Differential Calculus

Partial Differentiation

(Homogeneous Functions and Euler's Theorem)

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Homogeneous Expression:

An expression of the form $a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + ... + a_ny^n$,

where each term of degree 'n' , is called **Homogeneous expression** in x and y and of degree or order 'n'.

Homogeneous Function:

If this expression equal to some quantity 'u' , then 'u' is called **Homogeneous Function** in x and y of degree 'n' .

Now $u = a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + ... + a_ny^n$

$$= x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_n\left(\frac{y}{x}\right)^n\right]$$

$$\Rightarrow u = x^n f\left(\frac{y}{x}\right).$$

Also, we can write a Homogeneous Function in x and y of degree 'n' as $u = y^n f\left(\frac{x}{y}\right)$.

Similarly, a Homogeneous Function in x ,y and z of degree 'n' can be written as

$$u = x^n F\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } u = y^n F\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } u = z^n F\left(\frac{x}{z}, \frac{y}{z}\right).$$

Here 'u' is dependent variable and x, y, z are independent variables.

Euler's Theorem:

Statement: If 'u' is a homogeneous function of x and y of degree 'n', then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \; .$$

Proof: Given 'u' is a homogeneous function in x and y of degree 'n'.

Then we may write $u = x^n f\left(\frac{y}{x}\right)$. (i)

Differentiating (i) partially w.r.t. x [keeping y as constant], we get

$$\Rightarrow \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$= -x^{n-2} y f'\left(\frac{y}{x}\right) + nx^{n-1} f\left(\frac{y}{x}\right).$$

Similarly, differentiating (i) partially w.r.t. y [keeping x as constant], we get

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right).$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[-x^{n-2} y f'\left(\frac{y}{x}\right) + nx^{n-1} f\left(\frac{y}{x}\right)\right] + y\left[x^{n-1} f'\left(\frac{y}{x}\right)\right]$

$$= n\left[x^n f\left(\frac{y}{x}\right)\right] = nu \; .$$

This completes the proof**.**

Extension of Euler's Theorem:

Statement: If 'u' is a homogeneous function of x and y of degree 'n', then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \; .$$

Proof: Since 'u' is a homogeneous function in x and y of degree 'n' then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$. [by Euler's Theorem] ..... (i)

Differentiating (i) partially w. r. t. x [keeping y as constant], we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}.$$

Multiplying by x, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=nx\frac{\partial u}{\partial x}\;\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=(n-1)x\frac{\partial u}{\partial x}. \qquad ......(ii)$$

Again, Differentiating (i) partially w. r. t. y [keeping x as constant], we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}.$$

Multiplying by y, we get

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=ny\frac{\partial u}{\partial y}\;\Rightarrow xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)y\frac{\partial u}{\partial y}. \qquad .....(iii)$$

Adding (ii) and (iii), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=(n-1)nu=n(n-1)u\;.$$

This completes the proof.

Now let us solve some problems related to the above-mentioned topics:

Q.No.1.: Verify Euler's theorem, when $z=x^3-3x^2y-y^3$**.**

Sol.: Since $z=x^3-3x^2y-y^3$

$$\therefore \frac{\partial z}{\partial x}=3x^2-6xy \text{ and } \frac{\partial z}{\partial y}=-3x^2-3y^2.$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(3x^2-6xy\right)+y\left(-3x^2-3y^2\right)=3\left(x^3-3x^2y-y^3\right)=3z.$$

$$\text{Also } z=x^3\left[1-3\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^3\right]=x^3 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ z is a homogeneous function of x and y of degree 3.

$\therefore$ By Euler's theorem, we get $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=3z$.

Hence, Euler's theorem is verified**.**

Q.No.2.: If $u=\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$, then prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{12}u$.

Sol.: Here $u=\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}=\left[\dfrac{x^{1/3}\left(1+\dfrac{y^{1/3}}{x^{1/3}}\right)}{x^{1/2}\left(1+\dfrac{y^{1/2}}{x^{1/2}}\right)}\right]^{1/2}=\left[x^{-1/6}\left\{\dfrac{1+\left(\dfrac{y}{x}\right)^{1/3}}{1+\left(\dfrac{y}{x}\right)^{1/2}}\right\}\right]^{1/2}$

$$=x^{-1/12}\left[\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right]^{1/2}=x^{-1/12}f\left(\frac{y}{x}\right).$$

$\Rightarrow$ u is a homogeneous function of x and y of degree $-\dfrac{1}{12}$.

