Differential Calculus

Indeterminate Forms

$0^0, \quad \infty^0, \quad 1^\infty$

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Indeterminate forms-Problems of $0^0, \infty^0, 1^\infty$:

Q.No.1.: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \underset{x\to 0}{\mathrm{Lt}}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$ $\qquad \left[1^\infty \text{ form}\right]$

Taking log of both sides, we get

$$\log y = \underset{x\to 0}{\mathrm{Lt}}\log\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \underset{x\to 0}{\mathrm{Lt}}\frac{1}{x}\log\left(\frac{\sin x}{x}\right) = \underset{x\to 0}{\mathrm{Lt}}\frac{\log\left(\frac{\sin x}{x}\right)}{x} \qquad \left[\frac{0}{0}\text{form}\right]$$

Now apply Cauchy's Rule, we get

$$\log y = \underset{x\to 0}{\mathrm{Lt}}\left(\frac{x}{\sin x}\right)\left(\frac{x\cos x-\sin x}{x^2}\right) \qquad \left[\because \underset{x\to 0}{\mathrm{Lt}}\frac{x}{\sin x}=1\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\frac{x\cos x-\sin x}{x^2} \qquad \left[\frac{0}{0}\text{form}\right]$$

Applying Cauchy's Rule again, we get

$$\log y = \underset{x\to 0}{\mathrm{Lt}}\frac{\cos x - x\sin x-\cos x}{2x} = \underset{x\to 0}{\mathrm{Lt}}\frac{-x\sin x}{2x} = \underset{x\to 0}{\mathrm{Lt}}\frac{-\sin x}{2} = 0$$

$\log y = 0$

$\therefore y = e^0 = 1$. Ans.

Q.No.2.: Evaluate $\displaystyle\operatorname*{Lt}_{x\to 1}\left(1-x^2\right)^{\frac{1}{\log(1-x)}}$.

Sol.: Let $\displaystyle y = \operatorname*{Lt}_{x\to 1}\left(1-x^2\right)^{\frac{1}{\log(1-x)}}$ $\left[0^0 \text{form}\right]$

Taking log of both sides, we get

$$\log y = \operatorname*{Lt}_{x\to 1}\log\left(1-x^2\right)^{\frac{1}{\log(1-x)}}$$

$$= \operatorname*{Lt}_{x\to 1}\frac{1}{\log(1-x)}.\log\left(1-x^2\right) \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

Applying Cauchy's Rule, we get

$$\log y = \operatorname*{Lt}_{x\to 1}\frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)} = \operatorname*{Lt}_{x\to 1}\frac{2x(1-x)}{\left(1-x^2\right)} = \operatorname*{Lt}_{x\to 1}\frac{2x}{(1+x)} = 1$$

$\log_e y = 1$

$\therefore y = e^1 = e$. Ans.

Q.No.3.: Evaluate $\displaystyle\operatorname*{Lt}_{x\to 0}\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$.

Sol.: Let $\displaystyle y = \operatorname*{Lt}_{x\to 0}\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$ $\left[1^\infty \text{form}\right]$

Taking log of both sides, we get

$$\log y = \operatorname*{Lt}_{x\to 0}\log\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]^{\frac{1}{x}}$$

$$= \operatorname*{Lt}_{x\to 0}\frac{\log\left[\frac{a_1^x + a_2^x + \dots\dots\dots + a_n^x}{n}\right]}{x}$$

$\left[\frac{0}{0}\text{form}\right]$

Applying Cauchy's Rule, we get

$$\log y = \underset{x\to 0}{\mathrm{Lt}} \frac{n}{a_1^x + a_2^x + \ldots\ldots + a_n^x} \times \frac{\left(a_1^x \log a_1 + a_2^x \log a_2 + \ldots\ldots\ldots + a_n^x \log a_n\right)}{n}$$

$$= \left(\frac{n}{n}\right) \cdot \frac{1}{n} \left(\log a_1 + \log a_2 + \ldots\ldots\ldots + \log a_n\right)$$

$$\log y = \log\left(a_1 a_2 \ldots\ldots\ldots + a_n\right)^{\frac{1}{n}}$$

$$\left[\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}\right]$$

$$y = \left(a_1 . a_2 \ldots\ldots\ldots a_n\right)^{\frac{1}{n}}$$. **Ans.**

Q.No.4.: Evaluate $\underset{x\to 0}{\mathrm{Lim}}(1+\tan x)^{\cot x}$.

