Differential Calculus

Indeterminate Forms

Cauchy's Rule or L'Hospital's Rule

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(ii) Indeterminate forms-Problems of $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty$.

Cauchy's Rule or L'Hospital's Rule:

Suppose we are interested to find the value of

$\left[\frac{f(x)}{\phi(x)}\right]$ at $x = a$, where $[f(x)]_{x=a} = f(a) = 0$ (i)

$[\phi(x)]_{x=a} = \phi(a) = 0$. (ii)

Then $\left[\frac{f(x)}{\phi(x)}\right]_{x=a}$ is of the form $\frac{0}{0}$.

Then by **L'Hospital's Rule, "**we differentiate the numerator and denominator w.r.t. x separately. If once again, we find indeterminate form $\frac{0}{0}$, we have further repetition of the process till we get some definite result".

Proof: The limiting value of $\left[\frac{f(x)}{\phi(x)}\right]_{x=a} = \underset{x\to a}{\mathrm{Lt}}\, \frac{f(x)}{\phi(x)}$

Putting $x = a + h$ in $\underset{x\to a}{\mathrm{Lt}} \left[\frac{f(x)}{\phi(x)}\right]$, we have when $x \to a$ then $h \to 0$

$$\therefore \ \underset{x\to a}{\mathrm{Lt}}\left[\frac{f(x)}{\phi(x)}\right] = \underset{h\to 0}{\mathrm{Lt}}\frac{f(a+h)}{\phi(a+h)}$$

Using Taylor's Theorem, we get

$$= \underset{h\to 0}{\mathrm{Lt}}\frac{f(a+h)}{\phi(a+h)} = \underset{h\to 0}{\mathrm{Lt}}\frac{f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+........}{\phi(a)+h\phi'(a)+\frac{h^2}{2!}\phi''(a)+.....}$$

$$= \underset{h\to 0}{\mathrm{Lt}}\frac{hf'(a)+\frac{h^2}{2!}f''(a)+........}{h\phi'(a)+\frac{h^2}{2!}\phi''(a)+.........}\left[\begin{array}{l}\because f(a)=0 \text{ and} \\ \phi(a)=0 \text{ from (i) and (ii)}\end{array}\right]$$

As $h \neq 0$, we have

$$\underset{h\to 0}{\mathrm{Lt}}\frac{f'(a)+\frac{h}{2!}f''(a)+........}{\phi'(a)+\frac{h}{2!}\phi''(a)+.........} = \frac{f'(a)}{\phi'(a)} = \underset{x\to a}{\mathrm{Lt}}\frac{f'(x)}{\phi'(x)}$$

In case both $f'(a)$ and $\phi'(a)$ are zero, the above process can be repeated and we shall get $\underset{x\to a}{\mathrm{Lt}}\frac{f(x)}{\phi(x)} = \frac{f''(a)}{\phi''(a)} = \underset{x\to a}{\mathrm{Lt}}\frac{f''(x)}{\phi''(x)}$ and like this we can have further repetition of the process till we get some definite results.

Note: Cauchy's rule is also be applicable to $\frac{\infty}{\infty}$ form**.**

Q.No.1: Evaluate $\underset{x\to 0}{\mathrm{Lim}}\frac{x\cos x-\log(x+1)}{x^2}$**.**

Sol.: $\underset{x\to 0}{\mathrm{Lim}}\frac{x\cos x-\log(1+x)}{x^2}$. $\left[\frac{0}{0} \text{ form}\right]$

Apply Cauchy's Rule (i.e. differentiate the numerator and denominator w.r.t. to x separately), we get

$$= \underset{x\to 0}{\mathrm{Lim}}\frac{\cos x - x\sin x - \frac{1}{1+x}}{2x} \quad \left[\frac{0}{0} \text{ form}\right]$$

Again apply Cauchy's Rule, we get

$$= \lim_{x\to 0}\frac{-\sin x-\sin x-x\cos x+\dfrac{1}{(1+x)^2}}{2}=\frac{1}{2}. \text{ Ans.}$$

Q.No.2: Evaluate $\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}$.

Sol.: $\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}\left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x^2}\times\frac{x}{\sin x}$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x^2}\left[\because \lim_{x\to 0}\frac{x}{\sin x}=1\right]$$

$$= \lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{2x}\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{2}=\frac{1+1+2}{2}=2. \text{ Ans.}$$

Q.No.3: Evaluate $\lim_{x\mapsto 1}\frac{x^x-x}{1-x+\log x}$.

Sol. $\lim_{x\mapsto 1}\frac{x^x-x}{1-x+\log x}\left[\frac{0}{0}\text{ form}\right]$

Apply Cauchy's Rule, we get

$$= \lim_{x\mapsto 1}\frac{x^x(1+\log x)-1}{0-1+\dfrac{1}{x}}\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{x^x(1/x)+(1+\log x)x^x(1+\log x)}{0-0-\dfrac{1}{x^2}}\left[\begin{array}{l}\therefore \text{Let } y=x^x\\ \log y=\log x^x=\log x\\ \text{Differentiate w.r.t. to x}\\ \dfrac{dy}{dx}=y(1+\log x)=x^x(1+\log x)\end{array}\right]$$

$= -2$ **Ans.**

Q.No.4: Find the values of a, b and c so that $\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$.

Sol.: $\lim\limits_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$.

This is of $\frac{0}{0}$ form, if $a-b+c=0$. (i)

Apply Cauchy's Rule, we get

$$\lim_{x\to 0}\frac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x}$$

This is of $\frac{0}{0}$ form, if $a-c=0$. (ii)

$$= \lim_{x\to 0}\frac{ae^x + b\cos x + ce^{-x}}{x(-\sin x) + \cos x + \cos x} = \lim_{x\to 0}\frac{ae^x + b\cos x + ce^{-x}}{-x\sin x + 2\cos x} = \frac{a+b+c}{2} = 2\,(\text{given})$$

$\Rightarrow a+b+c=4$.(iii)

Solving (i), (ii) and (iii), for a, b, c, we get

$a = 1$, $b = 2$, and $c = 1$. Ans.

Q.No.5: Evaluate $\lim\limits_{x\to 0}\log_x \sin x$.

Sol.: $\operatorname*{Lt}\limits_{x\to 0}\frac{\log_e \sin x}{\log_e x}$. $\left[\frac{\infty}{\infty}\text{form}\right]$

Applying Cauchy's Rule, we get

$$= \operatorname*{Lt}_{x\to 0}\frac{\frac{1}{\sin x}\times\cos x}{\frac{1}{x}} = \operatorname*{Lt}_{x\to 0} x\cot x = \operatorname*{Lt}_{x\to 0}\frac{x}{\tan x} = 1\text{. Ans.} \qquad \left[\because \operatorname*{Lt}_{x\to 0}\frac{x}{\tan x} = 1\right]$$

Q.No.6: Evaluate $\operatorname*{Lt}\limits_{x\to 1}\sec\frac{\pi}{2x}.\log x$.

