4^{th} Topic

Double Integrals

[Double Integrals in Polar co-ordinates]

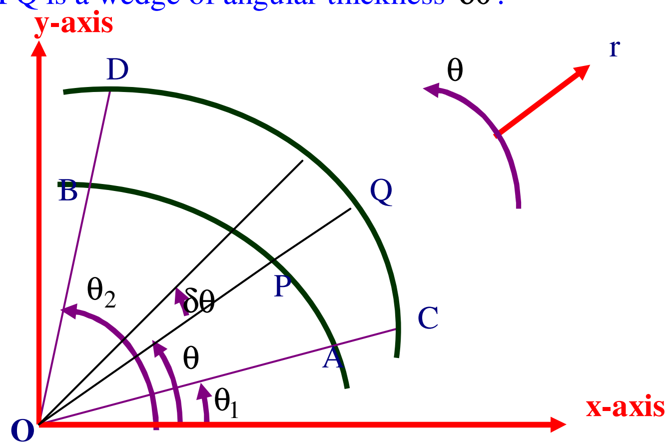
(Last updated on 15-07-2013)

(05 Solved problems and 00 Home assignments)

Evaluation of Double Integrals in Polar co-ordinates:

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants. Figure illustrated the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

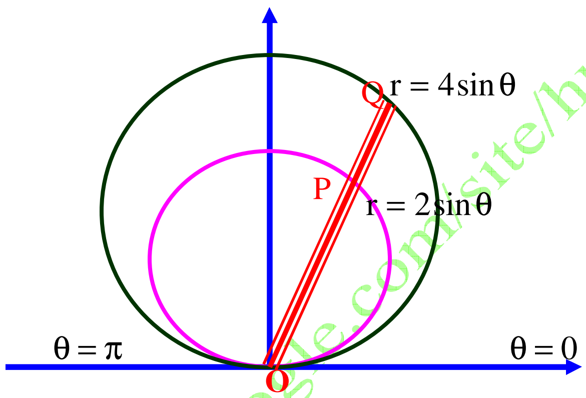


Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is along PQ from P to Q while the integration w. r. t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integrating is the area ACDB. The order of integration may changed with appropriate changes in the limits.

Q.No.1.: Calculate $\iint r^3 drd\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol.: Given circles are $r = 2\sin\theta$ and $r = 4\sin\theta$, as shown in the figure.



The area between these circles is the region of integration.

If we integrate first w. r. t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi}\left(\int_{2\sin\theta}^{4\sin\theta} r^3 dr\right)d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = 60\int_0^{\pi}\sin^4\theta d\theta = 60\times 2\int_0^{\pi/2}\sin^4\theta d\theta$$

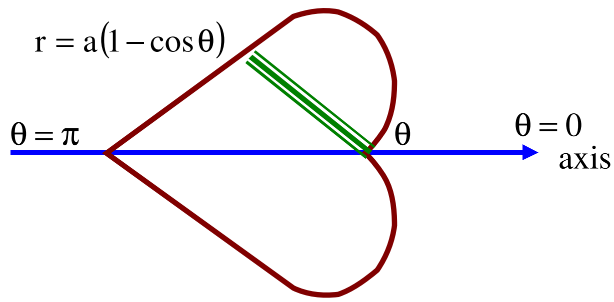
$$= 120\times\frac{3.1}{4.2}.\frac{\pi}{2} = 22.5\pi \text{.Ans.}$$

Q.No.2.: Evaluate $\iint r\sin\theta drd\theta$ over the cardioids $r = a(1-\cos\theta)$ above the initial line.

Sol.: The cardioids equation is $r = a(1-\cos\theta)$

The integral $\iint r\sin\theta drd\theta$ above initial line is

$$I = \int_0^{\pi}\left(\int_0^{a(1-\cos\theta)} r\sin\theta dr\right)d\theta = \int_0^{\pi}\left(\int_0^{a(1-\cos\theta)} rdr\right)\sin\theta d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)}\sin\theta d\theta$$



$$= \int_0^{\pi}\frac{a^2(1-\cos\theta)^2}{2}\sin\theta d\theta = \int_0^{\pi}.\frac{a^2\left(2\sin^2\frac{\theta}{2}\right)^2}{2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = \int_0^{\pi}4a^2\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 4a^2\int_0^{\pi}\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = 4a^2.2\int_0^{\pi/2}\sin^5\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = 4a^2.2\frac{4.2}{6.4.2}\times 1 = \frac{4a^2}{3}.\text{ Ans.}$$

Q.No.3.: Sketch the region of integration of $\int_a^{ae^{\pi/4}}\int_{2\log(r/a)}^{\pi/2} f(r,\theta)rdrd\theta$ and change the order of integration.

