8^{th} Topic

Matrices

Cayley-Hamilton Theorem

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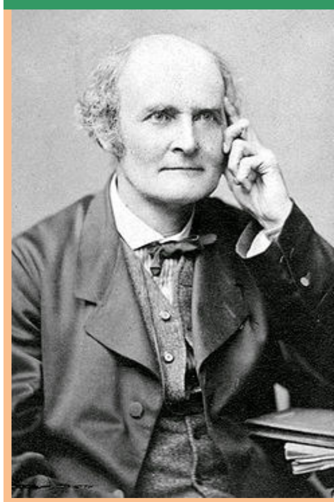
CAYLEY-HAMILTON THEOREM



Arthur Cayley

(16 August 1821 – 26 January 1895)

Sir William Rowan Hamilton

(4 August 1805 – 2 September 1865)

Cayley learned about matrices while attending one of Hamilton's lectures in Dublin, and later they both created their Cayley-Hamilton Theorem.

Statement: "Every square matrix over the real or complex field satisfies its own characteristic equation".

i.e., if the characteristic equation for the n^{th} order square matrix A is $|A - \lambda I| = 0$

$$\Rightarrow (-1)^n \lambda^n + \left[(-1)^{n-1} b_1\right]\lambda^{n-1} + \left[(-1)^{n-2} b_2\right]\lambda^{n-2} + \ldots\ldots\ldots + \left[(-1)^{n-n} b_n\right] = 0$$

$$\Rightarrow (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n\right] = 0$$

$$\Rightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n = 0.$$

Then $\boxed{\mathbf{A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots\ldots + a_nI = O}}$.

Proof: As we know, matrix $A - \lambda I$ is characteristic matrix of A.

This matrix can be written as

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{bmatrix}$$

This matrix shows that the elements of $A - \lambda I$ are at most of the 1^{st} degree in λ.

$\therefore$ The elements of Adj $(A - \lambda I)$ are ordinary polynomials in λ of degree $(n-1)$ or less.

$$\begin{bmatrix} \lambda^3 + 2\lambda + 1 & 4\lambda^3 + 5\lambda + 3 & 2\lambda + 9 \\ \lambda^3 + 5 & \lambda^3 + 2\lambda & 2\lambda^3 + 5\lambda + 7 \\ 3\lambda^3 + 2\lambda + 6 & \lambda^3 + 4 & \lambda^3 + 8\lambda + 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\lambda^3 + \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\lambda^2 + \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\lambda + \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

Now Adj $(A - \lambda I)$ can be written as matrix polynomials in λ, and is given by

$$\text{Adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1},$$

where $B_0, B_1, \ldots\ldots\ldots\ldots, B_{n-1}$ are matrices of the type $n \times n$, whose elements are functions of a_{ij}'s. [the elements of A].

Now, since A adj$A = |A| I_n$

Replacing A by $A - \lambda I$, we obtain

$$(A - \lambda I) \text{ Adj. } (A - \lambda I) = |A - \lambda I| I_n$$

$$\Rightarrow (A - \lambda I)\left[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}\right] = (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + \ldots\ldots\ldots + a_n\right] I_n$$

Comparing coefficients of the like powers of λ on both sides, we get

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

....

$$AB_{n-1} = (-1)^n a_n I$$

Pre-multiplying these successively by $A^n,\ A^{n-1}, \ldots\ldots\ldots, A,\ I$ and adding, we get

$$O = (-1)^n \left[A^n + a_1 A^{n-1} + \ldots\ldots\ldots\ldots + a_n I\right]$$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots + a_n I = O. \qquad \text{(i)}$$

i.e., Every square matrix satisfies its own characteristic equation.

This completes the proof.

Another method of finding the inverse:

If A be a non-singular matrix $\Rightarrow |A| \neq 0$.

