1st Topic

Fourier Series

Importance, Definitions of Fourier series

Euler's formulae

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Introduction:

Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist, best known for initiating the investigation of Fourier series and their application to problems of heat transfer. The Fourier transform and Fourier's Law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.

Fourier series introduced in 1807 by Fourier (after works by Euler and Danial Bernoulli) was one of the most important developments in applied mathematics. Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.

In mathematics, a Fourier series decomposes a periodic function or periodic signal into a sum of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis. Fourier series were introduced by Joseph Fourier for the purpose of solving the heat equation in a metal plate.

Heat Equation:

The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function $u(x,y,z,t)$ of three spatial variables (x,y,z) and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

or equivalently

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

where α is a constant.

Note: The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

Solution of heat equation prior to Fourier's work:

Prior to Fourier's work, there was no known solution to the heat equation in a general situation, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolutions. This superposition or linear combination is called the Fourier series.

Fourier series is named in honour of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli. He applied this technique to find the solution of the heat equation, publishing his initial results in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides* and 1811, and publishing his *Théorie analytique de la chaleur* in 1822.

Original motivation:

Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems.

Applications:

The Fourier series has many applications in

- communication engineering,
- electrical engineering,
- vibration analysis,
- acoustics,
- optics,
- signal processing,
- image processing,
- quantum mechanics, and
- econometrics.

Fourier series is also very useful in the study of

- heat conduction,
- mechanics,
- concentration of chemicals and pollutants (impurities),
- electrostatics, and
- in areas unheard of in Fourier's days such as computing and
- CAT scan (computer assisted tomography-medical technology that uses X-Rays and computers to produce 3-dimensional images of the human body).
- Fourier series is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms.

Additional validity:

As we know, Taylor's series expansion is valid only for functions, which are continuous and differentiable. But Fourier series is possible not only for continuous functions, but for periodic functions, functions discontinuous in their values and derivatives.

Further, because of periodic nature, Fourier series constructed for one period is valid for all values.

Drawbacks in Fourier's days:

From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century.

Later, Dirichlet and Riemann expressed Fourier's results with greater accuracy and formality.

Periodic functions:

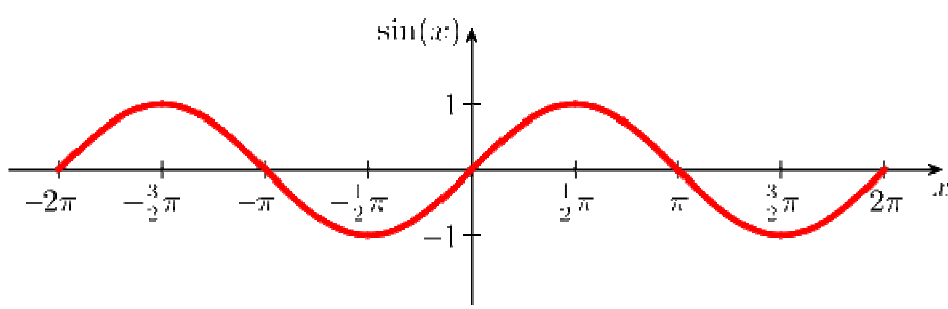
A function f(x) which satisfies the relation $f(x+T)=f(x)$ for all x and for some positive number T, is called a **periodic function**. The smallest positive number T, for which this relation holds, is called the **period** of f(x).

If T is the period, then $f(x)=f(x+T)=f(x+2T)=\dots\dots\dots=f(x+nT)=\dots..$

Also $f(x)=f(x-T)=f(x-2T)=\dots\dots\dots=f(x-nT)=\dots..$

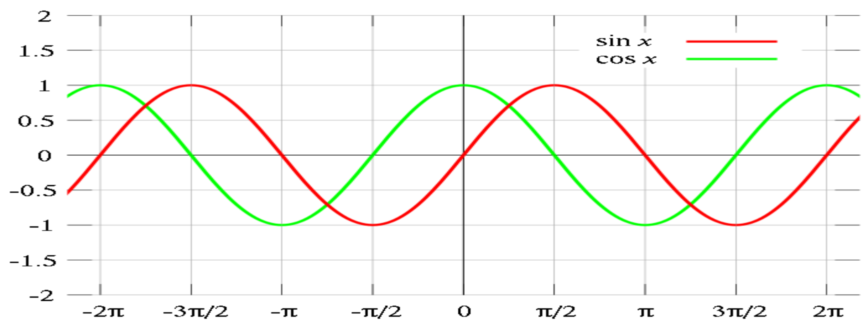
$\therefore f(x)=f(x\pm nT)$, where n is a positive integer.

Thus, f(x) repeats itself after periods of T.



A graph of the sine function, showing two complete periods.

Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry. Specifically, a function *f* is periodic with period *P* if the graph of *f* is invariant under translation in the *x*-direction by a distance of *P*. This definition of periodic can be extended to other geometric shapes and patterns, such as periodic tessellations of the plane.



A plot of *f*(*x*) = sin(*x*) and *g*(*x*) = cos(*x*); both functions are periodic with period 2π.

Aperiodic functions:

A function that is not periodic is called **aperiodic**.

Trigonometric series:

Trigonometric series is a functional series of the form

$$\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots$$

or $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$,

where the coefficients a_0, a_n, b_n $(n = 1,2,3,\ldots\ldots)$ are called the **coefficients**.

Fourier series:

Most of the single valued functions, which occur in many physical and engineering problems, can be expressed in the form

$$\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots$$

within a desired range of values of the variable.

Then, such a series is known as the Fourier series.

The individual terms in Fourier series are known as harmonics.

Euler's Formulae: [Fourier-Euler Formulae]

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx},$$

where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$,

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx.$$

Leonhard Paul Euler
(17-04-1707 to 18-09-1783)

These formulae of a_0, a_n, b_n are known as **Euler's Formulae.**

Note: For getting more symmetric formulae for the coefficients, we write $\frac{a_0}{2}$ instead of a_0.

To establish these formulae, the following definite integrals will be required:

1. $$\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = \left|\frac{\sin nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

2. $$\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = -\left|\frac{\cos nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

3. $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x]dx$$
$$= \frac{1}{2}\left|\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \left|\frac{x}{2} + \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

5. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

6. $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = \left|\frac{\cos 2nx}{2n}\right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (n \neq 0)$$

7. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

8. $$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \left|\frac{x}{2} - \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi \qquad (n \neq 0)$$

Proof: Let $f(x)$ be represented in the interval $(\alpha, \ \alpha+2\pi)$ by Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \,. \qquad \text{(i)}$$

In finding the coefficients a_0, a_n and b_n, we assume that the series on the RHS of (i) is uniformly convergent for $\alpha < x < \alpha + 2\pi$ and it can be integrated term by term in the given interval.

To determine the coefficient a_0:

Integrate both sides of (i) w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$. Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)dx$$

$$=\frac{1}{2}a_0(\alpha+2\pi-\alpha)+0+0=a_0\pi\,. \qquad \text{[by integrals (1) and (2) above]}$$

Hence $\boxed{a_0=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$.

To determine the coefficient a_n for n = 1, 2, 3,.......:

Multiply each side of (i) by $\cos nx$ and integrate w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$.

Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\cos nx\,dx$$
$$+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\cos nx\,dx$$

$$=0+\pi a_n+0. \qquad \text{[by integrals (1),(3), (4), (5) and (6)]}$$

Hence $\boxed{a_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$.

To determine the coefficient b_n for n = 1, 2, 3,......:

Multiply each side of (i) by $\sin nx$ and integrate w.r.t. x from $x=\alpha$ to $x=\alpha+2\pi$.

Then, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$
$$+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$=0+0+\pi b_n. \qquad \text{[by integrals (2),(5), (6), (7) and (8)]}$$

Hence $\boxed{b_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$.

Thus $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx$ and $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$.

