Quantum Weak Measurement

A Review of AAV and SSD Contributions

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Abstract

This report reviews the theory of quantum weak measurement based on the works of Aharonov, Albert, and Vaidman (AAV) and later analysis by Duck, Stevenson, and Sudarshan (SSD). Starting from the von Neumann measurement model, it explains how weak coupling between a system and measuring device yields the *weak value*, which can exceed the observable's eigenvalue range under nearly orthogonal pre- and post-selections. SSD's treatment refines AAV's approximation and confirms its validity in the weak regime. The equivalence of both results is demonstrated analytically. Experimental realizations through the *Stern–Gerlach setup* and an *optical analog* are discussed, showing how weak value amplification enables detection of extremely small physical effects.

Theory

AAV's discussion begins with the standard von Neumann measurement model. The key idea of **von Neumann** was, one can see that it is possible in principle to correlate the state of a microscopic quantum system with the value of a macroscopic classical variable, and we may take it for granted that we can perceive the value of the classical variable.

During the brief measurement interval, the interaction Hamiltonian can be assumed to dominate all other terms in the total Hamiltonian. Under this assumption, the coupling between the quantum system and the pointer is described by an interaction Hamiltonian of the form:

$$\boxed{\hat{H} = -\,g(t)\,\hat{A}\hat{q}} \tag{1}$$

where $g(t)$ is a normalized coupling function defined over the measurement duration, $\hat{A}$ is Observable operator of the system (e.g., a spin operator) and $\hat{q}$ is Position operator of the pointer.

Let the measuring device start in state $|\phi\rangle$ with a Gaussian momentum-space wave function $\phi(p)$ centered at $p = 0$ and width Δp. Its position-space wave function $\phi(q)$, the Fourier transform of $\phi(p)$, is also Gaussian.

$$\phi(q) \equiv \langle q|\phi\rangle = \exp\left[-\frac{q^2}{4\Delta^2}\right],$$

$$\phi(p) \equiv \langle p|\phi\rangle = \exp\left[-\Delta^2 p^2\right], \tag{2}$$

where

$$\Delta q \equiv \Delta, \qquad \Delta p = \frac{1}{2\Delta},$$

Note: The system–pointer interaction is represented by a **unitary evolution**, since the combined system is isolated during the measurement. According to the Schrodinger, its time evolution is $U = \exp\left(-\frac{i}{\hbar}\hat{H}_{\text{int}}t\right)$, ensuring **conservation of total probability** (norm preservation).

say, Preselected state,

$$|\psi_{in}\rangle = \sum_n \alpha_n |a_n\rangle \qquad \text{(in superposition of eigensates of } \hat{A})$$

Postselected state,

After a weak interaction, a strong measurement is performed and only the outcomes corresponding to a chosen

final state $|\psi_f\rangle$ are kept.
say,

$$|\psi_f\rangle = \sum_j \beta_j |a_j\rangle \qquad \text{(in superposition of some eigenstates of } \hat{A})$$

After Postselection, the complete system evolve as ,

$$\langle\psi_f|\Psi\rangle = \langle\psi_f| \hat{U}(|\psi_{in}\rangle \otimes |\phi\rangle)$$

$$= \langle\psi_f| e^{i\hat{q}\hat{A}} |\psi_{in}\rangle \otimes |\phi\rangle$$

$$= \sum_n \sum_j \alpha_n \beta_j^* \, \delta_{j,n} \, e^{i\hat{q}a_n} |\phi\rangle$$

$$= \sum_n \alpha_n \beta_n^* \, e^{ia_n\hat{q}} \, |\phi\rangle$$

now, projecting on pointer's momentum, we get

$$\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \langle p| e^{ia_n\hat{q}} |\phi\rangle$$

$$= \sum_n \alpha_n \beta_n^* \, \phi(p - a_n)$$

$$= \sum_n \alpha_n \beta_n^* \exp\left[-\frac{(p-a_n)^2}{4(1/2\Delta)^2}\right] \qquad \text{(No approximation used till now)}$$

$$\implies \boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* e^{-\Delta^2(p-a_n)^2}} \qquad \text{(SSD's Result)}$$

