7th Topic

Matrices

Characteristic Equations, Eigen Values and Eigen Vectors, Orthogonal vectors

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Characteristic matrix:

Let $A = \lfloor a_{ij} \rfloor_{n\times n}$ be any square matrix of order n and λ be a scalar. Then the matrix

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & .... & a_{1n} \\ a_{21} & a_{22}-\lambda & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn}-\lambda \end{bmatrix}$$

is called the characteristic matrix of A, where I is the unit matrix of the order n.

Characteristic polynomial:

The determinant ofcharacteristic matrixis called the characteristic polynomial.

or

The determinant

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & .... & a_{1n} \\ a_{21} & a_{22}-\lambda & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn}-\lambda \end{vmatrix},$$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A.

Characteristic equation:

The equation $|A-\lambda I|=0$, is called the characteristic equation of A.

Characteristic roots:

The roots of characteristic equation, i.e. the roots of $|A-\lambda I|=0$, are called the characteristic roots or latent roots or characteristic values or eigen values or proper values of the matrix A.

Spectrum:

The set of all eigen values of A is called the spectrum of A.

Remarks:

If λ is a characteristic root of the matrix A, then $|A-\lambda I|=0$

$\Rightarrow$ The matrix $A-\lambda I$ is singular.

Therefore, $\exists$ a non-zero vector X (i.e. X$\neq$O),s.t.

$$(A-\lambda I)X=O \Rightarrow AX=\lambda X.$$

Characteristic vectors:

If λ is a characteristic root of an $n\times n$ matrix A, then a non-zero vector X (i.e. X$\neq$ O),s.t. $AX=\lambda X$, is called a characteristic vector or eigen vector or latent vector of A corresponding to the characteristic root λ.

Relation between Characteristic roots and Characteristic vectors:

Theorem 1: Prove that, if λ is aneigenvalue of a matrix A if and only if there exists a non-zero vector X such that $AX=\lambda X$.

Proof: Suppose λ is aneigen value of the matrix A.

Then $|A-\lambda I|=0 \Rightarrow$ The matrix $A-\lambda I$ is singular.

Therefore, the matrix equation $(A-\lambda I)X=O$ possesses a non-zero solution,

i.e., $\exists$ a non-zero vector X s.t. $(A-\lambda I)X=O \Rightarrow AX=\lambda X$.

Converse Part:

Conversely, suppose there exists anon-zero vector X such that $AX=\lambda X$,

i.e., $(A-\lambda I)X=O$.

Since, the matrix equation $(A-\lambda I)X = O$ possesses a non-zero solution,

$\Rightarrow$ The coefficient matrix $A-\lambda I$ must be singular, i.e., $|A-\lambda I| = 0$.

Hence, λ is the eigenvalue of the matrix A.

This completes the proof.

Theorem 2.: Prove that, if X is aneigen vector of a matrix A, then X cannot correspond to more than one eigen values of A.

Proof: Let X be aneigen vector of a matrix A corresponding to two eigenvalues λ_1 and λ_2.

Then

$AX = \lambda_1 X$ and $AX = \lambda_2 X$.

Therefore $\lambda_1 X = \lambda_2 X$.

$\Rightarrow (\lambda_1 - \lambda_2)X = O \Rightarrow \lambda_1 - \lambda_2 = 0\ [\because X \neq O]$

$\Rightarrow \lambda_1 = \lambda_2$.

This completes the proof.

Properties of eigen values:

Property No.(1): Show that the sum of eigen values of a matrix is the sum of the elements of the principal diagonal and the product of the eigen values of a matrix A is equal to its determinant.

Proof: Consider the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ of order 3.

$$\therefore |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \lambda(........) + (.........). \quad \text{(i)}$$

Also, if λ_1, λ_2 and λ_3 be the eigen values of A, then

$$|A-\lambda I| = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$= -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \lambda(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) + (\lambda_1\lambda_2\lambda_3). \quad \text{(ii)}$$

(i). Equating R. H. S. of (i) and (ii) and comparing the coefficients of λ^2, we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

(ii). Putting $\lambda = 0$ in (ii), we get $|A| = \lambda_1\lambda_2\lambda_3$. Hence, this proves the results.

Property No. (2): If λ is an eigen value of a matrix A, then show that $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Proof: Let λ be an eigen value of A and X be corresponding eigen vector.

Then $AX = \lambda X$.

Pre-multiplying by A^{-1}, we get

$$X = A^{-1}(\lambda X) = \lambda(A^{-1}X) \Rightarrow \frac{1}{\lambda}X = A^{-1}X$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X \qquad [\because A^{-1} \text{exist} \Rightarrow A \text{ is non-singular } \Rightarrow \lambda \neq 0]$$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1} and X is the corresponding eigen vector.

Property No.(3): If λ is an eigen values of an orthogonal matrix, then show that $\frac{1}{\lambda}$ is also its eigen value.

Proof: Since we know that if λ is an eigen value of a matrix A, then

$\frac{1}{\lambda}$ is an eigen value of A^{-1}.

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A' $\qquad [\because$ A is orthogonal matrix, i.e., $AA' = I \Rightarrow A^{-1} = A'$]

But the matrices A and A' have same eigen values

$[\because$ the det. $|A - \lambda I|$ and $|A' - \lambda I|$ are the same]

Hence, $\frac{1}{\lambda}$ is also an eigen value of A.

Property No. (4): Show that if $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then A^2 has the latent roots $\lambda_1^2, \lambda_2^2, \ldots\ldots\ldots, \lambda_n^2$.

Sol.: Let λ be a latent root of the matrix A

Then $\exists$ a non-zero vector X s.t. $AX = \lambda X$. (i)

Pre-multiplying both sides by A, we get

$$\Rightarrow A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2 X$$

Since X is a non-zero vector, therefore λ^2 is a latent root of the matrix A^2.

$\therefore$ If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then $\lambda_1^2, \lambda_2^2, \ldots\ldots\ldots, \lambda_n^2$ are the latent roots of the A^2.

Property No. (5): Show that if $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A, then A^3 has the latent roots $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots, \lambda_n^3$.

Proof: Let λ be a latent root of the matrix A then $\exists$ a non-zero vector X s.t.

$AX = \lambda X$. (i)

Pre-multiplying both sides by A, we get

$$A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2X.$$

Again pre-multiplying both sides by A, we get

$$A\left(A^2X\right) = A\left(\lambda^2X\right) \Rightarrow A^3X = \lambda^2(AX) = \lambda^2(\lambda X) = \lambda^3X.$$

Since X is a non-zero vector, therefore λ^3 is a latent root of the matrix A^3.

$\therefore$ If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the latent roots of a matrix A,

then $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots, \lambda_n^3$ are the latent roots of the A^3.

This completes the proof.

Property No. (6): If $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n$ are the eigen values of a matrix A, then show that A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots\ldots, \lambda_n^m$.

[m being positive integer]

Proof: Let λ_i be aneigen value of A and X_i be the corresponding eigen vector.

Then $AX_i = \lambda_i X_i$.

