9th Topic

Matrices

Reduction to Diagonal form (Modal Matrix)

Reduction to Quadratic form to Canonical form

Nature. Index, signature of Quadratic form

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Reduction to Diagonal form:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.

Proof:

Let A be a square matrix of order 3.

Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values

and $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1 X_2 X_3] = [AX_1 \ AX_2 \ AX_3] = [\lambda_1 X_1 \ \lambda_2 X_2 \ \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD,$$

where D is the diagonal matrix.

$\therefore P^{-1}AP = P^{-1}PD = D$,

which proves the theorem.

Remarks:

1. The matrix P, which diagonalises A is called the **modal matrix** of A and the resulting diagonal matrix D is known as a **spectral matrix** of A.
2. The diagonal matrix has the eigen values of A as its diagonal elements.
3. The matrix P is found by grouping the eigen vectors of A into a square matrix.

Similarities of matrices:

A square matrix $\hat{A}$ of order n is called **similar** to a square matrix A of order n if $\hat{A} = P^{-1}AP$ for some non-singular $n \times n$ matrix P.

Similarity Transformation: This transformation of a matrix A by a non-singular matrix P to $\hat{A}$ is called a **similarity transformation.**

Remarks:

1. If the matrix $\hat{A}$ is similar to the matrix A, then $\hat{A}$ has the same eigen values of A.
2. If **X** is an eigen vector of A, then $Y = P^{-1}X$ is an eigen vector of $\hat{A}$ corresponding to the same eigen value.

Powers of a matrix:

Result: Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.

Proof: Let A be the square matrix.

Then, a non-singular matrix P can be found such that $D = P^{-1}AP$.

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$. $\quad [\because PP^{-1} = I]$

Similarly, $D^3 = P^{-1}A^3P$ and in general $D^n = P^{-1}A^nP$. (i)

To obtain A^n:

Pre-multiply (i) by P and post-multiply by P^{-1}, we get

$PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n$ which gives A^n.

Thus, $A^n = PD^nP^{-1}$, where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$.

Working procedure:

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the normal matrix A.
3. Find the diagonal matrix D from $D = P^{-1}DP$.
4. Obtain A^n from $A^n = PDP^{-1}$.

Quadratic Forms:

Definition: A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

For examples:

(i) $ax^2 + 2hxy + by^2$

(ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

Theorem: Every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = X'AX$, so that the matrix A is always symmetric, where $A = \lfloor a_{ij} \rfloor$ and $X = [x_1,\ x_2 \ldots\ldots, x_n]$.

Proof: In n-variables $x_1, x_2, \ldots\ldots\ldots\ldots x_n$, the general quadratic form is $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j$.

In the expansion, the co-efficient of $x_ix_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ij}$ where $a_{ij} = a_{ji}$ and $a_{ii} = b_{ii}$

$$\therefore \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j, \text{ where } a_{ij} = \frac{1}{2}(b_{ij} + b_{ji}).$$

Hence, every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j = X'AX$, so that the matrix A is always symmetric, where $A = [a_{ij}]$ and $X = [x_1,\ x_2 \ldots\ldots, x_n]$.

Now writing the above said examples of quadratic forms in matrix form, we get

(i). $ax^2 + 2hxy + by^2 = [xy]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(ii). $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = [x \quad y \quad z]\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

(iii). $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$

$$= [x \quad y \quad z \quad w]\begin{bmatrix} a & h & f & \ell \\ h & b & g & m \\ f & g & c & n \\ \ell & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Linear Transformation of a Quadratic form:

Let $X'AX$ be a quadratic form in n-variables and let $X = PY$, (i)

where P is a non-singular matrix, be the non-singular transformation.

From (i), we get $X' = (PY)' = Y'P'$.

Thus $X'AX = Y'P'APY = Y'(P'AP)Y = Y'BY$, (ii)

where $B = P'AP$.

Therefore, $Y'BY$ is also a quadratic form in n-variables.

Hence, it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$ and $B = P'AP$.

Note:

(i) Here $B' = (P'AP)' = P'AP = B$.

(ii) $\rho(B) = \rho(A)$.

$\therefore$ A and B are **congruent matrices**.

Canonical Form:

If a **real quadratic form** be expressed as a **sum or difference of the square of new variables** by means of any real non-singular linear transformation, then the later quadratic expression is called a canonical form of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j$ can be reduced to the quadratic form

$Y'BY = \sum_{i=1}^{n} \lambda_i y_1^2$ by a non-singular linear transformation X = PY, then $Y'BY$ is called the **canonical form** of the given one.

$\therefore$ If $B = P'AP = \text{diag.}(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$,

then $X'AX = Y'BY = \sum_{i=1}^{n} \lambda_i y_1^2$.

Remarks:

1. Here some of λ_1 (eigen values) may be positive or negative or zero.
2. A quadratic form is said to be real if the elements of the symmetric matrix are real.
3. If $\rho(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

Index and Signature of the quadratic form:

Index:

The number p of positive terms in the canonical form is called the index of the quadratic form.

Signature:

(The number of terms) – (The number of negative terms)

i.e., $p - (r - p) = 2p - r$ is called signature of the quadratic form, where $\rho(A) = r$.

Definite, Semi-definite and Indefinite Real Quadratic form:

Let $X'AX$ be real quadratic form in n-variables $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ with rank r and index p.

Then, we say that the quadratic form is

(i) positive definite if r = n, p = r

(ii) negative definite if r = n, p = 0

(iii) positive semi-definite if r < n, p = r and

(iv) negative semi-definite if $r < n$, $p = 0$.

If the canonical form has both positive and negative terms, the quadratic form is said to be indefinite.

Remarks: If $X'AX$ is positive definite then $|A| > 0$.

OR

Nature of Quadratic Form:

A real quadratic form $X'AX$ in a variables said to be

(i) Positive definite if all the eigen values of $A > 0$.

