3rd Topic

Fourier Series

Functions having points of discontinuity

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Functions having points of discontinuity:

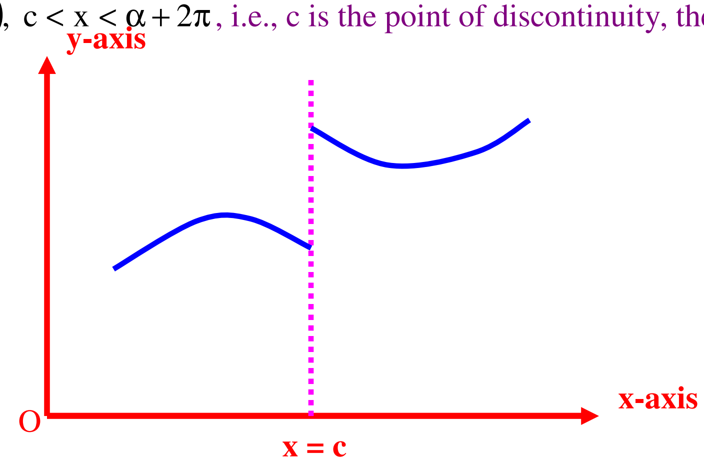
In deriving the Euler's formulae for $a_0,\ a_n,\ b_n$, it was assumed that f(x) was continuous. But in its place, if a function have a finite number of points of finite discontinuity, i.e., its graph consist of a finite number of different curves given by different equations, even then such a function is expressible as a Fourier series.

Example:

If in an interval $(\alpha,\ \alpha+2\pi)$, f(x) is defined by

$$f(x)=\phi(x),\quad \alpha<x<c$$

$$=\psi(x),\ c<x<\alpha+2\pi,$$ i.e., c is the point of discontinuity, then



$$\boxed{a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)dx + \int_{c}^{\alpha+2\pi} \psi(x)dx\right]},$$

$$\boxed{a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\cos nxdx + \int_{c}^{\alpha+2\pi} \psi(x)\cos nxdx\right]},$$

$$\boxed{b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\sin nxdx + \int_{c}^{\alpha+2\pi} \psi(x)\sin nxdx\right]}.$$

Value of f(x) at a point of finite discontinuity:

At a point of finite discontinuity x = c, there is finite jump in the graph of function (see fig.). Both the limits on the left [i.e., $f(c-0)$] and the limit on the right [i.e., $f(c+0)$] exit and are different. At such a point, the value of the function f(x) is the **arithmetic mean** of these two limits, i.e., at $x = c$,

$$\boxed{f(x) = \frac{1}{2}[f(c-0) + f(c+0)]}.$$

Now let us develop some Fourier series of functions having some points of discontinuity:

Q.No.1.: Find the Fourier series expansion for f(x), if

$$f(x) = -\pi, \quad -\pi < x < 0,$$
$$= x, \quad 0 < x < \pi,$$

and hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)dx + \int_{0}^{\pi} xdx\right] = \frac{1}{\pi}\left[-\pi|x|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2}$;

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nxdx + \int_{0}^{\pi} x\cos nxdx\right] = \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore a_1 = \frac{-2}{\pi.1^2},\ \ a_2 = 0,\ \ a_3 = \frac{-2}{\pi.3^2},\ \ a_4 = 0,\ \ a_5 = \frac{-2}{\pi.5^2},\ \ \ldots\ldots\ldots \text{ etc.}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx dx + \int_{0}^{\pi} x \sin nx dx\right] = \frac{1}{\pi}\left[\left|\frac{\pi \cos nx}{n}\right|_{-\pi}^{0} + \left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi) = \frac{1}{n}\left[1 - 2(-1)^n\right].$$

$$\therefore b_1 = 3,\ \ b_2 = -\frac{1}{2},\ \ b_3 = 1,\ \ b_4 = -\frac{1}{4},\ \ \ldots\ldots\ldots\ldots\text{etc.}$$

