1^{st} Topic

Triple Integrals

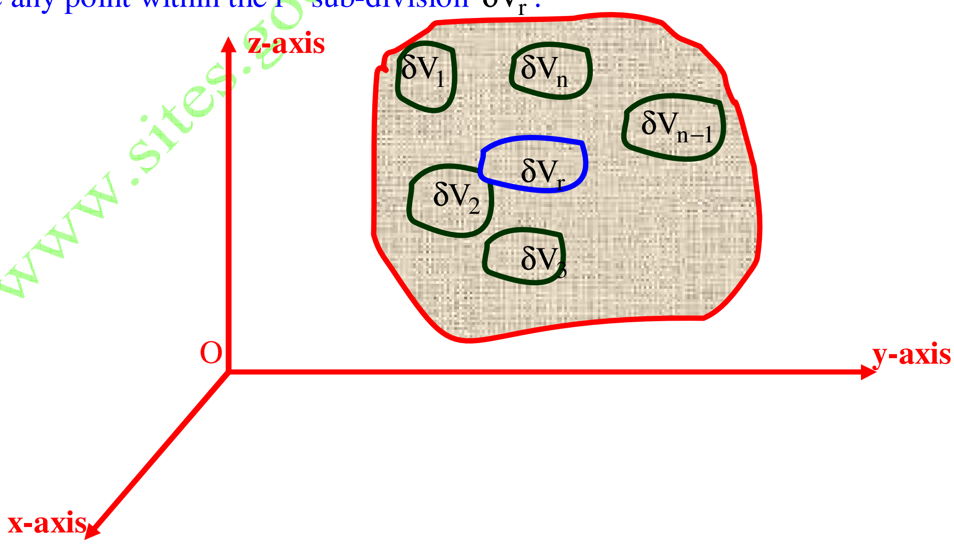
[where limits are given]

(Last updated on 15-07-2013)

(15 Solved problems and 00 Home assignment)

Triple Integrals:

Consider a function f(x, y, z) is defined at every point of the 3-demensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_r, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the r^{th} sub-division δV_r.



Now consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$.

The limit of this sum, if it exist, as $n \to \infty$ and consequently $\delta V_r \to 0$ is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\int f(x, y, z)dV$.

For purposes of the evaluation, it can be expressed as the repeated integrals

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)dxdydz .$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constant or functions of x and y, then this integral is evaluated as follows:

First $f(x, y, z)$ is integrating w. r. t. z between the limits z_1, and z_2 keeping x and y fixed. The resulting expression is integrated w. r. t. y between the limits y_1, and y_2 keeping x constant. The result just obtained is finally integrated w. r. t. x from x_1, and x_2 .

Thus $I = \left[\int_{x_1}^{x_2} \left\{ \int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz \right) dy \right\} dx \right]$,

where the integration is carried out from the innermost bracket to the outermost bracket. This order of integration may be different for different type of limits.

Here we will discuss those problems in triple integrals, where limits are given. By observing the limits, we will decide the order of integration. Since limits are given, so rough sketch of the region of integration is not required.

Q.No.1.: Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)dxdydz$.

Sol.: We have $I = \int_{-1}^{1} \left\{ \int_{0}^{z} \left(\int_{x-z}^{x+z} (x + y + z)dy \right) dx \right\} dz = \int_{-1}^{1} \left(\int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx \right) dz$

$$= \int_{-1}^{1} \left(\int_{0}^{z} \left\{ x[(x+z)-(x-z)] + \frac{1}{2}\left[(x+z)^2 - (x-z)^2\right] + [(x+z)-(x-z)]z \right\} dx \right) dz$$

$$= \int_{-1}^{1}\left(\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}4xz\right]dx\right)dz$$

$$= 2\int_{-1}^{1}\left|\frac{x^2z}{2}+z^2x+\frac{x^2z}{2}\right|_{0}^{z}dz = 2\int_{-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0.\ \text{Ans.}$$

Q.No.2.: Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{(1-x^2)}}\int_{0}^{\sqrt{(1-x^2-y^2)}} xyzdxdydz$.

