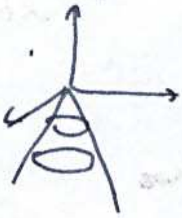
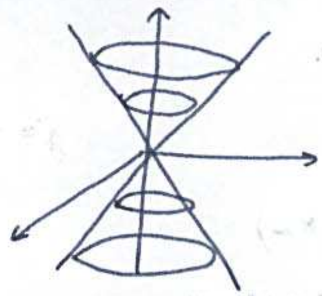
$z=-\sqrt{x^2+y^2}$



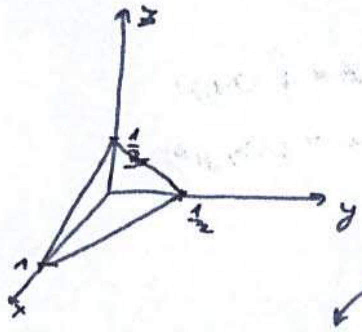
$z=3-\sqrt{x^2+y^2}$

$z^2=x^2+y^2 \to z=\pm\sqrt{x^2+y^2}$



$\boxed{ax+by+cz=d}$ sík

Pl: $x+2y+3z=1$



$\boxed{x^2+y^2+z^2=R^2}$ gömb

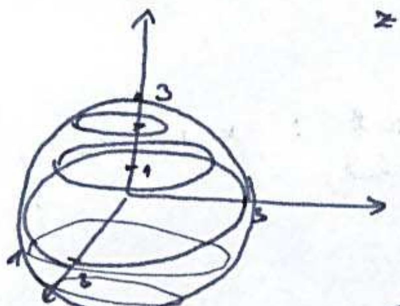
Pl: $x^2+y^2+z^2=9$

$z=0$ $x^2+y^2=9$

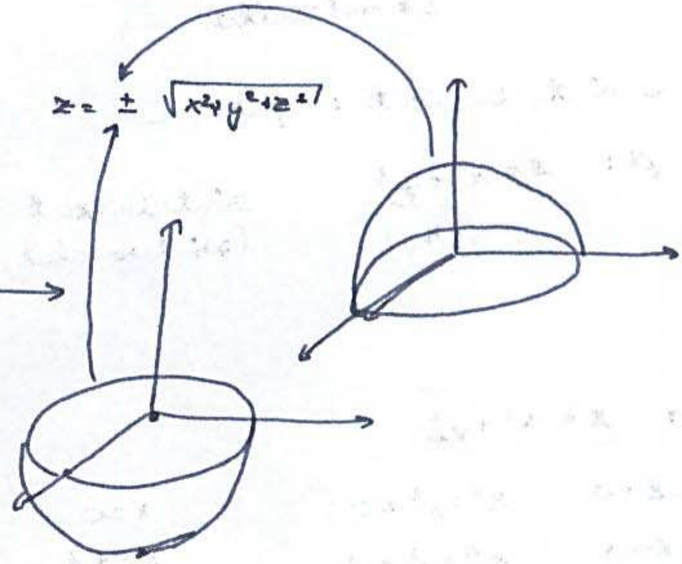
$z=\pm 1$ $x^2+y^2=8$

$z=\pm 2$ $x^2+y^2=5$

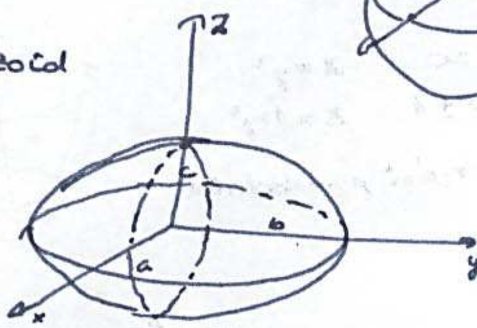
$z=\pm 3$ $x^2+y^2=0$



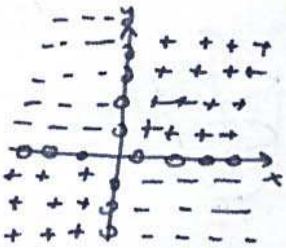
$z=\pm\sqrt{x^2+y^2+z^2}$

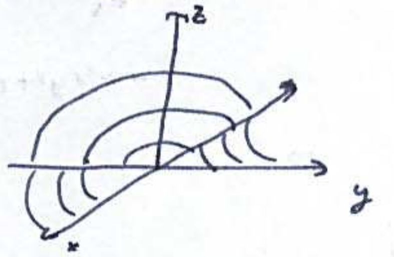


$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$ ellipszoid



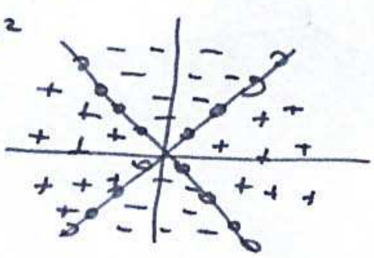
$z=xy$





nyeregfelület

$z=x^2-y^2$



ugyanaz, elforgatva

Háromvált. fv: $u=x^2+y^2+z^2$ szintfelület: $x^2+y^2+z^2=0$

csak szintvonalakat tudok ábrázolni, 4D tér

$(x_n,y_n,z_n)=\left(\frac{n+1}{n-1},\left(1+\frac{2}{n}\right)^n,\frac{n^2-1}{n^3+3}\right)\to(1,e^2,0)$

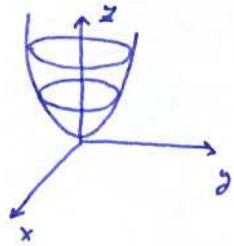
Egy pontsorozat akkor és csak akkor konv, ha egyenként konv.

Többváltozós függvények

$f(x,y)$

pl: $f(x,y) = x^2+y^2 = z$

3 D-s térben felület



forgásparaboloid

$u = f(x,y,z)$

$f(x_1, x_2, \dots, x_n)$

$f(\underline{r}) = f(\underline{x})$

skalár$_t$ - vektorhoz

skalár - vektor fv.

($\underline{r}(t)$ (pl. hely az idő függvénye)

vektor-skalár fv.)

$\underline{v}(\underline{r})$: vektor-vektor fv.

differenciálhatósága, deriválása: nehéz

az egyváltozósból indulok ki

$$\cancel{\lim_{\Delta \underline{r} \to 0} \frac{f(\underline{r}+\Delta \underline{r}) - f(\underline{r})}{\Delta \underline{r}}}$$

vektorral nem osztható!

grad f
gradiens vektor

$$\Delta f = f(\underline{r}+\Delta \underline{r}) - f(\underline{r}) = A \cdot \Delta \underline{r} + \underbrace{\varepsilon(\Delta \underline{r})}_{\to \underline{0},\ \text{ha}\ \Delta \underline{r} \to \underline{0}} \cdot \Delta \underline{r}$$

Csak a szükséges és elégséges tétellel tudjuk definiálni, ha a független változó vektor

$\boxed{\text{totális deriválhatóság}}$

Parciális deriválás: egy változó kivételével az összeset rögzítem.

$$f(x,y) = x \cdot e^{x^2+y} + x^3 + y^4 = x\, e^{x^2} \cdot e^y + x^3 + y^4$$

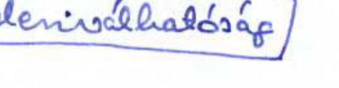
$$f'_x(x,y) = e^y\left(1 \cdot e^{x^2} + x \cdot e^{x^2} \cdot 2x\right) + 3x^2 + 0$$

$$f'_y(x,y) = x \cdot e^{x^2} \cdot e^y + 0 + 4y^3$$

$$\text{grad} f = f'_x \underline{i} + f'_y \underline{j}$$

$$\text{grad} f(0,1) = f'_x(0,1)\, \underline{i} + f'_y(0,1)\, \underline{j} = e \cdot \underline{i} + 4\underline{j}$$

grad f mindig merőleges a szintalakzatokra és a növekvő fv-ű szintalakzatra mutat.



szintvonalak

— o —

Differenciálegyenletek (de)

A fv, a független változók és az ezek szerinti ~~foga~~ deriváltak közötti kapcsolat

Közönséges differenciálegyenletek : 1 fgt változó (x)

implicit alak $\rightarrow F(x, y, y', y'', \dots y^{(n)}) = 0$ ← n-edrendű de

explicit alak $\rightarrow y^{(n)} = f(x, y, y', \dots y^{(n-1)})$

Általános mo:

$y = g(x, c_1, c_2, \dots c_n)$ expl. alakban az n-edrendű pmes de. megoldása

$G(x, y, C_1, C_2, \dots C_n) = 0$ megoldás implicit megoldása (megoldásgörbe, integrálgörbe)

Konstansok meghatározása

- kezdeti feltételekkel : $y(x_0) = y_{0,0}$, $y'(x_0) = y_{0,1}$, $\vdots$, $y^{(n-1)}(x_0) = y_{0,n-1}$ — $x_0, y_{0,0}, \dots y_{0,n-1}$ adott
  ↘ egyetlen görbe lesz, ha létezik
  - egzisztenciális probléma
  - unicitási probléma
- peremfeltételek: (pl a húr mozgása)

Elsőrendű de.

$F(x, y, y') = 0$

$y' = f(x, y)$

k. é. p. (kezdeti érték probléma)

$y(x_0) = y_0$ x_0, y_0 adott

Van-e olyan $y(x)$, amit behelyettesítve kielégíti a de-t. (és hány van?)

Szétválasztható változójú de. (szeparálható, szeparábilis)

$y' = \varphi(x, y)$ $\boxed{y' = f(x) \cdot g(y)}$ szétválasztható egy csak x-től és egy csak y-tól függő fv szorzatára.

(Pl) $y' = 2x + 2$

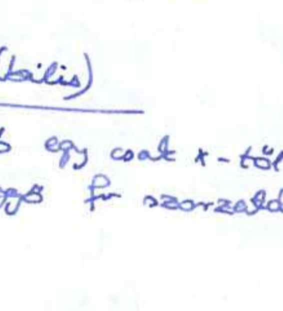
a) Ált. mo?

b) $y(1) = 5$ kezdeti feltételt kielégítő partikuláris megoldás?

c) Keresse meg azon integrálgörbét, amely érinti az $y = 8x - 20$ egyenest!

a) $\boxed{y = \int (2x+2)\,dx = \boxed{x^2 + 2x + C}}$ ált. mo $C \in \mathbb{R}$

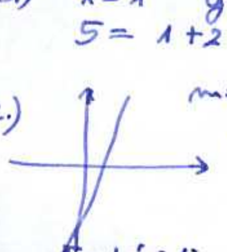
$y = (x+1)^2 + C - 1$ ∞ sok görbe



b.) $x = 1$ $y = 5$

$5 = 1 + 2 + C \Rightarrow C = 2$: $y = x^2 + 2x + 2$ a keresett part. mo (egyetlen fv.)

c.)



Kesb érinti

$m = 8$

$y' = 8 = 2x_0 + 2$

$x_0 = 3$

$y_0 = 8x - 20 \,|_{x=3}$

$y_0 = 4$

$y(3) = 4$

$4 = 3^2 + 6 + C \Rightarrow C = -11$

Keresett part. mo: $y = x^2 + 2x - 11$

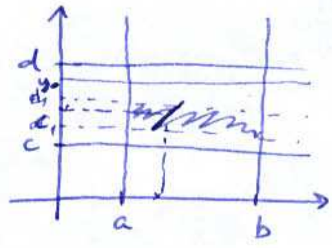
Anal II

2008. 02. 13.
Kónya

$y' = f(x) \cdot g(x)$

$y(x_0) = y_0$

$f \in C^0_{(a,b)}\ ,\ g \in C^0_{(c,d)}$



$y(x)$ mo: $\quad y'(x) \equiv f(x) \cdot g(y(x))$

$y_0 \in (c, d)$

$g(y_0) = 0 \quad : \quad y \equiv y_0$ mo.

(egyensúlyi helyzet)

$(c_1, d_1) \subset (c, d)$ – n már $g(y) \neq 0$ $\quad \forall$ ezen részbe eső kzp. egyértelműen megoldható $(\neg B)$

$\frac{dy}{dx} = f(x) \cdot g(y)$

$g(y) \neq 0 \quad \boxed{\int \frac{1}{g(y)} \cdot dy = \int f(x)\, dx}$ ált. m.o. $\quad$ (az integráljel nélkül egy differenciál)

a jobboldalon a mo-ban kell a $+C$.

