2^{nd} Topic

Matrices

Definitions of special types of Matrices-II

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Here, we will discuss some more definitions:

1. **Transpose of a matrix and their properties**
2. **Conjugate of a matrix and their properties**
3. **Transposed conjugate of a matrix and their properties**
4. **Symmetric, skew-symmetric matrices and their properties**

Complex Matrices:

5. **Hermitian, skew-Hermitian matrices and their properties**
6. **Normal matrix, Orthogonal (orthonormal) matrix and Unitary matrix**

Transpose of a matrix:

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$. Then the $n\times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by the symbol A' or A^T.

Symbolically: If $A = \left[a_{ij}\right]_{m\times n}$ then $A' = \left[b_{ji}\right]_{n\times m}$,

where $b_{ji} = a_{ij}$,

i.e., $(j,\ i)^{th}$ element of A' is the $(i,\ j)^{th}$ element of A.

Transposition: **The operation of interchanging rows with columns is called transposition.**

Example: The transpose of 3×4 matrix $A=\begin{bmatrix}1&2&3&4\\2&3&4&1\\3&4&2&1\end{bmatrix}_{3\times4}$ is the 4×3 matrix

$$A'=\begin{bmatrix}1&2&3\\2&3&4\\3&4&2\\4&1&1\end{bmatrix}_{4\times3}.$$

The first row of A is the first column of A'. The second row of A is the second column of A'. The third row of A is the third column of A'.

Properties of the transpose of matrix:

Theorem: If A' and B' be the transposes of A and B respectively, then

(i) $(A')'=A$,

(ii) $(A+B)'=A'+B'$, A and B being of the same size,

(iii) $(kA)'=kA'$, k being any complex number,

(iv) $(AB)'=B'A'$, A and B being comfortable to multiplication.

Proof:

(i). Let A be an $m\times n$ matrix.

Then A' will be an $n\times m$ matrix.

Therefore, $(A')'$ will be an $m\times n$ matrix.

Thus, the matrices A and $(A')'$ are the same type.

Also, the $(i,\ j)^{th}$ element of $(A')'=$ the $(j,\ i)^{th}$ element of $A'=$ the $(i,\ j)^{th}$ element of A.

Hence $A=(A')'$.

(ii). Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$.

Then A+B will be a matrix of the type $m\times n$ and consequently $(A+B)'$ will be matrix of the type $n\times m$.

Again, A' and B' are both $n\times m$ matrices.

Therefore, the sum $A'+B'$ exist and will also be a matrix of the type $n\times m$.

Further, $(j,\ i)^{th}$ element of $(A+B)'=$ the $(i,\ j)^{th}$ element of $A+B=a_{ij}+b_{ij}$

= the $(i,\ j)^{th}$ element of A + the $(i,\ j)^{th}$ element of B

= the $(j,\ i)^{th}$ element of A' + the $(j,\ i)^{th}$ element of B'

= the $(j,\ i)^{th}$ element of $A'+B'$.

Thus the matrices $(A+B)'$ and $A'+B'$ are the same type and their $(j,\ i)^{th}$ elements are equal. Hence $(A+B)'=A'+B'$.

(iii). Let $A=\left[a_{ij}\right]_{m\times n}$. If k is any complex number, then kA will also be an $m\times n$ matrix and consequently $(kA)'$ will be an $n\times m$ matrix.

Again A' will be an $n\times m$ matrix and therefore kA' will also be $n\times m$ matrix .

Further, the $(j,\ i)^{th}$ element of $(kA)'=$ the $(i,\ j)^{th}$ element of $kA=k.(i, j)^{th}$ element of A

$=k.(j,\ i)^{th}$ element of $A'=$ the $(j,\ i)^{th}$ element of kA'.

Thus, the matrices $(kA)'$ and kA' are the same size and their $(j,\ i)^{th}$ elements are equal. Hence $(kA)'=kA'$.

(iv). Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$,

then $A'=\left[c_{ji}\right]_{n\times m}$, where $c_{ji}=a_{ij}$ and $B'=\left[d_{kj}\right]_{p\times n}$, where $d_{kj}=b_{jk}$.

The matrix AB will be of the type $m\times p$.

Therefore the matrix $(AB)'$ will be of the type $p\times m$.

Again the matrix A' will be of the type $n\times m$ and the matrix B' will be of the type $p\times n$.

Therefore, the product $B'A'$ exists and will be a matrix of the type $p\times m$.

Thus, the matrices (AB) and $(k,\ i)^{th}$ are of the same type.

Now the $(k,\ i)^{th}$ element of $(AB)'=$ the $(i,\ k)^{th}$ element of $AB=\sum_{j=1}^{n} a_{ij}.b_{jk}$

$=\sum_{j=1}^{n} c_{ji}d_{kj}=\sum_{j=1}^{n} d_{kj}c_{ji}=$ the $(k,\ i)^{th}$ element of $B'A'$.

