5^{th} Topic

Double Integrals

(Change of variables)

(Last updated on 15-07-2013)

(20 Solved problems and 04 Home assignments)

Change of variables:

The evaluation of the double integrals is greatly simplified by a suitable change of variables. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

In a double integrals:

Let the variables x, y in the double integral $\iint_R f(x, y)dxdy$ be changed to the new variables u. v by the transformation

$$x = \phi(u, v), \quad y = \psi(u, v),$$

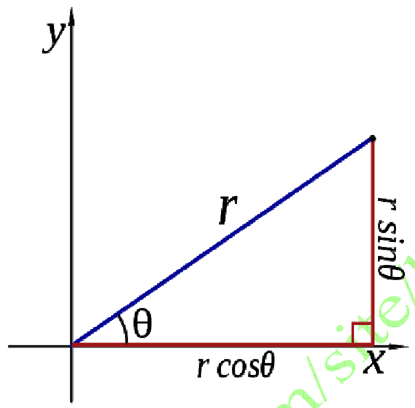
where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)]|J|dudv, \qquad (1)$$

where $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} (\neq 0)$ is the **Jacobian** of transformation from (x, y) to (u, v) co-ordinates.

Particular case:

CONVERSION OF CARTESION TO POLAR SYSTEM



A diagram illustrating the relationship between polar and Cartesian coordinates.

To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

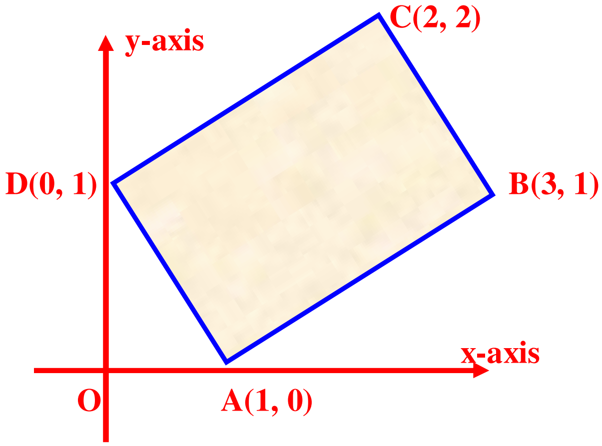
$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

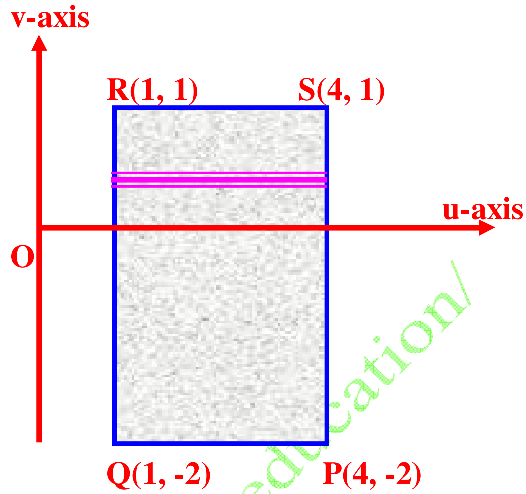
Then $\iint\limits_{R_{xy}} f(x, y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Q.No.1.: Evaluate $\iint\limits_{R} (x + y)^2 dxdy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

Sol.: The region R, i.e. parallelogram ABCD in the xy-plane becomes the region R', i.e. rectangle RSPQ in the uv-plane, as shown in the figure, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \tag{i}$$





From (i), we have $x = \frac{1}{3}(2u + v)$, $y = \frac{1}{3}(u - v)$.

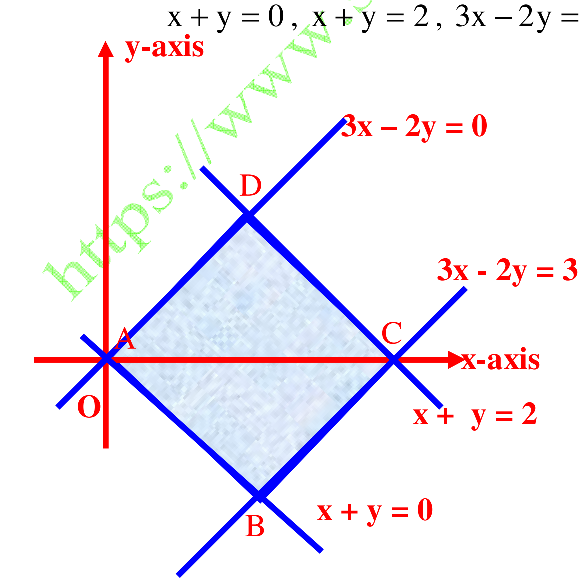
$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{3}{9} = -\frac{1}{3}.$$

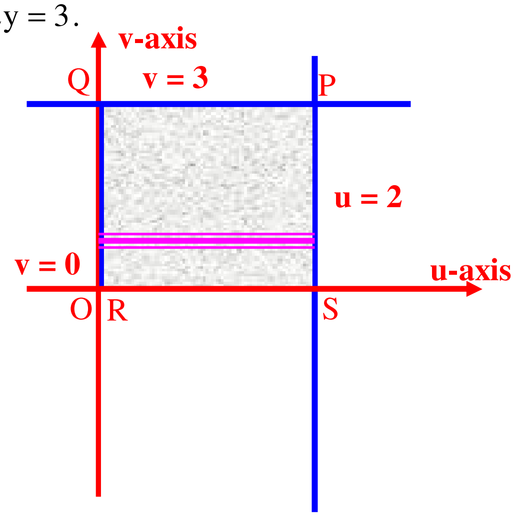
Thus $|J| = \frac{1}{3}$.

