Some problems on

1st & 2nd Topics

Matrices

Problems on definitions of special types of Matrices

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Now let us use the various definitions of special types of matrices in the following problems:

Q.No.1.: Evaluate $3A-4B$, where $A=\begin{bmatrix}3 & -4 & 6\\ 5 & 1 & 7\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Sol.: Here $A=\begin{bmatrix}3 & -4 & 6\\ 5 & 1 & 7\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Therefore $3A=\begin{bmatrix}9 & -12 & 18\\ 15 & 3 & 21\end{bmatrix}$ and $4B=\begin{bmatrix}4 & 0 & 4\\ 8 & 0 & 12\end{bmatrix}$

Now $3A-4B=\begin{bmatrix}9-4 & -12-0 & 18-4\\ 15-8 & 3-0 & 21-12\end{bmatrix}=\begin{bmatrix}5 & -12 & 14\\ 7 & 3 & 9\end{bmatrix}$. Ans.

Q.No.2.: If $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & -2\\ -1 & 0\\ 2 & -1\end{bmatrix}$, form the product of AB.

Is BA defined ?

Sol.: Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB defined and

$$AB = \begin{bmatrix} 0.1+1.(-1)+2.2 & 0.(-2)+1.0+2.(-1) \\ 1.1+2.(-1)+3.2 & 1.(-2)+2.0+3.(-1) \\ 2.1+3.(-1)+4.2 & 2.(-2)+3.0+4.(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}.$$

Again, since the number of columns of B $\neq$ the number of rows of A

$\therefore$ The product BA is not possible.

Q.No.3.: If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$,

compute AB and BA and show that $AB \neq BA$.

Sol.: Here $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$.

$$\text{Now } AB = \begin{bmatrix} 1.2+3.1+0.(-1) & 1.3+3.2+0.1 & 1.4+3.3+0.2 \\ (-1).2+2.1+1.(-1) & (-1).3+2.1+1.1 & (-1).4+2.3+1.2 \\ 0.2+0.1+2.(-1) & 0.3+0.2+2.1 & 0.4+0.3+2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 2.1+3.(-1)+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2 \\ 1.1+2.(-1)+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2 \\ (-1).1+1.(-1)+2.0 & (-1).3+1.2+2.0 & (-1).0+1.1+2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}.$$

Hence $AB \neq BA$.

Q.No.4.: Prove that $A^3 - 4A^2 - 3A + (11)I = O$, where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Sol.: Here $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

$$\text{Now } A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix},$$

and $A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}.$$

$$\therefore A^3 - 4A^2 - 3A + (11)I = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11, & 37-28-9+0, & 26-20-6+0 \\ 10-4-6-0, & 5-16+0+11, & 1-4+3+0 \\ 35-32-3+0, & 42-36-6+0, & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \text{. Ans.}$$

Q. No.5: Which of the following matrices are singular:

$$\textbf{(i)} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}, \textbf{(ii)} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{bmatrix}, \textbf{(iii)} \begin{bmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{bmatrix}.$$

Sol.: (i). Here the given matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$.

Since, we know that a matrix A is said to be singular if $|A| = 0$.

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} = 1(4-6) - 2(4-2) + 3(3-1) = -2-4+6 = 0.$$

Hence, the given matrix A is singular.

(ii). Here the given matrix is $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{bmatrix}$.

Now $|B| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 25 \end{vmatrix} = 1(100-72) - 1(50-24) + 1(18-12) = 28-16+6 = 18 \neq 0$

Now since $|B| \neq 0$. Hence, the given matrix B is non-singular.

(iii). Here the given matrix is $C = \begin{bmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{bmatrix}$.

Now $|C| = \begin{vmatrix} 2 & 5 & 19 \\ 1 & -2 & -4 \\ -3 & 2 & 0 \end{vmatrix} = 2(0+8) - 5(0-12) + 19(2-6) = 16 + 60 - 76 = 0$

Hence, the given matrix C is singular.

Q.No.6.: For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?

Sol.: Here the given matrix is $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$.

Now a matrix is said to be singular is $|A| = 0$.