(ii) Negative definite if all the eigen values of $A < 0$.

(iii) Positive semi-definite if all the eigen values of $A \geq 0$ and at least one eigen value = 0.

(iv) Negative semi-definite if all the eigen values of $A \leq 0$ and at least one eigen value = 0.

(v) Indefinite if some of the eigen values of A are positive and others negative.

Law-of-Inertia of Quadratic form:

Statement:

"The index of real quadratic form is invariant under real non-singular transformation".

Reduction to Canonical form by Orthogonal Transformation:

Let $X'AX$ be a given quadratic form. The modal matrix B of A is that matrix whose columns are characteristic vectors of A. If B represent the orthogonal matrix of A (the normalized modal matrix of A whose column vectors are pair-wise orthogonal),

then $X = BY$ will reduce $X'AX$ to $Y'DY$,

where $D = \text{diag}\left(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\right)$ and $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are characteristic roots of A.

Remarks: This method works successfully if the characteristic vectors A are linearly dependent which are pairwise orthogonal.

Determination of real symmetric matrix C of the quadratic form:

Q.No.1.: Find a real symmetric matrix C of the **quadratic form**

$$Q = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3.$$

Sol.: The coefficient matrix of Q is $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$,

Thus C = symmetric matrix = $\frac{1}{2}\left[A + A^T\right]$.

$$C = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

Remarks: The simplest way writing C is

1. Put coefficients of square terms as the diagonal elements.

2. Place $\frac{1}{2}$ of a_{ij}, the coefficients of x_i, x_j, x_{ij} and the remaining $\frac{1}{2}$ of a_{ij}, at c_{ji}, i.e.,

$c_{ij} = c_{ji} = \frac{1}{2}a_{ij}$ such that $c_{ij} + c_{ji} = \frac{1}{2}\left(a_{ij} + a_{ij}\right) = a_{ij}$.

Determine the nature, index and signature

Q.No.1.: Determine the nature, index and signature of the quadratic form

$$2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3.$$

Sol.: The real symmetric matrix A associated with the quadratic form is

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$

Its characteristic equation is $\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0 \Rightarrow (\lambda - 1)\left(\lambda - \left(3 + \sqrt{8}\right)\right)\left(\lambda - \left(3 - \sqrt{8}\right)\right) = 0.$$

The eigen values are $\lambda =$ 1, 0.1715, 3.1715, which are all positive.

Since, we know that if all the eigen values of A > 0, then the quadratic form is positive definite

So, here quadratic form is positive define.

Index: 3, Signature: $3-0=3$.

Q.No.2.: Find the nature, index and signature of quadratic form $2x_1x_2+2x_1x_3+2x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$.

Its characteristic equation is $\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0$.

$\Rightarrow \lambda^3-3\lambda-2=0 \Rightarrow (\lambda+1)^2(\lambda-2)=0$.

The eigen values are $2,\ -1,\ -1$, some are positive and some are negative.

So quadratic form is indefinite.

Index: 1, Signature: $1-2=-1$.

Q.No.3.: Identify the nature, index and signature of the quadratic form $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_3x_1-4x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}.$$

Its characteristic equation is $|A-\lambda I|=\begin{vmatrix}1-\lambda & -2 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=\lambda^2(\lambda-6)=0$.

Eigen values are $\lambda=$ 0, 0, 6.

So quadratic form is positive semi definite.

Index: 3, Signature: 3.

Q.No.4.: Classify the quadratic form and find the index and signature of $-3x_1^2-3x_2^2-3x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$.

Sol.: The real symmetric matrix A associated with the quadratic form is

$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$.

Its characteristic equation is $\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 + 9\lambda^2 + 24\lambda + 16 = (\lambda+1)(\lambda+4)^2 = 0$.

All the eigen values $-1, \ -4, \ -4$, are negative.

So quadratic form is negative definite.

Index: 0, Signature: $0-3=-3$

Note: $Q = 3x_1^2 + 3x_2^2 - 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ is positive definite.

Reduction to diagonal form

Q.No.1.: Find the matrix P which **diagonalises** the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify that

$P^{-1}AP = D$, where D is diagonal matrix , hence find A^6.

Sol.: *Since we know, if a square matrix A of or order n has n linearly independent eigen vectors, then a matrix P can be found such that* $P^{-1}AP$ *is a diagonal matrix.*

A is diagonalizable by P whose columns are the linearly independent eigen vectors of A.

The characteristic equation of A is $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0$.

So $\lambda = 2, \ 5$ are two distinct eigen values of A.

For $\lambda = 2$: $2x_1 + x_2 = 0$, $x_2 = -2x_1$, $X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For $\lambda = 5$: $-x_1 + x_2 = 0$, $x_2 = x_1$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, the matrix P which diagonalises A is $P = [X_1, \ X_2] = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

Verification: Since $P^{-1}=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$.

Therefore $P^{-1}AP=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2 & -2\\ 10 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$

$$P^{-1}AP=\frac{1}{3}\begin{bmatrix}6 & 0\\ 0 & 15\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}=D=\text{diagonal matrix}$$

D contain eigen values 2, 5 as diagonal elements.

To find A^6:

$$A^6=PD^6P^{-1}=\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}2^6 & 0\\ 0 & 5^6\end{bmatrix}\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$$

$$A^6=\frac{1}{3}\begin{bmatrix}64 & 15625\\ 128 & 15625\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}31314 & 15561\\ 31122 & 15753\end{bmatrix}$$

$$\therefore A^6=\begin{bmatrix}10438 & 5187\\ 10374 & 5251\end{bmatrix}. \text{ Ans.}$$

Q.No.2.: Define modal matrix & spectral matrix of a matrix.

Reduce the matrix $A=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$ into a diagonal matrix, by finding its modal matrix P, and hence write its spectral matrix.