Thus, the matrices $(AB)'$ and $B'A'$ are the same size and their $(k, i)^{th}$ element is equal. Hence $(AB)' = B'A'$.

The above law is called the reversal law for transposes, i.e., the transpose of the product of the transposes taken in reverse order.

Conjugate of a matrix:

Definition: The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Symbolically: If $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$,

where $\bar{a}_{ij}$ denotes the conjugate complex of a_{ij}.

If A be a matrix over the field of real numbers, then obviously $\bar{A}$ coincide with A.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ i & 6 & 9-i \end{bmatrix}$.

Properties of the conjugate of a matrix:

Theorem: If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively, then

(i) $\overline{(\bar{A})} = A$,

(ii) $\overline{(A+B)} = \bar{A} + \bar{B}$,

(iii) $\overline{(kA)} = \bar{k}\,\bar{A}$, k being any complex number,

(iv) $\overline{(AB)} = \bar{A}\,\bar{B}$, A and B conformable to multiplication.

Proof:

(i). Let $A = [a_{ij}]_{m\times n}$.

Then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$, where $\bar{a}_{ij}$ is the conjugate complex of a_{ij}.

Obviously, both A and $\overline{(\bar{A})}$ are matrices of the same type $m \times n$.

The $(i,\ j)^{th}$ element of $\overline{(\bar{A})}$ = the conjugate complex of $(i,\ j)^{th}$ element of $\bar{A}$

= the conjugate complex of $\bar{a}_{ij}$ = $\overline{(\bar{a}_{ij})}$ = a_{ij} = the $(i,\ j)^{th}$ element of A.

Hence $\overline{\left(\overline{A}\right)} = A$.

(ii). Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$.

Then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$ and $\overline{B} = \left[\overline{b}_{ij}\right]_{m\times n}$.

First we see both $\overline{(A+B)}$ and $\overline{A}+\overline{B}$ are $m\times n$ matrices.

Again the $(i,\ j)^{th}$ element of $\overline{(A+B)}$ = the conjugate complex of $(i,\ j)^{th}$ element of A + B

= the conjugate complex of $a_{ij}+b_{ij}$ = $\overline{\left(a_{ij}+b_{ij}\right)} = \overline{a}_{ij}+\overline{b}_{ij}$

= the $(i,\ j)^{th}$ element of $\overline{A}$ + the $(i,\ j)^{th}$ element of $\overline{B}$

= the $(i,\ j)^{th}$ element of $\overline{A}+\overline{B}$.

Hence $\overline{(A+B)} = \overline{A}+\overline{B}$.

(iii). Let $A = \left[a_{ij}\right]_{m\times n}$.

If k is any complex number, then both $\overline{(kA)}$ and $\overline{k}\ \overline{A}$ will be $m\times n$ matrices.

The $(i,\ j)^{th}$ element of $\overline{(kA)}$

= the conjugate complex of the $(i,\ j)^{th}$ element of kA

= the conjugate complex of $ka_{ij} = \overline{\left(ka_{ij}\right)} = \overline{k}\ \overline{a}_{ij}$

$= \overline{k}$. the $(i,\ j)^{th}$ element of $\overline{A}$ = the $(i,\ j)^{th}$ element of $\overline{k}\ \overline{A}$.

Hence $\overline{(kA)} = \overline{k}\ \overline{A}$.

(iv). Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{jk}\right]_{n\times p}$

Then $\overline{A} = \left[\overline{a}_{ij}\right]_{m\times n}$ and $\overline{B} = \left[\overline{b}_{jk}\right]_{n\times p}$.

First we see that both the matrices $\overline{(AB)}$ and $\overline{A}\ \overline{B}$ are of the type $m\times p$.

Again the $(i,\ k)^{th}$ element of $\overline{(AB)}$

= the conjugate complex of the $(i,\ k)^{th}$ element of AB

$= \text{the conjugate complex of } \sum_{j=1}^{n} a_{ij}b_{jk}$

$$= \overline{\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)} = \sum_{j=1}^{n} \overline{a_{ij}b_{jk}} = \sum_{j=1}^{n} \bar{a}_{ij}\bar{b}_{jk}$$

$= \text{the } (i,\ k)^{th} \text{ element of } \bar{A}\,\bar{B}$.

Hence $\overline{(AB)} = \bar{A}\,\bar{B}$. **Hermitian conjugate**, or **transjugate**

Transposed conjugate of a matrix or Hermitian conjugate or Hermitian transpose or Adjoint matrix or Transjugate :

Definition: The **conjugate transpose**, **Hermitian transpose**, or **adjoint matrix** of an *m*-by-*n* matrix *A* with complex entries is the *n*-by-*m* matrix A^* obtained from *A* by taking the transpose and then taking the complex conjugate of each entry (i.e., negating their imaginary parts but not their real parts).

or

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or by A^*.