INTEGRÁLÁST ÁTNÉZNI! (3-4. o. és 10 fejezet)

(Pl) $\quad y' = x \cdot e^{x^2 + 2y} = \underbrace{x \cdot e^{x^2}}_{f(x)} \cdot \underbrace{e^{2y}}_{g(y)}$ $\qquad f \in C^0_{\mathbb{R}} \quad g \in C^0_{\mathbb{R}}$ (mindenütt folytonos)

$g(y) \neq 0$

$\Downarrow$

a teljes síkon $\forall$ pontban pontosan 1 mo görbe halad át

$\frac{dy}{dx} = x e^{x^2} e^{2y}$

$\int \underbrace{\frac{1}{e^{2y}}}_{e^{-2y}} dy = \frac{1}{2} \int 2x e^{x^2} dx$

$\boxed{\frac{e^{-2y}}{-2} = \frac{1}{2} e^{x} + c}$

$e^{-2y} = -e^{x^2} + C$

$-2y = \ln(-e^{x^2} + c)$

$\boxed{y = -\frac{1}{2} \ln(-e^{x^2} + c)}$

$y(0) = 0$

$0 = -\frac{1}{2} \ln(-1 + c)$

$-1 + C = 1$

$C = 2$

$y(0) = 0 : y = \frac{1}{2} \ln(-e^{x^2} + 2)$

(Pl) $\quad y' = \underbrace{(2x+1) e^{-3x}}_{f(x)} \cdot \underbrace{(y^2 + 2y + 5)}_{g(y)}$ $\qquad f \in C^0_{\mathbb{R}}\ ,\ g \in C^0_{\mathbb{R}}$

$g(y) > 0$

$\frac{dy}{dx} = \dots$

$\int \frac{1}{y^2 + 2y + 5}\, dy = \int (2x+1)\, e^{-3x}\, dx$

$\frac{1}{(y+1)^2 + 4}\, dy = \frac{1}{4} \cdot \int \frac{1}{1 + \left(\frac{y+1}{2}\right)^2} dy \quad \Big\} \quad \begin{array}{ll} u = 2x+1 & v' = e^{-3x} \\ u' = 2 & v = \frac{e^{-3x}}{-3} \end{array}$ $\qquad J := -\frac{1}{3}(2x+1) \cdot e^{-3x} + \frac{2}{3} \int e^{-3x}\, dx$

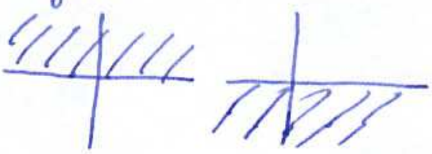
$= \boxed{\frac{1}{4} \cdot \frac{\operatorname{arctg} \frac{y+1}{2}}{\frac{1}{2}} = -\frac{1}{3}(2x+1) e^{-3x} + \frac{2}{3} \cdot \frac{e^{-3x}}{-3}}$

(Pl) $y' = \frac{\operatorname{sh} y}{\operatorname{ch} y}$ $f(x) \equiv 1$ $f \in C^0_{\mathbb{R}}$ $g(y)=0:$ $\operatorname{sh} y = 0$

$g(y) = \frac{\operatorname{sh} y}{\operatorname{ch} y}$ $g \in C^0_{\mathbb{R}}$

$y \equiv 0$ mo

$y \neq 0$



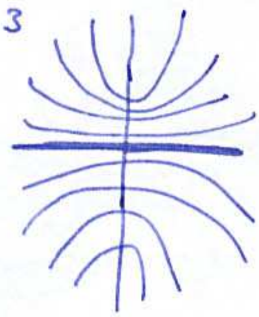
$$\int \frac{\operatorname{ch} y}{\operatorname{sh} y}\,dy = \int dx$$

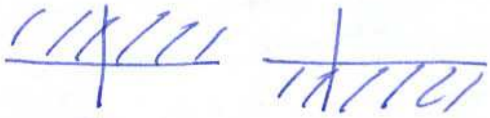
$\frac{f'}{f}$

$$\boxed{\ln|\operatorname{sh} y| = x + C}$$

(Pl) $\frac{dx}{dy} = y' = xy$ a) Ált. mo. b) $y(0) = 3$

$f(x) = x$
$g(y) = y$ fg folyt.
$g(y) = 0 \Rightarrow y \equiv 0$ mo.



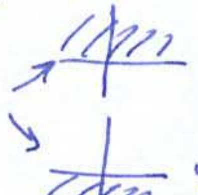


$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$|y| = e^{\frac{x^2}{2}+c}$$

$$|y| = \underbrace{e^c}_{>0} \cdot e^{\frac{x^2}{2}}$$



$y = e^c \cdot e^{\frac{x^2}{2}}$

$y = -e^c \cdot e^{\frac{x^2}{2}}$

és $y \equiv 0$ is mo.

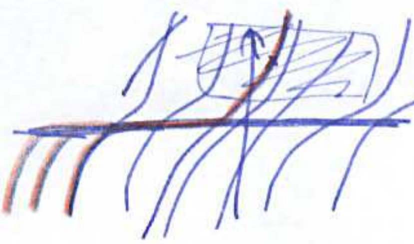
$\Rightarrow y = C \cdot e^{\frac{x^2}{2}}$ $C \in \mathbb{R}$ (mindhárom eset 1 alakban írva: $c>0, c=0, c<0$)

b) $y(0) = 3$
$3 = C \cdot e^0 \Rightarrow C = 3$ $y = 3e^{\frac{x^2}{2}}$
$y(3) = 0$ $y \equiv 0$

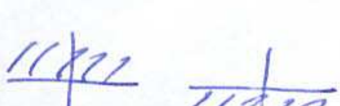
(Pl) $y' = 3 \cdot \sqrt[3]{y^2}$ $y(2) = 0$

$f(x) \equiv 1$ $g(y) = 3\sqrt[3]{y^2}$ folyt. $\mathbb{R}$-en.

$g(y) = 0$ $y \equiv 0$ mo



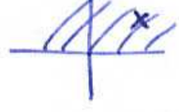
ez a görbesereg a mo.



$$\int \frac{1}{3} y^{-\frac{2}{3}}\,dy = \int dx$$

$$\cancel{\frac{1}{3}} \cdot \frac{y^{\frac{1}{3}}}{\cancel{\frac{1}{3}}} = x + C$$

$$y = (x+c)^3$$



$y(1) = 8$
$8 = (1+c)^3 \Rightarrow c = 1$
$y = (x+1)^3$

De az adott ponton nem csak az az egy görbe halad át:

$$y = \begin{cases} (x+1)^3, \text{ ha } x \geq -1 \\ 0, \text{ ha } -5 < x < -1 \\ (x+5)^3, \text{ ha } x \leq -5 \end{cases}$$

egymáshoz deriválhatóan csatlakoznak.

$\rightarrow$ sok megoldásgörbe lenne, ami átmegy az adott ponton

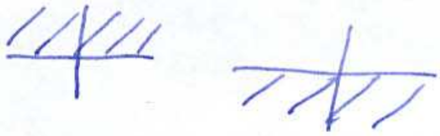
$\downarrow$

feloldás: a tétel csak az adott (~~[illegible]~~ szűkített) területre vonatkozik

Anal

2008. 02. 14.
Kónya

(Pl) $y' = \underbrace{(2x+1)^5}_{f(x)} \underbrace{\frac{(y^2+1)^2}{y}}_{g(y)}$ $y \neq 0$



$f \in C^0_{\mathbb{R}}$
g folyt, ha $y \neq 0$

k.é.p. egyértelműen megoldható
$(g(y) \neq 0)$

$\frac{dx}{dy} = (2x+1)^5 \cdot \frac{(y^2+1)^2}{y}$

$\int \frac{y}{(y^2+1)^2}\, dy = \int (2x+1)^5\, dx$

$\frac{1}{2}\int \underbrace{2y \cdot (y^2+1)^{-2}}_{f' \cdot f^{-2}}\, dy = \frac{1}{2}\int \underbrace{2 \cdot (2x+1)^5}_{f' \cdot f^5}\, dx$

$\boxed{\frac{(y^2+1)^{-1}}{-1} = \frac{(2x+1)^6}{6} + C}$ ← így jó, nem vonnak le

$y^2+1 = \frac{1}{-\frac{1}{6}(2x+1)^6 - C}$

$y = \pm\sqrt{\frac{1}{-\frac{1}{6}(2x+1)^6 - C} - 1}$

↑ enélkül pontlevonás (úgy tűnne, hogy 2 mo. van)

(Pl) $y(-1) = -1$ $(y(x) = ?)$

$-\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = -\frac{2}{3}$

$y = -1 = (-)\sqrt{\frac{1}{-\frac{1}{6}(2x+1)^6 + \frac{2}{3}} - 1}$

Lineáris elsőrendű de.

$y' + g(x)y = f(x)$

$g \in C^0_{(\alpha,\beta)}$, $f \in C^0_{(\alpha,\beta)}$

$\forall$ $y(x_0) = y_0$, $x_0 \in (\alpha, \beta)$ és y_0 tetsz. k.é.p. egyértelműen megoldható.

$\underbrace{y_{iá}}_{\text{inhomogén egyenlet általános mo-a}} = \underbrace{y_H}_{\text{homogén egyenlet mo-a}} + \underbrace{y_{ip}}_{\text{inhomogén egyenlet egy adott megoldása}}$

(H): $y' + g(x)y = 0$

(Pl) $y' - \frac{e^x}{e^x+1} \cdot y = 2x \cdot e^x$

(H): $y' - \frac{e^x}{e^x+1}\, y = 0$

(Pl) $2x\, y' + (x^2+1)y = x^3 \underset{\text{ha}}{\Rightarrow} y' + \frac{x^2+1}{2x}\, y = \frac{x^3}{2x}$ $x \neq 0$

(PL) $y' + 2xy = 4x$ (szétválasztható változójú is!) $\quad y' = 2x(2-y)$ ...

$y_{ip} = 2$

Homogén általános megoldása

(H): $y' + g(x)\cdot y = 0$

$\frac{dy}{dx} = y' = -g(x)\cdot y \qquad y \equiv 0$ mo

$y \neq 0 \qquad \int \frac{1}{y}\,dy = \int -g(x)\,dx$

$\ln|y| = G(x) + C_1 \qquad C_1 \in \mathbb{R}$

$|y| = e^{G(x)+C_1}$

$|y| = \underbrace{e^{C_1}}_{K>0} \cdot e^{G(x)}$

$y>0$: $y = K\cdot e^{G(x)}$ $\qquad$ $y<0$: $y = -K\cdot e^{G(x)}$ $\qquad y \equiv 0$ is mo

$y_H = C\cdot e^{G(x)} \qquad C \in \mathbb{R}$

Tehát $y_H = C\cdot \varphi(x) = C\cdot \underbrace{Y(x)}_{\neq 0} \left(= C_1\cdot Y_1(x)\right.$

(Lin. n-edrendű d.e.: $y_H = C_1 Y_1 + C_2 Y_2 + \dots + C_n Y_n$, $C_i \in \mathbb{R}$, $Y_1, Y_2, \dots Y_n$ a homogén egyenlet lin fgt megoldásai.)

(I): $y_{ip} = c(x)\cdot \underbrace{\varphi(x)}_{adott} = c(x)\cdot y(x)$ állandók variálása TB mindig van $c(x)$

$y' - \frac{2x}{x^2+1}\, y = x^2 \qquad y(0) = 4 \qquad y(x) = ?$

$y_{iá} = y_H + y_{ip}$ (mert lineáris)

(H): $y' - \frac{2x}{x^2+1}\, y = 0$

$y' = \frac{2x}{x^2+1}\, y$

$y \equiv 0$ mo

$y \neq 0 \qquad \int \frac{1}{y}\,dy = \int \frac{2x}{x^2+1}\,dx$

$\ln|y| = \ln(x^2+1) + C_1$

$|y| = e^{\ln(x^2+1)+C_1}$

$|y| = (x^2+1)\cdot e^{C_1}$

$\left.\begin{array}{l} y = \pm e^{C_1}(x^2+1) \\ y \equiv 0 \text{ mo} \end{array}\right\} \; y_H = C\cdot(x^2+1) \qquad C \in \mathbb{R}$

$y_{ip} = c(x)(x^2+1)$ állandó variálás

$c(x) = ?$

$y'_{ip} = c'(x)(x^2+1) + c(x)\cdot 2x$

I: $y' - \frac{2x}{x^2+1}\, y = x^2$

ezekbe kell behelyettesíteni

$\left(c'(x)\cdot(x^2+1) + \cancel{c\cdot 2x}\right) - \cancel{\frac{2x}{x^2+1}\cdot c\,(x^2+1)} = x^2$ $\quad$ ez is egy d.e. ⇒ valaminek ki kell esni! c-s tag

$c' = \frac{x^2}{x^2+1} \Rightarrow c(x) = \int \frac{x^2}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x - \operatorname{arctg} x$

nem írok konstanst, mert 1 mo-t keresek

$y_{ip} = (x - \operatorname{arctg} x)(x^2+1)$

$y_{iá} = y_H + y_{ip} = C(x^2+1) + (x - \operatorname{arctg} x)(x^2+1)$

Anal

2008.02.1
Kónya

$y(0)=4$

$y_{id} = C(x^2+1) + (x - \operatorname{arctg} x)(x^2+1)$

$4 = C + 0$

$C = 4$

$y = 4(x^2+1) + (x - \operatorname{arctg} x)(x^2+1)$

(70.) $y' - \frac{4}{x} y = x^3 + 1 \qquad x \neq 0$

$y_{id} = y_H + y_{ip}$

(H): $y' - \frac{4}{x} y = 0$

$\frac{dy}{dx} = y' = \frac{4}{x} y \qquad y_H = C \cdot \underbrace{\varphi(x)}_{\neq 0} \Rightarrow$ elég egyetlen mo-t keresni

$\int \frac{1}{y}\, dy = \int \frac{4}{x}\, dx$

$\ln y = 4 \ln x$

$y = x^4 = \varphi(x)$

$y_H = C \cdot x^4$

$C \in \mathbb{R}$

$y_{ip} = c(x)\, x^4$

$y'_{ip} = c' x^4 + c \cdot 4x^3$

$(c' x^4 + \cancel{c \cdot 4x^3}) - \cancel{\frac{4}{x} \cdot c x^4} = x^3 + 1$

$c' = \frac{1}{x} + \frac{1}{x^4}$

$c(x) = \ln|x| + \frac{x^{-3}}{-3}$

$y_{id} = Cx^4 + (x^4 \ln|x| - 3x) \leftarrow y_{ip} = \left(\ln|x| - \frac{3}{x^3}\right) \cdot x^4$

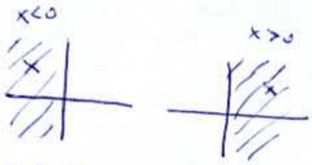
2008.02.20.
Kónya

(Pl.) $y' + \frac{y}{x} = 6x^4 \qquad x \neq 0$

$y_{id} = y_{hd} + y_{ip}$

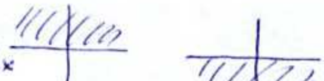
(H): $y' + \frac{y}{x} = 0$

$y' = \frac{dy}{dx} = -y \cdot \frac{1}{x}$



$y \equiv 0$ mo.