Sol.: $\operatorname*{Lt}\limits_{x\to 1}\sec\frac{\pi}{2x}.\log x$ $\qquad (\infty\times 0)$form

$$= \operatorname*{Lt}_{x\to 1}\frac{\log x}{\cos\left(\frac{\pi}{2x}\right)}.\left[\frac{0}{0}\text{ form}\right]$$

Applying Cauchy's Rule, we get

$$= \underset{x\to1}{\mathrm{Lt}} \frac{\frac{1}{x}}{-\sin\left(\frac{\pi}{2x}\right)\times\frac{\pi}{2}\times\left(-\frac{1}{x^2}\right)} = \underset{x\to1}{\mathrm{Lt}} \frac{2\times1\times x^2}{\pi\times x\times\sin\left(\frac{x}{2x}\right)} = \underset{x\to1}{\mathrm{Lt}} \frac{2\times x}{\pi\times\sin\left(\frac{\pi}{2x}\right)} = \frac{2}{\pi}\text{. Ans.}$$

Q.No.7: Evaluate $\underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\cot^2 x\right]$.

Sol.: We know that $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+........$

$$\therefore \underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{\tan^2 x}\right] = \underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+......\right)^{-2}\right]$$

$$= \underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-x^{-2}\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+......\right)^{-2}\right]$$

$$= \underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{x^2}\left\{1-2\left(\frac{x^2}{3}+\frac{2}{15}x^4+.....\right)+\text{terms of higher powers of } x\right\}\right]$$

$$= \underset{x\to0}{\mathrm{Lt}}\left[\frac{1}{x^2}-\frac{1}{x^2}+\frac{2}{3}+\text{terms containing } x\right] = \frac{2}{3}\text{. Ans.}$$

Similar Problem: Evaluate $\underset{x\to0}{\mathrm{Lt}}\left(\cot^2 x-\frac{1}{x^2}\right)$ $(\infty-\infty)$ form.

Sol.: $\underset{x\to0}{\mathrm{Lt}}\left(\cot^2 x-\frac{1}{x^2}\right) = \underset{x\to0}{\mathrm{Lt}}\left(\frac{1}{\tan^2 x}-\frac{1}{x^2}\right) = \underset{x\to0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^2\tan^2 x}$

$$= \underset{x\to0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4}\left(\frac{x}{\tan x}\right)^2$$

$$= \underset{x\to0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4}(1)^2 \left[\because \underset{x\to0}{\mathrm{Lt}}\frac{x}{\tan x}=1\right]$$

$$= \underset{x\to0}{\mathrm{Lt}}\frac{x^2-\tan^2 x}{x^4}\left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to0}{\mathrm{Lt}}\frac{2x-2\tan x\sec^2 x}{4x^3} = \underset{x\to0}{\mathrm{Lt}}\frac{2x-2\tan x\left(1+\tan^2 x\right)}{2x^3}$$

$$= \underset{x\to0}{\mathrm{Lt}}\frac{x-\tan x-\tan^2 x}{2x^3}\left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-\sec^2 x - 3\tan^2 x \sec^2 x}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-\left(1+\tan^2 x\right)-3\tan^2 x\left(1+\tan^2 x\right)}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{1-1-\tan^2 x-3\tan^2 x-3\tan^4 x}{6x^2}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{-4\tan^4 x-3\tan^4 x}{6x^2}$$

$$= -\underset{x\to 0}{\mathrm{Lt}} \frac{4+3\tan^2 x}{6}\left(\frac{\tan x}{x}\right)^2$$

$$= \frac{-4+0}{6}(1)^3 = \frac{-4}{6} = \frac{-2}{3}. \text{ Ans.}$$

Q.No.8: Find the value of $\underset{x\to 0}{\mathrm{Lt}} \frac{x^3.e^{x^4/4}-\sin^{3/2}\left(x^2\right)}{x^7}$.

Sol.: As $x \to 0$, the required limit takes the indeterminate form $\frac{0}{0}$. The denominator here is x^7 and the application of Cauchy's Rule will required us to differentiate the nominator and denominator at least seven times to come to the true value of the limit, which will be cumbersome.

We therefore, use the method of expansion by Macaulurin's Theory, which is very convenient.

Thus, using the expansion e^x

$$e^{\frac{x^4}{4}} = 1+\frac{x^4}{4}+\frac{1}{2!}\left(\frac{x^4}{4}\right)^2 \text{..........} = 1+\frac{x^4}{4}+\frac{x^8}{32}+\text{..........}$$

And using the series for $\sin x$

$$\left(\sin x^2\right)^{3/2} = \left[x^2-\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^5}{5!}-\text{......}\right]^{3/2} = x^3\left[1-\frac{x^4}{6}+\text{........}\right]^{3/2},$$

Now using Binomial Theorem, we get

$$\left(\sin x^2\right)^{3/2} = x^3\left[1-\frac{3}{2}\left(\frac{x^4}{6}-.......\right)+.....\right] = x^3\left[1-\frac{x^4}{4}+........\right]$$

$$\therefore \operatorname*{Lt}_{x\to 0}\frac{x^3.e^{x^4/4}-\sin^{3/2}\left(x^2\right)}{x^7} = \operatorname*{Lt}_{x\to 0}\frac{x^3\left[1+\frac{x^4}{4}+\frac{x^8}{32}+.......\right]-x^3\left[1-\frac{x^4}{4}+........\right]}{x^7}$$

$$= \operatorname*{Lt}_{x\to 0}\left[\frac{1}{2}+\text{terms containing } x\right] = \frac{1}{2}. \text{ Ans.}$$

Q.No.9.: Evaluate $\operatorname*{Lim}_{x\to 0}\frac{e^x-1-\sin x}{x^2}$.

Sol.: $\operatorname*{Lim}_{x\to 0}\frac{e^x-1-\sin x}{x^2}$. $\qquad\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to 0}\frac{e^x-\cos x}{2x}. \qquad \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to 0}\frac{e^x+\sin x}{2} = \frac{1}{2}.\text{Ans.}$$

Q.No.10.: Evaluate **(a)** $\operatorname*{Lim}_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$,

(b) $\operatorname*{Lim}_{x\to 0}\frac{e^x--e^{-x}-2x}{\tan x - x}$.

Sol.:(a) $\operatorname*{Lim}_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$. $\qquad\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \operatorname*{Lim}_{x\to 0}\frac{e^{ax}(a)-e^{-ax}(-a)}{\frac{1}{1+bx}.b} = \operatorname*{Lim}_{x\to 0}\frac{a\left(e^{ax}+e^{-ax}\right)(1+bx)}{b} = \frac{2a}{b}.\text{Ans.}$$

(b) $\operatorname*{Lim}_{x\to 0}\frac{e^x--e^{-x}-2x}{\tan x - x}$. $\qquad\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x - e^{-x}(-1) - 2}{\sec^2 x - 1} = \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{\sec^2 x - 1}. \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x + e^{-x}(-1)}{2\sec x.\sec x\tan x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{2\sec^2 x\tan x}. \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{e^x - e^{-x}(-1)}{2\left(\sec^2 x\sec^2 x + \tan x.2\sec x\sec x\tan x\right)}.$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x}}{2\left(\sec^4 x + 2\sec^2 x\tan^2 x\right)} = \frac{1+1}{2(1+0)} = 1. \text{ Ans.}$$

Q.No.11.: Evaluate $\lim_{x\to 0}\frac{x - \tan x}{x^3}$.

