3^{rd} Topic

Double Integrals

(Change of order of Integration)

(Last updated on 15-07-2013)

(26 Solved problems and 00 Home assignments)

Change of order of Integration:

In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly.

Thus $\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$.

In a double integral with variable limits, the change of order of integrationchanges the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration, to a certain extent makes easy, the evaluation of a double integral. The following examples will make these ideas clear.

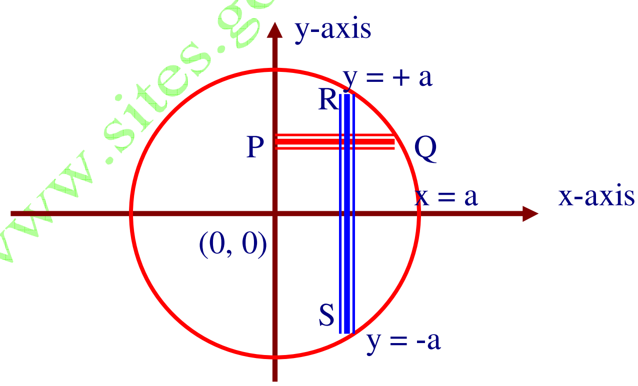
Here we will discuss those problems in double integrals, where limits are given, but we have to change the order of integration. So in this type of problems rough sketch of the region of integration is required. Let us clear this concept with the help of problems given below.

Q.No.1.: Change the order of integration in the integral $I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2-y^2)}} f(x, y) dx dy$.

Sol.: Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2 - y^2)}$ (i. e. to the circle $x^2 + y^2 = a^2$) and strip slides from $y = -a$ to $y = a$.

This shaded semi-circle area is therefore, the region of integration, as shown in the figure.

On changing the order of integration, we first integrate w. r. t. y along a vertical strip RS which extends from $R\left[y = -\sqrt{(a^2 - x^2)}\right]$ to $S\left[y = \sqrt{(a^2 - x^2)}\right]$. To cover the given region, we then integrate w. r. t. x from x = 0 to x = a.



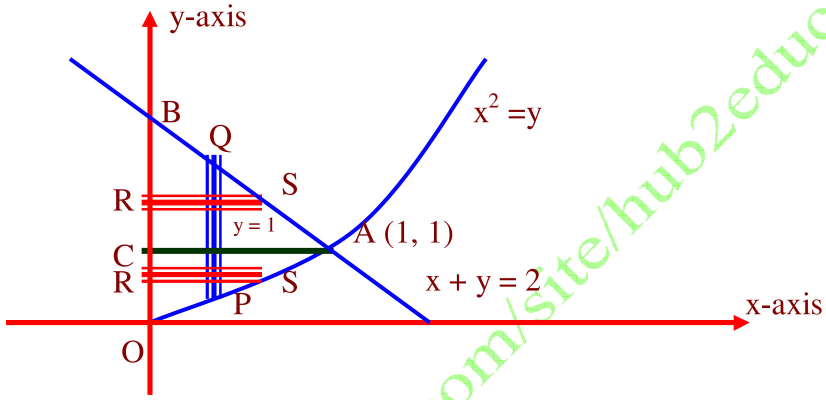
Thus, on reversing the order of integration, we get

$$I = \int_{0}^{a} \left(\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y) dy \right) dx \text{. Ans.}$$

Q.No.2.: Change the order of integration in $I = \int_{0}^{1} \int_{x^2}^{2-x} xy\, dx\, dy$, and hence evaluate the same.

Sol.: Here the integration is first w. r. t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from x = 0 to x = 1, giving the region of integration as the curvilinear triangle OAB(as shown in figure).

When we change the order of integration, we first integrate w. r. t. x along a horizontal strip RS and that requires the splitting up of the region OAB into two parts by the line AC(y = 1), the curvilinear triangle OAC and the triangle ABC.



For the region OAC, the limits of integration for x are from x = 0 to $x = \sqrt{y}$ and those for y are from y = 0 to y = 1. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 \left(\int_0^{\sqrt{y}} xydx \right) dy.$$

For the region ABC, the limits of integration for x are from x = 0 to $x = 2 - y$ and for those for y are from y = 1 to y = 2. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 \left(\int_0^{2-y} xydx \right) dy.$$

Hence, on reversing the order of integration, we get

$$I = \int_0^1 \left(\int_0^{\sqrt{y}} xydx \right) dy + \int_1^2 \left(\int_0^{2-y} xydx \right) dy = \int_0^1 \left| \frac{x^2}{2}.y \right|_0^{\sqrt{y}} dy + \int_1^2 \left| \frac{x^2}{2}.y \right|_0^{2-y} dy$$

$$=\frac{1}{2}\int_0^1 y^2 dy+\frac{1}{2}\int_1^2 y(2-y)^2 dy=\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1+\frac{1}{2}\left[4.\frac{y^2}{2}+\frac{y^4}{4}-4.\frac{y^3}{3}\right]_1^2$$

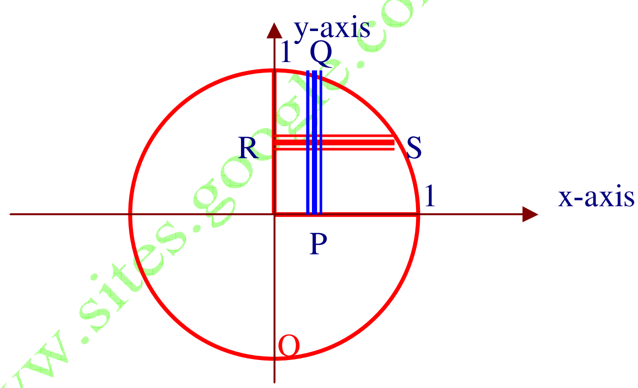
$$=\frac{1}{2}\left[\frac{1}{3}\right]+\frac{1}{2}\left[\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]=\frac{1}{6}+\frac{1}{2}\left(\frac{4}{3}-\frac{11}{12}\right)=\frac{1}{6}+\frac{1}{2}\left(\frac{5}{12}\right)$$

$$=\frac{1}{6}+\frac{5}{24}=\frac{3}{8}.\text{ Ans.}$$

Q.No.3.: Evaluate the integral by changing the order of integration $\int_0^1 \int_0^{\sqrt{(1-x^2)}} y^2 dxdy$.

Sol.: Given integral is $I=\int_0^1 \int_0^{\sqrt{(1-x^2)}} y^2 dxdy$.

The strip PQ parallel to y-axis is made from y = 0 to $y=\sqrt{1-x^2}$ and this strip PQ slides from x = 0 to x = 1 (radius of the circle)



Now when we change the order of integration, we make strip RS parallel to x-axis, we see that strip is made from x = 0 to $x=\sqrt{1-y^2}$, and it slides from y = 0 to y = 1

Hence, on reversing the order of integration, we get

$$I=\int_0^1\left(\int_0^{\sqrt{1-y^2}} y^2 dx\right)dy=\int_0^1\left(y^2|x|_0^{\sqrt{1-y^2}}\right)dy=\int_0^1 y^2\left(\sqrt{1-y^2}\right)dy$$