Hence, the given integral $= \iint_{R'} u^2 |J| du dv = \frac{1}{3} \int_{-2}^{1} \left(\int_{1}^{4} u^2 du \right) dv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 . |v|_{-2}^{1} = 21$. Ans.

Q.No.2.: Evaluate $\iint_R (x + y)^2 dx dy$, where R is the region bounded by parallelogram

$$x + y = 0,\ x + y = 2,\ 3x - 2y = 0,\ 3x - 2y = 3.$$





Sol.: By changing the variables x, y to the new variables u, v, by the substitution (transformation) $x+y=u$, $3x-2y=v$, then the region R, i. e. parallelogram ABCD in the xy-plane becomes the region R', i. e. rectangle RSPQ in the uv-plane, as shown in the figure, by taking $x+y=u$, $3x-2y=v$. (i)

From (i), we have $x=\frac{1}{5}(2u+v)$, $y=\frac{1}{5}(3u-v)$.

$$\therefore J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{2}{5} & \frac{1}{5}\\ \frac{3}{5} & -\frac{1}{5}\end{vmatrix}=-\frac{2}{25}-\frac{3}{25}=-\frac{5}{25}=-\frac{1}{5}.$$

Thus $|J|=\frac{1}{5}$.

Since, $u=x+y=0$ and $u=x+y=2$. Thus u varies from 0 to 2.

Also since $3x-2y=v=0$, $3x-2y=v=3$. Thus v varies from 0 to 3.

Thus the given integral in terms of new variables u, v is $\iint_R (x+y)^2 dxdy=\iint_{R'} u^2|J|dudv$

$$=\frac{1}{5}\int_0^3\left(\int_0^2 u^2du\right)dv=\frac{1}{5}\left|\frac{u^3}{3}\right|_0^2 .|v|_0^3=\frac{24}{15}=\frac{8}{5}.\text{ Ans.}$$

Q.No.3.: Evaluate $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy$ by changing to polar co-ordinates.

Hence show that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

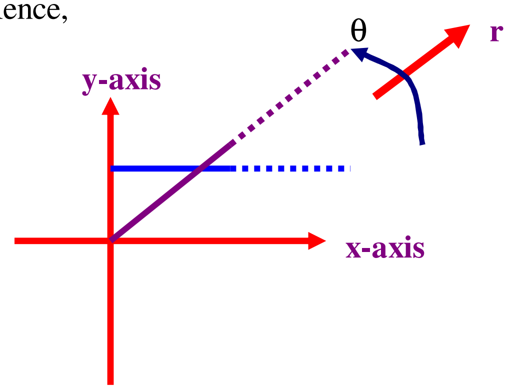
Sol.: To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x=r\cos\theta$, $y=r\sin\theta$ and

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left(\cos^2\theta+\sin^2\theta\right)=r.$$

Then $\iint_{R_{xy}} f(x,y)dxdy=\iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$. Hence,



$$I=\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}dxdy=\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}rdrd\theta=-\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{\infty} e^{-r^2}(-2r)dr\right\}d\theta$$

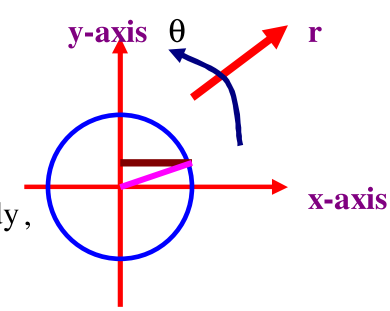
$$=-\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^{\infty}d\theta=\frac{1}{2}\int_0^{\pi/2}d\theta=\frac{\pi}{4}.\text{ Ans.} \qquad \text{(i)}$$

Also $I=\int_0^{\infty} e^{-x^2}dx\times\int_0^{\infty} e^{-y^2}dy=\left\{\int_0^{\infty} e^{-x^2}dx\right\}^2$, when $y=x$. (ii)

Thus, from (i) and (ii), we have $\int_0^{\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$. Ans.

Q.No.4.: Evaluate the integral by changing to polar co-ordinates

$$\int_0^a \int_0^{\sqrt{\left(a^2-y^2\right)}}\left(x^2+y^2\right)dxdy.$$



Sol.: We have to evaluate the integral $I=\int_0^a\left(\int_0^{\sqrt{\left(a^2-y^2\right)}}\left(x^2+y^2\right)dx\right)dy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x=r\cos\theta$,

$$y=r\sin\theta \text{ and } J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left(\cos^2\theta+\sin^2\theta\right)=r.$$

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

Also when $x = 0, r = 0;\quad x = \sqrt{a^2 - y^2}, r = a$

$y = 0,\ \theta = 0;\ y = a,\ \theta = \frac{\pi}{2}$

$$\therefore \int_0^a \left(\int_0^{\sqrt{(a^2-y^2)}} (x^2 + y^2)dx \right) dy = \int_0^{\pi/2} \left(\int_0^a r^2.rdr \right) d\theta = \int_0^{\pi/2} \left[\int_0^a r^3.dr \right] d\theta = \int_0^{\pi/2} \left| \frac{r^4}{4} \right|_0^a d\theta$$

$$= \int_0^{\pi/2} \left(\frac{a^4}{4} \right) d\theta = \left(\frac{a^4}{4} \right) \int_0^{\pi/2} d\theta = \frac{a^4}{4} \times \left(\frac{\pi}{2} - 0 \right) = \frac{\pi a^4}{8}. \text{ Ans.}$$

Q.No.5.: Evaluate the integral by changing to polar co-ordinates $\int_0^a \int_y^a \frac{x^2dxdy}{\sqrt{x^2 + y^2}}$.

