Social Network Analysis

Game Theory – II

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Modeling Network Traffic using Game Theory

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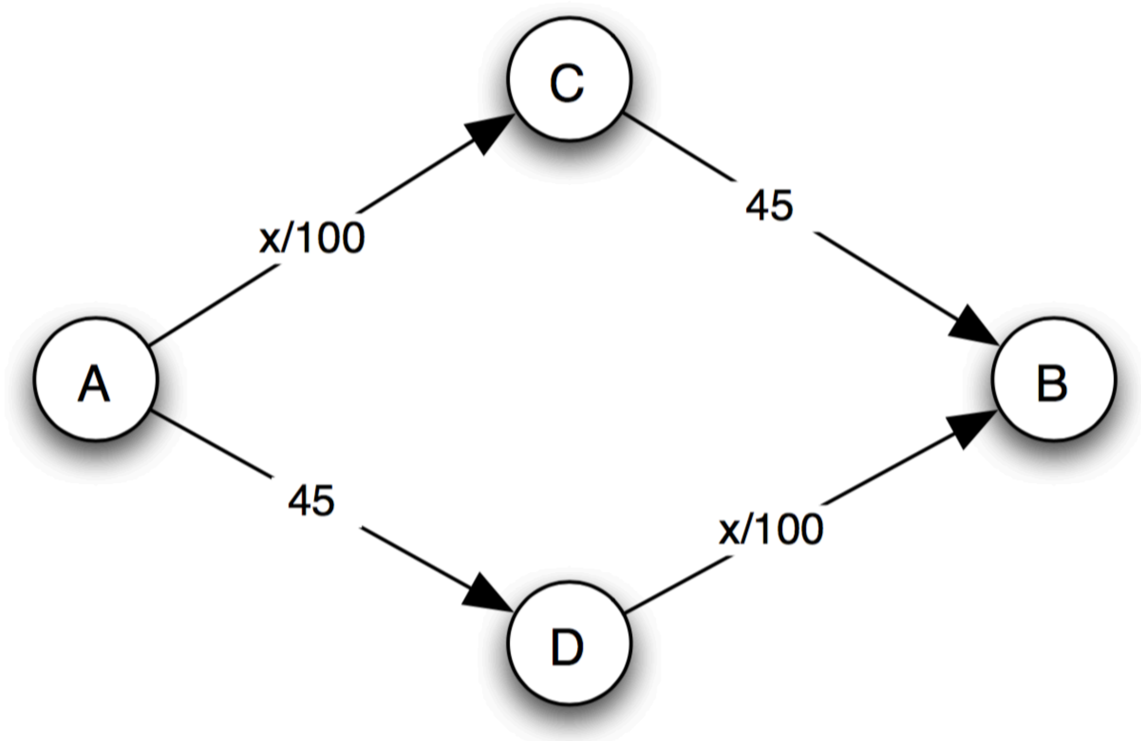
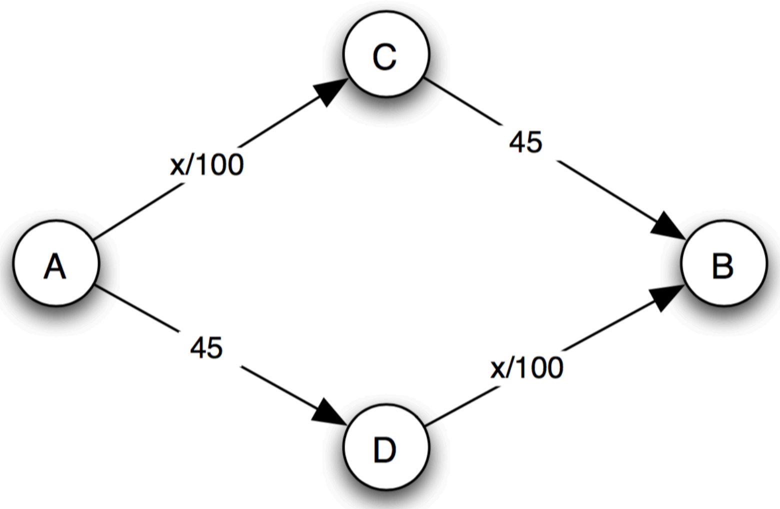
Networks



- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it. When 4000 cars need to get from A to B, they divide evenly over the two routes at equilibrium, and the travel time is 65 minutes.

Networks

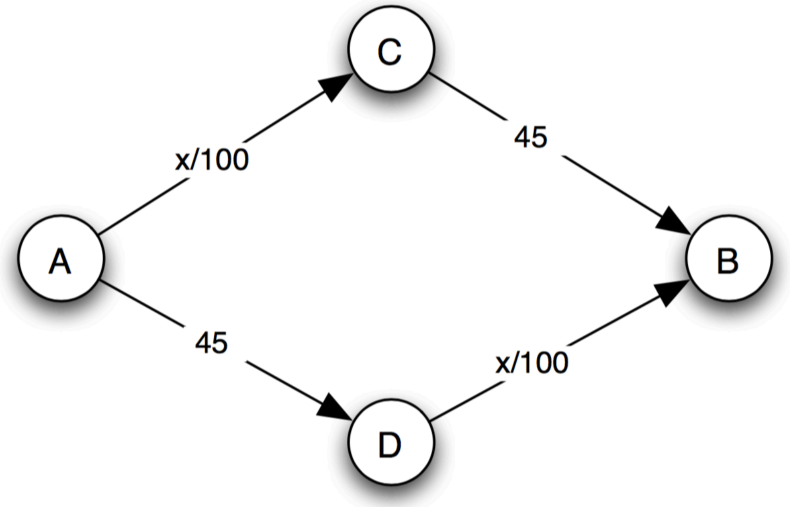


- Suppose that 4000 cars want to get from A to B as part of the morning commute. There are two possible routes that each car can choose: the upper route through C, or the lower route through D.

- For example, if each car takes the upper route (through C), then the total travel time for everyone is 85 minutes, since 4000/100 + 45 = 85.

