6^{th} Topic

Double Integrals

[Area enclosed by plane curves]

(Last updated on 15-07-2013)

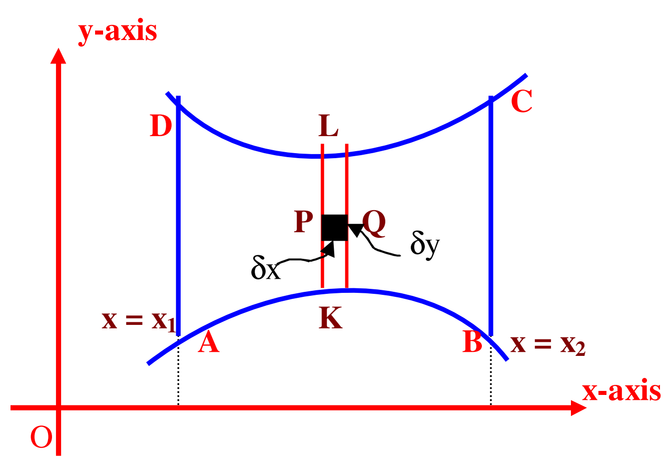
(15 Solved problems and 00 Home assignments)

Area enclosed by plane curves:

Cartesian co-ordinates:

Case1a.:

Consider the area enclosed by the curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = x_1, x = x_2$. Divide this area into vertical strips of width δx. If P(x, y), $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.



$\therefore$ Area of the strip $KL = \lim\limits_{\delta y \to 0} \sum \delta x \delta y$.

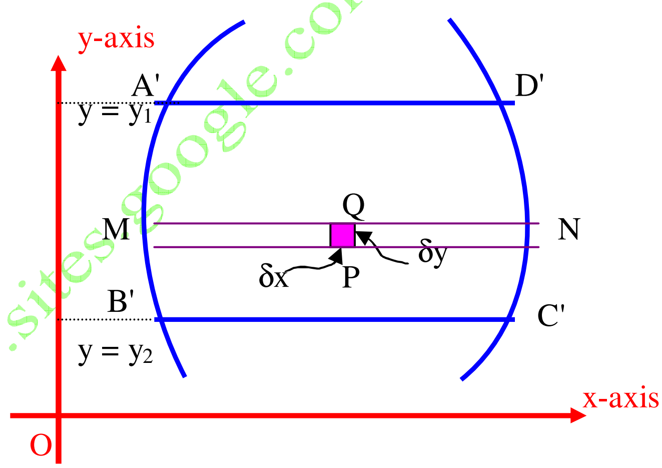
Since for all rectangles in this strip δx in the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$\therefore$ Area of the strip $KL = \delta x \lim\limits_{\delta y \to 0} \sum\limits_{f_1(x)}^{f_2(x)} \delta y = \delta x \int\limits_{f_1(x)}^{f_2(x)} dy$.

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area ABCD

$$= \lim_{\delta x \to 0} \sum_{x_1}^{x_2} \delta x . \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

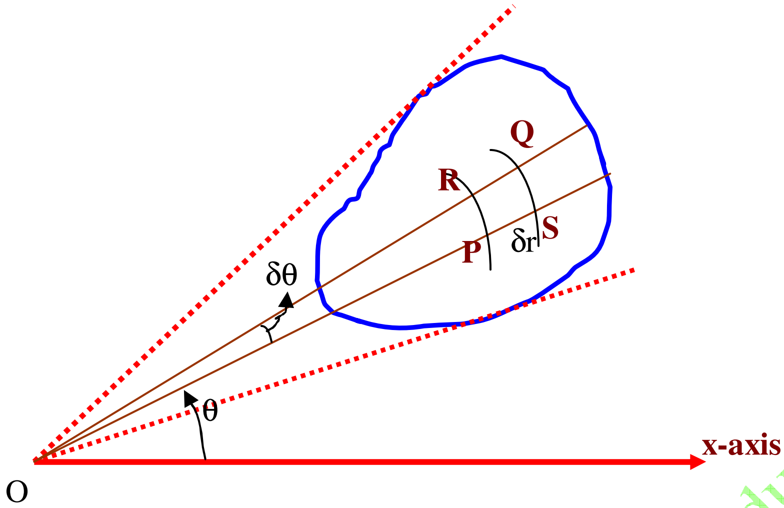
Case1b.: Similarly, dividing the area $A'B'C'D'$ as in the figure, into horizontal strips of width δy, we get the area $A'B'C'D' = \int\limits_{y_1}^{y_2} \int\limits_{f_1(y)}^{f_2(y)} dxdy$.



