Differential Calculus

Partial Differentiation

(Total Differentials, Explicit Function, Implicit Functions and Total Differential Coefficient)

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Explicit Function:

A function, where the dependent variable say y, is **expressed** in terms of the independent variable say x, then that function is called ***explicit function***.

Example: $y = 4x^3 + 3x^2 + 5x + 9$.

Implicit Functions:

A function, where one of the various variables **cannot be expressed** explicitly in terms of the other variables, then that function is called ***implicit function***.

Example: Consider the relation $x^3 + y^3 + 3axy = 0$.

In this case, we obtain $\frac{dy}{dx}$ by differentiating throughout w.r.t. x.

Total Differentials:

Let u be a function of x and y i.e. u = f(x , y).

Then the total differential of u is defined and written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\,.$$

Similarly, if u be a function of x , y and z i.e. u = f(x, y, z).

Then the total differential of u is defined and written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\,.$$

Evaluation of $\frac{dy}{dx}$ for an implicit function:

Let u = f(x, y) be an implicit function $\Rightarrow u = f(x,y) = 0$ or const.

$$\Rightarrow du = 0\,. \qquad \text{....(i)}$$

Also when u be a function of x and y, i.e. u = f(x , y),

then the total differential of u is defined and written as $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\,.$ ....(ii)

From (i) and (ii), we get $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{u_x}{u_y}$.

Total Differential Coefficient:

Let $u = f(x,y)$, where $x = \varphi(t)$, $y = \psi(t)$.

Then u is ultimately a function of t.

Then the total differential coefficient of u w.r.t. t is defined and written as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}\,.$$

Similarly, if $u = f(x,y,z)$, where $x = \varphi(t)$, $y = \psi(t)$ and $z = \xi(t)$.

Then the total differential coefficient of u w.r.t. t is defined and written as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}\,.$$

Remark: $u = f(x,y)$ and t = x, then from $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$, we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx},$$

where $\frac{du}{dx}$ is the total differential coefficient of u w.r.t. x.

Now let us solve some problems related to Total Differentials and Total Differential Coefficient:

Q.No.1: Find the total differential of u in the following cases:-

(i) $u = \sqrt{x+y}$ and **(ii)** $u = \log\left(x^2 + y^2\right)$.

Sol.: (i) Since $u = \sqrt{x+y}$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x+y}} \text{ and } \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x+y}}.$$

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{1}{2\sqrt{x+y}}(dx + dy)$.

(ii) Since $u = \log\left(x^2 + y^2\right) \therefore \frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$ and $\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$.

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{2xdx + 2ydy}{x^2 + y^2}$.

Q.No.2: If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$.

Sol.: Given $x^3 + y^3 - 3axy = 0$. So let $u = x^3 + y^3 - 3axy = 0$.

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial u}{\partial y} = 3y^2 - 3ax.$$

Hence $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{\left(3x^2 - 3ay\right)}{\left(3y^2 - 3ax\right)} = \frac{ay - x^2}{y^2 - ax}$.

Q.No.3: If $u = \sin^{-1}(x - y)$, where $x = 3t,\ y = 4t^3$. Prove that the total differential coefficient of u w. r. t. t is equal to $3\left(1 - t^2\right)^{-1/2}$.

Sol.: Given $u = \sin^{-1}(x - y)$, where $x = 3t, y = 4t^3$.

$$\therefore \frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-(x-y)^2}}\ ,\ \frac{\partial u}{\partial y}=\frac{-1}{\sqrt{1-(x-y)^2}}\ ,\ \frac{du}{dx}=3 \text{ and } \frac{du}{dy}=12t^2.$$

Then total differential coefficient of u w. r. t. t is

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt}=\frac{1}{\sqrt{1-(x-y)^2}}.3+\frac{-1}{\sqrt{1-(x-y)^2}}.12t^2=\frac{3\left(1-4t^2\right)}{\sqrt{1-\left(3t-4t^3\right)^2}}=\frac{3\left(1-4t^2\right)}{\sqrt{\left(1-4t^2\right)^2\left(1-t^2\right)}}$$

$$\Rightarrow \frac{du}{dt}=3\left(1-t^2\right)^{-1/2}.$$

Q.No.4: If $u=\sin\left(x^2+y^2\right)$, where $a^2x^2+b^2y^2=c^2$, find the total differential coefficient of u w. r. t. x.

