Social Network Analysis

Game Theory

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Games

- Game Theory is designed to address situations in which the outcome of a person's decision depends not just on how they choose among several options, but also on the choices made by the people they are interacting with.
  - Choosing how to target a soccer penalty kick and choosing how to defend against it can be modelled using game theory.
  - Pricing of a new product when other firms have similar new products.
  - Choosing a route on the Internet or through a transportation network.
  - Deciding whether to adopt an aggressive or a passive stance in international relations…

What is a Game?

- You are a college student. You have an exam, and a presentation.
- You can either study for the exam or prepare for the presentation, but not both.
- *Exam:* If you study, then your expected grade is a 92, while if you don't study, then your expected grade is an 80.
- *Preso:* If both you and your partner prepare for the presentation, then the presentation will go extremely well, and your expected joint grade is a 100.

  If just one of you prepares (and the other doesn't), you'll get an expected joint grade of 92; and if neither of you prepares, your expected joint grade is 84.
- Your partner is also in the same situation. You are unable to contact each other.

What is a Game?

- Both of you are interested in maximizing the average grade you get.

- If both of you prepare for the presentation, you'll both get 100 on the presentation and 80 on the exam, for an average of 90.

- If both of you study for the exam, you'll both get 92 on the exam and 84 on the presentation, for an average of 88.

- If one of you studies for the exam while the other prepares for the presentation, the result is as follows.
  - The one who prepares for the presentation gets a 92 on the presentation but only an 80 on the exam, for an average of 86.

  - On the other hand, the one who studies for the exam still gets a 92 on the presentation — since it's a joint grade, this person benefits from the fact that one of the two of you prepared for it. This person also get a 92 on the exam, through studying, and so gets an average of 92.

Exam or Presentation?

Players

Strategies

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Payoffs

Assumptions

- Everything that a player cares about is summarized in the player's payoffs.
  - A player who is altruistic may care about both his or her own benefits, and the other player's benefit.
  - If so, then the payoffs should reflect this: should constitute a complete description of each player's happiness with each of the possible outcomes of the game.
- Each player knows everything about the structure of the game.
- Each player knows his or her own list of possible strategies.
- Each player also knows the strategies available to this other player, and what his or her payoff will be for any choice of strategies. (Games with complete information)

Assumptions

- Each individual chooses a strategy to maximize her own payoffs given her beliefs about the strategy used by the other player.
- This model of individual behavior, which is usually called rationality, actually combines two ideas.
- The first idea is that each player wants to maximize her own payoff.
- The second idea is that each player actually succeeds in selecting the optimal strategy.

Exam or Presentation?

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

First, if you knew your partner was going to study for the exam,

then you would get a payoff of 88 by also studying, and a payoff of only 86 by preparing for the presentation.

So in this case, you should study for the exam.

Exam or Presentation?

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

On the other hand, if you knew that your partner was going to prepare for the presentation,

then you'd get a payoff of 90 by also preparing for the presentation, but a payoff of 92 by studying for the exam.

So in this case too, you should study for the exam.

Exam or Presentation?

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

This approach of considering each of your partner's options separately turns out to be a very useful way of analyzing the present situation: it reveals that no matter what your partner does, you should study for the exam.

Exam or Presentation?

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

- When a player has a strategy that is strictly better than all other options regardless of what the other player does, we will refer to it as a strictly dominant strategy.

- When a player has a strictly dominant strategy, we should expect that they will definitely play it.

- So, we should expect that the outcome will be for both of you to study, each getting an average grade of 88.

Exam or Presentation?

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- If you and your partner could somehow agree that you would both prepare for the presentation, you would each get an average grade of 90 — in other words, you would each be better off.
- Even though you both understand this, this payoff of 90 cannot be achieved by rational play.
- Even if you were to personally commit to preparing for the presentation, your partner would still have an incentive to study for the exam so as to achieve a still-higher payoff of 92 for himself.

Exam or Presentation?

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- The payoffs must truly reflect everything each player values in the outcome.
- If you care about the grade that your partner received as well, or
- If you care about the fact that your partner will be angry at you for not preparing for the joint presentation,
- then, this should be reflected in the payoffs, and will influence the outcome.

Prisoner's Dilemma

Suspect 2 (columns), Suspect 1 (rows)

	NC	C
NC	$-1, -1$	$-10, 0$
C	$0, -10$	$-4, -4$

Prisoner's Dilemma

- The Prisoner's Dilemma has been the subject of a huge amount of literature since its introduction in the early 1950s.
- It serves as a highly streamlined depiction of the difficulty in establishing cooperation in the face of individual self-interest.
- While no model this simple can precisely capture complex scenarios in the real world, the Prisoner's Dilemma has been used as an interpretive framework for many different real-world situations.
- For example, the use of performance-enhancing drugs in professional sports has been modeled as a case of the Prisoner's Dilemma game.

Performance-Enhancing Drugs

- The athletes are the players, and the two possible strategies are to use performance-enhancing drugs or not.
- If you use drugs while your opponent doesn't, you'll get an advantage in the competition, but you'll suffer long-term harm (and may get caught).
- If we consider a sport where it is difficult to detect the use of such drugs, and we assume athletes in such a sport view the downside as a smaller factor than the benefits in competition.
- Then, we can capture the situation with numerical payoffs that might look as follows.

Performance-Enhancing Drugs

		Athlete 2: Don't Use Drugs	Athlete 2: Use Drugs
Athlete 1	Don't Use Drugs	3, 3	1, 4
	Use Drugs	4, 1	2, 2

- The best outcome (with a payoff of 4) is to use drugs when your opponent doesn't, since then you maximize your chances of winning.
- However, the payoff to both using drugs (2) is worse than the payoff to both not using drugs (3), since in both cases you're evenly matched, but in the former case you're also causing harm to yourself.
- We can now see that using drugs is a strictly dominant strategy, and so we have a situation where the players use drugs even though they understand that there's a better outcome for both of them.

Prisoner's Dilemma

- More generally, situations of this type are often referred to as arms races, in which two competitors use an increasingly dangerous arsenal of weapons simply to remain evenly matched.

- The performance-enhancing drugs play the role of the weapons, but the Prisoner's Dilemma has also been used to interpret literal arms races between opposing nations, where the weapons correspond to the nations' military arsenals.