$y \neq 0$: $\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$



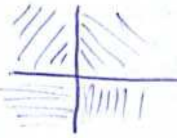
$$\left(\begin{array}{l} \ln y = \underbrace{-\ln x + \ln c}_{\ln \frac{c}{x}} \\ y = \frac{c}{x} \qquad c \in \mathbb{R} \end{array} \right)$$

← Ezt nem szeressük, mert az $y \leq 0$-t nem tudjuk megmagyarázni, csak a vége igaz.

1. mo: $\ln|y| = \ln|x| + c_1$

$|y| = \underbrace{e^{-\ln|x| + c_1}}_{e^{c_1} \cdot e^{\ln \frac{1}{|x|}}}$

$|y| = \underbrace{e^{c_1}}_{>0} \cdot \frac{1}{|x|}$



$$\left. \begin{array}{l} y = \pm e^{c_1} \cdot \frac{1}{x} \\ y \equiv 0 \end{array} \right\} y_H = C \cdot \frac{1}{x} \quad x \in \mathbb{R}$$

2. mo: $y_H = C \cdot \varphi(x)$

Elég egyetlen φ

$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$

$\ln y = -\ln x$ C-t nem írok, mert elég 1 mo!

$y = \frac{1}{x} = \varphi(x)$

$y_H = \frac{C}{x}$

Homogén egyenlet általános mo-a:

$y' + \frac{y}{x} = 6x^4 \qquad y_H = \frac{C}{x}$

$\left(\cancel{y_H = C \cdot e^{-\int g(x)\,dx}} \right)$ így tilos neked! $\qquad$ H $y' + g(x)\,y = 0$

$y_{ip} = \frac{c(x)}{x}$

$y'_{ip} = \frac{c' \cdot x - c \cdot 1}{x^2} \longrightarrow$ (I): $\frac{c'x - c}{x^2} + \frac{c}{x^2} = 6x^4$

$\frac{c'}{x} - \cancel{\frac{c}{x^2}} + \cancel{\frac{c}{x^2}} = 6x^4$

$c' = 6x^5 \Rightarrow c(x) = x^6$ $\qquad$ a konstans mostó

$y_{ip} = \frac{x^6}{x} = x^5$

$$\boxed{y_{id} = \frac{c}{x} + x^5 \qquad c \in \mathbb{R}}$$

$y(-1) = 6$

$6 = -c - 1 \quad \Rightarrow y = \frac{-7}{x} + x^5$

$c = -7$

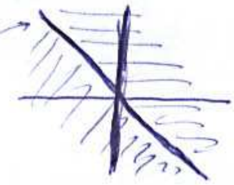
példatár: 1.6-nem kell, a többi jó!

Új változó bevezetése

Pl

$$y' = \frac{y-x}{x+y} \qquad u = \frac{y}{x} \qquad x \neq 0$$

$y \neq -x$



$$y' = \frac{\frac{y}{x}-1}{1+\frac{y}{x}} = f\left(\frac{y}{x}\right)$$

$$y = u \cdot x = u(x) \cdot x$$

$$y' = u' \cdot x + u \cdot 1$$

$$u' \cdot x + u = \frac{u-1}{1+u}$$

$$\frac{du}{dx} = u' = \underbrace{\left(\frac{u-1}{1+u} - u\right)}_{\frac{u-1-u-u^2}{1+u} = \underbrace{\frac{-(u^2+1)}{1+u}}_{\neq 0}} \cdot \frac{1}{x}$$

$$\int \underbrace{\frac{u+1}{u^2+1}}_{\frac{1}{2}\frac{2u}{u^2+1} + \frac{1}{u^2+1}} du = -\int \frac{1}{x}\, dx$$

$$\frac{1}{2}\ln(u^2+1) + \operatorname{arctg} u = -\ln|x| + C$$

$$\boxed{\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+1\right) + \operatorname{arctg}\frac{y}{x} = -\ln|x| + C \qquad C \in \mathbb{R}}$$

Pl a) $y' = 2x+y$

b) $y' = \dfrac{1}{2x+y}$

Szükség esetén alkalmazza az $u = 2x+y$ helyettesítést.

a) lin. elsőrendű diff. egy.

mo: HF

b) $u = 2x+y$

$y = u - 2x$

$y' = u' - 2$

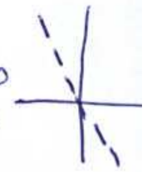
$u' - 2 = \dfrac{1}{u}$ szétválasztható változójú

$\dfrac{du}{dx} = u' = 2 + \dfrac{1}{u}$

$\dfrac{du}{dx} = \dfrac{2u+1}{u}$ $\quad u \neq 0$

$2x+y \neq 0$

$y \neq -2x$



$u \equiv -\frac{1}{2}$ mo

$2x+y = -\frac{1}{2}$

$y = -2x - \frac{1}{2}$ mo

$u \neq -\frac{1}{2}$

$$\int \frac{u}{2u+1}\, du = \int dx$$

$$\frac{1}{2}\int \frac{(2u+1)-1}{2u+1}\, du = x + C$$

$$\frac{1}{2}\left(u - \frac{\ln|2u+1|}{2}\right) = x + C$$

$u = 2x+y$

$$\frac{1}{2}\left((2x+y) - \frac{1}{2}\ln|4x+2y+1|\right) = x + C \quad \text{ill.} \quad y = -2x - \frac{1}{2}$$

Pl $\quad xyy' + y^2 = x^2 \qquad u = y^2$

$$u' = 2y \cdot y'$$

$$x \cdot \frac{u'}{2} + u = x^2$$

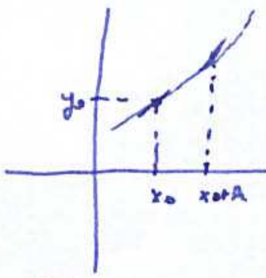
$$u' + \frac{2u}{x} = 2x \quad \text{lin. elsőr. d.e.}$$

$x \neq 0$

$u_{ált} = u_H + u_{ip}$ $\quad$ u-ban lineáris,

$y^2 = \ldots$ $\quad$ de y-ban NEM!

Grafikus megoldás (Euler-módszer) $y' = f(x,y)$, $y(x_0) = y_0$



$y'(x_0) = f(x_0, y_0)$
$y(x_0+h) = y'e(x_0+h)$

Továbbfejlesztve: Runge-Kutta:

bizonyos pontokban felírjuk az értéket, majd közelítő polinommal a többiben is lubát is becsül.

(Pl) a) Rajzolja fel az $y' - e^{3y} = x$ de-nek azt az izoklínáját, amelynek pontjaiban teljesül a lok sz.é. létezésének szüks. feltétele. Jelölje be az irdnymezőt ezen izoklína néhány pontjában az $x_0 = 1$, $y_0 = 0$ pontban.

b.) Mely pontokban van lok. minimuma a megoldásoknak?

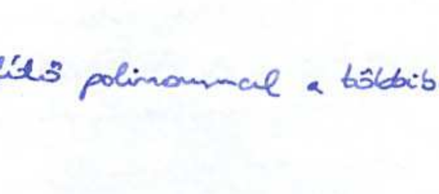
c.) Az $y = y(x)$, $x \in \mathbb{R}$ megoldása a fenti de-nek, akárhányszor differenciálható és átmegy a $(-\frac{4}{3}, 0)$ ponton. Van-e ennek a megoldásnak inflexiója az $x = -\frac{4}{3}$ helyen?

a) $y' = e^{3y} + x = f(x,y) = K$

Az izoklína egyenlete $e^{3y} + x = K$

Lok. sz.é $\exists$-nek szüks. felt. $y' = 0 \Rightarrow K = 0$

$e^{3y} + x \Rightarrow 3y = \ln(-x) \Rightarrow y = \frac{1}{3}\ln(-x)$



csak ezekben a pontokban lehet lok. szé

$x_0 = 1$ $y_0 = 0$

b) $y'(1) = e^0 + 1 = 2$

$y''(x) = e^{3y} \cdot 3y' + 1$

$y'|_{K=0} = 1$ $y = \frac{1}{3}\ln(-x)$ pontjaiban $y' = 0$ és $y''(1) > 0 \Rightarrow$ lok. min van. ezekben a pontokban.

Sehol nincs lokális maximum.

c.) $y' = e^{3y} + x$

$y(-\frac{4}{3}) = 0$

$y'(-\frac{4}{3}) = f(-\frac{4}{3}, 0) = e^0 - \frac{4}{3} = -\frac{1}{3}$

$y'' = e^{3y} \cdot 3y' + 1$

$x = -\frac{4}{3}$
$y = 0$
$y' = -\frac{1}{3}$

$y''(-\frac{4}{3}) = e^0 \cdot 3(-\frac{1}{3}) + 1 = 0 \Rightarrow$ lehet infl. pont.

$\Rightarrow y''' = (e^{3y} \cdot 3y') \cdot 3y' + e^{3y} \cdot (3y'') + 0$

$y'''(-\frac{4}{3}) = \underbrace{e^0 \cdot (-\frac{1}{3} \cdot 3) \cdot 3 \cdot (-\frac{1}{3})}_{1} + e^0 \cdot 3 \cdot 0 + 0 = 1 \neq 0$

$\left.\begin{array}{l} y''(-\frac{4}{3}) = 0 \\ y'''(-\frac{4}{3}) \neq 0 \end{array}\right\} \Rightarrow x_0 = -\frac{4}{3}$-ban inflexiós pont van

(Pé) $3y^2y' - 3y^3 = e^{3x} + x$ $\quad y(0)=1$ $\quad u = y^3$ helyettesítéssel dolgozzon (20 p.)

(Pé) Az $u = y^3 + 2x$ helyettesítést hajtsa végre az alábbi k.é.p-n!

$$3y^2y' = (y^3 + 2x + 1)^3 \cos \pi x - 2 \qquad y(1) = -1$$

Milyen de-hez jutottunk (Ne oldja meg) (7p)

(Pé) Hajtsa végre az $u = xy^3$ helyettesítést!

$$3xy^2y' - y^3 = x^3$$ (A kapott egyenletet ne oldja meg)

(Pé) Írja fel az $y' + y^2 + x^2 + 1 = 0$ izoklínáinak egyenletét!
Van-e a megoldásoknak szélsőértéke?
Vizsgálja meg a megoldásgörbéknek az 1. síknegyedbe $(x>0, y>0)$ eső részét monotonitás szempontjából.
Számítsa ki az $x_0 = 0$ $y_0 = 1$ ponton átmenő megoldás 1. és 2. deriváltját az x_0-ban.
Írja fel ezen megoldás $x=0$ pontjában másodrendű Taylor polinomját!

$$y' = \boxed{\underbrace{-y^2 - x^2 - 1}_{f(x,y)} = k} \qquad x^2 + y^2 = -(k+1)$$

$$k + 1 \le 0$$

$k \le -1 \Rightarrow y' \le -1 \Rightarrow$ nincs a mo-nak szélsőé-, mert nem teljesül a szüks. felt. $y' \le -1 \ne 0$

A megoldásgörbék szig. mon. csökkenőek: $y' < 0$

$y' = -y^2 - x^2 - 1$

$y(0) = 1$

$y'(0) = -1^2 + 0 - 1 = -2$

$y'' = -2 \cdot y \cdot y' - 2x - 0$ $\qquad y''(0) = -2 \cdot 1 \cdot (-2) - 2 \cdot 0 = 4$
$x=0$, $y=1$, $y'=-2$

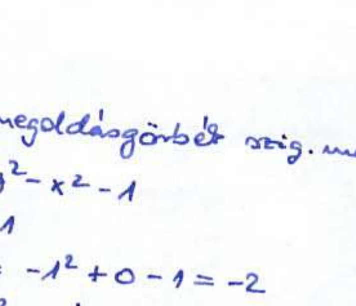
$$T_n(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \frac{y'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{y^{(k)}(x_0)}{k!}(x-x_0)^k$$

$y(x)$ x_0 pontbeli n-edrendű T. pn. (megoldásnak?)

Most $x_0 = 0$ $\qquad T_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 = 1 + (-2)x + \frac{4}{2!}x^2 = 1 - 2x + 2x^2$

(Pé) $y' = \frac{y}{x}$

31 ábra, 26.o.



$y = x \qquad y' = \frac{x}{x} = 1$

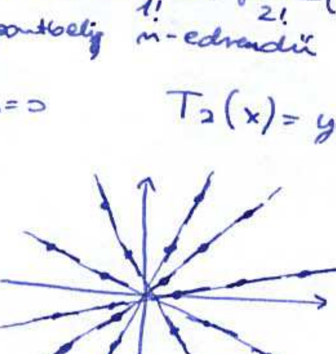
$y = 2x \qquad y' = \frac{2x}{x} = 2$

$\vdots$

$y = mx \qquad y' = \frac{mx}{x} = m$

(Pé) $y' = x - y^2$

$x - y^2 = 0$



<u>Analízis</u>

2008.02.21
Kónya

n-edrendű lin. d.e.-k

$y_{iá} = y_H + y_{ip}$

$y_{id} = \sum_{i=1}^{n} C_i Y_i(x) = C_1 Y_1(x) + \ldots + C_n Y_n(x) \qquad C_i \in \mathbb{R}$

(Pé) Mutassa meg, hogy az $Y_1(x) = e^{-2x}$, $Y_2(x) = e^{3x}$ megoldja az alábbi de-t!