Sol.: We have $I = \int_{0}^{1} x\left[\int_{0}^{\sqrt{(1-x^2)}} y\left\{\int_{0}^{\sqrt{(1-x^2-y^2)}} zdz\right\}dy\right]dx = \int_{0}^{1} x\left\{\int_{0}^{\sqrt{(1-x^2)}} y.\left|\frac{z^2}{2}\right|_{0}^{\sqrt{(1-x^2-y^2)}} dy\right\}dx$

$$= \int_{0}^{1} x\left\{\int_{0}^{\sqrt{(1-x^2)}} y.\frac{1}{2}\left(1-x^2-y^2\right)dy\right\}dx = \frac{1}{2}\int_{0}^{1} x\left|\left(1-x^2\right)\frac{y^2}{2}-\frac{y^4}{4}\right|_{0}^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8}\int_{0}^{1}\left[\left(1-x^2\right)^2.2x-\left(1-x^2\right)^4.x\right]dx = \frac{1}{8}\int_{0}^{1}\left(x-2x^3+x^5\right)dx$$

$$= \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_{0}^{1} = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}.\ \text{Ans.}$$

Q.No.3.: Evaluate $\int_{0}^{1}\int_{0}^{2}\int_{1}^{2} x^2yz\,dz\,dy\,dx$.

Sol.: We have $I = \int_{0}^{1}\int_{0}^{2}\int_{1}^{2} x^2yz\,dz\,dy\,dx = \int_{0}^{1}\left[\int_{0}^{2}\left(\int_{1}^{2} x^2yzdz\right)dy\right]dx$

$$= \int_{0}^{1}\left[\int_{0}^{2}\left(\frac{x^2yz^2}{2}\right)_{1}^{2}dy\right]dx = \int_{0}^{1}\left[\int_{0}^{2}\left(2x^2y-\frac{x^2y}{2}\right)dy\right]dx$$

$$= \int_{0}^{1}\left[\frac{2x^2y^2}{2}-\frac{x^2y^2}{4}\right]_{0}^{2}dx = \int_{0}^{1}\left[4x^2-x^2\right]dx = \int_{0}^{1} 3x^2dx$$

$$= \left[\frac{3x^3}{3}\right]_{0}^{1} = \left[x^3\right]_{0}^{1} = [1-0] = 1.\ \text{Ans.}$$

Q.No.4.: Evaluate $\int\limits_{-c}^{c}\int\limits_{-b}^{b}\int\limits_{-a}^{a}\left(x^2+y^2+z^2\right)dx\,dy\,dz$.

Sol.: We have $I=\int\limits_{-b}^{c}\left\{\int\limits_{-a}^{b}\left(\int\limits_{-a}^{a}\left(x^2+y^2+z^2\right)dx\right)dy\right\}dz=8\int\limits_{0}^{c}\left\{\int\limits_{0}^{b}\left[\frac{x^3}{3}+y^2x+z^2x\right]_0^a dy\right\}dz$

$$=8\int\limits_{0}^{c}\left\{\int\limits_{0}^{b}\left[\frac{a^3}{3}+ay^2+az^2\right]dy\right\}dz=8\int\limits_{0}^{c}\left[\frac{a^3}{3}y+\frac{ay^3}{3}+az^2y\right]_0^b dz$$

$$=8\int\limits_{0}^{c}\left[\frac{a^3b}{3}y+\frac{ab^3}{3}+abz^2\right]dz=8\left[\frac{a^3b}{3}z+\frac{ab^3}{3}z+\frac{abz^3}{3}\right]_0^c dz$$

$$=\frac{8}{3}\left[a^3bc+ab^3c+abc^3\right]=\frac{8}{3}abc\left[a^2+b^2+c^2\right].\ \text{Ans.}$$

Q.No.5.: Evaluate $\int\limits_{0}^{4}\int\limits_{0}^{2\sqrt{z}}\int\limits_{0}^{\sqrt{\left(4z-x^2\right)}}dy\,dx\,dz$.