$\therefore$ By Euler's theorem, we have $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu=-\dfrac{1}{12}u$.

Hence the result.

Q.No.3.: If $u=f\left(\dfrac{y}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

Sol.: Here $u=f\left(\dfrac{y}{x}\right)=x^0 f\left(\dfrac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0.u=0.$$

Hence the result.

Q.No.4.: If $u=xyf\left(\dfrac{y}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\left[xyf\left(\dfrac{y}{x}\right)\right]$.

Sol.: Here $u=xyf\left(\dfrac{y}{x}\right)=x^2\left[\dfrac{y}{x}f\left(\dfrac{y}{x}\right)\right]=x^2F\left(\dfrac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 2.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=2.u=2\left[xyf\left(\frac{y}{x}\right)\right].$$

Hence the result.

Q.No.5.: If $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$, find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Sol.: Here $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=x^0\left[\sin^{-1}\left\{\frac{1}{\frac{y}{x}}\right\}+\tan^{-1}\left(\frac{y}{x}\right)\right]=x^0 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0u=0.$$

Hence the result.

Q.No.6.: Verify Euler's Theorem on homogeneous functions in the following cases:-

(i) $f(x,y)=\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}$, **(ii)** $u=f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}$

(iii) $z=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{y}\right)$, **(iv)** $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

(v) $z=x^4\log\left(\frac{y}{x}\right)$.

Sol.: (i) Here $f(x,y)=\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}=x^{1/20}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right]=x^{1/20}f\left(\frac{y}{x}\right)$

$\Rightarrow f(x,y)$ is a homogeneous function of x and y of degree $\frac{1}{20}$.

Hence by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf=\frac{1}{20}\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}. \qquad \text{(i)}$$

Again since $f(x,y)=\dfrac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)}$

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial f}{\partial x}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}x^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}x^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

Multiplying by x, we get

$$x\frac{\partial f}{\partial x}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}x^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}x^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}y^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}y^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

Multiplying by y, we get

$$y\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}y^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}y^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \text{(iii)}$$

Adding (ii) and (iii), we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\frac{1}{4}\left(x^{1/4}+y^{1/4}\right)-\left(x^{1/4}+y^{1/4}\right)\frac{1}{5}\left(x^{1/5}+y^{1/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2}$$

$$=\frac{1}{20}\frac{\left(x^{1/4}+y^{1/4}\right)}{\left(x^{1/5}+y^{1/5}\right)} \qquad \text{(iv)}$$

Hence, the Euler's Theorem is verified**.**

(ii) Here $u=f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}=x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}+\left(\frac{z}{x}\right)^{1/2}\right]=x^{1/2}f\left(\frac{y}{x},\frac{z}{x}\right)$

$\Rightarrow u=f(x,y,z)$ is a homogeneous function of x ,y and z of degree $\dfrac{1}{2}$.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=\frac{1}{2}u=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Again since $u = f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$

Differentiating partially w. r. t x , y and z respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}, \quad \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}} \text{ and } \frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\frac{1}{2\sqrt{x}} + y\frac{1}{2\sqrt{y}} + z\frac{1}{2\sqrt{z}} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right).$$

Hence, the Euler's Theorem is verified**.**

(iii) Here $z = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{y}\right) = y^0 \tan^{-1}\left[\sqrt{\left(\frac{x}{y}\right)^2 + 1}\right] = y^0 f\left(\frac{x}{y}\right).$

$\Rightarrow z$ is a homogeneous function of x ,y and z of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 0.z = 0.$$

Again since $z = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{y}\right).$

