AMC Prep

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1 2010 AMC 12A Problem 17

Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

$$\textbf{(A)}\ \frac{47}{72} \qquad \textbf{(B)}\ \frac{37}{56} \qquad \textbf{(C)}\ \frac{2}{3} \qquad \textbf{(D)}\ \frac{49}{72} \qquad \textbf{(E)}\ \frac{39}{56}$$

1.1 Solution

Consider the following cases.

1. If Bernardo picks a 9, his number will always be greater than Silvia's. The probability of this is

$$\frac{1}{9} + \frac{8}{9} \cdot \frac{1}{8} + \frac{7}{9} \cdot \frac{1}{7} = \frac{3}{9} = \frac{1}{3}.$$

2. If Bernardo does not pick 9 ($\frac{2}{3}$), both Bernardo and Silvia are picking from the same set of numbers ($\{1..8\}$), so it may be tempting to say that the probability of one being greater than the other is exactly $\frac{1}{2}$. This would be on the right track, HOWEVER, it doesn't account for the possibility that they pick the same number.

   a. Since they are each choosing 3 numbers out of 8, and order doesn't matter, the number of possible numbers they can choose is

$$\binom{8}{3} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{6} = 56.$$

b. Then the probability that they choose the same number is just $\frac{1}{5}6$ (i.e. Bernardo can choose any number, there's a $\frac{1}{56}$ chance that Silvia choose the same number, or vice versa).

c. Then, to the probabilty that either one is greater than the other, we find 1 minus the probability that they are both the same (the probability that they are different), and then divide that by 2 (the probability that Bernardo specifically is greater).

$$\frac{1-\frac{1}{56}}{2}=\frac{55}{112}$$

.

Adding these two cases together,

$$\frac{1}{3}+\frac{2}{3}\times\frac{55}{112}=\frac{1}{3}+\frac{1}{3}\times\frac{55}{56}=\frac{1}{3}\left(1+\frac{55}{56}\right)=\frac{1}{3}\times\frac{111}{56}=\frac{37}{56}$$

.

Therefore, the answer is **(B)** $\frac{37}{56}$.

1.2 Key Insight

The probability of $X>Y$ is the same as the probability of $Y>X$ if X and Y are chosen from the same distribution. You can find this probability by removing the probability that $X=Y$ and dividing by 2.

2 2010 AMC 12A Problem 18

A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2\leq x\leq 2$, $-2\leq y\leq 2$ at each step?

(A) 92 **(B)** 144 **(C)** 1568 **(D)** 1698 **(E)** $12,800$

2.1 Solution

Nothing much to this problem, just chug through the numbers until you get **(C)** 1698. There are more elegant ways to do this with combinatorics, but it's a number 18, I can do some arithmetic at this point.

3 2010 AMC 12A Problem 19

Each of 2010 boxes in a line contains a single red marble, and for $1\leq k\leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

(A) 45 **(B)** 63 **(C)** 64 **(D)** 201 **(E)** 1005

3.1 Solution

Since the n-th box contains 1 red marble, and n white marbles, for a total of $n+1$ marbles. Isabella stops after drawing n marbles if and only if she draws a white marble from every single box, all the way up until the last one, where she draws a red marble. The probability of this is

$$P(n) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \dots \cdot \frac{n-1}{n} \cdot \frac{1}{n+1}.$$

Notice that the second-last term has a denominator of n – because there are n balls in the $n-1$-th box, and $n-1$ of those are white, and 1 is red. The last term is $\frac{1}{n+1}$ because there is only 1 red ball (the stopping condition) and the n-th box has $n+1$ balls total.

Notice that everything cancels, leaving,

$$P(n) = \frac{1}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

The problem asks for when $P(n) < \frac{1}{2010}$, so let's substitute

$$\begin{aligned} P(n) &< \frac{1}{2010} \\ \frac{1}{n(n+1)} &< \frac{1}{2010} \\ n(n+1) &> 2010. \end{aligned}$$

Some quick experimentation reveals that $44 \times 45 = 1980$ and $45 \times 46 = 2070$, so the answer is **(A) 45**.

