7th Topic

Double Integrals

[Volume of solids by double integrals]

(Last updated on 15-07-2013)

(17 Solved problems and 00 Home assignments)

Evaluation of Volume by double integrals:

Consider a surface $z = f(x, y)$. (i)

Let the orthogonal projection on xy-plane of its portion S' be the area S given by $\phi(x, y) = 0$. (ii)

Now (ii) represents a cylinder with generators parallel to z-axis and guiding curve given by (ii). Let V be the volume of this cylinder between S and S'.

Divide S into **elementary rectangles** of area $\delta x\, \delta y$ by drawing lines parallel to x and y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ Volume of this **prism** between S and the given surface $z = f(x, y)$ is $(z\ \delta x\, \delta y)$.

Hence, the **volume of the solid cylinder** on S as base, bounded by the given surface with generators parallel to the z-axis

$$V = \operatorname*{Lt}_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum \sum z\, \delta x\, \delta y = \iint z\,dxdy = \iint f(x, y)\,dxdy,$$

where the integration is carried over the area S.

Remarks: While using polar co-ordinates, divide S into elements of area $r\, \delta\theta\, \delta r$.

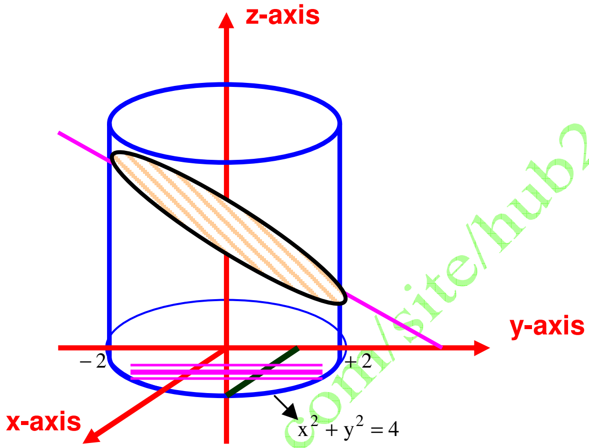
$\therefore$ By replacing dxdy by $r d\theta dr$, we get the required volume $= \iint z r d\theta dr$.

Q.No.1.: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Sol.: The required volume $= \iint zdxdy = \iint (4-y)dxdy$,

where the integration is carried over the area of circle $x^2 + y^2 = 4$.

Let us suppose strip is parallel to x-axis, then to cover the whole circle, x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ and y varies from -2 to 2.



$$\therefore \text{ Required volume} = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y)dxdy = 2\int_{-2}^{2} \left(\int_{0}^{\sqrt{(4-y^2)}} (4-y)dx \right) dy$$

$$= 2\int_{-2}^{2} (4-y)[x]_0^{\sqrt{(4-y^2)}} dy = 2\int_{-2}^{2} (4-y)\sqrt{(4-y^2)}dy$$

$$= 2\int_{-2}^{2} 4(4-y^2)dy - 2\int_{-2}^{2} y\sqrt{(4-y^2)}dy$$

$$= 8\int_{-2}^{2} \sqrt{(4-y^2)}dy - 0. \left[\begin{array}{l} \text{The second term vanishes as the} \\ \text{integrand is an odd function.} \end{array} \right]$$

Put $y = 2\sin\theta$ so that $dy = 2\cos\theta d\theta$.

And as y varies from -2 to 2, θ varies $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

