# 5th Topic

## Matrices

### Consistency of linear system of equations

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## Consistency of linear system of equations:

Let

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = k_m \end{array}\right\}, \qquad \text{(i)}$$

be a system of m-non-homogenous equations in n-unknowns $x_1, x_2, \dots\dots\dots, x_n$.

If we write $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_m \end{bmatrix}_{m\times 1}$

$\therefore$ (i) $\Rightarrow AX = B$.

**Solution of linear system of equations:**

Any set of values of $x_1, x_2, \dots\dots\dots, x_n$ which simultaneously satisfy all these equations is called a solution of the system of equations (i).

**Consistent and inconsistent:**

When the system of equations has one or more solutions, then the equations are said to be consistent, otherwise, they are said to be inconsistent.

## Augmented matrix:

**The matrix** $K = [A \vdots B] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} & \vdots & k_1 \\ a_{21} & a_{22} & .... & a_{2n} & \vdots & k_2 \\ ..... & ..... & .... & .... & \vdots & .... \\ a_{m1} & a_{m2} & .... & a_{mn} & \vdots & k_m \end{bmatrix}$ **is called the augmented matrix of the given system of equations.**

## *Theorem of consistency*

**Statement: The system of equations AX = B is consistent, i.e., possesses a solution iff the coefficient matrix A and the augmented matrix K = (A:B) are of the same rank. Otherwise, the system is inconsistent.**

**Proof:** Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + .......... + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + ........ + a_{2n}x_n &= k_2 \\ .......................................................& \\ a_{m1}x_1 + a_{m2}x_2 + ..... + a_{mn}x_n &= k_m \end{aligned}\right\}, \qquad \text{(i)}$$

be a system of m-non-homogenous equations in n-unknowns $x_1, x_2, ........, x_n$.

If we write $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} k_1 \\ k_2 \\ .... \\ k_m \end{bmatrix}_{m\times 1}$

$\therefore$ (i) $\Rightarrow AX = B$.

**Now we consider the following two possible cases:**

**Case I.:** When the rank of A = the rank of K = r ( r$\le$ the smaller of numbers m and n).

Then by, suitable row operations, the system of equations AX = B can be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + ....... + b_{1n}x_n &= \ell_1 \\ 0x_1 + b_{22}x_2 + .......... + b_{2n}x_n &= \ell_2 \\ ....................................................& \\ 0x_1 + 0x_2 + ............... + b_{rn}x_n &= \ell_r \end{aligned}\right\} \qquad \text{(ii)}$$

and the remaining $(m-r)$ equations being all of the form

$0.x_1 + 0.x_2 + .......... + 0.x_n = 0$.

The equations (ii) will have a solution, by choosing ( n – r ) unknowns arbitrary.

The solution will be unique only when r = n.

Hence, the equations (i) are consistent, i.e., possesses solution.

**Case II.:** When the rank of A (i.e. r) < the rank of K.

In Particular, let the rank of K be r + 1.

Then, by suitable row operations, the esystem of equations AX = B can reduced to

$$\left.\begin{array}{l} b_{11}x_1 + b_{12}x_2 + \ldots\ldots + b_{1n}x_n = \ell_1, \\ 0x_1 + b_{22}x_2 + \ldots\ldots\ldots + b_{2n}x_n = \ell_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0x_1 + 0x_2 + \ldots\ldots\ldots\ldots\ldots + b_{rn}x_n = \ell_r, \\ 0x_1 + 0x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + 0x_n = \ell_{r+1}, \end{array}\right\}$$

and the remaining $[m-(r+1)]$ equations are of the form

$$0.x_1 + 0.x_2 + \ldots\ldots\ldots + 0.x_n = 0.$$

Clearly, the $(r+1)^{th}$ equation can not be satisfied by any set of values for the unknowns.

Hence, the equations (i) are inconsistent, i.e., does not possess a solution.

This completes the proof.

## Procedure to test the consistency of a system of equations in n-unknown:

**Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A = r and rank of K = $r'$.**

**(i)** **If $r \neq r'$, then the equations are inconsistent, i.e., there is no solution.**

**(ii)** $r = r' = n$ **, then the equations are consistent and there is a unique solution.**

**(iii)** $r = r' < n$ **, then the equations are consistent and there are infinite number of solutions, by choosing to (n – r) unknowns arbitrary.**

## System of linear homogeneous equations:

**Consider the homogeneous linear equations**

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{array}\right\}. \qquad \text{(i)}$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary operations.

**Case (i):If the rank of A = n:**

Then the equations (i) have only a trivial solution

$$x_1 = x_2 = \ldots\ldots\ldots\ldots\ldots = x_n = 0.$$

**If the rank of A $< n$ :**

Then the equations (i) have $(n-r)$ independent solutions and r cannot be $> n$.

> **Remarks:**The number of linearly independent solutions is $(n-r)$ means, if arbitrary values are assigned to $(n-r)$of the variables, the values of the remaining variables can be uniquely found.

**Case (ii):When m < n:**

(i.e. the number of equations is less than the number of variables)

Then the solution is always other than $x_1 = x_2 = \ldots\ldots\ldots\ldots\ldots = x_n = 0$.

**Case (iii):When m = n:**

(i.e. the number of equations = the number of variables).

Then the necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots\ldots\ldots\ldots\ldots = x_n = 0$, is that the determinant of the coefficient matrix is zero. In this case the equations are said to be consistent and such a solution is called non-trivial solution. The determinant is called the eliminant of the equation.

**Now let us examine the consistency of the following system of equations:**

## System of homogenous equations

**Q.No.1.:** Solve the equations:

**(i)** $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$,

**(ii)** $4x + 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$, $2x + y + w = 0$.

**Sol.: (i).** Here coefficient matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$.

Find the rank of the coefficient matrix A:

Operating $R_2 \to R_2 - 3R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix}$.

Operating $R_3 \to R_3 - 7R_1 - 2R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus the rank of A = 3= the number of variables (i.e. r = n).

$\therefore$ The equations have only a trivial solution x = y = z = 0.

**(ii).** Here coefficient matrix A is $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

Find the rank of the coefficient matrix:

Operating $R_2 \to R_2 - \frac{3}{2}R_1,\ R_3 \to R_3 - \frac{1}{2}R_1$, we get $A \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Operating $R_3 \to R_3 + \frac{1}{5}R_2$, we get $A \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus the rank of A = 2 < the number of variable (i.e. $r < n$)

$\therefore$ Number of independent solutions = $4 - 2 = 2$. Also the given system is equivalent to

$4x + 2y + z + 3w = 0$, $z + w = 0$.

$\therefore$ We have $z = -w$ and $y = -2x - w$,

which give an infinite number of non-trivial solutions, by choosing the values of x and w arbitrary.

**Q.No.2.:**Find the values of $\lambda$ for which the equations

$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$,

$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$,

$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$, are consistent, and find the ratios of x : y : z, when $\lambda$ has the smallest of these values. What happens when $\lambda$ has the greater of these values.

**Sol.:** The given equations will be consistent, if $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$.

Operating $R_2 \to R_2 - R_1$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$.

Operating $C_3 \to C_3 + C_2$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$.

$\Rightarrow (\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda+1) \end{vmatrix} = 0 \Rightarrow 2(\lambda-3)[(\lambda-1)(3\lambda-1)-(5\lambda+1)] = 0 \Rightarrow 6\lambda(\lambda-3)^2 = 0$

$\Rightarrow \lambda = 0$ or 3.