Sol.: We have $I=\int\limits_{0}^{4}\left\{\int\limits_{0}^{2\sqrt{z}}\left(\int\limits_{0}^{\sqrt{4z-x^2}}dy\right)dx\right\}dz=\int\limits_{0}^{4}\left\{\int\limits_{0}^{2\sqrt{z}}\left[y\right]_0^{\sqrt{4z-x^2}}dx\right\}dz$

$$=\int\limits_{0}^{4}\left\{\int\limits_{0}^{2\sqrt{z}}\sqrt{4z-x^2}\,dx\right\}dz$$

Put $2\sqrt{z}=\rho\Rightarrow 4z=\rho^2$.

$\therefore I=\int\limits_{0}^{4}\left\{\int\limits_{0}^{\rho}\sqrt{\left(\rho^2-x^2\right)}dx\right\}dz$.

Put $x=\rho\sin\theta\Rightarrow dx=\rho\cos\theta d\theta$.

When $x=0,\ \theta=0;\quad x=\rho,\ \theta=\frac{\pi}{2}$.

$\therefore I=\int\limits_{0}^{4}\left\{\int\limits_{0}^{\pi/2}\sqrt{\rho^2-\rho^2\sin^2\theta}.\rho\cos\theta d\theta\right\}dz=\int\limits_{0}^{4}\left\{\int\limits_{0}^{\pi/2}\rho^2\cos^2\theta d\theta\right\}dz=\int\limits_{0}^{4}4z\left(\frac{1}{2}\times\frac{\pi}{2}\right)dz$

$$= \pi\int_0^4 z\,dz = \pi\left[\frac{z^2}{2}\right]_0^4 = \pi.\frac{16}{2} = 8\pi. \text{ Ans.}$$

Q.No.6.: Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}dz\,dy\,dx$.

Sol.: We have $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}dz\,dy\,dx = \int_0^a \left[\int_0^x \left(\int_0^{x+y} e^{x+y+z}dz\right)dy\right]dx$

$$= \int_0^a \left[\int_0^x \left(e^{2(x+y)} - e^{x+y}\right)dy\right]dx = \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right]dx = \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x\right]_0^a$$

$$= \frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^a - \frac{1}{8} + 1 + \frac{1}{2} - 1$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}. \text{ Ans.}$$

Q.No.7.: Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy$.

Sol.: We have $I = \int_1^e \left\{\int_1^{\log y}\left(\int_1^{e^x} \log z\,dz\right)dx\right\}dy = \int_1^e dy \int_1^{\log y} dx\left(\int_1^{e^x} \log z\,dz\right)$

$$= \int_1^e dy \int_1^{\log y} dx\left[\log z.z.z\right]_1^{e^x} = \int_1^e dy \int_1^{\log y} dx\left[\log e^x.e^x - 1\log z\right]$$

$$= \int_1^e dy \int_1^{\log y} dx\left[x.e^x - e^x + 1\right]dx = \int_1^e dy\left[x.e^x - e^x - e^x + x\right]_1^{\log y}$$

$$= \int_1^e dy\left[x.e^x - 2e^x + x\right]_1^{\log y}$$

$$= \int_1^e \left(\log y e^{\log y} - 2e^{\log y} + \log y - 1.e^1 + 2e^1 - 1\right)dy$$

$$=\int_1^e (y\log y-2y+\log y-e-1).dy$$

$$=\left[\frac{y^2}{2}\log y-\frac{y^2}{4}+y\log y-y-y^2+y-y\right]_1^e$$

$$=\left(\frac{e^2}{2}-\frac{e^2}{4}+e-e-e^2+e^2-e\right)-\left(-\frac{1}{4}-1-1+e-1\right)$$

$$=\frac{e^2}{4}-2e+\frac{13}{4}=\frac{1}{4}\left(e^2-8e+13\right).\ \text{Ans.}$$

Q.No.8.: Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\left(a^2-r^2\right)/a} rdz\,dr\,d\theta$.

