Differential Calculus

Partial Differentiation

(Partial Differential Coefficient)

Prepared by

Dr. Sunil

NIT Hamirpur (HP)

Introduction

- **Partial differentiation is the process of finding partial derivatives.**

  Let u be a function of x and y i.e. u = f(x, y).
- **A partial derivative of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constant.**
- **All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the other variables are treated (temporarily) as constants.**

Differential Coefficient:

If y is a function of only one independent variable, say x, then we can write

$$y = f(x).$$

Then, the rate of change of y w.r.t. x i.e. the derivative of y w.r.t. x is defined as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(y+\delta y)-y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

where δy is the change or increment of y corresponding to the increment δx of the independent variable x.

Partial Differential Coefficient:

Let u be a function of x and y i.e. u = f(x, y).

Then the partial differential coefficient of u (i.e. f(x, y) w.r.t. x (keeping y as constant) is defined and written as

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} = u_x = f_x = \frac{\partial f}{\partial x}.$$

Similarly, the partial differential coefficient of u (i.e. f(x, y) w.r.t. y (keeping x as constant) is defined and written as

$$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = u_y = f_y = \frac{\partial f}{\partial y}.$$

Similarly, we can find

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right), \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Also, it can be verified that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Notation:

The partial derivative $\frac{\partial u}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or $f_x(x, y, z)$ or f_x or $D_x f$ or $f_1(x, y, z)$, where the subscripts x and 1 denote the variable w.r.t. x which the partial differentiation is carried out.

Thus, we can have $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = f_y(x, y, z) = f_y = D_y f = f_2(x, y, z)$ etc.

The value of a partial derivative at a point (a, b, c) is denoted by

$$\left.\frac{\partial u}{\partial x}\right|_{x=a, y=b, z=c} = \left.\frac{\partial u}{\partial x}\right|_{(a,b,c)} = f_x(a, b, c).$$

Geometrical Interpretation of partial derivatives:

(Geometrical interpretation of a partial derivative of a function of two variables)

$z = f(x, y)$ represents the **equation of surface** in xyz-coordinate system. Let APB be the curve, which is drawn on a plane through any point P on the surface parallel to the xz-plane.

As point P moves along the curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the 'rate at which z changes w.r.t. x'.

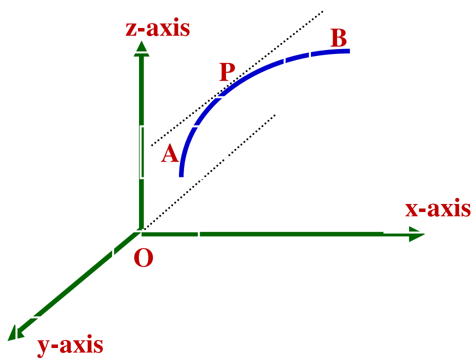

Figure 1

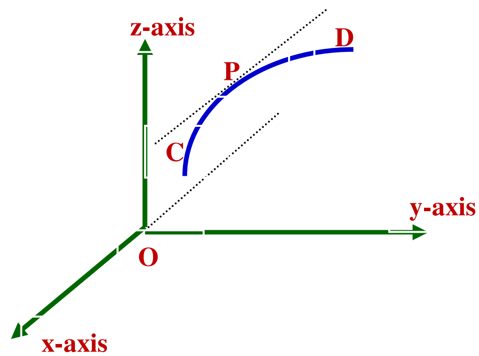

Figure 2

Thus $\dfrac{\partial z}{\partial x} = \tan\alpha$ = slope of the curve APB at the point P (see fig.1).

Similarly, $\dfrac{\partial z}{\partial y} = \tan\beta$ = slope of the curve CPD at the point P (see fig.2).

Higher Order Parallel Derivatives:

Partial derivatives of higher order, of a function f(x, y, z) are calculated by successive differentiate. Thus, if u = f(x, y, z) then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = f_{11}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = f_{21},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = f_{12}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = f_{22},$$

$$\frac{\partial^3 u}{\partial z^2 \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial z \partial y}\right) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right] = f_{yzz} = f_{233},$$

$$\frac{\partial^4 u}{\partial x \partial y \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial y \partial z^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z^2}\right)\right] = f_{zzyx} = f_{3321}.$$

The partial derivative $\frac{\partial f}{\partial x}$ obtained by differentiating once in known as first order partial derivative, while $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ which are obtained by differentiating twice are known as second order derivatives. 3rd order, 4th order derivatives involve 3, 4, times differentiation respectively.

Note 1: The crossed or mixed partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are, in general, equal

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

i.e. the order of differentiation is immaterial if the derivatives involved are continuous.

Note 2: In the subscript notation, the subscript are written in the same order in which differentiation is carried out, while in '∂' notation the order is opposite, for example

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy}.$$

Note 3: A function of 2 variables has two first order derivatives, four second order derivatives and 2^{n} of n^{th} order derivatives. A function of m independent variables will have m^{n} derivatives of order n.

Now let us solve some problems related to the above-mentioned topics:

Q.No.1.: If $u = \tan^{-1}\left(\frac{y}{x}\right)$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Sol.: Here $u = \tan^{-1}\left(\frac{y}{x}\right)$.

Since $\frac{\partial u}{\partial x}$ = the p. d. coefficient of u w. r. t. x (keeping y as constant)

$$= \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2).0-(2x)(-y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \qquad \text{....(i)}$$

Similarly, $\frac{\partial u}{\partial y}$ = the p. d. coefficient of u w. r. t. y (keeping x as constant)

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2).0-(2y)(x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \qquad \text{....(ii)}$$

Adding (i) and (ii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0.$$

This completes the proof.

Q.No.2.: If $u = f(x+ay)+\phi(x-ay)$, then prove that $\frac{\partial^2 u}{\partial y^2} = a^2.\frac{\partial^2 u}{\partial x^2}$.

Sol.: Here $u = f(x+ay)+\phi(x-ay)$.

$$\therefore \frac{\partial u}{\partial x} = f'(x+ay)+\phi'(x-ay) \text{ and } \frac{\partial^2 u}{\partial x^2} = f''(x+ay)+\phi''(x-ay)$$

Also $\frac{\partial u}{\partial y} = f'(x+ay)(a)+\phi'(x-ay)(-a)$

and $\frac{\partial^2 u}{\partial y^2} = f''(x+ay)(a^2)+\varphi''(x-ay)(-a)^2$.

$$\frac{\partial^2 u}{\partial y^2} = (a^2)[f''(x+ay)+\phi''(x-ay)] = a^2.\frac{\partial^2 u}{\partial x^2}.$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = a^2.\frac{\partial^2 u}{\partial x^2}.$$

This completes the proof.

Q.No.3: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist.

Sol.: Now $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{y\to0} \dfrac{0.y^3}{0+y^6} = \lim\limits_{y\to0} 0 = 0$. ...(i)

Again $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{x\to0} \dfrac{x.0}{x^2+0} = \lim\limits_{x\to0} 0 = 0$ ....(ii)

Let $(x,y)\to(0,0)$ along the curve $x = my^3$, where m is a constant.

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6} = \lim\limits_{y\to0} \dfrac{my^3.y^3}{m^2y^6+y^6} = m\lim\limits_{y\to0} \dfrac{y^6}{y^6(m^2+1)} = \dfrac{m}{m^2+1}\lim\limits_{y\to0} 1 = \dfrac{m}{m^2+1}$. (iii)

From (i) and (ii) given limit is zero as $(x, y)\to(0, 0)$ separately.

But from (iii) limit is not zero, but is different for different values of m.

Hence the given limit does not exist**.**

Q.No.4: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist**.**

Sol.: Now $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to0} \dfrac{0.y}{0+y^2} = \lim\limits_{y\to0} 0 = 0$. ...(i)

Again $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to0} \dfrac{x^2.0}{x^4+0} = \lim\limits_{x\to0} 0 = 0$. ....(ii)

Let $(x,y)\to(0,0)$ along the curve $x = \sqrt{my}$,where m is a constant.

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to0} \dfrac{my.y}{m^2y^2+y^2} = m\lim\limits_{y\to0} \dfrac{y^2}{y^2(m^2+1)} = \dfrac{m}{m^2+1}\lim\limits_{y\to0} 1 = \dfrac{m}{m^2+1}$ (iii)

From (i) and (ii) given limit is zero as $(x, y)\to(0, 0)$ separately.

But from (iii) limit is not zero, but is different for different values of m.