Hence substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots \text{(ii)}$$

2nd Part.:

Putting x = 0 in (ii), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\infty\right)$ (iii)

Now f(x) is discontinuous at x = 0. Also since $f(0-0) = -\pi$ and $f(0+0) = 0$.

$$\therefore f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = -\frac{\pi}{2}$$

Hence, (iii) take the form

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}.$$

Q.No.2.: Find the Fourier series to represent the function f(x) given by

$$f(x) = x, \quad \text{for } 0 \le x \le \pi,$$
$$= 2\pi - x \quad \text{for } \pi \le x \le 2\pi.$$

and hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\infty = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\left[\int_0^{\pi} x.dx+\int_{\pi}^{2\pi}(2\pi-x)dx\right]=\frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi}+\left|2\pi x-\frac{x^2}{2}\right|_{\pi}^{2\pi}\right]$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+(2\pi)^2-\frac{(2\pi)^2}{2}-2\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[5\pi^2-4\pi^2\right]=\frac{\pi^2}{\pi}=\pi\,.$$

$$a_n=\frac{1}{\pi}\left[\int_0^{\pi} x\cos nx dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx dx\right]$$

$$=\frac{1}{\pi}\left[\left|\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right|_0^{\pi}+\left|\frac{2\pi\sin nx}{n}-\frac{x\sin nx}{n}-\frac{\cos nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}-\frac{\cos 2n\pi}{n^2}+\frac{\cos n\pi}{n^2}\right]=\frac{1}{\pi}\left[\frac{2\cos n\pi}{n^2}-\frac{\cos 2n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$=\frac{1}{\pi n^2}\left[2\cos n\pi-\cos 2n\pi-1\right]=\frac{1}{\pi n^2}\left[2(-1)^n-(-1)^{2n}-1\right]=\frac{2}{\pi n^2}\left[(-1)^n-1\right].$$

$\therefore\ a_1=\frac{1}{\pi(1)^2}[-2-1-1]=\frac{-4}{\pi(1)^2},\quad a_2=0,\quad a_3=\frac{1}{\pi(3)^2}(-4),\quad a_4=0,$ ....etc.

$$b_n=\frac{1}{\pi}\left[\int_0^{\pi} x\sin nx dx+\int_{\pi}^{2\pi}(2\pi-x)\sin nx dx\right]$$

$$=\frac{1}{\pi}\left[\left|\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right|_0^{\pi}+\left|\frac{-2\pi\cos nx}{n}+\frac{x\cos nx}{n}-\frac{\sin nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}-\frac{2\pi\cos 2n\pi}{n}+\frac{2\pi\cos 2n\pi}{n}+\frac{2\pi\cos n\pi}{n}-\frac{\pi\cos n\pi}{n}\right]=0$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\ldots\ldots\right). \qquad \text{(ii)}$$

2nd Part:

Putting $x=\pi$, we get

$$f(\pi)=\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\right). \qquad \text{(iii)}$$

Now f(x) is discontinuous at $x=\pi$

$\therefore f(\pi-0)=\pi$ and $f(\pi+0)=2\pi-\pi=\pi$

$\therefore f(\pi)=\frac{1}{2}[f(\pi-0)+f(\pi+0)]=\frac{2\pi}{2}=\pi$.

Hence, (iii) takes the form

$$\pi=\frac{\pi}{2}+\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty\right]\Rightarrow\frac{\pi}{2}\times\frac{\pi}{4}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\infty.$$

Q.No.3.: An alternating current after passing through a rectifier has the form

$i=I_0\sin x$ for $0\le x\le\pi$,

$=0$ for $\pi\le x\le 2\pi$,

where I_0 is the maximum and the period is 2π.

