AAV's Paper on Weak Measurement

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Von Neumann Measurement Model

Von Neumann idea was, one can see that it is possible in principle to correlate the state of a microscopic quantum system with the value of a macroscopic classical variable, and we may take it for granted that we can perceived the value of the classical variable.

Hamiltonian:

$$\hat{H} = \hat{H}_0 + \frac{1}{2m}\hat{P}^2 + \lambda \hat{A}\hat{q}$$

- $\hat{H}_0$: System unperturbed Hamiltonian.
- $\frac{\hat{P}^2}{2m}$: Free pointer Hamiltonian.
- $\hat{A}$: Observable operator.
- $\hat{q}$: Pointer position.

Interaction Hamiltonian

Note: If the measurement interaction time is short enough, the free evolution of both the system and the pointer can be considered approximately constant during the interaction, and hence can be neglected.

During the measurement :

$$\hat{H}_{\text{int}} = -\, g(t)\, \hat{A}\hat{q}, \qquad \text{where } g(t) \text{ is a normalized function.}$$

Initial and Final State

Initial state:

$$|\Psi(0)\rangle = |\psi_{in}\rangle \otimes |\phi\rangle$$

After interaction:

$$\begin{aligned}
|\Psi\rangle &= \hat{U}(t)\ |\Psi(0)\rangle \\
&= e^{i\hat{A}\hat{q}}\big(|\psi_{in}\rangle \otimes |\phi\rangle\big) \\
&= e^{i\hat{A}\hat{q}}\Big(\sum_i c_i\,|a_i\rangle \otimes |\phi\rangle\Big) \\
&= \sum_i e^{ia_i\hat{q}} c_i\,|a_i\rangle \otimes |\phi\rangle \\
&= \sum_i \Big(c_i\,|a_i\rangle \otimes e^{ia_i\hat{q}}\,|\phi\rangle\Big)
\end{aligned}$$

This is a **unitary evolution**, preserving total probability.

Momentum Representation

Projecting onto momentum eigenstate $|p\rangle$:

$$\begin{aligned}\langle p|\Psi\rangle &= \langle p|\left\{\sum_i \Big(c_i\,|a_i\rangle \otimes e^{ia_i\hat{q}}\,|\phi\rangle\Big)\right\}\\ &= \sum_i \Big(c_i\,|a_i\rangle \otimes \langle p|\,e^{ia_i\hat{q}}\,|\phi\rangle\Big)\\ &= \sum_i \Big(c_i\,|a_i\rangle \otimes \phi(p-a_i)\Big)\\ &= \sum_i \Big(c_i\,|a_i\rangle \otimes e^{-\frac{(p-a_i)^2}{4\delta^2}}\Big)\end{aligned}$$

Weak vs Strong Measurement Regime

- **Strong regime:** Shift in momentum $> \delta \Rightarrow$ Gaussians are well-separated.
- **Weak regime:** Shift in momentum $< \delta \Rightarrow$ Gaussians strongly overlap.

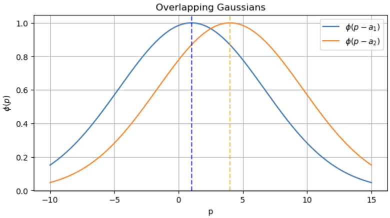

Figure: Weak regime: overlapping Gaussians

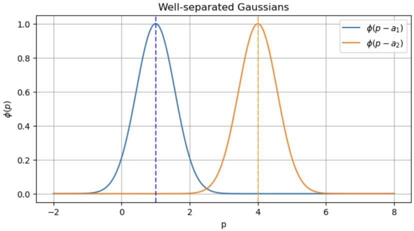

Figure: Strong regime: well-separated Gaussians

Idea of Postselection

the outcomes corresponding to a chosen final state $|\psi_f\rangle$ are kept.
Initially,

$$|\Psi(0)\rangle = |\psi_{in}\rangle \otimes |\phi\rangle$$

After interaction,

$$|\Psi\rangle = \hat{U}(t)\, |\Psi(0)\rangle$$

Now, postselection:

$$\begin{aligned}
\langle\psi_f|\Psi\rangle &= \langle\psi_f|\, \hat{U}\, |\Psi(0)\rangle \\
&= \langle\psi_f| \left(e^{i\hat{A}\hat{q}}\, |\psi_{in}\rangle \otimes |\phi\rangle \right) \\
&= \sum_i c_i \,\langle\psi_f|a_i\rangle \otimes e^{ia_i\hat{q}}\, |\phi\rangle
\end{aligned}$$

To observe shift in pointer's momentum:

$$\langle p|\Psi\rangle = \sum_i c_i \,\langle\psi_f|a_i\rangle\, \phi(p - a_i)$$

what is the difference from the previous case?