Putting $y=\sin\theta$ to solve $\int y^2\sqrt{1-y^2}dy$

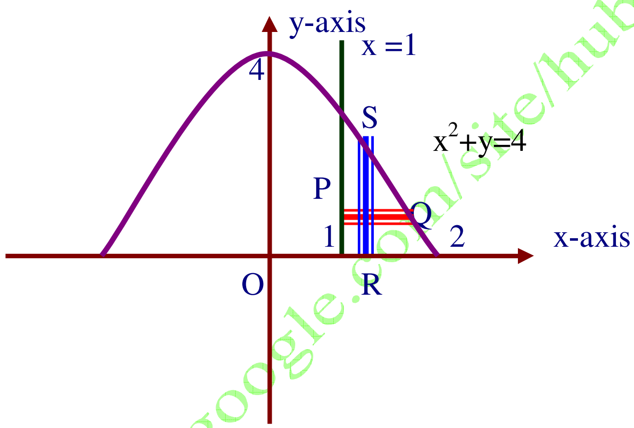
Also when $y = 0,\ \theta = 0;\quad y = 1,\ \theta = \dfrac{\pi}{2}$ and $\dfrac{dy}{d\theta} = \cos\theta$.

$$\therefore I = \int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta = \frac{1.1}{4.2}.\frac{\pi}{2} = \frac{\pi}{16}.\ \text{Ans.}$$

Q.No.4.: Evaluate the following integral by changing the order of integration

$$\int_0^3 \left(\int_1^{\sqrt{(4-y)}} (x+y)dx \right) dy.$$

Sol.: Here the strip PQ parallel to x-axis is made from x = 1 to $x = \sqrt{4-y}$ and this strip slides from y = 0 to y = 3.



Now when we change the order of integration, we make strip parallel to y-axis, which is made from y = 0 to $y = 4 - x^2$ and slides from x = 1 to x = 2.

Hence, on reversing the order of integration, we get

$$I = \int_1^2 \left(\int_0^{4-x^2} (x+y)dy \right) dx = \int_1^2 \left| xy + \frac{y^2}{2} \right|_0^{4-x^2} dx$$

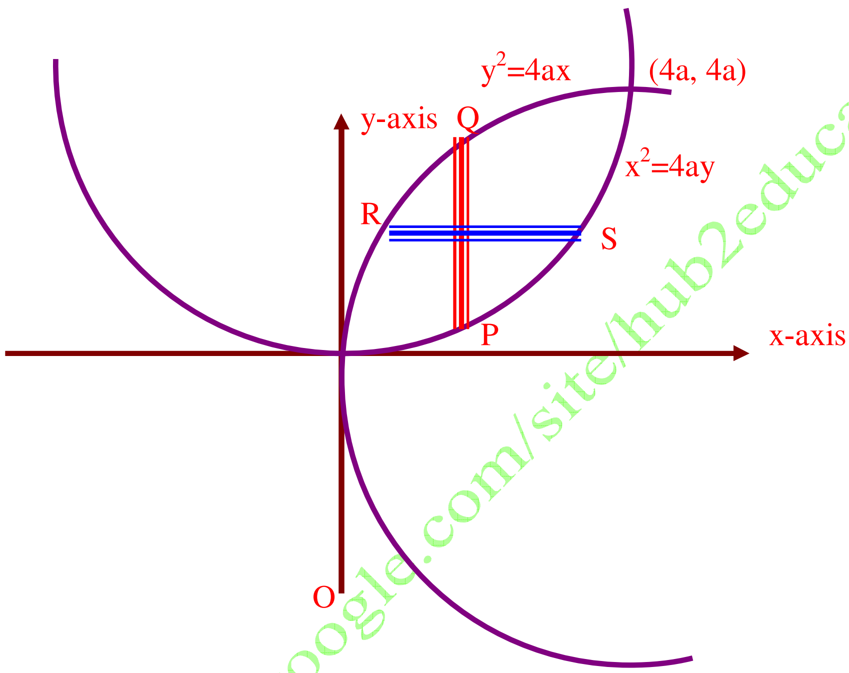
$$= \int_1^2 \left[x\left(4-x^2\right) + \frac{\left(16 + x^4 - 8x^2\right)}{2} \right] dx = \left[4.\frac{x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{5} - 4.\frac{x^3}{3} \right]_1^2$$

$$= 6 - \frac{15}{4} + 8 + \frac{31}{10} - \frac{28}{3} = \frac{360 - 225 + 480 + 186 - 560}{60} = \frac{241}{60} = 4.02.\ \text{Ans.}$$

Q.No.5.: Evaluate the integral by changing the order of integration $\displaystyle\int_0^{4a} \left(\int_{x^2/4a}^{2\sqrt{ax}} dy \right) dx$.

Sol.: Here the strip PQ parallel to y-axis is made from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and this strip PQ found above slides fromx = 0 to x = 4a.

When we changing the order of integration, we make strip RS parallel to x-axis and this strip is made from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ and slides from y = 0 to y = 4a.



Hence, on reversing the order of integration, we get

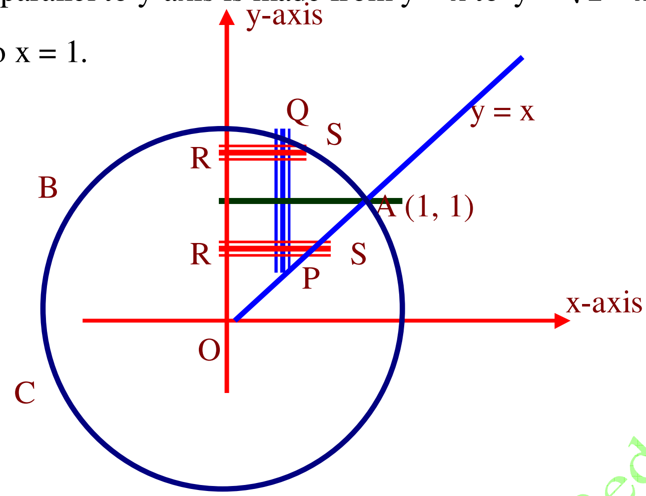
$$I = \int_0^{4a}\left(\int_{y^2/4a}^{2\sqrt{ay}} dx\right)dy = \int_0^{4a}\left[2\sqrt{a}\ \ y^{1/2} - \frac{1}{4a}y^2\right]dy = 2\sqrt{a}\left(\frac{y^{3/2}}{\frac{3}{2}}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{1}{12a}.64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3}.\ \text{Ans.}$$

Q.No.6.: Evaluate the integral by changing the order of integration

$$\int_0^1\left(\int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{x^2+y^2}}dy\right)dx.$$

Sol.: Here the strip PQ parallel to y-axis is made from y = x to $y = \sqrt{2 - x^2}$, and this strip PQ slides from x = 0 to x = 1.



When we change the order of integration,strip RS has nonviable character.

When we change the order of integration, we first integrate w. r. t. x along a horizontal strip RS and that requires the splitting up of the region OAB into two parts by the line AC(y = 1), the triangle OAC and thecurvilinear triangle ABC.

For the region OAC, the limits of integration for x are from x = 0 to $x = y$ and those for y are from y = 0 to y = 1. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 \left(\int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy .$$

For the region ABC, the limits of integration for x are from x = 0 to $x = \sqrt{2 - y^2}$ and for those for y are from y = 1 to $y = \sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy .$$

Hence, on reversing the order of integration, we get

$$I = \int_0^1 \left(\int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy + \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx \right) dy \quad .$$

Now $I_1 = \frac{1}{2}\int_0^1 \left(\int_0^y \frac{2x}{\sqrt{x^2+y^2}}dx\right)dy = \frac{1}{2}\int_0^1 2\left|\sqrt{x^2+y^2}\right|_0^y dy = \int_0^1 \left(\sqrt{2}\ y - y\right)dy$

$= \left[\sqrt{2}\frac{y^2}{2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{\sqrt{2}} - \frac{1}{2}$,

and $I_2 = \frac{1}{2}\int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{2x}{\sqrt{x^2+y^2}}dx\right)dy = \frac{1}{2}\int_1^{\sqrt{2}} 2\left|\sqrt{x^2+y^2}\right|_0^{\sqrt{2-y^2}} dy = \int_1^{\sqrt{2}} \left|\sqrt{2}-y\right|dy$

$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = 2-\sqrt{2}-\left(1-\frac{1}{2}\right) = \frac{3}{2}-\sqrt{2}$.