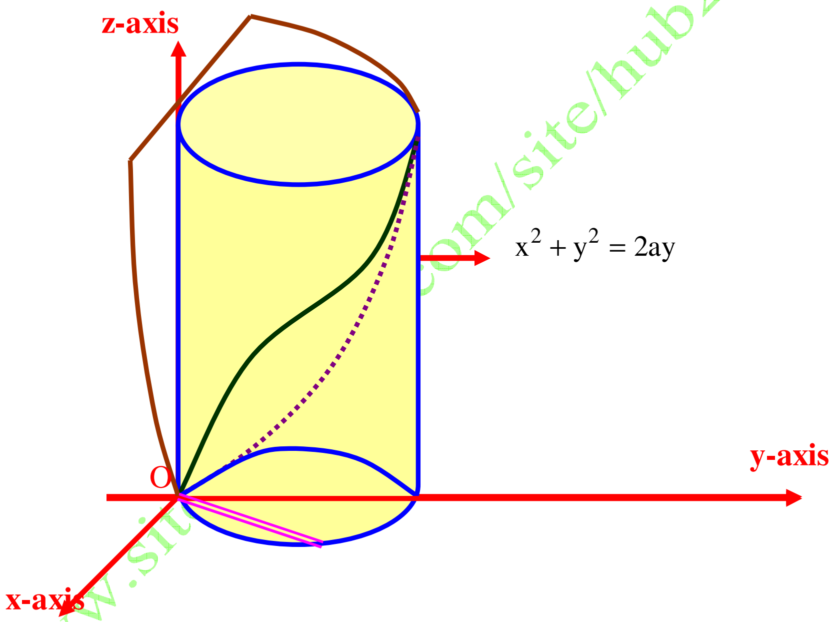
$\therefore$ Required volume $= 8\int_{-\pi/2}^{\pi/2} 2\cos\theta . 2\cos\theta d\theta = 32\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = 64\int_{0}^{\pi/2} \cos^2\theta d\theta$

$= 64 \times \frac{1}{2} \times \frac{\pi}{2} = 16\pi$. Cubic units. Ans.

Q.No.2.: Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

Sol.: The required volume $V = \iint zdxdy = \iint \frac{x^2 + y^2}{a} dxdy$,

over the circle $x^2 + y^2 = 2ay$.



To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r .$$

Then $\iint\limits_{R_{xy}} f(x, y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

$\therefore$ Paraboloid $x^2 + y^2 = az \Rightarrow z = \frac{r^2}{a}$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover the circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π.

$$\therefore \text{Required volume } = \iint \frac{x^2+y^2}{a}dxdy = \int_0^{\pi}\int_0^{2a\sin\theta}\frac{r^2}{a}.r\,d\theta\,dr = \frac{1}{a}\int_0^{\pi}\left(\int_0^{2a\sin\theta} r^3dr\right)d\theta$$

$$= \frac{1}{a}\int_0^{\pi}\left(\left|\frac{r^4}{4}\right|_0^{2a\sin\theta}\right)d\theta = 4a^3\int_0^{\pi}\sin^4\theta d\theta = 4a^3.2\int_0^{\pi/2}\sin^4\theta d\theta$$

$$= 8a^3\times\left(\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right) = \frac{3\pi a^3}{2}.$$ Cubic units Ans.

Q.No.3.: Find the volume bounded by the xy-plane, the paraboloid $2z = x^2+y^2$ and the cylinder $x^2+y^2=4$.

Sol.: Required volume is found by integrating $z = \frac{x^2+y^2}{2}$ over $x^2+y^2=4$.

i. e. $V = \iint zdxdy = \iint\limits_{x^2+y^2\le4}\frac{x^2+y^2}{2}dxdy$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r.$$

Then $\iint\limits_{R_{xy}} f(x,y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Paraboloid $2z = x^2+y^2 \Rightarrow z = \frac{x^2+y^2}{2} = \frac{r^2}{2}$ and

cylinder $x^2+y^2 = 4 \Rightarrow r^2 = 4$, $\therefore r = 2, -2$ (Rejected) $\therefore r = 2$

To cover full circle, r varies from 0 to 2 and θ varies from 0 to 2π

$$V = \int_0^{2\pi}\left(\int_0^2\frac{r^2}{2}rdr\right)d\theta \Rightarrow \frac{1}{2}\int_0^{2\pi}\left(\int_0^2 r^3dr\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left|\frac{r^4}{4}\right|_0^2 d\theta = \frac{1}{2}\int_0^{2\pi}4d\theta$$

$$= 2\int_0^{2\pi} d\theta = 2\times 2\pi = 4\pi \text{. Cubic units. Ans.}$$

Q.No.4.: Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

Sol.: The required volume $= 2\iint zdxdy = 2\iint \sqrt{2ax}dxdy$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ),

we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Then $\iint\limits_{R_{xy}} f(x, y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Now $x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r[r - 2a\cos\theta] = 0$.