Difference from Previous Case

To see the difference, let's consider the postselected state:

$$|\psi_f\rangle = \sum_j c_j |a_j\rangle$$

Then the pointer state after postselection is:

$$\langle p|\Psi\rangle = \sum_i c_i \sum_j c_j^* \langle a_j|a_i\rangle \; \phi(p - a_i), \quad i \in \{1,2,3,4\}, \, j \in \{2,3\}$$

Expanding term by term:

$$\begin{aligned}
\langle p|\Psi\rangle &= c_1 \sum_j c_j^* \langle a_j|a_1\rangle \, \phi(p - a_1) + c_2 \sum_j c_j^* \langle a_j|a_2\rangle \, \phi(p - a_2) \\
&\quad + c_3 \sum_j c_j^* \langle a_j|a_3\rangle \, \phi(p - a_3) + c_4 \sum_j c_j^* \langle a_j|a_4\rangle \, \phi(p - a_4) \\
&= |c_2|^2 \phi(p - a_2) + |c_3|^2 \phi(p - a_3)
\end{aligned}$$

Difference from Previous Case

Interpretation: Only the Gaussians shifted by eigenvalues a_2 and a_3 contribute.

Difference: Previously, the pointer shift considered all eigenvalues a_i, giving $\langle \hat{A} \rangle$. Here, postselection restricts the data, so the average shift is generally $\neq \langle \hat{A} \rangle$.

Why Average?

Since we are in the weak regime, a single weak measurement provides almost no information $(\Delta p \gg A)$. But by repeating the experiment many times, we can reconstruct the full distribution and determine the $\langle A \rangle_w$ with arbitrary precision.

Note: This is still within the eigenvalue range.

AAV's Result

Now, comes the Genius,

$$
\begin{aligned}
\langle \psi_f | \Psi \rangle &= \langle \psi_f | \left(\hat{U} (| \psi_{in} \rangle \otimes | \phi \rangle) \right) \\
&= \underbrace{\langle \psi_f | e^{i \hat{A} \hat{q}} | \psi_{in} \rangle}_{\text{scalar}} \otimes | \phi \rangle \\
&= \langle \psi_f | \left(\sum_{n=0}^{\infty} \frac{(\hat{A} \hat{q})^n}{n!} \right) | \psi_{in} \rangle | \phi \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \psi_f | (\hat{A}^n \hat{q}^n) | \psi_{in} \rangle | \phi \rangle
\end{aligned}
$$

(since $[\hat{A}, \hat{q}] = 0$, so, $(\hat{A} \hat{q})^n = \hat{A}^n \hat{q}^n$)

AAV's Result

$$\langle\psi_f|\Psi\rangle = \langle\psi_f|\psi_{in}\rangle \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \frac{\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle}{\langle\psi_f|\psi_{in}\rangle} \hat{q}^n \right\} |\phi\rangle$$

$$= \langle\psi_f|\psi_{in}\rangle \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \langle\hat{A}\rangle_w \hat{q}^n \right\} |\phi\rangle$$

$$= \langle\psi_f|\psi_{in}\rangle \left\{ 1 + \langle\hat{A}\rangle_w \hat{q} + \frac{\langle\hat{A}^2\rangle_w \hat{q}^2}{2!} + \dots \right\} |\phi\rangle$$

if expansion is truncated for $n \geq 2$, then ,

$$\langle p|\Psi\rangle = \langle\psi_f|\psi_{in}\rangle \; e^{i\langle\hat{A}\rangle_w \hat{q}} \; \langle p|\phi\rangle$$

$$= \langle\psi_f|\psi_{in}\rangle \quad \underbrace{\phi(p - A_w)}_{\text{shift by weak value}}$$

$$\implies \boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \; e^{-\Delta^2 (p - A_w)^2}}$$

AAV's result

where,

$$\langle \hat{A} \rangle_w = \frac{\langle \psi_f | \, \hat{A} \, | \psi_{in} \rangle}{\langle \psi_f | \psi_{in} \rangle}$$

CASE 1: $|\psi_f\rangle = |\psi_{in}\rangle$

$$\langle \hat{A} \rangle_w = \sum_i |c_i|^2 a_i \quad \text{(expectation value)}$$

CASE 2: $|\psi_f\rangle$ nearly orthogonal to $|\psi_{in}\rangle$, i.e. $\langle \psi_f | \psi_{in} \rangle \to 0$

$$\langle \hat{A} \rangle_w \to \infty$$

Weak values are **not bounded** by the eigenvalue range because they are ratios of complex amplitudes. When pre- and post-selected states are nearly orthogonal, the denominator can be arbitrarily small, making the weak value arbitrarily large or complex.