Case2.: Polar co-ordinates:

Consider an area A enclosed by a curve whose equation is in polar co-ordinates. Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S.

Since arc $PR = r\delta\theta$ and $PS = \delta r$.



$\therefore$ Area of curvilinear rectangle PRQS is approximately $= PR\,.\,PS = r\delta\theta\,.\,\delta r$.

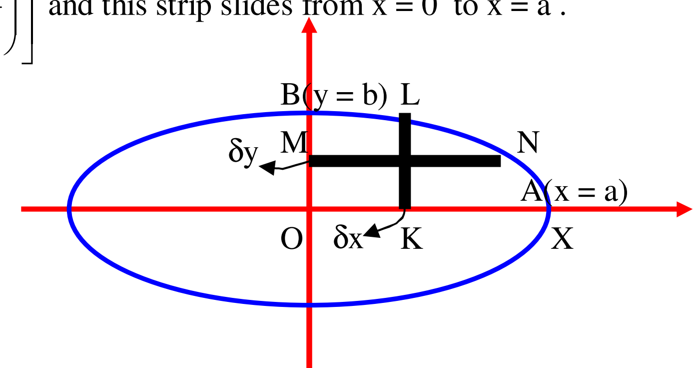
If the whole area is divided into such curvilinear rectangles, the sum $\sum\sum r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence, $A = \lim\limits_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \sum\sum r\delta\theta\delta r = \iint r\,d\theta\,dr$,

where the limits are to be so chosen as it cover the entire area.

Q.No.1.: Find, by double integration, the area of a plate in the form of a quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Sol.: Here we suppose that the strip is parallel to the y-axis, therefore y varies from K(y = 0) to $L\left[y = b\sqrt{\left(1 - \dfrac{x^2}{a^2}\right)}\right]$ and this strip slides from $x = 0$ to $x = a$.



$$\therefore \text{The required area} = \int_0^a \left(\int_0^{b\sqrt{1 - x^2/a^2}} dy \right) dx = \int_0^a \left([y]_0^{b\sqrt{1 - x^2/a^2}} \right) dx = \frac{b}{a}\int_0^a \sqrt{\left(a^2 - x^2\right)}\,dx$$

Now put $x = a\sin t$, $dx = a\cos dt$ and when $x = 0$, $t = 0$; when $x = a$, $t = \frac{\pi}{2}$.

Hence the required area $= \frac{b}{a}\int_0^{\pi/2} \left(a^2 - a^2\sin^2 t\right) a\cos t dt$

$$= \frac{b}{a}\int_0^{\pi/2} a^2\cos^2 t dt = ab\left(\frac{1}{2}\times\frac{\pi}{2}\right) = \frac{\pi ab}{4}.$$ Square units. Ans.

Second Method: Here we suppose that the strip is parallel to the x-axis, therefore x varies from M(x = 0) to $N\left[x = a\sqrt{\left(1 - \frac{y^2}{b^2}\right)}\right]$ and this strip slides from y = 0 to y = b.

$\therefore$ The required area $= \int_0^b dy \int_0^{a\sqrt{1-y^2/b^2}} dx = \int_0^b dy[x]_0^{a\sqrt{1-y^2/b^2}} = \frac{a}{b}\int_0^b \sqrt{\left(b^2 - y^2\right)} dy$

Now put $x = a\sin t$, $dx = a\cos dt$ and when $x = 0$, $t = 0$; when $x = a$, $t = \frac{\pi}{2}$.