Sol.: Let $y = \underset{x\to 0}{\mathrm{Lim}}(1+\tan x)^{\cot x}$. $[1^\infty \text{ form}]$

Taking log on both sides, we get

$$\log y = \log\left[\underset{x\to 0}{\mathrm{Lim}}(1+\tan x)^{\cot x}\right] = \left[\underset{x\to 0}{\mathrm{Lim}}\left\{\log(1+\tan x)^{\cot x}\right\}\right] = \left[\underset{x\to 0}{\mathrm{Lim}}\{\cot x \log(1+\tan x)\}\right]$$

$$= \underset{x\to 0}{\mathrm{Lim}}\left[\frac{\log(1+\tan x)}{\tan x}\right] = \underset{x\to 0}{\mathrm{Lim}}\frac{1}{\tan x}\left[\tan x - \frac{\tan^2 x}{2} + \frac{\tan^3 x}{3} - \ldots\ldots\ldots\infty\right]$$

$$= \underset{x\to 0}{\mathrm{Lim}}\left[1 - \frac{\tan x}{2} + \frac{\tan^2 x}{3} - \ldots\ldots\ldots\infty\right] = 1.$$

$\therefore y = e^1 = e$.

Hence $\underset{x\to 0}{\mathrm{Lim}}(1+\tan x)^{\cot x} = e$. **Ans.**

Q.No.5.: Evaluate $\underset{x\to 0}{\mathrm{Lim}}(\cos x)^{\frac{1}{x^2}}$.

Sol.: Let $y = \underset{x\to 0}{\mathrm{Lim}}(\cos x)^{\frac{1}{x^2}}$. $[1^\infty \text{ form}]$

Taking log on both sides, we get

$$\therefore \log y = \lim_{x\to 0} \log(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} \frac{1}{x^2}\log(\cos x) = \lim_{x\to 0} \frac{\log(\cos x)}{x^2}.$$

$$\left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0} \frac{\left[\frac{1}{\cos x}\sin x\right]}{2x} = \lim_{x\to 0}\left(-\frac{\tan x}{2x}\right). \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\left(-\frac{\sec^2 x}{2}\right) = -\frac{1}{2}.$$

$$\therefore y = e^{-1/2}.$$

Hence $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-1/2}$. Ans.

Q.No.6.: Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

Sol.: Let $y = \lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$. $\qquad [\infty^0 \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{1}{x}\right)^{\tan x} = \lim_{x\to 0}\tan x\log\left(\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)x\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} x\log\left(\frac{1}{x}\right) \qquad \left[\therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right]$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}. \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{x\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0}(x) = 0.$$

$\therefore y = e^0 = 1$.

Hence $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x} = 1$ .Ans.

Q.No.7.: Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

Sol.: Let $y = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. $[1^\infty \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}. \quad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{x}{\tan x}.\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x} = \lim_{x\to 0}\left(\frac{x}{\tan x}.\frac{x\sec^2 x - \tan x}{2x^3}\right)$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^3}. \quad \left[\frac{0}{0}\text{form}\right] \quad \left[\because \lim_{x\to 0}\left(\frac{x}{\tan x}\right) = 1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\sec^2 x + x.2\sec x\sec x\tan x - \sec^2 x}{6x^2} = \lim_{x\to 0}\frac{2x\sec^2 x\tan x}{6x^2}.$$

$$= \lim_{x\to 0}\frac{\sec^2 x\tan x}{3x} = \lim_{x\to 0}\frac{\sec^2 x}{3}.\frac{\tan x}{x} = \frac{\sec^2 0}{3} = \frac{1}{3}.$$

$$\left[\because \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right]$$

$\therefore y = e^{1/3}$.