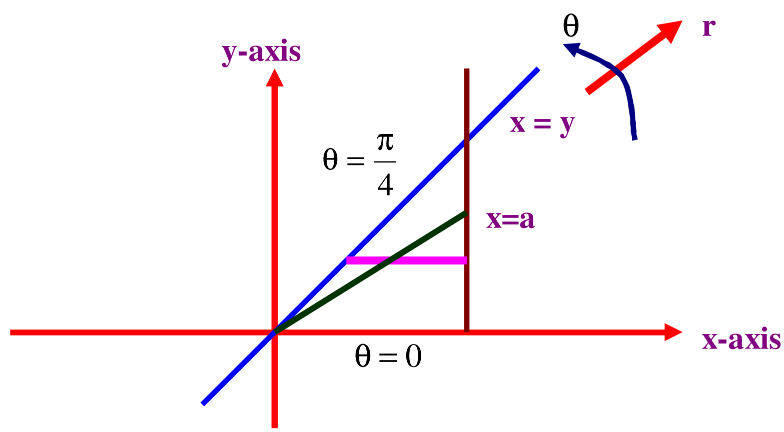
Sol.: We have to evaluate the integral $I = \int_0^a \int_y^a \frac{x^2dxdy}{\sqrt{x^2 + y^2}}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$



Also when x varies from y to a, r varies from 0 to $\frac{a}{\cos\theta}$, $\quad [\because x = r\cos\theta]$

And as y varies from 0 to a, θ varies from 0 to $\dfrac{\pi}{4}$.

$$\therefore \int_0^a\left(\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dx\right)dy = \int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{r^2\cos^2\theta}{r}rdrd\theta = \int_0^{\pi/4}\left[\int_0^{a/\cos\theta} r^2dr\right]\cos^2\theta d\theta$$

$$= \int_0^{\pi/4}\left|\frac{r^3}{3}\right|_0^{a/\cos\theta}\cos^2\theta d\theta = \int_0^{\pi/4}\left(\frac{1}{3}\frac{a^3}{\cos^3\theta}-0\right)\cos^2\theta d\theta$$

$$= \int_0^{\pi/4}\frac{1}{3}a^3\sec\theta d\theta = \frac{a^3}{3}\left[\log\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)-\log(1+0)\right]_0^{\pi/4} = \frac{a^3}{3}\log\left(1+\sqrt{2}\right). \text{ Ans.}$$

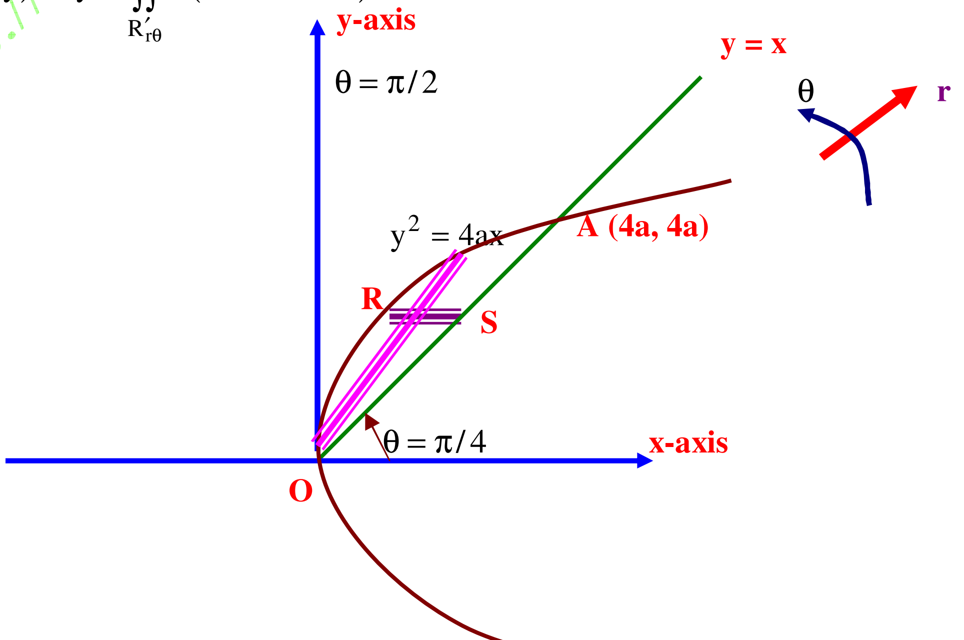
Q.No.6.: Evaluate the integral by changing to polar co-ordinates

$$\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}dxdy.$$

Sol.: We have to evaluate the integral $I = \int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}dxdy$, by changing Cartesian co-ordinates to Polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r$.

$$\therefore \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$



Now $\dfrac{x^2-y^2}{x^2+y^2}=\dfrac{r^2\cos^2\theta-r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}=\dfrac{r^2\cos 2\theta}{r^2}=\cos 2\theta$.

