

Q.1) Define the following matrices with examples.

1) Transpose of a matrix and their properties

Sol.) Transpose is an operation of matrices which flips a matrix over its diagonal i.e. it interchanges rows and columns. It is denoted by  $A^T$  (also by  $A'$ ).  $\Rightarrow$  if  $A = [a_{ij}]_{m \times n} \Rightarrow A^T = [a_{ji}]_{n \times m}$

Properties:

$$\textcircled{1} \quad (A^T)^T = A$$

$$\textcircled{5} \quad (kA)' = kA'$$

$$\textcircled{2} \quad (A+B)^T = A^T + B^T$$

$$\textcircled{3} \quad (AB)^T = B^T A^T$$

$$\textcircled{4} \quad \det|A^T| = \det|A|.$$

Eg:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 8 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 3 & 8 \\ 3 & 5 & 5 \end{bmatrix}$$

2) Conjugate of a matrix and their properties.

Conjugate of a matrix is an operation which replaces each element by its complex conjugate. It is denoted by  $\bar{A}$   
i.e. if  $A = [a_{ij}]_{m \times n} \Rightarrow \bar{A} = [\bar{a}_{ij}]_{m \times n}$ .

Properties:-

$$\textcircled{1} \quad (\bar{\bar{A}}) = A.$$

$$\textcircled{3} \quad (\bar{kA}) = k(\bar{A})$$

$$\textcircled{2} \quad (\overline{A+B}) = \bar{A} + \bar{B}$$

$$\textcircled{4} \quad (\bar{AB}) = \bar{A}\bar{B}$$

$$A = \begin{bmatrix} 1+2i & 3 & 4i \\ 1+i & 2+i & 3+i \\ 4 & 5 & i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-2i & 3 & -4i \\ 1-i & 2-i & 3-i \\ 4 & 5 & -i \end{bmatrix}$$

③

Transposed conjugate of a matrix and their properties.

It is an operation of matrices which changes a matrix by ~~its~~ conjugate of the transpose of the matrix or transpose of the conjugate of the matrix. i.e  $[a_{ij}] = [\bar{a}_{ji}]$

It is denoted by  $A^0$  or by  $A^*$ .

Properties:

$$\textcircled{1} \quad (A^0)^0 = A.$$

$$\textcircled{2} \quad (A+B)^0 = A^0 + B^0$$

$$\textcircled{3} \quad (kA)^0 = \bar{k} A^0$$

$$\textcircled{4} \quad (AB)^0 = B^0 \cdot A^0$$

Eg.  $A = \begin{bmatrix} 1+i & i & 2+i \\ 4i & 5i & 3+i \\ 1+2i & 1+5i & 2i \end{bmatrix} \Rightarrow A^0 = \begin{bmatrix} 1-i & -4i & 1-2i \\ -i & -5i & 1-5i \\ 2-i & 3-i & -2i \end{bmatrix}$

④ Symmetric, skew symmetric matrix and their properties.

(i) Symmetric: A <sup>square</sup> matrix which is equal to its transpose.  
i.e If  $a_{ij} = a_{ji}$  for all  $i, j$ , then a square matrix is called symmetric.

Eg.  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$  is a symmetric matrix.

(ii) Skew-symmetric: A square matrix whose transpose is its negative.

i.e. If  $a_{ij} = -a_{ji}$  for all  $i, j$ , then the square matrix is called skew symmetric matrix.

Eg

$$A = \begin{bmatrix} 0 & i & 2 \\ -i & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -i & -2 \\ i & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

as  $A^T = -A$

$\Rightarrow A$  is a skew-symmetric matrix

Properties of symmetric and skew-symmetric matrix:

- ① If  $A$  is a symmetric / skew-symmetric, then  $KA$  is also symmetric / skew-symmetric.

i.e  $(KA)' = KA' = KA$  for symmetric.

$(KA)' = K(-A)$  for skew-symmetric

- ② If  $A, B$  are symmetric (skew-symmetric), then so is  $A+B$ .

$(A+B)' = A+B$  for symmetric

$(A+B)' = -(A+B) = -A-B$  for skew-symmetric.

- ③ If  $AB=BA$  &  $A, B$  are symmetric matrices, then  $AB$  will be a symmetric matrix.

- ④ For any square matrix  $A$ ,  $A-A'$  is a skew-symmetric and  $A+A'$  is a symmetric matrix.