So r varies from 0 to $2a\cos\theta$ and θ varies from 0 to π.

$\therefore$ Required volume $= 2\iint \sqrt{2ax}dxdy = 2\int_0^{\pi}\int_0^{2a\cos\theta} \sqrt{2ar\cos\theta}rdrd\theta$

$$= 4\int_0^{\pi/2}\int_0^{(2a\cos\theta)} \sqrt{2a\cos\theta}r^{3/2}drd\theta = 4\int_0^{\pi/2}\left[\int_0^{2a\cos\theta} r^{3/2}\,dr\right]\sqrt{2a\cos\theta}d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{2ar\cos\theta}\left[\frac{r^{5/2}}{\frac{5}{2}}\right]_0^{2a\cos\theta} d\theta = 4\int_0^{\pi/2} \sqrt{2ar\cos\theta}\,\frac{2}{5}\left[(2a)^{5/2}\cos^{5/2}\theta\right]d\theta$$

$$= \frac{2^6}{5}a^3\left[\int_0^{\pi/2} \cos^3\theta d\theta\right] = \frac{64a^3}{5}.\frac{2}{3\times 1} = \frac{128}{15}a^3 \text{. Cubic units. Ans.}$$

Q.No.5.: Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.

Sol.: The required volume $= \iint zdxdy = \iint \frac{x^2 + y^2}{2a}dxdy$

To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ), we have put $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Then $\iint\limits_{R_{xy}} f(x, y)dxdy = \iint\limits_{R'_{r\theta}} f(r\cos\theta, r\sin\theta).r\,dr\,d\theta$.

Since $x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r[r - 2a\cos\theta] = 0$

To cover the circle r varies from 0 to $2a\cos\theta$ and θ varies from 0 to π.

$$\therefore \text{Required volume } = \int_0^{\pi}\int_0^{2a\cos\theta} \frac{r^2}{2a} r\,drd\theta = \frac{1}{2a}\int_0^{\pi} \left|\frac{r^4}{4}\right|_0^{2a\cos\theta} d\theta = \frac{1}{2a}\times\frac{16a^4}{4}\int_0^{\pi} \cos^4\theta d\theta$$

$$= 2a^3 \times 2\int_0^{\pi/2} \cos^4\theta d\theta = 4a^3 \times \frac{3}{4\times 2}\times\frac{\pi}{2} = \frac{3\pi a^3}{4}. \text{ Cubic units. Ans.}$$

Q.No.6.: Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, and $y = -a$, $y = a$.

Sol.: Required volume $= \iint zdxdy = \int_{-a}^{a}\int_{-a}^{a}\left(x^2 + y^2\right)dxdy = \int_{-a}^{a}\left|x^2y + \frac{y^3}{3}\right|_{-a}^{a} dx$

$$= \int_{-a}^{a}\left(x^2a + \frac{a^3}{3} + x^2a + \frac{a^3}{3}\right)dx = 2\int_{-a}^{a}\left(x^2a + \frac{a^3}{3}\right)dx$$

$$= 2\left|\frac{x^3}{3}a + \frac{a^3}{3}x\right|_{-a}^{a} = \frac{2}{3}\left(a^4 + a^4 + a^4 + a^4\right) = \frac{8}{3}a^4. \text{ Cubic units. Ans.}$$

Q.No.7.: Find the volume V of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.

Sol.: $V = \iint\limits_{R} zdxdy$.

R is a region defined by $x^2 + y^2 - 2ay = 0$.

Putting $z = \sqrt{4a^2 - \left(x^2 + y^2\right)}$

$$V = \iint_R \sqrt{4a^2 - \left(x^2 + y^2\right)}\,dxdy\,.$$

Putting $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

$$V = 2\int_0^{\pi}\int_0^{2a\sin\theta} \sqrt{4a^2 - \left(r^2\cos^2\theta + r^2\sin^2\theta\right)}\,rdrd\theta = 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\,rdrd\theta$$