Strong vs Weak measurement

If the pointer's momentum spread δ is small compared to the spacing between the eigenvalues a_n, the measuring device ends up in well-separated peaks, each centered at an eigenvalue a_n (**Strong Measurement**). In the limit $\delta \to 0$, this corresponds to an ideal measurement where the outcome is always one of the eigenvalues, the probability of obtaining a_n is $|\alpha_n|^2$, and the system collapses to the corresponding eigenstate $|A = a_n\rangle$.
However, in the opposite limit, in which δ is much larger than the spread of the a_n's. AAV refer to this case as a **"weak measurement."** The final state of the "measuring" device is then a superposition of strongly overlapping, broad Gaussians. (SSD's Result)

Note: any single *weak measurement* gives almost no information, since $\Delta p \gg \langle A\rangle$. However, by repeating the experiment many times, one can map out the whole distribution and thus determine the centroid $\langle A\rangle$ to any desired accuracy.

AAV's work

$$\langle\psi_f|\Psi\rangle = \langle\psi_f| \left(\hat{U}(|\psi_{in}\rangle \otimes |\phi\rangle)\right)$$

$$= \underbrace{\langle\psi_f| e^{i\hat{A}\hat{q}} |\psi_{in}\rangle}_{\text{scalar}} \otimes |\phi\rangle$$

$$= \langle\psi_f| \left(\sum_{n=0}^{\infty} \frac{(\hat{A}\hat{q})^n}{n!}\right) |\psi_{in}\rangle |\phi\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle\psi_f| (\hat{A}^n \hat{q}^n) |\psi_{in}\rangle |\phi\rangle \qquad \text{(since } [\hat{A}, \hat{q}] = 0, \text{ so, } (\hat{A}\hat{q})^n = \hat{A}^n \hat{q}^n \text{)}$$

$$\langle\psi_f|\Psi\rangle = \langle\psi_f|\psi_{in}\rangle \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\frac{\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle}{\langle\psi_f|\psi_{in}\rangle}\hat{q}^n\right\}|\phi\rangle$$

$$= \langle\psi_f|\psi_{in}\rangle \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\langle\hat{A}\rangle_w\hat{q}^n\right\}|\phi\rangle$$

$$= \langle\psi_f|\psi_{in}\rangle\left\{1 + \langle\hat{A}\rangle_w\hat{q} + \frac{\langle\hat{A}^2\rangle_w\hat{q}^2}{2!} + \dots\right\}|\phi\rangle$$

if expansion is truncated for $n \geq 2$, then ,

$$\langle p|\Psi\rangle = \langle\psi_f|\psi_{in}\rangle\ e^{i\langle\hat{A}\rangle_w\hat{q}}\ \langle p|\phi\rangle$$
$$= \langle\psi_f|\psi_{in}\rangle\ \underbrace{\phi(p - A_w)}_{\text{shift by weak value}}$$

$$\implies \boxed{\langle p|\Psi\rangle = \sum_n \alpha_n\beta_n^*\ e^{-\Delta^2(p - A_w)^2}} \qquad \text{AAV's result}$$

$$\langle\hat{A}\rangle_w = \frac{\langle\psi_f|\,\hat{A}\,|\psi_{in}\rangle}{\langle\psi_f|\psi_{in}\rangle}$$

Weak values are **not bounded** by the eigenvalue range because they are not probability-weighted averages but ratios of complex amplitudes. When the pre- and post-selected states are nearly orthogonal, the denominator can be arbitrarily small, making the weak value arbitrarily large (or even complex).

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Accurate Truncation condition (by SSD) :
condition is,

$$\frac{i\hat{q}^2}{n!}\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle \ll i\hat{q}\langle\psi_f|\,\hat{A}\,|\psi_{in}\rangle \qquad ,\forall\ n \geq 2$$

reduces to,

$$\implies \boxed{\Delta \ll \min_{n\geq 2}\left|\frac{\langle\psi_f|\,\hat{A}\,|\psi_{in}\rangle}{\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle}\right|^{\frac{1}{n-1}}}$$

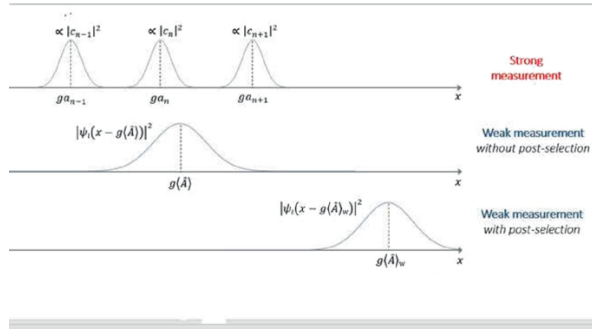this is the accurate condition for the validity of AAV's Result !