Truncation Condition and Pointer Shift

Truncation condition:

$$\frac{\langle \hat{A}^n \rangle_w \, \hat{q}^n}{n!} \ll 1, \qquad \forall n \geq 2$$

$$\frac{\langle \hat{A}^n \rangle_w}{n!} (\Delta)^n \ll 1, \qquad \forall n \geq 2$$

Note:

$$\hat{q} \sim \Delta \sim \frac{1}{\delta}, \quad \text{since } \delta\,\Delta \geq \frac{\hbar}{2} \quad \Rightarrow \quad \delta \sim \frac{1}{\Delta}$$

where:

- δ: spread in pointer's momentum
- Δ: spread in pointer's position

Hence, the condition becomes:

$$\Delta^n \ll \frac{1}{\langle \hat{A}^n \rangle_w}, \quad \boxed{\Delta \ll \left(\frac{\langle \psi_f | \psi_{in} \rangle}{\langle \psi_f | \, \hat{A}^n \, | \psi_{in} \rangle} \right)^{1/n}}, \qquad \forall n \geq 2$$

Pictorial Summary

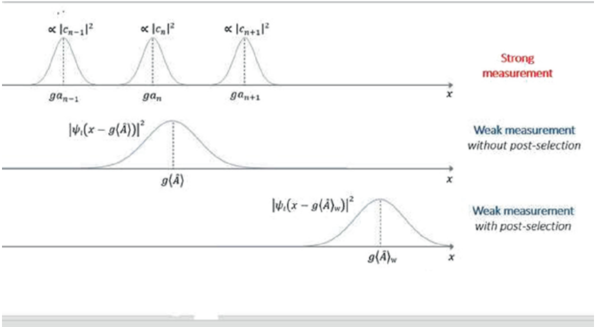

Figure: Schematic Summary

Weak Stern–Gerlach Experiment

This idea was proposed by Aharonov and his team !

Stern–Gerlach Experiment Setup

A beam of spin-$\frac{1}{2}$ particles travels along the y-direction with spins prepared in the xz-plane at an angle α to the x-axis.

- Spatial wavefunction is Gaussian in z with width Δ, giving momentum spread $\delta = \frac{1}{2\Delta}$.
- Beam passes through a weak SG magnet: $\delta p_z \ll \Delta p_z$.
- Wavefunction becomes a superposition of two slightly shifted Gaussians correlated with $\sigma_z = \pm 1$, still strongly overlapping.

Measurement scheme

- Beam then enters a strong SG magnet along x, separating σ_x eigenstates.
- Only $\sigma_x = +1$ is post-selected.
- The beam travels to a distant screen.
- Displacement along z due to average momentum from weak interaction exceeds initial uncertainty Δz.
- Measured displacement yields the weak value of σ_z.

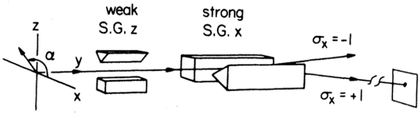

Figure: Stern–Gerlach setup layout for the AAV experiment.

Coupling Hamiltonian for SG Setup

Potential energy: $H = -\vec{\mu} \cdot \vec{B}$.

$$
\begin{aligned}
\vec{\mu} &= g\mu_B \frac{\hat{\vec{S}}}{\hbar} && \text{(Bohr magneton)} \\
&= \mu_B \hat{\vec{\sigma}} && (g = 2 \text{ for electron}) \\
\implies \mu_z &= \mu_B \hat{\sigma}_z
\end{aligned}
$$

Magnetic field along z:

$$B_z(z) \approx B_0 + \frac{\partial B_z}{\partial z} z$$

$$\hat{H} = -\mu_B \hat{\sigma}_z \frac{\partial B_z}{\partial z} \hat{z} \quad \Rightarrow \quad \boxed{\hat{H} = -g(t)\mu_B \hat{\sigma}_z \frac{\partial B_z}{\partial z} \hat{z}}$$

Overall System State

$$|\Psi\rangle = \underbrace{\left(\frac{1}{2\pi\Delta^2}\right)^{3/4} \exp\left[-\frac{x^2+y^2+z^2}{4\Delta^2}\right]}_{\text{Gaussian in 3D position}} \underbrace{\exp(-iP_0 y)}_{\text{momentum kick along } y} \quad |\uparrow_{\text{in}}\rangle$$