Hence $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}$.Ans.

Q.No.8.: Evaluate $\lim\limits_{x\to 0}(\cot x)^{\sin x}$.

Sol.: Let $y = \lim\limits_{x\to 0}(\cot x)^{\sin x}$. [$\infty^0$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log(\cot x)^{\sin x} = \lim_{x\to 0}\sin x\log(\cot x) = \lim_{x\to 0}\frac{\log(\cot x)}{\operatorname{cosec} x}.$$

$\left[\dfrac{\infty}{\infty}\text{form}\right]$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\frac{1}{\cot x}\operatorname{cosec}^2 x}{\operatorname{cosec} x\cot x} = \lim_{x\to 0}\frac{\operatorname{cosec} x}{\cot^2 x} = \lim_{x\to 0}\frac{\sin^2 x}{\sin x\cos^2 x} = \lim_{x\to 0}\frac{\sin x}{\cos^2 x} = 0.$$

$\therefore y = e^0 = 1$.

Hence $\lim\limits_{x\to 0}(\cot x)^{\sin x} = 1$.Ans.

Q.No.9.: Evaluate $\lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{\frac{1}{x}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x + b^x}{2}\right). \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{2}{a^x + b^x}\left(\frac{a^x\log x + b^x\log x}{2}\right) = \frac{\log a + \log b}{2} = \frac{1}{2}\log(ab) = \log(ab)^{1/2}.$$

$\therefore y = e^{\log(ab)^{1/2}} = \sqrt{ab}$.

Hence $\lim_{x\to0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$. Ans.

Q.No.10.: Evaluate $\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$.

Sol.: Let $y = \lim_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = \lim_{x\to0}\frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \lim_{x\to0}\frac{\log\left(\frac{\tan x}{x}\right)}{x}. \qquad \left[\frac{0}{0}\text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to0}\frac{\left[\frac{x}{\tan x}.\frac{x\sec^2 x - \tan x}{x^2}\right]}{2} = \lim_{x\to0}\left(\frac{x}{\tan x}.\frac{x\sec^2 x-\tan x}{2x^2}\right)$$

$$= \lim_{x\to0}\frac{x\sec^2 x-\tan x}{2x^2}. \qquad \left[\frac{0}{0}\text{form}\right] \qquad \left[\because \lim_{x\to0}\left(\frac{x}{\tan x}\right)=1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to0}\frac{\sec^2 x + x.2\sec x\sec x\tan x - \sec^2 x}{4x} = \lim_{x\to0}\frac{2x\sec^2 x\tan x}{4x}.$$

$$= \lim_{x\to0}\frac{x\sec^2 x}{2}.\frac{\tan x}{x} = \frac{0.\sec^2 0}{2} = 0. \qquad \left[\because \lim_{x\to0}\left(\frac{\tan x}{x}\right)=1\right]$$

$\therefore y = e^0 = 1$.

Hence $\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1$ .Ans.

Q.No.11.: Evaluate $\lim_{x\to0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$.

Sol.: Let $y = \lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$. [$1^\infty$ form]

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}\log\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sinh x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\sinh x}{x}\right)}{x^2}.$$

$\left[\frac{0}{0}\text{form}\right]$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{\left[\frac{x}{\sinh x}.\frac{x\cosh x - \sinh x}{x^2}\right]}{2x} = \lim_{x\to 0}\left(\frac{x}{\sinh x}.\frac{x\cosh x - \sinh x}{2x^3}\right)$$

$$= \lim_{x\to 0}\frac{x\cosh x - \sinh x}{2x^3}. \qquad \left[\frac{0}{0}\text{form}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{x}{\sinh x}\right) = 1\right]$$

Again, apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{x\sinh x + \cosh x - \cosh x}{6x^2} = \lim_{x\to 0}\frac{x\sinh x}{6x^2} = \lim_{x\to 0}\frac{1}{6}\frac{\sinh x}{x} = \frac{1}{6}.$$

$\therefore y = e^{1/6}$.