Here $|A| = \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix}$

$$= (3-x)[(4-x)(-1-x)+4] - 2[2(-1-x)+2] + 2[-8+2(4-x)]$$

$$= (3-x)(-4-4x+x+x^2+4) - 2(-2-2x+2) + 2(-8+8-2x)$$

$$= -9x + 3x^2 + 3x^2 - x^3 + 4x - 4x = -x^3 + 6x^2 - 9x = -x(x^2 + 6x - 9)$$

$$= -x\left[(x-3)^2\right].$$

Now $|A| = 0 \Rightarrow -x\left[(x-3)^2\right] = 0 \Rightarrow x = 0$ and $x = 3$. Ans.

Q.No.7.: Find the values of x, y, z and a, which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}.$$

Sol.: As the given matrices are equal, equating the elements of both the matrices, we get

$x + 3 = 0$; $2y + x = -7$; $z - 1 = 3$; $4a - 6 = 2a$.

$\therefore x = -3, \quad y = -2, \quad z = 4, \quad a = 3$. Ans.

Q.No.8.: Find x, y, z and w, given that:

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}.$$

Sol.: Given

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ -1+z+w & 2w+3 \end{bmatrix}$$

Now, both the matrices are equal, equating the elements of both the matrices, we get

$3x = x + 4 \qquad \Rightarrow x = 2$

$3y = x + y + 6 \qquad \Rightarrow y = 4$

$3w = 2w + 3 \qquad \Rightarrow w = 3$

$3z = -1 + z + w \qquad \Rightarrow z = 1$. Ans.

Q.No.9.: Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exist. Find x and y.

Sol.: Since the order of A is $x \times (x+5)$ and order of B is $y \times (11 - y)$.

Since AB exist $\Rightarrow x + 5 = y \Rightarrow x - y = -5$. (i)

Also BA exist $\Rightarrow 11 - y = x \Rightarrow x + y = 11$. (ii)

Solving (i) and (ii), we get

$2x = 6 \Rightarrow x = 3$. Ans.

$\therefore y = 8$. Ans.

Q.No.10.: If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$. Calculate the product AB.

Sol.: Here given $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$. (i)

and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$. (ii)

Adding (i) and (ii), we get $2A = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$.

Subtracting (i) and (ii), we get $2B = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$.

$$\therefore AB=\begin{bmatrix}2&0\\2&2\end{bmatrix}\begin{bmatrix}-1&-1\\1&-2\end{bmatrix}=\begin{bmatrix}-2+0&-2+0\\-2+2&-2-4\end{bmatrix}=\begin{bmatrix}-2&-2\\0&-6\end{bmatrix}=-2\begin{bmatrix}1&1\\0&3\end{bmatrix}. \text{ Ans.}$$

Q.No.11.: If $A=\begin{bmatrix}1&2&3&4\\2&0&1&2\\3&1&0&5\end{bmatrix}_{3\times4}$ and $B=\begin{bmatrix}2&1&0\\3&2&1\\1&0&1\end{bmatrix}_{3\times3}$,

find AB or BA, whichever exist.

Sol.: Here AB does not exist because the number of columns in A is not equal to the number of rows in B and BA exist because the number of columns in B is equal to the number of rows in A.

$$\text{Now } BA=\begin{bmatrix}2&1&0\\3&2&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\2&0&1&2\\3&1&0&5\end{bmatrix}=\begin{bmatrix}2+2+0&4+0+0&6+1+0&8+2+0\\3+4+3&6+0+1&8+2+0&12+4+5\\1+0+3&2+0+1&3+0+0&5+0+4\end{bmatrix}$$

$$\Rightarrow BA=\begin{bmatrix}4&4&7&10\\10&7&11&21\\4&3&3&9\end{bmatrix}. \text{ Ans.}$$

Q.No.12.: If $A=\begin{bmatrix}1&2\\-2&3\end{bmatrix}$, $B=\begin{bmatrix}2&1\\2&3\end{bmatrix}$ and $C=\begin{bmatrix}-3&1\\2&0\end{bmatrix}$,

verify that $(AB)C=A(BC)$ and $A(B+C)=AB+AC$.

