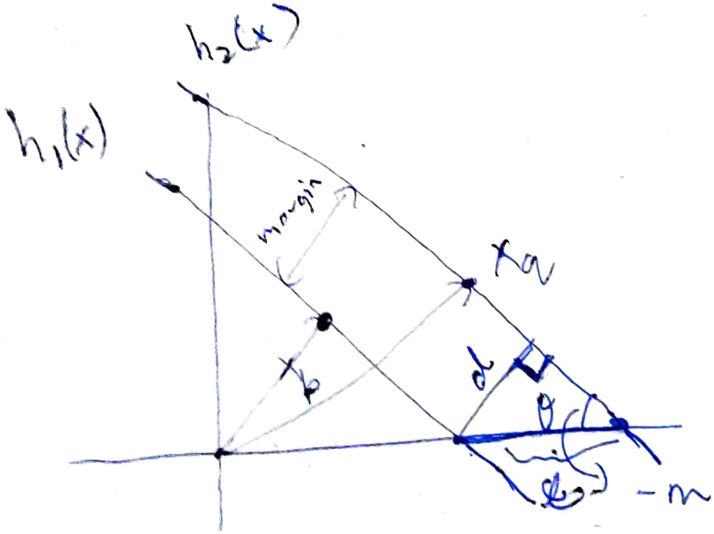
Case I (a). Support vectors x_p & x_q.

$x_p(x_{p1}, x_{p2}) \in$ Class 1 ; $x_q(x_{q1}, x_{q2}) \in$ Class 2

$h(x) : w0 + w1x1 + w2x2 = 0$

Expression of margin btw parallel lines passing through x_p & x_q



$$h_1(x_p) = (w0)_p + w1\, x_{p1} + w2\, x_{p2} = 0$$
$$\Rightarrow (w_0)_p = -w_1 x_{p1} - w_2 x_{p2}$$
$$h_2(x_q) = (w_0)_q + w_1 x_{q1} + w_2 x_{q2} = 0$$
$$\Rightarrow (w_0)_q = -w_1 x_{q1} - w_2 x_{q2}$$

Hence $h1(x) = w_1(x_1 - x_{p1}) + w_2(x_2 - x_{p2})$
$$h_2(x) = w_1(x_1 - x_{q1}) + w_2(x_2 - x_{q2})$$

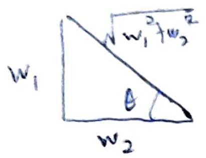
Hence intercept of $h_1(x)$ & $h_2(x)$ are

$\text{intercept}(h_1(x)) = \dfrac{+w_1 x_{p_1} + w_2 x_{p_2}}{w_2}$ & $\text{intercept}(h_2(x)) = \dfrac{w_1 x_{a_1} + w_2 x_{a_2}}{w_2}$

$$\therefore \ l = \left| \frac{w_1 x_{a_1} + w_2 x_{a_2} - w_1 x_{b_1} - w_2 x_{b_2}}{w_2} \right|$$

$$\tan\theta = -m = \frac{w_1}{w_2}$$

$$d = l \sin\theta = \left| \frac{w_1}{w_2}(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2}) \right| \sin\theta$$



$$\sin\theta = \frac{w_1}{\sqrt{w_1^2+w_2^2}}$$

$$\therefore \ d = \left| \frac{w_1}{w_2}(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2}) \right| \frac{w_1}{\sqrt{w_1^2+w_2^2}}$$

b) Best margin slope. ($\because$ Hard margin classification given in previous Question).

Problem is $\max d = \max\limits_{w_1, w_2} \left| \frac{w_1}{w_2}(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2}) \right| \frac{w_1}{\sqrt{w_1^2+w_2^2}}$

parameter are w_1 & w_2 & Support vectors x_1 & x_2 are fixed.

Also constraint.

$$y_i (w_0 + w_1 x_{1i} + w_2 x_{2i}) \geq 1 \quad \forall i \in \{1, 2, \ldots, m\}$$

~~Using Lagrange Multipliers.~~

$\max\limits_{w_1, w_2} d$ ~~is same as~~ $\min\limits_{w_1, w_2} \left(\frac{1}{2d}\right)$

$\min\limits_{w_1, w_2} l = \frac{1}{2d^2}$ ~~$= \frac{1}{2}\left(\frac{w_1}{w_2}(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2})\right)$~~

Using Lagrange Multipliers.

$$l = d - \sum_{i=1}^{m} \lambda_i (y_i (w_0 + w_1 x_{1,i} + w_2 x_{2,i}))$$

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$l = \cancel{\ldots}$

$$\left|\frac{w_1}{w_2}(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2})\right| \frac{w_1}{\sqrt{w_1^2 + w_2^2}} - \sum_{i=1}^{m} \lambda_i \left(y_i (w_0 + w_1 x_{1,i} + w_2 x_{2,i}) \right)$$

$\frac{\partial l}{\partial w_1}$

Assume ~~x_a~~ h_2 distance of $h_2(x)$ from origin $>$ distance of $h_1(x)$ from origin

This removes mod $|\ |$

$$\therefore l = \frac{w_1^2}{w_2\sqrt{w_1^2+w_2^2}}(x_{a_1} - x_{b_1}) + \frac{w_1(x_{a_2} - x_{b_2})}{\sqrt{w_1^2+w_2^2}} - \sum_{i=1}^{m} \lambda_i \left(y_i (w_0 + w_1 x_{1,i} + w_2 x_{2,i}) \right)$$

to differentiate substitute first part.