Obviously, the conjugate of the transpose of A is the same as the transpose of the conjugate of A, i.e., $\overline{(A')} = (\overline{A})' = A^{\theta}$.

Symbolically: If $A = \left[a_{ij}\right]_{m\times n}$, then $A^{\theta} = \left[b_{ji}\right]_{n\times m}$,

where $b_{ji} = \bar{a}_{ij}$,

i.e., $(j,\ i)^{th}$ element of A^{θ} = the conjugate complex of the $(i,\ j)^{th}$ element of A.

Example: If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$

then $A' = \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$ and $\overline{(A')} = A^{\theta} = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$.

Motivation for developing conjugate transpose:

The conjugate transpose can be motivated by noting that complex numbers can be usefully represented by 2×2 skew-symmetric matrices, obeying matrix addition and multiplication:

$$a+ib \equiv \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

An *m*-by-*n* matrix of complex numbers could therefore equally well be represented by a *2m*-by-*2n* matrix of real numbers. It therefore arises very naturally that when transposing such a matrix which is made up of complex numbers, one may in the process also have to take the complex conjugate of each entry.

Properties of transposed conjugate of a matrix:

Theorem: If A^θ and B^θ be the transposed conjugates of A and B respectively, then

(i) $\left(A^\theta\right)^\theta = A$ **,**

(ii) $\left(A+B\right)^\theta = A^\theta + B^\theta$, A and B being the same size,

(iii) $\left(kA\right)^\theta = \bar{k}\ A^\theta$ **,** k being any complex number,

(iv) $\left(AB\right)^\theta = B^\theta A^\theta$ **,** A and B being conformable to multiplication.

Proof: **(i).** $\left(A^\theta\right) = \overline{\left[\left\{\left(\bar{A}\right)'\right\}'\right]} = \overline{\left(\bar{A}\right)} = A$,

since $\left\{\left(\bar{A}\right)'\right\}' = A$.

(ii). $\left(A+B\right)^\theta = \overline{\left\{\left(A+B\right)'\right\}}\ \overline{\left(A'+B'\right)} = \overline{\left(A'\right)} + \overline{\left(B'\right)} = A^\theta + B^\theta$**.**

(iii). $\left(kA\right)^\theta = \overline{\left\{\left(kA\right)'\right\}} = \overline{\left(kA'\right)} = \bar{k}\overline{\left(A'\right)} = \bar{k}A^\theta$**.**

(iv). $\left(AB\right)^\theta = \overline{\left\{\left(AB\right)'\right\}} = \overline{\left(B'A'\right)} = \overline{\left(B'\right)}\overline{\left(A'\right)} = B^\theta A^\theta$**.**

Thus, the reversal law holds for the transposed conjugate also.

Symmetric and skew-symmetric matrices:

Symmetric matrix:

Definition: A **symmetric matrix** is a square matrix that is equal to its transpose.

or

A square matrix $A=\left[a_{ij}\right]$ is said to be symmetric if its $(i,\ j)^{th}$ element is the same as its $(j,\ i)^{th}$ element.

Symbolically: If $a_{ij}=a_{ji}$ for all j, i, then a square matrix $A=\left[a_{ij}\right]$ is said to be symmetric.

Examples: The matrices $\begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & s \end{bmatrix}$, $\begin{bmatrix} 1 & i & -2i \\ i & -2 & 4 \\ -2i & 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ are symmetric matrices.

Theorem: A necessary and sufficient condition for a matrix A to be symmetric is that A and A' are equal.

Proof: Necessary condition:

Let $A=\left[a_{ij}\right]$ to be an n-rowed symmetric matrix. Then $a_{ij}=a_{ji}$.

To show $A=A'$.

Now A' will also be an n-rowed square matrix.

Also the $(i,\ j)^{th}$ element of $A'=$ the $(j,\ i)^{th}$ element of $A=a_{ji}$

$$=a_{ij}=\text{the }(i,\ j)^{th}\text{ element of A.}$$

Hence $A'=A$.

Sufficient condition:

Let if $A'=A$, then A must be a square matrix.

To show; A is symmetric.

Also $(i,\ j)^{th}$ element of A = the $(i,\ j)^{th}$ element of A' $\quad [\because A=A']$

$$=\text{the }(j,\ i)^{th}\text{ element of A.}$$

Hence A is a symmetric matrix.

Skew-symmetric matrix or Antisymmetric matrix or antimetric matrix:

Definition: A **skew-symmetric** (or **antisymmetric** or **antimetric**) **matrix** is a square matrix *A* whose transpose is also its negative

A square matrix $A=\left[a_{ij}\right]$ is said to be skew-symmetric if the $(i,\ j)^{th}$ element of A is negative of the $(j,\ i)^{th}$ element of A.

$$\text{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)=\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\text{Skew}(\bar{a})\bar{x}=\bar{a}\times\bar{x}$$

Symbolically: If $a_{ij}=-a_{ji}$ for all i, j, then a square matrix $A=\left[a_{ij}\right]$ is said to be skew-symmetric.

