3rd Topic

Triple Integrals

[Change of variables]

(Last updated on 15-07-2013)

(13 Solved problems and 00 Home assignment)

Change of variables:

The evaluation of the triple integrals is greatly simplified by a suitable change of variables. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

For triple integrals:

Let the variables x, y, z in the triple integral

$$\iiint_R f(x,\ y,\ z)\,dxdydz \qquad \text{(i)}$$

be changed to the new variables u, v, w by the transformation

$$x = \phi(u,\ v,\ w),\quad y = \psi(u,\ v,\ w),\ z = \varphi(u,\ v,\ w),$$

where $\phi(u,\ v,\ w)$, $\psi(u,\ v,\ w)$, $\varphi(u,\ v,\ w)$ are continuous and have continuous first order derivatives in some region R'_{uvw} which corresponds to the region R_{xyz}.

The formula corresponds to (i) is

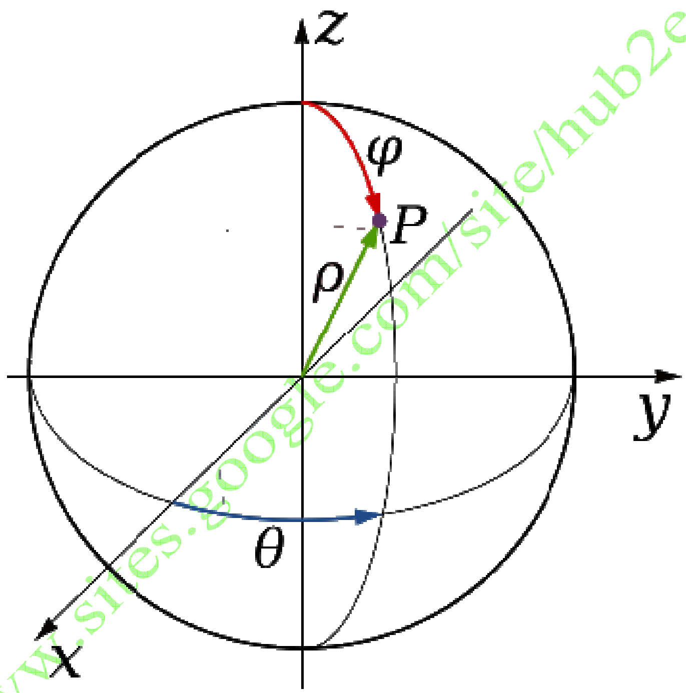
$$\iiint_{R_{xyz}} f(x,y,z)\,dxdydz = \iiint_{R'_{uvw}} f[x(u,v,w),y(u,v,w),z(u,v,w)]|J|\,dudvdw\ ,$$

where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \; (\neq 0)$

is the **Jacobian** of transformation from (x, y, z) to (u, v, w) co-ordinates.

Particular cases:

(I) CONVERSION OF RECTANGULAR TO SPHERICAL SYSTEM



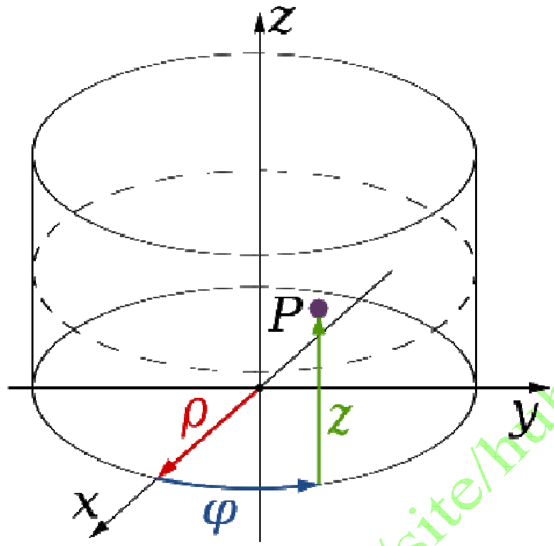
Spherical Coordinates

To change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint\limits_{R_{x,y,z}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

(II) CONVERSION OF RECTANGULAR TO CYLINDRICALSYSTEM



Cylindrical coordinates.