Express i as a Fourier series.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\left[\int_0^{\pi}I_0\sin x dx+\int_{\pi}^{2\pi}0.dx\right]=\frac{I_0}{\pi}[-\cos x]_0^{\pi}=\frac{I_0}{\pi}\times 2=\frac{2I_0}{\pi}$.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx=\frac{1}{\pi}\left[\int_0^{\pi}I_0\sin x\cos nx dx+0\right]=\frac{I_0}{2\pi}\int_0^{\pi}2\sin x\cos nx dx$$

$$=\frac{I_0}{\pi}\left[\int_0^{\pi}\sin x\cos nx dx\right]=\frac{I_0}{2\pi}\left[\int_0^{\pi}2\sin x\cos nx dx\right]$$

$$=\frac{I_0}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$= \frac{I_0}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)}\right]_0^{\pi} + \frac{I_0}{2\pi}\left[\frac{-\cos(1-n)x}{(1-n)}\right]_0^{\pi}, \qquad (n \neq 1)$$

$$= \begin{cases} \frac{I_0}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right], & \text{when n is odd} \\ \frac{I_0}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right], & \text{when n is even} \end{cases}$$

$$= \begin{cases} 0\ , & \text{when n is odd} \\ -\frac{2I_0}{\pi(n^2-1)}\ , & \text{when n is even} \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cos x dx = \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x dx = \frac{I_0}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} I_0 \sin x \sin nx dx = \frac{1}{2\pi}\int_0^{2\pi} I_0[\cos(1-n)x - \cos(1+n)x]dx$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)}\right]_0^{\pi} - \left[\frac{\sin(1+n)x}{(1+n)}\right]_0^{\pi} = 0 \text{ for } n > 1.$$

When n = 1, we get

$$b_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x.\sin x.dx = \frac{1}{\pi}\int_0^{\pi} I_0 \sin^2 x.dx = \frac{1}{\pi}\int_0^{\pi} I_0\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin nx}{2}\right]_0^{\pi} = \frac{I_0}{2}(\pi - 0 - 0 + 0) = \frac{\pi I_0}{2\pi} = \frac{I_0}{2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}, \text{ by supposing } n = 2m.$$

Q.No.4.: If $f(x) = 0, \quad \text{for } -\pi < x < 0,$

$= \sin x, \quad \text{for } 0 < x < \pi,$

prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}.$

Hence, show that **(i)** $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{3.7}+..........=\frac{1}{2}$,

(ii) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{3.7}-..........\infty=\frac{1}{4}(\pi-2)$.

Sol.: The Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}0.dx+\int_{0}^{\pi}\sin xdx\right]=\frac{1}{\pi}[-\cos x]_0^{\pi}=\frac{1}{\pi}[-(-1)-(-1)]=\frac{2}{\pi}$,

$$a_n=\frac{1}{\pi}\left[\int_0^{\pi}\sin x\cos nxdx\right]=\frac{1}{2\pi}\left[\int_0^{\pi}2\sin x\cos nxdx\right]$$

$$=\frac{1}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$=\frac{1}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)}\right]_0^{\pi}+\frac{1}{2\pi}\left[\frac{-\cos(1-n)x}{(1-n)}\right]_0^{\pi}, \qquad (n\neq 1)$$

$$=\begin{cases}\frac{1}{2\pi}\left[-\frac{1}{1+n}-\frac{1}{1-n}+\frac{1}{1+n}+\frac{1}{1-n}\right], & \text{when n is odd}\\ \frac{1}{2\pi}\left[\frac{1}{1+n}+\frac{1}{1-n}+\frac{1}{1+n}+\frac{1}{1-n}\right], & \text{when n is even}\end{cases}$$

$$=\begin{cases}0\,, & \text{when n is odd}\\ -\frac{2}{\pi(n^2-1)}\,, & \text{when n is even}\end{cases}$$