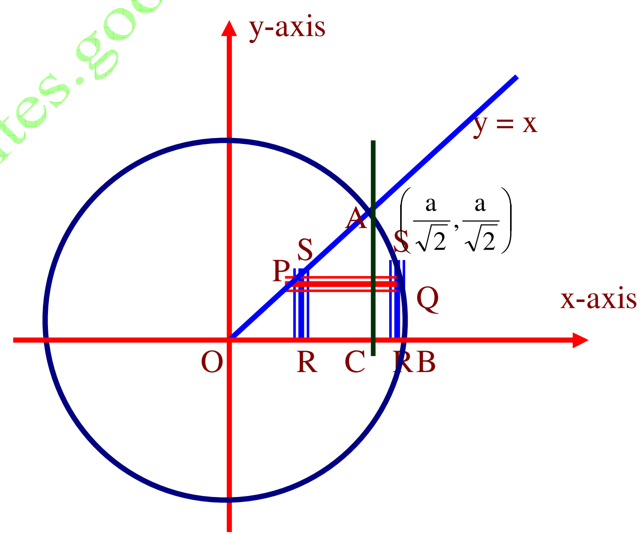
$\therefore$ The required integrals is

$I = I_1 + I_2 = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{3}{2}-\sqrt{2}\right) = \frac{1}{\sqrt{2}} + 1 - \sqrt{2} = \frac{1+\sqrt{2}-2}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$. Ans.

Q.No.7.: Evaluate the integral by changing the order of integration

$$\int_0^{a/\sqrt{2}} \left(\int_y^{\sqrt{(a^2-y^2)}} \log\left(x^2+y^2\right)dx\right)dy \qquad (a > 0).$$

Sol.:



Here the strip PQ parallel to x-axis is made from x = y to $x = \sqrt{a^2-y^2}$ and this strip slides from $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

When we change the order of integration, strip RS has nonviable character.

When we change the order of integration, we first integrate w. r. t. y along a vertical strip RS and that requires the splitting up of the region OAB into two partsby the line AC($y = \frac{a}{\sqrt{2}}$),, the triangle OAC and thecurvilinear triangle ABC.

For the region OAC, the limits of integration for y are from y = 0 to $y = x$ and those for y are from x = 0 to $x = \frac{a}{\sqrt{2}}$. So the contribution to I from the region OAC is

$$I_1 = \int_0^{a/\sqrt{2}} \left[\int_0^x \log\left(x^2 + y^2\right) dy \right] dx .$$

For the region ABC, the limits of integration for y are from y = 0 to $y = \sqrt{a^2 - x^2}$ and for those for x are from $x = \frac{a}{\sqrt{2}}$ to x =a. So the contribution to I from the region ABC is

$$I_2 = \int_{a/\sqrt{2}}^{a} \left[\int_0^{\sqrt{a^2-x^2}} \log\left(x^2 + y^2\right) dy \right] dx .$$

Hence, on reversing the order of integration, we get

$$I = \int_0^{a/\sqrt{2}} \left(\int_y^{\sqrt{a^2-y^2}} \log\left(x^2 + y^2\right) dx \right) dy$$

$$= \int_0^{a/\sqrt{2}} \left[\int_0^x \log\left(x^2 + y^2\right) dy \right] dx + \int_{a/\sqrt{2}}^{a} \left[\int_0^{\sqrt{a^2-x^2}} \log\left(x^2 + y^2\right) dy \right] dx = I_1 + I_2$$

$\Rightarrow I = I_1 + I_2$ [say]

where $I_1 = \int_0^{a/\sqrt{2}} \left[\int_0^x \log\left(x^2 + y^2\right) dy \right] dx$ and $I_2 = \int_{a/\sqrt{2}}^{a} \left[\int_0^{\sqrt{a^2-x^2}} \log\left(x^2 + y^2\right) dy \right] dx$

Now first evaluate $\int \log\left(x^2 + y^2\right) dy$: Integrating by parts, we get

$$\int \log\left(x^2+y^2\right)dy = \left\{\log\left(x^2+y^2\right)\right\}y - \int \frac{2y}{x^2+y^2}.ydy$$

$$= y\log\left(x^2+y^2\right) - 2\int \frac{y^2+x^2-x^2}{x^2+y^2}dy = y\log\left(x^2+y^2\right) - 2\int \left\{1-\frac{x^2}{x^2+y^2}\right\}dy$$

$$= y\log\left(x^2+y^2\right) - 2\left\{y - x^2.\left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)\right\} = y\log\left(x^2+y^2\right) - 2y + 2x\tan^{-1}\frac{y}{x}$$

Now $I_1 = \int_0^{a/\sqrt{2}} \left[y\log\left(x^2+y^2\right) - 2y + 2x\tan^{-1}\frac{y}{x}\right]_0^x dx$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2x^2 - 2x + 2x\tan^{-1}(1) - 0\right]dx$$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2 - 2x\log x - 2x + \frac{2x\pi}{4}\right]dx$$

$$= \left[\log 2.\frac{x^2}{2} - \left\{2\log x.\frac{x^2}{2} - \int_0^{a/\sqrt{2}} \frac{2}{x}.\frac{x^2}{2}dx\right\} - 2\frac{x^2}{2} + \frac{\pi}{2}.\frac{x^2}{2}\right]_0^{a/\sqrt{2}}$$

$$= \left[\left(\log 2 - 2 + \frac{\pi}{2} + 2\log x\right)\frac{x^2}{2}\right]_0^{a/\sqrt{2}} - \int_0^{a/\sqrt{2}} \frac{2}{x}.\frac{x^2}{2}dx$$

$$= \left(\log 2 - 2 + \frac{\pi}{2} + 2\log\frac{a}{\sqrt{2}}\right)\frac{a^2}{4} - \left[\frac{x^2}{2}\right]_0^{a/\sqrt{2}} = \left(\log 2 - 2 + \frac{\pi}{2} + 2\log\frac{a}{\sqrt{2}}\right)\frac{a^2}{4} - \frac{a^2}{4}$$

$$= \frac{a^2}{8}\left(2\log 2 - 4 + \pi + 4\log\left(\frac{a}{\sqrt{2}}\right) - 2\right) = \frac{a^2}{8}\left(2\log 2 + 4\log a - 2\log 2 - 6 + \pi\right)$$

$$I_1 = \frac{a^2}{8}\{\pi + 4\log a - 6\} \qquad \text{(i)}$$

Similarly, $I_2 = \int_{a/\sqrt{2}}^{a} \left[y\log\left(x^2+y^2\right) - 2y + 2x\tan^{-1}\frac{y}{x}\right]_0^{\sqrt{a^2-x^2}} dx$

$$= \int_{a/\sqrt{2}}^{a} \left\{\sqrt{a^2-x^2}\log a^2 - 2\sqrt{a^2-x^2} + 2x\tan^{-1}\frac{\sqrt{a^2+x^2}}{x}\right\}dx$$

$$=\left(\log a^2-2\right)\int_{a/\sqrt{2}}^{a}\sqrt{a^2-x^2}\,dx+2\int_{a/\sqrt{2}}^{a}x\tan^{-1}\frac{\sqrt{a^2+x^2}}{x}dx$$

$$=\left(\log a^2-2\right)\int_{a/\sqrt{2}}^{a}\sqrt{a^2-x^2}\,dx+I_3,\quad \text{where } I_3=2\int_{a\sqrt{2}}^{a}x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}dx$$

$$I_2=\left[\left(\log a^2-2\right)\times\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}+I_3.$$

Now evaluate $I_3=2\int_{a\sqrt{2}}^{a}x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}dx$.

