4^{th} Topic

Triple Integrals

[Volume by triple integrals]

(Last updated on 15-07-2013)

(16 Solved problems and 00 Home assignment)

Volume of solids as triple integrals:

Divide the given solid by planes parallel to the co-ordinate planes into rectangular parallelopiped of volume $\delta x\, \delta y\, \delta z$.

$\therefore$ The total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \sum \sum \sum \delta x\, \delta y\, \delta z = \iiint dx\, dy\, dz$,

with appropriate limits of integration.

Q.No.1.: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol.: Let A be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\therefore A = \left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$$

$$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \Rightarrow \frac{x^2}{a^2} \le 1, \ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \ \text{ and } \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

$$\Rightarrow x^2 \le a^2, \quad y^2 \le b^2\left(1 - \frac{x^2}{a^2}\right) \ \text{ and } \ z^2 \le c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$\Rightarrow -a \le x \le a\,,\ \ -b\sqrt{1-\dfrac{x^2}{a^2}} \le y \le b\sqrt{1-\dfrac{x^2}{a^2}}$ and $-c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} \le z \le c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}$.

$$\therefore A = \left\{ \begin{array}{c} (x,\ y,\ z)\ :\ -a \le x \le a,\ -b\sqrt{1-\dfrac{x^2}{a^2}} \le y \le b\sqrt{1-\dfrac{x^2}{a^2}}, \\ -c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} \le z \le c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} \end{array} \right\}$$

Hence the volume of the whole ellipsoid $= \iiint dxdydz$

$$= 8\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} \left\{ \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dz \right\} dy \right] dx$$

$$= 8\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} |z|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dy \right] dx = 8c\int_0^a \left[\int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)} dy \right] dx$$

$$= \frac{8c}{b}\int_0^a \left[\int_0^{\rho} \sqrt{(\rho^2-y^2)}\, dy \right] dx \quad \text{where, we put } b\sqrt{\left(1-\frac{x^2}{a^2}\right)} = \rho\,.$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{(\rho^2-y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho} \right]_0^{\rho} dx = \frac{8c}{b}\int_0^a \left(\frac{b^2}{2}\left\{1-\frac{x^2}{a^2}\right\}\frac{\pi}{2} \right) dx$$

$$= 2\pi bc\int_0^a \left(1-\frac{x^2}{a^2}\right) dx = 2\pi bc\left[x-\frac{x^2}{3a^2}\right]_0^a = \frac{4\pi abc}{3}.\ \text{Cubic units. Ans.}$$

or

Sol.: Volume of the ellipsoid $= \displaystyle\iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} 1dxdydz$.

Put $\dfrac{x}{a} = u,\ \ \dfrac{y}{b} = v,\ \ \dfrac{z}{c} = w$.

The given region transforms into the region

$D' = \{(u,\ v,\ w) : u^2+v^2+w^2 \le 1\}$

$\therefore J = \dfrac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)} = abc\,. \qquad \therefore |J| = abc$

Volume of the ellipsoid $= \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} 1dxdydz = \iiint\limits_{u^2+v^2+w^2\le 1} 1.abc.dudvdw$

$$= abc \iiint\limits_{u^2+v^2+w^2\le 1} dudvdw$$

To change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ), we have put $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and

$$J = \frac{\partial(u, v, w)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Then $\iiint\limits_{R_{uvw}} f(u,v,w)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore V = abc\int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^1 r^2dr\right)\sin\theta d\theta\right]d\phi = abc\int_0^{2\pi}\left[\int_0^{\pi}\left(\frac{r^3}{3}\right)_0^1 \sin\theta d\theta\right]d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\frac{1}{3}\sin\theta d\theta\right)d\phi = \frac{abc}{3}\int_0^{2\pi}(-\cos\theta)_0^{\pi}d\phi = -\frac{abc}{3}\int_0^{2\pi}[\cos\pi - \cos 0]d\phi$$

$$= \frac{2abc}{3}\int_0^{2\pi}1d\phi = \frac{2abc}{3}[\phi]_0^{2\pi} = \frac{2abc}{3}(2\pi) = \frac{4\pi}{3}abc\text{. Cubic units. Ans.}$$

Q.No.2.: Find the volume of the tetrahedron bounded by the co-ordinate planes and plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Or

Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, a, b, c are positive.