Sol.: Now $AB=\begin{bmatrix}1&2\\-2&3\end{bmatrix}\begin{bmatrix}2&1\\2&3\end{bmatrix}=\begin{bmatrix}2+4&1+6\\-4+6&-2+9\end{bmatrix}=\begin{bmatrix}6&7\\2&7\end{bmatrix}$.

$$\therefore (AB)C=\begin{bmatrix}6&7\\2&7\end{bmatrix}\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}-18+14&6+0\\-6+14&2+0\end{bmatrix}=\begin{bmatrix}-4&6\\8&2\end{bmatrix}. \qquad \text{(i)}$$

$$\text{Now } BC=\begin{bmatrix}2&1\\2&3\end{bmatrix}\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}-6+2&2+0\\-6+6&2+0\end{bmatrix}=\begin{bmatrix}-4&2\\2&2\end{bmatrix}.$$

$$\therefore A(BC)=\begin{bmatrix}1&2\\-2&3\end{bmatrix}\begin{bmatrix}-4&2\\2&2\end{bmatrix}=\begin{bmatrix}-4+0&2+4\\8+0&-4+6\end{bmatrix}=\begin{bmatrix}-4&6\\8&2\end{bmatrix}. \qquad \text{(ii)}$$

From (i) and (ii), we get $(AB)C=A(BC)$.

$$\text{Now } B+C=\begin{bmatrix}2&1\\2&3\end{bmatrix}+\begin{bmatrix}-3&1\\1&0\end{bmatrix}=\begin{bmatrix}-1&2\\4&3\end{bmatrix}$$

$$\therefore A(B+C)=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-1 & 2\\ 4 & 3\end{bmatrix}=\begin{bmatrix}-1+8 & 2+6\\ 2+12 & -4+9\end{bmatrix}=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}. \qquad \text{(iii)}$$

$$\text{Now } AC=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-3+4 & 1+0\\ 6+6 & -2+0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}$$

$$\therefore AB+AC=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}+\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}. \qquad \text{(iv)}$$

From(iii) and (iv), we get $A(B+C)=AB+AC$.

Hence verified.

Q.No.13.: Evaluate **(i)** $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$,

(ii) $\begin{bmatrix}2 & 1 & -1\\ 4 & -5 & 6\\ -3 & 7 & 3\end{bmatrix}\times\begin{bmatrix}3 & 1\\ -6 & 4\\ -2 & 5\end{bmatrix}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}$,

(iii) $\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}\times\begin{bmatrix}4 & 5 & 2\end{bmatrix}\times\begin{bmatrix}2\\ -3\\ 5\end{bmatrix}\times\begin{bmatrix}3 & 2\end{bmatrix}$.

Sol.: (i). $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}ax+hy+gz & hx+by+fz & gx+fy+zc\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$=\left[ax^2+hxy+gxz+hxy+by^2+fzy+gzx+fyz+z^2c\right]$$

$$=\left[ax^2+by^2+cz^2+2hxy+2fyz+2gzx\right]. \text{ Ans.}$$

(ii). Now $\begin{bmatrix}2 & 1 & -1\\ 4 & -5 & 6\\ -3 & 7 & 3\end{bmatrix}_{3\times 3}\times\begin{bmatrix}3 & 1\\ -6 & 4\\ -2 & 5\end{bmatrix}_{3\times 2}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}_{2\times 2}$

$$=\begin{bmatrix}6-6+2 & 2+4-5\\ 12+30-12 & 4-20+30\\ -9-42-6 & -3+28+15\end{bmatrix}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 30 & 14\\ -57 & 40\end{bmatrix}_{3\times 2}\times\begin{bmatrix}5 & 3\\ -2 & 1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix} 10-2 & 6+1 \\ 150-28 & 90+14 \\ -285-80 & -171+40 \end{bmatrix}=\begin{bmatrix} 8 & 7 \\ 122 & 104 \\ -365 & -131 \end{bmatrix}. \text{ Ans.}$$

(iii). Now $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}_{3\times1} \times \begin{bmatrix} 4 & 5 & 2 \end{bmatrix}_{1\times3} \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}_{3\times1} \times \begin{bmatrix} 3 & 2 \end{bmatrix}_{1\times2}$

$$=\begin{bmatrix} 4 & 5 & 2 \\ -8 & -10 & -4 \\ 12 & 15 & 6 \end{bmatrix}_{3\times3} \times \begin{bmatrix} 6 & 4 \\ -9 & -6 \\ 15 & 10 \end{bmatrix}_{3\times2}$$

$$=\begin{bmatrix} 24-45+30 & 16-30+20 \\ -48+90-60 & -32+60-40 \\ 72-135+90 & 48-90+60 \end{bmatrix}=\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}. \text{ Ans.}$$