Sol.: We have $\displaystyle I=\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\left(a^2-r^2\right)/a} rdz\,dr\,d\theta=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}\left(\int_0^{\left(a^2-r^2\right)/a} r.dz\right)dr\right]d\theta$

$$=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}(rz)_0^{\left(a^2-r^2\right)/a}.dr\right]d\theta=\int_0^{\pi/2}\left[\int_0^{a\sin\theta}\frac{r\left(a^2-r^2\right)}{a}.dr\right]d\theta$$

$$=\int_0^{\pi/2}\left[a\frac{r^2}{2}-\frac{r^4}{4a}\right]_0^{a\sin\theta}d\theta=\int_0^{\pi/2}\left[\frac{a}{2}\left(a^2\sin^2\theta\right)-\frac{a^4\sin^4\theta}{4a}\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{a^3}{2}\sin^2\theta-\frac{a^3}{4}\sin^4\theta\right]d\theta=\frac{a^3}{2}\left[\frac{1}{2}\times\frac{\pi}{2}\right]-\frac{a^3}{4}\left[\frac{3.1}{4.2}\times\frac{\pi}{2}\right]$$

$$=\frac{a^3\pi}{8}-\frac{3a^3\pi}{64}=\left(\frac{8-3}{64}\right)a^3\pi=\frac{5a^3\pi}{64}.\ \text{Ans.}$$

Q.No.9.: Evaluate $\displaystyle\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy$.

Sol.: $\displaystyle\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy=\int_0^1\left[\int_{y^2}^1\left(\int_0^{1-x} xdz\right)dx\right]dy=\int_0^1\left[\int_{y^2}^1 (xz)_0^{1-x}dx\right]dy$

$$= \int_0^1 \left[\int_{y^2}^1 x(1-x)dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy = \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{y^4}{2} - \frac{y^6}{3} \right) \right] dy$$

$$= \left[\frac{y}{2} - \frac{y}{3} - \frac{y^5}{10} + \frac{y^7}{21} \right]_0^1 = \left[\frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{21} \right]$$

$$= \frac{105 - 70 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}. \text{ Ans.}$$

Q.No.10.: Evaluate $\int_{-3}^{3} \int_0^1 \int_1^2 (x+y+z) dz dy dx$.

Sol.: $I = \int_{-3}^{3} \int_0^1 \int_1^2 (x+y+z) dz dy dx = \int_{-3}^{3} \left[\int_0^1 \left\{ \int_1^2 (x+y+z) dz \right\} dy \right] dx$

$$= \int_{-3}^{3} \left[\int_0^1 \left\{ xz + yz + \frac{z^2}{2} \right\}_1^2 dy \right] dx = \int_{-3}^{3} \left[\int_0^1 (2x+2y+2) - \left(x + y + \frac{1}{2} \right) dy \right] dx$$

$$= \int_{-3}^{3} \left[2xy + y^2 + 2y - xy - \frac{y^2}{2} - \frac{y}{2} \right]_0^1 dx = \int_{-3}^{3} \left(2x + 1 + 2 - x - \frac{1}{2} - \frac{1}{2} \right) dx$$

$$= \int_{-3}^{3} (x+3-1) dx = \int_{-3}^{3} (x+2) dx = \left[\frac{x^2}{2} + 2x \right]_{-3}^{3} = \left(\frac{9}{2} + 6 \right) - \left(\frac{9}{2} - 6 \right)$$

$$= \frac{21}{2} + \frac{3}{2} = \frac{24}{2} = 12. \text{ Ans.}$$

Q.No.11.: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx$.

Sol.: $$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \frac{y}{2}\sqrt{1-x^2-y^2} + \frac{\left(\sqrt{1-x^2}\right)^2}{2} \sin^{-1}\frac{y}{\sqrt{1-x^2}} \right] dx$$

$$= \int_0^1 \left[\frac{\sqrt{1-x^2}}{2} \times 0 + \frac{1-x^2}{2} \sin^{-1}\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \right] dx = \int_0^1 \left[\frac{\left(1-x^2\right)}{2} \sin^{-1} 1 \right] dx$$

$$= \int_0^1 \left[\frac{\left(1-x^2\right)}{2} \times \frac{\pi}{2} \right] dx = \frac{\pi}{4} \int_0^1 \left(1-x^2\right) dx = \frac{\pi}{4}\left[x - \frac{x^3}{3} \right]_0^1$$

$$= \frac{\pi}{4}\left[1-\frac{1}{3}\right] = \frac{\pi}{4} \times \frac{2}{3} = \frac{\pi}{6}.\ \text{Ans.}$$

Q.No.12.: Evaluate $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{mx} z^2 dzdydx$.