- ⑤ If  $A$  is a symmetric or unsymmetric matrix, then  $B'AB$  will be symmetric or unsymmetric respectively.

- ⑥ Every matrix can be represented as a sum of a symmetric & unsymmetric matrix.

→ Hermitian & skew Hermitian matrices and their properties.

∴ (i) Hermitian matrix: A square matrix  $A = [a_{ij}]$  is said to be Hermitian if the  $(i,j)^{\text{th}}$  element of  $A$  is equal to the conjugate complex of  $(j,i)^{\text{th}}$  element of  $A$ , i.e  $a_{ij} = \bar{a}_{ji}$  for all  $i, j$ .

e.g.

$$\begin{bmatrix} 1 & 2+3i & -4i \\ 2-3i & 2 & 2 \\ 4i & 2 & 3 \end{bmatrix}$$

(ii) Skew Hermitian matrix: A square matrix  $A = [a_{ij}]$  is said to be skew Hermitian if  $(i,j)^{\text{th}}$  element of  $A$  is equal to the negative of the conjugate complex of  $(j,i)^{\text{th}}$  element of  $A$ , i.e  $a_{ij} = -\bar{a}_{ji}$  for all  $i, j$ .

e.g.

$$\begin{bmatrix} 0 & 2+3i & -4i \\ 2-3i & 0 & 2 \\ 4i & -2 & -8i \end{bmatrix}$$

Properties of Hermitian and skew Hermitian matrix :-

- ① If  $A$  is a Hermitian matrix, then  $iA$  is skew Hermitian matrix.
- ② If  $A$  is a skew Hermitian matrix, then  $A$  is a Hermitian matrix.
- ③ If  $A, B$  are Hermitian / skew Hermitian then so is  $A+B$ .

i.e  $(A+B)^H = A^H + B^H = A+B$  Hermitian

$$(A+B)^D = A^D + B^D = -A-B$$
 Skew Hermitian.

- ④ If  $P, Q$  are Hermitian then  $AB+BA$  is Hermitian &  $AB-BA$  is skew Hermitian.

- ⑤ If  $A$  be any square matrix, then  $A+A^{\circ}$ ,  $AA^{\circ}$ ,  $A^{\circ}A$  are all Hermitian and  $A-A^{\circ}$  is skew Hermitian.
- ⑥  $B^{\circ}AB$  is Hermitian / skew Hermitian accordingly as  $A$  is Hermitian or skew Hermitian.
- 7) Every square matrix can be represented as the sum of a Hermitian matrix and a skew Hermitian matrix.  
i.e for  $A$  being a square matrix  
let  $P = A+A^{\circ}$ ,  $Q = A-A^{\circ}$   
 $\Rightarrow A = \frac{1}{2}[P+Q]$ .
- 8) Every real symmetric matrix is Hermitian.
- 9)  $\bar{A}$  is Hermitian / skew Hermitian accordingly as  $A$  is Hermitian or skew-Hermitian.
- 10) Every square matrix  $A$  can be expressed as  $P+iQ$  where  $P$  &  $Q$  are Hermitian matrices.  
for a sq. matrix  $A$ ,  $P = \frac{1}{2}(A+A^{\circ})$ ,  $Q = \frac{1}{2i}(A-A^{\circ})$ .  
 $A = P+iQ$ .
- 11) For every Hermitian matrix  $A$  can be written as  $A=B+iC$ , where  $B$  is real and symmetric &  $C$  is real and skew symmetric.  
for sq. matrix  $A$

$$B = \frac{1}{2}(A+\bar{A}) \quad C = \frac{1}{2i}(A-\bar{A}).$$

$$A = B+iC.$$

⑥ Normal matrix, Orthogonal (orthonormal) matrix and Unitary matrix.

(i) Normal matrix: A square matrix  $A$  is said to be normal matrix if  $AA^{\theta} = A^{\theta}A$ .

Eg.  $A = \begin{bmatrix} 1 & 10 \\ 0 & 11 \\ 1 & 0 \end{bmatrix} \Rightarrow AA^* = \begin{bmatrix} 2 & 11 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = A^*A$

Properties:

- 1) Among complex matrices, all unitary, Hermitian and skew-Hermitian matrices are normal.
- 2) Among real matrices, all orthogonal, symmetric, skew-symmetric matrices are normal.
- 3) If 2 normal matrices are commute then their sum or product is also a normal matrix.