Pre-multiplying both sides by A, we get

$$A^2X_i = A(\lambda_i X_i) = \lambda i(AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i.$$

Similarly, $A^3X_i = \lambda_i^3 X_i$.

In general, $A^m X_i = \lambda_i^m X_i$.

Thus, λ_i^m is an eigen value of A^m.

Hence $\lambda_1^m,\ \lambda_2^m, \ldots\ldots\ldots, \lambda_n^m$ are eigen values of A^m.

Property No. (7): If λ be aneigen value of a non-singular matrix A.

Show that $\dfrac{|A|}{\lambda}$ is an eigen value of matrix adj. A.

Proof: Since λ be an eigen value of a non-singular matrix A $\Rightarrow \lambda \neq 0$.

Also λ is an eigen value of A then $\exists$ a non-zero vector X. s. t. $AX = \lambda X$. (i)

Pre-multiplying both sides by A, we get

$$(\text{Adj}\, A)(AX) = (\text{Adj}\, A)(\lambda X) \Rightarrow [(\text{Adj}\, A)A]X = \lambda[(\text{Adj}\, A)X]$$

$$\Rightarrow (|A|I)X = \lambda(\text{Adj}\ A)X \left[\because A^{-1} = \frac{\text{Adj}\, A}{|A|} \Rightarrow \text{Adj}\, A.A = |A|\, I\right]$$

$$\Rightarrow |A|X = \lambda(\text{Adj}\, A)X \Rightarrow \frac{|A|}{\lambda}X = (\text{Adj}\, A)X. \qquad [\because \lambda \neq 0]$$

$$\Rightarrow (\text{Adj}\, A)X = \frac{|A|}{\lambda}X.$$

Since X is a non-zero vector, therefore $\dfrac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

Property No. (8): Show that the eigen values of a triangular matrix A are equal to the elements of the principal diagonal of A.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ 0 & a_{22} & \ldots. & a_{2n} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ 0 & 0 & \ldots. & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

Then $|A - \lambda I| = \begin{vmatrix} (a_{11} - \lambda) & a_{12} & \ldots. & a_{1n} \\ 0 & (a_{22} - \lambda) & \ldots. & a_{2n} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ 0 & 0 & \ldots. & (a_{nn} - \lambda) \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)\ldots\ldots.(a_{nn} - \lambda).$

$\therefore$ The roots of the equation $|A - \lambda I| = 0$ are $\lambda = a_{11}, a_{22}, \ldots\ldots\ldots., a_{nn}$.

Hence, the eigen valuesof A are $a_{11}, a_{22}, \ldots\ldots\ldots., a_{nn}$.

And as we define A,these are the diagonal elements of A.

This completes the proof.

Property No. (9).: Show that the eigen values of a unitary matrix have the absolute value 1.

or

Show that the eigen values of a unitary matrix are of unit modulus.

Proof: Suppose A is a unitary matrix $\Rightarrow A^{\theta}A = I$.

Let λ be an eigen value of A and X be corresponding eigen vector then $AX = \lambda X$. (i)

Taking conjugate transpose of both sides of (i), we get

$$(AX)^{\theta} = (\lambda X)^{\theta} \Rightarrow X^{\theta}A^{\theta} = \bar{\lambda}X^{\theta}. \qquad \text{(ii)}$$

From (i) and (ii), we have

$$(X^{\theta}A^{\theta})(AX) = (\bar{\lambda}X^{\theta})(\lambda X)$$

$$\Rightarrow (X^{\theta}A^{\theta})(AX) = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}(A^{\theta}A)X = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}IX = \bar{\lambda}\lambda X^{\theta}X$$

$$\Rightarrow X^{\theta}X = \bar{\lambda}\lambda X^{\theta}X \Rightarrow X^{\theta}X(\lambda\bar{\lambda} - 1) = O. \qquad \text{(iii)}$$

Since $X^{\theta}X \neq O$, (since $X \neq O$),

$\therefore$ (iii) gives $\lambda\bar{\lambda} - 1 = 0 \Rightarrow \lambda\bar{\lambda} = 1 \Rightarrow |\lambda|^2 = 1$.

Thus $|\lambda| = 1 \Rightarrow$ The eigen values of a unitary matrix have the absolute value 1.

This completes the proof.

Property No. (10): Show that the characteristic roots of Hermitian matrix are real.

Proof: Let λ be an eigen value of a Hermitian matrix A and X be the corresponding eigen vector.

Then $AX = \lambda X$. (i)

Pre-multiplying both sides of (i) by X^{θ}, we get

$$X^{\theta}(AX) = X^{\theta}(\lambda X) \Rightarrow X^{\theta}AX = \lambda X^{\theta}X. \qquad \text{(ii)}$$

Taking transpose conjugate of both sides of (ii), we get

$$(X^{\theta}AX)^{\theta} = (\lambda X^{\theta}X)^{\theta} \Rightarrow X^{\theta}A^{\theta}(X^{\theta})^{\theta} = \bar{\lambda}X^{\theta}(X^{\theta})^{\theta}$$

$$\Rightarrow X^{\theta}AX = \bar{\lambda}X^{\theta}X. \qquad \text{(iii)}$$

$$\left[\because (X^{\theta})^{\theta} = X \text{ and } A^{\theta} = A, A \text{ being Hermitian}\right]$$

From (ii) and (iii), we have

$\lambda X^{\theta}X = \bar{\lambda} X^{\theta}X \Rightarrow (\lambda - \bar{\lambda})X^{\theta}X = O$.

But X is not a zero vector. $\therefore X^{\theta}X \neq 0$.

Hence $\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

This completes the proof.

Property No. (11): Show that the characteristic roots of a Skew-Hermitian matrix are either pure imaginary or zero.

Proof: Suppose A is a Skew-Hermitian matrix. Then iA is Hermitian.

Let λ be a characteristic root of A and X be corresponding eigen vector. Then

$AX = \lambda X$.

Pre-multiplying both sides by i, we get $(iA)X = (i\lambda)X$

$\Rightarrow (i\lambda)$ is a characteristic root of iA, which is Hermitian.

Hence $(i\lambda)$ is real.

Therefore, either λ must be zero or pure imaginary.

Now let us solve some more important results:

Result No.1.: Show that the matrices A and A' have the same eigen values.

Sol.: We have $(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$.

$\therefore |(A - \lambda I)'| = |A' - \lambda I|$

$\Rightarrow |(A - \lambda I)| = |A' - \lambda I| \; [\because |B'| = |B|]$

$\therefore |(A - \lambda I)| = 0$ if and only if $|A' - \lambda I| = 0$

i.e., λ is an eigen value of A if and only if λ is an eigen value of A'.

This completes the proof.

Result No.2.: Show that the characteristic roots of A^{θ} are the conjugates of the characteristic roots of A.