Putting $4a^2 - r^2 = t^2 \Rightarrow -2rdr = 2tdt$

$$V = 4\int_0^{\pi/2}\int_{2a}^{2a\cos\theta} t(-tdt)d\theta = 4\int_0^{\pi/2}\left[\frac{-t^3}{3}\right]_{2a}^{2a\cos\theta} d\theta = 4\int_0^{\pi/2}\frac{(2a)^3}{3}\left[1 - \cos^3\theta\right]d\theta$$

$$= 4\times\frac{8a^3}{3}\int_0^{\pi/2} d\theta - 4\times\frac{8a^3}{3}\int_0^{\pi/2}\cos^2\theta.\cos\theta d\theta = 4\left[\frac{8a^3}{3}\left(\frac{\pi}{2}\right) - \frac{8a^3}{3}\int_0^{\pi/2}\left[1 - \sin^2\theta\right]\cos\theta d\theta\right]$$

$$= 4\left[\frac{8a^3}{3}.\frac{\pi}{2} - \frac{8a^3}{3}\int_0^{\pi/2}\cos\theta d\theta + \frac{8a^3}{3}\int_0^{\pi/2}\sin^2\theta\cos\theta d\theta\right]$$

$$= 4\left[\frac{8a^3}{3}.\frac{\pi}{2} - \frac{8a^3}{3} + \frac{8a^3}{3}.\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\right] = 4\times\frac{8a^3}{3}\left[\frac{\pi}{2} - 1 + \frac{1}{3}\right]$$

$$= 4\times\frac{8a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right] = \frac{16a^3}{3}\left[\pi - \frac{4}{3}\right].\ \text{Ans.}$$

Q.No.8.: Find, by double integration, the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol.: The volume of the required ellipsoid is equal to 8 times the volume of ellipsoid in any one octant (say XOY).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,.$$

For plane XOY: $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$.

Required volume = $8\int_0^a ydx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} zdydx = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dydx$

Putting $\sqrt{1-\frac{x^2}{a^2}} = t$

$$V = 8\int_0^a \left(\int_0^{bt} c\sqrt{t^2-\frac{y^2}{b^2}}dy\right)dx = 8\int_0^a \left(\int_0^{bt} \frac{c}{b}\sqrt{(bt)^2-y^2}\,dy\right)dx$$

$$= \frac{8c}{b}\left[\frac{y}{2}\sqrt{(bt)^2-y^2}+\frac{(bt)^2}{2}\sin^{-1}\frac{y}{bt}\right]_0^{bt} dx = \frac{8c}{b}\int_0^a \left[\frac{(bt)^2}{2}\times\frac{\pi}{2}\right]dx = \frac{dc}{b}\times\frac{b^2}{2}\times\frac{\pi}{2}\int_0^a t^2dx$$

$$= \frac{8bc}{4}\times\pi\int_0^a \left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\int_0^a dx - \frac{2\pi bc}{a^2}\int_0^a x^2dx = 2\pi bc[x]_0^a - \frac{2\pi bc}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= 2\pi abc - \frac{2\pi abc}{3} = \frac{4}{3}\pi abc \text{ Cubic units.}$$

Q.No.9.: Find, by double integration, the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol.: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the given equation of tetrahedron.

$$\Rightarrow z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

For plane XOY: $z = 0$, $\frac{x}{a}+\frac{y}{b}=1$

$$\text{Volume} = \int_0^a \int_0^{b(1-\frac{x}{a})} zdydx = \int_0^a \left(\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right)dx$$

$$= \int_0^a \left[c\left(b-\frac{xb}{a}\right)-\frac{cx}{a}\left(b-\frac{xb}{a}\right)-\frac{c}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= bc[x]_0^b - \frac{bc}{2a}\left[x^2\right]_0^a - \frac{bc}{2a}\left[x^2\right]_0^a + \frac{bc}{3a^2}\left[x^3\right]_0^a + \left[-\left(\frac{bc}{2}[x]_0^a - \frac{bc}{2\times 3a^2}\left[x^3\right]_0^a - \frac{bc}{2a}\left[x^2\right]_0^a\right)\right]$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} \mp \left(\frac{abc}{2} + \frac{abc}{2\times 3} - \frac{abc}{2} \right)$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} - \frac{abc}{2} - \frac{abc}{2\times 3} + \frac{abc}{2}$$

$$= abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} - \frac{abc}{2} - \frac{abc}{6} + \frac{abc}{2} = \frac{abc}{6}. \quad \text{Cubic units.}$$

Q.No.10.: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol.: $z = \sqrt{a^2 - x^2}$, $y = \sqrt{a^2 - x^2}$.

