Differential Calculus

Maxima and Minima

(Maxima and minima in case of two or more variables)

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Maxima and Minima of functions of two variables:

Let $z = f(x, y)$ be a function of two independent variables x and y.

Relative maximum: $f(x, y)$ is said to have a relative maximum at a point (a, b) if

$$f(a,b) > f(a+h, b+k) \Rightarrow f(a+h, b+k) - f(a,b) < 0$$

for small positive or negative values of 'h' and 'k'

or

If the value of the function f (x, y) at (a, b), i.e., $f(a, b)$ is **greater** than the value of the function f at all points in some small neighbourhood of (a, b), then $f(x, y)$ is said to have a relative maximum at a point (a, b).

Relative minimum: $f(x, y)$ is said to have a relative minimum at a point (a, b) if

$$f(a,b) < f(a+h, b+k) \Rightarrow f(a+h, b+k) - f(a,b) > 0$$

for small positive or negative values of 'h' and 'k'.

or

If the value of the function f at (a, b), i.e., $f(a, b)$ is **smaller** than the value of the function f at all points in some small neighbourhood of (a, b), then $f(x, y)$ is said to have a relative minimum at a point (a, b).

Notation: Denoting $\left[f(a+h,b+k)-f(a,b)\right]$ by $\Delta f(a,b)$ or simply by Δ

i.e. $\Delta=\left[f(a+h,b+k)-f(a,b)\right]$.

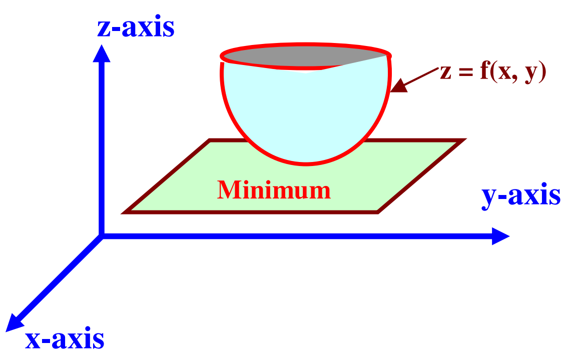
Then, $f(x,y)$ has the **maximum** at (a,b) if Δ has the **negative sign** for all small values of h, k; i.e. $\Delta<0$.

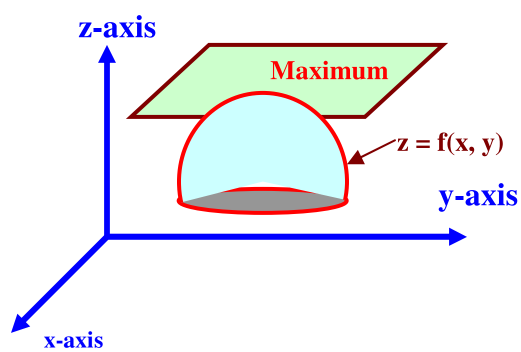
Similarly, $f(x,y)$ has the **minimum** at (a,b) if Δ has the **positive sign** for all small values of h, k; i.e. $\Delta>0$.

Extremum:

Extremum is a point, which is either a maximum or a minimum. The value of the function f at an extremum (maximum or minimum) point is known as **extremum (maximum or minimum) value** of the function f.

Geometrically, $z=f(x,y)$ represents a surface. The maximum is a point on the surface (hill top) from which the surface descends (comes down) in every direction towards the xy-plane. The minimum is the bottom of depression from which the surface ascends (climbs up) in every direction towards the xy-plane. In either case, tangent plane to the surface at a maximum or minimum point is horizontal (parallel to xy-plane) and perpendicular to z-axis.

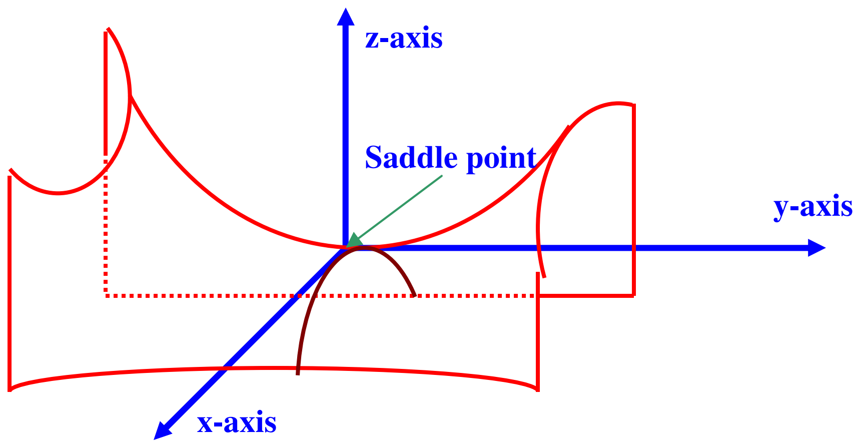




Saddle point:

Saddle point or minimax is a point where function is **neither maximum nor minimum**. At such point, f is maximum in one direction while minimum in other direction.

Geometrically, such a surface (looks like the leather seat on back of a horse) forms a ridge rising in one direction and falling in another direction.



Example: $z = xy$, hyperbolic paraboloid has a saddle point at the origin.

Necessary and sufficient conditions for extrema of a function f of two variables:

Since $\Delta = [f(a+h, b+k) - f(a,b)]$.

Expanding $f(a+h, b+k)$ by Taylor's theorem, we obtain

$$f(a+h,b+k) = f(a,b) + [hf_x(a,b) + kf_y(a,b)] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \ldots\ldots \qquad \text{(i)}$$

Since h and k are small, so neglecting higher order terms of h^2, hk, k^2, etc. Then the above expression reduces to

$$f(a+h,b+k) = f(a,b) + [hf_x(a,b) + kf_y(a,b)]$$

Thus $\Delta = [f(a+h,b+k) - f(a,b)] = hf_x(a,b) + kf_y(a,b)$. (ii)

Necessary conditions:

The necessary condition for $f(x,y)$ has the **maximum or minimum** at (a,b)

i.e. Δ has the negative or positive sign for all small values of h, k is

$f_x(a,b)=0$ and $f_y(a,b)=0$,

even though 'h' and 'k' can take both positive and negative values.

Method for developing the sufficient conditions:

With $f_x(a,b)=0$ and $f_y(a,b)=0$,

$$\Delta = f(a+h, b+k) - f(a,b) = \left[hf_x(a,b) + kf_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

reduces to

$$\Delta = f(a+h, b+k) - f(a,b) = \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

Denote $f_{xx}(a,b) = r,\ \ f_{xy}(a,b) = s,\ \ f_{yy}(a,b) = t$, we get

$$\Delta = f(a+h, b+k) - f(a,b) = \frac{1}{2!}\left[h^2 r + 2hks + k^2 t\right]. \qquad \text{(iii)}$$

From this expression, we observe that the nature of the sign of Δ depends on the nature of sign of $h^2 r + 2hks + k^2 t$.

Thus sign of Δ = sign of $\left[h^2 r + 2hks + k^2 t\right]$ = sign of $\left[\dfrac{h^2 r^2 + 2hkrs + k^2 rt}{r}\right]$

$$= \text{sign of} \left[\frac{(hr+ks)^2 + k^2\left(rt - s^2\right)}{r}\right]. \qquad \text{(iv)}$$

If $rt - s^2 > 0$, then the numerator of RHS of (iv) is positive.

In this case, sign of Δ = sign of r.

Thus $\Delta < 0$ if $rt - s^2 > 0$ and $r < 0$

and $\Delta > 0$ if $rt - s^2 > 0$ and $r > 0$.

Therefore, the **sufficient (Lagrange's) conditions** for extrema are:

I. f attains (has) a **maximum** at (a, b) if $rt - s^2 > 0$ and $r < 0$

II. f attains (has) a **minimum** at (a, b) if $rt - s^2 > 0$ and $r > 0$

III. **Saddle point:** If $rt - s^2 < 0$, then $\Delta < 0$ or $\Delta > 0$ depending on 'h' and 'k'. Therefore, f has a saddle point (minimax) at (a, b) if $rt - s^2 < 0$.

IV. **Failure case:** If $rt - s^2 = 0$, then further investigation is needed to determine the nature of the function f.

FINAL CONCLUSIONS:

A function of two variables $u = f(x, y)$ is said to have a maximum or minimum if

(1) $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$,

which gives the values of x say x = a and y say y = b, for which the function is maximum or minimum.

(2) $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} > \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$ at these values i.e. x = a and y = b.

i.e. $\left[f_{xx} . f_{yy} - (f_{xy})^2\right]_{\substack{x=a \\ y=b}} > 0$ i.e. $rt - s^2 > 0$.

(3a) Then the function $f(x, y)$ will have a maximum value if apart from the above conditions (1) and (2), also $\dfrac{\partial^2 f}{\partial x^2} < 0$ i.e. $r < 0$ at x = a and y = b.

(3b) Similarly, the function $f(x, y)$ will have a minimum value if apart from the above conditions (1) and (2), also $\dfrac{\partial^2 f}{\partial x^2} > 0$ i.e. $r > 0$ at x = a and y = b.

Remarks:

(a) If $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$ i.e. $rt - s^2 = 0$ at x = a and y = b,

Then the function $f(x, y)$ is neither a maximum nor a minimum.

(b) Again if $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} < \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$ i.e. $rt - s^2 < 0$ at x = a and y = b,

Then such a point where $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} < \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$ holds, is called a "Saddle point".

Method of finding Extrema of f(x, y):

1. Solving $f_x = 0$ and $f_y = 0$ yields **critical** or **stationary point** P of f.

2. Calculate $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$ at the critical point.

3. **a.** f attains (has) a **maximum** at (a, b) if $rt - s^2 > 0$ and $r < 0$

b. f attains (has) a **minimum** at (a, b) if $rt - s^2 > 0$ and $r > 0$

c. Saddle point: If $rt - s^2 < 0$, then $\Delta < 0$ or $\Delta > 0$ depending on 'h' and 'k'.

Therefore, f has a **saddle point** (minimax) at (a, b) if $rt - s^2 < 0$.

d. Failure case: If $rt - s^2 = 0$, then further investigation is needed to determine the nature of the function f.

Remarks: Extrema occur only at stationary points. However, stationary points need not be extrema.

Now let us solve some problems, where we have to evaluate the maxima or minima of the given function:

Q.No.1.: Find the values of x and y for which $u = x^2 + y^2 + 6x + 12$ has a minimum value and find this minimum value.

Sol.: Since $u(x, y) = x^2 + y^2 + 6x + 12$.

Then $\frac{\partial u}{\partial x} = 2x + 6$ and $\frac{\partial u}{\partial y} = 2y$.

For u (x, y) to be minimum or maximum, we have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

$\Rightarrow 2x + 6 = 0$ and $2y = 0$.

$\Rightarrow x = -3$ and $y = 0$.