(ii) Unitary matrix: A square matrix is said to be unitary matrix  $AA^{\theta} = I = A^{\theta}A$ .

Eg.  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow AA^{\theta} = I = A^{\theta}A$ .

Properties:

- 1) Inverse of a unitary matrix is unitary matrix.
- 2) Transpose of a unitary matrix is unitary matrix.
- 3) Product of two unitary matrices is a unitary matrix.

(iii) Orthogonal matrix: An orthogonal matrix is a square matrix with real entries whose columns (or rows) are orthogonal unit vectors (i.e orthogonal). Because the columns are unit vectors in addition to being orthogonal, some use orthonormal to describe such matrices.

Or.

for a square matrix  $A$  if  $AA' = I = A'A$

Eg.  $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$   $A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$  is

$$AA' = I = A'A.$$

Properties :

- ① Inverse of an orthogonal matrix is orthogonal.
- ② Transpose of an orthogonal matrix is orthogonal.
- ③ Product of two orthogonal matrix is an orthogonal matrix.

Q2) Show that  $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$  is a Hermitian matrix.

Sol) let  $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$A$  will be a Hermitian matrix if  $A = \bar{A}'$

$$\bar{A}' = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

$$(\bar{A}') = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

as  $(\bar{A})' = A$

$\Rightarrow A$  is a Hermitian matrix

Q3) Factorize the matrix  $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$  into  $LU$ , where  $L$  is

lower triangular matrix and  $U$  is upper triangular matrix.

Sol)

$$\text{let } L = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \quad U = \begin{bmatrix} m & n & o \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$$

$$LU = A \Rightarrow \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \begin{bmatrix} m & n & o \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} am & an & ao \\ dm & dn+bp & do+bq \\ em & en+fp & eo+fq+cr \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$am = 5 \quad an = -2 \quad ao = 1$$

$$dm = 7 \quad dn + bp = 1 \quad do + bq = -5$$

$$em = 3 \quad en + fp = 7 \quad eo + fq + cr = 4.$$

we have 12 variable & 9 eqn

$\Rightarrow$  we have infinite solutions (We can choose 3 unknown arbitrary and find others).

$$\text{let } a = b = c = 1.$$

$$\Rightarrow m = 5, n = -2, o = 1, d = \frac{7}{5}$$

$$e = \frac{3}{5}, \quad 9 \cdot \frac{7}{5}(-2) + p = 1, \quad 9 \cdot \frac{7}{5}(1) + 1(q) = -5. \\ p = \frac{19}{5} \quad \Rightarrow q = \frac{-32}{5}$$

$$\frac{3}{5}(2) + f\left(\frac{19}{5}\right) = 7 \quad \frac{3}{5}(1) + \frac{41}{19} \cdot \left(\frac{-32}{5}\right) + r = 4.$$

$$\Rightarrow f = \frac{41}{19}, \quad \Rightarrow r = \frac{327}{19}$$

Q.4 Given that  $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ , show that  $(I-A)(I+A)^{-1}$  is a unitary matrix

$$\underline{\text{Sol.4}} \quad I+A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix} \quad |I+A| = 1 - (-1-4) = 6.$$

$(I+A)$  is invertible.

$$\Rightarrow (I+A)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$$

$$I-A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}.$$

$$\Rightarrow (I-A)(I+A)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1-1-4 & -2(1+2i) \\ 2(1-2i) & -1-4+1 \end{bmatrix}$$

$$(I-A)(I+A)^{-1} = \frac{1}{6} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \quad -\textcircled{1}$$

Its conjugate transpose.  $= \frac{1}{6} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} \quad -\textcircled{2}$

$(I-A)(I+A)^{-1}$  is unitary if Product of  $\textcircled{1}$  and  $\textcircled{2}$  is  $I$ .

$$\text{Product of } \textcircled{1} \text{ and } \textcircled{2} = \frac{1}{36} \begin{bmatrix} -4 & -(2+4i) \\ 2-4i & -4 \end{bmatrix} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 16+4+16 & -4(2+4i)+4(2+4i) \\ -4(2-4i)+4(2-4i) & 4+16+16 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I.$$

$\Rightarrow (I-A)(I+A)^{-1}$  is a unitary matrix.

