Differential Calculus

Maxima and Minima

(Lagrange's method of undetermined multipliers)

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Lagrange's Method of undetermined multipliers:

Q.: Write a short note on "Lagrange's Method of undetermined multipliers".

What are the advantages and disadvantages of this method.

In many practical and theoretical problems, it is required to find the maximum or minimum of a function of several variables, where the variables are connected by some given relation or condition known as a constraint condition(s). Thus, if f(x, y, z) is a function of 3 independent variables, where x, y, z are related by a known constraint (restriction) $\phi(x, y, z) = 0$, then the problem of constrained extrema consists of finding the Extrema of

$$u = f(x, y, z) \qquad \text{(i)}$$

subject to constraint condition $\phi(x, y, z) = 0$. (ii)

Now this type of problem can be solved by

(a) Elimination method

(b) Implicit differentiation

(c) Lagrange's multiplier's method

(a) Elimination method: In this method, the constraint condition (ii) is solved for one variable, say z, in terms of the other variables x and y. Then z is eliminated from f(x, y, z), we get a function of two variables x and y only. The disadvantage of this method is that many times, (ii) may not be solvable and in case of solution also the amount of algebra will be generally huge.

(b) Implicit differentiation: In this method, no elimination of variables is done but derivative are eliminated by calculating them through implicit differentiation. This method also suffers due to more labour involved.

Remarks: If $u = f(x, y)$ then the function is said to have a maximum or minimum if

$f_x = 0$, $f_y = 0$, at (a, b).

For a max. : $f_{xx} < 0$, $f_{xx}.f_{yy} > (f_{xy})^2$ at (a, b).

For a min. : $f_{xx} > 0$, $f_{xx}.f_{yy} > (f_{xy})^2$ at (a, b).

For a saddle point, we have $f_{xx}.f_{yy} < (f_{xy})^2$ at (a, b).

If $f_{xx}.f_{yy} = (f_{xy})^2$ at (a, b). Then further investigations are necessary.

In the above case, the variables x, y are **independent**.

There are several situations, where function to be maximized or minimized, depend upon variables, which are **not independent**, but are inter-related by one or more constraint conditions. So in that situation, we may use Lagrange's method.

(c) Lagrange's method: The very useful "Lagrange's method of undetermined multiplier's" introduces an additional unknown constant λ known as Lagrange's multiplier.

Let f(x, y, z) be a function of x, y, z which is to be examined for maximum or minimum value.

Let the variables x, y, z be connected by the relation $\phi(x, y, z) = 0$. (i)

For f(x, y, z) to have a maximum or minimum value, the necessary conditions are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0.$$

$$\therefore \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \qquad \text{(ii)}$$

Also, from (i), taking differential, we get $\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0$. (iii)

Multiplying (iii) by a parameter λ and adding to (ii), we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}\right)dz=0.$$

This equation will hold good if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}=0,\quad \frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}=0,\quad \frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}=0. \qquad \text{(iv)}$$

These equations together with constraint condition (i), give the values of x, y, z and λ for a maximum or minimum.

Remarks: The above equations can be easily obtained by considering Lagrange's (auxiliary) function.

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

and considering the stationary values of F(x, y, z).

For stationary values of F(x, y, z), dF = 0.

$$\Rightarrow\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}\right)dz=0.$$

$$\Rightarrow\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}=0,\quad \frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}=0,\quad \frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}=0.$$

Advantages:

1. The stationary value f(x, y, z) can be determined from (i) and (iv) even without determining x, y, z explicitly.

2. This method can be extended to a function of several 'n' variables $x_1, x_2, x_3, \ldots\ldots, x_n$ and subject to many (more than one) 'm' constraints by forming the auxiliary equation

$$F(x_1,x_2,\ldots\ldots,x_n)=f(x_1,x_2,\ldots\ldots,x_n)+\sum_{i=1}^{m}\lambda_i\phi_i(x_1,x_2,\ldots\ldots,x_n).$$

The stationary values are obtained by solving the (n + m) equations consisting of

n equations $\frac{\partial F}{\partial x_i}=0$, for i = 1, 2, 3, ….,n and

m constraint conditions, $\phi_i=0$ for i = 1, 2, 3, ….,m.

Disadvantages:

1. Nature of the stationary points cannot be determined. Further investigation needed. This means that Lagrange's method does not enable us to find weather there is maximum or minimum. This fact is determined from the physical considerations of the problem.
2. Equations (iv) are only necessary conditions but not sufficient.

Now let us find maxima and minima of some problems:

Q.No.1: A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimension of the box requiring least material for its constructions.

Sol.: Let x, y and z ft. be the edges of the box and S be its surface.

Then $S = xy + 2yz + 2zx$ (i)

and $xyz = 32$ (ii)

Write $F = xy + 2yz + 2zx + \lambda(xyz - 32)$

Then $\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0$ (iii)

$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0$ (iv)

$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0$ (v)

Multiply (iii) by x and (iv) by y and subtracting, we get

$2zx - 2zy = 0 \Rightarrow x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (ii)]

Again multiply (iv) with y and (v) with z and subtracting, we get $y = 2z$.

Hence the dimension of the box is $x = y = 2z = 4$. (vi)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with the large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (vi). Hence S is minimum when x = y = 4 ft. and z = 2 ft.

Q.No.2.: Of all the rectangular parallelopipeds which have sides parallel to the co-ordinate planes and which are inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, find the dimensions of that one which has the largest possible volume.

or

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol.: Let the edges of the parallelopiped be 2x , 2y and 2z which are parallel to the axes.

Then its volume V = 8xyz.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0. \qquad \text{...(i)}$$

Write $F=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$

Then $\frac{\partial F}{\partial x}=8yz+\lambda\left(\frac{2x}{a^2}\right)=0$ ...(ii)

$$\frac{\partial F}{\partial y}=8zx+\lambda\left(\frac{2y}{b^2}\right)=0 \qquad \text{...(iii)}$$

$$\frac{\partial F}{\partial z}=8xy+\lambda\left(\frac{2z}{c^2}\right)=0 \qquad \text{...(iv)}$$

Equating the values of λ from (ii) and (iii), we get $\frac{x^2}{a^2}=\frac{y^2}{b^2}$.

Similarly from (iii) and (iv), we obtain $\frac{y^2}{b^2}=\frac{z^2}{c^2}$.

$\therefore \frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$.

Substituting these in (i), we get $\frac{x^2}{a^2}=\frac{1}{3}$.

i.e. $\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$. These give $x=\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$, $z=\frac{c}{\sqrt{3}}$ .Ans.

When x = 0, the parallelopiped is just a rectangular sheet and as such its volume V = 0.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (v).

Hence the greatest volume $= \dfrac{8abc}{3\sqrt{3}}$ cubic units.

Q.No.3.: A rectangular tank open at the top is to have a volume of 4 cubic meters . Find its dimensions so that material used is minimum.

Sol.: Let x , y and z mts. be the dimensions of the rectangular tank open at the top so that material for construction will be least if surface area is least.

Let surface area, $S = f(x, y, z) = xy + 2yz + 2zx$ . ....(i)

Also given volume $= xyz = 4$. ...(ii)

Write $F = xy + 2yz + 2zx + \lambda(xyz - 4)$

Then $\dfrac{\partial F}{\partial x} = y + 2z + \lambda yz = 0$ ...(iii)

$\dfrac{\partial F}{\partial y} = x + 2z + \lambda zx = 0$ ...(iv)

$\dfrac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0$ ...(v)

Multiplying (iii) by x and (iv) by y and subtracting, we get

$2zx - 2zy = 0 \Rightarrow x = y$.

[The value z = 0 is neglected, as it will not satisfy (ii)]

Again multiplying (iv) by y and (v) by z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are x = y = 2z = 2. i.e. x = 2m.,y = 2m. and z = 1m.Ans.

Q.No.4.: Of all the rectangular boxes having the same surface area, find that one, which encloses the maximum volume. Use Lagrange's method.

Sol.: Let its sides be x, y, z

$S.A. = 2(xy + yz + zx)$, $V = xyz$.

$F = xyz + \lambda(2xy + 2yz + 2zx - k) = xyz + 2\lambda xy + 2\lambda yz + 2\lambda zx - k\lambda$

$\dfrac{\partial F}{\partial x} = yz + 2\lambda y + 2\lambda z = 0$ (i)

$$\frac{\partial F}{\partial y} = xz + 2\lambda x + 2\lambda z = 0 \tag{ii}$$

$$\frac{\partial F}{\partial z} = xy + 2\lambda x + 2\lambda y = 0 \tag{iii}$$

From (i) and (ii), we get

$$(y - x)z + 2\lambda(y - x) = 0 \Rightarrow (y - x) + (2\lambda + z) = 0$$

$$\Rightarrow y = x, \qquad \lambda = \frac{-z}{2}.$$

From (ii) and (iii), we get

$$(z - y)x + 2\lambda(z - y) = 0$$

$$\Rightarrow z = y, \qquad \lambda = \frac{-x}{2}. \Rightarrow x = y = z.$$

Hence the rectangle box with maximum volume is a cube. Ans.