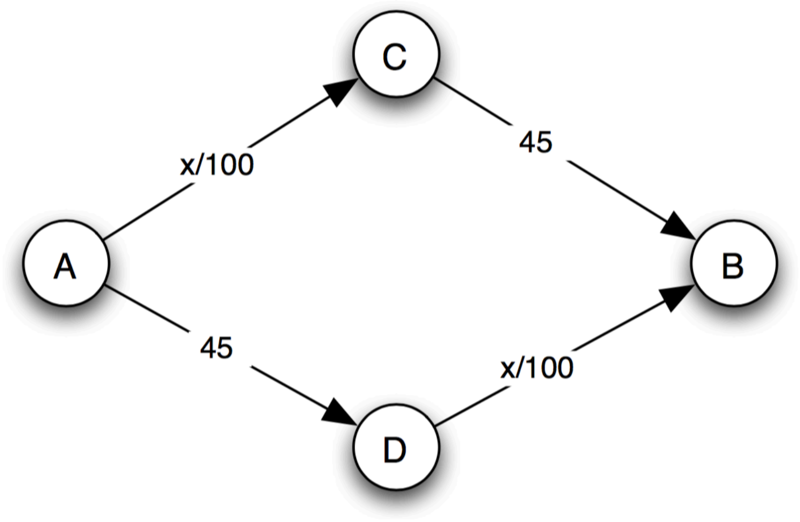
- The same is true if everyone takes the lower route. On the other hand, if the cars divide up evenly between the two routes, so that each carries 2000 cars, then the total travel time for people on both routes is 2000/100 + 45 = 65.

Networks



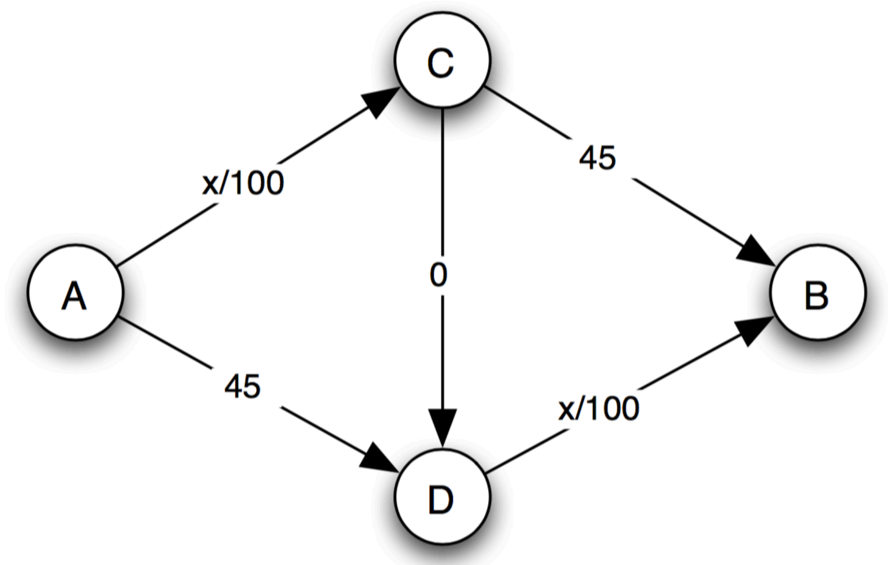
- The traffic model we've described is really a game in which the players correspond to the drivers, and each player's possible strategies consist of the possible routes from A to B.
- Each player only has two strategies; but in larger networks, there could be many strategies for each player. The payoff for a player is the negative of his or her travel time (we use the negative since large travel times are bad).
- A Nash equilibrium is still a list of strategies, one for each player, so that each player's strategy is a best response to all the others.

Networks



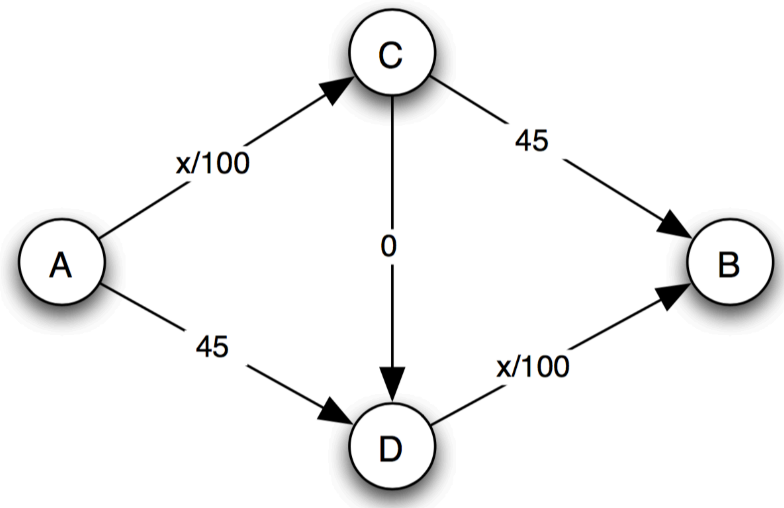
- In this traffic game, there is generally not a dominant strategy;
- Either route has the potential to be the best choice for a player if all the other players are using the other route.
- Any list of strategies in which the drivers balance themselves evenly between the two routes (2000 on each) is a Nash equilibrium, and these are the only Nash equilibria.

Networks



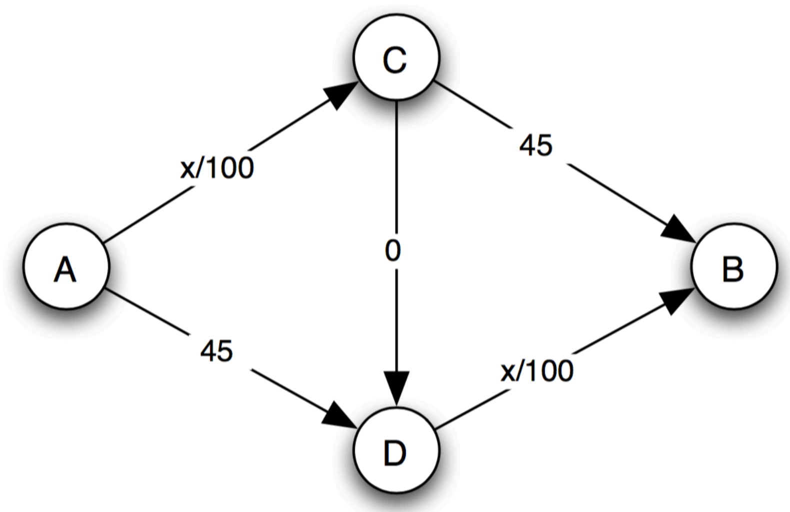
- The highway network from the previous figure, after a very fast edge has been added from C to D.
- At equilibrium, every driver uses the route through both C and D.

Networks



- The travel time for every driver is 80 (since 4000/100 + 0 + 4000/100 = 80).
- No driver can benefit by changing their route: with traffic snaking through C and D the way it is, any other route would now take 85 minutes.
- It's the only equilibrium: the edge from C to D has in fact made the route through C and D a dominant strategy for all drivers: regardless of the current traffic pattern, you gain by switching your route to go through C and D.

Networks



- The route through C and D acts like a “vortex” that draws all drivers into it — to the detriment of all.

- In the new network there is no way, given individually self-interested behaviour by the drivers, to get back to the even-balance solution that was better for everyone.