When $n=1$, we have

$$a_1=\frac{1}{\pi}\int_0^{\pi}\sin x\cos xdx=\frac{1}{2\pi}\int_0^{\pi}\sin 2xdx=\frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}=0\,.$$

$$b_n=\frac{1}{\pi}\int_0^{\pi}\sin x\sin nxdx=\frac{1}{2\pi}\int_0^{\pi}2\sin x\sin nxdx$$

$$=\frac{1}{2\pi}\int_0^{\pi}2\cos(1-n)xdx-\frac{1}{2\pi}\int_0^{\pi}2\cos(1+n)xdx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)}\right]_0^{\pi} - \frac{1}{2\pi}\left[\frac{\sin(1+n)x}{(1+n)}\right]_0^{\pi} = 0 - 0 = 0.$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x dx = \frac{1}{\pi}\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2\pi}[\pi - 0 - 0 + 0] = \frac{\pi}{2\pi} = \frac{1}{2}.$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\ldots\right] + \frac{\sin x}{2}$$

$$\Rightarrow f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2-1}. \qquad \text{(ii)}$$

This is the required Fourier series expansion.

IInd part:

If x = 0, then we get

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{4m^2-1}$$

$$\Rightarrow \frac{1}{2} = \sum_{m=1}^{\infty} \frac{1}{4m^2-1} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)(2m+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{3.7} + \ldots\ldots\ldots = \frac{1}{2}.$$

IIIrd part:

Now putting $x = \frac{\pi}{2}$ in (ii), we get

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos m\pi}{4m^2-1} \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$\Rightarrow \frac{\pi-2}{4} = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)(2m+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots\ldots\ldots\right)$$

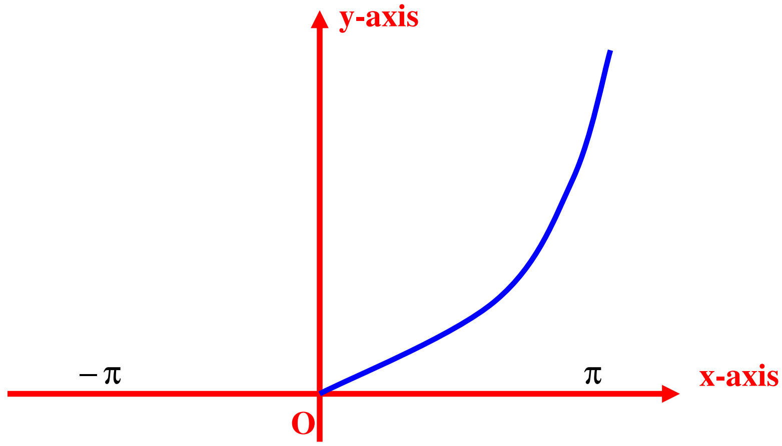
$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots$$

Q.No.5.: Draw the graph of the function $f(x) = 0, \quad -\pi < x < 0,$

$$= x^2, \quad 0 < x < \pi.$$

If $f(2\pi + x) = f(x)$, obtain Fourier series of f(x).

Sol.:



The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} x^2 dx\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\left[0 + \int_0^{\pi} x^2 \cos nx dx\right] = \frac{1}{\pi}\left[\left|x^2 \frac{\sin nx}{n}\right|_0^{\pi} - \int_0^{\pi} 2x.\frac{\sin nx}{n}.dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\left[\left|\frac{-x\cos nx}{n^2}\right|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n^2} dx\right]\right] = \frac{-2}{\pi}\left[\left(-x\frac{\cos nx}{n^2}\right)_0^{\pi} + \left(\frac{\sin nx}{n^3}\right)_0^{\pi}\right]$$

$$= \frac{-2}{\pi}\left[\frac{-\pi(-1)^n}{n^2} + 0\right] = \frac{2}{\pi}\left[\frac{\pi(-1)^n}{n^2}\right] = \frac{2(-1)^n}{n^2}.$$

$$\therefore a_1 = \frac{-2}{1^2}, \quad a_2 = \frac{2}{2^2}, \quad a_3 = \frac{-2}{3^2}, \qquad \text{...........etc.}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx.dx + \int_0^{\pi} x^2 \sin nx.dx\right] = \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} 2x\frac{\cos nx}{n} dx\right]$$

