8/11/21.

NP- Completeness.

Algorithms : efficiency

n bits to specify the input.

$O(n^c)$ where c is a constant } polynomial time

$O(n)$, n^2, n^3 ...

2^n $\quad$ n^{10^6}

$n \sim 100$

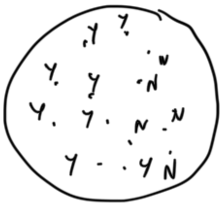
$2^{\sqrt{n}}$, $n^{\log n}$ ... : inefficient.

Q: Given an algorithmic problem, ?? is there an efficient algorithm for it ?

solution??

"Complexity theory" $\qquad$ NP- Completeness....

Definition: Decision Problems : A problem is called a decision problem if for every input, the answer is either YES or NO.



All possible inputs

φ : $(T \wedge x_2 \wedge x_3) \vee (F \wedge \bar{x}_2 \wedge x_3)$.

φ' : formula obtained by substituting $x_1 = T$ in φ.

φ'' : $x_1 = F$

Ex: (I) PRIMALITY ✓ Given a number n, is n prime? ✓

✓(II) FLOW Given a graph G with edge capacities u_e, does it have a max-flow of value T? ✓

(III) ✓ SATISFIABILITY : Boolean formula

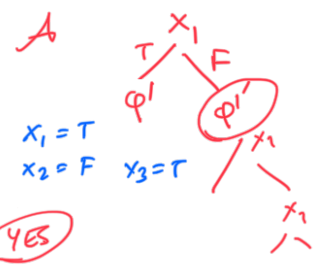
φ : Boolean variables $\rightarrow$ T/F

$x_1, \dots, x_n \rightarrow$ T/F

$\rightarrow \varphi = (x_1 \wedge x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3)$ $\rightarrow$ YES

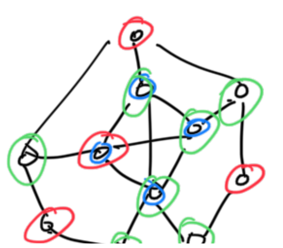
AND $\qquad$ OR $\qquad$ negation

$x_1 = T$, $x_2 = F$, $x_3 = T$



Q: Is there a setting of T/F values to the variables $x_1, \dots, x_n$ such that φ is True.

(IV) CLIQUE: G is an undirected graph. A subset of vertices is said to form a clique if there is an edge between every pair of them.



Find the largest clique in G. ✓

|| Given a graph G, $k \geq 0$  | does G have a clique of size $\geq k$? ||

(V) INDEPENDENT SET : G is an undirected graph. A subset of vertices is said to be an independent set if no pair of vertices have an edge between them.

Given a graph G, $k \geq 0$: does G have an IS of size $\geq k$?

(VI) VERTEX COVER : Given a graph G, a subset S of vertices is called a Vertex Cover if every edge has at least one end point in S.

S

Given a graph G, $k \geq 0$: does G have a VC of size $\leq k$?

(VII) SET COVER: X : any set $(X, \underline{S_1, S_2, \ldots, S_n})$ $S_i \subseteq X$

A sub collection $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ is called a set cover if $S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_k} = X$.

$k = 10$

$\boxed{S_1}$ $(X - S_1, \underbrace{S_2, \ldots, S_n})$

$k-1$

Q : Does $(X, S_1, \ldots, S_n)$ have a set cover consisting of $\leq k$ sets ?

(VIII) Subset Sum : Given n numbers (positive integer) $x_1, x_2, \ldots, x_n$ and a target T is there a subset of $x_1, \ldots, x_n$ which add to exactly T. ?

A

(IX) Hamiltonian Cycle : A graph G is said to be Hamiltonian if there is a cycle which contains all the vertices.

A : $x_2, \ldots, x_n$, $T - x_1$

Yes

No : $x_2, \ldots, x_n, T$

Q : given a graph G, is it Hamiltonian ??

x_2 : $x_1, x_3, \ldots, x_n, T - x_2$?

(X) - Decision vs Optimization / A more complex solution.

Is it enough to solve them ?? Yes.

SET-COVER $(X, \underbrace{S_1, \ldots, S_n})$ $\binom{n}{k} = n^k$.

Decision Version (k) ← Optimization Version : find a sub-collection which is a min. set cover.