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{x^2 + y^2}{y^2}} \cdot \frac{1}{y}\left(\frac{1}{2\sqrt{x^2 + y^2}}.2x\right) = \frac{xy}{\left(x^2 + 2y^2\right)} \cdot \frac{1}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{x^2 + y^2}{y^2}} \cdot \left(\frac{\frac{y}{2}\frac{1}{\sqrt{x^2 + y^2}}.2y - \sqrt{x^2 + y^2}}{y^2}\right) = \frac{1}{\left(x^2 + 2y^2\right)} \cdot \frac{y^2 - \left(x^2 + y^2\right)}{\sqrt{x^2 + y^2}}$$

$$= \frac{1}{\left(x^2 + 2y^2\right)} \cdot \frac{\left(-x^2\right)}{\sqrt{x^2 + y^2}}.$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[\frac{xy}{\left(x^2 + 2y^2\right)} \cdot \frac{1}{\sqrt{x^2 + y^2}}\right] + y\left[\frac{1}{\left(x^2 + 2y^2\right)} \cdot \frac{\left(-x^2\right)}{\sqrt{x^2 + y^2}}\right] = 0.$$

Hence, the Euler's Theorem is verified**.**

(iv) Here $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = x^0\left[\sin^{-1}\left\{\frac{1}{\frac{y}{x}}\right\} + \tan^{-1}\left(\frac{y}{x}\right)\right] = x^0 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0u = 0.$$

Again since $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{-y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}.$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}\right] + y\left[\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}\right] = 0.$$

Hence, the Euler's Theorem is verified.

(v) Here $z = x^4 \log\left(\frac{y}{x}\right) = x^4 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ z is a homogeneous function of x and y of degree 4.

Hence by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 4z = 4x^4\log\left(\frac{y}{x}\right).$$

Again since $z = x^4 \log\left(\frac{y}{x}\right)$.

Differentiating partially w. r. t x and y respectively, we get

$$\frac{\partial z}{\partial x} = 4x^3.\log\left(\frac{y}{x}\right) + x^4.\frac{1}{\frac{y}{x}}\left(\frac{-y}{x^2}\right) = 4x^3.\log\left(\frac{y}{x}\right) - x^3 \text{ and } \frac{\partial z}{\partial y} = x^4.\frac{1}{\frac{y}{x}}\left(\frac{1}{x}\right) = \frac{x^4}{y}$$

Hence, the Euler's Theorem is verified**.**

Q.No.7.: If $u = \log\frac{x^4 + y^4}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Sol.: Here $u = \log\frac{x^4 + y^4}{x + y} \Rightarrow e^u = \frac{x^4 + y^4}{x + y}$.

e^u is a homogeneous function of x and y of degree 3.

Hence by Euler's theorem, we have $x\frac{\partial(e^u)}{\partial x} + y\frac{\partial(e^u)}{\partial y} = ne^u = 3e^u$.

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u.$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Hence the result**.**

Q.No.8.: If $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.

Sol.: Here $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) \Rightarrow \sin u = \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) = x^0\left[\frac{1 - \left(\frac{y}{x}\right)^{1/2}}{1 + \left(\frac{y}{x}\right)^{1/2}}\right] = x^0 f\left(\frac{y}{x}\right)$.

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree 0.

Hence by Euler's theorem, we have

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u = 0.\sin u = 0.$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0 \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Hence $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.

Hence the result**.**

Q.No.9.: If $u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\varphi\left(\frac{y}{x}\right)$.

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ **.**

Sol.:(i) Here $u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right) = u_1 + u_2$ (say)

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y}$.

Now since u_1 and u_2 are homogeneous function of x and y of degree 1 and 0 respectively. Then by Euler's theorem, we have

$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = nu_1 = x\varphi\left(\frac{y}{x}\right)$ and $x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = nu_2 = 0$.

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = x\varphi\left(\frac{y}{x}\right) + 0$.

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\varphi\left(\frac{y}{x}\right)$.

Hence the result**.**

(ii) Again, since u_1 and u_2 are homogeneous function of x and y of degree 1 and 0 respectively. Then, by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1(1-1).x\varphi\left(\frac{y}{x}\right) = 0.$$

Hence $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Hence the result**.**

Q.No.10.: If $u = \left(x^2 + y^2\right)^{1/3}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2}{9}u$ **.**

Sol.: Here $u = \left(x^2 + y^2\right)^{1/3} = x^{2/3}\left[1 + \left(\frac{y}{x}\right)^2\right]^{1/3} = x^{2/3}f\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree $\frac{2}{3}$.