$$= \frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n} + \frac{2}{n}\cdot\left[\left|x.\frac{\sin nx}{n}\right|_0^\pi - \int_0^\pi \frac{\sin nx}{n}dx\right]\right]$$

$$= \frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n} + \frac{2}{n}\left(0 + \frac{\cos nx}{n^2}\right)_0^\pi\right] = (-1)^n\left[\frac{-\pi}{n} + \frac{2}{n^3\pi}\right] - \frac{2}{n^3\pi}.$$

$$\therefore b_1 = -\frac{1}{\pi}\left(\frac{4}{1^2} - \frac{\pi^2}{1}\right),\ \ b_2 = \frac{1}{\pi}\left(-\frac{\pi^2}{2}\right),\ \ b_3 = -\frac{1}{\pi}\left(\frac{4}{3^2} - \frac{\pi^2}{3}\right), \text{........ etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = \frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \text{............}\right)$$

$$-\frac{1}{\pi}\left\{\left(\frac{2}{1^3} - \frac{\pi^2}{1}\right)\sin x - \left(-\frac{\pi^2}{2}\right)\sin 2x + \left(\frac{4}{3^3} - \frac{\pi^2}{3}\right)\sin 3x - \text{.....}\right\}. \textbf{ Ans.}$$

Q.No.6.: Find the Fourier series of the following function,

$$f(x) = x^2, \qquad 0 \le x \le \pi$$

$$= -x^2, \ \ -\pi \le x \le 0.$$

Sol.: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x^2dx + \int_0^\pi x^2dx\right] = 0;$

$$a_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x^2\cos nxdx + \int_0^\pi x^2\cos nxdx\right]$$

Now $\int x^2\cos nxdx = x^2\frac{\sin nx}{n} - \int 2x.\frac{\sin nx}{n}.dx = x^2\frac{\sin nx}{n} - \frac{2}{n}\int x\sin nxdx$

$$= x^2\frac{\sin nx}{n} - \frac{2}{n}\left[\frac{-x\cos nx}{n} - \int \frac{\cos nx}{n}dx\right]$$

$$= x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2}{n^2}\int \cos nxdx$$

$$= x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2}{n^3}\sin nxdx\ .$$

$$-\int_{-\pi}^{0} x^2\cos nxdx = \left[-x^2\frac{\sin nx}{n} - 2x\frac{\sin nx}{n^2} + \frac{2}{n^3}\sin nx\right]_{-\pi}^{0}$$

$$= 0 - \left[-\pi^2\frac{\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2}{n^3}\sin n\pi\right]$$

$$= \frac{\pi^2\sin n\pi}{n} - \frac{2\pi\cos n\pi}{n^2} + \frac{2}{n^3}\sin n\pi$$

$$\int_{0}^{\pi} x^2\cos nxdx = \left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_{0}^{\pi}$$

$$= \left(\pi^2\frac{\sin nx}{n} + 2\pi\frac{\cos n\pi}{n^2} - \frac{2}{n^3}\sin n\pi\right) - 0$$

Thus $a_n = \frac{1}{\pi}\left(\frac{2\pi^2\sin n\pi}{n}\right) = 2\pi\frac{\sin n\pi}{n} = 0.$

$\therefore\ a_1 = 2\pi\sin\pi = 0,\ a_2 = \frac{2\pi\sin 2\pi}{2} = 0,\ a_3 = \frac{2\pi\sin 3\pi}{3} = 0,$ ....etc.

