%$, $\frac{\partial\theta}{\theta}=-1\%$, $\theta=30°$, we get

$$\frac{\delta a}{a}=-\left[\frac{\left[2(\cos 30°)1\%-\sin 30°\times 1\%\times 30°+\frac{\cos(2\times 30°)}{2}1\%-\frac{(\sin 60°)}{2}\times 1\%\times 30°\right]}{\cos 30°+\frac{\cos 60°}{4}}\right]$$

$$\therefore\frac{\delta a}{a}=-\left[\frac{\left(2\times\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{2}\times\frac{\pi}{6}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\times\frac{1}{2}\times\frac{\pi}{6}\right)}{\left(\frac{\sqrt{3}}{2}+\frac{1}{8}\right)}\right]\%=-1.5\%$$. Hence prove.

Q.No.23.: In a plane triangle ABC, if the sides and angles receive small variations, prove that $\delta a\cos C+\delta c\cos A=0$; b, B being constant.

Sol.: To prove: $\delta a\cos C+\delta c\cos A=0$**,** b and B as constants

Here using projection formula: $b=a\cos C+c\cos A$.

Differentiating, we get

$$db=da\cos C+a(-\sin C)dC+dc\cos A+c(-\sin A)dA$$

$$0=da\cos C+a(-\sin C)dC+dc\cos A-c\sin A dA$$

Now $A+B=\pi-C$

$$dA+dB=-dC\Rightarrow dA=-dC$$

$$0=da\cos C+a\sin C dA+dc\cos A-c\sin A dA$$

Now using sin formula: $\frac{\sin A}{a}=\frac{\sin C}{c}\Rightarrow c\sin A=a\sin C$

$$\therefore 0=da\cos C+dc\cos A+c\sin A dA-c\sin A dA$$

$$\Rightarrow da\cos C+dc\cos A=0.$$

Hence this proves the result.

Q.No.24.: The side a of a triangle ABC is calculated from b, c, A. Small errors db, dc, dA occur in the measured values of b,c, and A respectively. Prove that the error in

a is given by $da = \cos B dc + \cos C db + b \sin C dA$.

Sol.: To prove: $da = \cos B dc + \cos C db + b \sin C dA$.

Here using projection formula: $a = b \cos C + c \cos B$.

Differentiating, we get

$da = db \cos C + dc \cos B - b \sin C dC - c \sin B dB$.

Using sine formula: $c \sin B = b \sin C$

$\therefore da = db \cos C + dc \cos B - b \sin C(dB + dC)$

But $B + C = \pi - A \Rightarrow dB + dC = -dA$.

$\therefore da = db \cos C + dc \cos B + b \sin C dA$.

Hence this proves the result.

Q.No.25.: Given the formula $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$. If x and y are both in the error by r %, prove that z is also in the error of r %**.**

Sol.: Since $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$.

Taking differentials, we get $-z^{-2}dz = -x^{-2}dx - y^{-2}dy$

$$\Rightarrow -\frac{1}{z}\left(\frac{dz}{z}\times 100\right) = -\frac{1}{x}\left(\frac{dx}{x}\times 100\right) - \frac{1}{y}\left(\frac{dy}{y}\times 100\right) = -\frac{r}{x} - \frac{r}{y} = -r\left(\frac{1}{x} + \frac{1}{y}\right) = -\frac{r}{z}$$

$\Rightarrow \left(\frac{dz}{z}\times 100\right) = r\,\%$ **.** Ans**.**

Hence z is also in the error of r %.

Q.No.26.: The quantity Q of water flowing over a notch is given by

$Q = \frac{8}{15}\times 0.64 \times \sqrt{2g}\times (H)^{5/2}$, where H is the head at the notch. What is the % age error in Q caused by measuring H as 0.198 instead of 0.2?

Sol.: Since $Q = \frac{8}{15}\times 0.64 \times \sqrt{2g}\times (H)^{5/2}$.

Taking log on both sides, we get

$\log Q = \log\frac{8}{15} + \log 0.64 + \log\sqrt{2g} + \frac{5}{2}\log H$.