Hence the given limit does not exist**.**

Q.No.5: If $f(x,y) = \dfrac{y^2+x^2}{y^2-x^2}$, find the limit of f(x, y) when approaches origin (0, 0) along the line y = mx, where m is constant**.**

Sol.: Let $(x,y)\to(0,0)$ along the curve $y = mx$ where m is a constant.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{y^2+x^2}{y^2-x^2} = \lim_{x\to 0} \frac{m^2x^2+.x^2}{m^2x^2-x^2} = \frac{m^2+1}{m^2-1}\lim_{x\to 0}\frac{x^2}{x^2} = \frac{m^2+1}{m^2-1}\lim_{x\to 0} 1 = \frac{m^2+1}{m^2-1}.\ \text{Ans.}$$

Q.No.6.: If $u = \frac{1}{r}$, where $r^2 = x^2+y^2+z^2$. Show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$.

Sol.: Since $r^2 = x^2+y^2+z^2$.

Differential partially w. r. t. x , we get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Now here $u = \frac{1}{r}$,

Differential partially w. r. t. x , we get $\frac{\partial u}{\partial x} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}.\frac{x}{r} = -\frac{x}{r^3}$.

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{r^3.1 - x.3r^2.\frac{\partial r}{\partial x}}{r^6} = -\frac{r^3-3r^2.\frac{x^2}{r}}{r^6} = -\frac{r^3-3rx^2}{r^6} = \frac{3x^2}{r^5}-\frac{1}{r^3} \qquad \text{...(i)}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{3y^2}{r^5}-\frac{1}{r^3}$ ...(ii),

$$\frac{\partial^2 u}{\partial z^2} = \frac{3z^2}{r^5}-\frac{1}{r^3} \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{3}{r^5}\left[x^2+y^2+z^2\right]-\frac{3}{r^3} = \frac{3}{r^5}.r^2-\frac{3}{r^3} = \frac{3}{r^3}-\frac{3}{r^3} = 0.$$

This completes the proof**.**

Q.No.7: If u = xyz, find d^2u.

Sol.: We know that if u = f(x, y, z), then

$$du = \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz = \left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)u$$

$$\therefore d^2 = d(du)$$

$$= \left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)u = \left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 u$$

$$=\left[(dx)^2\frac{\partial^2}{\partial x^2}+(dy)^2\frac{\partial^2}{\partial y^2}+(dz)^2\frac{\partial^2}{\partial z^2}+2dxdy\frac{\partial^2}{\partial x\partial y}+2dydz\frac{\partial^2}{\partial y\partial z}+2dzdx\frac{\partial^2}{\partial z\partial x}\right]u$$

$$=\frac{\partial^2 u}{\partial x^2}(dx)^2+\frac{\partial^2 u}{\partial y^2}(dy)^2+\frac{\partial^2 u}{\partial z^2}(dz)^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+2\frac{\partial^2 u}{\partial y\partial z}dydz+2\frac{\partial^2 u}{\partial z\partial x}dzdx \qquad \text{(i)}$$

Here $u = xyz$

$$\frac{\partial u}{\partial x}=yz,\quad \frac{\partial u}{\partial y}=zx,\quad \frac{\partial u}{\partial z}=xy.$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial z^2}=0.$$

$$\frac{\partial^2 u}{\partial x\partial y}=z,\quad \frac{\partial^2 u}{\partial y\partial z}=x,\quad \frac{\partial^2 u}{\partial z\partial x}=y.$$

$\therefore$ From (i), we have $d^2u = 2zdxdy + 2xdydz + 2ydzdx$.

Q.No.8: Evaluate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, when **(a)** $u = x^y$ and **(b)** $xy + yu + ux = 1$.

Sol.: **(a)** Given $u = x^y$. ...(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^y\right)=yx^{y-1} \text{ and } \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^y\right)=x^y\log x\,.\ \text{Ans.}$$

(b) Given $xy + yu + ux = 1 \Rightarrow u(x+y) = 1-xy \quad \Rightarrow u = \dfrac{1-xy}{x+y}$ ...(ii)

Differentiate (ii) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1-xy}{x+y}\right)=\frac{(x+y)(-y)-(1-xy).1}{(x+y)^2}=-\frac{\left(1+y^2\right)}{(x+y)^2}$$

$$\text{and } \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1-xy}{x+y}\right)=\frac{(x+y)(-x)-(1-xy).1}{(x+y)^2}=-\frac{\left(1+x^2\right)}{(x+y)^2}.\ \text{Ans.}$$

Q.No.9: Verify that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$, where u is equal to

(i) $\log(y\sin x + x\sin y)$, **(ii)** $\log\left(\dfrac{x^2+y^2}{xy}\right)$,

(iii) $\log\tan\left(\frac{x}{y}\right)$ and **(iv)** $x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$.

Sol.:(i) Here $u=\log(y\sin x+x\sin y)$. ...(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{(y\cos x+\sin y)}{(y\sin x+x\sin y)}. \quad ....(ii)$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{(y\sin x+x\sin y)(\cos x+\cos y)-(y\cos x+\sin y)(\sin x+x\cos y)}{(y\sin x+x\sin y)^2}. \quad (iii)$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{(\sin x+x\cos y)}{(y\sin x+x\sin y)}. \quad (iv)$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=\frac{(y\sin x+x\sin y)(\cos x+\cos y)-(\sin x+x\cos y)(y\cos x+\sin y)}{(y\sin x+x\sin y)^2}. \quad (v)$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

This completes the proof.

(ii) Here $u=\log\left(\frac{x^2+y^2}{xy}\right)$. ...(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{\frac{x^2+y^2}{xy}}.\frac{xy(2x)-(x^2+y^2)y}{(xy)^2}=\frac{1}{x^2+y^2}.\frac{x^2y-y^3}{xy}=\frac{x^2-y^2}{x(x^2+y^2)}. \quad (ii)$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{(x^3+y^2x)(-2y)-(x^2-y^2)(2xy)}{(x^3+y^2x)^2}=-\frac{4x^3y}{(x^3+y^2x)^2}=-\frac{4xy}{(x^2+y^2)^2}. \quad (iii)$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{1}{\frac{x^2+y^2}{xy}}\cdot\frac{xy(2y)-\left(x^2+y^2\right)x}{(xy)^2}=\frac{1}{x^2+y^2}\cdot\frac{xy^2-x^3}{xy}=\frac{y^2-x^2}{y\left(x^2+y^2\right)}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=\frac{\left(yx^2+y^3\right)(-2x)-\left(y^2-x^2\right)(2xy)}{\left(yx^2+y^3\right)^2}=-\frac{4xy^3}{\left(yx^2+y^3\right)^2}=-\frac{4xy}{\left(x^2+y^2\right)^2}. \quad \text{..(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

This completes the proof.

(iii) Here $u=\log\tan\left(\frac{x}{y}\right)$. ....(i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{\tan\frac{x}{y}}.\sec^2\frac{x}{y}.\frac{1}{y}=\frac{\sec^2\frac{x}{y}}{y\tan\frac{x}{y}}. \qquad \text{....(ii)}$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{y\tan\frac{x}{y}.\frac{\partial}{\partial y}\left(\sec^2\frac{x}{y}\right)-\sec^2\frac{x}{y}.\frac{\partial}{\partial y}\left(y\tan\frac{x}{y}\right)}{y^2\tan^2\frac{x}{y}}$$

$$=\frac{x\sec^2\frac{x}{y}\tan\frac{x}{y}-3x\sec^2\frac{x}{y}\tan^2\frac{x}{y}}{y^3\tan^2\frac{x}{y}}. \qquad \text{(iii)}$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=\frac{1}{\tan\frac{x}{y}}.\sec^2\frac{x}{y}.\left(-\frac{x}{y^2}\right)=-\frac{x}{y^2}.\frac{\sec^2\frac{x}{y}}{\tan\frac{x}{y}}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = -\frac{y^2 \tan\frac{x}{y} . \frac{\partial}{\partial x}\left(x \sec^2\frac{x}{y}\right) - x\sec^2\frac{x}{y} . \frac{\partial}{\partial y}\left(y^2 \tan\frac{x}{y}\right)}{y^4 \tan^2\frac{x}{y}}$$

$$= \frac{x\sec^2\frac{x}{y}\tan\frac{x}{y} - 3x\sec^2\frac{x}{y}\tan^2\frac{x}{y}}{y^3\tan^2\frac{x}{y}}. \qquad \text{(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$**.**

This completes the proof**.**

(iv) Here $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$**.** (i)

Differentiate (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+\frac{y^2}{x^2}} . \left(-\frac{y}{x^2}\right) + \left[2x\tan^{-1}\frac{y}{x} - y^2 . \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)\right]$$

$$= -\frac{x^2 y}{x^2+y^2} + 2x\tan^{-1}\frac{y}{x} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\frac{y}{x} - \frac{x^2 y + y^3}{x^2+y^2} = 2x\tan^{-1}\frac{y}{x} - y\,. \qquad \text{(ii)}$$

Differentiate (ii) partially w. r. t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = 2x . \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{2x^2 - x^2 - y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad \text{(iii)}$$

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}} . \frac{1}{x} - \left[2y\tan^{-1}\frac{x}{y} + y^2 . \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)\right].$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = \frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} = \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\therefore \frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}. \qquad \text{(iv)}$$

Differentiate (iv) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right] = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}.\frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}. \qquad \text{(v)}$$

Hence from (iii) and (v), we get $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$**.**

This completes the proof**.**

Q.No.10: If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$**.**

Sol.: Since $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$.