Q.No.5.: Find the maximum value of $400xyz^2$ on a sphere $x^2 + y^2 + z^2 = 1$.

or

The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

Sol.: Let $f(x, y, z) = 400xyz^2$ . ....(i)

Also the given equation of sphere is $x^2 + y^2 + z^2 = 1$. ...(ii)

Now f is to be maximized subject to the constraint $x^2 + y^2 + z^2 = 1$ i.e. $\phi(x, y, z) = 0$.

Let $F = f + \lambda\phi$ where λ is Lagranges's multipliers.

$$\therefore \; F = 400xyz^2 + \lambda\left(x^2 + y^2 + z^2 - 1\right).$$

Then $\frac{\partial F}{\partial x} = 400yz^2 + \lambda 2x = 0$ ...(iii)

$$\frac{\partial F}{\partial y} = 400xz^2 + \lambda 2y = 0 \qquad \text{...(iv)}$$

$$\frac{\partial F}{\partial z} = 800xyz + \lambda 2z = 0 \qquad \text{...(v)}$$

Multiplying (iii) by x and (iv) by y and subtracting, we get

$2\lambda\left(x^2 - y^2\right) = 0 \Rightarrow x = y$.[The value $x = -y$ is neglected] ...(vi)

Multiplying (iv) by 2y and (v) by z and subtracting, we get

$$2\lambda\left(2y^2 - z^2\right) = 0 \Rightarrow z = \sqrt{2}y. \qquad \text{...(vii)}$$

From (vi) and (vii), we get $x = y = \frac{1}{\sqrt{2}}z$.

Substitute these values in (ii), we get

$$x^2 + x^2 + 2x^2 = 1 \Rightarrow x = \frac{1}{2}.$$

$$\therefore x = \frac{1}{2},\ y = \frac{1}{2},\ z = \sqrt{2}.\frac{1}{2} = \frac{1}{\sqrt{2}}.$$

Maximum value of $f(x, y, z) = 400xyz^2 = 400 \times \frac{1}{2}.\frac{1}{2}.\left(\frac{1}{\sqrt{2}}\right)^2 = 50$. Ans.

Q.No.6.: Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

or

Find the maximum and minimum distances of the point (3, 4, 12) from the unit sphere with centre at origin.

Sol.: Let (x, y, z) be any point on the sphere.

Distance of the point A(3, 4, 12) from (x, y, z) is given by $\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$

subject to the condition that $\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. (ii)

Consider Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

For stationary value, $dF = 0 \Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$.

$$\Rightarrow [2(x-3) + 2\lambda x]dx + [2(y-4) + 2\lambda y]dy + [2(z-12) + 2\lambda z]dz = 0$$

$$\Rightarrow 2(x-3) + 2\lambda x = 0, \qquad \text{(iii)}$$

$$2(y-4) + 2\lambda y = 0, \qquad \text{(iv)}$$

$2(z-12)+2\lambda z=0$. (v)

Multiplying (iii) by x, (iv) by y, and (v) by z and adding, we get

$2(x^2+y^2+z^2)-6x-8y-24z+2\lambda(x^2+y^2+z^2)=0$

$\Rightarrow 2-6x-8y-24z+2\lambda$ [using (ii)]

$\Rightarrow 3x+4y+12z=1+\lambda$. (vi)

From (iii), (iv) and (v), we get

$x=\dfrac{3}{1+\lambda}$, $y=\dfrac{4}{1+\lambda}$, $z=\dfrac{12}{1+\lambda}$.

Putting these values in (vi), we get

$\dfrac{9}{1+\lambda}+\dfrac{16}{1+\lambda}+\dfrac{144}{1+\lambda}=1+\lambda \quad \Rightarrow (1+\lambda)^2=169 \Rightarrow 1+\lambda=\pm 13$

$\therefore \lambda=12$ or -14.

When $\lambda=12$, $x=\dfrac{3}{13}$, $y=\dfrac{4}{13}$, $z=\dfrac{12}{13}$.

When $\lambda=-14$, $x=-\dfrac{3}{13}$, $y=-\dfrac{4}{13}$, $z=-\dfrac{12}{113}$.

Thus we get two points $P\left(\dfrac{3}{13},\dfrac{4}{13},\dfrac{12}{13}\right)$ and $Q\left(-\dfrac{3}{13},-\dfrac{4}{13},-\dfrac{12}{13}\right)$

on the sphere which are at a maximum or minimum distance from the given point A.

Now $AP=\sqrt{\left(3-\dfrac{3}{13}\right)^2+\left(4-\dfrac{4}{13}\right)^2+\left(12-\dfrac{12}{13}\right)}=12$

$AQ=\sqrt{\left(3+\dfrac{3}{13}\right)^2+\left(4+\dfrac{4}{13}\right)^2+\left(12+\dfrac{12}{13}\right)}=14$.

$\therefore P\left(\dfrac{3}{13},\dfrac{4}{13},\dfrac{12}{13}\right)$ is at a minimum distance from A and the minimum distance = 12.

$Q\left(-\dfrac{3}{13},-\dfrac{4}{13},-\dfrac{12}{13}\right)$ is at a maximum distance from A and the maximum distance = 14.

Q.No.7.: Prove that the stationary values of $u=\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}$, where

$\ell x+my+nz=0$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ are the roots of equation

$$\frac{\ell^2 a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0.$$

Sol.: Consider Lagrange's function,

$$F(x,y,z)=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda(\ell x+my+nz)+\mu\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right).$$

For stationary values, $dF=0$

$$\Rightarrow\left(\frac{2x}{a^4}+\lambda\ell+\frac{2\mu x}{a^2}\right)dx+\left(\frac{2y}{b^4}+\lambda m+\frac{2\mu y}{b^2}\right)dy+\left(\frac{2z}{c^4}+\lambda n+\frac{2\mu z}{c^2}\right)=0$$

$$\Rightarrow\frac{2x}{a^4}+\lambda\ell+\frac{2\mu x}{a^2}=0, \tag{i}$$

$$\frac{2y}{b^4}+\lambda m+\frac{2\mu y}{b^2}=0, \tag{ii}$$

$$\frac{2z}{c^4}+\lambda n+\frac{2\mu z}{c^2}=0. \tag{iii}$$

Multiplying (i), (ii), (iii) by x, y, z respectively and adding, we get

$$2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda(\ell x+my+nz)+2\mu\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

$\Rightarrow 2u+\lambda(0)+2\mu(1)=0$ [From given relation]

$\Rightarrow \mu=-u$.

$\therefore$ Equation (i) becomes $\frac{2x}{a^4}+\lambda\ell-\frac{2ux}{a^2}=0 \Rightarrow \frac{2x}{a^4}(1-a^2u)=-\lambda\ell \Rightarrow x=-\frac{\lambda\ell a^4}{2(1-a^2u)}$.

Similarly, $y=-\frac{\lambda m b^4}{2(1-b^2u)}$, $z=-\frac{\lambda n c^4}{2(1-c^2u)}$.

To eliminate λ between them, multiply these values of x, y, z by ℓ, m, n respectively and add. Then $\ell x+my+nz=\frac{\lambda}{2}\left[\frac{\ell^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}\right]$

Since $\ell x+my+nz=0$, we have $\frac{\ell^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$,

which is a quadratic in u and gives two stationary values of u.

Q.No.8: Use Lagrange's method to determine the minimum distance from the origin to the plane $3x + 2y + z = 12$.

Sol.: The distance of any point P(x, y, z) on the plane $3x + 2y + z - 12 = 0$ from the origin is given by $\sqrt{x^2 + y^2 + z^2}$

or the square of the distance is equal to $x^2 + y^2 + z^2$.

Let $f(x, y, z) = x^2 + y^2 + z^2$. (i)

Now f is to be maximized or minimized subject to the constraint.

$3x + 2y + z - 12 = 0$ i. e. $\phi(x, y, z) = 0$. (ii)

Let $F = f + \lambda\phi$, where λ is Lagrange's multipliers. Since f is to be extremised, we have

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + 3\lambda = 0 \Rightarrow x = \frac{-3}{2}\lambda.$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + 2\lambda = 0 \Rightarrow y = -\lambda.$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \Rightarrow z = \frac{-\lambda}{2}.$$

Putting the values of x, y, z in (ii), we get

$$3\left(\frac{-3}{2}\lambda\right) + 2(-\lambda) + \left(\frac{-\lambda}{2}\right) - 12 = 0.$$

$$\Rightarrow \lambda = \frac{-12}{7} \Rightarrow x = \frac{18}{7}, \quad y = \frac{12}{7}, \quad z = \frac{6}{7}.$$

$$\therefore f(x, y, z) = x^2 + y^2 + z^2 = \left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2.$$

Minimum distance $= \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{12}{\sqrt{14}}$. Ans.

Distance required has to be minimum, since the maximum distance can be as large as we like i.e., infinity.

Q.No.9: Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line

$y = \sqrt{3}(2 - x)$.

