AMC Prep

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1 2010 AMC 12B Problem 11

A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

$$\textbf{(A)}\ \frac{1}{10} \qquad \textbf{(B)}\ \frac{1}{9} \qquad \textbf{(C)}\ \frac{1}{7} \qquad \textbf{(D)}\ \frac{1}{6} \qquad \textbf{(E)}\ \frac{1}{5}$$

1.1 Solution

We are interested in four digit palindromes (10,000 is not a palindrome, and there are no other 5 digit numbers in the range). Notice that such a number is of the form $abba$ – in other words, this is an "outer number", a, and an inner nunmber, b.

Because the outer number a cannot be zero (because then it would be less than 1000), there are 9 choices for it (the digits 1 through 9, inclusive). On the other hand, there are 10 choices for the inner number b, so there's a total of 90 palindromes between 1000 and 10,000.

Now, write the palindrome $abba$ as

$$10^3a + 10^2b + 10b + a.$$

We want to calculate the number of possible choices of a and b such that this is divisble by 7.

$$10^3a + 10^2b + 10b + a \equiv 0 \mod 7.$$

Factor.

$$1001a + 110b \equiv 0 \mod 7$$

Finally, just know that the prime factorization of $1001 = 7 \times 11 \times 13$ and $110 = 5 \times 11 \times 12.

$$(7 \cdot 11 \cdot 13)a + (5 \cdot 11 \cdot 12)b \equiv 0 \mod 7$$

Since the a term will always have a factor of 7, there are two cases where the overall expression is divisible by 7.

1. When $b = 0$, because the LHS of the congruence reduces to $(7 \cdot 11 \cdot 13)a$, which is always divisble by 7. This is the sequence of palindromes $1001, 20002, 3003, \ldots$. There are 9 possible values, because as we established earlier, there are 9 choices for a.
2. When $b = 7$. Because the first term is always divisible by 7, for the entire expression to be divisible by 7, second term must also have a factor of 7. This is only possible when $b = 7$, leading to another 9 possible palindromes, one for each choice of a.

Therefore, in total, there are $9 + 9 = 18$ palindromes divisible by 7 between 1000 and 10,000, so the answer is $\frac{18}{90} = \frac{1}{5}$, **(E)**.

2 2010 AMC 12B Problem 12

For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40?$$

(A) 8 **(B)** 16 **(C)** 32 **(D)** 256 **(E)** 1024

2.1 Solution

This is a straightforward application of the change-of-base formula.

$$\begin{aligned}
\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4(x^2) + \log_8(x^3) + \log_{16}(x^4) &= 40 \\
\frac{1}{2}\frac{\log_2 x}{\log_2 \sqrt{2}} + \log_2 x + \frac{2\log_2 x}{\log_2 4} + \frac{3\log_2 x}{\log_2 8} + \frac{4\log_2 x}{\log_2 16} &= 40 \\
\log_2 x + \log_2 x + \log_2 x + \log_2 x + \log_2 x &= 40 \\
5\log_2 x &= 40 \\
\log_2 x &= 8 \\
x &= 256 \qquad \textbf{(D)}
\end{aligned}$$

3 2010 AMC 12B Problem 13

In $\triangle ABC$, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{2}$ **(E)** $2\sqrt{3}$

3.1 Solution

Because cos and sin can never be greater than 1, for them to add up to 2, they must both be exactly 1.

$$\begin{aligned}\cos(2A - B) &= 1\\ \sin(A + B) &= 1\end{aligned}$$

Instead of being stupid like me and trying to expand these out with sum and difference formulas, you can just directly solve for the angles now. $\arccos(1) = 0$ and $\arcsin(1) = \frac{\pi}{2}$, so

$$\begin{aligned}2A - B &= 0\\ A + B &= \frac{\pi}{2}\end{aligned}$$

Solving the system of equations,

$$\begin{aligned}2A - B &= 0\\ 2A &= B\\ A + B &= \frac{\pi}{2}\\ 3A &= \frac{\pi}{2}\\ A &= \frac{\pi}{6} = 30^\circ\\ B &= 2A = 60^\circ\\ C &= 90^\circ\end{aligned}$$

This is a $30 - 60 - 90$ triangle. Since AB(the side opposite C, the 90° angle) $= 4$, $AC = 2$, and $BC = 2\sqrt{3}$ **(E)**.