Study of Lattice Gauge Theories and its applications

MASTER OF SCIENCE

by

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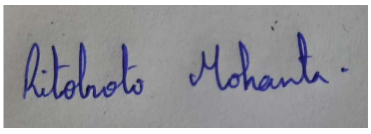
Declaration

I Ritobroto Mohanta 5th year Masters student of IISER-K, hereby declare that is MS report on 'Study of Lattice Gauge Theories and its applications' is my own work. It all the calculations and proofs done here were either done by me or checked and verified by me. Even the concepts that already exists on the literature, I tried to make the arguments and calculations as rigorous as possible on my own. I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree.



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Certificate of Supervisor

Date: 23.05.2022

This is to certify that the thesis titled "Study of Lattice Gauge Theories and its applications" submitted by **Mr. Ritobroto Mohanta**, ID number: **17MS151** dated 21st May, a student of the Department of Physical Science of the BS-MS Programme is based upon his/her own research work under my supervision.

I also certify, to the best of my knowledge, that neither the thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before. In my opinion, the thesis fulfils the requirement for the award of the degree of Master of Science.

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Contents

Chapter 1

Introduction

In this report we try to understand why one might do Lattice Gauge Theories and what are it's applications. We start with simple free theories with no interaction and try to see what information we can extract from the Lagrangian. It is fairly straightforward solving for solution for free theories, as they are linear doing a fourier transform simply diagonalizes the linear operator. We can compute objects like 2-point correlation function which can be used to extract the mass parameter of the theory. For free theories the 2-point functions can be easily solved analytically by fourier transforming the Green's function, but for interacting theories it not possible. To solve for 2-point function we have to compute this $\int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q e^{iS[q]} q(t_1)q(t_2)$. This represents N^d number of integrations for N number of lattices in d dimension, which is computationally very heavy, so we use random sampling techniques like Monte Carlo simulations for interacting theories.

When we discretize space-time into lattice, we place bosons and fermions in the lattice sites. For bosons it is trivial, but for fermions when we place them in the lattice and study the propagator, we see there are 16 extra poles in 4D. 16 extra poles means 16 extra physical fermions on the lattice than in the continuum case. To resolve this issue several solutions has been devised, two of them have been disicussed here namely Wilson fermion and Staggered fermion. Each of which has its own advantages and drawbacks, details have been discussed in Chapter-5. There are other solutions too like Domain Wall fermions which introduces an extra dimension.

Once we understand how to deal with bosons and fermions on the lattice, we can include interactions with gauge theories. Here we introduce a new term which is the gauge field which influences the dynamics of fermionic fields. The gauge fields exists on the links of the lattice unlike the fermions which resides on the lattice sites. We can interpret it as the gauge fields mediating interaction between fermions between two different sites (This makes sense and is aligned with the interpretation we have from perturbation theory). Once we get the lattice lagrangian we can complute observables using efficient numerical techniques like Monte Carlo simulations.

Quantum Link Models is a way of treating Lattice Gauge Theories where we treat gauge fields /and fermionic as quantum operators. The lattice is constructed as the fock space formed by tensor product of Hilbert space on each link. If we have the Hilbert space in each

link as finite dimensional, the Hilbert space of the whole lattice will be finite as the lattice size is finite. in QLM we take finite dimensional representation of the Hilbert space and study them. The main advantage of using this formalism is that it is suitable for quantum simulation(0,),(0,). We can engineer appropriate Hamiltonian in quantum systems which will simulate the systems from Gauge Theories. **This can be run in quantum computers. In this report we have tried to develop the formal mathematical framework of Abelian-Quantum Link Models.**

Chapter 2

Important Concepts

2.1 Path-integral formulation

We will start the report by deriving some of the basic entities of Quantum Field Theory, and understand the motivation behind defining these entities. A good point to start is by asking what is the probability amplitude that a quantum mechanical particle at q_i will travel to say q_f in time t (given a hamiltonian H)? We will define it as Z_t-

$$Z_t = P(q_f(t)|q_i(0)) = \langle q_i(0)|q_f(t)\rangle$$

$$Z_t = \langle q_i(t)|e^{-iHt}|q_f(t)\rangle \tag{2.1}$$

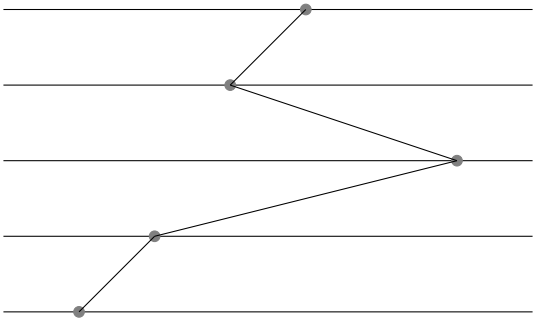
Since the Hamiltonian (H) is generator of time translation, $\langle q_i(t)| = \langle q_i(0)|\, e^{-iHt}$. Now we can divide e^{-iHt} into smaller time steps (since $[H, H] = 0$).

$$eqn(2.1) = \langle q_i|e^{-iH\epsilon}..e^{-iH\epsilon}|q_f\rangle$$

Note that H commutes with itself at different times only if it is time-independent.

Where $\epsilon \to 0$ and $N \times \epsilon = t$. The above step is basically dividing time t into smaller ϵ time steps. Inserting $N-1$ identities between each $e^{iH\epsilon}$ gives-

$$(\prod_{k=1}^{N-1} \int dq_k) \prod_{k=1}^{N-1} \langle q_k|e^{-iH\epsilon}|q_{k+1}\rangle \tag{2.2}$$



Now lets calculate the object $\langle q_k|e^{-iH\epsilon}|q_{k+1}\rangle$ from eqn(2.2). We know a general Quantum Mechanics Hamiltonian is of the form $H = \frac{p^2}{2m} + V$, if we put it into the equation-

$$\langle q_k|e^{-iH\epsilon}|q_{k+1}\rangle = \langle q_k|e^{-i(\frac{p^2}{2m}+V)\epsilon}|q_{k+1}\rangle \tag{2.3}$$

To compute $e^{-i(\frac{p^2}{2m}+V)\epsilon}$, let's first consider this term $e^{-i\epsilon\frac{p^2}{2m}}e^{-i\epsilon V}$ (remember these terms are operators, so we can't trivially break them into individual product)-

$$e^{-i\epsilon\frac{\hat{p}^2}{2m}}\cdot e^{-i\epsilon\hat{V}} = \left(1-i\epsilon\frac{\hat{p}^2}{2m}+\mathbf{O}(\epsilon^2)\right)\left(1-i\epsilon\hat{V}+\mathbf{O}(\epsilon^2)\right)$$

$$= 1-i\epsilon\left(\frac{p^2}{2m}+V\right)+\mathbf{O}(\epsilon^2)$$

Now, let's Taylor expand $e^{-i(\frac{p^2}{2m}+V)\epsilon}$

$$e^{-i(\frac{p^2}{2m}+V)\epsilon} = 1-i\epsilon\left(\frac{p^2}{2m}+V\right)+\mathbf{O}(\epsilon^2)$$

We see up to order $\mathbf{O}(\epsilon)-$

$$e^{-i(\frac{p^2}{2m}+V)\epsilon} = e^{-i\epsilon\frac{p^2}{2m}}\cdot e^{-i\epsilon V} \tag{2.4}$$

From eqn(2.3) and eqn(2.4) we see (Using Baker-Campbell-Hausdorff lemma)-

$$\begin{aligned}\langle q_k|e^{-i(\frac{p^2}{2m}+V)\epsilon}|q_{k+1}\rangle &= \langle q_k|e^{-i\epsilon\frac{p^2}{2m}}\cdot e^{-i\epsilon V}|q_{k+1}\rangle\\ &= \langle q_k|e^{-i\epsilon\frac{p^2}{2m}}(\int dp\,|p\rangle\,\langle p|)e^{-i\epsilon V}|q_{k+1}\rangle\\ &= \int dp\,\langle q_k|\,e^{-i\epsilon\frac{\hat{p}^2}{2m}}\,|p\rangle\,\langle p|\,e^{-i\epsilon\hat{V}}\,|q_{k+1}\rangle\\ &= \int dp e^{-i\epsilon\frac{p^2}{2m}}e^{-i\epsilon V(q_{k+1})}\,\langle q_k|p\rangle\,\langle p|q_{k+1}\rangle\\ &= \int dp e^{-i\epsilon\frac{p^2}{2m}}e^{-i\epsilon V(q_{k+1})}(Ne^{+ip\cdot q_k})(Ne^{-q_{k+1}\cdot p})\end{aligned} \tag{2.5}$$

The above is the Gaussian integral. so, simplifying eqn(2.5), we find

$$\langle q_k|e^{-i(\frac{p^2}{2m}+V)\epsilon}|q_{k+1}\rangle = \sqrt{\frac{\pi}{i\epsilon 2\pi}}\exp\left(i\epsilon[\frac{m}{2}(\frac{q_{k+1}-q_k}{\epsilon})^2-V(q_k)]\right) \tag{2.6}$$

Also note that this is a highly oscillatory integral in real-time, so analytic continuation is essential to make the integral converge. Putting this into our original equation of interest eqn(2.6), we find-

$$(\prod_{k=1}^{N-1}\int dq_k)\prod_{k=1}^{N-1}\sqrt{\frac{1}{2i\epsilon}}\exp\left(i\epsilon[\frac{m}{2}(\frac{q_{k+1}-q_k}{\epsilon})^2-V(q_k)]\right)$$

Now, q_k's and $V(q_k)$'s are not operators anymore, we can just change the product of exponentials into exponential of sum-

$$(\prod_{k=1}^{N-1}\sqrt{\frac{1}{2i\epsilon}}\int dq_k)\exp\left(i\epsilon\sum_{k=1}^{N-1}(\frac{m}{2}(\frac{q_{k+1}-q_k}{\epsilon})^2-V(q_k))\right) \tag{2.7}$$

We recognize $\frac{m}{2}(\frac{q_{k+1}-q_k}{\epsilon})^2$ is the the kinetic energy $T(q_k)$; and taking limit $\epsilon \to dt$ and $\sum_{k=1}^{N-1} \to \int_{t_0}^{t}$, we get-

$$\int_{q(t_0)}^{q(t)} \mathcal{D}q e^{i\int dt(T(q)-V(q))} = \int_{q(t_0)}^{q(t)} \mathcal{D}q e^{iS[q(t)]} \tag{2.8}$$

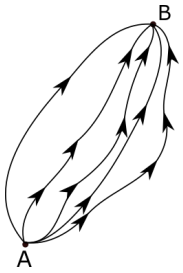


Figure 2.1: Each path taken by the particle

If we time to be imaginary $dt \to id\tau$, we get $\int \mathcal{D}q e^{iS[q,t]} = \int \mathcal{D}q e^{-S[q,\tau]}$. So now what is the interpretation of this equation? We initially started with calculating the probability of a particle at position $q = q_i$ at time $t_0 = 0$, propagating to q_f at time t. Whose equation we found out to be the expression in eqn(2.8), this is not an ordinary integral, it is a path integral. Where the integral is done over all possible paths where each paths contributes a phase factor $e^{iS[q,\dot{q},t]}$ to the integral. So the particle starting at q_i takes all possible paths to travel to q_f in time t and each path contributes a phase factor proportional to the action of that path.

Another thing to take away from eqn(2.8), is once we make time imaginary-

$$Z_\tau = U(x_f, \tau_f; x_i, \tau_i) = \int \mathcal{D}q e^{-S[q,\tau]} \tag{2.9}$$

We can see that for higher values of action, the contribution to the partition function is exponentially decreasing. Thus the maximum contribution to the partition function will come from the minimum action value, which is just the classical path.

2.2 Correlation function

To prove the connection between path integral correlation functions and time ordered products, we first evaluate the 2-point correlator which is defined as: $\langle q_f, t_f | T[\hat{q}(t_1)\hat{q}(t_2)] | q_i, t_i \rangle$ in terms of path integral. Here $T[q(\hat{t_1})q(\hat{t_2})]$ is defined as the time ordered product. So without loss of generality we can just take $t_2 > t_1$, the above equation just becomes-

$$\langle q_f, t_f | T[\hat{q}(t_1)\hat{q}(t_2)] | q_i, t_i \rangle = \langle q_f, t_f | \hat{q}(t_2)\hat{q}(t_1) | q_i, t_i \rangle \tag{2.10}$$

Inserting two identity matrices $I_1 = \int |q_1, t_1\rangle \langle q_1, t_1|$ and $I_2 = \int |q_2, t_2\rangle \langle q_2, t_2|$ in between $\hat{q}(t_1)$ and $\hat{q}(t_2)$, we get-

$$\begin{aligned} \langle q_f, t_f | T[\hat{q}(t_2)\hat{q}(t_1)] | q_i, t_i \rangle = \int dq_2 \int dq_1 \, \langle q_f, t_f | \hat{q}(t_2) | q_2, t_2 \rangle \\ \times \langle q_2, t_2 | q_1, t_1 \rangle \, \langle q_1, t_1 | \hat{q}(t_1) | q_i, t_i \rangle \end{aligned} \tag{2.11}$$

$$= \int dq_2 \int dq_1 \cdot q_1 q_2 \, \langle q_f, t_f | q_2, t_2 \rangle \, \langle q_2, t_2 | q_1, t_1 \rangle \, \langle q_1, t_1 | q_i, t_i \rangle \tag{2.12}$$

Using the result from eqn(2.1) and eqn(2.8), and putting it into the above equation-

$$\begin{aligned} \langle q_f, t_f | T[\hat{q}(t_2)\hat{q}(t_1)] | q_i, t_i \rangle = \int dq_2 \int dq_1 q_1 q_2 \\ \times C\Big(\int_{q(t_f)}^{q(t_2)} \mathcal{D}q e^{iS[q,t]} \int_{q(t_2)}^{q(t_1)} \mathcal{D}q e^{iS[q,t]} \int_{q(t_1)}^{q(t_i)} \mathcal{D}q e^{iS[q,t]} \Big) \end{aligned} \tag{2.13}$$

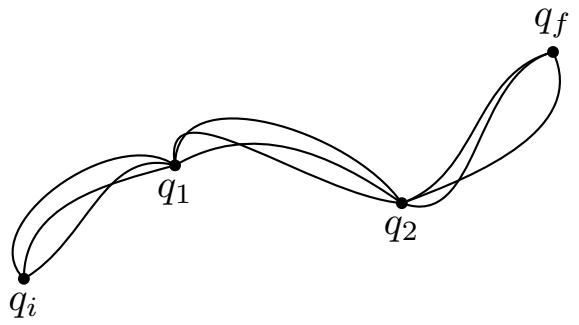


Figure 2.2: Do not forget!

The term inside the bracket is basically the product of probability of the particle travelling from q_i to q_1 to q_2 and finally to q_f: $P(q_i(t_i)|q_1(t_1)) \times P(q_1(t_1)|q_2(t_2)) \times P(q_2(t_2)|q_f(t_f))$. This can be clearly seen from the figure below. But we are integrating over with a weight factor $q_1 \cdot q_2$ over all possible values of q i.e. $q_1, q_2 \in \mathbb{R}^3$. Thus eqn(2.13) can be written as-

$$\langle q_f, t_f | T[\hat{q}(t_2)\hat{q}(t_1)] | q_i, t_i \rangle = C \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q e^{iS[q]} q(t_1) q(t_2) \tag{2.14}$$

In QFT we take the value in $\phi(x)$ as an individual degree of freedom, so instead of discretizing only time we discretize all space-time points, and $\phi(\mathbf{x}_i)$ is the value of the field taken at $\mathbf{x}_i \in M_{1+3}$ (1+3 dimension Minkowski space-time). Therefore instead of eqn(2.10), in case of fields, we will have $\langle 0 | T[\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)] | 0 \rangle$. And in the end instead of integrating over variables as in eqn(2.8), we would integrate over ∞ number of variables $\phi(\mathbf{x})$, where $\mathbf{x}$ is a 4-vector in Minkowski space-time.