Sol.: $\lim_{x\to 0}\frac{x - \tan x}{x^3}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{1 - \sec^2 x}{3x^2} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-2\sec x.\sec x\tan x}{6x} = \lim_{x\to 0}\frac{-\sec^2 x\tan x}{3x} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0} -\frac{\sec^2 x.\sec^2 x + \tan x.2\sec x.\sec x\tan x}{3}$$

$$= \lim_{x\to 0} -\frac{\sec^4 x + 2\sec^2 x\tan^2 x}{3} = -\frac{1+0}{3} = -\frac{1}{3}. \text{ Ans.}$$

Q.No.12.: Evaluate $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$.

Sol.: $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{x.e^x + e^x.1 - \frac{1}{x+1}}{2x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{x.e^x + e^x.1 + e^x + \frac{1}{(x+1)^2}}{2} = \frac{0.e^0 + e^0.1 + e^0 + \frac{1}{(1+0)^2}}{2} = \frac{3}{2} \text{ . Ans.}$$

Q.No.13.: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$.

Sol.: $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{6}}{x^5}\left[\dfrac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\cos x - 1 + \frac{3x^2}{6}}{5x^4}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-\sin x + x}{20x^3}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-\cos x + 1}{60x^2}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\sin x}{120x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\cos x}{120} = \frac{1}{120}\text{ . Ans.}$$

Q.No.14.: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

Sol.: $\lim\limits_{x\to 0}\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}\left[\dfrac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{2\cos 2x + 4\sin x\cos x - 2\cos x}{-\sin x + \cos x\sin x}$$

$$= \lim_{x\to 0}\frac{2\cos 2x + 2\sin 2x - 2\cos x}{-\sin x + \sin 2x}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Again using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{-4\sin 2x + 4\cos 2x + 2\sin x}{-\cos x + 2\cos 2x} = \frac{-0+4.1+0}{-1+2} = 4\text{ . Ans.}$$

Q.No.15.: Evaluate $\lim_{x\to\infty}\frac{\log x}{x^n}\ (n>0)$.

Sol.: $\lim_{x\to\infty}\frac{\log x}{x^n}\left[\frac{\infty}{\infty}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{nx^{n-1}} = \lim_{x\to\infty}\frac{1}{nx^n} = \frac{1}{\infty} = 0\text{ . Ans.}$$

Q.No.16.: Evaluate $\mathrm{Lt}_{x\to 0}\frac{x-\sin x}{\tan^3 x}$.

Sol.: $\mathrm{Lt}_{x\to 0}\frac{x-\sin x}{\tan^3 x}$

Applying Cauchy's rule, we get

$$\mathrm{Lt}_{x\to 0}\frac{1-\cos x}{3\tan^2 x.\sec^2 x}$$

Again applying Cauchy's rule, we get

$$\lim_{x\to 0}\frac{\sin x}{3\left[\left(\sec^4 x.2\tan x\right)+\left(\tan^3 x2\sec^2 x\right)\right]} = \lim_{x\to 0}\frac{\sin x}{6\tan x\sec^2 x\left(\tan^2 x+\sec^2 x\right)}$$

$$= \lim_{x\to 0}\frac{1}{6\sec^3 x\left(\tan^2 x+\sec^2 x\right)} = \frac{1}{6}\text{ . Ans.}$$

Q.No.17.: Evaluate $\mathrm{Lt}_{x\to 0}\left(\frac{1}{x^2}-\mathrm{cosec}^2 x\right)$.

Sol.: $\mathrm{Lt}_{x\to 0}\left(\frac{1}{x^2}-\mathrm{cosec}^2 x\right) = \mathrm{Lt}_{x\to 0}\left(\frac{1}{x^2}-\sin^{-2} x\right)$

$$= \lim_{x\to 0}\left(\frac{1}{x^2} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots\dots\dots\right)^{-2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\dots\dots\dots\right)^{-2}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x^2}\left(1 + 2\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \dots\dots\dots\dots\right)\right)\right]$$

$$= \lim_{x\to 0}\left(-\frac{2}{3!} + \frac{2x^4}{5!}\dots\dots\dots\dots\right) = -\frac{2}{3!} = -\frac{1}{3}\text{. Ans.}$$

Q.No.18.: Evaluate $\operatorname{Lt}_{x\to 0}\left(\frac{1}{x} - \cot x\right)$.

Sol.: $\operatorname{Lt}_{x\to 0}\left(\frac{1}{x} - \cot x\right) = \operatorname{Lt}_{x\to 0}\left(\frac{1}{x} - \frac{\cos x}{\sin x}\right) = \operatorname{Lt}_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} = \operatorname{Lt}_{x\to 0}\frac{\sin x - x\cos x}{x^2\left(\frac{\sin x}{x}\right)}$

$$= \operatorname{Lt}_{x\to 0}\frac{\sin x - x\cos x}{x^2}$$

Applying Cauchy's rule, $\therefore$ above equation is $\left[\frac{0}{0}\right]$ form, we get

$$= \operatorname{Lt}_{x\to 0}\frac{\cos x - \cos x + x\sin x}{2x} = \operatorname{Lt}_{x\to 0}\frac{\sin x}{2} = 0\text{. Ans}$$

Q.No.19.: Evaluate $\operatorname{Lt}_{x\to 0}\left(\frac{a}{x} - \cot\frac{x}{a}\right)$.

Sol.: $\operatorname{Lt}_{x\to 0}\left(\frac{a}{x} - \cot\frac{x}{a}\right) = \operatorname{Lt}_{x\to 0}\left(\frac{a}{x} - \frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right) = \lim_{x\to 0}\frac{a\sin\frac{x}{a} - x\cos\frac{x}{a}}{x\sin\frac{x}{a}}$

$$= \lim_{x\to 0}\frac{a\sin\frac{x}{a} - x\cos\frac{x}{a}}{\frac{x^2}{a}\frac{\sin\frac{x}{a}}{\frac{x}{a}}} = \lim_{x\to 0}\frac{a\sin\frac{x}{a} - x\cos\frac{x}{a}}{\frac{x^2}{a}} = \lim_{x\to 0}\frac{a^2\sin\frac{x}{a} - ax\cos\frac{x}{a}}{x^2}$$

Applying L hospital's rule, $\therefore$ above equation is $\left[\frac{0}{0}\right]$ form, we get

$$= \lim_{x\to 0}\frac{a\cos\frac{x}{a} - a\cos\frac{x}{a} + x\sin\frac{x}{a}}{2x} = \lim_{x\to 0}\frac{x\sin\frac{x}{a}}{2x} = 0\text{. Ans.}$$

Q.No.20.: Evaluate **(a)** $\lim_{x\to 0}\frac{\cot x - \frac{1}{x}}{x}$, **(b)** $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$**.**