Also $\left[\frac{\partial^2 u}{\partial x^2}\right]_{[-3,0]} = 2$, $\left[\frac{\partial^2 u}{\partial y^2}\right]_{[-3,0]} = 2$ and $\left[\frac{\partial^2 u}{\partial x \partial y}\right]_{[-3,0]} = 0$.

$$\therefore \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 2 \times 2 - 0 = 4 > 0 \text{ at } x = -3 \text{ and } y = 0.$$

And also $\left[\dfrac{\partial^2 u}{\partial x^2}\right]_{[-3,0]} = 2 > 0$.

Thus the function is minimum at $x = -3$ and $y = 0$.

And this minimum value $[u(x,y)]_{min} = \left[x^2 + y^2 + 6x + 12\right]_{[-3,0]} = 9 - 18 + 12 = 3$. Ans.

Q.No.2.: If $f(x,y) = 6xy + 9$, find the values of x and y for which $f(x,y)$ has a stationary value.

Sol.: Since $f(x,y) = 6xy + 9$.

Then $\dfrac{\partial f}{\partial x} = 6y$ and $\dfrac{\partial f}{\partial y} = 6x$.

For f (x, y) to be minimum or maximum, we have $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$.

$\Rightarrow 6y = 0$ and $6x = 0$.

$\Rightarrow y = 0$ and $x = 0$.

Also $\left[\dfrac{\partial^2 f}{\partial x^2}\right]_{[0,0]} = 0$, $\left[\dfrac{\partial^2 f}{\partial y^2}\right]_{[0,0]} = 0$ and $\left[\dfrac{\partial^2 f}{\partial x \partial y}\right]_{[0,0]} = 6$.

$$\therefore \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0 - 6^2 = -36 < 0 \text{ at } x = 0 \text{ and } y = 0.$$

Thus $x = 0$ and $y = 0$ does not stand for maxima and minima.

Thus the given function has a stationary value at $x = 0$ and $y = 0$.

Q.No.3.: Find the maximum and minimum values of

(a) $x^3 + y^3 - 3axy$.

(b) $\sin x . \sin y . \sin(x + y)$.

Sol.: (a) Let $u = x^3 + y^3 - 3axy$.

Partially differentiating u w. r. t. x and y, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay \quad \text{and} \quad \frac{\partial u}{\partial y} = 3y^2 - 3ax$$

For u (x, y) to be minimum or maximum, we have $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0$.

$\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow 3x^2 = 3ay \Rightarrow x^2 = ay$. (i)

$\frac{\partial u}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow 3y^2 = 3ax \Rightarrow y^2 = ax$. (ii)

Squaring (i), we get $x^4 = a^2y^2$ (iii)

Substituting the value of y^2 from (ii) in (iii), we get $x^4 = a^2ax \Rightarrow x^3 = a^3 \Rightarrow x = a$

$\therefore x^2 = ay \Rightarrow a^2 = ay \Rightarrow y = a$.

$\frac{\partial^2 u}{\partial x^2} = 6x$, $\frac{\partial^2 u}{\partial y^2} = 6y$, $\frac{\partial^2 u}{\partial x \partial y} = -3a$

$$\therefore \left(\frac{\partial^2 u}{\partial x^2}\right).\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 6x\,.\,6y - 9a^2 = 36xy - 9a^2$$

Substituting xy as a^2, we get

$$\left(\frac{\partial^2 u}{\partial x^2}\right).\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 36a^2 - 9a^2 = 27a^2 > 0.$$

To find max. and min. value which u will attain.

Put $\frac{\partial^2 u}{\partial x^2} = 0$, we get

$\frac{\partial^2 u}{\partial x^2} = 6x = 6a$

$6a < 0$ if $a < 0$ and $6a > 0$ if $a > 0$. Ans.

(b) Let $P(x, y) = \sin x . \sin y . \sin(x + y)$

$F_x = \frac{\partial F}{\partial x} = \sin y\{\cos x . \sin(x + y) + \sin x . \cos(x + y)\} = \sin y . \sin(2x + y)$ (i)

$F_y = \frac{\partial F}{\partial x} = \sin x\{\cos y . \sin(x + y) + \sin y . \cos(x + y)\} = \sin x . \sin(x + 2y)$ (ii)

For max. or min. values of F(x, y), we have (i) and (ii) equal to zero.

$F_x = 0 \Rightarrow \sin y . \sin(2x + y) = 0$ (iii)

$F_y = 0 \Rightarrow \sin x . \sin(x + 2y) = 0$ (iv)

From (iii) and (iv), we have

$\sin y = 0$ and $\sin x = 0$

$\Rightarrow x = 0$ and $y = 0$, which is not possible.

We have also

$x + 2y = \pi$ or 2π (v)

$2x + y = \pi$ or 2π (vi)

Taking $x + 2y = 2\pi$

$2x + y = 2\pi$

Solving the above equation we have $x = \dfrac{2\pi}{3}$, $y = \dfrac{2\pi}{3}$

Now taking $x + 2y = \pi$

$2x + y = \pi$

Solving the above equations, we get $x = \dfrac{\pi}{3}$, $y = \dfrac{\pi}{3}$.

So the points of extremum are $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ and $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$.

Let $r = F_{xx}$, $t = F_{yy}$ and $s = F_{xy}$.

So $F_{xx} = \sin y.\sin(2x + y).2 = 2\sin y.\cos(2x + y)$

$F_{yy} = 2\sin y.\cos(x + 2y)$

$F_{xy} = \cos y.\sin(2x + y) + \cos(2x + y).\sin y = \sin(2x + 2y)$.

Now at points $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, we have to calculate the

$F_{xx}.F_{yy} - F_{xy}^2$

So $F_{xx} = 2 \times \dfrac{\sqrt{3}}{2} \times -1 = -\sqrt{3}$

$F_{yy} = 2 \times \dfrac{\sqrt{3}}{2} \times -1 = -\sqrt{3}$

$F_{xy} = \dfrac{-\sqrt{3}}{2}$.

Now $\left(-\sqrt{3}\right)\left(-\sqrt{3}\right)-\left(\frac{-\sqrt{3}}{2}\right)^2 \Rightarrow 3-\frac{3}{4}=\frac{9}{4}>0$

So that $F_{xx}.F_{yy}-F_{xy}^2>0$ and $F_{xx}=-\sqrt{3}$.

Hence F(x, y) has maximum at $\left(x=\frac{\pi}{3}, y=\frac{\pi}{3}\right)$.

Maximum value of F(x, y) $\sin\frac{\pi}{3}.\sin\frac{\pi}{3}.\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{8}$.

Now taking points $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$, we have to calculate

$F_{xx}.F_{yy}-F_{xy}^2$

So $F_{xx}=2\times\frac{\sqrt{3}}{2}\times 1$

$F_{yy}=2\times\frac{\sqrt{3}}{2}\times 1$

$F_{xy}=\frac{\sqrt{3}}{2}$.

Now $\left(2\times\frac{\sqrt{3}}{2}\times 1\right)\left(2\times\frac{\sqrt{3}}{2}\times 1\right).-\left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow 3-\frac{3}{4}=\frac{9}{4}>0$

So that $F_{xx}.F_{yy}-F_{xy}^2>0$ and also $F_{xx}=\sqrt{3}>0$.

Hence the point $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is the point of minimum.

Minimum value of F(x, y) $\sin\frac{2\pi}{3}.\sin\frac{2\pi}{3}.\sin\frac{4\pi}{3}=\frac{-3\sqrt{3}}{8}$.

So the maximum and minimum values of function F(x, y) is $\frac{3\sqrt{3}}{8}$ and $\frac{-3\sqrt{3}}{8}$ respectively.

Q.No.4.: Show that the minimum value of $f(x,y)=xy+\frac{a^3}{x}+\frac{a^3}{y}$ is $3a^2$.

Sol.: Given function is $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$.

$\therefore f_x = \frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$, $f_y = \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$.

For f (x, y) to be minimum or maximum, we have $f_x = 0$ and $f_y = 0$.

If $f_x = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \Rightarrow y = \frac{a^3}{x^2}$ (i)

If $f_y = 0 \Rightarrow x - \frac{a^3}{y^2} = 0 \Rightarrow x = \frac{a^3}{y^2}$ (ii)

From (i) and (ii), we get

$$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = 0 \Rightarrow x - \frac{a^3.x^2}{a^3.a^3} = 0 \Rightarrow x\left(1 - \frac{x^3}{a^3}\right) = 0$$

As $x \neq 0 \Rightarrow 1 - \frac{x^3}{a^3} = 0$, because when x = 0, y can not be defined.

$\Rightarrow x^3 = a^3 \Rightarrow x = a$

$\therefore$ From (i), we get $y = \frac{a^3}{a^2} = a$.

Hence extreme point is (a, a)

Now, $f_{xx} = 0 - \frac{(-2).a^3}{x^3} = \frac{2a^3}{x^3}$,

$f_{yy} = \frac{\partial^2 F}{\partial y^2} = 0 - \frac{(-2).a^3}{y^3} = \frac{2a^3}{y^3}$,

$f_{xy} = \frac{\partial^2 F}{\partial x \partial y} = 1 - 0 = 1$.

At point (a, a) $f_{xx} = \frac{2a^3}{a^3} = 2$, $f_{yy} = \frac{2a^3}{a^3} = 2$.

Now, $f_{xx} \times f_{yy} - (f_{xy})^2 = 2 \times 2 - (1)^2 = 4 - 1 = 3 > 0$.

Also $f_{xx} = 2 > 0$.

$\therefore$ Minimum value of $f(x, y)$ is $a.a+\frac{a^3}{a}+\frac{a^3}{a}=a^2+a^2+a^2=3a^2$. Ans.

Q.No.5.: Find co-ordinates of a point P(x, y) such that sum of squares of its distances from rectangular axis of reference and line $x+y=8$ is minimum.

Sol.: We know that distance of a point P(x, y) from any line $ax+by+c=0$ is given by

$$\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

$$\therefore u=x^2y^2+\frac{(x+y-8)^2}{2}$$

For maxima or minima, we have $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0$.

$$\therefore 2x+\frac{2(x+y-8)}{2}=0 \qquad \text{and} \qquad 2y+\frac{2(x+y-8)}{2}=0$$

$$3x+y=8 \qquad \text{(i)} \qquad x+3y=8 \qquad \text{(ii)}$$

Solving (i) and (ii), we get $x=2$ and $y=2$.

$\therefore$ At x = 2, y = 2, the given function have max. or min. value.

Also we know that for minima, we have

$$\frac{\partial^2 u}{\partial x^2}>0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2}>0.$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right]_{\substack{x=2\\y=2}}=3>0 \quad \text{and} \quad \left.\frac{\partial^2 u}{\partial y^2}\right]_{\substack{x=2\\y=2}}=3>0.$$

$\therefore$ Thus the function have minimum value at x = 2, y = 2.