Required volume $= 8\iint zdydx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\,dxdy$

$$= 8\int_0^a \sqrt{a^2 - x^2}\,[y]_a^{\sqrt{a^2-x^2}}\,dx = 8\int_0^a \left(a^2 - x^2\right)dx = 8a^2[x]_0^a - \frac{8}{3}\left[x^3\right]_0^a$$

$$= 8a^3 - \frac{8a^3}{3} = \frac{16}{3}a^3. \quad \text{Cubic units}$$

Q.No.11.: Find, by double integration, the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$.

Sol.: The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $z = a^2 - y^2 - x^2$. On the account of symmetry of the sphere, half of it lies above the plane XOY and half below it.

$\therefore$ Required volume $= 2\int\int zdydx$,

where $z = \sqrt{\left(a^2 - y^2 - x^2\right)}$, and the region of integration is the area inside the circle $x^2 + y^2 = ay$.

On the account of symmetry, the volume above the two parts of circle $x^2 + y^2 = ay$ in the first and the second quadrants are equal.

Total volume required $= 2\times 2\int\int_R \sqrt{\left(a^2 - y^2 - x^2\right)}dydx$

where R is half of the circle $x^2 + y^2 = ay$ lying in the first quadrant.

Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$.

Equation of the circle $x^2 + y^2 = ay$ becomes

$r^2 = ar\sin\theta \Rightarrow r = a\sin\theta$

Thus the region of integration is bounded by $r = 0, r = a\sin\theta$ and $\theta = 0, \theta = \dfrac{\pi}{2}$.

$\therefore$ Required volume $V = 4\int_0^{\pi/2}\int_0^{a\sin\theta}\sqrt{a^2 - r^2}\; r\,dr d\theta$

Now put $a^2 - r^2 = t^2 \Rightarrow rdr = -t$

$$V = 4\int_0^{\pi/2}\int_a^{a\cos\theta} t^{2/2}(-t)dtd\theta = 4\int_0^{\pi/2}\int_a^{a\cos\theta} t^2 dtd\theta = -\frac{4}{3}\int_0^{\pi/2}\left[t^3\right]_a^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left(a^3\cos^3\theta - a^3\right)d\theta = -\frac{4}{3}\left[a^3 \times \frac{2.1}{3.1} - a^3 \times \frac{\pi}{2}\right] = \frac{2}{9}a^3[3\pi - 4].$$ Cubic units

Q.No.12.: Find, by double integration, the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

Sol.: The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $z = a^2 - y^2 - x^2$. On the account of symmetry of the sphere, half of it lies above the plane XOY and half below it.

$\therefore$ Required volume $= 2\int\int zdydx$,

where $z = \sqrt{\left(a^2 - y^2 - x^2\right)}$, and the region of integration is the area inside the circle $x^2 + y^2 = ax$.

On the account of symmetry, the volume above the two parts of circle $x^2 + y^2 = ay$ in the first and the second quadrants are equal.

Total volume required $= 2 \times 2\int\int_R \sqrt{\left(a^2 - y^2 - x^2\right)}dydx$

where R is half of the circle $x^2 + y^2 = ax$ lying in the first quadrant.

Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$.

Equation of the circle $x^2 + y^2 = ax$ becomes

$r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$

Thus the region of integration is bounded by $r = 0, r = a\cos\theta$ and $\theta = 0, \theta = \frac{\pi}{2}$.