Taking differentials, we get $\dfrac{\delta Q}{Q}=0+0+0+\dfrac{5}{2}.\dfrac{\delta H}{H}$

$$\Rightarrow \frac{\delta Q}{Q}\times 100=\frac{5}{2}.\left(\frac{\delta H}{H}\times 100\right)=\frac{5}{2}.\left(\frac{0.002}{0.2}\times 100\right)=\frac{5}{2} \qquad [\because \delta H=0.2-0.198=0.002]$$

Hence %age error in $Q = 2.5\,\%$ **.** Ans**.**

Q.No.27.: A closed rectangular box with unequal sides a, b, c has its edges slightly altered in length by amount δa, δb and δc respectively. Show that its volume and surface area remain unchanged then $\dfrac{\delta a}{a^2(b-c)}=\dfrac{\delta b}{b^2(c-a)}=\dfrac{\delta c}{c^2(a-b)}$**.**

Sol.: Given volume of a rectangular box is $V = abc$.

Taking log on both sides, we get

$\log V=\log a+\log b+\log c$

Taking differentials, we get

$\dfrac{\delta V}{V}=\dfrac{\delta a}{a}+\dfrac{\delta b}{b}+\dfrac{\delta c}{c}$. Now since $\delta V=0\Rightarrow \dfrac{\delta a}{a}+\dfrac{\delta b}{b}+\dfrac{\delta c}{c}=0$

$$\Rightarrow \frac{\delta a}{a}=-\frac{\delta b}{b}-\frac{\delta c}{c}\Rightarrow \delta a=-\left(\frac{a}{b}\delta b+\frac{a}{c}\delta c\right)$$

Also $S=2(ab+bc+ca)$

Taking differentials, we get

$0=2(a\delta b+b\delta a+b\delta c+c\delta b+c\delta a+a\delta c)$ (since $\delta S=0$)

$$\Rightarrow (a+c)\delta b+(a+b)\delta c+(b+c)\delta a=0$$

$$\Rightarrow \delta a=-\frac{(a+c)\delta b+(a+b)\delta c}{(b+c)}\Rightarrow \frac{a}{b}\delta b+\frac{a}{c}\delta c=\frac{(a+c)\delta b+(a+b)\delta c}{(b+c)}$$

$$\Rightarrow \left[\frac{a}{b}-\frac{a+c}{b+c}\right]\delta b=\left[\frac{a+b}{b+c}-\frac{a}{c}\right]\delta c$$

$$\Rightarrow \frac{\delta b}{b^2(c-a)}=\frac{\delta c}{c^2(a-b)}.$$

Similarly $\dfrac{\delta a}{a^2(b-c)}=\dfrac{\delta c}{c^2(a-b)}$.

Hence $\dfrac{\delta a}{a^2(b-c)}=\dfrac{\delta b}{b^2(c-a)}=\dfrac{\delta c}{c^2(a-b)}$**.** Hence prove.

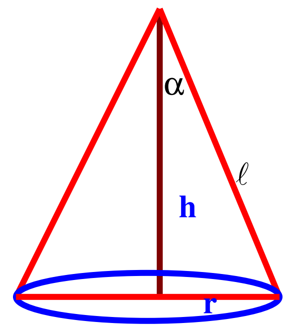
Q.No.28.: The height h and semi-vertical angle a of a cone are measured and from then A the total area of the surface of the cone including the base is calculated. If h and a are in error by small quantity δh and δa respectively. Find the corresponding error in the area. Show further that, if $\alpha = \frac{\pi}{6}$, an error of +1% in h will be approximately compensated by an error of -0.33^{o} in α.

Sol.: Base radius $r = h\tan\alpha$.

Slant height $\ell = h\sec\alpha$.

Area of base $= \pi r^2$.

Area of curved surface $= \pi r\ell$.



Total surface area $A = \pi r^2 + \pi r\ell = \pi r(r+\ell) = \pi r\left(r + \sqrt{h^2+r^2}\right)$

$$= \pi . h\tan\alpha\left(h\tan\alpha + \sqrt{h^2 + h^2\tan^2\alpha}\right) = \pi h\tan\alpha(h\tan\alpha + h\sec\alpha)$$

$$= \pi h^2\tan\alpha(\tan\alpha + \sec\alpha). \qquad \Big| = f(h,\alpha)$$

$$\therefore \ \delta A = \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial \alpha}\delta\alpha$$

$$= 2\pi h\left(\tan^2\alpha + \tan\alpha\sec\alpha\right)\delta h + \pi h^2\left(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha\right)\delta\alpha$$

$$= 2\pi h\tan\alpha(\tan\alpha+\sec\alpha)\delta h + \pi h^2\sec\alpha\left(2\tan\alpha\sec\alpha + \sec^2\alpha + \tan^2\alpha\right)\delta\alpha, \qquad \text{(i)}$$

which gives the error in A.