Hence (2, 2) be the required coordinates of the point such that sum of squares of its distances from rectangular axis of reference and line $x+y=8$ is minimum. Ans.

Q.No.6.: Divide ‘a’ into three parts such that their products be a maximum.

Sol.: Let ‘a’ be the sum of three no. x, y, z.

Therefore $a=x+y+z \Rightarrow z=a-(x+y)$

Their product can be written as

$$P=xyz \Rightarrow P=xy[a-(x+y)] \Rightarrow P=xay-x^2y-y^2x.$$

For maximum or minimum values F_x, F_y are zero.

$$\therefore F_x = \frac{\partial P}{\partial x} = ay - 2xy - y^2 = 0 \Rightarrow a - 2x - y = 0 \Rightarrow a = 2x + y \Rightarrow y = a - 2x \qquad \text{(i)}$$

$$F_y = \frac{\partial P}{\partial y} = xa - x^2 - 2yx = 0 \Rightarrow a - x - 2y = 0 \Rightarrow a = 2y + x \Rightarrow x = a - 2y \qquad \text{(ii)}$$

From (i) and (ii), we get

$$y = a - 2(a - 2y) \Rightarrow y = a - 2a + 4y \Rightarrow 3y = a \Rightarrow y = \frac{a}{3}$$

Putting this value in (ii), we get

$$x = a - \frac{2a}{3} \Rightarrow x = \frac{a}{3}.$$

$$\therefore z = a - \left(\frac{a}{3} + \frac{a}{3}\right) = a - \frac{2a}{3} \Rightarrow z = \frac{a}{3}.$$

Differentiating F_x, F_y again

$$\therefore F_{xx} = \frac{\partial^2 P}{\partial x^2} = -2y = -\frac{2}{3}a$$

$$F_{xx} = \frac{\partial^2 P}{\partial y^2} = -2x = -\frac{2}{3}a$$

also $F_{xy} = \dfrac{\partial^2 P}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial P}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(ya - 2xy - y^2\right) = a - 2x - 2y$.

Putting the value of x and y.

$$\frac{\partial^2 P}{\partial x \partial y} = a - \frac{2a}{3} - \frac{2a}{3} = \frac{3a - 4a}{3} = \frac{-a}{3}$$

$$\therefore \frac{\partial^2 P}{\partial x^2}.\frac{\partial^2 P}{\partial y^2} - \left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 = \frac{-2a}{3}.\frac{-2a}{3} - \frac{a^2}{9} = \frac{4a^2 - a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$$

Therefore the product is maximum.

Hence ‘a’ can be divided into three parts as $\left(\dfrac{a}{3}, \dfrac{a}{3}, \dfrac{a}{3}\right)$.

Q.No.7.: Divide 120 into three parts so that the sum of the products taken two at a time shall be maximum.

Sol.: Let $120 = x + y + z$. ...(i)

and $P = xy + yz + zx$. ...(ii)

From (i), we get $z = 120 - x - y$.

$\therefore P = xy + y(120 - x - y) + x(120 - x - y)$.

Then $\dfrac{\partial P}{\partial x} = y + y(-1) + 120 - 2x - y = 120 - 2x - y$

and $\dfrac{\partial P}{\partial y} = x + 120 - x - 2y - x = 120 - x - 2y$.

For maximum or minimum vales both $\dfrac{\partial P}{\partial x} = 0$ and $\dfrac{\partial P}{\partial y} = 0$.

$\Rightarrow 120 - 2x - y = 0$ and $120 - x - 2y = 0$.

Solving these two equations, we get

$x = 40$, $y = 40$ and $z = 40$.

Also $\left[\dfrac{\partial^2 P}{\partial x^2}\right]_{[40,40,40]} = -2$, $\left[\dfrac{\partial^2 P}{\partial y^2}\right]_{[40,40,40]} = -2$ and $\left[\dfrac{\partial^2 P}{\partial x \partial y}\right]_{[40,40,40]} = -1$.

$\therefore \dfrac{\partial^2 P}{\partial x^2} \cdot \dfrac{\partial^2 P}{\partial y^2} - \left(\dfrac{\partial^2 P}{\partial x \partial y}\right)^2 = (-2) \times (-2) - (-1)^2 = 3 > 0$ at $x = 40$, $y = 40$ and $z = 40$.

And also $\left[\dfrac{\partial^2 P}{\partial x^2}\right]_{[40,40,40]} = -2 < 0$

Thus the function is maximum at $x = 40$, $y = 40$ and $z = 40$.

Q.No.8.: Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

Sol.: Let x, y and z be the dimensions of the rectangular solid and D be the diameter of the given sphere.

The diagonal of the solid will be diameter of the sphere.

$\therefore D^2 = x^2 + y^2 + z^2 \Rightarrow D = \sqrt{x^2 + y^2 + z^2}$. ...(i)

Let V be the volume of the solid

$\therefore V = xyz = xy\sqrt{D^2 - x^2 - y^2} \Rightarrow V^2 = x^2y^2\left(D^2 - x^2 - y^2\right) = D^2x^2y^2 - x^4y^2 - x^2y^4$.

Let $V^2 = u$. $\therefore u = D^2x^2y^2 - x^4y^2 - x^2y^4$. ...(ii)

Differentiate (ii) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x} = 2D^2xy^2 - 4x^3y^2 - 2xy^4 = 2xy^2\left(D^2 - 2x^2 - y^2\right) \quad \text{...(iii)}$$

$$\frac{\partial u}{\partial y} = 2D^2x^2y - 2x^4y - 4x^2y^3 = 2x^2y\left(D^2 - x^2 - 2y^2\right) \quad \text{..(iv)}$$

For max. and min. of u, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$$\therefore \frac{\partial u}{\partial x} = 2xy^2\left(D^2 - 2x^2 - y^2\right) = 0$$

and $\frac{\partial u}{\partial y} = 2x^2y\left(D^2 - x^2 - 2y^2\right) = 0$

But $x \neq 0,\ y \neq 0$ being lengths

$$\left(D^2 - 2x^2 - y^2\right) = 0 \quad \text{...(v)}$$

$$\left(D^2 - x^2 - 2y^2\right) = 0 \quad \text{...(vi)}$$

From (v) $y^2 = D^2 - 2x^2$ and put in (vi), we get

$$D^2 - x^2 + 4x^3 - 2D^2 = 0 \Rightarrow 3x^2 = D^2 \Rightarrow x^2 = \frac{D^2}{3} \Rightarrow x = \frac{D}{\sqrt{3}}.$$

Now $y^2 = D^2 - 2x^2 = D^2 - \frac{2D^2}{3} = \frac{D^2}{3} \Rightarrow y = \frac{D}{\sqrt{3}}$; (y cannot be < 0)

Again differentiate (iii) partially w. r. t. x and y separately and (iv) w. r. t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = 2D^2y^2 - 12x^2y^2 - 2y^4 \ ,$$

$$\frac{\partial^2 u}{\partial x \partial y} = 4D^2xy - 8x^3y - 8xy^3,$$

$$\frac{\partial^2 u}{\partial y^2} = 2D^2x^2 - 2x^4 - 12x^2y^2.$$

$$A = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\left(\frac{D}{\sqrt{3}}, \frac{D}{\sqrt{3}}\right)} = 2D^2.\frac{D^2}{3} - \frac{12D^2}{3}.\frac{D^2}{3} - \frac{2D^4}{9} = -\frac{8D^4}{9}$$

$$B=\left(\frac{\partial^2 u}{\partial x\partial y}\right)_{\left(\frac{D}{\sqrt{3}},\frac{D}{\sqrt{3}}\right)}=4D^2.\frac{D}{\sqrt{3}}.\frac{D}{\sqrt{3}}-8\frac{D^3}{3\sqrt{3}}.\frac{D}{\sqrt{3}}-8\frac{D}{\sqrt{3}}.\frac{D^3}{3\sqrt{3}}=-4\frac{D^4}{9}$$

$$C=\left(\frac{\partial^2 u}{\partial y^2}\right)_{\left(\frac{D}{\sqrt{3}},\frac{D}{\sqrt{3}}\right)}=2D^2.\frac{D^2}{3}-\frac{2D^4}{9}-\frac{12D^2}{3}.\frac{D^2}{3}=-\frac{8D^4}{9}$$

Now $AC-B^2=\left(-\frac{8D^4}{9}\right)\left(-\frac{8D^4}{9}\right)-\left(\frac{4D^4}{9}\right)=\frac{64D^8}{81}-\frac{16D^8}{81}>0$ (+ ve)

Also A is negative for $x=y=\frac{D}{\sqrt{3}}$.

$\therefore$ u is maximum when $x=y=\frac{D}{\sqrt{3}}$.

$\therefore$ V^2 or V is maximum value for $x=y=\frac{D}{\sqrt{3}}$.

From (i), we get $z=\sqrt{D^2-x^2-y^2}=\sqrt{D^2-\frac{D^2}{3}-\frac{D^2}{3}}=\frac{D}{\sqrt{3}}$

$\therefore x=y=z=\frac{D}{\sqrt{3}}$ i.e. length = breadth = height.

Hence for maximum volume solid is a cube.

Q.No.9: The temperature T at any point (x, y, z) in space is defined by $T=400xyz^2$.

Find the maximum temperature on the surface of a unit sphere $x^2+y^2+z^2=1$.

Sol.: Given equation of the sphere is $x^2+y^2+z^2=1\Rightarrow z^2=1-x^2-y^2$ (i)

Given $T=400xyz^2\Rightarrow T=400xy\left(1-x^2-y^2\right)\Rightarrow T=400xy-400x^3y-400xy^3$ (ii)

Now, $\frac{\partial T}{\partial x}=0\Rightarrow 400y-1200x^2y-400y^3=0\Rightarrow y^2+3x^2-1=0\Rightarrow y^2=1-3x^2$ (iii)

$\frac{\partial T}{\partial y}=0\Rightarrow 400y-400x^3-1200xy^2=0\Rightarrow x^2+3y^2-1=0$

$\Rightarrow x^2+3\left(1-3x^2\right)-1=0$ [using (ii)]

$\Rightarrow x^2+3-9x^2-1=0\Rightarrow x^2=\frac{1}{4}\Rightarrow x=\frac{1}{2}$

Putting in (iii), we get $y = \frac{1}{2}$.

Putting in (i), we get $z^2 = \frac{1}{2}$.

Now to check Maxima and Minima

$$\frac{\partial^2 T}{\partial x^2} = -2400x \quad \text{and} \quad \frac{\partial^2 T}{\partial y^2} = -2400y$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=\frac{1}{2}} = -1200 \quad \text{and} \quad \left(\frac{\partial^2 T}{\partial y^2}\right)_{y=\frac{1}{2}} = -1200$$

Also $\frac{\partial^2 T}{\partial x \partial y} = 400 - 2400x \Rightarrow \left.\frac{\partial^2 T}{\partial x \partial y}\right|_{x=\frac{1}{2}} = -800$

Clearly $\frac{\partial^2 T}{\partial x^2} \frac{\partial^2 T}{\partial y^2} > \left(\frac{\partial^2 T}{\partial x \partial y}\right)^2$,

Now $\left(\frac{\partial^2 T}{\partial x^2}\right)_{x=\frac{1}{2},\ y=\frac{1}{2}} = -2400 \times \frac{1}{2} = 1200 < 0$.