**(a).** When $\lambda = 0$, the equations become $-x+y=0$ (i)

$-x-2y+3z=0$ (ii)

$2x+y-3z=0$ (iii)

Solving (ii) and (iii), we get $\dfrac{x}{6-3} = \dfrac{y}{6-3} = \dfrac{z}{-1+4}$.

Hence x = y = z.

**(b).** When $\lambda = 3$, equations become identical.

**Q.No.3.:** Determine the values of $\lambda$ for which the following set of equations may possess non-trivial solution:

$3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$.

For each permissible value of $\lambda$, determine the general solution.

**Sol.:** Given equations are $3x_1 + x_2 - \lambda x_3 = 0$, (i)

$4x_1 - 2x_2 - 3x_3 = 0$, (ii)

$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. (iii)

The given system of equations will be consistent, if $|A| = 0 \Rightarrow \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$.

$\Rightarrow 3(-2\lambda + 12) - 1(4 + 6\lambda) - \lambda(16 + 4\lambda) = 0$

$\Rightarrow \lambda^2 - \lambda + 9\lambda - 9 = 0 \Rightarrow \lambda = 1, \quad -9$.

For $\lambda = 1$, equations (i), (ii) and (iii), becomes

$3x_1 + x_2 - x_3 = 0$ (iv)

$4x_1 - 2x_2 - 3x_3 = 0$ (v)

$2x_1 + 4x_2 + x_3 = 0$ (vi)

By (iv) and (vi), we get

$5x_1 + 5x_2 = 0$

$\Rightarrow x_1 = -x_2 = k$ (say)

$\therefore x_1 = k, \quad x_2 = -k$

Value of k put in equation (iv), we get

$3k - k - x_3 = 0 \Rightarrow x_3 = 2k$.

When $\lambda = 1$. Solution is $x_1 = k$, $x_2 = -k$ and $x_3 = 2k$. Ans.

For $\lambda = -9$, equations (i), (ii) and (iii), becomes

$3x_1 + x_2 + 9x_3 = 0$ (vii)

$4x_1 - 2x_2 - 3x_3 = 0$ (viii)

$-18x_1 + 4x_2 - 9x_3 = 0$. (ix)

By equation (vii) and (viii), we get

$$\frac{x_1}{-3+18} = \frac{x_2}{36+9} = \frac{x_3}{-6-4} = k$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{9} = \frac{x_3}{-2} = k \Rightarrow x_1 = 3k, \quad x_2 = 9k, \quad x_3 = -2k.$$

Hence we calculated

For $\lambda = -9$, $x_1 = 3k$, $x_2 = 9k$, $x_3 = -2k$,

For $\lambda = 1$, $x_1 = k$, $x_2 = -k$ and $x_3 = 2k$, be the required general solution.

**Q.No.4.:** Solve completely the system of equations

$x + y - 2z + 3w = 0$, $x - 2y + z - w = 0$,

$4x + y - 5z + 8w = 0$, $5x - 7y + 2z - w = 0$.

**Sol.:** The matrix form of the given system of equations is $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 5R_1$, we get

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Operating $R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$, we get $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + y - 2z + 3w = 0$, (i)

$-3y + 3z - 4w = 0$. (ii)

Suppose $z = \lambda$ and $w = \mu$

Now put the values of z and w in equation (ii), we get

$$-3y + 3\lambda - 4\mu = 0 \Rightarrow -3y = 4\mu - 3\lambda \Rightarrow y = \frac{3\lambda - 4\mu}{3} \Rightarrow y = \lambda - \frac{4}{3}\mu .$$

Put the value of y in equation (i), we get

$$x + \frac{(3\lambda - 4\mu)}{3} - 2\lambda + 3\mu = 0 \Rightarrow 3x + 3\lambda - 4\mu - 6\lambda + 9\mu = 0$$

$$\Rightarrow 3x - 3\lambda + 5\mu = 0 \Rightarrow x = \lambda - \frac{5}{3}\mu .$$

Thus $x = \lambda - \frac{5}{3}\mu$, $y = \lambda - \frac{4}{3}\mu$, $z = \lambda$ and $w = \mu$ be the required solution.

**Q.No.5.:** Solve the equations

$x_1 + 3x_2 + 2x_3 = 0$, $2x_1 - 3x_2 + 3x_3 = 0$,

$3x_1 - 5x_2 + 4x_3 = 0$, $x_1 + 17x_2 + 4x_3 = 0$.

**Sol.:** In matrix notation, the given system of equations can be written as AX = 0

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Operating $R_1 - 2R_1$, $R_3 - 3R_1$, $R_4 - R_1$, we get $A \approx \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$.

Operating $R_3 - 2R_2$, $R_4 + 2R_2$, we get $A \approx \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Operating $R_1 + 2R_2$, we get $A \approx \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\rho(A) = 2 <$ number of unknowns.

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$x_1 - 11x_2 = 0$, $-7x_2 - x_3 = 0$

i.e., $x_1 = 11k$, $x_2 = k$, $x_3 = 7k$, where k is any number. Different values of k give different solutions.

## System of non-homogenous equations

**Q.No.1.:** Test the consistency and solve

$5x + 3y + +7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.

**Sol.:** The given set of equations can be written as $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$.

Operating $R_1 \to 3R_1,\ R_2 \to 5R_2$, we get $\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$.

Operating $R_1 \to \frac{7}{3}R_1, R_3 \to 5R_3, R_2 \to \frac{1}{11}R_2$, we get $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1 + R_2, R_1 \to \frac{1}{7}R_1$, we get $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$.

Here the ranks of coefficient matrix A = the rank the augmented matrix K = 2.

Hence, the equations are consistent.

Also the given system is equivalent to

$5x + 3y + 7z = 4$, $11y - z = 3$.

$\therefore\ y = \frac{3}{11} + \frac{z}{11}$ and $x = \frac{7}{11} - \frac{16}{11}z$, where z is parameter.

Thus, we have infinite number of solutions by choosing one unknown arbitrary.

If we put z = 0, we get

$x = \frac{7}{11},\ y = \frac{3}{11}$, which is a particular solution.

**Q.No.2.:** Investigate for consistency of the following equations and if possible find the solutions:

$4x - 2y + 6z = 8,\ \ x - y + 3z = -1,\ \ 15x - 3y + 9z = 21$.

**Sol.:** Here $\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$ is the matrix representation of the given equations.

Now operating $R_1 \to \frac{R_1}{2},\ R_3 \to \frac{R_3}{2}$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -3 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$.

Operating $R_2 \to 2R_2 - R_1$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -9 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$.

Operating $R_2 \to \dfrac{R_2}{3}$, $R_3 \to 2R_3 - 5R_1$, we get $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$. (i)

Here the rank of coefficient matrix A = 2 = the rank of augmented matrix $K < 3$.

Hence the given system of equations is consistent and we have infinite number of solutions.

Now (i) $\Rightarrow 2x - y + 3z = 4$, $y - 3z = -2$.

Let $z = k$ arbitrary number, hence

$y = 3k - 2$ and $2x - 3k + 2 + 3z = 4$

$\Rightarrow 2x - 3k + 2 + 3k = 4 \Rightarrow 2x = 2 \Rightarrow x = 1$.

Hence $x = 1$, $y = 3k - 2$ and $z = k$ for all k,

which gives an infinite no. of non-trivial solutions.