$\therefore$ Required volume $V = 4\int_0^{\pi/2}\int_0^{a\cos\theta}\sqrt{a^2 - r^2}\; r\,dr d\theta$

$$V = \frac{4}{-2}\int_0^{\pi/2}\int_a^{a\cos\theta}\sqrt{a^2-r^2}(-2r)drd\theta = -2\int_0^{\pi/2}\left[\frac{\left(a^2-r^2\right)}{3/2}\right]_a^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right)d\theta = -\frac{4}{3}\left[a^3 \times \frac{2.1}{3.1} - a^3 \times \frac{\pi}{2}\right] = \frac{2}{9}a^3[3\pi - 4].$$ Cubic units

Q.No.13.: Find, by double integration, the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Sol.: $z = \sqrt{x^2 + y^2 - 1}$, $y = \sqrt{4 - x^2}$

Volume $= 2\int\int_R z dydx = 2\int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{x^2 + y^2 - 1}dydx$

Putting $r\cos\theta = x$, $r\sin\theta = y$ and $|J| = r$

$x^2 + y^2 = r^2$

Also $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$.

$$V = 2\int_0^{2\pi}\int_1^2\sqrt{r^2 - 1}\; rdrd\theta.$$

Putting $r^2 - 1 = t^2 \Rightarrow 2rdr = 2tdt$.

$$V = 2\int_0^{2\pi}\int_0^{\sqrt{3}} t^2 dtd\theta \Rightarrow \int_0^{\pi}\left[\frac{t^3}{3}\right]_0^{\sqrt{3}} d\theta = 2\int_0^{2\pi}\sqrt{3}d\theta = 4\sqrt{3}\pi.$$ Cubic units.

Q.No.14.: Find, by double integration, the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.

Sol.: Given $z = x + y$, $4x^2 + 9y^2 = 36$.

Required volume $= \int\int_R z\,dy\,dx = \int_0^3 \left[\int_0^{\sqrt{\frac{36-4x^2}{9}}} (x+y)\,dy\right]dx$

$= \int_0^3 \left[x\frac{\sqrt{36-4x^2}}{3} + \frac{1}{2}\frac{(36-4x^2)}{9}\right]dx = \frac{1}{3}\int_0^3 x\sqrt{36-4x^2}\,dx + \frac{1}{18}\int_0^3 (36-4x^2)\,dx$

I II

For I, putting $36-4x^2 = t^2$, $-8x\,dx = 2t\,dt \Rightarrow x\,dx = -\frac{t}{4}dt$

$V = -\frac{1}{3}\int_6^0 \frac{t^2}{4}dt + 2[x]_0^3 - \frac{4}{18}\times\frac{1}{3}\left[x^3\right]_0^3 = \frac{1}{3}\int_0^6 \frac{t^2}{4}dt + 6 - \frac{4}{18}\times\frac{1}{3}[27]$

$= \frac{1}{12}\times\frac{1}{3}\times 6\times 6\times 6 + 6 - \frac{4}{18}\times\frac{1}{3}\times 3\times 3\times 3 = 6+6-2 = 10$. Cubic units

Q.No.15.: Find, by double integration, the volume bounded by the plane z = 0, surface $z = x^2+y^2+2$ and the cylinder $x^2+y^2 = 4$.

Sol.: Given $z = x^2+y^2+2$, $x^2+y^2 = 4$

Volume of required region $= 4\int\int_R z\,dy\,dx$

$V = 4\int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left[x^2+y^2+2\right]dy\right)dx = 4\int_0^2 \left[x^2[y]_0^{\sqrt{4-x^2}} + \frac{1}{3}\left[y^3\right]_0^{\sqrt{4-x^2}} + 2[y]_0^{\sqrt{4-x^2}}\right]dx$

$= 4\int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + 2\sqrt{4-x^2}\right]dx$

I II III

For I and II

Putting $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$= 4\int_0^{\pi/2} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos d\theta + \frac{4}{3}\int_0^{\pi/2} (4-4\sin^2\theta)^{3/2} 2\cos\theta + 2\times 4\int_0^2 \sqrt{4-x^2}$

$= 4\int_0^{\pi/2} 16\sin^2\theta\cos^2\theta\,d\theta + \frac{4}{3}\int_0^{\pi/2} 8\cos^3\theta . 2\cos\theta\,d\theta + 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$

$$= 64\int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta + \frac{32}{3}\times 2\int_0^{\pi/2} \cos^4\theta + 8\left[2\times\frac{\pi}{2}\right]$$

$$= 64\times\frac{1.1}{4.2}\times\frac{\pi}{2} + \frac{64}{3}\times\frac{3.1}{4.2}\times\frac{\pi}{2} + 8\pi = 4\pi + 4\pi + 8\pi = 16\pi.$$ Cubic units

Q.No.16.: Find, by double integration, the volume bounded by the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.