Q.5) Determine the rank of the following matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Sol5) let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$$R_3 \rightarrow R_3 - 2R_1, \quad R_2 \rightarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

minors :  $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$(2-0) = 2$$

$$+ (2-6)$$

$$= + (1-3)$$

$$= 8$$

$$= 4$$

as none of minors = 0.

$\Rightarrow$  rank of given matrix = 2.

Q.6)

Use Gauss-Jordan method to find inverse of the following matrices:

$$(i) \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Sol: (i) Given matrix is  $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ .

Writing the same matrix side by side with the unit matrix of order 3,

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow 2R_3 - 5R_1, \quad \left[ \begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -5 & 0 & 2 \end{array} \right]$$

$$R_3 \rightarrow 2R_3 + R_2, \quad \left[ \begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array} \right]$$

$$R_2 \rightarrow R_2 + R_3, \quad R_1 \rightarrow R_1 - R_3 \quad \text{we get} \quad \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 11 & -1 & -4 \\ 0 & 2 & 0 & -10 & 2 & 4 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array} \right]$$

$$R_1 \rightarrow 2R_1 - R_2, \quad R_2 \rightarrow \frac{1}{2}R_2, \quad R_3 \rightarrow (-1)R_3$$

$$\left[ \begin{array}{ccc|ccc} 4 & 0 & 0 & 32 & -4 & -12 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array} \right]$$

$$R_1 \rightarrow \frac{1}{4}R_1, \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array} \right] \Rightarrow \text{inverse is} \quad \left[ \begin{array}{ccc} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{array} \right]$$

(ii) Given matrix  $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Writing the same matrix side by side with the unitmatrix of order 3.

$$\begin{bmatrix} 8 & 4 & 3 : & 1 & 0 & 0 \\ 2 & 1 & 1 : & 0 & 1 & 0 \\ 1 & 2 & 1 : & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 - R_2, \quad R_2 \rightarrow 4R_2 - R_1, \quad \begin{bmatrix} 8 & 4 & 3 : & 1 & 0 & 0 \\ 0 & 0 & 1 : & -1 & 4 & 0 \\ 0 & 3 & 1 : & 0 & -1 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_3 \quad \begin{bmatrix} 8 & 4 & 3 : & 1 & 0 & 0 \\ 0 & -3 & 0 : & -1 & 5 & -2 \\ 0 & 3 & 1 : & 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_2 + R_3 \quad \begin{bmatrix} 8 & 4 & 3 : & 1 & 0 & 0 \\ 0 & -3 & 0 : & -1 & 5 & -2 \\ 0 & 0 & 1 : & -1 & 4 & 0 \end{bmatrix} \quad R_2 \rightarrow \left(\frac{1}{3}\right)R_2 \quad \begin{bmatrix} 8 & 4 & 3 : & 1 & 0 & 0 \\ 0 & 1 & 0 : & y_3 - \frac{5}{3} & 2 & y_3 \\ 0 & 0 & 1 : & -1 & 4 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 4R_2 - 3R_3$$

$$\begin{bmatrix} 8 & 0 & 0 : & y_3 & -16y_3 & -8y_3 \\ 0 & 1 & 0 : & y_3 & -5y_3 & 2y_3 \\ 0 & 0 & 1 : & -1 & 4 & 0 \end{bmatrix} \quad R_1 \rightarrow \frac{R_1}{8} \quad \begin{bmatrix} 1 & 0 & 0 : & y_3 & -2y_3 & -1y_3 \\ 0 & 1 & 0 : & y_3 & -5y_3 & 2y_3 \\ 0 & 0 & 1 : & -1 & 4 & 0 \end{bmatrix}$$

Inverse =  $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$

Q.7) If  $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$  find  $A^{-1}$ . Also find two non-singular matrices  $P$  and  $Q$  such that  $PAQ = I$ , where  $I$  is the unit matrix and verify that  $A^{-1} = QP$ .

Sol)

$$|A| = 3(-3+4) - 2(-3+4) = 3-2=1.$$

$$\text{Adj}|A| = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}|A|}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Now

$A = PAQ$ ,  $P$  &  $Q$  are 2 singular unit matrix of some order as of  $A$ .

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$l_3 \rightarrow l_3 - 4l_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = A^{-1}$$

$$\Rightarrow QP = A^{-1}$$

Hence proved.

