2^{nd} Topic

Fourier Series

Conditions for a Fourier Expansion

(Dirichlet's conditions)

Prepared by:
Prof. Sunil
Department of Mathematics
NIT Hamirpur (HP)

Dirichlet's conditions:

Dirichlet conditions are sufficient conditions for a real-valued, periodic function f(x) to be equal the sum of its Fourier series at each point where f is continuous. Moreover, the behavior of the Fourier series at points of discontinuity is determined as well. These conditions are named after Johann Peter Gustav Lejeune Dirichlet.

Johann Peter Gustav Lejeune Dirichlet

13-02-1805 to 05-05-1859

The conditions are:

- f(x) must have a finite number of extrema in any given interval
- f(x) must have a finite number of discontinuities in any given interval
- f(x) must be absolutely integrable over a period.
- f(x) must be bounded

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions. All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function f(x) can be developed as a Fourier series

$$\boxed{\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx},$$

where $a_0,\ a_n,\ b_n$ **are constants, provided the given function have satisfied the following conditions, which are known as Dirichlet's conditions.**

(i) f (x) is a periodic, single-valued and finite;

(ii) f (x) has a finite number of discontinuities in any one period;

(iii) f (x) has at the most a finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left

i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$,

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectively.

In fact the problem of expressing any function f(x) as a Fourier series depends upon evaluation of the integrals

$$\frac{1}{\pi}\int f(x)\cos nx dx\ ;\quad \frac{1}{\pi}\int f(x)\sin nx dx\ ,$$

within the limits $(0,\ 2\pi)$, $(-\pi,\ \pi)$ or $(\alpha,\ \alpha+2\pi)$ as according as f (x) is defined for every value of x in $(0,\ 2\pi)$, $(-\pi,\ \pi)$ or $(\alpha,\ \alpha+2\pi)$.

Now let us examine whether the following functions can be expended in Fourier series in the given interval.

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

Q.No.1: cosec x.

Sol.: Now this function has **infinite values** at $x = -\pi,\ 0,\ +\pi$, so we **cannot** develop a Fourier series of this function within this interval.

In this case Dirichlet's conditions no. 1, i.e., f (x) is finite is not satisfied.

Q.No.2: $\sin\frac{1}{x}$.

Sol.: Since this function is **not single valued** at $x = 0$ so we **cannot** develop a Fourier series of this function within this interval.

In this case Dirichlet's conditions no. 1, i.e., f (x) is single valued is not satisfied.

Q.No.3: $f(x) = \frac{(m+1)}{m},\ \frac{\pi}{m+1} < |x| \le \frac{\pi}{m},\ m = 1, 2, 3, \ldots\ldots\ldots\ldots\infty$.

Sol.: **Yes**, we can develop a Fourier series of this function because it **satisfies all** the Dirichlet's conditions, i.e.,

(i) f (x) is a periodic, single-valued and finite;

(ii) f (x) has a finite number of discontinuities in any one period;

(iii) f (x) has at the most a finite number of maxima and minima.

Q.No.4.: Is it possible to write the Fourier sine series for the function $f(x) = \cos x$, over the interval $(-\ell, \ell)$?

Sol.: For half range sine series f(x) must be defined in the interval $(0, \ell)$.

Hence, we cannot develop the Fourier half range sine series for $f(x) = \cos x$, over the interval $(-\ell, \ell)$.

****

****

****