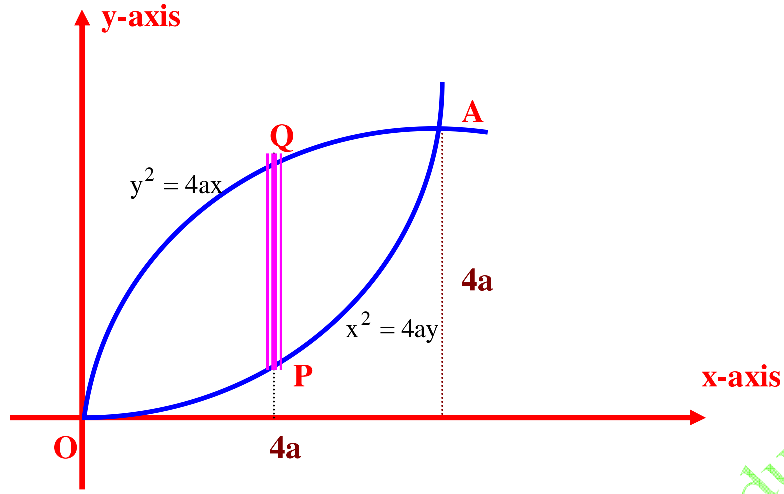
Hence the required area $= \frac{b}{a}\int_0^{\pi/2} \left(a^2 - a^2\sin^2 t\right) a\cos t dt$

$$= \frac{b}{a}\int_0^{\pi/2} a^2\cos^2 t dt = ab\left(\frac{1}{2}\times\frac{\pi}{2}\right) = \frac{\pi ab}{4}.$$ Square units. Ans.

Remarks: *The change of the order of integration does not in any way affect the value of the area.*

Q.No.2.: Show, by double integration, that area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Sol.: Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O(0, 0) and A(4a, 4a). Here we suppose that the strip is parallel to the y-axis, therefore y varies from P to Q i. e. from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{(ax)}$ and this strip slides from x = 0 to x = 4a.

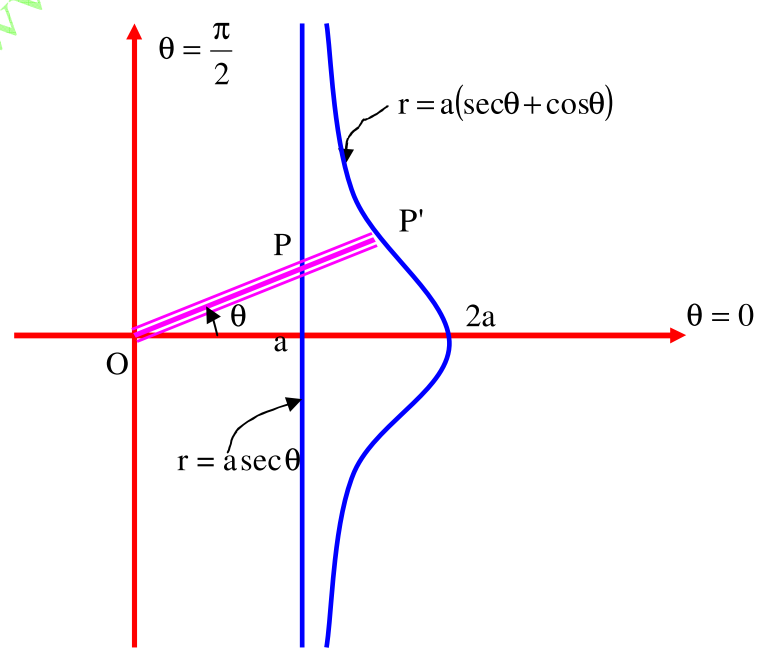


$$\therefore \text{The required area} = \int_0^{4a}\left(\int_{x^2/4a}^{2\sqrt{(ax)}} dy\right)dx = \int_0^{4a}\left[2\sqrt{(ax)} - \frac{x^2}{4a}\right]dx$$

$$= \left|2\sqrt{a}.\frac{2}{3}x^{3/2} - \frac{1}{4a}.\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2. \text{ Square units.}$$

Q.No.3.: Calculate the area, by double integration, included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.

Sol.: The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$. Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\frac{\pi}{2}$.



