Q.1) Define the following matrices with examples.

1. Transpose of a matrix and their properties

Sol.) Transpose is an operation of matrix which flips a matrix over its diagonal i.e. it interchanges rows and columns. It is denoted by A^T (also by A'). $\Rightarrow$ if $A = [a_{ij}]_{m \times n} \Rightarrow A^T = [a_{ji}]_{m \times n}$

Properties:

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(AB)^T = B^T A^T$

④ $\det|A^T| = \det|A|$

⑤ $(KA)' = KA'$

Eg:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 9 & 8 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 3 & 8 \\ 3 & 5 & 5 \end{bmatrix}$$

2. Conjugate of a matrix and their properties.

Conjugate of a matrix is an operation which replaces each element by its complex conjugate. It is denoted by $\bar{A}$

i.e. if $A = [a_{ij}]_{m \times n} \Rightarrow \bar{A} = [\bar{a}_{ij}]_{m \times n}$.

Properties:-

① $(\bar{\bar{A}}) = A$.

② $(\overline{A+B}) = \bar{A} + \bar{B}$

③ $(\overline{KA}) = K(\bar{A})$

④ $(\overline{AB}) = \bar{A}\bar{B}$

$$A = \begin{bmatrix} 1+2i & 3 & 4i \\ 1+i & 2+i & 3+i \\ 4 & 5 & i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 1-2i & 3 & -4i \\ 1-i & 2-i & 3-i \\ 4 & 5 & -i \end{bmatrix}$$

(3) Transposed conjugate of a matrix and their properties.

It is an operation of matrices which changes a matrix by conjugate of the transpose of the matrix or transpose of the conjugate of the matrix. i.e $[a_{ij}] = [\bar{a}_{ji}]$

It is denoted by A^{θ} or by A^{*}.

Properties:

(1) $(A^{\theta})^{\theta} = A.$

(2) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

(3) $(KA)^{\theta} = \bar{K} A^{\theta}$

(4) $(AB)^{\theta} = B^{\theta} . A^{\theta}$

Eg.

$$A = \begin{bmatrix} 1+i & i & 2+i \\ 4i & 5i & 3+i \\ 1+2i & 1+5i & 2i \end{bmatrix} \Rightarrow A^{\theta} = \begin{bmatrix} 1-i & -4i & 1-2i \\ -i & -5i & 1-5i \\ 2-i & 3-i & -2i \end{bmatrix}$$

(4) Symmetric, skew symmetric matrix and their properties.

(i) <u>Symmetric</u>: A square matrix which is equal to its transpose.
i.e If $a_{ij} = a_{ji}$ for all i, j, then a square matrix is called symmetric.

Eg

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$ is a symmetric matrix.

(ii) Skew-symmetric: A square matrix whose transpose is its negative.
i.e. If $a_{ij} = -a_{ji}$ for all i, j, then the square matrix is called skew symmetric matrix.

eg $A = \begin{bmatrix} 0 & i & 2 \\ -i & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -i & -2 \\ i & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$

as $A^T = -A$

$\Rightarrow A$ is a skew symmetrix matrix

Properties of symmetric and skew symmetric matrix:

① If A is a symmetric / skew symmetric, then KA is also symmetric / skew symmetric.

i.e $(KA)' = KA' = KA$ for symmetric.

$(KA)' = K(-A)$ for skew symmetric

② If A, B are symmetric (skew-symmetric), Then so is $A+B$.

$(A+B)' = A+B$ for symmetric

$(A+B)' = -(A+B) = -A-B$ for skew symmetric.

③ If $AB = BA$ & A, B are symmetric matrices, then AB will be a symmetric matrix.

④ For any square matrix A, $A-A'$ is a skew symmetric and $A+A'$ is a symmetric matrix.

⑤ If A is a symmetric or unsymmetric matrix, then $B'AB$ will be symmetric or unsymmetric respectively.

⑥ Every matrix can be represented as a sum of a symmetric & unsymmetric matrix.

⇒ Hermition & skew Hermitian matrices and their properties.