$\therefore$ The function T has maxima at the calculated values of x, y and z^2.

Put the values of x, y and z^2 in $T = 400xyz^2$, we get

$T = 400 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \Rightarrow T = 50$ units. Ans.

which is the maximum temperature on the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Q.No.10.: A rectangular tank open at the top and is to hold a given volume . Find the dimensions of the box requiring least material, for its construction.

Sol.: Let x , y and z ft. be the dimensions of the rectangular tank open at the top so that material for construction will be least if surface area is least.

Let surface area, $S = F(x, y, z) = xy + 2yz + 2zx$. ....(i)

Also given volume $= xyz = V$. ...(ii)

Eliminating z from (i) with the help of (ii), we get

$S = xy + 2(y+x)\dfrac{V}{xy} = xy + 2V\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$.

$\therefore \dfrac{\partial S}{\partial x} = y - \dfrac{2V}{x^2} = 0$ and $\dfrac{\partial S}{\partial y} = x - \dfrac{2V}{y^2} = 0$. ...(iii)

Solving these, we get $x = y = 2z$. ...(iv)

Now $r = \dfrac{\partial^2 S}{\partial x^2} = \dfrac{4V}{x^3}$, $s = \dfrac{\partial^2 S}{\partial x \partial y} = 1$, $t = \dfrac{\partial^2 S}{\partial y^2} = \dfrac{4V}{y^3}$.

At $x = y$, $rt - s^2 = \dfrac{4V}{x^3}.\dfrac{4V}{x^3} - 1 = \dfrac{16V^2 - x^6}{x^6} = \dfrac{(64-1)x^6}{x^6}$ = always + ve and r is also + ve.

Henve S is minimum for $x = y = 2z$.

Also from (ii) , $x = y = (2V)^{1/3}$. ...(v)

Thus from (iv) and (v), we get

length = breadth = twice height = (2 volume)$^{1/3}$.

Q.No.11.: Find the point on the surface $z^2 = xy + 1$, nearest to the origin.

Sol.: Let (x, y, z) be the point lies on the surface $z^2 = xy + 1$,which is nearest to the origin.

$\therefore$ Distance between origin and this point $= \sqrt{x^2 + y^2 + z^2}$. Squaring both sides, we get

$(\text{Distance})^2 = u(\text{say}) = x^2 + y^2 + z^2$ subject to the condition $z^2 = xy + 1$. ...(i)

$\therefore u = x^2 + y^2 + xy + 1$.

Then $\dfrac{\partial u}{\partial x} = 2x + y$ and $\dfrac{\partial u}{\partial y} = 2y + x$.

For maximum or minimum vales both $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0$.

$\Rightarrow 2x + y = 0$ and $2y + x = 0$.

$\Rightarrow x = -\dfrac{y}{2}$ and $y = -\dfrac{x}{2}$. $\Rightarrow x = 0$ and $y = 0$.

Also $\left[\dfrac{\partial^2 u}{\partial x^2}\right]_{[0,0]} = 2$, $\left[\dfrac{\partial^2 u}{\partial y^2}\right]_{[0,0]} = 2$ and $\left[\dfrac{\partial^2 u}{\partial x \partial y}\right]_{[0,0]} = 1$.

$$\therefore \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 2\times 2-1=3>0 \text{ at } x=0 \text{ and } y=0.$$

And also $\left[\frac{\partial^2 u}{\partial x^2}\right]_{[0,0]} = 2>0$. Also from (i), $z=\pm 1$.

Thus the function is minimum at $x=0$, $y=0$ and $z=\pm 1$.

Thus $(0,0,1)$ and $(0,0,-1)$ are the points on the surface $z^2 = xy+1$, which are nearest to the origin.

Q.No.12.: A tent having the form of a cylinder, surmounted by a cone is to contain a given volume. Prove that for the canvas to be minimum, height of the cone is twice that of the cylinder.

Sol.: Let the height of the cone is = y

Let the height of the cylinder is = x

Then, we have to prove, $y = 2x$

Let v be the total volume of the tent.

Then, $v = \pi r^2 x + \frac{1}{3}\pi r^2 y$ (i)

If S is the total surface area of the tent, then

$S = 2\pi r x + \pi r \ell \Rightarrow S = 2\pi r x + \pi r\sqrt{r^2+y^2}$ (ii)

Putting the value of x from (i) in (ii), we get

$$S = 2\pi r\left(\frac{v}{\pi r^2}-\frac{y}{3}\right)+\pi r\sqrt{r^2+y^2}$$

$$\therefore \frac{\partial S}{\partial y} = \frac{-2\pi r}{3}+\frac{\pi r}{2\sqrt{r^2+y^2}}\times 2y.$$

For maxima and minima, $\frac{\partial S}{\partial y}=0$.

$$\therefore \frac{-2\pi r}{3}+\frac{\pi r y}{\sqrt{r^2+y^2}}=0 \Rightarrow \left(\frac{2}{3}\right)^2=\left(\frac{y}{\sqrt{r^2+y^2}}\right)^2 \Rightarrow 4r^2+4y^2=9y^2$$

$\Rightarrow 5y^2 = 4r^2 \Rightarrow y = \frac{2}{\sqrt{5}}r$ (iii)

Differentiating (ii) partially w. r. t. r, we get

$$\frac{\partial S}{\partial r}=\frac{\partial}{\partial r}\left[2\pi r\times\left(\frac{v}{\pi r^2}-\frac{y}{3}\right)\right]+\frac{\partial}{\partial r}\left(\pi r\sqrt{r^2+y^2}\right)$$

$$=\frac{-2v}{r^2}-\frac{2\pi y}{3}+\pi\sqrt{r^2+y^2}+\frac{\pi r^2}{\sqrt{r^2+y^2}}.$$

For max. or min., $\frac{\partial S}{\partial r}=0$

$$\Rightarrow\frac{-2v}{r^2}-\frac{2\pi y}{3}+\pi\sqrt{r^2+y^2}+\frac{\pi^2}{\sqrt{r^2+y^2}}=0$$

$$\Rightarrow\frac{-2\pi r}{\pi r^2}\left(x+\frac{y}{3}\right)-\frac{2y}{3}+\sqrt{r^2+y^2}+\frac{r^2}{\sqrt{r^2+y^2}}=0$$

Putting $y=\frac{2}{\sqrt{5}}r$ in above equation, we get

$$\Rightarrow -2x-\frac{4}{3\sqrt{5}}r-\frac{4}{3\sqrt{5}}r+\frac{9}{3\sqrt{5}}r+\frac{\sqrt{5}}{3}r=0\Rightarrow 2x=\left(\frac{1}{3\sqrt{5}}+\frac{5}{3\sqrt{5}}\right)r=\frac{6}{3\sqrt{5}}r$$

$\Rightarrow x=\frac{1}{\sqrt{5}}r=\frac{y}{2}\Rightarrow y=2x$, which is the required proof.

Now, $$\frac{\partial^2 S}{\partial y^2}=\frac{\sqrt{r^2+y^2}\,\pi\ r}{\left(r^2+y^2\right)}-\pi r\,y\times\frac{y}{\sqrt{r^2+y^2}}=\frac{\pi r\left(r^2+y^2-y^2\right)}{\left(r^2+y^2\right)^{3/2}}$$

$$=\frac{\pi r^3}{\left(r^2+y^2\right)^{3/2}}>0 \qquad \text{(iv)}$$

$$\frac{\partial^2 S}{\partial r^2}=\frac{2v}{r^4}\times 2r+\frac{2\pi r}{\sqrt{r^2+y^2}}+\frac{\dfrac{2\pi r\sqrt{r^2+y^2}-\pi r^2}{r^2+y^2}}{\sqrt{r^2+y^2}}$$

$$=\frac{4v}{r^3}+\frac{2\pi r}{\sqrt{r^2+y^2}}+\frac{\pi r^3+2\pi r\,y^2}{\left(r^2+y^2\right)^{3/2}}>0 \qquad \text{(v)}$$

$$\frac{\partial^2 S}{\partial y\partial r}=\frac{\partial S}{\partial r}\left(\frac{\partial S}{\partial y}\right)=\frac{\partial S}{\partial r}\left[\frac{-2\pi r}{3}+\frac{\pi r\,y}{\sqrt{r^2+y^2}}\right]$$

$$= \frac{-2r}{3} + \frac{\sqrt{r^2+y^2} \times \pi y - \pi r y \times \frac{r}{\sqrt{r^2+y^2}}}{r^2+y^2} = \frac{-2\pi}{3} + \frac{\pi y\left(r^2+y^2\right) - \pi r^2 y}{\left(r^2+y^2\right)^{3/2}}$$

$$= \frac{-2\pi}{3} + \frac{\pi y^3}{\left(r^2+y^2\right)^{3/2}}.$$

Let $r = \frac{\partial^2 S}{\partial y^2}$, $t = \frac{\partial^2 S}{\partial r^2}$ and $f = \frac{\partial^2 S}{\partial y \partial r}$

$$r\,t - f^2 = \frac{\pi r^3}{\left(r^2+y^2\right)^{3/2}} \times \left[\frac{4v}{r^3} + \frac{2\pi r}{\sqrt{r^2+y^2}} + \frac{\pi r^2 + 2\pi r y^2}{\left(r^2+y^2\right)^{3/2}}\right] - \left[\frac{-2\pi}{3} + \frac{\pi y^2}{\left(r^2+y^2\right)^{3/2}}\right]^2$$

at $y = \frac{2}{\sqrt{5}} r$

$$r\,t - f^2 = \frac{\pi r^3}{\left(r^2 + \frac{4}{5} r^2\right)^{3/2}} \times \left[\frac{4v}{r^3} + \frac{2\pi r}{\sqrt{r^2 + \frac{4}{5} r^2}} + \frac{\pi r^2 + 2\pi r y^2}{\left(r^2+y^2\right)^{3/2}} + \frac{\pi r\left(r^2 + 2\pi \frac{4}{5} r^2\right)}{\left(\sqrt{r^2 + \frac{4}{5} r^2}\right)^3}\right]$$

$$- \left[\frac{2\pi}{3} + \frac{\pi \times \frac{4}{5\sqrt{5}} r^3}{\left(r^2 + \frac{4}{5} r^2\right)^{3/2}}\right]$$

v at $y = \frac{2}{\sqrt{5}} r$ and $x = \frac{r}{\sqrt{5}} = \pi r^2 \left(\frac{1}{\sqrt{5}} + \frac{2}{3\sqrt{5}}\right) r = \frac{\sqrt{5}}{3} \pi r^3$.

$$r\,t - f^2 = \frac{5\sqrt{5}}{27}\left[\frac{4\sqrt{5}}{3}\pi + \frac{2\sqrt{5}}{3}\pi + \frac{13\sqrt{5}}{3}\pi\right] - \left[\frac{-2\pi}{3} + \frac{4}{27}\pi\right]$$

$$= \frac{5\sqrt{5}}{27}\left[\frac{19\sqrt{5}}{3}\pi\right] + \frac{14\pi}{27} > 0 \qquad \text{(vi)}$$

From (iv), (v) and (vi), we find that the conditions satisfied from these equations is for minima.

