3rd Topic

Matrices

Rank, Inverse and Normal form of a Matrix

Gauss-Jordan method of finding the inverse

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Rank of a Matrix:

Definition: **A matrix is said to be of rank r, when**

(i) it has at least one non-zero minor of order r, and

(ii) every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

Another definition:

A number r is said to be the rank of a matrix A if it possesses the following two properties:

1. **There is atleast one square sub-matrix of A of order r, whose determinant is not equal to zero,**
2. **If the matrix A contains any square sub-matrix of order (r +1), then the determinant of every square sub-matrix of A of order (r +1) should be zero.**

Remarks: **If a matrix has a non-zero minor of order r, its rank is** $\geq r$ **.**

If all minors of a matrix of order (r + 1) are zero, its rank is $\leq r$ **.**

Minor: In linear algebra, a **minor** of a matrix **A** is the determinant of some smaller square matrix, cut down from **A** by removing one or more of its rows or columns. Minors obtained by removing just one row and one column from square matrices (**first minors**) are required for calculating matrix cofactors, which in turn are useful for computing both the determinant and inverse of square matrices.

Elementary transformations of a matrix:

The following operations, three of which refer to rows and three to columns are known as elementary transformations:

I. **The interchange of any two rows (columns).**

II. **The multiplication of any row (column) by a non-zero number.**

III. **The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).**

Notation:

The elementary row transformation will be denoted by the following symbols:

(i) $R_i \leftrightarrow R_j$ for the interchange of the i^{th} and j^{th} rows.

(ii) $R_i \to kR_i$ for multiplication of the i^{th} row by k.

(iii) $R_i \to R_i + pR_j$ for addition to the i^{th} row, p times the j^{th} row.

The corresponding column transformation will be denoted by writing C in place of R.

Elementary transformations do not change either the order or a rank of a matrix. While the value of the minors may get changed by the transformation (i) and (ii), their zero or non-zero character remains unaffected.

Equivalent matrix:

Definition: Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol ~ is used for equivalence.

Example: Determine the rank of the following matrices: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Obviously, the 3rd order minor of A vanishes.

Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$.

$\therefore \rho(A) = 2$.

Now since the rank of a matrix is the largest order of any non-vanishing minor of the matrix

Hence, the rank of the given matrix is 2.

Elementary matrices:

Definition: An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.

Examples of elementary matrices obtained from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are $R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23}$; $kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem:

Statement: Every elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrix.

Remarks: Consider the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Then $R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$.

So a pre-multiplication by R_{23} has interchanged the 2^{nd} and 3^{rd} rows of A. Similarly pre-multiplication by kR_2 will multiply the 2^{nd} row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2^{nd} row of A to its 1^{st} row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A.

Similarly, it can also be seen that post-multiplication will perform the elementary column transformations.

Normal form of a matrix:

Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A. (i)

Remarks:

(i) The rank of the matrix A is r if and only if it can be reduced to the normal form (i).

(ii) Since each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result:

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that PAQ = $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n, respectively.

Example: For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find non-singular matrices P and Q such that PAQ is in the normal form.

Sol.: We write $A = I\,A\,I \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (past factor) of A to the same.

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_3 \to C_3 - C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 + R_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$,

which is the required normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$. Ans.

Gauss-Jordan method of finding the inverse:

Johann Carl Friedrich Gauss

30 April 1777 – 23 February 1855

German mathematician and scientist

Wilhelm Jordan

1 March 1842 – 17 April 1899

German geodesist

It is named after Carl Friedrich Gauss and Wilhelm Jordan, because it is a modification of Gaussian elimination as described by Jordan in 1887. However, the method also appears in an article by Clasen published in the same year. Jordan and Clasen probably discovered Gauss–Jordan elimination independently.

Statement: Those elementary row transformations, which reduce a given square matrix A to the unit matrix, when applied to unit matrix I, give the inverse of A.

Proof: Let the successive row transformations, which reduce A to I, result from pre-multiplication by the elementary matrices $R_1,\ R_2, \ldots\ldots\ldots, R_i$ so that

$$R_i R_{i-1} \ldots\ldots\ldots\ldots R_2 R_1 A = I$$

$$\therefore R_i R_{i-1} \ldots\ldots\ldots\ldots R_2 R_1 A A^{-1} = I A^{-1}$$

$$\Rightarrow R_i R_{i-1} \ldots\ldots\ldots\ldots R_2 R_1 I = A^{-1}. \qquad \left[\because AA^{-1} = I\right]$$

Hence the result.

