1^{st} Topic

Matrices

Definitions of special types of Matrices-I

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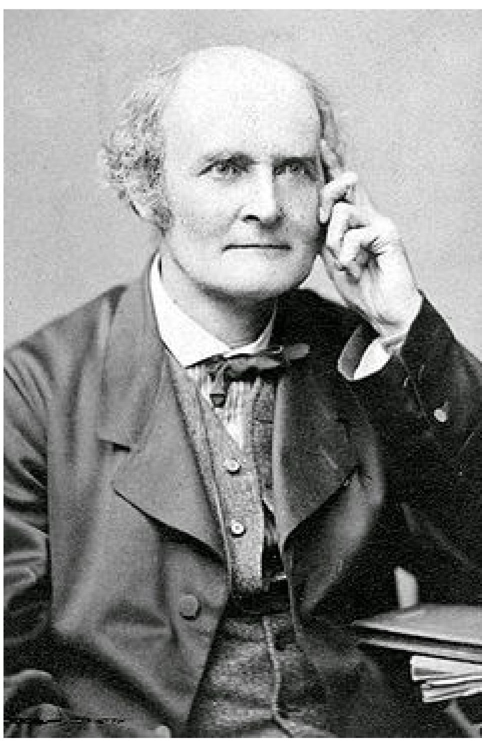
Introduction:

Cayley, a British mathematician discovered matrices in the year 1860. But it was not until the twentieth century was well advanced that engineers heard of them.



Arthur Cayley (16 August 1821 – 26 January 1895) was a British mathematician.
As a child, Cayley enjoyed solving complex math problems for enjoyment.
He entered Trinity College, Cambridge, where he excelled in Greek, French, German, and Italian, as well as mathematics. He worked as a lawyer for 14 years.
He proved the **Cayley-Hamilton theorem**—that every square matrix is a root of its own characteristic polynomial.

He was the first to define the concept of a group in the modern way—as a set with a binary operation satisfying certain laws.

During this period of his life, extending over fourteen years, Cayley produced/published between two and three hundred research papers.

A matrix is a rectangular array of numbers. These days, however, such arrays (matrices) have been found to be of great utility in

- **many branches of Applied mathematics such as algebraic and differential equations,**
- **mechanics,**
- **theory of electric circuits,**
- **nuclear physics,**
- **aerodynamics and**
- **astronomy.**
- **In many cases, they form the co-efficients of linear transformations or systems of linear equations arising, for instance, from electric network, frameworks in mechanics, curve fitting in statistics and transportation problems.**

Matrices are useful because they enable us to consider an array of many numbers as a single object, denote it by a single symbol, and perform calculations with these symbols in a very compact form. The mathematical shorthand thus obtained is very elegant and powerful and is suitable for various practical engineering problems. It entered engineering mathematics over seventy years ago and is of increasing importance of various engineering branches.

Therefore, it is necessary for the young engineers to learn the elements of matrix algebra in order to keep up with the fast development of physics and engineering.

Applications:

Matrices find many applications.

- **Physics** makes use of matrices in various domains, for example in geometrical optics and matrix mechanics; the latter led to studying in more detail matrices with an infinite number of rows and columns.
- **Graph theory** uses matrices to keep track of distances between pairs of vertices in a graph.
- **Computer graphics** uses matrices to project 3-dimensional space onto a 2-dimensional screen.

- **Matrix calculus** generalizes classical analytical notions such as derivatives of functions or exponentials to matrices. The latter is a recurring need in solving ordinary differential equations.
- **Serialism and dodecaphonism** are musical movements of the 20th century that use a square mathematical matrix to determine the pattern of music intervals.
- A major branch of **numerical analysis** is devoted to the development of efficient algorithms for matrix computations, a subject that is centuries old but still an active area of research.
- **Matrix decomposition methods** simplify computations, both theoretically and practically. For sparse matrices, specifically tailored algorithms can provide speedups; such matrices, for example, arise in the finite element method.

