SSD's Paper on Weak Measurement

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Initial and Final State

say, Preselected state,

$$|\psi_{in}\rangle = \sum_n \alpha_n |a_n\rangle \qquad \text{(in superposition of eigensates of } \hat{A}\text{)}$$

Postselected state,
After a weak interaction, a strong measurement is performed and only the outcomes corresponding to a chosen final state $|\psi_f\rangle$ are kept.
say,

$$|\psi_f\rangle = \sum_j \beta_j |a_j\rangle \qquad \text{(in superposition of some eigenstates of } \hat{A}\text{)}$$

Unitary time evolution

After Postselection, the complete system evolve as ,

$$
\begin{aligned}
\langle \psi_f | \Psi \rangle &= \langle \psi_f | \hat{U} (|\psi_{in}\rangle \otimes |\phi\rangle) \\
&= \langle \psi_f | e^{i\hat{q}\hat{A}} |\psi_{in}\rangle \otimes |\phi\rangle \\
&= \sum_n \sum_j \alpha_n \beta_j^* \ \delta_{j,n} \ e^{i\hat{q}a_n} |\phi\rangle \\
&= \sum_n \alpha_n \beta_n^* \ e^{ia_n\hat{q}} \ \ |\phi\rangle
\end{aligned}
$$

Momentum Representation

now, projecting on pointer's momentum, we get

$$\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \langle p| e^{ia_n \hat{q}} |\phi\rangle$$

$$= \sum_n \alpha_n \beta_n^* \, \phi(p - a_n)$$

$$= \sum_n \alpha_n \beta_n^* \exp\left[-\frac{(p-a_n)^2}{4(1/2\Delta)^2}\right] \qquad \text{(No approximation used till now)}$$

$$\Longrightarrow \boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* e^{-\Delta^2 (p-a_n)^2}} \qquad \text{(SSD's Result)}$$

Weak vs Strong Measurement Regime

Thus, if δ(pointer's momentum spread) is small compared to the spacing between the eigenvalues a_n, the measuring device ends up in a state consisting of well-separated peaks, each centered at one of the eigenvalues a_n. In the limit $\delta \to 0$, this corresponds to an **Strong Measurement** with the following properties:

1. The measurement always yields one of the eigenvalues a_n.
2. The probability of obtaining a_n is $|\alpha_n|^2$.
3. If the measurement yields a_n, then the quantum system is left in the corresponding eigenstate $|A = a_n\rangle$.

However, in the opposite limit, in which δ is much larger than the spread of the a_n's. AAV refer to this case as a **"weak measurement."** The final state of the "measuring" device is then a superposition of strongly overlapping broad Gaussians. (SSD's Result)

The Paradox

- from AAV's paper,

$$\boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \; e^{-\Delta^2 (p - A_w)^2}}$$

- from Duck's paper,

$$\boxed{\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* e^{-\Delta^2 (p - a_n)^2}}$$

Paradox :

How can $|\Psi\rangle$ be a single gaussian peaked at A_w, while simultaneously being a superposition of gaussians each peaked at value a_n, where A_w may well be much greater than any of the a_n's? How can a large shift be produced from a superposition of small shifts?

-The resolution of the paradox lies in the fact that the superposition of Gaussian involves complex coefficient here.(we will see how both results are same in **Stern-Garlach Revisit** section)

Modified AAV's Truncation condition

$$\frac{i\hat{q}^2}{n!}\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle \ll i\hat{q}\,\langle\psi_f|\,\hat{A}\,|\psi_{in}\rangle < \langle\psi_f|\psi_{in}\rangle \qquad ,\forall \text{ n} \geq 2$$

so, one and only conddition is,

$$\frac{i\hat{q}^2}{n!}\langle\psi_f|\,\hat{A}^n\,|\psi_{in}\rangle \ll i\hat{q}\,\langle\psi_f|\,\hat{A}\,|\psi_{in}\rangle \qquad ,\forall \text{ n} \geq 2$$

coparing order,

$$\Delta^n\langle\hat{A}^n\rangle_w \ll \Delta\langle\hat{A}\rangle_w$$

$$\implies \boxed{\Delta \ll \min_{n\geq 2}\left|\frac{\langle\psi_f|\,\hat{A}\,|\psi_{\text{in}}\rangle}{\langle\psi_f|\,\hat{A}^n\,|\psi_{\text{in}}\rangle}\right|^{\frac{1}{n-1}}}$$

this is more accurate condition for the validity of AAV's Result !