For practical evaluation of A^{-1}, the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example: Using the Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$$

Sol.: Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right].$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$.

Operate $R_2 \to \frac{1}{2}R_2$ and $R_3 \to \frac{1}{2}R_3$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$.

Operating $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + R_2$, we get $\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

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Operate $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$ and $R_3 \to \left(-\frac{1}{2}\right)R_3$, we get

$$\sim\left[\begin{array}{ccccccc} 1 & 0 & 0 & : & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & : & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right].$$

Hence, the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$. Ans.

Problems for finding rank of a matrix:

Q.No.1.: Prove that the row equivalent matrices have the same rank.

Sol.: Let A be any $m \times n$ matrix. Let B be a matrix row equivalent to A.

Since B is obtainable from A by a finite chain of E-row operations and every E-row operation is equivalent to pre-multiplication by the corresponding E-matrix, there exist E-matrices $E_1, E_2, \ldots\ldots\ldots, E_k$ each of the type $m \times m$ such that

$B = E_k E_{k-1} \ldots\ldots\ldots E_2 E_1 A \Rightarrow B = PA$,

where $P = E_k E_{k-1} \ldots\ldots\ldots E_2 E_1$ is a non-singular matrix of the type $m \times m$.

Let us write $$B = PA = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1m} \\ p_{21} & p_{22} & \ldots & p_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ p_{m1} & p_{m2} & \ldots & p_{mm} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \ldots \\ \ldots \\ R_m \end{bmatrix}, \qquad \text{(i)}$$

where the matrix A has been expressed as a matrix of its row sub-matrices $R_1, R_2, \ldots\ldots\ldots, R_m$.

From the product of the matrices on R. H. S. of (i) we observe that the rows of the matrix B are

$p_{11}R_1 + p_{12}R_2 + \ldots\ldots\ldots + p_{1m}R_m$,

$p_{21}R_1 + p_{22}R_2 + \dots\dots\dots + p_{2m}R_m$,

....

....

$p_{m1}R_1 + p_{m2}R_2 + \dots\dots\dots + p_{mm}R_m$.

Thus, we see that the rows of B are all linear combinations of the rows $R_1, R_2, \dots\dots\dots, R_m$ of A.

Therefore, every member of the row space of B is also a member of the row space of A.

Similarly, by writing $A = P^{-1}B$ and giving the same reasoning we can prove that every member of the row space of A is also a member of the row space of B.

Therefore the row space of A and B are identical.

Thus we see that elementary row operations do not alter the row space of a matrix remains invariant under E-rows transformations.

Note: From the above theorem we also conclude that pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.

Q.No.2.: Determine the rank of the following matrices: $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Sol.: Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

Operating $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Operating $R_3 \to R_3 - 3R_2, R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Operating $C_3 \to C_3 + 3C_2, C_4 \to C_4 + C_2$, we get $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Obviously, the 4^{th} order minor of A is zero. Also every third order minor of A is zero. But, of all the 2^{nd} order minors, only $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. $\therefore \rho(A) = 2$.

Hence, the rank of the given matrix is 2.

Q.No.3: Determine the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

Operating $R_3 \to 2R_3 - R_2$, $R_2 \to \left(-\frac{1}{2}R_2\right)$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

Operating $R_2 + 2R_1$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$.

$\therefore \rho(A) = 2$

Hence, the rank of the given matrix is 2.

Q.No.4: Determine the rank of the matrix **(i)** $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.

Sol.: (i) Let $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$.

Operating $C_1 \to C_1 - C_4$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 2 & 2 & 4 & 3 \end{bmatrix}$.

Operating $R_3 \to R_3 - 2R_1$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 4 & -2 & 1 \end{bmatrix}$

Operating $R_3 \to R_3 - R_2$, we get $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now since all 3×3 matrices are singular $\because \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{vmatrix} = 0$ and $\begin{vmatrix} -1 & 3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$.

Now $\because \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} \neq 0$. Hence, the rank of the given matrix is 2.