Here $u = \log\left(x^3 + y^3 + z^3\right)$. .(i)

Differentiate (i) partially w. r. t. x ,y and z separately, we get

$$\frac{\partial u}{\partial x} = \frac{3\left(x^2 - yz\right)}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3\left(y^2 - xz\right)}{x^3+y^3+z^3-3xyz} \text{ and } \frac{\partial u}{\partial z} = \frac{3\left(z^2 - xy\right)}{x^3+y^3+z^3-3xyz},$$

$$\therefore \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{\left(x^3+y^3+z^3-3xyz\right)} = \frac{3}{(x+y+z)}.$$

$$\text{Hence } \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}.$$

Hence $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$**.**

This completes the proof**.**

Q.No.11: If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$**.**

Sol.: Here $u = e^{xyz}$. Now $\dfrac{\partial u}{\partial z} = \dfrac{\partial}{\partial z}\left(e^{xyz}\right) = e^{xyz}xy$.

$$\therefore \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial}{\partial y}\left[e^{xyz}xy\right] = xy\left(e^{xyz}xz\right) + e^{xyz}x = x^2yze^{xyz} + e^{xyz}x = \left(x^2yz + x\right)e^{xyz}$$

And hence $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 u}{\partial y \partial z}\right] = \dfrac{\partial}{\partial x}\left[\left(x^2yz + x\right)e^{xyz}\right] = [2xyz + 1]e^{xyz} + \left[x^2yz + x\right]e^{xyz}yz$

$$= \left[2xyz + 1 + x^2y^2z^2 + xyz\right]e^{xyz} = \left[x^2y^2z^2 + 3xyz + 1\right]e^{xyz}.$$

This completes the proof**.**

Q.No.12: If $u = z = \left(1 - 2xy + y^2\right)^{-1/2}$, prove that

$$\textbf{(i)}\ x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = y^2z^3\ ,\ \textbf{(ii)}\ \frac{\partial}{\partial x}\left[\left(1 - x^2\right)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0.$$

Sol.: (i) Here $z = \left(1 - 2xy + y^2\right)^{-1/2}$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial z}{\partial x} = -\frac{1}{2}\left(1 - 2xy + y^2\right)^{-3/2}(-2y) = y\left(1 - 2xy + y^2\right)^{-3/2}.$$

and $\dfrac{\partial z}{\partial y} = -\dfrac{1}{2}\left(1 - 2xy + y^2\right)^{-3/2}(-2x + 2y) = (x - y)\left(1 - 2xy + y^2\right)^{-3/2}$.

Hence $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = x\left[y\left(1 - 2xy + y^2\right)^{-3/2}\right] - y\left[(x - y)\left(1 - 2xy + y^2\right)^{-3/2}\right]$

$$= \left(1 - 2xy + y^2\right)^{-3/2}\left[xy - xy + y^2\right] = y^2z^3.$$

This completes the proof**.**

(ii) To show: $\dfrac{\partial}{\partial x}\left[\left(1 - x^2\right)\dfrac{\partial u}{\partial x}\right] + \dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial u}{\partial y}\right) = 0$.

Here $u = \left(1 - 2xy + y^2\right)^{-1/2}$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x} = y\left(1 - 2xy + y^2\right)^{-3/2} \text{ and } \frac{\partial u}{\partial y} = (x - y)\left(1 - 2xy + y^2\right)^{-3/2}.$$

Now $\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial x}\left[\left(1-x^2\right)y\left(1-2xy+y^2\right)^{-3/2}\right]$

$$=y\left[\left(1-x^2\right)\frac{\partial}{\partial x}\left(1-2xy+y^2\right)^{-3/2}+\left(1-2xy+y^2\right)^{-3/2}\frac{\partial}{\partial x}\left(1-x^2\right)\right]$$

$$=y\left[\left(1-x^2\right)\left(-\frac{3}{2}\right)\left(1-2xy+y^2\right)^{-5/2}(-2y)+\left(1-2xy+y^2\right)^{-3/2}(-2x)\right]$$

$$=y\left[\frac{3y\left(1-x^2\right)}{\left(1-2xy+y^2\right)^{5/2}}-\frac{2x}{\left(1-2xy+y^2\right)^{3/2}}\right]=y\left[\frac{3y-3x^2y-2x+4x^2y-2xy^2}{\left(1-2xy+y^2\right)^{5/2}}\right]$$

$$\therefore \frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]=\frac{y\left(3y-2x+x^2y-2xy^2\right)}{\left(1-2xy+y^2\right)^{5/2}}.$$

Again $\frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial y}\left[y^2(x-y)\left(1-2xy+y^2\right)^{-3/2}\right]$

$$=\frac{\partial}{\partial y}\left(xy^2-y^3\right)\left(1-2xy+y^2\right)^{-3/2}$$

$$=\left(1-2xy+y^2\right)^{-3/2}\left(2xy-3y^2\right)+\left(xy^2-y^3\right)\left(-\frac{3}{2}\right)\left(1-2xy+y^2\right)^{-5/2}(-2x+2y)$$

$$=\frac{2xy-3y^2}{\left(1-2xy+y^2\right)^{3/2}}+\frac{3\left(xy^2-y^3\right)(x-y)}{\left(1-2xy+y^2\right)^{5/2}}=\frac{\left(2xy-3y^2\right)\left(1-2xy+y^2\right)+3\left(xy^2-y^3\right)(x-y)}{\left(1-2xy+y^2\right)^{5/2}}$$

$$=\frac{2xy-4x^2y^2+2xy^3-3y^2+6xy^3-3y^4+3y^2x^2-6xy^3+3y^4}{\left(1-2xy+y^2\right)^{5/2}}$$

$$=\frac{2xy-3y^2-x^2y^2+2xy^3}{\left(1-2xy+y^2\right)^{5/2}}=\frac{-y\left(3y-2x+x^2y-2xy^2\right)}{\left(1-2xy+y^2\right)^{5/2}}$$

or $\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right].$

Hence $\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=0$**.**

This completes the proof **.**

Q.No.13: If $u=\tan^{-1}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right]$, prove that $\frac{\partial^2 u}{\partial x\partial y}=\left(1+x^2+y^2\right)^{-3/2}$**.**

Sol.: Here $u = \tan^{-1}\left[\dfrac{xy}{\sqrt{1+x^2+y^2}}\right]$. ...(i)

Differentiate (i) partially w. r. t. y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\left[\frac{xy}{\sqrt{\left(1+x^2+y^2\right)}}\right]$$

$$= \frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\cdot\frac{\sqrt{1+x^2+y^2}.x - xy.\dfrac{1}{2\sqrt{1+x^2+y^2}}.2y}{\left(1+x^2+y^2\right)}$$

$$= \frac{\left(1+x^2+y^2\right)}{1+x^2+y^2+x^2y^2}\cdot\frac{x\left(1+x^2+y^2\right)-xy^2}{\left(1+x^2+y^2\right)\sqrt{1+x^2+y^2}} = \frac{x+x^3+xy^2-xy^2}{\left(1+x^2+y^2+x^2y^2\right)\sqrt{1+x^2+y^2}}$$

$$= \frac{x+x^3}{\left(1+x^2+y^2+x^2y^2\right)\sqrt{1+x^2+y^2}} = \frac{x\left(1+x^2\right)}{\left\{\left(1+x^2\right)+y^2\left(1+x^2\right)\right\}\sqrt{1+x^2+y^2}}$$

$$= \frac{x}{\left(1+y^2\right)\sqrt{1+x^2+y^2}}. \qquad \text{...(ii)}$$

Differentiate (ii) partially w. r. t. x ,we get

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{x}{\left(1+y^2\right)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{\sqrt{1+x^2+y^2}\left(1+y^2\right).1 - x\left\{\left(1+y^2\right).\dfrac{2x}{2\sqrt{1+x^2+y^2}}\right\}}{\left(1+y^2\right)^2\left(1+x^2+y^2\right)} = \frac{\left(1+x^2+y^2\right)\left(1+y^2\right)-x^2\left(1+y^2\right)}{\left(1+y^2\right)^2\left(1+x^2+y^2\right)\sqrt{1+x^2+y^2}}$$

$$= \frac{\left(1+y^2\right)\left(1+x^2+y^2-x^2\right)}{\left(1+y^2\right)^2\left(1+x^2+y^2\right)^{3/2}} = \frac{\left(1+y^2\right)^2}{\left(1+y^2\right)^2\left(1+x^2+y^2\right)^{3/2}} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}.$$

Hence $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{1}{\left(1+x^2+y^2\right)^{3/2}}$.