- This phenomenon — that adding resources to a transportation network can sometimes hurt performance at equilibrium — was first articulated by Dietrich Braess in 1968, and it has become known as Braess’s Paradox.

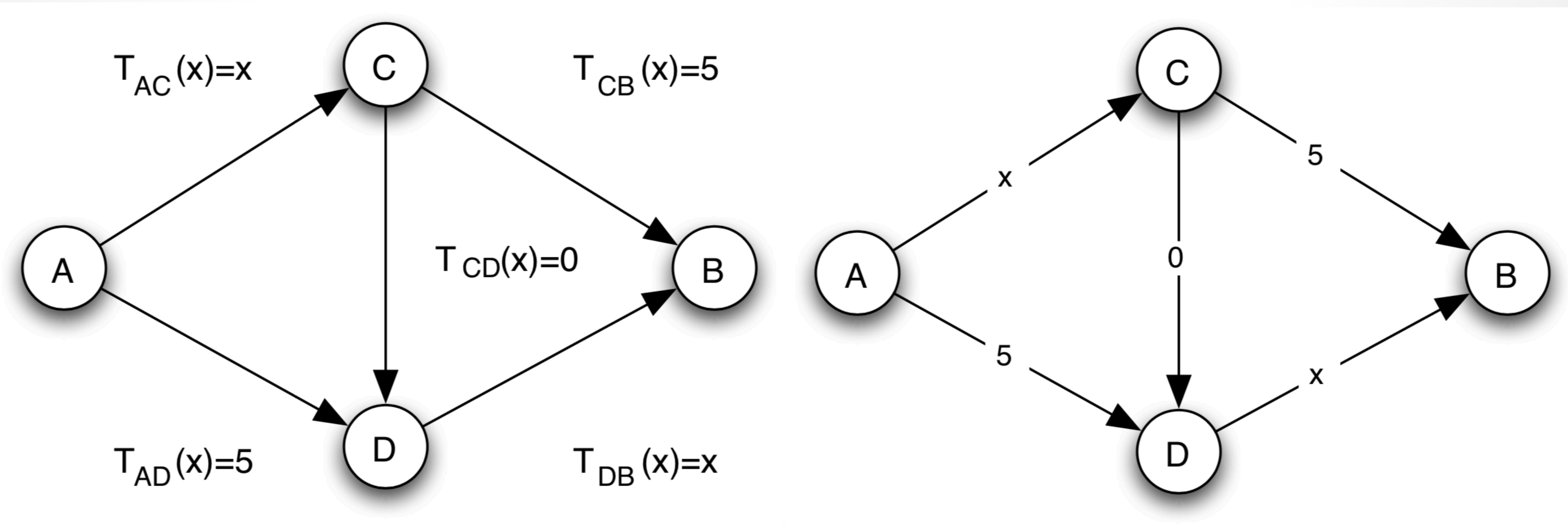
Braess's Paradox

- In Seoul, Korea, where the destruction of a six-lane highway to build a public park actually improved travel time into and out of the city (even though traffic volume stayed roughly the same before and after the change)
- We all have an informal sense that "upgrading" a network has to be a good thing, and so it is surprising when it turns out to make things worse.
- We could ask how bad Braess's Paradox can be for networks in general: how much larger can the equilibrium travel time be after the addition of an edge, relative to what it was before?
- Tim Roughgarden and Éva Tardos can be used to show that if we add edges to a network with an equilibrium pattern of traffic, there is always an equilibrium in the new network whose travel time is no more than 4/3 times as large.

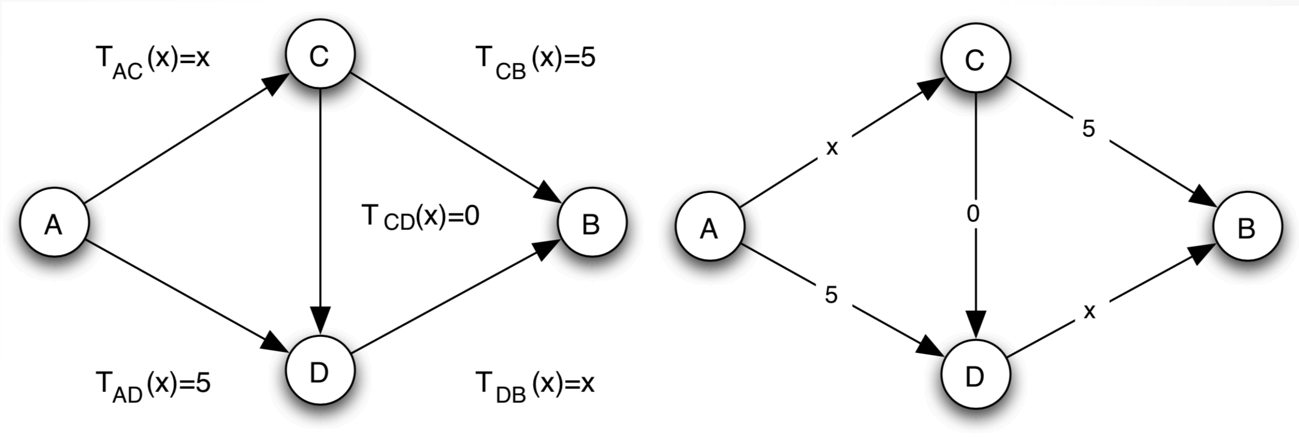
The Social Cost of Traffic at Equilibrium

- The Braess Paradox is one aspect of a larger phenomenon, which is that network traffic at equilibrium may not be socially optimal.

- We will now try to quantify how far from optimal traffic can be at equilibrium.



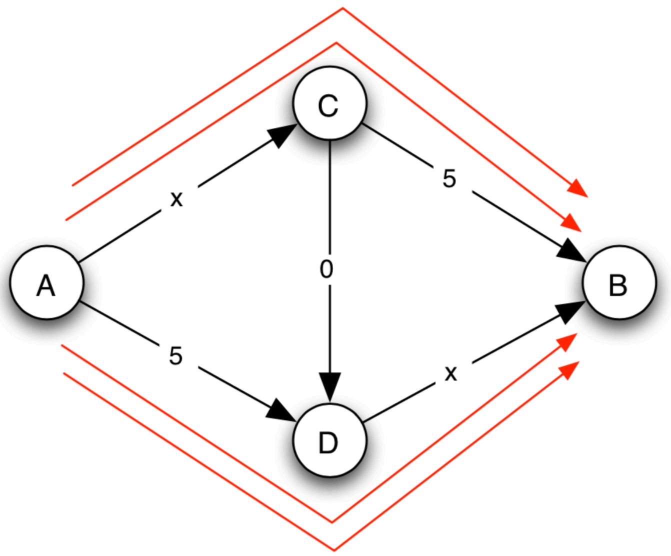
The Social Cost of Traffic at Equilibrium