$y'' - y' - 6y = 0$

Írja fel az ált. mo-t.

b/ $Y_1 = e^{-2x}$ $\quad -1/ Y_1' = -2e^{-2x}$ $\quad 1/ Y_1'' = 4e^{-2x}$

$\cancel{Y_2 = 3e^{3x}}$ $\quad \cancel{Y_2' = 3e^{3x}}$

$e^{-2x}(-6+2+4) = 0$ tényleg kielégíti

Hasonlóan Y_2-re.

$y_H = C_1 \cdot e^{-2x} + C_2 \cdot e^{3x} \qquad C_1, C_2 \in \mathbb{R}$

Homogén n-edrendű lineáris de.

(Pl.) $y^{V}+3y^{IV}-10y'''=0$

$y=e^{\lambda x}$

$y'=\lambda e^{\lambda x}$

$y''=\lambda^2 e^{\lambda x}$

$-10 \cdot \; y'''=\lambda^3 e^{\lambda x}$

$3 \cdot \; y^{IV}=\lambda^4 e^{\lambda x}$

$1 \cdot \; y^{V}=\lambda^5 e^{\lambda x}$

$\underbrace{e^{\lambda x}}_{\neq 0}(\lambda^5+3\lambda^4-10\lambda^3)=0$

$\lambda^5+3\lambda^4-10\lambda^3=0$ karakterisztikus egyenlet

$\lambda^3(\lambda^2+3\lambda-10)=0$

$(\lambda-0)^3$

$\lambda_1,\lambda_2,\lambda_3=0$

$\lambda_{4,5}=\frac{-3\pm\sqrt{9+40}}{2} < \begin{matrix} 2=\lambda_4 \\ -5=\lambda_5 \end{matrix}$

$e^{0x}\equiv 1,\ x\cdot 1,\ x^2\cdot 1,\ e^{2x},\ e^{-5x}$

$y_H=\sum_{i=1}^{5} C_i\cdot Y_i(x) = C_1+C_2x+C_3x^2+C_4e^{2x}+C_5e^{-5x}$ $\quad C_i\in\mathbb{R}$

(Pl.) $y'''+4y''+4y'=0$

$y=e^{\lambda x}$

$\lambda^3+4\lambda^2+4\lambda=\lambda(\lambda+2)^2=0$

$\lambda_1=0 \quad \lambda_2=-2 \quad \lambda_3=-2$

$$\boxed{y_H=C_1\cdot\underbrace{e^{0x}}_{1}+C_2\cdot xe^{-2x}+C_3\cdot x^2e^{-2x}}$$

(Pl.) $y''-2y'+26y=0$

$y=e^{\lambda x}$

$\lambda^2-2\lambda+26=0$

$\lambda_{1,2}=\frac{2\pm\sqrt{4-104}}{2}=\frac{2\pm\sqrt{-100}}{2}=\frac{2\pm 10\overbrace{\sqrt{-1}}^{\pm j}}{2}=1\pm 5j$

$Y_1=e^{\lambda_1 x}=e^{(1+j5)x}=e^{x+j\cdot 5x}=e^x\cdot e^{j(5x)}=e^x(\cos 5x+j\sin 5x)=e^x\cdot\cos 5x+e^x\cdot j\sin 5x$

$\mathrm{Re}\, Y_1=e^x\cdot\cos 5x$

$\mathrm{Im}\, Y_2=e^x\cdot\sin 5x$

$y_H=C_1\cdot e^x\cdot\cos 5x+C_2\cdot e^x\cdot\sin 5x$

(Pl.) Írjon fel egy olyan legalacsonyabb rendű valós konstans együtthatós homogén lineáris de-t, melynek megoldásai:

a) $e^{-2x},\ e^x$

b) $3e^{-2x}+5e^x$

c) $4xe^{-2x}-3e^{-5x}$

d.) $x^2,\ xe^{-2x}$

e.) $3-x,\ e^{-x}\sin 3x$

f.) $x\sin 2x$

a) $\lambda_1=-2 \quad \lambda_2=1$

$y_H=C_1e^{-2x}+C_2e^x$

$\overset{p=1}{1}(\lambda+2)(\lambda-1)=0$

$\lambda^2+\lambda-2=0$

$y''+y'-2y=0$

b.) u.a., mint az a) csak most a lineáris kombinációkat adták meg

$C_1=3 \quad C_2=5$

c.) $\lambda_1=-2 \quad \underset{\uparrow}{\lambda_2=-2} \quad \lambda_3=-5$

mert megvan szorozva x-szel

$(\lambda+2)^2(\lambda+5)=0$

$\lambda^3+9\lambda^2+24\lambda+20=0$

$y'''+9y''+24y'+20y=0$

d) $\lambda_1=0 \quad \lambda_2=0 \quad \lambda_3=0$

$\lambda_4=-2 \quad \lambda_5=-2$

$\lambda^3(\lambda+2)^2=0$

$\vdots$

e) $\lambda_1=0 \quad \lambda_2=0$

$e^{\alpha x}\cdot\cos\beta x;\ e^{\alpha x}\cdot\sin\beta x \Rightarrow \lambda_3=-1+j3 \quad \lambda_4=-1-j\cdot 3$ $\quad \alpha=-1 \quad \beta=3$ $\quad -1- \rightarrow \lambda^2(\lambda-(-1+j\cdot 3))(\lambda-(-1-j\cdot 3))=0\ldots$

f.) $\overset{\sin 2xi}{e^{\alpha x}} \cdot \sin \beta x$

$\alpha = 0 \quad \beta = 2$ $\qquad \lambda_1 = 0 + j\cdot 2 \quad \lambda_2 = 0 - j\cdot 2$

$\lambda_3 = j\cdot 2 \quad \lambda_4 = -j2$

Pl. $y'' + y' - 2y = 4x^2 + 7$ $\qquad y(0) = \frac{1}{2} \quad y'(0) = 0$

$y_{iá} = y_H + y_{ip}$

(H): $y = e^{\lambda x}$

$\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda_1 = -2 \quad \lambda_2 = 1$

$y_H = C_1 e^{-2x} + C_2 e^x$

$C_1, C_2 \in \mathbb{R}$

(I): $-2 \,|\, y_{ip} = Ax^2 + Bx + C$

$(f(x) = 4x^2 - 7)$

$1 \,|\, y'_{ip} = 2Ax + B$

$1 \,|\, y''_{ip} = 2A$

$x^2(-2A) + x(-2B + 2A) + (-2C + B + 2A) = 4x^2 - 7$

$-2A = 4$

$A = -2$

$-2B + 2A = 0$

$B = A = -2$

$-2C + B + 2A = -7$

$C = -\frac{1}{2}(-7 + 2 + 4) = \frac{1}{2}$

$y_{ip} = -2x^2 - 2x + \frac{1}{2}$

$y_{iá} = C_1 \cdot e^{-2x} + C_2 e^x + \left(-2x^2 - 2x + \frac{1}{2}\right)$

$y(0) = \frac{1}{2} \qquad \frac{1}{2} = C_1 + C_2 + \frac{1}{2}$

$y'(0) = 0 \qquad -2C_1 \cdot e^{-2x} + C_2 \cdot e^x - 4x - 2 = y'_{iá}$

$0 = -2C_1 + C_2 - 2$

$\frac{1}{2} = 3C_1 + \frac{5}{2} \Rightarrow C_1 = -\frac{2}{3} \qquad C_2 = \frac{2}{3}$

$$y = -\frac{2}{3} \cdot e^{-2x} + \frac{2}{3} e^x - 2x^2 - 2x + \frac{1}{2}$$

(M) $y' - 3y = \cos(3x)$ ler, de KONSTANS EH-s $\Rightarrow$ n-edrendű módszere alkalmazha f(x) megfelelő

$y_{id} = y_H + y_{ip}$

(H) $y' - 3y = 0 \qquad y = e^{\lambda x}$

$\lambda - 3 = 0 \Rightarrow \lambda = 3$

$y_H = C \cdot e^{3x}$

$f(x) = \cos 3x$: $\quad -3 \,|\, y_{ip} = \underline{A \cos 3x} + B \cdot \sin 3x$

$\qquad 1 \,|\, y'_{ip} = -3A \sin 3x + \underline{3B \cdot \cos 3x}$

$\cos 3x(-3A + 3B) + \sin 3x(-3B - 3A) = \cos 3x$

$$\begin{cases} -3A + 3B = 1 \\ -3B - 3A = 0 \Rightarrow A = -B \end{cases}$$

$6B = 1 \Rightarrow B = \frac{1}{6} \qquad A = -\frac{1}{6}$

$y_{ip} = -\frac{1}{6} \cos 3x + \frac{1}{6} \cdot \sin 3x$

$y_{iA} = C e^{3x} - \frac{1}{6} \cos 3x + \frac{1}{6} \sin 3x$

(Pl.) $y'' - 3y' + 2y = 3e^{2x}$

$y_{id} = y_H + y_{ip}$

(H): $y = e^{\lambda x}$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0 \qquad \lambda_1 = 1, \ \lambda_2 = 2$

$y_H = C_1 e^x + C_2 e^{2x}$

$f(x) = 3e^{2x}$

$y_{ip} = A e^{2x}$ külső rez (van y_H-ban e^{2x}-es alak)

$2 \,|\, y_{ip} = \underline{A x e^{2x}}$

$-3 \,|\, y'_{ip} = A \cdot e^{2x} + \underline{A \cdot x \cdot e^{2x} \cdot 2}$

$1 \,|\, y''_{ip} = 2A e^{2x} + 2A e^{2x} + \underline{2Ax e^{2x} \cdot 2}$

$x e^{2x} \underbrace{(2A - 6A + 4A)}_{=0} + e^{2x}(-3A + 4A) = 3 \cdot e^{2x}$

$A = 3$

$y_{id} = C_1 e^x + C_2 e^{2x} + 3x e^{2x}$

(Pl.) $y''' + 5y'' = 14 e^{2x} - 60x$

(H): $\lambda^3 + 5\lambda^2 = 0$

$\lambda^2(\lambda + 5) = 0 \qquad \lambda_{1,2} = 0$ Belső rez. $\qquad \lambda_3 = -5$

$y_H = C_1 + C_2 x + C_3 e^{-5x}$

y_{ip}: $f(x) = (14 e^{2x}) + (-60x)$

$y_{ip} = (A e^{2x}) + (Bx + C)x^2$ (eddig volt külső rez, mert volt benne x-es és konstans tag, már nincs)

$= A e^{2x} + Bx^3 + Cx^2$

$y'_{ip} = 2A e^{2x} + 3Bx^2 + 2Cx$

$5 \,|\, y''_{ip} = \underline{4A e^{2x}} + \underline{6Bx} + 2C$

$1 \,|\, y'''_{ip} = \underline{8A e^{2x}} + 6B$

$e^{2x}(20A + 8A) + x(30B) + (10C + 6B) = 14 e^{2x} - 60x$

$28A = 14 \qquad 30B = -60 \qquad 10C + 6B = 0$

$A = \frac{1}{2} \qquad B = -2 \qquad 10C = 12 \Rightarrow C = \frac{6}{5}$

$y_{id} = y_H + y_{ip} = C_1 + C_2 x + C_3 e^{-5x} + \underbrace{\frac{1}{2} e^{2x} - 2x^3 + \frac{6}{5} x^2}_{y_{ip}}$

(Pl) $y'' + 4y' + 4y = 4\,\mathrm{sh}\,2x$

$y_{ià} = y_H + y_{ip}$

(H): $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0 \Rightarrow \lambda_{1,2} = -2 \Rightarrow y_H = C_1 e^{-2x} + C_2 x e^{-2x}$ — meg kell szorozni, hogy lin. fgt. legyen. belső rez.

$f(x) = 4\,\mathrm{sh}\,2x = 4 \cdot \dfrac{e^{2x} - e^{-2x}}{2} = (2e^{2x}) + (-2e^{2x})$

$y_{ip} = (A e^{2x}) + (B e^{-2x}) x^2 \quad \ldots$

külső rez.

(Pl) a) Oldja meg az α par. függvényében az alábbi de-t: $y'' - 3y' + \alpha^2 y = 0$

b.) $y'' - 3y' = 18 \cos 3x + 6 e^{3x}$

$y(x) = ?$

a) $\lambda^2 - 3\lambda + \alpha^2 = 0 \qquad \lambda_{1,2} = \dfrac{3 \pm \sqrt{9 - 4\alpha^2}}{2}$

$D = 0 \qquad 9 - 4\alpha^2 = 0$

$|\alpha| = \frac{3}{2} \Rightarrow \alpha = \pm\frac{3}{2}$ belső rez.; $\lambda_{1,2} = \frac{3}{2}$

$y_H = C_1 e^{\frac{3}{2}x} + C_2 x e^{\frac{3}{2}x}$

$D > 0 \qquad 9 - 4\alpha^2 > 0 \qquad \frac{3}{2} > \alpha > -\frac{3}{2} \qquad \alpha \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ 2 kül. valós gyök

$y_H = C_1 e^{\frac{3+\sqrt{9-4\alpha^2}}{2}x} + C_2 e^{\frac{3-\sqrt{9-4\alpha^2}}{2}x}$

$D < 0 \qquad 9 - 4\alpha^2 < 0 \qquad \alpha^2 > \frac{9}{4} \qquad |\alpha| > \frac{3}{2}$

$\lambda_{1,2} = \dfrac{3 + \sqrt{(-1)(4\alpha^2 - 9)}}{2} = \dfrac{3 \pm j\sqrt{4\alpha^2 - 9}}{2} = \frac{3}{2} \pm j \dfrac{\sqrt{4\alpha^2-9}}{2}$

$y_H = C_1 e^{\frac{3}{2}x} \cos \dfrac{\sqrt{4\alpha^2 - 9}}{2} x + C_2 e^{\frac{3}{2}x} \sin \dfrac{\sqrt{4\alpha^2-9}}{2} x$

b.) (H): $\lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$

$y_H = C_1 + C_2 e^{3x} \qquad \lambda_1 = 0 \quad \lambda_2 = 3$

$e^0 = 1$

$f(x) = (18 \cos 3x) + (6 e^{3x})$

$y_{ip} = \cancel{A e^{3x} + B e^{3x}} (A \cdot \cos 3x + B \sin 3x) + (C\, e^{3x})$

$\vdots$

$y_{id} = C_1 + C_2 \cdot e^{3x} - \cos 3x - \sin 3x + 2x e^{3x}$

Hányados- és gyökkritérium

$\sum_1^\infty a_n \quad a_n > 0$

$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = c$, $\lim_{n\to\infty} \sqrt[n]{a_n} = c$ } $c < 1$: $\sum_1^\infty a_n$ konv. ; $c > 1$: $\sum_1^\infty a_n$ div. ; $c = 1$: ? $\quad$ ($c = \infty$ is lehet)

(Pl) $\sum_1^\infty \dfrac{n^2}{5^{n+1}}$

$\sqrt[n]{a_n} = \sqrt[n]{\dfrac{n^2}{5^{n+1}}} = \dfrac{\sqrt[n]{n^2}}{\sqrt[n]{5^n \cdot 5}} = \dfrac{(\sqrt[n]{n})^2}{\sqrt[n]{5} \cdot 5} \to \dfrac{1}{5} = c < 1 \Rightarrow \sum_1^\infty a_n$ konv.