**Q.No.3.:** Test for consistency and solve:

**(i)** $2x - 3y + 7x = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$,

**(ii)** $x + 2y + z = 3$, $2x + 3y + 2z = 5$, $3x - 5y + 5z = 2$, $3x + 9y - z = 4$,

**(iii)** $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$.

**Sol.: (i)** We have $AX = B \Rightarrow \begin{bmatrix} 2 & -3 & -7 \\ 3 & -1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$.

Operating $R_1 \to 3R_1 - 2R_2$, $R_3 \to R_3 - R_1$, we get $\begin{bmatrix} 0 & -11 & 27 \\ 3 & 1 & -3 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 13 \\ 27 \end{bmatrix}$.

Operating $R_3 \to R_3 + 2R_1$, we get $\begin{bmatrix} 0 & -11 & 27 \\ 3 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 13 \\ 27 \end{bmatrix}$.

Here $\rho(A) = 2 \neq \rho(K) = 3$.

This shows that the given system of equations is not consistent, i.e., no solution for these equations.

**(ii).** Given equations are

$x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$.

Now we have $AX=B \Rightarrow \begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ 3 & -5 & 5\\ 3 & 9 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 5\\ 2\\ 4\end{bmatrix}$.

Operating $R_2 \to R_2-2R_1$, $R_3 \to R_3-3R_1$ and $R_4 \to R_4-3R_1$, we get

$$\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & -11 & 2\\ 0 & 3 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -1\\ -7\\ -5\end{bmatrix}.$$

Operating $R_3 \to R_3-11R_1$, $R_4 \to R_4+3R_1$, we get $\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 0 & 2\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -1\\ 4\\ -8\end{bmatrix}$

Operating $R_4 \to R_4+2R_3$, we get $\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -1\\ 4\\ 0\end{bmatrix}$. (i)

Here $\rho(A)=3=\rho(K)=$ no. of unknowns.

Hence, the given system of equations is consistent and there is only unique solution.

Now (i) $\Rightarrow x+2y+z=3$,

$-y=-1 \Rightarrow y=1$,

$2z=4 \Rightarrow z=2$.

Now putting y and z in the equation, we get $x=-1$.

Hence, solution is $x=-1$, $y=1$ and $z=2$. Ans.

**(iii).** Given equation are

$2x+6y+11=0$, $6x+20y-6z+3=0$, $6y-18z+1=0$.

Now we have AX= B $\Rightarrow \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$.

Operating $R_2 \to R_2 - 3R_1$, we get $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$.

Operating $R_3 \to R_3 - 3R_2$, we get $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$.

Here $\rho(A) = 2 \neq \rho(K) = 3$.

This shows that the given system of equations is not consistent, i.e. no solution for these equations.

**Q.No.4.:** Test for consistency and solve:

$2x_1 - 2x_2 + 4x_3 + 3x_4 = 9$, $x_1 - x_2 + 2x_3 + 2x_4 = 6$,

$2x_1 - 2x_2 + x_3 + 2x_4 = 3$, $x_1 - x_2 + x_4 = 2$

**Sol.:** Apply elementary row operation on $[A|B]$.

Since $[A|B] = \left[\begin{array}{cccc|c} 2 & -2 & 4 & 3 & 9 \\ 1 & -1 & 2 & 2 & 6 \\ 2 & -2 & 1 & 2 & 3 \\ 1 & -1 & 0 & 1 & 2 \end{array}\right]$.

Operating $R_{12}$, $R_{21(-2)}$, $R_{41(-1)}$, $R_{31(-2)}$, $R_{2(-1)}$, $R_{3(-1)}$, $R_{4(-1)}$, we get

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 9 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right].$$

Operating $R_{34(-1)}$, we get $[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right]$.

Operating $R_{32}$, $R_{43}$, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 0 & 1 & 5\\ 0 & 0 & 2 & 1 & 4\\ 0 & 0 & 0 & 1 & 3\end{array}\right]$.

Operating $R_{32(-2)}$, $R_{3(-1)}$, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 1 & 3\end{array}\right]$.

Operating $R_{43(-1)}$, $R_{4(-1)}$, we get $[A|B]=\left[\begin{array}{cccc|c}1 & -1 & 2 & 2 & 6\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 0 & 3\end{array}\right]$.

Rank of (A) $=3\neq 4$ = rank of $[A|B]$.

So the given system in inconsistent and therefore has no solution.

**Q.No.5.:** Solve the system of equations:

$2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$,

$4x_1+3x_2+3x_3-3x_4=-1$, $2x_1+2x_2-x_3+x_4=10$.

**Sol.:** In matrix notation, the given system of equations can be written as AX = B

where $A=\begin{bmatrix}2 & 1 & 2 & 1\\ 6 & -6 & 6 & 12\\ 4 & 3 & 3 & -3\\ 2 & 2 & -1 & 1\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$, $B=\begin{bmatrix}6\\ 36\\ -1\\ 10\end{bmatrix}$.

Augmented matrix $[A\vdots B]=\left[\begin{array}{cccc|c}2 & 1 & 2 & 1 & 6\\ 6 & -6 & 6 & 12 & 35\\ 4 & 3 & 3 & -3 & -1\\ 2 & 2 & -1 & 1 & 10\end{array}\right]$.

Operating $R_2-3R_1, R_3-2R_1, R_4-R_1$, we get $[A\vdots B]=\left[\begin{array}{cccc|c}2 & 1 & 2 & 1 & 6\\ 0 & -9 & 0 & 9 & 18\\ 0 & 1 & -1 & -5 & -13\\ 0 & 1 & -3 & 1 & 4\end{array}\right]$.

Operating $-\frac{1}{9}R_2$, we get $[A \vdots B] = \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 1 & -1 & -5 & \vdots & -13 \\ 0 & 1 & -3 & 0 & \vdots & 4 \end{bmatrix}$.

Operating $R_1 - R_2, R_3 - R_2, R_4 - R_2$, we get $[A \vdots B] = \begin{bmatrix} 2 & 0 & 2 & 1 & \vdots & 8 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & -3 & 1 & \vdots & 6 \end{bmatrix}$.

Operating $R_4 - 3R_3, \frac{1}{2}R_1$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 4 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 13 & \vdots & 39 \end{bmatrix}$.

Operating $R_1 + R_3, \frac{1}{13}R_4$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & -3 & \vdots & -7 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Operating $R_1 + 3R_4, R_2 + R_4,\ R_3 + 4R_4$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & -1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Operating $(-1)R_3$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Hence $x_1 = 2$, $x_2 = 1$, $x_3 = 1$, $x_4 = 3$.

**Q.No.6.:** Using matrix method, show that the equations:

$3x + 3y + 2z = 1,\ x + 2y = 4,\ 10y + 3z = 2,\ 2x - 3y - z = 5$

are consistent and hence obtain the solutions for x, y and z.

**Sol.:** In matrix notation, the given system of equations can be written as AX = B where $A=\begin{bmatrix}3 & 3 & 2\\ 1 & 2 & 0\\ 0 & 10 & 3\\ 2 & -3 & -1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}1\\ 4\\ -2\\ 5\end{bmatrix}$.

Augmented matrix $[A \vdots B]=\begin{bmatrix}3 & 3 & 2 & \vdots & 1\\ 1 & 2 & 0 & \vdots & 4\\ 0 & 10 & 3 & \vdots & -2\\ 2 & -3 & -1 & \vdots & 5\end{bmatrix}$.