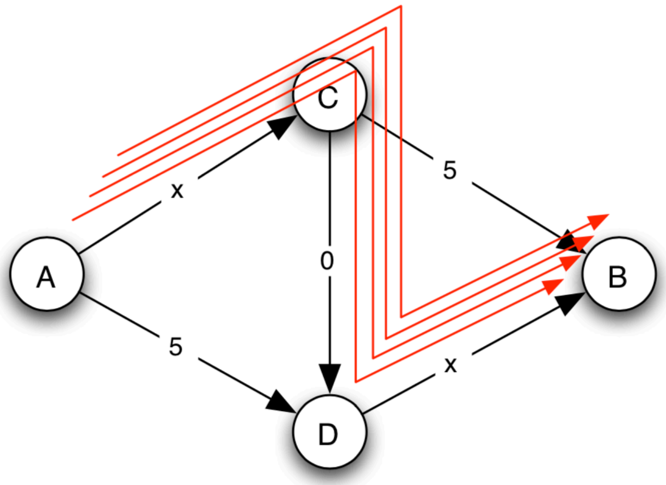


- Each edge e has a travel-time function Te(x) which gives the time it takes all drivers to cross the edge when there are x drivers using it.
- We will assume that all travel-time functions are linear in the amount of traffic, so that $T_e(x) = a_e x + b_e$ for some choice of numbers a_e and b_e that are either positive or zero.
- A traffic pattern is simply a choice of a path by each driver, and the social cost of a given traffic pattern is the sum of the travel times incurred by all drivers when they use this traffic pattern.

The Social Cost of Traffic at Equilibrium

- Two different traffic patterns on the network with 4 drivers.
- Each driver requires 7 units of time to get to their destination, and so the social cost is 28.
- This achieves the minimum possible social cost, is socially optimal.
- The second traffic pattern is the unique Nash equilibrium, and it has a larger social cost of 32.





The Social Cost of Traffic at Equilibrium

- Two questions:
  - In any network (with linear travel-time functions), is there always an equilibrium traffic pattern?
  - Whether there always exists an equilibrium traffic pattern whose social cost is not *much* more than the social optimum.
- Answers:
  - Yes, there is always an equilibrium traffic pattern.
  - Social cost is *at most twice* more than the social optimum.

The Social Cost of Traffic at Equilibrium

- Best Response Dynamics
- We will prove that an equilibrium exists by analysing the following procedure that explicitly searches for one.
- The procedure starts from any traffic pattern. If it is an equilibrium, we are done.
- Otherwise, there is at least one driver whose best response, given what everyone else is doing, is some alternate path providing a strictly lower travel time. We pick one such driver and have him switch to this alternate path.
- We now have a new traffic pattern and we again check whether it is an equilibrium — if it isn't, then we have some driver switch to his best response, and we continue in this fashion.

The Social Cost of Traffic at Equilibrium

- Best Response Dynamics

- The key is to show that in any instance of our traffic game, best-response dynamics must eventually stop at an equilibrium.

- Why should it? Remember matching pennies gave larger cycles?

- This could happen in the traffic game as well: one at a time, drivers shift their routes to ones that are better for them, thus increasing the delay for another driver, who then switches and continues the cascade.

- We now show that best-response dynamics must always terminate in an equilibrium, thus proving not only that equilibria exist but also that they can be reached by a simple process in which drivers constantly update what they're doing according to best responses.

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- A useful analysis technique is to define some kind of progress measure that tracks the process as it operates, and to show that eventually enough "progress" will be made that the process must stop.

- It is natural to think of the social cost of the current traffic pattern as a possible progress measure, but in fact the social cost is not so useful for this purpose.

- Some best-response updates by drivers can make the social cost better (for example, if a driver leaves a congested road for a relatively empty one),..

- ..but others can make it worse (as in the sequence of best-response updates that shifts the traffic pattern from the social optimum to the inferior equilibrium in the Braess Paradox).

- So in general, as best-response dynamics runs, the social cost of the current traffic pattern can oscillate between going up and going down.

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

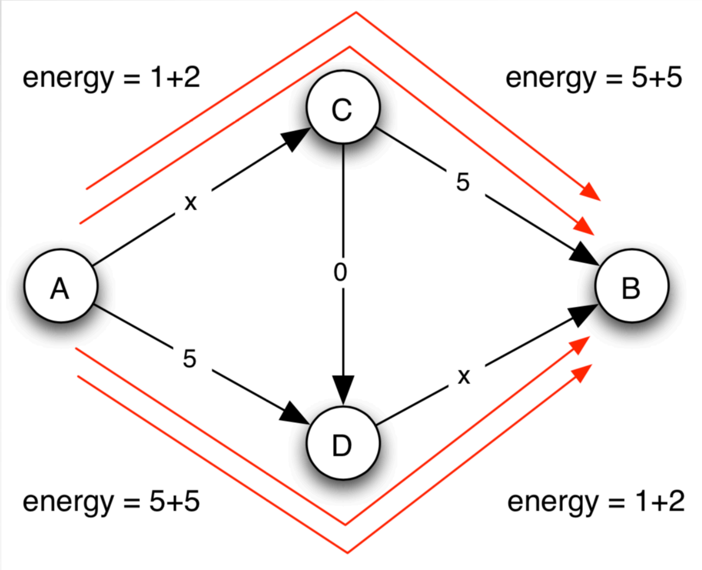
- Instead, we're going to define an alternate quantity that strictly decreases with each best-response update, and so it can be used to track the progress of best-response dynamics.
- This quantity as the potential energy of a traffic pattern. The potential energy of a traffic pattern is defined edge-by-edge, as follows.
- If an edge e currently has x drivers on it, then we define the potential energy of this edge to be:

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

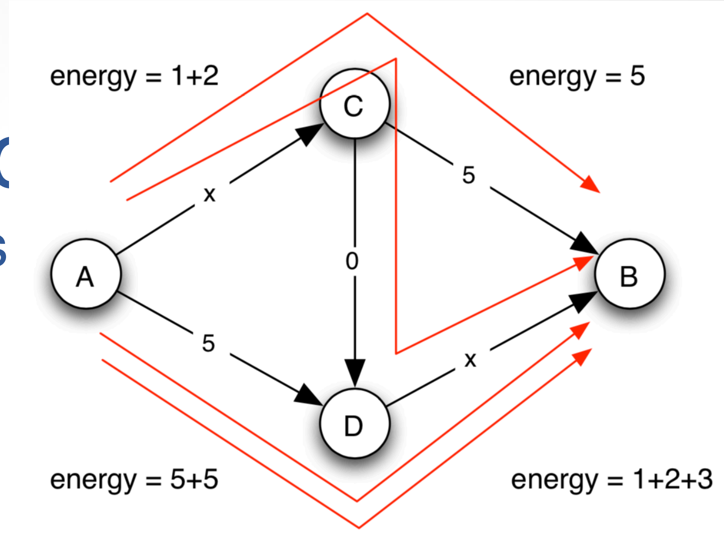
- If an edge has no drivers on it, its potential energy is defined to be 0.
- The potential energy of a traffic pattern is then simply the sum of the potential energies of all the edges, with their current number of drivers in this traffic pattern.