Result: Show that the diagonal elements of a skew-symmetric matrix are all zero:

Proof: If A is a skew-symmetric matrix, then $a_{ij}=-a_{ji}$. [by definition]

$\therefore a_{ii}=-a_{ii}$ for all values of i .

$\Rightarrow 2a_{ii}=0 \Rightarrow a_{ii}=0$.

Thus, the **diagonal elements** of a skew-symmetric matrix are **all zero**.

Examples: The matrices $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Theorem: A necessary and sufficient condition for a matrix A to be skew-symmetric is that $A=-A'$.

Proof: Necessary condition:

Let A be an n-rowed skew-symmetric matrix. Then $a_{ij}=-a_{ji}$

To show : $A=A'$.

Now $-A$ and A' are both n-rowed square matrices.

Also the $(i,\ j)^{th}$ element of $A' =$ the $(j,\ i)^{th}$ element of A

$$= a_{ji} = -a_{ij} = \text{the } (i,\ j)^{th} \text{ element of } -A.$$

Hence $A' = -A$.

Sufficient condition:

Let $A' = -A$, then A must be a square matrix.

To show: A is skew-symmetric matrix.

Now the $(i,\ j)^{th}$ element of A= the negative of the $(i,\ j)^{th}$ element of A' $[\because A = -A']$

$$= \text{the negative of the } (j,\ i)^{th} \text{ element of A.}$$

Hence, A is a skew-symmetric matrix.

Some important properties of symmetric and skew-symmetric matrices:

(1). If A is a symmetric (skew-symmetric) matrix, then show that kA is also symmetric (skew symmetric).

Proof: (i). Let A be symmetric matrix. Then $A' = A$.

We have $(kA)' = kA'$ $\quad [\because A' = A]$

$= kA.$

Since $(kA)' = kA$, therefore kA is a symmetric matrix.

(ii). Let A be skew symmetric matrix. Then $A' = -A$.

We have $(kA)' = kA' = k(-A)$ $\quad [\because A' = -A]$

$= -(kA)$.

Since $(kA)' = -(kA)$, therefore, kA is a skew-symmetric matrix.

(2). If A, B are symmetric (skew-symmetric), then so is also A + B.

Proof: (i). Let A and B be two symmetric matrices of the same order.

Then $A' = A$ and $B' = B$.

Now $(A+B)' = A' + B' = A + B$.

Since $(A+B)' = A + B$, therefore, A + B is a symmetric matrix.

(ii). Let A and B be two skew-symmetric matrices of the same order.

Then $A' = -A$ and $B' = -B$.

Now $(A+B)' = A' + B' = (-A) + (-B) = -(A+B$.

Since $(A+B)' = -(A+B)$, therefore, A + B is a skew-symmetric matrix.

(3). If A and B are symmetric matrices, then show that AB is symmetric if and only if A and B commute i. e. AB = BA.

Proof: It is given that A and B are two symmetric matrices.

Therefore $A' = A$ and $B' = B$.

Now suppose that AB = BA.

Then to prove that AB is symmetric.

We have $(AB)' = B'A' = BA$ $[\because A' = A, \quad B' = B]$

$= AB$ $[\because AB = BA]$

Since $(AB)' = AB$, therefore AB is symmetric matrix.

Conversely, suppose that AB is a symmetric matrix.

Then to prove that AB + BA.

We have $AB = (AB)'$ $[\because \text{AB is a symmetric matrix}]$

$= B'A' = BA$.

(4). If A be any matrix, then prove that AA' and $A'A$ are both symmetric matrices.

Proof: Let A be any matrix.

We have $(AA')' = (A')'A'$ [by reversal law for transposes]

$= AA'$ $[\because (A')' = A]$

Since $(AA')' = AA'$, therefore AA' is a symmetric matrix.

Again $(AA')' = A'A$, therefore $A'A$ is a symmetric matrix.

(5). If A be any square matrix, then show that $A + A'$ is a symmetric and $A - A'$ is skew-symmetric.

Proof: We have $(A + A')' = A' + (A')' = A' + A = A + A'$.

Hence $A + A'$ is symmetric.

Again $(A - A')' = A' - (A')' = A' - A = -(A - A')$.

Hence, $A - A'$ is skew-symmetric.

(6). Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Proof: Case I. Let A be a symmetric matrix. Then $A'=A$.

Now $(B'Ab)'=B'A'(B')'$, by the reversal law for the transposes

$$=B'A'B \qquad [\text{since } (B')'=B]$$

$$=B'AB.$$

Hence, $B'AB$ is symmetric.

Case II. Let A be a skew symmetric matrix. Then $A'=-A$.