Spin states:

$$|\uparrow_{\mathsf{in}}\rangle = \cos\left(\frac{\pi/2-\alpha}{2}\right)|\uparrow_z\rangle + \sin\left(\frac{\pi/2-\alpha}{2}\right)|\downarrow_z\rangle$$

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$

Prerequisite

The eigenstates of spin along an arbitrary direction $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ are given by:

$$|{\uparrow_n}\rangle = \cos\left(\frac{\theta}{2}\right)|{\uparrow_z}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|{\downarrow_z}\rangle,$$

$$|{\downarrow_n}\rangle = -e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|{\uparrow_z}\rangle + \cos\left(\frac{\theta}{2}\right)|{\downarrow_z}\rangle.$$

These are eigenstates of the spin operator along $\mathbf{n}$,

$$\hat{S}_n|{\uparrow_n}\rangle = +\frac{\hbar}{2}|{\uparrow_n}\rangle, \qquad \hat{S}_n|{\downarrow_n}\rangle = -\frac{\hbar}{2}|{\downarrow_n}\rangle.$$

Weak Value Calculation

Let's calculate the weak value:

$$(\hat{\sigma}_z)_w = \frac{\langle\uparrow_x|\,\hat{\sigma}_z\,|\uparrow_{\text{in}}\rangle}{\langle\uparrow_x|\uparrow_{\text{in}}\rangle}$$

Since we know $|\uparrow_{\text{in}}\rangle$ and $|\uparrow_x\rangle$:

$$\langle\uparrow_x|\uparrow_{\text{in}}\rangle = \cos\left(\frac{\alpha}{2}\right)$$

$$\hat{\sigma}_z\,|\uparrow_{\text{in}}\rangle = \cos\left(\frac{\pi/2-\alpha}{2}\right)|\uparrow_z\rangle - \sin\left(\frac{\pi/2-\alpha}{2}\right)|\downarrow_z\rangle$$

$$\begin{aligned}\langle\uparrow_x|\,\hat{\sigma}_z\,|\uparrow_{\text{in}}\rangle &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi/2-\alpha}{2} - \sin\frac{\pi/2-\alpha}{2}\right)\\ &= \sin\frac{\alpha}{2}\end{aligned}$$

$$(\hat{\sigma}_z)_w = \frac{\sin(\alpha/2)}{\cos(\alpha/2)} = \tan\frac{\alpha}{2}$$

Pointer Momentum Shift

From theory, pointer momentum shift:

$$\begin{aligned}\delta p_z &= \mu \frac{\partial B_z}{\partial z} \sigma_z \quad (\hat{z} \text{ is translator in momentum}) \\ &= \mu \frac{\partial B_z}{\partial z} \langle \sigma_z \rangle_w \quad (\text{postselection}) \\ &= \mu \frac{\partial B_z}{\partial z} \tan \frac{\alpha}{2}\end{aligned}$$

Note: At $\alpha = \pi$, $\tan(\alpha/2) \to \infty$
Condition for weak measurement:

$$\boxed{\delta p_z = \mu \frac{\partial B_z}{\partial z} \min \left\{ |\tan(\alpha/2)|, 1 \right\} \ll \Delta p_z \sim \frac{1}{2\Delta}}$$

Measuring Pointer Shift

For a long distance l, such that $\delta z > \Delta$, the shift on the screen can be measured:

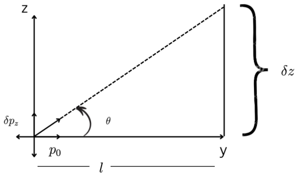

Figure: δp_z measurement schematics

Relation between shift and momentum:

$$\tan(\theta) = \frac{\delta z}{l} = \frac{\delta p_z}{p_0} \quad \Longrightarrow \quad \boxed{\delta p_z = \frac{\delta z}{l} p_0}$$

Why Do This?

- Measuring δp_z gives the pointer's momentum shift.
- Due to postselection, this corresponds to the weak value $\langle \sigma_z \rangle_w$.
- For nearly orthogonal pre- and post-selected states, $\langle \sigma_z \rangle_w$ can exceed the eigenvalue range.

Question?

In general, $A_w \in \mathbb{C}$, so we write

$$A_w = \mathrm{Re}(A_w) + i\ \mathrm{Im}(A_w)$$

- $\mathrm{Re}(A_w)$ governs the **shift in the pointer's mean position**.
- What does $\mathrm{Im}(A_w)$ represent?