Revisiting Stern–Gerlach Experiment

Goal : To show Duck's result and AAV's result are same under weak interaction

Stern–Gerlach Revisit

We know from AAV's paper,

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)|\uparrow_z\rangle + \frac{1}{\sqrt{2}}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)|\downarrow_z\rangle$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow_z\rangle + |\downarrow_z\rangle\big)$$

$$A_w = \langle\sigma_z\rangle_w = \tan\frac{\alpha}{2}, \qquad \left(\lambda = \mu\frac{\partial B_z}{\partial z}\right)$$

$$\begin{aligned}\langle p|\Psi\rangle &= \langle\psi_f|\psi_{\text{in}}\rangle \exp\left[-\Delta^2\big(p_z - A_w\big)^2\right] \\ &= \cos\frac{\alpha}{2}\exp\left[-\Delta^2\big(p_z - \lambda\tan\frac{\alpha}{2}\big)^2\right]\end{aligned}$$

Stern–Gerlach Revisit

Since interesting effects occur at $\alpha = \pi$, for a sufficiently small Δ we can go close to $\alpha = \pi$. Let $\alpha = \pi - 2\epsilon$, where $\epsilon \ll 1$.
Then AAV's result reduces to:

$$\langle p|\Psi\rangle = \cos\frac{\alpha}{2}\ \exp\left[-\Delta^2\left(p_z - \lambda\tan\frac{\alpha}{2}\right)^2\right]$$
$$= \epsilon\ \exp\left[-\Delta^2\left(p_z - \frac{\lambda}{\epsilon}\right)^2\right]$$

Validity condition: shift $\ll$ spread

$$\frac{\lambda}{\epsilon} \ll \Delta p_z = \frac{1}{2\Delta} \quad \Longrightarrow \quad \boxed{\lambda\Delta \ll \epsilon \ll 1}$$

Stern–Gerlach Revisit

Now, Duck's result reduces to:

$$\langle p|\Psi\rangle = \sum_n \alpha_n \beta_n^* \exp\Big[-\Delta^2 (p_z - \lambda a_n)^2\Big]$$

$$= \alpha_1 \beta_1^* e^{-\Delta^2 (p_z - \lambda a_1)^2} + \alpha_2 \beta_2^* e^{-\Delta^2 (p_z - \lambda a_2)^2}$$

since,

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}\Big[\big(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\big)|{\uparrow_z}\rangle + \big(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\big)|{\downarrow_z}\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[(\epsilon + 1)|+\rangle + (\epsilon - 1)|-\rangle\Big]$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\big(|{\uparrow_z}\rangle + |{\downarrow_z}\rangle\big)$$

Stern–Gerlach Revisit

On substituting, we get,

$$\langle p|\Psi\rangle = \left(\frac{1}{\sqrt{2}}(\epsilon+1)\right)\frac{1}{\sqrt{2}}e^{-\Delta^2(p_z-\lambda)^2} + \left(\frac{1}{\sqrt{2}}(\epsilon-1)\right)\frac{1}{\sqrt{2}}e^{-\Delta^2(p_z+\lambda)^2}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\left[(\epsilon+1)e^{-\Delta^2(p_z-\lambda)^2} - (1-\epsilon)e^{-\Delta^2(p_z+\lambda)^2}\right]$$

Summery

$$\boxed{\langle p|\Psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left[(\epsilon+1)e^{-\Delta^2(p_z-\lambda)^2} - (1-\epsilon)e^{-\Delta^2(p_z+\lambda)^2}\right]} \tag{1}$$

(No condition applied)

$$\boxed{\langle p|\Psi\rangle = \epsilon \, \exp\left[-\Delta^2\left(p_z - \frac{\lambda}{\epsilon}\right)^2\right]} \tag{2}$$

(AAV's truncation condition applied)

Our claim: *Duck's result approaches AAV's result in the weak regime condition.*

Proof

We start from Duck's expression:

$$\langle p|\Psi\rangle = \frac{1}{2}\Big[(\epsilon+1)e^{-\Delta^2(p_z-\lambda)^2} - (1-\epsilon)e^{-\Delta^2(p_z+\lambda)^2}\Big].$$

We will show that this expression approaches the AAV form in the *weak regime*.
Factor out a common Gaussian envelope:

$$\begin{aligned}\langle p|\Psi\rangle &= \frac{e^{-\Delta^2 p_z^2}}{2}\Big[(\epsilon+1)e^{2\Delta^2 p_z\lambda-\Delta^2\lambda^2} - (1-\epsilon)e^{-2\Delta^2 p_z\lambda-\Delta^2\lambda^2}\Big] \\ &= \frac{e^{-\Delta^2(p_z^2+\lambda^2)}}{2}\Big[(\epsilon+1)e^{2\Delta^2 p_z\lambda} - (1-\epsilon)e^{-2\Delta^2 p_z\lambda}\Big].\end{aligned}$$