2.3 Some key results

One can also write two point euclidean correlator of two operators O_1 and O_2 as-

$$\langle \hat{O}_2(t) \hat{O}_1(0) \rangle_T = \frac{1}{Z_T} Tr[e^{-(T-t)\hat{H}} \hat{O}_2 e^{-t\hat{H}} \hat{O}_1] \tag{2.15}$$

Here $Tr[A] := \sum_i \langle e_i | A | e_i \rangle$ ($|e_i\rangle$ has to form a complete basis of the Hilbert space) and $Z_T = Tr[e^{-tH}]$. Using this definition our aim in this section would be to derive the relation

in the equation below-

$$\lim_{T\to\infty}\frac{1}{Z_T}Tr[e^{-(T-t)\hat{H}}\hat{O}_2e^{-t\hat{H}}\hat{O}_1]=\sum_n\langle 0|\,\hat{O}_2\,|n\rangle\,\langle n|\,\hat{O}_1\,|0\rangle\,e^{-tE_n} \tag{2.16}$$

The left hand side of eqn(2.14) is the correlator of the operators O_1 and O_2 where $\hat{H}$ is the Hamiltonian of the system and $|n\rangle$ are the eigenstates of the Hamiltonian. Since we have summed over all the eigenenergies of the hamiltonian (e^{-tE_n}) on the right hand side of the equation, this relation shows that from the correlator we can extract information about energy spectrum of a theory. This would be studied more extensively in the next chapter.

To derive eqn(2.15) we first compute Z_T:

$$Z_T=\sum_n\langle n|\,e^{-T\hat{H}}\,|n\rangle=\sum_n e^{-TE_n}$$

Here we have used the orthonormality relation of the basis states $\langle n|m\rangle=\delta_{mn}$ and assumed $E_1\leq E_2\leq ..\leq E_n$ without loss of generality. Similarly we can write the euclidean correlator as-

$$\langle\hat{O}_2(t)\hat{O}_1(0)\rangle_T=\frac{1}{Z_T}\sum_m\langle m|\,e^{-(T-t)\hat{H}}\hat{O}_2e^{-t\hat{H}}\hat{O}_1\,|m\rangle$$

Inserting Identity $I=\sum_n|n\rangle\langle n|$ in between $\hat{O}_2$ and e^{-tH}, we get

$$\begin{aligned}&=\frac{1}{Z_T}\sum_{m,n}\langle m|\,e^{-(T-t)\hat{H}}\hat{O}_2\,|n\rangle\,\langle n|\,e^{-t\hat{H}}\hat{O}_1\,|m\rangle\\&=\frac{1}{Z_T}\sum_{m,n}e^{-(T-t)E_m}\,\langle m|\,\hat{O}_2\,|n\rangle\,e^{-tE_n}\,\langle n|\,\hat{O}_1\,|m\rangle\end{aligned}$$

Now putting the value of Z_T into the equation and dividing both numerator and denominator with e^{-TE_0}- ground state energy we get-

$$\langle\hat{O}_2(t)\hat{O}_1(0)\rangle_T=\frac{\sum_{m,n}\langle m|\,\hat{O}_2\,|n\rangle\,\langle n|\,\hat{O}_1\,|m\rangle\,e^{-(T-t)\Delta E_m}e^{-tE_n}}{1+e^{T\Delta E_1}+e^{T\Delta E_2}...} \tag{2.17}$$

Where $\Delta E_n=E_n-E_0$. From here we can see that the euclidean correlator depends only on the difference of energies with respect to the ground state. Now if we take the limit $T\to 0$, since $E_n\geq E_0$ all the terms with $m>0$ will dissapear, leaving us with the desired equation-

$$\lim_{T\to\infty}\langle\hat{O}_2(t)\hat{O}_1(0)\rangle_T=\sum_n\langle 0|\,\hat{O}_2\,|n\rangle\,\langle n|\,\hat{O}_1\,|0\rangle\,e^{-tE_n} \tag{2.18}$$

Also notice if we have a Hermitian operator $\hat{O}$ then eqn(2.18) becomes-

$$\lim_{T\to\infty}\langle\hat{O}(t)\hat{O}(0)\rangle_T=\sum_n|\,\langle 0|\,\hat{O}\,|n\rangle\,|^2e^{-tE_n} \tag{2.19}$$

Chapter 3

Klein-Gordon Propagator

3.1 Propagator calculation

In the eqn(2.14), if we replace $t \to i\tau$, we'll obtain $\int_{\phi(\mathbf{x}_i)}^{\phi(\mathbf{x}_f)} \mathcal{D}\phi(\mathbf{x}) e^{-S[\phi(\mathbf{x})]}$. Since $S \geq 0$, we can see the maximum contribution will come from the path where action is minimized, the classical path. This path can be computed using the Euler-Lagrange equation-

$$\frac{\partial \mathcal{L}}{\partial \phi(\mathbf{x})} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(\mathbf{x})}\right) = 0$$

For Klein-Gordon Lagrangian $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2$, the E-L equation:

$$(\partial_\mu \partial^\mu - m^2)\phi(\mathbf{x}) = 0 \tag{3.1}$$

To solve this, let's first take the discrete version of the above equation and break space-time into a lattice of finite size.

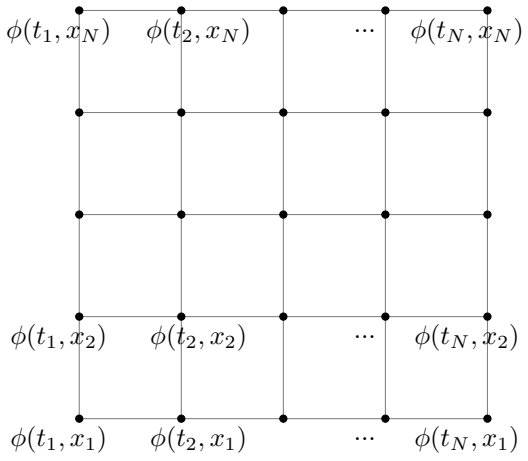


Figure 3.1: Discretizing space-time into N-lattice points, w/ distance 'a' between each.

Denoting (t, x, y, z) as (x^0, x^1, x^2, x^3). $\phi(\mathbf{x})$ can be represented as a vector-

$$\phi(\mathbf{x}) = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ . \\ . \\ \phi(x_n) \end{pmatrix}$$

Where $\phi(x_i^\mu) = \phi(x_{N^\mu+i})$. The operators can be written as linear transformations, to compute the linear operatior of the differential equation eqn(3.1), let's first consider this object $\partial_0\phi(\mathbf{x})$. In lattice case this would be given as-

$$\partial_0\phi(\mathbf{x}) = \frac{\phi(x_{i+1}) - \phi(x_i)}{a} = \frac{1}{a}(\delta_{i+1,j} - \delta_{i,j})\phi(x_j)$$

$$\implies M_{\partial_0} = \frac{1}{a}(\delta_{i+1,j} - \delta_{i,j})$$

In general,

$$M_{\partial_\mu} = \frac{1}{a}(\delta_{N^\mu+i,j} - \delta_{i,j}) \tag{3.2}$$

Defining $M_{\partial_\mu\partial^\mu} := M_{\partial_\mu}^t \cdot M_{\partial_\mu}$

$$M_{\partial_\mu\partial^\mu} = \frac{1}{a^2}\sum_k (\delta_{i-n^\mu,k} - \delta_{i,k})(\delta_{k+n^\mu,j} - \delta_{k,j})$$

$$= \frac{1}{a^2}(2\delta_{i,j} - \delta_{i-N^\mu,j} - \delta_{i+N^\mu,j}) \tag{3.3}$$

If we take the unitary matrix $U_{ij} = e^{ix_ik_j}$ (k_j is in some lattice) and do a similarity transformation, we see that matrices $M_{\partial_\mu\partial^\mu}$'s are diagonalized (denoting $\tilde{M}_{\partial_\mu\partial^\mu}$ as the new diagnolized matrix).

$$(UM_{\partial_\mu\partial^\mu}U^\dagger)_{ij} = \frac{1}{a^2}\sum_{k,l} e^{ix_i\cdot k_l}(M_{\partial_\mu\partial^\mu})_{lk}e^{-ix_k\cdot k_j}$$

$$= \frac{1}{a^2}\sum_l e^{ix_i\cdot k_l}(2e^{ix_l\cdot k_j} - e^{ix_{l+1}\cdot k_j} - e^{ix_{l-1}\cdot k_j})$$

$$= \frac{1}{a^2}(2 - e^{-ia\cdot k_j} - e^{ia\cdot k_j})\sum_l e^{i(x_i-x_j)\cdot k_l}$$

$$\therefore \tilde{M}_{\partial_\mu\partial^\mu} = \frac{1}{a^2}(2 - e^{-ia\cdot k_j} - e^{ia\cdot k_j})\delta_{ij} \tag{3.4}$$

If we define $M_{\partial_\mu\partial^\mu+m^2}$ as $M_{\partial_\mu\partial} + M_{m^2}$, we find-

$$\tilde{M}^{-1}_{(\partial_\mu\partial^\mu+m^2)} = \frac{a^2\cdot\delta_{ij}}{2 - e^{-ia\cdot k_j} - e^{ia\cdot k_j} + m^2a^2} \tag{3.5}$$

So what is the point calculating this? The above expression is the diagonalized form of the Green's function of discrete Klein-Gordon operator. Green's function can be used to find

the solution of linear differential equation of form $Du(x) = f(x)$ (where D is the differential operator and f(x) is known). If $G(x, y)$ is the Green's function of the differential operator, then it can be written-

$$u(x) = \int dy G(x, y) f(y)$$

So, if we have knowlwgde of Green's function, using the above expression, we can find the solution to the inhomogeneous differential equation. And for homogeneous DE's, the solution is the kernel of the Green's function- $u(x) = ker(G(x, y))$. This process relies upon the linearity of the operator D.

From eqn(3.5), we find

$$M_{ij} = \frac{a^2}{(2\pi)^4} \sum_l \frac{e^{ik_l \cdot (x_i - x_j)}}{4 \sum_\mu \sin^2 (a \cdot k_\mu / 2) + a^2 m^2} \tag{3.6}$$

If we take the continuum limit, $\lim a \to 0$ we get back the continuous version of Klein-Gordon propagator

$$< \phi(x)\phi(y) >= \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 + m^2}$$

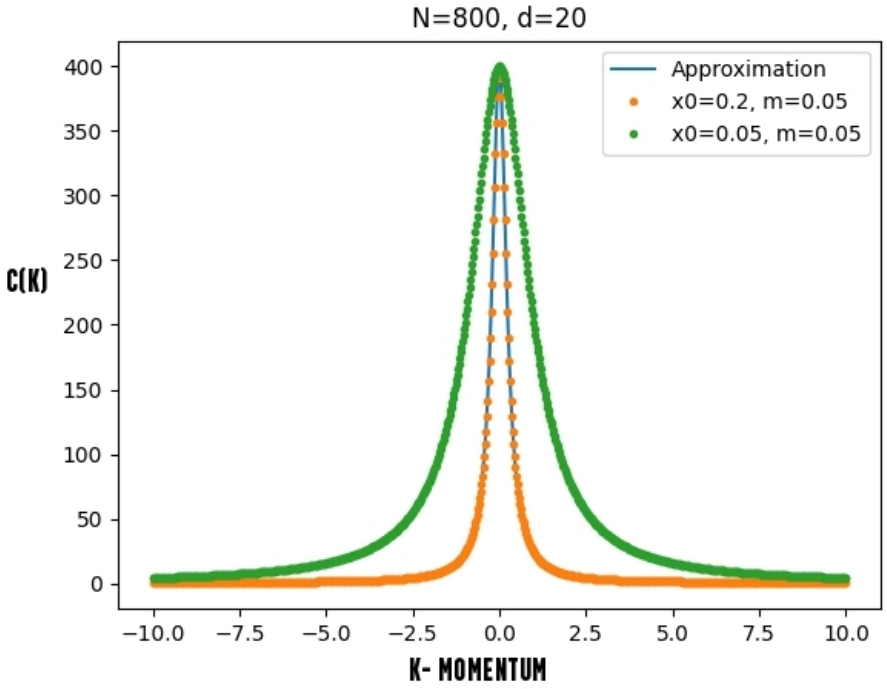


Figure 3.2: Graph of the Klein Gordon propagator.

In fig(3.2) we have plotted diagonalized propagator $C(k)$ against k. The orange and green graphs are the propagator values with lattice spacing a=0.2 and a=0.05 respectively, with total number of lattice points N=800 and mass m=0.05. The blue graph is the continuous approximation for a=0.2. From here we can observe that the propagator becomes more

spread out as we decrease the lattice spacing, so as we take a tending to 0 it is expected to become flatter.

3.2 Propagator Analysis

We have obtained the expression for propagator of Klien-Gordon field, we can now go ahead and analyze what information about the theory can we extract using the propagator. Although we have obtained the plot of the diagonalized propagator in fig(3.2), we can't extract legible information from it. To do so we can plot $\log(C(x))$ with x and see how it behaves.

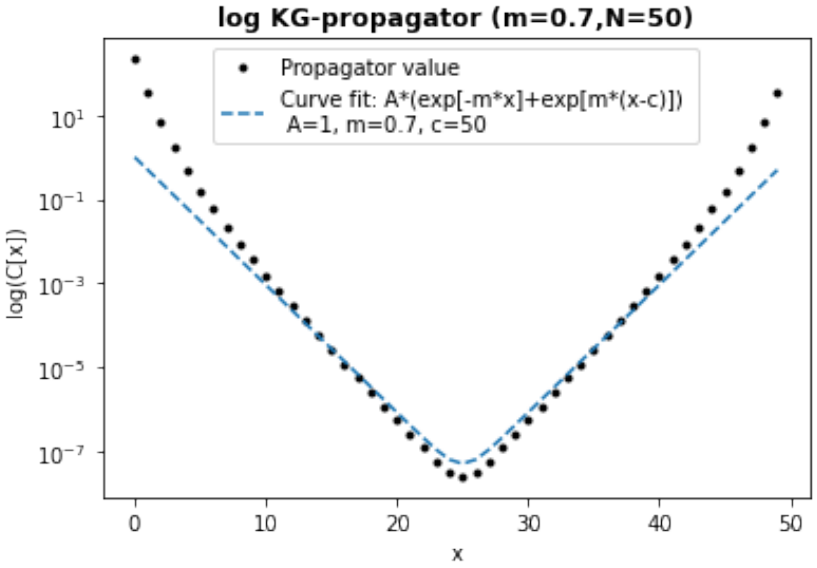


Figure 3.3: Log of Klien-Gordan propagator; Lattice points=50

We see from the graph of fig(3.3) that the propagator behaves linearly when plotted in a semi-log graph. We can go ahead and try to understand why is this the case. To do so we have to use the formula from eqn(2.18). The propagator we are calculating is just the two point function where $\hat{O}_1, \hat{O}_1$ are the position operators $\hat{O}_1 = \hat{\phi}(x)$ and $\hat{O}_2 = \hat{\phi}(y)$. Writing in terms of position operators we get-

$$\begin{aligned}\langle\hat{\phi}(\mathbf{0}, n_t)\hat{\phi}(\mathbf{0}, 0))\rangle = \langle 0|\,\hat{q}(n_t)\,|1\rangle\,\langle 1|\hat{q}(0)\,|0\rangle\,e^{-n_t E_1} + \langle 0|\,\hat{q}(n_t)\,|2\rangle\,\langle 2|\,\hat{q}(0)\,|0\rangle\,e^{-n_t E_2} + ... \\ = A_0 e^{-n_t E_0} + A_1 e^{-n_t E_1} + ..\end{aligned} \tag{3.7}$$

For finite size lattice theories we have finite dimensional Hamiltonian's thus the energy eigenvalues are also discrete and finite. Thus for higher values of n_t's if $E_0 << E_1$, where E_0 is the ground state energy. Thus we will have propagator $C(n_t)$-

$$C(n_t) \approx A_0 e^{-n_t E_0} \tag{3.8}$$

If the operators couple to single particle intermediate states, the masses m_k of those particles will correspond to the low-lying energy values, i.e., $E_k = m_k$. Thus plotting it in a

semi-log graph would give us a straight line: $y = mx + c$ with slope m as the mass of a single particle of the theory. But looking at the graph we have plotted it seems the propagator initially linearly decreases up till the half point $n_t = N/2$, and then it linearly increases again. This is because we have taken periodic boundary condition, point n_t and $N - n_t$ are identical. In B. Lang, it is given "For mesons propagation in nt and (N nt) is identical up to a possible relative minus sign", this also explains the data the curve fits with $\approx e^{n_t} + e^{N-n_t}$.

