2nd Topic

Double Integrals

[where limits are not given, but region of integration is given]

(Last updated on 15-07-2013)

(04 Solved problems and 00 Home assignments)

Here we will discuss those problems in double integrals, where limits are not given, but region of integration is given.

Since limits are not given, so rough sketch of the region of integration is required. So in that case, we can integrate first w.r.t. x or y depends upon our desire.

If we suppose, strip is parallel to x-axis (horizontal strip), then integrate w.r.t. x first and then w.r.t. y, whereas if we suppose strip is parallel to y-axis (vertical strip), then integrate w.r.t. y first and then w.r.t. x.

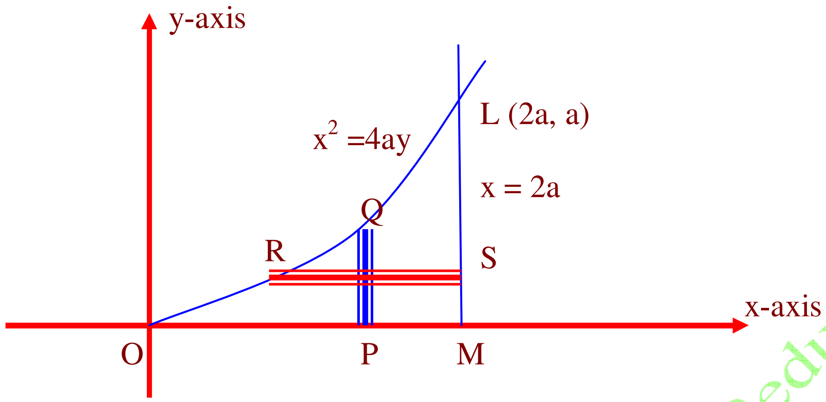
Let us clear this concept with the help of problems given below.

Q.No.1.: Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate

$$x = 2a \text{ and the curve } x^2 = 4ay.$$

Sol.: **First way to solve this problem:** Let us consider the strip, parallel to y-axis

The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. This figure shows the domain A which is the area OML.



Now let us suppose strip is parallel to y-axis. In that case integrating first over a vertical strip PQ, w.r.t. y from $P(y=0)$ to $Q\left(y = \frac{x^2}{4a}\right)$ on the parabola and then w.r.t. x from x = 0 to x = 2a, we have

$$\iint_A xydxdy = \int_0^{2a}\left(\int_0^{x^2/4a} xydy\right)dx = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5 dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}. \text{ Ans.}$$

Second way to solve this problem: Let us consider the strip, parallel to x-axis

Now let us suppose strip is parallel to x-axis. In that case integrating first over a horizontal strip RS, w. r. t. x from $R\left(x = 2\sqrt{ay}\right)$ on the parabola to $S(x = 2a)$ and then w. r. t. y from y = 0 to y = a, we get

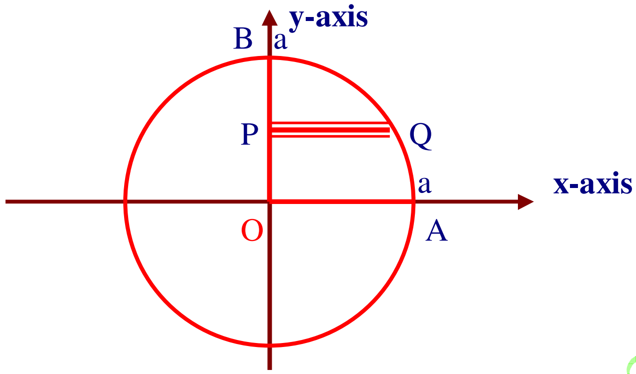
$$\iint_A xydxdy = \int_0^{a}\left(\int_{2\sqrt{(ay)}}^{2a} xydx\right)dy = \int_0^{a} y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^{a}\left(ay - y^2\right)dy = 2a\left|\frac{ay^2}{2} - \frac{y^3}{3}\right|_0^{a} = \frac{a^4}{3}. \text{ Ans.}$$

Q.No.2.: Evaluate the integral $\iint xydxdy$ over the positive quadrant of the circle

$x^2 + y^2 = a^2$.

Sol.: The region OAB, represents the positive quadrant of the circle $x^2 + y^2 = a^2$.



Let us suppose that the strip is parallel to x-axis, then in this region, x varies from 0 to $\sqrt{a^2 - y^2}$ and y varies from 0 to a,. Hence

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-y^2}} xydx \right) dy = \int_0^a \left[\frac{x^2 y}{2} \right]_0^{\sqrt{a^2-y^2}} dy = \int_0^a y \frac{\sqrt{\left(a^2 - y^2\right)^2}}{2} dy = \frac{1}{2} \int_0^a y\left(a^2 - y^2\right) dy$$

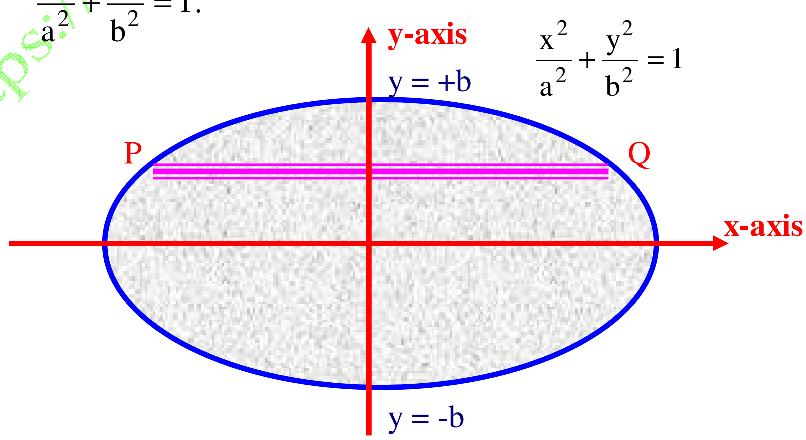
$$= \frac{1}{2} \int_0^a \left(a^2 y - y^3\right) dy = \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a = \frac{1}{2} \left[\frac{a^2 (a)^2}{2} - \frac{\left(a^4\right)}{4} \right] - (0 - 0)$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] = \frac{a^4}{8} \text{ .Ans.}$$