Q.No.14.: Prove that the product of two matrices $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Sol.: Here product of two matrices $=\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$=\begin{bmatrix} \cos^2\theta\cos^2\phi+\cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\phi\cos^2\phi+\sin^2\theta\cos\phi\sin\phi & \cos\theta\cos\phi\sin\theta\sin\phi+\sin^2\theta\sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta\cos\phi[\cos(\theta-\phi)] & \cos\theta\sin\phi[\cos(\theta-\phi)] \\ \cos\phi\sin\theta[\cos(\theta-\phi)] & \sin\theta\sin\phi[\cos(\theta-\phi)] \end{bmatrix}.$$

Now if above matrix is a null matrix, then

$$\cos(\theta-\phi)=0 \Rightarrow \theta-\phi=(2n+1)\frac{\pi}{2} \Rightarrow \theta=\phi+(2n+1)\frac{\pi}{2}.$$

Hence, θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

This is the required result.

Q.No.15.: If $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$, show that $I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$.

Sol.: Now $I+A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$. (i)

and $I-A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}$

$$\therefore (I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}\begin{bmatrix}\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{-2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\\ \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{2\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{-2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{\tan\frac{\alpha}{2}-\tan^3\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\\ \frac{-\tan\frac{\alpha}{2}+\tan^3\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{2\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\end{bmatrix}=\begin{bmatrix}\frac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{-\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}{1+\tan^2\frac{\alpha}{2}}\\ \frac{\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}{1+\tan^2\frac{\alpha}{2}} & \frac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\end{bmatrix}$$

$$=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}. \qquad \text{(ii)}$$

From (i) and (ii), we get $I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$.

This completes the proof.

Q.No.16.: If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$, where I is a unit matrix of second order.

Sol.: Given $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$ $\therefore A^2=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$,

$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$ and $7\,I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $A^2 - 5A + 7I = O$. This completes the proof.

Q.No.17.: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and I is the unit matrix of order 3,

evaluate $A^2 - 3A + 9\,I$.

Sol.: Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$.

$$\therefore A^2 = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix},$$

$3A = \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$ and $9\,I = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$\therefore A^2 - 3A + 9\,I = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 1 & 2 \\ 5 & -5 & 4 \\ 2 & 8 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$

Hence $A^2 - 3A + 9\,I = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$. Ans.

Q.No.18.: If $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}$,

verify the result $(A+B)^2 = A^2 + BA + AB + B^2$.

Sol.: Now $A+B=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}+\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}$

$$\therefore (A+B)^2=\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}\begin{bmatrix}4 & 1 & 0\\ 2 & 0 & 5\\ 4 & -2 & 4\end{bmatrix}=\begin{bmatrix}16+2+0 & 4+0+0 & 0+5+0\\ 4+0+20 & 2+0-10 & 0+0+20\\ 16-4+16 & 4+0-8 & 0-10+16\end{bmatrix}$$

$$=\begin{bmatrix}18 & 4 & 5\\ 28 & -8 & 20\\ 28 & -4 & 6\end{bmatrix}. \qquad \text{(i)}$$

Also $$A^2=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}=\begin{bmatrix}1+4+0 & 2+0-1 & -1+6-2\\ 2+0+0 & 4+0+3 & -2+0+6\\ 0+2+0 & 0+0+2 & 0+3+4\end{bmatrix}=\begin{bmatrix}5 & 1 & 3\\ 2 & 7 & 4\\ 2 & 2 & 7\end{bmatrix},$$

$$B^2=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}9+0+4 & -3+0-3 & 3-2+2\\ 0+0+8 & 0+0-6 & 0+0+4\\ 12+0+8 & -4+0-6 & 4-6+4\end{bmatrix}=\begin{bmatrix}13 & -6 & 3\\ 8 & -6 & 4\\ 20 & -10 & 2\end{bmatrix},$$

$$AB=\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}3+0-4 & -1+0+3 & 1+4-2\\ 6+0+12 & -2+0-9 & 2+0+6\\ 0+0+8 & 0+0-6 & 0+2+4\end{bmatrix}=\begin{bmatrix}-1 & 2 & 3\\ 18 & -11 & 8\\ 8 & -6 & 6\end{bmatrix},$$