Putting $\alpha = \frac{\pi}{6}$ and $\delta h = 1\%$ of $h = \frac{h}{100}$ in (i), we have

$$\delta A = 2\pi h.\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right).\frac{h}{100} + \pi h^2 . \frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}.\frac{2}{\sqrt{3}} + \frac{4}{4} + \frac{1}{3}\right)\delta\alpha = \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2\delta\alpha.$$

Since the error in h is to be compensated by the error in α $\Rightarrow \delta A = 0$

$$\Rightarrow \frac{1}{100} + \sqrt{3}\,\delta\alpha = 0 \Rightarrow \delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians}$$

$$\Rightarrow \delta\alpha = -\frac{0.01}{1.732}\times 57.3 \text{ degree} \qquad \left[\because 1\,\text{radian} = 57.3^{o}\text{ nearly}\right]$$

$$= -0.33 \text{ degree.}$$

Q.No.29.: At a distance of 30 meter from the foot of the tower the elevation of its top is

$30°$. If the possible error in measuring the distance and elevation are 2cm. and 0.05degrees. Find the approximate error in calculating the height.

Sol.: $h = x\tan\alpha$.

Taking log on both sides, we get

$\log h = \log x + \log\tan x$

Differentiating, we get

$$\frac{\delta h}{h} = \frac{\delta x}{x} + \frac{\sec^2\alpha}{\tan\alpha}.\delta x \Rightarrow \delta h = \frac{h}{x}\delta x + h\frac{\sec^2\alpha}{\tan\alpha}.\delta x = \tan\alpha.\delta x + x\sec^2\alpha.\delta\alpha$$

Given $\delta x = 0.02$, $\delta\alpha = 0.05° = 0.05.\frac{\pi}{180}$ rad.

$$\delta h = \tan 30°(0.02) + 30.\sec^2 30°\left(0.05.\frac{\pi}{180}\right) = 0.0464\text{ m} = 4.64\text{ cm. Ans.}$$

Q.No.30.: Find the %age error in the area of an ellipse if one % error is made in measuring the major and minor axes.

Sol.: Area of an ellipse $A = \pi ab$.

Taking log on both sides, we get

$\log A = \log\pi + \log a + \log b$

Differentiating, we get

$$\frac{\delta A}{A} = \frac{\delta a}{a} + \frac{\delta b}{b} \Rightarrow \frac{\delta A}{A}\times 100 = \frac{\delta a}{a}\times 100 + \frac{\delta b}{b}\times 100 = 1\% + 1\% = 2\%.$$

$\therefore$ %age error in area of an ellipse = 2%. Ans.

Q.No.31.: Two sides a, b of a triangle and included angle C are measured. Show that the error δc in the computed length of third side c due to a small error in the angle C is given by $\delta c = a\sin B\delta C$.

Sol.: Given $c^2 = a^2 + b^2 - 2ab\cos C$

Differentiating, we get

$$2c\delta c = 2a\delta a + 2b\delta b - 2[\delta a.b\cos C + a\delta b\cos C + ab(-\sin C)\delta C]$$

As $\delta a = \delta b = 0$

$\therefore$ $2c\delta c = -2[ab(-\sin C)\delta C] \Rightarrow c\delta c = ab\sin C.\delta C$

By sin law in ΔABC, $b\sin C = c\sin B$

$\therefore$ $c\delta c = ac\sin B.\delta C \Rightarrow \delta c = a\delta C.\sin B$.

Hence this proves the result.

Q.No.32.: Let $T = 2\pi\sqrt{\frac{\ell}{g}}$. Find the maximum %age error in T due to possible error of 1% in ℓ and g respectively.

Sol.: Given $T = 2\pi\sqrt{\frac{\ell}{g}}$.