⇒ (i) <u>Hermition matrix:</u> A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i,j)^{th}$ element of A is equal to the conjugate complex of $(j,i)^{th}$ element of A,

i.e $a_{ij} = \overline{a_{ji}}$ for all i, j.

eg.
$$\begin{bmatrix} 1 & 2+3i & -4i \\ 2-3i & 2 & 2 \\ 4i & 2 & 3 \end{bmatrix}$$

(ii) <u>Skew Hermition matrix:</u> A square matrix $A = [a_{ij}]$ is said to be skew hermition if $(i,j)^{th}$ element of A is equal to the negative of the conjugate complex of $(j,i)^{th}$ element of A, i.e $a_{ij} = -\overline{a_{ji}}$ for all i, j.

e.g.
$$\begin{bmatrix} 0 & 2+3i & -4i \\ 2-3i & 0 & 2 \\ 4i & -2 & -8i \end{bmatrix}$$

<u>Properties of Hermition and skew Hermition matrix:-</u>

① If A is a Hermition matrix, then iA is skew Hermitian matrix.

② If A is a Skew Hermition matrix, then A is a Hermition matrix

③ If A, B are Hermition / skew Hermitian then so is A + B.

i.e
$$(A+B)^\theta = A^\theta + B^\theta = A + B \qquad \text{Hermition}$$
$$(A+B)^\theta = A^\theta + B^\theta = -A - B \qquad \text{Skew Hermition.}$$

④ If A, B are Hermition then AB + BA is hermition & AB - BA is skew Hermition.

5) If A be any square matrix, then $A+A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A-A^\theta$ is skew Hermitian.

6) $B^\theta AB$ is hermitian / skew hermitian accordingly as A is Hermitian or skew Hermitian.

7) Every square matrix can be represented as the sum of a Hermitian matrix and a skew Hermitian matrix.

i.e for A being a square matrix

let $P = A+A^\theta$, $Q = A-A^\theta$

$\Rightarrow A = \frac{1}{2}[P+Q]$.

8) Every real symmetric matrix is Hermitian.

9) $\bar{A}$ is hermitian / skew hermitian accordingly as A is Hermitian or skew-Hermitian.

10) Every square matrix A can be expressed as $P+iQ$ where P & Q are Hermitian matrices.

for a sq. matrix A, $P = \frac{1}{2}(A+A^\theta)$, $Q = \frac{1}{2i}(A-A^\theta)$.

$A = P + iQ$.

11) For every Hermitian matrix A can be written as $A = B+iC$, where B is real and symmetric & C is real and skew symmetric.

for sq. matrix A

$B = \frac{1}{2}(A+\bar{A})$ $C = \frac{1}{2i}(A-\bar{A})$.

$A = B+iC$.

⑥ Normal matrix, Orthogonal (orthonormal) matrix and Unitary matrix.

(i) Normal matrix : A square matrix A is said to be normal matrix if $AA^\theta = A^\theta A$.

eg.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow AA^* = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = A^* A$$

Properties:

1) Among complex matrices, all unitary, Hermitian and skew-Hermitian matrices are normal.

2) Among real matrices, all orthogonal, symmetric, skew-symmetric matrices are normal.

3) If 2 normal matrices are commute then their sum or product is also a normal matrix.

(ii) ~~Orthogonal~~ Unitary matrix : A square matrix is said to be unitary matrix $AA^\theta = I = A^\theta A$.

eg
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad AA^\theta = I = A^\theta A.$$

Properties:

① Inverse of a unitary matrix is unitary. matrix.
② Transpose of a unitary matrix is unitary matrix.
③ Product of two unitary matrices is a unitary matrix.

(iii) Orthogonal matrix : An orthogonal matrix is a square matrix with real entries whose columns (or rows) are orthogonal unit vectors (i.e orthogonal). Because the columns are unit vectors in addition to being orthogonal, some use orthonormal to describe such matrices.

Or.

for a square matrix A if $AA' = I = A'A$

eg.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ is

$$AA' = I = A'A.$$

Properties :

1. Inverse of an orthogonal matrix is orthgonal.
2. Transpose of an orthogonal matrix is orthogonal.
3. Product of two Orthogonal matrix is an orthogonal matrix.

Q.2) Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.

Sol) let $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

A will be a Hermitian matrix if $A = \bar{A}'$

$$\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

as $(\bar{A})' = A$

$\Rightarrow$ A is a Hermitian matrix

Q3) Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into LU, where L is lower triangular matrix and U is upper triangular matrix.