Sol.: $$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\int_0^{mx} z^2 dz \right] dydx = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3} \right]_0^{mx} dydx$$

$$= \int_0^a \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\frac{m^3x^3}{3} \right) dy \right] dx = \int_0^a \frac{m^3x^3}{3} \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \right] dx = \int_0^a \frac{m^3x^3}{3} [y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{m^3x^3}{3} \left[\sqrt{a^2-x^2} + \sqrt{a^2-x^2} \right] dx = \int_0^a \left[\frac{2}{3} m^3 x^3 \sqrt{a^2-x^2} \right] dx$$

$$= \frac{2}{3} m^3 \int_0^a \left(x^3 \sqrt{a^2-x^2} \right) dx$$

Put $x = a\sin\theta \qquad \therefore dx = a\cos\theta d\theta$

$$= \frac{2}{3} m^3 \int_0^{\pi/2} a^3 \sin^3\theta \sqrt{a^2 - a^2\sin^2\theta}.a\cos\theta d\theta = \frac{2}{3} m^3 \int_0^{\pi/2} a^5 \sin^3\theta \sqrt{1-\sin^2\theta}.\cos\theta d\theta$$

$$= \frac{2}{3}a^5m^3 \int_0^{\pi/2} \sin^3\theta.\cos^2\theta d\theta$$

Now $\cos\theta = t$, $\therefore -\sin\theta d\theta = dt$

$$= \frac{2}{3}a^5m^3 \int_0^1 -\left(1-t^2\right)t^2 dt = \frac{2}{3}a^5m^3 \int_0^1 \left(t^2 - t^4\right)dt = \frac{2}{3}m^3a^5\left(\frac{t^3}{3} - \frac{t^5}{5}\right)_0^1 dt$$

$$= \frac{2}{3}m^3a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{3}m^3a^5\left(\frac{5-3}{15}\right) = \frac{2}{3}m^3a^5\left(\frac{2}{15}\right) = \frac{4m^3a^5}{45}. \text{ Ans.}$$

Q. No. 13.: Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z} dzdydx$.

Sol.: $I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z} dzdydx = \int_0^{\log 2}\int_0^{x} \left[e^{x+y+z}\right]_0^{x+\log y} dydx$

$$= \int_0^{\log 2}\int_0^{x} \left[e^{x+y+x+\log y} - e^{x+y}\right]dydx = \int_0^{\log 2}\int_0^{x} \left[e^{2x}e^{y}.y - e^{x}.e^{y}\right]dydx$$

$$= \int_0^{\log 2} e^{2x}\int_0^{x}\left[e^{y}.ydy - e^{x}\int_0^{x} e^{y}dy\right]dx = \int_0^{\log 2}\left[e^{2x}\left(y.e^{y} - e^{y}\right)_0^x - e^{x}\left(e^{y}\right)_0^x\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}\left(x.e^{x} - e^{x} + 1\right) - e^{x}\left(e^{x} - 1\right)\right]dx = \int_0^{\log 2}\left[xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^{x}\right]dx$$

$$= \int_0^{\log 2} xe^{3x}dx - \int_0^{\log 2} e^{3x} + \int_0^{\log 2} e^{x} = \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right]_0^{\log 2} - \left[\frac{e^{3x}}{3}\right]_0^{\log 2} + \left[e^{x}\right]_0^{\log 2}$$

$$= \left[\frac{8\log 2}{3} - \frac{8}{9} + \frac{1}{9}\right] - \left[\frac{8}{3} - \frac{1}{3}\right] + [2-1] = \left[\frac{8\log 2}{3} - \frac{7}{9}\right] - \left[\frac{7}{3}\right] + 1$$