Sol.: 1st Part: We know that if a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Modal matrix: The matrix P, which diagonalises A is called the **modal matrix** of A.

Spectral matrix: The resulting diagonal matrix D is known as a **spectral matrix** of A.

2nd Part:

The characteristic equation of A is $\begin{vmatrix}1-\lambda & 0\\ 2 & -1-\lambda\end{vmatrix}=0$

$$\Rightarrow(1-\lambda)(-1-\lambda)=0\Rightarrow(1-\lambda)(1+\lambda)=0\Rightarrow\lambda=1,-1.$$

So eigen values of A are $\lambda=1,-1$.

For $\lambda=-1$, we have $2x_1=0\Rightarrow x_1=0$

Thus $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $\lambda = 1$, we have $2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$

Thus $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, the **modal matrix** is $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Spectral matrix is $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Also $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Verification: $A = PDP^{-1}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Q.No.3.: **Diagonalise** $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^8. Find the modal matrix.

Sol.: The non-singular square matrix P containing eigen vectors of A as columns, diagonalises A.

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-3)(\lambda-4) = 0$.

So eigen values of A are $\lambda = -1,\ 3,\ 4$.

For $\lambda = -1$, we have

$$2x_1 + 6x_2 + x_3 = 0$$
$$x_1 + 3x_2 + 0 = 0$$
$$4x_3 = 0$$

$\therefore x_3 = 0\ x_1 = -3x_2$. Thus $X_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 3$, we have $-2x_1 + 6x_2 + x_3 = 0$

$$x_1 - x_2 = 0$$

$\therefore x_3 = x_2,\ x_3 = -4x_2$. Thus $X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$.

For $\lambda = 4$, we have $-3x_1 + 6x_2 + x_3 = 0$

$$x_2 - 2x_2 = 0$$

$$-x_3 = 0$$

$\therefore x_3 = 0,\ x_2 = 2x_2$. Thus $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$ is the modal matrix.

To find P^{-1}: Now $\begin{bmatrix} -3 & 1 & 2 & : & 1 & 0 & 0 \\ 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -4 & 0 & : & 0 & 0 & 1 \end{bmatrix}$

Operating $R_{12},\ R_{21(3)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 4 & 5 & : & 1 & 3 & 0 \\ 0 & -4 & 0 & : & 0 & 0 & 1 \end{bmatrix}$

Operating $R_{32(1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 4 & 5 & : & 1 & 3 & 0 \\ 0 & 0 & 5 & : & 1 & 3 & 1 \end{bmatrix}$

Operating $R_{2\left(\frac{1}{4}\right)},\ R_{3\left(\frac{1}{5}\right)}$, we get $\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & : & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Operating $R_{23\left(-\frac{5}{4}\right)}$, $R_{3(-1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 0 & : & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & : & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Operating $R_{12(-1)}$, we get $\sim \begin{bmatrix} 1 & 1 & 0 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{20} \\ 0 & 1 & 0 & : & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Thus $P^{-1} = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$.

Diagonalisation:

$$D = P^{-1}AP = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -15 \\ 16 & 48 & 16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -20 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 80 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

To find A^8 :

Now $A^8 = PDP^{-1} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} (-1)^8 & 0 & 0 \\ 0 & 3^8 & 0 \\ 0 & 0 & 4^8 \end{bmatrix}\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 65536 \end{bmatrix}\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$=\frac{1}{20}\begin{bmatrix}-3 & 6561 & 131072\\ 1 & 6561 & 65536\\ 0 & -26244 & 0\end{bmatrix}\left[\begin{bmatrix}-4 & 8 & 1\\ 0 & 0 & 5\\ 4 & 12 & 4\end{bmatrix}\right]=\frac{1}{20}\begin{bmatrix}524300 & 1572840 & 491480\\ 262140 & 786440 & 229340\\ 0 & 0 & 131220\end{bmatrix}$$

$$A^8=\begin{bmatrix}26215 & 78642 & 24574\\ 13107 & 39322 & 11467\\ 0 & 0 & 6561\end{bmatrix}. \text{ Ans}$$

Q.No.4.: Find a matrix P, which transforms the matrix $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ to **diagonal form**.

Hence, calculate A^4.

Sol.: *Since we know, if a square matrix A of or order n has n linearly independent eigen vectors, then a matrix P can be found such that* $P^{-1}AP$ *is a diagonal matrix.*

The eigen values of A are $-2,\ 3,\ 6$ and

the eigen vectors are $(-1,\ 0,\ 0),\ (1,\ -1,\ 1),\ (1,\ 2,\ 1)$.

Writing these eigen vectors as the three columns, the required transformation matrix (modal matrix) is

$$P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}.$$

To find P^{-1}: $|P|=\begin{vmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ (say).

$A_1-3,\ B_1=2,\ C_1=1,\ A_2=0,\ B_2=-2,\ C_2=2,\ A_3=3,\ B_2=3,\ C_3=1$.

Also $|P|=a_1A_1+b_1B_1+c_1C_1=6$.

$$\therefore P^{-1}=\frac{1}{|P|}\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}.$$

Thus $D = P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

$$\therefore D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}.$$

Hence, $A^4 = PD^4P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}.$$

Reduction of quadratic form to Canonical form by linear transformation

Q.No.1.: Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into **canonical form**.

or

Diagonalise the **quadratic form** $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by linear transformation and write the linear transformation.

or

Reduce the **quadratic form** $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into "sum of squares".

Sol.: The given quadratic form can be written as $X'AX$,

where $X' = [x, y, z]$ and the symmetric matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$.

Let us reduce A into diagonal matrix.