Taking log on both sides, we get

$$\log T = \log 2\pi + \frac{1}{2}\log \ell - \frac{1}{2}\log g.$$

Differentiating, we get

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta \ell}{\ell} - \frac{1}{2}\frac{\delta g}{g} = \frac{1}{2}\left(\frac{\delta \ell}{\ell} - \frac{\delta g}{g}\right)$$

$$\frac{\delta T}{T}\times 100 = \%\text{age error in T} = \frac{1}{2}\left(\frac{\delta \ell}{\ell} - \frac{\delta g}{g}\right)\times 100.$$

Maximum %age error in $T = \frac{1}{2}\left(\pm\frac{\delta \ell}{\ell} \pm \frac{\delta g}{g}\right).100 = \frac{1}{2}(\pm 1 \pm 2) = \pm 1.5\%$. Ans.

Q.No.33.: Let $R = \frac{V^2 \sin 2\theta}{g}$, find the %age error in R due to an error of 1% in v and $\frac{1}{2}\%$ in θ.

Sol.: Given $R = \frac{V^2 \sin 2\theta}{g}$. (i)

Given $\frac{dv}{v}\times 100 = 1\%$, $\frac{d\theta}{\theta}\times 100 = \frac{1}{2}\%$

Taking log on both sides of (i), we get

$$\log R = 2\log V + \log \sin 2\theta - \log g$$

Differentiate on both sides, we get

$$\frac{dR}{R} = 2\frac{dV}{V} + \frac{\cos 2\theta}{\sin 2\theta}.2d\theta.$$

Multiplying by 100, we get

$$\frac{dR}{R}\times 100 = 2\left(\frac{dV}{V}\times 100\right) + (\theta\cot 2\theta)\left(\frac{d\theta}{\theta}\times 100\right) = 2.1 + \theta\cot 2\theta.\frac{1}{2} = 2 + \frac{\theta}{2}\cot 2\theta. \text{ Ans.}$$

Q.No.34.: If $S = \dfrac{A}{A-W}$, find the maximum relative error in S and maximum error in S.

If the values of A and W are 1.1 and 0.6 respectively with possible error in 0.01 and 0.02 in A and W respectively.

Sol.: Given $S = \dfrac{A}{A-W}$. (i)

Now differentiating (i) w. r. t. to A, we get

$$\frac{\partial S}{\partial A} = \frac{(A-W).1 - A.1}{(A-W)^2} = \frac{A-W-A}{(A-W)^2} = \frac{-W}{(A-W)^2} \qquad \text{(ii)}$$

Differentiating (i) w. r. t. W, we get

$$\frac{\partial S}{\partial W} = \frac{(A-W).0 - A.(-1)}{(A-W)^2} = \frac{A}{(A-W)^2} \qquad \text{(iii)}$$

We know that $dS = \dfrac{\partial S}{\partial A}dA + \dfrac{\partial S}{\partial W}dW$ (iv)

Now putting the values of (ii) and (iii) in (iv), we get

$$dS = \frac{-W}{(A-W)^2}dA + \frac{A}{(A-W)^2}dW$$

Now given $A = 1.1$, $W = 0.6$, $dA = 0.01$, $dW = 0.02$.

Maximum error in S=

$$dS = \frac{.6}{(1.1-0.6)^2}\times 0.01 + \frac{1.1}{(1.1-0.6)^2}\times 0.02 = \frac{0.6}{0.25}\times 0.01 + \frac{1.1}{0.25}\times 0.02$$

$$= 0.024 + 0.088 = 0.112.$$

$\therefore$ Maximum error in S = 0.112. Ans.

Maximum relative error in S is given by $\dfrac{dS}{S}$.

Now $S = \dfrac{A}{A-W} = \dfrac{1.1}{1.1-0.5} = \dfrac{1.1}{0.5} = 2.2$.

$$\frac{dS}{S} = \frac{0.112}{2.2} = 0.0509 = 0.51$$

$\therefore$ Maximum relative error in S = 0.51. Ans.

Q.No.35.: Use differentials to compute $f(0.9,\ -1.2)$ approximately where

$$f(x, y) = \tan^{-1}(xy).$$

Sol.: Given $f(0.9,\ -1.2)$.

Let $x = 1$, $\delta x = -0.1$,

$y = -1$, $\delta y = -0.2$.

$\therefore\ f(x, y) = \tan^{-1}(-1)$

Let $f(x, y) = \theta$, $\quad \therefore \theta = \dfrac{3\pi}{4}$.

