Differential Calculus

Partial Differentiation

[Transformation of independent variables (Composite Functions), Jacobian, Properties of Jacobians]

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Composite function:

If $u = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots)$ and the independent variables $x_1, x_2, x_3, \ldots\ldots\ldots$ are further functions of other variables $t_1, t_2, t_3, \ldots\ldots\ldots$.

by the relations, $x_1 = \phi(t_1, t_2, t_3, \ldots\ldots\ldots\ldots\ldots)$, $x_2 = \psi(t_1, t_2, t_3, \ldots\ldots\ldots\ldots)$ etc.

Then u is said to be a **composite function** of the variables $t_1, t_2, t_3, \ldots\ldots\ldots$,

For example if u to be function of x, y, i.e. $u = f(x, y)$ and further if x ,y are function of t_1, t_2, i.e. $x = \phi(t_1, t_2)$ and if $y = \psi(t_1, t_2,)$.

Then u is a **composite function** of variables t_1, t_2,

Transformation of independent variables:

Now the necessary formulae for changing of independent variables are obtained:

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}, \quad \frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}, \ldots\ldots\ldots\ldots$$

Further, if $u = f(x, y)$ and if $t_1 = f_1(x, y)$ and $t_2 = f_2(x, y)$.

Then the transformation equations are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial y}.$$

Expansion:

Extending the above results, we may obtain.

In case $u=f(x,y,z)$ and $x=\varphi_1(t_1,t_2,t_3)$, $y=\varphi_2(t_1,t_2,t_3)$, $z=\varphi_3(t_1,t_2,t_3)$.

Then the transformation equations are

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_1},$$

$$\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_2},$$

$$\frac{\partial u}{\partial t_3}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_3}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_3}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t_3}.$$

Further, if $u=f(x,y,z)$, $t_1=f_1(x,y,z)$ $t_2=f_2(x,y,z)$ and $t_3=f_3(x,y,z)$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial x},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial y}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial y},$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial z}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial z}+\frac{\partial u}{\partial t_3}.\frac{\partial t_3}{\partial z}.$$

Jacobian:

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}$,

is called the functional determinant or **Jacobian** of u, v with respect to x, y, and is denoted by the symbol $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u,\ v)}{\partial(x,\ y)}$.

Similarly, if u, v, w are functions of three independent variables x, y, z, then the Jacobian of u, v, w with respect to x, y, z is $J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u,\ v, w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$.

Properties of Jacobians:

I. If u, v are functions of r, s where r, s are functions of x, y

then $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}$.

Proof: Since u, v are composite functions of x, y

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} = u_r r_x + u_s s_x,$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} = u_r r_y + u_s s_y,$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} = v_r r_x + v_s s_x,$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} = v_r r_y + v_s s_y.$$

Now $\frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} . \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$.

Interchanging rows and columns in the second determinant, we get

$$\begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} . \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

II. If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y, with respect to u, v, then $J_1 J_2 = 1$ i. e. $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$.

Proof: Let u = u(x, y) and v = v(x, y), so that u and v are functions of x, y.

Suppose on solving for x and y, we get x = $\phi(u, v)$ and y = $\psi(u, v)$.

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial u} = u_x x_u + u_y y_u$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial v} = u_x x_v + u_y y_v$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial u} = v_x x_u + v_y y_u$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

Now $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} . \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$.

Interchanging rows and columns in the second determinant, we get

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} . \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Now let us solve some more problems:

Q.No.1.: If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$, evaluate $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Sol.: Given $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$.

Now $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2},\quad \frac{\partial\theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{vmatrix}$$

$$=\frac{x^2}{\left(x^2+y^2\right)^{3/2}}+\frac{y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{1}{\sqrt{x^2+y^2}}.$$

Q.No.2.: If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$,

show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$.

Sol.: $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\cos\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

Taking out common factor (r from second column and $r\sin\theta$ from third column)

$$=r^2\sin\theta\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{vmatrix}$$

Expanding by third row

$$=r^2\sin\theta\left\{\cos\theta\begin{vmatrix}\cos\theta\cos\phi & -\sin\phi\\ \cos\theta\sin\phi & \cos\phi\end{vmatrix}+\sin\theta\begin{vmatrix}\sin\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\phi\end{vmatrix}\right\}$$

$$=r^2\sin\theta\left[\cos\theta\left(\cos\theta\cos^2\phi+\cos\theta\sin^2\phi\right)+\sin\theta\left(\sin\theta\cos^2\phi+\sin\theta\sin^2\phi\right)\right]$$

$$=r^2\sin\theta\left(\cos^2\theta+\sin^2\theta\right)=r^2\sin\theta$$

Q.No.3.: If $u=f(y-z,z-x,x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Sol.: Suppose $u_1=y-z, u_2=z-x, u_3=x-y$. (i)

$\therefore u=f(y-z,z-x,x-y)$ becomes $u=f(u_1,u_2,u_3)$. (ii)

From (i) and (ii) we conclude that u is composite function of x, y, z.