Now $(B'AB)'=B'A'(B')'=B'A'B=B'(-A)B$,

$$=-(B'A)B=-B'AB$$

Hence, $B'AB$ is skew-symmetric.

(7). Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof: Let A be any square matrix.

We can write $A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')=P+Q$, say,

where $P=\frac{1}{2}(A+A')$ and $Q=\frac{1}{2}(A-A')$.

We have $P'=\left\{\frac{1}{2}(A+A')\right\}'=\frac{1}{2}(A+A')' \qquad [\because (kA)'=kA']$

$$=\frac{1}{2}\{A'+(A')'\} \qquad [\because (A+B)'=A'+B']$$

$$=\frac{1}{2}(A'+A) \qquad [\because (A')'=A]$$

$$=\frac{1}{2}(A'+A)=P.$$

Therefore, P is symmetric matrix.

Again $Q'=\left\{\frac{1}{2}(A-A')\right\}'=\frac{1}{2}(A-A')'=\frac{1}{2}(A-A')'=\frac{1}{2}\{A'-(A')'\}$

$$=\frac{1}{2}(A'-A)=-\frac{1}{2}(A-A')=-Q.$$

Therefore, Q is a skew-symmetric matrix.

Thus we have expressed the square matrix A as the sum of a symmetric and a skew-symmetric matrix.

To prove: The above representation is unique.

Let A= R + S be another such representation of A, where R is symmetric and S skew-symmetric.

Then to prove that R = P and S = Q.

We have $A' = (R+S)' = R' + S' = R - S \qquad [\because R' = R \text{ and } S' = -S]$

$\therefore A + A' = 2R$ and $A' - A = 2S$.

This gives $R = \frac{1}{2}(A + A')$ and $S = \frac{1}{2}(A - A')$.

Thus, R = P and S + Q.

Therefore, the representation is unique.

Thus, every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Hermitian and skew-Hermitian matrices:

Hermitian matrix (or Self-adjoint matrix):

 

Charles Hermite

(December 24, 1822 – January 14, 1901) ,

(French mathematician)

Definition: A square matrix $A = \left[a_{ij}\right]$ is said to be Hermitian if the $(i,\ j)^{th}$ element of A is equal to the conjugate complex of the $(j,\ i)^{th}$ element of A,

i.e., $a_{ij} = \bar{a}_{ji}$ for all i, j.

or

A Hermitian matrix (or self-adjoint matrix) is a square matrix with complex entries which is equal to its own conjugate transpose

Examples: The matrices $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, $\begin{bmatrix} 1 & 2-3i & 3+4i \\ 2+3i & 0 & 4-5i \\ 3-4i & 4+5i & 2 \end{bmatrix}$ are Hermitian matrices.

Result: Show that every diagonal element of a Hermitian matrix must be real.

Proof: If A is a Hermitian matrix, then $a_{ii} = \bar{a}_{ii}$. [by definition]

$\Rightarrow a_{ii}$ is real for all i.

Thus, **every diagonal element** of a Hermitian matrix must be **real**.

Remarks: A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Thus, a Hermitian matrix is a generalization of a real symmetric matrix as every real symmetric matrix is Hermitian.

Obviously, a necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$.

Skew-Hermitian matrix or Antihermitian matrix:

Definition: A square matrix $A = \left[a_{ij}\right]$ is said to be skew-Hermitian if the $(i,\ j)^{th}$ element of A is equal to the negative of the conjugate complex of the $(j,\ i)^{th}$ element of A, i.e., if $a_{ij} = -\bar{a}_{ji}$ for all i and j.

or

A square matrix with complex entries is said to be **skew-Hermitian** or **antihermitian** if its conjugate transpose is equal to its negative

Result: Show that the diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zero.

Proof: If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii}$. [by definition]

$\Rightarrow a_{ii} + \bar{a}_{ii} = 0$

i.e., a_{ii} must be either a pure imaginary number or must be zero.

Thus, the **diagonal elements** of a skew-Hermitian matrix must be **pure imaginary** numbers or **zero**.

Examples: The matrices $\begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 3+4i \\ -3+4i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

Remarks: A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.

Thus, a skew Hermitian matrix is a generalization of real skew symmetric matrix.

Obviously, a necessary and sufficient condition for a matrix A to be skew-Hermitian is that $A^{\theta} = -A$.

Some important properties of Hermitian and skew Hermitian matrices:

(1). If A is Hermitian matrix, show that iA is skew-Hermitian.

Proof: Let A be a Hermitian matrix. Then $A^{\theta} = A$.

We have $(iA)^{\theta} = \bar{i}A^{\theta}$ $\left[\because (kA)^{\theta} = \bar{k}A^{\theta}\right]$

$=(-i)A^{\theta}$ $\quad [\because \bar{i}=-i]$

$=-(iA^{\theta})=-(iA)$ $\quad [\because A^{\theta}=A]$.

Since $(iA)^{\theta}=-(iA)$, therefore iA is a skew-Hermitian matrix.