Sol.: Given Integral is $I = \int_a^{ae^{\pi/4}}\left(\int_{2\log(r/a)}^{\pi/2} f(r,\theta)d\theta\right)rdr$

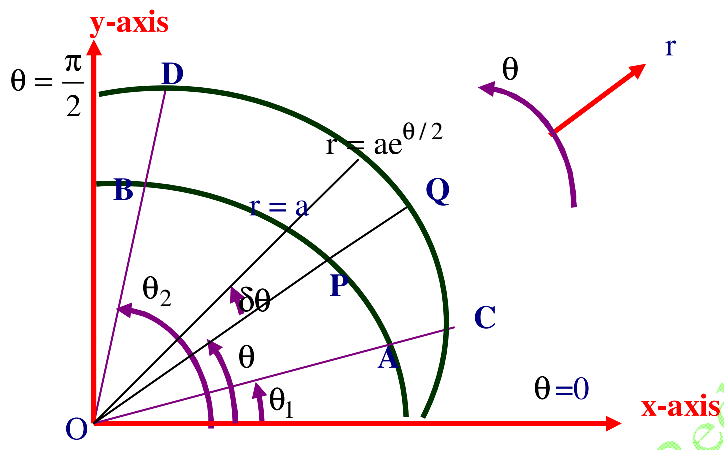
The region of integration is bounded by the curve

$\theta = 2\log\frac{r}{a}$, $\theta = \frac{\pi}{2}$, and $r = a$, $r = ae^{\pi/4}$.

Also when $r = a \Rightarrow \frac{r}{a} = 1$, then $\theta = 2\log\left(\frac{r}{a}\right) = 2\log(1) = 0$.

and when $r = ae^{\pi/4} \Rightarrow \frac{r}{a} = e^{\pi/4}$ and $\theta = 2\log\frac{r}{a} \Rightarrow \frac{r}{a} = e^{\theta/2}$

Then $e^{\theta/2} = e^{\pi/4} \Rightarrow \theta = \frac{\pi}{2}$.



Thus we change the order of integration r varies from $r = a$ to $r = ae^{\theta/2}$ and θ varies from $\theta = 0$ to $\Rightarrow \theta = \frac{\pi}{2}$.

Hence, on reversing the order of integration, we get

$$I = \int_0^{\pi/2} \left[\int_a^{ae^{\theta/2}} f(r,\ \theta) r dr \right] d\theta. \text{ Ans.}$$

Q.No.4.: Show that $\iint_R r^2 \sin\theta dr d\theta = \frac{2a^3}{3}$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

Sol.: The given semi-circle is $r = 2a\cos\theta$.

The shaded semicircle shown in the figure is the region of integration. If we integrate first w. r. t. r, then its limits are from r = 0 to $r = 2a\cos\theta$ and to cover the whole area of the semicircle θ varies from 0 to $\frac{\pi}{2}$.

Thus the integral is

$$I = \int_0^{\pi/2} \left[\int_0^{2a\cos\theta} r^2 dr \right] \sin\theta d\theta = \int_0^{\pi/2} \left[\left(\frac{r^3}{3} \right)_0^{2a\cos\theta} \right] \sin\theta d\theta = \int_0^{\pi/2} \left[\frac{8a^3\cos^3\theta}{3} \sin\theta \right] d\theta$$

$$= \frac{-8a^3}{3}\int_0^{\pi/2}\cos^3\theta\, d(\cos\theta) = \frac{-8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{-8a^3}{3}\cdot\frac{1}{4}[0-1]$$

$= \dfrac{-2a^3}{3}(-1) = \dfrac{2a^3}{3}$, which proves the result.

Q.No.5.: Evaluate $\iint \dfrac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Sol.: Symmetry: Curve is symmetric about the pole as even power of the r.

Limits: No position of the curve lies between $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$.

Region of integration is the area bounded by the curve.

$r = 0,\quad r = a\sqrt{\cos 2\theta}$ and $\theta = \dfrac{-\pi}{4},\quad \theta = \dfrac{\pi}{4}$

Thus $$I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\left[\int_0^{a\sqrt{\cos 2\theta}}\left(\frac{1}{2}\frac{2r\,dr}{\sqrt{a^2+r^2}}dr\right)\right]d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}\cdot\frac{2.\left(a^2+r^2\right)}{1}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4}\left[\left(a^2+r^2\right)^{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\left(a^2+a^2\cos 2\theta\right)^{1/2}-a\right]d\theta = a\int_{-\pi/4}^{\pi/4}\left[(1+\cos 2\theta)^{1/2}-1\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[\left(2\cos^2\theta\right)^{1/2}-1\right]d\theta = a\int_{-\pi/4}^{\pi/4}\left[\left(\sqrt{2}\cos\theta-1\right)\right]d\theta$$

$$= 2a\int_0^{\pi/4}\left(\sqrt{2}\cos\theta-1\right)d\theta = 2a\left[\sqrt{2}\sin\theta-\theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right] = 2a\left[1-\frac{\pi}{4}\right] = a\left(2-\frac{\pi}{2}\right).\ \text{Ans.}$$

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