Hence by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=\frac{2}{3}\left(\frac{2}{3}-1\right)=-\frac{2}{9}u\ .$$

Hence the result.

Q.No.11.: If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

Sol.: Here $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow \sin u(=z,\text{say})=\left(\frac{x^2+y^2}{x+y}\right)=x\left[\frac{1+\frac{y^2}{x^2}}{1+\frac{y}{x}}\right]=x^1 f\left(\frac{y}{x}\right)$.

$\Rightarrow$ z is a homogeneous function of x and y of degree 1.

Then, by Euler's theorem, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=z$.

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u\ .$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u\ .$$

Hence the result.

Q.No.12.: If $u=\sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$, then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u$.

(ii) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{\tan u}{144}\left(13+\tan^2 u\right)$.

Sol.: **(i)** Here $u=\sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$

$$\Rightarrow \sin u = \left[\frac{x^{1/3}\left(1+\frac{y^{1/3}}{x^{1/3}}\right)}{x^{1/2}\left(1+\frac{y^{1/2}}{x^{1/2}}\right)}\right]^{1/2} = \left[x^{-1/6}\left\{\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right\}\right]^{1/2} = x^{-1/12}\left[\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right]^{1/2} = x^{-1/12} f\left(\frac{y}{x}\right).$$

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree $-\frac{1}{12}$.

Then, by Euler's theorem, we have

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u = -\frac{1}{12}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = -\frac{1}{12}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \; . \qquad \text{...(i)}$$

Hence the result.

(ii) Differentiating (i) w. r. t. x partially, we get $xu_{xx} + u_x + yu_{xy} = -\frac{1}{12}\sec^2 u(u_x)$

Multiplying by x, we get

$$x^2u_{xx} + xu_x + xyu_{xy} = -\frac{1}{12}\sec^2 u(xu_x). \qquad \text{.....(ii)}$$

Differentiating (i) w. r. t. y partially, we get

$$xu_{xy} + u_y + yu_{yy} = -\frac{1}{12}\sec^2 u(u_y).$$

Multiplying by y, we get

$$yxu_{xy} + yu_y + y^2u_{yy} = -\frac{1}{12}\sec^2 u(yu_y). \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{1}{12}\sec^2 u(xu_x + yu_y) - (xu_x + yu_y) = \left(-\frac{1}{12}\sec^2 u - 1\right)(xu_x + yu_y)$$

$$= \left(\frac{1}{12}\sec^2 u + 1\right)\left(\frac{1}{12}\tan u\right) = \frac{1}{144}\sec^2 u\tan u + \frac{1}{12}\tan u$$

$$= \frac{1}{144}\left(1+\tan^2 u\right)\tan u + \frac{1}{12}\tan u = \frac{1}{144}\tan u\left(1+\tan^2 u + 12\right)$$

$$=\frac{1}{144}\tan u\left(13+\tan^2 u\right).$$

Hence the result.

Q.No.13.: If $u=\frac{\left(x^2+y^2\right)^m}{2m-1}+x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$, show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2m\left(x^2+y^2\right)^m.$$

Sol.: Here $u=\frac{\left(x^2+y^2\right)^m}{2m-1}+x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)=u_1+u_2+u_3$ (say)

Then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}+x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}+x\frac{\partial u_3}{\partial x}+y\frac{\partial u_3}{\partial y}$.

Now since u_1, u_2 and u_3 are homogeneous function of x and y of degree $2m$, 1 and 0 respectively. Then by Euler's theorem, we have

$$x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}=nu_1=2mu_1,\ x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=nu_2=u_2 \text{ and } x\frac{\partial u_3}{\partial x}+y\frac{\partial u_3}{\partial y}=nu_3=0.$$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}+x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}+x\frac{\partial u_3}{\partial x}+y\frac{\partial u_3}{\partial y}=2m.u_1+1.u_2$.

Again, since u_1, u_2 and u_3 are homogeneous function of x and y of degree 2m, 1 and 0 respectively.