Sol.: Let A be the region bounded by the four planes of the tetrahedron.

$$\therefore A = \left\{(x, y, z) : x \ge 0, y \ge 0, z \ge 0 \text{ and } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\right\}$$

$$\because \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1$$

$$\Rightarrow \frac{x}{a}\le 1,\ \frac{x}{a}+\frac{y}{b}\le 1 \ \text{ and } \ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1$$

$$\Rightarrow x\le a,\ y\le b\left(1-\frac{x}{a}\right) \ \text{ and } \ x\le c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$\therefore A=\left\{(x,y,z):0\le x\le a, 0\le y\le b\left(1-\frac{x}{a}\right),\quad 0\le z\le c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right\}$$

$$\therefore \text{The required volume } =\int_0^a\left[\int_0^{b(1-x/a)}\left\{\int_0^{c(1-x/a-y/b)}dz\right\}dy\right]dx$$

$$=\int_0^a\left[\int_0^{b(1-x/a)}\left|z\right|_0^{c(1-x/a-y/b)}dy\right]dx=\int_0^a\left[\int_0^{b(1-x/a)}c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right]dx$$

$$=c\int_0^a\left[\int_0^{b(1-x/a)}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right]dx=\int_0^a\frac{c\left[\left(1-\frac{x}{a}-\frac{y}{b}\right)^2\right]_0^{b\left(1-\frac{x}{a}\right)}}{2\left(-\frac{1}{b}\right)}dx$$

$$=-\frac{bc}{2}\int_0^a\left[0-\left(1-\frac{x}{a}\right)^2\right]dx=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx$$

$$=\frac{bc}{2}\frac{\left[\left(1-\frac{x}{a}\right)^3\right]_0^a}{3\left(-\frac{1}{a}\right)}=-\frac{abc}{6}[0-1]=\frac{abc}{6}.\ \text{Cubic unit. Ans.}$$

Q.No.3.: Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1.$$

Sol.: The volume of the solid $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$ is

$$V = \iiint\limits_{\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}\le 1} 1dxdydz .$$

Put $\left(\frac{x}{a}\right)^{1/3} = u$, $\left(\frac{y}{b}\right)^{1/3} = v$, $\left(\frac{z}{c}\right)^{1/3} = w$.

$\therefore$ The given region transforms into the region $D' = \{(u, v, w) : u^2 + v^2 + w^2 \le 1\}$

$\therefore \frac{x}{a} = u^3, \frac{y}{b} = v^3, \frac{z}{c} = w^3$

$\Rightarrow x = au^3, \ y = bv^3, \ z = cw^3$ and

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3au^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3cw^2 \end{vmatrix} = 27abc\, u^2v^2w^2 .$$

$$\therefore V = \iiint\limits_{u^2+v^2+w^2\le 1} 27abc\, u^2v^2w^2 dudvdw = 27abc \iiint\limits_{u^2+v^2+w^2\le 1} u^2v^2w^2 dudvdw . \qquad \text{(i)}$$

To change rectangular co-ordinates (u, v, w) to spherical polar co-ordinates (r, θ, ϕ), we have put $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and

$$J = \frac{\partial(u, v, w)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Then $\iiint\limits_{R_{uvw}} f(u,v,w)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$$\therefore \ V = 27abc \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin^2\theta\cos^2\phi.r^2\sin^2\theta\sin^2\phi.r^2\cos^2\theta.r^2\sin\theta.drd\theta d\phi$$

$$= 27abc \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^8\sin^5\theta\cos^2\theta.\cos^2\phi\sin^2\phi..drd\theta d\phi$$

$$=\frac{27abc}{9}\int_0^{2\pi}\left[\left(\int_0^{\pi}\sin^5\theta\cos^2\theta d\theta\right)\right]\cos^2\phi\sin^2\phi d\phi \qquad \left[\because \int_0^1 r^8 dr=\left(\frac{r^9}{9}\right)_0^1=\frac{1}{9}\right]$$

$$=\frac{27abc}{9}\int_0^{2\pi}\left[\left(2\int_0^{\pi/2}\sin^5\theta\cos^2\theta d\theta\right)\right]\cos^2\phi\sin^2\phi d\phi$$

$$=\frac{27abc}{9}\int_0^{2\pi}\left[2.\frac{4.2.1}{7.5.3.1}\right]\cos^2\phi\sin^2\phi d\phi=\frac{27abc}{9}\int_0^{2\pi}\left[\frac{16}{105}\right]\cos^2\phi\sin^2\phi d\phi$$

$$=\frac{16}{35}abc\int_0^{2\pi}\cos^2\phi\sin^2\phi d\phi=\frac{64abc}{35}\int_0^{\pi/2}\cos^2\phi\sin^2\phi d\phi$$

$$=\frac{64abc}{35}\frac{1.1}{4.2}\times\frac{\pi}{2}=\frac{4\pi abc}{35}.$$ Cubic units. Ans.

Q.No.4.: Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ax$.

or

Find the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$

Sol.: The required volume is easily found by changing to cylindrical co-ordinates (ρ,ϕ,z).

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ,ϕ,z), we have put $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$ and

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial z}\end{vmatrix}=\begin{vmatrix}\cos\phi & -\rho\sin\phi & 0\\ \sin\phi & \rho\cos\phi & 0\\ 0 & 0 & 1\end{vmatrix}=\rho\left(\cos^2\phi+\sin^2\phi\right)=\rho.$$

Then $\iiint_{R_{xyz}} f(x,y,z)dxdydz=\iiint_{R'_{\rho\theta z}} f(\rho\cos\phi,\rho\sin\phi,z).\rho d\rho d\phi dz$.