Let $x=a\cos\theta \therefore dx=-a\sin\theta d\theta$

Now when $x=\frac{a}{\sqrt{2}}\Rightarrow a\cos\theta=\frac{a}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4}$

and $x=a\Rightarrow a\cos\theta=a\Rightarrow\theta=0$.

$$\therefore I_3=2\int_{\pi/4}^{0}a\cos\theta\tan^{-1}\frac{\sqrt{a^2\left(1-\cos^2\theta\right)}}{a\cos\theta}\times-a\sin\theta d\theta=-a^2\int_{\pi/4}^{0}2\sin\theta\cos\theta\tan^{-1}(\tan\theta)d\theta$$

$$=-a^2\int_{\pi/4}^{0}\theta\sin 2\theta d\theta=a^2\int_{0}^{\pi/4}\theta\sin 2\theta d\theta=a^2\left[\frac{-\theta\cos 2\theta}{2}\right]_0^{\pi/4}+a^2\int_0^{\pi/4}\frac{1.\cos 2\theta}{2}d\theta$$

$$=a^2[0-0]_0^{\pi/4}+a^2\left[\frac{\sin 2\theta}{4}\right]_0^{\pi/4}=\frac{a^2}{4}[1-0]=\frac{a^2}{4}.$$

$$\text{Thus}\quad I_2=\left[\left(\log a^2-2\right)\times\left(\left(\frac{a}{2}\times 0\right)+\left(\frac{a^2}{2}.\frac{\pi}{2}\right)-\frac{a}{2\sqrt{2}}.\frac{a}{\sqrt{2}}-\frac{a^2}{2}.\frac{\pi}{4}\right)\right]+\frac{a^2}{4}$$

$$=(2\log a-2)\left(\frac{\pi a^2}{8}-\frac{a^2}{4}\right)+\frac{a^2}{4}\qquad\text{(ii)}$$

By (i) and (ii), we get

$$I=\left\{\frac{a^2}{8}\{\pi+4\log a-6\}\right\}+\left\{(2\log a-2)\left(\frac{\pi a^2}{8}-\frac{a^2}{4}\right)+\frac{a^2}{4}\right\}$$

$$I=\frac{a^2}{8}\{\pi+4\log a-6\}+\frac{a^2}{8}\{2\pi\log a-2\pi-4\log a+4+2\}$$

$$=\frac{a^2}{8}\{\pi+4\log a-6+2\pi\log a-2\pi-4\log a+6\}=\frac{a^2}{8}\{2\pi\log a-\pi\}$$

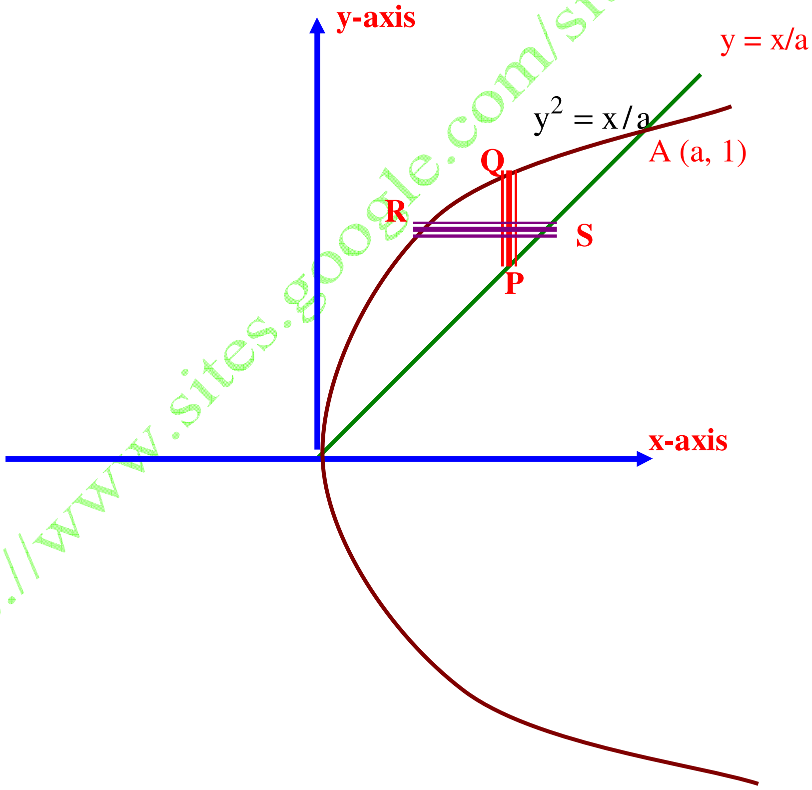
$$\therefore I=\frac{\pi a^2}{4}\left(\log a-\frac{1}{2}\right). \text{ Ans.}$$

Q.No.8.: Evaluate the integral by changing the order of integration

$$\int_0^a\left(\int_{x/a}^{\sqrt{(x/a)}}\left(x^2+y^2\right)dy\right)dx \quad .$$

Sol.: The strip PQ parallel to y-axis is made from $y=\frac{x}{a}$ to $y=\sqrt{\frac{x}{a}}$ and this strip PQ slides from $x = 0$ to $x = a$.

When we change the order of integration, the strip is changed from PQ to RS which is now parallel to x-axis, is made from $x=ay^2$ to $x = ay$ and slides from $y = 0$ and $y = 1$.



Hence, on reversing the order of integration, we get

$$I=\int_0^1\left[\int_{ay^2}^{ay}\left(x^2+y^2\right)dx\right]dy=\int_0^1\left[\frac{x^3}{3}+y^2x\right]_{ay^2}^{ay}dy$$

$$=\int_0^1\left[\frac{a^3y^3-a^3y^6}{3}\right]dy+\int_0^1\left[ay^3-ay^4\right]dy=\frac{a^3}{3}\left[\frac{y^4}{4}-\frac{y^7}{7}\right]_0^1+a\left[\frac{y^4}{4}-\frac{y^5}{5}\right]_0^1$$

$$=\frac{a^3}{28}+\frac{a}{20}.\ \text{Ans.}$$

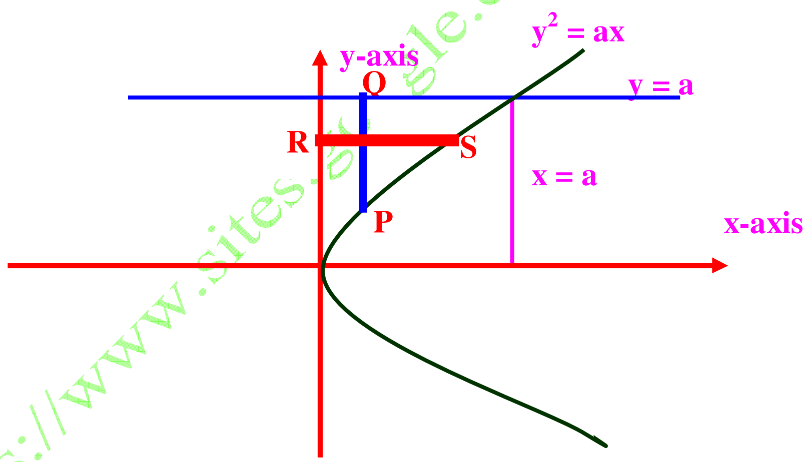
Q.No.9.: Evaluate the integral by changing the order of integration

$$\int_0^a\int_{\sqrt{ax}}^a\frac{y^2dydx}{\sqrt{y^4-a^2x^2}}.$$

Sol.: Given integral is $\int_0^a\int_{\sqrt{ax}}^a\frac{y^2dydx}{\sqrt{y^4-a^2x^2}}=\int_0^a\left(\int_{\sqrt{ax}}^a\frac{y^2}{\sqrt{y^4-a^2x^2}}dy\right)dx$

The strip PQ parallel to y-axis is made from $y=\sqrt{ax}$ to $y=a$, and this strip PQ slides from x = 0 to x = a.