This completes the proof**.**

Q.No.14: If $z^2+t^2-4x+y^2=0$ and $z^3+t^3-2x^3+3y=0$;

Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial t}{\partial x}$.

Sol.: Here $z^2 + t^2 - 4x + y^2 = 0$ and $z^3 + t^3 - 2x^3 + 3y = 0$.

Differentiate partially the given equations w. r. t. x, considering z and t as function of x, we get

$$2z\frac{\partial z}{\partial x} + 2t\frac{\partial t}{\partial x} - 4 = 0$$

and $3z^2\frac{\partial z}{\partial x} + 3t^2\frac{\partial t}{\partial x} - -6x^2 = 0$.

Solve these equations simultaneously for $\frac{\partial z}{\partial x}$ and $\frac{\partial t}{\partial x}$.

$$\frac{\frac{\partial z}{\partial x}}{2t.\left(-6x^2\right) + 4.3.t^2} = \frac{\frac{\partial t}{\partial x}}{-12z^2 + 12zx^2} = \frac{1}{6zt^2 - 6tz^2}.$$

$$\Rightarrow \frac{\frac{\partial z}{\partial x}}{12t\left(t - x^2\right)} = \frac{\frac{\partial t}{\partial x}}{12z\left(x^2 - z\right)} = \frac{1}{6tz(t - z)}.$$

Considering $\frac{\frac{\partial z}{\partial x}}{12t\left(t - x^2\right)} = \frac{1}{6tz(t - z)}$ and $\frac{\frac{\partial t}{\partial x}}{12z\left(x^2 - z\right)} = \frac{1}{6tz(t - z)}$.

We get $\frac{\partial z}{\partial x} = \frac{12t\left(t - x^2\right)}{6tz(t - z)} = \frac{2\left(x^2 - t\right)}{z(z - t)}$ and $\frac{\partial t}{\partial x} = \frac{12z\left(x^2 - z\right)}{6tz(t - z)} = \frac{2\left(x^2 - z\right)}{t(t - z)}$. Ans.

Q.No.15: If $u = \frac{ke^{\frac{-x^2}{4a^2y}}}{\sqrt{y}}$, then prove that $\frac{\partial u}{\partial y} = a^2\frac{\partial^2 u}{\partial x^2}$.

Sol.: Here $u = \frac{ke^{\frac{-x^2}{4a^2y}}}{\sqrt{y}}$, then $\frac{\partial u}{\partial y} = \frac{k}{\sqrt{y}}.e^{\frac{-x^2}{4a^2y}}\left(\frac{x^2}{4a^2y^2}\right) + k\left(-\frac{1}{2y^{3/2}}\right).e^{\frac{-x^2}{4a^2y}}$

$$= ke^{\frac{-x^2}{4a^2y}}\left[\frac{x^2}{4a^2y^{5/2}} - \frac{1}{2y^{3/2}}\right].$$

Also $\dfrac{\partial u}{\partial x}=\dfrac{k}{\sqrt{y}}e^{\frac{-x^2}{4a^2y}}\left(\dfrac{-2x}{4a^2y}\right)=-\dfrac{kx}{2a^2y^{3/2}}e^{\frac{-x^2}{4a^2y}}$

and $\therefore \dfrac{\partial^2 u}{\partial x^2}=-\dfrac{k}{2a^2y^{3/2}}.e^{\frac{-x^2}{4a^2y}}-\dfrac{kx}{2a^2y^{3/2}}.e^{\frac{-x^2}{4a^2y}}\left(\dfrac{-2x}{4a^2y}\right)=ke^{\frac{-x^2}{4a^2y}}\left[\dfrac{x^2}{4a^4y^{5/2}}-\dfrac{1}{2a^2y^{3/2}}\right]$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{a^2}\dfrac{\partial u}{\partial y}$, hence $\dfrac{\partial u}{\partial y}=a^2\dfrac{\partial^2 u}{\partial x^2}$.

This completes the proof**.**

Q.No.16: If $\theta=t^n e^{-\frac{r^2}{4t}}$, find what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$**.**

Sol.: Here $\theta=t^n e^{-\frac{r^2}{4t}}$. ....(i)

Differentiate (i) partially w. r. t. r, we get

$$\frac{\partial\theta}{\partial r}=\frac{\partial}{\partial r}\left[t^n e^{-\frac{r^2}{4t}}\right]=t^n.\frac{\partial}{\partial r}\left[e^{-\frac{r^2}{4t}}\right]=t^n.e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)=-\frac{1}{2}t^{n-1}r.e^{-\frac{r^2}{4t}}.$$

$$r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}t^{n-1}r^3.e^{-\frac{r^2}{4t}}. \quad ...(ii)$$

Differentiate (ii) partially w. r. t. r, we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial}{\partial r}\left[-\frac{1}{2}t^{n-1}r^3e^{-\frac{r^2}{4t}}\right]=-\frac{t^{n-1}}{2}\frac{\partial}{\partial r}\left[r^3e^{-\frac{r^2}{4t}}\right]$$

$$=-\frac{t^{n-1}}{2}\left\{3r^2.e^{-\frac{r^2}{4t}}+r^3.e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right\}=-\frac{t^{n-1}}{2}\left\{\left(3r^2-\frac{r^4}{2t}\right)e^{-\frac{r^2}{4t}}\right\}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{t^{n-1}}{2r^2}\left\{\left(3r^2-\frac{r^4}{2t}\right)e^{-\frac{r^2}{4t}}\right\} \quad ...(iii)$$

Now $\dfrac{\partial\theta}{\partial t}=\dfrac{\partial}{\partial t}\left[t^n e^{-\frac{r^2}{4t}}\right]=t^n.e^{-\frac{r^2}{4t}}\left(\dfrac{r^2}{4t^2}\right)+nt^{n-1}.e^{-\frac{r^2}{4t}}=e^{-\frac{r^2}{4t}}\left[\dfrac{r^2}{4}t^{n-2}+nt^{n-1}\right]$

$$= e^{-\frac{r^2}{4t}}\left[t^{n-1}\left(\frac{r^2}{4t}+n\right)\right]. \qquad \text{...(iv)}$$

But $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t} \Rightarrow -\frac{t^{n-1}}{2r^2}\left\{\left(3r^2-\frac{r^4}{2t}\right)e^{-\frac{r^2}{4t}}\right\}=e^{-\frac{r^2}{4t}}\left[t^{n-1}\left(\frac{r^2}{4t}+n\right)\right]$

$\Rightarrow -\frac{1}{2r^2}\left\{\left(3r^2-\frac{r^4}{2t}\right)\right\}=\left[\left(\frac{r^2}{4t}+n\right)\right]. \Rightarrow -\frac{1}{2}\left\{\left(3-\frac{r^2}{2t}\right)\right\}=\left[\left(\frac{r^2}{4t}+n\right)\right]$

$\Rightarrow -\frac{3}{2}=n$. Hence $n=-\frac{3}{2}$. Ans.

Q.No.17: If $u=Ae^{-gx}\sin(nt-gx)$, where A , g , n are positive constants, satisfies the heat conduction equation $\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$, then prove that $g=\sqrt{\frac{n}{2\mu}}$.

or

The equation $\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u=Ae^{-gx}\sin(nt-gx)$, where A , g , n are positive constants then $g=\sqrt{\frac{n}{2\mu}}$.

Sol.: Here $u=Ae^{-gx}\sin(nt-gx)$,we have $\frac{\partial u}{\partial t}=Ae^{-gx}\cos(nt-gx)n$.

Also $\frac{\partial u}{\partial x}=A\left[e^{-gx}(-g)\sin(nt-gx)+e^{-gx}\cos(nt-gx)(-g)\right]$

$$=A(-g)\left[e^{-gx}\sin(nt-gx)+e^{-gx}\cos(nt-gx)\right]$$

$$=-Age^{-gx}\left[\sin(nt-gx)+\cos(nt-gx)\right]$$

and $\frac{\partial^2 u}{\partial x^2}=-Ag\left[e^{-gx}\{\cos(nt-gx)(-g)-\sin(nt-gx)(-g)\}\right.$

$$\left.+\{\sin(nt-gx)+\cos(nt-gx)\}e^{-gx}(-g)\right]$$

$$=-Age^{-gx}(-g)\left[\cos(nt-gx)-\sin(nt-gx)+\sin(nt-gx)+\cos(nt-gx)\right]$$

$$=-Age^{-gx}(-g)\left[2\cos(nt-gx)\right]=2Ag^2e^{-gx}\cos(nt-gx).$$

Also given $\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$ $\Rightarrow Ae^{-gx}\cos(nt-gx)n=\mu 2Ag^2e^{-gx}\cos(nt-gx)$

$$\Rightarrow g^2=\frac{n}{2\mu}\text{ . Hence }\therefore g=\sqrt{\frac{n}{2\mu}}\text{ .}$$

This completes the proof.