Since $|A - \lambda I| = (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n\right]$

$$\Rightarrow |A| = (-1)^n a_n \Rightarrow a_n \neq 0.$$

Pre-multiplying (i) by A^{-1}, we get

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots\ldots\ldots\ldots\ldots + a_n A^{-1} = O.$$

$$\Rightarrow \boxed{A^{-1} = -\frac{1}{a_n}\left[A^{n-1} + a_1 A^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} I\right]}. \qquad \text{(since } a_n \neq 0\text{)}.$$

Now let us understand this important theorem by the following problems:

Q.No.1.: Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$ as a linear polynomial in A.

Sol.: **Find: Characteristic roots**

The characteristic equation of the matrix A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda-5)(\lambda+1) = 0. \qquad \text{(i)}$$

The roots of this equation are $\lambda = 5, -1$ and these are the characteristic roots of A.

By Cayley-Hamilton theorem, the matrix A must satisfy its characteristic equation (i) so we must have

$$A^2 - 4A - 5I = O. \qquad \text{(ii)}$$

Verification of Cayley-Hamilton theorem:

Since $A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$.

Therefore $A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

This verifies the theorem.

Find: Inverse of A

Now multiplying (ii) by A^{-1}, we get

$$A^2A^{-1} - 4AA^{-1} - 5IA^{-1} = OA^{-1} \Rightarrow A - 4I - 5A^{-1} = O \Rightarrow A^{-1} = \frac{1}{5}(A - 4I).$$

Now $A - 4I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$.

Express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$ **as a linear polynomial in A:**

Now (ii) $\Rightarrow A^2 = 4A + 5I$.

Multiplying by A^3, we get $A^5 = 4A^4 + 5A^3$.

Multiplying by A^2, we get $A^4 = 4A^3 + 5A^2$.

Multiplying by A, we get $A^3 = 4A^2 + 5A$.

Now $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\ I$

$= (4A^4 + 5A^3) - 4A^4 - 7(4A^2 + 5A) + 11A^2 - A - 10\ I$

$= 5A^3 - 17A^2 - 36A - 10\ I = 5(4A^2 + 5A) - 17A^2 - 36A - 10\ I$

$= 3A^2 - 11A - 10\ I = 3A^2 - 12A + A - 15\ I + 5\,I = 3\left(A^2 - 4A - 5I\right) + A + 5I = A + 5\,I$,

which is a linear polynomial in A.

Q.No.2.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and, hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol.: Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Let λ be the eigen value of the matrix A, then $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix}$.

Operating $C_3 \to C_3 + C_2$, we get

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & \lambda-1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Operating $R_2 \to R_2 + R_3$, we get

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)^2 - 1\right] = (1-\lambda)\left[4 + \lambda^2 - 4\lambda - 1\right] = -\lambda^3 + 5\lambda^2 - 7\lambda + 3.$$

$\therefore |A - \lambda I| = \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ is the characteristic equation of the matrix A. (i)

Then by Cayley-Hamilton theorem, the matrix A must satisfy (i), we have

$A^3 - 5A^2 + 7A - 3I = O$. (ii)

From (ii), we get

$A^3 = 5A^2 - 7A + 3I$, $\therefore$ $A^4 = 5A^3 - 7A^2 + 3A$ and $A^8 = 5A^7 - 7A^6 + 3A^5$ and

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= \left(5A^7 - 7A^6 + 3A^5\right) - 5A^7 + 7A^6 - 3A^5 + \left(5A^3 - 7A^2 + 3A\right) - 5A^3 + 8A^2 - 2A + I$

$$= A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}. \text{ Ans.}$$

Finding inverse by Cayley-Hamilton Theorem

Q.No.3.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$, and

hence find its inverse.

Sol.: **Find: Characteristic Equation**

The characteristic equation is $A = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 20\lambda + 8 = 0$.

which is the required characteristic equation of A.