We know that $A = I_3AI_3$, i.e., $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operating $R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 - \frac{4}{3}R_1$, we get

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $C_2 \to C_2 - \frac{2}{3}C_1$, $C_3 \to C_3 - \frac{4}{3}C_1$, we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Operating $R_3 \to R_3 + R_2$, we get $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operating $C_3 \to C_3 + C_2$, we get $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \text{Diag.}\left(3, -\frac{4}{3}, -1\right) = P'AP$.

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = [y_1, y_2, y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - \frac{4}{3}y_2^2 - y_3^2.$$

Here Rank of A = 3, Index = 1, Signature = $1-2=-1$.

Remarks: In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is X = PY

i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$,

i.e., $x = y_1 - \frac{2}{3}y_2 - 2y_3$, $y = y_2 + y_3$, $z = y_3$.

Q.No.2.: Reduce the **quadratic form** $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ into the **"sum of squares".**

Sol.: The matrix form of the given quadratic form is $X'AX$,

where $X' = [x \quad y \quad z \quad w]$ and $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

Let us reduce A to the diagonal matrix.

We know that $A = I_4 A I_4 \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $R_{21} - R_1$, $R_3 + 2R_1$, we get $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating $C_2 - C_1$, $C_3 + 2C_1$, we get $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $R_3 + \frac{2}{5}R_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $C_3 + \frac{2}{5}C_2$ we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 2 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $R_4 + \frac{15}{14}C_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Operating, $C_4 + \frac{15}{14}C_3$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & \frac{12}{7} \\ 0 & 1 & \frac{2}{5} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{15}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

i.e., $\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) = P'AP$.

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = Y'\,\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right)Y = [y_1\ \ y_2\ \ y_3\ \ y_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5}y_3^2 - \frac{17}{14}y_4^2,$$

which is the sum of squares.

Remarks: Here rank of A = 4

Index = 2

Signature = 2 – 2 = 0.

Reduction of quadratic form to Canonical form by Orthogonal Transformation:

Q.No.1.: Reduce $8x^2+7y^2+3z^2-12xy+4xz-8yz$ into **canonical form** by **orthogonal transformation.**

Sol.: The matrix of the quadratic form is $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

The characteristic of A are given by $|A-\lambda I|=0$

i.e., $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}=0 \Rightarrow \lambda(\lambda-15)=0$.

$\therefore \lambda=0,\ 3,\ 15$.

Characteristic vector for $\lambda=0$ is given by $[A-(0)I]X=O$.

i.e., $8x_1-6x_2+2x_3=0$

$-6x_1+7x_2-4x_3=0$

$2x_1-4x_2+3x_3=0$.

Solving first two, we get $\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{3}$ giving the eigen vector $X_1=(1,2,2)'$.

When $\lambda=3$, the corresponding characteristic vector is given by $[A-(3)I]X=O$.

i.e., $5x_1-6x_2+2x_3=0$

$-6x_1+4x_2-4x_3=0$

$2x_1-4x_2=0$

Solving any two equations, we get $X_2=(2,1,-2)'$.

Similarly, characteristic vector corresponding to $\lambda=15$ is $X_3=(2,-2,1)'$.

Now X_1, X_2, X_3 are pairwise orthogonal, i.e., $X_1 . X_2 = X_2 . X_3 = X_3 . X_1 = 0$.

$\therefore$ The normalized modal matrix is $B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$.

Now B is the orthogonal matrix i.e., $B^{-1} = B^T$ and $|B| = 1$.

Now $B^{-1}AB = D$ = diag (0, 3, 15)

$$\Rightarrow \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Now $X'AX = Y'\left(B^{-1}AB\right)Y = Y'DY = [y_1, \ y_2, \ y_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3y_2^2 + 15y_3^2$,

which is the required canonical form

Note: Here the orthogonal transformation is X = BY

Rank of quadratic form = 2

Index = 2

Signature = 2, it is a positive semi-definite.

Q.No.2.: Reduce $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ into **canonical form** by **orthogonal transformation.**

Sol.: The matrix of the quadratic form is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

The characteristic roots are given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$, which on solving gives $\lambda = 8, \ 2, \ 2$.

The vector corresponding to $\lambda = 8$ is given by $[A - 8I]X = O$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving any two of the equations, we get the vectors as $[2,\ -1, 1]'$.

The characteristic vector for $\lambda = 2$ is given by $[A - 2I]X = O$, which reduces to single equation

$2x_1 - x_2 + x_3 = 0$.

Putting $x_1 = 0$, we get $\dfrac{x_2}{1} = \dfrac{x_3}{1}$ or vectors is $[0,\ 1, 1]'$.

Again by putting $x_2 = 0$, we get $\dfrac{x_1}{1} = \dfrac{x_3}{-2}$ or the vectors $[2,\ 0, -2]'$

Now $X_1 = [2,\ -1, 1]'$; $X_2 = [0,\ 1, 1]'$ and $X_3 = [2,\ 0, -2]'$

Here X_1, X_2, X_3 are not pairwise orthogonal.

$\because X_1.X_2 = 0; X_2.X_3 \neq 0$ and $X_3 . X_1 = 0$

To get X_3 orthogonal to X_2 assume a vector $[u, v, w]'$ orthogonal to X_2 also satisfying

$2x_1 - x_2 + x_3 = 0$ i.e. $2u - v + w = 0$ and $0.u + 1.v + 1.w = 0$

Solving $[u, v, w]' = [1,\ 1, -1]' = X_3$ so that $X_1.X_2 = X_3 = X_3.X_1 = 0$.

$$\therefore \text{ The normalized modal matrix is } B = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

Now B is orthogonal matrix and $|B| = 1$.

i.e. $B' = B^{-1}$ and $B^{-1}AB = D$, where $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\therefore X'AX = Y'\left(B^{-1}AB\right)Y = Y'AB = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2,$$

which is the required canonical form.

Here Rank of the quadratic form is 3, Index = 3, signature = 3. It is positive definite.