...st of Traffic

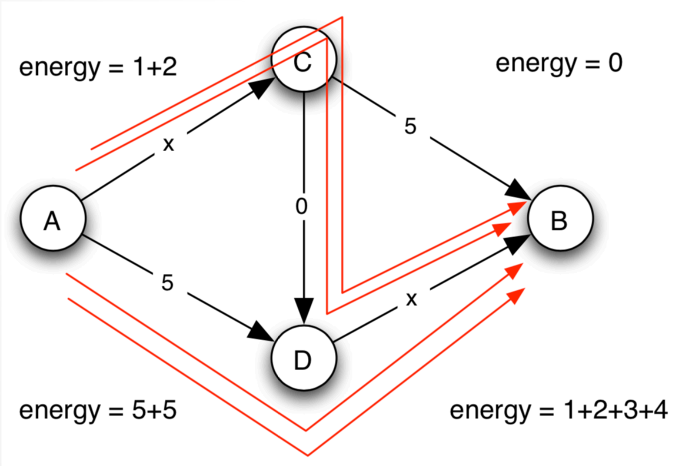
...onse Dynamics



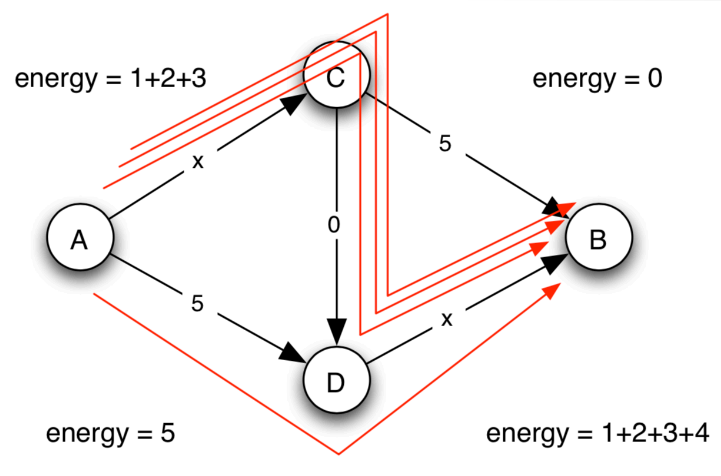
potential energy = 26



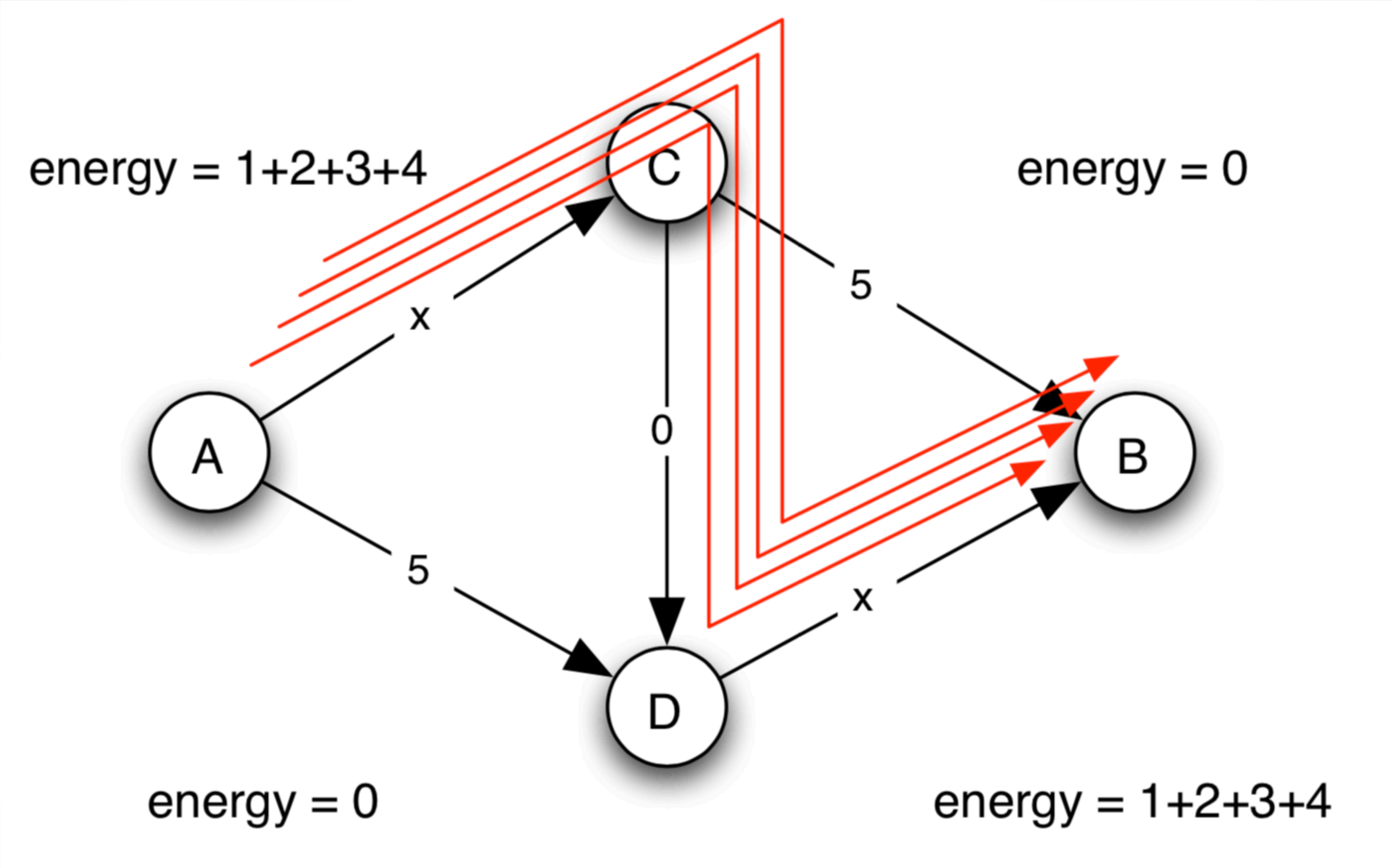
potential energy = 24



potential energy = 23



potential energy = 21



potential energy = 20

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- Notice that the potential energy of an edge e with x drivers is not the total travel time experienced by the drivers that cross it.