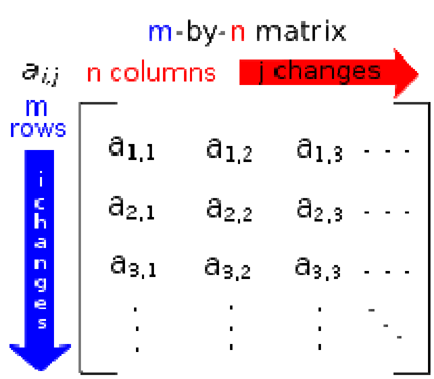
Matrix:

Definition: A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an $m \times n$ matrix.

[to be read as 'm by n' matrix.]

As $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}.$$



Specific entries of a matrix are often referenced by using pairs of subscripts.

In a compact form, the above matrix is represented by $A = \left[a_{ij}\right]_{m\times n}$, or simply by $\left[a_{ij}\right]_{m\times n}$, where i = 1, 2, ……,m, j = 1, 2, ……., n

We write the general element of the matrix and enclose it in brackets of the type [ ] or of the type ( ).

Elements of matrix: The numbers a_{11}, a_{12}, ….etc. of this rectangular array are called the elements of matrix. The element a_{ij} belongs to i^{th} row and j^{th} column and is sometimes called the $(i, j)^{th}$ element of the matrix.

Thus, in the element a_{ij} the first suffix i will always denote the number of the row and the second suffix j, the number of the column in which the element occurs.

Remarks: In a matrix, the number of rows and columns need not be equal.

SPECIAL TYPES OF MATRICES:

Square matrix:

Definition: An $m\times n$ matrix for which m = n (i.e., the number of rows is equal to the number of columns) is called a square matrix of order n.

It is also called an n-rowed square matrix.

Thus, in a square matrix, we have the same number of rows and columns.

Diagonal elements: The elements a_{ij} of square matrix $A = \left[a_{ij}\right]_{m\times n}$ for which i = j, i.e., the elements a_{11}, a_{22}, a_{33}, ……………, a_{nn} are called the **diagonal elements** and the line along which they lie is called the **principal diagonal** of the matrix.

Example: The matrix $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 \\ 5 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}_{4\times 4}$ is a square matrix of order 4.

The elements 0, 3, 1, 2 constitute the principal diagonal of this matrix.

Unit matrix or Identity matrix:

Definition: A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements is equal to zero is called a unit matrix or an identity matrix and is denoted by I. I_n **will denote a unit matrix of order n.**

Symbolically, a square matrix $A=\left[a_{ij}\right]$ **is a unit matrix if** $a_{ij}=1$ **when i = j**

and $a_{ij}=0$ **when** $i \neq j$**.**

Example: The matrices $I_4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$, $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

are unit matrices of order 4, 3, 2, respectively.

Null matrix or Zero matrix:

Definition: The $m\times n$ **matrix, whose elements are all 0, is called the null matrix (or zero matrix) of the type** $m\times n$**.**

It is usually denoted by O or more clearly $O_{m,n}$. Often a null matrix is simply denoted by the symbol 0 read as zero.

Example: The matrices $\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}_{3\times5}$ and $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}_{3\times3}$

are zero matrices of the type 3×5 and 3×3, respectively.

Row matrix and Column matrix:

Row matrix:

Any $1\times n$ **matrix, which has only one row and n columns, is called a row matrix or a row vector.**

Column matrix:

Any $m\times1$ **matrix, which has m rows and only one column, is a column matrix or a column vector.**

Examples: The matrix $X = \begin{bmatrix} 2 & 7 & -8 & 5 \end{bmatrix}_{1\times4}$ is a row matrix of the type 1×4.

The matrix $Y = \begin{bmatrix} 2 \\ -9 \\ 11 \end{bmatrix}_{3\times1}$ is a column matrix of the type 3×1.