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Now let us solve some problems:

Q.No.1.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} .$$

or

Evaluate $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$,

the integral being extended to the positive octant of the sphere $x^2+y^2+z^2=1$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

Also $\sqrt{1-x^2-y^2-z^2} = \sqrt{1-r^2}$.

$$\therefore I = \int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta}{\sqrt{1-r^2}} drd\theta d\phi$$

$$= \int_0^{\pi/2}\left[\int_0^{\pi/2}\left\{\int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr\right\}\sin\theta d\theta\right]d\phi$$

Now evaluate $\int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}} = \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t dt = \int_0^{\pi/2}\sin^2 t dt = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$.

Here we put $r = \sin t$, $\therefore dr = \cos t dt$. And as $r \to 0, t \to 0$ and $r \to 1, t \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}\left[\int_0^{\pi/2}\frac{\pi}{4}\sin\theta d\theta\right]d\phi = \frac{\pi}{4}\int_0^{\pi/2}\left[-\cos\phi\Big|_0^{\pi/2}\right]d\phi = -\frac{\pi}{4}\int_0^{\pi/2}\left[\cos\frac{\pi}{2}-\cos 0\right]d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi = \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}. \text{ Ans.}$$

Q.No.2.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dxdydz}{\left(1+x^2+y^2+z^2\right)^2}.$$

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

To cover the whole region, r varies from 0 to ∞, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to $\frac{\pi}{2}$.

Also $r^2 = x^2 + y^2 + z^2$.

Hence $I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dxdydz}{\left(1+x^2+y^2+z^2\right)^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty}\frac{1}{\left(1+r^2\right)^2}r^2\sin\theta drd\theta d\phi$

$$= \int_0^{\pi/2}\left\{\int_0^{\pi/2}\left(\int_0^{\infty}\frac{r^2}{\left(1+r^2\right)^2}dr\right)\sin\theta d\theta\right\}d\phi$$

$$= \int_0^{\pi/2}\left[\int_0^{\pi/2}\left\{\int_0^{\infty}\left(\frac{\left(1+r^2\right)}{\left(1+r^2\right)^2} - \frac{1}{\left(1+r^2\right)^2}\right)dr\right\}\sin\theta d\theta\right]d\phi \qquad \text{(i)}$$

Now first evaluate $\int_0^{\infty}\left(\frac{\left(1+r^2\right)}{\left(1+r^2\right)^2} - \frac{1}{\left(1+r^2\right)^2}\right)dr = \int_0^{\infty}\frac{1}{1+r^2}dr - \int_0^{\infty}\frac{1}{\left(1+r^2\right)^2}dr$

Let $r = \tan\theta \Rightarrow dr = \sec^2\theta d\theta$. Now here θ varies from 0 to $\frac{\pi}{2}$.

$$= \left[\tan^{-1} r\right]_0^{\infty} - \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta} d\theta = \frac{\pi}{2} - \int_0^{\pi/2} \cos^2\theta d\theta = \frac{\pi}{2} - \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}.$$

Now putting the value in (i), we get

$$= \frac{\pi}{4}\int_0^{\pi/2}\left(\int_0^{\pi/2} \sin\theta d\theta\right) d\phi = \frac{\pi}{4}\int_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{\pi}{4}\int_0^{\pi/2} 1.(d\phi) = \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Q.No.3.: Evaluate $\iiint (ax+by+cz)^2 dxdydz$, throughout the sphere $x^2+y^2+z^2=1$, using spherical polar co-ordinates.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x, y, z) dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \left(a^2x^2 + b^2y^2 + 2abxy + c^2z^2 + 2czax + 2czby\right) dxdydz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^2r^2\sin^2\theta\cos^2\phi + b^2r^2\sin^2\theta\sin^2\phi + 2abr^2\sin^2\theta\cos\phi\sin\theta + c^2r^2\cos^2\theta$$

$$+ 2car^2\sin\theta\cos\theta\cos\phi + 2bcr^2\sin\theta\cos\theta\cos\phi) r^2\sin\theta drd\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^2r^2\sin^2\theta\cos^2\phi + b^2r^2\sin^2\theta\sin^2\phi + abr^2\sin^2\theta\sin 2\theta + c^2r^2\cos^2\theta$$

$$+ car^2\sin\phi\sin\theta + bcr^2\sin 2\theta\sin\phi) r^2\sin\theta drd\theta d\phi$$