These formulae of a_0, a_n, b_n are known as **Euler's (or Fourier-Euler) formulae.**

The coefficients a_0, a_n and b_n, are known as **Fourier coefficients** of f(x).

Remarks:

1.: Putting $\alpha = 0$, the interval becomes $0 < x < 2\pi$, and the formula (1) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx .$$

2.: Putting $\alpha = -\pi$, the interval becomes $-\pi < x < \pi$ and the formula (1) take the form

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx .$$

Q.: What is the significance of the coefficient a_0?

Ans.: a_0 is an additive constant, i.e. changing it results in a shift of the graph in y-direction. Furthermore, $\frac{a_0}{2}$ is the mean value of the function represented by the series, taken over the interval $[0, 2\pi]$.

Note: Practically all interesting functions with period 2π may be written in this form. Interpreting x as time, the coefficients a_n, b_n may be interpreted as representing the contributions of frequency n to a given signal.

Now let us expand the following functions as a Fourier series. In all these problems, f(x) is assumed to have the period 2π.

Q.No.1.: Expand in a Fourier series, the function $f(x)=x$ in the interval $0<x<2\pi$.

Sol.: Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$ (i)

be the required Fourier series.

Here $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}xdx=\frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{2\pi}=\frac{1}{\pi}\frac{\left(4\pi^2-0\right)}{2}=2\pi.$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{2\pi}\underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\cos nx}\,dx$$

$$=\frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}=\frac{1}{\pi}\left[\left(2\pi(0)+\frac{\cos 2n\pi}{n^2}\right)-\left(0+\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{\pi}\left[\frac{1}{n^2}-\frac{1}{n^2}\right]=0.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_0^{2\pi}\underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\sin nx}\,dx \qquad \left[\int\mu vdx=\mu.v_1-\mu' v_2+\mu''.v_3-\mu'''.v_4+....\right]$$

$$=\frac{1}{\pi}\left[(x).\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-2\pi\frac{1}{n}+0-(-0+0)\right]=-\frac{2}{n}. \qquad \left[\begin{array}{l}\cos 2n\pi=(-1)^{2n}=1\\ \sin 2n\pi=0\end{array}\right]$$

Hence, from (i), we get

$$f(x)=\frac{1}{2}.2\pi+\sum_{n=1}^{\infty}\left(0+\left(\frac{-2}{n}\right)\sin nx\right)$$

$\boxed{\Rightarrow f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}}$, is the required Fourier series.

Q.No.2.: Prove that for all values of x between $-\pi$ and π,

$$\frac{1}{2}x=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+.........$$

Sol.: Here $f(x)=\frac{1}{2}x, \quad -\pi<x<\pi$.

As f(x) is an odd function.

Hence, the required Fourier series is $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$. (i)

Now $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}x\sin nx dx$

$$=\frac{1}{\pi}\int_0^{\pi}\underset{\text{I}}{x}\sin nx\,dx \qquad \left[\because \int_{-\pi}^{\pi} x\sin nx dx=2\int_0^{\pi} x\sin nx dx \;\; (x\sin nx \text{ is even function})\right]$$

$$=\frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n\right]=-\frac{(-1)^n}{n}=\frac{(-1)^{n+1}}{n} \qquad \left[\begin{array}{l}\sin n\pi=0,\ n\in Z\\ \cos n\pi=(-1)^n,\ n\in Z\end{array}\right]$$

Hence, from (i), we get

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$\boxed{\Rightarrow f(x)=\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\ldots\ldots}$, is the required Fourier series.

Q.No.3.: Prove that $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \quad -\pi<x<\pi$.

Hence, show that **(i)** $\sum\frac{1}{n^2}=\frac{\pi^2}{6}$,

(ii) $\sum\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$,

(iii) $\sum\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

or

Develop a Fourier series for the function $f(x)=x^2$ in the interval $-\pi<x<\pi$.

Hence, show that **(i)** $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, **(ii)** $\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, **(iii)** $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\left[\left(\frac{-x\cos nx}{n}\right)_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2.0}{n} + \frac{2\pi}{n^2}(-1)^n - \frac{2}{n^3}(0) - \frac{\pi^2.0}{n} + \frac{2\pi}{n^2}(-1)^n + \frac{2}{n^3}.0\right] = \frac{4(-1)^n\pi}{\pi n^2} = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx = \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nx dx\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n}(-1)^n + \frac{2\pi.0}{n^2} + \frac{2}{n^3}(-1)^n + \frac{\pi^2(-1)^n}{n} - \frac{2\pi.0}{n^2} - \frac{2(-1)^n}{n^3}\right] = 0.$$

$\Rightarrow b_n = 0$.

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = x^2$ for $-\pi < x < \pi$ as

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x^2 = \frac{2\pi^2}{3.2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + 0$$

$$\boxed{\Rightarrow x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx}. \qquad \text{(ii)}$$

To show (i): $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Putting $x = \pi$ in equation (i), we obtain

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right]$$

$$\Rightarrow 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right] = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$$

$\boxed{\Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}}$,which is the required result.

To show (ii): $\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

Putting x = 0 in (i), we get

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\right] \Rightarrow 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\right] = -\frac{\pi^2}{3}$$

$$\Rightarrow 4\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{3}$$

$\boxed{\Rightarrow \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}}$, which is the required result.

To show (iii): $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Adding results (i) and (ii), we get

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\dots\dots\right)+\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\dots\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\Rightarrow 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots\dots\dots\right)=\frac{\pi^2}{4}\Rightarrow\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots\dots\dots\right)=\frac{\pi^2}{8}$$

$\boxed{\Rightarrow \sum \frac{1}{(2n-1)^2}=\frac{\pi^2}{8}}$, which is the required result.

Q.No.4.: $f(x)=x+x^2$ for $-\pi<x<\pi$ and $f(x)=\pi^2$ for $x=\pm\pi$.

Expand f(x) in Fourier series and show that

$$x+x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right\}.$$

Hence, show that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\dots=\frac{\pi^2}{12}$.

Sol.: The Fourier series is given by

$$f(x)=x+x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(1+2x)\frac{\sin nx}{n}dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\left[\frac{-(2x+1)}{n}\frac{\cos nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{2\cos nx}{n^2}dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\left(x+x^2\right)\frac{\sin nx}{n}+\frac{(1+2x)\cos nx}{n^2}\right]_{-\pi}^{\pi}-\left[\frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}\right\}$$

$$=\frac{1}{\pi}\left[\frac{\left(\pi+\pi^2\right)\sin n\pi}{n}+\frac{(1+2\pi)\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}\right]$$

$$-\frac{1}{\pi}\left[\frac{\left(-\pi+\pi^2\right)\sin(-n\pi)}{n}+\frac{(1-2\pi)\cos(-n\pi)}{n^2}-\frac{2\sin(-n\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+2\pi)(-1)^n}{n^2} - \frac{(1-2\pi)(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+2\pi-1+2\pi)}{n^2}(-1)^n\right] = \frac{1}{\pi}\times\frac{4\pi}{n^2}(-1)^n = \frac{4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx = \frac{1}{\pi}\left\{\left[-(x+x^2)\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(1+2x)\frac{\cos nx}{n}dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-(x+x^2)\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \left[\frac{(1+2x)\sin nx}{n^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2\sin nx}{n^2}dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{-(x+x^2)\cos nx}{n} + \frac{(1+2x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(\pi+\pi^2)\cos n\pi}{n} + \frac{(1+2\pi)\sin n\pi}{n^2} + \frac{2}{n^3}\cos n\pi\right]$$

$$-\frac{1}{\pi}\left[\frac{-(-\pi+\pi^2)\cos(-n\pi)}{n} + \frac{(1-2\pi)\sin(-n\pi)}{n^2} + \frac{2}{n^3}\cos(-n\pi)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^n(\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n\right) - \left(\frac{-(-1)^n(-\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n(\pi+\pi^2)}{n} + \frac{2}{n^3}(-1)^n + \frac{(-\pi+\pi^2)}{n}(-1)^n - \frac{2}{n^3}(-1)^n\right]$$

$$= \frac{1}{n\pi}\left[(-\pi-\pi^2-\pi+\pi^2)(-1)^n\right] = -\frac{1}{\pi}\times\frac{2\pi}{n}(-1)^n.$$

$$\Rightarrow b_n = \frac{-2}{n}(-1)^n.$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of $f(x) = x + x^2$, in the range $-\pi < x < \pi$, as

$$x + x^2 = \frac{1}{2}\times\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n\sin nx$$

$$\boxed{\Rightarrow x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]}.$$

Deduction: Put x = 0, in above, we get

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow 0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\ldots\right)$$

$$\boxed{\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots}$$

Q.No.5.: Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.