Sub-matrix of a matrix:

Definition: Any matrix obtained by omitting some rows and columns from a given $(m\times n)$ matrix A is called a sub-matrix of A.

The matrix A itself is a sub-matrix of A as it can obtained from A by omitting no rows or no columns.

Example: The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ is a sub-matrix of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 7 & 11 & 6 & 5 \\ 0 & 2 & 1 & 8 \end{bmatrix}$

because it can be obtained from A by omitting the second row and the fourth column.

Principal sub-matrix of a matrix:

Definition: A square sub-matrix of a square matrix A is called a principal sub-matrix, if its diagonal elements are also the diagonal elements of matrix A. Principal sub-matrices are obtained only by omitting corresponding rows and columns.

Equality of two matrices:

Definition: Two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are said to be equal if

(i) they are of the same size and

(ii) the elements in the corresponding places of the two matrices are the same,

i.e., $a_{ij} = b_{ij}$ for each pair of the subscripts i and j.

If two matrices A and B are equal, then we write A = B.

If two matrices A and B are not equal, then we write $A \neq B$.

If two matrices are not of same size, they cannot be equal.

Example: Are the following matrices equal:

$$A=\begin{bmatrix}1 & 3 & -1\\ 2 & 4 & 2\\ 3 & 0 & 7\\ 1 & 0 & 9\end{bmatrix},\ B=\begin{bmatrix}1 & 3 & -1\\ 2 & 4 & 2\\ 3 & 0 & 7\\ 1 & 0 & 9\end{bmatrix}?$$

Sol.: The matrix A is of the type 4×3 and the matrix B is of the type 4×3. Also the corresponding elements of A and B are equal. Hence, A = B.

Addition of matrices:

Definition: Let A and B be two matrices of the same type $m\times n$. Then their sum (to be denoted by A + B) is defined to be the matrix of the type $m\times n$ obtained by adding the corresponding elements of A and B.

Symbolically, if $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$, then $A+B=\left[a_{ij}+b_{ij}\right]_{m\times n}$.

Note that A + B is also a matrix of the type $m\times n$.

More clearly we can say that if

$$A=\begin{bmatrix}a_{11} & a_{12} & .... & a_{1n}\\ a_{21} & a_{22} & .... & a_{2n}\\ .... & .... & .... & ....\\ a_{m1} & a_{m2} & .... & a_{mn}\end{bmatrix}_{m\times n} \text{ and } B=\begin{bmatrix}b_{11} & b_{12} & .... & b_{1n}\\ b_{21} & b_{22} & .... & b_{2n}\\ .... & .... & .... & ....\\ b_{m1} & b_{m2} & .... & b_{mn}\end{bmatrix}_{m\times n}.$$

$$\text{Then } A+B=\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12} & .... & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & .... & a_{2n}+b_{2n}\\ .... & .... & .... & ....\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & ...... & a_{mn}+b_{mn}\end{bmatrix}_{m\times n}.$$

Example: If $A=\begin{bmatrix}3 & 2 & -1\\ 4 & -3 & 1\end{bmatrix}_{2\times3}$ and $B=\begin{bmatrix}1 & -2 & 7\\ 3 & 2 & -1\end{bmatrix}_{2\times3}$,

then $A+B=\begin{bmatrix}3+1 & 2-2 & -1+7\\ 4+3 & -3+2 & 1-1\end{bmatrix}=\begin{bmatrix}4 & 0 & 6\\ 7 & -1 & 0\end{bmatrix}_{2\times3}$.