Then the equation of the cylinder becomes $\rho=a\cos\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown in the above region for which z varies from 0 to $\sqrt{\left(a^2-\rho^2\right)}$, ρ varies from 0 to $a\cos\phi$ and ϕ varies from 0 to π.

$$\therefore \text{Required volume} = 2\int_0^{\pi}\left\{\int_0^{a\cos\phi}\left(\int_0^{\sqrt{\left(a^2-\rho^2\right)}} dz\right)\rho d\rho\right\}d\phi = 2\int_0^{\pi}\left(\int_0^{a\cos\phi}\rho\sqrt{\left(a^2-\rho^2\right)}d\rho\right)d\phi$$

$$= 2\int_0^{\pi}\left[-\frac{1}{3}\left(a^2-\rho^2\right)^{3/2}\right]_0^{a\cos\phi}d\phi = \frac{2a^3}{3}\int_0^{\pi}\left(1-\sin^3\phi\right)d\phi$$

$$\left[\because \sin 3\phi = 3\sin\phi - 4\sin^3\phi \Rightarrow \sin^3\phi = \frac{3\sin\phi-\sin 3\phi}{4}\right]$$

$$= \frac{2a^3}{3}\int_0^{\pi}\left(1-\frac{3\sin\phi-\sin 3\phi}{4}\right)d\phi = \frac{2a^3}{3}\left(\pi-\frac{1}{4}\left(6-\frac{2}{3}\right)\right) = \frac{2a^3}{3}\left(\pi-\frac{4}{3}\right)$$

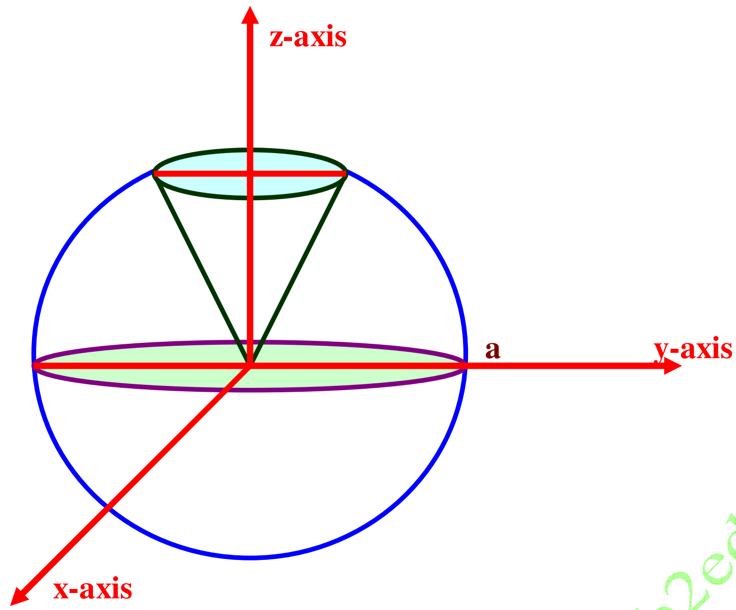
$$= \frac{2a^3}{9}(3\pi-4).$$ Cubic units. Ans.

Q.No.5.: Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$ above xy-plane.

Sol.: The required volume $V = \iiint_R dxdydz$.

To change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$



Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).r^2\sin\theta drd\theta d\phi$.

$\therefore x^2 + y^2 + z^2 = a^2 \Rightarrow r^2 = a^2$ and $x^2 + y^2 = z^2 \Rightarrow r^2\sin^2\theta = r^2\cos^2\theta$

$\Rightarrow r$ varies from 0 to a, θ varies from 0 to $\frac{\pi}{4}$, ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \text{Required volume} = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^a r^2\sin\theta dr\, d\theta d\phi = 4\int_0^{\pi/2}\int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^a \sin\theta\, d\theta d\phi$$

$$= \frac{4a^3}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sin\theta\, d\theta d\phi = \frac{4a^3}{3}\int_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/4} d\phi$$

$$= \frac{4a^3}{3}\int_0^{\pi/2}\left(1-\frac{1}{\sqrt{2}}\right)d\phi = 4\frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)\frac{\pi}{2}$$

$$= 2\frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)\pi = \frac{a^3}{3}\left(2-\sqrt{2}\right)\pi.$$ Cubic units. Ans.

Q.No.6.: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol.: The required volume $V = \iiint dxdydz$.

Since $x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2$.

$\Rightarrow z$ varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$.

Also $x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$.

$\Rightarrow$ y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$.

Now $x^2 = a^2$, by putting $y = 0$ and $z = 0$

$\Rightarrow$ x varies from $-a$ to a.

$$\therefore V = \iiint dxdydz = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\right\}dy\right]dx = 8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left\{\int_0^{\sqrt{a^2-x^2}} dz\right\}dy\right]dx$$

$$= 8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2}}\, dy\right]dx = 8\int_0^a\left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\right)dx$$

$$= 8\int_0^a \sqrt{a^2-x^2}\,[y]_0^{\sqrt{a^2-x^2}}\, dx = 8\int_0^a \left(a^2 - x^2\right)dx = 8\left|a^2x - \frac{x^3}{3}\right|_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right) = \frac{16a^3}{3}.$$ Cubic units. Ans.