Sol.: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points on the circle and straight line respectively such that the distance 'd' between P and Q is given by

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Now d is minimum if,

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} \quad \text{and} \quad \frac{\partial F}{\partial y_1} = 0 = \frac{\partial F}{\partial y_2},$$

where F is the Lagrange's function given by

$$F = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right] + \lambda_1\left(x_1^2 + y_1^2 - 1\right) + \lambda_2\left(y_2 + \sqrt{3}x_2 - 2\sqrt{3}\right)$$

$$\frac{\partial F}{\partial x_1} = 0, \text{ gives } -2(x_2 - x_1) + 2\lambda_1 x_1 = 0, \tag{i}$$

$$\frac{\partial F}{\partial x_2} = 0, \text{ gives } 2(x_2 - x_1) + \sqrt{3}\lambda_2 = 0. \tag{ii}$$

Similarly, $\frac{\partial F}{\partial y_1} = 0$, and $\frac{\partial F}{\partial x_2} = 0$

$$-2(y_2 - y_1) + 2\lambda_1 y_1 = 0, \tag{iii}$$

$$2(y_2 - y_1) + \lambda_2 = 0. \tag{iv}$$

The equations (i), (ii), (iii) and (iv) along with the equations of circle $x_1^2 + y_1^2 = 1$ and straight line $y_2 = \sqrt{3}(2 - x)$, constitute six equations, which on solving give $x_1, x_2, y_1, y_2, \lambda_1$ and λ_2.

We have from (i) and (iii) $\frac{y_2}{x_2} = \frac{y_1}{x_1}$. (v)

From (v) it is clear that the shortest distance lies on the line passing through the origin and meeting the point P and Q.

From (ii) and (iv), we get

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{\sqrt{3}}. \tag{vi}$$

Equation (vi) shows that the straight line joining P and Q and passing through the origin, makes an angle of 30^o with x-axis.

Thus the line where the minimum distance lies is given by

$$y = \frac{1}{\sqrt{3}}x. \tag{vii}$$

As Q lies on $y = 2\sqrt{3} - \sqrt{3}x$ and also on (vii), we have coordinates of Q on solving for x and y as

$$x = \frac{3}{2}, \quad y = \frac{\sqrt{3}}{2} \qquad \Rightarrow Q = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right).$$

The circle $x^2 + y^2 = 1$ also meets the lines $y = \frac{1}{\sqrt{3}}x$.

Solving for x and y from $x^2 + y^2 = 1$ and $y = \frac{1}{\sqrt{3}}x$.

We have $x = \pm\frac{\sqrt{3}}{2}, \quad y = \pm\frac{1}{2}$.

$\therefore$ Points on the circle where the line (vii) meets are

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$$

$$\therefore P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$d^2 = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 4 - 2\sqrt{3}$$

$\Rightarrow d = \sqrt{4 - 2\sqrt{3}}$, the required minimum distance.

The point $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$ gives maximum distance.

Q.No.10: A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas for its construction is to be minimum for the tent to have a given capacity.

Sol.: Let V be the volume enclosed by the tent and S be its surface area.

Then $V = \text{Cuboid}(ABCD, A'B'C'D') + \text{Pyramid } K, A'B'C'D$

$$= x^2y + \frac{1}{3}x^2h = x^2\left(y + \frac{h}{3}\right)$$

$$S = 4(ABGF) + 4\Delta KGH = 4xy + 4\frac{1}{2}(x.\ KM)$$

$$= 4xy + x\sqrt{\left(x^2 + 4h^2\right)} \qquad \left[\because KM = \sqrt{\left(KL^2 + LM^2\right)} = \sqrt{h^2 + \left(\frac{x}{2}\right)^2}\right]$$

For constant V, we have

$$\delta V = 2x\left(y + \frac{h}{3}\right)\delta x + x^2(\delta y) + \frac{x^2}{3}\delta h = 0.$$

For minimum S, we have

$$\delta S = \left[4y + \sqrt{\left(x^2 + 4h^2\right)} + x.\frac{1}{2}\left(x^2 + 4h^2\right)^{-1/2}.2x\right]\delta x + 4x\delta y + x.\frac{1}{2}\left(x^2 + 4h^2\right)^{-1/2}.8h\delta h = 0.$$

By Lagrange's method,

$$\left[4y + \sqrt{\left(x^2 + 4h^2\right)} + x^2\left(x^2 + 4h^2\right)^{-1/2}\right] + \lambda.2x\left(y + \frac{h}{3}\right) = 0 \qquad \text{(i)}$$

$$4x + \lambda.x^2 = 0 \qquad \text{(ii)}$$

$$4hx\left(x^2 + 4h^2\right)^{-1/2} + \lambda.\frac{x^2}{2} = 0 \qquad \text{(iii)}$$

(ii) gives $\lambda = -\frac{4}{x}$. Then (iii) becomes

$$4hx\left(x^2 + 4h^2\right)^{-1/2} - \frac{4x}{3} = 0 \Rightarrow x = \sqrt{5}h.$$

Now putting $x = \sqrt{5}$, $\lambda = -\frac{4}{x}$ in (i), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x}.2x\left(y + \frac{h}{3}\right) = 0 \Rightarrow 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0$$

$$\Rightarrow y = \frac{h}{2}.$$

Q.No.11: If xyz = 8, find the value of x, y for which $u = \frac{5xyz}{(x + 2y + 4z)}$ is a maximum.

Sol.: Let $f(x, y, z) = \frac{5xyz}{(x + 2y + 4z)}$. ....(i)

Also given $xyz = 8$ ...(ii)

Now f is to be maximized subject to the constraint $xyz - 8 = 0$ i.e. $\phi(x, y, z) = 0$.

Let $F = f + \lambda\phi$ where λ is Lagrange's multipliers.

$\therefore\ F = \dfrac{5xyz}{(x+2y+4z)} + \lambda(xyz-8)$.

Then $\dfrac{\partial F}{\partial x} = \dfrac{(x+2y+4z).5yz+5xyz}{(x+2y+4z)^2} + \lambda yz = 0$ ...(iii)

$\dfrac{\partial F}{\partial y} = \dfrac{(x+2y+4z).5zx+10xyz}{(x+2y+4z)^2} + \lambda zx = 0$ ...(iv)

$\dfrac{\partial F}{\partial z} = \dfrac{(x+2y+4z).5xy+20xyz}{(x+2y+4z)^2} + \lambda xy = 0$ ...(v)

Multiplying (iii) by x and (iv) by y and subtracting, we get

$x = 2y$ ...(vi)

Multiplying (iv) by y and (v) by z and subtracting, we get

$y = 2z$ ...(vii)

From (vi) and (vii) , we get $x = 2y = 4z$

Substitute these values in (ii), we get

$\therefore\ xyz = 8 \Rightarrow x.\dfrac{x}{2}.\dfrac{x}{4} = 8 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$\therefore x = 4,\ y = 2,\ z = 1$. Ans.

Q.No.12: Find the minimum value of $x^2 + y^2 + z^2$, given that

(i) $xyz = a^3$ (ii) $ax + by + cz = p$

Sol.: (i) Let $f(x,y,z) = x^2 + y^2 + z^2$. ....(i)

Also given $xyz = a^3$ ...(ii)

Now f is to be maximized subject to the constraint $\phi(x,y,z) = xyz - a^3 = 0$.

Let $F = f + \lambda\phi$ where λ is Lagranges's multipliers.

$\therefore\ F = x^2 + y^2 + z^2 + \lambda(xyz - a^3)$.

Then $\dfrac{\partial F}{\partial x} = 2x + \lambda yz = 0$ ...(iii)

$\dfrac{\partial F}{\partial y} = 2y + \lambda zx = 0$ ...(iv)

$$\frac{\partial F}{\partial z} = 2z + \lambda xy = 0 \qquad ...(v)$$

Multiplying (iii) by x and (iv) by y and subtracting, we get

$$x^2 = y^2 \qquad ...(vi)$$

Multiplying (iv) by y and (v) by z and subtracting, we get

$$y^2 = z^2 \qquad ...(vii)$$

From (vi) and (vii) , we get $x^2 = y^2 = z^2$

$$\Rightarrow x = y = z$$

Substitute these values in (ii), we get

$\therefore\ xyz = a^3 \Rightarrow x.x.x = a^2 \ \Rightarrow\ x^3 = a^3 \ \Rightarrow\ x = a$

$\therefore x = a,\ y = a,\ z = a$.

Hence the minimum value of $x^2 + y^2 + z^2 = a^2 + a^2 + a^2 = 3a^2$. Ans.

(ii) Let $f(x, y, z) = x^2 + y^2 + z^2$ . ....(i)

Also given $ax + by + cz = p$ ...(ii)

Now f is to be maximized subject to the constraint $\phi(x, y, z) = ax + by + cz - p = 0$.

Let $F = f + \lambda\phi$ where λ is Lagranges's multipliers.

$\therefore\ F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$.