Q.No.3.: Find the orthogonal transformation which transforms the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to canonical form (or "sum of squares form" or "principal axes form"). Determine the index, signature and nature of the quadratic form.

Sol.: Let $X = [x_1x_2x_3]^T$, $Y = [y_1y_2y_3]^T$.

Let P be the non-singular orthogonal matrix, containing the three eigen vectors of the coefficient matrix A of the given quadratic form. Then $X = \hat{P}Y$ is the required non-singular linear transformation that transforms (reduces) the given quadratics form to canonical form. Here $\hat{P}$ is the normalized modal matrix P.

The coefficient matrix A of the given quadratic form is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Its characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda - 1)(\lambda - 2)(\lambda - 4) = 0.$$

So there are three distinct real eigen values $\lambda = 1,\ 2,\ 4$ of A.

For $\lambda = 1$:

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{matrix} \sim \begin{matrix} 2x_2 = x_3 \\ x_2 = 2x_3 \end{matrix}$$

$\therefore x_2 = x_3 = 0$, $x_1 =$ arbitrary,

The eigen vector X_1 associated with $\lambda = 1$ is $X_1 = [1 \quad 0 \quad 0]^T$.

For $\lambda = 2$:

$-x_1 + 0 + 0 = 0$, $x_2 - x_3 = 0$, $-x_2 + x_3 = 0$

$\therefore \quad x_1 = 0$, $x_2 = x_3$

The eigen vector X_1 associated with $\lambda = 2$ is $X_2 = [0 \quad 1 \quad 1]^T$.

For $\lambda = 3$

$$\begin{matrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{matrix} \sim \begin{matrix} x_1 = 0 \\ x_2 = -x_3 \end{matrix}$$

The eigen vector X_1 associated with $\lambda = 3$ is $X_3 = [0 \quad 1 \quad -1]^T$.

Thus, the nodal matrix P is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

The norm of eigen vector X_1 is

$\|X_1\| = \sqrt{1^2 + 0 + 0} = 1$,

$\|X_2\| = \sqrt{0 + 1^2 + 1^2} = \sqrt{2}$,

$\|X_3\| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Then, the normalized modal matrix $\hat{P}$ is $\hat{P} = \begin{bmatrix} \frac{1}{1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

To find inverse of P:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \begin{matrix} R_{32(-1)} \\ R_{3\left(-\frac{1}{2}\right)} \\ R_{23(-1)} \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Thus $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ and the normalized P^{-1} is $P^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Diagonalisation:

$$\hat{P}^{-1} A \hat{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Then $\hat{P}^{-1} A \hat{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D =$ diagonal matrix

with the eigen values of A as the diagonal elements.

Transformation (Reduction) to canonical form:

Quadratic form (QF)

$$Q = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

Put $X = \hat{P} Y$ and $X^T = \left(\hat{P} Y\right)^T = Y^T \hat{P}^T$.

So $Q = X^T A X = Y^T \hat{P}^T A \hat{P} Y = Y^T \left(\hat{P}^T A \hat{P}\right) Y$.

But we know that $\hat{P}$ is an orthogonal matrix, because

$$\hat{P}\hat{P}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus $\hat{P}^T = \hat{P}^{-1}$

So $Q.F. = X^T AX = Y^T \left(\hat{P}^{-1} A \hat{P}\right) Y$.

But through Diagonalisation $\hat{P}^{-1} A \hat{P} = D$.

Therefore $Q = X^T A X = Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= \begin{bmatrix} y_1 & 2.y_2 & 4y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2.$$

This is the required canonical form (or sum of squares form).

Orthogonal transformation:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x3 \end{bmatrix} = \hat{P} Y = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

So $x_1 = y_1$, $x_2 = \frac{1}{\sqrt{2}}(y_2 + y_3)$, $x_3 = \frac{1}{\sqrt{2}}(y_2 - y_3)$ is the orthogonal transformation which reduces the QF to the canonical form.

Index is 3 for the QF since the number of positive terms in canonical form is 3 i.e. S = 3, Rank $r = 3$. The number of variables is n = 3.

Signature of the QF is $2s - r = 6 - 3 = 3$ (difference between number of positive terms and negative terms in CF).

The given QF is positive definite because $r = 3 = n$ and $s = 3 = n$.

Q.No.4.: Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the **canonical form**. Also specify the matrix of transformation.

Sol.: The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

Its characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$,

which gives $\lambda = 2, 3, 6$ as its eigen values.

Hence, the given quadratic form reduces to the canonical form

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$ i.e. $2x^2 + 3y^2 + 6z^2$.

To find the matrix of transformation from $[A-\lambda I]X = 0$, we obtain the equations

$(3-\lambda)x - y + z = 0$; $-x + (5-\lambda)y - z = 0$; $x - y + (3-\lambda)z = 0$

Now corresponding to $\lambda = 2$, we get $x - y + z = 0$, $-x + 3y - z = 0$ and $x - y + z = 0$,

whence $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$

$\therefore$ The eigen vector is $(1, 0, -1)$ and its normalized form is $\left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$

Similarly corresponding to $\lambda = 3$, the eigen vectors is (1, 1, 1) and its normalized form is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Finally, corresponding to $\lambda = 6$, the eigen vectors is $(1,\ -2,\ 1)$ and its normalized form is $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$..

Hence, the matrix of transformation is $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

Index of the quadratic form = 3. Its signature is also 3.

Q.No.5.: If $X_1 = \frac{1}{3}[2 \quad -1 \quad 2]^T$ and $X_2 = k[3 \quad -4 \quad -5]^T$,

where $k = \frac{1}{\sqrt{50}}$, construct an orthogonal matrix $A = [X_1 \quad X_2 \quad X_3]$.