$\therefore$ For minimum canvas to be used $y = 2x$. Ans.

Q.No.13.: Find the stationary value of $u = x^2 + y^2 + z^2$. If $xy + yz + zx = 3a^2$.

{Without Lagrange Theorem}

Sol.: $u = x^2 + y^2 + z^2$ (i)

$xy + yz + zx = 3a^2$ (ii)

From (ii), we get

$$z = \frac{3a^2 - xy}{x + y} \qquad \text{(iii)}$$

Putting above value of z in (i), we get

$$u = x^2 + y^2 + \left[\frac{3a^2 - xy}{x + y}\right]^2.$$

$$\frac{\partial u}{\partial x} = 2x + 2\left[\frac{3a^2 - xy}{x + y}\right]\left[\frac{(x + y).(-y) - (1)(3a^2 - xy)}{(x + y)^2}\right]$$

$$= 2x + \frac{2(3a^2 - y^2)(-xy - y^2 - 3a^2 + xy)}{(x + y)^3} = 2x - \frac{2(3a^2 - y^2)(y^2 + 3a^2)}{(x + y)^3}$$

$$= 2x - \frac{2(9a^4 - y^4)}{(x + y)^3} \qquad \text{(iv)}$$

and $$\frac{\partial u}{\partial y} = 2y + 2\left[\frac{3a^2 - y^2}{x + y}\right]\left[\frac{-x(x + y) - (3a^2 - xy)}{(x + y)^2}\right]$$

$$= 2y - \frac{2(9a^4 - x^4)}{(x + y)^3} \qquad \text{(v)}$$

Now putting $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$

$$2x - \frac{2(9a^4 - y^4)}{(x + y)^3} = 0$$

$$\Rightarrow x(x^3 + y^3 + 3xy^2 + 3x^2y) - (9a^4 - y^4) = 0$$

$$\Rightarrow x^4 + y^4 + xy^3 + 3x^2y^2 + 3xy^3 - 9a^4 = 0 \qquad \text{(vi)}$$

Similarly $x^4 + y^4 + yx^3 + 3x^2y^2 + 3yx^3 - 9a^4 = 0$ (vii)

Subtracting (vi) and (vii), we get

$$xy^3 - yx^3 + 3yx^3 - 3xy^3 = 0 \Rightarrow yx\left(x^2 - y^2\right) = 0$$

$$\Rightarrow yx(x-y)(x+y) = 0$$

Now, **Case 1.** $x = 0$. From (vi), we get

$$y^4 = 9a^4 \Rightarrow y = \sqrt{3}a.$$

Now putting above value in (iii), we get

$$z = \frac{3a^2}{\sqrt{3}a} = \sqrt{3}a.$$

Now, $x^2 + y^2 + z^2 = 0 + 3a^2 + 3a^2 \Rightarrow x^2 + y^2 + z^2 = 6a^2$ (viii)

Case 2. $y = 0$. From (vi), we get

$$y^4 = 9a^4 \Rightarrow y = \sqrt{3}a.$$

and from (iii), we get $z = \sqrt{3}a$

Now from (i), we get

Now, $x^2 + y^2 + z^2 = 3a^2 + 0 + 3a^2 \Rightarrow x^2 + y^2 + z^2 = 6a^2$ (ix)

Case 3. Either $x = -y$.

It is not possible as it does not satisfy (iii)

Case 4. If x = y.

From (vi), we get

$$x^4 + x^4 + x^4 + 3x^4 + 3x^4 - 9a^4 = 0 \Rightarrow 9x^4 = 9a^4.$$

Using (iii), we get $z = \dfrac{3a^2 - a^2}{2a} = a$.

$\therefore x = y = z = a$ (x)

Now, according to definition of stationary value f(a, b) is said to be stationary value of f(x, y).

If $f_x[a, b] = 0$ and $f_y[a, b] = 0$

i. e. the function is stationary at [a, b].

Stationary value of function will be

$u = x^2 + y^2 + z^2 \Rightarrow u = a^2 + a^2 + a^2 = 3a^2$ (xi)

From (viii), (ix) and (xi) stationary value of u are $6a^2$ and $3a^2$. Ans.

Q.No.14.: A rectangular box open at the top is to have volume equal to 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Sol.: Let x , y and z ft. be the dimensions of the rectangular box open at the top so that material for construction will be least if surface area is least.

Let surface area, $S = F(x, y, z) = xy + 2yz + 2zx$ . ....(i)

Also given volume $= xyz = 32$. ....(ii)

Eliminating z from (i) with the help of (ii), we get

$$S = xy + 2(y + x)\frac{32}{xy} = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right) .$$

$$\therefore \frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 \text{ and } \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0 .$$

Solving these, we get $x = y = 4$.

$$\text{Now } r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3} \text{ , } s = \frac{\partial^2 S}{\partial x \partial y} = 1 \text{ , } t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3} .$$

At $x = y = 4$, $rt - s^2 = (2 \times 2) - 1 = +$ ve and r is also + ve.

Hence S is minimum for x = y = 4. Then from (ii), z = 2.

Hence the dimensions of the box are x = 4ft.,y = 4ft. and z = 2ft.Ans.

Q.No.15.: Prove that maximum value of $\cos A \cos B \cos C = \frac{1}{8}$.

Sol.: $\cos A \cos B \cos C = \cos A \cos B \cos[W - (A + B)]$

$$= -\cos A \operatorname{coa} B \cos(A + B) \qquad [\because A + B + C = W]$$

$$f(A, B) = -\operatorname{coa} A \cos B \cos(A + B)$$

$$f_A = -\cos B[-\sin A.\cos(A + B) - \cos A.\sin(A + B)]$$

$$= \cos B[\sin A.\cos(A + B) + \cos A.\sin(A + B)] = \cos B \sin(2A + B)$$

$$f_B = -\cos A[-\sin B.\cos(A + B) - \cos B.\sin(A + B)]$$

$$= \cos A[\sin B.\cos(A + B) + \cos B.\sin(A + B)] = \cos A \sin(A + 2B)$$

$$r = f_{AA} = 2\cos B.\cos(2A + B)$$

$$s = f_{AB} = -\sin A \sin(A + 2B) + \cos A \cos(A + 2B) = \cos(A + A + 2B) = \cos(2A + 2B)$$

$t = f_{BB} = 2\cos A \cos(A + 2B)$.

For maximum and minimum values $f_A = 0$ and $f_B = 0$

$\cos B \sin(2A + B) = 0$

$\cos A \sin(A + 2B) = 0$

From above equations, we get the following four pairs of equations

(i) $\cos B = 0$, $\cos A = 0$, $B = \frac{W}{2}$, $A = \frac{W}{2}$.

It is not possible as two angles in same triangle cannot be $90°$.

(ii) $\cos B = 0$, $\sin(A + 2B) = 0$, $B = \frac{W}{2}$, $A + 2B = W \Rightarrow A = 0$.

Not possible as in same triangle no angle can be zero.

(iii) Similarly, we reject pair

$\sin(2A + B) = 0$, $\cos A = 0$,

$\sin(2A + B) = 0$

$2A + B = 0$ (i)

$A + 2B = 0$ (ii)

Solving (i) and (ii), we get

$A = B = \frac{W}{3}$, $A = B = \frac{W}{3}$

$r = 2\cos\frac{W}{3}\cos W = -1$,

$s = \cos\frac{4W}{3} = \cos\left(W + \frac{W}{3}\right) = \frac{-1}{2}$,

$t = 2\cos\frac{W}{3}\cos W = -1$.

$rt - s^2 > \frac{3}{4} > 0$ and $r < 0$.

Hence f(A, B) is maximum at $A = B = \frac{W}{3}$.

and maximum value $= -\cos\frac{W}{3}\cos\frac{W}{3}\cos\frac{2W}{3} = \frac{1}{8}$. Ans.

Q.No.17.: If x, y, z be the perpendicular from any point within the triangle on the sides a, b, c of a triangle ABC of area Δ. Show that the minimum value of

$$x^2+y^2+z^2=\frac{4\Delta^2}{a^2+b^2+c^2}.$$

Sol.: Δ **=** Area of the triangle ABC = area of three small triangles with in the ΔABC.

$$\Delta=\frac{1}{2}ax+\frac{1}{2}by+\frac{1}{2}cz$$

$$\therefore ax+by+cz=2\Delta \qquad \text{(i)}$$

Let $u=x^2+y^2+z^2$ (ii)

From (i) $cz=2\Delta-ax-by\Rightarrow z=\frac{1}{c}(2\Delta-ax-by)$

Put the value of z in (ii), we get

$$u=x^2+y^2+\frac{1}{c^2}(2\Delta-ax-by)^2 \qquad \text{(iii)}$$

Now differentiate (iii) w. r. t. x, we get

$$\frac{\partial u}{\partial x}=2x+\frac{2}{c^2}(2\Delta-ax-by)(-a).$$

For max. or min. $\frac{\partial u}{\partial x}=0$

$$2x+\frac{2}{c^2}(2\Delta-ax-by)(-a)=0$$

$$\frac{c^2x}{a}=2\Delta-ax-by, \qquad \text{(iv)}$$

Now, again differentiating w. r. t. y, we get

$$\frac{\partial u}{\partial y}=2y+\frac{2}{c^2}(2\Delta-ax-by)(-b)=0$$

$$\frac{c^2y}{b}=2\Delta-ax-by \qquad \text{(v)}$$

From (iv) and (v), we get

$$\frac{c^2x}{a}=\frac{c^2y}{b}\Rightarrow y=\frac{b}{a}x \qquad \text{(vi)}$$

Put the value of y in (iv), we get

$$\frac{c^2x}{a} = 2\Delta - ax - b\left(\frac{b}{a}\right)x \Rightarrow c^2x = 2\Delta a - a^2x - b^2x$$

$$\Rightarrow x = \frac{2\Delta a}{a^2 + b^2 + c^2}$$

Putting the value of x in (vi), we get

$$y = \frac{2\Delta a}{a^2 + b^2 + c^2} \times \frac{b}{a} \Rightarrow y = \frac{2\Delta b}{a^2 + b^2 + c^2}$$

Similarly, $z = \dfrac{2\Delta c}{a^2 + b^2 + c^2}$

Putting these values of x, y, z in (ii), we get

$$u = x^2 + y^2 + z^2$$

$$= \left(\frac{2\Delta a}{a^2 + b^2 + c^2}\right)^2 + \left(\frac{2\Delta b}{a^2 + b^2 + c^2}\right)^2 + \left(\frac{2\Delta c}{a^2 + b^2 + c^2}\right)^2 = \frac{4\Delta^2\left(a^2 + b^2 + c^2\right)}{\left(a^2 + b^2 + c^2\right)^2}$$

$$= \frac{4\Delta^2}{\left(a^2 + b^2 + c^2\right)}.$$ Ans.