Q.No.7.: Find the volume bounded by the cylinder $x^2 + y^2 = 4$, and the hyperboloid

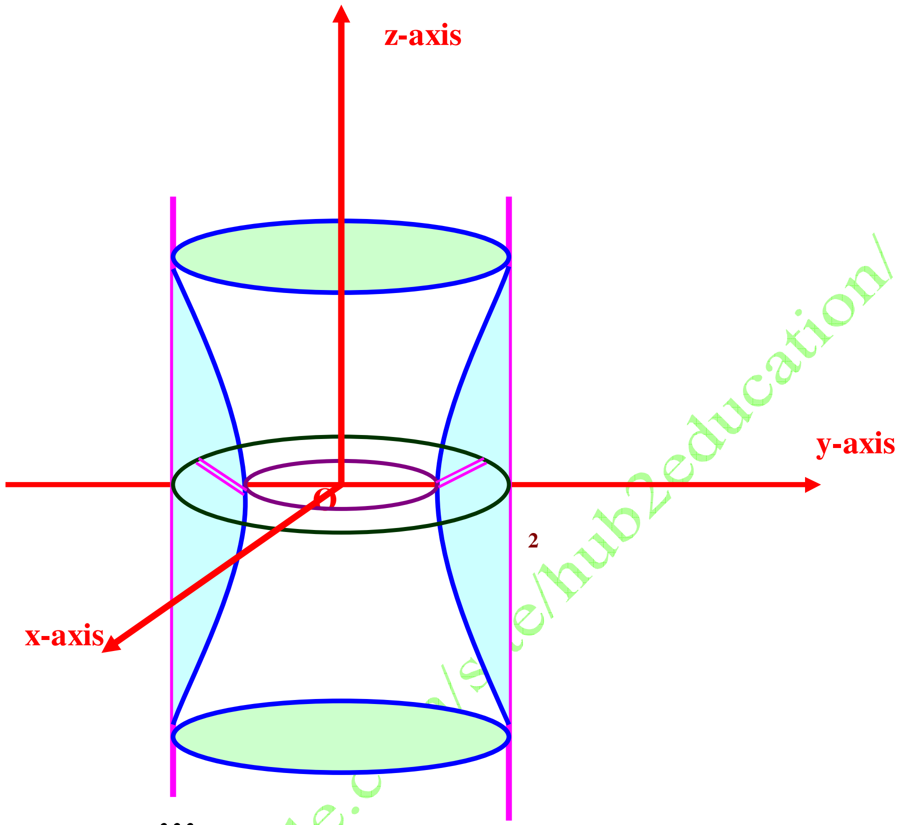
$x^2 + y^2 - z^2 = 1$.

Sol.: The required volume $V = \iiint dxdydz$.

To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z),

we have put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho.$$



Then $\iiint\limits_{R_{xyz}} f(x, y, z)dxdydz = \iiint\limits_{R'_{\rho\theta z}} f(\rho\cos\phi, \rho\sin\phi, z).\rho d\rho d\phi dz$.

Then the equation of hyperboloid $x^2 + y^2 - z^2 = 1 \Rightarrow \rho^2 - z^2 = 1$ and that of cylinder $x^2 + y^2 = 4 \Rightarrow \rho^2 = 4$.

The volume inside the cylinder bounded by the hyperboloid is twice the volume above the xy-plane. For which z varies from 0 to $\sqrt{\rho^2 - 1}$, ρ varies from 1 to 2, and ϕ varies from 0 to 2π.

$$\therefore \text{Required volume} = 2\int_0^{2\pi}\left[\int_1^2\left(\int_0^{\sqrt{\rho^2-1}} dz\right)\rho\, d\rho\right]d\phi = 2\int_0^{2\pi}\left[\int_1^2 \rho\sqrt{\rho^2-1}d\rho\right]d\phi$$

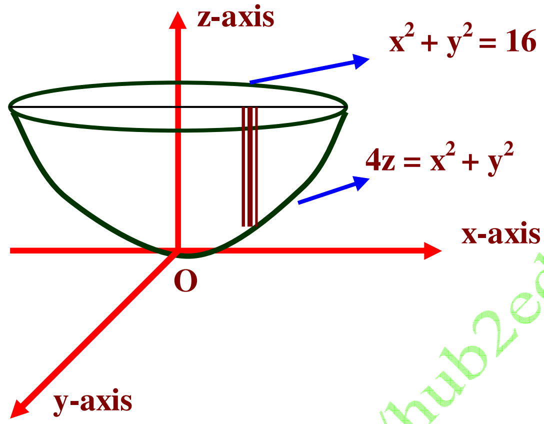
Put $t^2 = \rho^2 - 1$ so that $tdt = \rho d\rho$.

