Differential Calculus

CURVE TRACING

(a) Cartesian form

(b) Polar form

(c) Parametric form

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Introduction:

Curve tracing is an analytical method of drawing an approximate shape of a curve, by the study of some of its important characteristics such as symmetry, intercepts, asymptotes, tangents, multiple points, region of existence, sign of the first and second derivatives.

Knowledge of curve tracing is useful in application of integration in finding length, area, volume etc.

Here, we will study the procedure for tracing of standard curves in the

(a) Cartesian form

(b) Polar form

(c) Parametric form

GENERAL PROCEDURE FOR TRACING OF CARTESIAN CURVES:

Plane algebraic curve of n^{th} degree is represented by

$$f(x, y) = ay^n + (bx + c)y^{n-1} + (dx^2 + ex + f)y^{n-2} + \dots\dots\dots + u_n(x) = 0, \qquad (i)$$

where a, b, c, d, f ……. are all constants and $u_n(x)$ is a polynomial in x of degree n.

General procedure for tracing the algebraic curve consists of the study of the following characteristics of the curve.

1. Symmetry
2. Origin
3. Tangents to the curve at the origin
4. x and y-Intercepts
5. Special points (Multiple points)
6. Sign of first derivative $\frac{dy}{dx}$
7. Sign of second derivative $\frac{d^2y}{dx^2}$
8. Imaginary values (Imaginary region)
9. Region
10. Asymptotes
11. Shape of the curve

1. Symmetry

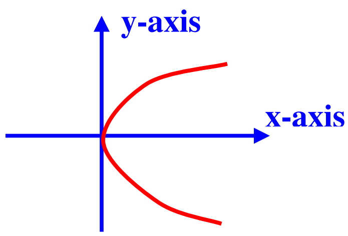
(i) Symmetric about x-axis:

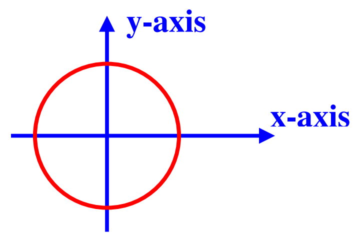
If even and only even powers of y occur in equation of the curve, then the curve is symmetrical about x-axis.

or

If the equation of the curve remains unchanged, when y is changed into $-y$, then the curve is symmetrical about x-axis.

This means that, if $f(x, -y) = f(x, y)$ then the curve is symmetrical about x-axis.





Examples: 1. The parabola $y^2 = 4ax$ is symmetrical about x-axis.

2. The circle $x^2 + y^2 = a^2$ is symmetrical about x-axis.

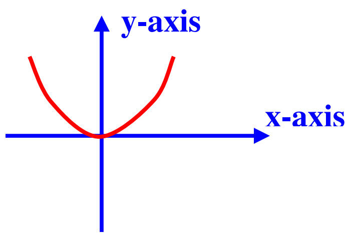
(ii) Symmetric about y-axis:

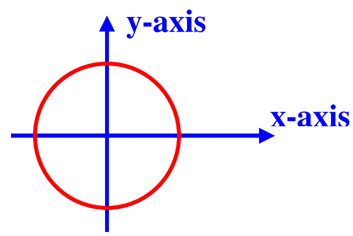
If even and only even powers of x occur in equation of the curve, then the curve is symmetrical about y-axis.

or

If the equation of the curve remains unchanged, when x is changed into $-x$, then the curve is symmetrical about y-axis.

This means that, if $f(-x,\ y) = f(x,\ y)$ then the curve is symmetrical about y-axis.





Examples: **1.** The parabola $x^2 = 4ay$ is symmetrical about y-axis.

2. The circle $x^2 + y^2 = a^2$ is symmetrical about y-axis.

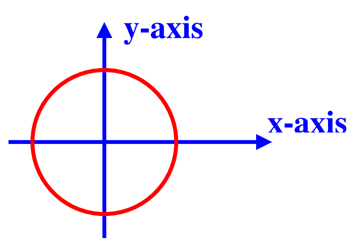
(iii) Symmetric about both x- and y-axes:

If even and only even powers of x and y occur in equation of the curve, then the curve is symmetrical about x- and y-axes.

or

If the equation of the curve remains unchanged, when x is changed into $-x$ and y is changed into $-y$, then the curve is symmetrical about x- and y-axes.

This means that, if $f(-x,\ -y) = f(x,\ y)$ then the curve is symmetrical about both the axes.



Example: **1.** The circle $x^2 + y^2 = a^2$ is symmetrical about both the axes.

(iv) Symmetric about origin:

If the equation remains unchanged, when x and y are replaced by $-x$ and $-y$, respectively, then the curve is also symmetrical about origin.

This means that, if $f(-x, -y) = f(x, y)$, then the curve is symmetrical about the origin.

Examples: **1.** The curve $x^5 + y^5 = 5a^2x^2y$ is symmetrical about origin.

2. The circle $x^2 + y^2 = a^2$ is symmetrical about origin.

(v) Symmetric about opposite quadrants:

If x and y are replaced by $-x$ and $-y$, respectively, the equation of the curve remains unchanged, then the curve is also symmetrical in opposite quadrants.

This means that, if $f(x, y) = f(-x, -y)$, then the curve is symmetrical about the opposite quadrant.

Examples: **1.** $xy = c^2$

2. Cubic parabola $y = x^3$

3. Circle $x^2 + y^2 = a^2$

4. Ellipse

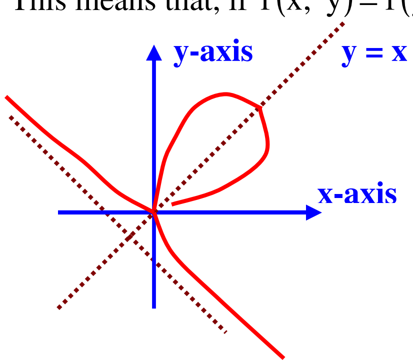
5. Hyperbola

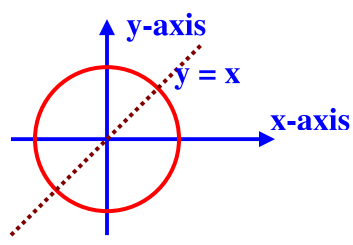
Remarks: Curve symmetric about both the axes is also symmetric about origin but not the converse (because of the presence of odd powers).