Q.8) Are the vectors  $x_1 = (1, 3, 4, 2)$ ,  $x_2 = (3, -5, 2, 2)$  and  $x_3 = (2, -1, 3, 2)$  linearly dependent? If so express one of these as a linear combination of the others.

Sol.8 since we know that, that vectors  $x_1, x_2, x_3$  are said to be linearly dependent, if 3 numbers  $\lambda_1, \lambda_2, \lambda_3$  not all zero s.t.  $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$ .

$$\lambda_1 (1, 3, 4, 2) + \lambda_2 (3, -5, 2, 2) + \lambda_3 (2, -1, 3, 2) = 0.$$

$$\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0, \quad 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0.$$

$$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0, \quad 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0.$$

Q.9

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \rightarrow R_2 - 3R_1$ ,  $R_3 \rightarrow R_3 - 4R_1$ ,  $R_4 \rightarrow R_4 - 2R_1$ , we get.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$$

$$2\lambda_2 + \lambda_3 = 0$$

$$\lambda_3 = -2\lambda_2 \text{ and } \lambda_1 = \lambda_2$$

$$\lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3$$

$\Rightarrow$  set of values which satisfies condition

are  $\lambda_1 = 1$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = -2$ . which are not zero.

$\Rightarrow$  the given vectors are linearly dependent.

$$\Rightarrow \text{Relation } x_1 + x_2 - 2x_3 = 0.$$

$$\Rightarrow \text{linear combination} \Rightarrow x_3 = \frac{x_1 + x_2}{2}$$

Q.9) Show that the system of equations  $AX=B$  is consistent i.e possesses a solution iff the coefficient matrix  $A$  and the augmented matrix  $K = (A:B)$  are of the same rank. Otherwise, the system is inconsistent.

1.9) Let

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = k_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = k_m \end{array} \right\} \quad \text{0}$$

be a system of  $m$  non-homogeneous equations in  $n$ -unknowns  $x_1, x_2, \dots, x_n$ .

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}_{m \times 1}$$

Case I rank of  $A$  = rank of  $K$  =  $\delta$  ( $\delta \leq \min(m, n)$ )

Then by suitable row operations, the system of equation  $AX=B$  can be reduced to

$$\left. \begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = l_1 \\ 0x_1 + b_{22}x_2 + \dots + b_{2n}x_n = l_2 \\ \vdots \\ 0x_1 + 0x_2 + \dots + b_{m\delta}x_n = l_\delta \end{array} \right\} - ii^{\circ}$$

and the remaining  $(m-\delta)$  eq<sup>n</sup> being all of the form.

$$0.x_1 + 0x_2 + \dots + 0.x_n = 0$$

The eq<sup>n</sup>'s (i) will have a solution, by choosing  $(n-\delta)$  unknowns arbitrary.

The solution will be unique only when  $\delta=n$ .

Hence the eq<sup>n</sup>'s (i) are consistent i.e possesses solution.

Case II When rank of A (i.e  $\delta$ ) < the rank of K.

In particular, let the rank of K be  $\gamma+1$ .

Then, by suitable row operations, the system of eq's,  $AX=B$  can be reduced to

$$\left. \begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = d_1 \\ b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n = d_2 \\ \vdots \\ b_{\gamma 1}x_1 + b_{\gamma 2}x_2 + \dots + b_{\gamma n}x_n = d_\gamma \\ 0x_1 + 0x_2 + \dots + 0x_n = d_{\gamma+1} \end{array} \right\}$$

and the remaining  $(m-\gamma+1)$  eq's being all of the form

$$0.x_1 + 0.x_2 + \dots + 0.x_n = 0.$$

The  $(\gamma+1)^{\text{th}}$  eq can not be satisfied by any set of values for the unknowns.

Hence, the equations (i) are inconsistent, i.e., does not possess a solution.

Q.10)

Investigate the values of  $K$  the eq's  $x+y+z=1$ ,  $2x+y+4z=k$ ,  $4x+y+10z=k^2$  have a solution and solve them completely in each case.