And as ρ varies from 1 to 2; and t varies from 0 to $\sqrt{3}$

$\therefore$ Required volume $= 2\int_0^{2\pi} \left| \frac{t^3}{3} \right|_0^{\sqrt{3}} d\phi = 2\times 2\times\sqrt{3}\,\pi = 4\sqrt{3}\,\pi$. Cubic units. Ans.

Q.No.8.: Find the volume cut from parabolic $4z = x^2 + y^2$ by the plane $z = 4$.

Sol.:



The volume is given by

$$v = 4\int_0^4 \left[\int_0^{\sqrt{16-x^2}} \left\{ \int_{\frac{x^2+y^2}{4}}^{4} dz \right\} dy \right] dx = 4\int_0^4 \left[\int_0^{\sqrt{16-x^2}} \left\{ 4 - \frac{x^2}{4} - \frac{y^2}{4} \right\} dy \right] dx$$

$$= 4\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) y - \frac{y^3}{12} \right]_0^{\sqrt{16-x^2}} dx = 4\int_0^4 \left[\left(4 - \frac{x^2}{4}\right)\sqrt{16-x^2} - \frac{\left(16-x^2\right)^{3/2}}{12} \right] dx$$

$$= 4\int_0^4 \left[\frac{1}{4}\left(16-x^2\right)\sqrt{16-x^2} - \frac{1}{12}\left(16-x^2\right)^{3/2} \right] dx = 4\int_0^4 \left[\frac{1}{4}\left(16-x^2\right)^{3/2} - \frac{1}{12}\left(16-x^2\right)^{3/2} \right] dx$$

$$= 4\int_0^4 \left[\frac{1}{6}\left(16-x^2\right)^{3/2} \right] dx = \frac{2}{3}\int_0^4 \left(16-x^2\right)^{3/2} dx$$

Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$ and $\theta = \frac{\pi}{2}$, when $x = 4$ and $\theta = 0$ when $x = 0$.

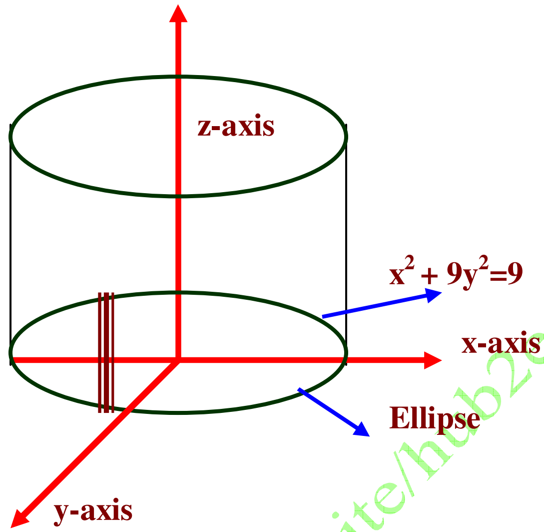
$$V = \frac{2}{3}\int_0^{\pi/2} (16)^{3/2} \cos^3\theta . 4\cos\theta d\theta = \frac{512}{3}\int_0^{\pi/2} \cos^4\theta d\theta = \frac{512}{3}.\frac{3.1}{4.2}.\frac{\pi}{2} = 32\pi$$

$\therefore$ Volume cut from paraboloid $4z = x^2 + y^2$ by plane $z = 4$ is given by 32π. Cubic units.

Q.No.9.: Find the volume bounded by the elliptic Paraboloids $z = x^2 + 9y^2$ and

$$z = 18 - x^2 - 9y^2.$$

Sol.:



The two surfaces intersect on the elliptic cylinder $x^2 + 9y^2 = z = 18 - x^2 - 9y^2$

$$\Rightarrow x^2 + 9y^2 = 9.$$

The projection of this volume onto xy-plane region D enclosed by ellipse having the same equation $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1^2$.

This volume can be covered as follows:

z: from $z_1(x, y) = x^2 + 9y^2$ to $z_2(x, y) = 18 - x^2 - 9y^2$

y: from $y_1(x, y) = -\sqrt{\frac{9 - x^2}{9}}$ to $y_2(x, y) = \sqrt{\frac{9 - x^2}{9}}$

x: from $x_1(x, y) = -3$ to $x_2(x, y) = 3$.

Thus the volume bounded by the elliptic Paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$ is

$$V = \int_{-3}^{3} \left\{ \int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}} \left(\int_{x^2+9y^2}^{18-x^2-9y^2} dz \right) dy \right\} dx$$

$$=\int_{-3}^{3}\left\{\int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\left\{\left(18-x^2-9y^2\right)-\left(x^2+9y^2\right)\right\}dy\right\}dx=2\int_{-3}^{3}\left\{\int_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\left(9-x^2-9y^2\right)dy\right\}dx$$

$$=2\int_{-3}^{3}\left\{\left(9y-x^2y-3y^3\right)\Big|_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}}\right\}dx=\frac{8}{9}\int_{-3}^{3}\left(9-x^2\right)^{3/2}dx=72\int_{0}^{\pi}\sin^4\theta d\theta,\ \text{where } x=3\cos\theta$$

$$=72\times 2\int_{0}^{\pi/2}\sin^4\theta d\theta=144\times\left(\frac{3\times 1}{4\times 2}\times\frac{\pi}{2}\right)=27\pi.\ \text{Cubic units.}$$

Q.No.10.: Find, by triple integration, the volume in the positive octant bounded by the coordinate planes and the plane $x+2y+3z=4$.