Since $y^2=4ax \Rightarrow r^2\sin^2\theta=4ar\cos\theta \Rightarrow r\left(r\sin^2\theta-4a\cos\theta\right)=0$

$\Rightarrow r=0$ and $r=\dfrac{4a\cos\theta}{\sin^2\theta}$. Thus

$$I=\int_{\pi/4}^{\pi/2}\left(\int_0^{4a\cos\theta/\sin^2\theta}\cos 2\theta r\,dr\right)d\theta=\int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta}\cos 2\theta\,d\theta$$

$$=8a^2\int_{\pi/4}^{\pi/2}\frac{\cos^2\theta}{\sin^4\theta}\left(\cos^2\theta-\sin^2\theta\right)d\theta=8a^2\int_{\pi/4}^{\pi/2}\left(\cot^4\theta-\cot^2\theta\right)d\theta$$

$$=8a^2\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\left(\operatorname{cosec}^2\theta-1\right)-\left(\operatorname{cosec}^2\theta-1\right)\right]d\theta$$

$$=8a^2\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\operatorname{cosec}^2\theta-\cot^2\theta-\left(\operatorname{cosec}^2\theta-1\right)\right]d\theta$$

$$=8a^2\left\{\left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}-\int_{\pi/4}^{\pi/2}\left(\operatorname{cosec}^2\theta-1\right)d\theta-\left[-\cot\theta-\theta\right]_{\pi/4}^{\pi/2}\right\}\left[\because \int[f(\theta)]^n f'(\theta)d\theta=\frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$=8a^2\left[\frac{-\cot^3\theta}{3}+\cot\theta+\theta+\cot\theta+\theta\right]_{\pi/4}^{\pi/2}$$

$$=8a^2\left\{\left[0+0+\frac{\pi}{2}+0+\frac{\pi}{2}\right]-\left[\frac{-1}{3}+1+\frac{\pi}{4}+1+\frac{\pi}{4}\right]\right\}$$

$$=8a^2\left[\pi-\left(-\frac{1}{3}+2+\frac{\pi}{2}\right)\right]=8\left[\frac{\pi}{2}-\frac{5}{3}\right]a^2.\ \text{Ans.}$$

Q.No.7.: Evaluate $\iint xy\left(x^2+y^2\right)^{n/2}dxdy$ over the positive quadrant of $x^2+y^2=4$ supposing $n+3>0$ by changing to polar co-ordinates.

Sol.: We have to evaluate the integral $I=\iint xy\left(x^2+y^2\right)^{n/2}dxdy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$y = r\sin\theta$ and $J = \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$.

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

Also in +ve quadrant of $x^2 + y^2 = 4$, r varies from 0 to 2 and θ varies from 0 to $\dfrac{\pi}{2}$.

$$I = \iint xy\left(x^2 + y^2\right)^{n/2}dxdy = \int_0^{\pi/2}\int_0^2 (r\cos\theta.\, r\sin\theta)r^n.r\,drd\theta$$

$$= \int_0^{\pi/2}\left[\int_0^2 r^{n+3}dr\right](\cos\theta\sin\theta)d\theta$$

$$= \int_0^{\pi/2}\left|\frac{r^{n+4}}{n+4}\right|_0^2 \cos\theta\sin\theta d\theta = \int_0^{\pi/2}\frac{2^{n+4}}{n+4}\cos\theta.\sin\theta d\theta = \frac{2^{n+4}}{n+4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

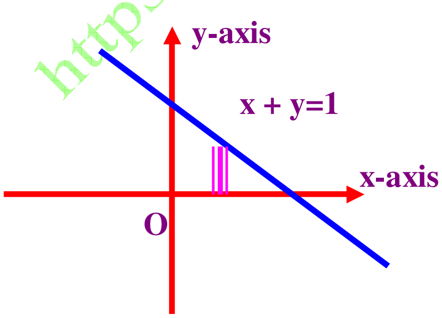
$$= \frac{2^{n+4}}{n+4}.\frac{1}{2} = \frac{2^{n+3}}{n+4}. \text{ Ans.}$$

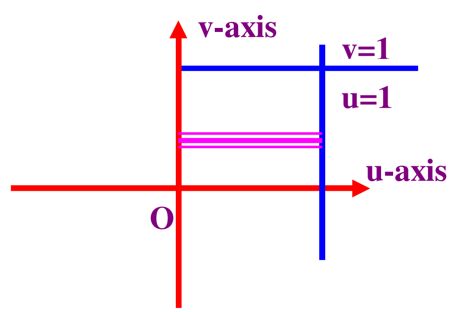
Q.No.8.: By using the transformation $x + y = u$, $y = uv$, show that

$$\int_0^1\left(\int_0^{1-x} e^{y/(x+y)}dy\right)dx = \frac{1}{2}(e-1).$$

Sol.: We have to evaluate $I = \displaystyle\int_0^1\left(\int_0^{1-x} e^{y/(x+y)}dy\right)dx$, by using the transformation

$x + y = u$, $y = uv$.