Sol.: We have $|A^{\theta} - \bar{\lambda} I| = |(A - \lambda I)^{\theta}| = \overline{|A - \lambda I|}$ [Note that $|B^{\theta}| = |\overline{(B')}| = \overline{|B'|} = \overline{|B|}$]

$\therefore |A^{\theta} - \lambda \bar{I}| = 0 \text{ iff } \overline{|A - \lambda I|} = 0$

$\Rightarrow |A^{\theta} - \bar{\lambda} I| = 0 \text{ iff } |A - \lambda I| = 0$ [$\because$ if z is a complex number, then $z = 0$ iff $\bar{z} = 0$]

$\Rightarrow \bar{\lambda}$ is an eigen values of A^θ if and only if λ is an eigen value of A.

Result No.3.: Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

Sol.: We have given 0 is an eigen value of $A \Rightarrow \lambda = 0$ satisfies the equation $|A - \lambda I| = 0$

$\Rightarrow |A| = 0 \Rightarrow$ A is singular.

Conversely, if A is singular $\Rightarrow |A| = 0 \Rightarrow \lambda = 0$ satisfy the equation $|A - \lambda I| = 0$

$\Rightarrow 0$ is an eigen value of A.

This completes the proof.

The process of finding the eigen values and eigen vectors of a matrix:

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n.

First we should write the characteristic equation of the matrix A, i.e., the equation $|A - \lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will give us the eigen values of the matrix A. If λ_1 is an eigen value of A, then the corresponding eigenvectors of A will be given by the non-zero vectors

$$X = [x_1, x_2, \ldots\ldots\ldots\ldots, x_n]'$$

satisfy the equation .

$$AX = \lambda_1 X \Rightarrow (A - \lambda_1 I)X = O .$$

Orthogonal Vectors:

Let X and Y be two real-n-vectors, then X is said to be orthogonal to Y if

$$X'Y = O .$$

Let X and Y be two complex-n-vectors, then X is said to be orthogonal to Y if

$$X^\theta Y = O .$$

Now let us solve some problems by using the properties of eigen values and eigen vectors:

Q.No.1.: Find the **sum and product** of the eigen values of $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

Sol.: Since, we know that the sum of the eigen values of a matrix is the sum of the elements of the principal diagonal and the product of the eigen values of a matrix is equal to its determinant.

Here $A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

If $\lambda_1, \lambda_2, \lambda_3$ be its eigen values of A, then $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 1 + 2 = 5$. Ans.

and $\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

$= 2(2-0) - 3(-4-1) + (-2)(0-1) = 4 + 15 + 2 = 21$. Ans.

Q.No.2.: Find the **product** of the eigen values of $\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Sol.: Since, we know that the product of the eigen values of a matrix is equal to its determinant.

Here $A = \begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

If $\lambda_1, \lambda_2, \lambda_3$ be its eigen values of A, then

$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{vmatrix} = 7\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} -6 & 2 \\ 6 & -1 \end{vmatrix} + 2\begin{vmatrix} -6 & -1 \\ 6 & 2 \end{vmatrix}$

$= 7(1-4) - 2(6-12) + 2(-12+6) = -21 + 12 - 12 = -21$. Ans.

Now let us solve some problems of evaluation of eigen values and eigen vectors:

Q.No.1.: Find the **eigen values and eigen vectors** of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Sol.: The characteristic equation is of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$.

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow (\lambda - 6)(\lambda - 1) = 0 \Rightarrow \lambda = 6,\ 1$.

Thus, the roots of this equation are $\lambda_1 = 6$, $\lambda_2 = 1$.

Therefore, the eigen values are 6 and 1.

The eigen vectors $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigen value 6 are given by the non-zero solution of the equation $(A - 6I)X_1 = O$

$$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 + R_1$, we get $\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The coefficient matrix of these equations is of rank 1. Therefore, these equations have $2-1$, i.e., 1 linearly independent solution. These equations reduced to the single equation $-x_1 + 4x_2 = 0$.

Obviously, $x_1 = 4$, and $x_2 = 1$ is a solution of this equation.

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen values 6. The set of all eigen vectors of A corresponding to the eigen values 6 is given by c_1X_1 where c_1 is any non-zero scalar.

The eigen vectors X_2 of A corresponding to the eigen value 1 is given by the non-zero solutions of the equation

$$(A - 1I)X_2 = O \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 + 4x_2 = 0, \quad x_1 + x_2 = 0.$$

From these $x_1 = -x_2$. Let us take $x_1 = 1$, $x_2 = -1$.

Then $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 1.

Every non-zero multiple of the vector X_2 is an eigen vector of A corresponding to the eigen value 1.

Q.No.2.: Find the **eigen values and eigen vectors** of the matrices:

(a) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol.: (a). Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. The characteristic equation of A is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$

$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0 \Rightarrow \lambda = 5, -2$.

If x, y, z be the components of eigen vector corresponding to eigenvalue λ.

Then $[A - \lambda I][X] = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$.

Put $\lambda = 5$, we get $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$.

Operating $R_2 \to 4R_2 - 3R_1$, we get

$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -4x + 4y = 0 \Rightarrow x - y = 0 \Rightarrow x = y = k$.

When $k = 1$, then $x = y = 1$.

Now putting $\lambda = -2$, we get

$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x + 4y = 0 \Rightarrow 3x + 4y = 0$.

Solving $x = 4, \quad y = -3$.

So eigen vectors are $(1, 1), \ (4, -3)$. Ans.

(b). Let $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$**.** The characteristic equation A is $|A - \lambda I| = 0$

$\Rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10 = 0 \Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \lambda = -1, \ 6$.

If x, y, be the components of eigen vector corresponding to eigen value λ.

Then $[A-\lambda I][X]=0 \Rightarrow \begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$. (i)

Putting $\lambda=-1$ in (i), we get $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$.

Operating $R_2 \to 2R_2-5R_1$, we get

$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x-2y=0 \Rightarrow x=y=k$.

When $k=1$, then $x=y=1$.

Now putting $\lambda=6$ in (i), we get $\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to R_2-R_1$, we get

$\begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -5x-2y=0 \Rightarrow 5x+2y=0$.

Solving, we get $x=2,\ y=-5$

Hence, the eigen vectors of A are $(1,\ 1)$ and $(2,\ -5)$. Ans.

Q.No.3.: (i) Find the eigen values and eigen vectors of $A=\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

(ii) Also find the eigen values and eigen vectors of $A^T=\begin{pmatrix} 8 & 2 \\ -4 & 2 \end{pmatrix}$.

(iii) Find the eigen values and eigen vectors of $A^{-1}=\frac{1}{24}\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}$.

(iv) Find the eigen values and eigen vectors of $B=kA$ where $k=-\frac{1}{2}$.

(v) Find the eigen values and eigen vectors of $A^2=\begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix}$.

(vi) Find the eigen values and eigen vectors of

$B=A\pm kI=\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}\pm k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 8\pm k & -4 \\ 2 & 2\pm k \end{pmatrix}$.

(vii) Find the eigen values and eigen vectors of $D=2A^2-\frac{1}{2}A+3I$.

(viii) Find the sum and product of eigen values of A.

Sol.: 1st Part: Find the eigen values and eigen vectors of $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

The eigen values are the roots of the characteristic equation

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (8-\lambda)(2-\lambda)+8=0 \Rightarrow \lambda^2 - 10\lambda + 24 = 0 \Rightarrow (\lambda-4)(\lambda-6)=0.$$

The two distinct eigen values are $\lambda = 4, 6$.