Taking this into the curve would behave approximately like-

$$C(n_t) \approx A_0 e^{-n_t E_0} + A_0 e^{-(N-n_t)E_0} \tag{3.9}$$

In the graph: fig(3.3) the blue dashed line represents fitting of the curve with fit eqn(3.9) and parameters $A_0 = 1, E_0 = 0.7, N = 50$. So we can extract the mass parameter by fitting the propagator with eqn(3.9), but it is apparent from the graph for lower values of n_t it doesn't behave linearly, it deviates from the dashed line. It is because for low n_t's, contributions from higher excited states will also be there which won't be negligible.

Another way we can extract the mass parameter is by plotting the slope of the log of the propagator which should give us a constant value if the log of propagator is linear.

$$\frac{d}{dn_t} \log C(n_t) := \frac{\log(C(n_t+1)) - \log(C(n_t))}{1} = \log\left(\frac{C(n_t+1)}{C(n_t)}\right) \tag{3.10}$$

So we can directly plot n_t vs $\log\left(\frac{C(n_t+1)}{C(n_t)}\right)$ to get the slope of plot in fig(3.3).

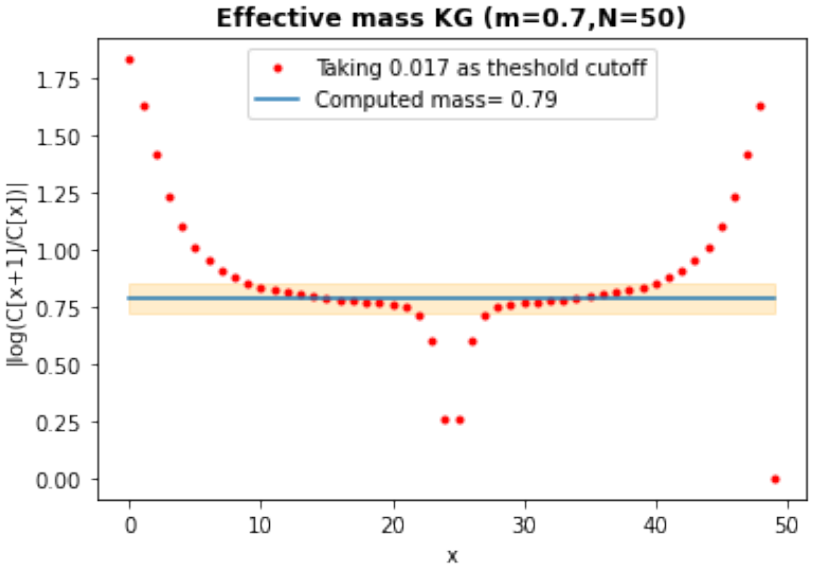


Figure 3.4: Absolute value of effective mass is plotted

It is much more apparent from the plot of effective mass: fig(3.4) the log of propagator doesn't behave linearly near the zero point, only for sufficiently large n_t's it converges to a constant value. Also for values near $n_t = N/2$ it deviates from linear behaviour since near

that point both contribution from e^{-mn_t} and $e^{-m(N-n_t)}$ comes into consideration, so it behaves non-linearly.

So to correctly extract the mass parameter of the theory we have to carefully only consider the points which gives near constant value for the effective mass, this has to be done with objective ... To pick the points for calculation of the mass parameters we have plotted the derivative of effective mass and only choose points with slope less than 0.017. It is shown in fig(3.5).

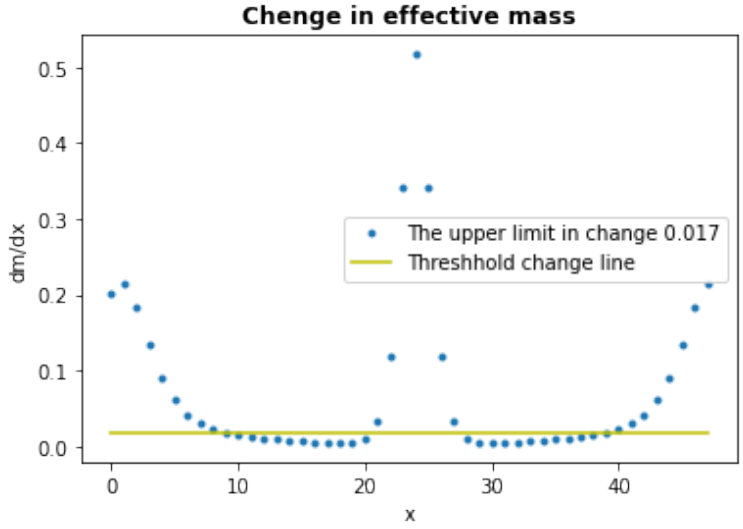


Figure 3.5: Change in effective mass

The graph in fig(3.5) visually shows the points chosen to take into consideration for calculating the mass parameter through effective mass graph. Taking only the points below the green line (which corresponds to $\frac{dm}{dx} = 0.017$) the mass parameters comes out to be $\mathbf{m \approx 0.79}$ which is not too far away from the mass that was input initially $m = 0.70$. It is clear that the mass calculated would be dependent on the cutoff change we impose, here we have taken it to be 0.017, but it was completely arbitrary. If we want more accuracy of the mass, we can take more number of lattice points, so we would have more number of points below the threshold limit to do analysis.

3.3 Klien-Gordon Schwinger Parametrization

In section 2.1 we had calculated the expression for Klien-Gordon propagator using the greens function definition and we obtained eqn(3.6) as the final expression which we had analyzed in section 2.2. eqn(3.6) involves 4 dimensional summation which takes some time to compute in a computer. We can do better and make the expression bit simpler.

To start the calculation, we will first do an approximation and write the summation of eqn(3.6) as integral, then the equation becomes (We can later see how well this approximation

works)-

$$C(\mathbf{0},\mathbf{n}) = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{e^{i\bar{k}\cdot n_i}}{4\sum_\mu \sin^2(\bar{k}_\mu/2) + \bar{m}^2} \tag{3.11}$$

Here $\bar{k}$ and $\bar{m}$ are defined as $\bar{k} = a\cdot k$, $\bar{m} = a\cdot m$ and $n_i = x_i/a$. The term $\frac{1}{4\sum_\mu \sin^2(\bar{k}_\mu/2)+\bar{m}^2}$ inside the integral can be re-written as-

$$\frac{1}{4\sum_\mu \sin^2(\bar{k}_\mu/2) + \bar{m}^2} = \int_0^\infty \exp\left[-(4\sum_\mu \sin^2(\bar{k}_\mu/2) + \bar{m}^2)t\right] dt$$

putting the above expression into eqn(3.11),

$$= \int_{BZ} \frac{d^4k}{(2\pi)^4} e^{i\bar{k}\cdot n_i} \int_0^\infty e^{-(4\sum_\mu \sin^2(\bar{k}_\mu/2)+\bar{m}^2)t} dt$$

$$= \int_0^\infty dt e^{-\bar{m}^2 t} \int_{BZ} \frac{d^4k}{(2\pi)^4} e^{i\bar{k}\cdot n_i} e^{-4\sum_\mu \sin^2(\bar{k}_\mu/2)t}$$

Notice $\bar{k}\cdot n_i$ is just compact way of writing $\sum_\mu \bar{k}_\mu n_i^\mu$. Now lets only consider the integral over d^4p for simplicity (denoting it as I)-

$$I = \int_{BZ} \frac{d^4k}{(2\pi)^4} e^{i\sum_\mu \bar{k}_\mu n_i^\mu} \cdot e^{-4\sum_\mu \sin^2(\bar{k}_\mu/2)t}$$

$$= \int_{BZ} \Big(\prod_{\mu=1}^4 \frac{dk_\mu}{(2\pi)}\Big)\Big(\prod_{\mu=1}^4 e^{i\bar{k}_\mu n_i^\mu} \cdot e^{-4\sin^2(\bar{k}_\mu/2)t}\Big)$$

$$= \prod_{\mu=1}^4 \int_{-\pi}^{\pi} \frac{dk_\mu}{(2\pi)} e^{i\bar{k}_\mu n_i^\mu} \cdot e^{-4t\sin^2(\bar{k}_\mu/2)} \tag{3.12}$$

Let us denote the integral inside the product as I_μ. from the above expression it is clear that: $I = I_1 \cdot I_2 \cdot I_3 \cdot I_4$, we could do this specifically because the integrand could be written as separate product of individual k variables. We now have I_μ as-

$$I_\mu = \int_{-\pi}^{\pi} \frac{dk_\mu}{(2\pi)} e^{i\bar{k}_\mu n_i^\mu} \cdot e^{-4t\sin^2(\bar{k}_\mu/2)} \tag{3.13}$$

The $e^{-4t\sin^2(\bar{k}_\mu/2)}$ part in the integral is an even function, and the part $e^{i\bar{k}_\mu n_i^\mu}$ can be broken into imaginary and real part which are odd and even respectively. And since the integral is done over $-\pi$ to π, the imaginary part of the integral will become zero, thus leaving only the real part-

$$I_\mu = 2\int_0^{\pi} \frac{dk_\mu}{(2\pi)} \cos(\bar{k}_\mu n_i^\mu) \cdot e^{-4t\sin^2(\bar{k}_\mu/2)} \tag{3.14}$$

Using the trigonometric identity $\sin^2(2x) = (1-\cos x)/2$ into eqn(3.14)

$$= 2\int_0^{\pi} \frac{dk_\mu}{(2\pi)} \cos(\bar{k}_\mu n_i^\mu) \cdot e^{-2t(1-\cos\bar{k}_\mu))}$$

$$= 2e^{-2t} \int_0^{\pi} \frac{dk_\mu}{(2\pi)} \cos\left(\bar{k}_\mu n_i^\mu\right) \cdot e^{2t\cos\bar{k}_\mu} \tag{3.15}$$

We can recognize the integral above as the Bessel function of second kind which is defined as $\mathbf{I}_\alpha(x) = \frac{1}{\pi}\int_0^{\pi} d\theta e^{x\cos\theta}\cos(\alpha x)$. Comparing this with eqn(3.15) we see $\alpha = n_i^\mu$ and $x = 2t$. We can then write-

$$I_\mu = e^{-2t}\mathbf{I}_{n_i^\mu}(2t) \tag{3.16}$$

Using this we can put it into our original equation of interest, we get-

$$C(\mathbf{0},\mathbf{n}) = \int_0^{\infty} dt e^{-\bar{m}^2 t} \prod_{\mu=1}^{4} \left(e^{-2t}\mathbf{I}_{n_i^\mu}(2t)\right)$$

Note since we had taken dimension of our space-time to be 4, the product is being done over index 1 to 4. We can take dimension of space-time to be anything and still the calculation will remain same, only that the product will be done from $\mu = 1, 2..d$. Thus we will attain a more general expression-

$$\boxed{C(\mathbf{0},\mathbf{n}) = \int_0^{\infty} dt e^{-\bar{m}^2 t} \prod_{\mu=1}^{d} \left(e^{-2t}\mathbf{I}_{n_i^\mu}(2t)\right)} \tag{3.17}$$

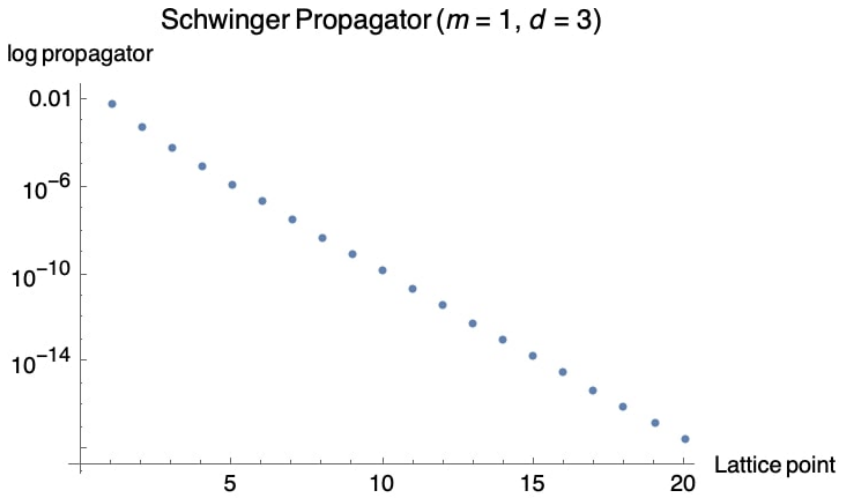


Figure 3.6: Caption

Chapter 4

Fermionic Propagator

Now let's go ahead and calculate the propagator for Dirac fermions the same way we did the KG propagator. This case is a little more sophisticated since we now have 4-component spinor field. The continuum spinor action is given by: $S_F = \int d^4x \bar{\psi}(\mathbf{x})(i\gamma_\mu \partial^\mu + m)\psi(\mathbf{x})$. Where $\psi(\mathbf{x})$ is of the form-

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \\ \psi_3(\mathbf{x}) \\ \psi_4(\mathbf{x}) \end{pmatrix}$$

And γ_μ are gamma matrices which satisfies the commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \times I$. Expanding $\psi(\mathbf{x})$ and γ_μ's in the action (repeated indices are summed over)-

$$S_F = \int d^4x \sum_{\alpha\beta} \psi^*_\eta(x)(\gamma_0)_{\alpha\eta}\Big(i(\gamma_\mu)_{\alpha\beta}\partial^\mu + m\delta_{\alpha\beta}\Big)\psi_\beta(x)$$

For lattice case, in the above equation we will have the following replacements $\int \to \sum$, $dx \to a$, $\partial_\mu \to M_{\partial_\mu} = \frac{\delta^{N^\mu+i,j} - \delta^{N^\mu+i,j}}{2a}$. Here indices i, j are 4-indices $i = (i_1, i_2, i_3, i_4) \in \prod_{j=0}^{4}\{1, .., N\}$.

$$S_F = \sum_{x_i \in \Lambda} a^4 \sum_{\alpha\beta} \psi^*_\eta(x_i)(\gamma_0)_{\alpha\eta}\Big(i\sum_\mu (\gamma_\mu)_{\alpha\beta} M_{\partial^\mu} + m\delta_{\alpha,\beta}\Big)\psi_\beta(x_i) \tag{4.1}$$

$$= \sum_{\alpha,\beta} a^4(\gamma_0)_{\alpha\eta} \sum_{x_i \in \Lambda} \psi^*_\eta(x_i)\Big(i\sum_\mu (\gamma_\mu)_{\alpha\beta} M_{\partial^\mu} + m\delta_{\alpha,\beta}\Big)\psi_\beta(x_i)$$

The part inside the $\sum_{x_i\in\Lambda}$ can be recognized to be in the form $v_1^\dagger A v_2$. Where $v_2 = \psi_\beta(x_i)$, $v_1 = \psi^*_\eta(x_i)$ and $A = i\sum_\mu (\gamma_\mu)_{\alpha\beta} M_{\partial^\mu} + m\delta_{\alpha,\beta}$. Let's try to diagonalize A-

$$A_{\alpha\beta} = i\sum_\mu (\gamma_\mu)_{\alpha\beta} M_{\partial^\mu} + m\delta_{\alpha,\beta} \tag{4.2}$$

Doing a change of basis under basis transformation $U_{ij} = e^{-ik_i \cdot x_j}$, A transforms to $\tilde{A} = UAU^\dagger$. From eqn(4.2) we find-

$$\tilde{A}^{ij} = U^{im} A^{ml} U^{lj*}$$

$$= \sum_{m,l} \Big[i \sum_{\mu} (\gamma_\mu)_{\alpha\beta} e^{-i(x_m \cdot k_i - x_l \cdot k_j)} \big(\frac{\delta^{m+N^\mu,l} - \delta^{m-N^\mu,l}}{2a} \big) + m\delta_{\alpha,\beta} e^{-i(x_m \cdot k_i - x_l \cdot k_j)} \delta^{i,j} \Big] \tag{4.3}$$