$$= \frac{8\log 2}{3} - \frac{7}{9} - \frac{7}{3} + 1 = \frac{8\log 2}{3} - \left(\frac{7+21-9}{9}\right) = \frac{8\log 2}{3} - \left(\frac{28-9}{9}\right)$$

$$= \frac{8\log 2}{3} - \frac{19}{9}. \text{ Ans.}$$

Q.No.14.: Evaluate $\int_0^4 \int_0^{\frac{12-3z}{4}} \int_0^{\frac{12-4y-3z}{6}} dxdydz$.

Sol.: $I=\int_0^4\int_0^{\frac{12-3z}{4}}\int_0^{\frac{12-4y-3z}{6}}dxdydz=\int_0^4\left[\int_0^{\frac{12-3z}{4}}\left\{\int_0^{\frac{12-4y-3z}{6}}dx\right\}dy\right]dz$

$$=\int_0^4\left[\int_0^{\frac{12-3z}{4}}\frac{12-4y-3z}{6}dy\right]dz=\frac{1}{6}\int_0^4 12[y]_0^{\frac{12-3z}{4}}-\frac{4}{2}\left[y^2\right]_0^{\frac{12-3z}{4}}-3z[y]_0^{\frac{12-3z}{4}}dz$$

$$=\frac{1}{6}\int_0^4 12\left(\frac{12-3z}{4}\right)-2\left(\frac{12-3z}{4}\right)^2-3z\left(\frac{12-3z}{4}\right)dz$$

$$=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[12-\frac{2(12-3z)}{4}-3z\right]dz=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[\frac{(24-12+3z-6z)}{2}\right]dz$$

$$=\frac{1}{6}\int_0^4\left(\frac{12-3z}{4}\right)\left[12-\frac{2(12-3z)}{4}-3z\right]dz=\frac{1}{8}\int_0^4\left[(4-z)\frac{(12-3z)}{2}\right]dz$$

$$=\frac{3}{16}\int_0^4(4-z)^2dz=\frac{3}{16}\left[\frac{-(4-z)^3}{3}\right]_0^4=\frac{3}{16}\times\frac{4^3}{3}=4.\text{ Ans.}$$

Q.No.15.: Evaluate $\int_0^1\int_0^{1-x}\int_0^{x+y}e^x dxdydz$.

Sol.: $I=\int_0^1\int_0^{1-x}\int_0^{x+y}e^x dxdydz=\int_0^1\left\{\int_0^{1-x}\left(\int_0^{x+y}e^x dz\right)dy\right\}dx$

$$=\int_0^1\left\{\int_0^{1-x}\left(e^x z\right)_0^{x+y}dy\right\}dx=\int_0^1\left\{\int_0^{1-x}e^x(x+y)dy\right\}dx=\int_0^1\left\{e^x\left(xy+\frac{y^2}{2}\right)_0^{1-x}\right\}dx$$

$$=\int_0^1\left\{e^x\left(x(1-x)+\frac{(1-x)^2}{2}\right)\right\}dx=\int_0^1\left\{e^x\left(x-x^2+\frac{1+x^2-2x}{2}\right)\right\}dx$$

$$=\int_0^1\left\{e^x\left(-\frac{x^2}{2}+\frac{1}{2}\right)\right\}dx=-\frac{1}{2}\left[\left(x^2e^x\right)_0^1-\int_0^1 2xe^x dx\right]+\left(\frac{e^x}{2}\right)_0^1$$

$$= -\frac{1}{2}\left[(e-0) - 2\left\{\left(xe^{x}\right)_0^1 - \int_0^1 e^{x}dx\right\}\right] + \left(\frac{e}{2} - \frac{1}{2}\right)$$

$$= -\frac{e}{2} + \{e - (e-1)\} + \left(\frac{e}{2} - \frac{1}{2}\right) = -\frac{e}{2} + 1 + \left(\frac{e}{2} - \frac{1}{2}\right) = \frac{1}{2}. \text{ Ans.}$$

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