6th Topic

Linear and Orthogonal Transformations

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Linear transformations:

Let (x, y) be co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ are given by

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\}. \qquad \text{(i)}$$

A more general transformation than (i) is

$$\left.\begin{aligned} x' &= a_1 x + b_1 y, \\ y' &= a_2 x + b_2 y \end{aligned}\right\}, \qquad \text{(ii)}$$

which in matrix notation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Such transformations as (i) and (ii), are called **linear transformations in two dimensions**.

$$\left.\begin{aligned} x' &= \ell_1 x + m_1 y + n_1 z \\ y' &= \ell_2 x + m_2 y + n_2 z \\ z' &= \ell_3 x + m_3 y + n_3 z \end{aligned}\right\} \qquad \text{(iii)}$$

give a linear transformation from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation Y=AX, where

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix},\quad A=\begin{bmatrix} a_1 & b_1 & c_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & k_n \end{bmatrix},\quad X=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \qquad \text{(iv)}$$

give linear transformation from n variables $x_1, x_2, \ldots\ldots, x_n$ to the variables $y_1, y_2, \ldots\ldots, y_n$, i.e., the transformation of the vector X to the vector Y

This transformation is called **linear** because the linear relations

(i) $A(X_1 + X_2) = AX_1 + AX_2$ and

(ii) $A(bX) = bAX$, hold for this transformation.

Singular and non-singular transformation:

If the transformation matrix A is singular, then the transformation is said to be singular, otherwise non-singular.

For a non-singular transformation Y = AX, we can also write the inverse transformation $X = A^{-1}Y$. A non-singular transformation is also called a regular transformation.

Remarks: If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by Y = AX and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by Z = BY, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by

Z = BY = B(AX) = (BA)X.

Orthogonal transformation:

The linear transformation Y = AX, where

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix},\quad A=\begin{bmatrix} a_1 & b_1 & c_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & k_n \end{bmatrix},\quad X=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

is said to be orthogonal if it transforms

$y_1^2 + y_2^2 + \ldots\ldots + y_n^2$ into $x_1^2 + x_2^2 + \ldots\ldots + x_n^2$.

The matrix A of this orthogonal transformation is called an **orthogonal matrix**.

Now $X'X = [x_1 x_2 \ldots\ldots x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$.

and similarly $Y'Y = y_1^2 + y_2^2 + \ldots\ldots + y_n^2$.

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$X'X = x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2 = Y'Y$

$= (AX)'(AX) = (X'A')(AX) = X'(A'A)X$, which is possible only if $A'A = I$.

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence, a square matrix A is said to be orthogonal if $AA' = A'A = I$.

Result 1.: If A is orthogonal, then show that A' and A^{-1} are also orthogonal.

Proof: A is orthogonal $\Rightarrow AA' = I$.

$\Rightarrow (A'A)' = I' \Rightarrow A'(A')' = I$

$\Rightarrow A'$ is orthogonal.

Again, A is orthogonal $\Rightarrow A'A = I \Rightarrow (A'A)^{-1} = I^{-1}$

$\Rightarrow A^{-1}(A')^{-1} = I \Rightarrow A^{-1}(A^{-1})' = I$ $\quad [(A')^{-1} = (A^{-1})']$

$\Rightarrow A^{-1}$ is orthogonal.

Result 2.: If A and B are orthogonal matrices, then prove that AB is also orthogonal.

Proof: Let A and B are both n-rowed square matrices, therefore AB is also n-rowed square matrix.

Since $|AB| = |A||B|$ and $|A| \neq 0$, also $|B| \neq 0$.

$\therefore |AB| \neq 0$.

Hence, AB is non-singular matrix.

Now $(AB)' = B'A'$.

$\therefore (AB)'(AB) = (B'A')(AB) = B'(A'A)B$

$= B'IB$ $\quad [\because A'A = I]$

$= B'B = I$ $\quad [\because B'B = I]$

Hence, AB is also an orthogonal matrix.

Result 3.: If A is orthogonal, then show that $|A| = \pm 1$.

Proof: If A is orthogonal matrix, then $AA' = 1$

$\Rightarrow |A||A'| = |I| \Rightarrow |A|.|A'| = I$ $\quad [\because \det(AB) = (\det A).(\det B)]$

$\Rightarrow |A|.|A| = I,$ $\quad [\because |A'| = |A|]$

$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$

$|A| \neq 0 \Rightarrow A$ is invertible.

Also then $A'A = I \Rightarrow A' = A^{-1}$.

Now, let us understand these transformations with the help of these problems:

Q.No.1.: Show that the transformation

$y_1 = 2x_1 + x_2 + x_3, \quad y_2 = x_1 + x_2 + 2x_3, \quad y_3 = x_1 - 2x_3$ is regular.