Sol.: Given $u=\sin\left(x^2+y^2\right)$, where $a^2x^2+b^2y^2=c^2$.

Let $f=a^2x^2+b^2y^2-c^2$.

$$\therefore \frac{\partial f}{\partial x}=2a^2x\,,\ \frac{\partial f}{\partial y}=2b^2y \text{ and } \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{a^2x}{b^2y}.$$

Since we know the total differential coefficient of u w. r. t. x. is

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}.$$

$$\Rightarrow \frac{du}{dx}=2x\cos\left(x^2+y^2\right)+2y.\cos\left(x^2+y^2\right)\left(-\frac{a^2x}{b^2y}\right)=2\left(1-\frac{a^2}{b^2}\right)x\cos\left(x^2+y^2\right).$$

Q.No.5: Find the total differentials in the following cases:

(a) $u=\left(2x^2-4y^3\right)^3$, **(b)** $u=\tan\frac{x}{y}$.

Sol.: Since we know the total differential of u is $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$.

(a) Here $u=\left(2x^2-4y^3\right)^3$.

$$\therefore \frac{\partial u}{\partial x}=3\left(2x^2-4y^3\right)^2.(4x)=12x\left(2x^2-4y^3\right)^2,$$

and $\frac{\partial u}{\partial y} = 3\left(2x^2 - 4y^3\right)^2\left(-12y^2\right) = -36y^2\left(2x^2 - 4y^3\right)^2$.

Hence $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 12x\left(2x^2 - 4y^3\right)^2 dx - 36y^2\left(2x^2 - 4y^3\right)^2 dy$

$= 12\left(2x^2 - 4y^3\right)^2\left(xdx - 3y^2dy\right)$. Ans.

(b) Here $u = \tan\frac{x}{y}$. $\therefore \frac{\partial u}{\partial x} = \sec^2\left(\frac{x}{y}\right).\left(\frac{1}{y}\right)$ and $\frac{\partial u}{\partial y} = \sec^2\left(\frac{x}{y}\right).\left(-\frac{x}{y^2}\right)$.

Hence $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \sec^2\left(\frac{x}{y}\right).\left(\frac{1}{y}\right)dx + \sec^2\left(\frac{x}{y}\right).\left(-\frac{x}{y^2}\right)dy$

$= \sec^2\left(\frac{x}{y}\right).\left(\frac{ydx - xdy}{y^2}\right)$. Ans**.**

Q.No.6: Find the total differential coefficient of $u = \sin\left(\frac{x}{y}\right)$, where $x = e^t$, $y = t^2$ w. r. t. t**.**

or

Given $u = \sin\left(\frac{x}{y}\right)$, where $x = e^t$, $y = t^2$, find $\frac{du}{dt}$ as a function of t.

Verify your result by direct substitution.

Sol.: We have $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = \left(\cos\frac{x}{y}\right)\frac{1}{y}.e^t + \left(\cos\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t$

$$= \cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^2} - 2\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^3} = \left(\frac{t-2}{t^3}\right)e^t\cos\left(\frac{e^t}{t^2}\right).$$

Also $u = \sin\left(\frac{x}{y}\right) = \sin\left(\frac{e^t}{t^2}\right)$.

$\therefore \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right).\frac{t^2e^t - e^t.2t}{t^4} = \left(\frac{t-2}{t^3}\right)e^t\cos\left(\frac{e^t}{t^2}\right)$ as before**.**

Q.No.7: If $u = x\log xy$, where $x^3 + y^3 + 3xy - 1 = 0$, find total differential coefficient of u w. r. t. x**.**

Sol.: Given $u = x\log xy$, where $x^3 + y^3 + 3xy - 1 = 0$.

Since we know that total differential coefficient of u w. r. t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}. \qquad .....(i)$$

Now $\frac{\partial u}{\partial x} = x.\left(\frac{1}{xy}.y\right) + 1.\log xy = 1 + \log xy$. Also $\frac{\partial u}{\partial y} = x.\left(\frac{1}{xy}.x\right) = \frac{x}{y}$.

Let $f = x^3 + y^3 + 3xy - 1$, then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\frac{(x^2 + y)}{(y^2 + x)}$.