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Figure 1: Pictorial Summery

The Paradox

- from AAV's paper,

$$\boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \ e^{-\Delta^2(p-A_w)^2}}$$ (single gaussian peaked at A_w)

- from Duck's paper,

$$\boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* e^{-\Delta^2(p-a_n)^2}}$$ (Superposition of gaussians each peaked at a_n)

Paradox :
How can $|\Psi\rangle$ be a single gaussian peaked at A_w, while simultaneously being a superposition of gaussians each peaked at value a_n, where A_w may well be much greater than any of the a_n's? How can a large shift be produced from a superposition of small shifts?
-The resolution of the paradox lies in the fact that the superposition of Gaussian involves complex coefficient here.

Stern-Gerlach

A beam of spin-$\frac{1}{2}$ particles travels along the y-direction with spins prepared in the xz-plane at an angle α to the x-axis. The particles' spatial wavefunction is Gaussian in z with width Δ, giving a momentum spread $\delta = \frac{1}{2\Delta}$. The beam first passes through a weak Stern–Gerlach magnet, where the field gradient is small so that the momentum shift $\delta p_z \ll \Delta p_z$. As a result, the spatial wavefunction becomes a superposition of two slightly shifted Gaussians correlated with $\sigma_z = \pm 1$, but still strongly overlapping. The beam then enters a strong Stern–Gerlach magnet aligned along x, which separates the σ_x eigenstates, and only the $\sigma_x = +1$ component is post-selected. This selected beam travels freely to a distant screen. The screen is placed sufficiently far so that the displacement in the z-direction due to the average momentum p_z acquired during the weak interaction exceeds the initial position uncertainty Δz. On the screen, a wide spot is obtained whose displacement along z is measured, and this displacement yields the weak value of σ_z.
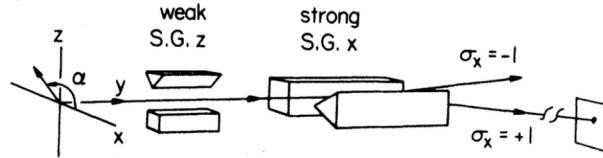

Figure 2: Stern-Gerlach setup layout for the AAV experiment.

For this case, the interaction Hamiltonian is given by,

$$\boxed{\hat{H} \ = \ -g(t)\mu_B\,\hat{\sigma}_z\,\frac{\partial B_z}{\partial z}\,\hat{z}}$$

which describes the coupling between the spin and the magnetic field gradient along the z-direction.

The overall wavefunction of the system can be expressed as,

$$|\Psi\rangle = \underbrace{\left(\frac{1}{2\pi\Delta^2}\right)^{3/4}\exp\left[-\left(\frac{x^2+y^2+z^2}{4\Delta^2}\right)\right]}_{\text{Gaussian in 3D position}}\underbrace{\exp(-iP_0 y)}_{\text{momentum kick } P_0 \text{ along } y}|\psi_{\text{in}}\rangle$$

representing a Gaussian beam moving along the y-axis with momentum P_0.

where,

$$|\psi_{\text{in}}\rangle = \cos\left(\frac{\pi/2 - \alpha}{2}\right)|\uparrow_z\rangle + \sin\left(\frac{\pi/2 - \alpha}{2}\right)e^{i0}|\downarrow_z\rangle$$

$$|\psi_f\rangle = |\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$$

Using these definitions, the weak value of $\hat{\sigma}_z$ becomes,

$$(\hat{\sigma}_z)_w = \frac{\langle\psi_f|\,\hat{\sigma}_z\,|\psi_{\text{in}}\rangle}{\langle\psi_f|\psi_{\text{in}}\rangle} \implies \boxed{(\hat{\sigma}_z)_w = \tan\left(\frac{\alpha}{2}\right)}.$$

from our theory we can directly say,

$$\langle p|\Psi\rangle = \langle\psi_f|\psi_{\text{in}}\rangle \, \exp\left[-\Delta^2(p_z - A_w)^2\right]$$

$$= \cos\frac{\alpha}{2} \, \exp\left[-\Delta^2(p_z - \lambda\tan\frac{\alpha}{2})^2\right] \qquad \left(\lambda = \mu\frac{\partial B_z}{\partial z}\right)$$