Q.No.18.: A rectangular strip $\ell \times b$ of metal is bent up at the sides to form a trough. Without ends. Find the width of the side bases and the angle through which the side must bent so that the trough may have a maximum capacity.

Sol.: Let the length ℓ of the strip is bent up by an angle of θ, the length of bent part of strip are of equal on both sides by x.

Let h be the height and y be the base.

$$\therefore \sin\theta = \frac{h}{x} \Rightarrow h = x\sin\theta, \qquad \cos\theta = \frac{y}{x} \Rightarrow h = x\cos\theta.$$

Let F be the capacity of the trough.

$$F = (\ell - 2x)b \times h + 2 \times \frac{1}{2} \times h \times y \times b = (\ell \times 2x)b \times x\sin\theta + x\sin\theta.x\cos\theta.b$$

$$= b\left(\ell x - 2x^2\right)\sin\theta + \frac{x^2}{2}b\sin 2\theta.$$

Now, $F_x = \dfrac{\partial F}{\partial x} = b(\ell \times 4x)\sin\theta + \dfrac{2 \times b\sin 2\theta}{2} = b(\ell \times 4x)\sin\theta + bx\sin 2\theta$

$$F_{xx} = \frac{\partial^2 F}{\partial x^2} = b(0-4)\sin\theta + b\sin 2\theta = -4b\sin\theta + b\sin 2\theta$$

$$F_\theta = \frac{\partial F}{\partial \theta} = b\left(\ell x - 2x^2\right)\cos\theta + \frac{x^2 b}{2}.2\cos 2\theta = b\left(\ell x - 2x^2\right)\cos\theta + bx^2\cos 2\theta$$

$$F_{\theta\theta} = \frac{\partial^2 F}{\partial \theta^2} = -b\left(\ell x - 2x^2\right)\sin\theta + 2bx^2\sin 2\theta$$

$$F_{x\theta} = \frac{\partial^2 F}{\partial x \partial \theta} = b(\ell - 4x).\cos\theta + 2bx\cos 2\theta .$$

For x be max. or min.

$F_x = 0$, $F_\theta = 0$

$F_x = 0 \Rightarrow b(\ell - 4x)\sin\theta = -bx2\sin\theta\cos\theta$.

$$\cos\theta = \frac{b(4x-\ell)}{2bx}.$$

Also, $F_\theta = 0$

$$\Rightarrow b\left(\ell x - 2x^2\right)\cos\theta + bx^2\left(2\cos^2\theta - 1\right) = 0$$

$$\Rightarrow b\left(\ell x - 2x^2\right)\frac{b(4x-\ell)}{2bx} + 2bx^2 \times \frac{b^2(4x-\ell)^2}{4x^2b^2} - bx^2 = 0$$

$$\Rightarrow \frac{bx(\ell-2x)(4x-\ell)}{2x} + \frac{b(4x-\ell)^2}{2} - bx^2 = 0 \quad \Rightarrow b\left[\frac{(\ell-2x)(4x-\ell)}{2} + b\frac{(4x-\ell)^2}{2}\right] = bx^2$$

$$\Rightarrow \frac{(4x-\ell)}{2}[(\ell-2x)+(4x-\ell)] = x^2 \quad \Rightarrow (4x-\ell).\frac{2x}{2} = x^2 \quad \Rightarrow \left[x^2 - x(4x-\ell)\right] = 0$$

$$\Rightarrow x[x-(4x-\ell)] = 0$$

As $x \neq 0$

$$x = (4x-\ell) = 0 \Rightarrow x - 4x + \ell = 0 \Rightarrow 3x = \ell \Rightarrow x = \frac{\ell}{3}.$$

When $x = \frac{\ell}{3}$.

$$\cos\theta = \frac{b\left(4\times\frac{\ell}{3} - \ell\right)}{2b.\frac{\ell}{3}} = \frac{\frac{4\ell - 3\ell}{3}}{\frac{2\ell}{3}} = \frac{\ell}{2\ell} = \frac{1}{2} = \cos 60.$$

$\therefore \theta = 60°$.

When $\theta = 60°$, $x = \frac{\ell}{3}$.

Now $F_{xx} = -4b\sin\theta + b\sin 2\theta = -4b\sin 60° + b\sin 120° = -4b \times \frac{\sqrt{3}}{2} + bx \times \frac{\sqrt{3}}{2}$

$$= -2b\sqrt{3} + \frac{b\sqrt{3}}{2} = \frac{-4b\sqrt{3} + b\sqrt{3}}{2} = \frac{-3b\sqrt{3}}{2} = \frac{-3\sqrt{3}\,b}{2}.$$

$$F_{\theta\theta} = -b\left(\ell x - 2x^2\right)\sin\theta - 2bx^2\sin 2\theta = -b\left(\ell.\frac{\ell}{3} - 2\times\frac{\ell^2}{9}\right)\sin 60° - 2b.\frac{\ell^2}{9}\sin 120°$$

$$= -b\left[\frac{\ell^2}{3} - \frac{2\ell^2}{9}\right]\times\frac{\sqrt{3}}{2} - \frac{2b\ell^2}{9}\times\frac{\sqrt{3}}{2} = -b\left(\frac{3\ell^2 - 2\ell^2}{9}\right)\frac{\sqrt{3}}{2} - \frac{2b\ell^2}{9}\times\frac{\sqrt{3}}{2}$$

$$= \frac{-b\ell^2\sqrt{3}}{18} = \frac{2\sqrt{3}b\ell^2}{18} = \frac{-3\sqrt{3}b\ell^2}{18} = \frac{-\sqrt{3}b\ell^2}{18}$$

$$F_{x\theta} = b(\ell - 4x).\cos\theta + 2bx\cos 2\theta = b\times\left(\ell - 4.\frac{-\ell}{3}\right)\times 60° + 2b.\frac{\ell}{3}\cos 120°$$

$$= b\times\left(\frac{-\ell}{3}\right)\times\frac{1}{2} - \frac{2b\ell}{3}\times\frac{1}{2} = \frac{-b\ell - 2b\ell}{6} = \frac{-3b\ell}{6} = \frac{-b\ell}{2}.$$

$$F_{xx}\times F_{\theta\theta} - F_{x\theta}^2 = \left(\frac{-3\sqrt{3}b}{2}\right)\times\left(\frac{-\sqrt{3}b\ell^2}{6}\right) - \left(\frac{-b\ell}{2}\right)^2 = \frac{3\times 3b^2\ell^2}{2\times 6} - \frac{b^2\ell^2}{4}$$

$$= \frac{3b^2\ell^2 - b^2\ell^2}{4} = \frac{2b^2\ell^2}{4}$$

$$\frac{b^2\ell^2}{2} > 0.$$

Also $\frac{-3\sqrt{3}\,b}{2} < 0$.

Hence the length have maximum capacity when $x = \frac{\ell}{3}$, $\theta = 60° = \frac{\pi}{3}$.

Hence width of bases $= \frac{\ell}{3}$ and angle $\frac{\pi}{3}$.

Q.No.19.: Divide 24 into three parts such that the continued product of the first, the square of 2^{nd} and the cube of the third may be a maximum.

Sol.: Let the three parts be x, y and $[24-(x+y)]$

And let, $f=[24-(x+y)]y^2x^3$

Then, $\dfrac{\partial f}{\partial x}=3x^2y^2[24-(x+y)]+(-1)y^2x^3=72x^2y^2-4x^3y^2-3x^2y^3$ (i)

$$\frac{\partial^2 f}{\partial x^2}=144xy^2-12x^2y^2-6xy^3$$

and $\dfrac{\partial f}{\partial y}=2yx^3[24-x-y]+(-1)y^2x^3=48yx^3-2yx^4-3y^2x^3$ (ii)

$$\frac{\partial^2 f}{\partial y^2}=48x^3-2x^4-3yx^3$$

From (i) we can get $\dfrac{\partial^2 f}{\partial x\partial y}=144x^2y-8x^3y-9x^2y^2$ (iii)

F has maximum and minimum, then $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$

From (i), we get

$$72x^2y^2-4x^3y^2-3x^2y^3=0 \qquad \left[\because \frac{\partial f}{\partial x}=0\right]$$

$\Rightarrow 72-4x-3y=0 \Rightarrow 4x+3y=72$ (iv)

From (ii), we get

$$48yx^3-2yx^4-3y^2x^3=0 \qquad \left[\because \frac{\partial f}{\partial y}=0\right]$$

$\Rightarrow 48-2x-3y=0 \Rightarrow 2x+3y=48$ (v)

Using (iv) and (v), we get

$x=12$ and $y=8$

Now f having maximum and minimum if at $x=12$ and $y=8$

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2>0$$

$$\Rightarrow \left[144\times12\times(8)^2-12\times(12)^2\times(8)^2-6\times12\times(8)^3\right]\left[48(12)^3-2(12)^4-6\times8(12)^3\right]$$
$$-\left[144\times(12)^2\times8-8\times(12)^3\times8-9\times(12)^2\times(8)^3\right]>0$$

$\Rightarrow 1528754688 > 0$

So, f having maximum and minimum at $x = 12$ and $y = 8$

Now f has maximum value if $\left[\dfrac{\partial^2 f}{\partial x^2}\right]_{\substack{x=12\\y=8}} < 0$

$\Rightarrow \left[144 \times 12 \times (8)^2 - 12 \times (12)^2 \times (8)^2 - 6 \times 12 \times (8)^3\right] = -36864 < 0$

So f having maximum and minimum if at $x = 12$ and $y = 8$.

So the required three parts are $12,\ 8\ \ [24 - (12 + 8)] = 12,\ 8,\ \ 4$. Ans.

Q.No.20.:. Examine $f(x) = x^3 + y^3 + 3xy$ for maximum and minimum values.

Sol.: For maxima and minima values. Solve two equations

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

i. e. $3x^2 + 3y = 0$ and $3y^2 + 3x = 0$

Solve these equations, we get

$x^2 + y = 0$ and $y^2 + x = 0$

$\Rightarrow y^4 + y = 0 \Rightarrow y = 0,\ -1$.

Similarly, $y = 0,\ -1$.

So, we get four points (0, 0), $(0, -1)$, $(-1,\ 0)$ and $(-1,\ -1)$.