Sol. 11)

Here the matrix form of the given system of eq's is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Operating  $R_2 \rightarrow R_2 - 2R_1$ ,  $R_3 \rightarrow -4R_1 + R_3$  we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ k^2-4 \end{bmatrix}$$

$R_3 \rightarrow \frac{R_3}{3}$  we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ \frac{k^2-4}{3} \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K-2 \\ \frac{K^2-4}{3} - K+2 \end{bmatrix}$$

$$x+y+z=1, \quad -y+2z=K-2 \quad \text{and} \quad 0=\frac{K^2-4}{3} - K+2.$$

This is only possible i.e. have solution if  $\left(\frac{K^2-4}{3}\right) - K+2=0$ .

$$\Rightarrow K^2-3K+2=0 \Rightarrow K=2, 1.$$

Case I

when  $K=2$ .

$$x+y+z=1.$$

$$y=2z.$$

$$\text{if } z=c \text{, } y=2z \Rightarrow y=2c$$

$$\text{and } x=1-3c.$$

$$\text{At } K=2, \quad x=1-3c, \quad y=2c, \quad z=c.$$

Case II when  $K=1$ , then  $-y+2c=-1 \Rightarrow y=1+2c$ .

$$x=1-2c-c=-3c$$

$$\text{At } K=1 \quad x=-3c, \quad y=1+2c, \quad z=c.$$

~~which is~~

Q10) Investigate the values of  $\lambda$  and  $\mu$  so that eq's.

$$2x+3y+5z=9, \quad 7x+3y-2z=8, \quad 2x+3y+\lambda z=\mu \text{ have.}$$

(i) no solution.

(ii) unique solution

(iii) an infinite number of solutions

Sol-10) The given set of eq's can be written as.

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$AX=B \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augmented matrix  $K = [A:B] \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$

operating  $R_3 \rightarrow R_3 - R_1$ ,  $R_2 \rightarrow 2R_2 - 7R_1$ ,

$$K \sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

(i) If  $\lambda \neq 5$ , we have rank of  $K=3 = \text{rank } A$

$\Rightarrow$  The given system of eq's is again consistent.

Also rank of  $A =$  the no. of unknowns.

$\Rightarrow$  The given system of eq's possesses a unique solution.

Thus  $\lambda \neq 5$ , the eq's possesses a unique solution for any value of  $\mu$ .

(ii) If  $\lambda=5$ , and  $\mu=9$  we have rank  $K = \text{rank } A$ .

$\Rightarrow$  The given system of equations is again consistent.

Also the rank of  $A <$  the number of unknowns.

$\Rightarrow$  The given system of eq's is inconsistent and possesses ~~no solution~~ infinite number of solutions.

(iii) If  $\lambda=5$  and  $\mu \neq 9$ , we have rank of  $K=3$ , rank of  $A=2$ ,

$\Rightarrow$  rank  $K \neq \text{rank } A$ .

$\Rightarrow$  The given system of eq's is inconsistent and possesses ~~infinite number~~ no solutions.

(12) show that the eq's  $3x + 4y + 5z = a$ ,  $4x + 5y + 6z = b$ ,  $5x + 6y + 7z = c$ . do not have a solution unless  $a+b+c=26$ .

$$\text{Let } B = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$AX = B \Rightarrow R_2 \rightarrow 3R_2 - 4R_1$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ c \end{bmatrix}$$

$$R_3 \rightarrow 3R_3 - 5R_1$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ 3c-5a \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{2}R_3 - R_2$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ \frac{3c-6b+3a}{2} \end{bmatrix}$$

If  $\frac{3c-6b+3a}{2} \neq 0$  then eq's are inconsistent.

If  $\rho(A) = \rho(K)$  then equations are consistent. This is possible only when  $\frac{3c-6b+3a}{2} = 0 \Rightarrow 3c-6b+3a=0$ .

$$\Rightarrow a+c=26.$$

The given equations do not have a solution unless  $a+c=26$ .

Q.13)

Define characteristic matrix, characteristic polynomial, characteristic equation, characteristic roots, characteristic vectors.

Sol)

Characteristic matrix : Let  $A = [a_{ij}]_{m \times n}$  be any square matrix of order  $n$  and  $\lambda$  be scalar. Then the matrix is called characteristic matrix, where  $I$  is the unit matrix of order  $n$ .

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

Characteristic Polynomial : The determinant of characteristic matrix is called the characteristic polynomial.

Characteristic Equation :- The equation  $|A - \lambda I| = 0$  is called characteristic equation.

Characteristic Roots :- The roots of characteristic equation i.e. roots of  $|A - \lambda I| = 0$  are called the characteristic roots.