Q.No.3.: Evaluate the integral $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol.:



Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow x^2=a^2\left(1-\frac{y^2}{b^2}\right) \Rightarrow x=\pm a\sqrt{1-\frac{y^2}{b^2}}$.

$$\therefore I=\int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}(x+y)^2dx\right]dy=\int_{-b}^{b}\left[\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}\left(x^2+y^2+2xy\right)dx\right]dy$$

$$=\int_{-b}^{b}\left[2\int_{0}^{a\sqrt{1-\frac{y^2}{b^2}}}\left(x^2+y^2\right)dx\right]dy=2\int_{-b}^{b}\left[\left(\frac{x^3}{3}+y^2x\right)_0^{a\sqrt{1-\frac{y^2}{b^2}}}\right]dy$$

$$=2\int_{-b}^{b}\left[\frac{a^3}{3}\left(1-\frac{y^2}{b^2}\right)^{3/2}+y^2a\sqrt{1-\frac{y^2}{b^2}}\right]dy$$

Since function is even, then we get

$$\Rightarrow I=4\int_{0}^{b}\left[\frac{a^3}{3}\left(1-\frac{y^2}{b^2}\right)^{3/2}+ay^2\sqrt{1-\frac{y^2}{b^2}}\right]dy$$

Put $y=b\sin\theta \Rightarrow dy=b\cos\theta d\theta$. Also when $y=0,\ \theta=0$ and when $y=b,\ \theta=\frac{\pi}{2}$.

Then $I=4\int_{0}^{\pi/2}\left[\frac{a^3}{3}\cos^3\theta+ab^2\sin^2\theta\cos\theta\right]b\cos\theta d\theta$

$$=4ab\left[\frac{a^2}{3}\int_{0}^{\pi/2}\cos^4\theta d\theta+b^2\int_{0}^{\pi/2}\sin^2\theta\cos^2\theta d\theta\right] \qquad \text{(i)}$$

Now first evaluate $\int_{0}^{\pi/2}\cos^4\theta d\theta=\frac{3.1}{4.2}.\frac{\pi}{2}=\frac{3\pi}{16}$,

and $\int_{0}^{\pi/2}\sin^2\theta\cos^2\theta d\theta=\frac{1.1}{4.2}.\frac{\pi}{2}=\frac{\pi}{16}$

Hence

$$I = 4ab\left[\frac{a^2}{3}.\frac{3\pi}{16} + b^2\frac{\pi}{16}\right] = \frac{ab\pi}{4}\left(a^2 + b^2\right).\text{ Ans.}$$

Q.No.4.: Evaluate the integral $\iint xy(x+y)dxdy$ over the area between $y = x^2$ and $y = x$.

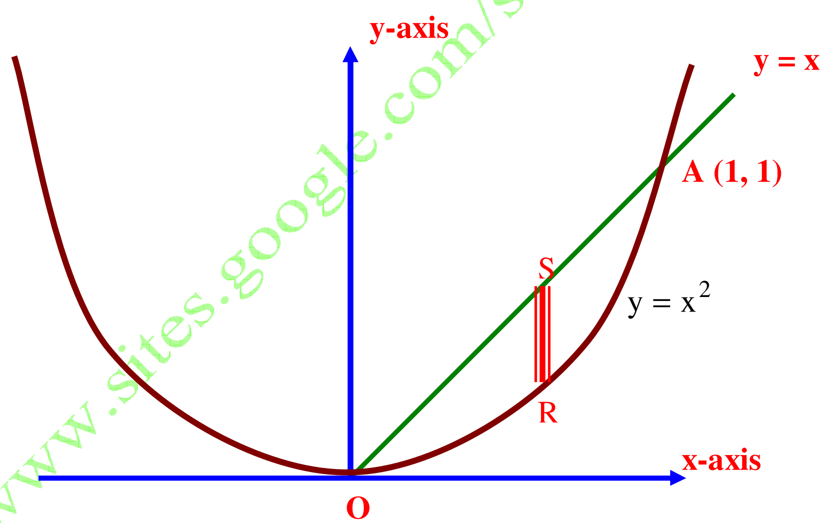
Sol.: Solving $y = x^2$ and $y = x$, we get $x = x^2 \Rightarrow x - x^2 = 0 \Rightarrow x(1-x) = 0$.

$\therefore x = 0$ and $x = 1$

When x = 0, y = 0 and x = 1, y = 1.

$\therefore$ The points of intersection are O(0, 0) and A (1, 1).

Let us suppose, the strip is parallel to y-axis. In that case integrating first over a horizontal strip RS, w. r. t. y from $y = x^2$ to $y = x$ and then w. r. t. x from x= 0 to x = 1, we get



$$I = \iint xy(x+y)dxdy = \int_0^1\left[\int_{x^2}^{x} xy(x+y)dy\right]dx = \int_0^1\left[\int_{x^2}^{x}\left(x^2y + xy^2\right)dy\right]dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx = \int_0^1\left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx = \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \left[\frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}\right] = [0.1 + 0.06667 - 0.07143 - 0.0417] = 0.5357\text{ . Ans.}$$

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