- Prisoner's Dilemma occurs only when the payoffs are aligned in a certain way.

- Even simple changes to a game can change it from an instance of the Prisoner's Dilemma to something more benign.

Exam or Presentation (v2.0)

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$98, 98$	$94, 96$
	Exam	$96, 94$	$92, 92$

Prisoner's Dilemma Variation

	No Confess	Confess
No Confess	-1, -1	-10, 0
Confess	0, -10	-4, -4

	No Confess	Confess
No Confess	-1, -1	-10, 0
Confess	0, -10	-10, -10

Traffic Problem

	Rules	No Rules
Rules	-5, -5	-30, -20
No Rules	-20, -30	-40, -40

	Rules	No Rules
Rules	-5, -5	-40, -20
No Rules	-20, -40	-30, -30

Best Responses and Dominant Strategies

- Best response: it is the best choice of one player, given a belief about what the other player will do.
- For instance, in the Exam-or-Presentation Game, we determined your *best choice in response to each possible choice* of your partner.
- If S is a strategy chosen by Player 1, and T is a strategy chosen by Player 2, then there is an entry in the payoff matrix corresponding to the pair of chosen strategies (S,T).
- We will write P1(S,T) to denote the payoff to Player 1 as a result of this pair of strategies, and P2(S,T) to denote the payoff to Player 2 as a result of this pair of strategies.

Best Responses and Dominant Strategies

- We say that a strategy S for Player 1 is a best response to a strategy T for Player 2 if S produces at least as good a payoff as any other strategy paired with T:

$$P_1(S,T) \geq P_1(S',T)$$

for all other strategies S' of Player 1.

- This definition allows for multiple different strategies of Player 1 to be tied as the best response to strategy T .

- A strategy S of Player 1 is a strict best response to a strategy T for Player 2 if S produces a strictly higher payoff than any other strategy paired with T:

$$P_1(S,T) > P_1(S',T)$$

for all other strategies S' of Player 1.

Best Responses and Dominant Strategies

- We say that a dominant strategy for Player 1 is a strategy that is a best response to every strategy of Player 2.
- We say that a strictly dominant strategy for Player 1 is a strategy that is a strict best response to every strategy of Player 2.
- If a player has a strictly dominant strategy, then we can expect him or her to use it.
- The notion of a dominant strategy is slightly weaker, since it can be tied as the best option against some opposing strategies. As a result, a player could potentially have multiple dominant strategies, in which case *it may not be obvious which one should be played.*

Game with single player dominant strategy

- There are two firms that are each planning to produce and market a new product; these two products will directly compete with each other.
- The population of consumers can be cleanly divided into two market segments: people who would only buy a low-priced version of the product, and people who would only buy an upscale version.
- This game has two players — Firm 1 and Firm 2 — and each has two possible strategies: to produce a low-priced product or an upscale one.
- People who would prefer a low-priced version account for 60% of the population, and people who would prefer an upscale version account for 40% of the population.
- When the two firms directly compete in a market segment, Firm 1 gets 80% of the sales and Firm 2 gets 20% of the sales.

Marketing Strategy

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- If the two firms market to different market segments, they each get all the sales in that segment.
- If both firms target the low-priced segment, then Firm 1 gets 80% of it, for a payoff of .48, and Firm 2 gets 20% of it, for a payoff of .12.

Marketing Strategy

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- For Firm 1, Low-Priced is a strict best response to each strategy of Firm 2. On the other hand, for Firm 2: Low-Priced is its best response when Firm 1 plays Upscale, and Upscale is its best response when Firm 1 plays Low-Priced.
- Firm 1 has a strictly dominant strategy, Firm 2 does not.
- Since Firm 1 has a strictly dominant strategy in Low-Priced, we can expect it will play it. Then, since Upscale is the strict best response by Firm 2 to Low-Priced, we can predict that Firm 2 will play Upscale.

Games, so far…

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

		Suspect 2: *NC*	Suspect 2: *C*
Suspect 1	*NC*	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	$.48, .12$	$.60, .40$
	Upscale	$.40, .60$	$.32, .08$

A 3-client game…

- When neither player in a two-player game has a strictly dominant strategy, we need some other way of predicting what is likely to happen.

- There are two firms that each hope to do business with one of three large clients, A, B, and C.
  - If the two firms approach the same client, then the client will give half its business to each.
  - Firm 1 is too small to attract business on its own, so if it approaches one client while Firm 2 approaches a different one, then Firm 1 gets a payoff of 0.

		Firm 2		
		A	B	C
	A	$4, 4$	$0, 2$	$0, 2$
Firm 1	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

hence 1 to each firm if it's split).

A 3-client game…

- When neither player in a two-player game has a strictly dominant strategy, we need some other way of predicting what is likely to happen.

- There are two firms that each hope to do business with one of three large clients, A, B, and C.
  - If the two firms approach the same client, then the client will give half its business to each.
  - Firm 1 is too small to attract business on its own, so if it approaches one

		Firm 2		
		A	B	C
Firm 1	A	4, 4	0, 2	0, 2
	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

4 to each firm if it's split), while doing business with B or C is worth 2 (and hence 1 to each firm if it's split).

A 3-client game…

		Firm 2		
		A	B	C
Firm 1	A	$4,4$	$0,2$	$0,2$
	B	$0,0$	$1,1$	$0,2$
	C	$0,0$	$0,2$	$1,1$

- For Firm 1, A is a strict best response to strategy A by Firm 2, B is a strict best response to B, and C is a strict best response to C. For Firm 2, A is a strict best response to strategy A by Firm 1, C is a strict best response to B, and B is a strict best response to C.

- How should we reason about the outcome of play in this game?

Nash Equilibrium

- In 1950, John Nash proposed a simple but powerful principle for reasoning about behaviour in general games, and its underlying premise is the following:

- Even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other.

- Suppose that Player 1 chooses a strategy S and Player 2 chooses a strategy T. We say that this pair of strategies (S,T) is a Nash equilibrium if S is a best response to T, and T is a best response to S.

- The idea is that if the players choose strategies that are best responses to each other, then no player has an incentive to deviate to an alternative strategy — so the system is in a kind of equilibrium state, with no force pushing it toward a different outcome.