(vi) Symmetric about the line y = x:

If on interchanging x and y the equation of the curve remains unaltered, then the curve is symmetrical about the line y = x.

This means that, if $f(x, y) = f(y, x)$, then the curve is symmetrical about the line y = x.





Examples: 1. The curve $x^3 + y^3 = 3axy$ is symmetrical about the line y = x.

2. The circle $x^2 + y^2 = a^2$ is symmetrical about the line y = x.

(vii) Symmetric about the line $y = -x$:

If on changing x to $-y$ and y to $-x$, the equation of a curve remains unchanged, then the curve is symmetrical about $y = -x$.

This means that, if $f(x, y) = f(-y, -x)$, then the curve is symmetrical about the line $y = -x$

Example: 1. The curve $x^3 - y^3 = 3axy$ is symmetrical about the $y = -x$.

2. Origin

If the co-ordinates of origin i.e. (0, 0) satisfy the given equation, then the curve passes through the origin.

This means that, if the equation of the curve has no constant term then it will pass through the origin.

Examples: 1. The curve $x^3 + y^3 = 3axy$ passes through the origin.

2. The curve $y^2(a - x) = x^3$ passes through the origin.

3. Tangents to the curve at the origin

If the curve passes through the origin, find the equations of the tangents at the origin.

Method for evaluating equation of tangents:

Equation of tangents is evaluated by equating the lowest degree term in the equation of the curve to zero.

Examples: 1. $y^2 = 4ax$. Here lowest degree term 4ax equated to zero gives x = 0 (y-axis) as tangent to the curve at origin.

2. $x^3 + y^3 = 3axy$. Here lowest degree terms 3axy equated to zero gives xy = 0 or x = 0 and y = 0 are the two tangents to the curve at origin.

3. $a^2y^2 = a^2x^2 - x^4$. Here lowest degree term $(y^2 - x^2)$ equating to zero gives $y = \pm x$ as the two tangents at origin.

4. x and y- Intercepts

- **x-intercept and y-intercept:** Find the points, where the curve meets the x-axis by substituting y = 0, in the equation of the curve. Also find the points where the curve meets the y-axis, by substituting $x = 0$.
- Find the **tangents** at these points; this can be done easily by shifting the origin to these points of intersection and equating the lowest degree term in the changed equation to zero.
- If the curve is symmetrical about the line y = x or $y = -x$. Find the points of intersection of curve with these lines also by putting $y = \pm x$, in the equation of the curve. Find the tangents at these points.

5. Special points (Multiple points)

Multiple point (or singular point):

A point, through which r branches of the curve pass is called a **multiple point** of the r^{th} order and has r tangents. Thus, at a double point two branches of the curve pass.

Double point is classified as a node, a cusp or an isolated (or conjugate) point.

If the tangents are **real and different**, then the point is called a **node.**

If the tangents are **real and coincident**, then point is called **a cusp**.

If the tangents are **imaginary**, then the point is called **an isolated point**.

Method for evaluating multiple points:

Multiple points are obtained by solving for (x, y) the three equations

$f_x(x, y) = 0,\ f_y(x, y) = 0,\ f(x, y) = 0$.

6. Sign of first derivative $\frac{dy}{dx}$

- In an interval $a \le x \le b$ if
  - $\frac{dy}{dx} > 0$, then the curve is increasing in [a, b]
  - $\frac{dy}{dx} < 0$, then the curve is decreasing in [a, b]

- If at $x = x_0, y = y_0$, $\frac{dy}{dx} = 0$, then (x_0, y_0) is a stationary point where maxima and minima can occur

Remarks:

At point where $\frac{dy}{dx} = 0$, the tangent is parallel to x-axis i.e., horizontal.

At point where $\frac{dy}{dx} = \infty$, the tangent is parallel to y-axis i.e., vertical.

7. Sign of second derivative $\frac{d^2y}{dx^2}$

- In an interval $a \le x \le b$ if
  - $\frac{d^2y}{dx^2} > 0$, then the curve is convex or concave upwards (holds water)
  - $\frac{d^2y}{dx^2} < 0$, then the curve is concave downwards (spills water)
  - A point at which $\frac{d^2y}{dx^2} = 0$ is known as an inflection point, where the curve changes the direction of concavity from downward to upward or vice- versa.

8. Imaginary values (Imaginary region)

Find the region, where no part of the curve lies. This region can be found by solving the given equation of the curve for one variable in terms of the other, say y in terms of x and then finding those values of x for which y becomes imaginary.

Thus, imaginary region is the region in which the curve does not exist. In such region y becomes imaginary (undefined) for values of x or vice-versa.

9. Region (Region of extent)

- Consider the variation of one of the variables, say y, as other say x varies, paying special attention, when x increases and finally approaches ∞.
- Similarly, observe the variation of y as x decreases and finally approaches $-\infty$.

In other way, **region of extent** is obtained by in terms of x or vice-versa.

Real horizontal extent is defined by values x for which y is defined.

Real vertical extent is defined by values of y for which x is defined.

10. Asymptotes

Find the asymptote of the curve, if any. Now first let us define asymptote.

Definition:

Asymptote to a curve is a straight line, which is tangent to the curve at infinity.

or

Asymptote is a straight line related to an infinite branch of a curve, such that its perpendicular distance, from a point, which moves to infinity along the curve, tends to zero.

Method for evaluating asymptote:

Asymptotes parallel to x-axis:

Asymptotes parallel to x-axis are obtained by equating the coefficients of the highest powers of x, to zero.