Sol) let $L = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$ $U = \begin{bmatrix} m & n & \theta \\ 0 & p & q \\ 0 & 0 & \gamma \end{bmatrix}$

$LU = A \Rightarrow \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \begin{bmatrix} m & n & \theta \\ 0 & p & q \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$

$$\begin{bmatrix} am & an & a\theta \\ dm & dn+bp & d\theta+bq \\ em & en+fp & e\theta+fq+c\gamma \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$am = 5$ $\quad an = -2$ $\quad a\theta = 1$

$dm = 7$ $\quad dn + bp = 1$ $\quad d\theta + bq = -5$

$em = 3$ $\quad en + fp = 7$ $\quad e\theta + fq + c\gamma = 4.$

we have 12 variable & 9 eq^n

$\Rightarrow$ we have infinite solutions (We can choose 3 unknown arbitary and find others).

let $a = b = c = 1.$

$\Rightarrow$ $m = 5$, $n = -2$, $\theta = 1.$, $d = \frac{7}{5}$

$e = \frac{3}{5}$, $\frac{7}{5}(-2) + p = 1$, $\frac{7}{5}(1) + 1(q) = -5.$

$p = \frac{19}{5}$ $\quad \Rightarrow q = \frac{-32}{5}$

$\frac{3}{5}(-2) + f\left(\frac{19}{5}\right) = 7$ $\quad \frac{3}{5}(1) + \frac{41}{19}\cdot\left(\frac{-32}{5}\right) + \gamma = 4.$

$\Rightarrow f = \frac{41}{19}$, $\quad \Rightarrow \gamma = \frac{327}{19}$

Q.4 Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix

Sol.4) $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$ $\quad |I+A| = 1 - (-1-4) = 6.$

$(I+A)$ is invertible.

$\Rightarrow (I+A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$

$I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}.$

$\Rightarrow (I-A)(I+A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 1-1-4 & -2(1+2i) \\ 2(1-2i) & -1-4+1 \end{bmatrix}$

$(I-A)(I+A)^{-1} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}. \quad -(1)$

Its conjugate transpose. $= \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} \quad -(2)$

$(I-A)(I+A)^{-1}$ is unitary
if Product of (1) and (2) is I.

Product of (1) and (2) $= \frac{1}{36}\begin{bmatrix} -4 & -(2+4i) \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$= \frac{1}{36}\begin{bmatrix} 16+4+16 & -4(2+4i)+4(2+4i) \\ -4(2-4i)+4(2-4i) & 4+16+16 \end{bmatrix}$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I.$$

$\Rightarrow (I-A)(I+A)^{-1}$ is a unitary matrix.

Q.5) Determine the rank of the following matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Sol:5) let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$ $\quad$ $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

minors : $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}$ $\quad \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}$

$|2-0| = 2$ $\quad |-2-6| = 8$ $\quad = |-1-3| = 4$

as none of minors $= 0$.

$\Rightarrow$ rank of given matrix $= 2$.

Q.6) Use Gauss-Jordan method to find inverse of the following matrices:

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Sol.6) (i) Given matrix is $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$.

writing the same matrix side by side with the unit matrix of order 3,

$$\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to 2R_3 - 5R_1$, $\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -5 & 0 & 2 \end{array}\right]$

$R_3 \to 2R_3 - R_2$, $\left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array}\right]$.

$R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$ we get $\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 11 & -1 & -4 \\ 0 & 2 & 0 & -10 & 2 & 4 \\ 0 & 0 & -1 & -10 & 1 & 4 \end{array}\right]$

$R_1 \to 2R_1 - R_2$, $R_2 \to \frac{1}{2}R_2$, $R_3 \to (-1)R_3$

$$\left[\begin{array}{ccc:ccc} 4 & 0 & 0 & 32 & -4 & -12 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$R_1 \to \frac{1}{4}R_1$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$ $\Rightarrow$ inverse is $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

(ii) Given matrix $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Writing the same matrix side by side with the unit matrix of order 3.

$$\left[\begin{array}{ccc:ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$, $R_2 \to 4R_2 - R_1$ $\left[\begin{array}{ccc:ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 3 & 1 & 0 & -1 & 2 \end{array}\right]$