Sol.: Equation of the given plane $x+2y+32=4\Rightarrow z=\dfrac{4-x-2y}{3}$

i.e. z varies from 0 to $\dfrac{4-x-2y}{3}$ and y varies from 0 to $\dfrac{4-x}{2}$ and similarly x varies from 0 to 4.

Required volume $=\int\int_R\int dzdydx=\int_0^4\int_0^{\frac{4-x}{2}}\int_0^{\frac{4-x-2y}{3}}dzdydx$

$$=\int_0^4\int_0^{\frac{4-x}{2}}\frac{4-x-2y}{3}dydx=\int_0^4\frac{4}{3}\left(\frac{4-x}{2}\right)-\frac{x}{3}\left(\frac{4-x}{2}\right)-\frac{2}{3}\times\frac{1}{2}\left(\frac{4-x}{2}\right)^2dx$$

$$=\int_0^4\frac{16-4x}{6}-\frac{4x-x^2}{6}-\frac{1}{3}\times\frac{16+x^2-8x}{2\times 2}dx$$

$$=\frac{16}{6}[x]_0^4-\frac{4}{6}\times\frac{1}{2}\left[x^2\right]_0^4-\frac{16}{2\times 6}[x]_0^4-\frac{1}{2\times 6}\times\frac{1}{3}\left[x^3\right]_0^4+\frac{8}{2\times 6}\times\frac{1}{2}\left[x^2\right]_0^4$$

$$=\frac{16}{6}\times 4-\frac{4}{12}\times 16-\frac{16}{12}\times 4-\frac{1}{18}\times\frac{64}{2}+\frac{8}{12}\times\frac{16}{2}$$

$$=\frac{32}{3}-\frac{16}{3}-\frac{16}{3}+\frac{32}{9}-\frac{16}{3}-\frac{16}{9}+\frac{16}{3}=\frac{32}{9}-\frac{16}{9}=\frac{16}{9}.\ \text{Cubic units}$$

Q.No.11.: Find, by triple integration, the volume of the region bounded by the paraboloid

$$az = x^2 + y^2 \text{ and the cylinder } x^2 + y^2 = R^2.$$

Sol.: Given equation of the paraboloid $az = x^2 + y^2 \Rightarrow z = \dfrac{x^2 + y^2}{a}$.

i.e. z varies from 0 to $\dfrac{x^2 + y^2}{a}$, similarly, y varies from 0 to $\sqrt{R^2 - x^2}$ and x varies from 0 to R.

$$\text{Volume required } = \int\int_R\int dzdydx = 4\int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{\frac{x^2+y^2}{a}} dzdydx = 4\int_0^R \int_0^{\sqrt{R^2-x^2}} \frac{x^2 + y^2}{a} dydx$$

$$= 4\int_0^R \left\{ \frac{x^2}{a}[y]_0^{\sqrt{R^2-x^2}} + \frac{1}{3a}\left[y^3\right]_0^{\sqrt{R^2-x^2}} \right\} dx = 4\int_0^R \left\{ \frac{x^2}{a}\sqrt{R^2 - x^2} + \frac{1}{3a}\left(R^2 - x^2\right)^{3/2} \right\} dx$$

Putting $x = R\sin\theta \Rightarrow dx = R\cos\theta d\theta$

$$V = 4\int_0^{\pi/2} \left\{ \frac{R^2}{a}\sin^2\theta\sqrt{R^2\left(1 - \sin^2\theta\right)}R\cos\theta + \frac{1}{3a}\left[R^2\left(1 - \sin^2\theta\right)\right]^{3/2} \right\} d\theta\, R\cos\theta$$

$$= \int_0^{\pi/2} \left(\frac{R^4}{a}\sin^2\theta\cos^2\theta + \frac{R^4}{3a}\cos^4\theta \right) d\theta = 4\left(\frac{R^4}{a}\frac{1.1}{4.2}\times\frac{\pi}{2} + \frac{R^4}{3a}\times\frac{3.1}{4.2}\times\frac{\pi}{2} \right)$$

$$= \frac{\pi R^4}{4a} + \frac{\pi R^4}{4a} = \frac{\pi R^4}{2a}. \text{ Cubic units}$$

Q.No.12.: Find, by triple integration, the volume of the sphere of radius a.