Then, by extension of Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=2m(2m-1)u_1+1(1-1)u_2=2m(2m-1)\frac{\left(x^2+y^2\right)^m}{2m-1}$$

Hence $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2m\left(x^2+y^2\right)^m$.

Hence the result.

Q.No.14.: If $u=\sin\left(\sqrt{x}+\sqrt{y}\right)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$$

Sol.: Here $u=\sin\left(\sqrt{x}+\sqrt{y}\right)$.

$$\Rightarrow \sin^{-1} u(= z, \text{say}) = \left(\sqrt{x} + \sqrt{y}\right) = x^{1/2}\left(1 + \left(\frac{y}{x}\right)^{1/2}\right) = x^{1/2} f\left(\frac{y}{x}\right).$$

$\Rightarrow$ z is a homogeneous function of x and y of degree $\frac{1}{2}$.

Then, by Euler's theorem, we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = \frac{1}{2}z$.

$$\Rightarrow x\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x} + y\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} = \frac{1}{2}\sin^{-1} u = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right).$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sqrt{1 - \sin^2\left(\sqrt{x} + \sqrt{y}\right)} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sqrt{\cos^2\left(\sqrt{x} + \sqrt{y}\right)}.$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\cos\left(\sqrt{x} + \sqrt{y}\right)$.

Hence the result.

Q.No.15.: If $z = \sin^{-1}\left(\sqrt{x^2 + y^2}\right)$, then show that $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = \tan^3 z$.

Sol.: Here $z = \sin^{-1}\left(\sqrt{x^2 + y^2}\right) \Rightarrow \sin z(= u, \text{say}) = \sqrt{x^2 + y^2} = x\sqrt{1 + \frac{y^2}{x^2}} = xf\left(\frac{y}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x and y of degree 1.

Then by Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u$.

$$\Rightarrow x\cos z\frac{\partial z}{\partial x} + y\cos z\frac{\partial z}{\partial y} = \sin z \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z. \qquad \text{(i)}$$

Differentiating (i) w. r. t. x partially, we get

$$xz_{xx} + z_x + yz_{xy} = \sec^2 z(z_x)$$

Multiplying by x, we get

$$x^2 z_{xx} + xz_x + xyz_{xy} = \sec^2 z(xz_x). \qquad \text{......(ii)}$$

Differentiating (i) w. r. t. y partially, we get

$$xz_{xy} + z_y + yz_{yy} = \sec^2 z(z_y)$$

Multiplying by y, we get

$$yxz_{xy} + yz_y + y^2z_{yy} = \sec^2 z\left(yz_y\right). \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get

$$x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} = \sec^2 z\left(xz_x + yz_y\right) - \left(xz_x + yz_y\right) = \sec^2 z(\tan z) - \tan z.$$

$$= \tan z\left(\sec^2 z - 1\right) = \tan z\left(\tan^2 z\right) = \tan^3 z.$$

Hence the result.

Q.No.16.: If $u + iv = (x \pm iy)^2$, and $w = \dfrac{u}{v}$, prove that $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 0$.

Sol.: Here $u + iv = (x \pm iy)^2 \Rightarrow u + iv = x^2 - y^2 \pm 2ixy$.

Thus $$w = \frac{u}{v} = \frac{x^2 - y^2}{2xy} = x^0\left[\frac{1 - \left(\frac{y}{x}\right)^2}{2\frac{y}{x}}\right] = x^0 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ w is a homogeneous function of x and y of degree 0.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = 0.w = 0.$$

Hence $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 0$.

This completes the proof.

Q.No.17.: If $u + iv = (ax \pm iby)^3$, show that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$,

(ii) $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 3v$,

(iii) $(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 0$ where $w = \dfrac{u}{v}$.

Sol.: (i) Here $u + iv = (ax \pm iby)^3 \Rightarrow u + iv = (a^3x^3 - 3ab^2xy^2) + i(b^3y^3 \pm 3a^2bx^2y)$.