Eigen vector corresponding to eigen value $\lambda = 4$

$$(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 18-4 & -4 \\ 2 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$4x_1 - 4x_2 = 0$

$2x_1 - 2x_2 = 0$

$\therefore x_1 = x_2 \; \overline{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

X_2corresponding $\lambda = 6$: $\begin{pmatrix} 8-6 & -4 \\ 2 & 2-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$2x_1 - 4x_2 = 0 \therefore x_1 = 2x_2$. $\overline{X}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2nd Part: Find the eigen values and eigen vectors of $A^T = \begin{pmatrix} 8 & 2 \\ -4 & 2 \end{pmatrix}$.

Characteristic equation $\begin{vmatrix} 8-\lambda & 2 \\ -4 & 2-\lambda \end{vmatrix} = 0$

Characteristic equation is $\lambda^2 - 10\lambda + 24 = 0$ same as the characteristic equation of A. Thus, the eigen values of A and A^T are same. However, the eigen vectors are not the same.

For $\lambda = 4$: $(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 8-4 & -2 \\ -4 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$4x_1 - +2x_2 = 0$. $\quad \therefore x_2 = -2x_1$.

$X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

X_2corresponding $\lambda = 6: \begin{pmatrix} 8-6 & 2 \\ -4 & 2-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$2x_1 + 2x_2 = 0$

$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

3rd Part: Find the eigen values and eigen vectors of $A^{-1} = \frac{1}{24}\begin{bmatrix} 2 & 4 \\ -2 & 8 \end{bmatrix}.$

Characteristic equation is $\left|A^{-1} - \lambda I\right| = 0$

$$\begin{vmatrix} \frac{1}{12} - \lambda & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \lambda \end{vmatrix} = \left(\frac{1}{12} - \lambda\right)\left(\frac{1}{3} - \lambda\right) + \frac{1}{12}.\frac{1}{6} = 0$$

$24\lambda^2 - 10\lambda + 1 = 0$, $\left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{1}{6}\right) = 0$.

The eigen values of A^{-1} are $\frac{1}{4}$, $\frac{1}{6}$ which are the reciprocal of 4, 6 of A.

Also the given vectors of A^{-1} and A are same

For $\lambda = \frac{1}{4}: \begin{pmatrix} \frac{1}{12} - \frac{1}{4} & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \frac{1}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-2x_1 + x_2 = 0 \therefore x_1 = x_2$. $\quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

For $\lambda = \frac{1}{6}: \begin{pmatrix} \frac{1}{12} - \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{12} & \frac{1}{3} - \frac{1}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-x_1 + 2x_2 = 0 \therefore x_1 = 2x_2$. $\quad X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

4th Part: Find the eigen values and eigen vectors of $B = kA$ where $k = -\frac{1}{2}$.

$$B = -\frac{1}{2}A = \begin{pmatrix} -4 & +2 \\ -1 & -1 \end{pmatrix}.$$

Characteristic equation of B is $|B - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = 0$

$(4+\lambda)(1+\lambda) + 2 = 0 \Rightarrow \lambda^2 + 5\lambda + 6 = 0$

So the eigen values of B are $-2,\ -3$, which are $-\frac{1}{2}$ times of eigen values 4, 6 of A. Also the eigen vectors of B and A are same.

For $\lambda = -2$: $\begin{bmatrix} -4+2 & 2 \\ -1 & -1+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $\therefore x_1 = x_2$. $X_1 = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -3$: $\begin{bmatrix} -4+3 & 2 \\ -1 & -1+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $-x_1 + 2x_2 = 0$. $X_2 = C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

5th Part: Find the eigen values and eigen vectors of $A^2 = \begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix}$.

Characteristic equation of A^2 is $\begin{vmatrix} 56-\lambda & -40 \\ 20 & -4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 52\lambda + 576 = (\lambda - 16)(\lambda - 36) = 0$

So eigen values of A^2 are 16, 36 which are square of the eigen values 4, 6 of A. Also the eigen vectors of A and A^2 are same.

For $\lambda = 16$: $\begin{bmatrix} 56-16 & -40 \\ 20 & -4-16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $\therefore x_1 = x_2$. $X_1 = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 36$: $\begin{bmatrix} 56-36 & -40 \\ 20 & -4-36 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. $x_1 - 2x_2 = 0 \therefore x_1 = 2x_2$. $X_2 = C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6th Part: Find the eigen values and eigen vectors of

$$B = A \pm kI = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix} \pm k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 \pm k & -4 \\ 2 & 2 \pm k \end{pmatrix}.$$

Characteristic equation of B is $|B - \lambda I| = 0$

$$\begin{vmatrix} 8 \pm k - \lambda & -4 \\ 2 & 2 \pm k - \lambda \end{vmatrix} = 0 \Rightarrow (8 \pm k - \lambda)(2 \pm k - \lambda) + 8 = 0$$

$\Rightarrow \lambda^2 - (10 \pm 2k)\lambda + (k^2 \pm 10k + 24) = 0$

Roots are $\frac{10+2}{2} \pm k$. i.e., $4 \pm k$ and $6 \pm k$ which are 4, 6 of A with $\pm k$.

Eigen vectors of B and A are same

For $\lambda = 4 \pm k$: $\begin{bmatrix} 8 \pm k - (4 \pm k) & -4 \\ 2 & 2 \pm k - (4 \pm k) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$ etc.

7th Part: Find the eigen values and eigen vectors of $D = 2A^2 - \frac{1}{2}A + 3I$.

$$D = 2\begin{pmatrix} 56 & -40 \\ 20 & -4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 111 & -78 \\ 39 & -6 \end{pmatrix}$$

Characteristic equation of D is $\begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^2 - 105\lambda + 2376 = (\lambda - 33)(\lambda - 72) = 0$.

Thus, the eigen values of D are 33, 72.

Note that $33 = 2.16 - \frac{1}{2}.4 + 3$ and $72 = 2.36 - \frac{1}{2}.6 + 3$ i.e., eigen value of D is $2\lambda^2 - \frac{1}{2}\lambda + 3$ where λ is the eigen value of A.

The eigen vectors of A and D are same.

For $\lambda = 33$: $\begin{bmatrix} 111-33 & -78 \\ 39 & -6-33 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow 78x_1 - 78x_2 = 0 \Rightarrow x_1 = x_2$ etc.

8th Part: Find the sum and product of eigen values of A.

Sum of eigen values of A = $4 + 6 = 10 =$ trace of A $= a_{11} + a_{22} = 8 + 10$.

Product of eigen values of $A = 4.6 = 24 = |A| = 16 + 8 = 24$.

Q.No.4.: Find the **characteristic roots and characteristic vectors** of the matrices:

(a) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, **(b)** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Sol.: (a). The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\{(7-\lambda)(3-\lambda)-16\}+6\{-6(3-\lambda)+8\}+2\{24-2(7-\lambda)\}=0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0 \Rightarrow \lambda(\lambda-3)(\lambda-15)=0$$

Hence, the characteristic roots of A are 0, 3 and 15.