$R_2 \to R_2 - R_3$ $\left[\begin{array}{ccc:ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 5 & -2 \\ 0 & 3 & 1 & 0 & -1 & 2 \end{array}\right]$

$R_3 \to R_2 + R_3$ $\left[\begin{array}{ccc:ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 5 & -2 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right]$ $R_2 \to \left(\frac{-1}{3}\right) R_2$ $\left[\begin{array}{ccc:ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right]$

$R_1 \to R_1 - 4R_2 - 3R_3$

$\left[\begin{array}{ccc:ccc} 8 & 0 & 0 & 8/3 & -16/3 & -8/3 \\ 0 & 1 & 0 & 1/3 & -5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right]$ $R_1 \to \frac{R_1}{8}$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & -5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right]$

$$\text{Inverse} = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{-1}{3} \\ \frac{1}{3} & -5/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$$

Q.7) If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1}. Also find two non-singular matrices P and Q such that $PAQ = I$, where I is the unit matrix and verify that $A^{-1} = QP$.

Sol)

$|A| = 3(-3+4) - 2(-3+4) = 3-2 = 1.$

$$\text{Adj}\,|A| = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}.$$

$$A^{-1} = \frac{\text{Adj}\,|A|}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Now

$A = PAQ$, P&Q are 2 singular unit matrix of some order as of A.

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + C_3$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 4C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}.$$

$$QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$= A^{-1}$$

$$\Rightarrow QP = A^{-1}$$

Hence proved.

Q-8) Are the vectors $x_1 = (1,3,4,2)$, $x_2 = (3,-5,2,2)$ and $x_3 = (2,-1,3,2)$ linearly dependent? If so express one of these as a linear combination of the others.

Sol 8 Since we know that, that vectors x_1, x_2, x_3 are said to be linearly dependent, if $\exists$ numbers $\lambda_1, \lambda_2, \lambda_3$ not all zero s.t. $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

$$\lambda_1(1,3,4,2) + \lambda_2(3,-5,2,2) + \lambda_3(2,-1,3,2) = 0.$$

$$\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0 \quad , \quad 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0.$$

$$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \quad , \quad 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0.$$

Q-

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$

$2\lambda_2 + \lambda_3 = 0$

$\lambda_3 = -2\lambda_2$ and $\lambda_1 = \lambda_2$

$\lambda_1 = \lambda_2 = \frac{-1}{2}\lambda_3$

$\Rightarrow$ set of values which satisfies condition are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -2$. which are not zero.

$\Rightarrow$ the given vectors are linearly dependent.

$\Rightarrow$ Relation $x_1 + x_2 - 2x_3 = 0$.

$\Rightarrow$ linear combination $\Rightarrow$ $x_3 = \dfrac{x_1 + x_2}{2}$

Q.9) Show that the system of equations $AX = B$ is consistent i.e possesses a solution iff the coefficient matrix A and the augmented matrix $K = (A:B)$ are of the same rank. Otherwise, the system is inconsistent.

1. a) Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= K_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= K_2 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_m &= K_m \end{aligned}\right\} \quad (i)$$

be a system of m non-homogeneous equations in n-unknowns $x_1, x_2 \cdots\cdots x_n$.

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}_{n\times 1}, \quad B = \begin{bmatrix} K_1 \\ K_2 \\ \cdots \\ K_m \end{bmatrix}_{m\times 1}$$

Case I rank of A = rank of K = r ($r \le$ smaller of m, n)

Then by suitable row operations, the system of equation $AX = B$ can be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots\cdots + b_{1n}x_n &= l_1 \\ 0x_1 + b_{22}x_2 + \cdots\cdots + b_{2n}x_n &= l_2 \\ \cdots\cdots\cdots\cdots \\ 0x_1 + 0x_2 + \cdots\cdots + b_{rn}x_n &= l_r \end{aligned}\right\} - (ii)$$

and the remaining $(m-r)$ eq^n being all of the form.

$0.x_1 + 0x_2 + \cdots\cdots + 0.x_n = 0$

The eq^n's (ii) will have a solution, by choosing $(n-r)$ unknowns arbitrary.

The solution will be unique only when $r = n$.

Hence the eq^{n}s (i) are consistent i.e possesses solution.

Case II When rank of A (i.e r) < the rank of K.

In Particular, let the rank of K be $r+1$.