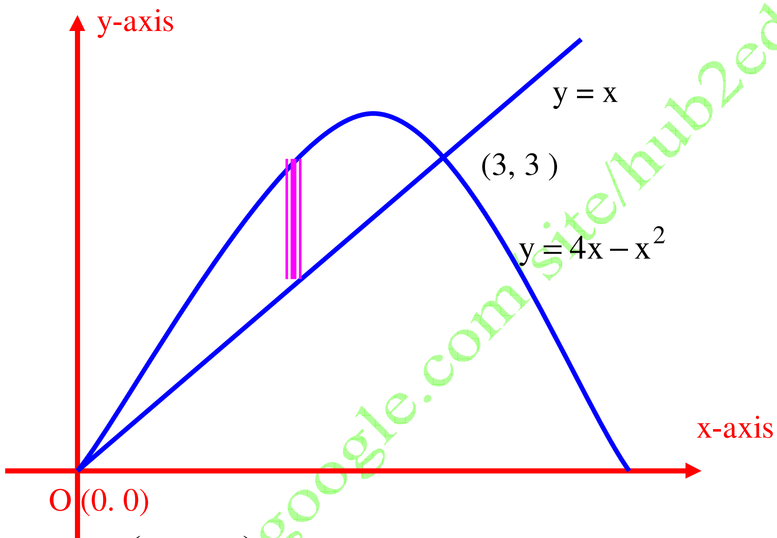
$\therefore$ The required area $= 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\text{coa}\theta)} r dr d\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$

$$= a^2\int_0^{\pi/2}\left(2+\cos^2\theta\right)d\theta = a^2\left[2.\frac{\pi}{2}+\frac{1}{2}.\frac{\pi}{2}\right] = a^2\left[\pi\left(1+\frac{1}{4}\right)\right] = \frac{5\pi a^2}{4}.$$

Square units. Ans.

Q.No.4.: Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol.:



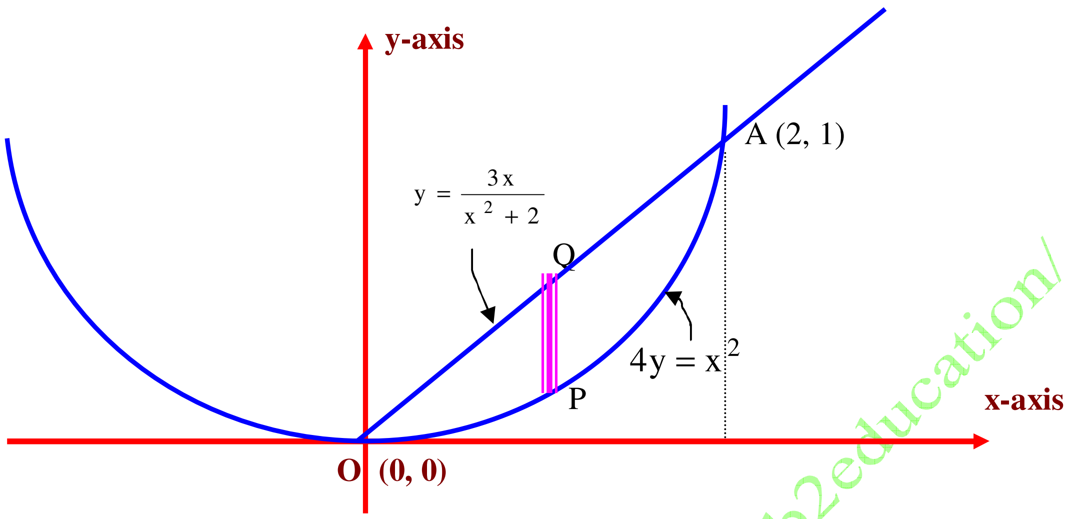
The required area $= \int_0^3\left(\int_x^{4x-x^2} dy\right)dx = \int_0^3 [y]_x^{4x-x^2} dx = \int_0^3 \left(4x - x^2 - x\right)dx$

$$= \int_0^3 \left(3x - x^2\right)dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2} = 4.5 \text{. Sq. units. Ans.}$$

Q.No.5.: Find, by double integration, the area enclosed by the curves $y = \frac{3x}{\left(x^2+2\right)}$ and $4y = x^2$.