Now $b_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x^2\sin nxdx + \int_{0}^{\pi} x^2\sin nxdx\right].$

Now $\int x^2\sin nxdx = -x^2\frac{\cos nx}{n} + \frac{2}{n}\int x\cos nxdx$

$$= -x^2\frac{\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} - \int\left(\frac{1-\sin nx}{n}\right)dx\right]$$

$$= -x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} - \frac{2}{n^2}\int \sin nxdx$$

$$= -x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}$$

Now $-\int_{-\pi}^{0} x^2\sin nxdx = \left[x^2\frac{\cos nx}{n} - \frac{2}{n^2}x\sin nx - \frac{2}{n^3}\cos nx\right]_{-\pi}^{0}$

$$= -\frac{2}{n^3} - \left[\frac{\pi^2\cos n\pi}{n} - \frac{2}{n^3}\cos n\pi\right] = \frac{-2}{n^3} - \frac{\pi^2\cos n\pi}{n} + \frac{2}{n^3}\cos n\pi$$

$$\int_0^{\pi} x^2 \sin nx dx = \left[-\frac{x^2 \cos nx}{n} + \frac{2}{n^2} x \sin nx + \frac{2}{n^3} \cos nx \right]_0^{\pi}$$

$$= \frac{-\pi^2 \cos n\pi}{n} + \frac{2\pi \sin n\pi}{n^2} + \frac{2}{n^3} \cos n\pi - \frac{2}{n^3}$$

$$= -\frac{2}{n^3} - \frac{\pi^2 \cos n\pi}{n} + 0 + \frac{2}{n^3} \cos n\pi .$$

Thus $b_n = \frac{1}{\pi}\left[\frac{-4}{n^3} - \frac{2\pi^2 \cos n\pi}{n} + \frac{4}{n^3} \cos n\pi \right]$.

$$\therefore\ b_1 = \frac{1}{\pi}\left[-4 - 2\pi^2 \cos \pi + 4 \cos \pi \right] = 2\left(\pi - \frac{4}{\pi} \right),$$

$$b_2 = \frac{1}{\pi}\left[\frac{-4}{8} - \frac{2\pi^2 \cos 2\pi}{2} + \frac{4}{8} \cos 2\pi \right] = -\pi ,$$

$$b_3 = \frac{1}{\pi}\left[\frac{-4}{27} - \frac{2\pi^2 \cos 3\pi}{3} + \frac{4}{27} \cos 3\pi \right] = \frac{2\pi}{3} - \frac{8\pi}{27} = \frac{2}{3}\left(\pi - \frac{4\pi}{9} \right),$$

$$b_4 = \frac{1}{\pi}\left[\frac{-4}{64} - \frac{2\pi^2 \cos 4\pi}{4} + \frac{4}{64} \cos 4\pi \right] = -\frac{\pi}{2}, \text{............etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = 2\left(\pi - \frac{4}{\pi} \right) \sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4\pi}{9} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \text{.............} \text{ Ans.}$$

Q.No.7.: If $f(x) = x$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$ and $f(x) = 0$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$, find the Fourier series of f(x).

Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Sol.: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \quad \text{(i)}$$

Here $a_0 = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} x dx + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} 0.dx = \frac{1}{\pi}\left[\frac{x^2}{2} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{4} \right] = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n} + \frac{1}{n}.\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx dx = \frac{1}{\pi}\left[x.\frac{-\cos nx}{n} + \frac{1}{n\pi}.\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2}$$

$$b_n = \frac{-1}{n}\left[\cos n\frac{n}{2}\right] + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

For $n = 1$, $b_1 = \frac{2}{\pi}$, $b_2 = 0 + \frac{1}{2}$, $b_3 = -\frac{2}{9\pi}$, $b_4 = -\frac{1}{4}$, $b_5 = \frac{2}{25\pi}$ etc.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left(\frac{-\cos n.\frac{\pi}{2}}{n} + \frac{2}{n^2\pi}\sin n\frac{\pi}{2}\right)\sin nx.$$

$$f(x) = \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x - ......$$

2nd Part:

At $x = \frac{\pi}{2}$, which is point of discontinuity,

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left[f\left(\frac{\pi}{2} - 0\right) + f\left(\frac{\pi}{2} + 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}.$$

Putting $x = \frac{\pi}{2}$ in the Fourier series expansion, we get

$$\frac{\pi}{4} = \frac{2}{\pi}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{2}{9\pi}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{2}{25\pi}\sin\frac{5\pi}{2} - ......$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi} + 0 + \frac{2}{9\pi} + 0 + \frac{2}{25\pi} + 0......$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25}......\right) \Rightarrow \frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}......\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Q.No.8.: Find the Fourier series to represent $f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$.