Then, by suitable row operations, the system of eq^{n}s, $AX=B$ can reduced to

$$\left.\begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots\dots + b_{1n}x_n = l_1 \\ 0x_1 + b_{22}x_2 + \dots\dots + b_{2n}x_n = l_2 \\ \dots\dots\dots\dots\dots \\ 0x_1 + 0x_2 + \dots\dots + b_{rn}x_n = l_r \\ 0x_1 + 0x_2 + \dots\dots + 0x_n = l_{r+1} \end{array}\right\}$$

and the remaining $(m-(r+1))$ eq^{n}s being all of the form

$0.x_1 + 0.x_2 + \dots\dots + 0.x_n = 0$.

The $(r+1)^{th}$ eq^n can not be satisfied by any set of values for the unknowns.

Hence, the equations (i) are inconsistent, i.e., does not possess a solution.

Q.11) Investigate the values of K the eq^{n}s $x+y+z=1$, $2x+y+4z=K$, $4x+y+10z=K^2$ have a solution and solve them completely in each case.

Sol.11)

Here the matrix form of the given system of eq^{n}s is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K \\ K^2 \end{bmatrix}$$

operating $R_2 \to R_2 - 2R_1$, $R_3 \to -4R_1 + R_3$ we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K-2 \\ K^2-4 \end{bmatrix}$

$R_3 \to \frac{R_3}{3}$ we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K-2 \\ \frac{K^2-4}{3} \end{bmatrix}$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ \frac{k^2-4}{3} - k + 2 \end{bmatrix}$$

$x+y+z=1$, $-y+2z = k-2$ and $0 = \frac{k^2-4}{3} - k+2$.

This is only possible i.e have solution if $\left(\frac{k^2-4}{3}\right) - k+2 = 0$.

$\Rightarrow k^2 - 3k + 2 = 0 \Rightarrow k = 2, 1.$

Case I when $k=2$.

$x+y+z=1.$

$y = 2z.$

if $z=c$, $y=2z \Rightarrow y = 2c$

and $x = 1-3c$.

At $k=2$, $x = 1-3c$, $y = 2c$, $z = 3$.

Case II when $k=1$, then $-y+2c = -1 \Rightarrow y = 1+2c$.

$x = 1-1-2c-c = -3c$

At $k=1$ $x = -3c$, $y = 1+2c$, $z = c$.

~~which is~~

Q10) Investigate the values of λ and μ so that eq^ns.

$2x+3y+5z = 9$, $7x+3y-2z = 8$, $2x+3y+\lambda z = \mu$ have

(i) no solution.

(ii) Unique solution

(iii) an infinite number of solutions

Sol.10) The given set of eq^{n}s can be written as.

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$AX = B \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

The augmented matrix $K = [A:B] \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$

operating $R_3 \to R_3 - R_1$, $R_2 \to 2R_2 - 7R_1$

$$K \sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

(i) If $\lambda \neq 5$, we have rank of $K = 3 =$ rank of A

$\Rightarrow$ The given system of eq^{n}s is ~~again~~ consistent.

Also rank of $A =$ the no. of unknown.

$\Rightarrow$ The given system of eq^{n}s posses a unique solution.

Thus $\lambda \neq 5$, the eq^{n}s possesses a unique solution for any value of μ.

(ii) If $\lambda = 5$, and $\mu = 9$ we have rank $K =$ rank A.

$\Rightarrow$ The given system of equations is again consistent.

Also the rank of $A <$ the numbers of unknowns.

$\Rightarrow$ The given system of eq^{n}s ~~is inconsistent and~~ possesses ~~no solution~~ infinite number of solutions.

(iii) If $\lambda = 5$ and $\mu \neq 9$, we have rank of $K = 3$, rank of $A = 2$,

$\Rightarrow$ rank $K \neq$ rank A.

$\Rightarrow$ The given system of eq^{n}s is inconsistent and possesses ~~an infinite number of~~ no solutions.

12) show that the eq^{n}s $3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$. do not have a solution unless $a+c=2b$.

Sol)

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$AX = B \Rightarrow R_2 \to 3R_2 - 4R_1$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ c \end{bmatrix}$$

$R_3 \to 3R_3 - 5R_1$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ 3c-5a \end{bmatrix}$$

$R_3 \to \frac{1}{2}R_3 - R_2$

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b-4a \\ \frac{3c-6b+3a}{2} \end{bmatrix}$$

If $\frac{3c-6b+3a}{2} \neq 0$ then eq^{n}s are inconsistent.

