1^{st} Topic

Double Integrals

(where limits are given)

(Last updated on 15-07-2013)

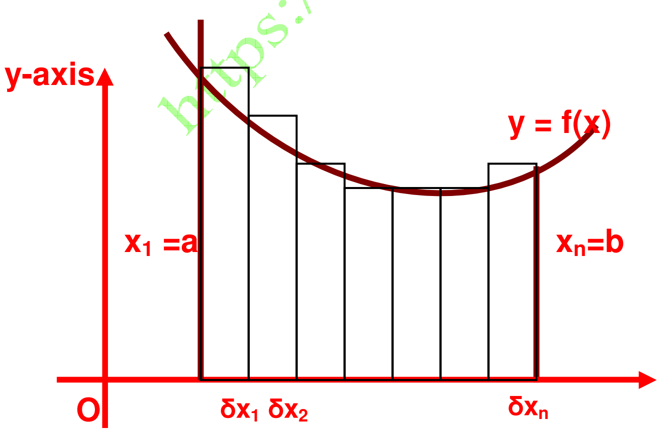
(10 Solved problems and 06 Home assignments)

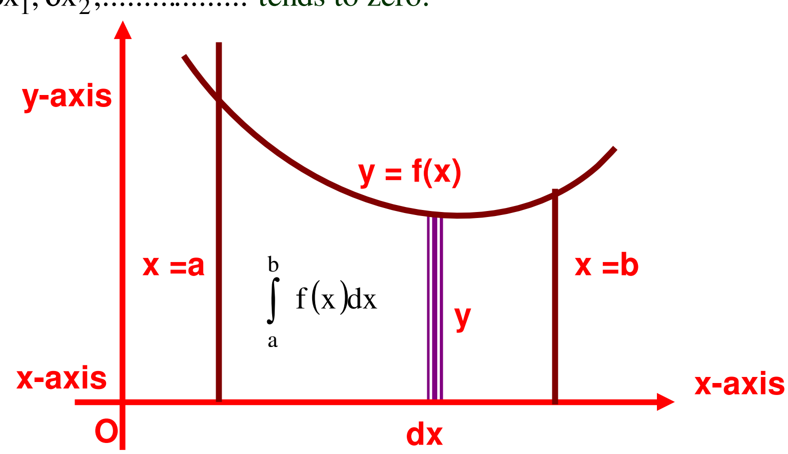
DOUBLE INTEGRALS:

The definite integral $\int_a^b f(x)dx$ is defined as the limit of the sum

$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots\ldots\ldots\ldots + f(x_n)\delta x_n$ i.e. $\sum_{r=1}^{n} f(x_r)\delta x_r$,

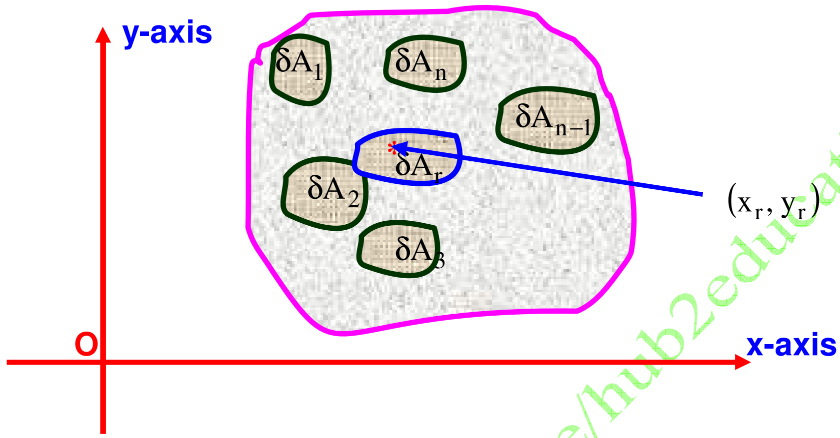
where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots\ldots\ldots\ldots$ tends to zero.





A double integral is its counterpart in two dimensions

Consider a function f (x, y) of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots\ldots\ldots, \delta A_n$. Let (x_r, y_r) be any point with in the r[th] elementary area δA_r.