Sol.: Let $X_3 = [a_1 \quad a_2 \quad a_3]^T$ be the undetermined vector. Since A is orthogonal, the columns vectors of A form an orthogonal system $X_i^T X_j = \delta_{ij}$

$$X_1^T X_2 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3k \\ -4k \\ -5k \end{bmatrix} = 2k + \frac{4}{3} - \frac{10}{3}k = 0 \text{, true}$$

$\therefore$ X_1 and X_2 are orthogonal.

$$X_1^T X_3 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{3}[2a_1 - a_2 + 2a_3] = 0 . \qquad \text{(i)}$$

$$X_2^T X_3 = [3k \quad -4k \quad -5k] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [3a_1 - 4a_2 - 5a_3]k = 0 . \qquad \text{(ii)}$$

Since X_3 should b e normalized

$$X_3^T X_3 = [a_1 \quad a_2 \quad a_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1^2 + a_2^2 + a_3^2$$

$$I = \|X_3\| = \sqrt{X_3^T X_3} = \sqrt{a_1^2 + a_2^2 + a_3^2} . \qquad \text{(iii)}$$

Solving (i), (ii), (iii), we get a_1, a_2, a_3

$$2a_1 - a_2 + 2a_3 = 0$$

$$3a_1 - 4a_2 + 5a_3 = 0$$

$$a_1^2 + a_2^2 + a_3^2 = 1$$

So $a_1 = -\frac{13}{5}a_3$, $\quad a_2 = -\frac{16}{5}a_3$, $\quad a_3^2 = \frac{25}{550}$ $\quad a_3 = \frac{1}{\sqrt{22}}$

$\therefore a_1 = -\frac{13}{5}k_1$, $\quad a_2 = -\frac{16}{5}k_1$, $a_3 = k_1$, $\quad$ where $k = \frac{1}{\sqrt{50}}$

Thus, the required orthogonal matrix A is $A=\begin{bmatrix} \frac{2}{3} & 3k & -\frac{13}{5}k_1 \\ -\frac{1}{3} & -4k & -\frac{16}{5}k_1 \\ \frac{2}{3} & -5k & k_1 \end{bmatrix}$.

Reduction of quadratic form to Canonical form by Lagrange's reduction trasformation:

Q.No.13.: By Lagrange's reduction transform the quadratic form X^TAX to "sum of squares" form for $A=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$.

Sol.: $QF=X^TAX=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$QF=\begin{bmatrix} x_1+2x_2+4x_3 & 2x_1+6x_2-2x_3 & 4x_1-2x_2+18x_3 \end{bmatrix}\times\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$=x_1^2+6x_2^2+18x_3^2+4x_1x_2+8x_1x_3-4x_2x_3$$

$$=\left[x_1^2+4x_1(x_2+2x_3)\right]+6x_2^2+18x_3^2-4x_2x_3$$

$$=\left[x_1^2+4x_1(x_2+2x_3)+2^2(x_2+2x_3)^2\right]-2^2(x_2+2x_3)^2+6x_2^2+18x_3^2-4x_2x_3$$

$$=[x_1+2(x_2+2x_3)]^2+2x_2^2+2x_3^2-20x_2x_3$$

$$=[x_1+2(x_2+2x_3)]^2+2\left[x_2^2-10x_2x_3\right]+2x_3^2$$

$$=[x_1+2(x_2+2x_3)]^2+2\left[x_2^2-10x_2x_3+5^2x_3^2\right]-2.5^2x_3^2+2x_3^2$$

$$=[x_1+2(x_2+2x_3)]^2+2[x_2-5x_3]^2-48x_3^2$$

$QF=y_1^2+2y_2^2-48y_3^2$,

where $y_1=x_1+2(x_2+2x_3)$, $y_2=x_2-5x_3$, $y_3=x_3$.

Index: $S=2$, $(n=3,\ r=3)$,

Signature: $2s-r=2.2-3=1\,(\text{or }\ 2-1=1)$.

Home Assignments

Reduction to diagonal form

Q.No.1.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$.

Q.No.2.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Ans.: Not diagonalizable since only one eigen vector $\begin{bmatrix} k \\ 0 \end{bmatrix}$ exists.

Q.No.3.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$.

Q.No.4.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$.

Q.No.5.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Ans.: $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

Q.No.6.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A=\begin{bmatrix}1 & 2\\ 5 & 4\end{bmatrix}$.

Ans.: $P=\begin{bmatrix}1 & 2\\ -1 & 5\end{bmatrix}$, $D=\begin{bmatrix}-1 & 0\\ 0 & 6\end{bmatrix}$.

Q.No.7.: Diagonalise the matrices. Find the modal matrix P which diagonalises (transforms) $A=\begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$, hence find A^5

Ans.: $P=\begin{bmatrix}1 & 1\\ -3 & 1\end{bmatrix}$, $D=\begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}$, $A^5=\begin{bmatrix}2344 & 781\\ 2343 & 782\end{bmatrix}$.

Q.No.8.: Diagonalise the matrices. Find the modal matrix P which diagonalises (transforms) $A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$.

Ans.: No real eigen values, $\lambda=1+i$, so not diagonalizable over real.

Modal matrix over complex $\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$, $D=\begin{bmatrix}1+i & 0\\ 0 & 1+i\end{bmatrix}$.

Q.No.9.: Diagonalise the matrices. Find the modal matrix P, which diagonalises (transforms) $A=\begin{bmatrix}7.3 & 0.2 & -3.7\\ -11.5 & 1.0 & 5.5\\ 17.7 & 1.8 & -9.3\end{bmatrix}$.

Ans.: Characteristic equation $\lambda^3+\lambda^2-12\lambda=0$, eigen values $3,\ -4,\ 0$.