The eigen vectors $X = [x_1, x_2, x_3]'$ of A corresponding to the eigen value 0 are given by the non-zero solutions of the equation $(A - 0I)X = O$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & -4 & 3 \\ -6 & -5 & 5 \\ 2 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (\text{ by } R_1 \to R_3)$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\text{by } R_2 \to +3R_1,\ R_3 \to R_3 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\text{by } R_3 \to R_3 + 2R_2)$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3-2=1$ linearly independent solution. Thus, there is only one linearly independent eigen vector corresponding to the eigen value 0. These equations can be written as

$2x_1 - 4x_2 + 3x_3 = 0$, $-5x_2 + 5x_3 = 0$.

From the last equation, we get $x_2 = x_3$.

Let us take $x_2 = 1$, $x_3 = 1$. Then, the first equation gives $x_1 = \frac{1}{2}$.

Therefore $X_1 = \begin{bmatrix} \frac{1}{2} & 1 & 1 \end{bmatrix}'$ is an eigen vector of A corresponding to the eigen vector 0.

If c_1 is any non-zero scalar, then e_1X_1 is also an eigen vector of A corresponding to the eigen value 0.

The eigen vector of A corresponding to the eigen value 3 are given by the non-zero solution of the equation

$$(A-3\ I)X = O \Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_1 \to R_1 + R_3)$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_2 - 6R_1,\ \ R_3 \to R_3 + 2R_1)$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_3 + \frac{1}{2}R_2)$$

The coefficient matrix of these equations is of rank 2.

Therefore, these equations have $3-2=1$ linearly independent solution.

These equations can be written as

$-x_1 - 2x_2 - 2x_3 = 0,\ 16x_2 + 8x_3 = 0$.

From the second equation we get $x_2 = -\frac{1}{2}x_3$.

Let us take $x_3 = 4$, $x_2 - 2$, then the first equation gives $x_1 = -4$.

Therefore, $X_2 = [-4 \ \ -2 \ \ 4]'$ is an eigen vector of A corresponding to eigen value 3. Every non-zero multiple of X_2 is an eigen vector of A corresponding to the eigen value 3.

The eigen vectors of A corresponding to the eigen value 15 are given by the non-zero solutions of the equation $A - 15\ I = O$.

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & -2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -7 & -6 & -2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{by } R_2 \to R_2 - 6R_1, \quad R_3 \to R3 + 2R_1)$$

The coefficient matrix of these equations is of rank 2.

Therefore, these equations have $3-2=1$ linearly independent solution.

These equations can be written as

$-x_1 + 2x_2 + 6x_3 = 0$, $20x_2 - 40x_3 = 0$.

The last equation gives $x_2 = -2x_3$.

Let us take $x_3 = 1$, $x_2 = -2$, then the first equation gives $x_1 = 2$.

Therefore $X_3 = [2 \quad -2 \quad 1]'$ is an eigen vector of A corresponding to the eigen value 15, if k is any non-zero scalar, then kX_3 is also an eigen vector of A corresponding to the eigen value 15.

(b). $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Let λ be the eigen value of A, then characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda)(2-\lambda) - 1(2-\lambda) = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \Rightarrow \lambda = 1,\ 2,\ 3$$

When $\lambda = 1$, we get $(A - \lambda I)X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x + z = 0$, $y = 0$, $x + z = 0$

By solving these equations, we get $x = 1$, $y = 0$, $z = -1$.

When $\lambda = 2$, we get $(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x = 0, \; y = k, \; z = 0$.

By solving these equations, we get $x = 0, \; y = 1, \; z = 0$.

When $\lambda = 3$, we get $(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x + z = 0, \; y = 0.$

By solving these equations, we get $x = 1, \; y = 0, \; z = 1$.

Hence, eigen vectors are $(1, 0, -1), \; (0, 1, 0), \; (1, 0, 1)$.

Q.No.5.: Find the **characteristic roots and characteristic vectors** of the matrices:

$$\textbf{(a)} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \; \textbf{(b)} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

(a). Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = (2-\lambda)\left(-\lambda + \lambda^2 - 12\right) = 0 \Rightarrow \lambda^3 + \lambda^2 - 14\lambda - 24 = 0$$

$\Rightarrow \lambda = 5, -3, -3$.

If x, y, z be the components of eigen vector corresponding to the eigen value λ. Then

$$(A - \lambda I)X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = 5$, we get $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -1 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -7x+2y-32=0 \Rightarrow 2x-4y-2=0 \Rightarrow -x-2y=52=0$

$\therefore\ x=1,\ y=2,\ \ z=-1$

When $\lambda=-3$, we get $\begin{bmatrix}1 & 2 & -3\\ 9 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow x+2y-32=0$,

$9x+4y-62=0$,

$-x-2y+32=0$.

Solving these equations, we get $\ x=-2,\ \ y=-1,\ \ z=0$

Hence, the vectors are $(-2,-1,0)$ and $(1,2,-1)$.

(b). $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$.

The characteristic equation of A is $|A-\lambda I|=0 \Rightarrow \begin{bmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

Operating $C_3 \to C_3+C_2$, we get

$$\begin{bmatrix}6-\lambda & -2 & 0\\ -2 & 3-\lambda & 2-\lambda\\ 2 & -1 & 2-\lambda\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \Rightarrow (2-\lambda)\begin{bmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & 1\\ 2 & -1 & 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

Operating $R_2 \to R_2-R_3$, we get

$$(2-\lambda)\begin{bmatrix}6-\lambda & -2 & 0\\ -4 & 4-\lambda & 0\\ 2 & -1 & 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \Rightarrow (2-\lambda)[(6-\lambda)(4-\lambda)-8]=0$$

$$\Rightarrow (2-\lambda)(\lambda^2-10\lambda+16)=0 \Rightarrow (2-\lambda)(\lambda-2)(\lambda-8)=0$$

Therefore, the characteristic roots of A are given by $\lambda=2,\ 2,\ 8$.

The characteristic vectors of A corresponding to the characteristic root 8 are given by the non-zero solutions of the equation $(A-8I)X=O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The coefficient matrix of these equations is of rank 2. Therefore, these equations possess $3-2=1$ linearly independent solution.

These equations can be written as

$-2x_1 - 2x_2 + 2x_3 = 0$, $-3x_2 - 3x_3 = 0$.

From the last equation, we get $x_2 = -x_3$. Let us take $x_3 = 1$, $x_2 = -1$. Then the first equation gives $x_1 = 2$.

Therefore, $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 8.

Every non-zero multiple of X_1 is also an eigen vector of A corresponding to the eigen value 8.

The eigen vectors of A corresponding to the eigen value 2 are given by the non-zero solution of the equation

$$(A - 32\,I)X = O \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_1 \leftrightarrow R_2$, we get $\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$, we get $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The coefficient matrix of these equations is of rank 1. Therefore, these equations possess $3-1=2$ linearly independent solution. We see that these equations reduce to the single equation

$2x_1 - x_2 - x_3 = 0$.

Obviously $X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are two linearly independent solutions of this equation.