Asymptotes parallel to y-axis:

Asymptotes parallel to y-axis are obtained by equating the coefficients of the highest powers of y, to zero.

Example: Let us consider the curve $y^2(a-x)=x^3$.

Equating to zero the coefficient of y^2, the highest degree term in y.

The asymptote parallel to y-axis is $x-a=0 \Rightarrow x=a$.

Oblique Asymptotes

- Let $y=mx+c$ be an asymptote.
- Put $y = mx + c$ in the given equation and we get the equation in terms of descending powers of x.
- Equate the coefficients of the powers of x to zero, separately.
- Solve these equations for m and c.
- Substitute these values of m and c in $y = mx + c$.
- Then, we get the required equation of asymptote.

Note that any two such equations in m and c will be sufficient to give the values of m and c.

Example: Let us consider the curve $x^3 + y^3 = 3axy$.

Let $y = mx + c$ be an asymptote.

Putting this value of y in $x^3 + y^3 = 3axy$, we get

$$x^3 + (mx+c)^3 = 3ax(mx+c) \Rightarrow x^3 + m^3x^3 + c^3 + 3cmx(mx+c) = 3amx^2 + 3acx$$

$$\Rightarrow x^3 + m^3x^3 + c^3 + 3cm^2x^2 + 3c^2mx - 3amx^2 - 3acx = 0$$

$$\Rightarrow x^3(1+m^3) + x^2(3cm^2 - 3am) + x(3c^2m - 3ac) + c^3 = 0.$$

This one is the equation in terms of descending powers of x.

Equate the coefficients of the powers of x to zero, separately.

Equating coefficient of x^3 equal to zero, we get

$$1 + m^3 = 0 \Rightarrow m = -1.$$

Equating coefficient of x^2 equal to zero, we get

$$3cm^2 - 3am = 0 \Rightarrow cm - a = 0 \Rightarrow cm = a \Rightarrow c = -a. \qquad [\because m = -1]$$

Putting these values of m and c in $y = mx + c$, we get

$y = -x - a \Rightarrow x + y + a = 0$ is the required asymptote.

11. Approximate shape of the curve

Draw the rough sketch of the curve (approximate shape of the curve).

Important Note:

For approximate shape of the curve, make use of salient features of the above procedure.

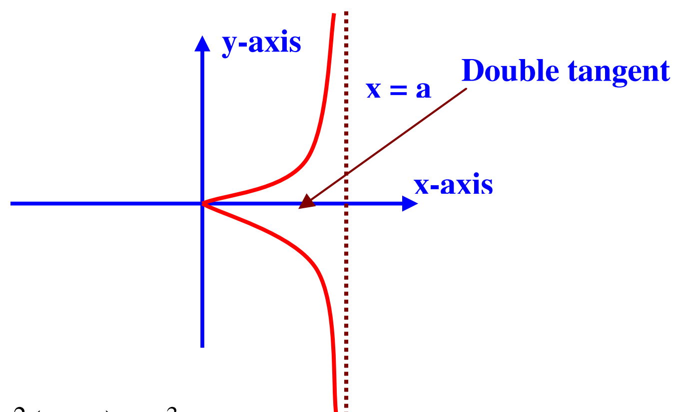
Now let us trace some important curves with their salient features:

1. **Cissoid:** $y^2(a-x)=x^3$
2. **Folium of Descartes:** $x^3+y^3=3axy$ or $x^3-y^3=3axy$
3. **Astroid or Four cusped hypocycloid:**

$$x^{2/3}+y^{2/3}=a^{2/3} \text{ or } \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$$

4. **Cycloid:** $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$
5. **Equiangular Spiral:** $r=ae^{\theta\cot\alpha}$

Q.No.1: Trace the following curve giving the salient points $y^2(a-x)=x^3$ **(Cissoid).**



Sol.: The given curve is $y^2(a-x)=x^3$. (i)

1. **Symmetry:**

The curve is symmetrical about x-axis as these are even and only even powers of y in the equation of the curve.

2. **Origin:**

As we know, if the co-ordinates of origin i.e. (0, 0) satisfy the given equation, then the curve passes through the origin. Here this curve passes through the origin.

3. **Tangents at origin:**

The equation of given curve can be written as

$$ay^2-xy^2=x^3 \Rightarrow x^3+xy^2-ay^2=0.$$

Equating lowest degree term to zero, we get $ay^2=0 \Rightarrow y^2=0 \Rightarrow y=0, y=0$.

$\Rightarrow$ The two tangents are real and coincident therefore the origin is **cusp**, in the present problem.

4. Intersection with coordinate axis:

When $x = 0 \Rightarrow y = 0$, when $y = 0 \Rightarrow x = 0$.

$\Rightarrow$ The curve does not intersect the coordinate axis.

The curve meets the coordinate axes only at origin.

5. Special points:

From the equation of the curve

$y = \dfrac{x^{3/2}}{\sqrt{a-x}}$ [taking positive sign for discussion]

$$\frac{dy}{dx} = \frac{\frac{3}{2}x^{1/2}(a-x)^{1/2} + \frac{1}{2}(a-x)^{-1/2}x^{3/2}}{(a-x)} = \frac{\sqrt{x}(3a-2x)}{2(a-x)^{3/2}}$$

$=0$ when $x = 0$ or $x = \dfrac{3a}{2}$.

Rejecting the value of $x = \dfrac{3a}{2}$, because y is imaginary when $x = \dfrac{3a}{2}$.

The tangent at $x = 0$ is parallel to x-axis.

6. Imaginary values:

If $x < 0$, y^2 becomes negative and y is imaginary.

Hence no part of the curve lies in the second and third quadrants.

Also, if $x >$ a, y becomes imaginary.

Hence no part of the curve lies beyond the point x = a.