Q.No.18: (a) Show that at the point for surface $x^x y^y z^z = \text{const.}$, where $x = y = z$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-1}{x\log(ex)}.$$

(b) If $u=e^{xyz}$; find the value of $\frac{\partial^3 u}{\partial x\partial y\partial z}$.

Sol.: (a) Given $x^x y^y z^z = \text{const.}$, where $x = y = z$.

Taking log both sides, we get

$x\log x+y\log y+z\log z=\log c$

Differentiating z partially w. r. t. x [keeping y as constant] , we get

$$(1+\log x)+(1+\log z)\frac{\partial z}{\partial x}=0\ \Rightarrow\frac{\partial z}{\partial x}=-\frac{1+\log x}{1+\log z}.\ \text{ Similarly, }\frac{\partial z}{\partial y}=-\frac{1+\log y}{1+\log z}.$$

$$\text{Now }\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]=\frac{\partial}{\partial z}\left[\frac{\partial z}{\partial x}\right]\frac{\partial z}{\partial y}=\frac{\partial}{\partial z}\left[-\frac{1+\log x}{1+\log z}\right]\times\left[-\frac{1+\log y}{1+\log z}\right]$$

$$=\frac{(1+\log z).0-(1+\log x)\frac{1}{z}}{(1+\log z)^2}\times\left[\frac{1+\log y}{1+\log z}\right]=-\frac{1}{z}\frac{(1+\log x)(1+\log y)}{(1+\log z)^3}$$

Since $x = y = z$,

$$\therefore\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x}\frac{(1+\log x)^2}{(1+\log x)^3}=-\frac{1}{x(1+\log x)}=\frac{-1}{x(\log e+\log x)}=\frac{-1}{x\log(ex)}.$$

Hence $\frac{\partial^2 z}{\partial x\partial y}=\frac{-1}{x\log(ex)}$. This completes the proof.

(b) Here $u=e^{xyz}$.

Now $\frac{\partial u}{\partial z}=\frac{\partial}{\partial z}\left(e^{xyz}\right)=e^{xyz}xy$.

$$\therefore \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial}{\partial y}\left[e^{xyz}xy\right] = xy\left(e^{xyz}xz\right) + e^{xyz}x = x^2yze^{xyz} + e^{xyz}x = \left(x^2yz + x\right)e^{xyz}$$

And hence $\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right] = \frac{\partial}{\partial x}\left[\left(x^2yz + x\right)e^{xyz}\right] = [2xyz + 1]e^{xyz} + \left[x^2yz + x\right]e^{xyz}yz$

$= \left[2xyz + 1 + x^2y^2z^2 + xyz\right]e^{xyz} = \left[x^2y^2z^2 + 3xyz + 1\right]e^{xyz}$. Ans.

Q.No.19: If $z = xf(x+y) + yg(x+y)$, show that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$.

Sol.: Since $z = xf(x+y) + yg(x+y)$. ...(i)

$$\therefore \frac{\partial z}{\partial x} = xf'(x+y) + f(x+y) + yg'(x+y) .$$

and $\therefore \frac{\partial^2 z}{\partial x^2} = f'(x+y) + xf''(x+y) + f'(x+y) + yg''(x+y)$. ...(ii)

Also $\frac{\partial z}{\partial y} = xf'(x+y) + yg'(x+y) + g(x+y)$.

and $\therefore \frac{\partial^2 z}{\partial y^2} = xf''(x+y) + yg''(x+y) + g'(x+y) + g'(x+y)$. ...(iii)

Now since $\frac{\partial z}{\partial x} = xf'(x+y) + f(x+y) + yg'(x+y)$.

$\therefore \frac{\partial^2 z}{\partial x \partial y} = xf''(x+y) + f'(x+y) + g'(x+y) + yg''(x+y)$. ...(iv)

Putting these values in $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$, we get

$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$. This completes the proof.

Q.No.20: If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Sol.: Since $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$. ...(i)

Differentiate (i) partially w. r. t. x, y and z separately, we get

$$\frac{\partial u}{\partial x}=\left(\frac{1}{y}-\frac{z}{x^2}\right),\ \frac{\partial u}{\partial y}=\left(\frac{1}{z}-\frac{x}{y^2}\right)\text{ and }\frac{\partial u}{\partial z}=\left(\frac{1}{x}-\frac{y}{z^2}\right).$$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x\left(\frac{1}{y}-\frac{z}{x^2}\right)+y\left(\frac{1}{z}-\frac{x}{y^2}\right)+z\left(\frac{1}{x}-\frac{y}{z^2}\right)=0$.

This completes the proof.

Q.No.21: If $u=e^{ax+by}\phi(ax-by)$, then prove that $b\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}=2abu$.

Sol.: Since $u=e^{ax+by}\phi(ax-by)$. ....(i)

Differentiate (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=e^{ax+by}a.\phi(ax-by)+e^{ax+by}.\phi'(ax-by)a\ ,$$

and $\frac{\partial u}{\partial y}=e^{ax+by}b.\phi(ax-by)+e^{ax+by}.\phi'(ax-by)(-b)$

Now $b\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}=2abe^{ax+by}\phi(ax-by)=2abu$.

This completes the proof.

Q.No.22: If $x=r\cos\theta$, $y=r\sin\theta$, then show that

(i) $\frac{\partial x}{\partial r}=\frac{\partial r}{\partial x}$, **(ii)** $r\frac{\partial\theta}{\partial x}=\frac{1}{r}\frac{\partial x}{\partial\theta}$.

Sol.: (i) Given $x=r\cos\theta$, $y=r\sin\theta\ \Rightarrow x^2+y^2=r^2$ ......(i)

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$2x+0=2r\frac{\partial r}{\partial x}\ \Rightarrow r\frac{\partial r}{\partial x}=x=r\cos\theta\ \Rightarrow\frac{\partial r}{\partial x}=\cos\theta \qquad ......(ii)$$

Also since $x=r\cos\theta\ \Rightarrow\frac{\partial x}{\partial r}=\cos\theta$. ....(iii)

Comparing (ii) and (iii) ,we get $\frac{\partial x}{\partial r}=\frac{\partial r}{\partial x}$. Ans.

This completes the proof.

(ii) To show : $r\frac{\partial\theta}{\partial x}=\frac{1}{r}\frac{\partial x}{\partial\theta}$.

Now since $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

$$\therefore \frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}.\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}.$$

Now $r\frac{\partial\theta}{\partial x} = r.\left(\frac{-y}{x^2+y^2}\right) = r.\left(\frac{-y}{r^2}\right) = \frac{-y}{r}$. ...(i)

since $x = r\cos\theta \therefore \frac{\partial x}{\partial\theta} = -r\sin\theta \Rightarrow \frac{1}{r}\frac{\partial x}{\partial\theta} = -\sin\theta = -\frac{y}{r}$. .....(ii)

Comparing (i) and (ii) ,we get $r\frac{\partial\theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial\theta}$. This completes the proof.

Q.No.23: If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

(ii) $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \quad (x \neq 0, y \neq 0)$

Sol.: (i) Given $x = r\cos\theta$, $y = r\sin\theta$.

[By looking at the answer we find that we need partial derivative of r w. r. t. x and y. Therefore, let us express r as an explicit function of x and y]

Squaring and adding $x = r\cos\theta$, $y = r\sin\theta$; we find that

$r^2 = x^2 + y^2$ i.e. $r = \sqrt{x^2+y^2}$. ...(i)

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x = \left(x^2+y^2\right)^{-1/2}.x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}. \quad ...(ii)$$

Similarly, differentiating (i) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2y = \left(x^2+y^2\right)^{-1/2}.y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}. \quad .(iii)$$

Again differentiating(ii) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial^2 r}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)=\frac{r\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(r)}{r^2}=\frac{r-x\frac{\partial r}{\partial x}}{r^2}=\frac{r-x.\frac{x}{r}}{r^2}=\frac{r^2-x^2}{r^3}=\frac{y^2}{r^3}.$$

Again differentiating(iii) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial^2 r}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{y}{r}\right)=\frac{r\frac{\partial}{\partial y}(y)-y\frac{\partial}{\partial y}(r)}{r^2}=\frac{r-y\frac{\partial r}{\partial y}}{r^2}=\frac{r-y.\frac{y}{r}}{r^2}=\frac{r^2-y^2}{r^3}=\frac{x^2}{r^3}.$$

$$\text{L.H.S.}=\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}+\frac{x^2}{r^3}=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}.$$

$$\text{R.H.S.}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]=\frac{1}{r}\left[\frac{x^2+y^2}{r^2}\right]=\frac{1}{r}\left[\frac{r^2}{r^2}\right]=\frac{1}{r}.$$

$\therefore$ L.H.S.= R.H.S. This completes the proof**.**

(ii) It is given that $x=r\cos\theta$, $y=r\sin\theta$. Dividing ,we get $\tan\theta=\frac{y}{x}$

$$\therefore \theta=\tan^{-1}\frac{y}{x}\ . \qquad ...(i)$$

Differentiating (i) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}. \qquad ...(ii)$$

Again differentiating (ii) w. r. t. x partially (keeping y as constant), we get

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)=\frac{(x^2+y^2)(0)-(-y).2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}. \qquad ...(iii)$$

Differentiating (i) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial\theta}{\partial y}=\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}=\frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}. \qquad ...(iv)$$

Again differentiating (iv) w. r. t. y partially (keeping x as constant), we get

$$\frac{\partial^2\theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{(x^2+y^2)(0)-(x).2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}. \qquad ...(v)$$

Adding (iv) and (v), we get

L.H.S. $=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{2xy}{(x^2+y^2)^2}-\frac{2xy}{(x^2+y^2)^2}=0=$ R.H.S. This completes the proof.