Then $\frac{\partial F}{\partial x} = 2x + \lambda a = 0$ ...(iii)

$$\frac{\partial F}{\partial y} = 2y + \lambda b = 0 \qquad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda c = 0 \qquad ...(v)$$

Multiplying (iii) by b and (iv) by a and subtracting, we get

$$ay = bx \qquad ...(vi)$$

Multiplying (iii) by c and (v) by a and subtracting, we get

$$az = cx \qquad ...(vii)$$

From (vi) and (vii) , we get $ax + by + cz = p$

$$\Rightarrow ax + b\frac{bx}{a} + c\frac{cx}{a} = p$$

$$\Rightarrow a^2x + b^2y + c^2z = ap$$

$$\Rightarrow x = \frac{ab}{a^2 + b^2 + c^2}$$

Similarly $y = \dfrac{bp}{a^2 + b^2 + c^2}$, $z = \dfrac{cp}{a^2 + b^2 + c^2}$

Hence the minimum value of $x^2 + y^2 + z^2 = \dfrac{p^2}{a^2 + b^2 + c^2}$. Ans.

Q.No.13: Find the extreme values of $x^2 + y^2 + z^2$, when $ax + by + cz = p$.

Sol.: Let $f(x, y, z) = x^2 + y^2 + z^2$

And $\phi(x, y, z) = ax + by + cz - p = 0$ (i)

$\therefore f(x, y, z)$ is extremised.

$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$\therefore F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - P)$

$\dfrac{\partial F}{\partial x} = 2x + a\lambda$, $\dfrac{\partial F}{\partial y} = 2y + b\lambda$ and $\dfrac{\partial F}{\partial z} = 2z + ac\lambda$

Now to find for extreme values

$\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$.

$\therefore 2x + a\lambda = 0$, $2y + b\lambda = 0$ and $2z + c\lambda = 0$

$\Rightarrow \lambda = \dfrac{-2x}{a}$, $\lambda = \dfrac{-2y}{b}$ and $\lambda = \dfrac{-2z}{c}$

$\therefore \dfrac{x}{y} = \dfrac{a}{b}$, $\dfrac{y}{z} = \dfrac{b}{c}$ and $\dfrac{z}{x} = \dfrac{c}{a}$.

Substituting these in (i), we get

$x = \dfrac{ap}{a^2 + b^2 + c^2}$, $y = \dfrac{bp}{a^2 + b^2 + c^2}$ and $z = \dfrac{cp}{a^2 + b^2 + c^2}$

$\therefore f(x, y, z) = x^2 + y^2 + z^2$ i. e. the minimum value.

$$\frac{a^2p^2}{\left(a^2+b^2+c^2\right)^2}+\frac{b^2p^2}{\left(a^2+b^2+c^2\right)^2}+\frac{c^2p^2}{\left(a^2+b^2+c^2\right)^2}=\frac{p^2}{a^2+b^2+c^2}$$. Ans.

Q.No.14.: Find a point (x_0, y_0, z_0) on the plane $ax+by+cz+d=0$, which is the nearest to the origin.

Sol.: Let L be the length from the origin.

$$\Rightarrow L=\sqrt{(x_0-0)^2+(y_0-0)^2+(z_0-0)^2}=\sqrt{x_0^2+y_0^2+z_0^2}$$

Now, L is to be minimized subject to the constant

$$ax_0+by_0+cz_0+d=0$$

$$\phi(x_0, y_0, z_0)=ax_0+by_0+cz_0+d=0$$

i. e. $F=L+\lambda\phi$

$$\Rightarrow F=\sqrt{x_0^2+y_0^2+z_0^2}+\lambda(ax_0+by_0+cz_0+d)$$

where λ is the Langrange's multiplier.

Since L is to be minimized, so we have

$$\frac{\delta F}{\delta x_0}=0,\quad \frac{\delta F}{\delta y_0}=0 \quad \text{and} \quad \frac{\delta F}{\delta z_0}=0$$

which gives (i)

$$\frac{y_0}{\sqrt{x_0^2+y_0^2+z_0^2}}+b\lambda=0 \qquad \text{(ii)}$$

and $$\frac{z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}+c\lambda=0 \qquad \text{(iii)}$$

Fro equation (i), $\lambda=\dfrac{-x_0}{a\sqrt{x_0^2+y_0^2+z_0^2}}$

Hence $y_0=\dfrac{bx_0}{a}$ and $z_0=\dfrac{cx_0}{a}$

Putting these values in the given equation, we get

$$ax_0+\frac{b^2x_0}{a}+\frac{c^2x_0}{a}+d=0\Rightarrow\left(a^2+b^2+c^2\right)x_0=ad$$

$$\Rightarrow x_0=\frac{-ad}{a^2+b^2+c^2},\quad y_0=\frac{-bd}{a^2+b^2+c^2} \quad \text{and} \quad z_0=\frac{-cd}{a^2+b^2+c^2}$$

$$\Rightarrow |x_0| = \frac{ad}{a^2+b^2+c^2}, \quad |y_0| = \frac{bd}{a^2+b^2+c^2} \text{ and } |z_0| = \frac{cd}{a^2+b^2+c^2}.$$

Putting these values in L finally

$$L = \sqrt{\frac{a^2d^2}{\left(a^2+b^2+c^2\right)^2} + \frac{b^2d^2}{\left(a^2+b^2+c^2\right)^2} + \frac{c^2d^2}{\left(a^2+b^2+c^2\right)^2}} = d\sqrt{\frac{a^2+b^2+c^2}{\left(a^2+b^2+c^2\right)^2}}$$

$$= \frac{d}{\sqrt{a^2+b^2+c^2}}.$$

As (x_0, y_0, z_0) approach towards infinity, the distance from origin also tends to infinity and when (x_0, y_0, z_0) approach towards origin, the distance gets shorter. Hence the values founds out of (x_0, y_0, z_0) gives the minimum distance from origin and that minimum distance is $\frac{d}{\sqrt{a^2+b^2+c^2}}$. Ans.

Q.No.15.: Find a point on the curve of intersection of surface $x+y+z=1$ and $z-xy=5$, which is nearest to the origin.

Sol.: Distance of point x, y, z from origin is given by $d = \sqrt{x^2+y^2+z^2}$.

So, let $f(x) = x^2+y^2+z^2$ (i)

As x, y, z lies on intersection of two curves it must satisfy both the equations.

$\phi(x) = x+y++z-1=0$ (ii)

$\psi(x) = z-xy-5=0$ (iii)

Using Lagrange's Multiplier method, let

$$F(x) = f(x) + \lambda_1(x) + \lambda_2(x) = x^2+y^2+z^2+\lambda_1(x+y+z-1)+\lambda_2(z-xy-5)$$

Now, $\frac{\partial F(x)}{\partial x} = 0 = 2x+\lambda_1+\lambda_2 y$ (iv)

$\frac{\partial F(x)}{\partial y} = 0 = 2y+\lambda_1-\lambda_2 y$ (v)

$\frac{\partial F(x)}{\partial z} = 0 = 2z+\lambda_1+\lambda_2$ (vi)

Subtracting (v) from (iv), we get

$(2+\lambda_2)(x+y)=0$

Either $x = y$ (vii) $\lambda_2 = -2$ (viii)

If $x = y$ or $z = 1-2y$

$\therefore$ In equation (iii) $z-2y-y^2-5=0 \Rightarrow y^2+2y+1=-3$

$\Rightarrow (y+1)^2 = -3$, which is not possible.

$\lambda_2 = -2$

$\therefore$ From equation (iv), $2(x+y)+\lambda_1 = 0$ (ix)

From equation (vi), $2z-2+\lambda_1 = 0$ (x)

Adding (ix) and (x), we get

$2(x+y+z)-2+2\lambda_1 = 0$ (xi)

From (ii) and (xi), we get $0 = 2\lambda_2 \Rightarrow \lambda_1 = 0$ (xii)

From (x) and (xii), we get $2z-2+0=0 \Rightarrow z=1$ (xiii)

From (ix), we get $x+y=0 \Rightarrow x=-y$ (xiv)

Using (xiii), (xiv) and (iii), we get $1+y^2-5=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$.

From (xiv), we get $\therefore x = -y = -(\pm 2) = \pm 2$.

$\therefore$ The points are $(2,-2,\ 1)$ and $(-2, 2,\ 1)$. Ans.

Q.No.16.: Show that all the rectangular parallelepipeds which have sides parallel to the coordinate planes and which are inscribed in the sphere $x^2+y^2+z^2=a^2$, one which has the maximum volume is a cube.

Sol.: Let the sides be ℓ, m, n. $\therefore$ Volume = ℓmn

From figure, $\ell^2+m^2+n^2=a^2$.

By Lagrange's method, we get

$\ell mn + \lambda(\ell^2+m^2+n^2-a^2) = 0$

$f_\ell = 0 \Rightarrow mn + 2\lambda\ell = 0$ (i)

$f_m = 0 \Rightarrow \ell n + 2\lambda m = 0$ (ii)

$f_n = 0 \Rightarrow m\ell + 2\lambda n = 0$ (iii)

From (i) and (ii), we get $\dfrac{mn}{\ell} = \dfrac{\ell n}{m} \Rightarrow \ell^2 = m^2 \Rightarrow \ell = m$

From (i) and (ii), we get $\frac{\ell n}{m}=\frac{m\ell}{n}\Rightarrow m^2=n^2\Rightarrow m=n$.

$\Rightarrow \ell=m=n\Rightarrow 3\ell^2=a^2\Rightarrow \ell=\frac{a}{\sqrt{3}}=m=n$.