$$BA=\begin{bmatrix}3 & -1 & 1\\ 0 & 0 & 2\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 2 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}=\begin{bmatrix}3-2+0 & 6+0+1 & -3-3+2\\ 0+0+0 & 0+0+2 & 0+0+4\\ 4-6+0 & 8+0+2 & -4-9+4\end{bmatrix}=\begin{bmatrix}1 & 7 & -4\\ 0 & 2 & 4\\ -2 & 10 & -9\end{bmatrix},$$

$$\therefore A^2+BA+AB+B^2=\begin{bmatrix}5 & 1 & 3\\ 2 & 7 & 4\\ 2 & 2 & 7\end{bmatrix}+\begin{bmatrix}1 & 7 & -4\\ 0 & 2 & 4\\ -2 & 10 & -9\end{bmatrix}+\begin{bmatrix}-1 & 2 & 3\\ 18 & -11 & 8\\ 8 & -6 & 6\end{bmatrix}+\begin{bmatrix}13 & -6 & 3\\ 8 & -6 & 4\\ 20 & -10 & 2\end{bmatrix}$$

$$=\begin{bmatrix} 5+1-1+13 & 1+7+2-6 & 3-4+3+3 \\ 2+0+18+8 & 7+2-11-6 & 4+4+8+4 \\ 2-2+8+20 & 2+10-6-10 & 7-9+6+2 \end{bmatrix}$$

$$=\begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}. \qquad \text{(i)}$$

From (i) and (ii), we get $(A+B)^2 = A^2 + BA + AB + B^2$.

Hence, the result is verified.

Q.No.19.: If $E=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + F^2E \neq E$.

Sol.: Now $EF=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

and $FE=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Ans.

Now $E^2=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E^2F=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$F^2=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F^2E=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$\therefore E^2F + F^2E=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow E^2F + F^2E \neq E$.

Q.No.20.: By mathematical induction, prove that if $A=\begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$, then

$$A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}.$$

Sol.: For n = 1, $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = \begin{bmatrix} 1+10.1 & -25.1 \\ 4.1 & -1-10.1 \end{bmatrix}$.

Thus, the result is true for n = 1.

Now, let us suppose that the result is true for n = k, then $A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$.

Now, we have to prove that the result is true for n = k + 1.

Now $A^{k+1} = A^k.A = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}\begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = \begin{bmatrix} 11+10k & -25-25k \\ 4k+4 & -9-10k \end{bmatrix}$

$$A^{k+1} = \begin{bmatrix} 1+10(k+1) & -15(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}.$$

Thus the result is also true for n = k + 1.

Hence, this proves the result.

Q.No.21.: If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$,

where n is a positive integer.

Sol.: For n = 1, $A^1 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos 1\alpha & \sin 1\alpha \\ -\sin 1\alpha & \cos 1\alpha \end{bmatrix}$.

Thus, the result is true for n = 1.

Now, let us suppose that the result is true for n = k, then $A^k = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}$.

Now, we have to prove that the result is true for n = k + 1.

Now $A^{k+1} = A^k.A = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha & \cos k\alpha\sin\alpha + \sin k\alpha\cos\alpha \\ -(\cos\alpha\sin k\alpha + \sin\alpha\cos k\alpha) & -\sin k\alpha\sin\alpha + \cos\alpha\cos k\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\alpha & \sin(k+1)\alpha \\ -\sin(k+1)\alpha & \cos(k+1)\alpha \end{bmatrix}.$$

Thus, the result is also true for n = k + 1.

Hence, this proves the result.

Q.No.22.: Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into LU, where L is lower triangular matrix and U is the upper triangular matrix.

Sol.: Let $L = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $U = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$ be the lower triangular matrix and upper triangular matrix respectively.

Now $LU = A \Rightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Equating, we get

$a_{11}b_{11} = 5$, $a_{11}b_{12} = -2$, $a_{11}b_{13} = 1$, $a_{21}b_{11} = 7$, $a_{21}b_{12} + a_{22}b_{22} = 1$,

$a_{21}b_{13} + a_{22}b_{23} = -5$, $a_{31}b_{11} = 3$, $a_{31}b_{12} + a_{32}b_{22} = 7$, $a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} = 4$.

Since, we have 9 equations and we have to find 12 unknowns, so we can choose 3 unknowns arbitrary.

In other way, we have infinite number of such type of matrices whose product is A.

Now let us suppose $a_{11} = a_{22} = a_{33} = 1$.