- Since there are x drivers each experiencing a travel time of $T_e(x)$, their total travel time is $xT_e(x)$, which is a different number.

- The potential energy, instead, is a sort of "cumulative" quantity in which we imagine drivers crossing the edge one by one, and each driver only "feels" the delay caused by himself and the drivers crossing the edge in front of him.

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

- Now, we need to prove that best-reponse dynamics comes to an end.

- Each step of best-response dynamics causes the potential energy of the current traffic pattern to strictly decrease. This is not true of the social cost.

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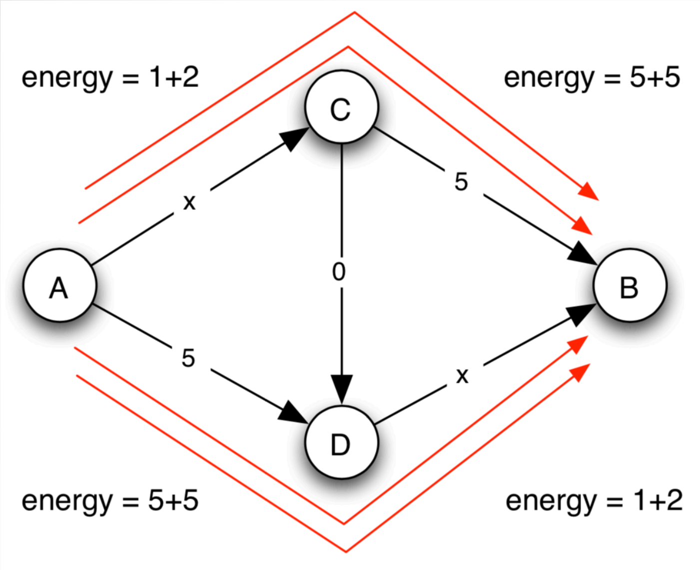
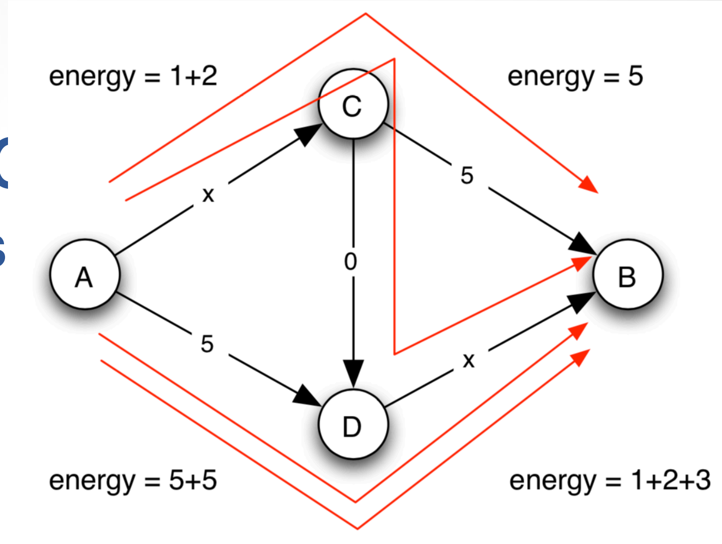
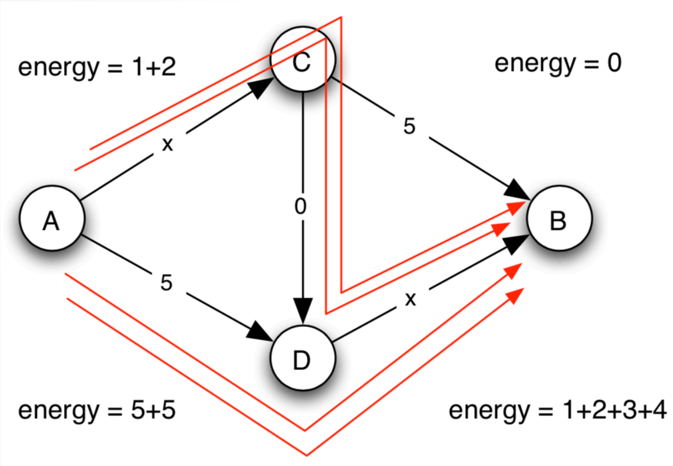
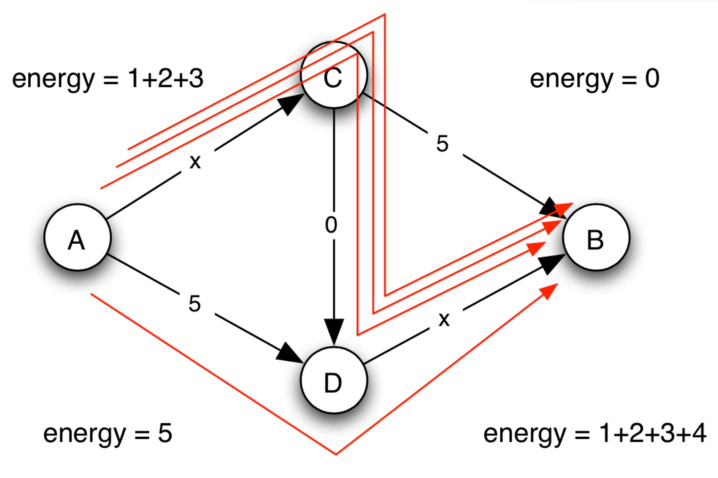