Nash Equilibrium

- In the Three-Client Game,
- If Firm 1 chooses A and Firm 2 chooses A, then we can check that Firm 1 is playing a best response to Firm 2's strategy, and Firm 2 is playing a best response to Firm 1's strategy.
- Hence, the pair of strategies (A, A) forms a Nash equilibrium.
- This is the only Nash equilibrium. (How?)

		Firm 2		
		A	B	C
	A	$4, 4$	$0, 2$	$0, 2$
Firm 1	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

Nash Equilibrium

- Two ways to find Nash equilibria:
  - Check all pairs of strategies, and ask for each one of them whether the individual strategies are best responses to each other.
  - Compute each player's best response(s) to each strategy of the other player, and then find strategies that are mutual best responses.

- For a game with a single Nash equilibrium, such as the 3-Client Game, it seems reasonable to predict that the players will play the strategies in this equilibrium..

- ..under any other play of the game, at least one player will not be using a best response to what the other is doing. (How?)

Games, so far…

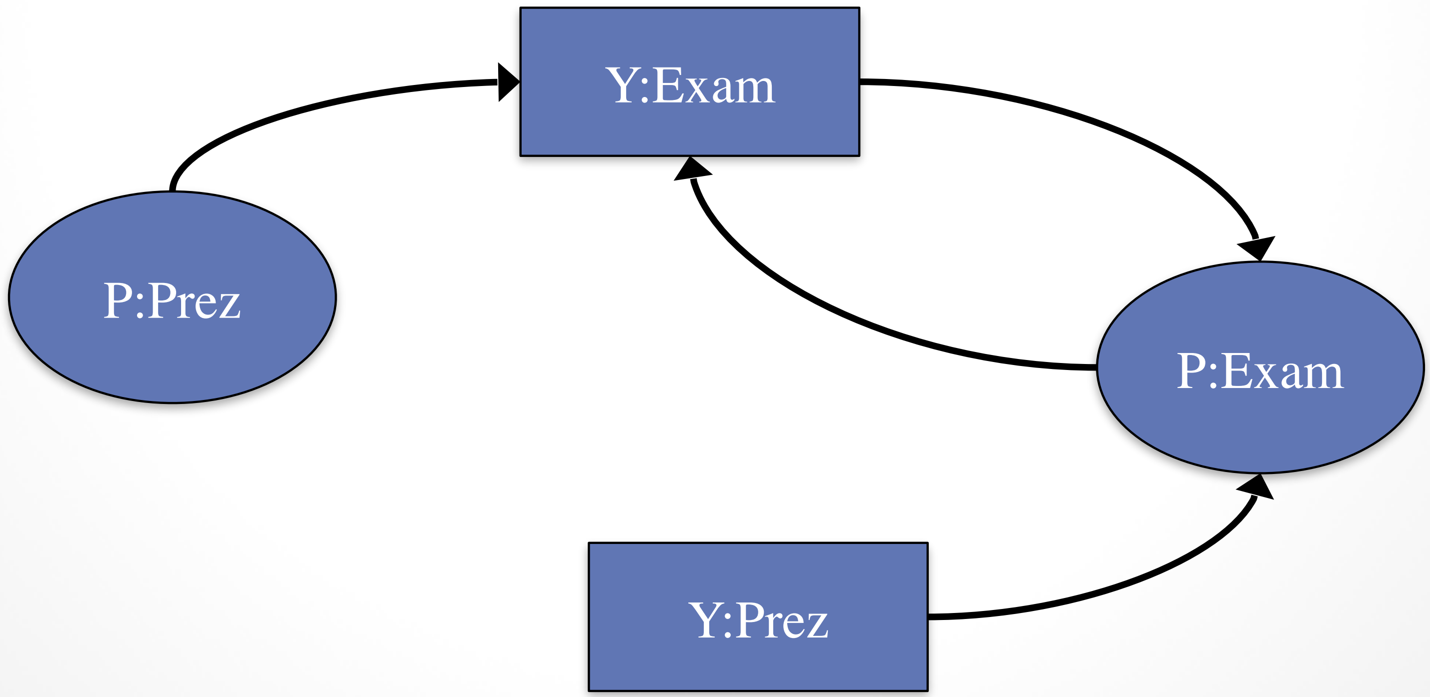
		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

		Suspect 2: *NC*	Suspect 2: *C*
Suspect 1	*NC*	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

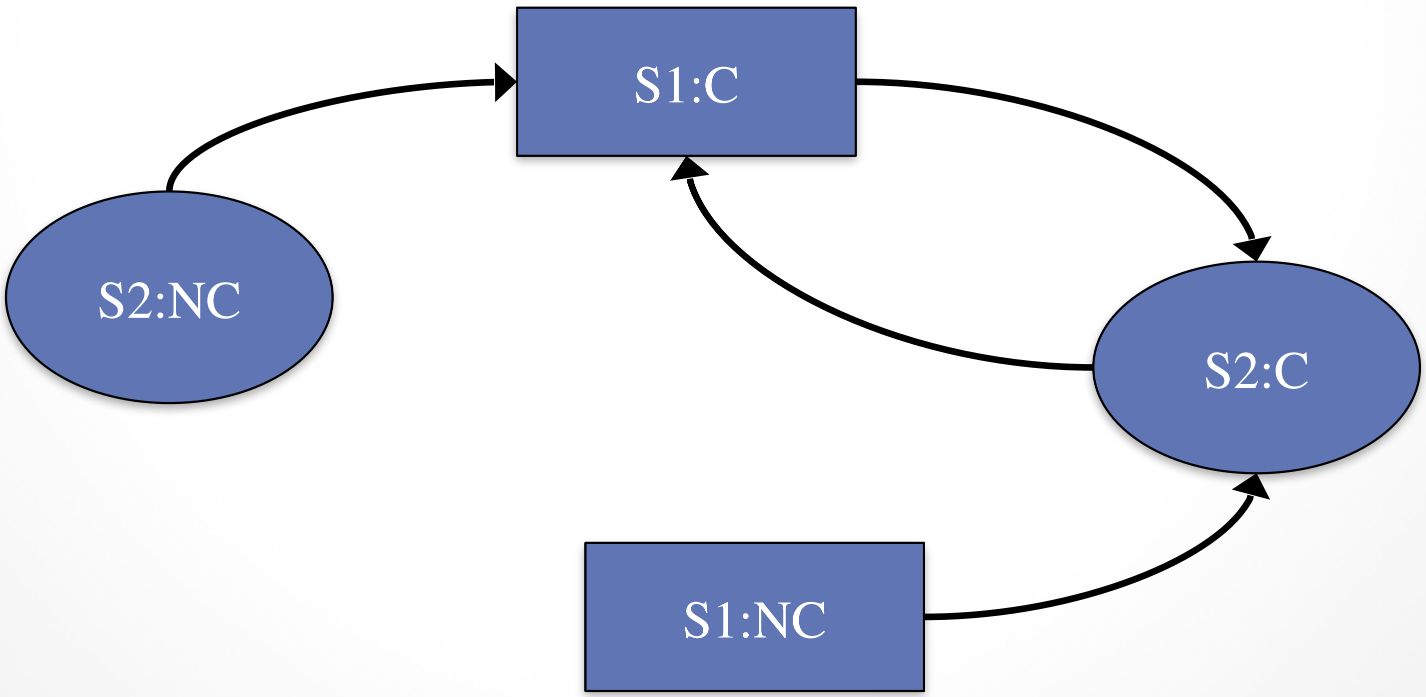
Best Response Graphs
Exam vs. Presentation