Proof

In the *weak regime*, $\Delta\lambda \ll 1$, we can use the Taylor expansion

$$e^{\pm 2\Delta^2 p_z \lambda} \approx 1 \pm 2\Delta^2 p_z \lambda.$$

Substituting gives:

$$\langle p|\Psi\rangle \approx \frac{e^{-\Delta^2(p_z^2+\lambda^2)}}{2}\Big[(\epsilon+1)(1+2\Delta^2 p_z\lambda) - (1-\epsilon)(1-2\Delta^2 p_z\lambda)\Big]$$

$$= e^{-\Delta^2(p_z^2+\lambda^2)}\big[\epsilon + 2\Delta^2 p_z\lambda\big].$$

Proof

Factor out ϵ and use $1 + x \approx e^x$ for small x:

$$\langle p|\Psi\rangle \approx \epsilon\, e^{-\Delta^2(p_z^2+\lambda^2)} \left(1 + \frac{2\Delta^2 p_z \lambda}{\epsilon}\right)$$

$$\approx \epsilon\, e^{-\Delta^2(p_z^2+\lambda^2)} \exp\left(\frac{2\Delta^2 p_z \lambda}{\epsilon}\right)$$

$$= \epsilon \exp\left[-\Delta^2\left(p_z^2 - \frac{2p_z\lambda}{\epsilon} + \lambda^2\right)\right].$$

Complete the square inside the exponent:

$$p_z^2 - \frac{2p_z\lambda}{\epsilon} + \lambda^2 = \left(p_z - \frac{\lambda}{\epsilon}\right)^2 + \left(\lambda^2 - \frac{\lambda^2}{\epsilon^2}\right).$$

Proof

Substituting this back gives:

$$\langle p|\Psi\rangle \approx \epsilon \, \exp\Big[-\Delta^2\Big(p_z - \frac{\lambda}{\epsilon}\Big)^2\Big] \, \exp\Big[-\Delta^2\Big(\lambda^2 - \frac{\lambda^2}{\epsilon^2}\Big)\Big].$$

The second exponential is p_z-independent and can be absorbed into normalization.
Thus, up to normalization:

$$\boxed{\langle p|\Psi\rangle \propto \epsilon \exp\Big[-\Delta^2\Big(p_z - \frac{\lambda}{\epsilon}\Big)^2\Big]}$$

which is precisely the AAV form in the weak regime. □

Important Note

Important Note:

AAV's result holds strictly within the *weak measurement regime*, while Duck's result is more general and naturally reduces to the AAV expression when the interaction is weak.

Claim: The truncation condition is precisely the weak measurement condition.
We will now show this through a simple proof.

Simple Proof

$$\frac{i\hat{q}^2}{n!} \langle\psi_f|\, \hat{A}^n \,|\psi_{in}\rangle \ll i\hat{q}\, \langle\psi_f|\, \hat{A}\,|\psi_{in}\rangle < \langle\psi_f|\psi_{in}\rangle \,, \qquad \forall\, n \geq 2$$

$$\implies \frac{\langle\psi_f|\, \hat{A}\,|\psi_{in}\rangle}{\langle\psi_f|\psi_{in}\rangle}\, \hat{q} \ll 1$$

$$\implies \langle\hat{A}\rangle_w\, \hat{q} \ll 1 \quad \implies \quad \boxed{\langle\hat{A}\rangle_w\, \Delta \ll 1} \qquad \text{(weak condition).}$$

Summery

1. **Weak measurement theory (AAV)** shows that by weakly coupling a system to a measuring device and postselecting a final state, one obtains a *weak value* that can lie far outside the observable's eigenvalue range.
2. In the **Stern–Gerlach realization**, the weak value of spin (σ_z) determines the pointer's momentum shift, which can become anomalously large when the pre- and post-selected states are nearly orthogonal.
3. **SSD's analysis** resolved the AAV paradox and established a refined truncation condition, proving that AAV's results naturally emerge as the weak-coupling limit of the full theory.

Optical Analog

The idea was proposed by Duck and his team !

Optical Analog Setup

An optical analog of the previous experiment is constructed using polarized light. A laser beam, expanded with lenses, provides the broad coherent source. A polarizer at angle α and an analyzer at angle β define the initial and final polarizations, while a weakly birefringent crystal serves as the weak measurement device.

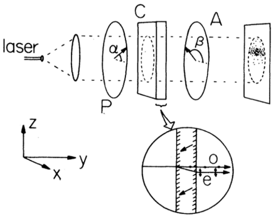

Figure: Schematic of the optical analog: a broad, coherent beam passes through a polarizer (P) and an analyzer (A), with a birefringent crystal (C) in between.