$\therefore b_{11} = 5$, $b_{12} = -2$, $b_{13} = 1$, $a_{21} = \frac{7}{5}$, $a_{31} = \frac{3}{5}$,

$\frac{7}{5} \times (-2) + b_{22} = 1 \Rightarrow b_{22} = 1 + \frac{14}{5} = \frac{19}{5}$,

$\frac{7}{5} \times 1 + 1 \times b_{23} = -5 \Rightarrow b_{23} = -5 - \frac{7}{5} = \frac{-32}{5}$,

$\frac{7}{5} \times (-2) + a_{32} \times \frac{19}{5} = 7 \Rightarrow \frac{19}{5} a_{32} = \frac{41}{5} \Rightarrow a_{32} = \frac{41}{19}$,

$\frac{3}{5}\times 1+\frac{41}{19}\times\frac{-32}{5}+b_{33}=4 \Rightarrow \frac{57-3112}{95}+b_{33}=4 \Rightarrow \frac{-251}{19}+b_{33}=4$,

$\Rightarrow b_{33}=4+\frac{251}{19}=\frac{76+251}{19} \Rightarrow b_{33}=\frac{327}{19}$.

Thus $\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & \frac{-32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$.

$\Rightarrow A = LU$.

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & \frac{-32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$ be the lower triangular and upper triangular matrices, respectively.

Q.No.23.: Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.

Sol.: A given matrix A is said to be Hermitian if $A = A^{\theta}$ or $A' = \overline{A}$.

Let $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$.

$\therefore \overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$.

Also $A' = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$.

$\therefore A' = \overline{A}$.

Hence, the given matrix is Hermitian.

Q.No.24.: If $A=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$.

Then show that A is Hermitian and iA is Skew-Hermitian.

Sol.: Since, here $A=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$.

Therefore $\overline{A}=\begin{bmatrix} 2 & 3-2i & -4 \\ 3+2i & 5 & -6i \\ -4 & 6i & 3 \end{bmatrix}$ and $\overline{A}'=\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}=A$.

Thus A is Hermitian.

Let $B=iA=i\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & +6i \\ -4 & -6i & 3 \end{bmatrix}=\begin{bmatrix} 2i & -2+3i & -4i \\ 2+3i & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$.

Therefore $\overline{B}=\begin{bmatrix} -2i & -2-3i & 4i \\ 2-3i & -5i & -6 \\ 4i & 6 & -3i \end{bmatrix}$ and $B^T=\begin{bmatrix} 2i & 2+3i & -4i \\ -2+3i & 5i & 6 \\ -4i & -6 & 3i \end{bmatrix}$.

Thus $\overline{B}=-B^T \Rightarrow B$ is Skew-Hermitian.

Q.No.25.: If $A=\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, shows that AA* is a Hermitian matrix, where A* is the conjugate transpose of A.

Sol.: We have $A'=\begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$ and $A^*=\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$.

$$\therefore AA^*=\begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$=\begin{bmatrix} 4-i^2+9+1-9i^2, & -10-5-3i-10+10i \\ -10+5i+3i-10-10i, & 25-i^2+16-4i^2 \end{bmatrix}$$

$=\begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}$, which is a Hermitian matrix.

Q.No.26.: Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

Sol.: A given matrix A is said to be unitary if $AA^{\theta} = I$.

Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$.

$\therefore \overline{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$ and $A^{\theta} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$.

Now $AA^{\theta} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} . \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+2 & 2-2 \\ 2-2 & 2+2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

$\therefore AA^{\theta} = I$.

Hence, the given matrix is unitary.

Q.No.27.: Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(1-A)(1+A)^{-1}$ is a unitary matrix.

or

If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, Obtain the matrix $(I-N)(I+N)^{-1}$, and show that it is unitary.

Sol.: $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$, $|I+A| = 1-(-1-4) = 6$.

$(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6$. Also $I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$$\therefore (I-A)(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \quad \text{(i)}$$

Its conjugate-transpose $= \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$ (ii)

$\therefore$ Product of (i) and (ii) $\dfrac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \dfrac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$.

Hence the result.

Or

Sol.: Since here $I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$.