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88



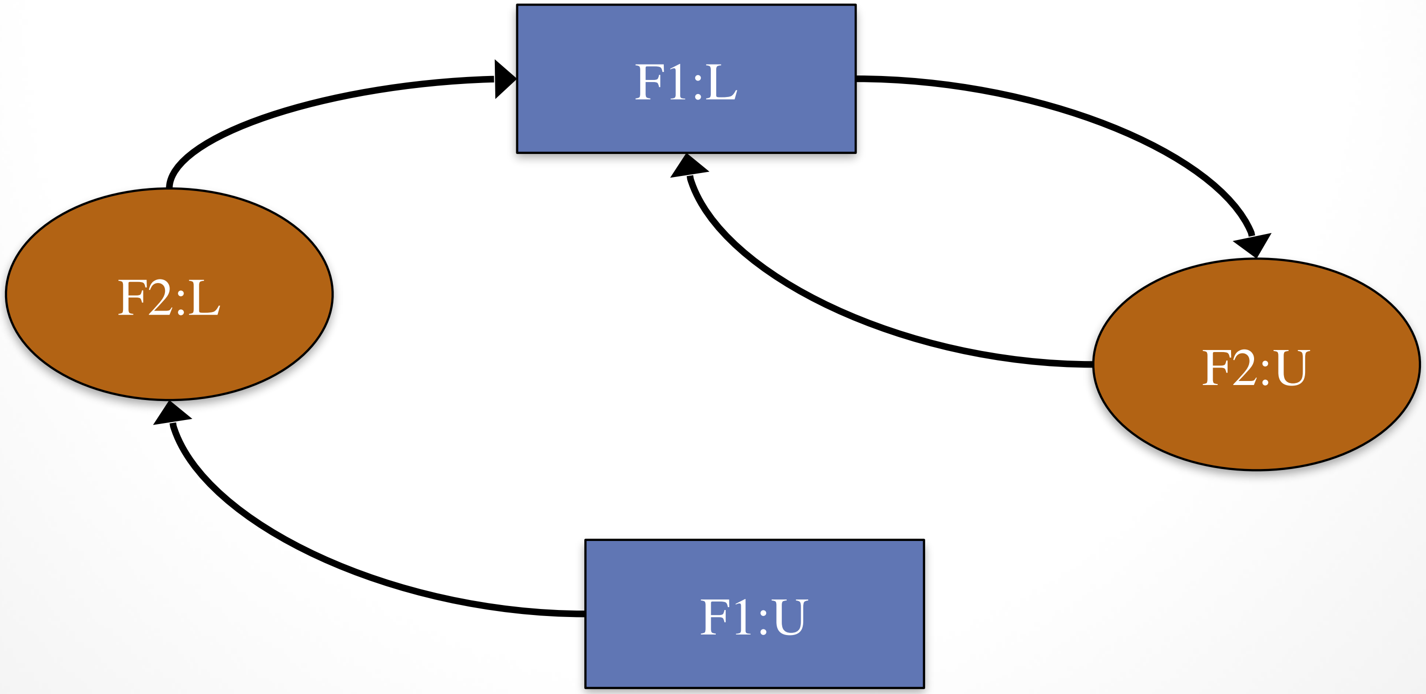
Best Response Graphs
Prisoner's Dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$