Thus $$u = (a^3x^3 - 3ab^2xy^2) = x^3\left[a^3 - 3ab^2\left(\frac{y}{x}\right)^2\right] = x^3 f\left(\frac{y}{x}\right).$$

$\Rightarrow$ u is a homogeneous function of x and y of degree 3.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=3.w\,.$$

Hence $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=3w$**.**

This completes the proof**.**

(ii) Here $u+iv=(ax\pm iby)^3 \;\Rightarrow u+iv=(a^3x^3-3ab^2xy^2)+i(b^3y^3\pm 3a^2bx^2y)$.

Thus $v=(b^3y^3\pm 3a^2bx^2y)=x^3\left[b^3\left(\dfrac{y}{x}\right)^3\pm 3a^2b\left(\dfrac{y}{x}\right)\right]=x^3f\left(\dfrac{y}{x}\right)$.

$\Rightarrow$ v is a homogeneous function of x and y of degree 3.

Then, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=3.v\,.$$

Hence $x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}=3v$.

This completes the proof**.**

(iii) Here $u+iv=(ax\pm iby)^3 \;\Rightarrow u+iv=(a^3x^3-3ab^2xy^2)+i(b^3y^3\pm 3a^2bx^2y)$.

$$\text{Thus}\quad w=\frac{u}{v}=\frac{a^3x^3-3ab^2xy^2}{b^3y^3\pm 3a^2bx^2y}=x^0\left[\frac{a^3-3ab^2\left(\dfrac{y}{x}\right)^2}{b^3\left(\dfrac{y}{x}\right)^3\pm 3a^2b\left(\dfrac{y}{x}\right)}\right]=x^0f\left(\frac{y}{x}\right).$$

$\Rightarrow$ w is a homogeneous function of x and y of degree 0.

Then, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=0.w=0\,.$$

Hence $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=0$.

This completes the proof**.**

Q.No.18.: If $v = \left(x^2 + y^2 + z^2\right)^{-1/2}$,then show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = -v$.

Sol.: Here $v = \left(x^2 + y^2 + z^2\right)^{-1/2} \Rightarrow v = x^{-1}\left[1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2\right]^{-1/2} = x^{-1} f\left(\frac{y}{x}, \frac{z}{x}\right)$

$\Rightarrow$ v is a homogeneous function of x ,y and z of degree -1.

Hence by Euler's theorem, we have

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nu = (-1).v = -v$.

Hence the result**.**

Q.No.19.: If $u = \sin^{-1}\left(\sqrt{x} + \sqrt{y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u.\cos 2u}{4\cos^3 u}.$$

Sol.: Here $u = \sin^{-1}\left(\sqrt{x} + \sqrt{y}\right) \Rightarrow \sin u = \left(\sqrt{x} + \sqrt{y}\right) = x^{1/2}\left(1 + \left(\frac{y}{x}\right)^{1/2}\right) = x^{1/2} f\left(\frac{y}{x}\right)$.

$\Rightarrow \sin u$ is a homogeneous function of x and y of degree $\frac{1}{2}$

$\therefore$ By Euler's theorem, $x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u = \frac{1}{2}\sin u$

$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u \ . \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$ . ...(i)

Differentiating (i) w. r. t. x partially, we get

$xu_{xx} + u_x + yu_{xy} = \frac{1}{2}\sec^2 u\left(u_x\right)$

Multiplying by x , we get

$x^2 u_{xx} + xu_x + xyu_{xy} = \frac{1}{2}\sec^2 u\left(xu_x\right)$ ......(ii)

Differentiating (i) w. r. t. y partially, we get

$xu_{xy} + u_y + yu_{yy} = \frac{1}{2}\sec^2 u\left(u_y\right)$

Multiplying by y , we get

$$yxu_{xy}+yu_y+y^2u_{yy}=\frac{1}{2}\sec^2 u\left(yu_y\right) \qquad ......(iii)$$

Adding (ii) and (iii), we get

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{1}{2}\sec^2 u\left(xu_x+yu_y\right)-\left(xu_x+yu_y\right)=\left(\frac{1}{2}\sec^2 u-1\right)\left(xu_x+yu_y\right)$$

$$=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)=\frac{1}{4}\sec^2 u\tan u-\frac{1}{2}\tan u$$

$$=\frac{1}{4}\tan u\left(\sec^2 u-2\right)=\frac{\tan u}{4}\left(\frac{1}{\cos^2 u}-2\right)=\frac{\tan u}{4}\frac{\left(1-2\cos^2 u\right)}{\cos^2 u}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-\sin u.\cos 2u}{4\cos^3 u}.$$