Now $\tan^{-1}(xy) = \theta = \dfrac{3\pi}{4}$.

Taking log on both sides, we get $\quad \log\theta = \log\tan^{-1}(xy)$.

Differentiating, we get

$$\frac{\delta\theta}{\theta} = \frac{1}{\tan^{-1}(xy)}\frac{x\delta y + y\delta x}{1 + x^2y^2} = \frac{1}{\tan^{-1}(xy)}\frac{-0.2 + 0.1}{1+1} = \frac{1}{\theta}\frac{-0.2+0.1}{1+1} \Rightarrow \delta\theta = -\frac{0.1}{2} = -0.05.$$

$\therefore f(0.9, -1.2) = f(x + \delta x, y + \delta y) = \theta + \delta\theta = \dfrac{3\pi}{4} - 0.05 = 2.307$. Ans.

Q.No.36.: If the sides and angles of a triangle ABC vary in such a manner that its circum-radius remains constant, prove that $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0$.

Sol.: To prove: $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0$.

We know that, the circum-radius R of a ΔABC is given by

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}.$$

Now $a = 2R\sin A$. [R is constant]

Differentiating, we get $\quad da = 2R\cos A dA \Rightarrow \dfrac{da}{\cos A} = 2RdA$.

Similarly $\quad db = 2R\cos B dB \Rightarrow \dfrac{db}{\cos B} = 2RdB$.

$$dc = 2R\cos C dC \Rightarrow \frac{dc}{\cos C} = 2RdC.$$

Adding, we get $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=2R(dA+dB+dC)=2Rd(A+B+C)$ (i)

Also $A+B+C=\pi$. $\Rightarrow d(A+B+C)=0$.

$\therefore$ From (i), we get $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=0$.

This completes the proof.

Q.No.37.: The area of a rectangle is found from measurements of side a and angle B and C. Prove that error in the calculated value of area due to small error $\delta a,\ \delta B,\ \delta C$ is given by $\left(\dfrac{2}{a}\delta a+\dfrac{c}{a}\dfrac{\delta B}{\sin B}+\dfrac{b}{a}\dfrac{\delta C}{\sin C}\right)\Delta$.

Sol.: We know that area $\Delta=\dfrac{1}{2}ab\sin C$ (i)

Now $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}\Rightarrow b=\dfrac{a\sin B}{\sin A}$

Putting in (i), we get

$$\Delta=\frac{1}{2}(a)\left(\frac{a\sin B}{\sin A}\right)\sin C=\frac{1}{2}a^2\frac{\sin B\sin C}{\sin[\pi-(B+C)]} \qquad \text{(ii)} \quad [\because A+B+C=\pi]$$

Taking log of (ii) on both sides, we get

$$\log\Delta=\log\frac{1}{2}+2\log a+\log\sin B+\log\sin C-\log\sin[(B+C)] \qquad \text{(iii)}$$

$$[\because \sin[\pi-(B+C)]=\sin(B+C)]$$

Differentiating (iii), we get

$$\frac{\delta\Delta}{\Delta}=\frac{2\delta a}{a}+\frac{\cos B}{\sin B}\delta B+\frac{\cos C}{\sin C}\delta C-\frac{\cos(B+C)}{\sin(B+C)}(\delta B+\delta C)$$

$$=\frac{2\delta a}{a}+\delta B\left[\frac{\cos B}{\sin B}-\frac{\cos(B+C)}{\sin(B+C)}\right]+\delta C\left[\frac{\cos C}{\sin C}-\frac{\cos(B+C)}{\sin(B+C)}\right]$$

$$=\frac{2\delta a}{a}+\delta B\left[\frac{\cos B\sin(B+C)-\sin B\cos(B+C)}{\sin(B+C)\sin B}\right]$$

$$+\delta C\left[\frac{\cos C\sin(B+C)-\sin C\cos(B+C)}{\sin(B+C)\sin C}\right]$$

$$= \frac{2\delta a}{a} + \delta B \frac{\sin C}{\sin B \sin(B+C)} + \frac{\delta C}{\sin C} \cdot \frac{\sin B}{\sin(B+C)} \qquad \text{(iv)}$$

$A + B + C = \pi \Rightarrow B + C = \pi - A \Rightarrow \sin(B+C) = \sin(\pi - A)$