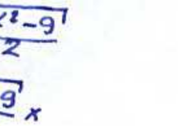
($n^2 < 2^n$ majoráns krit.

$n^k a^n \to 0$, $|a| < 1$, $k \in \mathbb{N}^+$

$n^2 \left(\frac{1}{2}\right)^n \to 0$

$0 < n^2 \left(\frac{1}{2}\right)^n < 1 \quad n \geq N_1$)

$(\sqrt[n]{a_n} \leq q < 1$



$\dfrac{a_{n+1}}{a_n} = \dfrac{\frac{(n+1)^2}{5^{n+2}}}{\frac{n^2}{5^{n+1}}} = \dfrac{(n+1)^2 \cdot 5^{n+1}}{n^2 \cdot 5^{n+2}} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{5} \Rightarrow \frac{1}{5} = c < 1 \Rightarrow \sum_1^\infty a_n$ konv.

(Pl) $\sum_1 \dfrac{2^n}{n^2 \cdot 7^{n+1}}$ $\qquad a_n \leq \dfrac{2^n}{1 \cdot 7^n}$ maj. kr. $\qquad \sum_1^\infty \left(\frac{2}{7}\right)^n$ konv. majoráló $\Rightarrow \sum_1^\infty a_n$ konv. (de gyök- és hányados kr. is)

(Pl) $\sum_1^\infty \dfrac{(n+1)^2}{5^n}$

$\sqrt[n]{a_n} = \dfrac{(\sqrt[n]{n+1})^2}{5} \to \frac{1}{5} = c < 1$ mert $1 \leq \sqrt[n]{1} \leq \sqrt[n]{n+1} \leq \sqrt[n]{n+n} = \sqrt[n]{2} \cdot \sqrt[n]{n}) \Rightarrow \sqrt[n]{n+1} \to 1$ rendőr elv

További hányados kr.

(Pl) $\sum_1 \left(\dfrac{4n+1}{4n-2}\right)^{n^2} \cdot \dfrac{1}{7^{n+1}} \quad \ldots$

Anal elm

2008. 03. 05.
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ZH konzultációk:
márc. 17. ápr. 14. V2.501 17⁰⁰

$\left(\sum_1^\infty \frac{(n+1)^2}{5^n} = \sum_2^\infty \frac{n^2}{5^{n-1}} = 5 \sum_2^\infty \frac{n^2}{5^n} \quad + \text{ gyökkr.} \right)$

Pl. $\sum^\infty \frac{n^2(2n+1)^{n^2}}{(2n+3)^{n^2}}$ $\quad \sqrt[n]{a_n} = \frac{(\sqrt[n]{n})^2 (2n+1)^n}{(2n+3)^n} = (\sqrt[n]{n})^2 \left(\frac{2n+1}{2n+3}\right)^n = (\sqrt[n]{n})^2 \frac{\left(1+\frac{1/2}{n}\right)^n}{\left(1+\frac{3/2}{n}\right)^n} \to$

$\to 1^2 \cdot \frac{e^{1/2}}{e^{3/2}} = \frac{1}{e} < 1 \Rightarrow \sum^\infty a_n$ konv.

Pl. $\sum^\infty \frac{(n+1)!}{n^{n+1}}$

$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)!}{(n+1)^{n+2}}}{\frac{(n+1)!}{n^{n+1}}} = \frac{(n+2)!}{(n+1)!} \frac{n^{n+1}}{(n+1)^{n+2}} = (n+2) \frac{n^{n+1}}{(n+1)^{n+2}} = \underbrace{\frac{(n+2)n}{(n+1)^2}}_{\frac{1+\frac{2}{n}}{(1+\frac{1}{n})^2}} \cdot \underbrace{\frac{n^n}{(n+1)^n}}_{\left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{(1+\frac{1}{n})^n}} \to 1 \cdot \frac{1}{e} < 1$

$\sum a_n$ konv.

Pl. $\sum^\infty \frac{7^{2n+1}}{(3+n)^n}$ hányados kr. is jó

$\sqrt[n]{a_n} = \sqrt[n]{\frac{7^{2n} \cdot 7}{(3+n)^n}} = \frac{49 \cdot \sqrt[n]{7}}{\underbrace{3+n}_{\to\infty}} \to 0 < 1 \Rightarrow \sum^\infty a_n$ konv.

Pl.
a) $\sum^\infty \left(\frac{n^2+1}{n^2+2}\right)^{n^2}$ $\quad a_n \to e^{-1} \Rightarrow \sum a_n$ divergens, nem teljesül a konv. szüks. feltétel. $a_n \not\to 0$

$a_n = \frac{\left(1+\frac{1}{n^2}\right)^{n^2}}{\left(1+\frac{2}{n^2}\right)^{n^2}} = \frac{e^1}{e^2} = e^{-1}$

b) $\sum^\infty \left(\frac{n^2+1}{n^2+2}\right)^{n^3}$ $\quad \sqrt[n]{b_n} = \sqrt[n]{\left(\frac{n^2+1}{n^2+2}\right)^{n^3}} = \left(\quad \right)^{n^3 \cdot \frac{1}{n}} \to \frac{1}{e} < 1 \Rightarrow \sum^\infty b_n$ konv.

Pl. a) $\sum^\infty \frac{n}{(n+4) \cdot 5^{n+2}}$ $\quad$ b) $\sum^\infty \left(\frac{n}{n+4}\right)^n \cdot \frac{1}{5^{n+2}}$ $\quad$ c) $\sum \left(\frac{n}{n+4}\right)^{n^2} \frac{1}{5^{n+2}}$

a) $a_n = \underbrace{\frac{n}{n+4}}_{<1} \cdot \frac{1}{5^{n+2}} < \frac{1}{5^n}$ $\quad \sum^\infty \left(\frac{1}{5}\right)^n$ konv. maj. sor $\Rightarrow \sum^\infty a_n$ konv.

$\left(\text{De: } \sum \frac{n+5}{n+4} \cdot \frac{1}{5^{n+2}} \quad a_n = \underbrace{\frac{n+5}{n+4}}_{\to 1} \cdot \frac{1}{5^{n+2}} \underset{n>N_0}{<} 1{,}1 \cdot \frac{1}{5^{n+2}} \dots \text{ konv.} \right)$

Vagy: hányados kr.

$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{(n+5)} \frac{(n+4)\, 5^{n+2}}{5^{n+3} \cdot n} = \frac{n+1}{n+5} \cdot \frac{n+4}{n} \cdot \frac{5^{n+2}}{5^{n+3}} = \frac{\overbrace{1+\frac{1}{n}}^{\to 1}}{1+\frac{5}{n}} \cdot \frac{\overbrace{1+\frac{4}{n}}^{\to 1}}{1} \cdot \frac{1}{5} \to \frac{1}{5} < 1 \Rightarrow \sum a_n$ konv.

b) $\sqrt[n]{b_n} = \underbrace{\frac{n}{n+4}}_{\to 1} \frac{1}{5} \cdot \underbrace{\frac{1}{\sqrt[n]{25}}}_{\to 1} \Rightarrow 1 \cdot \frac{1}{5} \cdot 1 = \frac{1}{5} < 1 \Rightarrow \sum^\infty b_n$ konv.

c) $\sqrt[n]{c_n} = \left(\frac{n}{n+4}\right)^n \cdot \frac{1}{5 \cdot \sqrt[n]{25}} \to e^{-4} \cdot \frac{1}{5} \cdot 1 = \frac{e^{-4}}{5} < 1 \Rightarrow \sum c_n$ konv.

Pl $\sum\limits_{1}^{\infty} \left(\frac{3n+1}{3n+3}\right)^{n^2+2n}$ $\quad \sqrt[n]{a_n} = \left(\frac{3n+1}{3n+3}\right)^{n+2} = \frac{\left(1+\frac{1/3}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n} \cdot \left(\frac{3n+1}{3n+3}\right)^2 \to \frac{e^{\frac{1}{3}}}{e^1} \cdot 1 = e^{-\frac{2}{3}} < 1$ $\Rightarrow \sum a_n$ konv.

Pl $\sum\limits_{1}^{\infty} \frac{n!\,3^n}{(2n)!}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)! \cdot 3^{n+1}\,(2n)!}{(2n+2)! \cdot n! \cdot 3^n} = \frac{(n+1)\cdot 3}{(2n+1)(2n+2)} = \frac{3n+3}{4n^2+6n+2} = \frac{n}{n^2} \cdot \frac{1+\frac{3}{n}}{1+\frac{6}{n}+\frac{2}{n^2}} \to 0 < 1$ $\Rightarrow \sum\limits_{1}^{\infty} a_n$ konv.

$= \frac{1}{n} \to 0$

$\Downarrow$

$\sum a_n$ konv.

Pl $\sum\limits_{1}^{\infty} \frac{3^n}{\binom{2n}{n}} = \sum\limits_{1} \frac{3^n}{\frac{(2n)!}{n!\cdot n!}} = \sum\limits_{1} \frac{(n!)^2 \cdot 3^n}{(2n)!}$ HF!

Pl $\sum\limits_{1}^{\infty} \frac{n^8+3}{2^n+2^{2n}}$ $\quad \left(\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b} \right.$

$a_n \leq \frac{n^8+3n^8}{4^n} = 4 \cdot \frac{n^8}{4^n} = b_n$

$\sqrt[n]{b_n} = \sqrt[n]{4} \cdot \frac{\sqrt[n]{n^8}}{4} \to \frac{1}{4} < 1 \Rightarrow \sum\limits_{1} b_n$ konv $\underset{\text{maj. kr.}}{\Rightarrow} \sum\limits_{1} a_n$ konv.

Pl $\sum\limits_{1}^{\infty} a_n$ konv.-e?

$a_n = \begin{cases} n^2\left(\frac{1}{3}\right)^n, & \text{ha } n \text{ ps} \\ \left(\frac{1}{2}\right)^n, & \text{ha } n \text{ pt} \end{cases}$ $\qquad \sum\limits_{1}^{\infty} a_n = \frac{1}{2} + 2^2 \cdot \left(\frac{1}{3}\right)^2 + \ldots$

$\sqrt[n]{a_n} = \begin{cases} \sqrt[n]{n^2} \cdot \frac{1}{3} \to \frac{1}{3}, & \text{ha } n \text{ ps} \\ \frac{1}{2} \to \frac{1}{2}, & \text{ha } n \text{ pt.} \end{cases}$ $\qquad \overline{\lim} \sqrt[n]{a_n} = \frac{1}{2} < 1 \Rightarrow \sum\limits_{1}^{\infty} a_n$ konv.

M $\sum\limits_{1}^{\infty} \frac{n^4+5}{n^7+4n^3+3}$ $\quad c = 1$ $\quad$ itt nem alkalmazható a gyök- és a hányados kr.

Lineáris rekurzió

$f: \mathbb{N} \to \mathbb{R} \quad f(n)$

Rekurzió: $f(n) = F(f(0), f(1), \ldots, f(n-1))$ $\quad f(0), f(1), \ldots, f(n-1)$ adottak.

Lineáris rekurzió: $\boxed{f(n) = \sum\limits_{i=1}^{m} a_i \cdot f(n-i)}$ $\quad a_i \in \mathbb{R}$ (adott)

$f(0), \ldots, f(m-1)$ adottak

Fibonacci-sorozat

$f(n) = f(n-1) + f(n-2)$ $\quad f(0), f(1)$ adott

Pl $f(0) = 0$

$f(1) = 1$ $\qquad 0, 1, 1, 2, 3, 5, 8, \ldots$

Sejtés: van q^n alakú megoldás ($q \neq 0$)

$f(n) = q^n$

$f(n-1) = q^{n-1}$ $\qquad$ Behelyettesítve: $q^n = q^{n-1} + q^{n-2} \quad / : q^{n-2} \quad (q \neq 0)$

$f(n-2) = q^{n-2}$

$q^2 = q + 1$

$q^2 - q - 1 = 0$

$q_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$\nearrow$
$\searrow$

Anal elm

2008.03.05
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Két megoldást kaptunk,

$$f_1(n) = q_1^n = \left(\frac{1+\sqrt{5}}{2}\right)^n \qquad f_2(n) = q_2^n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Belátható, hogy az összes megoldás $f(n) = C_1 \cdot q_1^n + C_2 \cdot q_2^n$ $\quad C_1, C_2 \in \mathbb{R}$ $\quad$ TB

Általánosságban:

$$f(n) = a_1 f(n-1) + a_2 f(n-2) \qquad a_1, a_2 \text{ adott}$$

$f(0), f(1)$ adott

Megoldásai lineáris teret alkotnak.