$$=\frac{1}{5}\int_0^{2\pi}\int_0^{\pi}(a^2\sin^3\theta\cos^2\phi+b^2\sin^3\theta\sin^2\phi+ab\sin^2\theta\sin 2\theta+c^2\cos^2\theta\sin$$

$$+ac\cos\phi\sin 2\theta+bc\sin 2\theta\sin\phi\sin\theta)d\theta d\phi$$

$$=\frac{1}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta 9(a^2\cos^2\phi+b^2\sin^3\theta\sin^2\phi+2ab\sin\phi\cos\phi+\int_0^{\pi}c\sin^2\theta\sin 2\theta)$$

$$+(a\cos\phi+b\sin\phi)+\int_0^{\pi}c^2\cos^2\theta\sin\theta)d\theta d\phi$$

$$=\frac{1}{5}\int_0^{2\pi}\left[\int_0^{\pi}\frac{3\sin\theta-\sin 3\theta}{4}(a\cos\phi+b\sin\phi)^2 d\phi+\int_0^{\pi}2c\sin^2\theta\cos\theta(a\cos\phi+b\sin\phi)d\theta\right.$$

$$\left.+\int_0^{\pi}c^2\left(\sin\theta-\sin^3\theta\right)d\theta\right]d\phi$$

$$=\frac{1}{5}\int_0^{2\pi}\left[\frac{1}{4}(a\cos\phi+b\sin\phi)^2\left\{(-3\cos\phi)_0^{\pi}+\left(\frac{\cos 3\theta}{3}\right)_0^{\pi}\right\}+\int_0^{\pi}2c\left(\cos\theta-\cos^3\theta\right)\right.$$

$$\left.(a\cos\phi+b\sin\phi)d\theta+\frac{c^2}{4}\int_0^{\pi}4\sin\theta-3\sin\theta+\sin 3\theta d\theta\right]d\phi$$

$$=\frac{1}{5}\int_0^{2\pi}\frac{4}{3}(a\cos\phi+b\sin\phi)^2-\frac{c}{2}(a\cos\phi+b\sin\phi)\left[\left(\frac{\sin 3\theta}{3}\right)_9^{\pi}+3(\sin\theta)_0^{\pi}\right]+\frac{c^2}{4}\left(2+\frac{2}{3}\right)$$

$$=\frac{1}{5}\int_0^{2\pi}\left(2a^2\cos\phi+b\sin\phi\right)+\frac{2}{3}c^2$$

$$=\frac{2}{15}\int_0^{2\pi}\left(2a^2\cos^2\phi+2b^2\sin^2\phi+4ab\cos\phi\sin\phi+c^2\right)d\phi$$

$$=\frac{2}{15}\int_0^{2\pi}\left[2a^2\left(\frac{1+\cos 2\phi}{2}\right)+2b^2\left(\frac{1-\sin 2\phi}{2}\right)+2ab\sin 2\phi+c^2\right]d\phi$$

$$= \frac{2}{15}\left[\begin{array}{c} 2.\frac{a^2}{2}\left\{(\phi)_0^{2\pi} + \left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right\} + 2.\frac{b^2}{2}\left\{(\phi)_0^{2\pi} - \left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right\} \\ -2ab\left(\frac{\cos 2\phi}{2}\right)_0^{2\pi} + c^2(\phi)_0^{2\pi} \end{array}\right]$$

$$= \frac{2}{15}\left[a^2 2\pi + b^2 2\pi - ab(1-1) + c^2 2\pi\right] = \frac{4}{15}\pi\left(a^2 + b^2 + c^2\right). \text{ Ans.}$$

Q.No. 4.: Find the value of $\iiint x^2 dxdydz$, taking throughout the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ using spherical polar co-ordinates..}$$

Sol.: Let $A = \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} x^2 dxdydz$

Putting $\frac{x}{a} = u$, $\frac{y}{b} = v$ and $\frac{z}{c} = w$.