Also deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots..$

Sol.: $f(x)=\frac{1}{2}a_0+\sum\limits_{n=1}^{\infty}\left(a_0\cos nx+b_n\sin nx\right)$ (i)

be the required series.

Here $a_0=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)=\frac{1}{\pi}\int\limits_{-\pi}^{0}-kdx+\frac{1}{\pi}\int\limits_{0}^{\pi}kdx$

$$=\frac{1}{\pi}\left|-kx\right|_{-\pi}^{0}+\frac{1}{\pi}\left|kx\right|_{0}^{\pi}=\frac{1}{\pi}[-k\pi]+\frac{1}{\pi}[k\pi]=0.$$

$$a_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int\limits_{-\pi}^{0}-k\cos nxdx+\frac{1}{\pi}\int\limits_{0}^{\pi}k\cos nxdx$$

$$=\frac{-k}{\pi}\left|\frac{\sin nx}{n}\right|_{-\pi}^{0}+\frac{k}{\pi}\left|\frac{\sin nx}{n}\right|_{0}^{\pi}=\frac{-k}{n\pi}(0)+\frac{k}{n\pi}(0)=0. \qquad [\sin n\pi=0,\ \ n\in Z]$$

$$b_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int\limits_{-\pi}^{0}-k\sin nxdx+\frac{1}{\pi}\int\limits_{0}^{\pi}k\sin nxdx$$

$$=\frac{-k}{\pi}\left|\frac{-\cos nx}{n}\right|_{-\pi}^{0}+\frac{k}{\pi}\left|\frac{-\cos nx}{n}\right|_{0}^{\pi}=\frac{k}{n\pi}\left[1-(-1)^n\right]+\frac{k}{n\pi}(-1)\left[(-1)^n-1\right]$$

$$=\frac{k}{n\pi}\left[1-(-1)^n-(-1)^n+1\right]=\frac{2k}{n\pi}\left[1-(-1)^n\right]$$

$$=\begin{cases}0, & n\text{ is even}\\ \frac{4k}{n\pi}, & n\text{ is odd}\end{cases}$$

Hence, from (i), we get

$$f(x)=\frac{1}{2}.0+\sum\limits_{n=0}^{\infty}\left(0+b_n\sin nx\right)=\sum\limits_{\substack{n=1\\ n\text{ is odd}}}^{\infty}\frac{4k}{n\pi}\sin x=\frac{4k}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\ldots..\right].$$

$$f(x)=\frac{4k}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\ldots..\right]$$

Deduction: Put $x = \frac{\pi}{2}$, in above, we get

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\right] \qquad \left[\sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\ldots.$$

Q.No.9.: Develop f(x) in a Fourier series in the interval $(-\pi, \pi)$ if

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}.$$

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_0 \cos nx + b_n \sin nx\right)$ (i)

be the required series.

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_{0}^{\pi} 1dx = \frac{1}{\pi}.\pi = 1$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx dx$$

$$= \frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_0^{\pi} = 0. \qquad [\sin n\pi = 0, \quad n \in Z]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx$$

$$= \frac{1}{\pi}\left|\frac{-\cos nx}{n}\right|_0^{\pi} = \frac{-1}{n\pi}\left[(-1)^n - 1\right] = \frac{1-(-1)^n}{n\pi}$$

$$= \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd} \end{cases}.$$

Hence, from (i), we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{2} + \sum_{n=odd}^{\infty} \frac{2}{n\pi}\sin nx = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \ldots\ldots\right).$$

Q.No.10.: Find the Fourier expansion of the function defined in one period by the

relation $f(x)=\begin{cases}1, & 0<x<\pi \\ 2, & \pi<x<2\pi\end{cases}$

and deduce that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+......$.