Find: Inverse of A

Now since by Cayley-Hamilton theorem, we have $A^3 - 20A + 8I = O$

$\Rightarrow A^2 - 20I + 8A^{-1} = O$

$$\Rightarrow A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2 = \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \text{ Ans.}$$

Q.No.4.: Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$.

Sol.: (i). Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

If λ be the eigen value of the matrix A, then characteristic equation of A is $|A-\lambda I|=0$.

$$\Rightarrow \begin{bmatrix} 5-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda)-9 \Rightarrow 10-5\lambda-2\lambda+\lambda^2-9=0$$

$$\Rightarrow \lambda^2-7\lambda+1=0.$$

Then, by Cayley-Hamilton theorem, we have $A^2-7A+1=0$.

Pre-multiplying both sides by A^{-1}, we get $A-7\ I+A^{-1}=O$

$$\Rightarrow A^{-1}=-A+7\ I=-\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -5+7 & -3+0 \\ -3+0 & -2+7 \end{bmatrix}=\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}. \text{ Ans.}$$

(ii). Let $A=\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

If λ be the eigen value of the matrix A, then characteristic equation of A is $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}=0 \qquad \Rightarrow (1-\lambda)^3-(1-\lambda)+3(-2-1+\lambda)=0$$

$$\Rightarrow (1-\lambda)^3-(1-\lambda)+3(\lambda-3)=0$$

$$\Rightarrow (1-\lambda)^3+4\lambda-10=0 \Rightarrow -\lambda^3+3\lambda^2+\lambda-9=0.$$

Then, by Cayley Hamilton theorem, we have $-A^3+3A^2+A-9\ I=O$.

Pre-multiplying both sides by A^{-1}, we get $-A^2+3A+I-9A^{-1}=O$

$$\Rightarrow 9A^{-1}=-A^2+3A+I \Rightarrow A^{-1}=\frac{1}{9}\left(-A^2+3A+I\right).$$

Now $A^2=\begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$

$$\therefore A^{-1}=\frac{1}{9}\left[-A^2+3A+1\right]=\frac{1}{9}\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}+\frac{3}{9}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}+\frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -4+3+1 & 3+0+0 & -6+9+0 \\ -3+6+0 & -2+3+1 & -4-3+0 \\ 0+3+0 & 2-3+0 & -5+3+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}. \text{ Ans.}$$

(iii). Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$.

If λ be the eigen value of matrix A, then characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ 2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)(-4-\lambda)-12\right]-(-4-\lambda+6)+3\left[-4-2(3-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[-12-3\lambda+4\lambda+\lambda^2-12\right]+(4+\lambda-6)-12-18+6\lambda=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2+\lambda-24\right]+(\lambda-2)-30+6\lambda=0$$

$$\Rightarrow \lambda^2+\lambda-24-\lambda^3-\lambda^2+24\lambda+\lambda-2-30+6\lambda=0$$

$$\Rightarrow -\lambda^3+32\lambda-56=0$$

Then, by Cayley-Hamilton theorem, we have $-A^3+32A-56I = O$.

Pre-multiplication both sides by A^{-1}, we have $-A^2+32I-56A^{-1} = O$

$$\Rightarrow -A^2+32I = 56A^{-1} \Rightarrow A^{-1} = \frac{1}{56}\left[-A^2+32I\right]$$

Now $A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1+1+6 & 1+3-12 & 3-3-12 \\ 1+3-6 & 1+9+12 & 3-9+12 \\ 2-4-8 & 2-12+16 & 6+12+16 \end{bmatrix} = \begin{bmatrix} 8 & -8 & -12 \\ -2 & 22 & 6 \\ -10 & 6 & 34 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{56}\left[-A^2+32I\right] = \frac{1}{56}\begin{bmatrix} -8+32 & 8 & 12 \\ 2 & -22+32 & -6 \\ 10 & -6 & -34+32 \end{bmatrix} = \frac{1}{56}\begin{bmatrix} 24 & 8 & 12 \\ 2 & 10 & -6 \\ 10 & -6 & -2 \end{bmatrix}.$$