Yes/No

[If there is an efficient algorithm $\boxed{A}$ to solve the decision version, then there is an efficient algorithm to solve the optimization version also.

By using A we can find the min. # sets in a set cover: $\boxed{k}$ ← ??

S_1 Does S_1 appear in the min. set cover?

$(X - S_1, \; S_2, \dots, S_n) \leftarrow k-1.$

$\frac{10}{::}$

Find Min Set Cover $(X, S_1, \dots, S_n)_k$ {

- if $(X - S_1, S_2, \dots, S_n)$ has a set cover of size $k-1$ ← use A

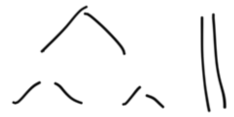
output S_1 and Find Min Set Cover $(X - S_1, S_2, \dots, S_n)_{k-1}$

else

Find Min Set Cover $(X, S_2, \dots, S_n)_k$ ←

}

Shall I take



$X, S_1, \dots, S_n$

$\downarrow A$

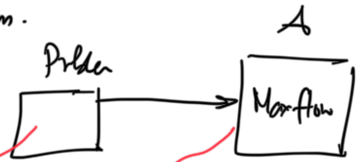
$X', S_2, \dots, S_n$

$\downarrow A$

⋮

Max n calls to A.

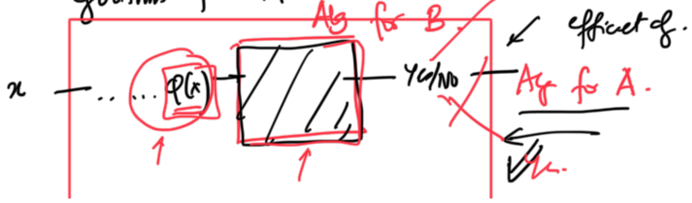
(II) **Reducibility**: You use the solution to one problem to solve another problem.



Let A, B be two decision problems.

We say that $A \leq_p B$ (A is reducible to B) if

"a polynomial time algorithm for B implies a polynomial time algorithm for A"



Reduction.

A $\qquad\qquad\qquad$ B

x : input for A $\xrightarrow{\varphi}$ $\varphi(x)$: input for B.

$\varphi(x) \rightarrow$ $\boxed{}$ Alg. for B $\rightarrow$ Yes/No

(i) φ can be implemented in poly. time.

(ii) Answer to x is YES $\Rightarrow$ Answer to $\varphi(x)$ is YES

" is NO $\Rightarrow$ " $\varphi(x)$ is NO

$p \Rightarrow q$

$\neg q \Rightarrow \neg p$.

$A \le_p B$ $\quad$ if B can be solved efficiently, then A can be solved efficiently.

($p \Rightarrow q$)

$\Updownarrow$

if A cannot be solved efficiently, then B can't be solved effic.

11/11/2021.

Major Exam : 17th 9 – 11:30. (11:30 – 12:00 upload time).

Syllabus: all the course, 75% questions from syllabus covered after Minor I : Divide & Conquer, DP, Flows, NP completeness.

- HW2 solution
- open notes from lectures but no books etc.

$A \le_p \boxed{B}$: if B has a polynomial time solution, then so does A.

Ex: (i) IS $\le_p$ Clique. $\quad$ $\mathcal{A}$ is a polynomial time alg. to solve CLIQUE.

IS $\longrightarrow$ CLIQUE

Input : $G = (V,E)$, k $\qquad$ Input' : $\bar{G}$, k

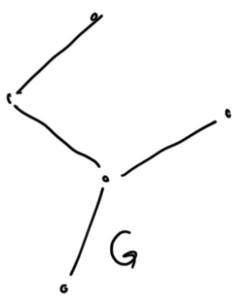
Q : Does G have an IS of size k ? $\qquad$ "opposite of G"

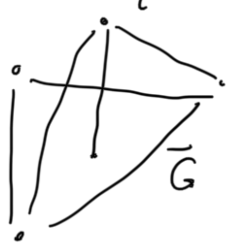
$\bar{G} = (V, \bar{E})$

$\bar{E} = \{(u,v) : (u,v) \notin E\}$

Time taken: $\boxed{\mathcal{A}}$ + time taken to Input $\rightarrow$ Input'

$n^2 \quad n^3 \quad n^4$





Answer for $\bar{G}$ $\equiv$ Answer for G.