Sol.: (a) $\lim_{x\to 0}\frac{\cot x - \frac{1}{x}}{x}\left[\frac{-\infty}{0}\text{form}\right]$

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{x} = \lim_{x\to 0}\frac{x\cos x - \sin x}{x^2\sin x} = \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^3}\right]\cdot\left[\frac{x}{\sin x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^3}\right].\ \because \lim_{x\to 0}\left[\frac{x}{\sin x}\right] = 1.\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\left[\frac{-x\sin x + \cos x - \cos x}{3x^2}\right] = \lim_{x\to 0}\left[-\frac{\sin x}{3x}\right] = -\frac{1}{3}\text{.Ans.}$$

(b) $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}\ [(0\times\infty)\text{form}]$

$$= \lim_{x\to 1}\frac{(x-1)}{\cot\frac{\pi x}{2}}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 1}\frac{1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}\text{. Ans.}$$

Q.No.21.: Evaluate $\mathop{\mathrm{Lt}}_{y\to\infty} y^2\left(1 - e^{-2gx/y^2}\right)$**.**

Sol.: $\mathop{\mathrm{Lt}}_{y\to\infty} y^2\left(1 - e^{-2gx/y^2}\right)$

Substituting $\lim_{y\to\infty} y^2 = \lim_{n\to 0}\frac{1}{n^2}$

$$\therefore \lim_{n\to 0}\frac{\left(1 - e^{-1gxn^2}\right)}{n^2}$$

$\therefore$ Using L'Hospital Rule, we get

$$\lim_{n\to 0}\frac{2gx.2n.e^{-2gxn^2}}{2n} = 2gx\text{ . Ans.}$$

Q.No.22.: Evaluate $\lim_{x\to a}\log\left(2-\frac{x}{a}\right)\cot(x-a)$.

Sol.: $\lim_{x\to a}\log\left(2-\frac{x}{a}\right)\cot(x-a)\ [(0\times\infty)\text{form}]$

$$=\lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\tan(x-a)}\ \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\times\left(-\frac{1}{a}\right)}{\text{sac}^2(x-a)}=-\frac{1}{a}\text{. Ans.}$$

Q.No.23.: Evaluate $\operatorname*{Lt}_{x\to 0}\frac{x^{1/2}\tan x}{\left(e^x-1\right)^{3/2}}$.

Sol.: $\operatorname*{Lt}_{x\to 0}\frac{x^{1/2}\tan x}{\left(e^x-1\right)^{3/2}}=\operatorname*{Lt}_{x\to 0}\frac{x^{3/2}\frac{\tan x}{x}}{\left(e^x-1\right)^{3/2}}=\lim_{x\to 0}\frac{x^{1/2}\tan x}{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\text{..........}\right)^{3/2}}$

$$=\lim_{x\to 0}\frac{x^{1/2}\tan x}{x.x^{1/2}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\text{..........}\right)}=\lim_{x\to 0}\frac{1}{\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\text{..........}\right)}=1\text{. Ans.}$$

Q.No.24.: Prove that $\lim_{n\to\infty}\frac{e^{1/n}+e^{2/n}+^{3/n}+\text{............}+e^{n/n}}{n}=e-1$

Sol.: TakingL.H.S. $=\lim_{n\to\infty}\frac{e^{1/n}+e^{2/n}+^{3/n}+\text{............}+e^{n/n}}{n}$

Here, the series given in numerator is in geometric progression,

where, first term, $a=e^{1/n}$,

common ratio, $r=e^{1/n}>1$,

number of terms = n.

The sum of series given in numerator is, δ_n

$$\therefore \delta_n = \frac{a\left(r^n - 1\right)}{r-1} = \frac{e^{1/n}\left\{\left(e^{1/n}\right)^n - 1\right\}}{e^{1/n} - 1} = \frac{e^{1/n}\{e-1\}}{e^{1/n}-1}$$

So, L.H.S.= $\lim\limits_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left(e^{1/n}-1\right)n} = \lim\limits_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{\left(1+\frac{1}{1!n}+\frac{1}{2!n^2}+\dots\dots\dots+\frac{1}{n!n^n}\right)-1\right\}.n}$

$$= \lim_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{\frac{1}{n}+\frac{1}{2!n^2}+\dots\dots\dots+\frac{1}{n!n^n}\right\}.n} = \lim_{n\to\infty} \frac{e^{1/n}\{e-1\}}{\left\{1+\frac{1}{2!n}+\dots\dots\dots+\frac{1}{n!n^{n-1}}\right\}}$$

$$= \lim_{n\to\infty} \frac{e^0\{e-1\}}{(1+0+\dots\dots\dots\dots\dots\dots)} = (e-1). = \text{R.H.S.}$$

Hence this completes the proof.

Q.No.25.: Prove that $\lim\limits_{x\to\infty} \frac{1^2+2^2+3^2+\dots\dots+x^2}{x^3} = \frac{1}{3}$**.**

Sol.: $\lim\limits_{x\to\infty} \frac{1^2+2^2+3^2+\dots\dots+x^2}{x^3} = \lim\limits_{x\to\infty} \frac{\sum x^2}{x^3} = \lim\limits_{x\to\infty} \frac{x(x+1)(2x+1)}{x^3}$

$$= \lim_{x\to\infty} \frac{x^3\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)}{x^3} = \frac{2}{6} = \frac{1}{3} \text{.Ans.}$$

Q.No.26.: Prove that $\mathop{Lt}\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}}\,\text{cosec}\sqrt{a^2-x^2} = \frac{1}{2a}$**.**

Sol.: Taking L.H.S. = $\mathop{Lt}\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}}\,\text{cosec}\sqrt{a^2-x^2} = \lim\limits_{x\to a} \frac{\sin^{-1}\sqrt{\frac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$

The given equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" (i. e. differentiate numerator and denominator w. r. t. x separately)

$$\text{L.H.S.} = \lim_{n\to} \frac{\frac{1}{\sqrt{1-\left(\sqrt{\frac{a-x}{a+x}}\right)^2}}\cdot\frac{1}{2\sqrt{\frac{a-x}{a+x}}}\cdot\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}}{\cos\sqrt{a^2-x^2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\cdot(-2x)}$$

$$= \lim_{x\to a} \frac{\frac{\sqrt{a+x}}{\sqrt{2x}}\cdot\frac{\sqrt{a+x}}{2\sqrt{a-x}}\cdot\frac{(-2a)}{(a+x)^2}}{\cos\sqrt{a^2-x^2}\cdot\frac{(-x)}{\sqrt{(a-x)(a+x)}}}$$

$$= \lim_{x\to a} \frac{(x+a).(2a).\sqrt{a-x}.\sqrt{a+x}}{x.\sqrt{2x}.2\sqrt{a-x}.(a+x)^2.\cos\sqrt{a^2-x^2}} = \lim_{x\to a} \frac{a.\sqrt{a+x}}{x.\sqrt{2x}.(a+x).\cos\sqrt{a^2-x^2}}$$

$$= \frac{a.\sqrt{2a}}{a.\sqrt{2a}.(2a).\cos 0} = \frac{1}{2a} = \text{R.H.S.}$$

Hence this completes the proof.