Therefore, X_2 and X_3 are two linearly independent eigen vectors of A corresponding to the eigen value 2.

If c_1, c_2 are scalars not both equal to zero, then $c_1X_2 + c_2X_3$ gives all the eigen vectors of A corresponding to the eigen value 2.

Q.No.6.: Find the **eigen values and eigen vectors** of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol.: The characteristic equation is $|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$.

Since $\lambda = -2$ satisfies it, we can write this equation as

$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$.

Thus, the roots of this equation are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$.

Therefore, the eigen values of A are $\lambda = -2,\ 3,\ 6$.

If x, y, z be the components of an eigen vector corresponding to the eigenvalue λ, we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \qquad \text{(i)}$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}.$$

Hence, the eigen vectors are $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i).

Hence, the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Q.No.7.: Find the eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

Sol.: For upper triangular, lower triangular and diagonal matrices, the eigen values are given by the diagonal elements.

Characteristic equation is $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0$.

So eigen values of A are 3, 2, 5 which are the diagonal elements of A.

Eigen vector X_1 for $\lambda = 3$: $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_2 + 4x_3 = 0$, $-x_2 + 6x_3 = 0$, $2x_3 = 0$

$\Rightarrow x_2 = 0$, $x_3 = 0$, $x_1 =$ arbitrary. $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigen vector X_2 for $\lambda = 2$: $x_1 + x_2 + 4x_3 = 0$, $6x_3 = 0$, $3x_3 = 0$

$\Rightarrow x_3 = 0$, $x_1 = -x_2$. $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Eigen vector $X_3$for $\lambda = 5: -2x_1 + x_2 + 4x_3 = 0$, $-3x_2 + 6x_3 = 0$,

$\Rightarrow x_1 = 3x_3$, $x_2 = 2x_3$. $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Q.No.8.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Determine whether the eigen vectors are orthogonal.

Sol.: Characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

So $\lambda =$ 1, 2, 3 are three distinct eigen values of A

For $\lambda = 1$: $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x_3 = 0$, $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$.

Let $x_1 = 1 \Rightarrow x_2 = -1$. Also $x_3 = 0$. Thus $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

For $\lambda = 2$: $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$x_1 + x_3 = 0 \Rightarrow x_3 = -x_1$. And $2x_1 + 2x_2 + x_3 = 0 \Rightarrow x_2 = \frac{1}{2}x_3$.

Let $x_1 = 2 \Rightarrow x_3 = -2$ and $x_2 = -1$. Thus $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

For $\lambda = 3$: $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x_1 = -x_2$, $x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = -\frac{1}{2}x_3$.

Let $x_1 = 1 \Rightarrow x_2 = -1$. Also $x_3 = -2$. Thus $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

Thus, there are three linearly independent eigen vectors X_1, X_2, X_3 corresponding to the three distinct eigen values.

Since $X_1^T X_2 = 3 \neq O$, $X_2^T X_3 = 5 \neq O$, $X_3^T X_1 = 2 \neq O$.

Therefore, no pair of eigen vectors are orthogonal.

Q.No.9.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2 = 0$.

So $\lambda =$ 1, 2, 2 are eigen values with $\lambda = 2$ repeated twice (double root) of multiplicity 2.

The algebraic multiplicity of the eigen values $\lambda = 2$ is 2.

For $\lambda = 1$: $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, $x_2 = -x_3$ $x_1 = -x_3$. $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda = 2$: $\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, $x_3 = 0$, $x_1 = 2x_2$. $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Thus, only one eigen vector X_2 corresponds to the repeated eigenvalue $\lambda = 2$.

The geometric multiplicity of eigen value $\lambda = 2$ is one.

Q.No.10.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$.

$\lambda = 1,\ 1,\ 1$ is an eigen value of algebraic multiplicity 3.

For $\lambda = 1$:

$-x_1 + x_2 = 0, \quad \therefore\ x_1 = x_2$

$-x_2 + x_3 = 0, \quad x_2 = x_3$

$x_1 - 3x_2 + 2x_3 = 0$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, only one eigen value X Corresponds to the thrice repeated eigenvalues $\lambda = 1$, so geometric multiplicity is one.

Q.No.11.: Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Determine the algebraic and geometric multiplicity.

Sol.: Characteristic equation is $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda-1)(\lambda-3) = 0$.

Thus $\lambda = 1,\ 1,\ 3$ is an eigen values of A.

So the algebraic multiplicity of eigenvalue $\lambda = 1$. Is two.

For $\lambda = 3$: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim\ x_3 = 0,\ x_1 = x_2$. $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 1$: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad n = 3,\ \ r = 1$

$n - r = 3 - 1 = 2$ = arbitrary

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

where x_2 and x_3 are arbitrary.

For a choice $x_2 = 0,\ x_3 =$ arbitrary.

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

For a choice of $x_2 \neq 0$, $x_3 = 0$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Thus, for the repeated eigenvalue $\lambda = 1$, there corresponds two linearly independent eigen vectors X_2 and X_3. So the geometric multiplicity of eigen value $\lambda = 1$ is 2.

Q.No.12.: Find the eigen values of orthogonal matrix $B = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Sol.: Characteristic equation of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0 \Rightarrow (\lambda - 3)^2(\lambda + 3) = 0.$$

The eigen values of A are 3, 3, -3, so the eigen values of $B = \frac{1}{3}A$ are 1, 1, -1.

Note that $\lambda = 1$ is an eigen value of B then its reciprocal $\frac{1}{\lambda} = \frac{1}{1} = 1$ is also an eigen values of B.

Q.No13.: Show that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ is Hermitian.

Find its eigen values and eigen vectors.

Sol.: Since here $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$.

Therefore $\overline{A} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$, $\overline{A}^T = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} = A$.

Thus A is Hermitian. (Note that the diagonal elements of A are real).

The characteristic equation for A is $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)^2 - (3+4i)(3-4i) = 4+\lambda^2 - 4\lambda - [9+16] = 0$$

$$\Rightarrow \lambda^4 - 4\lambda - 21 = (\lambda+3)(\lambda-7) = 0\,.$$

Eigen values of A, Hermitian matrix are real $-3,\ 7$.

For $\lambda = -3$: $\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$$x_1 = -\left(\frac{3+4i}{5}\right)x_2\,.$$

The eigen vector corresponding to $\lambda = -3$ is $X_1 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$.

For $\lambda = 7$: $\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$$x_1 = \left(\frac{3+4i}{5}\right)x_2\,.$$

The eigen vector corresponding to $\lambda = 7$ is $X_1 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$.

Q.No.14.: Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is Skew-Hermitian and also unitary. Find the eigen values and eigen vectors.

Sol.: $\overline{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$, $\overline{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -A$.

Thus, A is Skew-Hermitian.

Consider $A\overline{A}^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ! \end{bmatrix} = I$.

Thus $\overline{A}^T = A^{-1}$,

i.e., A is unitary matrix also.

The characteristic equation of A is $|A-\lambda I| = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow (i-\lambda)(\lambda^2+1) = \lambda^3 - i\lambda^2 + \lambda - i = 0 \Rightarrow (\lambda+i)(\lambda-i)^2 = 0$.