Note: at $\alpha = \pi \implies \tan(\alpha/2) \to \infty$

therefore,

$$\boxed{\delta p_z \;=\; \mu\,\frac{\partial B_z}{\partial z}\,\min\left\{\left|\tan\frac{\alpha}{2}\right|,\,1\right\} \;\ll\; \Delta p_z \sim \frac{1}{2\Delta}}$$

the particle travels a distance l along the y-axis while experiencing a small deflection δz in the z-direction. From the following geometry, $\tan(\theta) = \delta z/l = \delta p_z/p_0$, giving $\delta p_z = (\delta z/l)p_0$. This relation shows that the shift in the pointer's momentum corresponds to the weak value, which can exceed the eigenvalue range when the pre- and post-selected states are nearly orthogonal.
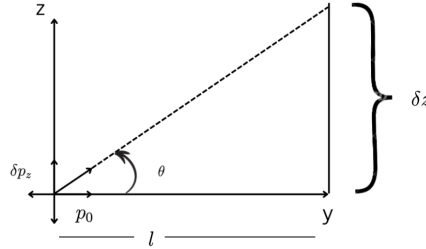

Figure 3: δp_z measurement schematics

Further analysis by SSD's paper

SSD's paper takes the analysis a step further, providing a deeper understanding of the AAV result. Since the interesting effects occur at $\alpha = \pi$, for a sufficiently small Δ we can approach $\alpha \approx \pi$. To analyze this more clearly, let us set $\alpha = \pi - 2\epsilon$ with $\epsilon \ll 1$.

then SSD's and AAV's result reduces to,

$$\langle p|\Psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[(\epsilon + 1)\,e^{-\Delta^2(p_z - \lambda)^2} - (1 - \epsilon)\,e^{-\Delta^2(p_z + \lambda)^2}\right] \quad \text{(No condition applied)}$$

$$\langle p|\Psi\rangle = \epsilon\,\exp\left[-\Delta^2\left(p_z - \frac{\lambda}{\epsilon}\right)^2\right] \qquad\qquad \text{(valid iff, shift} \ll \text{spread, i.e. } \lambda\Delta \ll \epsilon \ll 1).$$

they showed through graphical analysis that, SSD's result approaches AAV's result in the weak regime condition. i.e, AAV's result holds strictly within the *weak measurement regime*, while Duck's result is more general and naturally reduces to the AAV expression when the interaction is weak.

Note: The truncation condition is precisely the weak measurement condition.

Optical Analog (proposed by SSD's paper)

> **Mathematical Analogy:** Electron's spin states $\leftrightarrow$ Light's polarization states.

An optical analog of the previous experiment is constructed using polarized light. A laser beam, expanded with lenses, provides the broad coherent source. A polarizer at angle α and an analyzer at angle β define the initial and final polarizations, while a weakly birefringent crystal serves as the weak measurement device. It introduces a small lateral displacement between the ordinary (x-polarized) and extraordinary (z-polarized) components, much smaller than the beam width and determined by the crystal properties.
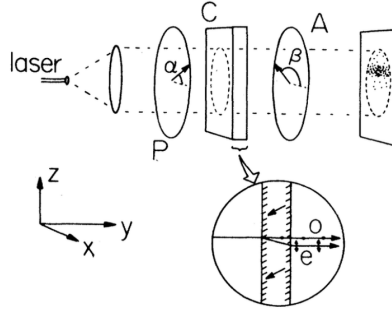

Figure 4: An optical analog schematic. A broad, coherent beam
passes through a polarizer (P) and an analyzer (A).
Between them, a birefringent crystal (C) is placed.

Note: The shift is lateral rather than angular (as in the Stern–Gerlach case), the analysis focuses on the spatial z-distribution instead of momentum.