Q.No.27.: Evaluate $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$.

Sol.: $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to y} \frac{x^y\left(\frac{y}{x}\right) - y^x(\log y)}{x^x(1+\log x) - 0} = \frac{y^y\left(\frac{y}{y}\right) - y^y(y\log y)}{y^y(1+\log y)} = \frac{y^y(1-\log y)}{y^y(1+\log y)} = \frac{1-\log y}{1+\log y}. \text{ Ans.}$$

Q.No.28.: Determine a, b, c such that

$$\underset{\theta\to 0}{\text{Lt}} \frac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1.$$

Sol.: $\underset{\theta\to 0}{\text{Lt}} \frac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1$

The given equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\text{Lt}} \frac{a + b\cos\theta + \theta(0 - b\sin\theta) - c\cos\theta}{5\theta^4}$$

$\therefore a+b-c=0 \Rightarrow a+b=c$. (i)

Again apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\text{Lt}} \frac{0-b\sin\theta-b\sin\theta-\theta b\cos\theta+c\sin\theta}{20.\theta^3}$$

The above equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\text{Lt}} \frac{-2b\cos\theta-b\cos\theta+\theta b\sin\theta+c\cos\theta}{60.\theta^2}$$

$\therefore -3b+c=0 \Rightarrow c=3b$.(ii)

The above equation is in the form $\left[\frac{0}{0}\right]$. So, apply "Cauchy's Rule" , we get

$$\underset{\theta\to 0}{\text{Lt}} \frac{2b\sin\theta+b\sin\theta+b\sin\theta+\theta b\cos\theta-c\sin\theta}{120.\theta}$$

$$\Rightarrow \left\{\frac{4b}{120}\frac{\sin\theta}{\theta}+\frac{\theta b\cos\theta}{120.\theta}-\frac{c\sin\theta}{120.\theta}\right\}=\frac{4b}{120}+\frac{b}{120}-\frac{c}{120}=1 \text{ (given)}$$

$\therefore 5b-c=120$. (iii)

$\therefore$ From (ii) and (iii), we get

$b=60,\quad c=180,\quad a=120$. Ans.

Q.No.29.: Find the values of a and b such that $\underset{x\to 0}{\text{Lim}}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$.

Sol.: $\underset{x\to 0}{\text{Lim}}\frac{x(1+a\cos x)-b\sin x}{x^3}=\underset{x\to 0}{\text{Lim}}\frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right)}{x^3}$

$$=\underset{x\to 0}{\text{Lim}}\frac{(1+a-b)x+\left(-\frac{a}{2}+\frac{b}{6}\right)x^3+...}{x^3}\quad\left[\frac{0}{0}\text{form}\right]$$

Since the given limit is equal to 1, we must have

$1+a-b=0$ (i)

and $-\frac{a}{2}+\frac{b}{6}=1$. (ii)

Solving (i) and (ii), we get

$a = -\frac{5}{2}$, $b = -\frac{3}{2}$. Ans.

Q.No.30.: Evaluate **(a)** $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\log\left(\theta-\frac{\pi}{2}\right)}{\tan\theta}$, **(b)** $\lim\limits_{\theta\to\frac{\pi}{3}} \frac{1-2\cos x}{\sin\left(x-\frac{\pi}{3}\right)}$.

Sol.: (a) $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\log\left(\theta-\frac{\pi}{2}\right)}{\tan\theta}$. $\left[\frac{\infty}{\infty}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\theta\to\frac{\pi}{2}} \frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2\theta} = \lim_{\theta\to\frac{\pi}{2}} \frac{\cos^2 x}{x-\frac{\pi}{2}} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\theta\to\frac{\pi}{2}} \frac{2\cos x(-\sin x)}{1} = \lim_{\theta\to\frac{\pi}{2}}(-\sin 2x) = 0\text{. Ans.}$$

(b) $\lim\limits_{\theta\to\frac{\pi}{3}} \frac{1-2\cos x}{\sin\left(x-\frac{\pi}{3}\right)} = \lim\limits_{\theta\to\frac{\pi}{3}} \frac{2\sin x}{\cos\left(x-\frac{\pi}{3}\right)}$

$\therefore$ Using L'Hospital'sRule, we get

$$= \frac{2.\left(\frac{\sqrt{3}}{2}\right)}{\cos 0} = \sqrt{3}\text{. Ans.}$$

Q.No.31.: Evaluate $\mathop{\mathrm{Lt}}\limits_{x\to a} \frac{\log(x-a)}{\log\left(e^x-e^a\right)}$.

Sol.: $\mathop{\mathrm{Lt}}\limits_{x\to a} \frac{\log(x-a)}{\log\left(e^x-e^a\right)} = \frac{\left(\frac{1}{x-a}\right)}{\left(\frac{e^x}{e^x-e^a}\right)} = \lim\limits_{x\to a} \frac{e^x-e^a}{e^x(x-a)}$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{0+e^a} = 1. \text{ Ans.}$$

Q.No.32.: Evaluate $\underset{x\to 0}{\text{Lt}}\ \frac{\log\sin 2x}{\sin x}$.

Sol.: $\underset{x\to 0}{\text{Lt}}\ \frac{\log\sin 2x}{\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\log(2\sin x\cos x)}{\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\log(2\sin x)+\log\cos x}{\sin x}$

$$= \lim_{x\to 0}\left[\frac{\log(2\sin x)}{\sin x}+\frac{\log\cos x}{\sin x}\right] = \lim_{x\to 0}\frac{\log(2\sin x)}{\sin x}+\lim_{x\to 0}\frac{\log\cos x}{\sin x}$$

The second limit is of $\frac{0}{0}$ form and can be evaluated with the L' Hospital's rule

$$\therefore \lim_{x\to 0}\frac{\log\cos x}{\sin x} = \lim_{x\to 0}\frac{-\tan x}{\cos x} = 0. \text{ Ans.}$$

Q.No.33.: Evaluate $\lim_{x\to 0} x\log\sin x$.

Sol.: $\lim_{x\to 0} x\log\sin x$ $[0\times\infty \text{ form}]$ $\left[\because \lim_{x\to 0}\log x \to -\infty\right]$

$$= \lim_{x\to 0}\frac{\log\sin x}{\frac{1}{x}} \left[\frac{\infty}{\infty}\text{form}\right]$$

Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = -\lim_{x\to 0} x^2\cot x = -\lim_{x\to 0}\frac{x^2}{\tan x} \left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{2x}{\sec^2 x} = 0. \text{ Ans.}$$

Q.No.34.: Evaluate $\lim_{x\to 0} x\log x$.

Sol.: $\lim_{x\to 0} x\log x$ $[0\times\infty \text{ form}]$

$$= \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} \left[\frac{\infty}{\infty}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}(-x) = 0 \text{. Ans.}$$

Q.No.35.: Evaluate $\underset{x\to\infty}{\mathrm{Lt}}\ x\tan\left(\frac{1}{x}\right)$.