Properties of matrix addition:

(i). Matrix addition is commutative:

If A and B be two $m \times n$ matrices, then $A + B = B + A$.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$, then

$A + B = \left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n} = \left[a_{ij} + b_{ij}\right]_{m\times n}$ [by definition of addition of two matrices]

$= \left[b_{ij} + a_{ij}\right]_{m\times n}$ [Since a_{ij} and b_{ij} are numbers and addition of numbers is commutative]

$= \left[b_{ij}\right]_{m\times n} + \left[a_{ij}\right]_{m\times n} = B + A.$ [by definition of addition of two matrices]

(ii). Matrix addition is associative:

If A, B and C be three matrices each of the type $m \times n$, then $(A + B) + C = A + (B + C)$.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$ and $C = \left[c_{ij}\right]_{m\times n}$

Then $(A + B) + C = \left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n} + \left[c_{ij}\right]_{m\times n}$

$= \left[a_{ij} + b_{ij}\right]_{m\times n} + \left[c_{ij}\right]_{m\times n}$ [by definition of A + B]

$= \left[\left(a_{ij} + b_{ij}\right) + c_{ij}\right]_{m\times n}$ [by definition of addition of matrices]

$= \left[a_{ij} + \left(b_{ij} + c_{ij}\right)\right]_{m\times n}$

Since a_{ij}, b_{ij} and c_{ij} are numbers and addition of numbers is associative, we get

$(A + B) + C = \left[a_{ij}\right]_{m\times n} + \left[b_{ij} + c_{ij}\right]_{m\times n}$ [by addition of two matrices]

$= \left[a_{ij}\right]_{m\times n} + \left(\left[b_{ij}\right]_{m\times n} + \left[c_{ij}\right]_{m\times n}\right) = A + (B + C)$.

(iii). Existence of additive identity:

If O be the $m \times n$ matrix each of whose elements is zero, then

$A + O = A = O + A$ for every $m \times n$ matrix A.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$. Then $A + O = \left[a_{ij} + 0\right]_{m\times n} = \left[a_{ij}\right]_{m\times n} = A$.

Also $O + A = \left[0 + a_{ij}\right]_{m\times n} = \left[a_{ij}\right]_{m\times n} = A$.

Thus the null matrix O of the type $m \times n$ acts as the identity element for addition in the set of all $m \times n$ matrices.

(iv). Existence of the additive inverse:

Negative of a matrix:

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$. Then, the negative of the matrix A is defined as the matrix $\left[-a_{ij}\right]_{m\times n}$ and is denoted by $-A$.

The matrix $-A$ is the additive inverse of the matrix A.

Obviously, $-A + A = O = A + (-A)$.

Here O is the null matrix of the type $m \times n$. It is identity element for matrix addition.

Subtraction of two matrices:

Definition: If A and B are two $m \times n$ matrices, then we defined $A - B = A + (-B)$.

Thus, the differences $A - B$ is obtained by subtracting from each element of A the corresponding element of B.

(v). Cancellation law hold good in the case of addition of matrices:

If A, B, C are three $m \times n$ matrices, then

$A + B = A + C \Rightarrow B = C$ (left cancellation law)

and $B + A = C + A \Rightarrow B = C$ (right cancellation law)

Proof: We have $A + B = A + C$

Adding $-A$ to both sides, we get

$-A + (A + B) = -A + (A + C)$

$\Rightarrow (-A + A) + B = (-A + A) + C$ [$\because$ Matrix addition is associative]

$\Rightarrow O + B = O + C$ [$\because -A + A = O$]

$\Rightarrow B = C$ $[\because O + B = B]$

Similarly, we can prove the right cancellation law.

(vi). The equation $A + X = O$ has a unique solution in the set of all $m \times n$ matrices.

Proof: Let A be an $m \times n$ matrix and let $X = -A$. Then X is also an $m \times n$ matrix.

We have $A + X = A + (-A) = O$

$\therefore X = -A$ is an $m \times n$ matrix such that $A + X = O$.

To show: The uniqueness of the solution.

Let X_1 and X_2 be two solutions of the equation A+ X = O. Then $A + X_1 = O$, and $A + X_2 = O$. Therefore, we have

$A + X_1 = A + X_2 \Rightarrow X_1 = X_2$, be left cancellation law.

Hence, the solution is unique.

Multiplication of a matrix by a scalar:

Definition: Let A be any $m \times n$ matrix and k any complex number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called the scalar multiple of A by k and is denoted by kA or Ak.