Now consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots\ldots\ldots\ldots + f(x_n, y_n)\delta A_n \quad \text{i.e.} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r.$$

Definition:

The limit of this sum, if it exists, as the number of sub-division increases indefinitely (i.e. $n \to \infty$) and consequently (as a result) the area of each sub-division (i.e. δA_r) decreases to zero (i.e. $\delta A_r \to 0$), is defined as the double integral of f (x, y) over the region R and is written as $\int_R f(x, y)dA$.

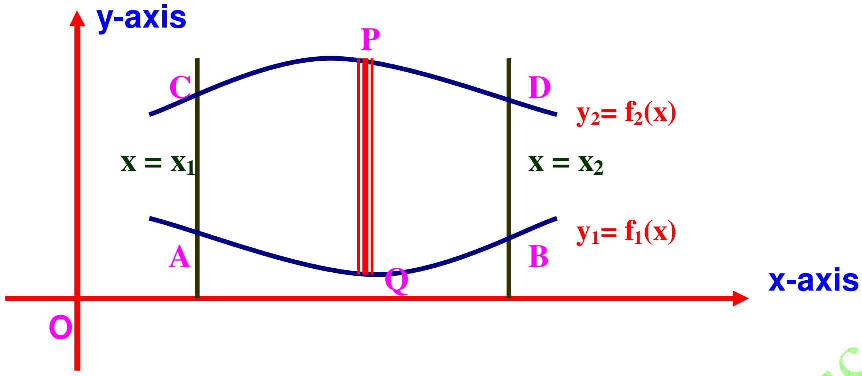
Thus $$\int_R f(x, y)dA = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r. \qquad \text{(i)}$$

For purpose of evaluation, (i) is expressed as the repeated Integrals

$$\int_R f(x, y)dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dxdy.$$

Its value is found as follows:

Case 1: When y_1, y_2 are functions of x and x_1, x_2 are constants.



Here f(x, y) is first integrated w. r. t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w. r. t. x within the limits x_1, x_2 i.e.

$$I_1 = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y) dy} dx},$$

where integration is carried from the inner to the outer rectangle.

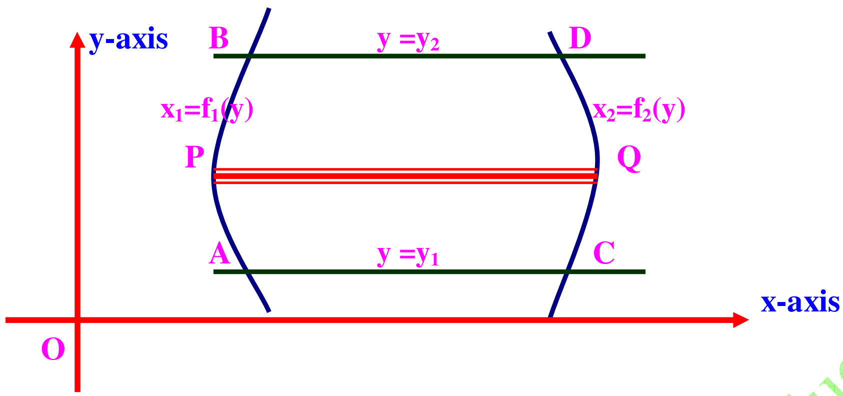
Geometrical illustration with Figure:

Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

The inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the strip from AC to BD.

Thus, the whole region of integration is the area ABDC.

Case 2: When x_1, x_2 are functions of y and y_1, y_2 are constants.



Here f(x,y) is first integrated w. r. t. x keeping y fixed, with in the limits x_1, x_2 and the resulting expression is integrated w. r. t. y between the limits y_1, y_2 i.e.

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)dx}\, dy},$$

Geometrical illustration with Figure:

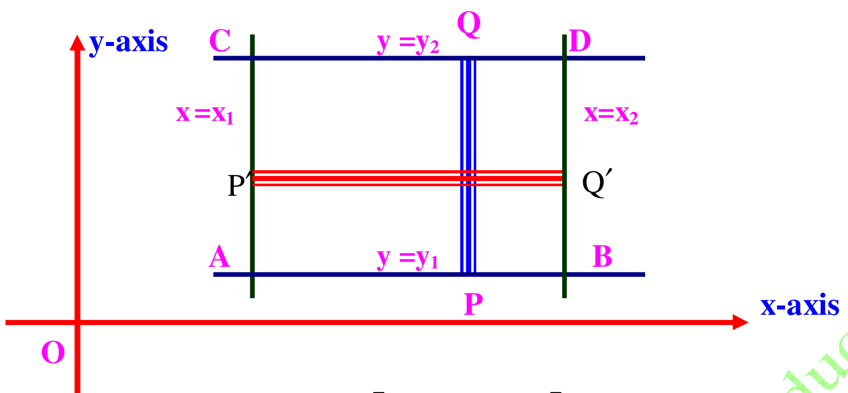
Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is horizontal strip of width dy.