Hence, show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$.

Sol.: The Fourier series is given by

$$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = = \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx dx = \frac{1}{\pi}\left[\left|(x - x^2)\frac{\sin nx}{n}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1 - 2x)\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\left|(x - x^2)\frac{\sin nx}{n}\right|_{-\pi}^{\pi} - \left|(1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(-2)\left(-\frac{\cos nx}{n^2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi} = \frac{-4(-1)^n}{n^2}.$$

$$\left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore a_1 = \frac{4}{1^2},\ a_2 = \frac{-4}{2^2},\ a_3 = \frac{4}{3^2},\ a_4 = \frac{-4}{4^2}, \ldots\ldots \text{ etc.}$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx dx$

$$= \frac{1}{\pi}\left[\left(x - x^2\right)\left(-\frac{\cos nx}{n}\right) - (1-2x)\times\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-2(-1)^n}{n}.$$

$$\therefore b_1 = \frac{2}{1},\ b_2 = \frac{-2}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{-2}{4}, \ldots\ldots \text{etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$ as

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\ldots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\ldots\right]. \text{ Ans.}$$

2nd Part:

Putting x = 0, in the above relation, we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots..\right)$$

$$\boxed{\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.. = \frac{\pi^2}{12}}$$, which is the required result.

Remarks:

In the above example, we have used the result $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$.

Q.No.6.: Obtain the Fourier series of $f(x) = \frac{(\pi - x)}{2}$ in the interval $(0,\ 2\pi)$.

Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

Sol.: Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$ (i)

be the required Fourier series. Then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)dx = \frac{1}{2\pi}\left[\frac{(\pi-x)^2}{2(-1)}\right]_0^{2\pi} = -\frac{1}{4\pi}\left(\pi^2 - \pi^2\right) = 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}\underset{\mathrm{I}}{(\pi - x)}\underset{\mathrm{II}}{\cos nx}\, dx$$

Integrating by parts, we get

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{2\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right] = 0.$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}\underset{\mathrm{I}}{(\pi - x)}\underset{\mathrm{II}}{\sin nx}\, dx$

Integrating by parts, we get

$$b_n = \frac{1}{2\pi}(\pi - x)\sin x dx = \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left((-\pi)\left(-\frac{1}{n}\right) - 0\right) - \left(-\frac{\pi}{n} - 0\right)\right] = \frac{1}{2n}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{1}{n}.$$

Hence, from (i), we get

$\boxed{f(x) = \frac{1}{2}.0 + \sum_{n=1}^{\infty}\left(\frac{1}{n}\sin nx\right) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}}$, is the required Fourier series.

2nd Part: Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..$

Put $x = \frac{\pi}{2}$, then Fourier series becomes

$$\frac{\pi - \frac{\pi}{2}}{2} = \frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{1}{n}\sin n\frac{\pi}{2}$$

$$\boxed{\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..}$$

Q.No.7.: If $f(x) = \left[\frac{\pi - x}{2}\right]^2$ in the range 0 to 2π, then show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

Also, deduce that **(i).** $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(ii). $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}$,

(iii). $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Sol.: The Fourier series is given by

$$f(x) = \left[\frac{\pi - x}{2}\right]^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \,. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\left[\int_0^{2\pi} \pi^2 dx - 2\pi\int_0^{2\pi} xdx + \int_0^{2\pi} x^2 dx\right]$

$$= \frac{1}{4\pi}\left[\pi^2 \left|x\right|_0^{2\pi} - 2\pi\left|\frac{x^2}{2}\right|_0^{2\pi} + \left|\frac{x^3}{3}\right|_0^{2\pi}\right]$$

$$= \frac{1}{4\pi}\left[2\pi^3 - 4\pi^3 + \frac{8\pi 3}{3}\right] = \frac{\pi^2}{2} - \pi^2 + \frac{2\pi^2}{3} = \frac{\pi^2}{6}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \text{coxnxdx} = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nxdx$$

$$= \frac{1}{4\pi}\left[\pi^2 \int_0^{2\pi} \cos nxdx - 2\pi\int_0^{2\pi} x\cos nxdx + \int_0^{2\pi} x^2 \cos nxdx\right]$$

$$= \frac{1}{4\pi}\left[\pi^2 \left|\frac{\sin nx}{n}\right|_0^{2\pi} - 2\pi\int_0^{2\pi} x\cos nxdx + \int_0^{2\pi} x^2 \cos nxdx\right].$$

Let $I_1 = \int_0^{2\pi} x\cos nxdx = \frac{x\sin nx}{n} - \frac{1}{n}\int_0^{2\pi} \sin nxdx = \left|\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right|_0^{2\pi}$

$$= \left[\frac{2\pi \sin n2\pi}{n} + \frac{\cos n2\pi}{n^2}\right] - \left[\frac{\cos(0)n}{n^2}\right] = \frac{1}{n^2} - \frac{1}{n^2} = 0.$$

$$I_2 = \int_0^{2\pi} x^2 \cos nxdx = \frac{x^2 \sin nx}{n} - \frac{2}{n}\int_0^{2\pi} x\sin nxdx = \left|\frac{x^2 \sin nx}{n}\right|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} x\sin nxdx \,.$$

Let $I_3 = \int_0^{2\pi} x \sin nx dx$

$$\therefore I_3 = \int_0^{2\pi} x \sin nx dx = \frac{-x\cos nx}{n} - \frac{1}{n}\int_0^{2\pi} \frac{1}{n}\int (-\cos nx)dx = \left| \frac{-x\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n} \right|_0^{2\pi}$$

$$\therefore I_2 = \left| \frac{x^2 \sin nx}{n} - \frac{2}{n}\left(\frac{-x\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n} \right) \right|_0^{2\pi} = \frac{4\pi}{n^2}.$$

Thus $a_n = \frac{1}{4\pi}\left[0 + \frac{4\pi}{n^2} + 0\right] = \frac{4\pi}{n^2} \times \frac{1}{4\pi} = \frac{1}{n^2}$.

It is clear that $b_n = 0 \Rightarrow \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx dx = 0$.

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$f(x) = \left[\frac{\pi - x}{2}\right]^2$ in the range 0 to 2π, as

$$\boxed{f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}}. \qquad \text{(ii)}$$

2nd Part:

(i) To show: $\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}$.

Putting x = 0 in (ii), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\right]$$

$$\Rightarrow \frac{\pi^2}{4} - \frac{\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\right]$$

$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is the required result.

(ii) To show: $\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots = \frac{\pi^2}{12}}$.

Put $x = \pi$ in (ii), we get

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{12}+\left(1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-..........\right)$$

$\Rightarrow \dfrac{\pi^2}{12}=1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+......$, which is the required result.

(iii) To show: $\boxed{\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+......=\dfrac{\pi^2}{8}}$.

Adding (i) and (ii), we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}-..........\right)+\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+..........\right)$$

$$\Rightarrow \frac{\pi^2}{4}=2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+......\right)$$

$\Rightarrow \dfrac{\pi^2}{8}=1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+......$, which is the required result.