Sol.: Equation of the sphere of radius a

$$x^2 + y^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

$$\text{Required volume } = 8\int\int_R\int dzdydx = 8\int_0^{\pi/2a} \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$x^2 + y^2 = r^2$$

$|J| = r$

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dydx = 8\int_0^{\pi/2}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta$$

$\Rightarrow a^2 - r^2 = t^2 \Rightarrow -2rdr = 2tdt \Rightarrow rdr = -tdt$.

$$V = 8\int_0^{\pi/2}\int_a^0 -t^2dt = 8\int_0^{\pi/2}\frac{1}{3}\left[t^3\right]_0^a d\theta = \frac{8}{3}\int_0^{\pi/2} a^3 d\theta = \frac{8}{3}a^3 \times \frac{\pi}{2} = \frac{4\pi a^3}{3}.$$ Cubic units

Q.No.13.: Find, by triple integration, the volume bounded above by the sphere $x^2+y^2+z^2 = 2a^2$ and below the paraboloid $az = x^2+y^2$.

Sol.: Equation of the given sphere is $x^2+y^2+z^2 = 2a^2$ and equation of the given paraboloid is $az = x^2+y^2$.

i.e. z varies from $z = \dfrac{x^2+y^2}{a}$ to $z = \sqrt{2a^2-x^2-y^2}$.

Now $x^2+y^2+z^2 = 2a^2$

$$\Rightarrow az + z^2 = 2a^2 \Rightarrow z^2 + az - 2a^2 = 0 \Rightarrow z = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = -2a,\ a$$

Since we have to find volume bounded above by the sphere $x^2+y^2+z^2 = 2a^2$ and below the paraboloid $az = x^2+y^2$. Thus z = -2a (rejected).

Thus equation of circle becomes $x^2+y^2+a^2 = 2a^2 \Rightarrow x^2+y^2 = a^2$

and y varies from $y = -\sqrt{a^2-x^2}$ to $y = \sqrt{a^2-x^2}$ and similarly x varies from $x = -a$ to $x = a$.

$$\text{Required volume} = \int\int_R\int dzdydx = \int_{-a}^{a}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\int_{\frac{x^2+y^2}{a}}^{\sqrt{2a^2-x^2-y^2}} dz\right)dy\right\}dx$$

$$= \int_{-a}^{a}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\sqrt{2a^2-x^2-y^2} - \frac{x^2+y^2}{a}\right)dy\right\}dx$$

Put $x = r\cos\theta,\ \ y = r\sin\theta,\ \ J = r$, we get

Required volume $=\int_0^{2\pi}\left\{\int_0^a\left(\sqrt{2a^2-r^2}-\frac{r^2}{a}\right)rdr\right\}d\theta=\int_0^{2\pi}\left\{\int_0^a\left(r\sqrt{2a^2-r^2}-\frac{r^3}{a}\right)dr\right\}d\theta$

$$=\int_0^{2\pi}\left\{\int_0^a\left(-\frac{1}{2}\left(\sqrt{2a^2-r^2}\right)(-2r)-\frac{r^3}{a}\right)dr\right\}d\theta=\int_0^{2\pi}\left\{\left(-\frac{1}{2}\frac{\left(2a^2-r^2\right)^{3/2}}{3/2}-\frac{r^4}{4a}\right)_0^a\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{1}{3}\left(2a^2-a^2\right)^{3/2}-\frac{a^4}{4a}\right)-\left(-\frac{1}{3}\left(2a^2-0\right)^{3/2}-\frac{0}{4a}\right)\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{1}{3}\left(a^2\right)^{3/2}-\frac{a^3}{4}\right)-\left(-\frac{1}{3}\left(2a^2\right)^{3/2}\right)\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\left(-\frac{a^3}{3}-\frac{a^3}{4}\right)-\left(-\frac{2\sqrt{2}a^3}{3}\right)\right\}d\theta=\int_0^{2\pi}\left\{-\frac{7a^3}{12}+\frac{2\sqrt{2}a^3}{3}\right\}d\theta$$

$$=\left\{-\frac{7a^3}{12}+\frac{2\sqrt{2}a^3}{3}\right\}2\pi=\left\{-\frac{7}{12}+\frac{2\sqrt{2}}{3}\right\}2\pi a^3=\left\{\frac{4\sqrt{2}}{3}-\frac{7}{6}\right\}\pi a^3.$$ Cubic units.

Q.No.14.: Find the volume bounded by $xy=z$, $z=0$ and $(x-1)^2+(y-1)^2=1$.

Sol.: Required volume $=\int\int_R\int dzdydx=\iint_{(x-1)^2+(y-1)^2\le 1}\left(\int_0^{xy}dz\right)dydx$

$$=\iint_{(x-1)^2+(y-1)^2\le 1}xydydx$$

Let $x-1=u$ and $y-1=v \Rightarrow dx=du,\ dy=dv$.