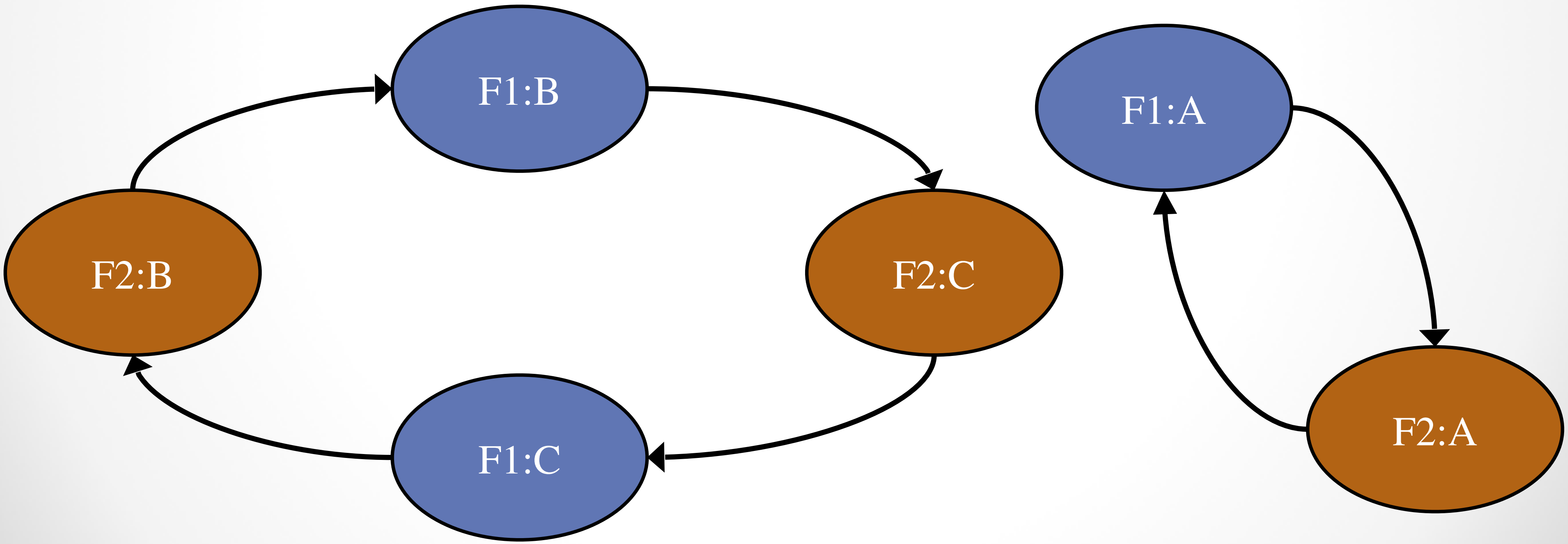
Best Response Graphs
Marketing Strategy

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08



Best Response Graph

		Firm 2 A	Firm 2 B	Firm 2 C
Firm 1	A	4, 4	0, 2	0, 2
Firm 1	B	0, 0	1, 1	0, 2
Firm 1	C	0, 0	0, 2	1, 1



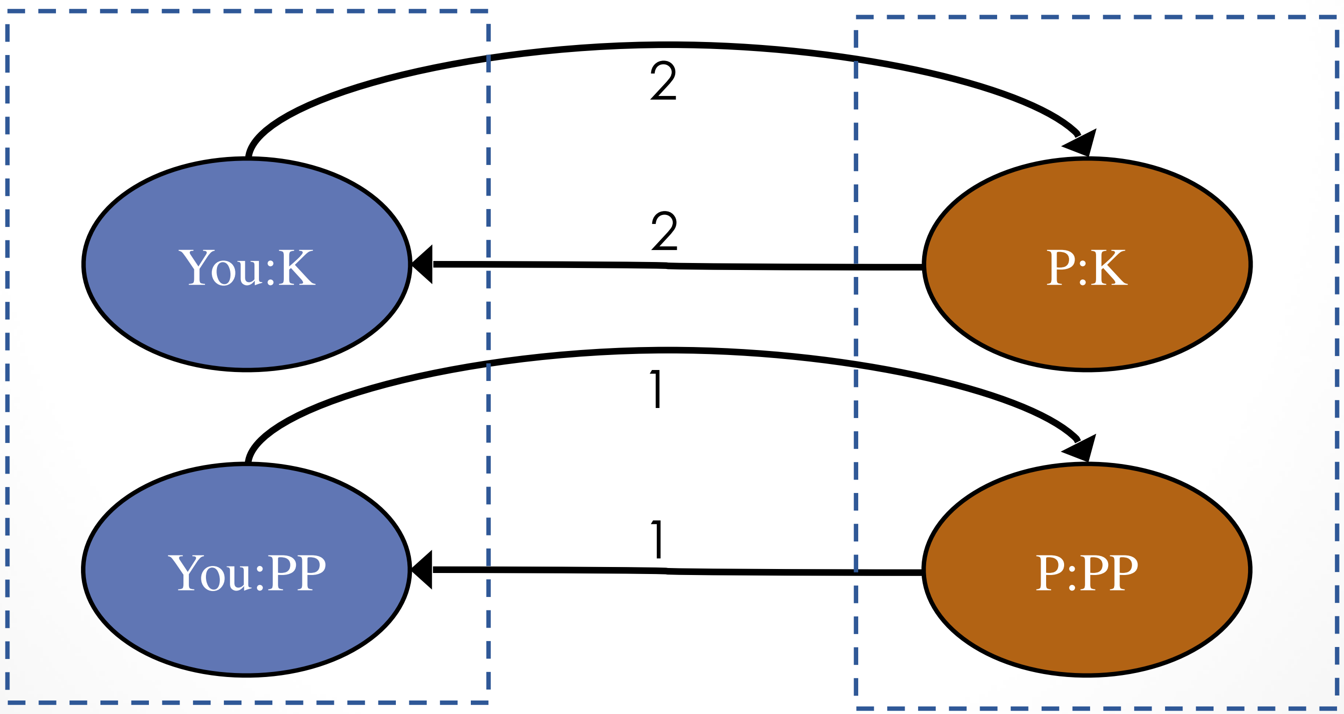
A Coordination Game

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

- You have to decide whether to prepare your half of the slides in PowerPoint or in Apple's Keynote software.
- Either would be fine, but it will be much easier to merge your slides together with your partner's if you use the same software.
- This is called a Coordination Game because the two players' shared goal is really to coordinate on the same strategy.
  - Two platoons in the same army need to decide whether to attack an enemy's left flank or right flank;
  - two people trying to find each other in a crowded mall need to decide whether to wait at the north end of the mall or at the south end.

Best Response Graph

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2



Multiple Equlibria: Coordination Games

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	$1, 1$	$0, 0$
You	*Keynote*	$0, 0$	$1, 1$

- The underlying difficulty is that the game has two Nash equilibria — (PowerPoint,PowerPoint) and (Keynote,Keynote).
- If the players fail to coordinate on one of the Nash equilibria, perhaps because one player expects PowerPoint to be played and the other expects Keynote, then they receive low payoffs.
- So what do the players do?

Focal Points

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

- Thomas Schelling introduced the idea of a focal point as a way to resolve this difficulty.
- He noted that in some games there are natural reasons (possibly outside the payoff structure of the game) that cause the players to focus on one of the Nash equilibria.
  - Drivers avoiding a collision. (social convention)

Focal Points in Unbalanced Coordination Games

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

- Here, Schelling's theory of focal points suggests that we can use a feature *intrinsic* to the game – higher payoffs for both the players.
- But what happens when the payoffs are as follows?

	Opera	Football
Opera	3,2	0,0
Football	0,0	2,3

Focal Points in Unbalanced Coordination Games

		Suspect 2 *NC*	Suspect 2 *C*
Suspect 1	*NC*	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

		Hunter 2 *Hunt Stag*	Hunter 2 *Hunt Hare*
Hunter 1	*Hunt Stag*	$4, 4$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

- This is quite similar to the Unbalanced Coordination Game, except that if the two players miscoordinate, the one who was trying for the higher-payoff outcome gets penalized more than the one who was trying for the lower-payoff outcome.