Then the inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this strip from AC to BD.

Thus the whole region of integration is the area ABDC.

Case 3: When both pairs of limits are constant.

The region of integration is the rectangle ABDC as shown in the figure.



$$I_1 = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y)\, dy \right] dx \ ,$$

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

$$I_2 = \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x, y)\, dx \right] dy \ ,$$

In I_2, we integrate along horizontal strip $P'Q'$ and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

Here we will discuss those problems in double integrals, where limits are given. By observing the limits, we will decide the order of integration. Since limits are given, so rough sketch of the region of integration is not required.

Q.No.1.: Evaluate the integral $\int_1^2 \int_1^3 xy^2 dxdy$.

Sol.: Let $I = \int_1^2 \left(\int_1^3 xy^2 dx \right) dy = \int_1^2 \left[\frac{x^2 y^2}{2} \right]_1^3 dy = \int_1^2 \left[\frac{(3)^2 y^2}{2} - \frac{y^2}{2} \right] dy$

$$= \int_1^2 \left(\frac{9y^2 - y}{2} \right) dy = \int_1^2 4y^2 dy = \left[\frac{4y^3}{3} \right]_1^2 = \frac{4}{3}(2)^3 - \frac{4}{3}(1)^3 = \frac{4 \times 8}{3} - \frac{4}{3} = \frac{28}{3} \text{ .Ans.}$$

Q.No.2.: Evaluate $\int_0^5 \int_0^{x^2} x\left(x^2+y^2\right)dxdy$.

Sol.: $I=\int_0^5\left(\int_0^{x^2}\left(x^3+xy^2\right)dy\right)dx=\int_0^5\left[x^3y+x.\frac{y^3}{3}\right]_0^{x^2}dx=\int_0^5\left[x^3.x^2+x.\frac{x^6}{3}\right]dx$

$$=\int_0^5\left(x^5+\frac{x^7}{3}\right)dx=\left|\frac{x^6}{6}+\frac{x^8}{24}\right|_0^5=5^6\left[\frac{1}{6}+\frac{5^2}{24}\right]=18880.2 \text{ nearly. Ans.}$$

Q.No.3.: Evaluate the integral $\int_0^1 \int_x^{\sqrt{x}} \left(x^2+y^2\right)dxdy$.

Sol.: Let $I=\int_0^1\left(\int_x^{\sqrt{x}}\left(x^2+y^2\right)dy\right)dx=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{\sqrt{x}}dx$

$$=\int_0^1\left[x^2\sqrt{x}+\frac{x^{3/2}}{3}-x^3-\frac{x^3}{3}\right]dx$$

$$=\int_0^1\left(x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right)dx=\left[\frac{x^{7/2}}{7/2}+\frac{1}{3}.\frac{x^{5/2}}{5/2}-\frac{x^4}{3}\right]_0^1$$

$$=\left[\frac{2}{7}(1)^{7/2}+\frac{2}{15}(1)^{5/2}-\frac{1}{3}\right]=\frac{2}{7}+\frac{2}{15}-\frac{1}{3}=\frac{30+14-35}{105}=\frac{9}{105}=\frac{3}{35}\text{. Ans.}$$

Q.No.4.: Evaluate the integral $\int_0^4 \int_0^{x^2} e^{y/x}dxdy$.

Sol.: Let $I=\int_0^4\left(\int_0^{x^2} e^{y/x}dy\right)dx=\int_0^4\left[xe^{y/x}\right]_0^{x^2}dx=\int_0^4\left[xe^{x^2/x}-xe^0\right]dx=\int_0^4\left[xe^x-x\right]dx$

$$=\int_0^4 xe^xdx-\int_0^4 xdx=x\int_0^4 e^xdx-\int_0^4\left[\frac{d}{dx}(x)\int e^xdx\right]-\left[\frac{x^2}{2}\right]_0^4$$

$$=\left[xe^x\right]_0^4-\left[e^x\right]_0^4-\left[\frac{x^2}{2}\right]_0^4=\left(4e^4-0\right)-\left(e^4-1\right)-\left[\frac{(4)^2}{2}-0\right]$$

$$=4e^4-e^4+1-8=3e^4-7\text{. Ans.}$$

Q.No.5.: Evaluate the integral $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dxdy}{1+x^2+y^2}$.