When we change the order of integration, the strip is changed from PQ to RS which is now parallel to x-axis, is made from x = 0 to $x=\frac{y^2}{a}$ and slides from y = 0 and y = a.



Hence, on reversing the order of integration, we get

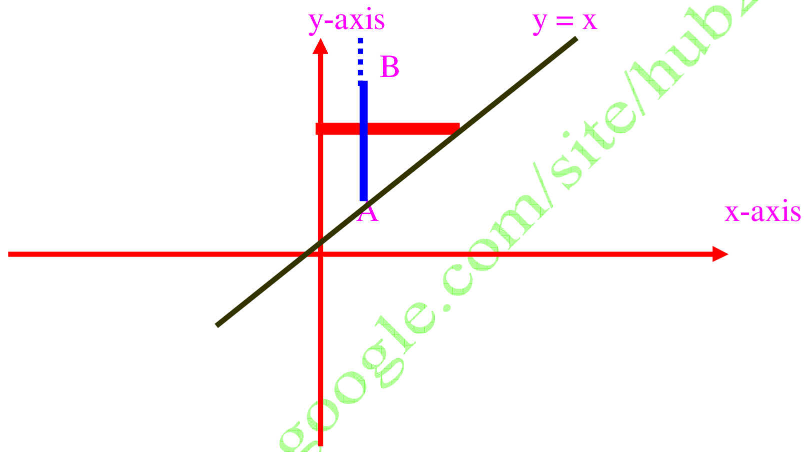
$$I=\int_0^a\left[\int_0^{y^2/a}\frac{dx}{\sqrt{1-\frac{a^2x^2}{y^4}}}\right]dy=\int_0^a\left[\frac{y^2}{a}\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a}dy$$

$$= \int_0^a \left[\frac{y^2}{a} \left\{ \sin^{-1}\left(\frac{a}{y^2} . \frac{y^2}{a} \right) - \sin^{-1} 0 \right\} \right] dy = \int_0^a \frac{y^2}{a} \sin^{-1} 1 dy = \int_0^a \frac{y^2}{a} . \frac{\pi}{2} dy$$

$$= \frac{\pi}{2a} \left[\frac{y^3}{3} \right]_0^a = \frac{\pi a^2}{6} . \text{ Ans.}$$

Q.No.10.: Evaluate the integral by changing the order of integration $\int_0^\infty \left(\int_x^\infty \frac{e^{-y}}{y} dy \right) dx$.

Sol.: The elementary strip given in the problem is a strip AB parallel to the y-axis whose one end lies on the line y = x and the other end extends to infinity (i.e. $y \to \infty$), and this strip slides from x = 0 to $x \to \infty$.



On changing the order of the above problem, we first integrate it along a horizontal strip which extends from x = 0 to x = y, and this strip slides from y = 0 to $y \to \infty$.

(The region of integration is the area above the line x = y which extends up to infinity to cover this area)

Hence, on reversing the order of integration, we get

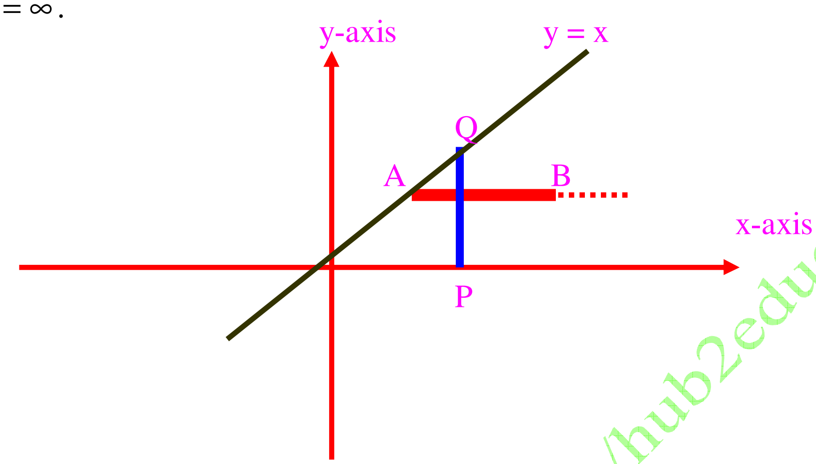
$$I = \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y} dx \right] dy = \int_0^\infty \left[\frac{e^{-y}}{y} . x \right]_0^y dy = \int_0^\infty \frac{e^{-y}}{y} (y - 0) dy$$

$$= \int_0^\infty e^{-y} dy = \left| e^{-y} \right|_\infty^0 = e^{-0} - e^{-\infty} = 1 - \frac{1}{e^\infty} = 1 - 0 = 1 . \text{ Ans.}$$

Q.No.11.: Evaluate the integral by changing the order of integration

$$\int_0^{\infty}\left(\int_0^{x} xe^{-x^2/y}dy\right)dx .$$

Sol.: The region of integration or the given integral is the area bounded by y = 0, y = x and x = 0, x = ∞.



The elementary strip given in the problem is a strip PQ parallel to the y-axis whose one end lies on the line y= 0 and other end lies on the line y = x and this strip slides from x = 0 to $x \to \infty$.

On changing the order of the above problem, we first integrate it along a horizontal strip which extends from x = 0 to $x \to \infty$, and this strip slides from y = 0 to $y \to \infty$.

(The region of integration is the area below the line x = y which extends up to infinity to cover this area)

Hence, on reversing the order of integration, we get

$$I = \int_0^{\infty}\left(\int_y^{\infty} xe^{-x^2/y}dx\right)dy$$

Let $e^{-x^2/y} = t \Rightarrow \frac{-2x}{y}e^{-x^2/y}dx = dt \Rightarrow xe^{-x^2/y}dx = -\frac{y}{2}dt$

Also when x = y, $t = e^{-y}$ and when x = ∞, t = 0.

$$\therefore I = \int_0^{\infty}\left(\int_y^{\infty} xe^{-x^2/y}dx\right)dy = \int_0^{\infty}\left(\int_{e^{-y}}^{0}\left(-\frac{y}{2}\right)dt\right)dy = \int_0^{\infty}\left[-\frac{y}{2}.t\right]_{e^{-y}}^{0} dy = \int_0^{\infty}\frac{1}{2}ye^{-y}dy$$

$$= \frac{1}{2}\left[-y.e^{-y} + \int e^{-y}dy\right]_0^{\infty} = \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^{\infty} = \frac{1}{2}. \text{ Ans.}$$

Q.No.12.: Change the order of integration in the following

(i) $\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dydx$ **,** **(ii).** $\int_{0}^{1}\int_{x^2}^{\sqrt{2-x^2}} f(x,y)dydx$.

Sol.: (i) The region of integration is one region bounded by the curves $x^2+y^2=1$ and $x+y=1$.