4 2010 AMC 12B Problem 11

A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{7}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{5}$

4.1 Solution

We are interested in four digit palindromes (10,000 is not a palindrome, and there are no other 5 digit numbers in the range). Notice that such a number is of the form *abba* – in other words, this is an "outer number", a, and an inner nunmber, b.

Because the outer number a cannot be zero (because then it would be less than 1000), there are 9 choices for it (the digits 1 through 9, inclusive). On the other hand, there are 10 choices for the inner number b, so there's a total of 90 palindromes between 1000 and 10,000.

Now, write the palindrome *abba* as

$$10^3a + 10^2b + 10b + a.$$

We want to calculate the number of possible choices of a and b such that this is divisble by 7.

$$10^3a + 10^2b + 10b + a \equiv 0 \mod 7.$$

Factor.

$$1001a + 110b \equiv 0 \mod 7$$

Finally, just know that the prime factorization of $1001 = 7 \times 11 \times 13$ and \$110 = 5 $\times$ 11 $\times$ 12.

$$(7 \cdot 11 \cdot 13)a + (5 \cdot 11 \cdot 12)b \equiv 0 \mod 7$$

Since the a term will always have a factor of 7, there are two cases where the overall expression is divisible by 7.

1. When $b = 0$, because the LHS of the congruence reduces to $(7 \cdot 11 \cdot 13)a$, which is always divisble by 7. This is the sequence of palindromes $1001, 20002, 3003, \ldots$. There are 9 possible values, because as we established earlier, there are 9 choices for a.
2. When $b = 7$. Because the first term is always divisible by 7, for the entire expression to be divisible by 7, second term must also have a factor of 7. This is only possible when $b = 7$, leading to another 9 possible palindromes, one for each choice of a.

Therefore, in total, there are $9 + 9 = 18$ palindromes divisible by 7 between 1000 and 10,000, so the answer is $\frac{18}{90} = \frac{1}{5}$, **(E)**.

5 2010 AMC 12B Problem 12

For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40?$$

$\textbf{(A)}\ 8 \qquad \textbf{(B)}\ 16 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 256 \qquad \textbf{(E)}\ 1024$

5.1 Solution

This is a straightforward application of the change-of-base formula.

$$\begin{aligned}
\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4(x^2) + \log_8(x^3) + \log_{16}(x^4) &= 40 \\
\frac{1}{2}\frac{\log_2 x}{\log_2 \sqrt{2}} + \log_2 x + \frac{2\log_2 x}{\log_2 4} + \frac{3\log_2 x}{\log_2 8} + \frac{4\log_2 x}{\log_2 16} &= 40 \\
\log_2 x + \log_2 x + \log_2 x + \log_2 x + \log_2 x &= 40 \\
5\log_2 x &= 40 \\
\log_2 x &= 8 \\
x &= 256 \qquad \textbf{(D)}
\end{aligned}$$

6 2010 AMC 12B Problem 13

In $\triangle ABC$, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?

$\textbf{(A)}\ \sqrt{2} \qquad \textbf{(B)}\ \sqrt{3} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 2\sqrt{2} \qquad \textbf{(E)}\ 2\sqrt{3}$

6.1 Solution

Because cos and sin can never be greater than 1, for them to add up to 2, they must both be exactly 1.

$$\begin{aligned}
\cos(2A - B) &= 1 \\
\sin(A + B) &= 1
\end{aligned}$$

Instead of being stupid like me and trying to expand these out with sum and difference formulas, you can just directly solve for the angles now. $\arccos(1) = 0$ and $\arcsin(1) = \frac{\pi}{2}$, so

$$
\begin{aligned}
2A - B &= 0 \\
A + B &= \frac{\pi}{2}
\end{aligned}
$$

Solving the system of equations,

$$
\begin{aligned}
2A - B &= 0 \\
2A &= B \\
A + B &= \frac{\pi}{2} \\
3A &= \frac{\pi}{2} \\
A &= \frac{\pi}{6} = 30^\circ \\
B &= 2A = 60^\circ \\
C &= 90^\circ
\end{aligned}
$$

This is a $30-60-90$ triangle. Since AB(the side opposite C, the 90° angle) $= 4$, $AC = 2$, and $BC = 2\sqrt{3}$ **(E)**.