$\therefore$ The maximum volume that can be inscribed is a cube. Ans.

Q.No.17.: If $f=a^3x^2+b^3y^2+c^2z^2$, where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, prove that stationary value of f is given by $x=\frac{a+b+c}{a}$, $y=\frac{a+b+c}{b}$ and $z=\frac{a+b+c}{c}$.

Sol.: Given that $f=a^3x^2+b^3y^2+c^2z^2$ (i)

Also $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ (ii)

$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0=\phi(x,y,z)$ [say]

To find the stationary value of f subject to considered $\phi(x,y,z)$

Let $F=f+\lambda\phi$ [Here λ is Lagrange's multiplier]

$\Rightarrow F=a^3x^2+b^3y^2+c^2z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$.

Since we have to find stationary value of f.

$\therefore \frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$ and $\frac{\partial F}{\partial z}=0$.

$\frac{\partial F}{\partial x}=2a^3x-\frac{\lambda}{x^2}=0\Rightarrow 2a^3x=\frac{\lambda}{x^2}\Rightarrow \lambda=2a^3x^3$ (iii)

$\frac{\partial F}{\partial y}=2b^3y^3=\lambda$ (iv) $\frac{\partial F}{\partial z}=2c^3z^3=\lambda$ (v)

From (iii), (iv) and (v), we get

$a^3x^3=b^3y^3=c^3z^3$

$\therefore x=\left(\frac{b}{a}\right)y=\left(\frac{c}{a}\right)z$.

Also $y=\left(\frac{a}{b}\right)x$ (vi) $z=\left(\frac{a}{c}\right)x$ (vii)

Using (vi) and (vii) in (ii), we get

$$\frac{1}{x}+\frac{b}{ax}+\frac{c}{ax}=1 \Rightarrow \frac{1}{x}\left[1+\frac{b}{a}+\frac{c}{a}\right]=1 \; x=\frac{a+b+c}{a}.$$

$$x=\left(\frac{b}{a}\right)y, \quad z=\left(\frac{b}{c}\right)y.$$

Using these two equations in (ii), we get

$$\frac{a}{by}+\frac{1}{y}+\frac{c}{by}=1 \Rightarrow \frac{1}{y}\left[\frac{a}{b}+1+\frac{c}{b}\right]=1 \Rightarrow y=\frac{a+b+c}{b}.$$

$$x=\left(\frac{c}{a}\right)z, \quad y=\left(\frac{c}{b}\right)z.$$

Using these two equations in (ii), we get

$$\frac{1}{\left(\frac{c}{a}\right)z}+\frac{1}{\left(\frac{c}{b}\right)z}+\frac{1}{z}=1 \Rightarrow \frac{a}{cz}+\frac{b}{cz}+\frac{1}{z}=1 \Rightarrow \frac{1}{z}\left[\frac{a}{c}+\frac{b}{c}+1\right]=1$$

$$\Rightarrow z=\frac{a+b+c}{c}.$$

$\therefore x=\dfrac{a+b+c}{a}$, $y=\dfrac{a+b+c}{b}$ and $z=\dfrac{a+b+c}{c}$ for stationary value of f. Hence this proves the result. Ans.

Q.No.18.: A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found by using Lagrange's method.

Sol.: Let x and y be two parts into which the given wire is cut so that x + y = b.

Suppose the piece of wire of length x is bent into a square so that each side is $\frac{x}{4}$ and thus the area of the square is $\frac{x}{4}.\frac{x}{4}=\frac{x^2}{16}$.

Suppose the wire of length y is bent into a circle with perimeter y. So the area of this circle so formed is

$$\pi(\text{radius})^2=\pi\left(\frac{y}{2\pi}\right)^2=\frac{\pi y^2}{4\pi^2}=\frac{y^2}{4\pi}. \qquad \text{Since } 2\pi(\text{radius})=y \Rightarrow \text{radius}=\frac{y}{2\pi}.$$

Thus to find the minimum of the sum of the two areas subject to the constraint that sum x +y = b.

Let $F = f + \lambda\phi$, where λ is Lagranges's multipliers.

$$F((x, y) = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x + y - b).$$

Then $\dfrac{\partial F}{\partial x} = \dfrac{x}{8} + \lambda = 0$

$$\frac{\partial F}{\partial y} = \frac{y}{2\pi} + \lambda = 0$$

Solving, we get $x = -8\lambda$, $y = -2\pi\lambda$.

Substituting these values in the constraint condition x +y = b, we get

$$-8\lambda - 2\pi\lambda = b \Rightarrow \lambda = -\frac{b}{8 + 2\pi}.$$

Thus $x = -8\lambda = \dfrac{8b}{8 + 2\pi}$ and $y = -2\pi\lambda = \dfrac{2\pi b}{8 + 2\pi}$.

Thus the least value of the sum of the areas of the square and circle is

$$F((x, y) = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right)_{\substack{x = \frac{8b}{8+2\pi} \\ y = \frac{2\pi b}{8+2\pi}}} = \frac{64b^2}{16(8 + 2\pi)^2} + \frac{4\pi^2 b^2}{4\pi(8 + 2\pi)^2} = \frac{4b^2}{4(4 + \pi)^2} + \frac{\pi b^2}{4(4 + \pi)^2}$$

$$= \frac{4b^2 + \pi b^2}{4(4 + \pi)^2} = \frac{b^2}{4(4 + \pi)}. \text{ Ans.}$$

Q.No.19.: Find (by the Lagrange's method) the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

Sol.: This is a constraint maximum problem where the function $f(x, y, z) = x^m y^n z^p$ is given, subjected to the constraint condition $x + y + z = a$.

So consider the Lagrange's auxiliary function

$F = f + \lambda\phi$, where λ is Lagrange's multiplier.

$$\Rightarrow F(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a). \qquad \text{(i)}$$

Differentiating (i) w.r.t x, y, z and equating to zero, we get

$$F_x = \frac{\partial F}{\partial x} = mx^{m-1} y^n z^p + \lambda = 0, \qquad \text{(ii)}$$

$$F_y = \frac{\partial F}{\partial y} = nx^m y^{n-1} z^p + \lambda = 0\,, \tag{iii}$$

$$F_z = \frac{\partial F}{\partial z} = px^m y^n z^{p-1} + \lambda = 0\,. \tag{iv}$$

Solving for x, y, z, we obtain $\frac{m}{x}f + \lambda = 0 \Rightarrow x = -\frac{mf}{\lambda}$.

Similarly, we can obtain $y = -\frac{nf}{\lambda}$ and $z = -\frac{pf}{\lambda}$.

Substituting these values in the given constraint condition, we have

$$x + y + z = -\left(\frac{m}{\lambda} + \frac{n}{\lambda} + \frac{p}{\lambda}\right)f = a\,.$$

Solving, we get the value of λ as $\lambda = -\frac{(m+n+p)}{a}f$.

Using this λ, we get

$$x = -\frac{mf}{\lambda} = -\frac{mf.(-a)}{f(m+n+p)} = \frac{am}{m+n+p}\,.$$

Similarly, $y = -\frac{nf}{\lambda} = \frac{an}{m+n+p}$ and $z = -\frac{pf}{\lambda} = \frac{ap}{m+n+p}$.

Thus, the maximum value is

$$x^m y^n z^p = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p = \frac{a^{m+n+p}.m^m.n^n.p^p}{(m+n+p)^{m+n+p}}\text{. Ans.}$$

Q.No.20.: Find (by the Lagrange's method) the maximum and minimum distances from the origin to the curve $3x^2 + 4xy + 6y^2 = 140$.

Sol.: The distance d from the origin (0, 0) to any point (x, y) is given by

$$x^2 + y^2 = f(x, y)\,.$$

To find: The extrema of f (x, y) subject to the condition that the point (x, y) lies on the curve $3x^2 + 4xy + 6y^2 = 140$.

So consider the Lagrange's auxiliary function

$F = f + \lambda\phi$, where λ is Lagrange's multiplier.

$$\Rightarrow F(x, y) = \left(x^2 + y^2\right) + \lambda\left(3x^2 + 4xy + 6y^2 - 140\right). \tag{i}$$

Differentiating (i) w.r.t x, y and equating to zero we get

$F_x = 2x + \lambda(6x + 4y) = 0$,

$F_y = 2y + \lambda(12y + 4x) = 0$.

Solving for $\lambda = -\dfrac{x}{(3x + 2y)} = -\dfrac{y}{(6y + 2x)}$,

$$-\lambda = \frac{x^2}{3x^2 + 2xy} = \frac{y^2}{6y^2 + 2xy} = \frac{x^2 + y^2}{3x^2 + 4xy + 6y^2}$$

$\therefore -\lambda = \dfrac{f}{140}$.

Substituting λ in $F_x = 0$ and $F_y = 0$, we get

$(140 - 3f)x - 2fy = 0$,

$-2fx + (140 - 6f)y = 0$.