Characteristic vectors : If  $\lambda$  is a characteristic root of an  $n \times n$  matrix  $A$ , then a non zero vector  $X$  (i.e.  $X \neq 0$ ), s.t.  $AX = \lambda X$ , is called a characteristic vector of  $A$  corresponding to characteristic root  $\lambda$ .

- number of  $a_{ij}$ 's [the elements of  $A$ ].

Q.14) If  $\lambda$  be an eigen value of a non-singular matrix  $A$ .  
show that  $\frac{|A|}{\lambda}$  is an eigen value of matrix  $\text{adj}\cdot A$ .

Sol.14) Proof: Since  $\lambda$  be an eigen value of a non-singular matrix  $A \Rightarrow \lambda \neq 0$ .  
Also  $\lambda$  is an eigen value of  $A$  then  $\exists$  a non zero vector  
 $X$  s.t.  $AX = \lambda X$ . Pre-multiplying both sides by  $A$ , we get.

$$(\text{adj} A)(AX) = (\text{adj} A)(\lambda X) \Rightarrow [(\text{adj} A)A]X = \lambda [(\text{adj} A)X].$$

$$|A| \cdot AX = \lambda (\text{adj} A)X \quad \left[ \because A^{-1} = \frac{\text{adj} A}{|A|} \Rightarrow \text{adj} A \cdot A = |A|I \right]$$

$$|A|X = \lambda (\text{adj} A)X \Rightarrow \frac{|A|X}{\lambda} = (\text{adj} A)X \quad [\because \lambda \neq 0]$$

$$(\text{adj} A)X = \frac{|A|X}{\lambda}$$

Since  $X$  is a non zero vector, therefore  $\frac{|A|}{\lambda}$  is an  
eigen value of the matrix  $\text{adj} A$ .  $\square$

Q.15) Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol.15) Characteristic equations is  $|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - 7\lambda^2 + 36 = 0.$$

Since  $\lambda = -2$  satisfies it, we can write this equation as

$$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6.$$

Therefore eigen values of  $A$  are  $\lambda = -2, 3, 6$ .

If  $x, y, z$  be the components of an eigen vector corresponding to the eigen value  $\lambda$ , we have.

$$[A - \lambda I] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \quad (i)$$

Putting  $\lambda = -2$ , we have  $3x + y + 3z = 0$ ,  $x + 7y + z = 0$ ,  $3x + y + 3z = 0$ .

The first and third equations being the same, we have from first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence, the eigen vectors are  $(-1, 0, 1)$ . Also every non-zero multiple of this vector is an eigen vector corresponding to  $\lambda = -2$ .

Similarly, the eigen vectors corresponding to  $\lambda = 3$  and  $\lambda = 6$  are the arbitrary non-zero multiples of the vectors  $(5, 1, 1)$  and  $(1, 2, 1)$  which are obtained from (i).

Hence, the three eigen vectors may be taken as  $(-1, 0, 1)$ ,  $(1, -1, 1)$ ,  $(1, 2, 1)$ .

Q.16) Show that any eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Sol.16) Proof: Let  $x_1$  and  $x_2$  be two eigen vectors corresponding to two distinct eigen values  $\lambda_1$  and  $\lambda_2$  of a unitary matrix  $A$ . Then by definition.

$$Ax_1 = \lambda_1 x_1, \quad Ax_2 = \lambda_2 x_2.$$

Since  $A$  is unitary matrix, then the eigen values have the absolute value 1.

$$\text{i.e. } |\lambda_1| = 1 \Rightarrow |\lambda_1|^2 = 1 \Rightarrow \lambda_1 \bar{\lambda}_1 = 1 \Rightarrow \bar{\lambda}_1 = \frac{1}{\lambda_1}$$

$$|\lambda_2| = 1 \Rightarrow |\lambda_2|^2 = 1 \Rightarrow \lambda_2 \bar{\lambda}_2 = 1 \Rightarrow \bar{\lambda}_2 = \frac{1}{\lambda_2}$$

Also  $AA^0 = I$

To show:  $x_1$  and  $x_2$  are orthogonal vectors i.e.,  $x_2^0 x_1 = 0$ .