To show: $\bar{G}$ has a clique of size k if and only if G has an IS of size k.

IMPORTANT: there are two statements here.

Pf: Suppose $\bar{G}$ has a clique of size k.

V' : set of k vertices in $\bar{G}$ which form a clique

$\longrightarrow$ in G, there is no edge between any pair of vertices in V'.

$\Rightarrow$ V' is an IS in G.

Suppose G has an IS of size k $\longrightarrow$ $\bar{G}$ has a clique of size k.

NO. $\longleftarrow$ NO

(II) CLIQUE $\le_p$ IS. ✓

$G, k \longrightarrow \bar{G}, k$

$\Rightarrow$ CLIQUE $\le_p$ VC

$\left.\begin{matrix} A \le_p B \\ B \le_p C \end{matrix}\right\} \Rightarrow A \le_p C$.

(III) IS $\le_p$ $\boxed{\text{VC}}$ A

$\boxed{\text{IS}}$ $\longrightarrow$ VC

Input $\qquad$ Input'

$G, k \longrightarrow G', k'$ Q: Does G' have a VC of size k'?

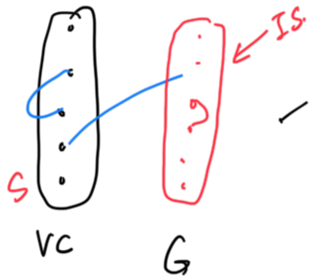
Q: Does G have IS of size k?

Yes	$\longleftarrow$	Yes
Yes	$\longrightarrow$	Yes

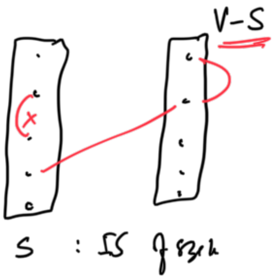
G has n vertices

IS $\longrightarrow$ VC

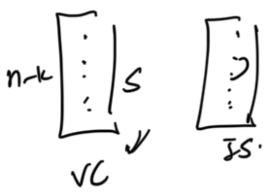
$G, k \qquad G, n-k$.



If G has an IS of size k, then G has a VC of size $n-k$. ✓



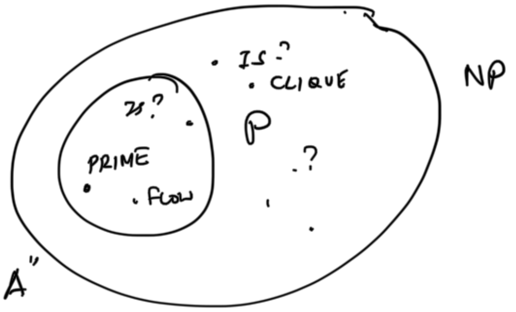
If G has a VC of size $n-k$, then G has a IS of size k.



Two classes of problems : $\mathbb{P}$ & NP.

P : set of decision problems for which there is a polynomial time alg.

NP : Non-deterministic poly. time.



A decision problem A is in the class NP if there is an efficient algorithm "$\mathcal{A}$ which can check a solution of A"

I : input.

take a subset S of vertices of size k, and check whether it is an IS.

$k^2 n$

G, k IS

Defn: (NP) A decision problem $A \in NP$ if there is a polynomial time alg $\mathcal{A}$ with the following property:

(i) Suppose x is an input for which the answer is YES then there is a string y such that $\mathcal{A}(x,y)$ outputs YES.

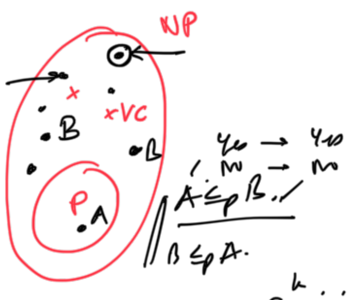
$\exists$; ↖ solution ; checks whether y is a valid soln for x.

(ii) If x is an input for which the answer is NO. then for all strings y $\mathcal{A}(x,y)$ outputs NO.

$\forall$

(1) $\mathbb{P} \subseteq NP$? Suppose $A \in \mathbb{P}$. $\mathcal{A}$ y = empty string

$\boxed{\mathbb{P} = NP ?}$

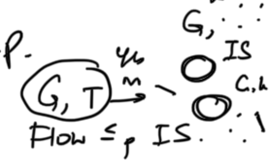


Problem : A

Want to know is $A \in P$ or not ??