Q.No.8.: Prove that for $-\pi<x<\pi$, $\dfrac{\left(\pi^2-x^2\right)x}{12}=\dfrac{\sin x}{1^2}-\dfrac{\sin 2x}{2^2}+......$.

Sol.: $f(x)=\dfrac{x\left(\pi^2-x^2\right)}{12}$, $-\pi<x<\pi$.

Now $f(-x)=-x\dfrac{\left(\pi^2-x^2\right)}{12}=-f(x)\Rightarrow$ f(x) is odd function.

$\therefore a_0=0,\quad a_n=0.$

Let $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$ (i)

be the required Fourier series.

$$\text{Now } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi} x\frac{\left(\pi^2-x^2\right)}{12}\sin nx dx$$

$$=\frac{1}{12\pi}\int_{-\pi}^{\pi}\left(\pi^2 x-x^3\right)\sin nx dx=\frac{1}{6\pi}\int_{0}^{\pi}\underset{\text{I}}{\left(\pi^2 x-x^3\right)}\underset{\text{II}}{\sin nx}\, dx$$

$\left[\left(\pi^2 x-x^3\right)\sin nx \text{ is even function}\right]$

Integrating by parts, we get

$$=\frac{1}{6\pi}\left[\left(\pi^2x-x^3\right)\left(\frac{-\cos nx}{n}\right)-\left(\pi^2-3x^3\right)\left(\frac{-\sin nx}{n^2}\right)+(-6x)\left(\frac{\cos nx}{n^3}\right)-(-6)\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$=\frac{1}{6\pi}\left[\left(0-0-\frac{6\pi(-1)^n}{n^3}-0\right)-(0)\right]=\frac{(-1)^{n+1}}{n^3} \qquad \left[\begin{array}{l}\sin n\pi=0,\ n\in Z\\ \sin 0=0\end{array}\right]$$

Hence, from (i), we get

$$\boxed{x\left(\frac{\pi^2-x^2}{12}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx=\frac{\sin x}{1^2}-\frac{\sin 2x}{2^2}+\ldots\ldots}.$$

Q.No.9.: Obtain the Fourier series to represent e^x in the interval $0<x<2\pi$.

Sol.: Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_0^{2\pi}e^x dx=\frac{1}{\pi}\left|e^x\right|_0^{2\pi}=\frac{\left(e^{2\pi}-1\right)}{\pi}$.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx=\frac{1}{\pi}\int_0^{2\pi}e^x\cos nx dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_0^{2\pi} \qquad \left[\because\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(1+0)-\frac{1}{1+n^2}(1+0)\right]=\frac{e^{2\pi}-1}{\pi\left(1+n^2\right)}.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx dx=\frac{1}{\pi}\int_0^{2\pi}e^x\sin nx dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx-n\cos nx)\right]_0^{2\pi} \qquad \left[\because\int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)\right]$$

$$=\frac{1}{\pi\left(1+n^2\right)}e^{2\pi}(0-n)-\frac{1}{\pi}\left[\frac{1}{1+n^2}(-n)\right]=\frac{n\left(1-e^{2\pi}\right)}{\pi\left(1+n^2\right)}.$$

Hence, from (i), we get

$$\boxed{f(x)=\frac{1}{2\pi}\left(e^{2\pi}-1\right)+\frac{e^{2\pi}-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2}-\frac{n\sin nx}{1+n^2}\right)}.$$

Q.No.10.: Find the Fourier series to represent e^x in the interval $(-\pi,\pi)$.

Sol.: Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}e^x dx=\frac{1}{\pi}\left|e^x\right|_{-\pi}^{\pi}=\frac{1}{\pi}\left(e^{\pi}-e^{-\pi}\right)=\frac{2}{\pi}\sinh\pi$

$$\left[\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}\right]$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}e^x\cos nxdx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx+n\sin nx\right)\right]_{-\pi}^{\pi}\qquad\left[\because\int e^{ax}\cos bxdx=\frac{e^{ax}}{a^2+b^2}\left(a\cos bx+b\sin bx\right)\atop \text{Here } a=1,\ b=n\right]$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left((-1)^n+0\right)-\frac{e^{-x}}{1+n^2}\left((-1)^n+0\right)\right]$$

$$=\frac{1}{\pi}\frac{(-1)^n}{\left(1+n^2\right)}\left(e^{\pi}-e^{-\pi}\right)\qquad\left[\cos(-n\pi)=\cos n\pi=(-1)^n\atop \text{Also } \sin n\pi=0,\ n\in Z\right]$$

$$=\frac{2\sinh\pi(-1)^n}{\pi\left(1+n^2\right)}.\qquad\left[\sinh\pi=\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

Further,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\sin nxdx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\sin nx-n\cos nx\right)\right]_{-\pi}^{\pi}\qquad\left[\because\int e^{ax}\cos bxdx=\frac{e^{ax}}{a^2+b^2}\left(a\cos bx+b\sin bx\right)\atop \text{Here } a=1,\ b=n\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(0-(-1)^n\right)-\frac{e^{-\pi}}{1+n^2}\left(-n(-1)^n\right)\right]=\frac{-(-1)^n n}{\pi\left(1+n^2\right)}\left(e^{\pi}-e^{-\pi}\right)=\frac{-2n(-1)^n\sinh\pi}{\pi\left(1+n^2\right)}.$$

Hence, from (i), we get

$$f(x)=\frac{1}{2}.\frac{2}{\pi}\sinh\pi+\sum_{n=1}^{\infty}\left(\frac{2\sinh\pi}{\pi\left(1+n^2\right)}(-1)^n\cos nx-\frac{2n(-1)^n}{\pi\left(1+n^2\right)}(\sinh x)\times\sin nx\right)$$

$$=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n\cos nx}{\left(1+n^2\right)}-\frac{n(-1)^n}{1+n^2}\sin nx\right)\right]$$

$$=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\left(\frac{1}{2}\cos x-\frac{\cos 2x}{5}+\frac{\cos 3x}{10}-.....\right)\right.$$

$$\left.-\left(\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x-.....\right)\right].$$

Q.No.11.: Obtain the Fourier series for $f(x)=e^{-x}$ in the interval $0<x<2\pi$.

Sol.: The Fourier series is given by

$$e^{-x}=\frac{a_0}{2}+\sum_{n=2}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\,. \qquad \text{(i)}$$

Here $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}dx=\frac{1}{\pi}\left|-e^{-x}\right|_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi}$.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\cos nxdx$$

$$=\frac{1}{\pi\left(n^2+1\right)}\left|e^{-x}(-\cos nx+n\sin nx)\right|_0^{2\pi}=\left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{1}{n^2+1}.$$

$$\therefore a_1=\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2=\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5},\ a_3=\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{10}\quad\text{.......etc.}$$

Finally, $b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_0^{2\pi}e^{-x}\sin nxdx$

$$=\frac{1}{\pi\left(n^2+1\right)}\left|e^{-x}(-\sin nx-n\cos nx)\right|_0^{2\pi}=\left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{n}{n^2+1}.$$

$$\therefore b_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{2}{5},\ b_3 = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{3}{10}, \ldots \text{ etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = e^{-x}$ in the interval $0 < x < 2\pi$ as

$$\boxed{e^{-x} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\begin{aligned}&\frac{1}{2}+\left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots\ldots\ldots\right)\\ &+\left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\ldots\ldots\ldots\right)\end{aligned}\right\}}\text{. Ans.}$$

Q.No.12.: Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$.

Hence, derive series for $\dfrac{\pi}{\sinh \pi}$.

Sol.: The Fourier series is given by

$$f(x) = e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx = \dfrac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \dfrac{1}{\pi}\left[-\dfrac{e^{-ax}}{a}\right]_{-\pi}^{\pi} = \dfrac{-1}{\pi a}\left[e^{-ax}\right]_{-\pi}^{\pi} = \dfrac{-1}{\pi a}\left[e^{-a\pi} - e^{a\pi}\right]$

$$= \left[\frac{e^{a\pi} - e^{-a\pi}}{a\pi}\right] = \frac{2\sinh a\pi}{a\pi}.$$

Since we know that $\sinh x = \dfrac{e^x - e^{-x}}{2} \Rightarrow 2\sinh x = e^x - e^{-x} \Rightarrow e^{a\pi} - e^{-a\pi} = 2\sinh a\pi$.