This system has non-trivial solution if $\begin{vmatrix} 140 - 3f & -2f \\ -2f & 140 - 6f \end{vmatrix} = 0$

$\Rightarrow (140 - 3f)(140 - 6f) - 4f^2 = 0$

$\Rightarrow 14f^2 - 1260f + 140^2 = 0 \quad \Rightarrow f^2 - 90f - 1400 = 0$

$\Rightarrow (f - 70)(f - 20) = 0 \quad\quad \Rightarrow f = 70,\ 20$.

Thus, the maximum and minimum distances are $\sqrt{70},\ \sqrt{20}$. Ans.

Q.No.21.: Find (by the Lagrange's method) the dimensions of a rectangular box of maximum capacity whose surface area is given when

(a) box is open at the top **(b)** box is closed.

Sol.: Let x, y, z be the dimensions of the rectangular box so that its volume V is

$V = xyz$. (i)

The total surface area of the box is

$nxy + 2yz + 2zx = S =$ given constant (ii)

Here n = 1, the box is open at the top

n = 2, the box is closed (on all sides)

The constrained maximum problem is to maximize V subject to constraint condition (ii).

So consider the Lagrange's auxiliary function

$F = f + \lambda\phi$, where λ is Lagrange's multiplier.

$\Rightarrow F(x, y, z) = xyz + \lambda(nxy + 2yz + 2zx - S)$. (iii)

Differentiating (iii) w.r.t x, y, z and equating to zero, we get

$F_x = yz + \lambda(ny + 2z) = 0$, (iv)

$F_y = xz + \lambda(nx + 2z) = 0$, (v)

$F_z = xy + \lambda(2y + 2x) = 0$. (vi)

Multiplying (iv), (v) and (vi) by x, y, z respectively and adding, we get

$3xyz + \lambda[2(nxy + 2yz + 2zx)] = 0$

$\Rightarrow 3.V + 2\lambda.S = 0$ (using (i) and (ii))

$\Rightarrow \lambda = -\dfrac{3V}{2S}$. (vii)

Substituting the value of λ from (vii) in (iv), (v) and (vi), we get

$yz - \dfrac{3V}{2S}(ny + 2z) = 0 \Rightarrow yz - \dfrac{3xyz}{2S}(ny + 2z) = 0 \Rightarrow nxy + 2xz = \dfrac{2S}{3}$. (viii)

Similarly, we can obtain $nxy + 2yz = \dfrac{2S}{3}$. (ix)

$2yz + 2zx = \dfrac{2S}{3}$ (x)

From $(viii) - (ix)$, we get $x = y$. (xi)

From $(ix) - (x)$, we get $ny = 2z$. (xii)

Substituting (ix) and (xii) in the given constraint condition (ii), we obtain

$n.x.x + 4x\dfrac{nx}{2} = S \Rightarrow 3nx^2 = S \Rightarrow x^2 = \dfrac{S}{3n}$.

Case a: When box is open at the top: n = 1.

Then $x^2 = \dfrac{S}{3} \Rightarrow x = \sqrt{\dfrac{S}{3}}$.

Also since $x = y$ and $ny = 2z$. Thus $x = y = \sqrt{\dfrac{S}{3}}$, $z = \dfrac{1}{2}\sqrt{\dfrac{S}{3}}$.

Hence the dimensions of the open top box are $x = y = \sqrt{\dfrac{S}{3}}$, $z = \dfrac{1}{2}\sqrt{\dfrac{S}{3}}$. Ans.

Case b: When the box is closed: n = 2.

Then $x^2 = \frac{S}{6} \Rightarrow x = \sqrt{\frac{S}{6}}$,

Also since $x = y$ and $ny = 2z$. Thus $x = y = z = \sqrt{\frac{S}{6}}$.

Hence the dimensions of the closed box are $x = y = z = \sqrt{\frac{S}{6}}$. Ans.

Q.No.22.: Suppose a closed rectangular box has length twice its breadth and has constant volume V. Determine the dimensions of the box requiring least surface area (sheet metal) by the Lagrange's method..

Sol.: Let x be the breadth so that the length is 2x and y be the height of the closed rectangular box.

Its volume is $x.2x.y = 2x^2y = V$ (given).

The surface area (6 faces) S is given by $S = 2(2x.x) + 2(2x.y) + 2(x.y) = 4x^2 + 6xy$.

Thus, the problem is to minimize $f(x, y) = 4x^2 + 6xy$, (i)

subject to the constraint condition $x^2y = \frac{V}{2}$ (known). (ii)

So consider the Lagrange's auxiliary function

$F = f + \lambda\phi$, where λ is Lagrange's multiplier.

$$\Rightarrow F(x, y) = 4x^2 + 6xy + \lambda\left(x^2y - \frac{V}{2}\right). \tag{iii}$$

Differentiating (iii) w.r.t x and y and equating to zero, we get

$$F_x = 8x + 6y + 2\lambda xy = 0, \tag{iv}$$

$$F_y = 6x + \lambda x^2 = 0. \tag{v}$$

Solving (v), $\lambda = -\frac{6}{x} \Rightarrow x = -\frac{6}{\lambda}$. (vi)

Substituting x from (vi) in (iv), we get

$$\frac{-48}{\lambda} + 6y - 12y = 0 \Rightarrow y = -\frac{8}{\lambda}. \tag{vii}$$

Substituting (vi) and (vii) in the given constraint condition (ii), we get

$$\lambda^3 = -\frac{576}{V} \Rightarrow \lambda = -\left(\frac{576}{V}\right)^{1/3}. \quad \text{(viii)}$$

Using (viii), from (vi) and (vii), we get

$$x = \frac{-6}{\lambda} = 6.\left(\frac{V}{576}\right)^{1/3} = \left(\frac{3V}{8}\right)^{1/3}, \quad \text{(ix)}$$

$$y = \frac{-8}{\lambda} = 8.\left(\frac{V}{576}\right)^{1/3} = \left(\frac{8V}{9}\right)^{1/3}. \quad \text{(x)}$$

The least surface area with these dimensions (ix) and (x) is

$$S = 4x^2 + 6xy = 4.\left(\frac{3V}{8}\right)^{2/3} + 6.\left(\frac{3V}{8}\right)^{1/3}\left(\frac{8V}{9}\right)^{1/3}.$$

On simplification

$S = \left(3^5 V^2\right)^{1/3} = \left(243V^2\right)^{1/3}$. Ans.

Q.No.23.: Find the extremum values of $\sqrt{x^2 + y^2}$ when $13x^2 - 10xy + 13y^2 = 72$ by the Lagrange's method.

Sol.: $F = \sqrt{x^2 + y^2} + \lambda\left(13x^2 - 10xy + 13y^2 - 72\right)$

$$\frac{\partial F}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} + \lambda(26x - 10y)$$

$\frac{\partial F}{\partial x} = 0$ for maxima or minima

$$\Rightarrow \lambda = \frac{-x}{\sqrt{x^2 + y^2}} \times \frac{1}{(26x - 10y)} \quad \text{(i)}$$

$$\frac{\partial F}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} + \lambda(26y - 10x)$$

$\frac{\partial F}{\partial y} = 0$ for maxima or minima

$$\Rightarrow \lambda = \frac{-y}{\sqrt{x^2 + y^2}} \times \frac{1}{(26y - 10x)} \quad \text{(ii)}$$

From (i) and (ii), we get

$$\frac{-x}{\sqrt{x^2+y^2}} \times \frac{1}{(26x-10y)} = \frac{-y}{\sqrt{x^2+y^2}} \times \frac{1}{26y-10x}$$

$$\sqrt{x^2+y^2}\left(26xy-10y^2\right) = \sqrt{x^2+y^2}\left(26xy-10y^2\right)$$

$$\Rightarrow y^2 = x^2 \Rightarrow x = \pm y$$

For $13x^2 - 10xy + 13y^2 = 72$

When $x = y$

$$13y^2 + 13y^2 - 10y^2 = 72$$

$$16y^2 = 72 \Rightarrow y^2 = \frac{9}{2}$$

when $x = -y$

$$13x^2 + 10x^2 + 13y^2 = 72$$

$$36x^2 = 72 \Rightarrow x^2 = 2$$

$$\sqrt{x^2+y^2} = \sqrt{2y^2} = \sqrt{9} = 3 \quad \text{and} \quad \sqrt{2x^2} = \sqrt{2 \times 2} = 2$$

Maximum value = 3,

Minimum value = 2.

Q.No.24.: Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ by the Lagrange's method.

Sol.: $F(x, y, z) = x^2 + y^2 + z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

$$\frac{\partial F}{\partial x} = 2x - \frac{\lambda}{x^2}, \quad \frac{\partial F}{\partial y} = 2y - \frac{\lambda}{y^2}, \quad \frac{\partial F}{\partial z} = 2z - \frac{\lambda}{z^2}.$$

For maxima and minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$$\Rightarrow x^3 = y^3 = z^3 = \frac{\lambda}{2}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = 1$$

$\Rightarrow \frac{3}{x} = 1 \Rightarrow x = 3$

$\therefore x = y = z = 3$

$\therefore x^2 + y^2 + z^2 = 3^2 + 3^2 + 3^2 = 27$

when, $x = y = z = 0$

$\therefore x^2 + y^2 + z^2 = 27$ is the minimum value.