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (i), we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} = (1 + \log xy) + \left(\frac{x}{y}\right).\left(-\frac{x^2 + y}{y^2 + x}\right) = 1 + \log xy - \frac{x(x^2 + y)}{y(y^2 + x)}$$. Ans.

Q.No.8: Find the total differential coefficient of x^2y w. r. t. x where x and y are connected by $x^2 + xy + y^2 = 1$.

Sol.: Let $u = x^2y$, where x and y are connected by $x^2 + xy + y^2 = 1$.

Since we know that total differential coefficient of u w. r. t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}. \qquad .....(i)$$

Now $\frac{\partial u}{\partial x} = 2xy$. Also $\frac{\partial u}{\partial y} = x^2$.

Let $f = x^2 + xy + y^2 - 1$,

then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(2x + y)}{(2y + x)} = -\frac{(2x + y)}{(x + 2y)}$.

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (i), we get

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}=2xy+x^2.\left(-\frac{2x+y}{x+2y}\right)=\frac{2xy(x+2y)-x^2(2x+y)}{(x+2y)}$$

$$\Rightarrow \frac{du}{dx}=\frac{x\left(xy+4y^2-2x^2\right)}{(x+2y)}. \text{ Ans.}$$

Q.No.9: (i) If $f(x,y)=0$, $\varphi(y,z)=0$,

show that $\frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial z}.\frac{dz}{dx}=\frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y}$.

(ii) If the curves If $f(x,y)=0$ and $\varphi(x,y)=0$ touch,

show that at the point of contact $\frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y}=\frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial x}$.

Sol.: (i) Given $f(x,y)=0$, $\varphi(y,z)=0 \Rightarrow df=0$ and $d\varphi=0$.

i.e. $\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$ and $\frac{\partial \varphi}{\partial y}dy+\frac{\partial \varphi}{\partial z}dz=0$.

$$\Rightarrow \frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad ....(i) \quad \text{and} \quad \frac{dz}{dy}=-\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial z}}. \quad .....(ii)$$

$$(i) \Rightarrow \frac{\partial f}{\partial y}.\frac{dy}{dx}=-\frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial y}.\frac{dy}{dz}.\frac{dz}{dx}=-\frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial y}.\left[-\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}\right].\frac{dz}{dx}=-\frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial z}.\frac{dz}{dx}=\frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y}.$$

(ii) Let the curves $f(x,y)=0$ and $\varphi(x,y)=0$ are touching at a point (a,b).

Now the slope of the tangent of the curve $f(x,y)=0$ at point of contact is

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \qquad ......(i)$$

and the slope of the tangent of the curve $\varphi(x,y)=0$ at point of contact is

$$\frac{dy}{dx} = -\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}. \qquad ....(ii)$$

Now since the curves $f(x, y) = 0$ and $\varphi(x, y) = 0$ are touching so that their slope of the tangents are same.

Hence from (i) and (ii), at the point of contact $\frac{\partial f}{\partial x}.\frac{\partial\varphi}{\partial y} = \frac{\partial f}{\partial y}.\frac{\partial\varphi}{\partial x}$.

Q.No.10: If $x^2 + y^2 + z^2 - 2xyz = 1$. Show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$.

Sol.: Given $x^2 + y^2 + z^2 - 2xyz = 1$.

Let $u = x^2 + y^2 + z^2 - 2xyz - 1 = 0$.

Here u be an implicit function $\Rightarrow du = 0$. ...(i)

Here u be a function of x , y and z, i.e. u = f(x, y, z).

Then the total differential of u is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0. \quad \text{[by (i)]} \qquad ....(ii)$$

Evaluate: $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Since $u = x^2 + y^2 + z^2 - 2xyz - 1$.

$$\therefore \frac{\partial u}{\partial x} = 2(x - yz), \ \frac{\partial u}{\partial y} = 2(y - xz) \ \text{ and } \ \frac{\partial u}{\partial z} = 2(z - xy).$$

Hence (ii) becomes $(x - yz)dx + (y - xz)dy + (z - xy)dz = 0$. ....(iii)

Find: $(x - yz)$, $(y - xz)$ and $(z - xy)$.