Also $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} e^{-ax}\cos nx dx = I$ (say).

Then $I = \dfrac{1}{\pi}\left[\cos nx\left(\dfrac{e^{-ax}}{-a}\right)\Bigg|_{-\pi}^{\pi} - \displaystyle\int_{-\pi}^{\pi}\left(\dfrac{e^{-ax}}{-a}\right)(-n\sin nx)\,dx\right]$

$$= \left(\frac{-e^{ax}\cos nx}{\pi a}\right)_{-\pi}^{\pi} - \frac{n}{\pi a}\left[\sin nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right).n\cos nx dx\right]$$

$$= \left[\frac{-e^{-ax}\cos nx}{a\pi} + \frac{n\sin nx.e^{-ax}}{a^2\pi}\right]_{-\pi}^{\pi} - \frac{n^2}{a^2\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$= \left[\frac{-e^{-ax}\cos nx}{a\pi} + \frac{n\sin nx e^{-ax}}{a^2\pi} \right]_{-\pi}^{\pi} - \frac{n^2}{a^2}[I] \qquad \left[\because I = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx \right]$$

$$\Rightarrow \left[I + \frac{n^2}{a^2}.I \right] = \left[\frac{n\sin nx e^{-ax}}{a^2\pi} - \frac{e^{-ax}\cos nx}{a\pi} \right]_{-\pi}^{\pi}$$

$$\Rightarrow I = \frac{a^2}{(a^2+n^2)}\left[\frac{n\sin nx e^{-ax}}{a^2\pi} - \frac{e^{-ax}\cos nx}{a\pi} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[ne^{-a\pi}\sin n\pi - ae^{-a\pi}\cos n\pi + n\sin n\pi e^{+a\pi} + ae^{+a\pi}\cos n\pi \right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[0 - ae^{-a\pi}(-1)^n - 0 + ae^{a\pi}(-1)^n \right] = \frac{1}{\pi(a^2+n^2)}\left[ae^{-a\pi} - ae^{+a\pi} \right](-1)^n$$

$$= \frac{(-1)^n a}{\pi(n^2+a^2)}[2\sinh a\pi].$$

Now put $n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots$, we get

$$a_1 = \frac{-2a\sinh a\pi}{\pi(a^2+1^2)}, \quad a_2 = \frac{2a\sinh a\pi}{\pi(a^2+2^2)}, \quad a_3 = \frac{-2a\sinh a\pi}{\pi(a^2+3^2)}, \quad \ldots\ldots\ldots\text{etc.}$$

Similarly, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx = I$ (say).

$$\therefore I = \frac{1}{\pi}\left[\sin nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right).n\cos nx dx \right]$$

$$= \left[\frac{-e^{-ax}\sin nx}{a\pi} \right]_{-\pi}^{\pi} + \frac{n}{a\pi}\left[\cos nx\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{e^{-ax}}{-a}\right)(-n)\sin nx dx \right]$$

$$= \left[\frac{-e^{-ax}\sin nx}{a\pi} \right]_{-\pi}^{\pi} - \left[\frac{ne^{-ax}\cos nx}{\pi a^2} \right]_{-\pi}^{\pi} - \frac{n^2}{a^2\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx.$$

$$\text{Thus}\left[I + \frac{n^2}{a^2}.I \right] = \left[\frac{-e^{-ax}\sin nx}{a\pi} - \frac{ne^{-ax}\cos nx}{\pi a^2} \right]_{-\pi}^{\pi}$$

$$\Rightarrow I = \frac{a^2}{a^2+n^2}\left[\frac{-e^{-ax}\sin nx}{a\pi} - \frac{ne^{-ax}\cos nx}{\pi a^2}\right]_{-\pi}^{\pi}$$

$$\therefore I = \frac{a^2}{\left(a^2+n^2\right)}\times\frac{1}{a^2\pi}\left[-ae^{-ax}\sin nx - ne^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi\left(a^2+n^2\right)}\left[-ae^{-a\pi}\sin n\pi - ne^{-a\pi}\cos n\pi - ae^{a\pi}\sin n\pi + ne^{a\pi}\cos n\pi\right]$$

$$= \frac{1}{\pi\left(a^2+n^2\right)}\left[0 - ne^{-a\pi}(-1)^n + 0 + ne^{a\pi}(-1)^n\right] = \frac{(-1)^n}{\pi\left(a^2+n^2\right)}\left[ne^{a\pi} - ne^{-a\pi}\right]$$

$$= \frac{n(-1)^n}{\pi\left(a^2+n^2\right)}\left[e^{a\pi} - e^{-a\pi}\right] \Rightarrow b_n = I = \frac{n(-1)^n 2\sinh a\pi}{\pi\left(a^2+n^2\right)}. \qquad \left[\because 2\sinh a\pi = e^{a\pi} - e^{-a\pi}\right]$$

Now putting $n = 1, 2, 3, \ldots\ldots\ldots.$, we get

$$b_1 = \frac{-2\sinh a\pi}{\pi\left(a^2+1^2\right)}, \quad b_2 = \frac{2(2\sinh a\pi)}{\pi\left(a^2+2^2\right)}, \quad b_3 = \frac{3(-2\sinh a\pi)}{\pi\left(a^2+3^2\right)} \ \ldots\ldots \text{ etc.}$$

Substituting the values of a_i's and b_i's in (i), we get the required Fourier series of

$f(x) = e^{-ax}$ from $x = -\pi$ to $x = \pi$ as

$$e^{-ax} = \frac{2\sinh a\pi}{2\pi a} - \frac{2\sinh \pi a}{\pi}\left[\frac{a\cos x}{\left(a^2+1^2\right)} - \frac{a\cos 2x}{\left(a^2+2^2\right)} + \frac{a\cos 3x}{\left(a^2+3^2\right)} + \ldots\ldots\ldots\ldots\right]$$

$$+ \frac{2\sinh \pi a}{2\pi a}\left[\frac{-\sin x}{\left(a^2+1^2\right)} + \frac{2\sin 2x}{\left(a^2+2^2\right)} - \frac{3\sin 3x}{\left(a^2+3^2\right)} + \ldots\ldots\ldots\ldots\right]$$

$$\boxed{\Rightarrow e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2a} - \frac{a\cos x}{a^2+1^2} + \frac{a\cos 2x}{a^2+2^2} - \frac{a\cos 3x}{a^2+3^2} + \ldots\ldots\ldots\ldots\right) \\ -\left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \ldots\ldots\ldots\ldots\right)\end{array}\right]}$$

2nd Part:

By putting $x = 0$, $a = 1$ in the above relation, we get

$$e^{-1(0)} = \frac{2\sinh \pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2} - \frac{\cos 0}{1^2+1^2} + \frac{\cos 0}{2^2+1^2} - \frac{\cos 0}{3^2+1^2} + \ldots\ldots\ldots\ldots\right) \\ -\left(\frac{\sin 0}{1^2+1^2} - \frac{2\sin 0}{2^2+1^2} + \frac{3\sin 0}{3^2+1^2} - \ldots\ldots\ldots\ldots\right)\end{array}\right]$$

$$\Rightarrow 1 = \frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1^2} - \frac{1}{3^2+1^2} + \ldots\ldots\right) + (0)\right]$$

$$\boxed{\Rightarrow \frac{\pi}{2\sinh\pi} = \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \ldots\ldots},$$

which is the required result.

Q.No.13.: Obtain the Fourier series expansion of $f(x) = e^{ax}$ in $(0, 2\pi)$.