Q.No.25.: Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

Sol.: $F = yx^2z^3 + \lambda(y + x + z - 24)$ $\left\{\begin{matrix} \because yx^2z^3 = \max. \\ y + x + z = 24 \end{matrix}\right\}$

$\frac{\partial F}{\partial x} = 2yxz^3 + \lambda$, $\frac{\partial F}{\partial y} = x^2z^3 + \lambda$, $\frac{\partial F}{\partial z} = 3yx^2z^2 + \lambda$.

For maxima and minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$\lambda = -2yxz^2$

$\lambda = -x^2z^3 \Rightarrow x^2z^3 = 2yxz^3 \Rightarrow x = 2y$

$\lambda = -3yx^2z^2 \Rightarrow 3yx^2z^2 = x^2z^3 \Rightarrow 3y = z$

$y + x + z = 24$

$2y + y + 3y = 24$

$y = 4$, $x = 8$, $z = 12$.

$\therefore$ maximum value of $yx^2 z^2 = 4.8^2 . 12^3$.

Q.No.26.: Determine the perpendicular distance of the point (a, b, c) from the plane $\ell x + my + nz = 0$ by the Lagrange's method.

Sol.: Perpendicular distance of a point (a, b, c) from a point (x, y,z) can be calculated as

$D = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

$f \Rightarrow D^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

$\phi \Rightarrow \ell x + my + nz = 0$

$F = f + \lambda\phi = (x-a)^2 + (y-b)^2 + (z-c)^2 + \lambda(\ell x + my + nz)$

$$\frac{\partial F}{\partial x} = 2(x-a) + \lambda\ell = 0, \; \frac{\partial F}{\partial y} = 2(y-b) + \lambda m = 0, \; \frac{\partial F}{\partial z} = 2(z-c) + \lambda n = 0.$$

For maxima and minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$$\Rightarrow x = a - \frac{\lambda\ell}{2}, \quad y = b - \frac{\lambda m}{2}, \quad z = \frac{-\lambda n}{2} + c.$$

Also, $\ell x + my + nz = 0$

$$\ell a - \frac{\lambda\ell^2}{2} + mb - \frac{\lambda m^2}{2} + nc - \frac{\lambda n^2}{2} = 0$$

$$\lambda = \frac{2(\ell a + mb + nc)}{\ell^2 + m^2 + n^2} \qquad \text{(i)}$$

$$D^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

$$\Rightarrow x - a = a - \frac{\lambda\ell}{2} - a = \frac{-\lambda\ell}{2}$$

$$y - b = b - \frac{\lambda m}{2} - b = \frac{-\lambda m}{2}$$

$$D^2 = \frac{\lambda^2\ell^2}{4} + \frac{\lambda^2 m^2}{4} + \frac{\lambda^2 n^2}{4} = \frac{\lambda^2}{4}\left(\ell^2 + m^2 + n^2\right)$$

$$D = \frac{\lambda}{2}\sqrt{\ell^2 + m^2 + n^2} = \frac{\ell a + mb + nc}{\ell^2 + m^2 + n^2} \times \sqrt{\ell^2 + m^2 + n^2} \qquad \text{[From (i)]}$$

$$D = \frac{\ell a + mb + nc}{\sqrt{\ell^2 + m^2 + n^2}}. \text{ Ans.}$$

Q.No.27.: Determine the point in the plane $3x - 4y + 5z = 50$ nearest to the origin by the Lagrange's method.

Sol.: Let (x, y, z) be a point on the given plane which is nearest to the origin.

Distance between (x, y, z) and (0, 0, 0) is given by

$$f = D^2 = x^2 + y^2 + z^2$$

$$\phi = 3x - 4y + 5z - 50$$

$$F = f + \lambda\phi = x^2 + y^2 + z^2 + \lambda(3x - 4y + 5z - 50)$$

$$\frac{\partial F}{\partial x} = 2x + 3\lambda, \; \frac{\partial F}{\partial y} = 2y + 4\lambda, \; \frac{\partial F}{\partial z} = 2z + 5\lambda.$$

For maxima and minima, $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$

$\Rightarrow x=\frac{-3z}{2}, \quad y=\frac{4z}{2}, \quad z=\frac{-5z}{2}.$

Also, $3x-4y+5z=50$

$\frac{-9\lambda}{2}-\frac{16\lambda}{2}-\frac{25\lambda}{2}=50$

$\Rightarrow \frac{-50\lambda}{2}=50 \Rightarrow \lambda=-2$

$x=\frac{-3}{2}(-2)=3, \quad y=\frac{4}{2}(-2)=-4, \quad z=\frac{-5}{2}(-2)=5.$

$\therefore$ $(3, \ -4, \ 5)$ is the point on the given plane which is nearest to the origin (0, 0, 0).

Q.No.28.: Determine the point on the paraboloid $z=x^2+y^2$ which is closest to the point $(3, \ -6, \ 4)$ by the Lagrange's method.

Sol.: Let (x, y, z) be the required point.

Distance between (x, y, z) and $(3, \ -6, \ 4)$ is given by

$D=\sqrt{(x-3)^2+(y+6)^2+(z-4)^2}$

$f=D^2=(x-3)^2+(y+6)^2+(z-4)^2$

$\phi=x^2+y^2-z$

$F=f+\lambda\phi$

$F=(x-3)^2+(y+6)^2+(z-4)^2+\lambda(x^2+y^2-z)$

$\frac{\partial F}{\partial x}=2(x-3)+2\lambda x, \ \frac{\partial F}{\partial y}=2(y+6)+2\lambda y, \ \frac{\partial F}{\partial z}=2(z-4)-\lambda.$

For maxima and minima, $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$

$\Rightarrow x=\frac{3}{1+\lambda}, \quad y=\frac{-6}{1+\lambda}, \quad z=\frac{z}{2}+4.$

Also, $x^2+y^2=z$

$$\frac{9}{(1+\lambda)^2}+\frac{36}{(1+\lambda)^2}=\frac{\lambda}{2}+4$$

$$\frac{45}{(1+\lambda)^2}=\frac{\lambda+8}{2}\Rightarrow\left(1+\lambda^2+2\lambda\right)(\lambda+8)=90$$

$$\Rightarrow\lambda+8+\lambda^3+8\lambda^2+2\lambda^2+16\lambda=90$$

$$\Rightarrow\lambda^3+10\lambda^2+17\lambda-82=0$$

$$\Rightarrow(\lambda-2)\left(\lambda^2+12\lambda+41\right)=0$$

$$\Rightarrow\lambda=2$$

$$x=\frac{3}{3}=1,\quad y=\frac{-6}{3}=-2,\quad z=1+4=5.$$

$\therefore$ $(1,\ -2,\ 5)$ is the point on the paraboloid $z=x^2+y^2$ which is nearest to the point $(3,\ -6,\ 4)$.

Q.No.29.: a. Find (by the Lagrange's method) the dimensions of the rectangular box, without top, of maximum capacity, whose surface is 108 square cm.

b. What are the dimensions, when the box is closed (on all sides)?

Sol.: (a). Let x, y, z, V, A be the length, breadth, height, volume, surface area of the rectangular box.

$\phi = A = xy+2yz+2xz-108$

A = 108 sq. cm.

f = V = x, y, z

$F=f+\lambda\phi=xyz+\lambda(xy+2yz+2xz-108)$.

$$\frac{\partial F}{\partial x}=yz+\lambda y+2\lambda z,\quad \frac{\partial F}{\partial y}=xz+2\lambda y\frac{z}{y}=xz+2\lambda z+\lambda x,\quad \frac{\partial F}{\partial z}=xy+2\lambda x+2\lambda y.$$

For maxima and minima, $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$

$$\Rightarrow y\lambda=-yz\Rightarrow\lambda=-z\Rightarrow z=-\lambda$$

and $xz+2\lambda z=0\Rightarrow x=-2\lambda$

and $xy+2\lambda x=0\Rightarrow y=-2\lambda$

Also, $xy+2yz+2xz=108$

$4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 108 \Rightarrow 12\lambda^2 = 108$

$\Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3$

When $\lambda = -3$

$z = 3,\quad x = 6,\quad y = 6.$

When $\lambda = 3$

$z = -3,\quad x = -3,\quad z = -3.$

Lengths cannot be negative.

$\therefore$ Dimensions of rectangular base are

$z = 3,\quad x = 6,\quad y = 6.$

Maximum Volume = 6 . 6. 3 = 108 cm^3.

(b). f = V = x, y, z

$\phi = A = 2xy + 2yz + 2xz = 108$

$\phi = 2xy + 2yz + 2xz - 108$

$F = f + \lambda\phi = xyz + \lambda(2xy + 2yz + 2zx - 108)$

$\frac{\partial F}{\partial x} = yz + 2\lambda y + 2\lambda z$, $\frac{\partial F}{\partial y} = xz + 2\lambda x + 2\lambda x$, $\frac{\partial F}{\partial z} = xy + 2\lambda y + 2\lambda x$.