Since we have given $x^2 + y^2 + z^2 - 2xyz = 1$

$$\Rightarrow x^2 - 2xyz = 1 - y^2 - z^2 \Rightarrow x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$\Rightarrow (x - yz)^2 = (1 - y^2)(1 - z^2) \ \Rightarrow (x - yz) = \sqrt{(1 - y^2)(1 - z^2)}.$$

Similarly, $(y - xz) = \sqrt{(1 - x^2)(1 - z^2)}$

and $\quad (z - xy) = \sqrt{(1-x^2)(1-y^2)}$.

Hence (iii) becomes $\sqrt{(1-y^2)(1-z^2)}dx + \sqrt{(1-x^2)(1-z^2)}dy + \sqrt{(1-x^2)(1-y^2)}dz = 0$.

Last step: Dividing by $\sqrt{(1-x^2)(1-y^2)(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0 .$$

Q.No.11: If $u = x^2 + y^2$, where $x = a\cos t$, $y = b\sin t$. Find $\frac{du}{dt}$ and verify the result.

Sol.: Given $u = x^2 + y^2$.

We have $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = 2x.a(-\sin t) + 2y.b\cos t = 2(yb\cos t - xa\sin t)$

Also $u = x^2 + y^2 = a^2\cos^2 t + b^2\sin^2 t$.

$$\therefore \frac{du}{dt} = a^2 2\cos t(-\sin t) + b^2 2\sin t(\cos t) = 2[-(a\cos t)(a\sin t) + (b\sin t)(b\cos t)]$$

$$= 2(yb\cos t - xa\sin t) \text{ as before.}$$

Q.No.12: If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$, then prove that

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly} .$$

Also find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Sol.: Let $f = ax^2 + by^2 + cz^2 - 1$ and $\phi = lx + my + nz$.

$\therefore f = 0 \Rightarrow df = 0$ and $\phi = 0 \Rightarrow d\phi = 0$.

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 2axdx + 2bydy + 2czdz = axdx + bydy + czdz = 0$ ....(i)

Also $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = ldx + mdy + ndz = 0$ ...(ii)

Solving (i) and (ii), we get

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly} .$$

Now consider $\frac{dx}{bny-cmz}=\frac{dy}{clz-anx}$ and $\frac{dx}{bny-cmz}=\frac{dz}{amx-bly}$.

$\therefore \frac{dy}{dx}=\frac{clz-anx}{bny-cmz}$ and $\frac{dz}{dx}=\frac{amx-bly}{bny-cmz}$. Ans.

Q.No.13: If $x^2y-e^x+x\sin z=0$ and $x^2+y^2+z^2=a^2$. Find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

Sol.: Let $f=x^2y-e^x+x\sin z=0$ and $\phi=x^2+y^2+z^2-a^2=0$.

$\therefore f=0 \Rightarrow df=0$ and $\phi=0 \Rightarrow d\phi=0$.

Now $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=(2xy-e^x+\sin z)dx+x^2dy+x\cos z dz=0$ ....(i)

Also $d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz=2xdx+2ydy+2zdz=xdx+ydy+zdz=0$ ...(ii)

Solving (i) and (ii), we get

$$\frac{dx}{x^2z-yx\cos z}=\frac{dy}{x^2\cos z-z(2xy-e^x+\sin z)}=\frac{dz}{y(2xy-e^x+\sin z)-x^3}.$$

Now consider $\frac{dx}{x^2z-yx\cos z}=\frac{dy}{x^2\cos z-z(2xy-e^x+\sin z)}$

and $\frac{dx}{x^2z-yx\cos z}=\frac{dz}{y(2xy-e^x+\sin z)-x^3}$.

We get $\frac{dy}{dx}=\frac{x^2\cos z-z(2xy-e^x+\sin z)}{x^2z-yx\cos z}$ and $\frac{dz}{dx}=\frac{y(2xy-e^x+\sin z)-x^3}{x^2z-yx\cos z}$ .Ans.

Q.No.14: If $x^y=e^{x-y}$, then prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.

Sol.: Given $x^y=e^{x-y}$.

Taking log on both sides, we get

$\log x^y=\log e^{x-y} \Rightarrow y\log x=x-y \Rightarrow y\log x-x+y=0$.

Let $u=y\log x-x+y$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$\therefore \frac{\partial u}{\partial x}=\frac{y}{x}-1=\frac{y-x}{x}$ and $\frac{\partial u}{\partial y}=(\log x+1)$.