$f(n) = q^n$ alakban keressük

$$f(n) = a_1 \cdot q^{n-1} + a_2 q^{n-2} \xRightarrow[q^{n-2} \neq 0]{} q^2 - a_1 q - a_2 = 0 \qquad q_1, q_2 \text{ ~~adott~~ } \text{ küb valós.}$$

$$f(n) = C_1 q_1^n + C_2 q_2^n$$

Ha $f(0), f(1)$ adott, akkor C_1, C_2 meghatározható.

Pl $f(n) = f(n-1) + f(n-2) \qquad f(0) = 0 \qquad f(1) = 1 \qquad 0, 1, 1, 2, 3, \ldots$

$$f(n) = \alpha q_1^n + \beta q_2^n = \alpha \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\left.\begin{array}{ll} n=0 & \alpha + \beta = f(0) = 0 \\ n=1 & \alpha \cdot q_1 + \beta \cdot q_2 = f(1) = 1 \end{array}\right\} \text{ lin. e.r.} \to \alpha = -\beta = \frac{1}{\sqrt{5}}$$

$$\boxed{f(n) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

$q_1 > 1 \quad |q_2| < 1$

$q_2^n \to 0$

Anal elm

2008.03.06
Kónya

(Pl.) $f(n) = 3f(n-1) - 2f(n-2)$

a) $f(0)=0 \quad f(1)=1$

b) $f(0)=1 \quad f(1)=1$

$f(n)=q^n \quad (q\neq 0)$

$$q^n = 3q^{n-1} - 2q^{n-2} \Big/ : q^{n-2} \neq 0$$

$$q^2 = 3q - 2 \Rightarrow q^2 - 3q - 2 = 0 \Rightarrow q_1 = 1 \quad q_2 = 2$$

Az összes megoldás: $f(n) = \alpha + \beta \cdot 2^n \qquad \alpha, \beta \in \mathbb{R}$

$n=0 \quad \alpha + \beta 2^0 = f(0)$

$n=1 \quad \alpha + \beta \cdot 2^1 = f(1)$ $\Big\} \Rightarrow \alpha = 2f(0) - f(1)$, $\beta = f(1) - f(0)$

a) $\alpha = -1,\ \beta = 1 \qquad f(n) = 2^n - 1$

b.) $\alpha = 1,\ \beta = 0 \qquad f(n) = 1$

(Pl) a) Adja meg az
$$f(n) = \frac{9}{2} f(n-1) - 2f(n-2)$$
rekurziót kielégítő sorozatokat.

b) Írja fel az $f(0)=0$, $f(1)=7$ kezdőértékhez tartozó megoldást

c) Van-e $\sigma(n)$ nagyságrendű megoldása?

a) $f(n) = q^n \mid q\neq 0 \qquad q^2 = \frac{9}{2}q - 2 \Rightarrow q^2 - \frac{9}{2}q + 2 = 0 \Rightarrow q_1 = 4 \quad q_2 = \frac{1}{2}$

$f(n) = C_1 4^n + C_2 \left(\frac{1}{2}\right)^n \qquad C_1, C_2 \in \mathbb{R}$

b). $n=0 \quad C_1 + C_2 = 0$

$n=1 \quad 4C_1 + \frac{1}{2}C_2 = 7$ $\Big\} \Rightarrow C_1 = 2 \quad C_2 = -2 \qquad f(n) = 2\cdot 4^n + 2\cdot\left(\frac{1}{2}\right)^n$

c.) $a_n = \sigma(b_n)$

$\exists C>0 \qquad |a_n| \leq C\cdot|b_n| \quad n>N$

$C_1 = 0$, C_2 tetszőleges $\quad (\sigma(n))$

Függvénysorok

Ismétlés:

(D) $a_n \to A$: $\forall\, \varepsilon>0$-hoz $\exists\, N(\varepsilon)$: $|a_n - A| < \varepsilon$, ha $n > N(\varepsilon)$

(T) Cauchy-féle konv. krit.: $\forall\, \varepsilon>0$-hoz $\exists\, M(\varepsilon)$: $|a_m - a_n| < \varepsilon$, ha $m,n > N(\varepsilon)$ $\quad m>n$, $m = n+k$

$|a_{n+k} - a_n| < \varepsilon$, ha $n > M(\varepsilon)$, $k\in\mathbb{N}^+$

(D) $\sum_{k=0}^{\infty} a_k = s$, ha $\exists \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = s$ részletösszeg

(T) Cauchy: $|s_{n+k} - s_n| = |a_{n+1} + a_{n+2} + \dots + a_{n+k}| < \varepsilon$, ha $n > M(\varepsilon)$, $k\in\mathbb{N}^+$

$f_0(x) + f_1(x) + \ldots + f_n(x) + \ldots = \sum_{k=0}^{\infty} f_k(x)$ $\qquad D = \bigcap_{k=0}^{\infty} D_{f_k}$ (ÉT)

Pl: $1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x} = s(x)$ $\qquad$ H: KT (konvergencia tartomány)

$|x| < 1$ $\qquad H \subseteq D$

$H = (-1, 1)$

$x_0 \in H$ (K.T.)

$$\sum_{k=0}^{\infty} f_k(x_0) = \lim_{n \to \infty} \sum_{k=0}^{n} f_k(x_0) = \lim_{n \to \infty} s_n(x_0) = s(x_0)$$

(D) $\forall \varepsilon > 0$ - hoz $\exists N(\varepsilon, x_0)$

$|s_n(x_0) - s(x_0)| < \varepsilon$, ha $n > N(\varepsilon, x_0)$ pontonkénti konvergencia

---

Öröklődnek-e bizonyos tulajdonságok $s(x)$-re?

$\sum_{k=0}^{\infty} s_k(x) = s(x)$

folytonos?

$\frac{d}{dx} \sum^{\infty} f_k(x) \overset{?}{=} \sum^{\infty} \frac{d}{dx} f_k(x)$

$\int_a^b \sum^{\infty} f_k(x)\, dx \overset{?}{=} \sum^{\infty} \int_a^b f_k(x)\, dx$

jogos kérdések, mert lehet ellenpéldát mutatni

Pl (ellenpélda)

$$\sum_0^{\infty} \frac{\sin^2 x}{(1+x^2)^n} = \sin^2 x + \sin^2 x \frac{1}{1+x^2} + \sin^2 x \frac{1}{(x^2+1)^2} + \ldots$$

folyt.

$$s(x) = \begin{cases} 0, & \text{ha } x = 0 \\ \sin^2 x \dfrac{1}{1 - \frac{1}{1+x^2}} = \dfrac{\sin^2 x}{x^2} \cdot (1+x^2), & \text{ha } x \neq 0 \end{cases}$$

$$\lim_{x \to 0} s(x) = \lim_{x \to 0} \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1} \cdot \underbrace{(1+x^2)}_{\to 1} = 1 \neq s(0) = 0$$

s nem folytonos $x=0$-ban

Egyenletes konvergencia

(D) A fv. sor egyenletesen konv. a H halmazon, ha $\exists$ H-n közös $N(\varepsilon)$, tehát $\forall \varepsilon > 0$ - hoz $\exists N(\varepsilon)$, hogy $|s_n(x) - s(x)| < \varepsilon$, ha $n > N(\varepsilon)$ $\forall x \in H$-ra.

Pl: $1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x} = s(x)$ $\qquad H = (-1, 1)$

Biz. be, hogy a fv.sor $\forall [-a, a] \subset (-1, 1)$ intervallumon egyenletesen konvergens. $\qquad$ $-1 \quad -a \quad 0 \quad a \quad 1$ $\qquad 1 > a > 0$

$s(x) = \sum_{k=0}^{\infty} x^k$

$$|s_n(x) - s(x)| = |s(x) - s_n(x)| = \left|\sum_{k=n+1}^{\infty} x^k\right| = \frac{x^{n+1}}{1-x} = \frac{|x|^{n+1}}{1-x} \leq \frac{a^{n+1}}{1-a} < \varepsilon$$

$x \in [-a, a]$ $\quad |x| \leq a$

$a^{n+1} < \varepsilon(\overset{>0}{1-a})$

$(n+1)\ln a < \ln(\varepsilon(1-a))$ $\quad / : \ln a < 0$

$n+1 > \frac{\ln(\varepsilon(1-a))}{\ln a}$

$n > \frac{\ln(\varepsilon(1-a))}{\ln a} - 1$

$N(\varepsilon) = \left[\frac{\ln(\varepsilon(1-a))}{\ln a} - 1\right]$

Tehát $\forall$ $[-a, a] \subset (-1, 1)$ intervallumon egyenletes a konvergencia, de belátható, hogy $[-1, 1]$-en NEM.

($\forall [a,b] \subset (-R, R)$-en egyenl. konv.)

Anal elm.

2008.03.12.
Kónya

Konzultáció: márc. 17 (H) $17-19^{00}$ V2.530 (ápr. 15.)

(Pl) Egyenletesen konv-e?

$$\sum_{n=1}^{\infty} \frac{n^3\cdot 2^{-x^2}\operatorname{arctg} n^7x}{(2+x^2)^n}$$

$\sum_{1}^{\infty} f_n(x)$. $|f_n(x)| \le b_n$ $\forall n$-re és $\forall x\in H$-ra és $\sum_{1}^{\infty} b_n$ konv. num. sor $\Rightarrow$

$\Rightarrow \sum_{1}^{\infty} f_n(x)$ egyenletesen konv H-n

$$|f_n(x)| = \frac{n^3\cdot 2^{-x^2}\,|\operatorname{arctg} n^7x|}{(2+x^2)^n} \le \frac{n^3\cdot 1\cdot \frac{\pi}{2}}{2^n} = \frac{\pi}{2}\cdot\frac{n^3}{2^n} = b_n$$

$\frac{\pi}{2}\sum^{\infty}\frac{n^3}{2^n}$ konv, mert $\sqrt[n]{b_n} = \frac{(\sqrt[n]{n})^3}{2} \to \frac{1}{2} < 1 \Rightarrow$ a sor egyenletesen konv. $(-\infty,\infty)$-en

(Pl) $\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ Egyenletesen konv-e $[-2,2]$-n?

$$|f_n(x)| = \left|\frac{x}{3}\right|^n \le \left(\frac{2}{3}\right)^n = b_n$$

$\sum b_n = \sum\left(\frac{2}{3}\right)^n$ konv. geom. sor. $\left(0<q=\frac{2}{3}<1\right)$

$\Downarrow$

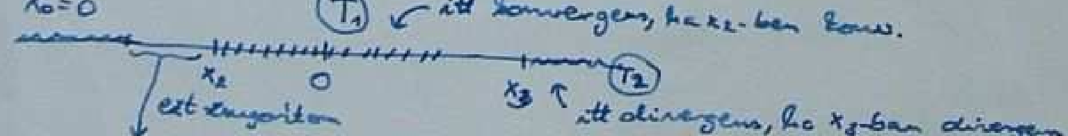
$\sum f_n(x)$ egyenl. konv. $H=[-2,2]$-n

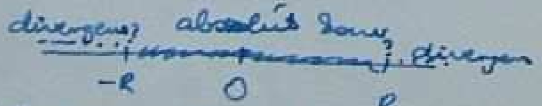
Hatványsorok

$$\sum_{k=0}^{\infty} a_k\cdot x^k = \sum_{n=0}^{\infty} a_n\cdot x^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)^n$$

$x_0 = 0$





(véges) $\infty > R > 0$ konv. sugár.

$R = 0$: csak az alappontban konv

$R = \infty$: $\forall x$-re konv.

$(x_0+R,\ x_0-R)$-ben konv $\quad |x-x_0|<R$

azon kívül div $\quad |x-x_0|>R$

végpontokat nem tudom $\quad |x-x_0|=R$

$R = ?$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$\boxed{\lim \sqrt[n]{|a_n|} := \alpha}$$

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| := \alpha}$$

$$„R = \frac{1}{\alpha}"$$

Pl $\sum_{n=1}^{\infty} \frac{n-1}{n+2}(x+2)^n \qquad a_n = \frac{n-1}{n+2} \qquad x_0 = -2$

$-3 \quad -2 \quad -1$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n}{n+3}}{\frac{n-1}{n+2}} = \frac{n^2+2n}{n^2+2n-3} = \underbrace{\frac{n^2}{n^2}}_{=1} \cdot \underbrace{\frac{1+\frac{2}{n}}{1+\frac{1}{n}-\frac{3}{n^2}}}_{\to 1} = 1 = \alpha \Rightarrow = \frac{1}{R} \Rightarrow R = 1$$

$|x+2| < 1$ abs. konv.

$|x+2| > 1$ div.

$|x+2| = 1$?

$x=-1: \sum_{1}^{\infty} \frac{n-1}{n+2} \cdot \underbrace{1^n}_{=1} = \sum \underbrace{\frac{n-1}{n+2}}_{\to 1 \neq 0}$ div, nem teljesül a szüks. felt.

$x=-3: \sum_{1}^{\infty} \underbrace{\frac{n-1}{n+2} \cdot (-1)^n}_{\not\to 0}$ div.