Symbolically, if $A = \left[a_{ij}\right]_{m\times n}$, then $kA = Ak = \left[ka_{ij}\right]_{m\times n}$.

Example: If $k = 2$ and $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}_{2\times 3}$,

then $2A = \begin{bmatrix} 2\times 3 & 2\times 2 & 2\times -1 \\ 2\times 4 & 2\times -3 & 2\times 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 8 & -6 & 2 \end{bmatrix}_{2\times 3}$.

Properties of multiplication of a matrix by a scalar:

Theorem (1): If A and B are two matrices each of the type $m \times n$, then

$$k(A + B) = kA + kB.$$

i.e., the scalar multiplication of matrices distributes over the addition of matrices.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$, then

$k(A + B) = k\left(\left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n}\right)$

$= k\left[a_{ij} + b_{ij}\right]_{m\times n}$ [by definition of addition of two matrices two matrices]

$= \left[k\left(a_{ij} + b_{ij}\right)\right]_{m\times n}$ [by definition of scalar multiplication]

$= \left[ka_{ij} + kb_{ij}\right]_{m\times n}$ [by distributive law of numbers]

$= \left[ka_{ij}\right]_{m\times n} + \left[kb_{ij}\right]_{m\times n} = k\left[a_{ij}\right]_{m\times n} + k\left[b_{ij}\right]_{m\times n} = kA + kB$.

Theorem (2): If p and q are two scalars and A is any $m \times n$ matrix, then

$$(p+q)A = pA + qA.$$

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$. Then

$(p+q)A = (p+q)\left[a_{ij}\right]_{m\times n} = \left[(p+q)a_{ij}\right]_{m\times n} = \left[pa_{ij} + qa_{ij}\right]_{m\times n}$

$= \left[pa_{ij}\right]_{m\times n} + \left[qa_{ij}\right]_{m\times n} = p\left[a_{ij}\right]_{m\times n} + q\left[a_{ij}\right]_{m\times n} = pA + qA$.

Theorem (3): If p and q are two scalars and A is any $m \times n$ matrix, then $p(qA) = (pq)A$.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$. Then

$p(qA) = p\left(q\left[a_{ij}\right]_{m\times n}\right) = p\left[qa_{ij}\right]_{m\times n} \left[p\left(qa_{ij}\right)\right]_{m\times n}$

[$\because$ multiplication of numbers is associative]

$= (pq)\left[a_{ij}\right]_{m\times n} = (pq)A$.

Theorem (4): If A be any $m \times n$ matrix and k be any scalar, then

$$(-k)A = -(kA) = k(-A).$$

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$. Then

$(-k)A = \left[(-k)a_{ij}\right]_{m\times n} = \left[-\left(ka_{ij}\right)\right]_{m\times n} = -\left[ka_{ij}\right]_{m\times n} = -(kA)$.

Also $(-k)A = \left[(-k)a_{ij}\right]_{m\times n} = \left[k\left(-a_{ij}\right)\right]_{m\times n} = k\left[-a_{ij}\right]_{m\times n} = k(-A)$.

Theorem (5): If A be any $m \times n$ matrix, then **(i)** $1A = A$ **(ii)** $(-1)A = -A$.

Proof: Let $A = \left[a_{ij}\right]_{m\times n}$. Then

$1A = \left[1a_{ij}\right]_{m\times n}$ [by definition of scalar multiplication]

$=\left[a_{ij}\right]_{m\times n}=A$ $\qquad [\because 1a_{ij}=a_{ij}]$.

Also $(-1)A=\left[(-1)a_{ij}\right]_{m\times n}=\left[-a_{ij}\right]_{m\times n}=-A$.

Theorem (6): If A and B are two $m\times n$ matrices, then $-(A+B)=-A-B$.