Hence $\lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{1/6}$.Ans.

Q.No.12.: Evaluate $\lim_{x\to 0}\frac{1 - x^x}{x\log x}$.

Sol.: Let $y = \lim_{x\to 0}\frac{1 - x^x}{x\log x}$ $\left[\frac{0}{0}\text{ form}\right]$

Applying Cauchy's Rule, we get

$$y = \lim_{x\to 0}\frac{-x^x(1+\log x)}{(1+\log x)} = \lim_{x\to 0} -x^x$$

Taking log of both sides, we get

$$\log y = \lim_{x\to 0} -x\log x = \lim_{x\to 0} -\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0$$

$\log y = 0$

$\therefore y = 1$. Ans.

Q.No.13.: Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{2/\pi}$.

Sol.: Let $y = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$. $[1^\infty \text{ form}]$

Taking log of both sides, we get

$$\log y = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right) \qquad [\infty\times 0 \text{ form}]$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\cdot\left(-\frac{1}{a}\right)}{-\frac{\pi}{2a}\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)} = \frac{2}{\pi}.$$

$\therefore y = e^{2/\pi}$.

Hence $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{2/\pi}$.Ans

Q.No.14.: Prove that $\operatorname{Lt}_{x\to 2}\left(8-x^3\right)^{\frac{1}{\log(2-x)}} = e$.

Sol.: Let $y = \operatorname{Lt}_{x\to 2}\left(8-x^3\right)^{\frac{1}{\log(2-x)}}$

Taking log on both sides, we get

$$\log y = \underset{x\to 2}{\text{Lt}} \log\left(8-x^3\right)^{\frac{1}{\log(2-x)}} = \underset{x\to 2}{\text{Lim}} \frac{\log\left(\sqrt{8}-x^3\right)}{\log\left(\sqrt{2}-x\right)}$$

Apply Cauchy's rule, we get

$$\log y = \underset{x\to 2}{\text{Lim}} \frac{\frac{-3x^2}{8-x^3}}{\frac{-1}{2-x}} = \underset{x\to 2}{\text{Lim}} \frac{3x^2}{4+3x+x^2} = \frac{12}{12} = 1$$

$$\log y = 1$$

$\therefore$ $y = e$. Ans.

Q.No.15.: Prove that $\underset{x\to 0}{\text{Lt}}\left[\frac{2(\cosh x-1)}{x^2}\right]^{\frac{1}{x^2}} = e^{\frac{1}{12}}$.

Sol.: Let $y = \underset{x\to 0}{\text{Lt}}\left[\frac{2(\cosh x-1)}{x^2}\right]^{\frac{1}{x^2}}$

$$\Rightarrow y = \underset{x\to 0}{\text{Lt}}\left[\frac{2}{x^2}\left(1+\frac{x^2}{2}+\frac{x^4}{4}+\text{..........}-1\right)\right]^{\frac{1}{x^2}}$$

Taking log on both sides, we get

$$\log y = \frac{\log\left(1+\frac{x^2}{12}+\text{...........}\infty\right)}{x^2} = \underset{x\to c}{\text{Lim}} \frac{\left(\frac{x}{6}+\frac{x^3}{3}+\text{........}\right)}{2x\left(1+\frac{x^2}{12}+\frac{x^4}{360}\text{.......}\right)}$$

$$\log y = \frac{1}{12}.$$

$y = e^{\frac{1}{12}}$. Ans.

Q.No.16.: Prove that $\underset{N\to\infty}{\text{Lt}}\left[\cos\frac{\beta}{N}\right]^{N^2} = e^{-\frac{1}{2}\beta^2}$.