(2). If A is skew-Hermitian matrix, show that iA is Hermitian.

Proof: Let A be a skew-Hermitian matrix. Then $A^{\theta}=-A$.

We have $(iA)^{\theta}=\bar{i}A^{\theta}=(-i)A^{\theta}=-(iA^{\theta})$

$=-\{i(-A)\}=-\{i(-A)\}$ $\quad [\because A^{\theta}=-A]$.

$=-\{-(iA)\}=iA$.

Since $(iA)^{\theta}=iA$, therefore iA is a Hermitian matrix.

(3). If A, B are Hermitian or skew-Hermitian, then so is also A + B.

Proof: (i). Let A and B be two Hermitian matrices of the same order.

Then $A^{\theta}=A$ and $B^{\theta}=B$.

Now $(A+B)^{\theta}=A^{\theta}+B^{\theta}=A+B$.

Since $(A+B)^{\theta}=A+B$, therefore A + B is a Hermitian matrix.

(ii). Let A and B be two skew-Hermitian matrices of the same order.

Then $A^{\theta}=-A$ and $B^{\theta}=-B$.

Now $(A+B)^{\theta}=A^{\theta}+B^{\theta}=-A+(-B)=-(A+B)$.

Since $(A+B)^{\theta}=-(A+B)$, therefore A + B is a skew-Hermitian matrix.

(4). A and B are Hermitian; show that $AB+BA$ is Hermitian and $AB-BA$ is skew-Hermitian.

Proof: Let A and B be two Hermitian matrices of the same order.

Then $A^{\theta}=A$ and $B^{\theta}=B$.

Now $(AB+BA)^{\theta}=(AB)^{\theta}+(BA)^{\theta}=B^{\theta}A^{\theta}+A^{\theta}B^{\theta}=BA+AB=AB+BA$.

Hence $AB+BA$ is Hermitian.

Again $(AB-BA)^{\theta}=(AB)^{\theta}-(BA)^{\theta}=B^{\theta}A^{\theta}-A^{\theta}B^{\theta}=BA-AB=-(AB-BA)$.

Hence $AB-BA$ is skew-Hermitian.

(5). If A be any square matrix, prove that $A+A^{\theta}$, AA^{θ}, $A^{\theta}A$ are all Hermitian and $A-A^{\theta}$ is skew-Hermitian.

Proof: The necessary and sufficient condition for a matrix A to be Hermitian is that A^{θ} and A are equal.

(i). $\left(A+A^{\theta}\right)^{\theta}=A^{\theta}+\left(A^{\theta}\right)^{\theta}=A^{\theta}+A=A+A^{\theta}$.

Hence $A+A^{\theta}$ is Hermitian.

(ii). $\left(AA^{\theta}\right)^{\theta}=\left(A^{\theta}\right)^{\theta}A^{\theta}$ [by the reversal law for conjugate transposes].

$=AA^{\theta}$.

Hence AA^{θ} is Hermitian.

(iii). $\left(A^{\theta}A\right)^{\theta}=A^{\theta}\left(A^{\theta}\right)^{\theta}=A^{\theta}A$.

Hence $A^{\theta}A$ is Hermitian.

(iv). The necessary and sufficient condition for a matrix A to be skew-Hermitian is that $-A$ and A^{θ} are equal.

Now $\left(A-A^{\theta}\right)^{\theta}=A^{\theta}-\left(A^{\theta}\right)^{\theta}=A^{\theta}-A=-\left(A-A^{\theta}\right)$

Hence $A-A^{\theta}$ is skew-Hermitian.

(6). Show that the matrix $B^{\theta}AB$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

Proof: Case I. Let A be a Hermitian matrix. Then $A^{\theta}=A$.

Now $B^{\theta}AB=B^{\theta}A^{\theta}\left(B^{\theta}\right)^{\theta}$, by reversal law for the conjugate transposes.

$=B^{\theta}A^{\theta}B=B^{\theta}AB$.

Hence $B^{\theta}AB$ is Hermitian matrix.

Case II. Let A be a skew-Hermitian matrix. Then $A^{\theta}=-A$.

Now $\left(B^{\theta}AB\right)^{\theta}=B^{\theta}A^{\theta}\left(B^{\theta}\right)^{\theta}=B^{\theta}A^{\theta}B=B^{\theta}\left(-A\right)B$

$=-\left(B^{\theta}A\right)B=-B^{\theta}AB$.

Hence $B^{\theta}AB$ is a skew-Hermitian.

(7). Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof: If A is any square matrix, then $A + A^{\theta}$ is Hermitian matrix and $A - A^{\theta}$ is a skew-Hermitian matrix.

Therefore $\frac{1}{2}\left(A + A^{\theta}\right)$ is a Hermitian and $\frac{1}{2}\left(A - A^{\theta}\right)$ is a skew-Hermitian matrix.