For maxima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$yz + 2\lambda y + 2\lambda z = 0$ (i)

$xz + 2\lambda x + 2\lambda z = 0$ (ii)

$xy + 2\lambda y + 2\lambda x = 0$ (iii)

Substituting (ii) from (i), we get

$(y - x)z + 2\lambda(y - x) = 0$

$\Rightarrow (y - x)(z + 2\lambda) = 0$

$\Rightarrow y = x \Rightarrow z = -2\lambda$

Substituting (iii) from (ii), we get

$x(z - y) + 2\lambda(z - y) = 0$

$\Rightarrow (z - y)(x + 2\lambda) = 0$

$\Rightarrow z = y \Rightarrow x = -2\lambda$

$\therefore x = y = z = -2\lambda$.

Also, $2xy + 2yz + 2xz = 108$

$\Rightarrow 2x^2 + 2x^2 + 2x^2 = 108$

$\Rightarrow x^2 = 18 \Rightarrow x = \sqrt{18}$.

$\therefore x = y = z = \sqrt{18}$ are the required dimensions of the rectangular base.

Q.No.30.: a. If the surface of the rectangular box, with open top, is 432 sq. cm, find (by the Lagrange's method) the dimensions of the box having maximum capacity (volume).

b. If the box is closed (on all sides), what are the dimensions?

Sol.: (a). Let x, y, z, V, A be the length, breadth, height, volume, surface area of the rectangular base respectively.

$f = V = xyz$

$A = xy + 2yz + 2zx$

$\phi = xy + 2yz + 2zx - 432$

$F = f + \lambda\phi = xyz + \lambda(xy + 2yz + 2zx - 432)$

$$\frac{\partial F}{\partial x} = yz + \frac{2}{2}\lambda y + 2\lambda z, \ \frac{\partial F}{\partial y} = xz + \lambda x + 2\lambda y \frac{z}{y}, \ \frac{\partial F}{\partial z} = xy + 2\lambda y + 2\lambda x.$$

For maxima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$yz + \lambda y + 2\lambda z = 0$ (i)

$xz + \lambda x + 2\lambda z = 0$ (ii)

$xy + 2\lambda y + 2\lambda x = 0$ (iii)

Substituting (ii) from (i), we get

$z(y - x) + \lambda(y - x) = 0$

$\Rightarrow (y - x)(z + \lambda) = 0$

$\Rightarrow y = x \Rightarrow z = -\lambda$.

Substituting (iii) from (ii), we get

$x(2z - y) + 2\lambda(2z - y) = 0$

$\Rightarrow (x+2\lambda)(2z-y)=0$

$x=-2\lambda \Rightarrow y=2z=-2\lambda$

$\therefore x=y=-2\lambda$ and $z=-\lambda$.

Also, $xy+2xz+2yz=432$

$4\lambda^2+2(-2\lambda)(-z)+2(-2z)(-z)=432$

$4\lambda^2+4\lambda^2+4\lambda^2=432$

$\lambda^2=36 \Rightarrow \lambda=\pm 6$

x = y =12 and z = 6.

$\therefore (12,\ \ 12,\ \ 6)$ are the required dimensions.

(b). $f=V=xyz$

$A=2xy+2yz+2xz=432$

$\phi=2xy+2yz+2xz-432$

$F=f+\lambda\phi=xyz+\lambda(2xy+2yz+2xz-432)$

$$\frac{\partial F}{\partial x}=yz+2\lambda y+2\lambda z \qquad \text{(i)}$$

$$\frac{\partial F}{\partial y}=xz+2\lambda x+2\lambda z \qquad \text{(ii)}$$

$$\frac{\partial F}{\partial z}=xy+2\lambda x+2\lambda y \qquad \text{(iii)}$$

For maxima, $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$

$z(y-x)+2\lambda(y-x)=0$ [From (i) and (ii)]

$\Rightarrow (z+2\lambda)(y-x)=0$

$\Rightarrow z=-2\lambda \Rightarrow y=x$

$z(z-y)+2\lambda(z-y)=0$ [From (ii) and (iii)]

$\Rightarrow (z-y)(x+2\lambda)=0$

$\Rightarrow z=y \Rightarrow x=-2\lambda$

$\therefore x=y=z=-2\lambda$

$A = 2xy + 2yz + 2xz = 432$

$2x^2 + 2x^2 + 2x^2 = 432$

$\Rightarrow 6x^2 = 432 \Rightarrow x^2 = 72 \Rightarrow x = \sqrt{72}$

$\therefore x = y = z = \sqrt{72}$ are the required dimensions of the rectangular box.

Q.No.31.: Find (by the Lagrange's method) the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$.

Sol.: Let x and y be the length and breadth of rectangle.

Area, $A = xy$

$f = xy$

$\phi = 4x^2 + 9y^2 - 36$

$F = f + \lambda\phi = xy + \lambda\left(4x^2 + 9y^2 - 36\right)$.

$\frac{\partial F}{\partial x} = y + 8\lambda x$, $\frac{\partial F}{\partial y} = x + 18\lambda y$.

For maxima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$

$y + 8\lambda x = 0$ (i)

$x + 18\lambda y = 0$ (ii)

Multiplying (i) by x and (ii) by y and subtracting

$\lambda\left(18y^2 - 8x^2\right) = 0$

$\Rightarrow 8x^2 = 18y^2 \Rightarrow 4x^2 = 9y^2 \Rightarrow 2x = 3y$.

Also, $4x^2 + 9y^2 = 36$

$4\left(\frac{9}{4}\right)y^2 + 9y^2 = 36 \Rightarrow 18y^2 = 36 \Rightarrow y^2 = 2 \Rightarrow y = \sqrt{2}$

$x = \frac{3\sqrt{2}}{2}$

$\therefore$ Length $= \frac{3\sqrt{2}}{2}$, Breadth $= \sqrt{2}$

Maximum area = xy = $\frac{3\sqrt{2}}{2} \times \sqrt{2} = 3$ sq. units.

Ans.: Length $= 3\sqrt{2}/2$, breadth: $\sqrt{2}$. Maximum area of the rectangle is 12 square units.

Q.No.32.: Find (by the Lagrange's method) the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid of revolution

$$4x^2 + 4y^2 + 9z^2 = 36.$$

Sol.: Let 2x, 2y, 2z be the dimensions of the parallelopiped

$$\phi = 4x^2 + 4y^2 + 9z^2 - 36$$

$$f = 8xyz$$

$$F = f + \lambda\phi = 8xyz + \lambda\left(4x^2 + 4y^2 + 9z^2 - 36\right)$$

$$\frac{\partial F}{\partial x} = 8yz + 8\lambda x, \quad \frac{\partial F}{\partial y} = 8xz + 8\lambda y, \quad \frac{\partial F}{\partial z} = 8xy + 18\lambda z.$$

For maxima and minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$$8yz + 8\lambda x = 0 \qquad \text{(i)}$$

$$8xz + 8\lambda y = 0 \qquad \text{(ii)}$$

$$8xy + 8\lambda z = 0 \qquad \text{(iii)}$$

Solving (i) and (ii), we get

$$8\lambda\left(x^2 - y^2\right) = 0$$

$$x = \pm y.$$

Solving (ii) and (iii), we get

$$\lambda\left(8x^2 - 18z^2\right) = 0 \Rightarrow 2x = 3z.$$

Also, $4x^2 + 4y^2 + 9z^2 = 36$

$$\Rightarrow 4x^2 + 4x^2 + 9 \times \left(\frac{4}{9}\right)x^2 = 36$$

$$\Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$$

$$\therefore x = y = \sqrt{3}, \quad z = \frac{2}{3}\sqrt{3}.$$

Volume of the parallelopiped = $8xyz = \sqrt{3}\times\sqrt{3}\times\frac{2\sqrt{3}}{3}\times 18 = 16\sqrt{3}$ cu. Unit.

Q.No.33.: Find (by the Lagrange's method) the dimensions of a rectangular box, with open top, of given capacity (volume) such that sheet metal (surface area) required is least.

Sol.: Let x and y be the dimension of rectangular box.

$\phi = V = xyz$

$f = A = xy + 2yz + 2zx$

$F = f + \lambda\phi = xy + 2yz + 2zx + \lambda(xyz - V)$

$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$, $\frac{\partial F}{\partial y} = x + 2z + \lambda xz$, $\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy$.

For maxima and minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$y + 2z + \lambda yz = 0$ (i)

$x + 2z + \lambda xz = 0$ (ii)

$2y + 2x + \lambda xy = 0$ (iii)

Solving (i) and (ii), we get

$(y - x) + \lambda z(y - x) = 0 \Rightarrow (y - x)(1 + \lambda z) = 0$

$\Rightarrow y = x \Rightarrow \lambda = \frac{-1}{z}$.

Solving (ii) and (iii), we get

$2(2x - y) + \lambda x(2z - y) = 0 \Rightarrow (2 + \lambda x)(2z - y) = 0$

$\Rightarrow y = 2z \Rightarrow x = \frac{-2}{\lambda}$

$\therefore x = y = 2z$

$V = xyz = x \times y \times \left(\frac{x}{2}\right) = \frac{x^3}{2}$

$\Rightarrow x = (2V)^{1/3}$

$\therefore$ Required dimensions are $x = y = 2z = (2V)^{1/3}$.

Home Assignments

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