8^{th} Topic

Double Integrals

[Volumes of solids of revolution]

(Cartesian and Polar co-ordinates)

(Last updated on 15-07-2013)

(02 Solved problems and 02 Home assignments)

Volumes of solids of revolution:

Cartesian co-ordinates:

Consider an elementary area $\delta x\ \delta y$ at the point P(x, y) of a plane area A.

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi\left[(y+\delta y)^2 - y^2\right]\delta x = 2\pi y\ \delta x\ \delta y,$$

nearly to the first powers of δy.

Hence, the total volume of the solid formed by the revolution of the area A about x-axis

$$= \iint_A 2\pi y\,dx\,dy.$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis

$$= \iint_A 2\pi x\,dx\,dy.$$

Polar co-ordinates:

In polar co-ordinates, the above formula for the volume becomes

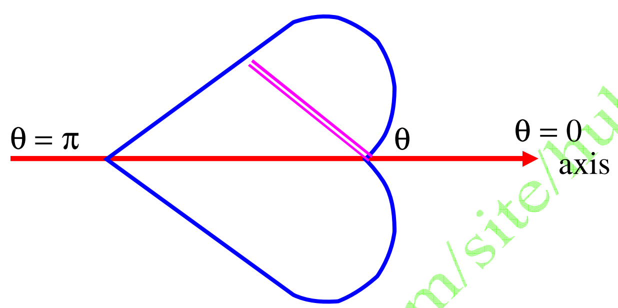
$$\iint_A 2\pi r\sin\theta . rd\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Q.No.1.: Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about its axis.

Sol.: In polar co-ordinates, the formula for evaluating the volume of revolution is

$$\iint_A 2\pi r\sin\theta . rd\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Here $r = a(1-\cos\theta)$.



$$\therefore \text{Required volume } = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta dr d\theta = 2\pi\int_0^{\pi} \left|\frac{r^3}{3}\right|_0^{a(1-\cos\theta)} \sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi} (1-\cos\theta)^3 . \sin\theta d\theta .$$

Put $1-\cos\theta = t$, so that $\sin\theta d\theta = dt$.

And when $\theta = 0$, $t = 0$, and when $\theta = \pi$, $t = 2$.

$$\therefore \text{Required volume of revolution } = \frac{2\pi a^3}{3}\int_0^2 t^3 dt$$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 dt = \frac{8\pi a^3}{3}. \text{ Cubic units. Ans.}$$

Q.No.2.: Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about its axis is $\frac{8\pi a^3}{3}$.

Sol.: In polar co-ordinates, the formula for evaluating the volume of revolution is

$$\iint_A 2\pi r\sin\theta . r d\theta dr = \iint_A 2\pi r^2 \sin\theta . d\theta dr .$$

Here $r = a(1+\cos\theta)$.

$$\therefore \text{Required volume of revolution} = \int_0^{\pi}\left(\int_0^{a(1+\cos\theta)} 2\pi r^2 dr\right)\sin\theta d\theta$$

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}\sin\theta d\theta = 2\pi\int_0^{\pi}\left[\frac{a^3(1+\cos\theta)^3}{3} - 0\right]\sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta d\theta .$$

Put $1+\cos\theta = t$, so that $-\sin\theta d\theta = dt$.

And when $\theta = 0$, $t = 2$, and when $\theta = \pi$, $t = 0$.

$$\therefore \text{Required volume of revolution} = -\frac{2\pi a^3}{3}\int_2^0 t^3 dt = \frac{2\pi a^3}{3}\int_0^2 t^3 dt$$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 dt = \frac{8\pi a^3}{3} . \text{ Cubic units. Ans.}$$

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Home Assignments

Q.No.1.: Find, by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the y-axis.

Ans.: $\frac{4}{3}\pi a^2 b$. Cubic units.

Q.No.2.: Find, by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

Ans.: $\frac{4}{3}\pi ab^2$. Cubic units.

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