Q.No.5.: Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

$$\textbf{(i)} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \textbf{(ii)} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}.$$

Sol.: (i). Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

If λ be the eigen value of matrix A, then characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-1(2-\lambda)+1\} + 1\{1-(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1) = 0 \quad \Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0.$$

Then, by Cayley-Hamilton's theorem, we get $-A^3 + 6A^2 - 9A + 4\ I = O$

Multiplying both sides by A^{-1}, we get $\qquad -A^2 + 6A - 9\ I + 4A^{-1} = O \qquad$ (i)

$$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9\ I\right].$$

First verify result (i):

Now $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$,

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore -A^3 + 6A^2 - 9A + 4I$

$$= -\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} - \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Hence Cayley-Hamilton theorem verified.

IInd: Find the inverse of A.

$$\text{Now } A^{-1} = \frac{1}{4}\left(A^2 - 6A + 9I\right) = \frac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{9}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \frac{6-12+9}{4} & \frac{-5+6+0}{4} & \frac{5-6+0}{4} \\ \frac{-5+6+0}{4} & \frac{6-12+9}{4} & \frac{-5+6+0}{4} \\ \frac{5-6+0}{4} & \frac{-5+6+0}{4} & \frac{6+2++9}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}. \text{ Ans.}$$

(ii). Given $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

If λ is the eigen value of A then characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 \begin{bmatrix} 7-\lambda & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow (1-\lambda)^2[(1-\lambda) - 2(1) - 2] = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 - \lambda + 3 - 6\lambda + 3\lambda^2 = 0 \quad \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

Now by Cayley-Hamilton theorem, we get $A^3 - 5A^2 + 7A - 3I = O$.

Now $A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$

$A^3 - 5A^2 + 7A - 3I$

$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 5\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Hence this proves the result.

Now $A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right] = \frac{1}{3}\left\{\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right\}$

$= \frac{1}{3}\begin{bmatrix} 25+7-35 & 8-10+0 & -8+10+0 \\ -24+35+0 & -7+5+7 & 8-10+0 \\ 24-30+0 & 8-10+0 & -7+5+7 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$. Ans.

Q.No.6.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtains the inverse of the given matrix.

Sol.: **Find: Characteristic Equation**

Given $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$.

If λ be an eigen value of matrix A, then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(1-\lambda)-6]-3[4(1-\lambda)-3]+7[8-(2-\lambda)]=0$$

$$\Rightarrow (1-\lambda)\left[2-2\lambda-\lambda+\lambda^2-6\right]-3(1-4\lambda)+7(6+\lambda)=0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2-3\lambda-4\right)-3+12\lambda+42+7\lambda=0$$

$$\Rightarrow \lambda^2-3\lambda-4-\lambda^3+3\lambda^2+4\lambda-3+12\lambda+42+7\lambda=0$$

$$\Rightarrow -\lambda^3+4\lambda^2+20\lambda+35=0 \qquad \Rightarrow \lambda^3-4\lambda^2-20\lambda-35=0,$$

which is the required characteristic equation.

To shows the above characteristic equation is satisfied by A.

i.e., $A^3-4A^2-20A-35\ I=O$.

$$\text{Now } A^2=\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}=\begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}=\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}=\begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$=\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\therefore A^3-4A^2-20A-35\ I$$

$$=\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}-4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}-20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}-35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0.$$

Find: Inverse of A

Since $A^3-4A^2-20A-35\ I=O$

Multiplying both sides by A^{-1}, we get $A^2-4A-20\ I-35\ A^1=O$

$$\Rightarrow A^{-1} = \frac{1}{35}\left[A^2 - 4A - 20\ I\right] = \frac{1}{35}\left\{\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{35}\begin{bmatrix} 20-4-20 & 23-12+0 & 23-28+0 \\ 15-16+0 & 22-8-20 & 37-12+0 \\ 10-4+0 & 9-8+0 & 14-4-20 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}. \text{ Ans.}$$

Q.No.7.: Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, find A^{-1}**.**

Determine A^8.