Hence $\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{y-x}{x}}{(\log x+1)}=\frac{x-y}{x(\log x+1)}=\frac{y\log x}{x(1+\log x)}$. ...(i) $[\because x-y=y\log x]$

Now since $y\log x=x-y \Rightarrow \frac{y}{x}\log x=1-\frac{y}{x} \Rightarrow \frac{y}{x}(1+\log x)=1 \Rightarrow \frac{y}{x}=\frac{1}{(1+\log x)}$

Substituting the value of $\frac{y}{x}$ in (i), we get

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}.$$

This completes the proof**.**

Q.No.15: Using partial differentiation, find $\frac{dy}{dx}$ when $x^y+y^x=C$.

Sol.: Given $x^y+y^x=C \Rightarrow x^y+y^x-C=0$

Let $f(x,y)=x^y+y^x-C$ .....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial f}{\partial x}=yx^{y-1}+y^x\log y \text{ and } \frac{\partial f}{\partial y}=x^y\log x+xy^{x-1}.$$

But $\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{y\left(x^{y-1}+y^{x-1}\log y\right)}{x\left(x^{y-1}\log x+y^{x-1}\right)}$. Ans**.**

Q.No.16.: Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if

$$u=x^2+y^2+z^2 \text{ and } x=e^{2t},\ y=e^{2t}\cos 3t,\ z=e^{2t}\sin 3t.$$

Sol.: Here u is a function of x, y, z are and x, y, z are in turn functions of t. Thus u is a function ‘t’ via the intermediate variables x, y, z. Then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$=2x.2e^{2t}+2y.\left(2e^{2t}\cos 3t-3e^{2t}\sin 3t\right)+2z\left(2e^{2t}\sin 3t+3e^{2t}\cos 3t\right)$$

Rewriting in terms of x, y, z

$$=2x.2.x+2.y(2y-3.z)+2z(2z+3y)$$

$$= 4\left(x^2 + y^2 + z^2\right)$$

or in terms of t

$$\frac{du}{dt} = 4\left(e^{4t} + e^{4t}\left(\cos^2 3t + \sin^2 3t\right)\right) = 8e^{4t}$$

Verification by direct submission:

$$u = x^2 + y^2 + z^2 = e^{4t} + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t = 2e^{4t}$$

$$\frac{du}{dt} = 8e^{4t}.$$

Q.No.17.: Find the total differential coefficient of $x^2 y$ w.r.t. x when x, y are connected by $x^2 + xy + y^2 = 1$.

Sol.: Let $u = x^2 y$, then the total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus the total differential coefficient of u w.r.t x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = 2xy + x^2\frac{dy}{dx}$$

From the Implicit relation $f = x^2 xy + y = 1$, we calculate

$$\frac{dy}{dx} = \frac{f_x}{f_y} = -\frac{2x + y}{x + 2y}$$

so $\dfrac{du}{dx} = 2xy + x^2.\dfrac{dy}{dx} = 2xy + x^2\left(-\dfrac{(2x+y)}{(x+2y)}\right)$

$$\frac{du}{dx} = 2xy - \frac{x^2(2x+y)}{(x+2y)}.$$

Q.No.18.: The altitude of the right circular cone is 15 cm and is increasing at 0.2 cm/sec. The radius of the base is 10 cm and is decreasing at 0.3 cm/sec. How fast is the volume changing?

Sol.: Let x be the radius and y be the altitude of the cone. So volume V of the right circular cone is $V=\frac{1}{3}\pi x^2 y$.

Since x and y are changing w.r.t time t, differentiate V w.r.t. t.

$$\frac{dV}{dt}=\frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$=\frac{1}{3}\pi\left(2xy\frac{dx}{dt}+x^2\frac{dy}{dt}\right)$$

It is given that $x = 10,\ y = 15,\ \frac{dx}{dt}=-0.3$ and $\frac{dy}{dt}=0.2$, substituting these values

$$\frac{dV}{dt}=\frac{1}{3}\pi\left[2.10.15(-0.3)+10^2(0.2)\right]=\frac{-70}{3}\pi\ \text{cm}^3/\text{sec}$$

i.e, volume is decreasing at the rate of $\frac{70\pi}{3}$.