Taking conjugate transpose of (ii), we get.

$$(Ax_2)^0 = (\lambda_2 x_2)^0 = x_2^0 A^0 = \bar{\lambda}_2 x_2^0 \quad \text{---(iii)}$$

From (i) and (iii) we get.

$$(x_2^0 A^0)(Ax_1) = (\bar{\lambda}_2 x_2^0)(\lambda_1 x_1)$$

$$x_2^0 (A^0 A) x_1 = \bar{\lambda}_2 \lambda_1 x_2^0 x_1$$

$$x_2^0 x_1 = \bar{\lambda}_2 \lambda_1 x_2^0 x_1 \quad [A^0 A = I]$$

$$x_2^0 x_1 [\bar{\lambda}_2 \lambda_1 - 1] = 0.$$

also  $\bar{\lambda}_2 = \frac{1}{\lambda_2}$

$$\Rightarrow x_2^0 x_1 \left[ \frac{\lambda_1}{\lambda_2} - 1 \right] = 0.$$

$$x_2^0 x_1 \left[ \frac{\lambda_1 - \lambda_2}{\lambda_2} \right] = 0 \quad \text{as } \lambda_1 \neq \lambda_2$$

$$\Rightarrow x_2^0 x_1 = 0.$$

Hence  $x_1$  and  $x_2$  are orthogonal vectors.

Q.17)

State and Prove Cayley Hamilton theorem.

Sol.17)

Cayley Hamilton Theorem :- Every square matrix over the real or complex field satisfies its own characteristic equation.  
i.e. if the characteristic equation for the  $n^{\text{th}}$  order square matrix  $A$  is  $|A - \lambda I| = 0$ .

$$\Rightarrow (-1)^n \lambda^n + [(-1)^{n-1} b_1] \lambda^{n-1} + [(-1)^{n-2} b_2] \lambda^{n-2} + \dots + [(-1)^0 b_n] = 0.$$

$$\Rightarrow (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n] = 0.$$

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n = 0.$$

$$\text{Then } A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0.$$

Proof: As we know, matrix  $A - \lambda I$  is a characteristic matrix of  $A$ .  
This matrix can be written as

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$

This matrix shows that the elements of  $A - \lambda I$  are at most of the 2<sup>nd</sup> degree in  $\lambda$ .

∴ Elements of  $\text{Adj}(A - \lambda I)$  are ordinary polynomials in  $\lambda$  of degree  $(n-1)$  or less.

$$\begin{bmatrix} \lambda^3 + 2\lambda + 1 & 4\lambda^3 + 5\lambda + 3 & 2\lambda + 9 \\ \lambda^3 + 5 & \lambda^3 + 2\lambda & 2\lambda^3 + 5\lambda + 7 \\ 3\lambda^3 + 2\lambda + 5 & \lambda^3 + 4 & \lambda^3 + 8\lambda + 3 \end{bmatrix} = \left[ \lambda^3 + \begin{bmatrix} & & \\ & & \end{bmatrix} \right] \lambda^2 + \left[ \begin{bmatrix} & & \\ & & \end{bmatrix} \right] \lambda + \left[ \begin{bmatrix} & & \\ & & \end{bmatrix} \right]$$

Now  $\text{Adj}(A - \lambda I)$  can be written as matrix polynomials in  $\lambda$ , + [ ] and is given by.

$$\text{Adj}(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \dots + B_{n-2} \lambda + B_{n-1}$$

where  $B_0, B_1, \dots, B_{n-1}$  are matrices of the type  $n \times n$ , whose elements are functions of  $a_{ij}$ 's [the elements of  $A$ ].

Now since  $A \text{adj} A = A/I_n$ .

Replacing  $A$  by  $A - \lambda I$ , we obtain.

$$(A - \lambda I) \text{adj}(A - \lambda I) = |A - \lambda I| I_n.$$

$$(A - \lambda I) [B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + B_{n-2} \lambda + B_{n-1}] = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + \dots + a_n] I_n$$

Comparing coefficients.

$$-IB_0 = (-1)^n I$$

$$AB_1 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - JB_2 = (-1)^n a_2 I.$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$AB_{n-1} = (-1)^n a_n I$$

Pre multiplying by  $A^n, A^{n-1}, \dots, A, I$  and adding successively, we get.

$$0 = (-1)^n [A^n + a_1 A^{n-1} + \dots + a_n I].$$

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0.$$

i.e Every square matrix satisfies its own characteristic equation.

Find the characteristic roots of matrix  $A = \begin{bmatrix} 1 & 4 \\ 1 & 7 \end{bmatrix}$