Sol.: The Fourier series is given by

$$f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}dx = \left.\frac{e^{ax}}{\pi}\right|_0^{2\pi} = \frac{e^{2ax}-1}{\pi}$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nxdx$$

Using $\int e^{ax}\cos bxdx = e^{ax}\frac{[a\cos bx + b\sin bx]}{(a^2+b^2)}$

$$a_n = \frac{1}{\pi}\left[e^{ax}\frac{(a\cos nx + n.\sin nx)}{(a^2+n^2)}\Bigg|_0^{2\pi}\right] = \frac{1}{\pi(a^2+n^2)}\left[ae^{2ax}\cos 2n\pi - e^0.\cos 0\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[ae^{2a\pi} - 1\right].$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nxdx = \frac{1}{\pi}\left[e^{ax}\frac{(a\sin nx - n\cos nx)}{(a^2+n^2)}\Bigg|_0^{2\pi}\right]$$

$$\left[\because \int e^{ax}\sin bxdx = e^{ax}\frac{[a\sin bx - b\cos bx]}{(a^2+b^2)}\right]$$

$$\therefore\ b_n = \frac{n}{\pi(a^2+n^2)}\left[-e^{2a\pi}\cos 2n\pi + 1\right] = \frac{n}{\pi(a^2+n^2)}\left[1 - e^{2a\pi}\right].$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$f(x) = e^{ax}$, in the range $0 < x < 2\pi$, as

$$\boxed{f(x)=\left(\frac{e^{2a\pi}-1}{\pi}\right)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{a^2+n^2}\right)(-n\sin nx)\right]+\left(\frac{ae^{2a\pi}-1}{\pi}\right)\sum_{n=1}^{\infty}\frac{\cos nx}{\left(a^2+n^2\right)}}. \text{ Ans.}$$

Q.No.14.: Find the Fourier series to represent e^{ax} in the interval $-\pi<x<\pi$.

Sol.: The Fourier series is given by

$$f(x)=e^{ax}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx. \qquad \text{(i)}$$

$$\text{Here } a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}dx=\frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi}=\frac{1}{\pi a}\left(e^{ax}-e^{-ax}\right)=\frac{2\sinh a\pi}{\pi a}.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nxdx=\frac{1}{\pi}\left[\frac{e^{ax}}{a^2+\pi^2}(a\cos nx+n\sin nx)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi\left(a^2+n^2\right)}\left[ae^{ax}\cos n\pi-ae^{-ax}\cos n\pi\right]=\frac{a\cos n\pi\left(e^{ax}-e^{-ax}\right)}{\pi\left(a^2+\pi^2\right)}=\frac{2a(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}.$$

Similarly, $b_n=\dfrac{2n(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}$.

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of

$$e^{ax}=\frac{\sinh a\pi}{a\pi}+\sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}\cos nx+\sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi\left(a^2+n^2\right)}\sin nx$$

$$\boxed{e^{ax}=\frac{2\sinh a\pi}{\pi}\left[\begin{array}{l}\left(\frac{1}{2a}-\frac{a\cos x}{a^2+1^2}+\frac{a\cos 2x}{a^2+2^2}-\frac{a\cos 3x}{a^2+3^2}+\ldots\ldots\right)\\ -\left(\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-\ldots\ldots\right)\end{array}\right]}. \text{ Ans.}$$

Q.No.15.: Expand $f(x)=x\sin x$, $0<x<2\pi$, in a Fourier series.

Sol.: The Fourier series is given by $f(x)=x\sin x=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos nx+\sum\limits_{n=1}^{\infty}b_n\sin nx$.(i)

$$\text{Here } a_0=\frac{1}{\pi}\int_{0}^{2\pi}x\sin xdx=\frac{1}{\pi}\left[x(-\cos x)-\int(-\cos x)dx\right]_0^{2\pi}$$

$$=\left[\frac{-x\cos x}{\pi}+\frac{\sin x}{\pi}\right]_0^{2\pi}=\frac{1}{\pi}\left[\sin x-x\cos x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 - 2\pi(+1) - 0 + 0\right] = \frac{-2x\pi}{\pi} = -2\,.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x.\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{2}{2}.x\sin x\cos nx dx = \frac{1}{2\pi}\int_0^{2\pi} x.2\sin x\cos nx dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(x+nx)+\sin(x-nx)\right]dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin x(1+n)dx + \frac{1}{2\pi}\int_0^{2\pi} x\sin x(1-n)dx$$

$$= \frac{1}{2\pi}\left[x\frac{(-\cos x(1+n))}{1+n} - \int_0^{2\pi}\frac{(-\cos x(1+n))}{1+n}dx\right]$$

$$+ \frac{1}{2\pi}\left[x\frac{(-\cos x(1-n))}{1-n} - \int_0^{2\pi}\frac{(-\cos x(1-n))}{1-n}dx\right]$$

$$= \left[\frac{-x\cos x(1+n)}{2\pi(1+n)} + \frac{\sin x(1+n)}{2\pi(n+1)^2}\right]_0^{2\pi} + \left[\frac{-x\cos x(1-n)}{2\pi(1-n)} + \frac{\sin x(1-n)}{2\pi(n-1)^2}\right]_0^{2\pi}$$

$$= \left[\frac{-2\pi\cos(n+1)2\pi}{2\pi(1+n)} + 0 + 0 - 0\right] + \left[\frac{-2\pi\cos(1-n)2\pi}{2\pi(1-n)} + 0 - 0 + 0\right]$$

$$= \left[\frac{(-1)(-1)^{2(n+1)}}{1+n} + \frac{(-1)(-1)^{2(1-n)}}{(1-n)}\right] = \frac{(-1)(-1)^{2n+2}(1-n) + (-1)(-1)^{2-2n}(1+n)}{(1-n^2)}$$

$$= \frac{-1+n-1-n}{1-n^2} = \frac{-2}{1-n^2} = \frac{2}{n^2-1}.\ (n \neq 1)$$

Thus $a_n = \frac{2}{n^2-1}.(n \neq 1)$.

When n = 1, then

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\sin 2x dx = \frac{1}{2\pi}\left[\frac{x(-\cos 2x)}{2} - \frac{\sin 2x}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{2\pi(-1)-0}{2}\right] = \frac{1}{2\pi}\left(\frac{-2\pi}{2}\right) = -\frac{1}{2}.$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx dx = 0$. $\left[\because\ x\sin x\sin nx \text{ is odd function}\right]$

When n = 1, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x . \sin x dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \frac{(1-\cos 2x)}{2} dx = \frac{1}{2\pi}\left[\frac{-x^2}{2}\right]_0^{2\pi} - \left[\frac{\sin 2x}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\times\left[\frac{4\pi^2}{2} - 0\right] - 0 = \frac{1}{2\pi}\times\frac{4\pi^2}{2} = \pi .$$

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of $f(x) = x\sin x$, in the range $0 < x < 2\pi$, as

$$\boxed{x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2 - 1}}.$$

Q.No.16.: Prove that, in the range $-\pi < x < \pi$,

$$\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2}\cos n\pi\right].$$

Sol.: The Fourier series is given by

$$f(x) = \cosh ax = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad \text{(i)}$$

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh ax dx = \frac{2\times 1}{\pi}\int_0^{\pi} \cosh ax dx$

$$= \frac{1}{\pi}\left[2\int_0^{\pi}\frac{e^{ax} + e^{-ax}}{2} dx\right] = \frac{2}{2\pi}\left[\int_0^{\pi} e^{ax} dx + \int_0^{\pi} e^{-ax} dx\right]$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a} - \frac{e^{-ax}}{a}\right|_0^{\pi} = \frac{1}{\pi a}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{2}{\pi a}\left[\frac{e^{a\pi} - e^{a\pi}}{2}\right] = \frac{2}{\pi a}\sinh a\pi .$$