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial u_1}.\frac{\partial u_1}{\partial x}+\frac{\partial u}{\partial u_2}.\frac{\partial u_2}{\partial x}+\frac{\partial u}{\partial u_3}.\frac{\partial u_3}{\partial x} \qquad \text{(iii)}$$

Now $\frac{\partial u_1}{\partial x}=0$, $\frac{\partial u_2}{\partial x}=-1$, $\frac{\partial u_3}{\partial x}=1$

$$\therefore \text{(iii) becomes } \frac{\partial u}{\partial x}=-\frac{\partial u}{\partial u_2}+\frac{\partial u}{\partial u_3}. \qquad \text{(iv)}$$

$$\text{Similarly } \frac{\partial u}{\partial y}=-\frac{\partial u}{\partial u_3}+\frac{\partial u}{\partial u_1}, \qquad \text{(v)}$$

$$\text{and } \frac{\partial u}{\partial z}=-\frac{\partial u}{\partial u_3}+\frac{\partial u}{\partial u_2}. \qquad \text{(vi)}$$

Adding (iv), (v) and (vi), we get

$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$, which is the required result**.**

Q.No.4.: If $w=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$,

$$\text{show that } \left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2.$$

Sol.: The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial r}=\frac{\partial w}{\partial x}.\cos\theta+\frac{\partial w}{\partial y}.\sin\theta$$

$$\Rightarrow \frac{\partial w}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta \qquad [\because w=f(x,y)] \qquad \text{(i)}$$

$$\text{Also } \frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta}=\frac{\partial w}{\partial x}(-r\sin\theta)+\frac{\partial w}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta}=-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta. \qquad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2.$$

Q.No.5.: If $z=\sqrt{x^2+y^2}$ and $x^3+y^3+3axy=5a^2$, find the value of $\frac{dz}{dx}$, when $x = y = a$.

Sol.: The given equation are of the form $z = f(x, y)$ and $\phi(x, y) = c$.

$\therefore$ z is the composite function of x.

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial x}.\frac{dx}{dx} + \frac{\partial z}{\partial y}.\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \qquad \text{(i)}$$

Now $\dfrac{\partial z}{\partial x} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}.2x = \dfrac{x}{\sqrt{x^2 + y^2}}$

Similarly, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$, we get

$$3x^2 + 3y^2.\frac{dy}{dx} + 3ay + 3ax.\frac{dy}{dx} = 0 \Rightarrow \left(y^2 + ax\right)\frac{dy}{dx} = -\left(x^2 + ay\right)$$

$$\therefore \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (i), we get $\dfrac{dz}{dx} = \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{y}{\sqrt{x^2 + y^2}}\left(-\dfrac{x^2 + ay}{y^2 + ax}\right)$

$$\left[\frac{dz}{dx}\right]_{\substack{x=a\\y=a}} = \frac{a}{\sqrt{a^2 + a^2}} - \frac{a}{\sqrt{a^2 + a^2}}.\frac{a^2 + a^2}{a^2 + a^2} = 0.$$

Q.No.6.: If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, find $\dfrac{du}{dx}$.

Sol.: Here u is a function of x, y and z while y and z are functions of x.

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial z}{\partial y}\frac{dz}{dx}$$

$$= e^y z.1 + xe^y z.\frac{1}{2}\left(a^2 - x^2\right)^{-1/2}(-2x) + xe^y.2\sin x\cos x$$

$$= e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x\sin 2x\right]. \text{ Ans.}$$

Q.No.7.: If $\varphi(x, y, z) = 0$, show that $\left(\dfrac{\partial y}{\partial z}\right)_x\left(\dfrac{\partial z}{\partial x}\right)_y\left(\dfrac{\partial x}{\partial y}\right)_z = -1$.

Sol.: The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}.$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}}.$$

Similarly, $\left(\frac{\partial x}{\partial y}\right)_y = -\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}}.$

Multiplying, we get $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = -1$. Hence prove.

Q.No.8.: Prove that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$,

where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$.

or

By changing the independent variables u and v to x by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$.

Sol.: Here z is a composite function of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{dx}{du} + \frac{\partial z}{\partial y}.\frac{dy}{du} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial u}(z) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)z \qquad \Rightarrow \frac{\partial}{\partial u} = \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}. \qquad \text{(i)}$$

Also $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{dx}{dv} + \frac{\partial z}{\partial y}.\frac{dy}{dv} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial v}(z)=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right)z \qquad \Rightarrow \frac{\partial}{\partial v}=-\sin\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}. \qquad \text{(ii)}$$

Now we shall make use of the equivalence of operations as given by (i) and (ii)

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)$$

$$=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}\ . \qquad \text{(iii)}$$

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right)$$

$$=\sin^2\alpha\frac{\partial^2 z}{\partial x^2}-\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y}-\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial y\partial x}+\cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$=\sin^2\alpha\frac{\partial^2 z}{\partial x^2}-2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\cos^2\alpha\frac{\partial^2 z}{\partial y^2}. \qquad \text{(iv)}$$

Adding (iii) and (iv), we get $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$. Hence prove.

Q.No.9: If $u=f(r,s)$, $r=x+y$, $s=x-y$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2\frac{\partial u}{\partial r}$.

Sol.: Since $u=f(r,s)$ and r, s are the function of x and y.

$\therefore$ u is the composite function of x and y.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \qquad \left[\because \frac{\partial r}{\partial x}=1 \text{ and } \frac{\partial s}{\partial x}=1\right] \qquad \text{(i)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\Rightarrow\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s} \qquad \left[\because \frac{\partial r}{\partial y}=1 \text{ and } \frac{\partial s}{\partial y}=-1\right] \qquad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}+\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2.\frac{\partial u}{\partial r}$$

Hence this proves the result.