Sol.: $\underset{x\to\infty}{\mathrm{Lt}}\ x\tan\left(\frac{1}{x}\right) = \underset{x\to\infty}{\mathrm{Lt}}\ x\frac{\sin\left(\frac{1}{x}\right)}{\cos\left(\frac{1}{x}\right)} = \left[\underset{\frac{1}{x}\to 0}{\mathrm{Lt}}\ \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right]\left[\underset{x\to\infty}{\mathrm{Lt}}\ \frac{1}{\cos\frac{1}{x}}\right] = 1\text{. Ans.}\qquad \begin{bmatrix} x\to\infty \\ \frac{1}{x}\to 0\end{bmatrix}$

Q.No.36.: Evaluate $\underset{x\to\infty}{\mathrm{Lt}}\left(a^{\frac{1}{x}}-1\right)x$.

Sol.: $\underset{x\to\infty}{\mathrm{Lt}}\left(a^{\frac{1}{x}}-1\right)x$

Let $\frac{1}{x} = y \ \therefore x\to\infty, \ \ y\to 0$.

Then $\underset{x\to\infty}{\mathrm{Lt}}\left(a^{\frac{1}{x}}-1\right)x = \underset{y\to 0}{\mathrm{Lt}}\ \frac{\left(a^y-1\right)}{y} = \underset{y\to 0}{\mathrm{Lt}}\ \frac{\left(a^y\log a\right)}{1} = \log a$. Ans.

Q.No.37.: Evaluate $\underset{x\to 0}{\mathrm{Lt}}\ \frac{A}{x^2}\left[\frac{\sin kx}{\sin \ell x}-\frac{k}{\ell}\right]$.

Sol.: $\underset{x\to 0}{\mathrm{Lt}}\ \frac{A}{x^2}\left[\frac{\sin kx}{\sin \ell x}-\frac{k}{\ell}\right] = \underset{x\to 0}{\mathrm{Lt}}\ \frac{A}{x^2}\left[\frac{\ell\sin kx - k\sin\ell x}{\ell\sin\ell x}\right]$

$$= \underset{x\to 0}{\mathrm{Lt}}\ \frac{A}{x^2}\left[\frac{\left(\ell kx-\frac{\ell k^3x^3}{3!}+\frac{\ell k^5x^5}{5!}-\text{........}\infty\right)-\left(k\ell x-\frac{k\ell^3x^3}{3!}+\frac{k\ell^5x^5}{5!}-\text{........}\infty\right)}{\left(\ell^2x-\frac{\ell^4x^3}{3!}+\frac{\ell^6x^5}{5!}\text{....................}\right)}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}}\ \frac{Ak\ell}{\ell^2}\left[\frac{\left(-\frac{k^2x}{3!}+\frac{k^4x^3}{5!}-\text{........}\infty\right)-\left(-\frac{\ell^2x}{3!}+\frac{k\ell^4x^5}{5!}-\text{........}\infty\right)}{\left(\ell^2x-\frac{\ell^4x^3}{3!}+\frac{\ell^6x^5}{5!}\text{....................}\right)}\right]$$

Using L'Hospital Rule, we get

$$= \underset{x\to 0}{\text{Lt}} \frac{Ak}{\ell}\left[\frac{\frac{-k^2}{3!}+\frac{3k^4x^2}{5!}-\ldots\ldots\ldots\ldots\infty+\frac{\ell^2}{3!}-\frac{5\ell^4x^4}{5!}-\ldots\ldots\ldots\ldots\infty}{1-\frac{3\ell^2x^2}{3!}+\frac{5\ell^4x^4}{5!}\ldots\ldots\ldots\ldots\infty}\right]$$

$$= \frac{Ak}{\ell}\left[\frac{\ell^2}{6}-\frac{k^2}{6}\right] = \frac{Ak}{6!}\left(\ell^2-k^2\right).\text{ Ans.}$$

Q.No.38.: Evaluate $\underset{x\to 1}{\text{Lim}}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$.

Sol.: $\underset{x\to 1}{\text{Lim}}\left[\frac{x}{x-1}-\frac{1}{\log x}\right] = \underset{x\to 1}{\text{Lim}}\left[\frac{x\log x-(x-1)}{(x-1)\log x}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$\underset{x\to 1}{\text{Lim}}\left[\frac{1+\log x-1}{\left(\frac{x-1}{x}\right)+\log x}\right] = \underset{x\to 1}{\text{Lim}}\left(\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}\right) = \frac{1}{2}.\text{ Ans.}$$

Q.No.39.: Find $\underset{x\to a}{\text{Lt}}\left[\frac{f'(x)}{f(x)-f(a)}-\frac{1}{x-a}\right]$.

Sol.: $\underset{x\to a}{\text{Lt}}\left[\frac{f'(x)}{f(x)-f(a)}-\frac{1}{x-a}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{f'(x)(x-a)-f(x)+f(a)}{f(x)-f(a)(x-a)}\right]$

$$= \underset{x\to a}{\text{Lt}}\left[\frac{f'(x).x-f'(x).a-f(x)+f(a)}{f'(x).x-f(a)x-f(x)a+f(a).a}\right]$$

$$= \underset{x\to a}{\text{Lt}}\left[\frac{x.f''(x)+f'(x)-af''(x)-f'(x)+f'(a)}{x.f'(x)-f(x)-xf'(a)+f(a)-af'(x)+af'(a)}\right]$$

$$= \underset{x\to a}{\text{Lt}}\left[\frac{x.f'''(x)+f''(x)-af'''(x)-f''(a)}{x.f''(x)-f'(x)+f'(x)-xf''(a)+f'(a)+f'(a)-af''(x)+af''(a)}\right]$$

Applying the limits, we get

$$= \left[\frac{a.f'''(a)+f''(a)-af'''(a)-f''(a)}{a.f''(a)-f'(a)+f'(a)-af''(a)+f'(a)+f'(a)-af''(a)+af''(a)}\right]$$

$$= \frac{2f''(a)}{4f'(a)} = \frac{f''(a)}{2f'(a)}.\text{Ans.}$$

Q.No.40.: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$.

Sol.: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$ $[\infty\times\infty \text{ form}]$

$$=\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to 0}\left[\frac{e^x-1}{x.e^x+(e^x-1).1}\right]=\lim_{x\to 0}\frac{e^x-1}{(x+1)e^x-1}\left[\frac{0}{0}\text{form}\right]$$

$\therefore$ Using L'Hospital Rule, we get

$$=\lim_{x\to 0}\frac{e^x}{(x+1).e^x+e^x}=\frac{1}{1+1}=\frac{1}{2}.\text{Ans.}$$

Q.No.41.: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$.

Sol.: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$ $[(\infty-\infty)\text{form}]$

$$=\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-......\right)\right]=\lim_{x\to 0}\left[\frac{1}{2}-\frac{1}{3}x+...\right]=\frac{1}{2}.\text{ Ans.}$$

Q.No.42.: Prove that $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=-\frac{e}{2}$.