Also, write down the inverse transformation.

Sol.: In matrix notation, the given transformation is Y = AX, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}.$$

Now $|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = 2(-2-0) - 1(-2-2) + 1(0-1) = -4 + 4 - 1 = -1 \neq 0$.

Thus, the matrix A is non-singular and hence the given transformation is non-singular or regular.

$\therefore$ The inverse transformation is given by $X = A^{-1}Y$,

where $A^{-1} = \dfrac{\operatorname{adj} A}{|A|} = \dfrac{1}{(-1)} \begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$.

$$\therefore X = A^{-1}Y \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$\Rightarrow x_1 = 2y_1 - 2y_2 - y_3; \; x_2 = -4y_1 + 5y_2 + 3y_3: \; x_3 = y_1 - y_2 - y_3,$

which is the required inverse transformation.

Q.No.2.: Represent each of the transformations

$x_1 = 3y_1 + 2y_2$, $y_1 = z_1 + 2z_2$, $x_2 = -y_1 + 4y_2$, $y_2 = 3z_1$,

by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

Sol.: A transformation from the variable x_1, x_2 to y_1, y_2 can be represented by

$$X = A_1Y, \text{ where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A_1 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Second transformation from the variable y_1, y_2 to x_1, x_2 can be represented by

$$Y = A_2Z, \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \ Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Given $x_1 = 3y_1 + 2y_2 = 3(z_1 + 2z_2) + 2(3z_1) = 3z_1 + 6z_2 + 6z_1 = 9z_1 + 6z_2$.

and $x_2 = -y_1 + 4y_2 = -(z_1 + 2z_2) + 4(3z_1) = -z_1 + -2z_2 + 12z_1 = 11z_1 - 2z_2$.

The composite transformation, which expresses x_1, x_2 in terms of z_1, z_2 by the use of matrices is $X = AZ$.

$$\text{where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}, \ z_2 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Q.No.3.: If $\xi = x\cos\alpha - y\sin\alpha$, $\eta = x\sin\alpha + y\cos\alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$.

Hence write the inverse transformation.

Sol.: Let the transformed matrix of the equations

$\xi = x\cos\alpha - y\sin\alpha$ and $\eta = x\sin\alpha + y\cos\alpha$ is A.

$$\therefore A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Thus, the given transformation can be written as $Y = AX$,

$$\text{where } Y = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \ A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now $|A| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$.

Thus, the given transformation matrix A is non-singular and hence the transformation is non-singular or regular.

$\therefore$ The inverse transformation is given by $X = A^{-1}Y$

Now $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

Thus, the inverse transformation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$.

Thus $x = \xi\cos\alpha + \eta\sin\alpha$,

$y = \xi(-\sin\alpha) + \eta\cos\alpha$

is the inverse transformation of the given transformation.

Q.No.4.: A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

Sol.: Given two transformation $Y = AX$ and $Z = BY$.

Now $Z = BY = B(AX) \Rightarrow Z = (BA)X$.

We have $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$.

$$\text{Now } BA = \begin{bmatrix} 2+0-1 & 1+1+2 & 0-2+1 \\ 2+0-3 & 1+2+6 & 0-4+3 \\ 2+0-5 & 1+3+10 & 0-6+5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}.$$

Now since $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\therefore Z = (BA)X \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 - x_3 \\ -x_1 + 9x_2 - x_3 \\ -3x_1 + 14x_2 - x_3 \end{bmatrix}.$$

$$\Rightarrow z_1 = x_1 + 4x_2 - x_3,$$

$$z_2 = -x_1 + 9x_2 - x_3,$$

$$z_3 = -3x_1 + 14x_2 - x_3,$$

which is the required transformation.

Q.No.5.: Verify that the following matrix is orthogonal:

$$\textbf{(i)} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \textbf{(ii)} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = A, \ A = I$.

(i). Here $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

$$\therefore AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, the given matrix A is orthogonal matrix.

(ii). Here $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$.

$$\therefore AA' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} . \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos^2\theta+\sin^2\theta & 0 & -\cos\theta\sin\theta+\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & 0 & \sin^2\theta+\cos^2\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I.$$

Hence the given matrix A is orthogonal matrix.

Q.No.6.: Prove that the following matrix is orthogonal:

$$\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

Sol.: Now, since we know that a matrix is said to be orthogonal if $AA' = A'A = I$.

Here $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.