Q.No.24: If $u=f(ax^2+2hxy+by^2)$ and $v=\varphi(ax^2+2hxy+by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

Sol.: Given $u=f(ax^2+2hxy+by^2)$ ...(i)

and $v=\varphi(ax^2+2hxy+by^2)$ ....(ii)

Differentiating (ii) partially w. r. t. x and y separately, we get

$$\frac{\partial v}{\partial x}=\varphi'(ax^2+2hxy+by^2).(2ax+2hy)=\varphi'.(2ax+2hy)$$

$$\frac{\partial v}{\partial y}=\varphi'(ax^2+2hxy+by^2).(2by+2hx)=\varphi'.(2by+2hx)$$

Now L.H.S.= $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial y}\left[f.\varphi'.(2ax+2hy)\right]$

$$=f'.(2by+2hx).\varphi'.(2ax+2hy)+f.\varphi''.(2by+2hx).(2ax+2hy)+f.\varphi'.2h$$

$$=(2ax+2hy).(2by+2hx).\left[f'\varphi'+f\varphi''\right]+2h.f.\varphi'. \qquad ....(iii)$$

R.H.S.= $\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}\left[f.\varphi'.(2by+2hx)\right]$

$$=f'.(2ax+2hy).\varphi'.(2by+2hx)+f.\varphi''.(2ax+2hy).(2by+2hx)+f.\varphi'.2h$$

$$=(2ax+2hy).(2by+2hx).\left[f'\varphi'+f\varphi''\right]+2h.f.\varphi'. \qquad (iv)$$

From (iii) and (iv), we have $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$. This completes the proof.

Q.No.25: If $u=(x^2-y^2)f(t)$, where $t=x\,y$, prove that

$$\frac{\partial^2 u}{\partial x\partial y}=(x^2-y^2)\left[tf''(t)+3f'(t)\right]$$

Sol.: Given $u=(x^2-y^2)f(t)=(x^2-y^2)f(xy)=x^2f(xy)-y^2f(xy)$. (i)

Differentiating (i) partially w. r. t. x and y separately, we get

$$\frac{\partial u}{\partial x}=\left[2x.f(xy)+x^2.f'(xy)y\right]-\left[y^2.f'(xy)y\right]=2xf(xy)+x^2yf'(xy)-y^3f'(xy)$$

$$\therefore \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial y}\left[2xf(xy)+x^2yf'(xy)-y^3f'(xy)\right]$$

$$=\left[2xf'(xy)x\right]+\left[x^2y.f''(xy)x+x^2.f'(xy)\right]-\left[y^3.f''(xy)x+3y^2.f'(xy)\right]$$

$$=\left[2x^2f'(t)\right]+\left[x^3yf''(t)+x^2f'(t)\right]-\left[y^3xf''(t)+3y^2f'(t)\right]$$

$$=3x^2f'(t)-3y^2f'(t)+\left(x^3y-y^3x\right)f''(t)$$

$$=3\left(x^2-y^2\right)f'(t)+xy\left(x^2-y^2\right)f''(t)$$

$$=\left(x^2-y^2\right)tf''(t)+\left(x^2-y^2\right)3f'(t)$$

Hence $\dfrac{\partial^2 u}{\partial x\partial y}=\left(x^2-y^2\right)\left[tf''(t)+3f'(t)\right]$. This completes the proof.

Q.No.26: If u and v are functions of x and y defined by $x=u+e^{-v}\sin u$,

$y=v+e^{-v}\cos u$, then prove that $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial x}$.

Sol.: Given $x=u+e^{-v}\sin u$ and $y=v+e^{-v}\cos u$.

Differentiating both the equations partially w. r .t. x and y separately, we get

$$1=\frac{\partial u}{\partial x}+e^{-v}\cos u\frac{\partial u}{\partial x}+e^{-v}\left(-\frac{\partial v}{\partial x}\right)\sin u \Rightarrow 1=\frac{\partial u}{\partial x}\left[1+e^{-v}\cos u\right]-e^{-v}\frac{\partial v}{\partial x}\sin u \qquad \text{(i)}$$

$$0=\frac{\partial u}{\partial y}+e^{-v}\cos u\frac{\partial u}{\partial y}+e^{-v}\left(-\frac{\partial v}{\partial y}\right)\sin u \Rightarrow 0=\frac{\partial u}{\partial y}\left[1+e^{-v}\cos u\right]-e^{-v}\frac{\partial v}{\partial y}\sin u \qquad \text{(ii)}$$

$$0=\frac{\partial v}{\partial x}+e^{-v}(-\sin u)\frac{\partial u}{\partial x}+e^{-v}\left(-\frac{\partial v}{\partial x}\right)\cos u \Rightarrow 0=\frac{\partial v}{\partial x}\left[1-e^{-v}\cos u\right]-e^{-v}\frac{\partial u}{\partial x}\sin u \qquad \text{(iii)}$$

$$1=\frac{\partial v}{\partial y}+e^{-v}(-\sin u)\frac{\partial u}{\partial y}+e^{-v}\left(-\frac{\partial v}{\partial y}\right)\cos u \Rightarrow 1=\frac{\partial v}{\partial y}\left[1-e^{-v}\cos u\right]-e^{-v}\frac{\partial u}{\partial y}\sin u \qquad \text{(iv)}$$

Multiplying (i) by $e^{-v}\sin u$ and (iii) by $\left[1+e^{-v}\cos u\right]$ and then adding, we get

$$\frac{\partial v}{\partial x}=\frac{e^{-v}\sin u}{1-e^{-2v}} \qquad \text{(v)}$$

Multiplying (ii) by $\left[1-e^{-v}\cos u\right]$ and (iv) by $e^{-v}\sin u$ and then adding, we get

$$\frac{\partial u}{\partial y} = \frac{e^{-v}\sin u}{1-e^{-2v}} \qquad \text{(vi)}$$

From (v) and (vi), we get

$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. This completes the proof.

Q.No.27: If $z(x+y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Sol.: Since $z(x+y) = x^2 + y^2 \Rightarrow z = \frac{x^2+y^2}{x+y}$. ...(i)

Differentiating (i) partially w. r. t. x and y separately, we get

$$\frac{\partial z}{\partial x} = \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y).2y - (x^2+y^2).1}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

Now L.H.S.= $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}\right]^2$

$$= \left[\frac{(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right]^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \left[\frac{2(x-y)(x+y)}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 = \frac{4(x-y)^2}{(x+y)^2}. \qquad \text{(ii)}$$

R.H.S.= $4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{(x^2-y^2+2xy)}{(x+y)^2} - \frac{(y^2-x^2+2xy)}{(x+y)^2}\right]$

$$= 4\left[\frac{(x^2+y^2+2xy)-(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right] = 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right]$$

$$= \frac{4(x-y)^2}{(x+y)^2}. \qquad \text{(iii)}$$

From (ii) and (iii), we have L.H.S.=R.H.S. This completes the proof.

Q.No.28: If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.: Since $u = x^y$. (i)

For $\dfrac{\partial^3 u}{\partial x^2 \partial y}$, first differentiate (i) partially w. r. t. y and then twice w. r. t. x

$\therefore \dfrac{\partial u}{\partial y} = x^y \log x$. Now differentiate twice w. r. t. x, we get

$\dfrac{\partial^2 u}{\partial x \partial y} = x^y . \dfrac{1}{x} + \log x . y x^{y-1} = x^{y-1} + y \log x^{y-1} = x^{y-1}(1 + y \log x)$ and

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial x \partial y}\right] = (1 + y \log x).(y-1)x^{y-2} + x^{y-1}.\frac{y}{x} = x^{y-2}[(1 + y \log x)(y-1) + y]. \quad \text{(ii)}$$

For $\dfrac{\partial^3 u}{\partial x \partial y \partial x}$, first differentiate (i) partially w. r. t. x , then y and then x

$\therefore \dfrac{\partial u}{\partial x} = y x^{y-1}$. Now differentiate partially w. r t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = y.x^{y-1} \log x + x^{y-1} = (1 + y \log x) x^{y-1}.$$

Now again differentiate partially w. r. t. x, we get

$$\frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial x}\right] = x^{y-2}[(1 + y \log x)(y-1) + y]. \quad \text{(iii)}$$

Hence from (ii) and (iii), $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$. This completes the proof.