Sol.: Let $y = \underset{N\to\infty}{\text{Lt}}\left[\cos\frac{\beta}{N}\right]^{N^2}$

$$y = \underset{N\to\infty}{\text{Lim.}}\left[1 - \frac{\beta^2}{2!N^2} + \frac{\beta^4}{4!N^4} + \dots\dots\dots\right]^{N^2}$$

Taking log on both sides, we get

$$\log y = \underset{N\to\infty}{\text{Lim.}}\, N^2 \log\left[1 - \frac{\beta^2}{2!N^2} + \frac{\beta^4}{4!N^4} + \dots\dots\dots\right]$$

$$\Rightarrow \log y = \underset{N\to\infty}{\text{Lim.}}\, N^2 \log\left[1 - \left(\frac{\beta^2}{2!N^2} - \frac{\beta^4}{4!N^4} + \dots\dots\dots\right)\right]$$

$$\Rightarrow \log y = -\underset{N\to\infty}{\text{Lim.}}\, N^2\left[\left(\frac{\beta^2}{2!N^2} - \frac{\beta^4}{4!N^4} + \dots\dots\dots\right) + \frac{1}{2}\left(\frac{\beta^2}{2!N^2} - \frac{\beta^4}{4!N^4} + \dots\dots\dots\right)^2 + \dots\dots\dots\right]$$

$$\Rightarrow \log y = -\underset{N\to\infty}{\text{Lim.}}\, N^2\left[\left(\frac{\beta^2}{2!N^2} - \frac{\beta^4}{4!N^4} + \dots\dots\dots\right) + \frac{\beta^4}{8N^4}\left(1 - \frac{\beta^2}{12N^2} + \dots\dots\dots\right)^2 + \dots\dots\dots\right]$$

$$\Rightarrow \log y = -\underset{N\to\infty}{\text{Lim.}}\, N^2\left[\left(\frac{\beta^2}{2!N^2} - \frac{\beta^4}{4!N^4} + \dots\dots\dots\right) + \frac{\beta^4}{8N^4}\left(1 - \frac{2\beta^2}{12N^2} + \dots\dots\dots\right) + \dots\dots\dots\right]$$

$$\Rightarrow \log y = -\underset{N\to\infty}{\text{Lim.}}\, N^2\left[\frac{\beta^2}{2!N^2} - \frac{1\beta^4}{12N^4} + \dots\dots\dots\dots\dots\dots\right]$$

$$\Rightarrow \log y = -\underset{N\to\infty}{\text{Lim.}}\left[\frac{\beta^2}{2!} - \frac{1\beta^4}{12N^2} + \dots\dots\dots\dots\dots\dots\right] \Rightarrow \log y = -\frac{\beta^2}{2}$$

$\therefore\ y = e^{-\frac{\beta^2}{2}}$ **.** Ans**.**

Q.No.17.: Prove that $\underset{x\to\infty}{\text{Lt}}\left(\dfrac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$ **.**

Sol.: Let $y = \underset{x\to\infty}{\text{Lt}}\left(\dfrac{ax+1}{ax-1}\right)^x$ [1^∞ form]

Taking log on both sides, we get

$$\log y = \underset{x\to\infty}{\text{Lt}}\ x \log\left(\frac{ax+1}{ax-1}\right) = \underset{x\to\infty}{\text{Lt}} \frac{\log\left[1 + \frac{1}{ax}\right]}{\frac{1}{x}\left[1 - \frac{1}{ax}\right]} = \underset{x\to\infty}{\text{Lt}} \frac{\log\left[1 + \frac{1}{ax}\right] - \log\left[1 - \frac{1}{ax}\right]}{\frac{1}{x}}$$

$$= \frac{\left[\frac{1}{ax} - \frac{1}{2a^2x^2} + \frac{1}{3a^3x^3} - \ldots\ldots\ldots\ldots\right] - \left[-\frac{1}{ax} - \frac{1}{2a^2x^2} - \frac{1}{3a^3x^3}\ldots\ldots\ldots\right]}{\frac{1}{x}}$$

$$\log y = \lim_{x\to\infty} \frac{2}{a}\left[1 + \frac{1}{3a^2x^2} + \ldots\ldots\right]$$

$$\log y = \frac{2}{a}.$$

$$\therefore\ y = e^{\frac{2}{a}}.\ \text{Ans.}$$

Q.No.18.: Evaluate $\displaystyle \mathop{\mathrm{Lt}}_{x\to\infty}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$**.**