Initially the strip was parallel to x-axis having its two ends on $x=-\sqrt{1-y^2}$ and $x=1-y$, sliding from 0 to 1.

Changing the order of integration:

Taking one strip parallel to y-axis. Now, the strip will slide parallel to y-axis having its two ends on y = 0 and $x^2+y^2=1$ from $x=-1$ to x = 0. This will give us the integral

$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x,y)dydx$. After that when it will coincide with y-axis it will start sliding with its two ends on y = 0 and $y=1-x$. In the positive direction of x., which will give us the integral $\int_{0}^{1}\int_{0}^{1-x} f(x,y)dydx$ as it moves from x = 0 to x = 1.

Thus the integral becomes

$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x,y)dydx+\int_{0}^{1}\int_{0}^{1-x} f(x,y)dydx.$$

(ii) The region of given interval is bounded by $y=x^2$,

$y=\sqrt{2-x^2}\Rightarrow y^2=2-x^2\Rightarrow x^2+y^2=\left(\sqrt{2}\right)^2$, and x = 0, x = 1.

The graph of the region is the area bounded by the curve is OAB. When we change the order of integration in the given interval, we have to first integrate w. r. t. x y as constant and then w. r. t. y. this is done by covering the area OAB by drawing an elementary strip PQ parallel to x-axis and then moving it parallel to x-axis as to cover whole of the area. On the strip PQ y is constant and x varies first from x = 0 to $y=\sqrt{y}$ and then from x = 0 to $x=\sqrt{2-y^2}$. And when the strip PQ moves parallel to x-axis, so as to cover the whole area, y vanishes first from y = 0 to y = 1 and then from y = 1 to $y=\sqrt{y}$.

Thus the integral becomes

$$\int_0^1\left[\int_{x^2}^{\sqrt{y}}(x,y)dx\right]dy+\int_0^{\sqrt{2}}\left[\int_{x^2}^{\sqrt{2-y^2}}(x,y)dx\right]dy\text{. Ans.}$$

Q.No.13.: Evaluate the following integral by changing the order of integration

$$\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}x^3ydxdy\,.$$

Sol.: The region of the given integral is bounded by x = 0 and $x=\frac{a}{b}\sqrt{b^2-y^2}$

$$\Rightarrow x^2b^2=a^2\left(b^2-y^2\right)=\frac{x^2b^2}{a^2}=b^2-y^2\Rightarrow\frac{x^2b^2}{a^2}+y^2=b^2$$

$$\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

This equation represent an ellipse.

Now to change the order

$$b^2x^2+a^2y^2=a^2b^2\Rightarrow a^2y^2=a^2b^2-b^2x^2\Rightarrow a^2y^2=b^2\left(a^2-x^2\right)$$

$$\Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}\,.$$

$$\therefore\text{Integral becomes }\int_0^a\left[\int_0^{\frac{a}{b}\sqrt{a^2-x^2}}x^3ydy\right]dx\text{. Ans.}$$

$$I=\int_0^a\left[\int_0^{\frac{a}{b}\sqrt{a^2-x^2}}x^3ydy\right]dx=\int_0^a\left[x^3\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\right]dx=\int_0^a\left\{\frac{x^3}{2}\left[\frac{b^2}{a^2}\left(a^2-x^2\right)\right]\right\}dx$$

$$=\int_0^a\left\{\frac{x^3}{2}\left[\frac{b^2\left(a^2-x^2\right)}{a^2}\right]\right\}dx=\int_0^a\left\{\frac{x^3}{2}\left[\frac{\left(a^2-b^2x^2\right)}{a^2}\right]\right\}dx=\frac{1}{2}\int_0^a\left\{x^3b^2-\frac{b^2x^5}{a^2}\right\}dx$$

$$=\frac{1}{2}\left[b^2\left[\frac{x^4}{4}\right]_0^a-\frac{b^2}{a^2}\left[\frac{x^6}{6}\right]_0^a\right]=\frac{1}{2}\left[\frac{b^2a^4}{4}-\frac{b^2}{a^2}\frac{a^2a^4}{6}\right]=\frac{1}{2}b^2a^4\left[\frac{1}{4}-\frac{1}{6}\right]$$

$$=\frac{1}{2}b^2a^4\times\frac{2}{24}=\frac{a^4b^2}{24}. \text{ Ans.}$$

Q.No.14.: Determine the limit of integration for $\iint\limits_R f(x,y)dydx$, where the region is bounded by $y = 0,\ y = 1-x^2$ and hence change the order.

Sol.: The region of the given integral is bounded by $y = 0$ and $y = 1-x^2 \Rightarrow x^2 = 1-y$.

Thus the shaded region is the region of integration. To find the limit of integral $\iint\limits_R f(x,y)dydx$.

Let us take an element strip parallel to y-axis. Now moving the strip parallel to y-axis so as to cover the whole of the area. On the strip parallel to y-axis, the x is constant and y-varies from y = 0 to $y = 1-x^2$ and when the strip is parallel to x-axis and is moved parallel to x-axis, so as to cover whole of the area.

Thus the integral becomes.

$$\int_{-1}^{1}\left[\int_{0}^{1-x^2} f(x,y)dy\right]dx.$$

Now we have to change the limit of integration (change of order).

Now taking a strip parallel to x-axis and moving it parallel to x-axis so as to cover the whole area. Now when we change the order of integration in the given integral, we have to first integrate w. r. t. x keeping y as constant and then w. r. t. y.

As y is constant. Therefore x varies from $x = -\sqrt{1-y}$ to $x = \sqrt{1-y}$, and when the strip is parallel to y-axis, the strip moved parallel to y-axis. So as to cover the whole area. Therefore y varies from y = 0 to y = 1.

Thus the equation becomes

$$\int_{0}^{1}\left[\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx\right]dy.$$

Q.No.15.: Evaluate the following integral by changing the order of integration

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)dxdy .$$

Sol.: $I=\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)dxdy$ [given]

In the given equation the elementary strip is parallel to x-axis (say AB) and is from x = 0 to $x=\sqrt{a^2-y^2}$

And from $x=\sqrt{a^2-y^2} \Rightarrow x^2=a^2-y^2$ which is a circle with radius O.

The same strip is from $y=-a$ to $y = a$.

So area of integration is the area covered by the semicircle (shaded portion)

Now on changing the order of integration we will first integrate w. r. t. y along the vertical strip CD. This vertical is from $y=-\sqrt{a^2-x^2}$ to $y=-\sqrt{a^2-x^2}$ along the strip parallel to x-axis, we will integrate from x = 0 to x = a.

So the changed order of integration is

$$\int_{0}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)dy\right]dx .$$

Q.No.16.: Evaluate the following integral by changing the order of integration

$$\int_{0}^{4}\int_{0}^{x(4-x)} dydx .$$

Sol.: Given $I=\int_{0}^{4}\int_{0}^{x(4-x)} dydx$.

Here strip is parallel to y-axis. Such PQ and extended from y = 0 to $y=x(4-x)$

{i. e. parabola $(y-4)=-(x-2)^2$ }., and this strip moves from x = 0 to x =4.

Now this shaded region is area of integration. On changing the order of integration we first integrate w. r. t. x along horizontal strip RS. Now, this strip will move

$(x-2)^2=(y-4)\Rightarrow(x-2)=\pm\sqrt{4-y}$ $x=2\pm\sqrt{4-y}$, from $x = 2-\sqrt{4-y}$ to $x = 2+\sqrt{4-y}$. So to cover the given region we then integrate w. r. t. y from y = 0 to y = 4.