Here $\delta_{\alpha\beta}$ (lower indices) is used for spinor indices, and δ^{ij} is used for space-time indices. Note $\{\gamma_\mu\}_{\alpha\beta}$ term has no m,j dependence, so we can take the summation inside. Dimensionaly it also makes sense since $[m] = \frac{1}{[a]}$. Now lets consider only the first two terms inside (the terms with $2a$ in the denominator), other terms would be irrelevant for now-

$$\sum_{m,l} e^{-i(x_m \cdot k_i - x_l \cdot k_j)} \big(\frac{\delta^{m+N^\mu,l} - \delta^{m-N^\mu,l}}{2a} \big)$$

$$= \sum_{m} \frac{1}{2a} (e^{-i(x_m \cdot k_i - x_{m+N^\mu} \cdot k_i)} - e^{i(x_m \cdot k_i - x_{m-N^\mu} \cdot k_j)})$$

$$= \frac{e^{i\bar{x}_\mu \cdot k_j} - e^{-i\bar{x}_\mu \cdot k_j}}{2a} \sum_{m} e^{i(k_i - k_j) \cdot x_m} = \frac{\sin(\bar{x}_\mu \cdot k_j)}{a} \delta^{ij} \tag{4.4}$$

Here $\bar{x}_\mu$ is defined as $\bar{x}_0 = (a, 0, 0, 0)$, etc. The last term can be evaluated similarly. After simplification putting everything onto eqn(4.3), we find-

$$\tilde{A}^{ij}_{\alpha\beta} = i(\gamma_\mu)_{\alpha\beta} \delta^{ij} \frac{\sin(\bar{x}_\mu \cdot k_j)}{a} + m\delta_{\alpha\beta}\delta^{ij}$$

Inverting the equation and then using the relation $(\gamma_\mu)^2 = \mathbf{I}$ we find-

$$(\tilde{A}^{ij}_{\alpha\beta})^{-1} = \frac{-i(\gamma_\mu)_{\alpha\beta}\delta^{ij} \sin(\bar{x}_\mu \cdot k_j)/a + m\delta_{\alpha\beta}\delta^{ij}}{\sum_\mu \sin^2(\bar{x}_\mu \cdot k_j)/a^2 + m^2} \tag{4.5}$$

If we now go ahead and plot the denominator of $(\tilde{A}^{ii}_{\alpha\beta})^{-1}$ w.r.t. k_i-

From the plot we can see the there are two poles and two half-poles from in the region $-\pi \leq k^i \leq \pi$. But the problem is that in continuum euclidian space for the propagator, we only obtain 1 singularity at the $k_i = 0$ (In 1 dimension). In 4 dimension we get 16 extra singularities- This is called the **fermion doubling problem**. Each singularity represents an on shell point so having 16 singularities means we have have 16 extra fermions in the lattice case. The doubling phenomenon must occur in a lattice regularization which respects the usual hermiticity, locality and translational invariance requirements, follows from a theorem by Nielsen and Ninomiya [Nielsen (1981)] which states that, under the above assumptions, one cannot solve the fermion doubling problem without breaking chiral symmetry for vanishing fermion mass.

4.1 Propagator Analysis

We do the same analysis of the propagator we did for the Klien-Gordon case. We first see how the propagator behaves on a semi-log plot. The difference here with the Klien-Gordon case is that here we have 16 different components of propagator for each index of gamma matrix α, β (see eqn(4.5)). We here have chosen $\alpha = 0, \beta = 0$ for our analysis, but whatever indices of α, β we choose, we would get the same result.

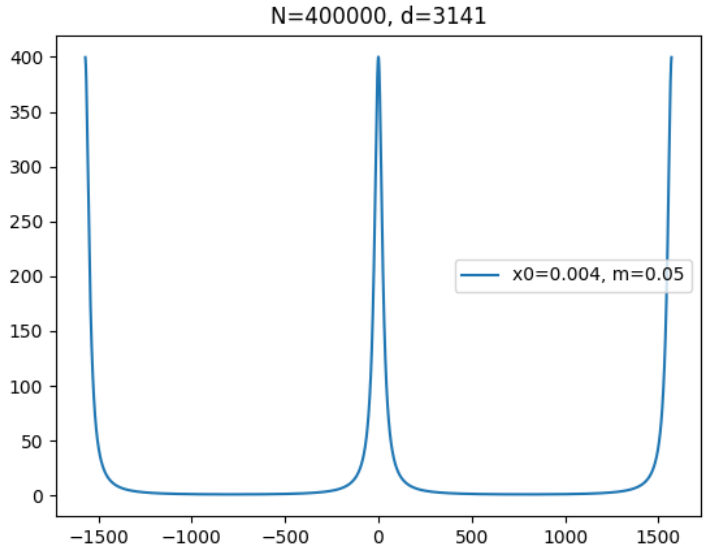


Figure 4.1: Denominator of the Fermionic propagator

Here the mass parameter input was $m = 1$ with lattice points $N = 70$. From the graph we can see the parameters $A = 10^{6.5}, m = 1, c = 70$ gives a good fit for eqn(3.9).

The plot of effective mas gives a peculiar behaviour, although the value of effective mass is near initially input mass: $m = 1$, but it oscillates around the 1 value. The curve doesn't look continuous at all, to resolve this problem we have taken the two adjacent points average and plotted it.

from fig(4.4) we can see taking the two point average makes the plot much smoother but at the expense of loosing half of the points. Now just like in KG case, to calculate the effective mass of the theory we only choose certain points, here we have considered threshold cutoff change = 0.011. Taking this the mass parameter from effective mass plot comes out to be $\mathbf{m \approx 1.017}$

4.2 Fermionic Schwinger Parametrization

In section 2.3 we obtained an alternative expression for the propagator using Laplace transform of diagonalized propagator. We can try to do the same thing and see whether it works for Fermionic action too. Instead of taking discrete summation if we write it as integral, we have the fermionic action as-

$$C_{\alpha,\beta}(\mathbf{0},\mathbf{n}) = \int_{BZ} d^4p \frac{e^{i\bar{p}\cdot n}}{i\gamma_{\alpha\beta}\cdot \sin(\bar{p}) + \bar{m}} \tag{4.6}$$

Here $a \cdot b$ is just compact notation for summation over lorentz index i.e. $\sum_\mu a_\mu b^\mu$. Also, $\bar{p} = ap$, $\bar{m} = am$. Now we can write the denominator as the following-

$$\frac{1}{i\sum \gamma^\mu_{\alpha\beta} \sin(\bar{p_\mu}) + \bar{m}} = \int_0^\infty \exp\left[-(i\sum_\mu \gamma^\mu_{\alpha\beta} \sin(\bar{p_\mu}) + \bar{m})t\right] dt \tag{4.7}$$

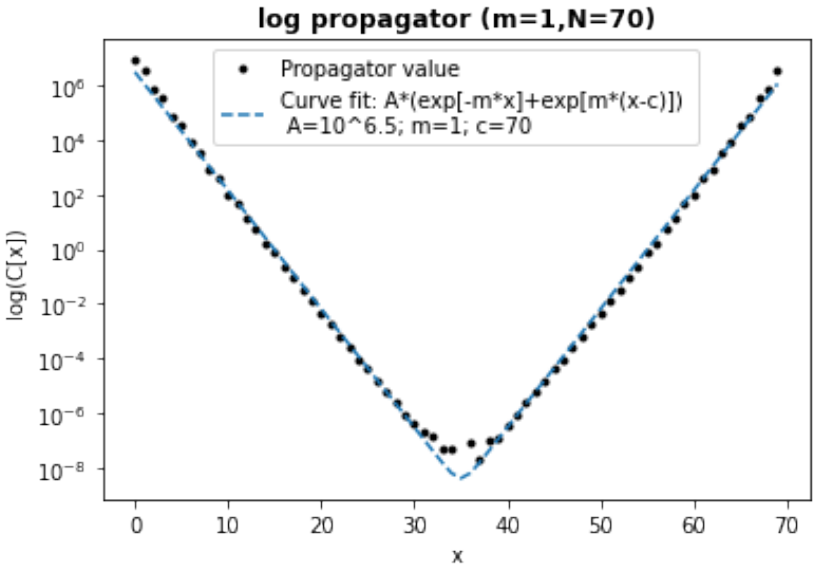


Figure 4.2: Here $\alpha = 0, \beta = 0$ component of propagator is considered

Substituting this into the above equation-

$$C_{\alpha,\beta}(\mathbf{0},\mathbf{n}) = \int_{BZ} d^4p e^{i\bar{p}\cdot n} \int_0^\infty e^{-(i\gamma_{\alpha\beta}\cdot\sin(\bar{p})+\bar{m})t} dt$$

$$= \int_0^\infty dt e^{-\bar{m}t} \int_{BZ} d^4p e^{i\bar{p}\cdot n} e^{-i\gamma_{\alpha\beta}\cdot\sin(\bar{p})t} \tag{4.8}$$

We can just now focus of the integral over d^4p and try to simplify it, we can just call it I

$$I = \int_{BZ} d^4p e^{i(\bar{p}\cdot n - \gamma_{\alpha\beta}\cdot\sin(\bar{p})t)} = \int_{BZ} d^4p e^{i\sum(\bar{p}_\mu n^\mu - \gamma^\mu_{\alpha\beta}\sin(\bar{p}_\mu)t)}$$

We can write exponential of sum as product of exponential

$$= \int_{BZ} \prod_\mu^4 dp_\mu \Big(\prod_\mu^4 e^{i(\bar{p}_\mu n^\mu - \gamma^\mu_{\alpha\beta}\sin(\bar{p}_\mu)t)}\Big)$$

$$= \prod_\mu^4 \int_{-\pi}^{\pi} dp_\mu e^{i(\bar{p}_\mu n^\mu - \gamma^\mu_{\alpha\beta}\sin(\bar{p}_\mu)t)} \tag{4.9}$$

Let's denote the integral inside the product as I_μ. We can write the integrand inside as sum of real and imaginary part. The imaginary part would be- $\sin\big(\bar{p}_\mu n^\mu - \gamma^\mu_{\alpha\beta}\sin(\bar{p}_\mu)t\big)$, which can be clearly seen is an odd function. Thus integral over $-\pi$ to π would give zero, and only the real part would give contribution.

$$I_\mu = \int_{-\pi}^{\pi} \cos[(\bar{p}_\mu n^\mu - \gamma^\mu_{\alpha\beta}\sin(\bar{p}_\mu))t] dp_\mu \tag{4.10}$$

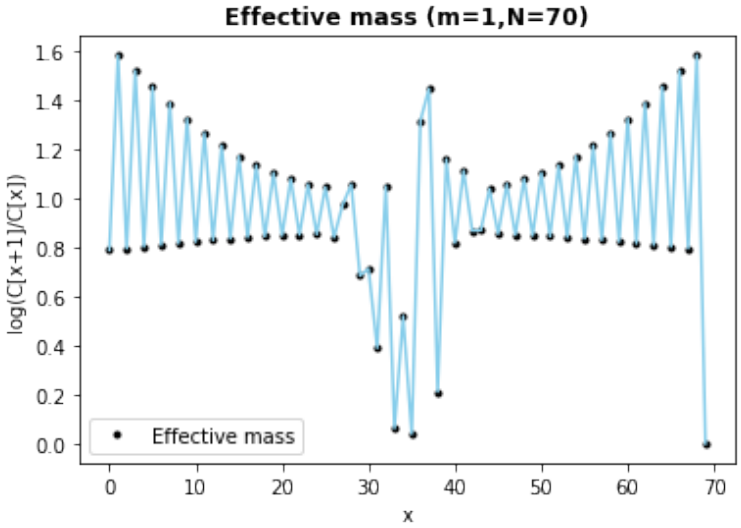


Figure 4.3: Effective mass plot

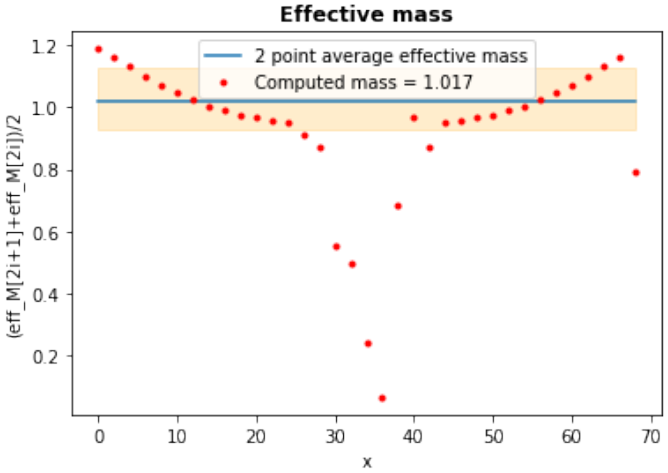


Figure 4.4: Two point average plot of effective mass

This we can recognize as the Bessel function $\mathbf{J}_k(x) = \int_0^\infty \cos(k\tau - x\sin(\tau))d\tau$. Comparing this and eqn(4.10) we see that $k = tn^\mu$ and $x = t\gamma^\mu_{\alpha\beta}$. Thus we can write eqn(4.8) as

$$\boxed{C_{\alpha,\beta}(\mathbf{0},\mathbf{n}) = -\int_{BZ} d^4p e^{-\bar{m}t} \prod_\mu^4 \mathbf{J}_{tn^\mu}(t\gamma^\mu_{\alpha\beta})} \tag{4.11}$$

Chapter 5

Solutions to the fermion doubling problem

5.1 Wilson fermion

To solve the doubling problem Wilson tweaked the fermionic action such that it shall also converge to the same continuum limit as $a \to 0$. He proposed this action-

$$S_F^{(W)} = S_F + \frac{r}{2} \sum_{x_i \in \Lambda} \sum_{\alpha} \psi_\alpha(x_i) \partial_\mu \partial^\mu \psi_\alpha(x_i) \tag{5.1}$$

Here r is the Wilson's parameter. If we go ahead and calculate the propagator, we find (the diagonalized version)-

$$(\tilde{B}^{ij}_{\alpha\beta})^{-1} = \frac{i(\gamma_\mu)_{\alpha\beta}\delta^{ij} \sin{(\bar{x}_\mu \cdot k_j)}/a + m(k_j)\delta_{\alpha\beta}\delta^{ij}}{\sum_\mu \sin^2(x_\mu \cdot k_j)/a^2 + m^2(k_j)}$$

Where $m(k_j) = m + \frac{2r}{a} \sum_\mu \sin^2(\frac{\bar{x}_\mu \cdot k_j}{2a})$. We can see the the new r term in the action introduces an extra term in the mass part. The case r = 0 corresponds to the naive fermions. This new part $\sin^2(\frac{\bar{x}_\mu \cdot k_j}{2a})$ part vanishes for $k_i = 0$, but gives a non-zero $(\frac{2r}{a})$ value at the end of the Brilloun zone. Thus removing the extra singularities.

This eliminates the fermion doubling problem, but at the expense that the chiral symmetry of the original action for m = 0 has been broken. This makes this scheme less attractive for studying such questions as spontaneous chiral symmetry breaking in QCD (which requires a fine tuning of the parameter m).

Let us define this -

$$j_{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$

Where $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, we can calculate-

$$\partial_\mu j_{\mu 5} = 2im\bar{\psi}\gamma^5\psi \tag{5.2}$$

We can see for $m = 0$ it leads to conserved current which leads to chirality conservation. But for Wilson fermions, even when we set mass (m=0), we have $m(p) \neq 0$. So chirality conservation doesn't hold in this case.