potential energy = 26
social cost = 28

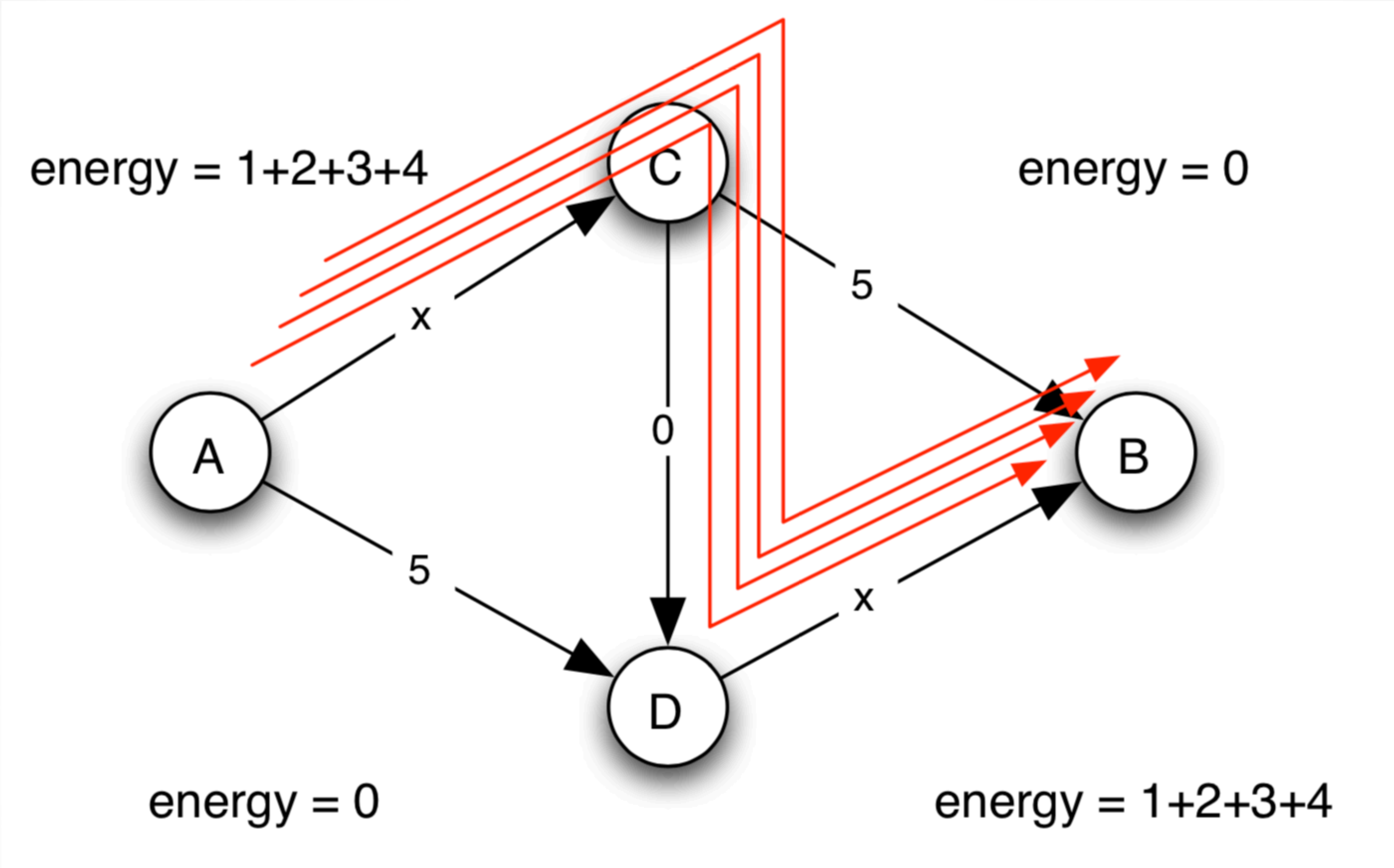
potential energy = 24
social cost = 28





potential energy = 23
social cost = 30

potential energy = 21
social cost = 30



potential energy = 20
social cost = 32

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- It is easy to track the change in potential energy through this sequence as follows. From one traffic pattern to the next, the only change is that one driver abandons his current path and switches to a new one.

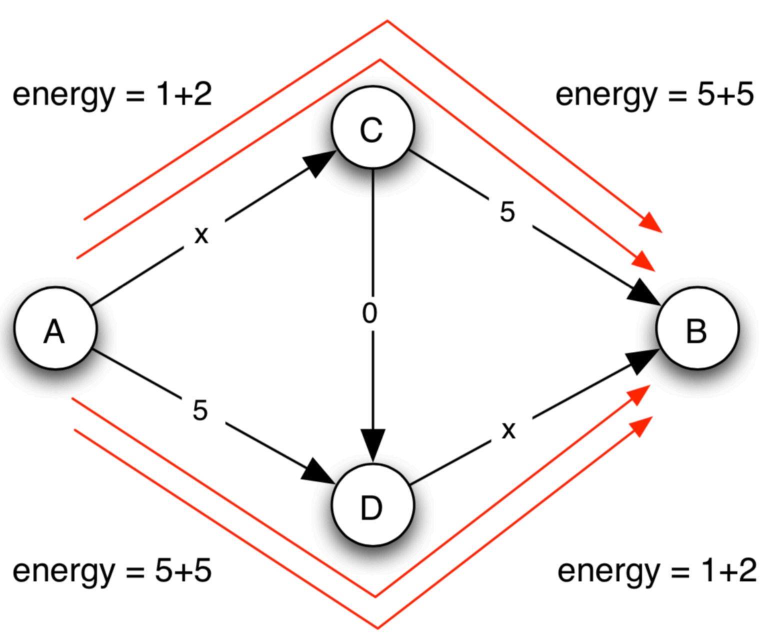
- This switch as a two-step process:
    - first the drivers abandons his current path, temporarily leaving the system;
    - then, the driver returns to the system by adopting a new path.

- This first step releases potential energy as the driver leaves the system, and the second step adds potential energy as he re-joins.

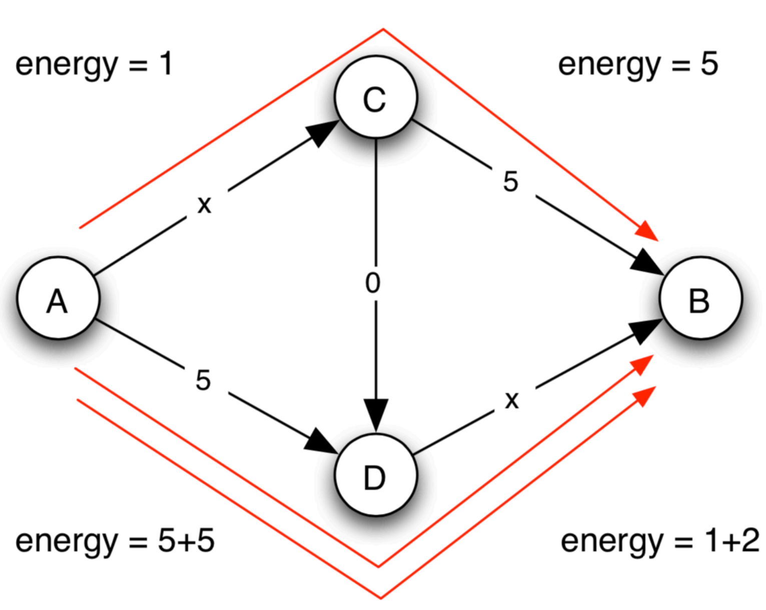
- What's the net change?