Sol.: The characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (\lambda - 1)(1 + \lambda) - 4 = 0 \Rightarrow \lambda^2 - 5 = 0.$$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 5I \Rightarrow A^2 - 5I = O$$

Thus A satisfies the characteristic equation.

To find A^{-1}, multiply $A^2 - 5I = O$ by A^{-1}, we get

$$A^{-1}.A^2 - 5A^{-1}I = O \quad \Rightarrow A - 5A^{-1} = O$$

So $A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

To find A^8, multiply $A^2 - 5I = 0$ by A^6, we get

$$A^6.A^2 - 5I.A^6 = O$$

$$A^8 = 5A^6 = 5.A^2.A^2.A^2 = 5.(5I)(5I)(5I)$$

$A^8 = 625$ I.

Q.No.8.: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and hence find the inverse of A. Find A^4.

Express $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$ as a quadratic polynomial in A. Find B.

Sol.: The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]+3[10-3(4-\lambda)] = 0,$$

$\Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$.

Cayley Hamilton theorem is verified if A satisfies the above characteristic equation,

i.e., $A^3 - 11A^2 - 4A + I = 0$.

Now $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$.

$A^3 = A.A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$.

Verification:

$A^3 - 11A^2 - 4A + I$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}:

From characteristic equation $A^{-1} = -A^2 + 11A + 4I$.

So $A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

To find A^4:

From Cayley-Hamilton theorem

$A^3 - 11A^2 - 4A + I = 0 \Rightarrow A^3 = 11A^2 + 4A - I$.

Multiplying both sides by A

$A^4 = A.A^3 = A(11A^2 + 4A - I) = 11A^3 + 4A^2 - A$

$$=11\begin{bmatrix}157 & 283 & 353\\ 283 & 510 & 636\\ 353 & 636 & 793\end{bmatrix}+\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}-\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}1782 & 3211 & 4004\\ 3211 & 5786 & 7215\\ 4004 & 7215 & 8997\end{bmatrix}$$

To find B:

Rewrite $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

$$= A^5\left(A^3 - 11A^2 - 4A + I\right) + A\left(A^3 - 11A^2 - 4A + 1\right) + A^2 + A + I$$

$$= A^5(0) + A(0) + A^2 + A + I.$$

Thus, the quadratic polynomial in A of B is $A^2 + A + I$.

Now $B = A^2 + A + I = \begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

$$B = \begin{bmatrix}16 & 27 & 34\\ 27 & 50 & 61\\ 34 & 61 & 77\end{bmatrix}.$$

Q.No.9.: Determine A^{-1}, A^{-2}, A^{-3} if $A = \begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}$.

Sol.: The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix}4-\lambda & 6 & 6\\ 1 & 3-\lambda & 2\\ -1 & -4 & -3-\lambda\end{vmatrix} = \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

It follows from Cayley-Hamilton theorem that

$$A^3 - 4A^2 - A + 4I = 0$$

Multiplying by A^{-1},

$$A^{-1}A^3 - 4A^{-1}A^2 - A^{-1}.A + A^{-1}4I = 0$$

Solving $A^{-1} = \dfrac{1}{4}\left(I + 4A - A^2\right)$

$$A^2 = A.A = \begin{bmatrix}16 & 18 & 18\\ 5 & 7 & 6\\ -5 & -6 & -5\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}4&6&6\\1&3&2\\-1&-4&-3\end{bmatrix}-\frac{1}{4}\begin{bmatrix}16&18&18\\5&7&6\\-5&-6&-5\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1&6&6\\-1&6&2\\1&-10&-6\end{bmatrix}.$$