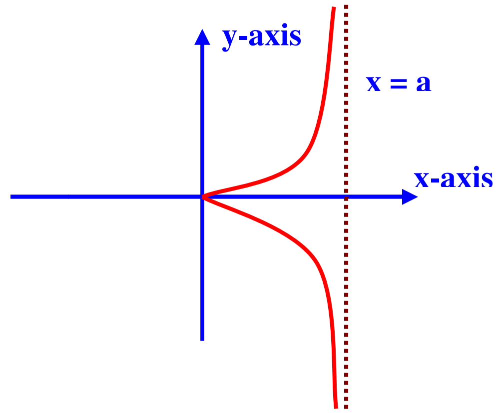
7. Asymptotes:

Equating to zero the coefficient of y^2, the highest degree term in y.

The asymptote parallel to y-axis is $x - a = 0 \Rightarrow x = a$

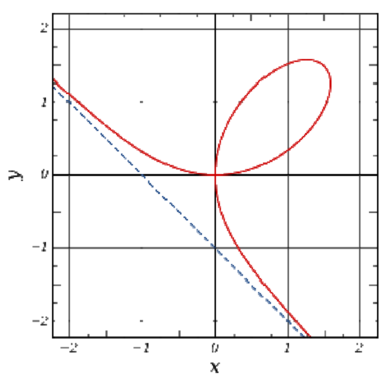
There is no asymptote parallel to x-axis.

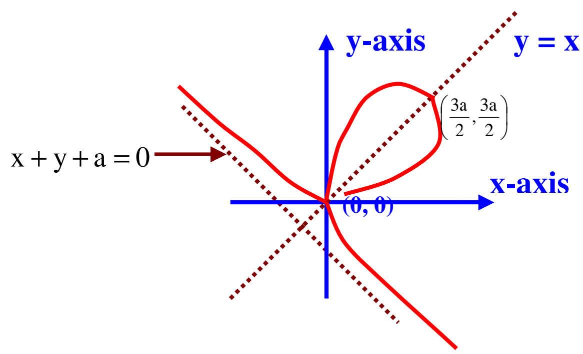
8. Shape:



Q.No.2.: Trace the following curve $x^3 + y^3 = 3axy$, by studying its important features/characteristics..

(Folium of Descartes).





The folium of Descartes with the parameter *a*=1

The curve was first proposed by **Descartes** in 1638.

Many famous curves have names from nature. "Folium" means leaf.

31 March 1596 – 11 February 1650

French philosopher, mathematician, physicist

Sol.: The given curve is $x^3 + y^3 = 3axy$.

1. Symmetry:

If we interchange x and y, the given equation does not alter, thus there is symmetry about the line y = x.

2. Origin:

As we know, if the co-ordinates of origin i.e. (0, 0) satisfy the given equation, then the curve passes through the origin. Here this curve passes through the origin.

3. Tangent at origin:

Tangents at (0, 0) are given by equating lowest degree terms to zero. Thus tangents at (0, 0) are given by

$3axy = 0 \Rightarrow x = 0$ and $y = 0$ $\qquad \because a \neq 0$

4. Intersection of the curve with co-ordinate axes:

The curve is $x^3 + y^3 = 3axy$

When x = 0, we get y = 0

When y = 0, we get x = 0.

Thus the curve does not intersect the co-ordinate axes.

5. Region in which the curve lies.

The given curve is $x^3 + y^3 = 3axy$.

When x and y both are positive, the equation is satisfied when x and y both are negative the equation is not satisfied, thus the curve does not exist in the third quadrant. When x is positive and y is negative, the equation is satisfied. When x is negative and y is positive, the equation is satisfied. Thus the curve exist in the first, second and fourth quadrant.

6. Asymptotes:

The curve has

(a). No asymptote parallel to x-axis.

(b). No. asymptotes parallel to y-axis.

(c). Let y = mx + c be an asymptote.

Putting this value of y in $x^3 + y^3 = 3axy$

We get $x^3 + (mx + c)^3 = 3ax(mx + c)$

$$\Rightarrow x^3 + m^3x^3 + c^3 + 3cmx(mx + c) = 3amx^2 + 3acx$$

$$\Rightarrow x^3 + m^3x^3 + c^3 + 3cm^2x^2 + 3c^2mx - 3amx^2 - 3acx = 0$$

$$\Rightarrow x^3(1 + m^3) + x^2(3cm^2 - 3am) + x(3c^2m - 3ac) + c^3 = 0.$$

Equating coefficient of x^3 equal to zero, we get

$$1 + m^3 = 0 \Rightarrow m = -1$$

Equating coefficient of x^2 equal to zero, we get

$$3cm^2 - 3am = 0 \Rightarrow cm - a = 0 \Rightarrow cm = a \Rightarrow c = -a \qquad [\because m = -1]$$

Putting these values of m and c in

$y = mx + c$ we get $y = -x - a$

$\Rightarrow x + y + a = 0$ is the required asymptote.

7. Intersection of the curve with the line y = x

$$x^3 + y^3 = 3axy \qquad \text{(i)}$$

$$y = x \qquad \text{(ii)}$$

Solving for x and y from (i) and (ii)

$$2x^3 = 3ax^2$$

$$2x = 3a$$

$$\Rightarrow x = \frac{3a}{2} \text{ and } y = \frac{3a}{2}.$$

Differentiate $x^3 + y^3 = 3axy$ w.r.t. x, we get

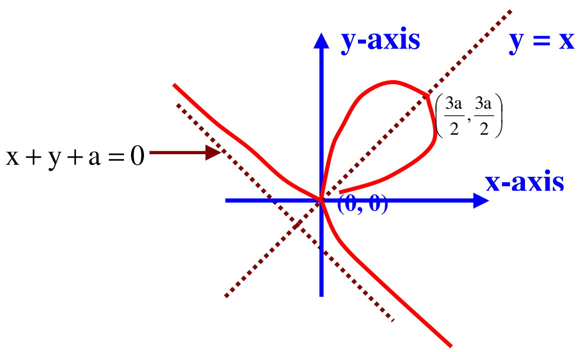
$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right] \Rightarrow x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} = ay$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = ay - x^2 \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

The value of $\frac{dy}{dx}$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \dfrac{a \times \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \times \frac{3a}{2}} = \dfrac{\frac{3}{2}a^2 - \frac{9}{4}a^2}{\frac{9a^2}{4} - \frac{3}{2}a^2} = -1.$

This means that the tangent to the curve at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ makes an angle of 135^o with the positive direction of x-axis, keeping in view all the points.