(ii). Let $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.

Operating $R_3 \to 3R_3 - R_2$, $R_2 \to \frac{1}{3}R_2$, we get $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, the rank of the given matrix is 2.

Q.No.5: Determine the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}.$$

Operating $R_4 \to R_4 - R_3$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

Operating $R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

As 4×4 matrix is singular. But 3×3 matrix like $\begin{vmatrix} 2 & 3 & 0 \\ 0 & -3 & 2 \\ -4 & -8 & 3 \end{vmatrix}$ is non-singular.

So the rank of given matrix is 3.

Q.No.6: Determine the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Operating $R_4 \to R_4 - 3R_1$, $R_3 \to R_3 - 3R_2$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & -6 & 3 & -4 \end{bmatrix}$,

Operating $R_2 = 2R_2 - R_1$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & 4 & 9 & 10 \\ 0 & -6 & 3 & 4 \end{bmatrix}$.

Operating $C_4 \to C_4 + C_3$, we get $A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & -5 & -3 & -10 \\ 0 & 4 & 9 & 19 \\ 0 & 6 & 3 & -1 \end{bmatrix}$.

Operating $C_2 \to C_2 - 6C_4, C_3 \to C_3 + 3C_4$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 22 & 9 & 10 \\ 0 & 0 & 3 & -4 \end{bmatrix}$.

Operating $R_3 \to R_3 + 2R_2$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 3 & -4 \end{bmatrix}$.

Operating $R_4 \to R_4 - R_3$, we get $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -11 & -3 & -7 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

As 4×4 matrix is singular. But 3×3 matrix is non-singular.

So the rank of the matrix is 3. Ans.

Q.No.7: Determine the rank of the matrix $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.

Sol.: Let $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1, R_4 \to R_4 - R_2$, we get $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 6 & 8 & 9 \\ 6 & 6 & 6 & 6 \\ 10 & 10 & 10 & 10 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \\ 6 & 6 & 6 & 6 \\ 10 & 10 & 10 & 10 \end{bmatrix}$.

Operating $C_1 \to C_1 - C_4, C_2 \to C_2 - C_4, C_3 \to C_3 - C_4$, we get $A = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

Operating $R_4 \to R_4 - 10R_2,\ \ R_3 \to R_3 - 6R_2$, we get $A = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence the rank of the given matrix is 2. Ans.

Q.No.8.: Find the rank of matrix

(i). $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ **(ii).** $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ **(iii).** $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Sol.: (i). Here $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ is a 2×4 matrix.

$\therefore \rho(A) \le 2$, the smaller of 2 and 4.

The second order minor $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \ne 0$ $\therefore \rho(A) = 2$.

(ii). Here $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ is a 3×4 matrix.

$\therefore \rho(A) \le 3$.

Operating C_{14}, we get $A = \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -5 \end{bmatrix}$

Operating $R_2 - R_1,\ R_3 - R_1$, we get $A = \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix}$

Operating $R_1 + 3R_2,\ R_3 + 2R_2$, we get $A = \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & -0 \end{bmatrix}$.

All the first order minors are zero but the second order minor

$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \ne 0$. $\therefore \rho(A) = 2$.

(iii). Here $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ is a 4×4 matrix. $\therefore \rho(A) \le 4$

Operating R_{12}, we get $A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Operating $C_2+C_1,\ \ C_3+2C_1,\ \ C_4+4C_1,$, we get $A=\begin{bmatrix}1&0&0&0\\2&5&3&7\\3&4&9&10\\6&9&12&17\end{bmatrix}$.

Operating $R_2-2R_1,\ \ R_3-3R_1,\ \ R_4-6R_1$, we get $A=\begin{bmatrix}1&0&0&0\\0&5&3&7\\0&4&9&10\\0&9&12&17\end{bmatrix}$.

Operating $R_2-R_3,\ \ \ R_4-2R_3$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&-6&-3\\0&4&9&10\\0&1&-6&-3\end{bmatrix}$.

Operating $C_3+6C_2,\ \ C_4+3C_2$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&4&33&22\\0&1&0&0\end{bmatrix}$.

Operating $R_3-4R_2,\ \ \ R_4-R_2$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&33&22\\0&0&0&0\end{bmatrix}$.