5.2 Staggered fermion

Another way to discretize fermions on the is through Staggered fermions or Kogut–Susskind fermions. Here the treatment is done in such a way so that we preserve chiral symmetry. We impose a local transformation $T(n)$ to $\psi(n)$-

$$\psi(n) \to T(n)\psi(n)^{'}$$
$$\bar{\psi}(n) \to \bar{\psi}(n)^{'}T(n)^{\dagger}$$

Here $x_n = (x_{n1}, x_{n2}, x_{n3}, x_{n4})$ is simply denoted as $n = (n_1, n_2, n_3, n_4)$. And the transformation $T(n)$ is defined in the following way-

$$T(n) = \gamma_1^{n_1}\gamma_2^{n_2}..\gamma_4^{n_4} \tag{5.3}$$

Imposing this transformation, if we calculate this term, using eqn(5.3) and the gamma matrices anti-commutation relation $\{\gamma_i, \gamma_j\} = \delta_{ij}$ we find-

$$\bar{\psi}(n)\gamma_3\psi(n \pm 3) = (-1)^{n_1+n_2}\bar{\psi}(n)^{'}\psi(n \pm 3)^{'} \tag{5.4}$$

For general $\gamma_\mu = 1, 2, ..k$, we will have $(-1)^{n_1+n_2..+n_{\mu-1}} \cdot \bar{\psi}(n)^{'}\psi(n \pm \mu)^{'}$. This result will be useful in calculating the transformed fermionic action. The original fermionic action we have

$$S_F[\psi, \bar{\psi}] = a^4 \sum_{n\in\Lambda} \bar{\psi}(n)(\sum_{\mu=1}^{4} \gamma_\mu \frac{\psi(n+\mu) - \psi(n-\mu)}{2a}) + m\bar{\psi}(n)\psi(n)$$

$$= a^4 \sum_{n\in\Lambda} \frac{1}{2a} \sum_{\mu=1}^{4} [\bar{\psi}(n)\gamma_\mu\psi(n+\mu) - \bar{\psi}(n)\gamma_\mu\psi(n-\mu)] + m\bar{\psi}(n)\psi(n)$$

Using the relation from eqn(5.4), we find

$$S_F^{'}[\psi^{'}, \bar{\psi}^{'}] = a^4 \sum_{n\in\Lambda} \frac{1}{2a} \sum_{\mu=1}^{4} \eta_\mu(n)[\bar{\psi}^{'}(n)\psi^{'}(n+\mu) - \bar{\psi}^{'}(n)\psi^{'}(n-\mu)] + m\bar{\psi}^{'}(n)\psi^{'}(n)$$

Where $\eta_\mu = (-1)^{n_1+n_2..+n_{\mu-1}}$. The existence of γ term mixes the spinor components of the dirac field. But since after this transformation the gamma matrices becomes essentially identity, all the spinor components of the field decouples with each other. So the terms $\bar{\psi}^{'}(n)\psi^{'}(n \pm \mu) = \sum_\alpha \bar{\psi}_\alpha^{'}(n)\psi_\alpha^{'}(n \pm \mu)$. Writing the field in terms of its components, we get the Dirac action-

$$S_F^{'}[\psi^{'}, \bar{\psi}^{'}] = a^4 \sum_{n\in\Lambda} \sum_{\alpha} \Big(\sum_{\mu=1}^{4} \eta_\mu(n)\bar{\psi}_\alpha^{'}(n) \frac{\psi_\alpha^{'}(n+\mu) - \psi_\alpha^{'}(n-\mu)}{2a} + m\bar{\psi}_\alpha^{'}(n)\psi_\alpha^{'}(n) \Big) \tag{5.5}$$

Here the Dirac matrices index runs over 1 to 4, but we can make it run over other number of numbers too. If we choose other representation than gamma matrices, as long as they satisfy the same anti-commutation relations the argument works fine. So in theory the α index can run from 1 to k. The simplest case is $\alpha = 1$, setting it to 1 and proceeding so we don't have to worry about the index. Eqn(5.5) then becomes-

$$S_F^{'}[\psi^{'}, \bar{\psi}^{'}] = a^4 \sum_{n\in\Lambda} \Big(\sum_{\mu=1}^{4} \eta_\mu(n)\bar{\psi}^{'}(n) \frac{\psi^{'}(n+\mu) - \psi^{'}(n-\mu)}{2a} + m\bar{\psi}^{'}(n)\psi^{'}(n) \Big) \tag{5.6}$$

A 4-component spinor field ψ somehow has reduced to a single component field ψ'.

Now using the property $\gamma_\mu^2 = \mathbf{I}$ one can easily notice:

$$T(n_1, n_2 + 2, n_3, n_4) = T(n_1, n_2, n_3, n_4) \tag{5.7}$$

So in 4-spacetime dimension we have only 16 different transformations, no matter how many lattice points are there. The transformation repeats itself after every two lattice points. This compels us to redefine the lattice indices in the following way.

$$n_\mu = 2r_\mu + \rho_\mu$$

Where r_μ runs from $1, 2..N/2$ and $\rho \in \{0, 1\}^d$. So now instead of denoting lattice points with n, we denote it with a tuple (r, ρ).

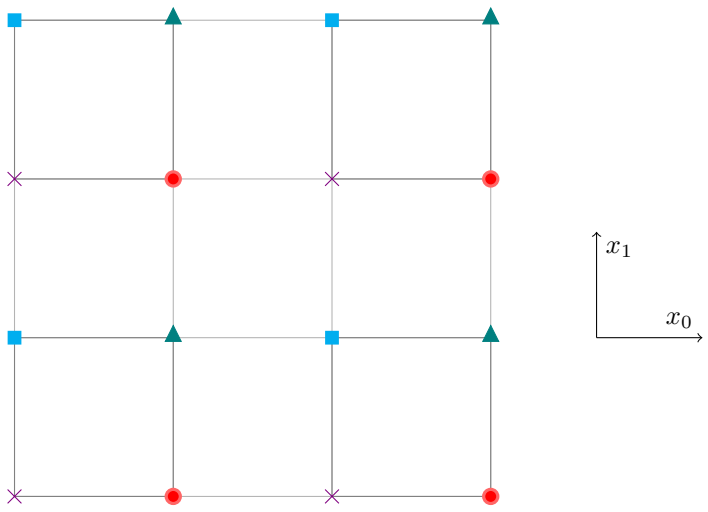


Figure 5.1: In 2D there are total 4 different transformations denoted by the four colours.

From the above figure we can see the the label r fixes the hypercube (in 2D it's just a square) and the label ρ fixes the vertex of the unit hypercube we are in. From eqn(5.7) we can also see that the transformation $T(n)$ only depends on the index ρ i.e. which site of the hypercube the point resides in. So now we can write the action in terms of new labels as-

$$\begin{aligned} S_F'[\psi', \bar{\psi}'] = a^4 \sum_{r,\rho,\mu} \frac{1}{2a} \eta_\mu(\rho) \bar{\psi}'(2r + \rho)[\psi'(2r + \rho + \mu) - \psi'(2r + \rho - \mu)] \\ + \sum_{r,\rho} m \bar{\psi}'(2r + \rho)\psi'(2r + \rho) \end{aligned} \tag{5.8}$$

Here we must remember that ρ index can only go from 0 to 1. So if we have $\rho_\mu = 1$ then, $\psi'(2r_\mu + 1 + \mu) = \psi'(2(r_\mu + 1))$. This can be summarized with the following equation (if we relabel $\psi'(2r + \rho) \to \psi'_\rho(r)$):

$$\psi'(2r + \rho + \mu) = \sum_{\rho'} [\delta_{\rho+\mu,\rho'} \psi'_{\rho'}(r) + \delta_{\rho-\mu,\rho'} \psi'_{\rho'}(r - \mu)]$$

In a similar way one obtains,

$$\psi^{'}(2r+\rho-\mu)=\sum_{\rho^{'}}[\delta_{\rho-\mu,\rho^{'}}\psi^{'}_{\rho^{'}}(r)+\delta_{\rho+\mu,\rho^{'}}\psi^{'}_{\rho^{'}}(r-\mu)]$$

Using the above results and inserting it into eqn(5.8), we get-

$$\begin{aligned}S^{'}_{F}[\psi^{'},\bar{\psi}^{'}]=\frac{1}{2}\sum_{r,\mu,\rho,\rho^{'}}\eta_{\mu}(\rho)\bar{\psi}^{'}_{\rho}(r)[\delta_{\rho+\mu,\rho^{'}}\partial^{L}_{\mu}\psi^{'}_{\rho^{'}}(r)+\delta_{\rho-\mu,\rho^{'}}\partial^{R}_{\mu}\psi^{'}_{\rho^{'}}(r)]\\ +m\sum_{r,\rho}\bar{\psi}^{'}_{\rho}(r)\psi^{'}_{\rho}(r)\end{aligned}\tag{5.9}$$

Here we have defined $\partial^{L}_{\mu}\psi^{'}_{\rho}(r):=\frac{1}{a}(\psi^{'}_{\rho}(r+\mu)-\psi^{'}_{\rho}(r))$ and $\partial^{R}_{\mu}\psi^{'}_{\rho}(r):=\frac{1}{a}(\psi^{'}_{\rho}(r)-\psi^{'}_{\rho}(r-\mu))$. These terms are like derivatives, ∂^{L}_{ρ} is the left hand derivative of $\psi^{'}_{\rho}(r)$ and ∂^{R}_{ρ} is the right hand derivative. The difference here with the naive fermions case is that here the difference of $\psi^{'}$ is taken for every second lattice site. Since the derivative taken w.r.t. r index (which is equal to $2n$). Also another thing we have done here is that we have absorbed label ρ (which we used earlier to label sites of a hypercube) into $\psi^{'}_{\rho}$. So now we have 2^{d} number of fields each from one vertex of hypercube, refer to fig(5.1), where d is the dimension of space.

Our main aim was to calculate the lattice propagator, we have obtained the expression for lattice action for the propagator. Now let's compute the 2-point correlator for the staggered fermionic action. We will start with the first term of the eqn(5.9) and try to diagonalize it-

$$\sum_{r,\mu,\rho,\rho^{'}}\eta_{\mu}(\rho)\bar{\psi}^{'}_{\rho}(r)\delta_{\rho+\mu,\rho^{'}}\partial^{L}_{\mu}\psi^{'}_{\rho^{'}}(r)=\sum_{\mu}\sum_{r,\rho,\rho^{'}}\bar{\psi}^{'}_{\rho}(r)\Big(\eta_{\mu}(\rho)\delta_{\rho+\mu,\rho^{'}}\partial^{L}_{\mu}\Big)\psi^{'}_{\rho^{'}}(r)\tag{5.10}$$

Once we got this form of the action term, we would like to diagonalize the operator ∂^{L}_{μ}. From its definition itself we can write-

$$\big(\partial^{L}_{\mu}\big)^{m,n}=(\frac{\delta^{m+N^{\mu},n}-\delta^{m,n}}{2a})\tag{5.11}$$

Notice we have $2a$ in the denominator, since the r index runs every second site the distance between two consecutive r's is $2a$. Doing a change of basis on eqn(5.11) with the same transformation matrix we had used for naive fermionic action, we get ∂^{L}_{μ} in terms of new basis as-

$$\big(\tilde{\partial^{L}_{\mu}}\big)^{m,n}=\delta^{m,n}\Big(\frac{e^{r_{\mu}\cdot k_{m}}-1}{2a}\Big)$$

In similar way we can obtain the diagonalized form of the ∂^{R}_{μ} from the second form. To simplify the expressions we define-

$$\Gamma^{\mu}_{\rho\rho^{'}}:=\eta_{\mu}(\rho)[\delta_{\rho+\mu,\rho^{'}}+\delta_{\rho-\mu,\rho^{'}}]\tag{5.12}$$

Now putting evrything we obtained into eqn(5.9), we get the diagonalized action for staggered fermions as (here $\psi(k)$ is the fourier transformed field of $\psi(r)$)-

$$S^{'}_{F}[\psi^{'},\bar{\psi}^{'}]=\frac{1}{2}\sum_{k,k^{'},\rho,\rho^{'}}\bar{\psi}^{'}_{\rho}(k)\Big(\sum_{\mu}\Gamma^{\mu}_{\rho,\rho^{'}}\delta^{m,n}\sin\Big(\frac{r_{\mu}\cdot k_{m}}{2a}\Big)+m\delta^{m,n}\delta_{\rho,\rho^{'}}\Big)\psi^{'}_{\rho^{'}}(k^{'})$$

Thus the diagonalized propagator for staggered fermions is-

$$X^{mn}_{\rho\rho'} = \sum_{\mu} \Gamma^{\mu}_{\rho,\rho'} \delta^{m,n} \sin\left(\frac{r_\mu \cdot k_m}{2a}\right) + m\delta^{m,n}\delta_{\rho,\rho'} \tag{5.13}$$

We can find out the inverse $(X^{mn}_{\rho\rho'})^{-1}$ using the relation $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$. We get-

$$\boxed{(X^{mn}_{\rho\rho'})^{-1} = \frac{-i\sum_\mu \Gamma^{\mu}_{\rho,\rho'} \delta^{m,n} \sin\left(r_\mu \cdot k_m/2a\right) + m\delta^{m,n}\delta_{\rho,\rho'}}{\sum_\mu \sin^2(r_\mu \cdot k_m/2a) + m^2}} \tag{5.14}$$

Chapter 6

Continuum Gauge theories

Gauge theories are a general way to impose a certain local symmetry in the theory. of interest i.e. the Lagrangian. We are interested in studying gauge theories because the Standard model itself is a $SU(3) \times SU(2) \times U(1)$ gauge theory. We will first look at continuum gauge theories in this this section, and then try to discretize it in the next one. The term gauge refers to any specific mathematical formalism to regulate redundant degrees of freedom in the Lagrangian of a physical system. The transformations between possible gauges, called gauge transformations, form a Lie group—referred to as the symmetry group or the gauge group of the theory. We define a gauge group as (say for $U(1)$ group)-

$$\mathcal{G} = \{g : M_{1+3} \to U(1)\} \tag{6.1}$$

At every point we have an element of the symmetry group acting on that space-time point. So under this local transformation, the terms like $\bar{\psi}\psi$ or $(\bar{\psi}\gamma_\mu\psi)^2$ are still invariant. But terms involving the derivative $\partial_\mu\psi$ or $\partial_\mu\bar{\psi}$ are no more invariant. To make it invariant we redefine the notion of derivative, but first we define the parallel transporter-

$$U(y, x) \in U(1) \qquad \forall x, y \in M_{1+3}$$

A parallel transporter is simply a recipe to compare fields between two points, and it depends on the path from x to y we are taking. Such that-

$$U(y, x)\psi(x) = \psi(y) \tag{6.2}$$

Under gauge transformation ($g \in \mathcal{G}$), the parallel transporter transforms as-

$$U(y, x) \to U'(y, x) = e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)}$$

Using the parallel transporter, we can redefine derivative ∂_μ as the covariant derivative-

$$D_\mu\psi(x) := \lim_{\epsilon\to 0} \frac{\psi(x + \epsilon e^\mu) - U(x + \epsilon e^\mu, x)\psi(x)}{\epsilon} \tag{6.3}$$

Here e^μ is a unit vector in μ direction. Now, if we Taylor expand $U(x + \epsilon e^\mu, x)$ around the point x, we find-

$$U(x + \epsilon e^\mu, x) = U(x, x) + i\epsilon e^\mu \partial_\mu U(x, x) + ... \tag{6.4}$$

We can simply label the term $\partial_\mu U(x,x)$ as $-ieA_\mu$. Using this and putting the above expression into eqn(6.3) (2^{nd} and higher order terms can be ignored as $\epsilon \to 0$) we get the expression for covariant derivative as:

$$\boxed{\mathrm{D}_\mu\psi(x) = \partial_\mu\psi(x) + ieA_\mu(x)\psi(x)} \tag{6.5}$$

Let's now see how $D_\mu(x)$ transforms under gauge transformation $g \in \mathcal{G}$. Using the definition in eqn(6.3) and eqn(6.2) we can show-

$$D_\mu(x)\psi(x) \to D'_\mu(x)\psi(x) = e^{i\alpha(x)}D_\mu(x)\psi(x)$$

One thing to remember, since we are considering $U(1)$ symmetry group the transformation is $e^{i\alpha(x)}$, in general for a symmetry group $g(x)$, it will be $D_\mu(x) \to D'_\mu(x) = g(x)D_\mu(x)$. Using this it is easy to show that we can construct non-trivial terms which are gauge invariant like-

$$\bar{\psi}\gamma^\mu D_\mu\psi, \quad \bar{\psi}(\gamma^\mu D_\mu)^2\psi \quad ..etc$$

Here $(D_\mu)^2$ meansD_μ acting on D_μ. To construct a Lagrangian with **Poincare invariance + Gauge invariance** we can make use of these terms. We can write the Lagrangian in terms of the following terms and it would work. But one thing we can recognize that A_μ itself is a vector field and it is static in all the above terms (It is called the gauge field). What if we want to introduce terms which will dynamically vary field A_μ? We have to add gauge invariant terms with ∂A_μ in it. Let's try to do that.