8. Shape: The shape of the curve is given as follows.



Q.No.3.: Trace the following curve giving the salient points $x^{2/3} + y^{2/3} = a^{2/3}$

(Astroid or Four cusped hypocycloid).

Sol.: The given equation $x^{2/3} + y^{2/3} = a^{2/3}$ can be written in the parametric form as

$x = a\cos^3\theta$ and $y = a\sin^3\theta$.

1. Symmetry:

Parametric form: $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Here x is even function and y is odd function. $\therefore$ Symmetrical about x-axis.

The parametric form can be written as

$x = a\sin^3\theta$, $y = a\cos^3\theta$.

($\because$ The value of x changes as θ is changed to $-\theta$ but the value of y remains unaltered as θ is changed to $-\theta$).

Here x is odd function and y is even function $\therefore$ Symmetrical about y-axis.

If we interchange x and y in $x^{2/3} + y^{2/3} = a^{2/3}$, the equation remains the same.

$\therefore$ there is symmetry about the line y = x.

2. Origin:

As we know, if the co-ordinates of origin i.e. (0, 0) satisfy the given equation, then the curve passes through the origin. Clearly, the curve does not pass through the origin.

3. Intersection of the curve with co-ordinate axes:

When x = 0, we get y = ± a.

When y = 0, we get x = ± a.

Thus, the curve meets x-axes at $(\pm a, 0)$ and y-axes at $(0, \pm a)$.

4. Shape of curve at (a, 0) and (0, a):

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Differentiating w.r.t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow y^{-1/3}\frac{dy}{dx} = -x^{-1/3}\ \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$\therefore \left(\frac{dy}{dx}\right)_{(a,0)} = 0$, the tangent is x-axis and (a, 0) is cusp as there is symmetry about x-axis.

$\left(\frac{dy}{dx}\right)_{(0,a)} = \infty$, the tangent is y-axis and (0, a) is cusp as there is symmetry about y-axis.

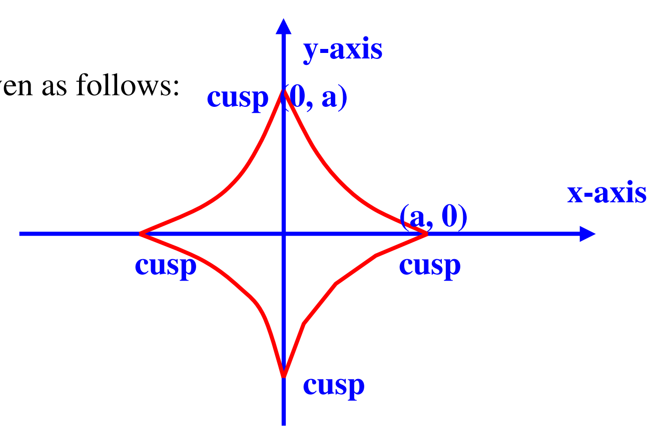
5. Region in which the curve lies:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$y^{2/3} = a^{2/3} - x^{2/3} \Rightarrow \left[y^{1/3}\right]^2 = a^{2/3} - x^{2/3}.$$

When x > a, y will be imaginary, $\therefore$ the curve does not exist when x > a. Similarly, the curve does not exist when y > a.

6. Shape: Shape of the curve is given as follows:



Q.No.4.: Trace the following curve giving the salient points

$$x = a(\theta + \sin\theta)$$

$y = a(1 + \cos\theta)$ from cusp to cusp. **(Cycloid)**

Sol.:

1. Symmetry:

Here x is an odd function and y is an even function

($\because$ The value of x changes as θ is changed to $-\theta$ but the value of y remains unaltered as θ is changed to $-\theta$).

$\therefore$ The curve is symmetrical about y-axis.

2. Origin:

There is no value of θ for which both x and y become zero simultaneously. Hence, the curve does not pass through (0, 0).

3. Tangents at $\theta = 0$ and $\theta = \pi$:

Since $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$.

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \quad \frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{-a\sin\theta}{a(1 + \cos\theta)} = -\tan\frac{\theta}{2}.$$

When $\theta = 0$: x = 0, y = 2a

i.e. at (0, 2a), $\frac{dy}{dx} = -\tan 0 = 0$.

i.e. the tangent at (0, 2a) is parallel to x-axis.

When $\theta = \pi$: $x = a\pi$, $y = 0$

i.e. at $(a\pi, 0)$ or $\theta = \pi$, $\frac{dy}{dx} = -\tan\frac{\pi}{2} = -\infty$.

i.e. the tangent is parallel to y-axis.

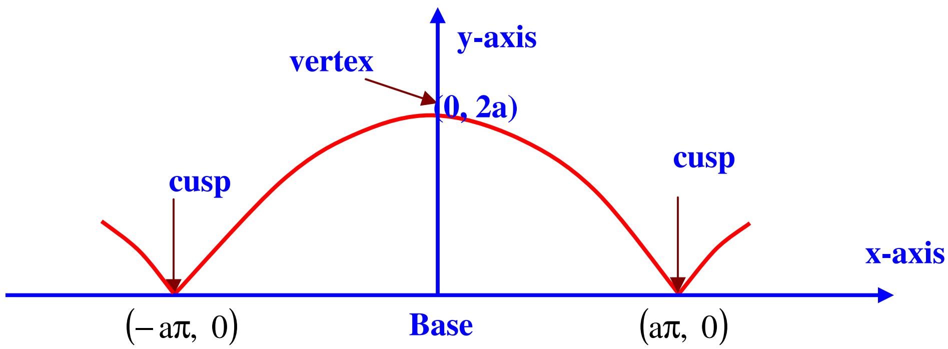
4. Table for values of θ, x and y is:

θ	$-\pi$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$-\frac{\pi}{3}$	0
x	$-a\pi$	$-a\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$	$-a\left(\frac{\pi}{6} + \frac{1}{2}\right)$	0	$a\left(\frac{\pi}{6} + \frac{1}{2}\right)$	$a\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$	$a\pi$

		$=-1.913a$					
y	0	$\frac{3a}{2}$	$a\left(1+\frac{\sqrt{3}}{2}\right)$	2a	$a\left(1+\frac{\sqrt{3}}{2}\right)$	$\frac{3a}{2}$	0

5. Shape: One arch will be traced with this table.

With the aid of all the points discussed before, shape of the cycloid will be as under.