Now, we have $A = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right) = P + Q$, say,

where P is Hermitian and Q skew-Hermitian.

Thus every square matrix can be expressed as the sum of a Hermitian and a skew-Hermitian matrix.

Let, Now, $A = R + S$ be another such representation of A, where R is Hermitian and S skew-Hermitian.

Then, $A^{\theta} = \left(R + S\right)^{\theta} = R^{\theta} + S^{\theta} = R - S$.

$\therefore R = \frac{1}{2}\left(A + A^{\theta}\right) = P$ and $S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$.

Thus the representation is unique.

(8). Show that every real symmetric matrix is Hermitian.

Proof: Let $A = \left[a_{ij}\right]_{n\times n}$ be a real symmetric matrix. Then $a_{ij} = a_{ji}$.

Since a_{ji} is a real number, therefore $\overline{a}_{ji} = a_{ji}$.

Consequently $a_{ij} = \overline{a}_{ji}$. Hence A is Hermitian.

(9). Prove that $\overline{A}$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

Proof: Case 1: Suppose A is Hermitian. Then $A^{\theta} = A$.

We are to prove that $\overline{A}$ is Hermitian.

We have $\left(\overline{A}\right)^{\theta} = \left[\overline{\left(\overline{A}\right)}\right]'$ [by definition of conjugate transpose]

$= \left(A\right)'$ $\left[\because \overline{\left(\overline{A}\right)} = A\right]$

$= \left(A^{\theta}\right)'$ $\left[\because A \text{ is Hermitian } \Rightarrow A = A^{\theta}\right]$

$$=\left[\left(\bar{A}\right)'\right]' \qquad \left[\because \left(A'\right)' = A\right]$$

Since $\left(\bar{A}\right)^{\theta} = \bar{A}$, therefore A is Hermitian.

Case 2: Now let us suppose that A is skew-Hermitian. Then $A^{\theta} = -A$.

We have $\left(\bar{A}\right)^{\theta} = \left[\overline{\left(\bar{A}\right)}\right]' = (A)' = \left(-A^{\theta}\right)' = -\left(A^{\theta}\right)' = -\left[\left(\bar{A}\right)\right]' = -\bar{A}$.

Therefore, $\bar{A}$ is also skew-Hermitian.

(10). Show that every square matrix A can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices.

Proof: Let $P = \frac{1}{2}\left(A + A^{\theta}\right)$ and $Q = \frac{1}{2i}\left(A - A^{\theta}\right)$.

Then $A = P + iQ$. (i)

Now $P^{\theta} = \left\{\frac{1}{2}\left(A + A^{\theta}\right)\right\}^{\theta} = \frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left\{A^{\theta} + \left(A^{\theta}\right)^{\theta}\right\} = \frac{1}{2}\left(A^{\theta} + A\right) = \frac{1}{2}\left(A + A^{\theta}\right) = P$.

$\therefore$ P is Hermitian matrix.

Also $Q^{\theta} = \left\{\frac{1}{2i}\left(A - A^{\theta}\right)\right\}^{\theta} = \overline{\left(\frac{1}{2i}\right)}\left(A - A^{\theta}\right)^{\theta} = -\frac{1}{2i}\left\{A^{\theta} - \left(A^{\theta}\right)^{\theta}\right\} = -\frac{1}{2i}\left(A^{\theta} - A\right)$

$$= \frac{1}{2i}\left(A - A^{\theta}\right) = Q.$$

$\therefore$ Q is also a Hermitian.

Thus A can be expressed in the form (i) where P and Q are Hermitian matrices.

To show that the expression (i) for A is unique.

Let $A = R + iS$, where R and S are both Hermitian Matrices.

We have $A^{\theta} = \left(R + iS\right)^{\theta} = R^{\theta} + \left(iS\right)^{\theta} = R^{\theta} + \bar{i}\, S^{\theta} = R^{\theta} - iS^{\theta}$

$= R - iS$ [$\because$ R and S both Hermitian]

$\therefore A + A^{\theta} = (R + iS) + (R - iS) = 2R$.

This gives $R = \frac{1}{2}\left(A + A^{\theta}\right) = P$.

Also $A - A^{\theta} = (R + iS) - (R - iS) = 2iS$.

This gives $S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$.

Hence expression (i) for A is unique.

(11). Prove that every Hermitian matrix A can be written as $A = B + iC$, where B is real and symmetric and C is real and skew-symmetric.

Proof: Let A be a Hermitian matrix. Then $A^{\theta} = A$.

Let us take $B = \frac{1}{2}\left(A + \bar{A}\right)$ and $C = \frac{1}{2i}\left(A - \bar{A}\right)$.

Then obviously both B and C are real matrices.