Sol.:



Let us suppose that the strip is parallel to y-axis. Then integrate w. r. t. y first and then w. r. t. x.

The required area $A = \int_0^2 \left(\int_{x^2/4}^{3x/(x^2+2)} dy \right) dx = \int_0^2 [y]_{x^2/4}^{3x/(x^2+2)} dx = \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4} \right) dx$.

Let $t = x^2 + 2 \Rightarrow dt = 2xdx \Rightarrow \frac{dt}{2} = xdx$.

$\therefore \int \frac{3x}{x^2+2} dx = \int \frac{3\frac{dt}{2}}{t} = \frac{3}{2} \int \frac{dt}{t} = \frac{3}{2} \log t$.

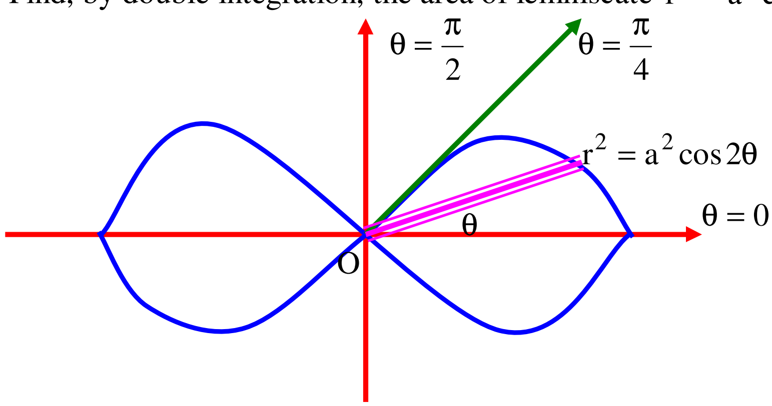
At x = 0, t = 2; x = 2, t = 6.

$$\therefore A = \int_0^2 \left(\int_{x^2/4}^{3x/(x^2+2)} dy \right) dx = \left[\frac{3}{2} \log_e t \right]_2^6 - \left[\frac{x^3}{12} \right]_0^2 = \frac{3}{2} [\log_e 6 - \log_e 2] - \frac{8}{12}$$

$$= \left(\frac{3}{2} \log_e \frac{6}{2} - \frac{2}{3} \right) = \left(\frac{3}{2} \log_e 3 - \frac{2}{3} \right)$$. Sq. units. Ans.

Q.No.6.: Find, by double integration, the area of lemniscate $r^2 = a^2 \cos 2\theta$.

Sol.:



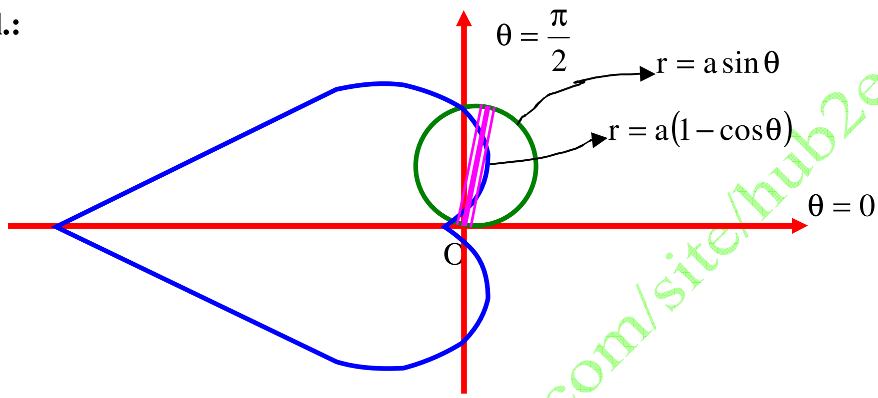
The required area $A = 4\times$[Area in the first quadrant]

$$= 4\times\int_0^{\pi/4}\left(\int_0^{a\sqrt{\cos 2\theta}} r dr\right)d\theta = 4.\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \frac{4}{2}\int_0^{\pi/4} a^2\cos 2\theta.d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2\left[\sin\frac{\pi}{2} - \sin 0\right] = a^2. \text{ Sq. units. Ans.}$$

Q.No.7.: Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Sol.:



The required area $A = \int_0^{\pi/2}\left(\int_{a(1-\cos\theta)}^{a\sin\theta} r dr\right)d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\left[a^2\sin^2\theta - a^2\left(1 - 2\cos\theta + \cos^2\theta\right)\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right)d\theta.$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2}\sin^2\theta d\theta - \int_0^{\pi/2} d\theta + \int_0^{\pi/2} 2\cos\theta d\theta - \int_0^{\pi/2}\cos^2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\left(\frac{1}{2}\times\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) + (2\times 1) - \left(\frac{1}{2}\times\frac{\pi}{2}\right)\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = a^2\left[1 - \frac{\pi}{4}\right]. \text{ Square units. Ans.}$$

Q.No.8.: Find the area between the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the axis.

Sol.: Since the x and y are under radical sign, x and y can take only positive values, therefore the curve lies in the first quadrant.

Now for $x = 0$, $y = a$ and $y - 0$, $x = a$ (here it is important that a is also positive)

Also $x = y = \frac{a}{4}$, satisfy the equation of the curve. Thus the curve can be plotted as shown in the figure.

To find the area, we have to calculate the following integral.

$$A = \int_0^a \left[\int_0^{\left(\sqrt{a}-\sqrt{x}\right)^2} dy \right] dx = \int_0^a [y]_0^{\left(\sqrt{a}-\sqrt{x}\right)^2} dx = \int_0^a \left(\sqrt{a}-\sqrt{x}\right)^2 dx$$

$$= \int_0^a \left(a + x - 2\sqrt{ax}\right) dx = \left[ax + \frac{x^2}{2} - 2\times\frac{2}{3}\sqrt{a}\,x^{3/2} \right]_0^a$$

$$= a^2 + \frac{a^2}{2} - \frac{4}{3}a^2 = \frac{a^2}{6}.$$ Square units. Ans.

Q.No.9.: Find, by double integration, the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.

Sol.: Equation of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Area required $\int_0^2 \int_{\frac{6-3y}{2}}^{\sqrt{\frac{36-9y^2}{4}}} dxdy$

$$A = \frac{1}{2}\int_0^2 \left[\sqrt{6^2 - (3y)^2} - (6-3y) \right] dy = \frac{3}{2}\int_0^2 \sqrt{2^2 - y^2}\,dy - \int_0^2 \frac{6-3y}{2} dy$$

$$= \frac{3}{2}\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} \right]_0^2 - \left[3y - \frac{3}{4}y^2 \right]_0^2 = \frac{3}{2}\left[2\times\frac{\pi}{2} \right] - [6-3]$$

$$= \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2).$$ Square units.

Q.No.10.: Find, by double integration, the smaller of the areas bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.

Sol.: Equation of the circle $x^2 + y^2 = 3^2$.

Area required $= \int_0^3 \int_{3-y}^{\sqrt{9-y^2}} dxdy = \int_0^3 \left[\sqrt{9-y^2} - (3-y)\right] dy$

$= \left[\frac{y}{2}\sqrt{9-y} + \frac{9}{2}\sin^{-1}\frac{y}{3}\right]_0^3 - \left[3y - \frac{y^2}{2}\right]_0^3 = \frac{9}{2} \times \frac{\pi}{2} - \left(9 - \frac{9}{2}\right) = \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2)$. Sq.units.

Q.No.11.: Find, by double integration, the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Sol.: Required area

$A = \int_{-1}^{3} \int_{x^2}^{2x+3} dydx = \int_{-1}^{3} (2x + 3 - x^2) dx = 2\left[\frac{x^2}{2}\right]_{-1}^{3} + 3[x]_{-1}^{3} - \left[\frac{x^3}{3}\right]_{-1}^{3}$

$= 2\left[\frac{9}{2} - \frac{1}{2}\right] + 3(3+1) - \left(\frac{27}{3} + \frac{1}{3}\right) = 8 + 12 - \frac{28}{3} = 20 - \frac{28}{3} = \frac{32}{3} = 10\frac{2}{3}$. Square units.

Q.No.12.: Find, by double integration, the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.