Putting this value in (iv), we get

$$\frac{\delta\Delta}{\Delta} = \frac{2\delta a}{a} + \frac{\delta B}{\sin B}\frac{\sin C}{\sin A} + \frac{\delta C}{\sin C} \cdot \frac{\sin B}{\sin A}. \qquad \text{(v)}$$

According to sine formula: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

$$\Rightarrow \frac{c}{a} = \frac{\sin C}{\sin A}, \quad \frac{b}{a} = \frac{\sin B}{\sin A}$$

Putting these values in (v), we get

$$\frac{\delta\Delta}{\Delta} = \frac{2\delta a}{a} + \frac{c}{a}\frac{\delta B}{\sin B} + \frac{b}{a}\frac{\delta C}{\sin C}$$

$\delta\Delta = \left(\frac{2}{a}\delta a + \frac{c}{a}\frac{\delta B}{\sin B} + \frac{b}{a}\frac{\delta C}{\sin C}\right)\Delta$, which is the required proof.

Q.No.38.: In a plane triangle, if the sides a, b be constant, prove that the variations of its angles are given by the relations

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}} = -\frac{dC}{c},$$

the letters have their usual significance.

Sol.: By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a \sin B = b \sin A$ (i)

Taking differentials, we get $a \cos B . dB = b \cos A . dA$

$$\Rightarrow \frac{dA}{a \cos B} = \frac{dB}{b \cos A} = \frac{dA + dB}{a \cos B + b \cos A}. \qquad \text{(ii)} \quad \left[\because \text{If } \frac{a}{b} = \frac{c}{d} \text{ then each } = \frac{a+c}{b+d}\right]$$

Now $a \cos B = a\sqrt{1 - \sin^2 B} = \sqrt{a^2 - a^2 \sin^2 B} = \sqrt{a^2 - b^2 \sin^2 A}$ [using (i)]

$b \cos A = b\sqrt{1 - \sin^2 A} = \sqrt{b^2 - b^2 \sin^2 A} = \sqrt{b^2 - a^2 \sin^2 B}$ [using (i)]

$a \cos B + b \cos A = c$ [By projection formula]

Also $A + B + C = \pi \Rightarrow A + B = \pi - C$ so that $dA + dB = -dC$

$$\therefore \text{ From (ii), } \frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}} = -\frac{dC}{c}.$$

Q.No.39.: If there is a small error δc in measuring the side c in a triangle, show that relative error in the area of the triangle is equal to

$$\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]\frac{\delta c}{4}.$$

or

If A be the area of a triangle, prove that the error in A resulting from a small error in 'c' is given by

$$\delta A=\frac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c.$$

Sol.: Let δA is the error in A, then relative error in A $=\dfrac{\delta A}{A}$.

Now to prove: $\dfrac{\delta A}{A}=\left[\dfrac{1}{s}+\dfrac{1}{s-a}+\dfrac{1}{s-b}-\dfrac{1}{s-c}\right]\dfrac{\delta c}{4}$.

or $\qquad \delta A=\dfrac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c$

Since, we know that $A=\sqrt{s(s-a)(s-b)(s-c)}$.

Taking log on both sides, we get

$$\log A=\frac{1}{2}\log s+\frac{1}{2}\log(s-a)+\frac{1}{2}\log(s-b)+\frac{1}{2}\log(s-c).$$

Differentiating on both sides, we get

$$\frac{\delta A}{A}=\frac{1}{2}\left[\frac{\delta s}{s}+\frac{\delta s}{s-a}+\frac{\delta s}{s-b}+\frac{\delta s-\delta c}{s-c}\right]. \qquad \text{(i)}$$

Since $s=\dfrac{a+b+c}{2}\Rightarrow \delta s=\dfrac{\delta c}{2}$.

Putting in (i), we get

$$\frac{\delta A}{A}=\frac{1}{4}\left[\frac{\delta c}{s}+\frac{\delta c}{s-a}+\frac{\delta c}{s-b}-\frac{\delta c}{s-c}\right]=\frac{\delta c}{4}\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]$$

$$\Rightarrow \delta A=\frac{A}{4}\left[s^{-1}+(s-a)^{-1}+(s-b)^{-1}-(s-c)^{-1}\right]\delta c.$$

Hence this proves the result.

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