Sol.: $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$. $\left[\frac{0}{0}\text{form}\right]\left[\because \lim_{x\to 0}(1+x)^{1/x}=e\right]$

We first evaluate $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$. $\therefore$

$$\log y=\frac{1}{x}\log(1+x)=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-..\right)=1+\left(-\frac{x}{2}+\frac{x^2}{3}-..\right)=1+z,$$

where $z=-\frac{x}{2}+\frac{x^2}{3}-...$.

$$\therefore\ y = e^{1+z} = e.e^{z} = e\left[1 + z + \frac{z^2}{2!} + ...\right]$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - ...\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - ...\right)^2 + ...\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - ......\right].$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0}\frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ...\right) - e}{x} = \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{11}{24}x + ...\right) = -\frac{1}{2}e \text{ .Ans.}$$

This completes the proof.

Q.No.43.: Evaluate $\underset{x\to 0}{\text{Lt}}\ \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$.

Sol.: $\underset{x\to 0}{\text{Lt}}\ \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

Using the expansion of $e^x \sin x$ and $\log(1-x)$, we get

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(x - \frac{x^3}{3!} + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - ....\right)}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\left(x + x^2 + \frac{x^3}{3} - x^4 + ...\right) - x - x^2}{x^2 - \left(x^2 - \frac{x^3}{2} - \frac{x^4}{3} + ...\right)} = \underset{x\to 0}{\text{Lt}}\ \frac{\left(\frac{x^3}{3} - x^4 + ...\right)}{\left(-\frac{1}{2} - \frac{x}{3} + ...\right)} = \frac{-2}{3}\text{. Ans.}$$

Q.No.44.: Prove that $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$.

Sol.: $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$. $\left[\frac{0}{0}\text{form}\right]\left[\because \lim_{x\to 0}(1+x)^{1/x} = e\right]$

We first evaluate $(1+x)^{1/x}$.

Let $y = (1+x)^{1/x}$.

$$\therefore \ \log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - ..\right) = 1 + \left(-\frac{x}{2} + \frac{x^2}{3} - ..\right) = 1 + z,$$

where $z = -\frac{x}{2} + \frac{x^2}{3} - \dots$.

$$\therefore \ y = e^{1+z} = e.e^z = e\left[1 + z + \frac{z^2}{2!} + ...\right]$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)^2 + ...\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - ......\right].$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \lim_{x\to 0}\frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots\right) - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0}\left[\frac{11}{24}e + \text{terms containing powers of } x\right] = \frac{11}{24}e.$$

This completes the proof.

Q.No.45.: Evaluate $\lim_{x\to 0}\frac{\tanh x - 2\sin x + x}{x^5}$**.**

Sol.: $\lim_{x\to 0}\frac{\tanh x - 2\sin x + x}{x^5}$

$$= \operatorname*{Lt}_{x\to 0}\frac{\dfrac{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)}{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right)} - 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....\right) + x}{x^5}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+.........\right)-2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.........\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)+x\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}{x^5\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{(1-2+1)x+\left(\frac{1}{6}-1+\frac{1}{3}+\frac{1}{2}\right)x^3+\left(\frac{1}{120}-\frac{2}{24}+\frac{2}{12}-\frac{2}{120}+\frac{1}{24}\right)x^5+(...)x^6+...........}{x^5\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+.........\right)}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{14}{120}x^5+(....)x^6+.............}{x^5} = \frac{14}{120} = \frac{7}{60}\text{. Ans.}$$

Q.No.46.: Evaluate $\underset{x\to 0}{\mathrm{Lim}} \frac{x\sin(\sin x)-\sin^2 x}{x^6}$.

Sol.: We make use of one standard series to obtain this limit

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-.........$$

$$\text{Now } \underset{x\to 0}{\mathrm{Lim}} \frac{x\sin(\sin x)-\sin^2 x}{x^6} = \underset{x\to 0}{\mathrm{Lim}} \frac{x\sin\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$$

$$= \underset{x\to 0}{\mathrm{Lim}} \frac{x\left\{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)-\frac{1}{3!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^3+\frac{1}{5!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^5+....\right\}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$$

$$= \underset{x\to 0}{\mathrm{Lim}} \frac{\left\{\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-......\right)-\frac{x^4}{3!}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)^3+\frac{x^6}{5!}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)^5+....\right\}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)^2}{x^6}$$

Expanding by Binomial expansion, we get

$$\left\{\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-....\right)-\frac{x^4}{3!}\left\{1-3\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)+...\right\}+\frac{x^6}{5!}\left\{1-5\left(\frac{x^2}{3!}-\frac{x^4}{5!}+....\right)+.....\right\}+..\right\}$$

$$=\lim_{x\to0}\frac{-x^2\left\{1-2\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+......\right)^2\right\}}{x^6}$$

$$=\lim_{x\to0}\frac{x^6\left(\frac{1}{120}+\frac{1}{12}+\frac{1}{120}-\frac{2}{120}-\frac{1}{36}\right)+(........)x^7+....}{x^6}$$

$$=\lim_{x\to0}\frac{\frac{1}{18}x^6+(........)x^7+....}{x^6}=\frac{1}{18}.\ \text{Ans.}$$

Q.No.47.: Evaluate $\lim\limits_{x\to0}\dfrac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}$.

Sol.: $\lim\limits_{x\to0}\dfrac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}$

We make use of two standard series to obtain this limit

$$e^{x\sin x}=1+x\sin x+\frac{(x\sin x)^2}{2!}+\frac{(x\sin x)^3}{3!}+\frac{(x\sin x)^4}{4!}+............$$

$$=1+x\sin x+\frac{x^2}{2!}(\sin x)^2+\frac{x^3}{3!}(\sin x)^3+\frac{x^4}{4!}(\sin x)^4+............$$

Now using expansion of $\sin x$

$$e^{x\sin x}=1+x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+..........\right)+\frac{x^2}{2!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+..........\right)^2$$

$$+\frac{x^3}{3!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+..........\right)^3+\frac{x^4}{4!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+..........\right)^4$$

$$=1+x^2\left(1-\frac{x^2}{6}+\frac{x^4}{120}-..........\right)+\frac{x^4}{2}\left(1-\frac{x^2}{6}+\frac{x^5}{120}-..........\right)^2$$

$$+\frac{x^6}{6}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-..........\right)^3+\frac{x^8}{24}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-..........\right)^4$$

Now expanding by Binomial theorem,

$$e^{x\sin x}=1+x^2\left(1-\frac{x^2}{6}+\frac{x^4}{120}+..........\right)+\frac{x^4}{2}\left[1-2\left(\frac{x^2}{6}+\frac{x^5}{120}+.........\right)\right]$$

$$+\frac{x^6}{6}\left[1-3\left(\frac{x^2}{6}+\frac{x^4}{120}+.........\right)\right]+...............$$

Collecting terms of same type

$$e^{x\sin x}=1+x^2+\frac{x^4}{3}+\frac{x^6}{120}............ \qquad \text{(i)}$$