Since we have given $x+y=u$ and $y=uv \therefore x=u(1-v)$, $y=uv$ and

$$\therefore |J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$$

Since $x+y=u$ and $\frac{y}{u}=v \Rightarrow \frac{y}{x+y}=v$, then

When $y=0$, $v=0$; $y=1-x$, $u=1$,

When $x=0$, $v=1$; $x=1$, $u=\frac{y}{v}=0$. (because at $x=1$: $y=0$)

Thus u and v varies from 0 to 1.

$$\therefore\ I = \int_0^1 \left(\int_0^{1-x} e^{y/(x+y)} dy \right) dx = \int_0^1 \int_0^1 e^v u du dv = \int_0^1 \left[\int_0^1 u du \right] e^v dv$$

$$= \int_0^1 \left| \frac{u^2}{2} \right|_0^1 e^v dv = \frac{1}{2} \int_0^1 e^v dv = \frac{1}{2} \left| e^v \right|_0^1 = \frac{1}{2}(e-1). \text{ Ans.}$$

Q.No.9.: Show that $\iint \frac{dxdy}{4-x^2-y^2} = \pi \log 3$, over the region between the concentric circle $x^2+y^2=1$ and $x^2+y^2=3$ by changing to polar co-ordinates.

Sol.: We have to evaluate the integral $I = \iint \frac{dxdy}{4-x^2-y^2}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y) dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dr\, d\theta.$$

$$\text{Thus } I = \int_0^{2\pi} \left[\int_1^{\sqrt{3}} \frac{rdr}{4-r} \right] d\theta$$

Putting $4-r^2 = t \Rightarrow -2rdr = dt \Rightarrow dr = -\dfrac{dt}{2r}$.

$\therefore$ The limits changed from $\sqrt{3}$ to 1 and 1 to 3.

$$\therefore I = \int_0^{2\pi}\left[-\int_3^1 \frac{r}{t}\frac{dt}{2r}\right]d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\log t\right]_3^1 dt = \frac{1}{2}\int_0^{2\pi}\log 3 d\theta = \frac{1}{2}\log 3\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\log 3[\theta]_0^{2\pi} = \frac{1}{2}\log 3.2\pi = \pi\log 3$$, which is the required proof.

Q.No.10.: Evaluate $\iint_R \left[\dfrac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy$ by changing to polar co-ordinates, over the positive quadrant of the circle $x^2+y^2=1$ by changing to polar co-ordinates.

Sol.: The region for integration is bounded by the curves.

$x=0,\ x=1,\ y=0,\ y=1,$ and $x^2+y^2=1$.

We have to evaluate the integral $\iint_R \left[\dfrac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x,y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

In the integral, we have

$$\iint_R \left[\frac{1-x^2-y^2}{1+x^2+y^2}\right]^{1/2} dxdy = \iint_R \left[\frac{1-r^2\cos^2\theta - r^2\sin^2\theta}{1+r^2\cos^2\theta + r^2\sin^2\theta}\right] rdrd\theta$$

$\therefore$ Limits required are $r=0,\ r=1$ and $\theta=0,\ \theta=\dfrac{\pi}{2}$ (positive quadrant).

Thus, we need to evaluate

$$I=\int_0^{\pi/2}\int_0^1\left[\frac{1-r^2\left(\cos^2\theta+\sin^2\theta\right)}{1+r^2\left(\cos^2\theta+\sin^2\theta\right)}\right]rdrd\theta=\int_0^{\pi/2}\left[\int_0^1\left[\frac{1-r^2}{1+r^2}\right]rdr\right]d\theta.$$

Substitute $r^2=\cos\phi\Rightarrow 2rdr=-\sin\phi d\phi$.

$\therefore$ New limits are

at $r=0\Rightarrow\cos\phi=0\Rightarrow\phi=\frac{\pi}{2}$, $r=1\Rightarrow\cos\phi=1\Rightarrow\phi=0$, .

So the value to be integrated is

$$I=\int_0^{\pi/2}\left[\frac{1}{2}\int_{\pi/2}^0\left[\frac{1-\cos\phi}{1+\cos\phi}\right]^{1/2}(-\sin\phi)d\phi\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_{\pi/2}^0\left[\frac{2\sin^2\frac{\phi}{2}}{2\cos^2\frac{\phi}{2}}\right]^{1/2}\sin\phi d\phi\right]d\theta \qquad \left[\because -\int_b^a f(x)dx=\int_a^b f(x)dx\right]$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}\left(\tan\frac{\phi}{2}.2\sin\frac{\phi}{2}.\cos\frac{\phi}{2}\right)d\phi\right]d\theta=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}2\sin^2\frac{\phi}{2}d\phi\right]d\theta$$

$$=\int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos\phi)d\phi\right]d\theta=\int_0^{\pi/2}\left[\frac{1}{2}[\phi-\sin\phi]_0^{\pi/2}\right]d\theta=\int_0^{\pi/2}\frac{1}{2}\left(\frac{\pi}{2}-1\right)d\theta$$

$$=\frac{1}{2}\left(\frac{\pi-2}{2}\right)[\theta]_0^{\pi/2}=\frac{1}{2}\left(\frac{\pi-2}{2}\right)\frac{\pi}{2}=\frac{\pi}{8}(\pi-2).\text{ Ans.}$$

Q.No.11.: Evaluate $\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}e^{-\left(x^2-y^2\right)}dy\right)dx$ by changing to polar co-ordinates.

Sol.: Let $y=\sqrt{a^2-x^2}\Rightarrow y^2=a^2-x^2\Rightarrow x^2+y^2=a^2$ $\therefore$ The graph is a circle.

We have to evaluate the integral $\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}e^{-\left(x^2-y^2\right)}dy\right)dx$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

$$\therefore I = \int_0^{\pi/2}\int_0^a e^{-r^2} rdrd\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^a 2e^{-r^2} rdrd\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2}\right]_0^a d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-a^2} - 1\right]d\theta$$

$$= -\frac{1}{2}\left(e^{-a^2} - 1\right)\int_0^{\pi/2} d\theta = -\frac{1}{2}\left(e^{-a^2} - 1\right)[\theta]_0^{\pi/2} = -\frac{1}{2}\left(e^{-a^2} - 1\right)\frac{\pi}{2} = -\frac{\pi}{4}\left(e^{-a^2} - 1\right).$$

Q.No.12.: Evaluate $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$ by changing the polar co-ordinates.