Sol.: Given $x + y + z = 3 \Rightarrow z = 3 - x - y$

$x^2 - y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}$

$$\text{Volume} = 4\iint_R zdydx = 4\int_0^1 \int_0^{R/\sqrt{1-x^2}} 3 - x - ydydx$$

$$= 4\int_0^1 3[y]_0^{\sqrt{1-x^2}} - x[y]_0^{\sqrt{1-x^2}} - \frac{1}{2}\left[y^2\right]_0^{\sqrt{1-x^2}} dx = 4\int_0^1 \underset{\text{I}}{3\sqrt{1-x^2}} - \underset{\text{II}}{x\sqrt{1-x^2}} - \underset{\text{III}}{\frac{1}{2}\left(1-x^2\right)}dx$$

For II, $1 - x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$.

$$V = 4.3\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 - 4\int_1^0 -t^2dt - \frac{4}{2}\left[x - \frac{1}{3}x^3\right]_0^1$$

$$= 4.3\left[\frac{1}{2}\times\frac{\pi}{2}\right] + 4\left[\frac{t^3}{3}\right]_1^0 - \frac{4}{2}\left[1 - \frac{1}{3}\right] = 4.3\times\frac{\pi}{4} - \frac{4}{3} - \frac{4}{3} = 4.\frac{3\pi}{4} - \frac{2.4}{3} = 3\pi - \frac{8}{3}$$ Cubic units

Q.No.17.:A rectangular prism is formed by the planes whose equations are ay = bx, y = 0 and $x = a$. Find, by double integration, the volume of this prism between the plane z = 0 and the surface $z = c + xy$.

Sol.: Volume = $4\iint_R zdydx$

$$V = \int_0^a \int_0^{\frac{b}{a}x} (c + xy)dydx = \int_0^a \left(c[y]_0^{\frac{bx}{a}} + \frac{x}{2}\left[y^2\right]_0^{\frac{bx}{a}}\right)dx = \int_0^a \left(\frac{bcx}{a} + \frac{b^2x^3}{2a^2}\right)dx$$

$$= \frac{bc}{a}\times\frac{1}{2}\left[x^2\right]_0^a + \frac{b^2}{2a^2}\times\frac{1}{4}\left[x^4\right]_0^a = \frac{bc}{2a}\times a^2 + \frac{b^2}{8a^2}\times a^4$$

$$=\frac{abc}{2}+\frac{a^2b^2}{8}=\frac{ab}{8}(4c+ab).\text{ Cubic units}$$

Q.No.18.: Find, by double integration, the volume of the sphere $x^2+y^2+z^2=9$.

Sol.: Required volume will be equal to 8times the volume of XOY, $z=0$

$$z=\sqrt{9-x^2-y^2}$$

$$\text{Volume }=8\int\int_R zdydx=8\int_0^3\int_0^{\sqrt{9-x^2}}\sqrt{9-x^2-y^2}dydx$$

Put $x=r\cos\theta$, $y=r\sin\theta$ and $|J|=r$

$$x^2+y^2=r^2$$

$$\text{Volume }=8\int_0^{\pi/2}\int_0^3\sqrt{9-r^2}\,rdrd\theta$$

Put $9-r^2=t^2\Rightarrow -2rdr=2tdt\Rightarrow rdr=-tdt$

$$V=8\int_0^{\pi/2}\int_3^0 -t^2dtd\theta=8\int_0^{\pi/2}\int_0^3 t^2dtd\theta=\frac{8}{3}\int_0^{\pi/2}\left[t^3\right]_0^3d\theta=8\int_0^{\pi/2}\int_0^3 t^2dtd\theta$$

$$=\frac{8}{3}\int_0^{\pi/2}\left[t^3\right]_0^3d\theta=\frac{8}{3}\times3^3\times\frac{\pi}{2}=36\pi.\text{ Cubic units}$$

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Home Assignments