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PROCEDURE FOR TRACING OF POLAR CURVES:

The equation of a curve in the form $r = f(\theta)$ or $f(r,\theta)=0$ is known as polar equation of the curve.

The procedure for tracing of such curves is fundamentally the same as for the tracing of polar curves is given by as under.

1. Symmetry

(a). If the equation of a curve remains unaltered when θ is changed to $-\theta$, then the curve is symmetrical about the initial line.

(b). If the equation of a curve remains unchanged when θ is changed to $\pi-\theta$ or θ is changed to $-\theta$ and r to $-r$, then the curve is symmetrical about the line through the pole and perpendicular to the initial line i.e. about the line $\theta=\frac{\pi}{2}$.

(c). If the equation of the curve remains unaltered when θ is changed to $\frac{\pi}{2}-\theta$, then the curve is symmetrical about the line $\theta=\frac{\pi}{4}$.

(d). If the equation of a curve remains unchanged when θ is changed to $\frac{3\pi}{2}-\theta$,

then the curve is symmetrical about the line $\theta=\frac{3\pi}{4}$.

(e). If the equation of a curve is remains unchanged when r is replaced by $-r$,

then the curve is symmetrical about the pole.

2. Pole

(a). Find if the pole lies on the curve. The pole will lie on the curve if for some real value of θ, we have $r=0$.

(b). If the pole lies on the curve, the values of θ for which $r=0$, give the tangents to the curve at the pole.

3. Determination of ϕ

[The angle between the radius vector and the tangent to the curve at a point on the curve]

(a). Find $\tan\phi=r\frac{d\theta}{dr}$. Then ϕ gives the direction of the tangent of the curve at a point.

(b). Find the points on the curve for which ϕ is 0 or $\frac{\pi}{2}$. The tangent being parallel or perpendicular to the initial line.

4. Limitation of the curve

(a). Let the least and the greatest value of r be a and b respectively, then the curve lies with in a circle of a radius b but outside the circle of radius a.

(b). Solve the given equation of the curve for r in terms of θ and find for what value of θ, r is imaginary. Let for $\alpha\le\theta\le\beta$, the value of r be imaginary then no part of the curve lies between the lines $\theta=\alpha$ and $\theta=\beta$.

5. Asymptotes

If for some value of θ, $r\to\infty$, then the asymptotes exist.

6. Region

(a). Find the variation of r for positive and negative values of θ, making values of θ for which r attains a maximum, minimum or zero value, when r is a periodic function of θ, the negative values of θ which need not to be considered and curve is traced for one period only.

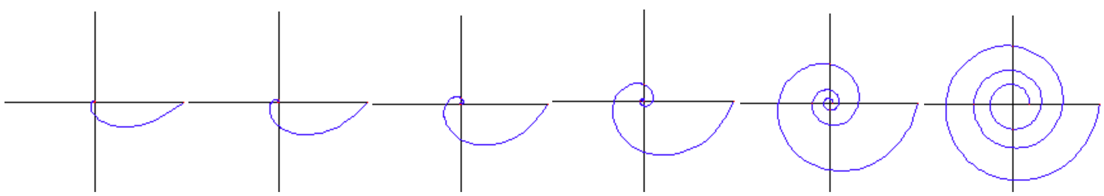
(b). Giving suitable values of θ, find the corresponding values of r to get some points on the curve. Find also ϕ for these values of θ.

7. Approximate shape of the curve

Draw the rough sketch of the curve (approximate shape of the curve).

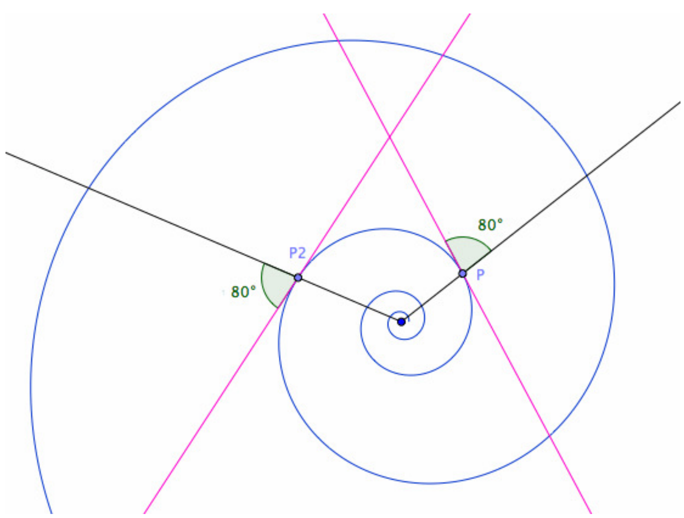
Q.No.5.: Trace the equiangular or Logarithmic spiral $r = ae^{\theta \cot \alpha}$, where a and α are constants.

A logarithmic spiral, equiangular spiral or growth spiral is a special kind of spiral curve which often appears in nature. The logarithmic spiral was first described by Descartes and later extensively investigated by Jakob Bernoulli, who called it Spira mirabilis, "the marvelous spiral".