Now $AA' = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}\times\frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$

$$=\frac{1}{9}\begin{bmatrix} 4+1+4 & -4+2+2 & -2-2+4 \\ -4+2+2 & 4+4+1 & 2-4+2 \\ -2-2+4 & 2-4+2 & 1+4+4 \end{bmatrix}=\frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I.$$

Hence, the given matrix is orthogonal.

Q.No.7.: Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal.

Sol.: Here $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Now $AA^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I$$

i.e. $A^T = A^{-1}$ $\therefore$ A is orthogonal.

Q.No.8.: Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ orthogonal? If not, can it be converted into orthogonal matrix?

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = A'A = I$.

Here $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$. $\therefore A' = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$.

$$\therefore AA' = \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I.$$

Hence, the given matrix is not orthogonal.

It can be converted into orthogonal matrix.

It means first row is divided by $\sqrt{2^2 + (-3)^2 + (1)^2} = \sqrt{14}$,

Second row is divided by $\sqrt{4^2 + (3)^2 + (1)^2} = \sqrt{26}$,

Third row is divided by $\sqrt{(-3)^2 + (1)^2 + (9)^2} = \sqrt{91}$

Hence, the orthogonal matrix is $\begin{bmatrix} \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{-3}{\sqrt{91}} & \frac{1}{\sqrt{91}} & \frac{9}{\sqrt{91}} \end{bmatrix}$.

Q.No.9.: Prove that $\begin{bmatrix} \ell & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & \ell & -m & 0 \\ -m & n & -\ell & 0 \end{bmatrix}$ is orthogonal when $\ell = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.

Sol.: Since, we know that a matrix is said to be orthogonal if $AA' = I$.

Now $AA' = \begin{bmatrix} \ell & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & \ell & -m & 0 \\ -m & n & -\ell & 0 \end{bmatrix} \begin{bmatrix} \ell & 0 & n & -m \\ m & 0 & \ell & n \\ n & 0 & -m & -\ell \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} \ell^2 + m^2 + n^2 + 0 & 0 & \ell n + m\ell - nm & -\ell m + mn - n\ell \\ 0 & 1 & 0 & 0 \\ n\ell + \ell m - mn & 0 & n^2 + \ell^2 + m^2 & -nm + \ell n + m\ell \\ -m\ell + nm - \ell n & 0 & -mn + n\ell + \ell m & m^2 + n^2 + \ell^2 \end{bmatrix}.$$

Putting the values of ℓ, m and n, we get $AA' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$.

Hence, the given matrix is orthogonal, if $\ell = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.

Q.No.10.: Determine a, b, c so that A is orthogonal, where $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$.

Sol.: Here $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$.

For orthogonal matrix, we have $AA^T = I$. Therefore

$$AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I$$

Solving $2b^2 - c^2 = 0$, $a^2 - b^2 - c^2 = 0$ (non-diagonal elements of I)

$c = \pm\sqrt{2}b$, $a^2 = b^2 + c^2 = b^2 + 2b^2 = 3b^2$, $a = \pm\sqrt{3}b$

From diagonal elements of I, we have

$4b^2 + c^2 = 1$, $4b^2 + 2b^2 = 1$.

$\therefore b = \pm\frac{1}{\sqrt{6}}$, $c = \pm\frac{1}{\sqrt{3}}$, $a = \pm\frac{1}{\sqrt{2}}$.

Q.No.11: Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$, $y_2 = -x_2 + 2x_3$,

$y_3 = 2x_1 + 4x_2 + 11x_3$.

Sol.: Let $Y = [y_1 \quad y_2 \quad y_3]^T$ and $X = [x_1 \quad x_2 \quad x_3]^T$.

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$. Here $|A| = -1$.

$$\text{Adj } A = \begin{bmatrix} -19 & -2 & 9 \\ 4 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

Thus, the inverse transformation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}Y = \frac{\text{adj}A}{|A|}Y = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 19y_1 + 2y_2 - 9y_3 \\ -4y_1 - y_2 + 2y_3 \\ -2y_1 + y_3 \end{bmatrix}.$$

Home Assignments

Q.No.1.: Show that the transformation $y_1 = x_1 - x_2 + x_3$, $y_2 = 3x_1 - x_2 + 2x_3$,

$y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular.

Also find the inverse transformation.

Ans.: $x_1 = \frac{1}{2}(y_1 + y_2 - y_3), x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3),\ x_3 = -2y_1 + y_3$.

Q.No.2.: Which of the following matrices is orthogonal?

$$\textbf{(i).}\ \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix},\ \textbf{(ii).}\ \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}.$$

Ans.: **(i).** Orthogonal, **(ii).** Not orthogonal.

Q.No.3.: Verify that the following matrix is orthogonal: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Q.No.4.: Verify that the following matrix is orthogonal: $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

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