Operating $\frac{1}{33}C_3$, we get $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&22\\0&0&0&0\end{bmatrix}$.

Operating C_4-22C_3, we get $A=\begin{bmatrix}1&0&0&:&0\\0&1&0&:&0\\0&0&1&:&0\\\ldots&\ldots&\ldots&:&\ldots\\0&0&0&:&0\end{bmatrix}=\begin{bmatrix}I_3&:&O_{3\times2}\\\ldots&\ldots&\ldots\\O_{1\times3}&:&O_{1\times1}\end{bmatrix}$.

$\therefore \rho(A)=3$.

Problems for inverse of a matrix by Gauss-Jordan method:

Q.No.1.: Use Gauss-Jordan method to find inverse of the following matrices:

$$\textbf{(i)} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}, \textbf{(ii)} \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \textbf{(iii)} \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \textbf{(iv)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Sol.: (i). Given matrix is $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$.

Writing the same matrix side by side with the unit matrix of order 3., we have

$$\begin{bmatrix} 2 & 1 & -1 : & 1 & 0 & 0 \\ 0 & 2 & 1 : & 0 & 1 & 0 \\ 5 & 2 & -3 : & 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to 2R_3 - 5R_1$, we get $\begin{bmatrix} 2 & 1 & -1 : & 1 & 0 & 0 \\ 0 & 2 & 1 : & 0 & 1 & 0 \\ 0 & -1 & -1 : & -5 & 0 & 2 \end{bmatrix}$.

Operating $R_3 \to 2R_3 - R_2$, we get $\begin{bmatrix} 2 & 1 & -1 : & 1 & 0 & 0 \\ 0 & 2 & 1 : & 0 & 1 & 0 \\ 0 & 0 & -1 : & -10 & 1 & 4 \end{bmatrix}$.

Operating $R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$, we get $\begin{bmatrix} 2 & 1 & 0 : & 11 & -1 & -4 \\ 0 & 2 & 0 : & -10 & 2 & 4 \\ 0 & 0 & -1 : & -10 & 1 & 4 \end{bmatrix}$.

Operating $R_1 \to 2R_1 - R_2$, $R_2 \to \frac{1}{2}R_2$, $R_3 \to (-1)R_3$, we get

$$\begin{bmatrix} 4 & 0 & 0 : & 32 & -4 & -12 \\ 0 & 1 & 0 : & -5 & 1 & 2 \\ 0 & 0 & 1 : & 10 & -1 & -4 \end{bmatrix}.$$

Operating $R_1 \to \frac{1}{4}R_1$, we get $\begin{bmatrix} 1 & 0 & 0 : & 8 & -1 & -3 \\ 0 & 1 & 0 : & -5 & 1 & 2 \\ 0 & 0 & 1 : & 10 & -1 & -4 \end{bmatrix}$.

Hence, the inverse of the given matrix is $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$. Ans.

(ii). Given $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Writing the same matrix side by side with the unit matrix of order 3., we have

$$\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 2 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 1 & : & 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to 2R_3 - R_2$, $R_2 \to 4R_2 - R_1$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & 0 & 1 & : & -1 & 4 & 0 \\ 0 & 3 & 1 & : & 0 & -1 & 2 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_3$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & -3 & 0 & : & -1 & 5 & -2 \\ 0 & 3 & 1 & : & 0 & -1 & 2 \end{bmatrix}$.

Operating $R_3 \to R_3 + R_2$, $R_2 \to \left(-\frac{1}{3}\right)R_2$, we get $\begin{bmatrix} 8 & 4 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to R_1 - 3R_3$, we get $\begin{bmatrix} 8 & 4 & 0 & : & 4 & -12 & 0 \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to R_1 - 4R_2$, we get $\begin{bmatrix} 8 & 0 & 0 & : & \frac{8}{3} & -\frac{16}{3} & -\frac{8}{3} \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$.

Operating $R_1 \to \frac{1}{8}R_1$, we get $\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & : & \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & : & -1 & 4 & 0 \end{bmatrix}$

Hence the inverse of given matrix is $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$.

$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$. Ans.

(iii). Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

According to Gauss-Jordan method, we have $A = IA$.

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_3 \to R_3 - R_1$, we get $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$.