Hence the result**.**

Q.No.20.: If $V=\tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)$, prove that

$$x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}=\sin 4V-\sin 2V\,\textbf{.}$$

Sol.: Here $V=\tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)\Rightarrow \tan V=\frac{x^3+y^3}{2x+3y}=x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{2+3\frac{y}{x}}\right]=x^2 f\left(\frac{y}{x}\right)=z$ (say).

$\Rightarrow$ z is a homogeneous function of x and y of degree 2.

Then by Euler's theorem, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=2z$.

$$\Rightarrow x\sec^2 V\frac{\partial V}{\partial x}+y\sec^2 V\frac{\partial V}{\partial y}=2\tan V$$

$$\Rightarrow x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\frac{2\tan V}{\sec^2 V}=2\frac{\sin V}{\cos V}.\cos^2 V=2\sin V\cos V=\sin 2V. \qquad ....(i)$$

Differentiating (i) partially w. r. t. x [keeping y as constant], we get

$$x\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}+y\frac{\partial^2 V}{\partial x\partial y}=\cos 2V.2\frac{\partial V}{\partial x}.$$

Multiplying by x, we get

$$x^2\frac{\partial^2 V}{\partial x^2}+x\frac{\partial V}{\partial x}+xy\frac{\partial^2 V}{\partial x\partial y}=2x\cos 2V\frac{\partial V}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 V}{\partial x^2}+xy\frac{\partial^2 V}{\partial x\partial y}=(2\cos 2V-1)x\frac{\partial V}{\partial x} \qquad ..(ii)$$

Again, Differentiating (i) partially w. r. t. y [keeping x as constant], we get

$$x\frac{\partial^2 V}{\partial y\partial x}+y\frac{\partial^2 V}{\partial y^2}+\frac{\partial V}{\partial y}=\cos 2V.2\frac{\partial V}{\partial y}.$$

Multiplying by y, we get

$$xy\frac{\partial^2 V}{\partial y\partial x}+y^2\frac{\partial^2 V}{\partial y^2}+y\frac{\partial V}{\partial y}=2y\cos 2V\frac{\partial V}{\partial y}$$

$$\Rightarrow xy\frac{\partial^2 V}{\partial y\partial x}+y^2\frac{\partial^2 V}{\partial y^2}=(2\cos 2V-1)y\frac{\partial V}{\partial y}. \qquad .....(iii)$$

Adding (ii) and (iii), we get

$$x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}=(2\cos 2V-1)\left[x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right]=(2\cos 2V-1)\sin 2V$$

$$=2\sin 2V\cos 2V-\sin 2V=\sin 4V-\sin 2V.$$

Hence $x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}=\sin 4V-\sin 2V$.

Hence the result**.**

Q.No.21.: If $u=x^n f\left(\frac{y}{x},\frac{z}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$ **.**

Sol.: Here $u=x^n f\left(\frac{y}{x},\frac{z}{x}\right)$.Let $\frac{y}{x}=t_1$, $\frac{z}{x}=t_2$

$$\therefore u=x^n f(t_1,t_2)=x^n f$$

Differentiating partially w. r. t x , y and z respectively, we get

$$\frac{\partial u}{\partial x}=nx^{n-1}f+x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial x}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial x}\right]=nx^{n-1}f+x^n\left[\frac{\partial f}{\partial t_1}\left(\frac{-y}{x^2}\right)+\frac{\partial f}{\partial t_2}\left(\frac{-z}{x^2}\right)\right]$$

$$\Rightarrow \frac{\partial u}{\partial x}=nx^{n-1}f-yx^{n-1}\frac{\partial f}{\partial t_1}-zx^{n-1}\frac{\partial f}{\partial t_2},$$

$$\frac{\partial u}{\partial y}=x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial y}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial y}\right]=x^n\left[\frac{\partial f}{\partial t_1}\left(\frac{1}{x}\right)\right]=x^{n-1}\frac{\partial f}{\partial t_1},$$