Mathematical Analogy

It introduces a small lateral displacement between the ordinary (x-polarized) and extraordinary (z-polarized) components, much smaller than the beam width and is determined by the crystal properties.

$$\boxed{\text{Electron's spin states} \leftrightarrow \text{Light's polarization states.}}$$

Initial Polarization State

Note: The shift observed is *lateral* rather than angular (as in the Stern–Gerlach case). Hence, the analysis focuses on the spatial z-distribution instead of momentum.

The input beam is assumed to have a wide Gaussian profile.
After passing through the polarizer, the state of the beam is

$$|\psi_{in}\rangle = \cos(\alpha)\,|\hat{x}\rangle + \sin(\alpha)\,|\hat{z}\rangle\,.$$

Following the theoretical model,

$$|\Psi_{in}\rangle = |\psi_{in}\rangle \otimes |\phi_{in}\rangle\,.$$

Projection on Pointer Position

Projecting on the pointer's position,

$$\begin{aligned} \langle q|\Psi_{in}\rangle &= |\psi_{in}\rangle \otimes \langle q|\phi_{in}\rangle \\ &= |\psi_{in}\rangle \; e^{-\frac{z^2}{4\Delta^2}} \\ &= (\cos(\alpha)\,|\hat{x}\rangle + \sin(\alpha)\,|\hat{z}\rangle)\, e^{-\delta^2 z^2}. \end{aligned}$$

This represents a Gaussian beam peaked at $z = 0$.

After Crystal

After passing through the birefringent crystal, the two orthogonal polarization components experience lateral displacements a_1 and a_2 along the $\hat{z}$-direction.

Hence, the field can be written as:

$$\langle q|\Psi\rangle = \underbrace{\cos(\alpha)\, e^{-\delta^2 (z-a_1)^2}}_{\text{Gaussian peaked at } z=a_1} |\hat{x}\rangle + \underbrace{\sin(\alpha)\, e^{-\delta^2 (z-a_2)^2}}_{\text{Gaussian peaked at } z=a_2} |\hat{z}\rangle\,.$$

Note

Note: The quantities a_1 and a_2 are not eigenvalues of polarization. They are physical displacements used as labels for the ordinary and extraordinary modes.

After Analyzer

After the analyzer at angle β, the projected field becomes:

$$\langle \psi_f | \Psi \rangle = \Big(\cos(\beta)\, |\hat{x}\rangle + \sin(\beta)\, |\hat{z}\rangle \Big) \cdot \Big(\cos(\alpha)\, e^{-\delta^2 (z-a_1)^2}\, |\hat{x}\rangle + \sin(\alpha)\, e^{-\delta^2 (z-$$

$$= \underbrace{\cos(\beta)\cos(\alpha)\, e^{-\delta^2 (z-a_1)^2} + \sin(\beta)\sin(\alpha)\, e^{-\delta^2 (z-a_2)^2}}_{\text{superposition of two Gaussians at } a_1, a_2}.$$

Near-Orthogonal Limit

Claim: In the near-orthogonal limit, this expression reduces to Duck's result.

$$\langle q|\Psi\rangle = \cos(\alpha+\beta)\left\{\frac{1}{2}\left[(1+\epsilon)\,e^{-\delta^2(z-a_1)^2} - (1-\epsilon)\,e^{-\delta^2(z-a_2)^2}\right]\right\}$$

for $\epsilon \ll 1$

Derivation of Duck's Form

$$\langle \psi_f | \Psi \rangle = \cos(\beta)\cos(\alpha)\, e^{-\delta^2 (z-a_1)^2} + \sin(\beta)\sin(\alpha)\, e^{-\delta^2 (z-a_2)^2}$$

$$= \frac{\cos(\alpha+\beta)}{2} \Bigg[\left(1 + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}\right) e^{-\delta^2 (z-a_1)^2} - \left(1 - \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}\right) e^{-\delta^2 (z-a_2}$$

For $\alpha = \pi/4, \;\; \beta = 3\pi/4 - \epsilon, \;\; \epsilon \ll 1$:

$$\langle q | \Psi \rangle = \cos(\alpha+\beta)\, \frac{1}{2} \left[(1+\epsilon) e^{-\delta^2 (z-a_1)^2} - (1-\epsilon) e^{-\delta^2 (z-a_2)^2} \right].$$

Weak Regime and Amplification

From our theoretical analysis, under the weak regime condition, this expression simplifies to a **single Gaussian** centered at the **weak value**.

Remarkable Result:

- The weak value can lie far outside the eigenvalue range.
- For example, in the spin case, it can reach values ~ 100 even though the eigenvalue limit is $1/2$.
- These tiny lateral shifts can thus be *amplified* under near-orthogonal pre- and post-selection.

This amplification allows detection of extremely small birefringence in the crystal.

Thank You