We can construct a term called plaquette operator using the parallel transporter as-

$$U_{box} = U(x, x+\epsilon\hat{\mu})U(x+\epsilon\hat{\mu}, x+\epsilon\hat{\mu}+\epsilon\hat{\nu})U(x+\epsilon\hat{\mu}+\epsilon\hat{\nu}, x+\epsilon\hat{\nu})U(x+\epsilon\hat{\nu}, x)$$

If we act U_{box} on $\psi(x)$, it can be thought of transporting $\psi(x)$ along a small square and returning to the same point x (it is shown in the diagram below). We can easily check that this term is invariant under gauge transformation.

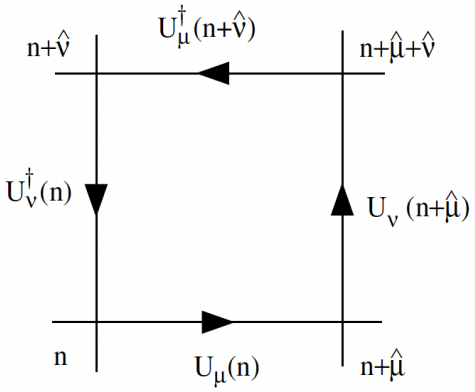


Figure 6.1: Plaquette operator

If we expand U_{box} in terms of A_μ, we can show-

$$U_{box}(x) = \exp\Big(-ie\epsilon[-A_\mu(x+\frac{\epsilon}{2}\hat{\mu}) - A_\nu(x+\frac{\epsilon}{2}\hat{\nu}+\epsilon\hat{\mu}) + A_\mu(x+\frac{\epsilon}{2}\hat{\mu}+\epsilon\hat{\nu}) + A_\nu(x+\frac{\epsilon}{2}\hat{\nu})]\Big)$$

$$\Rightarrow U_{box}(x) = 1 - i\epsilon^2 e[\partial_\mu A_\nu - \partial_\nu A_\mu] + \mathbf{O}(\epsilon^3)$$

Since $U_{box}(x)$ is gauge invariant, each coefficient of ϵ will also be gauge invariant. Thus we find a new gauge invariant term (coefficient of ϵ^2), we call it the anti-symmetric electromagnetic tensor-

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Now we have found a gauge invariant term with ∂A_μ in it which will give rise to dynamics of A_μ term. To make it Lorentz invariant, we have again Lorentz contract it $F^{\mu\nu}F_{\mu\nu}$.

Using all the information from above, and combining all the Lorentz + Gauge invariant terms we have computed, we can construct the simplest Lagrangian-

$$\boxed{\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^\mu D_\mu\psi + m\bar{\psi}\psi} \tag{6.6}$$

This is the Lagrangian for QED, it is a $U(1)$ Gauge theory with Poincare symmetry. This Lagrangian governs the dynamics of interaction between fermions and photons, it describes how light and matter interact. Now we can go ahead and try to do this construction for more general non-abelian gauge theories.

Non-abelian gauge theories (Yang Mills theory)

In the previous part we had derived a Lagrangian which is invariant under local $U(1)$ transformation. To derive Lagrangian invariant under general non-abelian group, we would just consider $SU(2)$ case. Exactly the same arguments would work for the general case.

For our theory to be invariant under $SU(2)$ we need to chose a representation. So consider $V(x) \in SU(2)$-

$$V(x) = \begin{pmatrix} v_{11}(x) & v_{12}(x) \\ v_{21}(x) & v_{22}(x) \end{pmatrix}$$

Such that $det(V) = 1$ and $VV^\dagger = V^\dagger V = 1$. So under local gauge transformation $\psi(x)$ transforms as-

$$\psi_j(x) = \sum_{k=1}^{2} v_{jk}(x)\psi_k(x)$$

Here we have introduced two independent spinor fields ψ_1, ψ_2, (these should not be confused with spinor field components) such that V acts on ψ_1, ψ_2 as described above. Form these spinor fields we can construct a larger field as-

$$\Psi(x) = \begin{pmatrix} (\psi_1(x)) \\ \\ (\psi_2(x)) \end{pmatrix}$$

$\Psi(x)$ under local gauge group transforms as $\Psi(x) \to \Psi'(x) = V(x)\Psi(x)$. From here it is easy to see that we can construct gauge invariant terms using ψ_1, ψ_2-

$$\sum_{j=1}^{2} \psi_j(x)\psi_j(x), \quad \sum_{j=1}^{2} \psi_j(x)\gamma_\mu\psi_j(x), \quad ..etc$$

But again these are terms with no derivative w.r.t. ψ_1 or ψ_2, so constructing a Lagrangian with these terms won't give dynamics to ψ. To do so we again have to construct parallel transporter. So we again define the parallel transporter $U(x, y) \in SU(2)$ such that $U(x, y)\psi_i(y) = \psi_i(x)$. From here we can define covariant derivative as (n^μ is unit 4-vector in μ direction)-

$$n^\mu D_\mu \Psi(x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\Psi(x + \epsilon n) - U(x + \epsilon n, x)\Psi(x)]$$

Again to compute this derivative we have to expand out $U(x + \epsilon n, x)$. We can do that by first recognizing that $SU(2)$ forms a Lie group under matrix multiplication, so we can write any element of the group $U(x, y) \in SU(2)$ in terms of its Lie algebra- $\mathbf{su}(2)$ by exponentiating it.

$$U = e^{iA(x)}$$

Here $A(x)$ is an element of the lie algebra $\mathbf{su}(2)$. And it satisfies the following properties $A_\dagger A = AA^\dagger = 1$ and $tr(A) = 0$ i.e. A is traceless Hermitian matrix. To proceed further, we choose a particular basis in $\mathbf{su}(2)$ namely the Pauli sigma matrices-

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices satisfy the following commutation relation- $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma^k$. So we can write the gauge field $A(x)$ in terms of the Pauli sigma matrices as-

$$A(x) = \sum_{j=1}^{3} A^j(x)\frac{\sigma^j}{2}$$

Now we can write $U(x + \epsilon n, x)$ in terms of $A(x)$-

$$U(x + \epsilon n, x) \approx 1 + ig\epsilon n^\mu \sum_j A^j_\mu(x)\frac{\sigma^j}{2} + \mathbf{O}(\epsilon^2)$$

Using the result of eqn(??) we construct the covariant derivative as-

$$\boxed{D_\mu = \partial_\mu - igA^i_\mu\sigma_i}$$

Now we can see whether $D_\mu\Psi$ transforms the right way. We can show for infinitesimal gauge transformation $V(x) = 1 + \frac{i\alpha^j\sigma^j}{2}$, the covariant derivative $D_\mu\Psi$ transforms as-

$$\begin{aligned} D_\mu\Psi(x) \to D'_\mu\Psi(x) &= (1 + \frac{i\alpha^j\sigma^j}{2})D_\mu\Psi(x) \\ &= V(x)D_\mu\Psi(x) \end{aligned}$$

Any general gauge transformation $V(x)$ can be generated by doing repeated infinitesimal transformations i.e. exponentiating $i\alpha^j\sigma^j/2$. Now it is easy to show that the term $\bar{\Psi}(x)D_\mu\Psi(x)$ is invariant under gauge transformation. We can now construct terms of Lagrangian with $\Psi(x)$ dynamics such as-

$$\bar{\Psi}(x)\gamma^\mu D_\mu\Psi(x), \quad \bar{\Psi}(x)(\gamma^\mu D_\mu)^2\Psi(x) \quad ..etc$$

Again we would like to construct terms with A_μ dynamics in it, so we have to include terms with ∂A_μ in it. To do so we notice under local gauge, the following term transforms as (using the result form eqn(??))-

$$[D_\mu, D_\nu]\Psi \to V(x)[D_\mu, D_\nu]\Psi \tag{6.7}$$

In technical terms $[D_\mu, D_\nu]$ is curvature if $SU(2)$ fibre bundle. If we now go ahead and calculate the term we can show it is in the following form-

$$[D_\mu, D_\nu] = igF^j_{\mu\nu}\frac{\sigma^j}{2} \tag{6.8}$$

Where,

$$F^j_{\mu\nu}\frac{\sigma^j}{2} = \partial_\mu A^j_\nu\frac{\sigma^j}{2} - \partial_\nu A^j_\mu\frac{\sigma^j}{2} - ig[A^j_\mu\frac{\sigma^j}{2}, A^k_\nu\frac{\sigma^k}{2}] \tag{6.9}$$

We can right away see that the difference here with the abelian case is the last term with the commutator which is quadratic in A_μ. When we construct lorentz invariant term, we have to square $F_{\mu\nu}$, this will introduce cubic and quartic terms A_μ into the lagrangian which would give rise to self interactions to the bosonic fields. This is not the case for abelian gauge theories.

From eqn(6.7) we see $F^j_{\mu\nu}\frac{\sigma^j}{2}$ transforms under local gauge transformation as-

$$F^j_{\mu\nu}\frac{\sigma^j}{2} \to V(x)(F^j_{\mu\nu}\frac{\sigma^j}{2})V^\dagger(x)$$

Since $V(x) \in U(N)$, taking the trace would give us a gauge invariant term which would be-

$$tr\big((F^j_{\mu\nu}\frac{\sigma^j}{2})(F^{\mu\nu,k}\frac{\sigma^k}{2})\big) \tag{6.10}$$

We now have everything to construct a Gauge invariant and Poincare invariant Lagrangian with ψ and A_μ dynamics-

$$\boxed{\mathcal{L} = \bar{\psi}(i\gamma^\mu D^\mu - m)\psi - \frac{1}{4}(F^j_{\mu\nu})^2} \tag{6.11}$$

6.0.1 Parallel transporter

In the previous part we had defined parallel transporter object to redefine the notion of derivative such that it transforms the right way under gauge transformation. Parallel transporter is an object which transports a tangent vector from one point to another point in the manifold. The parallel transporter is therefore a map $\Delta_C : T_pM \to T_qM$ where C is a curve from p to q.

$U(y,x)$ has the property $U(y,x)\psi(x) = \psi(y)$. From eqn(6.4) we have-

$$U(x+\epsilon n, x) = 1 - ie\epsilon n_\mu A^\mu(x) + \mathbf{O}(\epsilon^2) \tag{6.12}$$

Where ϵ is tending towards zero. Now say we want to go from point x to point y through the path C, we can divide the path into n points. And first go from $x = x_0$ to x_1, x_1 to x_2 like so on and finally to $y = x_{n+1}$.

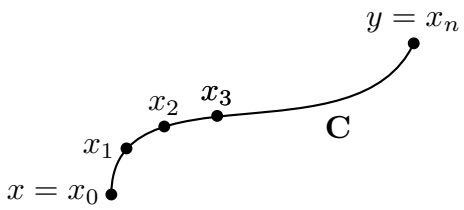


Figure 6.2: Breaking path **C** from x to y into n-components

Breaking path **C** into n small pieces, we can write;

$$U(y,x) = \lim_{n\to\infty} U(x_n, x_{n-1})..U(x_2,x_1)U(x_1,x_0) \tag{6.13}$$

$$= \lim_{n\to\infty} \prod_{k=0}^{n} U(x_{k+1}, x_k) = \lim_{n\to\infty} \prod_{k=0}^{n} (1 - ie\epsilon n_\mu A_\mu(x_k))$$

$$= \lim_{n\to\infty} \left(1 - ie\epsilon n^\mu \sum_{k=0}^{n} A_\mu(x_k) + (-ie\epsilon n^\mu)^2 \sum_{k=0}^{n} \sum_{k'>k} A_\mu(x_k) A_\mu(x_{k'}) + ..\right)$$

$$= \lim_{n\to\infty} \left(1 - ie\epsilon n^\mu \sum_{k=0}^{n} A_\mu(x_k) + \frac{(-ie\epsilon n^\mu)^2}{2} \sum_{k,k'=0}^{n} A_\mu(x_k) A_\mu(x_{k'}) + ..\right) \tag{6.14}$$

As limit $n \to 0$ we can take $\sum \to \int$ and $\epsilon n^\mu \to ds$. Doing this substitution we can write eqn(6.14) as-

$$U(y,x) = 1 + (-ie)\int A \cdot ds + \frac{(-ie)^2}{2}(\int A \cdot ds)^2 + .. \tag{6.15}$$

This can be written in compact form as

$$U(y,x) = \exp\left(-ie\int_{\mathbf{C}} A \cdot ds\right) \tag{6.16}$$

From eqn(6.16) it is clear is highly dependent on the curve of path taken, as the line integral along the path **C**. The parallel transporter transports the vector in a parallel fashion along the manifold. This can be visually seen from the figure shown below.

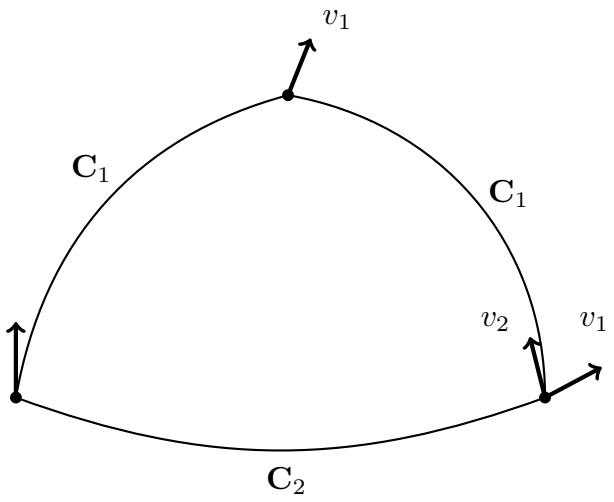


Figure 6.3: v_1 and v_2 parallel transported along $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively

Chapter 7

Lattice Gauge Theories

7.1 Lagrangian Lattice Gauge Theory

In the previous section we have developed the formalism for imposing local gauge symmetry in our continuum theory and from chapter-2 and 3 we already know how to place fermions and bosons on the lattice. In this section we will try to develop formalism to impose the gauge symmetry in a lattice.

A general action can be divided into two parts **Fermionic part+Gauge part**:

$$S = S_F + S_G$$

We have already derived some of the different fermionic actions (S_F) on the lattice in Chapter 2 and Chapter 3, given in eqn(4.1), eqn(5.1) and eqn(5.9). Imposing gauge symmetry will firstly change the definition of derivative into all these equations where all ∂_μ will change to $D_\mu := \partial_\mu + ieA_\mu$. And secondly introduce a gauge action term S_G which will give the dynamics to the gauge fields.

Denoting $\psi(n)$ as the value of field ψ at lattice point $n \in \Lambda$. Under gauge transformation G, $\psi(n)$ will transform as:-

$$\begin{aligned}\psi(n) \to \psi'(n) = G(n)\psi(n)\\ \bar{\psi}(n) \to \bar{\psi}'(n) = \bar{\psi}(n)G^\dagger(n)\end{aligned}$$

Since now space-time points are not continuous anymore, parallel transporter can only take a field form one lattice point to its adjacent point, so we can construct any parallel transporter in terms of unit transporter $U_{n+\hat{\mu},n}$ and $U^\dagger_{n+\hat{\mu},n}$ such that-

$$\begin{aligned}U_{\hat{\mu}}(n)\psi(n) = \psi(n+\hat{\mu})\\ U_{\hat{\mu}}(n)^\dagger\psi(n+\mu) = \psi(n)\end{aligned} \tag{7.1}$$

Diagrammatically it looks like this-
From here it is clear-

$$U^\dagger_{\hat{\mu}}(n) = U_{-\hat{\mu}}(n+\hat{\mu})$$

We redefine fermionic action on the lattice using this such that it is gauge invariant (We use naive fermionic discretization action, the same will apply for Wilson and Staggered fermions

Figure 7.1: Action of parallel transporter on a lattice

too)-

$$S_F = a^4 \sum_{n\in\Lambda} \bar{\psi}(n)\Big(\sum_{\mu} \frac{U_{\hat{\mu}}(n)\psi(n+\hat{\mu}) - U_{-\hat{\mu}}(n)\psi(n-\hat{\mu})}{2a} + m\psi(n)\Big) \tag{7.2}$$

To construct the the action term with gauge field dynamics we first consider the following properties of the parallel transporter. Let P be a path of k links on the lattice connecting the points $n = n_0$ and $m = n_k$. We define the ordered product-

$$P[U] = U_{\hat{\mu}_0}(n_0)U_{\hat{\mu}_1}(n_1)..U_{\hat{\mu}_k}(n_k) = \prod_{(n,\mu)\in P} U_{\hat{\mu}}(n) \tag{7.3}$$

U basically connects the path between site $n = n_0$ and $m = n_k$ along the path. This is just the lattice version of the path we had constructed in eqn(6.16).