		Your Partner *Presentation*	Your Partner *Exam*
You	*Presentation*	$90, 90$	$82, 88$
	Exam	$88, 82$	$88, 88$

Multiple Equilibria: Hawk-Dove Game

- Multiple Nash equilibria also arise in a different but equally fundamental kind of game, in which the players engage in a kind of "anti-coordination" activity.

- Two animals are engaged in a contest to decide how a piece of food will be divided between them.

- Each animal can choose to behave aggressively (the Hawk strategy) or passively (the Dove strategy).

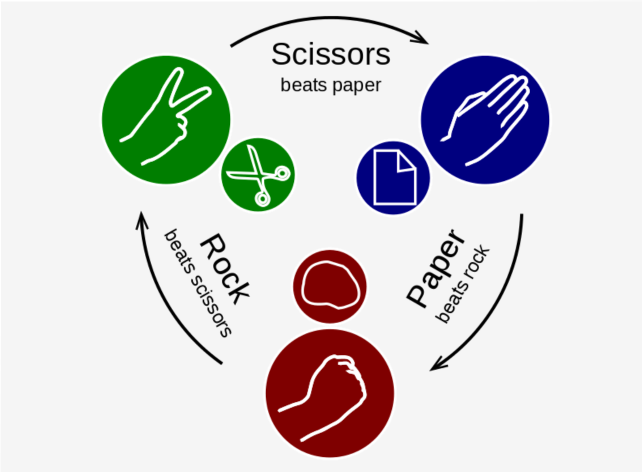
		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

- This game has two Nash equilibria: (D, H) and (H, D). Without knowing more about the animals we cannot predict which of these equilibria will be played.

Mixed Equilibria

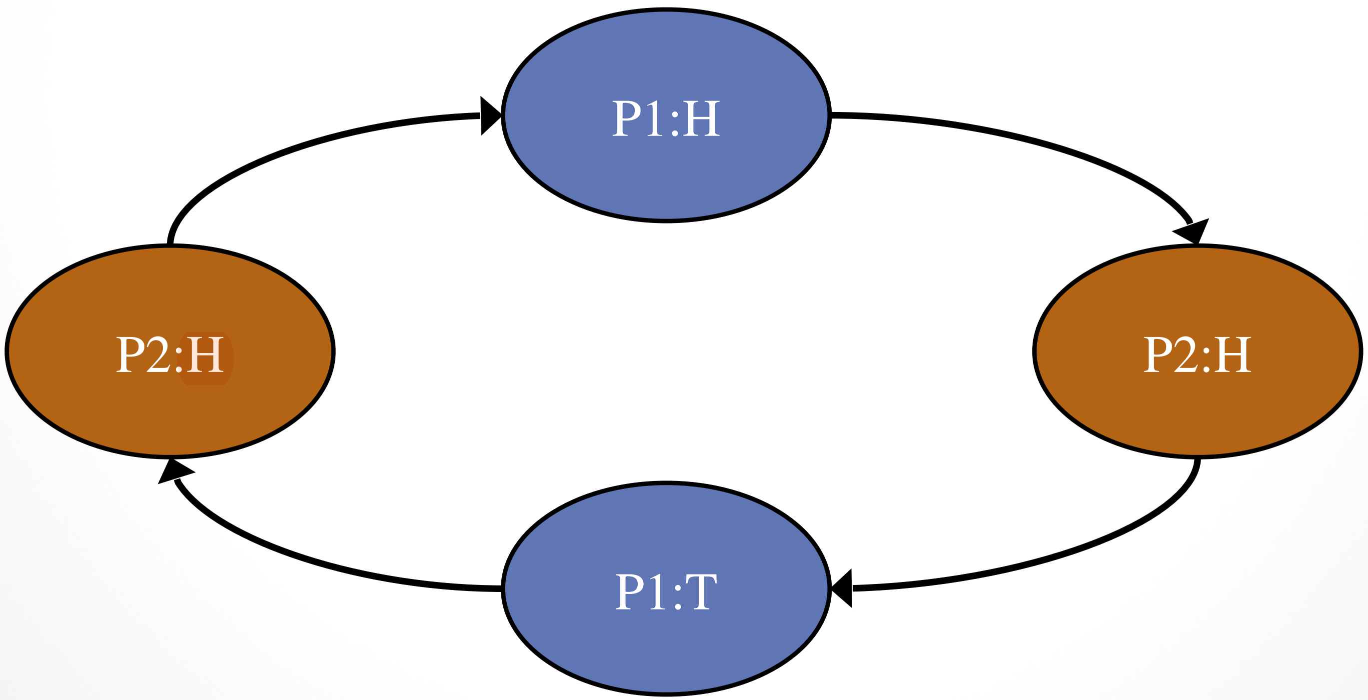
		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- There are games that have no Nash equilibria.
- This is an example of a zero-sum game.
- No pair of strategies is a best response for each other.
- Players generally try to make it difficult for their opponents to predict what they will play.



Best Response Graph

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$



Mixed Strategies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- We shouldn't treat the strategies as simply H or T, but as ways of randomizing one's behavior between H and T.
- Randomized behavior: Each player is not actually choosing H or T directly, but rather is choosing a probability with which she will play H.
- A given number p means that Player 1 is committing to play H with probability p, and T with probability 1 – p. ($0 \le p \le 1$)
- Similarly, the possible strategies for Player 2 are numbers q ($0 \le q \le 1$), representing the probability that Player 2 will play H.

Mixed Strategies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- By allowing randomization, we have actually changed the game.
- It no longer consists of two strategies by each player, but instead a set of strategies corresponding to the interval of numbers between 0 and 1.
- We will refer to these as mixed strategies, since they involve *"mixing" between the options H and T* .
- The set of mixed strategies still includes the original two options of committing to definitely play H or T (p = 1 or p = 0).
- We call these pure strategies.

Mixed Strategies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- With this new set of strategies, we also need to determine the new set of payoffs.
- Payoffs are now random quantities: each player will get +1 with some probability, and will get –1 with the remaining probability.
- When payoffs were numbers, it was obvious how to rank them: bigger was better.
- We want a principled way to say that one random outcome is better than another.