Sol.: Let $f(x)=\dfrac{1}{2}a_0+\sum\limits_{n=1}^{\infty}\left(a_0\cos nx+b_n\sin nx\right)$ (i)

be the required Fourier series.

Here $a_0=\dfrac{1}{\pi}\int\limits_0^{2\pi}f(x)dx=\dfrac{1}{\pi}\int\limits_0^{\pi}1.dx+\dfrac{1}{\pi}\int\limits_{\pi}^{2\pi}2dx=\dfrac{1}{\pi}.\pi+\dfrac{2}{\pi}(2\pi-\pi)=3$.

$$a_n=\frac{1}{\pi}\int\limits_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int\limits_0^{\pi}\cos nxdx+\frac{1}{\pi}\int\limits_{\pi}^{2\pi}2\cos nxdx$$

$$=\frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_0^{\pi}+\frac{2}{\pi}\left|\frac{\sin nx}{n}\right|_0^{2\pi}=0.$$

$$b_n=\frac{1}{\pi}\int\limits_0^{2\pi}f(x)\sin nxdx=\frac{1}{\pi}\int\limits_0^{2\pi}\sin nxdx+\frac{2}{\pi}\int\limits_{\pi}^{2\pi}\sin nxdx$$

$$=\frac{1}{\pi}\left|\frac{\cos nx}{n}\right|_0^{\pi}+\frac{2}{\pi}\left|\frac{-\cos nx}{n}\right|_0^{2\pi}=\frac{-1}{n\pi}\left[(-1)^n-0\right]-\frac{-2}{n\pi}\left[1-(-1)^n\right].$$

$$=\frac{1}{n\pi}\left[(-1)^n-1\right]=\begin{cases}0, & \text{n is even}\\ \dfrac{-2}{n\pi}, & \text{n is odd}\end{cases}$$

Hence, from (i), we get

$$f(x)=\frac{3}{2}+\sum_{n=1}^{\infty}\left(0+b_n\sin nx\right)=\frac{3}{2}-\frac{2}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\sin nx=\frac{3}{2}-\frac{2}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+.......\right).$$

Deduction: Put $x=\dfrac{\pi}{2}$

$$f(\pi/2)=\frac{3}{2}-\frac{2}{\pi}\left(1+\left(-\frac{1}{3}\right)+.......\right).$$

$$\Rightarrow\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...............$$

Home Assignments

Q.No.1.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = a\sin t, \text{ if } 0 \le t \le \pi, \qquad \textbf{(Half wave rectifier)}$$
$$= 0 \qquad \text{if } \pi \le t \le 2\pi.$$

Deduce $\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$

Ans.: $f(x) = \frac{a}{\pi} + \frac{1}{2} a\sin x - \frac{2a}{n}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$.

Put $t = \pi$, then $0 = a\sin\pi = \frac{a}{\pi} + \frac{a}{2}\sin\pi - \frac{2a}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\pi}{4n^2 - 1}$.

Q.No.2.: Find the Fourier expansion of the **Modified saw-toothed wave form**

$$f(x) = 0 \text{ for } -\pi < x \le 0,$$
$$= x \text{ for } 0 < x \le \pi.$$

Hence, deduce $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + ...\right)$. Put $x = 0$.

Q.No.3.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = 2x \text{ when } 0 \le x \le \pi,$$
$$= x \quad \text{when } -\pi < x \le 0.$$

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ....\right)$.

Q.No.4.: Find the Fourier expansion of the function defined in one period by the relation

$$f(x) = -x \text{ if } -\pi < x \le 0,$$
$$= 0 \quad \text{if } 0 < x \le \pi.$$

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty} (-1)^{n-1}\frac{\sin nx}{n}$.

Q.No.5.: Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \quad \text{if} \ -\pi < x \le 0,$$
$$\quad = -2 \quad \text{if} \quad 0 < x \le \pi.$$

Ans.: $f(x) = -\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$.

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