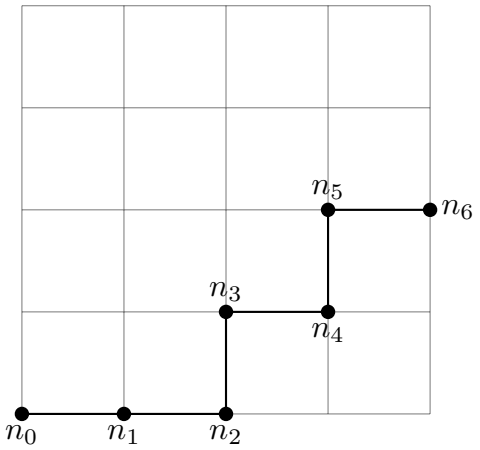


Figure 7.2: A path taken from site $n = n_0$ to $m = n_6$

From the transformation properties of U given it is clear that under gauge transformation $P[U]$ transforms as-

$$P[U] \to P'[U] = V(n)P[U]V^{\dagger}(m)$$

Where $V(n)$ is the gauge transformation at site n. Like for the single link term, from such a product of link variables P[U] a gauge-invariant quantity can be constructed by taking a closed path with $n = m$ and taking the trace of it.

$$tr[P[U]] \to tr[P'[U]] = tr[V(n)P[U]V^{\dagger}(n)] = tr[P[U]] \tag{7.4}$$

Now we have established that we can construct gauge entities taking trace of closed loops of parallel transporters, we can use it to construct action involving gauge fields. The simplest

loop we can construct in a 2D lattice square lattice is a square, we call it a plaquette operator, we define it as-

$$U_{\mu\nu}(n) := U_{\hat{\mu}}(n)U_{\hat{\nu}}(n+\hat{\mu})U_{-\hat{\mu}}(n+\hat{\mu}+\hat{\nu})U_{-\hat{\nu}}(n+\hat{\nu})$$

We define Wilson gauge action as-

$$S_G = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} Re[\mathbf{1} - U_{\mu\nu}] \tag{7.5}$$

Claim: As $\lim a \to 0$ Wilson gauge action $S_G = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} Re[\mathbf{1} - U_{\mu\nu}]$ converges to continuum gauge action $F_{\mu\nu}F^{\mu\nu}$.

Proof- We know the expression of $F_{\mu\nu}$ in terms of the elements of lie algebra A_μ, A_ν. So we would like to write the lattice version of parallel transporter in terms of them and then take the continuous limit.

We have derived the expression for parallel transporter in eqn(6.16). Using this unit parallel transporter at site n along μ direction can be written as-

$$U_\mu(n) = \exp\left(ie\int_n^{n+\mu} A_\mu(x)dx_\mu\right)$$

We are assuming lattice spacing is small, so we can take $A_\mu(x)$ to be (almost) constant, using this we can reduce the above expression into-

$$U_\mu(n) = \exp(iaA_\mu(n)) \tag{7.6}$$

We have the plaquette operator-

$$U_{\mu\nu}(n) := U_{\hat{\mu}}(n)U_{\hat{\nu}}(n+\hat{\mu})U_{-\hat{\mu}}(n+\hat{\mu}+\hat{\nu})U_{-\hat{\nu}}(n+\hat{\nu})$$

Using the Baker–Campbell–Hausdorff formula, we can write $U_{\mu\nu}$ as-

$$\begin{aligned}
U_{\mu\nu}(n) &= \exp\left(iaA_\mu(n) + iaA_\nu(n+\mu) - \frac{a^2}{2}[A_\mu(n), A\nu(n+\mu)]\right) \\
&\times \exp\left(-iaA_\mu(n+\nu) - iaA_\nu(n) - \frac{a^2}{2}[A_\mu(n+\nu), A\nu(n)]\right)
\end{aligned}$$

$$\begin{aligned}
&= exp(iaA_\mu(n) + iaA_\nu(n+\mu) - \frac{a^2}{2}[A_\mu(n), A\nu(n+\mu)] \\
&\quad -iaA_\mu(n+\nu) - iaA_\nu(n) - \frac{a^2}{2}[A_\mu(n+\nu), A\nu(n)] \\
&\quad +\frac{a^2}{2}[A_\nu(n+\mu), A_\mu(n+\nu)] + \frac{a^2}{2}[A_\mu(n), A_\nu(n)] \\
&+\frac{a^2}{2}[A_\mu(n), A_\mu(n+\nu)] + \frac{a^2}{2}[A_\nu(n+\mu), A_\nu(n)] + \mathbf{O}(a^3))
\end{aligned} \tag{7.7}$$

Terms like $A_\mu(n+\nu)$ can be Taylor expanded in the form $A_\mu(n+\nu) = A_\mu(n) + a\partial_\nu A\mu(n)$. Substituting this simplifies the above equation into-

$$U_{\mu\nu}(n) = \exp\left[ia^2(\partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) + i[A_\mu(n), A_\nu(n)]) + \mathbf{O}(a^3)\right] \tag{7.8}$$

The term inside the bracket looks familiar, it is $F_{\mu\nu}$ itself. Thus Taylor expanding the exponential gives us-

$$U_{\mu\nu}(n) = \mathbf{1} + ia^2 F_{\mu\nu} + \mathbf{O}(a^3)$$

Thus we can write,

$$S_G = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} Re[\mathbf{1} - U_{\mu\nu}] = \frac{a^4}{2g^2}\sum_{n\in\Lambda}\sum_{\nu,\mu} Tr[F^2_{\mu\nu}(n)] + \mathbf{O}(a^2) \tag{7.9}$$

Taking the limit $a \to 0$ we get the continuous gauge action-

$$S = \frac{1}{2g^2}\int d^4x \sum_{\mu,\nu} Tr[F^2_{\mu\nu}] \tag{7.10}$$

■

We have established that the lattice action eqn(7.5) is indeed the correct action, as it converges to the right continuum action. Thus we write any action with **Dirac fermions** and **Local Gauge symmetry** on a lattice as-

$$\boxed{S = a^4\sum_{n\in\Lambda}\bar{\psi}(n)\left(\sum_{\mu}\frac{U_{\hat{\mu}}(n)\psi(n+\hat{\mu}) - U_{-\hat{\mu}}(n)\psi(n-\hat{\mu})}{2a} + m\psi(n)\right) + \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} Re[\mathbf{1} - U_{\mu\nu}]} \tag{7.11}$$

7.2 Hamiltonian Lattice Gauge Theory

In the last section we developed the Lagrangian formalism for Lattice Gauge theories where we had derived the action for local gauge theory with fermions on it. In the Lagrangian formalism, the most important entity is the action (and/or Lagrangian). Once we fix the action we can study everything about the theory from there, both classical and quantum. If the system is classical, we can study the time evolution using the Euler-Lagrange equation. In the case of quantum system we have every possible path contributing to the probability of transition, with each path contributing a phase e^{iS} to the probability of transition. From here we can calculate entities like partition function $Z = \int_{x_0}^{x} \mathcal{D}x e^{iS[\mathbf{x}]}$ and expectation value of an observable using $< O >= \frac{1}{Z}\int \mathcal{D}x O(\mathbf{x}) e^{iS[\mathbf{x}]}$.

In Hamiltonian formalism the treatment is done a bit differently(0,). To obtain time evolution of any observable in Hamiltonian formalism, we take the commutator with the Hamiltonian $i\partial_t \hat{O} = [\hat{O}, \hat{H}]$ and the expectation value can be computed using $< O >= \langle\Psi|\,\hat{O}\,|\Psi\rangle$. In this part we will try to develop Hamiltonian formalism for lattice gauge theories.

If we have the Lagrangian of the theory from classical mechanics we know wwe can find the Hamiltonian doing Legendre transform $H = \sum_i p_i\dot{q}_i - L$ where $p_i = \frac{\partial L}{\partial \dot{q}}$. Let us construct the Hamiltonian formulation of the pure gauge part of the action S_G. If we want to take time derivative of the canonical variables, we need to take time to be continuous. To do so We will set the fields to be static in the time direction $\hat{0}$ i.e.

$$U_{\hat{0}}(n) \to U'_{\hat{0}}(n) = V(n)U_{\hat{0}}(n)V^{\dagger}(n+\hat{0}) = \mathbf{1} \tag{7.12}$$

Justification to why we take fields to be static. If we take time to be continuous, the object $U_0(n)$ which is the parallel transporter in the 0 (time) direction would make no sense, since in time direction we have no lattice. So we don't

Since we are taking time direction to be continuous, the sum over lattice will become-

$$\sum_{n\in\Lambda} a^4 = \int dx_0 \sum_n a^3$$

Here n denotes all space lattice points (excluding time). Now the action term can be divided into two parts, one with the temporal index included in it and one without the temporal index. The ones without the temporal index will remain same, but the one with temporal index will become-

$$\frac{2}{g^2}\sum_n \sum_{\mu=1}^{3} Re[tr[\mathbf{1} - U_{\hat{\mu}}(n)U_{\hat{0}}(n+\mu)U^{\dagger}_{\hat{\mu}}(n+0)U^{\dagger}_{\hat{0}}(n)]]$$

$$= \frac{2}{g^2}\sum_n Re[tr[\mathbf{1} - U_{\hat{\mu}}(n)U^{\dagger}_{\hat{\mu}}(n+0)]] \tag{7.13}$$

Using this we can write-

$$-2Re[Tr[U_{\hat{\mu}}(n)U^{\dagger}_{\hat{\mu}}(n+0)]] = Tr[(U_{\hat{\mu}}(n+0) - U_{\hat{\mu}}(n))(U^{\dagger}_{\hat{\mu}}(n+0) - U^{\dagger}_{\hat{\mu}}(n))] + Const.$$

One can simply expand the term in the right side, and see we get the equation on the left hand side. And the constant we see is **2**. We can take the time continuum limit by rewriting this term as a time-derivative (multiplying and dividing by a^2)-

$$\frac{a^2}{g^2}\sum_n Tr[(\frac{U_{\hat{\mu}}(n+0) - U_{\hat{\mu}}(n)}{a})(\frac{U^{\dagger}_{\hat{\mu}}(n+0) - U^{\dagger}_{\hat{\mu}}(n)}{a})] \tag{7.14}$$

As $\lim a \to 0$ in time direction, these terms becomes time derivatives, being reduced to-

$$\frac{a^2}{g^2}\sum_n Tr[\partial_t U_\mu(n)\partial_t U^{\dagger}_{\mu}(n)] \tag{7.15}$$

Thus the action becomes-

$$S = \int dt\big(\frac{a^2}{g^2}\sum_n Tr[\partial_t U_\mu(n)\partial_t U^{\dagger}_{\mu}(n)] - \frac{2}{g^2}\sum_n Re[\mathbf{1} - U_{\mu\nu}]\big) \tag{7.16}$$

From here it is clear that the Lagrangian is-

$$\boxed{L = \frac{a^2}{g^2}\sum_n Tr[\partial_t U_\mu(n)\partial_t U^{\dagger}_{\mu}(n)] - \frac{2}{g^2}\sum_n Re[\mathbf{1} - U_{\mu\nu}]} \tag{7.17}$$

From here using the formula $H = \sum_i p_i\dot{q}_i - L$ we can easily show-

$$H = \frac{a}{g^2}\sum_n Tr[\dot{U}_\mu(n)\dot{U}^{\dagger}_{\mu}(n)] - \frac{2}{ag^2}\sum_{\square} Re[Tr[U_{\square} + U^{\dagger}_{\square}]]$$

Here we denoted $U_{\mu\nu} = U_{\square}$. Notice if we are considering $U(1)$ gauge group, then the group manifold would be S^1 and would be generated by only generator, in this case generator of any gauge transformation in link is $E_{n,\mu}$ (in the link n, μ). We define the following entity-

$$E_{n,\mu} = \frac{\partial L}{\partial \dot{U}_{n,\mu}}(iU_{n,\mu}) + \frac{\partial L}{\partial \dot{U}^{\dagger}_{n,\mu}}(-iU_{n,\mu})^{\dagger} \tag{7.18}$$

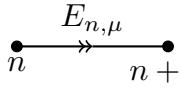


Figure 7.3: $E_{n,\mu}$ is the generator of gauge transformation at the link $n, n+\mu$

Since $U(1)$ has one generator only, we can write E^a just as E.

Now using the identities-

$$U^{\dagger}U = 1 \implies \dot{U}^{\dagger}U + U^{\dagger}\dot{U} = 0$$

And (from U(N) algebra),

$$\tau^a_{ij}\tau^a_{kl} = 2\delta_{ij}\delta_{jk} - \delta_{ij}\delta_{kl}$$

We can show-

$$E^2_{n,\mu} = \dot{U}_{n,\mu}\dot{U}^{\dagger}_{n,\mu} \tag{7.19}$$

Thus we eqn(7.18) becomes-

$$\boxed{H = \frac{a}{g^2}\sum_{n,\mu} E_{n,\mu}E_{n,\mu} - \frac{2}{ag^2}\sum_{\square}(U_{\square} + U^{\dagger}_{\square})} \tag{7.20}$$

We have obtained the Hamiltonian for the pure gauge part, to treat it quantum mechanically we have to raise the variables to operators. Here we are considering the simplest case, only $U(1)$ gauge theory (without any fermions). Once we have quantized it, we have to define the commutation relation of the operators.

Gauge transformations and Gauss law

The generators of gauge transformation exists on the link which are $E_{n,\mu}$ given in fig(7.3). And since it generates the the gauge transformation (Which is in this case $U(1)$-Gauge Theory), a general gauge transformation on the link can be written by-

$$\mathcal{G}_{n,\mu} = \exp(i\alpha_{n,\mu}E_{n,\mu})$$

Where $\alpha_{n,\mu}$ is the parameter that we can set for the gauge transformation. We can take values of $\alpha_{n,\mu}$ from S^1 which is given in the fig(7.4). This is because S^1 is diffeomorphic to $U(1)$.

Now since we have the generator of $U(1)$ which in this case is $E_{n,\mu}$, we can find its eigenstates and eigenvalues. Using E, if we define the following object

$$G_n = \sum_{k=1}^{d}(E_{n-k,k} - E_{n,k}) \tag{7.21}$$

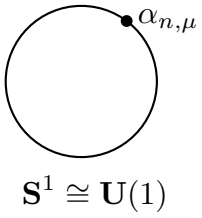


Figure 7.4: We can choose parameter $\alpha_{n,\mu}$ from $\mathbf{S}^1$

Calling it the Gauss operator. Notice that the Gauss operator resides on the lattice sites rather than the links. And from here we can construct Gauss Operator of the whole Hilbert space of the lattice. It would be in a Fock space formed by the tensor product of the individual Hilbert spaces on the lattice points $\mathcal{H}_\Lambda = \bigotimes_{n\in\Lambda} \mathcal{H}_n$. This is simply written as-

$$G_\Lambda = \bigotimes_{n\in\Lambda} G_n$$

In the similar way we can construct Fock space formed by joining Hilbert spaces of all the links, which would be $\mathcal{H} = \bigotimes_{<>} \mathcal{H}_{<>}$ (where $\mathcal{H}_{<>}$ represents the Hilbert space on the link '<>'). On that Hilbert space we can also define operators (like): $E = \bigotimes_{(n,\mu)} E_{(n,\mu)}$. We can create states which are eigenvectors of the operators like E or G_Λ and study them.

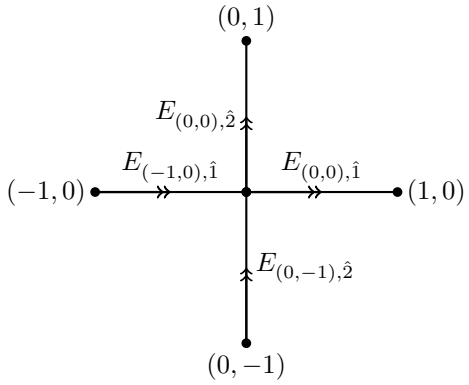


Figure 7.5: Pictorial representation of Gauss operator $G_{(0,0)}$ in 2D lattice

We have constructed G Gauss operator because it has the special property on commuting with the Hamiltonian i.e. $[G_\Lambda, H] = 0$ and thus we can have simultaneous eigenstates of G_Λ and H. We can study the eigenstates of the Hamiltonian in question in terms of the eigenstates of Gauss Operator G_Λ.