Q.No.10: If $u=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, then

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Sol.: Here u is a composite function of r and θ

So we have

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r} \quad \text{since } \frac{\partial x}{\partial r}=\cos\theta,\ \frac{\partial y}{\partial r}=\sin\theta$$

$$=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

By squaring, we get

$$\left(\frac{\partial u}{\partial r}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta+2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\cos\theta.\sin\theta. \qquad \text{(i)}$$

Similarly we can get

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}.\frac{\partial u}{\partial \theta} \quad \text{since } \frac{\partial x}{\partial \theta}=-r\sin\theta,\ \frac{\partial y}{\partial \theta}=r\cos\theta$$

$$=-r\frac{\partial u}{\partial x}.\sin\theta+r\frac{\partial u}{\partial y}.\cos\theta$$

By squaring, we get

$$\left(\frac{\partial u}{\partial \theta}\right)^2=\left[-r^2\left(\frac{\partial u}{\partial x}\right)^2.\sin^2\theta+\left(\frac{\partial u}{\partial y}\right)^2.\cos^2\theta-2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\sin\theta\cos\theta\right]$$

$$\Rightarrow \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta-2\left(\frac{\partial u}{\partial x}\right).\left(\frac{\partial u}{\partial y}\right)\sin\theta\cos\theta. \qquad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2\theta+\sin^2\theta\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Hence this proves the result**.**

Q.No.11: If z be a function of x and y, and u and v be two other variables, such that

$u = \ell x + my$, $v = \ell y - mx$. Show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\ell^2 + m^2\right)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right),$$ assuming that z is a function of u and v.

Sol.: Let us assume that z is a function of u and v.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}.\ell + \frac{\partial z}{\partial v}.(-m) = \ell\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}$$

Let $\frac{\partial z}{\partial x} = f$. Since f is a composite function of x and y. Noting that f is also a function of u and v.

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}.\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right).\frac{\partial v}{\partial x} \qquad \left[\because \text{By putting } f = \frac{\partial z}{\partial x}\right]$$

$$= \frac{\partial}{\partial u}.\left(\ell\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\ell\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\ell\frac{\partial^2 z}{\partial u^2} - m\frac{\partial^2 z}{\partial u\partial v}\right)\frac{\partial u}{\partial x} + \left(\ell\frac{\partial^2 z}{\partial v\partial u} - m\frac{\partial^2 z}{\partial v^2}\right).\frac{\partial v}{\partial x}\ . \tag{i}$$

Similarly $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = m\frac{\partial z}{\partial u} + \ell\frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial u}.\left(\frac{\partial z}{\partial y}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right).\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = .\left(m\frac{\partial^2 z}{\partial u^2} + \ell\frac{\partial^2 z}{\partial u\partial v}\right)\frac{\partial u}{\partial y} + \left(m\frac{\partial^2 z}{\partial u\partial v} + \ell\frac{\partial^2 z}{\partial v^2}\right).\frac{\partial v}{\partial y}\ . \tag{ii}$$

By adding (i) and (ii) we get,

$$\left(\frac{\partial^2 z}{\partial x^2}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial u}{\partial x}\left(\ell\frac{\partial^2 z}{\partial u^2} - m\frac{\partial^2 z}{\partial u\partial v}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\ell\frac{\partial^2 z}{\partial u\partial v} - m\frac{\partial^2 z}{\partial v^2}\right)$$

$$+\left(\frac{\partial u}{\partial y}\right)\left(m\frac{\partial^2 z}{\partial v^2}+\ell\frac{\partial^2 z}{\partial u\partial v}\right)+\left(\frac{\partial v}{\partial y}\right)\left(m\frac{\partial^2 z}{\partial u\partial v}+\ell\frac{\partial^2 z}{\partial v^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\ell^2\frac{\partial^2 z}{\partial u^2}-\ell m\frac{\partial^2 z}{\partial u\partial v}-\ell m\frac{\partial^2 z}{\partial u\partial v}+m^2\frac{\partial^2 z}{\partial v^2}$$

$$+m^2\frac{\partial^2 z}{\partial v^2}+\ell m\frac{\partial^2 z}{\partial u\partial v}+\ell m\frac{\partial^2 z}{\partial u\partial v}+\ell^2\frac{\partial^2 z}{\partial v^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\ell^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right).$$

Hence this proves the result.

Q.No.12: If $z=f(u,v)$ and $u=x^2-2xy-y^2$ and $v=y$. Show that

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=(x-y)\frac{\partial z}{\partial v}.$$

Sol.: Clearly z is a composite function of x and y

$$\therefore \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}(2x-2y)+\frac{\partial z}{\partial v}(0)$$

$$\Rightarrow \frac{\partial z}{\partial x}=2(x-y)\frac{\partial z}{\partial u}. \qquad \text{(i)}$$

Also

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}(-2x-2y)+\frac{\partial z}{\partial v}(1)$$

$$\Rightarrow \frac{\partial z}{\partial y}=-2(x+y)\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}. \qquad \text{(ii)}$$

Taking L.H.S., we get

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=\left[(x+y)2(x-y)\frac{\partial z}{\partial u}\right]+\left[(x-y)\left\{(-2)(x+y)\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right\}\right]$$

$$=2(x+y)(x-y)\frac{\partial z}{\partial u}-2(x-y)(x+y)\frac{\partial z}{\partial u}+(x-y)\frac{\partial z}{\partial v}$$

$$=(x-y)\frac{\partial z}{\partial v}=\text{R.H.S.}$$

Hence this proves the result.

Q.No.13: Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates.

Sol.: The relations connecting Cartesian co-ordinates (x, y) with polar co-ordinates (r, θ) are $x = r\cos\theta$, $y = r\sin\theta$.