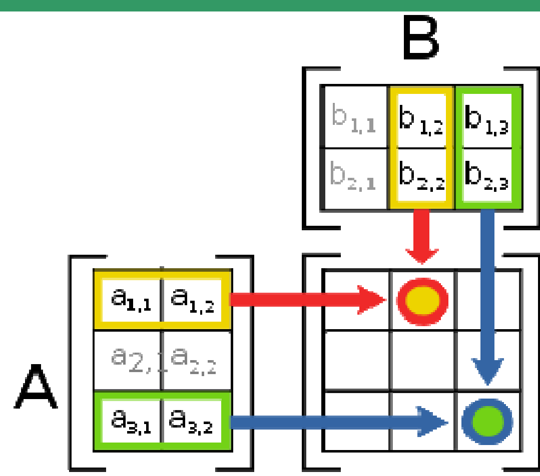
Proof: we have $-(A+B)=(-1)(A+B)$ [by (ii) of theorem 5]

$=(-1)A+(-1)B$ [by theorem (1)]

$=-A+(-B)$ [by theorem 5]

$=-A-B$.

MULTIPLICATION OF TWO MATRICES:



Schematic depiction of the matrix product AB of two matrices A and B.

Definition: Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{jk}\right]_{n\times p}$ be two matrices such that the number of column in A is equal to the number of rows in B.

Then, the $m\times p$ matrix $C=\left[c_{ik}\right]_{m\times p}$ such that

$$c_{ik}=\sum_{j=1}^{n}a_{ij}b_{jk}=a_{i1}b_{1k}+a_{i2}b_{2k}+................+a_{in}b_{nk},$$

is called the product of the matrices A and B in that order and we write C=AB.

Remarks: If the product AB exists, then it is not necessary that the product BA will also exist.

Conformable for multiplication: The product AB of two matrices A and B exists if and only if the number of columns in A is equal to the number of rows in B, then two such matrices are said to be conformable for multiplication.

Properties of matrix multiplication:

(i). Matrix multiplication is associative, if conformability is assured:

i.e., A(BC)=(AB)C if A, B, C are $m\times n$, $n\times p$, $p\times q$ matrices respectively.

Proof: Let $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{jk}\right]_{n\times p}$ and $C=\left[c_{k\ell}\right]_{p\times q}$.

Then $AB=\left[u_{ik}\right]_{m\times p}$ is an $m\times p$ matrix, where $u_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$. (i)

Also $BC=\left[v_{j\ell}\right]_{n\times q}$ is an $n\times q$ matrix, where $v_{j\ell}=\sum_{k=1}^{p} b_{jk}c_{k\ell}$. (ii)

Now A(BC) is an $m\times q$ matrix and (AB)C is also an $m\times q$ matrix.

Let A(BC) $=\left[w_{i\ell}\right]_{m\times q}$ where $w_{i\ell}$ is the $(i,\ \ell)^{th}$ element of A(BC).

Then $w_{i\ell}=\sum_{j=1}^{n} a_{ij}v_{j\ell}=\sum_{j=1}^{n}\left[a_{ij}\left\{\sum_{k=1}^{p} b_{jk}c_{k\ell}\right\}\right]$ [Putting the value of $v_{j\ell}$ from (ii)]

$=\sum_{k=1}^{p}\left[\left\{\sum_{j=1}^{n} a_{ij}b_{jk}\right\}c_{k\ell}\right]$ [$\because$ finite summations can be interchanged]

$=\sum_{k=1}^{p} u_{ik}c_{k\ell}$ [From (i)]

= the $(i,\ \ell)^{th}$ element of (AB)C.

Therefore, by the equality of two matrices, we have A(BC) = (AB)C.

(ii). Multiplication of matrices is distributive with respect to addition of matrices.

i.e., A(B+C) = AB+AC, where A, B, C are $m\times n$, $n\times p$, $n\times p$ matrices, respectively.

Proof: $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{jk}\right]_{n\times p}$ and $C=\left[c_{jk}\right]_{n\times p}$..

Then both A(B+C) and AB+AC are $m\times p$ matrices.