$\therefore dx = adu$, $dy = bdv$ and $dz = cdw$

$$A = \iiint\limits_{u^2+v^2+w^2\le 1} a^2u^2 adu.bdv.cdw = a^3bc \iiint\limits_{u^2+v^2+w^2\le 1} u^2 du.dv.dw = \iiint\limits_{R_{uvw}} f(u, v, w)dudvdw.$$

Now we have to solve this problem by changing rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ), we have put $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and

$$J = \frac{\partial(u, v, w)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint\limits_{R_{uvw}} f(u, v, w)dudvdw = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore A = a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^2\sin^2\theta\cos^2\phi r^2\sin\theta dr\right)d\theta\right]d\phi$$

$$= a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^4 dr\right).\sin^3\theta d\theta\right]\cos^2\phi d\phi = a^3bc\int_0^{2\pi}\left[\int_0^{\pi}\left(\left|\frac{r^5}{5}\right|_0^1\right).\sin^3\theta d\theta\right]\cos^2\phi d\phi$$

$$= \frac{a^3bc}{5}\int_0^{2\pi}\left[\int_0^{\pi}\sin^3\theta d\theta\right]\cos^2\phi d\phi = \frac{a^3bc}{5}\int_0^{2\pi}\left[\int_0^{\pi}\frac{3\sin\theta - \sin 3\theta}{4}d\theta\right]\cos^2\phi.d\phi$$

$$= \frac{a^3bc}{20}\int_0^{2\pi}\left[3(-\cos\theta)_0^{\pi} - \left(\frac{-\cos 3\theta}{3}\right)_0^{\pi}\right]\cos^2\phi.d\phi = \frac{a^3bc}{20}\int_0^{2\pi}\left[3(1+1) - \left(\frac{1+1}{3}\right)\right]\cos^2\phi.d\phi$$

$$= \frac{a^3bc}{20}\int_0^{2\pi}\left(6 - \frac{2}{3}\right)\cos^2\phi.d\phi = \frac{a^3bc}{20}\int_0^{2\pi}\frac{18-2}{3}\cos^2\phi.d\phi = \frac{a^3bc}{20}\times\frac{16}{3}\int_0^{2\pi}\frac{1+\cos 2\phi}{2}d\phi$$

$$= \frac{a^3bc}{20}\times\frac{16}{3}\times\frac{1}{2}\left[(\phi)_0^{2\pi} + \left(\frac{\sin 2\phi}{2}\right)_0^{2\pi}\right] = \frac{a^3bc}{20}\times\frac{16}{3}\times\frac{1}{2}\times 2\pi = \frac{4\pi a^3bc}{15}\text{ Ans.}$$

Q.No.5.: Evaluate $\iiint \frac{1}{x^2+y^2+z^2}dxdydz$ throughout the volume of the sphere

$x^2+y^2+z^2 = a^2$ using spherical polar co-ordinates..

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x,y,z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore I = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}\frac{r^2\sin\theta}{r^2}drd\theta d\phi = 8\int_0^{\pi/2}\left[\int_0^{\pi/2}\left(\int_0^{a}\frac{r^2\sin\theta}{r^2}dr\right)d\theta\right]d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\left[r\sin\theta\right]_0^a d\theta d\phi = 8\int_0^{\pi/2}\left[\int_0^{\pi/2} r\sin\theta d\theta\right]d\phi = 8\int_0^{\pi/2} d\phi = 8a\times\frac{\pi}{2} = 4\pi a\text{. Ans.}$$

Q.No.6.: Evaluate $\iiint xyzdxdydz$ throughout the positive octant of the sphere

$x^2+y^2+z^2=a^2$ using spherical polar co-ordinates.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta .$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore \iiint xyzdxdydz = \int_0^{\pi/2}\left[\int_0^{\pi/2}\left(\int_0^a r^5 dr\right)\sin^3\theta\cos\theta d\theta\right]\cos\phi\sin\phi d\phi$$

$$= \int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{a^6}{6}\sin^3\theta\cos\theta d\theta\right)\cos\phi\sin\phi d\phi = \int_0^{\pi/2}\frac{a^6}{24}\cos\phi\sin\phi d\phi$$

$$= \frac{a^6}{24}\int_0^{\pi/2}\cos\phi\sin\phi d\phi = \frac{a^6}{48}\int_0^{\pi/2}\sin 2\phi d\phi = \frac{a^6}{48}\left|\frac{-\cos 2\phi}{2}\right|_0^{\pi/2}$$

$$= \frac{a^6}{48}+\left(-\frac{1}{2}+\frac{1}{2}\right) = \frac{a^6}{48}\text{. Ans.}$$

Q.No.7.: Evaluate the following integral by changing to spherical polar co-ordinates:

$$\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_{\sqrt{(x^2+y^2)}}^1 \frac{dzdydx}{\sqrt{(x^2+y^2+z^2)}}.$$

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and in this case

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

Now $x^2 + y^2 + z^2 = r^2$.