Q.No.29: If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$, where u is a function of x , y , z ; prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Sol.: Since $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$.

Now differentiate partially w. r. t. x, we get

$$\frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} + \frac{-y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{-z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} - \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} - \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{(a^2+u)2x - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} = \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2}$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \frac{x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} + \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} + \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2}$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

Similarly $\frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$,

$$\frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)} \div \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

Now L.H.S.= $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{\left\{\frac{2x}{(a^2+u)}\right\}^2 + \left\{\frac{2y}{(b^2+u)}\right\}^2 + \left\{\frac{2z}{(c^2+u)}\right\}^2}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2}$

$$= \frac{4}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}$$

$$\text{R.H.S.}= 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{2\left[x.\frac{2x}{(a^2+u)^2}+y.\frac{2y}{(b^2+u)^2}+z.\frac{2z}{(c^2+u)^2}\right]}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}$$

$$=\frac{4}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}$$

$$=\text{L.H.S.}$$

Hence $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$**.** This completes the proof**.**

Q.No.30: If $v=(x^2+y^2+z^2)^{-1/2}$. Show that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$ **.**

Sol.: Since $v=(x^2+y^2+z^2)^{-1/2}$, we have

$$\frac{\partial v}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.2x=-x(x^2+y^2+z^2)^{-3/2}.$$

and

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]=-\left[1.(x^2+y^2+z^2)^{-3/2}+x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}.2x\right]$$

$$=-(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2-3x^2\right]=(x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2) \quad ..(i)$$

Similarly, $\frac{\partial^2 v}{\partial y^2}=(x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$. ...(ii)

and $\frac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$. ...(iii)

Adding (i), (ii) and (iii), we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}(0)=0 \text{ .}$$

This completes the proof**.**

Q.No.31: If $V = r^m$, $r^2 = x^2 + y^2 + z^2$, then show that

$$V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}.$$

Sol.: Since $r^2 = x^2 + y^2 + z^2$ $\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Now $V = r^m$ $\therefore \frac{\partial V}{\partial x} = mr^{m-1}.\frac{x}{r} = mxr^{m-2}$ and

$$\therefore \frac{\partial^2 V}{\partial x^2} = m\left[r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right] = m\left[r^{m-2} + x(m-2)r^{m-3}\frac{x}{r}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = m\left[r^{m-2} + (m-2)x^2 r^{m-4}\right]. \qquad ......(i)$$

Similarly, $\frac{\partial^2 V}{\partial y^2} = m\left[r^{m-2} + (m-2)y^2 r^{m-4}\right]$ ......(ii)

and $\frac{\partial^2 V}{\partial z^2} = m\left[r^{m-2} + (m-2)z^2 r^{m-4}\right].$ .....(iii)

Adding (i), (ii) and (iii), we get

$$V_{xx} + V_{yy} + V_{zz} = m\left[3r^{m-2} + (m-2)r^2 r^{m-4}\right] = m\left[r^{m-2}(3+m-2)\right] = m(m+1)r^{m-2}.$$

This completes the proof.

Q.No.32: If $u = \log(\tan x + \tan y + \tan z)$, then prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

Sol.: Here $u = \log(\tan x + \tan y + \tan z)$. ....(i)

Differentiate (i) partially w. r. t. x, y and z separately, we get

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}, \quad \frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}.$$

Now L.H.S.= $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z}$

$$= \frac{2\sin x\cos x.\frac{1}{\cos^2 x} + 2\sin y\cos y.\frac{1}{\cos^2 y} + 2\sin z\cos z.\frac{1}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2 = \text{R.H.S.}$$

This completes the proof.

Q.No.33: If $u = \frac{xy(x^2 - y^2)}{x^2 + y^2}$; $u(0,0) = 0$, show that $\frac{\partial^2 u}{\partial x \partial y} \neq \frac{\partial^2 u}{\partial y \partial x}$ at $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$.

Sol.: For $(x, y) \neq (0,0)$, $u(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$ (given) ....(i)

Differentiating (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{xy(x^2 - y^2)}{x^2 + y^2}\right] = y\frac{\partial}{\partial x}\left[\frac{x^3 - xy^2}{x^2 + y^2}\right] = y\left[\frac{(x^2 + y^2)(3x^2 - y^2) - (x^3 - xy^2)2x}{(x^2 + y^2)^2}\right]$$

$$= y\left[\frac{3x^4 + 2x^2y^2 - y^4 - 2x^4 + 2x^2y^2}{(x^2 + y^2)^2}\right] = y\left[\frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}\right]$$

$\therefore$ For $(x, y) \neq (0,0)$, $\frac{\partial u}{\partial x} = u_x(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$. ...(ii)

For $\frac{\partial u}{\partial x}(0,0)$, let us consider $\frac{\partial u}{\partial x}(0,0) = \lim_{\delta x \to 0} \frac{u(\delta x, 0) - u(0,0)}{\delta x} = \lim_{\delta x \to 0} \frac{0 - 0}{\delta x} = 0$.

which exists. $\therefore \frac{\partial u}{\partial x}(0,0) = 0$.

For the existence of $u_{yx}(0,0)$, i.e. $\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]_{(0,0)}$

Consider $\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]_{(0,0)} = \lim_{\delta y \to 0} \frac{u_x(0, \delta y) - u_x(0,0)}{\delta y} = \lim_{\delta y \to 0} \frac{-\delta y - 0}{\delta y} = -1$, which exists.

$\therefore \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]_{(0,0)} = -1$. ...(iii)

Again because for $(x, y) \neq (0,0)$, $u(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$ (given) ....(i)

Differentiating (i) partially w. r. t. x, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{xy(x^2 - y^2)}{x^2 + y^2}\right] = x\frac{\partial}{\partial y}\left[\frac{yx^2 - y^3}{x^2 + y^2}\right] = x\left[\frac{(x^2 + y^2)(x^2 - 3y^2) - (x^2y - y^3)2y}{(x^2 + y^2)^2}\right]$$

$$= x\left[\frac{x^4 - 2x^2y^2 - 3y^4 - 2x^2x^2 + 2y^4}{\left(x^2+y^2\right)^2}\right] = x\left[\frac{x^4 - 4x^2y^2 - y^4}{\left(x^2+y^2\right)^2}\right]$$

$\therefore$ For $(x, y) \neq (0,0)$, $\dfrac{\partial u}{\partial y} = u_y(x, y) = \dfrac{x\left(x^4 - 4x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2}$. ...(iv)

For $\dfrac{\partial u}{\partial y}(0,0)$, let us consider $\dfrac{\partial u}{\partial y}(0,0) = \underset{\delta y \to 0}{\text{Lim}} \dfrac{f(0, \delta y) - f(0,0)}{\delta y} = \underset{\delta y \to 0}{\text{Lim}} \dfrac{0-0}{\delta y} = 0$.

which exists. $\therefore \dfrac{\partial u}{\partial y}(0,0) = 0$. For the existence of $u_{xy}(0,0)$, i.e. $\dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial y}\right]_{(0,0)}$

Consider $\dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial y}\right]_{(0,0)} = \underset{\delta x \to 0}{\text{Lim}} \dfrac{u_y(\delta x, 0) - u_y(0,0)}{\delta x} = \underset{\delta y \to 0}{\text{Lim}} \dfrac{\delta x - 0}{\delta x} = 1$, which exists.

$\therefore \dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial y}\right]_{(0,0)} = 1$. ...(v)

$\therefore$ From (iii) and (v), we get $\dfrac{\partial^2 u}{\partial x \partial y} \neq \dfrac{\partial^2 u}{\partial y \partial x}$ at $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$.

i.e. $u_{yx}(0,0) \neq u_{xy}(0,0)$.

This completes the proof.

Q.No.34: If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$.