with $\tan\theta = \frac{w_1}{w_2}$ and apply chain rule

$$l = \left(\tan\theta (x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2})\right)\sin\theta - \sum_{i=1}^{m} \lambda_i \left(y_i (w_0 + w_1 x_{1,i} + w_2 x_{2,i}) \right)$$

$$\frac{\partial l}{\partial w_1} = \frac{\partial l}{\partial \theta}\frac{\partial \theta}{\partial w_1} = \sec^2\theta (x_{a_1} - x_{b_1}) \sin\theta \frac{\partial \theta}{\partial w_1} + \left(\tan\theta(x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2})\right)\cos\theta \frac{\partial \theta}{\partial w_1}$$

$$- \sum_{i=1}^{m} \lambda_i (y_i)(x_{1,i}) = 0 \qquad - (1)$$

Now $\frac{\partial \theta}{\partial w_1} = \frac{\partial}{\partial w_1}\tan^{-1}\left(\frac{w_1}{w_2}\right) = \frac{1}{1 + \left(\frac{w_1}{w_2}\right)^2} = \frac{w_2^2}{w_1^2 + w_2^2}$

$\frac{\partial \theta}{\partial w_2} = \frac{\partial}{\partial w_2}\cot^{-1}\left(\frac{w_2}{w_1}\right) = \frac{-1}{1 + \left(\frac{w_2}{w_1}\right)^2} = \frac{-w_1^2}{w_1^2 + w_2^2}$

Now substituting in (1)

$$\frac{\partial l}{\partial w_1} = \tan\theta\sec\theta(x_{a_1} - x_{b_1})\frac{w_2^2}{w_1^2 + w_2^2} + \left[\sin\theta(x_{a_1} - x_{b_1}) + \cos\theta(x_{a_2} - x_{b_2})\right]\frac{w_2^2}{w_1^2 + w_2^2}$$

$$- \sum_{i=1}^{m} \lambda_i (y_i)(x_{1,i}) = 0$$

Or $\frac{\partial l}{\partial w_1} = \frac{w_1}{w_2} \frac{\sqrt{w_1^2+w_2^2}}{w_2} (x_{a_1} - x_{b_1}) \frac{w_2^2}{w_1^2+w_2^2} +$

$$\left[\frac{w_1}{\sqrt{w_1^2+w_2^2}} (x_{a_1} - x_{b_1}) + \frac{w_2}{\sqrt{w_1^2+w_2^2}} (x_{a_2} - x_{b_2}) \right] \frac{w_2^2}{w_1^2+w_2^2}$$

$$- \sum_{i=1}^{n} \lambda_i (y_i)(x_{1,i}) = 0$$

$$\Rightarrow \frac{w_1}{\sqrt{w_1^2+w_2^2}} (x_{a_1} - x_{b_1}) + \frac{w_1 w_2^2}{\sqrt{w_1^2+w_2^2}} (x_{a_1} - x_{b_1}) + \frac{w_2^3}{(w_1^2+w_2^2)^{3/2}} (x_{a_2} - x_{b_2})$$

$$- \sum_{i=1}^{n} \lambda_i (y_i)(x_{1,i}) = 0 \quad - \text{(a)}$$

Now $\frac{\partial l}{\partial w_2} = \sec^2\theta (x_{a_1} - x_{b_1}) \sin\theta \frac{\partial \theta}{\partial w_2} + (\tan\theta (x_{a_1} - x_{b_1}) + (x_{a_2} - x_{b_2})) \cot \frac{\partial \theta}{\partial w_2}$.

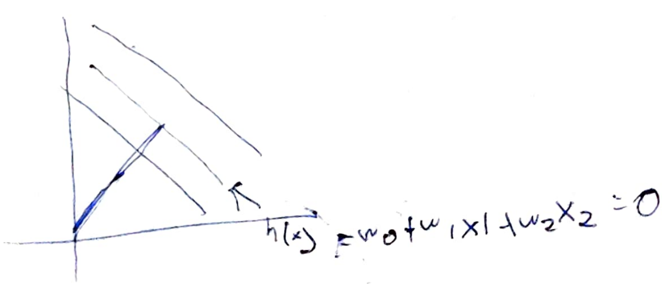
$$- \sum_{i=1}^{m} \lambda_i y_i w_2 x_{2,i} = 0$$

$\Rightarrow \frac{w_1}{w_2}$ ⋮

$$\frac{\partial l}{\partial \lambda} = 0$$

solve $\frac{\partial l}{\partial w_1}, \frac{\partial l}{\partial w_2}$ & $\frac{\partial l}{\partial \lambda}$ to get answer

Case 2:



$h(x) = w_0 + w_1 x_1 + w_2 x_2 = 0$

~~$w = (w_1, w_2)$~~

$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$x = [x_1, x_2]$

$b = w_0$.

$$w \cdot x + b = 0$$

Here distance of optimal hyperplane from origin is

$$\left| \frac{-b}{\|w\|} \right|$$

$H_1 = \frac{-1-b}{\|w\|}$

$H_2 = \frac{1-b}{\|w\|}$

~~The distance of~~

$M = \frac{1-b}{\|w\|} - \frac{(-1-b)}{\|w\|} = \frac{2}{\|w\|} = \frac{2}{\sqrt{w_1^2+w_2^2}}$

the margin $= \frac{M}{2} = \frac{1}{\sqrt{w_1^2+w_2^2}}$