If $\rho(A) = \rho(K)$ then equations are consistent. This is possible only when $\frac{3c-6b+3a}{2} = 0 \Rightarrow 3c-6b+3a = 0$.

$\Rightarrow a+c = 2b$.

The given equations do not have a solution unless $a+c=2b$.

Q.13) Define characteristic matrix, characteristic polynomial, characteristic equation, characteristic roots, characteristic vector.

Sol) Characteristic matrix : Let $A = [a_{ij}]_{n \times n}$ be any square matrix of order n and λ be scalar, Then the matrix is called characteristic matrix, where I is the unit matrix of order n.

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} - \lambda \end{bmatrix}$$

Characteristic Polynomial: The determinant of characteristic matrix is called the characteristic polynomial.

Characteristic Equation :- The equation $|A - \lambda I| = 0$ is called characteristic equation.

Characteristic Roots :- The roots of characteristic equation i.e roots of $|A - \lambda I| = 0$ are called the characteristic roots.

Characteristic vectors: If λ is a characteristic root of an $n \times n$ matrix A, then a non zero vector X (i.e $X \neq 0$), s.t. $AX = \lambda X$, is called a characteristic vector of A corresponding to characteristic root λ.

functions of a_{ij}'s [the elements of A].

Q.14) If λ be an eigen value of a non-singular matrix A. Show that $\frac{|A|}{\lambda}$ is an eigen value of matrix adj. A.

Sol.14) Proof: Since λ be an eigen value of a non-singular matrix $A \Rightarrow \lambda \neq 0$. Also λ is an eigen value of A then $\exists$ a non zero vector X s.t. $AX = \lambda X$. Pre-multiplying both sides by A, we get.

$$(\operatorname{Adj} A)(AX) = (\operatorname{Adj} A)(\lambda X) \Rightarrow [(\operatorname{Adj} A)A]X = \lambda[(\operatorname{Adj} A)X].$$

$$|A| \cdot AX = \lambda(\operatorname{adj} A)X \qquad \left[\because A^{-1} = \frac{\operatorname{Adj} A}{|A|} \Rightarrow \operatorname{adj} A \cdot A = |A| I\right]$$

$$|A| X = \lambda(\operatorname{Adj} A)X \Rightarrow \frac{|A| X}{\lambda} = (\operatorname{Adj} A)X \qquad [\because \lambda \neq 0]$$

$$(\operatorname{Adj} A)X = \frac{|A|}{\lambda} X$$

Since X is a non zero vector, therefore $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj$A$.

Q.15) Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol.15) Characteristic equations is $|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - 7\lambda^2 + 36 = 0.$$

Since $\lambda = -2$ satisfies it, we can write this equation as

$$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda_1 = -2, \quad \lambda_2 = 3, \quad \lambda_3 = 6.$$

Therefore eigen values of A are $\lambda = -2, 3, 6$.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have.

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \qquad (i)$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence, the eigen vectors are $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i)

Hence, the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Q.16) Show that any eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Sol.16) Proof: Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1 and λ_2 of a unitary matrix A. Then by definition.

$$AX_1 = \lambda_1 X_1 \quad , \quad AX_2 = \lambda_2 X_2 .$$

Since A is unitary matrix, then the eigen values have the absolute value 1.

i.e $\therefore |\lambda_1| = 1 \Rightarrow |\lambda_1|^2 = 1 \Rightarrow \lambda_1 \bar{\lambda}_1 = 1 \Rightarrow \bar{\lambda}_1 = \frac{1}{\lambda_1}$

$|\lambda_2| = 1 \Rightarrow |\lambda_2|^2 = 1 \Rightarrow \lambda_2 \bar{\lambda}_2 = 1 \Rightarrow \bar{\lambda}_2 = \frac{1}{\lambda_2}$

Also $AA^\theta = I$

To show: X_1 and X_2 are orthogonal vectors i.e, $X_2^\theta X_1 = 0$.