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$, bounded by the plane z = 1 in the positive octant.

Since z = 1 in the positive octant $\Rightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta$.

Hence, r varies from 0 to $\sec\theta$, θ varies from 0 to $\frac{\pi}{4}$, and ϕ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ The given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{1}{r}.r^2\sin\theta drd\theta d\phi = \int_0^{\pi/2}\left(\int_0^{\pi/4}\left|\frac{r^2}{2}\right|_0^{\sec\theta}\sin\theta d\theta\right)d\phi = \int_0^{\pi/2}\left[\int_0^{\pi/4}\frac{\sec^2\theta}{2}\sin\theta d\theta\right]d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\int_0^{\pi/4}\sec\theta.\tan\theta d\theta\right)d\phi = \frac{1}{2}\int_0^{\pi/2}\left([\sec\theta]_0^{\pi/4}\right)d\phi$$

$$= \frac{\left(\sqrt{2}-1\right)}{2}\int_0^{\pi/2} d\phi = \frac{\left(\sqrt{2}-1\right)\pi}{4}. \text{ Ans.}$$

(II) CONVERSION OF RECTANGULAR TO CYLINDRICALSYSTEM

Q.No.8.: Evaluate the following integral by changing to cylindrical co-ordinates:

$\iiint z^2 dxdydz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z),

we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

$\therefore \rho^2 = a^2$, $\rho^2 = z$ and $z = 0$

So here ρ varies from 0 to a, z varies from 0 to ρ^2 and ϕ varies from 0 to 2π.

$$\therefore I = \iiint z^2 dxdydz = \int_0^{2\pi}\int_0^{a}\int_0^{\rho^2} z^2 \rho d\rho d\phi dz = \int_0^{2\pi}\left[\int_0^{a}\left\{\int_0^{\rho^2} z^2 dz\right\}\rho d\rho\right]d\phi$$

$$= \int_0^{2\pi}\left[\int_0^{a}\left[\frac{z^3}{3}\right]_0^{\rho^2}\rho d\rho\right]d\phi$$

$$= \int_0^{2\pi}\left[\int_0^{a}\frac{\rho^7}{3}d\rho\right]d\phi = \int_0^{2\pi}\left[\frac{\rho^8}{24}\right]_0^{a}d\phi = \int_0^{2\pi}\frac{a^8}{24}d\phi = \frac{\pi a^8}{12}. \text{ Ans.}$$

Q.No.9.: By transforming into cylindrical coordinates evaluate the integral

$\iiint \left(x^2 + y^2 + z^2\right)dxdydz$ taken over the region $0 \le z \le x^2 + y^2 \le 1$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

$$\int_0^1 \iint\limits_R \left(x^2 + y^2 + z^2\right)dxdydz = \int_0^1 \int_0^{2\pi} \int_0^1 \left(r^2\cos^2\theta + r^2\sin^2\theta + z^2\right)rdrd\theta dz$$

where R: circular region bounded by the circle of radius one and centre at origin: $x^2 + y^2 = 1$, so that r varies from 0 to 1 and θ varies from 0 to 2π.

Thus $\int_0^1 \int_0^{2\pi} \int_0^1 \left(r^2\cos^2\theta + r^2\sin^2\theta + z^2\right)rdrd\theta dz = \int_0^1 \int_0^{2\pi} \int_0^1 \left(r^3 + rz^2\right)drd\theta dz$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + \frac{r^2}{2}z^2\right)_0^1 d\theta dz = \int_0^1 \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2}z^2\right)d\theta dz = 2\pi.\int_0^1 \left(\frac{1}{4} + \frac{1}{2}z^2\right)dz$$

$$= 2\pi.\left(\frac{z}{4} + \frac{1}{2}\frac{z^3}{3}\right)_0^1 = 2\pi.\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5\pi}{6} \text{ Ans.}$$

Q.No.10.: By transforming into cylindrical coordinates evaluate the integral

$\iiint\limits_V \left(x^2 + y^2\right)dxdydz$ taken over the region V bounded by the paraboloid

$z = 9 - x^2 - y^2$ and the plane $z = 0$.