$H = (-3, -1)$

Pl $\sum_{n=1}^{\infty} \underbrace{\frac{n-1}{(n+2)!}}_{a_n} (x-3)^n \qquad x_0 = 3$

3

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{(n+3)!} \cdot \frac{(n+2)!}{n-1} = \frac{n}{(n+3)(n-1)} = \frac{1}{\underbrace{\left(1+\frac{3}{n}\right)}_{1}\underbrace{(n-1)}_{\infty}} \to 0 = \alpha \qquad R = \infty$$

$\Downarrow$

$\forall x$-re konv.

Pl $\sum \underbrace{\frac{(-2)^n}{\sqrt{n}}}_{a_n} (x-1)^n \qquad x=1$

$\frac{1}{2} \quad 1 \quad \frac{3}{2}$ (? ?)

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n}{\sqrt{n}}} = \frac{2}{\sqrt[n]{\sqrt{n}}} \to 2 = \alpha = \frac{1}{R} \Rightarrow R = \frac{1}{2}$$

$x = \frac{1}{2} \quad \sum \frac{(-2)^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{\frac{1}{2}}}$ div $\qquad \alpha = \frac{1}{2} < 1$

$x = \frac{3}{2} \quad \sum \frac{(-2)^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum \frac{(-1)^n}{\sqrt{n}}$ konv. Leibniz sor.

$H = \left(\frac{1}{2}, \frac{3}{2}\right]$

<u>Anal elm</u>

2008.03.13.
Kötya

(Pl) $\sum_1^\infty n^3\left(-\frac{x}{7}\right)^n = \sum a_n x^n = \sum \underbrace{n^3\left(-\frac{1}{7}\right)^n}_{a_n} \cdot x^n \quad x_0=0$ K.T.?

? konv ? (-7, 0, 7)

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|n^3\cdot\left(-\frac{1}{7}\right)^n\right|} = \lim_{n\to\infty}\left(\sqrt[n]{n}\right)^3\cdot\frac{1}{7} = \frac{1}{7} \Rightarrow R=7$

$x=7: \quad \sum n^3\cdot(-1)^n\left(\frac{1}{7}\right)^n\cdot(7)^n = \sum (-1)^n\cdot n^3$ div. $\not\to 0$

$x=-7: \quad \sum n^3\cdot(-1)^n\left(\frac{1}{7}\right)^n\cdot\underbrace{(-7)^n}_{(-1)^n(7)^n} = \sum n^3$ div $\not\to 0$

$K.T = (-7,7)$

(Pl) $\sum \underbrace{\left(1+\frac{1}{n}\right)^{n^2}}_{a_n} x^n$ $R=?$

$\sqrt[n]{|a_n|} = \left|\left(1+\frac{1}{n}\right)^n\right| \to e \Rightarrow R=\frac{1}{e}$ ($-\frac{1}{e}$, 0, $\frac{1}{e}$)

(Pl) $\sum_1^\infty \underbrace{\frac{n^n}{n!}}_{a_n} x^n \quad (x_0=0)$ $R=?$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \Rightarrow R=\frac{1}{e}$

(Pl) $\sum \frac{(2-4x)^n}{n+1} = \sum_0^\infty a_n(x-x_0)^n = \sum \underbrace{\frac{(-4)^n}{n+1}}_{\frac{(-1)^n\cdot 4^n}{n+1}=a_n}\left(x-\frac{1}{2}\right)^n \quad x_0=\frac{1}{2}$ K.T.=?

$\sqrt[n]{|a_n|} = \frac{4}{\sqrt[n]{n+1}} \to 4 = \frac{1}{R} \Rightarrow R=\frac{1}{4}$, $\sqrt[n]{n+1}\to 1$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}}{n+2}\cdot\frac{n+1}{4^n} = 4\cdot\frac{n+1}{n+2} = 4\cdot\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 4$

Mert: $\underset{\downarrow 1}{\sqrt[n]{1}} \le \underset{\downarrow 1}{\sqrt[n]{n+1}} \le \sqrt[n]{n+n} = \sqrt[n]{2}\cdot\sqrt[n]{n} \to 1$

? ? ($\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$)

K.T: $\left(\frac{1}{4},\frac{3}{4}\right]$

$x=\frac{1}{4} \quad \sum_1^\infty (-1)^n\frac{4^n}{n+1}\cdot\left(-\frac{1}{4}\right)^n = \sum_1^\infty \frac{1}{n+1} = \sum_2^\infty \frac{1}{n}$ div (harmonikus sor)

$x=\frac{3}{4} \quad \sum_1 (-1)^n\cdot\frac{4^n}{n+1}\cdot\left(\frac{1}{4}\right)^n = \sum (-1)^n\frac{1}{n+1}$ felt. konv. (Leibniz-sor)

A.K.T (abszolút): $\left(\frac{1}{4},\frac{3}{4}\right)$

(Pl.) $\sum^\infty \frac{4^n}{n}x^{2n}$

$\left(a_n = \begin{cases}0, \text{ ha } n \text{ pt}\\ \frac{4^{n/2}}{n/2}, \text{ ha } n \text{ ps}\end{cases} \quad \sqrt[n]{|a_n|}\right)$ $\quad \sqrt[2n]{\frac{4^n}{n}} = \sqrt{\sqrt[n]{\frac{4^n}{n}}}$

Jobb megoldás:

$u=x^2$

$\sum^\infty \underbrace{\frac{4^n}{n}}_{a_n} u^n \quad \sqrt[n]{|a_n|} = \frac{4}{\sqrt[n]{n}} \to 4 = \frac{1}{R} \Rightarrow R=\frac{1}{4}$ ($-\frac{1}{4}$, 0, $\frac{1}{4}$, u)

$u=-\frac{1}{4} \quad \sum \frac{4^n}{n}\left(-\frac{1}{4}\right)^n = \sum (-1)^n\frac{1}{n}$ konv (Leibniz sor)

$u=\frac{1}{4} \quad \sum \frac{4^n}{n}\left(\frac{1}{4}\right)^n = \sum \frac{1}{n}$ div

K.T: $\quad -\frac{1}{4} \le u < \frac{1}{4} \Rightarrow -\frac{1}{4} \le \boxed{x^2 < \frac{1}{4}} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$ ($-\frac{1}{2}$, 0, $\frac{1}{2}$)

ez fölösleges

(Pl) a) $\sum^{\infty} \frac{(x+2)^n}{2n+8^n}$ b) $\sum \frac{(x+2)^{2n}}{2n+8^n}$ $(x+2)^2 = u$

(Pl) $f(x) = \sum_{n=1}^{\infty} (n+1)\, x^n = ?$ $g(x) = \sum_{n=1}^{\infty} (n+1)\, x^{n+2} = ?$

$-R \quad 0 \quad R$ $[0,x] \subset (-R,R)$

$$f_1(x) = \int_0^x f(x)\,dx = \int_0^x \sum_{n=1}^{\infty} (n+1)\, x^n\,dx = \sum_{n=1}^{\infty} \int_0^x (n+1)\, x^n\,dx = \underbrace{\sum_{n=1}^{\infty} (n+1)\frac{x^{n+1}}{n+1}}_{x^2+x^3+\dots}\Bigg|_0^x = \sum_{n=1}^{\infty} x^{n+1} - 0 = \frac{x^2}{1-x}$$

geom sor, konv, ha $R = 1$, mert nem változik integrálásnál, deriválásnál.

$$f_1(x) = \frac{x^2}{1-x}$$

$$f_1'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x) = \left(\frac{x^2}{1-x}\right)' = \frac{2x(1-x) - x^2(-1)}{(1-x)^2}$$

$R=1$

$|x|<1$

$$g(x) = \sum_{n=1}^{\infty} (n+1)\, x^2 x^n \underset{x\neq 0}{=} x^2 \underbrace{\sum_{n=1}^{\infty} (n+1)\, x^n}_{f(x)} = \dots$$

$g(0) = 0$ (behelyettesítéssel)

2009.03.19.
Kónya

Anal elm.

(Pl) $f(x) = \sum_{k=1}^{\infty} (k+1)x^k = \sum_{m=1}^{\infty} (m+1)x^m = \_\_\_ = \frac{-x^2+2x}{(1-x)^2}$ $\quad R=1$

$h(x) = \sum_{k=0}^{\infty} (k+2)x^k = ?$

1. mo: $h(x) = \underbrace{\sum_{k=0}^{\infty}(k+1)x^k}_{f(x)+1} + \underbrace{\sum_{k=0}^{\infty} x^k}_{\frac{1}{1-x}} = \dots$

2. mo $h(x) = \sum_{k=0}^{\infty}(k+2)x^k$

$h_1(x) = x \cdot h(x)$

$h(0) = 2 + 3x + 4x^2 + 5x^3 + \dots \Big|_{x=0} = 2$

$h_1(x)\Big|_{x\neq 0} = x \cdot \sum_{k=0}^{\infty}(k+2)x^k = \sum_{k=0}^{\infty}(k+2)x^{k+1}$

$h_2(x) = \int_0^x h_1(x) = \int_0^x \sum_{k=0}^{\infty}(k+2)x^{k+1}dx = \sum_{k=0}^{\infty}\int_0^x (k+2)x^{k+1}dx =$

$[0,x] \subset [-R,R]$

$= \sum_{k=0}^{\infty} k+2 \, \frac{x^{k+2}}{k+2}\Big|_0^x = \sum_{k=0}^{\infty} x^{k+2} - 0^{k+2} = \frac{x^2}{1-x}$ $\quad R=1$

$h_2(x) = \frac{x^2}{1-x}$

$h_2'(x) = \left(\frac{x^2}{1-x}\right)' = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{-x^2+2x}{(1-x)^2} = h_1(x)$

$h(x) = \frac{h_1(x)}{x} = \frac{2x - x^2}{(1-x)^2 x} = \underline{\underline{\frac{2-x}{(1-x)^2}}}$ $\quad R=1$

(Pl) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+2} = ? = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \dots =$

$f(0) = \frac{1}{2}$

$x \neq 0$ $\quad f_1(x) = x^2 \cdot f(x) = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n+2} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2}$

$f_1'(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2} = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{x^{n+2}}{n+2} = \sum_{n=0}^{\infty} \cancel{(n+2)}\frac{x^{n+1}}{\cancel{n+2}} = \frac{x}{1-x}$

$R=1$

$\left(\begin{array}{l} \int f_1'(x)\,dx = \int \frac{x}{1-x}\,dx \\ f_1(x) = \dots + C \end{array} \right)$ $\quad \int_0^x f_1'(x)\,dx = f_1(x)\Big|_0^x = f_1(x) - \underbrace{f_1(0)}_{0} = f_1(x) = \int_0^x \frac{x}{1-x}\,dx = -1\int_0^x \frac{1-x-1}{1-x}\,dx =$

$= -\int_0^x 1 - \frac{1}{1-x}\,dx = \cancel{[x + \ln(1-x)]_0^x} = -(x + \ln(1-x))\Big|_0^x = -(x + \ln(1-x))$

$f_1(x) = -(x + \ln(1-x)) = x^2 f(x)$

$f(x) = -\frac{x + \ln(1-x)}{x^2}$

$f(x) = \begin{cases} -\frac{x+\ln(1-x)}{x^2}, & \text{ha } x \neq 0 \\ \frac{1}{2}, & \text{ha } x = 0 \end{cases}$

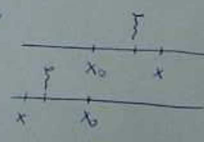
HF $\lim_{x\to 0} f(x) = ?$

Taylor-polinom:

$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$T_n(x_0) = f(x_0)$

$f(x) \approx T_n(x)$ $\qquad H = R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-x_0)^{n+1}$



(Pl) $f(x) = e^{\frac{x}{2}} - 3x^2 + 5$

a) $T_3(x) = ?$, ha $x_0 = 0$

b.) $x \in (0,1]$ esetben $f(x) \approx T_3(x)$ közelítéssel élünk. Adjon becslést az elkövetett hibára!