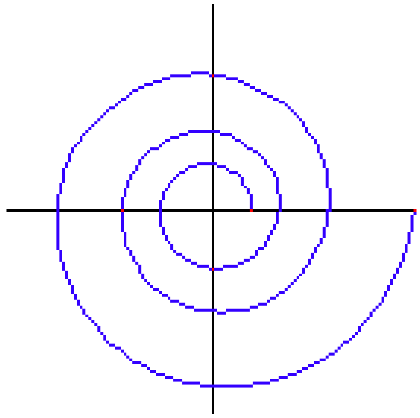


Equiangular spirals with 40°, 50°, 60°, 70°, 80° and 85°. (left to right)

A special case of equiangular spiral is the circle, where the constant angle is 90°.



A example of equiangular spiral with angle 80°.



Descartes discovered the Logarithmic Spiral, also know as the Equiangular Spiral in 1638 while studying dynamics.

Sol.: Given equation of curve is $r = ae^{\theta \cot \alpha}$.

1. Symmetry:

No symmetry about initial line and no symmetry about pole.

2. Origin:

When $\theta = 0$, $r = a$ i.e. curve does not pass through origin.

3. $r = ae^{\theta \cot \alpha}$

Suppose θ to be positive, as the magnitude of θ increases the corresponding values of r will increase.

Again suppose θ to be negative, as the magnitude of θ increases in the negative direction the corresponding values of r will decrease.

4. Angle between radius vector and tangent at any point is constant:

We know that $\tan \phi = r\frac{d\theta}{dr}$,

where ϕ is the angle between radius vector and tangent at any point of the curve.

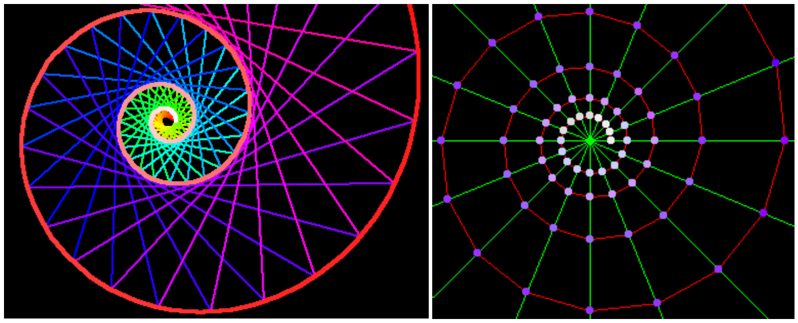
Since $r = aee^{\theta \cot \alpha}$. $\therefore \frac{dr}{d\theta} = ae^{\theta \cot \alpha} \times \cot \alpha = r \cot \alpha$

$$\therefore \frac{d\theta}{dr} = \frac{1}{r \cot \alpha} = \frac{\tan \alpha}{r},$$

$$\text{and } r\frac{d\theta}{dr} = \tan \alpha.$$

$\therefore \tan \phi = \tan \alpha \Rightarrow \phi = \alpha$, which means angle between radius vector and tangent at any point is constant.

Shape of the curve is shown as follows:



Spiral in nature

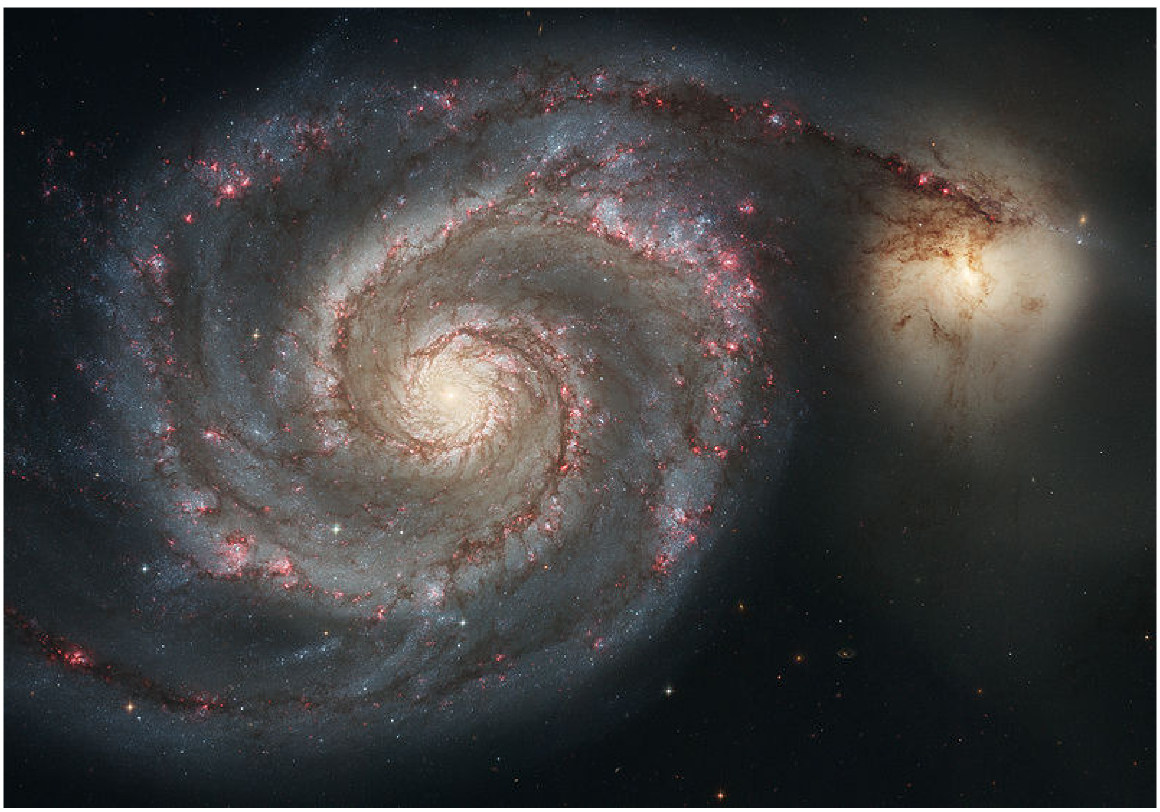
Spiral is the basis for many natural growths.