The Social Cost of Traffic at Equilibrium

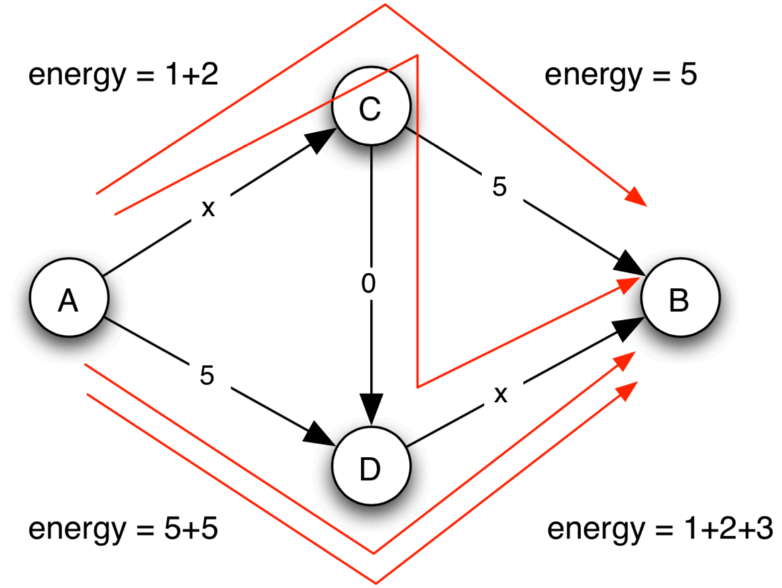
Analyzing Best-Response Dynamics Via Potential Energy



potential energy = 26



potential energy = 26 - 7 = 19



potential energy = 19 + 5 = 24

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- Abandoning the upper path releases 2 + 5 = 7 units of potential energy, while adopting the zigzag path puts 2 + 0 + 3 units of potential energy back into the system. The resulting change is a decrease of 2.
- Notice that the decrease of 7 is simply the travel time the driver was experiencing on the path he abandoned, and the subsequent increase of 5 is the travel time the driver now experiences on the path he has adopted.
- This relationship is in fact true for any network and any best response by a driver. Specifically, the potential energy of edge e with x drivers is

$$\text{Energy}(e) = T_e(1) + T_e(2) + \dots + T_e(x)$$

- When one of these drivers leaves it drops to

$$\text{Energy}(e) = T_e(1) + T_e(2) + \dots + T_e(x-1)$$

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- Hence the change in potential energy on edge e is $T_e(x)$, exactly the travel time that the driver was experiencing on e.

- Summing this over all edges used by the driver, we see that the potential energy released when a driver abandons his current path is exactly equal to the travel time the driver was experiencing.

- By the same reasoning, when a driver adopts a new path, the potential energy on each edge e he joins increases to

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x+1)$$

- .. and the increase of $T_e(x + 1)$ is exactly the new travel time the driver experiences on this edge.

- Hence, the potential energy added to the system when a driver adopts a new path is exactly equal to the travel time the driver now experiences.

The Social Cost of Traffic at Equilibrium

Analyzing Best-Response Dynamics Via Potential Energy

- It follows when a driver switches paths, the net change in potential energy is simply his new travel time minus his old travel time.
- But in best-response dynamics, a driver only changes paths when it causes his travel time to decrease…
- So the change in potential energy is negative for any best-response move.
- This establishes what we wanted to show: that the potential energy in the system strictly decreases throughout best-response dynamics.
- As argued above, since the potential energy cannot decrease forever, best-response dynamics must therefore eventually come to an end, at a traffic pattern in equilibrium.

The Social Cost of Traffic at Equilibrium

Comparing Equilibrium Traffic to the Social Optimum

- Having shown that an equilibrium traffic pattern always exists, we now consider how its travel time compares to that of a socially optimal traffic pattern.

- The basic idea is to establish a relationship between the potential energy of an edge and the total travel time of all drivers crossing the edge.

- We denote the potential energy of an edge by Energy(e), and we recall that when there are x drivers,

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

- On the other hand, each of the x drivers experiences a travel time of $T_e(x)$, and so the total travel time experienced by all drivers on the edge is

$$\text{Total-Travel-Time}(e) = xT_e(x)$$

The Social Cost of Traffic at Equilibrium

Comparing Equilibrium Traffic to the Social Optimum

- For purposes of comparison with the potential energy, it is useful to write this as follows:

$$\text{Total-Travel-Time}(e) = T_e(x) + T_e(x) + \ldots + T_e(x)$$

- Since the potential energy and the total travel time each have x terms, but the terms in the latter expression are at least as large as the terms in the former, we have

$$\text{Energy}(e) \leq \text{Total-Travel-Time}(e)$$

- Since T_e is a linear function, we have

$$\begin{aligned} T_e(1) + T_e(2) + \ldots + T_e(x) &= a_e\,(1 + 2 + \ldots + x) + b_e(x) \\ &= a_e\,x\,(x+1)/2 + b_e x \\ &= x\,[a_e\,(x+1)/2 + b_e)] \\ &= x/2\,[a_e x + a_e + 2b_e)] \\ &= ½\,x\,[(a_e x + b_e) + a_e + b_e] \\ &\geq ½\,x(T_e x) \end{aligned}$$

$$\text{Energy}(e) \geq ½\ \text{Total-Travel-Time}(e)$$

The Social Cost of Traffic at Equilibrium

Relating the Travel Time at Equilibrium and Social Optimality

- We now use this relationship between potential energy and total travel to relate the equilibrium and socially optimal traffic patterns.
- Let Z be a traffic pattern; we define Energy(Z) to be the total potential energy of all edges when drivers follow the traffic pattern Z.
- We write Social-Cost(Z) to denote the social cost of the traffic pattern; recall that this is the sum of the travel times experienced by all drivers.
- Summing the social cost edge-by-edge, Social-Cost(Z) is the sum of the total travel times on all the edges.
- Applying our relationships between potential energy and travel time on an edge-by-edge basis, we see that the same relationships govern the potential energy and social cost of a traffic pattern:

$$\tfrac{1}{2}\,\text{Social-Cost}(Z) \leq \text{Energy}(Z) \leq \text{Social-Cost}(Z)$$

The Social Cost of Traffic at Equilibrium

Relating the Travel Time at Equilibrium and Social Optimality

- Now, suppose that we start from a socially optimal traffic pattern Z, and we then allow best-response dynamics to run until they stop at an equilibrium traffic pattern Z'.

- The potential energy decreases as best-response dynamics moves from Z to Z', and so:

$$\text{Energy}(Z') \leq \text{Energy}(Z)$$

- The quantitative relationships between energies and social cost say that:

$$\text{Social-Cost}(Z') \leq 2 \cdot \text{Energy}(Z')$$

$$\text{Energy}(Z) \leq \text{Social-Cost}(Z)$$

- Now we just chain these inequalities together:

$$\text{Social-Cost}(Z') \leq 2 \cdot \text{Energy}(Z') \leq 2 \cdot \text{Energy}(Z) \leq 2 \cdot \text{Social-Cost}(Z)$$