The eigen values of A are $\lambda = -i,\ i,\ i$ which are purely imaginary (for Skew-Hermitian) and are of absolute value unity (i.e. $|-i| = |i| = 1$)

For $\lambda = -i$: $\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Solving $x_1 = 0,\ \ x_2 = -x_3$.

Thus the eigen vector corresponding to $\lambda = -i$ is $X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda = i$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Solving $x_1 =$ arbitrary, $x_2 = x_3$.

Choose x_1, so that two linearly independent eigen vectors are obtained (with $x_1 = 0$, $x_2 = 1$ and $x_1 = 1$, $x_2 = 0$)

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Q.No.15.: Find the Hermitian form H for

$A = \begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}$ with $X = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$.

Sol.: Since $H = \overline{X}^T A X = \begin{bmatrix} -i & 1 & i \end{bmatrix}\begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}\begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$

$$= \begin{bmatrix} -i & 1+1-2 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix} = 1, \text{ real.}$$

Q.No.16.: Determine the Skew-Hermitian form S for

$$A = \begin{pmatrix} 2i & 3i \\ 3i & 0 \end{pmatrix} \text{with } X = \begin{bmatrix} 4i \\ -5 \end{bmatrix}.$$

Sol.: Since $S = \overline{X}^T AX = \begin{bmatrix} -4i & -5 \end{bmatrix} \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix} \begin{bmatrix} 4i \\ -5 \end{bmatrix}$

$$= \begin{pmatrix} 8-15i & 12 \end{pmatrix} \begin{pmatrix} 4i \\ -5 \end{pmatrix} = 32i + 60 - 60 = 32i \text{, purely imaginary.}$$

Orthogonal Vectors:

Let X and Y be two real-n-vectors, then X is said to be orthogonal to Y if

$$X'Y = O.$$

Let X and Y be two complex-n-vectors, then X is said to be orthogonal to Y if

$$X^{\theta}Y = O.$$

Q.No.1.: For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two unequal eigen values λ_1 and λ_2 of a symmetrical square matrix A. Then, by definition

$AX_1 = \lambda_1 X_1$ (i)

and $AX_2 = \lambda_2 X_2$ (ii)

Since A is symmetrical square matrix therefore $A' = A$.

Also $\lambda_1 \neq \lambda_2$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2' X_1 = O$.

Now $\lambda_1 X_2' X_1 = X_2'(\lambda_1 X_1) = X_2'(AX_1) = (X_2' A) X_1 = (X_2' A') X_1$

$$= (AX_2)' X_1 = (\lambda_2 X_2)' X_1 = \lambda_2 X_2' X_1$$

$\Rightarrow \lambda_1 X_2' X_1 = \lambda_2 X_2' X_1 \Rightarrow (\lambda_1 - \lambda_2) X_2' X_1 = O$.

But $\lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) \neq 0$.

Thus $X_2' X_1 = O$.

Hence X_1 and X_2 are orthogonal vectors.

Q.No.2.: Show that any eigen vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal.

or

Show that the eigen vectors X_i, X_j corresponding to two distinct eigen values λ_i, λ_j of a Hermitian matrix H are orthogonal, i.e. $\overline{X}_i^T X_j = 0$.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1 and λ_2 of a Hermitian matrix A. Then by definition

$$AX_1 = \lambda_1 X_1 \qquad \text{(i)}$$

and $$AX_2 = \lambda_2 X_2. \qquad \text{(ii)}$$

Since A is Hermitian matrix, then both the eigen values are real $\Rightarrow \lambda_1, \lambda_2$ are real.

Also $A^\theta = A$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2^\theta X_1 = O$.

Now $\lambda_1 X_2^\theta X_1 = X_2^\theta(\lambda_1 X_1) = X_2^\theta(AX_1) = \left(X_2^\theta A\right)X_1 = \left(X_2^\theta A^\theta\right)X_1$

$$= (AX_2)^\theta X_1 = (\lambda_2 X_2)^\theta X_1 = \overline{\lambda}_2 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1 \qquad [\because \lambda_2 \text{ is real}]$$

$\Rightarrow \lambda_1 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1 \Rightarrow (\lambda_1 - \lambda_2) X_2^\theta X_1 = O$.

But $\lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) \neq 0$.

Thus $X_2^\theta X_1 = O$.

Hence, X_1 and X_2 are orthogonal vectors.

Q.No.3.: Show that any eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Proof: Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1 and λ_2 of a unitary matrix A. Then by definition

$$AX_1 = \lambda_1 X_1 \qquad \text{(i)}$$

and $AX_2 = \lambda_2 X_2$. (ii)

Since A is unitary matrix, then the eigen values have the absolute value 1.

i.e. $\therefore |\lambda_1| = 1 \Rightarrow |\lambda_1|^2 = 1 \Rightarrow \lambda_1\bar{\lambda}_1 = 1 \Rightarrow \bar{\lambda}_1 = \frac{1}{\lambda_1}$

$|\lambda_2| = 1 \Rightarrow |\lambda_2|^2 = 1 \Rightarrow \lambda_2\bar{\lambda}_2 = 1 \Rightarrow \bar{\lambda}_2 = \frac{1}{\lambda_2}$

Also $AA^{\theta} = I$.

To show: X_1 and X_2 are orthogonal vectors, i.e., $X_2^{\theta}X_1 = O$.

Taking conjugate transpose of (ii), we get

$$(AX_2)^{\theta} = (\lambda_2 X_2)^{\theta} \Rightarrow X_2{}^{\theta}A^{\theta} = \bar{\lambda}_2 X_2{}^{\theta}. \qquad \text{(iii)}$$

From (i) and (iii), we get

$$\left(X_2{}^{\theta}A^{\theta}\right)(AX_1) = \left(\bar{\lambda}_2 X_2{}^{\theta}\right)(\lambda_1 X_1)$$

$$\Rightarrow X_2{}^{\theta}\left(A^{\theta}A\right)X_1 = \bar{\lambda}_2\lambda_1 X_2{}^{\theta}X_1$$

$$\Rightarrow \left(1 - \bar{\lambda}_2\lambda_1\right)X_2{}^{\theta}X_1 = O. \qquad \text{(iv)}$$

Also $\bar{\lambda}_2 = \frac{1}{\lambda_2}$. (iv)

Thus, from (iv), we get

$$\left(1 - \frac{\lambda_1}{\lambda_2}\right)X_2{}^{\theta}X_1 = O \Rightarrow \left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right)X_2{}^{\theta}X_1 = O.$$

But $\lambda_2 \neq \lambda_1 \Rightarrow \lambda_2 - \lambda_1 \neq 0$.

Thus $X_2^{\theta}X_1 = O$.

Hence, X_1 and X_2 are orthogonal vectors.

Home Assignments:

Use of properties:

Q.No.1.: Show that, if λ is a characteristic root of the matrix A, then $\lambda + k$ is a characteristic root of the $A + kI$.