Taking conjugate transpose of (ii), we get.

$$(AX_2)^\theta = (\lambda_2 X_2)^\theta = X_2^\theta A^\theta = \bar{\lambda}_2 X_2^\theta \qquad \text{—(iii)}$$

From (i) and (iii) we get.

$$(X_2^\theta A^\theta)(AX_1) = (\bar{\lambda}_2 X_2^\theta)(\lambda_1 X_1)$$

$$X_2^\theta (A^\theta A) X_1 = \bar{\lambda}_2 \lambda_1 X_2^\theta X_1$$

$$X_2^\theta X_1 = \bar{\lambda}_2 \lambda_1 X_2^\theta X_1 \qquad [A^\theta A = I]$$

$$X_2^\theta X_1 [\bar{\lambda}_2 \lambda_1 - 1] = 0.$$

also $\bar{\lambda}_2 = \frac{1}{\lambda_2}$

$$\Rightarrow X_2^\theta X_1 \left[\frac{\lambda_1}{\lambda_2} - 1\right] = 0.$$

$$X_2^\theta X_1 \left[\frac{\lambda_1 - \lambda_2}{\lambda_2}\right] = 0 \qquad \text{as } \lambda_1 \neq \lambda_2$$

$$\Rightarrow X_2^\theta X_1 = 0.$$

Hence X_1 and X_2 are orthogonal vectors

Q.17) State and Prove Cayley Hamilton Theorem.

Sol.17) Cayley Hamilton Theorem:- Every square matrix over the real or complex field satisfies its own characteristic equation.

i.e. if the characteristic equation for the n^{th} order square matrix A is $|A-\lambda I| = 0$.

$$\Rightarrow (-1)^n \lambda^n + [(-1)^{n-1} b_1]\lambda^{n-1} + [(-1)^{n-2} b_2]\lambda^{n-2} + \dots + [(-1)^{n-n} b_n] = 0$$

$$\Rightarrow (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n] = 0.$$

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n = 0.$$

Then $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$.

Proof:- As we know, matrix $A-\lambda I$ is a characteristic matrix of A. This matrix can be written as

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

This matrix shows that the elements of $A-\lambda I$ are at most of the 1^{st} Degree in λ.

$\therefore$ Elements of $Adj(A-\lambda I)$ are ordinary polynomials in λ of degree $(n-1)$ or less.

$$\begin{bmatrix} \lambda^3+2\lambda+1 & 4\lambda^3+5\lambda+3 & 2\lambda+9 \\ \lambda^3+5 & \lambda^3+2\lambda & 2\lambda^3+5\lambda+7 \\ 3\lambda^3+2\lambda+1 & \lambda^3+4 & \lambda^3+8\lambda+3 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\lambda^3 + \begin{bmatrix} \quad \end{bmatrix}\lambda^2 + \begin{bmatrix} \quad \end{bmatrix}\lambda + \begin{bmatrix} \quad \end{bmatrix}$$

Now $Adj(A-\lambda I)$ can be written as matrix polynomials in λ, and is given by.

$$Adj(A-\lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \dots + B_{n-2}\lambda + B_{n-1}$$

where $B_0, B_1, \dots, B_{n-1}$ are matrices of the type $n \times n$, whose elements are functions of a_{ij}'s [the elements of A].

Now since $A \, adj A = |A| I_n$.

Replacing A by $A-\lambda I$, we obtain.

$$(A-\lambda I) \, adj \, (A-\lambda I) = |A-\lambda I| I_n.$$

$$(A-\lambda I)\left[B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_{n-2}\lambda + B_{n-1}\right] = (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + \dots + a_n\right]I_0$$

Comparing coefficients.

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I.$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$AB_{n-1} = (-1)^n a_n I$$

Pre multiplying by $A^n, A^{n-1}, \dots A, I$ and adding successively, we get.

$$0 = (-1)^n\left[A^n + a_1 A^{n-1} + \dots + a_n I\right].$$

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0.$$

i.e Every square matrix satisfies its own characteristic equation.

Q.18) Find the characteristic roots of matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton for this matrix. Find the inverse of matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

Sol.18) Characteristic eq^n of matrix A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0.$$

$$\lambda^2 - 4\lambda - 5 = 0.$$

$$(\lambda-5)(\lambda+1) = 0 \qquad -①$$

roots of eq^n are $\lambda = 5, -1$. and these are the characteristic roots of A.