Thus $a_0 = \frac{2}{\pi a}\sinh a\pi = \frac{2a^2}{\pi}\sinh a\pi\left[\frac{1}{a^2}\right]$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cosh ax)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{ax} + e^{-ax}}{2}\right)\cos nx dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{ax}\cos nx dx + \int_{-\pi}^{\pi} e^{-ax}\cos nx dx\right]. \qquad \text{(i)}$$

Let $I_1 = \int_{-\pi}^{\pi} e^{ax}\cos nx dx = \left[\frac{e^{ax}}{a}\cos nx\right]_{-\pi}^{\pi} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}(-\sin nx)dx$

$$= \left[\frac{e^{ax}}{a}\cos nx + \frac{n}{a}\left(\sin nx \frac{e^{ax}}{a} - \frac{n}{a}I_1\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{I\left(n^2+a^2\right)}{a^2} = \left[\frac{e^{ax}}{a}\cos nx + \frac{ne^{ax}}{a^2}\sin nx\right]_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = \left[\frac{e^{ax}}{n^2+a^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi} = \frac{1}{n^2+a^2}\left[e^{a\pi}.a(-1)^n - e^{-a\pi}.a(-1)^n\right]$$

$$= \frac{2a(-1)}{n^2+a^2}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] = \frac{2a(-1)^n \sinh a\pi}{\left(a^2+n^2\right)}.$$

Similarly, $I_2 = \int_{-\pi}^{\pi} e^{-ax}\cos nx dx = \left[\frac{e^{-ax}}{n^2+a^2}(-a\cos nx - n\sin nx)\right]_{-\pi}^{\pi}$

$$= \frac{1}{n^2+a^2}\left[e^{-a\pi}(-a)(-1)^n - e^{a\pi}(-a)(-1)^n\right]$$

$$= \frac{e^{-a\pi}}{n^2+a^2}\left[-a(-1)^n\right] - \left[\frac{e^{-a(-\pi)}}{n^2+a^2}(-a)(-1)^n\right]$$

$$= \frac{2(-1)^n(-a)}{n^2+a^2}\left[\frac{e^{-a\pi}-e^{a\pi}}{2}\right] = \frac{2a(-1)^n}{n^2+a^2}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] = \frac{+2a(-1)^n}{\left(n^2+a^2\right)}\sinh a\pi.$$

Thus $I_1 + I_2 = \frac{4a(-1)^n}{\left(n^2+a^2\right)}\sinh a\pi.$

On keeping the values of $I_1 + I_2$ in equation (i), we get

$$a_n = \frac{2a(-1)^n \sinh a\pi}{\pi\left(n^2+a^2\right)}.$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\left(e^{ax}+e^{-ax}\right)}{2}\sin nx dx = 0$. [as odd function]

Substituting the values of a_0, a_n and b_n in (i), we get the required Fourier series of $f(x) = \cosh ax$, in the range $-\pi < x < \pi$, as

$$\boxed{\cosh ax = \frac{2a\sinh a\pi}{\pi}\left[\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(a^2+n^2\right)}\cos nx\right]}.$$

Q.No.17.: Obtain the Fourier series for $\sqrt{1-\cos 2x}$ in the interval $(0, 2\pi)$.

Sol.: Let $f(x) = \sqrt{1-\cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2}|\sin x|$.

Also $f(-x) = \sqrt{2}|\sin(-x)| = \sqrt{2}|-\sin x| = \sqrt{2}|\sin x| = f(x)$. $\left[\begin{array}{l}|\alpha x| = |\alpha||x| \\ |-1| = 1\end{array}\right]$

$\Rightarrow f(x)$ is even function

$\therefore b_n = 0 \forall n$

Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, (i)

be the required Fourier series for f(x).

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}(\sin x)dx = \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi}|\sin x| + \int_{\pi}^{2\pi}|\sin x|dx\right]$

$= \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi}\sin x dx + \int_{\pi}^{2\pi}(-\sin x)dx\right]$ $\left[\begin{array}{l}|x| = \begin{cases}x, & x > 0 \\ -x, & x < 0\end{cases} \\ \text{For } 0 < x < \pi,\ \sin x = +\text{ve and} \\ \text{For } \pi < x < 2\pi,\ \sin x = -\text{ve}\end{array}\right]$

$= \frac{\sqrt{2}}{\pi}\left[|-\cos x|_0^{\pi} + |\cos x|_{\pi}^{2\pi}\right] = \frac{\sqrt{2}}{\pi}(-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi)$

$= \frac{\sqrt{2}}{\pi}(1+1+1+1) = \frac{4\sqrt{2}}{\pi}$. (ii)

Also $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{\sqrt{2}}{\pi}\int_0^{2\pi}|\sin x|\cos nx dx$

$$= \frac{\sqrt{2}}{\pi}\left[\int_0^{\pi} \sin x \cos nx dx + \int_{\pi}^{2\pi} -\sin x \cos nx dx\right]$$

$$= \frac{\sqrt{2}}{\pi}\frac{1}{2}\int_0^{\pi} 2\cos nx \sin x dx - \frac{\sqrt{2}}{\pi}.\frac{1}{2}\int_{\pi}^{2\pi} 2\cos nx \sin x dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]dx - \frac{1}{\sqrt{2}\pi}\int_{\pi}^{2\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} - \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left(\frac{-1}{n+1} + \frac{1}{n+2}\right)\right]$$

$$-\left[\left(\frac{-\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right) - \left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{2}{\sqrt{2}\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\begin{array}{l} \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1};\ \text{n is even} \\ \frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1};\ \text{n is odd} \end{array}\right.$$

$$= \frac{\sqrt{2}}{\pi}\left[\begin{array}{ll} \frac{2}{n+1} - \frac{2}{n-1}, & \text{n is even} \\ 0, & \text{n is odd} \end{array}\right.$$

$$= \frac{-4\sqrt{2}}{\pi(n^2-1)}, \quad \text{n is even.}$$

Take $n = 2m$, we get $a_n = \dfrac{-4\sqrt{2}}{\pi(4m^2-1)}$, $m = 1, 2, \ldots..$ (iii)

Putting the values of a_0 from (ii) and a_n from (iii) in (i), we get

$$f(x)=\frac{1}{2}.\frac{4\sqrt{2}}{\pi}+\sum_{m=1}^{\infty}\frac{-4\sqrt{2}}{\pi\left(4m^2-1\right)}\cos 2mx=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}+\sum_{m=1}^{\infty}\frac{\cos 2mx}{\pi\left(4m^2-1\right)}$$

Thus $\boxed{\sqrt{1-\cos x}=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}}$.

Q.No.18.: Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi<x<\pi$.

Sol.: Here $f(x)=\sqrt{1-\cos x}=\sqrt{2\sin^2\frac{x}{2}}=\sqrt{2}\left|\sin\frac{x}{2}\right|$.

Now $f(-x)=\sqrt{2}\left|\sin\left(-\frac{x}{2}\right)\right|=\sqrt{2}\left|\sin-\frac{x}{2}\right|$ $\qquad [\therefore \sin(-\theta)=-\sin\theta]$

$=\sqrt{2}\left|\sin\frac{x}{2}\right|=f(x)$. $\qquad [|\alpha x|=|\alpha||x|]$

$\Rightarrow f(x)$ is even function. $\therefore b_n=0$.