Sol.: We have to solve this problem by changing rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z).

As we know, when we change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho .$$

Then $\iiint_{R_{xyz}} f(x, y, z)dxdydz = \iiint_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Now $I = \iiint \left(\rho^2\right)\rho dz d\rho d\phi$

Now $z = 9 - x^2 - y^2$, $z = 9 - \rho^2$ and $z = 0$

At $z = 0$, $\rho^2 = 9 \Rightarrow \rho = 3$

$$\therefore I = \int_0^{2\pi}\int_0^{3}\int_0^{9-\rho^2} \left(\rho^2\right)\rho dz d\rho d\phi$$

$$= \int_0^{2\pi}\int_0^{3} [z]_0^{9-\rho^2}.\rho^3.d\rho.d\phi = \int_0^{2\pi}\left(\int_0^{3}\left(9-\rho^2\right).\rho^3.d\rho\right)d\phi = \int_0^{2\pi}\left(\int_0^{3}\left(9\rho^3 - \rho^5\right)d\rho\right).d\phi$$

$$= \int_0^{2\pi}\left[\frac{9\rho^4}{4} - \frac{\rho^6}{6}\right]_0^3 d\phi = \int_0^{2\pi}\left[\frac{9(81)}{4} - \frac{81\times 9}{6}\right]d\phi = \frac{243}{4}\times 2\pi = \frac{243\pi}{2}. \text{ Ans.}$$

(III) CONVERSION OF RECTANGULAR TO ANY OTHERSYSTEM

Q.No.10.: Using the transformation $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate the integral $\iiint [xyz(1 - x - y - z)]^{1/2} dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1 - v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian = $J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\iiint [xyz(1-x-y-z)]^{1/2}\,dxdydz = \int_0^1\int_0^1\int_0^1 [u(1-v).uv(1-w).uvw(1-u)]^{1/2}.u^2v\,dudvdw$$

Integrating w.r.t. u, we get

$$I = \int_0^1\int_0^1 [(1-v)v(1-w)(vw)]^{1/2}.v\left(\int_0^1 [u^3(1-u)]^{1/2}u^2du\right)dvdw$$

$$\Rightarrow \int_0^1\int_0^1 \left[v^4(1-v)w(1-w)\right]^{1/2}dv.dw \times \int_0^1 [u^7(1-u)]^{1/2}du \quad \text{(iv)}$$

Let $u = \sin^2\theta \Rightarrow du = 2\sin\theta d\theta$

$$\text{So } \int_0^1 \left[u^7(1-u)\right]^{1/2}du = \int_0^{\pi/2} (\sin^7\theta\cos\theta)2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} 2.\sin^8\theta\cos^2\theta d\theta = 2\int_0^{\pi/2}\left(\sin^8\theta - \sin^{10}\theta\right)d\theta$$

$$= 2.\left[\frac{7\times5\times3\times1}{8\times6\times4\times2}.\frac{\pi}{2} - \frac{9\times7\times5\times3\times1}{10\times8\times6\times4\times2}.\frac{\pi}{2}\right] = \frac{7\pi}{256}.$$

Putting this in (iv), we get

$$\frac{7\pi}{256}\int_0^1 [w(1-w)]^{1/2}dw \times \int_0^1 \left[v^4(1-v)\right]^{1/2}dv \qquad \text{(v)}$$

Let $v = \sin^2\theta \Rightarrow dv = 2\sin\theta\cos\theta d\theta$

$$\therefore \int_0^1 [v^4(1-v)]^{1/2}dv = 2\int_0^{\pi/2} \sin^5\theta\cos^2\theta d\theta$$

$$= 2\int_0^{\pi/2}\left(\sin^5\theta - \sin^7\theta\right)d\theta = \left[\frac{4\times2}{5\times3\times1} - \frac{6\times4\times2}{5\times7\times3\times1}\right] = \frac{16}{105}.$$