Squaring and adding, we get $r^2 = x^2 + y^2$.

Dividing, we get $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2)}} = \frac{r\cos\theta}{r} = \cos\theta \quad \text{and}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{\sqrt{(x^2 + y^2)^2}} = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\text{Now } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}.\cos\theta - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}.\left(\frac{\partial u}{\partial x}\right) = \frac{\partial f}{\partial x}, \text{ where } f = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}.\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}.\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial r\partial \theta} - \frac{\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial \theta\partial r} + \frac{\sin^2\theta}{\partial^2 r}.\frac{\partial^2 u}{\partial \theta^2}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial \theta} + \frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial \theta}. \qquad \text{(i)}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}.\left(\frac{\partial u}{\partial y}\right) = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta.\sin\theta}{r}.\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}-\frac{\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}. \qquad \text{(ii)}$$

Adding (i) and (ii) we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 u}{\partial r^2}+\frac{\cos^2\theta+\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\cos^2\theta+\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

$$=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0.$$

Hence this proves the result.

Q.No.14: If $v=r^3$ and $r^2=x^2+y^2+z^2$ then show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}.$$

Sol.: Let $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}.\frac{\partial r}{\partial x}=3r^2.\frac{x}{r}=3rx$

$$\Rightarrow\frac{\partial^2 v}{\partial x^2}=3r+3x.\frac{\partial r}{\partial x}=3r+3x.\frac{x}{r}=\frac{3\left(r^2+x^2\right)}{r}. \qquad \text{(i)}$$

Similarly we can find

$$\frac{\partial^2 v}{\partial y^2}=\frac{3\left(r^2+y^2\right)}{r} \qquad \text{(ii)}$$

$$\frac{\partial^2 v}{\partial z^2}=\frac{3\left(r^2+z^2\right)}{r}. \qquad \text{(iii)}$$

By adding (i), (ii) and (iii), we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{3\left(3r^2+x^2+y^2+z^2\right)}{r}=\frac{3\left(3r^2+r^2\right)}{r}=\frac{3\times 4r^2}{r}=12r.$$

(iv)

By differentiating $v=r^3$ w. r. t. r, we get

$$\frac{dv}{dr}=3r^2.$$

Again differentiating, we get $\frac{d^2v}{dr^2} = 6r$

$\therefore$ Let R. H. S. $\frac{d^2v}{dr^2} + \frac{2}{r}.\frac{dv}{dr} = 6r + \frac{2}{r}.3r^2 = 6r + 6r = 12r$. (v)

Hence from (iv) and (v), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}$$

Hence this proves the result**.**

Q.No.15: If $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}.$$

Sol.: Since z is a composite function of u and v

Thus $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}.\cos\alpha. + \frac{\partial z}{\partial y}.\sin\alpha = f$

Now, $\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \cos\alpha\left(\cos\alpha\frac{\partial^2 z}{\partial x^2} + \sin\alpha\frac{\partial^2 z}{\partial x\partial y}\right) + \sin\alpha\left(\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\frac{\partial^2 z}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2 z\frac{\partial^2 z}{\partial y^2}. \quad \text{(i)}$$

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y} = g$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}$$

$$= -\sin\alpha\left(-\sin\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\frac{\partial^2 z}{\partial x\partial y}\right) + \cos\alpha\left(-\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos\alpha\frac{\partial^2 z}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}. \quad \text{(ii)}$$

Now by adding (i) and (ii), we get

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\left(\cos^2\alpha+\sin^2\alpha\right)\frac{\partial^2 z}{\partial x^2}+\left(\cos^2\alpha+\sin^2\alpha\right)\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

Hence this proves the result.

Q.No.16: If $f(p,t,v)=0$. Prove that $\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{t=c}=-1$.

Sol.: When $v=c$ then

$f_1=f(p,t,v)=f(p,t,c)=f(p,t)=0$

Now $$\left(\frac{dp}{dt}\right)_{v=c}=\frac{-\partial f_1/\partial t}{\partial f_1/\partial p}$$

(i)

Similarly $$\left(\frac{dt}{dv}\right)_{p=c}=\frac{-\partial f_2/\partial v}{\partial f_2/\partial t} \quad \text{(ii)}$$

and $$\left(\frac{dv}{dp}\right)_{t=c}=\frac{-\partial f_3/\partial p}{\partial f_3/\partial v} \quad \text{(iii)}$$

Multiplying (i), (ii) and (iii), we get

$$\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{p=c}=\frac{-\partial f_1/\partial t}{\partial f_1/\partial p}\times\frac{-\partial f_2/\partial v}{\partial f_2/\partial t}\times\frac{-\partial f_3/\partial p}{\partial f_3/\partial v}$$

$$\Rightarrow\left(\frac{\partial f}{\partial p}\right)_{v,t=0}=\left(\frac{\partial f_1}{\partial p}\right)_{v=c}=\left(\frac{\partial f_3}{\partial p}\right)_{t=c}$$

Similarly $\frac{\partial f_1}{\partial t}=\frac{\partial f_2}{\partial t}$ and $\frac{\partial f_2}{\partial v}=\frac{\partial f_3}{\partial v}$

Thus, we get

$$\left(\frac{dp}{dt}\right)_{v=c}\times\left(\frac{dt}{dv}\right)_{p=c}\times\left(\frac{dv}{dp}\right)_{t=c}=\frac{\partial f_1}{\partial t}\times\frac{1}{\partial f_1/\partial p}\times\frac{\partial f_2}{\partial v}=-1=\text{R. H. S.}.$$

Hence this proves the result.