$$\frac{\partial u}{\partial z}=x^n\left[\frac{\partial f}{\partial t_1}.\frac{\partial t_1}{\partial z}+\frac{\partial f}{\partial t_2}.\frac{\partial t_2}{\partial z}\right]=x^n\left[\frac{\partial f}{\partial t_2}\left(\frac{1}{x}\right)\right]=x^{n-1}\frac{\partial f}{\partial t_2}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x\left[nx^{n-1}f-yx^{n-1}\frac{\partial f}{\partial t_1}-zx^{n-1}\frac{\partial f}{\partial t_2}\right]+y\left[x^{n-1}\frac{\partial f}{\partial t_1}\right]+z\left[x^{n-1}\frac{\partial f}{\partial t_2}\right].$$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nx^nf(t_1,t_2)=nu$.

Hence the result**.**

Q.No.22.: If $u=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$**.**

Sol.: Here $u=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}=x^{-2}\left[1+\frac{1}{\frac{y}{x}}-\frac{\log\left(\frac{y}{x}\right)}{1+\frac{y^2}{x^2}}\right]=x^{-2}f\left(\frac{y}{x}\right).$

$\Rightarrow$ u is a homogeneous function of x and y of degree -2.

Hence by Euler's theorem, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u$.

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$**.**

Q.No.23.: If $f(x,y,z)=\log\left(\frac{x^5+y^5+z^5}{x+y+z}\right)$, show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4$**.**

Sol.: Here $f(x,y,z)=\log\left(\frac{x^5+y^5+z^5}{x+y+z}\right)\Rightarrow e^f=\frac{x^5+y^5+z^5}{x+y+z}$ ...(i)

$$\Rightarrow e^f=x^4\left[\frac{1+\left(\frac{y}{x}\right)^5+\left(\frac{z}{x}\right)^5}{1+\frac{y}{x}+\frac{z}{x}}\right]=x^4f\left(\frac{y}{x},\frac{z}{x}\right).$$

$\Rightarrow$ e^f is a homogeneous function of x, y and z of degree 4.

Hence by Euler's theorem, we have

$$x\frac{\partial\left(e^f\right)}{\partial x}+y\frac{\partial\left(e^f\right)}{\partial y}+z\frac{\partial\left(e^f\right)}{\partial z}=4e^f .$$

$$\Rightarrow xe^f\frac{\partial f}{\partial x}+ye^f\frac{\partial f}{\partial y}+ze^f\frac{\partial f}{\partial z}=4e^f \Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4 .$$

Hence the result**.**

Q.No.24.: If $u=\sin\left(\frac{x^2-y^2+z^2}{xy-yz-zx}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Sol.: Here $u=\sin\left(\frac{x^2-y^2+z^2}{xy-yz-zx}\right)=x^0\sin\left[\frac{1-\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}{\frac{y}{x}-\frac{y}{x}.\frac{z}{x}-\frac{z}{x}}\right]=x^0 f\left(\frac{y}{x},\frac{z}{x}\right)$.

$\Rightarrow$ u is a homogeneous function of x ,y and z of degree 0.

Then by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=0.u=0 .$$

Hence the result**.**

Q.No.25: Given that F(u) = V(x , y , z), where V is a homogeneous function of x , y , z of degree n , then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{F(u)}{F^{/}(u)}$.

Sol.: Here V(x , y , z) is a homogeneous function of x , y , z of degree n , then by Euler's theorem $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=nV \Rightarrow x\frac{\partial F(u)}{\partial x}+y\frac{\partial F(u)}{\partial y}+z\frac{\partial F(u)}{\partial z}=nF(u)$

$$\Rightarrow xF'(u)\frac{\partial u}{\partial x}+yF'(u)\frac{\partial u}{\partial y}+zF'(u)\frac{\partial u}{\partial z}=nF(u)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{F(u)}{F^{/}(u)} .$$

Hence the result**.**

Thank you

NEXT TOPIC

Total Differentials,

Explicit Function, Implicit Functions

and

Total Differential Coefficient

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