Then the required volume $=\iint_{u^2+v^2\le 1}(u+1)(v+1)dudv$

Put $u=r\cos\theta$, $v=r\sin\theta$, $J=r$, we get

the required volume $=\int_0^{2\pi}\int_0^1(r\cos\theta+1)(r\sin\theta+1)rdrd\theta$

$$=\int_0^{2\pi}\int_0^1\left[r^3\cos\theta\sin\theta+r^2(\cos\theta+\sin\theta)+r\right]drd\theta$$

$$=\int_0^{2\pi}\left[\frac{r^4}{4}\cos\theta\sin\theta+\frac{r^3}{3}(\cos\theta+\sin\theta)+\frac{r^2}{2}\right]_0^1 d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{4}\cos\theta\sin\theta+\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]d\theta=\int_0^{2\pi}\left[\frac{2\cos\theta\sin\theta}{8}+\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]d\theta$$

$$=\int_0^{2\pi}\left[\frac{\sin 2\theta}{8}+\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]d\theta=\left[\frac{\cos 2\theta}{16}+\frac{1}{3}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right]_0^{2\pi}$$

$$=\left[\frac{(0-0)}{16}+\frac{1}{3}\{(0-0)-(1-1)\}+\frac{1}{2}(2\pi-0)\right]=\pi \text{ Cubic units. Ans.}$$

Q.No.15.: Compute the volume of solid bounded by planes, $2x+3y+4z=12$, xy-plane and the cylinder $x^2+y^2=1$.

Sol.: Required volume $=\int\int_R\int dzdydx=\iint_{x^2+y^2\le 1}\left(\int_0^{\frac{1}{4}(12-2x-3y)}dz\right)dydx$

$$=\iint_{x^2+y^2\le 1}\frac{1}{4}(12-2x-3y)dydx=\int_{-1}^{+1}\left[\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\frac{1}{4}(12-2x-3y)dy\right]dx$$

$$=\frac{1}{4}\int_{-1}^{+1}\left[12y-2xy-3\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}dx$$

$$=\frac{1}{4}\int_{-1}^{+1}\left[\left(12\sqrt{1-x^2}-2x\sqrt{1-x^2}-3\frac{\left(\sqrt{1-x^2}\right)^2}{2}\right)-\left(-12\sqrt{1-x^2}+2x\sqrt{1-x^2}-3\frac{\left(-\sqrt{1-x^2}\right)^2}{2}\right)\right]dx$$

$$=\frac{1}{4}\int_{-1}^{+1}\left[24\sqrt{1-x^2}-4x\sqrt{1-x^2}\right]dx=\int_{-1}^{+1}\left[6\sqrt{1-x^2}-x\sqrt{1-x^2}\right]dx$$

$$=\int_{-1}^{+1}6\sqrt{1-x^2}dx-\int_{-1}^{+1}x\sqrt{1-x^2}dx=6\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{-1}^{+1}+\left[\frac{1}{2}\frac{\left(1-x^2\right)^{3/2}}{3/2}\right]_{-1}^{+1}$$

$$=6\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]+[0-0]=6.\frac{2\pi}{4}=3\pi \text{ Cubic units. Ans.}$$

Q.No.15.: Compute the volume in the first octant bounded by the cylinder $x=4-y^2$ and the planes $z = y, x = 0, z = 0$.

Sol.: Required volume $= \int \int_R \int dzdydx = \int_0^4 \int_0^{\sqrt{4-x}} \left(\int_0^y dz \right) dydx$

$= \int_0^4 \int_0^{\sqrt{4-x}} [z]_0^y \, dydx = \int_0^4 \left(\int_0^{\sqrt{4-x}} ydy \right) dx$

$= \int_0^4 \left[\frac{y^2}{2} \right]_0^{\sqrt{4-x}} dx = \int_0^4 \frac{\left(\sqrt{4-x}\right)^2}{2} dx = \int_0^4 \frac{4-x}{2} dx$

$= \frac{1}{2}\left(4x - \frac{x^2}{2} \right)_0^4 = \frac{1}{2}\left(4.4 - \frac{4^2}{2} \right) = \frac{1}{2}(16-8)$

$= 4$ Cubic units. Ans.

Q.No.16.: Find the volume cut from the sphere of radius b and the cone $\varphi = \alpha$. Hence deduce the volumes of the hemisphere and sphere (by triple integrals).

Sol.: Volume $= \iiint \delta x \delta y \delta z$

We can solve this problem by changing rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

As we know, when we change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), we have put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi,)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta .$$

Now $V = 2 \times \int_0^\alpha \int_0^\pi \left(\int_0^b r^2 \right) \sin\phi \delta x \delta\theta \delta\phi = 2\left(\frac{b^3}{3}\right) \int_0^\alpha \int_0^\pi \sin\phi (\delta\theta)(\delta\phi)$

$$= \frac{2b^3}{3} \int_0^\alpha [\theta]_0^\pi \sin\phi \delta\phi = \frac{2\pi b^3}{3}\left[-(\cos\phi)_0^\alpha \right] = \frac{2b^3\pi}{3}(1-\cos\alpha) = \frac{2b^3}{3}(1-\cos\alpha)\pi$$

For volume of the hemisphere, put $\alpha = \frac{\pi}{2}$, we get $V = \frac{2b^3}{3}\pi$. Ans.

For volume of the sphere, put $\alpha = \frac{\pi}{2}$, we get $V = \frac{2b^3}{3}\pi(1-\cos\pi) = \frac{4\pi b^3}{3}$. Ans.

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Home Assignments