Sol.: Area required $= \int\int_R dxdy$

$A = \int_{-2}^{2} \int_{\frac{4-y^2}{4}}^{4-y^2} dxdy = \int_{-2}^{2} \left[4 - y^2 - \left(\frac{4-y^2}{4}\right)\right] dy = \int_{-2}^{2} \frac{16 - 4y^2 - 4 + y^2}{4} dy$

$= \int_{-2}^{2} \frac{12 - 3y^2}{4} dy = 3[y]_{-2}^{2} - \frac{3}{4 \times 3}\left[y^3\right]^2 = 3(2+2) - \frac{1}{4}(8+8) = 12 - 4 = 8$. Square units.

Q.No.13.: Find, by double integration, the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol.: Area required $= 2\int\int_R rdrd\theta = 2\int_0^{\pi/2} \int_{2\sin\theta}^{4\sin\theta} rdrd\theta = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta$

$= 2\int_0^{\pi/2} \frac{16\sin^2\theta - 4\sin^2\theta}{2} d\theta = 2\int_0^{\pi/2} 6\sin^2\theta d\theta$.

$$=12\int_0^{\pi/2}\sin^2\theta d\theta = 12\times\frac{1}{2}\times\frac{\pi}{2}=3\pi \text{ Square units.}$$

Q.No.14.: Find, by double integration, the area outside the circles $r=a$ and inside the cardioids $r=a(1+\cos\theta)$.

Sol.: Required area $=2\int\int_R rdrd\theta$

$$A=2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} rdrd\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \frac{2}{2}\int_0^{\pi/2}\left[a^2(1+\cos\theta)^2-a^2\right]d\theta$$

$$=\frac{2a^2}{2}\int_0^{\pi/2}\left(1+\cos^2\theta+2\cos\theta-1\right)d\theta = \frac{2a^2}{2}\int_0^{\pi/2}\cos^2\theta + 2a^2\int_0^{\pi/2} 2\cos\theta$$

$$=\frac{2a^2}{2}\times\frac{1}{2}\times\frac{\pi}{2}+\frac{2a^2}{2}\times 2[\sin\theta]_0^{\pi/2} = \frac{\pi a^2}{4}+2a^2=\frac{a^2}{4}(\pi+8). \text{ Square units.}$$

Q.No.15.: Find, by double integration, the area of the curvilinear quadrilateral bounded by four parabolas $y^2=ax,\ y^2=bx,\ x^2=cy,\ x^2=dy$.

Sol.: Area required $=\iint_R dxdy$

Given parabolas are

$y^2=ax,\ y^2=bx,\ x^2=cy,\ x^2=dy$ (i, ii, iii, iv)

Now substituting $y^2 = u^3x$ and $x^2 = v^3 y$

Now from (i) we know

$ax = u^3x,\quad u^3 = a$ and $x = 0\quad u = a^{1/3}$ (A)

Also from (ii)

$bx = u^3x,\quad u = b^{1/3}$ (B)

and from (iii)

$cy = u^3y,\quad v = c^{1/3}$ (C)

From (iv)

$dy = v^3y,\quad v = d^{1/3}$ (D)

Considering $b > a$ and $d > c$

Now from A, B, C and D

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v^2 & 2uv\\ 2uv & u^2\end{vmatrix}=u^2v^2-4u^2v^2-3u^2v^2.$$

$$\Rightarrow |J|=3u^2v^2.$$

$$\therefore A=\int_{c^{1/3}}^{d^{1/3}}\left(\int_{a^{1/3}}^{b^{1/3}}3u^2v^2du\right)dv=3\int_{c^{1/3}}^{d^{1/3}}\left(\int_{a^{1/3}}^{b^{1/3}}u^2v^2du\right)dv=3\int_{c^{1/3}}^{d^{1/3}}\left(\frac{u^3}{3}\right)_{a^{1/3}}^{b^{1/3}}v^2dv$$

$$=\frac{3}{3}\int_{c^{1/3}}^{d^{1/3}}(b-a)v^2dv=(b-a)\left(\frac{v^3}{3}\right)_{e^{1/3}}^{d^{1/3}}=\frac{(b-a)(d-c)}{3}.\text{ Square units. Ans.}$$

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Home Assignments