Home Assignments

Q.No.1.: Find $\frac{du}{dt}$ when $u=\sin\left(\frac{x}{y}\right)$ and $x=e^t$, $y=t^2$. Verify the result by direct substitution.

Ans.: $\frac{t-2}{t^3}e^t\cos\left(\frac{e^t}{t^2}\right)$.

Q.No.2.: Find $\frac{du}{dt}$ given $u=\sin^{-1}(x-y)$, $x = 3t$, $y=4t^3$. Verify the result by direct substitution.

Ans.: $3\left(1-t^2\right)^{-1/2}$

Q.No.3.: If $u = x^3ye^z$ where $x = t$, $y=t^2$ and $z = \text{In}\, t$, find $\frac{du}{dt}$ at $t = 2$.

Ans.: $6t^5$; 192.

Q.No.4.: Find $\frac{du}{dt}$, if $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $x = e^{t} - e^{-t}$ and $y = e^{t} + e^{-t}$.

Ans.: $\frac{-2}{e^{2t} + e^{-2t}}$

Q.No.5.: If x, y are related by $x^2 - y^2 = 2$ and $u = \tan\left(x^2 + y^2\right)$, find $\frac{du}{dx}$.

Ans.: $4x\sec^2\left(2x^2 - 2\right)$.

Q.No.6.: If $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $y = x^4$ find $\frac{du}{dx}$ at x = 1.

Ans.: $\frac{3x^2}{1 + x^6}$; $\frac{3}{2}$ at x = 1.

Q.No.7.: In order that the function $u = 2xy - 3x^2 y$ remains constant. What should be the rate of change of y (w.r.t. t) given that x increases at the rate of 2cm/sec at the instant when x = 3 cm and y = 1 cm.

Ans.: $\frac{dy}{dt} = -\frac{32}{21}\,\text{cm/sec}$; y must decrease at the rate of $\frac{32}{21}$ cm/sec.

Q.No.8.: Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 ft/sec. and 0.5 ft/sect respectively.

Ans.: 6.5 sq. ft/sec.

Q.No.9.: Find (**a**). $\frac{dz}{dx}$ and **(b).** $\frac{dz}{dy}$, given $z = xy^2 + x^2 y$, $y = \text{In } x$.

Ans.: **(a).** Here x is the independent variable

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = y^2 + 2xy + 2y + x$$

(b). Here y is the independent variable

$$\frac{dz}{dy} = \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{dx}{dy} = xy^2 + 2x^2 y + 2xy + x^2$$

Q.No.10.: Find the differential of the function $f(x, y) = x\cos y - y\cos x$.

Ans.: $df = (\cos y + y\sin x)dx - (x\sin y + \cos x)dy$

Q.No.11.: Find the differential of the function $u(x, y, z)) = e^{xyz}$.

Ans.: $du = e^{xyz}\left(yzdx + zxdy + xydz\right)$.

Q.No.12.: Find $\dfrac{du}{dt}$ for the functions $u = x^2 - y^2,\ x = e^t \cos t,\ \ y = e^t \sin t$ at $t = 0$.

Ans.: $2e^{2t}\left(\cos 2t - \sin 2t\right)$; At $t = 0,\ \dfrac{du}{dt} = 2$

Q.No.13.: Find $\dfrac{du}{dt}$ for the functions $u = \text{In}\left(x + y + z\right);\ \ x = e^{-t},\ \ y = \sin t,\ \ z = \cos t$.

Ans.: $\dfrac{\cos t - \sin t - e^{-t}}{\cos t + \sin t + e^{-t}}$

Q.No.14.: Find $\dfrac{du}{dt}$ for the functions $u = \sin\left(e^x + y\right),\ \ x = f(t),\ \ y = g(t)$.

Ans.: $\dfrac{du}{dt} = \left[\cos\left(e^x + y\right)\right]e^x f'(t) + \left[\cos\left(e^x + y\right)\right]g'(t)$.

Q.No.15.: Find $\dfrac{du}{dt}$ for the functions $u = x^y$ when $y = \tan^{-1} t,\ \ x = \sin t$.

Ans.: $y.x^{y-1}\cos t + x^y \text{In}\, x.\dfrac{1}{1+t^2}$.

Thank you

NEXT TOPIC

Transformation of independent variables (Composite Functions),

Jacobian, Properties of Jacobians

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