Q.No.2.: If $\lambda_1, \lambda_2, \ldots\ldots\lambda_n$ are eigen values of a matrix A, then A^m has the eigen values $\lambda_1{}^m, \lambda_2{}^m, \ldots\ldots\lambda_n{}^m$ (m being a positive integer).

Q.No.3.: Find the sum and product of the eigen value of

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Ans.: Sum = trace = 2 +1 +2 = 5, Product = $|A| = 21$.

Find the eigen values and eigen vectors of 2×2 matrices:

Q.No.1.: Find the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Ans.: $5, -2$, $(1, 1)$, $4, -3$.

Q.No.2.: Find the eigen value and eigen vector of $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Ans.: $\lambda^2 + 7\lambda + 6 = 0$, $\lambda = -1, -6$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Q.No.3.: Find the eigen value and eigen vector of $\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$.

Ans.: $10, -10$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Q.No.4.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

Ans.: $2, -1$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Q.No.5.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Ans.: $4, \ -1, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Find the eigen values and eigen vectors of 3×3 matrices:

Q.No.1.: Find the eigen values and eigen vectors of the matrices:

(i). $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(ii).** $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Ans.: **(i).** $1, 1, 3;$ $(1,-2,1), (1,-1,0), (1,1,0)$ **(ii).** $2, 3, 5;$ $(1,-1,0), (1,0,0), (2,0,1)$.

Q.No.2.: Find the eigen value and eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Ans.: $5, \ -3, \ -3,$ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Q.No.3.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Ans.: $\lambda^3 - 7\lambda^2 + 36 = 0, \ \lambda = -2, \ 3, \ 6,$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Q.No.4.: Find the eigen value and eigen vector of $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

Ans.: $(\lambda - 1)^3 = 0, \ \lambda = 1, \ 1, \ 1,$ $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

Q.No.5.: Find the eigen value and eigen vector of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Ans.: $\lambda^3 - 18\lambda^2 + 45\lambda = 0, \ \ \lambda = 0, \ 3, \ 15$, $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Q.No.6.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Ans.: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0, \ \ \lambda = 5, \ 1, \ 1$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Q.No.7.: Find the eigen value and eigen vector of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Ans.: $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0, \ \ \lambda = 2, \ 2, \ 3$, For $\lambda = 2, \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$, For $\lambda = 3, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Q.No.8.: Find the eigen value and eigen vector of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Ans.: $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0, \ \ \lambda = 2, \ 2, \ 8$, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, For $\lambda = 8, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Q.No.9.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Ans.: $(\lambda - 2)^3 = 0, \ \ \lambda = 2, \ 2, \ 2$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Q.No.10.: Find the eigen value and eigen vector of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Ans.: $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0,\ \ \lambda = 2,\ 2,\ -2,$ For $\lambda = 2$, $[0 \quad 1 \quad 1]^T$

For $\lambda = -2$, $[-4 \quad -1 \quad 7]^T$.

Q.No.11.: Find the eigen value and eigen vector of $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$.

Ans.: $(\lambda + 5)(\lambda - 2)^2 = 0,\ \ \lambda = 5,\ 2,\ 2,$ For $\lambda = 5$, $X_1 = [3 \quad 2 \quad 4]^T$

For $\lambda = 2$, $X_2 = [1 \quad 3 \quad -1]^T$.

Q.No.12.: Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are = 1 each.

Find the eigen values of A^{-1}.

Ans.: 1, 1, $\frac{1}{5}$.

Find the eigen values and eigen vectors of 4×4 matrices:

Q.No.1.: Find the eigen value and eigen vector of $\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$.

Ans.: $\lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = 0,\ \ \lambda = 2,\ 1,\ 1, 1,$ For $\lambda = 2$, $\begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}$, For $\lambda = 1$, $\begin{bmatrix} 3 \\ 6 \\ -4 \\ -5 \end{bmatrix}$.

Find the eigen values and eigen vectors of SPECIAL matrices:

Q.No.1.: Show that eigen values of the skew-symmetric matrix

$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ are purely imaginary or zero.

Ans.: Eigen values are $0,\ -25i,\ 25i$.

Q.No.2.: Prove that $A=\begin{bmatrix}4 & 1-3i\\ 1+3i & 7\end{bmatrix}$ is Hermitian matrix. Find its eigen values.

Ans.: Characteristic equation: $\lambda^2-11\lambda+18=0$, eigen values 9, 2.

Q.No.3.: Find the eigen vectors of the Hermitian matrix $A=\begin{pmatrix}a & b+ic\\ b-ic & k\end{pmatrix}$.

Ans.: $\lambda_{1,2}=\dfrac{\left[(a+k)\pm(a-k)^2+4\left(b^2+c^2\right)\right]}{2}$

Eigen vectors: $\begin{bmatrix}\dfrac{-\left(b^2+c^2\right)}{(a-\lambda)(b-ic)} & 1\end{bmatrix}^T_{\text{at } \lambda=\lambda_1\lambda_2}$.

Q.No.4.: Find the Hermitian form of $A=\begin{bmatrix}3 & 2-i\\ 2+i & 4\end{bmatrix}$ with $X=\begin{bmatrix}1+i\\ 2i\end{bmatrix}$.

Ans.: 34.

Q.No.5.: Find the Hermitian form of $A=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$, $X=\begin{pmatrix}1\\ i\end{pmatrix}$.

Ans.: -2.

Q.No.6.: Show that $B=\begin{bmatrix}3i & 2+i\\ -2+i & -i\end{bmatrix}$ is Skew-Hermitian. Find its eigen values.

Ans.: Characteristic equation: $\lambda^2-2i\lambda+8=0$, eigen values $4i,\ -2i$.

Q.No.7.: Find the eigen vectors of the Skew Hermitian matrix $A=\begin{bmatrix}2i & 3i\\ 3i & 0\end{bmatrix}$.

Ans.: $\lambda_{1,2}=\left(1\pm\sqrt{10}\right)i$, eigen vectors: $\begin{pmatrix}1\pm\dfrac{\sqrt{10}-1}{3}\end{pmatrix}^T$.

Q.No.8.: Find the Skew-Hermitian form for

(a) $A=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ with $X=\begin{pmatrix}1\\ i\end{pmatrix}$,

(b). $A=\begin{pmatrix}2i & 4\\ -4 & 0\end{pmatrix}$ with $X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$.

Ans.: (a). 0, (b). $2i|x_1|^2 + 8i\ \mathrm{Im}(\bar{x}_1 x_2)$.

Q.No.9.: Find the Skew- Hermitian form for $A = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}$ with $X = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Ans.: 16i.

Q.No.10.: $C = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$ is unitary matrix. Find its eigen values.

Ans.: $\lambda^2 - i\lambda - 1 = 0$, $\lambda = \left(\sqrt{3} + i\right)/2$, $\left(-\sqrt{3} + i\right)/2$.

Q.No.11.: Show that the column (and also row) vectors of the unitary matrix

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$ form an orthogonal system.

Q.No.12.: Determine the eigen values and eigen vectors of the unitary matrix $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$.

Ans.: Eigen values 1, -1, eigen vectors $\left[1i \pm i\sqrt{2}\right]^T$.

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