For maxima or minima 2^{nd} condition is

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \Rightarrow 6x.6y > 9 \Rightarrow xy > \frac{1}{4}.$$

Only point $(-1,\ -1)$ satisfies this condition .

Now for maxima $f_{xx} < 0$ at point $(-1,\ -1)$ will $\Rightarrow$ that this is maxima.

$f_{xx} = 6x = -6 < 0$ at $(-1,\ -1)$. So this point is a maximum value and no minimum value of function.

Now, condition for saddle point.

i. e. $f_{xx}.f_{yy} < (f_{xy})^2 \Rightarrow xy < \dfrac{1}{4}$.

Three points (0, 0), $(0,-1)$, $(-1,\ 0)$ satisfies the above equation.

Therefore these points are saddle points for the curve x^3+y^3+3xy .

Q.No.21.: Find the shortest and longest distance of the point $(1,2,-1)$ to the sphere

$$x^2+y^2+z^2=24.$$

Sol.: Let us consider a point A(x, y, z) on the surface of the sphere.

$\therefore$ AP $=\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

Let, $F=(x-1)^2+(y-2)^2+(z+1)^2=x^2-2x+1+y^2-4y+4+z^2+2z+1$

$$=x^2+y^2+2z-2x-4y+2z+6 \qquad \text{(i)}$$

Now the given equation of sphere is $x^2+y^2+z^2=24$ (ii)

From (i) and (ii), we get

$F=24-2x-4y+2z+6=30-2x-4y+2z$

also from (ii), we get $z=\pm\sqrt{24-x^2-y^2}$

When $z=\sqrt{24-x^2-y^2}$ i. e. positive, then

$F=30-2x-4y+2\sqrt{24-x^2-y^2}$

$$\therefore F_x=-2+\frac{2(-2x)}{2\sqrt{24-x^2-y^2}}=\frac{(-2x)}{\sqrt{24-x^2-y^2}}-2 \qquad \text{(iii)}$$

$$F_y=\frac{2(-2y)}{2\sqrt{24-x^2-y^2}}-4=\frac{(-2y)}{\sqrt{24-x^2-y^2}}-4 \qquad \text{(iv)}$$

For maximum or minimum.

$F_x=0$ and $F_y=0$

$$\frac{(-2x)}{\sqrt{24-x^2-y^2}}-2=0\Rightarrow\frac{-2x}{\sqrt{24-x^2-y^2}}=2\Rightarrow -x=\sqrt{24-x^2-y^2}$$

Squaring both sides, we get

$$x^2=24-x^2-y^2\Rightarrow 2x^2+y^2=24 \qquad \text{(v)}$$

$$\frac{(-2y)}{\sqrt{24-x^2-y^2}}-4=0\Rightarrow -y=2\sqrt{24-x^2-y^2}$$

Squaring both sides, we get

$$y^2 = 4\left(24 - x^2 - y^2\right) \Rightarrow 4x^2 + 5y^2 96 \qquad \text{(vi)}$$

Multiplying (v) by 2 and subtracting from (vi), we get

$$3y^2 = 48 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4 .$$

But from (iv) y should be negative

$$\therefore y = -4$$

$$x = \frac{1}{2}\sqrt{24-16} = \sqrt{4} = 2 .$$

But from (iii) x should be negative. $\therefore x = -4$

$$\therefore z = \sqrt{24-(-2)^2-(-4)^2} = \sqrt{24-4-16} = \sqrt{4} = 2 .$$

Hence the points are $(-2, \ -4, \ 2)$.

Now $F_{xx} = \dfrac{-48+2y^2}{\left(24-x^2-y^2\right)^{3/2}}$.

$$F_{xx} \ \text{at} \ (-2, \ -4, \ 2) = \frac{-48+32}{(24-4-16)^{3/2}} = \frac{-16}{8} = -2 .$$

$$F_{yy} \ \text{at} \ (-2, \ -4, \ 2) = \frac{-48+2x^2}{\left(24-x^2-16\right)^{3/2}} = \frac{-48+8}{8} = -5 .$$

$$F_{xy} \ \text{at} \ (-2, \ -4, \ 2) = \frac{-2xy}{\left(24-x^2-16\right)^{3/2}} = \frac{-2\times 8}{8} = -2 .$$

For maxima or minima

$$F_{xx}.F_{yy} - \left(F_{xy}\right)^2 > 0$$

$\Rightarrow (-2)(-5)-(-2)^2 > 0 \Rightarrow 10-4 > 0 \Rightarrow 6 > 0$, which is true.

Also $F_{xx} = -2 < 0$.

$\therefore (-2, \ -4, \ 2)$ are the points of maxima for F.

$$F_{max.} = (-2-1)^2 + (-4-2)^2 + (2+1)^2 \ 9+36+9 = 54 .$$

When F is maximum then AP is also maximum.

$$\therefore AP_{max.} = \sqrt{54} .$$

Now, when $z = -\sqrt{24 + x^2 - y^2}$ i. e. negative.

Then, $F = 30 - 2x - 4y - 2\sqrt{24 - x^2 - y^2}$

$$\therefore F_x = \frac{2x}{\sqrt{24 - x^2 - y^2}} - 2 \qquad \text{(vii)}$$

$$F_y = \frac{2y}{\sqrt{24 - x^2 - y^2}} - 4 \qquad \text{(viii)}$$

For max. or min.

$F_x = 0$ and $F_y = 0$

$$\frac{(-2x)}{\sqrt{24 - x^2 - y^2}} - 2 = 0 \Rightarrow \frac{-2x}{\sqrt{24 - x^2 - y^2}} = 2 \Rightarrow -x = \sqrt{24 - x^2 - y^2}$$

Squaring both sides, we get

$$x^2 = 24 - x^2 - y^2 \Rightarrow 2x^2 + y^2 = 24$$

$$\frac{(-2y)}{\sqrt{24 - x^2 - y^2}} - 4 = 0 \Rightarrow -y = 2\sqrt{24 - x^2 - y^2}$$

Squaring both sides, we get

$$y^2 = 4\left(24 - x^2 - y^2\right) \Rightarrow 4x^2 + 5y^2 96$$

Solving above equations, we get

$x = \pm 2$ and $y = \pm 2$

But from (vii) and (viii) x and y should be positive

$\therefore x = 2$ and $y = 2$ and $z = -2$.

Hence the points are $(2,\ 4,\ -2)$

$$F_{xx} \text{ at } (2,\ 4,\ -2) = 2\left[\frac{24 - y^2}{\left(24 - x^2 - y^2\right)^{3/2}}\right]. = 2\left[\frac{24 - 16}{(24 - 4 - 16)^{3/2}}\right] = 2$$

$$F_{yy} \text{ at } (2,\ 4,\ -2) = 2\left[\frac{24 - x^2}{\left(24 - x^2 - y^2\right)^{3/2}}\right] = 5.$$

$$F_{xy} \text{ at } (2,\ 4,\ -2)\ \frac{2xy}{24 - x^2 - y^2} = \frac{16}{8} = 2.$$

Now, for max. or mini.

$F_{xx}.F_{yy} - (F_{xy})^2 > 0$

$\therefore 10 - 4 > 0 \Rightarrow 6 > 0$, which is true.

Also, $F_{xx} = 2 > 0$.

$\therefore (2,\ 4,\ -2)$ are the point of minimum for F.

when F is minimum AP is also minimum

$\therefore AP_{min.} = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{6}$.

$\therefore$ The longest distance of point $(1, 2, -1)$ from sphere $x^2 + y^2 + z^2 = 24$ is $\sqrt{54}$ and the shortest distance is $\sqrt{6}$. Ans.

Q.No.22.: Find the extreme values of $x^2 + y^2 + z^2$, when $ax + by + cz = P$.

Sol.: A maximum or minimum value is called extreme value of the function.

Let $u = x^2 + y^2 + z^2$ (i)

And also $ax + by + cz = P$

$$\therefore z = \frac{P - ax - by}{c}$$

Putting in (i), we get

$$u = x^2 + y^2 + \frac{1}{c^2}(P - ax - by)^2 \qquad \text{(ii)}$$

Differentiate partially (ii) w. r. t. x, we get

$$\frac{\partial u}{\partial x} = 2x + \frac{2}{c^2}(P - ax - by) \times -a$$

Differentiate partially (ii) w. r. t. y, we get

$$\frac{\partial u}{\partial y} = 2y + \frac{2}{c^2}(P - ax - by) \times -b$$

for max. and mini. Values both

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$$2x + \frac{2}{c^2}(-a)(P - ax - by) = 0 \qquad \text{(iii)}$$

$$2y - \frac{2b}{c^2}(P - ax - by) = 0 \qquad \text{(iv)}$$

Solving (iii) and (iv), we get

$$x = \frac{aP - aby}{c^2 + a^2}$$

Put in (iv), we get

$$c^2 y - bP + ab\left(\frac{aP - aby}{c^2 + a^2}\right) + b^2 y = 0 \Rightarrow (c^2 + a^2)(c^2 + b^2)y + ab(aP - aby) = bP(c^2 + a^2)$$

$$\therefore y = \frac{bP}{a^2 + b^2 + c^2},$$

and also $x = \dfrac{aP}{a^2 + b^2 + c^2}$, $z = \dfrac{cP}{a^2 + b^2 + c^2}$.

Now $\dfrac{\partial^2 u}{\partial x^2} = 2 - \dfrac{2a}{c^2}(-a) = 2 + \dfrac{2a^2}{c^2}$ and $\dfrac{\partial^2 u}{\partial y^2} = 2 + \dfrac{2b^2}{c^2}$, $\dfrac{\partial^2 u}{\partial x \partial y^2} = \dfrac{ab}{c^2}$

$$\therefore \frac{\partial^2 u}{\partial x^2}.\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x.\partial y}\right)^2 \Rightarrow 4 + 2\frac{(b^2 + a^2)}{c^2} - \frac{a^2 b^2}{c^4} > 0$$

Clearly $\dfrac{\partial^2 u}{\partial x^2}.\dfrac{\partial^2 u}{\partial y^2} > \left(\dfrac{\partial^2 u}{\partial x.\partial y}\right)^2 > 0$

Also $\dfrac{\partial^2 u}{\partial x^2} = 2 + \dfrac{2b^2}{c^2} > 0$.

$\therefore$ u has minimum value at $x = \dfrac{aP}{a^2 + b^2 + c^2}$, $y = \dfrac{bP}{a^2 + b^2 + c^2}$ and $z = \dfrac{cP}{a^2 + b^2 + c^2}$.

And minimum value of $x^2 + y^2 + z^2$

$$= \frac{a^2 P^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2 P^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2 P^2}{(a^2 + b^2 + c^2)^2} = \frac{P^2}{a^2 + b^2 + c^2}. \text{ Ans.}$$

Q.No.23.: Find the maximum and minimum values of

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$

Sol.: Differentiating f partially w.r.t. x and y, we get

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72,$$

$f_y = \frac{\partial f}{\partial y} = 6xy - 30y$.