Operating $R_3 \to R_3 - R_2$, $R_2 \to R_2 - 2R_1$, we get $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix} A$.

Operating $R_1 \to R_1 + R_2$, $R_3 \to 2R_3 - R_2$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 2 & -3 & -2 \end{bmatrix} A$.

Operating $R_3 \to \frac{1}{5} R_3$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A$.

Operating $R_1 \to R_1 + R_3$, $R_2 \to R_2 + 3R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ \frac{6}{5} & -\frac{4}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A$.

Operating $R_2 \to -\frac{R_2}{2}$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} A \Rightarrow I = A^{-1}A$.

$\therefore A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$. Ans.

(iv). Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

According to Gauss-Jordan method, we have $A = IA \quad \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$.

Operating $R_2 \to R_2 + R_1$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$.

Operating $R_3 \to -2R_1 - R_3$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} A$.

Operating $R_3 \to -R_3 + R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A$.

Operating $R_2 \to -R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A$.

Operating $R_3 \to \dfrac{R_3}{-4}$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_1 \to R_1 - R_2$, we get, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_1 \to R_1 - R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

Operating $R_2 \to R_2 - R_3$, we get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \text{ Ans.}$$

Q.No.2.: Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row operations.

Sol.: Writing the given matrix side by side with unit matrix I_3, we get

$$[A \ : \ I_3] = \begin{bmatrix} 0 & 1 & 2 & : & 1 & 0 & 0 \\ 1 & 2 & 3 & : & 0 & 1 & 0 \\ 3 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

Operating R_{12}, we get $=\begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 3 & 1 & 1 & : & 1 & 0 & 1 \end{bmatrix}$

Operating $R_3 - 3R_1$, we get $=\begin{bmatrix} 1 & 2 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & -5 & -8 & : & 0 & -3 & 1 \end{bmatrix}$

Operating $R_1 - 2R_2,\ R_3 + 5R_2$, we get $=\begin{bmatrix} 1 & 0 & -1 & : & -2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 0 & 2 & : & 5 & -3 & 1 \end{bmatrix}$

Operating $\frac{1}{2}R_3$, we get $=\begin{bmatrix} 1 & 0 & -1 & : & -2 & 1 & 0 \\ 0 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 0 & 1 & : & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

Operating $R_1 + R_2,\ R_2 - 2R_3$, we get $=\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & : & -4 & 3 & -1 \\ 0 & 0 & 1 & : & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} I_3 & : & A^{-1} \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

Problems on normal form:

Q.No.1.: If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1}. Also find two non-singular matrices P and Q such that PAQ = I, where I is the unit matrix and verify that $A^{-1} = QP$.

Sol.: Here $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

Part I: To find A^{-1}, $A_{11}=1,\ A_{12}=-2,\ A_{13}=-2$,

$A_{21}=-1,\ A_{22}=3,\ A_{23}=3$,

$A_{31}=0,\ A_{32}=-4,\ A_{33}=-3$.

$$\therefore \text{Adj. A} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}.$$

$$\text{Now } |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4)-2(-3+4)=3-2=1.$$

$$\therefore A^{-1} = \frac{\text{Adj. A}}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}. \text{ Ans.}$$

Part II: Since A= PAQ, where P and Q are two non-singular unit matrices of order 3 each.

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_1 \to R_1 - R_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 \to C_2 + C_3$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Operating $C_3 \to C_3 - 4C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$.

$\Rightarrow I = PAQ$, where $P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$.

Part III: Verification: $A^{-1} = Q\ P$.

Now $RHS = QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

Now $LHS = A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

$\therefore$ L.H.S. = R.H.S.

Hence $A^{-1} = QP$.

Q.No.2.: Reduce the following matrices to the normal form and hence find their ranks.

(i) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

Sol.: (i). Let $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$.

Operating $C_2 \to C_2 - \frac{1}{8}C_1$, $C_3 \to C_3 - \frac{3}{8}C_1$, $C_4 \to C_4 - \frac{6}{8}C_1$, we get

$A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ -8 & 0 & 0 & 10 \end{bmatrix}$.

Operating $R_1 \to \frac{1}{8}R_1$, $R_3 \to R_3 + R_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

Operating $R_3 \to \frac{1}{10}R_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_4 \to C_4 - \frac{2}{3}C_2$, $C_3 \to C_3 - \frac{2}{3}C_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to \frac{1}{3}R_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_3 \leftrightarrow C_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

which is the required normal form $[I_3 \quad O]$.