So, we get

$$I = \int_0^4 \left[\int_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dx\right] dy = \int_0^4 [x]_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} dy = \int_0^4 \left[2+\sqrt{4-y}-2+\sqrt{4-y}\right] dy$$

$$= \int_0^4 2\sqrt{4-y}\,dy = 2.\int_0^4 \sqrt{4-y}\,dy.$$

Put $4-y = t$, $\quad \dfrac{dt}{dy} = -1 \Rightarrow -dt = dy$

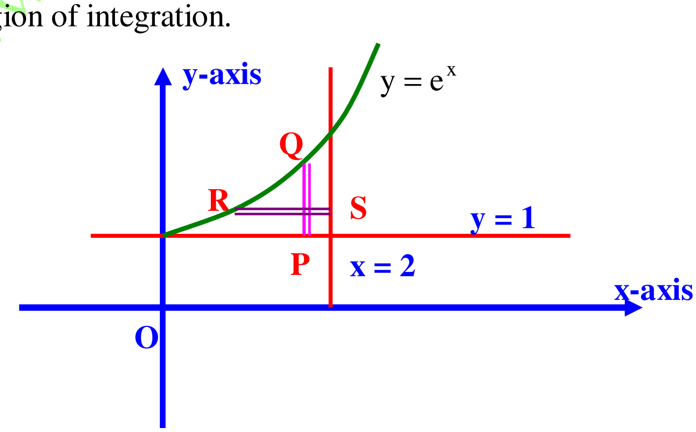
$$\Rightarrow I = -2\int_0^4 \sqrt{t}\,dt = -2\left[\frac{t}{\frac{1}{2}+1}\right]_0^4 = \frac{-2\times 2}{3}\left[t^{\frac{3}{2}}\right]_0^4 = \frac{-4}{3}\left(4^{3/2}\right) = \frac{-4\times 8}{3} = \frac{-32}{3}.$$

Now neglect negative sign, we get $\dfrac{32}{3}$ sq. unit is the required result.

Q.No.17.: Evaluate the following integral by changing the order of integration

$$\int_0^2 \int_1^{e^x} dydx.$$

Sol.: Here the elementary strip is parallel to y-axis (such as PQ) and extends from $y_1 = 1$ to $y_2 = e^x$ and this strip slides from $x_1 = 0$ to $x_2 = 2$. This shaded semicircle, area is therefore, the region of integration.



On changing the order of integration, we first integrate w. r. t. x along a strip RS which extends from. $R[x_1 = \log y]$. To cover the given region, we then integrate w. r. t. y from $y_1 = 1$ to $y_2 = e^2$.

So the changed order of integration is $\int_1^{e^2}\int_{\log y}^{2} dxdy = \int_1^{e^2}\left(\int_{\log y}^{2} dx\right)dy$.

$$I = \int_1^{e^2}\left[\int_{\log y}^{2} dx\right]dy = \int_1^{e^2}[x]_{\log y}^{2} dy = \int_1^{e^2}(2-\log y)dy = \int_1^{e^2} 2dy - \int_1^{e^2} \log y dy$$

$$=(2e^2-2)-\int_1^{e^2} \log y.1dy = (2e^2-2)-\left[\log y\int dy - \int \frac{d}{dy}(\log y)\int 1.dy\right]$$

$$=(2e^2-2)-[y\log y - y]_1^{e^2} = (2e^2-2)-[e^2\log e^2 - e^2 - 1\log 1 + 1]$$

$$=2e^2-2-2e^2+e^2-1 = e^2-3 \quad \left[\begin{array}{l}\because \log 1 = 0\\ \log e^2 = 2\end{array}\right]$$

$$\int_1^{e^2}\int_{\log y}^{2} dxdy = e^2-3. \text{ Ans.}$$

Q.No.18.: Evaluate the following integral by changing the order of integration

$$\int_0^1\int_{e^x}^{e}\frac{1}{\log y}dydx.$$

Sol.: $I = \int_0^1\int_{e^x}^{e}\frac{dy}{\log y}dx$.

First we will have to change the order of integral. Taking strip parallel to x-axis, we get the limits from 0 to log y and for dy limits to be taken from 1 to e.

$$I = \int_1^{e}\int_0^{\log y}\frac{dx}{\log y}dy = \int_1^{e}\frac{x}{\log y}dy \quad [\text{Where } x = \log y]$$

$$=\int_1^{e}\frac{dy}{\log y}[x]_0^{\log y} = \int_1^{e}\frac{\log y}{\log y} = \int_1^{e} dy = [y]_1^{e} = e-1. \text{ Ans.}$$

Q.No.19.: Evaluate $\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}$ by changing the orderof integration.

Sol.: $I=\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}=\int_0^a \left[\int_y^a \frac{x}{x^2+y^2}dx\right]dy$.

Here $x=y$, $x=a$ and $y=a$, $y=0$.

As strip AB moves from 0 to x.

And ‘x’ changes from 0 to a

$$\therefore I=\int_0^a \left[x\int_0^x \frac{dy}{x^2+y^2}\right]dx=\int_0^a \left[x\int_0^x \frac{1}{x}\tan^{1}\frac{y}{x}\right]dx=\int_0^a \left[x\times\frac{1}{x}\tan^{1}\frac{y}{x}\right]_0^x dx$$

$$=\int_0^a \left(\tan^{1}\frac{x}{x}-\tan^{-1}0\right)dx=\int_0^a \frac{\pi}{4}dx=\frac{\pi}{4}\int_0^a dx=\frac{\pi}{4}[x]_0^a=\frac{\pi}{4}a. \text{ Ans.}$$

Q.No.20.: Evaluate $\int_0^a \int_{y^2/a}^{y} \frac{ydxdy}{(a-x)\sqrt{ax-y^2}}=\frac{1}{2}\pi a$ by changing the orderof integration.

Sol.: The required area of integration is OABC. In the problem we have to first integrate first w. r. t. a horizontal strip PQ w. r. t. x on the parabola y^2-ax and line y = x. The required region g integration is OABC.

To solve this problem, we have to change the order g integration i. e. we first integrate w. r. t. y along vertical strip $P'Q'$ and then split the area OABPO from that area. Thus for region OABCO, the limit of integration w. r. t. $y=0$, $y=\sqrt{ax}$ and y = 0 and y = x and then for x it is for x = 0 to x = a.

Thus the required area.

$$A=\int_0^a \int_{y^2/a}^{y} \frac{ydxdy}{(a-x)\sqrt{ax-y^2}}=\int_0^a \int_0^{\sqrt{ax}} \frac{ydydx}{(a-x)\sqrt{ax-y^2}}-\int_0^a \int_0^x \frac{ydydx}{(a-x)\sqrt{ax-y^2}}$$

$$=\int_0^a \left[\frac{-\sqrt{ax-y^2}}{a-x}\right]_0^{\sqrt{ax}} dx-\int_0^a \left[\frac{-\sqrt{ax-y^2}}{a-x}\right]_0^x dx$$

$$= \int_0^a \left[\frac{-\sqrt{ax-y^2}}{a-x} + \frac{\sqrt{ax}}{a-x} \right] dx - \int_0^a \left[\frac{-\sqrt{ax-y^2}}{a-x} + \frac{\sqrt{ax}}{a-x} \right] dx$$

$$= \int_0^a \frac{\sqrt{ax}}{a-x} - \frac{\sqrt{ax}}{a-x} + \frac{-\sqrt{ax-y^2}}{a-x} dx = \int_0^a \sqrt{x}\frac{\sqrt{a-x}}{a-x} dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}} dx$$

We have the property of definite integral.