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+...............$$

$$\cosh\left(\sqrt{2}x\right)=1+\frac{2x^2}{2!}+\frac{4x^4}{4!}+\frac{8x^6}{6!}+...............$$

$$=1+x^2+\frac{x^4}{6}+\frac{x^6}{90}+............... \qquad \text{(ii)}$$

$$\operatorname*{Lt}_{x\to 0}\frac{e^{x\sin x}-\cosh\left(\sqrt{2}x\right)}{x^4}$$

$$=\operatorname*{Lt}_{x\to 0}\frac{\left(1+x^2+\frac{x^4}{3}+\frac{x^6}{120}+......\right)-\left(1+x^2+\frac{x^4}{6}+\frac{x^6}{90}+......\right)}{x^4}$$

$$=\operatorname*{Lt}_{x\to 0}\frac{\left(\frac{x^4}{3}+\frac{x^6}{120}+......\right)-\left(\frac{x^4}{6}+\frac{x^6}{90}+......\right)}{x^4}$$

Neglecting terms having powers more than 4

$$=\operatorname*{Lt}_{x\to 0}\frac{x^4\left(\frac{1}{3}-\frac{1}{6}\right)}{x^4}=\frac{1}{6}. \text{ Ans.}$$

Q.No.48.: Prove that $\operatorname*{Lim}_{x\to 0}\frac{2x^2-2e^{x^2}+2\cos x^{3/2}+\sin^3 x}{x^4}=-1$.

Sol.: $\operatorname*{Lim}_{x\to 0}\frac{2x^2-2e^{x^2}+2\cos x^{3/2}+\sin^3 x}{x^4}=-1$

We use $\sin x$ series for expansion of $\sin^3 x$

$$\sin^3 x = \left[x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - \ldots\ldots\ldots\right]^3 = x^3\left[1 - \frac{(x)^3}{6} + \frac{(x)^5}{120} - \ldots\ldots\ldots\right]^3$$

Using Binomial Theorem

$$= x^3\left[1 - \frac{(x)^3}{2} + \frac{(x)^5}{40} - \ldots\ldots\ldots\right]$$

Similarly

$$\cos x^{3/2} = 1 - \frac{\left(x^{3/2}\right)^2}{2!} + \frac{\left(x^{3/2}\right)^4}{4!} - \frac{\left(x^{3/2}\right)^6}{6!} + \ldots\ldots\ldots\ldots\ldots = 1 - \frac{x^3}{2} + \frac{x^6}{24} - \ldots\ldots\ldots\ldots\ldots$$

$$\therefore \lim_{x\to 0} \frac{2x^2 - 2e^{x^2} + 2\cos x^{3/2} + \sin^3 x}{x^4}$$

$$= \lim_{x\to 0} \frac{2x^2 - 2\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\ldots\right) + 2\left(1 - \frac{x^3}{2} + \frac{x^6}{24} - \ldots\ldots\ldots\right) + x^3\left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \ldots\ldots\ldots\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{-\left(x^4 + \frac{x^6}{3} + \ldots\ldots\right) + \left(\frac{x^6}{24} - \ldots\ldots\ldots\right) + x^4\left(\frac{x}{2} + \frac{x^3}{40} - \ldots\ldots\ldots\right)}{x^4} = -1. \text{ Ans.}$$

Q.No.49.: Evaluate $\lim_{x\to 0} \frac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}$.

Sol.: $\lim_{x\to 0} \frac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}$

$$= \lim_{x\to 0} \frac{1 + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right) - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\right)}{x + \frac{x^3}{3} + \frac{2x^5}{15} - x\ldots\ldots\ldots}$$

$$= \lim_{x\to 0} \frac{x\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right) - \left(\frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots\right) - \left(\frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\right)}{\frac{x^3}{3} + \frac{2x^5}{15} - x\ldots\ldots}$$

Neglecting terms greater than x^3

$$= \lim_{x\to 0}\frac{\left(-\frac{x^3}{2}-\frac{x^3}{3}\right)}{\frac{x^3}{3}} = -\frac{5}{2}\text{. Ans.}$$

Q.No.50.: The current i in a circuit containing an inductance L, a capacitance C and an alternator of angular frequency ω and maximum e.m.f. E., is given by

$i = \frac{\omega E}{L\left(n^2-\omega^2\right)}\left(\cos\omega t - \cos nt\right)$ where $n = \frac{1}{\sqrt{LC}}$. Find the limiting form of the expression for i, when $\omega \to n$.

Sol.: Since $\lim_{\omega\to n}\frac{\omega E}{L\left(n^2-\omega^2\right)}\left(\cos\omega t - \cos nt\right)\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Using L'Hospital Rule, we get

$$= \lim_{\omega\to n}\frac{E\left(\cos\omega t - \cos nt\right) + E\omega t\left(-\sin\omega t\right)}{-2\omega L}$$

$$= \frac{E\left(\cos nt - \cos nt\right) + Ent\left(-\sin nt\right)}{-2nL}$$

$$= \frac{E\times 0 - Ent\left(\sin nt\right)}{-2nL} = \frac{Et}{2L}\sin nt\text{. Ans.}$$

Q.No.51.: A column of length ℓ has a vertical load P and horizontal load F at the top, and the transverse deflection is given by

$D = \frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right]$, where $m^2 = \frac{P}{EI}$. Show that as $P \to 0$, $D \to \frac{F\ell^3}{3EI}$.

Sol.: Given $D = \frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right]$, where $m^2 = \frac{P}{EI}$.

$$\text{Now} \lim_{P\to 0} D = \lim_{P\to 0}\frac{F\ell}{P}\left[\frac{\tan m\ell}{m\ell} - 1\right] = \lim_{P\to 0}\frac{F\ell}{P}\left[\frac{\left(m\ell + \frac{m^3\ell^3}{3} + \frac{2}{15}m^5\ell^5 + \ldots\ldots\right)}{m\ell} - 1\right]$$

$$= \lim_{P\to 0}\frac{F\ell}{P}\left[\left(1 + \frac{m^2\ell^2}{3} + \frac{2}{15}m^4\ell^4 + \ldots\ldots\right) - 1\right] = \lim_{P\to 0}\frac{F\ell}{P}\left[\left(\frac{m^2\ell^2}{3} + \frac{2}{15}m^4\ell^4 + \ldots\ldots\right)\right]$$

$$= \lim_{P\to 0}\frac{F\ell}{P}\left[\frac{1}{3}\frac{P\ell^2}{EI}+\frac{2}{15}\left(\frac{P}{EI}\right)^2\ell^4+\ldots\ldots\right]=\lim_{P\to 0}\frac{F\ell}{P}\times\frac{P\ell^2}{3EI}\left[1+\frac{2}{05}\frac{P}{EI}\ell^2+\ldots\ldots\right]$$

$$= \lim_{P\to 0}\frac{F\ell^3}{3EI}\left[1+\frac{2}{05}\frac{P}{EI}\ell^2+\ldots\ldots\right]=\frac{F\ell^3}{3EI}.$$

Thus as $P\to 0$, $D\to\frac{F\ell^3}{3EI}$.

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