Sol.: The region of integration is the area of integration is the area bounded by the curve, $x = a,\ y = 0,\ y = a$.

We have to evaluate the integral $\int_0^a \left(\int_y^a \frac{xd}{x^2 + y^2} dx\right)dy$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R_{xy}} f(x, y)dxdy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta.$$

$\therefore$ In polar co-ordinates area of integration is bounded by curves

$r\cos\theta = a \Rightarrow r = a\sec\theta$

$r\sin\theta = 0 \Rightarrow \theta = 0$ and $r = 0$

$r\sin\theta = r\cos\theta \Rightarrow \theta = \frac{\pi}{4}$.

$$I=\int_0^a\left(\int_y^a \frac{x}{x^2+y^2}dx\right)dy=\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r^2\cos\theta}{r^2}.dr.d\theta=\int_0^{\pi/4}\int_0^{a\sec\theta}\cos\theta.dr.d\theta$$

$$=\int_0^{\pi/4}\left[\cos\theta\int_0^{a\sec\theta}dr\right]d\theta=\int_0^{\pi/4}\left[\cos\theta[r]_0^{a\sec\theta}\right]d\theta=\int_0^{\pi/4}(\cos\theta\times\sec\theta)d\theta$$

$$=\int_0^{\pi/4}a d\theta=a[\theta]_0^{\pi/4}=\frac{\pi}{4}a\text{ . Ans.}$$

Q.No.13 .: Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{xdydx}{\sqrt{x^2+y^2}}$ by changing to polar co-ordinates.

Sol.: Let $I=\int_0^2\left(\int_0^{\sqrt{2x-x^2}}\frac{xd}{\sqrt{x^2+y^2}}dy\right)dx$

Taking $y=\sqrt{2x-x}$ (i)

$\Rightarrow (x-1)^2+y^2=1^2$ which is the equation of circle having its centre at (1,0) and a radius of 1 unit, we can draw the curve represented by the given integral as shown in Fig. 1.

Comparing the given integral I with the general form represented by $\int_{x_1}^{x_2}\left[\int_{y_1}^{y_2}f(x,y)\right]dx$,

we can conclude that the units of variable 'y' are given in terms of 'x' and hence the integration should be first carried and on dy, taking an elemental ship parallel to y-axis.

We have to evaluate the integral $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{xdydx}{\sqrt{x^2+y^2}}$,

by changing cartesian co-ordinates to polar co-ordinates.

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have $x=r\cos\theta$,

$$y=r\sin\theta \text{ and } J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left(\cos^2\theta+\sin^2\theta\right)=r\,.$$

$$\therefore \iint_{R_{xy}} f(x,y)dxdy=\iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta\,.$$

Using (i) and (ii), we write r in terms of θ,

$$r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$$

$$\therefore r^2\left(\sin^2 + \cos^2\theta\right) = 2r\cos\theta \Rightarrow r = 2\cos\theta. \qquad \text{(iii)}$$

Thus Fig (i) can be redrawn as in Fig. (ii)

Using (ii) and (iii), the integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dxdy \text{ can be changed to}$$

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} (r\cos\theta,\ r\sin\theta)J\left(\frac{x, y}{r, \theta}\right)drd\theta.$$

$$\therefore I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta.\, xr\, drd\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \int_0^{\pi/2}\left[\int_0^{2\cos\theta} r\cos\theta dr\right]d\theta$$

$$= \int_0^{\pi/2} 2\cos^3\theta d\theta \qquad \left[\begin{array}{l}\because \cos 3A = 4\cos^3 A - 3\cos A \\ \Rightarrow \cos^3 A = \dfrac{\cos 3A + 3\cos A}{A}\end{array}\right]$$

$$= \int_0^{\pi/2}\left(\cos 3\theta + 3\cos\theta\right)d\theta = \left[\frac{\sin 3\theta}{6} + \frac{3\sin\theta}{2}\right]_0^{\pi/2} = \frac{4}{3}. \text{ Ans.}$$

Q.No.14.: Transform the following to cartesian form and hence evaluate

$$\int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\, drd\theta.$$

Sol.: We have evaluate the integral $I = \int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\, drd\theta$,

by changing polar co-ordinates to Cartesian co-ordinates.

To change polar co-ordinates (r, θ) to Cartesian co-ordinates (x, y), we have $x = r\cos\theta$,

$$y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta = \iint_{R_{xy}} f(x, y)dxdy.$$

$$\therefore I = \int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta drd\theta = \int_0^{\pi}\int_0^{a} r\sin\theta.r\cos\theta.rdrd\theta = \int_0^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xydxdy$$

[Here we suppose that the strip is parallel to x-axis]

$$I = \int_0^a \left[\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdx\right]ydy = \int_0^a \left[\left|\frac{x^2}{2}\right|_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\right]ydy$$

$$= \frac{1}{2}\int_0^1 \left[\left(a^2 - y^2\right) - \left(a^2 - y^2\right)\right]ydy = \int_0^a 0dy = 0. \text{ Ans.}$$

Q.No.15.: Evaluate $\iint_D (y - x)dxdy$, where D is the region in xy-plane bounded by the straight lines $y = x + 1$, $y = x - x$, $y = -\frac{1}{3}x + \frac{7}{3}$, $y = -\frac{1}{3}x + 5$ using the transformation $u = y - x$ and $v = y + \frac{x}{3}$.