Operating $R_{12}$, we get $[A \vdots B]=\begin{bmatrix}1 & 2 & 0 & \vdots & 4\\ 3 & 3 & 2 & \vdots & 1\\ 0 & 10 & 3 & \vdots & -2\\ 2 & -3 & -1 & \vdots & 5\end{bmatrix}$.

Operating $R_2-3R_1$, $R_4-2R_1$ we get $[A \vdots B]=\begin{bmatrix}1 & 2 & 0 & \vdots & 4\\ 0 & -3 & 2 & \vdots & -11\\ 0 & 10 & 3 & \vdots & -2\\ 0 & -7 & -1 & \vdots & -3\end{bmatrix}$.

Operating $R_3+3R_2$, $R_4-2R_2$ we get $[A \vdots B]=\begin{bmatrix}1 & 2 & 0 & \vdots & 4\\ 0 & -3 & 2 & \vdots & -11\\ 0 & 1 & 9 & \vdots & -35\\ 0 & -1 & -5 & \vdots & 19\end{bmatrix}$.

Operating $R_1-2R_3$, $R_2+3R_3$, $R_4+R_3$ we get $[A \vdots B]=\begin{bmatrix}1 & 0 & -18 & \vdots & 74\\ 0 & 0 & 29 & \vdots & -116\\ 0 & 1 & 9 & \vdots & -35\\ 0 & 0 & 4 & \vdots & -16\end{bmatrix}$.

Operating $R_{23}$, $\frac{1}{4}R_4$, we get $[A \vdots B]=\begin{bmatrix}1 & 0 & -18 & \vdots & 74\\ 0 & 0 & 9 & \vdots & -35\\ 0 & 1 & 29 & \vdots & -116\\ 0 & 0 & 1 & \vdots & -4\end{bmatrix}$

Operating $R_1+18R_4$, $R_2-9R_4$, $R_3-29R_4$ we get $[A \vdots B]=\begin{bmatrix}1 & 0 & 0 & \vdots & 2\\ 0 & 1 & 0 & \vdots & 1\\ 0 & 0 & 0 & \vdots & 0\\ 0 & 0 & 1 & \vdots & -4\end{bmatrix}$

Operating $R_{34}$, we get $[A \vdots B] = \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\rho(A) = \rho(A:B) = 3 =$ number of unknowns.

$\Rightarrow$ The given system of equations is consistent and the solution is $x = 2,\ y = 1,\ z = -4$.

**Q.No.7.:** Test for consistency and solve:

$3x + 3y + 2z = 1,\ x + 2y = 4, 10y + 3z = -2,\ 2x - 3y - z = 5$.

**Sol.:** $[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$.

Operating $R_{12}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$.

Operating $R_{21(-3)}$, $R_{41(-2)}$ , we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & 3 \end{array}\right]$.

Operating $R_{2\left(-\frac{1}{3}\right)}$, $R_{32(-10)}$ , $R_{42(7)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & \frac{29}{3} & -\frac{116}{3} \\ 0 & 0 & -\frac{17}{3} & \frac{68}{3} \end{array}\right]$.

Operating $R_{3\left(\frac{3}{29}\right)}$, $R_{43\left(\frac{17}{3}\right)}$ , we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -\frac{116}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$.

r(A) = 3 = $[A|B]$ = n = number of variables.

The system is consistent and has unique solution.

Solving, we get $z = -\frac{116}{29} = -4$.

$y - \frac{2}{3}z = \frac{11}{3} \Rightarrow y = \frac{11}{3} + \frac{2}{3}(-4) = 1$

$x + 2y + 0 \Rightarrow x = 4 - 2 = 2$.

i.e., $x = 2,\ y = 1,\ z = -4$.

**Q.No.8.:**Solve $x_1 + x_2 - x_3 = 0$, $2x_1 - x_2 + x_3 = 3$, $4x_1 + 2x_2 - 2x_3 = 2$.

**Sol.:** By applying elementary row operation

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right].$$

Operating $R_{21(-2)}, R_{31(-4)}$, we get $[A|B] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{bmatrix}$.

Operating $R_{2\left(-\frac{1}{3}\right)}, R_{3\left(-\frac{1}{2}\right)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$.

Operating $R_{32(-1)}$, we get $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

r(A) = 2 = $[A|B]$ < 3 = n = number of variables.

The system is consistent but has infinite numbers of solutions in terms of $n - r = 3 - 2 = 1$ variable.

Choose $x_3 = k$ = arbitrary constant

Solving $x_2 - x_3 = 1 \Rightarrow x_3 = x_3 - 1 = k - 1$.

$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = -x_2 + x_3 = -k + 1 + k = 1$.

Thus the solutions are

$x_1 = 1,\ x_2 = k - 1,\ x_3 = k$, where k is arbitrary.

**Q.No.9.:** Solve, with the help of matrices, the simultaneous equations:

$x+y+z=3,\quad x+2y+3z=4,\quad x+4y+9z=6.$

**Sol.:** In this question, there is no restriction that the solution must be obtained by finding $A^{-1}$.

Now here augmented matrix $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 1 & 2 & 3 & : & 4\\ 1 & 4 & 9 & : & 6\end{bmatrix}$.

Operating $R_2-R_1,\quad R_3-R_1$, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 3 & 8 & : & 3\end{bmatrix}$.

Operating $R_3-3R_2$, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 0 & 2 & : & 0\end{bmatrix}$.

Operating $\frac{1}{2}R_3$, we get $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 3\\ 0 & 1 & 2 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

Operating $R_2-R_3,\quad R_2-2R_3$, we get $[A:B]=\begin{bmatrix}1 & 1 & 0 & : & 3\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

Operating $R_1-R_2$, we get $[A:B]=\begin{bmatrix}1 & 0 & 0 & : & 2\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & 0\end{bmatrix}$.

$\therefore x=2,\ y=1,\ z=0.$

This method is especially useful when the number of unknown is 4, since $|A|$ is order of 4 and the co-factor of its various elements are determinants of order 3.

**Q.No.10.:** Investigate the values of $\lambda$ and $\mu$ so that the equations

$2x+3y+5z=9,\quad 7x+3y-2z=8,\quad 2x+3y+\lambda z=\mu$, have

**(i)** no solution, **(ii)** a unique solution and **(iii)** an infinite number of solutions.

**Sol.:** The given set of equations can be written as $\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & -2\\ 2 & 3 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 8\\ \mu\end{bmatrix}$.

$$\Rightarrow AX = B \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}.$$

The augmented matrix $K = [A : B] \begin{bmatrix} 2 & 3 & 5 & \vdots & 9 \\ 7 & 3 & -2 & \vdots & 8 \\ 2 & 3 & \lambda & \vdots & \mu \end{bmatrix}$.

Operating $R_3 \to R_3 - R_1$, $R_2 \to 2R_2 - 7R_1$, we get $K \sim \begin{bmatrix} 2 & 3 & 5 & \vdots & 9 \\ 0 & -15 & -39 & \vdots & -47 \\ 0 & 0 & \lambda - 5 & \vdots & \mu - 9 \end{bmatrix}$

**(i).** If $\lambda \neq 5$, we have rank of K = 3 = rank of A [i.e. $r = r'$]

$\Rightarrow$ The given system of equations is consistent.

Also the rank of A = the number of unknowns.

$\Rightarrow$ The given system of equations posses a unique solution.

