2nd Topic

Triple Integrals

[where limits are not given, but the region of integration is given]

(Last updated on 15-07-2013)

(02 Solved problems and 00 Home assignment)

Here we will discuss those problems in triple integrals, where limits are not given. Now when limits are not given then how we can evaluate the limits without rough sketch of the region of integration. With the basic understanding of the problems given below, we can easily evaluate the limits in future.

Q.No.1.: Evaluate $\iiint (x+y+z)dxdydz$ over the tetrahedron bounded by the planes

$$x=0,\ y=0,\ z=0 \text{ and } x+y+z=1\ .$$

Sol.: The region here is a tetrahedron bounded by the planes

$$x=0,\ y=0,\ z=0,\ x+y+z=1\,.$$

$\because x+y+z\le 1$

$\Rightarrow x\le 1,\ \ (x+y)\le 1,\ \ (x+y+z)\le 1$

$\Rightarrow x\le 1,\ \ y\le (1-x),\ \ \ z\le (1-x-y)$

$\therefore R=\{(x,y,z),\ 0\le x\le 1,\ 0\le y\le (1-x),\ 0\le z\le (1-x-y)\ \}.$

$$\therefore I = \iiint_V (x+y+z)\,dxdydz = \int_0^1 \left[\int_0^{1-x}\left\{\int_0^{1-x-y}(x+y+z)dz\right\}dy\right]dx$$

$$= \int_0^1 \left\{\int_0^{1-x}\left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dy\right\}dx \qquad \left[\because \int (az+b)^n dz = \frac{(az+b)^{n+1}}{(n+1)a}(n \neq -1)\right]$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^{1-x}\left[1-(x+y)^2\right]dy\right)dx = \frac{1}{2}\int_0^1 \left(y - \frac{(x+y)^3}{3}\right)_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\left(1-x-\frac{1}{3}\right)-\left(0-\frac{x^3}{3}\right)\right]dx = \frac{1}{2}\int_0^1 \left(\frac{2}{3}-x+\frac{x^3}{3}\right)dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3}-\frac{1}{2}+\frac{1}{12}\right] = \frac{1}{2}\left(\frac{8-6+1}{12}\right) = \frac{3}{24} = \frac{1}{8}.\ \text{Ans.}$$

Q.No.2.: Compute triple integral $\iiint \frac{dxdydz}{(1+x+y+z)^3}$, if the region of integration is bounded by the co-ordinate planes and the plane $x+y+z=1$.

Sol.: The region here is a bounded by the co-ordinate planes and the plane $x+y+z=1$.

i.e. $x=0,\ y=0,\ z=0,\ x+y+z=1$.

$\because x+y+z \le 1$

$\Rightarrow x \le 1,\ (x+y) \le 1,\ (x+y+z) \le 1$

$\Rightarrow x \le 1,\ y \le (1-x),\ z \le (1-x-y)$

$\therefore R = \{(x, y, z),\ 0 \le x \le 1,\ 0 \le y \le (1-x),\ 0 \le z \le (1-x-y)\}$.

Thus $I = \int_0^1 \left\{\int_0^{1-x}\left(\int_0^{1-x-y}\frac{1}{(1+x+y+z)^3}dz\right)dy\right\}dx = \int_0^1\left\{\int_0^{1-x}\left|-\frac{1}{2(1+x+y+z)^2}\right|_0^{1-x-y} dy\right\}dx$

$$= -\frac{1}{2}\int_0^1\left\{\int_0^{1-x}\left(\frac{1}{(1+x+y+1-x-y)^2}-\frac{1}{(1+x+y)^2}\right)dy\right\}dx$$

$$= -\frac{1}{2}\int_0^1\left\{\int_0^{1-x}\left(\frac{1}{4}-\frac{1}{(1+x+y)^2}\right)dy\right\}dx = -\frac{1}{2}\int_0^1\left\{\left(\frac{1}{4}\left|y\right|_0^{1-x} + \left|\frac{1}{(1+x+y)}\right|_0^{1-x}\right)\right\}dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{(1-x)}{4} + \frac{1}{1+x+1-x} - \frac{1}{1+x}\right)dx = -\frac{1}{2}\int_0^1 \left(\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{1+x}\right)dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{1+x}\right)dx = -\frac{1}{2}\left[\frac{3}{4}(x)_0^1 - \frac{1}{4}\left(\frac{x^2}{2}\right)_0^1 - \left|\log(1+x)_0^1\right|\right]$$

$$= -\frac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] = \frac{1}{2}\log 2 - \frac{5}{16}. \text{ Ans.}$$

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