Sol.: Given transformations are $u = y - x$ and $v = y + \frac{x}{3}$

$$\Rightarrow -\frac{4}{3}x = u - v, \ \frac{4}{3}y = \frac{1}{3}u + v \Rightarrow y = \frac{3}{4}\left(\frac{u}{3} + v\right)$$

Here $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -3/4 & 1/4 \\ 3/4 & 3/4 \end{vmatrix} = -\frac{9}{12} - \frac{3}{16} = -\frac{12}{16} = -\frac{3}{4} \Rightarrow |J| = \frac{3}{4}$

As given, $y - x = 1$, $y - x = -3$

$u = -3$, $v = 1$

$-3 \le u \le 1$,

Again, $y + \frac{1}{3}x = \frac{7}{3}$, $y + \frac{1}{3}x = 5$

$v = \frac{7}{3}$, $v = 5$

$\frac{7}{3} \le v \le 5$

We will now integrate,

$$\iint_D (y - x)dydx = \int_{7/3}^{5}\int_{-3}^{1} u|J|dudv = \int_{7/3}^{5}\left[\frac{u^2}{2}\right]_{-3}^{1}\left(\frac{3}{4}\right)dv = \left(-\frac{8}{2}\times\frac{3}{4}\right)[v]_{7/3}^{5}$$

$$-\frac{8}{2}\times\frac{3}{4}\times\left(5-\frac{7}{3}\right)=-8\text{. Ans.}$$

Q.No.16.: Evaluate $\int\limits_0^1\int\limits_0^{1-x} e^{\frac{y}{(x+y)}}dydx$, using the transformation $x+y=u$ and $y=uv$.

Sol.: Given transformations are $x+y=u$ and $y=uv$.

Here $J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=u\Rightarrow|J|=u$.

$$\iint e^v ududv=\int\limits_0^1\left(\int\limits_0^1 e^v udu\right)dv=\int\limits_0^1\left[e^v\frac{u^2}{2}\right]_0^1 dv=\frac{1}{2}\int\limits_d^1 e^v dv=\frac{e-1}{2}\text{. Ans.}$$

Q.No.17.: Evaluate $\iint\limits_D\left[xy(1-x-y)\right]^{1/2}dxdy$, where D is the region in bounded by the Δ with sides $x=0$, $y=0$, $x+y=1$ using the transformation $u=x+y$ and $uv=y$.

Sol.: Given transformations are $x+y=u$ and $y=uv$.

$x+uv=u\Rightarrow x=u(1-v)$

$$\iint\limits_D u(1-v)uv(1-u)^{1/2}ududv=\int\limits_0^1\int\limits_0^1 u^2(1-v)(1-u)^{1/2}vdudv$$

$$=\int\limits_0^1\left[\int\limits_0^1\left(v-v^2\right)dv\right]u^2(1-u)^{1/2}du=\int\limits_0^1\left[\frac{v^2}{2}-\frac{v^3}{3}\right]_0^1 u^3(1-u)^{1/2}du$$

$$=\int\limits_0^1\left[\frac{1}{2}-\frac{1}{3}\right]u^3(1-u)^{1/2}du=\int\limits_0^1 u^3(1-u)^{1/2}\frac{1}{6}du=\frac{1}{6}\int\limits_0^1 u^3(1-u)^{1/2}du$$

Put $u=\sin^2\theta$, $u^3=\sin^6\theta$

$du=2\sin\theta\cos\theta d\theta$

$\theta_1=0$, $\theta_2=\frac{\pi}{2}$

$$=\frac{1}{6}\int\limits_0^{\pi/2}\sin^6\theta(\cos\theta)2\sin\theta\cos\theta d\theta=\frac{1}{3}\int\limits_0^{\pi/2}\sin^7\theta\cos^2\theta d\theta$$

$$=\frac{1}{3}\left[\frac{6.1}{9.7.5.3}\right]=\frac{1}{3}\left[\frac{6}{9\times 105}\right]=\frac{2}{945}\text{. Ans.}$$

Q.No.18.: Evaluate $\iint\limits_{R} (x-y)^4 e^{x+y} dxdy$, where R is the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1) using the transformation $x + y = u, \quad x - y = v$.

Sol.: Given transformations are $x + y = u, \quad x - y = v$.
Solving above two, we get

$$x = \frac{u+v}{2}, \quad y = \frac{v-v}{2}.$$

Now Jacobion $$\frac{J(x,y)}{(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} \quad -\frac{1}{4} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}.$$

Plotting graph taking x and y as coordinate axes

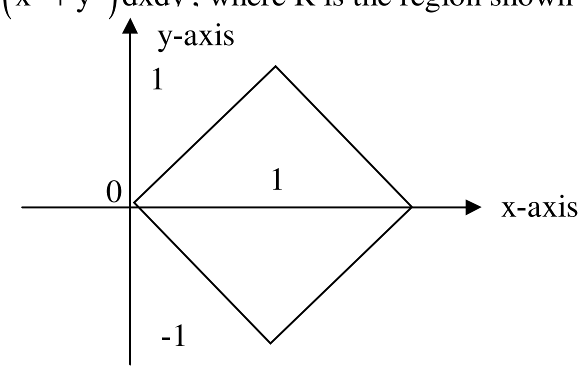
Now plotting graph taking u and v as axes.