The input beam is assumed to have a wide Gaussian profile.
After passing through polarizer, the preselected state becomes,

$$|\psi_{in}\rangle = \cos(\alpha)\,|\hat{x}\rangle + \sin(\alpha)\,|\hat{z}\rangle$$

following the theory, we can say,

$$|\Psi_{in}\rangle = |\psi_{in}\rangle \otimes |\phi_{in}\rangle$$

now, projecting on pointer's position,

$$\langle q|\Psi_{in}\rangle = |\psi_{in}\rangle \otimes \langle q|\phi_{in}\rangle$$

$$= |\psi_{in}\rangle\, e^{-\frac{z^2}{4\Delta^2}}$$

$$= (\cos(\alpha)\,|\hat{x}\rangle + \sin(\alpha)\,|\hat{z}\rangle)\, e^{-\delta^2 z^2} \qquad \text{,gaussian peaked at } z = 0$$

After passing through the crystal, the two orthogonal polarization components experience lateral displacements a_1 and a_2 along the $\hat{z}$-direction. Hence, we can express the field as:

$$\langle q|\Psi\rangle = \underbrace{\cos(\alpha)\, e^{-\delta^2(z-a_1)^2}}_{\text{gaussian peaked at } z = a_1} |\hat{x}\rangle + \underbrace{\sin(\alpha)\, e^{-\delta^2(z-a_2)^2}}_{\text{gaussian peaked at } z = a_2} |\hat{z}\rangle$$

after passing through Analyser,

$$\langle \psi_f | \Psi \rangle = \Big(\cos(\beta) \, |\hat{x}\rangle + \sin(\beta) \, |\hat{z}\rangle \Big) \cdot \Big(\cos(\alpha) \; e^{-\delta^2 (z-a_1)^2} \, |\hat{x}\rangle + \sin(\alpha) \; e^{-\delta^2 (z-a_2)^2} \, |\hat{z}\rangle \Big)$$

$$= \underbrace{\cos(\beta)\cos(\alpha) \; e^{-\delta^2 (z-a_1)^2} + \; \sin(\beta)\sin(\alpha) \; e^{-\delta^2 (z-a_2)^2}}_{\text{superposition of two gaussian peaked at } a_1 \text{ and } a_2}$$

Claim: In the near-orthogonal limit, this expression reduces to Duck's result.

$$\langle q | \Psi \rangle = \cos(\alpha + \beta) \left\{ \frac{1}{2} \Big[(1+\epsilon) \; e^{-\delta^2 (z-a_1)^2} - (1-\epsilon) \; e^{-\delta^2 (z-a_2)^2} \Big] \right\} \qquad , \epsilon \ll 1$$

> **Proof**
>
> $$\langle \psi_f | \Psi \rangle = \cos(\beta)\cos(\alpha) e^{-\delta^2 (z-a_1)^2} + \sin(\beta)\sin(\alpha) e^{-\delta^2 (z-a_2)^2}$$
>
> $$= \frac{\cos(\alpha+\beta)}{2} \left[\frac{2\cos(\alpha)\cos(\beta)}{\cos(\alpha+\beta)} e^{-\delta^2 (z-a_1)^2} + \frac{2\sin(\alpha)\sin(\beta)}{\cos(\alpha+\beta)} e^{-\delta^2 (z-a_2)^2} \right]$$
>
> $$= \frac{\cos(\alpha+\beta)}{2} \left[\left(1 + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} \right) e^{-\delta^2 (z-a_1)^2} - \left(1 - \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} \right) e^{-\delta^2 (z-a_2)^2} \right]$$
>
> $$= \langle q | \Psi \rangle = \cos(\alpha+\beta) \frac{1}{2} \Big[(1+\epsilon) e^{-\delta^2 (z-a_1)^2} - (1-\epsilon) e^{-\delta^2 (z-a_2)^2} \Big],$$
>
> $$\text{for } \alpha = \pi/4, \; \beta = 3\pi/4 - \epsilon, \; \epsilon \ll 1$$

From our theoretical analysis, under the weak regime condition, this expression simplifies to a single Gaussian centered at the weak value. Remarkably, this weak value can lie far outside the range of the eigenvalues (for instance, in the spin case, a value of 100 is possible, whereas the maximum allowed eigenvalue is $1/2$). Consequently, these tiny lateral shifts can be significantly amplified—though still smaller than the overall position spread—when operating in the weak regime with nearly orthogonal pre- and post-selected states. This amplification makes it possible to detect extremely small birefringence in the crystal.

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