[Note that if $z = x + iy$ is a complex number, then $\frac{1}{2}\left(z + \bar{z}\right)$ is real and also $\frac{1}{2}\left(z - \bar{z}\right)$ is real]

Now we can write $A = \frac{1}{2}\left(A + \bar{A}\right) + i\left[\frac{1}{2i}\left(A - \bar{A}\right)\right] = B + iC$.

It remains to show that B is symmetric and C is skew-symmetric. We have

$$B' = \left[\frac{1}{2}\left(A + \bar{A}\right)\right]' = \frac{1}{2}\left(A + \bar{A}\right)' = \frac{1}{2}\left[A' + \left(\bar{A}\right)'\right] = \frac{1}{2}\left(A' + A^{\theta}\right) = \frac{1}{2}\left[\left(A^{\theta}\right)' + A\right]\left[\because A^{\theta} = A\right]$$

$$= \frac{1}{2}\left[\left\{\left(\bar{A}\right)'\right\}' + A\right] = \frac{1}{2}\left(\bar{A} + A\right) = B.$$

$\therefore$ B is symmetric.

Also $$C' = \left[\frac{1}{2i}\left(A - \bar{A}\right)\right]' = \frac{1}{2i}\left(A - \bar{A}\right)' = \frac{1}{2i}\left[A' - \left(\bar{A}\right)'\right] = \frac{1}{2i}\left(A' - A^{\theta}\right) = \frac{1}{2i}\left[\left(A^{\theta}\right)' - A\right]$$

$$= \frac{1}{2i}\left(\bar{A} - A\right) = -\frac{1}{2i}\left(A - \bar{A}\right) = -C.$$

$\therefore$ C is symmetric.

Hence the result.

Orthogonal matrix or Orthonormal matrix:

Definition: An **orthogonal matrix** is a square matrix with real entries whose columns (or rows) are orthogonal unit vectors (i.e., orthonormal). Because the columns are unit vectors in addition to being orthogonal, some people use the term **orthonormal** to describe such matrices.

or

A square matrix A is said to be orthogonal matrix if $AA' = I = A'A$.

Another way to define orthogonal matrix:

A square matrix A is said to be orthogonal matrix if $A' = A^{-1}$.

An orthogonal matrix is the real specialization of a unitary matrix, and thus always a normal matrix.

Normal matrix:

Definition: A square matrix A is said to be normal matrix if $AA^\theta = A^\theta A$.

Among complex matrices, all unitary, Hermitian, and skew-Hermitian matrices are normal. Likewise, among real matrices, all orthogonal, symmetric, and skew-symmetric matrices are normal.

However, it is *not* the case that all normal matrices are either unitary or (skew-) Hermitian. As an example, the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is normal because

$$AA^* = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A^*A$$

The matrix A is neither unitary, Hermitian, nor skew-Hermitian.

The sum or product of two normal matrices is not necessarily normal. If they commute, however, then this is true.

If A is both a triangular matrix and a normal matrix, then A is diagonal. This can be seen by looking at the diagonal entries of A^*A and AA^*, where A is a normal, triangular matrix.

Unitary matrix:

Definition: A square matrix A is said to be unitary matrix if $AA^\theta = I = A^\theta A$.

Another way to define unitary matrix:

A square matrix A is said to be unitary matrix if $\overline{A}' = A^{-1}$.

This is a **generalization** of the orthogonal matrix in the complex field.

Some important properties of unitary matrix:

1. Inverse of a unitary matrix is unitary.

Proof: If U is a unitary matrix, then

$\bar{U}' = U^{-1} \Rightarrow U' = \overline{U^{-1}}$

$\therefore \left[\left(U^{-1}\right)^{-1}\right]' = \overline{U^{-1}}$

Writing $U^{-1} = V$, we have $\left[V^{-1}\right]' = \bar{V} \Rightarrow V^{-1} = \bar{V}'$

Thus $V\left(= U^{-1}\right)$ is also unitary.

Remark: Inverse of an orthogonal matrix is orthogonal.

2. Transpose of s unitary matrix is unitary.

Proof: If U is a unitary matrix, $\bar{U}' = U^{-1} \Rightarrow \left(\bar{U}'\right) = U^{-1} \Rightarrow \left[\left(\bar{U}'\right)\right]' = \left[U^{-1}\right]' = [U']^{-1}$

Writing $U' = V$, we have $\bar{V}' = V^{-1}$

Thus V (i.e. U') is also unitary.

Remark: Transpose of an orthogonal matrix is orthogonal.

3. Product of two unitary matrices is a unitary matrix.

Proof: If U and V are unitary matrices then

$U' = \bar{U}^{-1}, \quad V' = \bar{V}^{-1}$

Now, $\left(\overline{UV}\right)^{-1} = \left(\bar{U}\ \bar{V}\right)^{-1} = \bar{V}^{-1}\bar{U}^{-1}$

$= V'U' = \left(UV\right)'$

Thus UV is unitary matrix.

Remark: Product of two orthogonal matrixes is an orthogonal matrix.

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