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$2A = \int_0^a \left(\frac{\sqrt{a-x}}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{a-x}} \right) dx = \int_0^a \left(\frac{a-x+x}{\sqrt{x}\sqrt{a-x}} \right) dx = \int_0^a \left(\frac{a}{\sqrt{x}\sqrt{a-x}} \right) dx.$$

Put $x = a\sin^2\theta$, $\therefore dx = 2a\sin\theta\cos\theta d\theta$.

$$2A = \int_0^{\pi/2} \frac{a \times 2a\sin\theta\cos\theta}{a\sin\theta\cos\theta} d\theta = \int_0^{\pi/2} 2a d\theta$$

$$\Rightarrow A = \int_0^{\pi/2} a d\theta = \frac{\pi}{2} a,$$ which is the required proof.

Q.No.21.: Show by an example that the interchangethe orderof integrationwill not always give the same result.

Sol.: Consider the following two integrals

(i) $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy$ (ii) $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx$

Let us evaluate (i):

$$\int_0^1 \frac{x-y}{(x+y)^3} dx = \int_0^1 \frac{x+y-2y}{(x+y)^3} dx = \int_0^1 \frac{1}{(x+y)^2} dx - \int_0^1 \frac{2y}{(x+y)^3} dx$$

$$= \frac{y}{(1+y)^2} - \frac{1}{(1+y)} = \frac{1}{(1+y)^2}.$$

Using the above result in (i), we get $\displaystyle\int_0^1 \frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y} \right]_{y=0}^{y=1} = -\frac{1}{2}$

So $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy = -\frac{1}{2}$. (A)

In similar manner, evaluating (ii), we get $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx = \frac{1}{2}$ (B)

From (A) and (B) we see that the interchange of the order of integration will not always give the same result.

Q.No.22.: Change the order of the integration and then evaluate: $\int_0^{2a} \int_0^{\sqrt{2ay-y^2}} dxdy$.

Sol.: The given integral is $\int_0^{2a} \left(\int_0^{\sqrt{2ay-y^2}} dx \right) dy$

$$= \int_0^a \left(\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} dy \right) dx = \int_0^a [y]_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} = 2\int_0^a \sqrt{a^2-x^2} dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$.

Also, when x = 0, $\theta = 0$ and when x = a, then $a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}$.

Thus $2\int_0^a \sqrt{a^2-x^2} dx = 2\int_0^{\pi/2} a^2\cos^2\theta d\theta = 2a^2 \times \frac{\pi}{4} = \frac{\pi a^2}{2}$. Ans.

Q.No.23.: Change the order of the integration and then evaluate:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} dydx$$

Sol.: The given integral is $\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} dy \right) dx$

$$= \int_0^a \left(\int_a^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx \right) dy$$

$$= \int_0^a \left(\frac{x}{2}\sqrt{a^2-x^2-y^2} + \left(\frac{\sqrt{a^2-y^2}}{2} \right)^2 \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}} \right) \right)_0^{\sqrt{a^2-y^2}} dy$$

$$=\int_0^a \frac{\pi}{2}\left(\frac{a^2-y^2}{2}\right)dy=\frac{\pi}{4}\int_0^a\left(a^2y-\frac{y^3}{3}\right)dy$$

$$=\frac{\pi}{4}\left(a^3-\frac{a^3}{3}\right)=\frac{\pi}{4}\left(\frac{2a^3}{3}\right)=\frac{\pi a^3}{6}\text{.Ans.}$$

Q.No.24.: Change the order of the integration and then evaluate: $\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$.

Sol.: The given integral is $\int_0^{\sqrt{2}}\left(\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydx\right)dy=\int_{-2}^{+2}\left(\int_0^{\sqrt{2-\frac{x^2}{2}}} ydy\right)dx=\int_{-2}^{+2}\left(\frac{y^2}{2}\right)_0^{\sqrt{2-\frac{x2}{2}}}dx$

$$=\int_{-2}^{+2}\frac{1}{2}\times\left(2-\frac{x^2}{2}\right)dx=\int_{-2}^{+2}\left(1-\frac{x^2}{4}\right)dx=\left|x-\frac{x^3}{12}\right|_{-2}^{+2}$$

$$=2-(-2)-\left(\frac{8}{12}-\left(-\frac{8}{12}\right)\right)=4-\left(2\times\frac{8}{12}\right)=4-\frac{4}{3}=\frac{8}{3}\text{. Ans.}$$

Q.No.25.: Change the order of the integration and then evaluate: $\int_0^a\int_{y^2/a}^{2a-y} xydxdy$

Sol.: The given integral is $\int_0^a\int_{y^2/a}^{2a-y} xydxdy$

Here $x=\frac{y^2}{a}\Rightarrow y^2=ax$ and $x=2a-y\Rightarrow x+y=2a$.

Initially strip is parallel to x-axis

For change of order consider a take strip which is parallel to y-axis.

So we have the following regions for integration

For I_1 y varies from 0 to $\sqrt{ax}$

x varies from 0 to a

For I_2 y varies from 0 to $2a-x$

x varies from a to 2a

$$\therefore\int_0^a\int_{y^2/a}^{2a-y} xydxdy=\int_a^a\left(\int_0^{\sqrt{ax}} xydy\right)dx+\int_a^{2a}\left(\int_0^{2a-x} xydy\right)dx$$

$$= \int_0^a \left|\frac{xy^2}{2}\right|_0^{\sqrt{ax}} dx + \int_0^{2a} \left|\frac{xy^2}{2}\right|_0^{2a-x} dx = \int_0^a \frac{ax^2}{2} dx + \int_a^{2a} \frac{x(2a-x)^2}{2} dx$$

$$= \left|\frac{ax^3}{6}\right|_0^6 + \int_a^{2a} \left(\frac{4a^2x + x^3 - 4ax^2}{2}\right) dx = \frac{a^4}{6} + \left|\frac{4a^2x^2}{4} + \frac{x^4}{8} + \frac{4ax^3}{6}\right|_a^{2a}$$

$$= \frac{a^4}{6} + \left[3a^4 + \frac{15}{8}a^4 - \frac{14}{3}a^4\right] = \frac{19}{6}a^4 - \frac{28a^4}{6} + \frac{15}{8}a^4 = -\frac{12a^4}{8} + \frac{15a^4}{8} = \frac{3a^4}{8}. \text{ Ans.}$$

Q.No.26.: Change the order of the integration and then evaluate: $\int_0^2 \int_{y^3}^{4\sqrt{2y}} y^2 dxdy$

Sol.: The given integral is $\int_0^2 \int_{y^3}^{4\sqrt{2y}} y^2 dxdy$

In the given problem, firstly, integration is done w.r.t. x and the w.r.t. y

Now let us take strip parallel to y-axis i.e. PQ

Now, PQ slides from $\frac{x^2}{32} \to x^{1/3}$

$$\text{Now} \int_0^2 \left(\int_{y^3}^{4\sqrt{2y}} y^2 dx\right) dy = \int_0^8 \left(\int_{x^2/32}^{x^{1/3}} y^2 dy\right) dx = \int_0^8 \left(\frac{y^3}{3}\right)_{x^2/32}^{x^{1/3}} dx$$

$$= \frac{1}{3}\int_0^8 \left(x - \frac{x^6}{(32)^3}\right) dx = \frac{1}{3}\left(\frac{x^2}{2} - \frac{x^7}{7.(32)^3}\right)_0^8 = \frac{1}{3}\left(32 - \frac{8^7}{7(32)^3}\right)$$

$$= \frac{32}{3}\left(1 - \frac{2}{7}\right) = \frac{32}{3} \times \frac{5}{7} = \frac{160}{21}. \text{ Ans.}$$

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