Suppose $B \notin P$ and $B \le_p A$: reduction gives $A \notin P$.

$A \notin P$



Flow $\le_p$ IS.

Is there any problem which is not in P ? What if $P \neq NP$? Yes!

[Cook's thm '71] If L is a problem in NP, 3-SAT.

$L \le_p$ SATISFIABILITY.

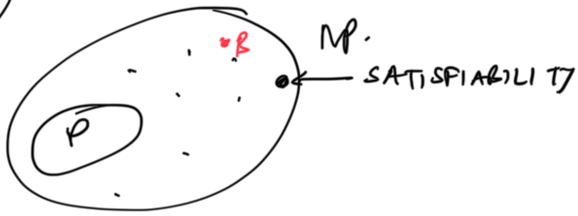
NP-Complete.

$A \le_p B$ — prog. for alg.

x → yes.
→ no.

⇒ if there is a poly. time alg. for SATISFIABILITY, then there is a poly time alg. for any $L \in NP$.

⇒ if $P \ne NP$, then SATISFIABILITY $\notin P$.



Def (NP-Completeness) A problem "B" is said to be NP-Complete
(i) $B \in NP$
(ii) $\forall$ problem $L \in NP$, $L \le_p B$.

Cook's th.
⇒ SATISFIABILITY is NP-C..

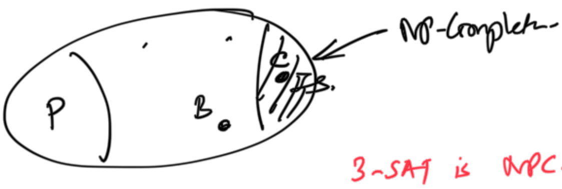
⇒ if B is NP-Complete & $P \ne NP$, then $B \notin P$.

We "believe" that $P \ne NP$.

$L \le_p$ SATISFIAB. ⇒ SATISFIABILITY $\notin P$.

SATISFIABILITY $\le_p$ B ⇒ B is also NP-complete!

NP



$C \le_p B$ & C is NP-Complete
⇒ B is also NP-Complete

3-SAT is NPC.
3-SATISFIABILITY

Thm: SATISFIABILITY (3SAT) $\le_p$ IS.
⇒ IS also NP-complete.

IS $\le_p$ CLIQUE ⇒ CLIQUE NP-Complete.
IS $\le_p$ VC ⇒ VC is NP-Complete.
IS $\le_p$ Hamilt. ⇒ ...

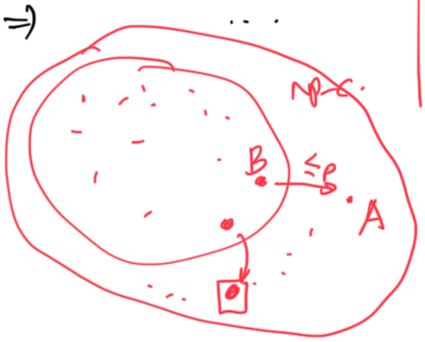
3-CNF Boolean Formula

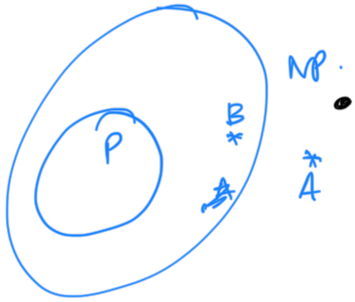
$[\underbrace{(x_1 \vee x_2 \vee \bar{x}_3)}_{C_1} \wedge \underbrace{(\quad)}_{C_2} \wedge \underbrace{(\quad)}_{C_3} \wedge (\quad)]$

$C_1 \wedge \dots \wedge C_m$ m clauses

OR of 3 variables.

$(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee \bar{x}_3) \wedge$
$(\_\,\_\,\_) \wedge (\_\,\_\,\_)$





NP-Hard!

A problem A is NP-Hard if $\forall L \in NP$, $L \le_p A$

NP-Complete B, $B \le_p A$.

(G, k) : G does not have a VC of size k. : Believed to be not in NP.

16th: 4 pm ??

Sample problem:

6