By Cayley Hamilton method, the matrix A must satisfy its characteristic roots of A. $\Rightarrow A^2 - 4A - 5I = 0$ -(ii).

Verification:-

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Inverse of A.

Multiplying (ii) by A^{-1}, we get.

$$A^2 A^{-1} - 4A.A^{-1} - 5A^{-1} = 0.$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}.$$

Express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A:

Now (ii) $\Rightarrow A^2 = 4A + 5I$.

Multiplying by A^3, we get $A^5 = 4A^4 + 5A^3$

Multiplying by A^2, we get $A^4 = 4A^3 + 5A^2$.

Multiplying by A, we get $A^3 = 4A^2 + 5A$.

Now $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= (4A^4 + 5A^3) - 4A^4 \cancel{+(4A^3+5A^2)} - 7(4A^2 + 5A) + 11A^2 - A - 10I.$

$= 5A^3 - 17A^2 - 36A - 10I = 5(4A^2 + 5A) - 17A^2 - 36A - 10I$

$= 3A^2 - 11A - 10I = 3A^2 - 12A + A - 15I = 3(A^2 - 4A - 5I) + A + 5I$

$= A + 5I$ which is linear polynomial in A.

Q.19) If a square matrix A of order n has n linearly independent eigen vectors, then show that a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Sol.19) Proof: Let A be a square matrix of order 3.

Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values.

and $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda X_1, \lambda X_2, \lambda X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD.$$

where D is the diagonal matrix.

$$P^{-1}AP = P^{-1}PD = D.$$

which proves the theorem.

Q.20) Reduce the quadratic form $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ into the "sum of squares"?

Sol.20) The matrix form of given quadratic form is $X'AX$.

Where $X' = [x\ y\ z\ w]$ and $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

Let us reduce A to the diagonal matrix.

We know that $A = I_4 A I_4 \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$ we get.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 + 2C_1$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{2}{5} R_2$ we get.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 8/5 & 2/5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + \frac{2}{5} C_2$, we get.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 2 & 14/5 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 8/5 & 2/5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 8/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$R_4 \to R_4 + \frac{15}{14} C_2$, we get.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & \frac{-17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 8/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 + \frac{15}{14} C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & \frac{-17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 8/5 & 12/7 \\ 0 & 1 & 2/5 & 3/7 \\ 0 & 0 & 1 & 15/14 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e $\text{diag}\left(1, -5, \frac{14}{5}, \frac{-17}{14}\right) = P'AP$.

∴ The canonical form of the given quadratic form is.

$$Y'(P'AP)Y = Y'\text{diag}\left(1, -5, \frac{14}{5}, \frac{-17}{14}\right)Y = [y_1\ y_2\ y_3\ y_4]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & \frac{-17}{14} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5}y_3^2 - \frac{17}{14}y_4^2$$

which is sum of squares.

Q.21) Reduce $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into canonical form by orthogonal transformation.

Sol 21) The matrix of the qudratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic of A are given by $|A - \lambda I| = 0$.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

∴ $\lambda = 0, 3, 15$.

Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]X = 0$.

i.e $8x_1 - 6x_2 + 2x_3 = 0$.

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$.

Solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$ giving the eigen vector $X_1 = (1,2,2)$.

When $\lambda = 3$, the characteristic vector is given by $[A-(3)I]X = 0$.

i.e
$$5x_1 - 6x_2 + 2x_3 = 0.$$
$$-6x_1 + 4x_2 - 4x_3 = 0.$$
$$2x_1 - 4x_2 = 0.$$

On Solving $X_2 = (2,1,-2)$

Similarly, characteristic vector corresponding to $\lambda = 15$ is $X_3 = (2,-2,1)$

Now X_1, X_2, X_3 are pairwise orthogonal i.e.

$$X_1 . X_2 = X_2 . X_3 = X_3 . X_1 = 0.$$

$\therefore$ The normalized modal matrix is $B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

Now B is the orthogonal matrix and $|B| = 1$.

i.e. $B^{-1} = B^T$ and $B^{-1}AB = D = \text{diag}(0,3,15)$

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$X'AX = Y'(B^{-1}AB)Y = Y'DY = [y_1, y_2, y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 15y_3^2$$

which is required canonical form.