Mixed Strategies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Suppose that Player 2 chooses the strategy q; that is, he commits to playing H with probability q and T with probability 1 – q.
  - Then if Player 1 chooses pure strategy H, she receives a payoff of –1 with probability q (since the two pennies match with probability q, in which event she loses), and she receives a payoff of +1 with probability 1 – q.
- So even if Player 1 uses a *pure* strategy, her payoffs can still be *random* due to the randomization employed by Player 2.
- In order to rank random payoffs numerically, we will calculate the expected value of the payoff.

Mixed Strategies

		Player 2 H	Player 2 T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- So for example, if Player 1 chooses the pure strategy H while Player 2 chooses a probability of q, as above, then the expected payoff to Player 1 is:

$$(-1)(q)+(1)(1-q) = 1-2q.$$

- Similarly, if Player 1 chooses the pure strategy T while Player 2 chooses a probability of q, then the expected payoff to Player 1 is

$$(1)(q)+(-1)(1-q) = 2q-1.$$

- Mixed-strategy version of the Matching Pennies game: strategies are probabilities of playing H, and payoffs are the expectations of the payoffs from the four pure outcomes (H,H),(H,T),(T,H), and (T,T).

- We can now ask whether there is a Nash equilibrium for this richer version of the game.

Equilibrium with Mixed Strategies

		Player 2 H	Player 2 T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Nash equilibrium (mixed- strategy version): it is a pair of strategies (now probabilities) so that each is a best response to the other.

- No pure strategy can be part of a Nash equilibrium for this game. So, both players have probabilities, $0 < p < 1$, $0 < q < 1$ as best responses.

- Player 1's best response should be to the strategy q used by Player 2:
  - Expected payoff by playing purely H: $1 - 2q$.
  - Expected payoff by playing purely T: $2q - 1$.

- If $1 - 2q \neq 2q - 1$, then one of the pure strategies H or T is in fact the unique best response by Player 1 to a play of q by Player 2.

Equilibrium with Mixed Strategies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- So, for Nash equilibrium for the mixed- strategy version of matching pennies

$$1 - 2q = 2q - 1$$

$$q = ½$$

- The situation is symmetric when we consider things from Player 2's point of view, so in any Nash equilibrium, we also have p = ½.

- The pair of strategies p = 1/2 and q = 1/2 is the only possibility for a Nash equilibrium.

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Suppose two people actually sit down to play Matching Pennies, and each of them actually commits to behaving randomly according to probabilities p and q respectively.
- If Player 1 believes that Player 2 will play H strictly more than half the time, then she should definitely play T – in which case Player 2 should not be playing H more than half the time.
- So the point is that the choice of q = 1/2 by Player 2 makes Player 1 indifferent between playing H or T: the strategy q = 1/2 is effectively "non-exploitable" by Player 1.
- This was our original intuition for introducing randomization: each player wants their behavior to be unpredictable to the other, so that their behaviour can't be taken advantage of.

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- This notion of indifference is a general principle behind the computation of mixed-strategy equilibria in two-player, two-strategy games when there are no equilibria involving pure strategies:
  - Each player should randomize so as to make the other player indifferent between their two alternatives.
- This way, neither player's behaviour can be exploited by a pure strategy, and the two choices of probabilities are best responses to each other.
- Nash's main mathematical result accompanying his definition of equilibrium was to prove that every game with finite number of players and finite number of strategies has at least one mixed-strategy equilibrium.

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Sometimes, particularly when the participants are genuinely playing a sport or game, the players may be actively randomizing their actions:
  - a tennis player may be randomly deciding whether to serve the ball up the center or out to the side of the court;
  - a card-player may be randomly deciding whether to bluff or not;
- In the case of Matching Pennies, with its unique mixed equilibrium, this means that it is enough for you to expect that when you meet an arbitrary person, they will play their side of Matching Pennies with a probability of 1/2.
- In this case, playing a probability of 1/2 makes sense for you too, and hence this choice of probabilities is self-reinforcing — it is in equilibrium — across the entire population.

Mixed Strategies: Examples and Empirical Analysis

		Defense	
		Defend Pass	*Defend Run*
Offense	*Pass*	$0, 0$	$10, -10$
	Run	$5, -5$	$0, 0$

- An attack-defense game with two players named "offense" and "defense" respectively, and where the attacker has a stronger option (pass) and a weaker option (run).

- Check that there is no Nash equilibrium where either player uses a pure strategy.

- First, suppose the defense chooses a probability of q for defending against the pass. Then the expected payoff to the offense from passing is:

$$(0)(q) + (10)(1 - q) = 10 - 10q,$$

- while the expected payoff to the offense from running is

$$(5)(q) + (0)(1 - q) = 5q.$$

Mixed Strategies: Examples and Empirical Analysis

		Defense: *Defend Pass*	Defense: *Defend Run*
Offense	*Pass*	$0, 0$	$10, -10$
	Run	$5, -5$	$0, 0$

- To make the offense indifferent between its two strategies, we need to set $10 - 10q = 5q$, and hence $q = 2/3$.

- Next, suppose the offense chooses a probability of p for passing. Then the expected payoff to the defense from defending against the pass is
$$(0)(p) + (-5)(1 - p) = 5p - 5,$$

- with the expected payoff to the defense from defending against the run is:
$$(-10)(p) + (0)(1 - p) = -10p.$$

- To make the defense indifferent between its two strategies, we need to set $5p - 5 = -10p$, and hence $p = 1/3$.

- The only possible probability values that can appear in a mixed-strategy equilibrium are $p = 1/3$ for the offense, and $q = 2/3$ for the defense.

Mixed Strategies: Examples and Empirical Analysis

		Defense: *Defend Pass*	Defense: *Defend Run*
Offense	*Pass*	$0, 0$	$10, -10$
	Run	$5, -5$	$0, 0$

- The expected payoff to the offense with these probabilities is 10/3, and the corresponding expected payoff to the defense is –10/3.
- Although passing is the offense's more powerful weapon, it uses it only 1/3 the time. Why?
- If the offense placed any higher probability on passing, then the defense's best response would be to always defend against the pass, and the offense would actually do worse in expectation.
- For example, if we take p = 1/2, the defense will always defend against the pass: (1/2) 0 + (1/2) (5) = 5/2 (< 10/3).
- The real power of passing as a strategy is to notice that in equilibrium, the defense is defending against the pass 2/3 of the time, even though the offense is using it only 1/3 of the time. So somehow the threat of passing is helping the offense, even though it uses it relatively rarely.