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}.$$

$$|I + N| = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix} = 1 - \left(4i^2 - 1\right) = 6.$$

$$\text{adj }(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}.$$

$$(I + N)^{-1} = \frac{1}{|I + N|}\text{adj}(I + N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore (I - N)(I + N)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A \text{ (say)}$$

$$A' = \frac{1}{6}\begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix}$$

$$\overline{(A')} = A^* = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$A^*A = \frac{1}{6}\begin{bmatrix} -2 & 2+4i \\ -2+4i & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$\Rightarrow A = (I - N)(I + N)^{-1}$ is unitary.

Q.No.28.: If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{i2\pi/3}$, then show that $S^{-1} = \dfrac{1}{3}\overline{S}$.

Sol.: Now $a = e^{i2\pi/3} = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = \omega$ (cube root of unity).

$$\therefore a^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

and $a^3 = e^{6i\pi/3} = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1 = \omega^3$.

$$\therefore S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}.$$

$$\text{Now } \overline{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \dfrac{1}{\omega^2} & \dfrac{1}{\omega} \\ 1 & \dfrac{1}{\omega} & \dfrac{1}{\omega^2} \end{bmatrix} \Rightarrow \overline{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}. \qquad [\because \omega^3 = 1]$$

$$\text{Also } |S| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = (\omega^4 - \omega^2) - (\omega^2 - \omega) + (\omega - \omega^2)$$

$$= (\omega - \omega^2) + (\omega - \omega^2) + (\omega - \omega^2) = 3(\omega - \omega^2)$$

$$\text{And Adj } S = \begin{bmatrix} \omega^4 - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix}$$

$$\therefore S^{-1} = \frac{\text{Adj } A}{|S|} = \frac{1}{3(\omega - \omega^2)} \begin{bmatrix} \omega^4 - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix}$$

$$= \frac{1}{3(\omega - \omega^2)} \begin{bmatrix} \omega - \omega^2 & \omega - \omega^2 & \omega - \omega^2 \\ \omega - \omega^2 & \omega^2 - 1 & 1 - \omega \\ \omega - \omega^2 & 1 - \omega & \omega^2 - 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\dfrac{1+\omega}{\omega} & \dfrac{1}{\omega} \\ 1 & \dfrac{1}{\omega} & -\dfrac{1+\omega}{\omega} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \qquad \begin{bmatrix} \because 1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega = -\omega^2 \\ \omega^3 = 1 \Rightarrow \dfrac{1}{\omega} = \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\overline{S}.$$

Thus $S^{-1} = \dfrac{1}{3}\overline{S}$.

Hence, this proves the result.

Home Assignments

Q.No.1.: Express A as the sum of a symmetric and skew-symmetric matrix where

$$A=\begin{bmatrix}4 & 2 & -3\\ 1 & 3 & -6\\ -5 & 0 & -7\end{bmatrix}$$

Ans.: $A+A^T=\frac{1}{2}\begin{bmatrix}8 & 3 & -8\\ 3 & 6 & -6\\ -8 & -6 & -14\end{bmatrix}$ symmetric,

$A-A^T=\frac{1}{2}\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & -6\\ -2 & 6 & 0\end{bmatrix}$ skew-symmetric.

Q.No.2.: Prove that the inverse of a non-singular symmetric matrix A is symmetric.

Q.No.3.: Write $A=\begin{bmatrix}3 & -4 & -1\\ 6 & 0 & -1\\ -3 & 13 & -4\end{bmatrix}$ as the sum of a symmetric R and skew-symmetric S.

Ans.: $R=\frac{1}{2}\left[A+A^T\right]=\begin{bmatrix}3 & 1 & -2\\ 1 & 0 & 6\\ -2 & 6 & -4\end{bmatrix}$, $S=\frac{1}{2}\left[A-A^T\right]=\begin{bmatrix}0 & -5 & 1\\ 5 & 0 & -7\\ -1 & 7 & 0\end{bmatrix}$.

Q.No.4.: Prove that the product AB of two symmetric matrices A and B is symmetric if AB = BA.

Q.No.5.: Determine for what values of numbers a and b, c = aA + bB is Skew-Hermitian given that A and B are Skew-Hermitian.

Ans.: both a and b must be real.

Q.No.6.: If $A=\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$, show that $(I-A)(1+A)^{-1}$ is a unitary matrix.

Q.No.7.: Show that $A=\begin{bmatrix}a+ic & -b+id\\ b+id & a+ic\end{bmatrix}$ is unitary matrix if $a^2+b^2+c^2+d^2=1$.

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