Modal matrix $=\begin{bmatrix}-1 & 1 & 2\\ 3 & -1 & 1\\ -1 & 3 & 4\end{bmatrix}$, $D=\begin{bmatrix}3 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & 0\end{bmatrix}$

Q.No.10.: Diagonalise the matrices. Find the modal matrix P which diagonalises (transforms) $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$, hence find A^4.

Ans.: Characteristic equation $(\lambda+2)(\lambda-3)(\lambda-6)=0$, $\lambda=-2,3,6$, Modal matrix $P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$, $D=\begin{bmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}$, $A^4=\begin{bmatrix}251 & 485 & 235\\ 485 & 1051 & 485\\ 235 & 485 & 251\end{bmatrix}$.

Q.No.11.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

$$\text{(transforms) } A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}.$$

Ans.: $\lambda^3-18\lambda^2+45\lambda=0$, $\lambda=0,3,15$, $P=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, $D=\begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{bmatrix}$.

Q.No.12.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

$$\text{(transforms) } A=\begin{bmatrix}7 & -2 & 1\\ -2 & 10 & -2\\ 1 & -2 & 7\end{bmatrix}.$$

Ans.: $\lambda^3-24\lambda^2+180\lambda-432=0$, $\lambda=6,6,12$, $P=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$.

Q.No.13.: Diagonalise the matrices. Find the modal matrix P, which diagonalises

$$\text{(transforms) } \begin{bmatrix}+1 & -3 & 2\\ -3 & 7 & -5\\ 2 & -5 & 8\end{bmatrix}.$$

Ans.: $\lambda=1,-2,18$, $P=\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}$.

Q.No.14.: Find A^8 for $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ -4 & 4 & 3\end{bmatrix}$.

Ans.: $(1-\lambda)(\lambda-2)(\lambda-3)=0$, $\lambda=1,2,3$, $P=\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 1\\ -2 & 0 & 1\end{bmatrix}$, $A^8=\begin{bmatrix}-12099 & 12355 & 6305\\ -12100 & 12356 & 6305\\ -13120 & 13120 & 6561\end{bmatrix}$.

Q.No.15.: Find A^5 for $A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$.

Ans.: $\lambda=0,1,2$, $P=\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & -1\\ 1 & 2 & -2\end{bmatrix}$, $A^5=\begin{bmatrix}191 & -64 & -127\\ 97 & -32 & -65\\ 190 & -64 & -126\end{bmatrix}$.

Q.No.16.: Find A^4 for $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$.

Ans.: $\lambda=2,3,6$, $P=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -2\\ -1 & 1 & 1\end{bmatrix}$, $A^4=\begin{bmatrix}251 & -405 & 235\\ -405 & 891 & -405\\ 235 & -405 & 251\end{bmatrix}$.

Problems related to quadratic form and canonical forms:

Q.No.1.: Write down the **quadratic forms** corresponding to following matrices:

(i) $\begin{bmatrix}2 & 4 & 5\\ 4 & 3 & 1\\ 6 & 1 & 1\end{bmatrix}$, **(ii)** $\begin{bmatrix}1 & 1 & -2 & 0\\ 1 & -4 & 0 & 0\\ -2 & 0 & 6 & -3\\ 0 & 0 & -3 & 2\end{bmatrix}$.

Ans.: **(i)** $2x^2+3y^2+z^2+8xy+2yz+10zx$

(ii) $x_1^2-4x_2^2+6x_3^2+2x_4^2+2x_1x_2-4x_1x_3-6x_1x_4$

Q.No.2.: Write down the matrices of the following **quadratic form**:

(i) $2x^2+3y^2+6xy$

(ii) $2x^2+5y^2-6z^2-2xy-yz+8zx$

(iii) $x_1^2+2x_2^2+3x_3^2+4x_4^2+2x_1x_2+4x_1x_3-6x_1x_4-4x_2x_3-8x_2x_4+12x_3x_4$

Ans.: **(i)** $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$, **(iii)** $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ 2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$.

Q.No.3.: Find real symmetric matrix C such that $Q = X^T CX$, where

$$Q = 6x_1^2 - 4x_1x_2 + 2x_2^2.$$

Ans.: $\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$.

Q.No.4.: Find real symmetric matrix C such that $Q = X^T CX$, where $Q = 2(x_1 - x_2)^2$.

Ans.: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Q.No.5.: Find real symmetric matrix C such that $Q = X^T CX$, where

$$Q = (x_1 + x_2 + x_3)^2.$$

Ans.: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Q.No.6.: Find real symmetric matrix C such that $Q = X^T CX$, where

$$Q = 4x_1x_3 + 2x_2x_3 + x_3^2.$$

Ans.: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Determine the nature, index and signature of the quadratic form

Q.No.1.: Determine the nature, index and signature of the quadratic form

$$x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2.$$

Ans.: Indefinite, Eigen value: 1, 1, -2, Index : 2, Signature : 1.

Q.No.2.: Determine the nature, index and signature of the quadratic form

$$5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2.$$

Ans.: Positive semi-definite, Eigen value: $5, 0, 5$, Index : 3, Signature : 3.

Q.No.3.: Determine the nature, index and signature of the quadratic form

$$x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1.$$

Ans.: Indefinite, Eigen value: $-2, 3, 6$, Index: 2, Signature : 1.

Q.No.4.: Determine the nature, index and signature of the quadratic form

$$3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2.$$

Ans.: Positive definite, Eigen value: $2, 3, 6$, Index : 3, Signature : 3.

Q.No.5.: Determine the nature, index and signature of the quadratic form

$$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_2.$$

Ans.: Positive semi-definite, Eigen value: $3, 0, 15$, Index : 3, Signature : 3.

Q.No.6.: Determine the nature, index and signature of the quadratic form

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$$

Ans.: Positive definite, Eigen value: $8, 2, 2$, Index : 3, Signature : 3.