Mixed Strategies: Examples and Empirical Analysis

- In 2002, Ignacio Palacios-Huerta undertook a large study of penalty kicks from the perspective of game theory.
- Penalty kicks capture the ingredients of two-player, two-strategy games remarkably faithfully.
- The kicker can aim the ball to the left or the right of the goal, and the goalie can dive to either the left or right as well.
- The ball moves to the goal fast enough that the decisions of the kicker and goalie are effectively being made simultaneously; and based on these decisions the kicker is likely to score or not.
- The structure of the game is very much like Matching Pennies: if the goalie dives in the direction where the ball is aimed, he has a good chance of blocking it; if the goalie dives in the wrong direction, it is very likely to go in the goal.

Mixed Strategies: Examples and Empirical Analysis

- Based on an analysis of roughly 1400 penalty kicks in professional football, Palacios-Huerta determined the empirical probability of scoring for each of the four basic outcomes: whether the kicker aims left or right, and whether the goalie dives left or right.

Goalie

Kicker		L	R
	L	$0.58, -0.58$	$0.95, -0.95$
	R	$0.93, -0.93$	$0.70, -0.70$

- Two differences with the matching pennies game:
  - A kicker has a reasonably good chance of scoring even when the goalie dives in the correct direction,
  - Kickers are generally right-footed, and so their chance of scoring is not completely symmetric between aiming left and aiming right.

- There is no equilibrium in pure strategies, and so we need to consider how players should randomize their behavior in playing this game.

Mixed Strategies: Examples and Empirical Analysis

- Using the principle of indifference as in previous examples, we see that if q is the probability that a goalie chooses L, we need to set q so as to make the kicker indifferent between his two options:

$$(.58)(q) + (.95)(1 - q) = (.93)(q) + (.70)(1 - q).$$

		Goalie	
		L	R
Kicker	L	$0.58, -0.58$	$0.95, -0.95$
	R	$0.93, -0.93$	$0.70, -0.70$

- Solving for q, we get q = .42. We can do the analogous calculation to obtain the value of p that makes the goalie indifferent, obtaining p = .39.

- The goalies dive left a .42 fraction of the time (matching the prediction to two decimal places), and the kickers aim left a .40 fraction of the time (coming within .01 of the prediction). [??]

Finding All Nash Equilibria (2-p,2-s)

- First, it is important to note that a game may have both pure-strategy and mixed-strategy equilibria.
- We should first check all four pure outcomes to look for pure equilibria.
- Then, to check whether there are any mixed-strategy equilibria, we need to see whether there are mixing probabilities p and q that are best responses to each other.
- If there is a mixed-strategy equilibrium, then we can determine Player 2's strategy (q) from the requirement that Player 1 randomizes.
- Player 1 will only randomize if his pure strategies have equal expected payoff. This equality of expected payoffs for Player 1 gives us one equation which we can solve to determine q.

Finding All Nash Equilibria (2-p,2-s)

- The same process gives an equation to solve for determining Player 2's strategy p.

- If both of the obtained values p and q are strictly between 0 and 1, and are thus legitimate mixed strategies, then we have a mixed-strategy equilibrium.

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	$1, 1$	$0, 0$
	Keynote	$0, 0$	$2, 2$

- Suppose that you place a probability of p strictly between 0 and 1 on PowerPoint, and your partner places a probability of q strictly between 0 and 1 on PowerPoint.

- Then you'll be indifferent between PowerPoint and Keynote if
  $(1)(q) + (0)(1 - q) = (0)(q) + (2)(1 - q)$, → q = 2/3. (→ p= 2/3)

Pareto-Optimality and Social Optimality

- In a Nash equilibrium, each player's strategy is a best response to the other player's strategies.

- The players are optimizing individually. This doesn't mean that, as a group, the players will necessarily reach an outcome that is in any sense good.

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- It is possible to classify outcomes in a game not just by their strategic or equilibrium properties, but also by whether they are "good for society."

Pareto-Optimality and Social Optimality

- Pareto-Optimality
  A choice of strategies — one by each player — is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
You	*Exam*	92, 86	88, 88

- If the players could jointly agree on what to do, and make this agreement binding, then surely they would prefer to move to this superior choice of strategies.

- The binding agreement is needed when this pair of strategies is not a Nash equilibrium.

Pareto-Optimality and Social Optimality

- Social-Optimality
  A choice of strategies — one by each player — is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

- The social optimum produces a combined payoff of 90 + 90 = 180. (Provided adding makes sense).

- Outcomes that are socially optimal must also be Pareto-optimal.
  - if such an outcome weren't Pareto-optimal, there would be a different outcome in which all payoffs were at least as large, and one was larger — and this would be an outcome with a larger sum of payoffs.

- A Pareto-optimal outcome need not be socially optimal.
  - For example, the Exam-or-Presentation Game has three outcomes that are Pareto-optimal, but only one of these is the social optimum.

Dominated Strategies

- The Facility Location Game

- Our example is a game in which two firms compete through their choice of locations. Suppose that two firms are each planning to open a store in one of six towns located along six consecutive exits on a highway.

Firm 2

		B	D	F
Firm 1	A	1, 5	2, 4	3, 3
	C	4, 2	3, 3	4, 2
	E	3, 3	2, 4	5, 1

Dominated Strategies

Firm 2

Firm 1	B	D	F
A	$1,5$	$2,4$	$3,3$
C	$4,2$	$3,3$	$4,2$
E	$3,3$	$2,4$	$5,1$

- In any situation where Firm 1 has the option of choosing A, it would receive a strictly higher payoff by choosing C.
- Similarly, F is a strictly dominated strategy for Firm 2: in any situation where Firm 1 has the option of choosing F, it would receive a strictly higher payoff by choosing D.

Dominated Strategies

		Firm 2	
		B	D
Firm 1	C	$4, 2$	$3, 3$
	E	$3, 3$	$2, 4$

- The strategies B and E weren't previously strictly dominated.
- But with A and F eliminated, the strategies B and E now are strictly dominated.

		Firm 2
		D
Firm 1	C	$3, 3$

- This is called iterative deletion of strictly dominated strategies,