Q.No.17: If $f(u,v)=0$, $u=\ell x+my+mz$ and $v=x^2+y^2+z^2$. Hence show that

$$(\ell y-mx)+(ny-mz)\frac{\partial z}{\partial x}+(\ell z-nx)\frac{\partial z}{\partial y}=0.$$

Sol.: Since f is a composite function of x, y, and z. Then we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}\Rightarrow\frac{\partial f}{\partial x}=\ell\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v} \qquad \text{(i)}$$

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial y}\Rightarrow\frac{\partial f}{\partial y}=m\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v} \qquad \text{(ii)}$$

and $$\frac{\partial f}{\partial z}=n\frac{\partial f}{\partial u}+2z\frac{\partial f}{\partial v} \qquad \text{(iii)}$$

Solving (i) and (ii), we get

$$\frac{\partial f}{\partial u}=\frac{y\dfrac{\partial f}{\partial x}-x\dfrac{\partial f}{\partial y}}{\ell y-mx} \text{ and } \frac{\partial f}{\partial v}=\frac{m\dfrac{\partial f}{\partial x}-\ell\dfrac{\partial f}{\partial y}}{2xm-2\ell y}$$

$$\therefore\frac{\partial f}{\partial z}=n\frac{\partial f}{\partial u}+2z\frac{\partial f}{\partial v}=n\left(\frac{y\dfrac{\partial f}{\partial x}-x\dfrac{\partial f}{\partial y}}{\ell y-mx}\right)+2z\left(\frac{m\dfrac{\partial f}{\partial x}-\ell\dfrac{\partial f}{\partial y}}{2(mx-\ell y)}\right)$$

$$\Rightarrow(\ell y-mx)\frac{\partial f}{\partial z}=ny\frac{\partial f}{\partial x}-nx\frac{\partial f}{\partial y}-mz\frac{\partial f}{\partial x}+z\ell\frac{\partial f}{\partial y}$$

$$\Rightarrow(\ell y-mx)\frac{\partial f}{\partial z}=(ny-mz)\frac{\partial f}{\partial x}+(\ell z-nx)\frac{\partial f}{\partial y}$$

$$\Rightarrow(\ell y-mx)-(ny-mz)\frac{\partial f/\partial x}{\partial f/\partial z}-(\ell z-nx)\frac{\partial f/\partial y}{\partial f/\partial z}=0$$

$$\Rightarrow(\ell y-mx)+(ny-mz)\frac{\partial z}{\partial x}+(\ell z-nx)\frac{\partial z}{\partial y}=0. \qquad \left[\begin{array}{l}\therefore\dfrac{\partial z}{\partial x}=-\dfrac{\partial f/\partial x}{\partial f/\partial z}\\ \dfrac{\partial z}{\partial y}=-\dfrac{\partial f/\partial y}{\partial f/\partial z}\end{array}\right]$$

Hence this proves the result**.**

Q.No.18.: If $z=f(x,y)$, $x=u+v$, $y=uv$, prove that

(i) $(u-v)\dfrac{\partial z}{\partial x}=u\dfrac{\partial z}{\partial u}-v\dfrac{\partial z}{\partial v}$**.**

(ii) $(u-v)\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial v}-\dfrac{\partial z}{\partial u}$**.**

Sol.: Here z is a composite function of u and v

Hence $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}=(1)\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y} \qquad \text{(i)}$$

Similarly we get

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}=(1)\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y} \qquad \text{(ii)}$$

Let $u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}=u\frac{\partial z}{\partial x}+uv\frac{\partial z}{\partial y}-v\frac{\partial z}{\partial x}-uv\frac{\partial z}{\partial y}=u\frac{\partial z}{\partial x}-v\frac{\partial z}{\partial x}\Rightarrow(u-v)\frac{\partial z}{\partial x}$.

Hence this prove the (i) relation.

Let us subtract (ii) from (i), we get

$$\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y}-\frac{\partial z}{\partial x}-v\frac{\partial z}{\partial y}=(u-v)\frac{\partial z}{\partial y}.$$

Hence this proves the (ii) relation.

Q.No.19.: If $z=f(r,s,t)$ and $r=\frac{x}{y},\ s=\frac{y}{z}$ and $t=\frac{z}{x}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

Sol.: Here $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial x}=\frac{\partial u}{\partial r}.\frac{1}{y}+\frac{\partial u}{\partial s}.(0)+\left(-\frac{z}{x^2}\right).\frac{\partial u}{\partial t}$

$$=\frac{1}{y}\frac{\partial u}{\partial r}-\frac{z}{x^2}\frac{\partial u}{\partial t}.$$

Similarly, we get

$$\frac{\partial u}{\partial y}=\frac{1}{z}\frac{\partial u}{\partial s}-\frac{x}{y^2}\frac{\partial u}{\partial r} \text{ and } \frac{\partial u}{\partial z}=\frac{1}{x}\frac{\partial u}{\partial s}-\frac{y}{z^2}\frac{\partial u}{\partial s}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t}+\frac{y}{z}\frac{\partial u}{\partial s}-\frac{x}{y}\frac{\partial u}{\partial r}+\frac{z}{x}\frac{\partial u}{\partial t}-\frac{y}{z}\frac{\partial u}{\partial s}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

Hence this proves the result.

