4th Topic

Matrices

Vectors and Linear Dependence

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Vectors:

Any quantity having n-components is called a vector of order n.

Therefore, the coefficients in linear equation or the elements in a row matrix or column matrix will form a vector. Thus, any n numbers $x_1, x_2, \ldots\ldots\ldots, x_n$ written in a particular order, constitute a vector x.

Linear dependent vectors:

The vectors $x_1, x_2, \ldots\ldots\ldots, x_n$ are said to be linearly dependent, if there exist n numbers $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ not all zero, such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots\ldots + \lambda_n x_n = 0. \tag{i}$$

Linear independent vectors:

If no such numbers, other than zero, exist, then the vectors are said to be linearly independent.

Now let us suppose vectors $x_1, x_2, \ldots\ldots\ldots, x_n$ are said to be linearly dependent, then there exist r numbers $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ not all zero.

Let us suppose $\lambda_1 \neq 0$, then we write (i) in the form

$$x_1 = -\frac{1}{\lambda_1}[\lambda_2 x_2 + \lambda_3 x_3 + \ldots\ldots\ldots\ldots\ldots + \lambda_n x_n]$$

$$\Rightarrow x_1 = \mu_2 x_2 + \mu_3 x_3 + \ldots\ldots\ldots\ldots\ldots + \mu_n x_n.$$

This means that the vector x_1 is said to be a linear combination of the vectors $x_2, \ldots\ldots\ldots, x_n$.

Now let us solve some problems:

Q.No.1.: Are the vectors $x_1 = (1, 3, 4, 2)$, $x_2 = (3,\ -5,\ 2,\ 2)$ and $x_3 = (2, -1, 3, 2)$ linearly dependent? If so express one of these as a linear combination of the others.

Sol.: Since we know that, the vectors x_1, x_2, x_3 are said to be L.D., if $\exists$ numbers $\lambda_1, \lambda_2, \lambda_3$ not all zero s.t. $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

The relation $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

$\Rightarrow \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = 0$,

$\Rightarrow \lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$, $3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$, $4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$, $2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$.

$$\approx \begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_3 \to 7R_3 - 5R_2$, $R_4 \to 5R_4 - 2R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to \left(-\frac{1}{7}\right) R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$,

$\quad 2\lambda_2 + \lambda_3 = 0$.

$\Rightarrow \lambda_3 = -2\lambda_2$ and $\lambda_1 = \lambda_2$

$\Rightarrow \lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3$.

Now these are satisfied by the values $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = -2$, which are not zero. Thus, the given vectors are linearly dependent.

Relation: Substituting these values in (i), we get $x_1 + x_2 - 2x_3 = 0$,

$\Rightarrow$ Any of the given vectors can be expressed as a linear combination of the others.

e.g. $x_1 = 2x_3 - x_2$.

Thus x_1 is a linear combination of x_2 and x_3

Remarks:

Applying elementary row operations to the vectors x_1, x_2, x_3, we see that the matrices

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 - 2x_3 \end{bmatrix},$$

have the same rank. The rank of B being 2, the rank of A is also 2.

Moreover, x_1, x_2 are linearly independent and x_3 can be expressed as a linear combination of x_1 and x_2 $\left[\because x_3 = \frac{1}{2}(x_1 + x_2)\right]$.

Similar results will hold for column operations and for any matrix. In general, we have the following results:

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors and remaining vectors (if any) can be expressed as a linear combination of these vectors.

Q.No.2: Are the following vectors linearly dependent. If so, find the relation between them:

(i) $(3, 2, 7)$, $(2, 4, 1)$, $(1, -2, 6)$,

(ii) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9),

(iii) $x_1 = (1, 2, 4)$, $x_2(2, -1, 3)$, $x_3 = (0, 1, 2)$, $x_4 = (-3, 7, 2)$.

Sol.: (i). Let $x_1 = (3, 2, 7)$, $x_2 = (2, 4, 1)$, $x_3 = (1, -2, 6)$.