Thus, $\lambda \neq 5$, the given equations possesses a unique solution for any value of $\mu$.

**(ii).** If $\lambda = 5$ and $\mu = 9$, we have rank K = rank A.

$\Rightarrow$ The given system of equations is again consistent.

Also the rank of A < the numbers of unknowns.

$\Rightarrow$ The given system of equations possesses an infinite number of solutions.

**(iii).** If $\lambda = 5$ and $\mu \neq 9$, we have rank of K = 3, and rank of A = 2 $\Rightarrow$ rank K $\neq$ rank A.

$\Rightarrow$ The given system of equations is inconsistent and possesses no solution.

**Q.No.11.:** Investigate for what values of $\lambda$ and $\mu$ the simultaneous equations

$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$, have

**(i)** no solution, **(ii)** a unique solution **(iii)** an infinite number of solutions.

**Sol.:** We have $AX = B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$.

The augmented matrix $K = [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get $K = [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$.

**(i).** If $\lambda = 3$ and $\mu \neq 10$, then $\rho(A) = 2 \neq \rho(K) = 3$

$\Rightarrow$ the given system of equations is inconsistent i.e., possesses no solution.

**(ii).** If $\lambda \neq 3$ and $\forall \mu$, then $\rho(A) = \rho(K) = 3$ = the number of unknowns.

$\Rightarrow$ The given system of equations is consistent, and possesses a unique solution.

Thus if $\lambda \neq 3$, $\forall \mu$, the given system of equations possesses a unique solution.

**(iii).** If $\lambda = 3$ and $\mu = 10$, then $\rho(A) = \rho(K) = 2 <$ the number of unknowns.

$\Rightarrow$ The given system of equations is again consistent and possesses an infinite number of solutions.

**Q.No.12.:** For what values of k the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case.

**Sol.:** Here the matrix form of the given system of equations is $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$.

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ k^2-4 \end{bmatrix}$.

Operating $R_3 \to \frac{1}{3} R_3$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ \frac{(k^2-4)}{3} \end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ \frac{(k^2-4)}{3} - k + 2 \end{bmatrix}$.

$\Rightarrow x+y+z=1,\ -y+2z=k-2$ and $0=\dfrac{k^2-4}{3}-k+2$.

This is only possible i. e. have solution if $\dfrac{\left(k^2-4\right)}{3}-k+2=0$

$\Rightarrow k^2-3k+2=0 \Rightarrow\ k=2,\ 1.$

**Case 1:** Let k = 2.

We have $x+y+z=1$, $-y+2z=k-2=0 \Rightarrow y=2z$

If z = c, then $-y+2c=0 \Rightarrow y=2c$

and $x=1-3c$.

$\therefore$ At k = 2, $x=1-3c$, $y=2c$, $z=c$,

which is the required solution when k = 1.

**Case 2:** Let k = 1, then $-y+2c=-1 \Rightarrow y=1+2c$

and $x=1-1-2c=-3c$.

$\therefore$ At k = 1, $x=-3c$, $y=1+2c$, $z=c$,

which is the required solution when k = 1.

**Q.No.13.:** Find the values of a and b for which the equations:

$x+ay+z=3$, $x+2y+2z=b$, $x+5y+3z=9$ are consistent.

Determine the solution in each case.

When will these equations have unique solution ?

**Sol.:** The matrix form of the given system of equations is $\begin{bmatrix}1 & a & 1\\ 1 & 2 & 2\\ 1 & 5 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ b\\ 9\end{bmatrix}$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{bmatrix}1 & a & 1\\ 1 & 2 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ b\\ 9-b\end{bmatrix}$.

Operating $R_2 \to R_2 - R_1$, we get $\begin{bmatrix}1 & a & 1\\ 0 & 2-a & 1\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ b-3\\ 9-b\end{bmatrix}$.

Operating $R_2 \to R_3 + R_2$, we get $\begin{bmatrix} 1 & a & 1 \\ 0 & 2-a & 0 \\ 0 & 1+a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ b-3 \\ 12-2b \end{bmatrix}$.

**Case (i):** When $a = -1,\ b = 6$, then equations will be consistent and have infinite number of solutions.

**Case (ii):** When $a = -1,\ b \neq 6$, then equations will be inconsistent.

**Case (iii):** When $a \neq -1\ \ \forall\ b$, then equations will be consistent and have a unique solutions.

**Q.No.14.:** Determine the values of a and b for which the system

$x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$

has (i) no solution (ii) unique solution (iii) infinite number of solutions.

Find the solution in case (ii) and (iii).

**Sol.:** $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$

Operating $R_{21(-1)}$, $R_{31(-2)}$, we get $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$

Operating $R_{32(-1)}$, we get $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$

**Case 1:** $a = 8,\ b \neq 15$, $r(A) = 2 \neq 3 = r[A|B]$, inconsistent, no solution.

**Case 2:** $a \neq 8$**,** b any value. r(A) = 3 = $[A|B]$ = n = number of variables, unique solution,

$z = \frac{b-15}{a-8}$.

$y = \frac{(3a-2b+6)}{(a-8)}$**,** $x = z = \frac{(b-15)}{(a-8)}$**.**

**Case 3:** a = 8, b = 15, r(A) = 2 = $[A|B]$< 3 = n. Infinite solutions with $n - r = 3 - 2 = 1$ arbitrary variable. x = k, $y = 3 - 2k$, z = k, with k arbitrary.

**Q.No.15.:** Show that the equations $3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$ do not have a solution unless $a+c=2b$.

**Sol.:** Let $A=\begin{bmatrix}3 & 4 & 5\\ 4 & 5 & 6\\ 5 & 6 & 7\end{bmatrix}$, $B=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.

Then the matrix form of the equations is $AX=B \Rightarrow \begin{bmatrix}3 & 4 & 5\\ 4 & 5 & 6\\ 5 & 6 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$.

Operating, $R_2 \to 3R_2-4R_1$, we get $\begin{bmatrix}3 & 4 & 5\\ 0 & -1 & -2\\ 5 & 6 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ 3b-4a\\ c\end{bmatrix}$.

Operating, $R_3 \to 3R_3-5R_1$, we get $\begin{bmatrix}3 & 4 & 5\\ 0 & -1 & -2\\ 0 & -2 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ 3b-4a\\ 3c-5a\end{bmatrix}$.

Operating, $R_3 \to \frac{1}{2}R_3-R_2$, we get $\begin{bmatrix}3 & 4 & 5\\ 0 & -1 & -2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ 3b-4a\\ \dfrac{3c-6b+3a}{2}\end{bmatrix}$.

If $\dfrac{3c-6b+3a}{2}\neq 0$, then equations are inconsistent.

If $\rho(A)=\rho(K)$, then equations are consistent. This is possible only when $\dfrac{3c-6b+3a}{2}=0 \Rightarrow 3c-6b+3a=0 \Rightarrow a+c=2b$.

Thus the given equations do not have a solution unless a + c =2b.

**Q.No.16.:** Show that if $\lambda \neq -5$, the system of equations

$3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ have a unique solution.

If $\lambda=-5$, show that the equations are consistent.

Determine the solution in each case.

**Sol.:** The matrix form of the given system of equations is $\begin{bmatrix}3 & -1 & 4\\ 1 & 2 & -3\\ 6 & 5 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -2\\ -3\end{bmatrix}$.