Multiplying A^{-1} by A^{-1}, we have

$$A^{-2}=A^{-1}A^{-1}=A^{-1}\frac{1}{4}\left[I+4A-A^2\right]=\frac{1}{4}\left[A^{-1}+4I-A\right]=\frac{1}{4}\begin{bmatrix}\frac{1}{4}&-\frac{9}{2}&-\frac{9}{2}\\-\frac{5}{4}&\frac{5}{2}&-\frac{3}{2}\\\frac{5}{4}&\frac{3}{2}&\frac{11}{2}\end{bmatrix}.$$

$$A^{-3}=A^{-1}A^{-2}=A^{-1}\left[A^{-1}+4I-A\right]\frac{1}{4}=\frac{1}{4}\left[A^{-2}+4A^{-1}-I\right]=\frac{1}{64}\begin{bmatrix}1&78&78\\-21&90&26\\21&-154&-90\end{bmatrix}$$

Home Assignments

Problems on verification of Cayley-Hamilton theorem

Q.No.1.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix}2&5\\1&-3\end{bmatrix}$.

Ans.: Characteristic polynomial: $\lambda^2+\lambda-11$.

Q.No.2.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix}2&-3\\7&-4\end{bmatrix}$.

Ans.: Characteristic polynomial: $\lambda^2+2\lambda+13$.

Q.No.3.: Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix}1&4&-3\\0&3&1\\0&2&-1\end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3-3\lambda^2-3\lambda+5=0$.

Q.No.4.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

Ans.: $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$.

Problems on finding the inverse by using Cayley-Hamilton theorem

Q.No.5.: Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, **(ii).** $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$.

Ans.: **(i).** $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ **(ii).** $\frac{1}{50}\begin{bmatrix} 8 & 20 & -7 \\ 40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$.

Q.No.6.: Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 6\lambda^2 - 9\lambda - 4 = 0$, $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Q.No.7.: Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$, $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$.

Problems on verifications

and

finding the inverse by using Cayley-Hamilton theorem

Q.No.8.: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

Ans.: $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$, $A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & 10 \end{bmatrix}$.

Q.No.9.: Verify Cayley-Hamilton theorem to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 20\lambda + 8 = 0$, $A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$.

Q.No.10.: Verify Cayley-Hamilton theorem and hence find A^{-1} for $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^4 - \lambda^3 - \lambda + 1 = 0$, $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

Problems on finding the matrix polynomials

Q.No.11.: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find A^{-1}.

Find $B = A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$.

Ans.: Characteristic equation: $\lambda^2 - 4\lambda - 5 = 0$, $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$,

$$B = A + 5\ I = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

Q.No.12.: If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find A^{-1}.

Find $B = A^8 - 5A^7 + 7A^6 - 3A^5 - 5A^3 + 8A^2 - 2A + I$

Ans.: Characteristic equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$,

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

Q.No.13.: Find $B = A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ if $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Ans.: Characteristic equation : $\lambda^2 - 4\lambda + 5 = 0$, $B = 5\ I - 4A = \begin{bmatrix} 1 & -8 \\ 4 & -7 \end{bmatrix}$.

Q.No.14.: Find A^{-1} and A^4 if $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$,

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix},\ A^4 = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}.$$

Q.No.15.: Compute A^{-1}, A^{-2}, A^3 and A^4 if $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$.

Ans.: Characteristic equation: $\lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0$, $A^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$,

$$A^{-2} = \frac{1}{121}\begin{bmatrix} -8 & -24 & 29 \\ 40 & -1 & -24 \\ -27 & 40 & -8 \end{bmatrix},\ A^3 = \begin{bmatrix} 42 & 31 & 29 \\ 45 & 39 & 31 \\ 53 & 45 & 42 \end{bmatrix},\ A^4 = \begin{bmatrix} 193 & 160 & 144 \\ 224 & 177 & 160 \\ 272 & 224 & 193 \end{bmatrix}.$$

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