Then $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$. (i)

$\Rightarrow \lambda_1(3, 2, 7) + \lambda_2(2, 4, 1) + \lambda_3(1, -2, 6) = 0$,

which is equivalent to

$3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$, $2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0$, $7\lambda_1 + \lambda_2 + 6\lambda_3 = 0$.

$$\Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & -2 \\ 7 & 1 & 6 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{7}{3}R_1$, we get $\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{8}{3} & -\frac{8}{3} \\ 0 & -\frac{11}{3} & \frac{11}{3} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_3 \to R_3 + \frac{11}{8}R_2$, we get $\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{8}{3} & -\frac{8}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow 3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ and $\frac{8}{3}\lambda_2 - \frac{8}{3}\lambda_3 = 0 \Rightarrow \lambda_2 = \lambda_3$.

Thus $3\lambda_1 + 3\lambda_3 = 0 \Rightarrow \lambda_1 = -\lambda_3$.

$\therefore \lambda_1 = -\lambda_2 = -\lambda_3$.

Putting $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -1$, which are not zero.

Thus, the given vectors are linearly dependent.

Relation: Putting these values in (i), we get $x_1 - x_2 - x_3 = 0$.

Hence $x_1 = x_2 + x_3$.

Hence, x_1 can be expressed in terms of x_2 and x_3.

(ii). Let $x_1 = (1, 1, 1, 3)$, $x_2 = (1, 2, 3, 4)$, $x_3 = (2, 3, 4, 9)$

Then $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$. (i)

$\lambda_1(1, 1, 1, 3) + \lambda_2(1, 2, 3, 4) + \lambda_3(2, 3, 4, 9) = 0$

which is equivalent to

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$,

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$,

$\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0$,

$3\lambda_1 + 4\lambda_2 + 9\lambda_3 = 0$.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1 + \lambda_2 + 2\lambda_3 = 0$, $\lambda_2 + \lambda_3 = 0$ and $2\lambda_3 = 0$.

$\Rightarrow \lambda_3 = 0$, $\lambda_2 = 0$ and $\lambda_1 = 0$.

Thus λ_1, λ_2 and λ_3 have value equal to 0.

Hence, it is linearly independent.

Relation: Since vectors are linearly independent, so there is no relation between them.

(iii). $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$, $x_4 = (-3, 7, 2)$.

Then $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$. (i)

$\Rightarrow \lambda_1(1, 2, 4) + \lambda_2(2, -1, 4) + \lambda_3(0, 1, 2) + \lambda_4(-3, 7, 2) = 0$,

which is equivalent to

$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$,

$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$,

$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\Rightarrow \lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$, $-5\lambda_2 + \lambda_3 - 13\lambda_4 = 0$, $\lambda_3 + \lambda_4 = 0$.

By solving $\lambda_1 = \frac{9}{5}\lambda_4$, $\lambda_2 = -\frac{12}{5}\lambda_4$, $\lambda_3 = -\lambda_4$.

Putting $\lambda_4 = 1$, we get $\lambda_1 = \frac{9}{5}$, $\lambda_2 = -\frac{12}{5}$, $\lambda_3 = -1$.

Thus, the given vectors are linearly dependent.

Relation: Now $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$$\Rightarrow \frac{9}{5}x_1 + \left(-\frac{12}{5}\right)x_2 + (-1)x_3 + (1)x_4 = 0$$

$$\Rightarrow 9x_1 - 12x_2 - 5x_3 + 5_4 = 0,$$

which is the required relation between these vectors.

Home Assignments

Q.No.1.: Are the following vectors linearly dependent? If so, find a relation between them.

(i). $x_1 = (1,3,2)$, $x_2 = (5,-2,1)$, $x_3 = (-7,13,4)$

(ii). $x_1 = (1,-1,3,2)$, $x_2 = (1,3,4,2)$, $x_3 = (3,-5,2,2)$

(iii). $x_1 = (2,3,1,-1)$, $x_2 = (2,3,1,-2)$, $x_3 = (4,6,2,1)$.

Ans.: (i). Yes, Relation: $3x_1 - 2x_2 - x_3 = 0$.

(ii). Yes, Relation: $2x_1 - x_2 - x_3 = 0$.

(iii). Yes, Relation: $5x_1 - 3x_2 - x_3 = 0$.

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