Operating $R_2 \to 3R_2 - R_1$, $R_3 \to R_3 - 2R_1$, we get $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 7 & \lambda-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -9 \end{bmatrix}$.

Operating, $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda+5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$.

**Case 1.** If $\lambda \neq -5$. Then $\rho(A) = 3 = \rho(K) =$ number of unknowns.

$\Rightarrow$ The system of equations is consistent and have a unique solution.

Then the unique solution is $z = 0$, $y = -\frac{9}{7}$, $x = \frac{4}{7}$.

**Case 2.** If $\lambda = -5$, then $\rho(A) = 2 = \rho(K) <$ number of unknowns = 3.

$\Rightarrow$ The system of equations is consistent and have infinite number of solutions.

Put $z = k$ for all values of k, then

$y = \frac{1}{7}(13k - 9)$, $x = \frac{1}{3}(y - 4z + 3) \Rightarrow x = \frac{1}{7}(4 - 5k)$.

Hence when $\lambda = -5$, then $x = \frac{1}{7}(4 - 5k)$, $y = \frac{1}{7}(13k - 9)$ and $z = k$ for all values of k, be the required solution.

**Q.No.17.:**Find the values of $\lambda$ for which the equations $(2-\lambda)x + 2y + 3 = 0$, $2x + (4-\lambda)y + 7 = 0$, $2x + 5y + (6-\lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of $\lambda$.

**Sol.:** Here coefficient matrix $A = \begin{bmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{bmatrix}$.

The given equations are consistent if $|A| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{vmatrix} = 0$.

Operating $R_3 \to R_3 - R_2$, we get $\begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 0 & 1+\lambda & -1-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 - R_2$, we get $\begin{vmatrix} -\lambda & -2+\lambda & -4 \\ 2 & 4-\lambda & 7 \\ 0 & 1+\lambda & -1-\lambda \end{vmatrix} = 0$.

Operating $C_2 \to C_2 + C_3$, we get $\begin{vmatrix} -\lambda & 6+\lambda & -4 \\ 2 & 11-\lambda & 7 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 + R_2$, we get $\begin{vmatrix} 2-\lambda & 5 & 3 \\ 2 & 11-\lambda & 7 \\ 0 & 0 & 1+\lambda \end{vmatrix} = 0$.

Now expanding the determinant, we get

$(1+\lambda)\{(2-\lambda)(11-\lambda)-10\} = 0 \Rightarrow (1+\lambda)(\lambda^2 - 13\lambda + 12) = 0$

$\Rightarrow$ Either $(\lambda+1) = 0$ or $\lambda^2 - 13\lambda + 12 = 0$.

$\Rightarrow \lambda^2 - 13\lambda + 12 = 0 \Rightarrow \lambda = 12,\ 1$.

Therefore the values of $\lambda = -1,\ 1,\ 12$.

**Case 1.** When $\lambda = -1$, the equations become

$3x + 2y + 3 = 0$,

$2x + 5y + 7 = 0$,

$2x + 5y + 7 = 0$.

On solving these equations, we get $x = -\frac{1}{11}$, $y = -\frac{15}{11}$. Ans.

**Case 2.** When $\lambda = 1$, the equations become

$x + 2y + 3 = 0$,

$2x + 3y + 7 = 0$,

$2x + 5y + 5 = 0$.

On solving these equations, we get $x = -5$, $y = 1$. Ans

**Case 3.** When $\lambda = 12$, the equations become

$-10x + 2y + 3 = 0$,

$2x + (-8y) + 7 = 0$,

$2x + 5y - 6 = 0$.

On solving these equations, we get $x = \dfrac{1}{2}$, $y = 1$. Ans.

**Q.No.18.:**Show that there are three real values of $\lambda$ for which the equations

$(a-\lambda)x + by + cz = 0$, $bx + (c-\lambda)y + az = 0$, $cx + ay + (b-\lambda)z = 0$

are simultaneously true and that the product of these values of $\lambda$ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

**Sol.:** Here the coefficient matrix $A = \begin{bmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{bmatrix}$.

These equations will be consistent if $|A| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$.

Operating $C_1 \to C_1 + C_2 + C_3$, we get $\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0$.

Taking $(a+b+c-\lambda)$ out side, we get $(a+b+c-\lambda)\begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$.

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$(a+b+c-\lambda)\begin{vmatrix} 0 & b-a & c-b+\lambda \\ 0 & c-\lambda-a & a-b+\lambda \\ 1 & a & b-\lambda \end{vmatrix} = 0.$$

On expanding, we get

$$(a+b+c-\lambda)\{(b-a)(a-b+\lambda) - (c-b+\lambda)(c-\lambda-a)\} = 0$$

$$\Rightarrow (a+b+c-\lambda)\left\{\left(ab - b^2 + b\lambda - a^2 + ab - a\lambda\right) - c^2 + c\lambda + ac + cb - b\lambda - ab - c\lambda + \lambda^2 + a\lambda\right\} = 0$$

$$\Rightarrow (a+b+c-\lambda)\left\{\lambda^2 - 2\left(a^2 + b^2 + c^2 - ab - ca - bc\right)\right\} = 0$$

Either $\lambda = a+b+c$ or $\lambda^2 - 2\left\{a^2 + b^2 + c^2 - (ab + ca + bc)\right\} = 0$.

$$\Rightarrow \lambda = \frac{\pm\sqrt{4\left(a^2+b^2+c^2-ab-bc-ca\right)}}{2} = \pm\sqrt{\left(a^2+b^2+c^2-ab-bc-ca\right)}.$$

Thus three roots are $\lambda_1 = a+b+c$, $\lambda_2 = \sqrt{a^2+b^2+c^2-ab-bc-ca}$ and

$\lambda_3 = -\sqrt{a^2+b^2+c^2-ab-bc-ca}$

Product of three roots of equation

$$\lambda_1\lambda_2\lambda_3 = -(a+b+c)\left[a^2+b^2+c^2-ab-bc-ca\right]. \qquad \text{(i)}$$

Now we have given $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Operating $R_1 \to R_1+R_2+R_3$, we get $D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$.

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get $(a+b+c)\begin{vmatrix} 0 & b-a & c-b \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$.

On expanding D, we get $D = (a+b+c)\begin{vmatrix} 0 & b-a & c-b \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$

$= (a+b+c)\{(b-a)(a-b)-(c-a)(c-b)\}$

$= (a+b+c)\left[\left(ba-b^2-a^2+ab\right)-\left(c^2-cb-ac+ab\right)\right]$

$= (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

$= -(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) \qquad \text{(ii)}$

Hence we have found that (i) and (ii) are equal.

Hence, it is proved that product of 3 values of $\lambda$ is equal to the $|D|$.

**Q.No.19.:**Show that the system of the equations $2x_1 - 2x_2 + x_3 = \lambda x_1$,

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can posses a non-trivial solution

only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

**Sol.:** Given equations are $(2-\lambda)x_1 - 2x_2 + x_3 = 0$, $2x_1 - (3+\lambda)x_2 + 2x_3 = 0$,

$-x_1+2x_2-\lambda x_3=0$.

The given system of equations will be consistent, if $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix}=0$.