Seashells have the geometry of equiangular spiral

A cauliflower exhibiting equiangular spiral and fractal geometry.

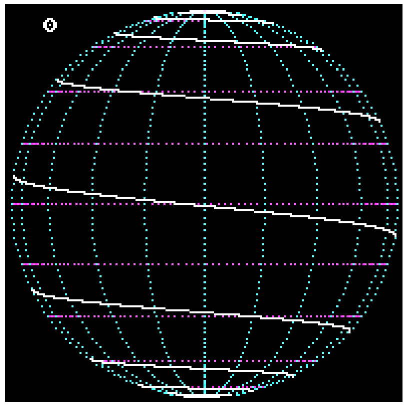
Logarithmic Spirals

A low pressure area over Iceland shows an approximately logarithmic spiral pattern



The arms of spiral galaxies often have the shape of a logarithmic spiral, here the Whirlpool Galaxy

Spherical spiral



A ***spherical spiral*** is the curve on a sphere traced by a ship traveling from one pole to the other while keeping a fixed angle (unequal to 0° and to 90°) with respect to the meridians of longitude, i.e. keeping the same bearing. The curve has an infinite number of revolutions, with the distance between them decreasing as the curve approaches either of the poles.

Q.No.3.: Trace the following curve giving the salient points $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

(Astroid or Four cusped hypocycloid).

In mathematics, an **astroid** is a particular type of curve: a hypocycloid with four cusps. Astroids are also superellipses: all astroids are scaled versions of the curve specified by the equation

$$x^{2/3} + y^{2/3} = 1.$$

Its modern name comes from the Greek word for "star". The curve had a variety of names, including **tetracuspid** (still used), cubocycloid, and paracycle. It is nearly identical in form to the evolute of an ellipse.

A circle of radius 1/4 rolls around inside a circle of radius 1 and a point on its circumference traces an astroid. A line segment of length 1 slides with one end on the x-axis and the other on the y-axis, so that it is tangent to the astroid (which is therefore an envelope). The parametric equations are

$$x = \cos^3\theta, \qquad y = \sin^3\theta.$$

An astroid created by a circle rolling inside a circle of radius a will have an area of

$$\frac{3}{8}\pi a^2$$

and a perimeter of 6a.

See demonstration:

Sol.: The given equation is $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$.

1. Symmetry:

The parametric form can be written as $x = a\cos^3\theta,\ \ y = b\sin^3\theta$.

Here x is even and y is odd. $\therefore$ symmetric about x-axis.

Also the parametric form can be written as $x = a\sin^3\theta,\ \ y = b\cos^3\theta$.

Here x is odd and y is even. $\therefore$ symmetric about y-axis.

Thus, the curve is symmetrical about both the axis, as there are even and only even powers of both x and y in the equation of the curve.

2. Origin:

As we know, if the co-ordinates of origin i.e. (0, 0) satisfy the given equation, then the curve passes through the origin. Clearly, the curve does not pass through the origin.

3. Intersection with the axes:

When x = 0, we get y = $\pm$ b.

When y = 0, we get x = $\pm$ a.

Thus, the curve meets x-axes at $(\pm a,0)$ and y-axes at $(0,\pm b)$.

4. Special Points: From the equation of the curve $\left(\frac{y}{b}\right)^{2/3}=1-\left(\frac{x}{a}\right)^{2/3}$

Differentiating w.r.t. x, we get $\frac{2}{3}\left(\frac{y}{b}\right)^{-1/3}\frac{1}{b}\frac{dy}{dx}=-\frac{2}{3}\left(\frac{x}{a}\right)^{-1/3}\frac{1}{a}$

$$\Rightarrow \frac{dy}{dx} = \frac{b}{a}\frac{\left(\frac{y}{b}\right)^{1/3}}{\left(\frac{x}{a}\right)^{1/3}} = -\left(\frac{b^2 y}{a^2 x}\right)^{1/3}$$

Now $\frac{dy}{dx} = 0$, when $y = 0$

From (i), when $y = 0,\ x \pm a$.

Hence, the tangents are parallel to x-axis at the points $(\pm a, 0)$

Also (i), when $x = 0,\quad y = \pm b$

Hence, the tangents are parallel to y-axis at the points $(0, \pm b)$

5. Imaginary values:

From (i), we have

when $|x| > a$, $\left(\frac{y}{b}\right)^{2/3} < 0$, $\quad \rightarrow y$ is imaginary.

Hence, no part of the curve lies beyond the lines $x = \pm b$.

6. Asymptotes:

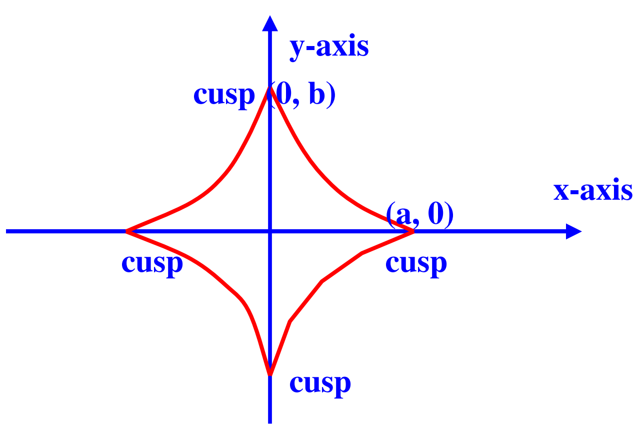
The curve has no asymptotes.

7. Region: From (i), when $x = 0,\ y = \pm b$

$y = 0,\ x = \pm a$.

Also as x increases from 0 to a, y decreases b to 0 in the first quadrant.

6. Shape: Shape of the curve is given as follows:



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