Sol.: Let $I = \int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right) dx = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} \tan^{-1} 0 \right] dx$$

$$= \int_0^1 \frac{\pi/4}{\sqrt{1+x^2}} dx = \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1} \right) \right]_0^1 = \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right) - \log 1 \right].$$ Ans.

Q.No.6.: Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \left(\frac{x^2-y^2}{x^2+y^2} \right) dxdy$.

Sol.: Let $I = \int_0^{4a} \left[\int_{\frac{y^2}{4a}}^{y} \left(\frac{x^2-y^2}{x^2+y^2} \right) dx \right] dy = \int_0^{4a} \left[\int_{\frac{y^2}{4a}}^{y} \left(\frac{x^2-y^2}{x^2+y^2} - 1 + 1 \right) dx \right] dy$

$$= \int_0^{4a} \left[\int_{\frac{y^2}{4a}}^{y} \left(\frac{-2y^2}{x^2+y^2} + 1 \right) dx \right] dy = \int_0^{4a} \left(\left[-2y^2 \times \frac{1}{y} \times \tan^{-1}\left(\frac{x}{y} \right) + x \right]_{\frac{y^2}{4a}}^{y} \right) dy$$

$$= \int_0^{4a} \left(\left(-2y \times \frac{\pi}{4} + y \right) - \left[-2y \times \tan^{-1}\left(\frac{y}{4a} \right) + \frac{y^2}{4a} \right] \right) dy$$

$$= \int_0^{4a} \left[-\frac{\pi y}{2} + y + 2y.\tan^{-1} \frac{y}{4a} - \frac{y^2}{4a} \right] dy$$

$$= \left[\frac{-\pi}{2} . \frac{y^2}{2} + \frac{y^2}{2} + \left\{ \left(y^2 + 16a^2 \right) \tan^{-1} \frac{y}{4a} - 4ay \right\} - \frac{y^3}{12a} \right]_0^{4a}$$

$$= \left[\frac{-\pi}{4} \times 16a^2 + \frac{16a^2}{2} + \left(16a^2 + 16a^2 \right) \times \frac{\pi}{4} - 16a^2 - \frac{64a^3}{12a} \right]$$

$$=\left[-4\pi a^2+8a^2+8\pi a^2-16a^2-\frac{16}{3}a^2\right]=a^2\left[4\pi-\frac{40}{3}\right]=8\left[\frac{\pi}{2}-\frac{5}{3}\right]a^2 \text{. Ans.}$$

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Home Assignments

Q.No.1.: Evaluate the integral $\int_0^{2a}\left(\int_0^{x^2/4a} xy\,dy\right)dx$.

Q.No.2.: Evaluate the integral $\int_0^{a}\left(\int_0^{\sqrt{a^2-y^2}} xy\,dx\right)dy$.

Q.No.3.: Evaluate the integral $\int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} (x+y)^2\,dx\right]dy$.

Q.No.4.: Evaluate the integral $\int_0^{1}\left[\int_{x^2}^{x} xy(x+y)\,dy\right]dx$.

Q.No.5.: Evaluate the integral $\int_1^{2}\left(\int_0^{4-x^2} (x+y)\,dy\right)dx$.

Q.No.6.: Evaluate the integral $\int_0^{4a}\left(\int_{y^2/4a}^{2\sqrt{ay}} dx\right)dy$.

Q.No.7.: Evaluate the integral $\int_0^{1}\left[\int_{ay^2}^{ay} \left(x^2+y^2\right)dx\right]dy$.

Q.No.8.: Evaluate the integral $\int_0^{a}\left[\int_0^{y^2/a} \frac{dx}{\sqrt{1-\frac{a^2x^2}{y^4}}}\right]dy$.

Q.No.9.: Evaluate the integral $\int_0^a \left[\int_0^{\frac{a}{b}\sqrt{a^2 - x^2}} x^3 y dy \right] dx$.

Q.No.10.: Evaluate the integral $I = \int_1^{e^2} \left[\int_{\log y}^{2} dx \right] dy$.

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