Q.No.20: If $z=f(x,y)$ and $x=r\cos\theta$, $y=r\sin\theta$ express the equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

in terms of $r\,\theta$. Is the equation in terms of r and θ valid at $r=0$.

Sol.: Let $x=r\cos\theta$ and $y=r\sin\theta\ \Rightarrow r=\sqrt{x^2+y^2}$

And $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = -\frac{y}{\left(\sqrt{x^2 + y^2}\right)^2} = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}.\cos\theta - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial f}{\partial x} \quad \text{where } f = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \cos\theta.\frac{\partial f}{\partial x} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$= \cos\theta.\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}.\frac{\partial}{\partial \theta}\left(\cos\theta.\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta.\cos\theta}{r}.\frac{\partial^2 u}{\partial r.\partial \theta} + \frac{\sin^2\theta}{r^2}.\frac{\partial^2 u}{\partial^2 \theta} + \frac{\sin^2\theta}{r}.\frac{\partial u}{\partial r}$$

$$+ \frac{\sin\theta\cos\theta}{r^2}.\frac{\partial u}{\partial \theta} - \frac{\sin\theta.\cos\theta}{r^2}.\frac{\partial u}{\partial \theta}. \qquad \text{(i)}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta.\cos\theta}{r}.\frac{\partial^2 u}{\partial r.\partial \theta} + \frac{\cos^2\theta}{r^2}.\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}.\frac{\partial u}{\partial r}$$

$$- \frac{\sin\theta\cos\theta}{r^2}.\frac{\partial u}{\partial r} + \frac{\sin\theta.\cos\theta}{r^2}.\frac{\partial u}{\partial r}. \qquad \text{(ii)}$$

By adding (i) and (ii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}.\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}.\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial u}{\partial r}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}.\frac{\partial^2 \theta}{\partial \theta^2} + \frac{1}{r}.\frac{\partial u}{\partial r}$$

From this equation, we get

$$r^2\frac{\partial^2 z}{\partial r^2}+r\frac{\partial z}{\partial r}+\frac{\partial^2 z}{\partial \theta^2}=0.$$

When r = 0 then we have

$\frac{\partial^2 z}{\partial \theta^2}=0$. Thus the equation is valid.

Hence this proves the result.

Q.No.21: If $x=u+v+w$, $y=u^2+v^2+w^2$, $z=u^3+v^3+w^3$ then prove that

$$\frac{\partial u}{\partial x}=\frac{vw}{(u-v)(u-w)}.$$

Sol.: Let $x=u+v+w$

By differentiating w. r. t. x, we get

$$\frac{\partial x}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\Rightarrow\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}=1 \qquad \text{(i)}$$

Also $y=u^2+v^2+w^2$

Again by differentiating partially w. r. t. x, we get

$$0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}+2w\frac{\partial w}{\partial x}\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=0 \qquad \text{(ii)}$$

and $z=u^3+v^3+w^3$

Again by differentiating partially w. r. t. x, we get

$$0=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x}+3w^2\frac{\partial w}{\partial x}\Rightarrow u^2\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial x}+w^2\frac{\partial w}{\partial x}=0 \qquad \text{(iii)}$$

Let $\frac{\partial u}{\partial x}=a$, $\frac{\partial v}{\partial x}=b$ and $\frac{\partial w}{\partial x}=c$

Putting these values in (i), (ii) and (iii), we get

$$a+b+c=1 \qquad \text{(iv)}$$

$$ua+vb+wc=0 \qquad \text{(v)}$$

$$u^2a+v^2b+w^2c=0 \qquad \text{(vi)}$$

$$a+b+c=1 \quad \Rightarrow wa+wb+wc=w \qquad \text{(vii)}$$

$$ua+vb+wc=0 \quad \Rightarrow wua+wvb+w^2c=0 \qquad \text{(viii)}$$

Now subtracting (v) from (vii), we get

$(w-u)a+(w-v)b=w$

Now subtracting (vi) from (viii), we get

$\left(wu-u^2\right)a+\left(wv-v^2\right)b=0$ i. e.

$(w-u)a+(w-v)b=w \Rightarrow v(w-u)a+v(w-v)b=wv$ (ix)

and $u(w-v)a-v(w-v)b=0$ (x)

By solving, we get

$$v(w-u)a-u(w-v)a=vw \Rightarrow (v-u)(w-v)a=vw \Rightarrow a=\frac{vw}{(v-u)(w-v)}$$

Hence $\frac{\partial u}{\partial x}=\frac{uw}{(u-v)(u-w)}$.

Hence this proves the result.

Q.No.22: If $x=\cosh\theta.\cos\phi$, $y=\sinh\theta.\sin\phi$ then show that

$$J\left(\frac{x,y}{\theta,\phi}\right)=\frac{1}{2}(\cosh 2\theta-\cos 2\phi).$$

Sol.: Let $J\left(\frac{x,y}{\theta,\phi}\right)=\begin{vmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\frac{\partial x}{\partial\theta}.\frac{\partial y}{\partial\phi}-\frac{\partial x}{\partial\phi}.\frac{\partial y}{\partial\theta}$

$$\Rightarrow \frac{\partial x}{\partial\theta}=\sinh\theta\cos\phi\,;\quad \frac{\partial x}{\partial\phi}=-\cosh\theta.\sin\phi$$

and $\frac{\partial y}{\partial\theta}=\cosh\theta\sin\phi\,;\quad \frac{\partial x}{\partial\phi}=\sinh\theta.\cos\phi$