Hence, rank of the matrix A is 3. Ans.

(ii). Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

Operating $C_4 \to C_3 - 2C_1$, $C_4 \to C_4 - C_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 2 & -1 \\ -2 & 2 & 12 & -2 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 2R_1$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & -2 \end{bmatrix}$.

Operating $C_3 \to C_3 + 2C_2$, $C_4 \to C_4 - C_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 16 & -4 \end{bmatrix}$.

Operating $R_3 \to R_3 + R_2$, $R_4 \to R_4 - 2R_2$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & -4 \end{bmatrix}$.

Operating $C_4 \to C_4 + \frac{4}{16} C_3$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$.

Operating $R_4 \to R_4 + \frac{1}{16} R_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Operating $R_3 \leftrightarrow R_4$, we get $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

which is the required normal form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$.

Hence, the rank of the matrix A is 3. Ans.

Q.No.3.: Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices:

(i) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Sol.: (i) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Since we know that $A = I.A.I \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 \to C_2 + C_1, C_3 \to C_3 + C_1$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix}1&0&0\\0&2&2\\0&4&4\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-3&0&1\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}.$$

Operating $R_3 \to R_3 - 2R_2$, we get $\begin{bmatrix}1&0&0\\0&2&2\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-1&-2&1\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}.$

Operating $R_2 \to \frac{R_2}{2}$, we get $\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-\frac{1}{2}&\frac{1}{2}&0\\-1&-2&1\end{bmatrix}A\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}.$

Operating $C_3 \to C_3 - C_2$, we get $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-\frac{1}{2}&\frac{1}{2}&0\\-1&-2&1\end{bmatrix}A\begin{bmatrix}1&1&0\\0&1&-1\\0&0&1\end{bmatrix}.$

$\therefore PAQ=\begin{bmatrix}I_2&0\\0&0\end{bmatrix}$, where $P=\begin{bmatrix}1&0&0\\-\frac{1}{2}&\frac{1}{2}&0\\-1&-2&1\end{bmatrix}$ and $Q=\begin{bmatrix}1&1&0\\0&1&-1\\0&0&1\end{bmatrix}$. Ans.

Also rank of the matrix A is 2.

(ii). Let $A=\begin{bmatrix}1&2&3&-2\\2&-2&1&3\\3&0&4&1\end{bmatrix}$.

Since we know that $A = I_3\,A\,I_4$, where I_3 and I_4 are the unit matrix of order 3 and 4 respectively.

$$\Rightarrow \begin{bmatrix}1&2&3&-2\\2&-2&1&3\\3&0&4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Operating $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 + 2C_1$, we get

$$\begin{bmatrix}1&2&0&0\\2&-6&-5&7\\1&-6&-5&7\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -6 & -5 & 7 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 \to \left(-\frac{1}{6}\right)C_2$, $C_3 \to \left(-\frac{1}{5}\right)C_3$, $C_4 \to \left(\frac{1}{7}\right)C_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{3}{5} & \frac{2}{7} \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

Operating $C_4 \to C_4 - C_2$, $C_3 \to C_3 - C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

Operating $R_2 \to R_2 - 2R_3$, $R_3 \to R_3 - R_1$ we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ, \text{ where } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}. \text{ Ans.}$$

Also rank of the matrix A is 2.

Home Assignments:

Q.No.1.: Find the rank of matrix

$$\textbf{(i).} \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix} \quad \textbf{(ii).} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \quad \textbf{(iii).} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}.$$

Ans.: (i). 2 **(ii).** 2 **(iii).** 3.

Q.No.2.: Find the rank of matrix

$$\textbf{(i).} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad \textbf{(ii).} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Ans.: (i). 2 **(ii).** 4 .

Q.No.3.: Use Gauss-Jordan method to find the inverse of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Ans.: $\begin{bmatrix} 3 & -1 & 1 \\ 15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

Q.No.4.: Reduce the matrices to normal form and hence find its rank

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Ans.: 3.

Q.No.5.: Determine the rank of the matrix $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ by reducing it to the normal form.

Ans.: 2.

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