We have $B+C=\left[b_{ik}+c_{jk}\right]_{n\times p}$.

$\therefore$ the $(i,\ k)^{th}$ element of $A(B+C)=\sum_{j=1}^{n} a_{ij}\left(b_{jk}+c_{jk}\right)=\sum_{j=1}^{n} \left\{a_{ij}b_{jk}+a_{ij}c_{jk}\right\}$

[since the multiplication of numbers is distributive w.r.t. addition of numbers]

$=\sum_{j=1}^{n} a_{ij}b_{jk}+\sum_{j=1}^{n} a_{ij}c_{jk}$

= the $(i,\ k)^{th}$ element of AB + the $(i,\ k)^{th}$ element of AC

= the $(i,\ k)^{th}$ element of AB+AC.

Hence, A(B+C) = AB+AC.

Remarks:

(1). It can be shown in a similar manner as above that (B+C)D=BD+CD, where B, C, D are matrices of suitable types so that the above equation is meaningful i. e., if B and C are $m\times n$ matrices then D should be an $n\times p$ matrix.

(2). Distributive law hold unconditionally for square matrices of order n, since conformability is always assured for them.

(3). ***The multiplication of matrices is not always commutative.***

(a). Whenever AB exist, it is not always necessary that BA should also exist. For example if A be a 5×4 matrix while B be 4×3 matrix then AB exist while BA does not exist.

(b). Whenever AB and BA both exist, it is always not necessary that they should be matrices of the same type. For example if A be 5×4 matrix while B be a 4×5 matrix, then AB exist and it is a 5×5 matrix. In this case BA also exists and it is a 4×4 matrix. Since the matrices AB and BA are not of the same size therefore we have $AB \neq BA$.

(c). Whenever AB and BA both exist and are matrices of the same type, it is not necessary that AB =BA. For example, if

$A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1.0+0.1 & 1.1+0.0\\ 0.0-1.1 & 0.1-1.0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$BA=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0.1+1.0 & 0.0-1.1\\ 1.1+0.0 & 1.0+0.1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Thus $AB \neq BA$.

(d). It is however does not imply that AB is never equal to BA. For example, if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}, \text{ Then } AB = \begin{bmatrix} -3 & 1 & 0 \\ 4 & -2 & -1 \\ -5 & 1 & 1 \end{bmatrix} = BA.$$

Triangular, Diagonal and Scalar matrices:

Upper triangular matrix:

Definition: A square matrix $A = \left[a_{ij}\right]$ is called an upper triangular matrix

if $a_{ij} = 0$ whenever $i > j$.

Thus, in an upper triangular matrix, all the elements below the principal diagonal are zero.

Example: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & a_{nn} \end{bmatrix}$, is an upper triangular matrix of the type $m \times n$.

Similarly, $A = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 8 \end{bmatrix}_{4\times 4}$, $B = \begin{bmatrix} 2 & -9 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$, are triangular matrices.

Lower triangular matrix:

Definition: A square matrix $A = \left[a_{ij}\right]$ is called an lower triangular matrix

if $a_{ij} = 0$ whenever $i < j$.

Thus, in a lower triangular matrix, all the elements above principal diagonal are zero.

Example: $\begin{bmatrix} a_{11} & 0 & 0 & ..... & 0 \\ a_{21} & a_{22} & 0 & ..... & 0 \\ a_{31} & a_{32} & a_{33} & ..... & 0 \\ ..... & ..... & ..... & ..... & ..... \\ a_{n1} & a_{n2} & a_{n3} & ..... & a_{nn} \end{bmatrix}_{n\times n}$, is a lower triangular matrix of the type $n \times n$.

Similarly, , $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 5 & 7 & 1 & 2 \end{bmatrix}_{4\times4}$ are lower triangular matrices.