$$\therefore J\left(\frac{x,y}{\theta,\phi}\right)=(\sinh\theta.\cos\phi).(\sinh\theta.\cos\phi)+(\cosh\theta.\sin\phi).(\cosh\theta.\sin\phi)$$

$$=\cos^2\phi.\sinh^2\theta+\cosh^2\theta\sin^2\phi$$

Now here $\sinh\theta=\frac{e^x-e^{-x}}{2}$ and $\cosh\theta=\frac{e^x+e^{-x}}{2}$

$$\Rightarrow \cosh^2\theta=\frac{e^{x^2}+e^{-x^2}+2e^{x-x}}{4}=\frac{e^{x^2}+e^{-x^2}}{4}+\frac{1}{2}$$

and $\sinh^2\theta = \dfrac{e^{x^2}+e^{-x^2}}{4} - \dfrac{1}{2}$

$$J\left(\frac{x, y}{\theta, \phi}\right) = \cos^2\phi\left(\frac{e^{x^2}+e^{-x^2}}{4} - \frac{1}{2}\right) + \sin^2\phi\left(\frac{e^{x^2}+e^{-x^2}}{4} + \frac{1}{2}\right)$$

$$= \frac{e^{x^2}+e^{-x^2}}{4}\left(\cos^2\phi + \sin^2\phi\right) - \frac{1}{2}\left(\cos^2\phi - \sin^2\phi\right)$$

$$= \frac{e^{x^2}+e^{-x^2}}{4} - \frac{1}{2}\cos 2\phi = \frac{1}{2}.\frac{e^{x^2}+e^{-x^2}}{2} - \frac{1}{2}\cos 2\phi$$

$$= \frac{1}{2}.\left(\frac{e^{x^2}+e^{-x^2}}{2} - \cos 2\phi\right) = \frac{1}{2}.\left(\cos 2h\theta - \cos 2\phi\right) \qquad \left[\because \frac{e^{x}+e^{-x}}{2} = \cosh\theta\right]$$

Hence this proves the result**.**

Q.No.23.: If $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.

Sol.: Here $\dfrac{\partial u}{\partial x} = -\dfrac{yz}{x^2}$, $\dfrac{\partial v}{\partial y} = -\dfrac{zx}{y^2}$ and $\dfrac{\partial w}{\partial z} = -\dfrac{xy}{z^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{z}{x}$, $\dfrac{\partial v}{\partial z} = \dfrac{x}{y}$ and $\dfrac{\partial w}{\partial x} = \dfrac{y}{z}$

and $\dfrac{\partial u}{\partial z} = \dfrac{y}{x}$, $\dfrac{\partial v}{\partial x} = \dfrac{z}{y}$ and $\dfrac{\partial w}{\partial y} = \dfrac{x}{z}$

$\therefore$ Taking L. H. S., we get

$$J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{z} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} = \frac{1}{x\,y\,z}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{zx}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$= \frac{1}{x\,y\,z}\left[-\frac{yz}{x}\left(\frac{(zx)(xy)}{zy} - x^2\right) - z(-xy - yx) + y(zx + zx)\right]$$

$$= \frac{1}{x\,y\,z}\left[(-yzx + yzx) - z(-2xy) + y(2zx)\right]$$

$$= \frac{1}{x\,y\,z}\left((0) + 2xyz + 2xyz\right) = \frac{1}{x\,y\,z}(4xyz) = 4 = \text{R. H. S.}.$$

Hence this proves the result.

Q.No.24.: If $x = r\cos\theta$, $y = r\sin\theta$, prove that $J\left(\frac{r,\theta}{x,y}\right) = \frac{1}{r}$.

Sol.: Given that $x = r\cos\theta$ (i)

And $y = r\sin\theta$ (ii)

From (i) and (ii), we get

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

So we get $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2}} = \dfrac{x}{\sqrt{x^2+y^2}} = \cos\theta$

And $\dfrac{\partial r}{\partial y} = \dfrac{2y}{2\sqrt{x^2+y^2}} = \dfrac{y}{\sqrt{x^2+y^2}} = \sin\theta$

Similarly $\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2}$

And $\dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\dfrac{1}{x} = \dfrac{x}{x^2+y^2}$

Let L. H. S.

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{\partial r}{\partial x}\cdot\frac{\partial \theta}{\partial y} - \frac{\partial \theta}{\partial x}\cdot\frac{\partial r}{\partial y} = \frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{y}{x^2+y^2}$$

$$= \frac{x^2}{x^2+y^2\sqrt{x^2+y^2}} + \frac{y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{x^2+y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$= \text{R. H. S.}.$

Hence this proves the result.

Q.No.25.: If $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$, show that $\dfrac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \rho$.

Sol.: Let $x = \rho\cos\theta$, $y = \rho\sin\theta$ and $z = z$

$$\Rightarrow \frac{\partial x}{\partial \rho} = \cos\theta, \frac{\partial y}{\partial \rho} = \sin\theta \text{ and } \frac{\partial z}{\partial \rho} = 0$$

$$\frac{\partial x}{\partial \theta} = -\rho\sin\theta, \quad \frac{\partial y}{\partial \theta} = \rho\cos\theta \text{ and } \frac{\partial z}{\partial \theta} = 0$$

and $\dfrac{\partial x}{\partial z} = 0$, $\dfrac{\partial y}{\partial z} = 0$ and $\dfrac{\partial z}{\partial z} = 1$

Taking L. H. S., we get

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\left(\rho\cos^2\theta + \rho\sin^2\theta\right)$$

$$= \rho\left(\cos^2\theta + \sin^2\theta\right) = \rho = \text{R. H. S.}.$$

Hence this proves the result.