$\Rightarrow (2-\lambda)(\lambda^2+3\lambda-4)+2(-2\lambda+2)+(4-\lambda-3)=0$

$\Rightarrow 4+2\lambda^2+6\lambda-8-\lambda^3-3\lambda^2+4\lambda(-1)+4-\lambda-3=3$

$\Rightarrow -\lambda^3-\lambda^2+5\lambda+(-3)=0 \Rightarrow \lambda^3+\lambda^2-5\lambda+3=0$

$\Rightarrow (\lambda-1)(\lambda^2+2\lambda-3)=0$

Thus $\lambda=1$ and $\lambda^2+2\lambda-3=0 \Rightarrow \lambda=-3,\ 1$.

For $\lambda=1$ and $\lambda=-3$ the given system of the equations are consistent and posses a non-trivial solution.

If we put $\lambda=1$ in the given equations, we get

$x_1-2x_2+x_3=0$,

$2x_1-4x_2+2x_3=0$,

$-x_1+2x_2-x_3=0$.

Let $x_1=a$, $x_3=b$, $\Rightarrow x_2=\dfrac{a+b}{2}$.

If we put $\lambda=-3$ in the given equations, we get

$5x_1-2x_2+x_3=0$,

$2x_1+2x_3=0$,

$-x_1+2x_2+3x_3=0$.

$\Rightarrow x_2=-2x_3 \Rightarrow x_3=-\dfrac{x_2}{2}$

$x_1=\dfrac{x_2}{2}=x_3=t$.

$x_1=t$, $x_2=-2t$, $x_3=t$ is the general solution.

**Q.No.20.:** Prove that the equations $5x+3y+2z=12$, $2x+4y+5z=2$,

$39x+43y+45z=c$ are incompatible unless $c=74$; and in that case the equations are satisfied by $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$, where t is any arbitrary quantity.

**Sol.:** The equations are $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$, $39x + 43y + 45z = c$.

The matrix form of these equations is $AX = B \Rightarrow \begin{bmatrix} 5 & 3 & 2 \\ 2 & 4 & 5 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ c \end{bmatrix}$.

Operating, $R_2 \to 5R_2 - 2R_1$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 14 & 21 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -14 \\ c \end{bmatrix}$.

Operating, $R_2 \to \frac{R_2}{7}$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ c \end{bmatrix}$.

Operating, $R_3 \to 5R_3 - 39R_3$, $R_2 \to \frac{R_2}{7}$ we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 98 & 147 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 5c - 468 \end{bmatrix}$.

Operating $R_3 \to R_3 - 49R_2$, we get $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 5c - 370 \end{bmatrix}$.

If $5c - 370 \neq 0 \Rightarrow c \neq 74$. The equations are inconsistent (or incompatible).

If $\rho(A) = \rho(K)$, then equations are consistent.

This is possible only when c = 74. Thus the equations are incompatible unless c = 74.

**IInd Part.** Now when c = 74, then

$5x + 3y + 2z = 12$ and $2y + 3z = -2$.

Now putting $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$ and c = 74 in the given equations, we obtain

$5x + 3y + 2z = 12$

$\Rightarrow 5(2 + t) + 3(2 - 3t) + 2(-2 + 2t) = 12 \Rightarrow 10 + 5t + 6 - 9t - 24 + 4t = 12 \Rightarrow 12 = 12.$

Hence equation is satisfied.

On putting the given values of x, y, z in the equation, we get

$2x + 4y + 5z = 2$

$\Rightarrow 2(2+t)+4(2-3t)+5(-2+2t)=2 \Rightarrow 2=2$.

Hence equation is satisfied.

On putting the given values of x, y, z in the equation, we get

$39x+43y+45z=c$

$\Rightarrow 39(2+t)+43(2-3t)+4(-2+2t)=74 \Rightarrow 164-90=74 \Rightarrow 74=74$.

**Q.No.21.:**If $b\ell = am-n$, $cm = bn-\ell$, $an = c\ell - m$, prove that $1+a^2+b^2+c^2=0$.

**Sol.:** Given $am-n-b\ell=0$, $cm-bn+\ell=0$, $-m-an+c\ell=0$.

Putting them into determinant form, we get $\begin{vmatrix} a & -1 & -b \\ c & -b & 1 \\ -1 & -a & c \end{vmatrix}=0$.

$\Rightarrow a(-bc+a)+1(c^2+1)-b(-ac-b)$

$\Rightarrow -abc+a^2+c^2+1+bac+b^2=0$

$\Rightarrow 1+a^2+b^2+c^2=0$.

Hence, this completes the proof.

**Q.No22.:** Solve by calculating the inverse by elementary row operations

$x_1+x_2+x_3+x_4=0$, $x_1+x_2+x_3-x_4=4$,

$x_1+x_2-x_3+x_4=-4$, $x_1-x_2+x_3+x_4=2$.

**Sol.:** The system is written as AX = B, where

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix},\quad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\quad B=\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

Inverse by elementary row operations

$$[A|I]=\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_{21(-1)}$, $R_{31(-1)}$, $R_{41(-1)}$ and $R_{2(-1)}$, $R_{3(-1)}$, $R_{4(-1)}$, we get

$$[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&0&0&2&1&-1&0&0\\0&0&2&0&1&0&-1&0\\0&2&0&0&1&0&0&-1\end{array}\right]$$

Operating $R_{24}$, $R_{2\left(\frac{1}{2}\right)}$, $R_{3\left(\frac{1}{2}\right)}$, $R_{4\left(\frac{1}{2}\right)}$, we get $[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&1&0&0&\frac{1}{2}&0&0&-\frac{1}{2}\\0&0&1&0&\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&0&1&\frac{1}{2}&\frac{1}{2}&0&-1\end{array}\right]$

Operating $R_{14(-1)}$, $R_{13(-1)}$, $R_{12(-1)}$, we get $[A|I]=\left[\begin{array}{cccc|cccc}1&1&1&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&1&0&0&\frac{1}{2}&0&0&-\frac{1}{2}\\0&0&1&0&\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&0&1&\frac{1}{2}&-\frac{1}{2}&0&0\end{array}\right]$.

Thus $A^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&0&0&-1\\1&0&-1&0\\1&-1&0&0\end{bmatrix}$.

The require solution is

$$X=A^{-1}B=\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&0&0&-1\\1&0&-1&0\\1&-1&0&0\end{bmatrix}\begin{bmatrix}0\\4\\-4\\2\end{bmatrix}=\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix}$$

i.e., $X_1=1$, $X_2=-1$, $X_3=2$, $X_4=-2$.

# Home Assignments:

## System of homogenous equations

**Q.No.1.:** Solve the equations

$x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$.

**Ans.:** $x = 11k$, $y=k$, $z=-7k$, where k is arbitrary.

**Q.No.2.:** Solve completely the system of equations

$3x+4y-z-6w=0$, $2x+3y+2z-3w=0$,

$2x+y-14z-9w=0$, $x+3y+13z+3w=0$.

**Ans.:** $x=11k_2+6k_1$, $y=-8k_2-3k_1$, $z=k_2$, $w=k_1$,

where $k_1$, $k_2$ are arbitrary constants.

**Q.No.3.:** Using the loop current method on a circuit, the following equations were obtained: $7i_1-4i_2=12$, $-4i_1+12i_2-6i_3=0$, $-6i_2+14i_3=0$.

By matrix method, solve for $i_1$, $i_2$ and $i_3$.

**Ans.:** $i_1=\dfrac{396}{175}$, $i_2=\dfrac{24}{25}$, $i_3=\dfrac{72}{175}$.

## System of non-homogenous equations

**Q.No.1.:** Test for consistency and solve:

$5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$.