Strictly triangular matrix:

Definition: **A triangular matrix** $A = \left[a_{ij}\right]_{n\times n}$ **is called strictly triangular if** a_{ii} **= 0 for i =1, 2,...., n.**

Diagonal Matrix:

Definition: **A square matrix** $A = \left[a_{ij}\right]_{n\times n}$**, whose elements above and below the principal diagonal are all zero, i.e.,** a_{ij} **= 0 for all** $i \neq j$**, is called diagonal matrix.**

Thus, a diagonal matrix is both upper and lower triangular. An n-rowed diagonal matrix whose diagonal elements in order are $d_1, d_2, d_3, ..., d_n$ **will often be denoted by the symbol** $\text{Diag}\left[d_1, d_2, ..., d_n\right]$**.**

Example: The matrices $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are diagonal matrices.

Scalar matrix:

Definition: **A diagonal matrix, whose diagonal elements are all equal, is called a scalar matrix.**

Example: A matrix $S=\begin{bmatrix} k & 0 & ..... & 0 \\ 0 & k & ..... & ... \\ 0 & 0 & ..... & ... \\ ..... & ..... & ...... & ..... \\ ..... & ..... & ...... & ..... \\ ..... & ..... & ..... & ..... \\ 0 & 0 & ..... & k \end{bmatrix}$ is an n-rowed scalar matrix each of whose diagonal elements is equal to k and A is any n-rowed square matrix, then AS = SA = kA, i.e., the pre-multiplication or the post-multiplication of A by S has the same effect as the multiplication of A by the scalar k. This is perhaps the motivation behind the name 'scalar matrix'.

As a particular case, if we take

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $S=\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$, then

$$AS=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\times\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}=kA .$$

Similarly, SA=kA. Hence SA= AS=kA.

If in place of S, we take $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

Then $AI_3=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Similarly, $I_3A=A$.

Hence, $AI_3=I_3A=A$.

Trace of a matrix:

Definition: Let A be a square matrix of order n. The sum of the elements of A lying along principal diagonal is called the trace of A.

We shall write the trace of A as tr A.

Thus, if $A=\left[a_{ij}\right]_{n\times n}$**, then** $\text{tr } A=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+\ldots\ldots..+a_{nn}$.

Properties of the trace of a matrix:

Theorem: Let A and B be two square matrices of order n and λ be a scalar. Then

(i). $\text{tr }(\lambda A)=\lambda \text{ tr } A$,

(ii). $\text{tr }(A+B)=\text{tr } A+\text{tr } B$,

(iii). $\text{tr }(AB)=\text{tr }(BA)$.

Proof: Let $A=\left[a_{ij}\right]_{n\times n}$ and $B=\left[b_{ij}\right]_{n\times n}$.

(i). We have $\lambda A=\left[\lambda a_{ij}\right]_{n\times n}$, by definition of multiplication of a matrix by a scalar.

$$\therefore \text{tr }(\lambda A)=\sum_{i=1}^{n} \lambda a_{ii}=\lambda\sum_{i=1}^{n} a_{ii}=\lambda \text{ tr } A.$$

(ii). We have $A+B=\left[a_{ij}+b_{ij}\right]_{n\times n}$.

$$\therefore \text{tr }(A+B)=\sum_{i=1}^{n}(a_{ii}+b_{ii})=\sum_{i=1}^{n} a_{ii}+\sum_{i=1}^{n} b_{ii}=\text{tr } A+\text{tr } B.$$

(iii). We have $AB=\left[c_{ij}\right]_{n\times n}$, where $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$.

Also $BA=\left[d_{ij}\right]_{n\times n}$ where $d_{ij}=\sum_{k=1}^{n} b_{ik}a_{kj}$.

Now $\text{tr }(AB)=\sum_{i=1}^{n} c_{ii}=\sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)=\sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{ki}$,

[Interchanging the order of summation in the last sum]

$$=\sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki}a_{ik}\right)=\sum_{k=1}^{n} d_{kk}=d_{11}+d_{22}+\ldots\ldots\ldots.+d_{nn}=\text{tr }(BA).$$

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