Sol.: Let $\displaystyle y = \mathop{\mathrm{Lt}}_{x\to\infty}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$

Taking log on both sides, we get

$$\log y = \frac{\log\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]}{x-a} = \frac{2\sqrt{xa}(x-a)}{a+x}\left[\sqrt{a}x^{-\frac{3}{2}} + \frac{x^{-\frac{1}{2}}}{\sqrt{a}}\right] = \frac{2a}{2a}\times 0$$

$$\log y = 0.$$

$$\therefore\ y = 1.\ \text{Ans.}$$

Q.No.19.: Prove that $\displaystyle \mathop{\mathrm{Lt}}_{x\to 0}\frac{1-x^{\sin x}}{x\log x} = -1$**.**

Sol.: Let $\displaystyle y = \mathop{\mathrm{Lt}}_{x\to 0}\frac{1-x^{\sin x}}{x\log x}$

$$y = \mathop{\mathrm{Lt}}_{x\to 0}\frac{1-x^{\sin x}.x^x}{x\log x} = \lim_{x\to 0}\frac{1-x^x}{x\log x} = \mathop{\mathrm{Lt}}_{x\to 0}\frac{-x^x(1+\log x)}{1+\log x}$$

$$y = \lim_{x\to 0} - x^x$$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0} - x\log x = \lim_{x\to 0}\frac{-\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty}\ \text{form}\right]$$

Applying Cauchy's rule, we get

$$\log y = \frac{\frac{-1}{x}}{\frac{-1}{x^2}} = x = 0.$$

$\therefore y = 1$. Ans.

Q.No.20.: Evaluate $\underset{m\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{m}\right)^m$.

Sol.: Let $y = \underset{m\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{m}\right)^m$

Taking log on both sides, we get

$$\log y = \underset{m\to\infty}{\mathrm{Lt}}\ m\log\left(\cos\frac{x}{m}\right) = \underset{m\to\infty}{\mathrm{Lt}}\frac{\log\left(\cos\frac{x}{m}\right)}{1/m}$$

$\left[\frac{0}{0}\ \text{form}\right]$

$$\log y = \underset{m\to\infty}{\mathrm{Lt}}\frac{\frac{1}{\cos\frac{x}{m}}\left(-\sin\frac{x}{m}\right)\left(-\frac{1}{m^2}\right)}{-\frac{1}{m^2}} = \underset{m\to\infty}{\mathrm{Lt}}\left(-\tan\frac{x}{m}\right) = 0$$

$\therefore y = e^0 = 1$. Ans.

Q.No.21.: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{x}\right)^{1-\cos x}$.

Sol.: Let $y = \underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{x}\right)^{1-\cos x}$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0}(\cos x - 1)\log x = \lim_{x\to 0}\frac{(\cos x - 1)}{x}(x\log x) = \lim_{x\to 0}\frac{(\cos x - 1)}{x} \qquad \left[\frac{0}{0}\ \text{form}\right]$$

Apply Cauchy's rule, we get

$$\log y = \lim_{x\to 0}\frac{-\sin x}{1} = 0$$

$\log y = 0 \therefore y = e^0 = 1$. Ans.

Q.No.22.: Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$.

Sol.: Let $y = \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$ $[\infty^0 \text{ form}]$

Taking log on both sides, we get

$$\log y = \lim_{x\to\frac{\pi}{2}}\cos x \log(\tan x) \quad [0\times\infty \text{ form}]$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\log(\tan x)}{\sec x} \quad \left[\frac{\infty}{\infty}\text{form}\right]$$

Apply Cauchy's rule, we get

$$= \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\tan x}.\sec^2 x}{\sec x.\tan x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin^2 x} = 0.$$

$\therefore y = e^0 = 1$. Ans.

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