Sol.: Given $\theta = t^n e^{-\frac{r^2}{4t}}$.

$$\frac{\partial \theta}{\partial r} = t^n.e^{-\frac{r^2}{4t}}.\left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\therefore r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3.t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)$$

Also $\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-\frac{r^2}{4t}}+t^ne^{-\frac{r^2}{4t}}.\left(\frac{r^2}{4t^2}\right)=t^{n-1}e^{-\frac{r^2}{4t}}.\left(n+\frac{r^2}{4t^2}\right).$

Since $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$ is given

$$\therefore \frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}.\left(\frac{r^2}{2t}-3\right)=t^{n-1}e^{-\frac{r^2}{4t}}.\left(n+\frac{r^2}{4t}\right)$$

$\Rightarrow \frac{r^2}{4t}-\frac{3}{2}=n+\frac{r^2}{4t}$ $\qquad\qquad$ $\therefore n=-\frac{3}{2}$. Ans.

Q.No.35: If $u=f(r)$, where $r^2=x^2+y^2$, prove that $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

Sol.: Given $r^2=x^2+y^2$. (i)

Differentiating partially w. r. t., we get $2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y}=\frac{x}{r}$.

Now $u=f(r)$ $\therefore \frac{\partial u}{\partial x}=f'(r).\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)$

Differentiating again w. r. t. x, we get

$$\frac{\partial^2u}{\partial x^2}=\frac{1}{r}f'(r)+x.\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r)+\frac{x}{r}f''(r).\frac{\partial r}{\partial x}$$

$$\left[\because -\frac{\partial}{\partial x}(uvw)=vw\frac{\partial}{\partial x}(u)+uw\frac{\partial}{\partial x}(v)+uv\frac{\partial}{\partial x}(w)\right]$$

$$=\frac{1}{r}f'(r)-\frac{x}{r^2}.\frac{x}{r}f'(r)+\frac{x}{r}.f''(r).\frac{x}{r}=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$$

$$=\frac{r^2-x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)=\frac{y^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r) \qquad \text{[using (i)]}$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{x^2}{r^3}f'(r)+\dfrac{y^2}{r^2}f''(r)$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{x^2+y^2}{r^3}f'(r)+\frac{x^2+y^2}{r^2}f''(r)=\frac{r^2}{r^3}f'(r)+\frac{r^2}{r^2}f''(r)$$

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f''(r)+\dfrac{1}{r}f'(r)$. Hence prove.

Q.No.36: If $x=e^{r\cos\theta}\cos(r\sin\theta)$ and $y=e^{r\cos\theta}\sin(r\sin\theta)$,

prove that $\dfrac{\partial x}{\partial r}=\dfrac{1}{r}.\dfrac{\partial y}{\partial \theta}$, $\dfrac{\partial y}{\partial r}=-\dfrac{1}{r}.\dfrac{\partial x}{\partial \theta}$.

Hence deduce that $\dfrac{\partial^2 x}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial x}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2}=0$.

Sol.: Given $x=e^{r\cos\theta}\cos(r\sin\theta)$.

$$\therefore \frac{\partial x}{\partial r}=e^{r\cos\theta}.\cos\theta.\cos(r\sin\theta)-e^{r\cos\theta}.\sin(r\sin\theta).\sin\theta$$

$$=e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$

$$=e^{r\cos\theta}\cos(\theta+r\sin\theta) \qquad \text{(i)}$$

$$\frac{\partial x}{\partial \theta}=e^{r\cos\theta}.(-r\sin\theta).\cos(r\sin\theta)-e^{r\cos\theta}.\sin(r\sin\theta).r\cos\theta$$

$$=-re^{r\cos\theta}[\sin\theta\cos(r\sin\theta)+\cos\theta\sin(r\sin\theta)]$$

$$=-re^{r\cos\theta}\sin(\theta+r\sin\theta) \qquad \text{(ii)}$$

Also $y=e^{r\cos\theta}\sin(r\sin\theta)$

$$\frac{\partial y}{\partial r}=e^{r\cos\theta}.\cos\theta.\sin(r\sin\theta)+e^{r\cos\theta}.\cos(r\sin\theta).\sin\theta$$

$$=e^{r\cos\theta}[\sin\theta\cos(r\sin\theta)+\cos\theta\sin(r\sin\theta)]$$

$$=e^{r\cos\theta}\sin(\theta+r\sin\theta) \qquad \text{(iii)}$$

$$\frac{\partial y}{\partial \theta}=e^{r\cos\theta}.(-r\sin\theta).\sin(r\sin\theta)+e^{r\cos\theta}.\cos(r\sin\theta).r\cos\theta$$

$$=re^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{(iv)}$$

From (i) and (iv), we get $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}.\dfrac{\partial y}{\partial \theta}$ (v)

From (ii) and (iii), we get $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r}.\dfrac{\partial x}{\partial \theta}$ (vi)

From (v), we get $\dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2}.\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}.\dfrac{\partial^2 y}{\partial r \partial \theta}$

From(vi), we get $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$

$$\therefore \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

$$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2}.\frac{\partial y}{\partial \theta} + \frac{1}{r}.\frac{\partial^2 y}{\partial r \partial \theta} + \frac{1}{r^2}.\frac{\partial y}{\partial \theta} - \frac{1}{r}.\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

Q.No.37: Prove that if $f(x, y) = \dfrac{1}{\sqrt{y}}.e^{-\frac{(x-a)^2}{4y}}$, then $f_{xy} = f_{yx}$.

Sol.: Given $f(x, y) = \dfrac{1}{\sqrt{y}}.e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}$.

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2}y^{-\frac{3}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= e^{-\frac{(x-a)^2}{4y}}.\left[-\frac{1}{2}y^{-\frac{3}{2}} + y^{-\frac{1}{2}}.\frac{(x-a)^2}{4y^2}\right] = \frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}\left[-2 + y^{-1}(x-a)^2\right]$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$=\frac{1}{4}y^{-\frac{3}{2}}\left\{e^{-\frac{(x-a)^2}{4y}}.\frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right].\left[-2+y^{-1}(x-a)^2\right]+e^{-\frac{(x-a)^2}{4y}}.2y^{-1}(x-a)\right\}$$

$$=\frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}\left\{-\frac{2(x-a)}{4y}\left[-2+y^{-1}(x-a)^2\right]+2y^{-1}(x-a)\right\}$$

$$=\frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{x-a}{y}\left\{-\frac{1}{2}\left[-2+y^{-1}(x-a)^2\right]+2\right\}$$

$$=\frac{1}{4}y^{-\frac{5}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right].$$

$$f_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=-\frac{1}{2}(x-a)\left[-\frac{3}{2}y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}+y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{(x-a)^2}{4y^2}\right]$$

$$=-\frac{1}{4}(x-a).y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}\left[-3+\frac{(x-a)^2}{2y}\right]$$

$$=\frac{1}{4}y^{-\frac{5}{2}}(x-a).e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right].$$

Hence $f_{xy}=f_{yx}$.

Q.No38.: Find the value of $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$ when $a^2x^2+b^2y^2-c^2z^2=0$.

Sol.: Here $a^2x^2+b^2y^2-c^2z^2=0\Rightarrow c^2z^2=a^2x^2+b^2y^2$

$$\therefore z^2=\frac{1}{c^2}\left(a^2x^2+b^2y^2\right) \qquad \text{(i)}$$

Differentiating (i) partially w.r.t. x, we get

$$2z\frac{\partial z}{\partial x}=\frac{1}{c^2}.2a^2x\Rightarrow\frac{\partial z}{\partial z}=\frac{a^2}{c^2}\left(\frac{x}{z}\right) \qquad \text{(ii)}$$

Differentiating (ii) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x^2}=\frac{a^2}{c^2}\left[\frac{z.1-x\dfrac{\partial z}{\partial x}}{z^2}\right]=\frac{a^2}{c^2z^2}\left[z-x\frac{a^2}{c^2}\left(\frac{x}{z}\right)\right]=\frac{a^2}{c^2z^2}\left[z-\frac{a^2x^2}{c^2z}\right]=\frac{a^2}{c^2z^2.c^2z}\left[c^2z^2-a^2x^2\right]$$

$$=\frac{a^2}{c^4z^3}\left(b^2y^2\right)\quad\left[\because a^2x^2+b^2y^2-c^2z^2=0\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=\frac{a^2b^2}{c^4}\frac{y^2}{z^3} \qquad \text{(iii)}$$

Similarly, $\therefore \dfrac{\partial^2 z}{\partial y^2}=\dfrac{a^2b^2}{c^4}\dfrac{x^2}{z^3}$ (iv)

Consider $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2}+\dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{a^2}.\dfrac{a^2b^2}{c^4}\dfrac{y^2}{z^3}+\dfrac{1}{b^2}.\dfrac{a^2b^2}{c^4}\dfrac{x^2}{z^3}=\dfrac{1}{c^4z^3}\left[b^2y^2+a^2x^2\right]$

$$=\frac{1}{c^4z^3}\left(c^2z^2\right)\quad\left[\because a^2x^2+b^2y^2-c^2z^2=0\right]$$

$\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2}+\dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{c^2z}$. Ans.

Thank you

NEXT TOPIC

Homogeneous Functions and Euler’s Theorem

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