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\sqrt{2}\left|\sin\frac{x}{2}\right|dx=\frac{2}{\pi}\int_{0}^{\pi}\sqrt{2}\left|\sin\frac{x}{2}\right|dx$

$=\frac{2^{3/2}}{\pi}\int_0^{\pi}\sin\frac{x}{2}dx$ $\qquad \left[|x|=\begin{cases}x, & x>0\\ -x, & x<0\end{cases}\right.$ For $0<x<\pi$, $\sin\frac{x}{2}$ is positive $\therefore \left|\sin\frac{x}{2}\right|=\sin\frac{x}{2}\Big]$

$$=\frac{2^{3/2}}{\pi}\frac{\left|-\cos\frac{x}{2}\right|_0^{\pi}}{1/2}=\frac{-2}{\pi}.2^{3/2}(0-1)=\frac{4.\sqrt{2}}{\pi}.$$

Also $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sqrt{2}\left|\sin\frac{x}{2}\right|\cos nxdx$ $\qquad \left[\left|\sin\frac{x}{2}\right|\cos nx \text{ is even function}\right]$

$=\frac{2}{\pi}.\sqrt{2}\int_0^{\pi}\left|\sin\frac{x}{2}\right|\cos nxdx-2.\frac{\sqrt{2}}{\pi}\int_0^{\pi}\sin\frac{x}{2}\cos nxdx$ $\quad$ { For $0<x<\pi$, sinx is positive}

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(\frac{1}{2}+n\right)x + \sin\left(\frac{1}{2}-n\right)x\right]dx \qquad [2\sin A\cos B = \sin(A+B)+\sin(A-B)]$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} - \frac{\cos\left(\frac{1}{2}-n\right)x}{\frac{1}{2}-n}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\left(\frac{-\cos\left(n+\frac{1}{2}\right)\pi}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\pi}{n-\frac{1}{2}}\right) - \left(\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\right]$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right) \qquad \left[\begin{array}{l}\cos(n\pi+\theta) = (-1)^n\cos\theta \\ \therefore \cos\left(n\pi+\frac{\pi}{2}\right) = (-1)^n\cos\frac{\pi}{2}\end{array}\right]$$

$$= \frac{-4\sqrt{2}}{\pi\left(4n^2-1\right)}.$$

Hence, from (i), we get

$$\sqrt{2}\sin\frac{x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx = \frac{1}{2}.\frac{4\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}}{\pi\left(4n^2-1\right)}.\cos nx$$

$$\boxed{\Rightarrow \sqrt{2}\sin\frac{x}{2} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}},$$

is the required Fourier series.

Q.No.19.: Express $f(x) = \cos wx$, $-\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence, prove that $\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2-\pi^2} + \frac{2\theta}{\theta^2-4\pi^2} + \ldots\ldots\ldots.$

Sol.: Here $f(x) = \cos wx$ is an even function. $\therefore b_n = 0$. $\qquad [\cos(-wx) = \cos wx]$

Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx$ (i)

be the required Fourier series.

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos wxdx = \frac{2}{\pi}\int_{0}^{\pi} \cos wxdx$ $\left[\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(x) \text{ is even}\right]$

$$= \frac{2}{\pi}\left|\frac{\sin wx}{w}\right|_0^{\pi} = \frac{2}{\pi w}(\sin w\pi).$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos wx \cos nxdx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \cos wx \cos xdx = \frac{1}{\pi}\int_{0}^{\pi} 2\cos wx \cos nxdx \qquad [\cos wx \cos x \text{ is even function}]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [\cos(w+n)x \cos(w-n)x]dx$$

$$[2\cos A \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{\pi}\left[\frac{\sin(w+n)x}{w+n} + \frac{\sin(w-n)x}{w-n}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(w+n)\pi}{w+n} + \frac{\sin(w-n)\pi}{w-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin w\pi \cos n\pi + \cos w\pi \sin n\pi}{w+n} + \frac{\sin w\pi \cos n\pi - \cos w\pi \sin n\pi}{w-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin w\pi \cos n\pi}{w+n} + \frac{\sin w\pi \cos n\pi}{w-n}\right] \qquad [\sin n\pi = 0]$$

$$= \frac{\sin w\pi \cos n\pi}{\pi}\left[\frac{1}{(w+n)} + \frac{1}{(w-n)}\right]$$

$$= \frac{\sin w\pi \cos n\pi}{\pi}\left[\frac{w-n+w+n}{(w+n)(w-n)}\right] = \frac{2w \sin w\pi \cos n\pi}{\pi(w^2-n^2)}$$

$$= \frac{2w(-1)^n \sin w\pi}{\pi(w^2-n^2)} = \frac{2w(-1)^{n+1} \sin w\pi}{\pi(w^2-n^2)}.$$

Hence, from (i), we get

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\sin w\pi}{w\pi} + \sum_{n=1}^{\infty} \frac{2w(-1)^{n+1} \sin w\pi}{\pi(n^2-w^2)} \cos nx$$

$$=\frac{\sin w\pi}{w\pi}+\frac{2w}{\pi}\sin w\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\left(n^2-w^2\right)}\cos nx$$

$$=\frac{\sin w\pi}{w\pi}+\frac{2w\sin w\pi}{\pi}\left(\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right)$$

$$=\frac{2w\sin w\pi}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right)$$

$$\boxed{\therefore \cos wx=\frac{2w\sin w\pi}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\frac{\cos 3x}{3^2-w^2}+\dots\dots\dots\right).}$$

Deduction : Take $x=\pi$, we get

$$\cos wx=2\sin w\pi\times\frac{w}{\pi}\left(\frac{1}{2w^2}+\frac{\cos x}{1^2-w^2}-\frac{\cos 2x}{2^2-w^2}+\dots\dots\dots\right) \qquad \begin{bmatrix}\cos\pi=-1\\ \cos 2\pi=1\end{bmatrix}$$

$$\Rightarrow \cot w\pi=\frac{1}{w\pi}-\frac{2w}{\left(1^2-w^2\right)\pi}-\frac{2w}{\left(2^2-w^2\right)\pi}-\dots\dots\dots$$

$$\boxed{\Rightarrow \cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\dots..} \qquad \begin{bmatrix}w\pi=\theta\\ w=\dfrac{\theta}{\pi}\end{bmatrix}.$$

Q.No.20.: Find the Fourier series for f(x) in the interval $(-\pi,\ \pi)$, when

$$f(x)=\begin{cases}n+x, & -\pi<x<0\\ n-x, & 0<x<\pi\end{cases}.$$

Sol.: $f(-x)=\begin{cases}n-x, & -\pi<x<0\\ n+x, & 0<x<\pi\end{cases}$

$$=\begin{cases}\pi-x, & 0<x<\pi\\ \pi+x, & -\pi<x<0\end{cases} \qquad [1<2<4\Rightarrow -1>-2>-4]$$

$=f(x)\Rightarrow$ f(x) is even function. $\therefore b_n=0.$

Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx$ (i)

be the required Fourier series.

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)dx$

$$= \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[nx - \frac{x^2}{2}\right]_{0}^{\pi} = \frac{1}{\pi}\left[0 - \left(-\pi^2 + \frac{\pi^2}{2}\right)\right] + \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right]$$

$$= \frac{1}{\pi}.\frac{\pi^2}{2} + \frac{1}{\pi}.\frac{\pi^2}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx dx$$

$$= \frac{1}{\pi}\left[(\pi + x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2} - \left(0 + \frac{\cos nx}{n^2}\right)\right] + \frac{1}{\pi}\left[\left(0 - \frac{\cos nx}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{2}{n^2} - \frac{2\cos n\pi}{n^2}\right) = \frac{2}{n^2\pi}[1 - \cos n\pi]$$

$$= \begin{cases} 0, & n \text{ is even} \\ \dfrac{4}{n^2\pi}, & n \text{ is odd} \end{cases}$$

Hence, from (i), we get

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}}^{\infty} \frac{4}{n^2\pi}\cos nx = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right)}.$$

Q.No.21.: Define Fourier series over the interval $-\pi$ to π.

Is it possible to write the Fourier sine series for the function $f(x) = \cos x$, over the interval $(-\ell, \ell)$?

Sol.: Let f(x) be a function defined in the interval $(-\pi, \pi)$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

is called Fourier series for f(x).

For half range sine series f(x) must be defined in the interval $(0, \ell)$.

Hence, we cannot find the Fourier half range sine series for $f(x) = \cos x$, over the interval $(-\ell, \ell)$.

Home Assignments

No assignment

(Students are advised to solve each problem before moving next topic)

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