Q.No.7.: Determine the nature, index and signature of the quadratic form

$$-4x_1^2 - 2x_2^2 - 13x_3^2 - 4x_1x_2 - 8x_2x_3 - 4x_1x_3.$$

Ans.: Negative definite, Index: 0, Signature : -3.

Q.No.8.: Determine the nature, index and signature of the quadratic form

$$-3x_1^2 - 3x_2^2 - 7x_3^2 - 6x_1x_2 - 6x_2x_3 - 6x_1x_3.$$

Ans.: Negative definite, Index : 0, Signature : -3.

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation:

Q.No.1.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $17x_1^2 - 30x_1x_2 + 17x_2^2$.

Ans.: $A = \begin{bmatrix} 17 & -15 \\ -15 & +17 \end{bmatrix}$, $\lambda = 2, 32$,

$P=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}$, CF : $2y_1^2+32y_2^2$.

Q.No.2.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix)

$$5x_1^2+26x_2^2+10x_3^2+4x_2x_3+6x_1x_2+14x_1x_3$$

Ans.: $A=\begin{bmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{bmatrix}$, $\lambda=5,\ \frac{121}{3},\ 0$, $P=\begin{bmatrix}1 & -\frac{3}{5} & -\frac{16}{11}\\ 0 & 1 & \frac{1}{11}\\ 0 & 0 & 1\end{bmatrix}$, CF $5y_1^2+\frac{121}{3}y_2^2$.

Q.No.3.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2(x_1x_2+x_2x_3+x_3x_1)$; nature of QF.

Ans.: $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & +1\\ 1 & +1 & 0\end{bmatrix}$, $\lambda=2,\ -1,\ -1$, $P=\begin{bmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}$,

CF : $2y_1^2-y_2^2-y_3^2$

Nature: Indefinite.

Q.No.4.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2\left(x_1^2+x_1x_2+x_2^2\right)$.

Ans.: $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$, $\lambda=1,\ 3$,

$P=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, CF: $y_1^2+3y_2^2$.

Q.No.5.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$, find index.

Ans.: $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$, $\lambda = 1, -1, -1$, $P = \begin{bmatrix} a & -3b & \frac{11c}{17} \\ 0 & b & \frac{2b}{17} \\ 0 & 0 & c \end{bmatrix}$,

where $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{17}}$, $c = \sqrt{\left(\frac{17}{81}\right)}$,

CF: $y_1^2 - y_2^2 - y_3^2$, Index = 1.

Q.No.6.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation. State matrix for transformation (i.e., modal matrix) $3x_1^2 - 2x_2^2 - x_3^2 - 4x_1x_2 + 12x_2x_3 + 8x_1x_3$.

Ans.: $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$, $\lambda = 3, 6, -9$,

$P = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, CF: $3y_1^2 + 6y_2^2 - 9y_3^2$.

Q.No.7.: Reduce the following **quadratic forms to canonical forms** or to sum of squares by **orthogonal transformation**. Write also the rank, index and signature.

(i) $2x^2 + 5y^2 + 3z^2 - 2xy - 2yz + zx$

(ii) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_3$

(iii) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$

(iv) $x^2 + 3y^2 + 3z^2 - 2yz$.

Ans.: Ans.: (i). $2y_1^2 + 3y_2^2 + 6y_3^2$**;** Rank = 3, Index = 3, signature = 3

(ii). $4y_1^2 + y_2^2 + y_3^2$**;** Rank = 3, Index = 3, signature = 3

(iii). $3y_1^2 + 6y_2^2 - 9y_3^2$**;** Rank = 3, Index = 2, signature = 1

(iv). $y_1^2 + 2y_2^2 - +4y_3^2$**;** Rank = 3, Index = 3, signature = 3

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by linear transformation:

Q.No.1.: Reduce the following **quadratic forms** to **canonical forms** or to sum of squares by linear transformation. Write also the rank, index and signature.

(i) $2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4zx$

(ii) $12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$

(iii) $2x^2 + 9y^2 + 6z^2 + 8xy + 8yz + 6zx$

(iv) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$**.**

Ans.: (i). $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$**;** Rank = 3, Index = 3, signature = 3

(ii). $12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$**;** Rank = 3, Index = 3, signature = 3

(iii). $2y_1^2 + y_2^2 - \frac{5}{2}y_3^2$**;** Rank = 3, Index = 2, signature = 1

(iv). $y_1^2 + 2y_2^2 - \frac{1}{2}y_3^2$**;** Rank = 3, Index = 2, signature = 1.

Reduction of quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method:

Q.No.1.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$.

Ans.: $(x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$**.**

Q.No.2.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$$2x_1^2 + 5x_2^2 + 19x_3^2 - 24x_4^2 + 8x_1x_2 + 12x_1x_3 + 8x_1x_4 + 18x_2x_3 - 8x_2x_4 - 16x_3x_4$$

.

Ans.: $2(x_1 + 2x_2 + 3x_3 + 2x_4)^2 - 3(x_2 + x_3 + 4x_4)^2 + 4(x_3 - 2x_4)^2$.

Q.No.3.: Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by Lagrange's Reduction method

$$2x_1^2 + 7x_2^2 + 5x_3^2 - 8x_1x_2 - 10x_2x_3 + 4x_1x_3.$$

Ans.: $2(x_1 - 2x_2 - x_3)^2 - (x_2 + x_3)^2 + 4x_3^2$.

Q.No.4.: By Lagrange's reduction transform the quadratic form X^TAX to sum of the squares form for $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 4 & 6 & 4 \\ 0 & 6 & 11 & 8 \\ 2 & 4 & 8 & 8 \end{bmatrix}$.

Ans.: $(x_1 - x_2 + 2x_3)^2 + 3(x_2 + 2x_3 + 2x_4)^2 - (x_3 + 4x_4)^2 + 8x_4^2$.

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