**Ans.:** $x=\dfrac{(7-16k)}{11}$, $y=\dfrac{(3+k)}{11}$, $z=k$, k arbitrary.

**Q.No.2.:** Test for consistency and solve:

$x_1+x_2-2x_3+x_4+3x_5=1$, $2x_1-x_3+2x_3+2x_4+6x_5=2$,

$3x_1+2x_2-4x_3-3x_4-9x_5=3$.

**Ans.:** $x_1=1$, $x_2=2a$, $x_3=a$, $x_4=-3b$, $x_5=b$, where a and b are arbitrary constants.

**Q.No.3.:** Test for consistency and solve:

$x_1+x_2+2x_3+x_4=5$, $2x_1+3x_2-x_3-2x_4=2$, $4x_1+5x_2+3x_3=7$

**Ans.:** No solution, system inconsistent.

**Q.No.4.:** Test for consistency and solve:

$2x_1+3x_2-x_3=1$, $3x_1-4x_2+3x_3=-1$,

$2x_1-x_2+2x_3=-3$, $3x_1+x_2-2x_3=4$.

**Ans.:** No solution, system inconsistent.

**Q.No.5.:** Test for consistency and solve:

$2x_1+2x_2+x_3=3$, $2x_1+x_2+x_3=0$, $6x_1+2x_2+4x_3=6$.

**Ans.:** No solution, system inconsistent.

**Q.No.6.:** Test for consistency and solve:

$7x+16y-7z=4,\quad 2x+5y-3z=-3,\quad x+y+2z=4.$

**Ans.:** No solution, system inconsistent.

**Q.No.7.:** Test for consistency and solve:

$x+y+z=4,\quad 2x+5y-2z=3,\quad x+7y-7z=5.$

**Ans.:** No solution, system inconsistent.

**Q.No.8.:** Test for consistency and solve:

$-x_1+x_2+2x_3=2,\ 3x_1-x_2+x_3=6,\ -x_1+3x_2+4x_3=4.$

**Ans.:** $x_1=1,\quad x_2=-1,\quad x_3=2$**,** Unique solution.

**Q.No.9.:** Test for consistency and solve:

$2x+y-z=0,\ 2x+5y+7z=52,\quad x+y+z=9.$

**Ans.:** Unique solution, x = 1, y = 3, z = 5.

**Q.No.10.:** Test for consistency and solve:

$x+y+z=6,\ 2x-3y+4z=8,\ x-y+2z=5.$

**Ans.:** $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

**Q.No.11.:**Show that the equations $x+2y-z=3$**,** $3x-y+2z=1$**,** $2x-2y+3z=2$**,**

$x-y+z=-1$ are consistent and solve them.

**Ans.:** $x=-1$**,** y = 4, z = 4.

**Q.No.12.:** Solve the following systems of equations by matrix method:

(i). $x+y+z=8,\ x-y+2z=6, 3x+5y-7z=14$

(ii). $x+y+z=6,\ x-y+2z=5,\ 3x+y+z=8$

(iii). $x+2y+3z=1,\ 2x+3y+2z=2,\ 3x+3y+4z=1.$

**Ans.:** (i). x = 5, $y=\frac{5}{3}$, $z=\frac{4}{3}$ (ii). x = 1, y = 2, z = 3 (iii) $x=-\frac{3}{7}$, $y=\frac{8}{7}$, $=-\frac{2}{7}$.

**Q.No.13.:**For what values of a and b do the equations $x+2y+3z=6$**,** $x+3y+5z=9$**,**

$2x+5y+az=b$ have (i) have a no solution (ii) a unique solution (iii) more than one solution.

**Ans.:** (i). a = 8, $b\neq 15$ (ii). $a\neq 8$, b may have any value (ii) a = 8, b =15.

**Q.No.14.:** Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solution for $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+az=b$.

**Ans.: (i).** No solution of a = 5, $b \neq 9$.

**(ii)** Unique solution $a \neq 5$, b any value.

**(iii)** Infinitely many solutions a = 5, b = 9.

**Q.No.15.:**Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solution for $x+y+z=6$, $x+2y+3z=10$, $x+2y+az=b$.

**Ans.: (i).** a = 3, $b \neq 10$ inconsistent

**(ii)** Unique solution $a \neq 3$, b any value.

**(iii)** Infinitely many solutions a = 3, b = 10.

**Q.No.16.:**Test for consistency $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$, where a, b, c are constants.

**Ans.: (i)** if $a+b+c \neq 0$, inconsistent.

**(ii).** $a+b+c=0$, infinite solution.

**Q.No.17.:**Find the value of k so that the equations $x+y+3z=0$, $4x+3y+kz=0$, $2x+y+2z=0$ have a non-trivial solution.

**Ans.:** k = 8.

**Q.No.18.:** Show that if $\lambda \neq -5$, the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$, have a unique solution. If $\lambda=-5$, show that the equations are consistent. Determine the solutions in each case.

**Ans.:** $\lambda \neq -5$, $x=\frac{4}{7}$, $y=-\frac{9}{7}$, $z=0$: $\lambda=-5$, $x=\frac{1}{7}(4-5)$, $y=\frac{1}{7}(13k-9)$, z = k for all k.

**Q.No.19.:**Solve using $A^{-1}$(inverse of the coefficient matrix):

$2x_1+x_2+5x_3+x_4=5$, $x_1+x_2-3x_3-4x_4=-1$, $3x_1+6x_2-2x_3+x_4=8$, $2x_1+2x_2+2x_3-3x_4=2$.

**Ans.:** $x_1 = 2,\ \ x_2 = \frac{1}{5},\ \ x_3 = 0,\ x_4 = \frac{4}{5}$, unique solution.

**Q.No.20.:** Write the following equations in matrix form AX = B and solve for X by finding $A^{-1}$:

(i). $2x - 2y + z = 1,\ \ x + 2y + 2z = 2,\ \ 2x + y - 2z = 7$

(ii). $2x_1 - x_2 + x_3 = 4,\ \ x_1 + x_2 + x_3 = 1,\ \ x_1 - 3x_2 - 2x_3 = 2$.

**Ans.:** (i). $x = 2,\ y = 1,\ z = -1$ (ii). $x_1 = 1,\ x_2 = -1,\ x_3 = 1$.

## Frequently asked questions and their replies:

**Q.: What are the rank conditions for consistency of a linear algebraic system?**

**Ans.:**Well, what is your definition of "rank"? The definition I would use is that the rank of a matrix is the number of non-zero rows left after you row-reduce the matrix. Obviously, that idea applies to non-square matrices. In fact, if you append a new column to a square matrix, to form the "augmented matrix", any non-zero row, after row-reduction, for the square matrix will still be non-zero for the augmented matrix- add values on the end can't destroy non-zero values already there. The only way the rank could be changed is if you have non-zero values in the new column on a row that is all zeroes except for that, so that the augmented rank has greater rank than the original matrix. That tells you that one of your matrices has reduced to 0x+ 0y+ 0z+ ...= a where a is non-zero and that is impossible. If there is no such case, you have at least one solution to each equation. Yes, the system is consistent if and only if the rank of the coefficient matrix is the same as the rank of the augmented matrix.

**Q.: Whatdo you mean by row reduction, please elaborate?**

**Ans.:**Row reduction refers to the algorithmic procedure of Gaussian elimination. There are three row-reduction techniques:

1. Swapping rows
2. Multiplying a row by a constant.
3. Adding a multiple of a row to another.

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