(x,y)	(u,v)
1, 0	1, 1
2, 1	3, 1
1, 2	3, −1
0,1	1, −1

Now $$\iint\limits_{R(x,y)} (x-y)^4 e^{x+y} dxdy = \iint\limits_{R(u,v)} v^4 e^u |J| dudv = \int_1^3 \left(\int_{-1}^{1} v^4 e^u \times \frac{1}{2} \times dv \right) du$$

$$= \int_1^3 \left| \frac{v^5}{5} \right|_{-1}^{1} \times \frac{1}{2} e^u du = \left(\frac{1}{5} + \frac{1}{5} \right) \times \frac{1}{2} \int_1^3 e^u du = \frac{2}{5} \times \frac{1}{2} \left| e^u \right|_1^3 = \left(\frac{e^3 - e}{5} \right).$$ Ans.

Q.No.19.: Evaluate $\iint\limits_{R} (x^2 + y^2) dxdv$, where R is the region shown in figure



Sol.: Point in (x, y) coordinates (0, 0), (1, 1), (2, 0), $(-1,1)$

Let $x + y = u$ and $x - y = v$

So point in $u-v$ coordinates (0, 0), (2, 0), (2, 2), (0, 2)

$$x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2.$$

so $|J| = 1/2$

$$\text{Now} \iint_R \left(x^2+y^2\right)dxdy = \iint_R \frac{(x+y)^2+(x-y)^2}{2}dxdy = \frac{1}{2}\int_0^2\int_0^2 \left(u^2+v^2\right)|J|\,dudv$$

$$= \frac{1}{4}\int_0^2\left(\int_0^2 \left(u^2+v^2\right)du\right)dv = \frac{1}{4}\int_0^2\left[\frac{u^3}{3}+uv^2\right]_0^2 dv = \frac{1}{4}\int_0^2\left(\frac{8}{3}+2v^2\right)dv = \frac{1}{4}\left[\frac{8}{3}v+\frac{2v^3}{3}\right]_0^2$$

$$= \frac{1}{4}\times\frac{4\times 8}{3} = \frac{8}{3}. \text{ Ans.}$$

Q.No.20.: Evaluate $\int_0^e \left(\int_{\alpha x}^{\beta x} f(x,y)dy\right)dx$, using the transformation $x = u - uv$ and $y = uv$

Sol.:. Consider $x = u - uv$, $y = uv$

Since from the given integral, we have

$x = 0$, $x = e$ and $y = \alpha x$, $y = \beta x$.

Substituting the values of x, y we get the values of u, v as

$$0 = u(1-v) \Rightarrow u = 0$$

$$e = u(1-v) \Rightarrow u = \frac{e}{1-v}$$

$$\text{Now } \alpha x = uv \Rightarrow \alpha\left(u-uv\right) = uv \Rightarrow v = \frac{\beta}{1+\beta}$$

$$\text{Finally, } \beta x = uv \Rightarrow u\alpha\left(1-v\right) = uv \Rightarrow \alpha - \alpha v = v \Rightarrow v = \frac{\alpha}{1+\alpha}$$

Also Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -v \\ v & u \end{vmatrix} = u - uv + uv = u$

$$\therefore \int_0^c \int_{\alpha x}^{\beta x} f(x,y)\,dydx = \int_{\alpha/1+\alpha}^{\beta/1+\beta} \int_0^{e/1-v} f(u-uv, uv)\,ududv.$$

Home Assignments

Q.No.1.: Evaluate $\iint_R xydxdy$, where R is the region in the first quadrant bounded by the hyperbola $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ and the circle $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$, $0 < a < b < c < d$, using the transformation $x^2 - y^2 = u$ and $x^2 + y^2 = v$.

Hint: Put $x^2 - y^2 = u$, $x^2 + y^2 = v$, $J = 8xy$

R* : rectangle $a^2 \le u \le b^2$, $c^2 \le v \le d^2$

Answer: $\frac{1}{8}\left(b^2 - a^2\right)\left(d^2 - c^2\right)$

Q.No.2.: Evaluate $\iint_D e^{(x-y)/(x+y)}dxdy$, D is the triangle bounded by x = 1, x = 1, y = x, using the transformation $x = v - uv$ and $y = uv$.

Hint: Use $x = v - uv$ and $y = uv$ to transform the double integral

Answer: $\dfrac{e^2 - 1}{4e}$

Q.No.3.: Evaluate $\int_0^c \int_0^b f(x,y)\,dydx$, using the transformation $x = u - uv$ and $y = uv$

Answer: $\int_0^c \int_0^b f(x,y)\,dydx = \int_0^{\frac{b}{b+c}} \int_0^{\frac{c}{1-v}} f(u-uv, uv)\,ududv + \int_{\frac{b}{b+c}}^{b} \int_0^{\frac{b}{v}} f(u-uv, uv)\,ududv.$

Q.No.4.: Evaluate $\int_0^\infty \int_0^\infty \dfrac{x^2 + y^2}{1 + \left(x^2 - y^2\right)^2} e^{-2xy}dxdy$, using the transformation $u = x^2 - y^2$, and $v = 2xy$.

Answer: $\int_0^\infty \int_0^\infty \frac{x^2+y^2}{1+\left(x^2-y^2\right)^2} e^{-2xy} dxdy = \frac{\pi}{4}$.

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