The stationary (critical) points are given by $f_x = 0$ and $f_y = 0$.

From $f_y = 6xy - 30y = 0 \Rightarrow 6y(x-5) = 0$.

Thus, either $y = 0$ or $x = 5$.

Since $f_x = 3x^2 + 3y^2 - 30x + 72 = 0$.

For $y = 0$, $3x^2 - 30x + 72 = 0 \Rightarrow x = 6$ or 4.

For $x = 5$, $75 + 3y^2 - 150 + 72 = 0 \Rightarrow y = \pm 1$

Thus, the four stationary points are given by

$(6, 0), (4, 0), (5, 1), (5, -1)$.

To determine the nature of these points, calculate f_{xx}, f_{yy} and f_{xy}, we obtain

$f_{xx} = A = 6x - 30$, $f_{xy} = B = 6y$, $f_{yy} = C = 6x - 30$.

Thus, $AC - B^2 = (6x-30)^2 - 36y^2 = 36\left[(x-5)^2 - y^2\right]$

1. At the stationary point (6, 0), we have $A = 36 - 30 = 6 > 0$ and $AC - B^2 = 36 > 0$. So (6, 0) is a minimum point of the given function f and the minimum value of f at (6, 0) is $6^3 + 0 - 15.36 + 72.6 = 108$.

2. At (4, 0): $A = 24 - 30 = -6 < 0$ and $AC - B^2 = 36 > 0$. So a maximum occurs at the point (4, 0) and the maximum value of f at (4, 0) is 112.

3. At (5, 1), $A = 0$, $AC - B^2 = -36 < 0$. So (5, 1) is a saddle point (It is neither maximum nor minimum).

4. At $(5, -1)$, $A = 0$, $AC = B^2 = -36 < 0$. So $(5, -1)$ is neither a maximum nor a minimum (it is a saddle point).

Q.No.24.: Find the shortest distance from origin to the surface $xyz^2 = 2$.

Sol.: Given equation of surface is $xyz^2 = 2$.

Let d be the distance from origin (0, 0, 0) to any point (x, y, z) of the given surface, then

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \Rightarrow d^2 = x^2 + y^2 + z^2$.

Eliminating z^2 using the equation of the surface $xyz^2 = 2$. So replace $z^2 = \frac{2}{xy}$.

$\therefore d^2 = x^2 + y^2 + \frac{2}{xy} = f(x, y)$.

Now $f_x = 2x - \frac{2}{x^2 y}$, $f_y = 2y - \frac{2}{xy^2}$.

The stationary (critical) points are given by $f_x = 0$ and $f_y = 0$.

Solving $f_x = 0$ and $f_y = 0$, we get

$\frac{x^3 y - 1}{x^2 y} = 0$ and $\frac{xy^3 - 1}{xy^2} = 0$

$x^3 y = 1 = xy^3 \Rightarrow xy\left(x^2 - y^2\right) = 0$.

Since $x \neq 0$, $y \neq 0$, so $x = \pm y = 1$.

Thus, the two stationary points are (1, 1) and $(-1, -1)$.

Now $f_{xx} = 2 + \frac{4}{x^3 y}$, $f_{yy} = 2 + \frac{4}{xy^3}$, $f_{xy} = \frac{2}{x^2 y^2}$.

At (1, 1): $f_{xx} = 6 > 0$, $f_{xx}.f_{yy} - f_{xy}^2 = 6.6 - 6 = 32 > 0$.

At $(-1, -1)$: $f_{xx} = +6.f_{xx} f_{yy} - f_{xy}^2 = 32$.

So minimum occurs at $(1,\ 1,\ \sqrt{2})$ and $(-1,\ -1,\ \sqrt{2})$.

Hence the shortest distance is $\sqrt{1^2 + 1^2 + \left(\sqrt{2}\right)^2} = \sqrt{4} = 2$. Ans.

Q.No.25.: The temperature T at any point (x, y, z)in space is T(x, y, z) = $kxyz^2$, where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol.: Given the temperature T at any point (x, y, z) in space is T(x, y, z) = $kxyz^2$.

Eliminating the variable z, using $z^2 = a^2 - x^2 - y^2$, we get

$T(x, y, z) = kxyz^2 = kxy(a^2 - x^2 - y^2) = F(x, y)$.

Now $F_x = ky(a^2 - 3x^2 - y^2)$, $F_y = kx(a^2 - x^2 - 3y^2)$.

The stationary (critical) points are given by $F_x = 0$ and $F_y = 0$.

$ky(a^2 - 3x^2 - y^2) = 0$, $kx(a^2 - x^2 - 3y^2) = 0$

$\Rightarrow 3x^2 + y^2 = a^2$, $x^2 + 3y^2 = a^2$ and x =0, y = 0

Solving, we get $x = y = \pm\frac{a}{2}$.

Also $F_{xx} = -6kxy$, $F_{yy} = -6kxy$, $F_{xy} = k\left(a^2 - 3x^2 - 3y^2\right)$.

At (0, 0), $F_{xx} = 0 = F_{yy}$, $F_{xy} = ka^2$.

$\therefore 0.0 - ka^2 < 0 \Rightarrow$ (0, 0) is a saddle point.

At both the points $\left(\frac{a}{2}, \frac{a}{2}\right)$ and $\left(\frac{-a}{2}, \frac{-a}{2}\right)$,

$$F_{xx} = -6\frac{ka^2}{4} < 0 \text{ and } F_{xx}.F_{yy} - F_{xy}^2 = \frac{9}{4}k^2a^4 - \frac{a^4k^2}{4} = 2k^2a^4 > 0.$$

$\therefore$ T attains a maximum value at both these points $\left(\frac{a}{2}, \frac{a}{2}\right)$ and $\left(\frac{-a}{2}, \frac{-a}{2}\right)$.

The maximum value of T is $k.\frac{a^2}{4}\left(\frac{a^2}{2}\right) = \frac{ka^4}{8}$. Ans.

Q.No.26.: Find the shortest distance between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

$$\text{and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Sol.: The given equations of lines are

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \qquad \text{(i)}$$

$$\text{and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \qquad \text{(ii)}$$

Equating each of the fractions of (i) to λ, we get $x = 3 + \lambda$, $y = 5 - 2\lambda$, $z = 7 + \lambda$.

Thus any point P on the first line (i) is given by

$(3 + \lambda,\ 5 - 2\lambda,\ 7 + \lambda)$.

Similarly, any point Q on the second line (ii) is $(-1 + 7\mu,\ -1 - 6\mu,\ -1 + \mu)$.

The distance between the given

$$PQ = \sqrt{(3+\lambda+1-7\mu)^2 + (5-2\lambda+1+6\mu)^2 + (7+\lambda+1-\mu)^2}\,.$$

Consider $f(\lambda,\mu) = (PQ)^2 = 6\lambda^2 + 86\mu^2 - 40\lambda\mu + 116$.

The problem is to find minimum value of f as a function of the two variables λ, μ.

$f_\lambda = 12\lambda - 40\mu$, $f_\mu = 172\mu - 40\lambda$.

Solving $12\lambda - 40\mu = 0$ and $172\mu - 40\lambda = 0$, we get

$\lambda = 0$ and $\mu = 0$ as the only stationary point.

$f_{\lambda\lambda} = 12$, $f_{\mu\mu} = 172$, $f_{\lambda\mu} = -40$.

Now $f_{\lambda\lambda} = 12 > 0$ and $\left(f_{\lambda\lambda} f_{\mu\mu} - f_{\lambda\mu}^2\right) = 12 \times 172 - (-40)^2 = 2064 - 1600 = 464 > 0$.

Thus a minimum occurs at $\lambda = 0, \ \mu = 0$.

The minimum (shortest) distance is given by

$PQ = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116} = 2\sqrt{29}$. Ans.

Home Assignments

Q.No.1.: Test the functions for maxima, minima and saddle points:

a. $x^4 + y^4 - x^2 - y^2 + 1$.

b. $x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$.

Ans.: a. Maximum at (0, 0)

Maximum value is 1.

Minima at four points $\left(\pm 1/\sqrt{2}, \ \pm 1/\sqrt{2}\right)$

Minimum value at these 4 points is $\frac{1}{2}$.

Saddle points at four points $\left(0, \ \pm 1/\sqrt{2}\right), \left(\pm 1/\sqrt{2}, \ 0\right)$

b. Maximum at (1, 1), minimum at $(-1, \ -1)$.

Q.No.2.: Find the extrema of f(x, y) : $\left(x^2 + y^2\right)e^{6x+2x^2}$.

Ans.: Minima at (0, 0) (minimum value 0) and at $(-1,\ 0)$(minimum value e^{-4}).

Saddle point at $\left(-\frac{1}{2},\ 0\right)$.

Q.No.3.: Examine the following function f(x, y) for extrema : $\sin x + \sin y + \sin(x+y)$.

Ans.: Maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, maximum value $\frac{3\sqrt{3}}{2}$.

Q.No.4.: Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

Ans.: Shortest distance is 1 (from 0, 0, 0) to the point $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ on the plane.

Q.No.5.: Find the shortest distance between the lines

$$\frac{x-2}{3} = \frac{y-6}{-2} = \frac{z-5}{-2}$$

and $$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-8}{6}.$$

Ans.: Shortest distance is 3 between the points (5, 4, 3) and (3, 2, 2).

Q.No.6.: If the perimeter of a triangle is constant, prove that the area of this triangle is maximum, when the triangle is equilateral.

Ans.: Maximum when $a = b = c = \frac{2s}{3}$.

Q.No.7.: Find the volume of the largest rectangular parallelopiped with edges parallel to the axes, that can be inscribed in the:

a. Sphere

b. Equation of the ellipsoid is $4x^2 + 4y^2 + 9z^2 = 36$.

Ans.: a. Volume : $\frac{8a^3}{3\sqrt{3}}$, $x = y = z = \frac{a}{\sqrt{3}}$

b. Volume: $16\sqrt{3}$ (with $a = 3,\ b = 3,\ c = 2$).

Q.No.8.: Find the dimensions of a rectangular box, with open top, so that the total surface area of the box is a minimum, given that the volume of the box is constant say V.

Ans.: $x = y = 2z = (2V)^{1/3}$.

Q.No.9.: Find the dimensions of the rectangular box, with open top, of maximum capacity whose surface area is 432 sq. cm.

Ans.: 12, 12, 6.

Q.No.10.: If the total surface area of a closed rectangular box is 108 sq. cm., find the dimensions of the box having maximum capacity.

Ans.: $\sqrt{18},\ \sqrt{18},\ \sqrt{18}$ **.**

Q.No.11.: Find absolute maximum and minimum values of the function

$f(x, y) = 3x^2 + y^2 - x$ over the region $2x^2 + y^2 \le 1$.

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