Q.No.26.: If $x = f(u, v)$, $y = \phi(u, v)$ are two functions which satisfy the equations

$\dfrac{\partial f}{\partial u} = \dfrac{\partial \phi}{\partial v}$, $\dfrac{\partial f}{\partial v} = -\dfrac{\partial \phi}{\partial u}$ and z is a function of x and y, then prove that

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

Sol.: Given that $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial u}$

$$\Rightarrow g = \frac{\partial z}{\partial x}.\frac{\partial f}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial \phi}{\partial u} \Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u} g$$

$$= \frac{\partial g}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}.\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial x \partial u} + \frac{\partial^2 z}{\partial x \partial y}.\frac{\partial \phi}{\partial u} + \frac{\partial^2 \theta}{\partial x \partial u}.\frac{\partial z}{\partial y}\right)$$

$$+\frac{\partial\phi}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial y\partial u}+\frac{\partial^2 z}{\partial y^2}.\frac{\partial\phi}{\partial u}+\frac{\partial^2\theta}{\partial y\partial u}.\frac{\partial z}{\partial y}\right).$$

Now we have $\frac{\partial^2 f}{\partial x\partial u}=\frac{\partial^2 f}{\partial u\partial x}-\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial u}(1)=0-\frac{\partial^2 f}{\partial y\partial v}$.

Similarly, we can have $\frac{\partial^2\phi}{\partial y\partial u}=0=\frac{\partial^2\phi}{\partial y\partial v}$.

So that

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial\phi}{\partial v}+\frac{\partial^2\phi}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial f}{\partial y\partial u}+\frac{\partial\phi}{\partial u}.\frac{\partial^2 z}{\partial y^2}\right) \quad \text{(i)}$$

Similarly, we can find

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial\phi}{\partial v}+\frac{\partial^2\phi}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial y\partial v}+\frac{\partial\phi}{\partial v}.\frac{\partial^2 z}{\partial y^2}\right) \quad \text{(ii)}$$

Since $\frac{\partial f}{\partial u}=\frac{\partial\phi}{\partial v}$ and $\frac{\partial f}{\partial v}=\frac{\partial\phi}{\partial u}$

Taking L. H. S., we get

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial f}{\partial v}-\frac{\partial^2 f}{\partial x\partial v}.\frac{\partial z}{\partial y}\right)-\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial u}.\frac{\partial z}{\partial x}-\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial y^2}\right)$$

$$+\frac{\partial f}{\partial v}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}.\frac{\partial f}{\partial u}+\frac{\partial^2 f}{\partial x\partial u}.\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial v}.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial v}.\frac{\partial z}{\partial x}+\frac{\partial f}{\partial u}.\frac{\partial^2 z}{\partial y^2}\right)$$

$$=\left(\frac{\partial f}{\partial u}\right)^2.\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial f}{\partial u}\right)^2.\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial f}{\partial v}\right)^2.\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial f}{\partial v}\right)^2.\frac{\partial^2 z}{\partial y^2}$$

$$+\frac{\partial f}{\partial v}\left(-\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial u\partial y}+\frac{\partial z}{\partial y}.\frac{\partial^2 f}{\partial u\partial x}\right)+\frac{\partial f}{\partial u}\left(-\frac{\partial z}{\partial x}.\frac{\partial^2 f}{\partial v\partial y}-\frac{\partial z}{\partial y}.\frac{\partial^2 f}{\partial v\partial x}\right)$$

$$\left[\because \frac{\partial f}{\partial u\partial x}=\frac{\partial f}{\partial v\partial x}=0\right]$$

$$=\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]+\frac{\partial f}{\partial v}\left(-\frac{\partial z}{\partial x}-\frac{\partial^2\phi}{\partial v\partial y}\right)+\frac{\partial f}{\partial u}\left(-\frac{\partial z}{\partial x}-\frac{\partial^2\phi}{\partial u\partial y}\right)$$

$$\left[\because \frac{\partial^2 u}{\partial v \partial y} = \frac{\partial^2 \phi}{\partial u \partial y} = 0\right]$$

$$= \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] = \text{R. H. S..}$$

Hence this proves the result**.**

Q.No.27: If $z = u^2 + v^2$, $x = u^2 - v^2$ and $y = uv$. Find the value of $\frac{\partial z}{\partial x}$**.**

Sol.: Here $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}2u + 2v\frac{\partial v}{\partial x}$

Now $u^2 - v^2 = x$.

Differentiating w.r.t. to x, we get

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = \frac{\partial x}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = 1\ , \qquad \text{(i)}$$

and $vu = y$.

Differentiating w. r. t. to x, we get

$$u.\frac{\partial v}{\partial x} + v.\frac{\partial u}{\partial x} = 0. \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\frac{\frac{\partial u}{\partial x}}{0+u} = \frac{\frac{\partial v}{\partial x}}{-v-0} = \frac{1}{2u^2+2v^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u}{2\left(u^2+v^2\right)} = \frac{u}{2z} \text{ and } \frac{\partial v}{\partial x} = \frac{-v}{2\left(u^2+v^2\right)} = \frac{-v}{2z}$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2u.\frac{u}{2z} - 2v.\frac{v}{2z} = \frac{u^2-v^2}{z} = \frac{x}{z}$$

Hence $\frac{\partial z}{\partial x} = \frac{x}{z}$**.** Ans**.**

Thank you

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