Let's try to study the eigenstates of G_Λ. Suppose we take a state $|\psi\rangle$ which gives eigenvalue 0 of G_Λ at point n which is $G_n |\psi\rangle = 0$. We can constuct this in terms of eigenstates of $E_{n,\mu}$'s. Let's say we have the state $|E_{n,\mu}(e_{n,\mu})\rangle$ which satisfies $E_{n,\mu} |E_{n,\mu}(e_{n,\mu})\rangle = e_{n,\mu} |E_{n,\mu}(e_{n,\mu})\rangle$. Using eqn(7.21) is easy to show that the state $|G_n\rangle = \bigotimes_\mu |E_{n,\mu}\rangle$ is an eigenstate of G_n which gives eigenvalue-

$$g_n = \sum_{k=1}^{d} (e_{n-k,k} - e_{n,k}) \tag{7.22}$$

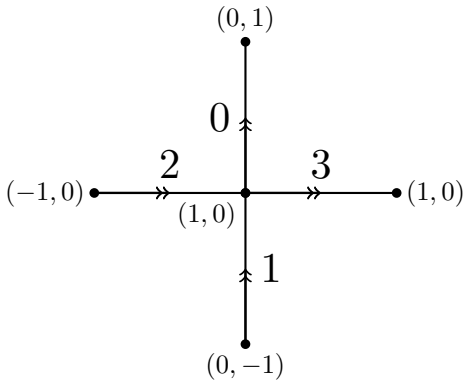


Figure 7.6: g_n in this case is equal to 0

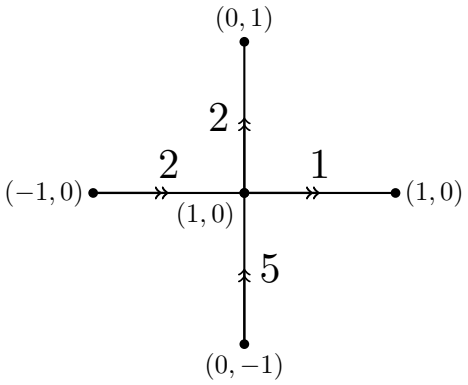


Figure 7.7: $g_n = 4$

This just says that the difference of the sum of eigenvalues of incoming links and outgoing links would be g_n. To see diagrammatically what it means, we consider see the figures. In the first one the sum of incoming E eigenvalues is 3 and outgoing eigenvalue is also 3 thus $g_n = 0$ which is the difference of these two. In the second figure sum of incoming eigenvalues is 7 and outgoing is 3.

Since the G_n commutes with the Hamiltonian and since the Hamiltonian is generator of time translation, we will have the eigenstates of G_n as invariant. Say if we choose a state with $g_n = 0$, it will remain at 0 for all times. This imposes a constraint condition on the choices of $|E_{n,\mu}\rangle$ namely in this case ($g_n = 0$), the sum of incoming eigenvalues into site n has to be equal to sum of outgoing eigenvalues. **This is called the Gauss Law** (0,).

$g_n = 0$ was an arbitrary choice, we could have chosen any other eigenvalue state with equal validity ($g_n = 1, 2..$ etc). But once we make a choice on g_n's we have to stick to it. This imposes a restriction on the the possible choice of states that we can consider. This is equivalent to doing a gauge fixing in the Lagrangian case. Choosing value of g fixes the space of possible states that are physical ($\mathcal{H}_{phys} \subseteq \mathcal{H}$).

Deriving important commutation relations

In order for the Hamiltonian to be gauge invariant, they should have the same transformation properties as their classical counterparts, i.e. they should transform as

$$U'_{n,\mu} \to \exp(-i\alpha_n E_{n,\mu}) U_{n,\mu} \exp(i\alpha_n E_{n,\mu}) = \exp(i\alpha_n) U_{n,\mu} \tag{7.23}$$

$$U'_{n,\mu} \to \exp(-i\alpha_{n+\mu} E_{n,\mu}) U_{n,\mu} \exp(i\alpha_{n+\mu} E_{n,\mu}) = \exp(i\alpha_{n+\mu}) U_{n,\mu} \tag{7.24}$$

From here we can derive the commutation relation between $E_{n,\mu}$ and $U_{n,\mu}$ from eqn(7.24) and eqn(7.23) by expanding the exponential upto order $\mathbf{O}(\alpha)$ -

$$[E_{n,\mu}, U_{n,\mu}] = U_{n,\mu} \tag{7.25}$$

Taking the hermitian conjugate of eqn(7.25) gives-

$$[E_{n,\mu}, U^\dagger_{n,\mu}] = -U^\dagger_{n,\mu} \tag{7.26}$$

Operators of separate links commute with each other.

Chapter 8

Intro To Quantum Link Models

In the last chapter we had derived formalism to treat Gauge Theories on a lattice. And saw how the treatment is done differently, for both Lagrangian and Hamiltonian framework. In the Hamiltonian formalism we treated the observables as field operators. To study the dynamics and expectation values of the observables we need to know the commutation relations that the observables. For the $U(1)$-gauge theory case we saw that the Hamiltonian can be written in terms of the electric field operator $E_{n,\mu}$ and the parallel transporter $U_{n,\mu}$ and we imposed commutation relation between them in eqn(7.25) and eqn(7.26)-

$$[E_{n,\mu}, U_{n,\mu}] = U_{n,\mu}$$

$$[E_{n,\mu}, U^{\dagger}_{n,\mu}] = -U^{\dagger}_{n,\mu}$$

This looks like a familiar commutation relation. If we take angular momentum operator in x, y and z direction to be L_x, L_y, L_z respectively. We remember the raising and lowering operators are defined as $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, satisfy the commutation relation-

$$[L^z, L^+] = L^+$$

$$[L^z, L^-] = -L^-$$

We can recognize that both set of commutators satisfy the same commutation relation. So we can construct a Hamiltonian by replacing by replacing L^z and L^+ in place of E and U respectively. This would make the new Hamiltonian-

$$H = \frac{a}{g^2} \sum_{n,\mu} L^z_{n,\mu} L^z_{n,\mu} - \frac{2}{ag^2} \sum_{\square} (L^+_{\square} + L^-_{\square}) \tag{8.1}$$

Here $L_{\square}$ is defined in similar way as $U_{\square}$. Now we have this Hamiltonian which consists of only angular momentum operator and the ladder operators and we know how to deal with them from quantum mechanics.

U(1) QLM for spin-1/2 states

Let us consider the case with only spin-$\frac{1}{2}$ states. Then we can only focus on the Hilbert space spanned by $\left|\frac{1}{2}\right\rangle$ and $\left|-\frac{1}{2}\right\rangle$ where $L_z \left|\pm\frac{1}{2}\right\rangle = \pm\frac{1}{2} \left|\pm\frac{1}{2}\right\rangle$.

Lets consider a simple lattice in 2-dimension with only four sites as given in the figure and try to find it's eigenstates and eigenvalues. The Hilbert space formed would be a $2 \times 2 \times 2 \times 2 = 16$ dimensional Hilbert space. We can take the spanning basis set $\left|\pm\frac{1}{2}\right\rangle_1 \otimes \left|\pm\frac{1}{2}\right\rangle_2 \otimes \left|\pm\frac{1}{2}\right\rangle_3 \otimes \left|\pm\frac{1}{2}\right\rangle_4$, and so on.

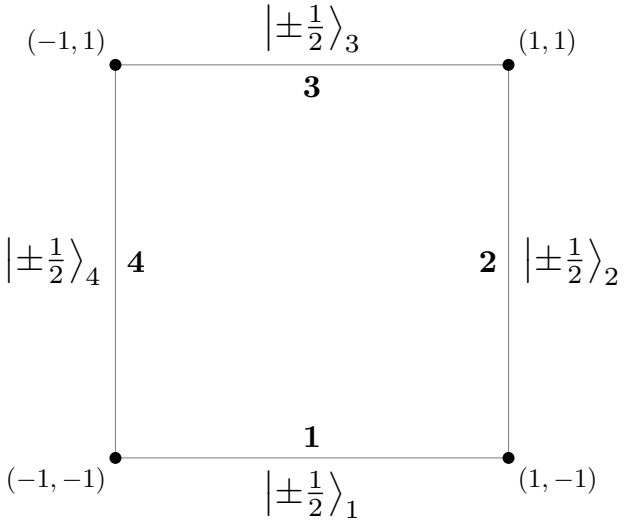


Figure 8.1: The basis set formed to span the Hilbert space

The links are labelled by $1, 2, 3$ and 4 correspondingly as given in fig(8.1). Using this label the Hamiltonian can be re-written as-

$$H = \frac{a}{g^2}\sum_i L_i^z L_i^z - \frac{2}{ag^2}(L_1^+ \otimes L_2^+ \otimes L_3^- \otimes L_4^- + h.c.)$$

Here L_1^z is defined as $L_1^z \otimes \mathbf{0} \otimes \mathbf{0} \otimes \mathbf{0}$. Using these definitions it is easy to show $L_i^z \left|\pm\frac{1}{2}\right\rangle_j = \pm\frac{1}{2}\delta_{ij}$. Therefore $\left|\pm\frac{1}{2}\right\rangle_j$ acting on the first term of the Hamiltonian will be-

$$\sum_i (L_i^z)^2 \left|\pm\frac{1}{2}\right\rangle_j = \sum_i \frac{1}{4}\delta_{ij} \left|\pm\frac{1}{2}\right\rangle_j = \frac{1}{4}\left|\pm\frac{1}{2}\right\rangle_j \tag{8.2}$$

This will hold true for all basis state and thus the term $\sum_i (L_i^z)^2$ is just Identity $\mathbf{I}/4$. All states are eigenstates of Identity operator, so we don't need to worry about this term while finding eigenstates.

Now wee can just focus on the term $L_\square$ and $L_\square^\dagger$. The corresponding are the eigenstates of $L = L_\square + L_\square^\dagger$.

Now the argument is since $U_\square + U_\square^\dagger$ is hermitian, it should be diagonalizable and thus we should have 16 eigenstates for the term. There are one set of states with eigenvalues 1 and one set with eigenvalues -1. For every link we have $L_i^+ + L_i^-$ operator which is a linear transformation in Hilberrt space $\mathcal{H} = \mathbb{C}^2$. $L_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus

$L_i^+ + L_i^- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenstates of this transformation are-

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8.3}$$

Let's call the state with eigenvalue 1 as $|\mathbf{1}\rangle$ and the state with eigenvalue -1 as $|\mathbf{\text{-}1}\rangle$ Using this we can form eigenstates of the whole Hamiltonian which would be all the combination of $|\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle$ or $|\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle$. From here it is clear we would have 16 eigenstates as we can take 16 possible combination $|\mathbf{1}\rangle$ or $|\mathbf{\text{-}1}\rangle$. The eigenvalue would be the peoduct of the individual eigenvalues.

But all of them doesn't satisfy Gauss law. For a state to be physical it has to satisfy the Gauss law. Imposing the condition $G_n |\psi\rangle = 0$; Gauss law at site $(0, 0)$ reads-

$$G_{(0,0)} = L_1^z \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes L_4^z$$

Where $\mathbf{I}$ is identity transformation in $\mathbb{C}^2$. The eigenstates of the Hamiltonian forms two subspace, one spanned by basis with eigenvalues 1 and one spanned by basis with eigenvalues -1, let's call them S_1 and S_{-1} respectively. Any linwar combination of of the basis in S_1 or S_{-1} is also a eigenstate of the Hamiltonian. We also notice $L_i^z |\mathbf{1}\rangle = |\mathbf{\text{-}1}\rangle$ and $L_i^z |\mathbf{\text{-}1}\rangle = |\mathbf{1}\rangle$.

Using all the information above we can show there are 8 total states satisfying Gauss law with $G_n |\psi\rangle = 0$, 4 of them reside on the subspace S_1 and 4 of them reside on S_{-1}.

Physical states residing on S_1:

$$|A_1\rangle = |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle$$

$$|A_2\rangle = |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle$$

$$|A_3\rangle = |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle$$

$$|A_4\rangle = |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle$$

Physical states residing on S_{-1}:

$$|B_1\rangle = |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle$$

$$|B_2\rangle = |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle + |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle$$

$$|B_3\rangle = |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle + |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle$$

$$|B_4\rangle = |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{\text{-}1}\rangle \otimes |\mathbf{1}\rangle + |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{\text{-}1}\rangle$$

Here $H |A_i\rangle = |A_i\rangle$ and $H |B_i\rangle = - |B_i\rangle$. And $G_n |A_i\rangle = G_n |B_i\rangle = 0 \ \forall n \in \Lambda$. Given below are pictorial representation of some of the states we have found.

We have done this analysis for spin-$\frac{1}{2}$ representation of angular momentum operator, we can also do this for spin-1 representation where each link represents a Hilbert space with

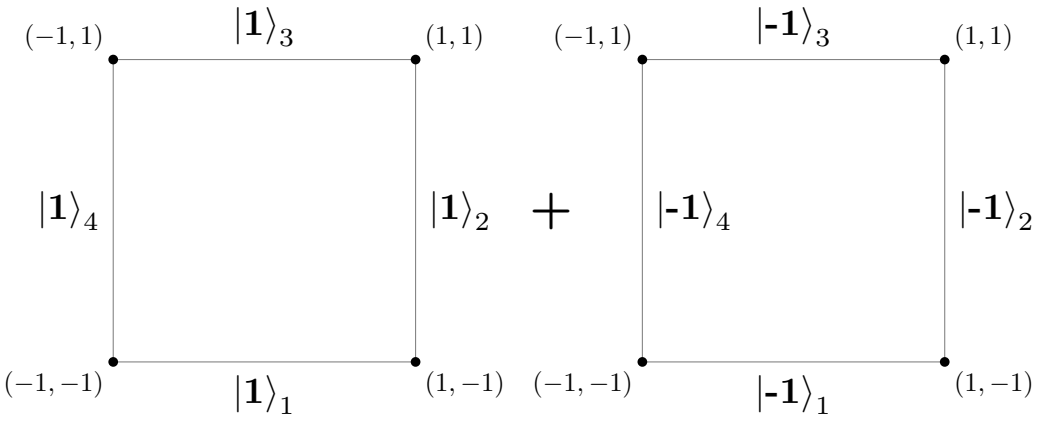


Figure 8.2: This physical state represents a state in S_1

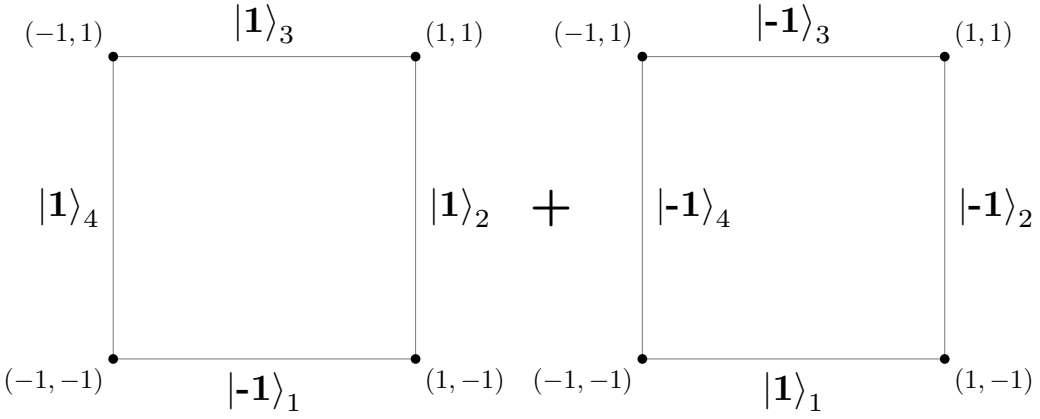


Figure 8.3: This physical state represents a state in S_{-1}

dimension 3 spanned bby $|-1\rangle$, $|0\rangle$ and $|1\rangle$.

The advantage of using this way of dealing with Gauge Theories like this is that we can create real spin-$\frac{1}{2}$ and simulate them in quantum systems.

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