Putting this in (v), we get

$$I = \frac{7\pi}{256}\times\frac{16}{105}\int_0^1 [w(1-w)]^{1/2}dw$$

Let $w = \sin^2\theta \Rightarrow dw = 2\sin\theta\cos\theta d\theta$

$$I = \frac{7\pi}{256}\times\frac{16}{105}\int_0^{\pi/2} .2\sin^2\theta\cos^2 d\theta = \frac{7\pi}{256}\times\frac{16}{105}\times 2\int_0^{\pi/2}\left(\sin^2\theta - \sin^4\theta\right)d\theta$$

$$= \frac{7\pi}{256}\times\frac{16}{105}\times 2\left[\frac{1}{2}\frac{\pi}{2} - \frac{3\times 1}{4\times 2}.\frac{\pi}{2}\right] = \frac{7\pi}{256}\times\frac{16}{105}\times\frac{\pi}{8} = \frac{\pi^2}{1920}$$

Hence $\iiint \left[xyz(1-x-y-z)\right]^{1/2} dxdydz = \dfrac{\pi^2}{1920}$. Ans.

Q.No.11.: Using the transformation u = x + y + z , uv = y + z, uvw = z, evaluate the integral $\iiint (x+y+z)^2 xyz\, dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0,\ y = 0,\ z = 0$ and x + y + z = 1.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1-v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian $= J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\int_0^1\int_0^1\int_0^1 (u)^2u^3v^2w(1-v)(1-w).u^2v\,dudvdw = \int_0^1\int_0^1\int_0^1 u^7v^3w(1-v)(1-w)dudvdw$$

$$= \int_0^1\int_0^1\int_0^1 u^7\left(v^3-v^4\right)(w-w^2)dudvdw = \int_0^1\left(\int_0^1\left(\int_0^1 u^2du\right)(v^2-v^4)dy\right)(w-w^2)dw$$

$$= \int_0^1\left(\int_0^1\left[\frac{u^8}{8}\right]_0^1(v^3-v^4)dv\right)(w-w^2)dw = \int_0^1\left(\int_0^1\frac{1}{8}\left(v^3-v^4\right)dv\right)(w-w^2)dw$$

$$= \int_0^1\frac{1}{8}\left[\frac{v^4}{4}-\frac{v^5}{5}\right]_0^1(w-w^2)dw = \int_0^1\frac{1}{8}\left(\frac{1}{4}-\frac{1}{5}\right)(w-w^2)dw = \int_0^1\frac{1}{8}\left(\frac{5-4}{20}-\right)(w-w^2)dw$$

$$=\int_0^1 \frac{1}{160}\left(w-w^2\right)dw=\frac{1}{160}\left[\frac{w^2}{2}-\frac{w^3}{3}\right]_0^1=\frac{1}{160}\left[\frac{1}{2}-\frac{1}{3}\right]=\frac{1}{168}\times\frac{1}{6}=\frac{1}{960}. \text{ Ans.}$$

Q.No.12.: Using the transformation $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate the integral $\iiint e^{(x+y+z)^3}dxdydz$ taken over the tetrahedral volume enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Sol.: Here we use the transformation

$u = x + y + z$ (i)

$uv = y + z$ (ii)

$uvw = z$ (iii)

Solving (i), (ii) and (iii), we get

$x = u(1-v)$

$y = uv(1-w)$

$z = uvw$ and Jacobian $= J = u^2v$

According to the problem u, v and w vary from 0 to 1 each.

So triple integral becomes:

$$\iiint e^{(x+y+z)^3}dxdydz=\iiint e^{[u(1-v)+4v(1-w)+4vw]^3}u^2vdudvdw$$

$$=\int_0^1\int_0^1\int_0^1 e^{u^3}u^2vdudvdw=\int_0^1\left\{\int_0^1\left(\int_0^1 e^{u^3}u^2vdv\right)dw\right\}dv=\frac{1}{2}\int_0^1 e^{u^3}u^2du$$

Put $u^3=t\Rightarrow 3u^2du=dt\Rightarrow u^3du=\frac{dt}{3}$.

When $u = 0$, $t = 0$, $u = 1$, $t = 1$

Then integral becomes

$$\iiint e^{(x+y+z)^3}dxdydz=\frac{1}{6}\int_0^1 e^t dt=\frac{1}{6}[e^t]_0^1=\frac{1}{6}\left[e^1-e^0\right]=\frac{e-1}{6}. \text{ Ans.}$$

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