$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 6 + \frac{1}{2}x + (\frac{1}{4} - 6)\frac{1}{2}x^2 + \frac{1}{8}\frac{1}{6}x^3$

$f'(x) = \frac{1}{2} \cdot e^{\frac{x}{2}} - 6x$

$f''(x) = \frac{1}{4}e^{\frac{x}{2}} - 6$

$f'''(x) = \frac{1}{8}e^{\frac{x}{2}}$

$f^{IV}(x) = \frac{1}{16}e^{\frac{x}{2}}$

b) $f(x) \approx T_3(x)$

$f(0{,}1) \approx T_3(0{,}1) = 6 + \frac{1}{2}\,0{,}1^2 \dots$

$H = f(0{,}1) - T_3(0{,}1) = \frac{f^{(4)}(T)}{4!}\, x^4 \Big|_{x=0{,}1} = \frac{\frac{1}{16}\cdot e^{T/2}}{24} \cdot 0{,}1^4$

$0 < H < \frac{1}{16\cdot 24} \cdot 0{,}1^4 \cdot \underbrace{e^{T/2}}_{< e^1 < 3} < \frac{0{,}1^4}{16\cdot 24} \cdot 3$

Anal elm

(Pl) $f(x)=x^3-8$ T. sor a) $x_0=0$ b.) $x_0=1$

a) $T(x)=-8+x^3=f(x)$

b.) $f(x)=(x-1)^3+3x^2-3x+1-8=(x-1)^3+3(x-1)^2+\underbrace{6x-3-3x-7}_{3x-10}=(x-1)^3+3(x-1)^2+3(x-1)-7=T(x)$

vagy $T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-x_0)^n$

$f(x)=x^3-8$ $f(1)=-7$

$f'(x)=3x^2$ $f'(1)=3$

$f^{(k)}(x)=0$ $k\geq 4$

(Pl) a) Tudjuk, hogy $\operatorname{arctg} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$; $R=1$

Írja fel az $f(x)=\operatorname{arctg}\frac{x^3}{8}$ T. sorát, ha $x_0=0$! $R=?$

b.) Adjon becslést $\int_0^1 \operatorname{arctg}\frac{x^3}{8}\,dx$ értékére, ha az integrálandó fv-t T_9-cel közelítjük.

a.) $f(x)=\left(\frac{x^3}{8}\right)-(\ )^3\cdot\frac{1}{3}+\frac{(\ )^5}{5}-\ldots=\frac{x^3}{8}-\frac{x^9}{3\cdot 8^3}+\frac{x^{15}}{5\cdot 8^5}-\ldots$

$\operatorname{arctg} u = u-\frac{u^3}{3}+\frac{u^5}{5}-\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{2n+1}$ $T_9(x)$ $|u|=\left|\frac{x^3}{8}\right|=\frac{|x^3|}{8}<1 \Rightarrow |x|^3<8$

$|u|<1$ $(|u|\leq 1)$ $|x|<2 \Rightarrow R=2$

$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{x^3}{8}\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)8^{2n+1}}x^{6n+3}$

b) $\int_0^1 \operatorname{arctg}\frac{x^3}{8}=\int_0^1\left(\frac{x^3}{8}-\frac{x^9}{3\cdot 8^3}+\frac{x^{15}}{5\cdot 8^5}-\ldots\right)dx=\left[\frac{x^4}{8\cdot 4}-\frac{x^{10}}{3\cdot 8^3\cdot 10}+\frac{x^{16}}{5\cdot 8^5\cdot 16}-\ldots\right]_0^1=$ $\left|\frac{x^3}{8}\right|<1\ldots$ $R=2$ $[0,1]\subseteq(-2,2)$

$=\underbrace{\frac{1}{32}-\frac{1}{30\cdot 8^3}}_{:=a}+\underbrace{\frac{1}{80\cdot 8^5}}_{:=b}-\ldots\approx a$

Leibniz-sor $|H|\leq|b|=b=\frac{1}{80\cdot 8^5}$

$T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x^1}{1!}+\frac{x^2}{2!}+\ldots$ $x\in\mathbb{R}$ $(R=\infty)$

$\operatorname{sh} x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ $x\in\mathbb{R}$

$\operatorname{ch} x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$ $x\in\mathbb{R}$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$ $x\in\mathbb{R}$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$ $x\in\mathbb{R}$

Pe $f(x)= x^3 \sqrt[3]{e^x}$; $x_0=0$

$e^u = 1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\dots \quad u\in\mathbb{R}$ $\quad u=\frac{c_1 x^k}{c_2}$

$\sqrt[3]{e^x} = e^{\frac{x}{3}} = 1+\frac{x}{3}+\frac{x^2}{2!\cdot 3^2}+\frac{x^3}{3!\cdot 3^3}+\frac{x^4}{4!\cdot 3^4}+\dots \quad R=\infty$

$f(x)= x^3+\frac{1}{3}\cdot x^4+\frac{1}{2!\cdot 3^2}\cdot x^5+\frac{1}{3!\cdot 3^3}\cdot x^6+\dots \quad R=\infty$

$f^{(100)}(0)=?$

$f(x)=\sum_{k=0}^{\infty} a_k\cdot x^k$ = hatványsor $\quad a_k=\frac{f^{(k)}(0)}{k!} \Rightarrow f^{(k)}(0)=k!\,a_k$

$f^{(100)}(0)=100!\cdot \underbrace{a_{100}}_{x^{100} \text{ együtthatója}} = 100!\cdot\frac{1}{97!\cdot 3^{97}}$

Pe. $f(x)=e^{9x}$

a) $x_0=0$ $\quad$ b) $x_0=-3$

$e^u = 1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\dots=\sum_{n=0}^{\infty}\frac{u^n}{n!} \quad u\in\mathbb{R}$

a) $x-x_0=x$

$e^{9x}=1+9x+\frac{(9x)^2}{2!}+\frac{(9x)^3}{3!}+\dots=\dots$ vagy $e^{9x}=\sum_{n=0}^{\infty}\frac{u^n}{n!}\Big|_{u=9x}=\sum_{n=0}^{\infty}\frac{(9x)^n}{n!}=\sum_{n=0}^{\infty}\frac{9^n}{n!}\cdot x^n$

$x\in\mathbb{R}$ $(R=\infty)$

b) $x_0=-3$: $x-x_0=x-(-3)=x+3$

$e^{9x}=e^{9(x+3)-27}=e^{-27}\cdot e^{9(x+3)}=e^{-27}\cdot\sum_{n=0}^{\infty}\frac{(9\cdot(x+3))^n}{n!}=\sum_{n=0}^{\infty}\underbrace{e^{-27}\cdot\frac{9^n}{n!}}_{a_n}(x+3)^n \quad x\in\mathbb{R}$

Pe $f(x)=\sin 5x^3$

a.) T. sor, $x_0=0$

b.) $\int_0^1 \sin 5x^3\,dx\approx ?$

$\left(e^u=1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\dots\right)$

$\sin u = u-\frac{u^3}{3!}+\frac{u^5}{5!}-\dots \quad u\in\mathbb{R}$

$\sin(5x^3)=(5x^3)-\frac{(5x^3)^3}{3!}+\frac{(5x^3)^5}{5!}-\dots=$

$=5x^3-\frac{5^3}{3!}x^9+\frac{5^5}{5!}x^{15}-\dots \quad x\in\mathbb{R}$

b) $\int_0^1 \sin 5x^3\,dx=\int_0^1 5x^3-\frac{5^3}{3!}x^9+\frac{5^5}{5!}x^{15}-\dots\,dx=\frac{5x^4}{4}-\frac{5^3}{3!\cdot 10}\cdot x^{10}+\frac{5^5}{5!\,16}\cdot x^{16}-\dots\Big|_0^1=$

$=\underbrace{\boxed{\frac{5}{4}-\frac{5^3}{3!\cdot 10}}}_{:=a}+\underbrace{\boxed{\frac{5^5}{5!\,16}}}_{:=b}-\dots\approx a$ | belátható, hogy Leibniz sor.

$|H|\leq b$

Anal elm

2008.03.27
Kónya

(Pl) $\lim\limits_{x\to 0} \dfrac{\cos 5x^2-1}{x^2\cdot \sin 2x^2} = ?$

$e^u = 1+u+\dfrac{u^2}{2!}+\dfrac{u^3}{3!}+\dots$

$\cos 5x^2 = 1-\dfrac{(5x^2)^2}{2!}+\dfrac{(5x^2)^4}{4!}-\dfrac{(\ )^6}{6!}+\dots = 1-\dfrac{5^2}{2!}x^4+\dfrac{5^4}{4!}x^8$

$\sin 2x^2 = 2x^2-\dfrac{(2x^2)^3}{3!}+\dfrac{(2x^2)^5}{5!}-\dots = 2x^2-\dfrac{2^3}{3!}x^6+\dfrac{2^5}{5!}x^{10}-\dots$

$\lim\limits_{x\to 0} \dfrac{-\frac{5^2}{2!}x^4+\dots}{2x^4-\frac{2^3}{3!}x^8+\frac{2^5}{5!}x^{12}-\dots} = \lim\limits_{x\to 0} \dfrac{x^4}{x^4}\cdot\dfrac{-\frac{5^2}{2!}+c_1x^4+c_2x^8+\dots}{2-d_1x^4+d_2x^8+\dots} = \dfrac{-\frac{5^2}{2!}}{2}$

(Pl) $f(x) = xe^{x^2}$ $\quad f^{(100)}(0) = ? \quad f^{(101)}(0) = ?$

$f(x) = x\sum\limits_{n=0}^{\infty}\dfrac{u^n}{n!}\Big|_{u=x^2} = x\sum\limits_{n=0}^{\infty}\dfrac{x^{2n}}{n!} = \sum \dfrac{1}{n!}x^{2n+1} \qquad R=\infty$

$f(x) = \sum\limits_{k=0}^{\infty} a_k\cdot x^k \qquad a_k = \dfrac{f^{(k)}(0)}{k!} \Rightarrow f^{(k)}(0) = k!\, a_k$

$\left(f(x) = x+0x^2+x^3+0x^4+\dfrac{1}{2!}x^5+\dots\right)$

$f^{(100)}(0) = 100!\, a_{100} = 0$
x^{100} el sem fordul... x^{100} el jön $=0$

$f^{(101)}(0) = 101!\,\underbrace{a_{101}}_{x^{101}} = 101!\,\dfrac{1}{50!}$

$2n+1 = 101 \Rightarrow n = 50$

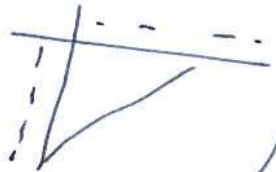
(Pl) g $f(x) = e^{-7x}\,\mathrm{sh}\, 4x \qquad$ T. sor $x_0=0$

a) $f(x) = e^{7x}\left(\dfrac{e^{4x}-e^{-4x}}{2}\right) = \dfrac{1}{2}\cdot e^{11x}-e^{3x} = \dots \qquad R=\infty$

b) $g(x) = e^{-x}\sin 2x \qquad$ T. sor $x_0=0$

b.) $e^{-x} = 1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\dots$

$\sin 2x = 2x-\dfrac{(2x)^3}{3!}+\dfrac{(2x)^5}{5!}-\dots$



(Pl) $f(x) = e^{x^2}$ Írja fel egy prim. sorfejtését.

$F(x) = \int\limits_0^x e^{x^2}dx = \int\limits_0^x e^{t^2}dt = \int\limits_0^x \sum\limits_{n=0}^{\infty}\dfrac{u^n}{n!}\Big|_{u=t^2}dt = \int\limits_0^x\sum\limits_{n=0}^{\infty}\dfrac{t^{2n}}{n!}dt \overset{R=\infty}{=} \sum\limits_{n=0}^{\infty}\dfrac{1}{n!}\int\limits_0^x t^{2n}dt =$

$= \sum\limits_{n=0}^{\infty}\dfrac{1}{n!}\,\dfrac{t^{2n+1}}{2n+1}\Big|_0^x = \sum\limits_{n=0}^{\infty}\dfrac{1}{n!}\cdot\dfrac{x^{2n+1}}{2n+1} \qquad R=\infty$

Binomiális sor

$f(x) = (1+x)^{\alpha} \quad \alpha \in \mathbb{R}$

Ha $\alpha = -1$ $\quad f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ $\quad$ K.T.: $|x| < 1$

(geometriai sor

Ha $\alpha \neq -1$: binomiális sor

2008 04.02
Kónya

Többváltozós fv.

$f: \mathbb{R}^m \to \mathbb{R}$ $\qquad y = f(\underline{x}) = f(\underline{r})$ skalár – vektoros

$y = f(x_1, x_2, \dots x_m)$

$\underline{x} = (x_1, x_2, \dots x_m) = \underline{r}$

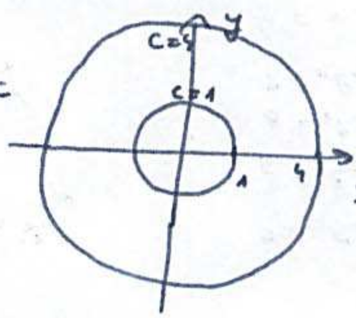
Spec. jelölések: $\quad m=2 \quad z = f(x,y)$

$\qquad m=3 \quad u = f(x,y,z)$

Ábrázolás: – szintvonallal
– síkmetszettel

(D) Szintalakzat: $f(\underline{x}) = c$

pl: $\underbrace{z = x^2+y^2}_{f(x,y)}$ $\quad$ szintalakzat (szintvonalak): $x^2+y^2=c$



pl: $\boxed{z = x^2+y^2}$

$z=0 \quad x^2+y^2=0$

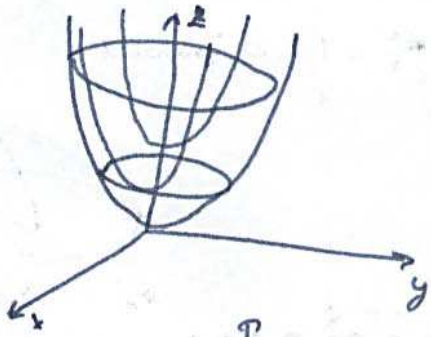
$z=1 \quad x^2+y^2=1$

$z=4 \quad x^2+y^2=4$

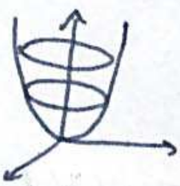
$x=0 \quad z=y^2$

$x=\pm 1 \quad z=1+y^2$

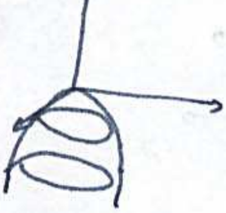
forgási paraboloid



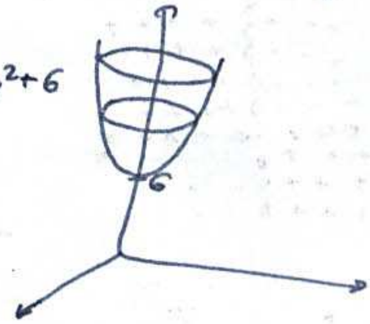
$z = 6x^2+6y^2$



$z = -2x^2-2y^2 = -2(x^2+y^2)$



$z = x^2+y^2+6$



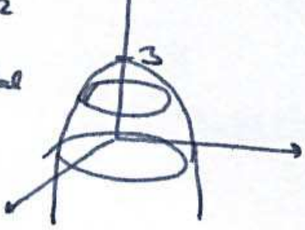
$z = 3-x^2-y^2$

Metszet: xy síkkal
$z=0$

$0 = 3-x^2-y^2$

$x^2+y^2 = (\sqrt{3})^2$

kör

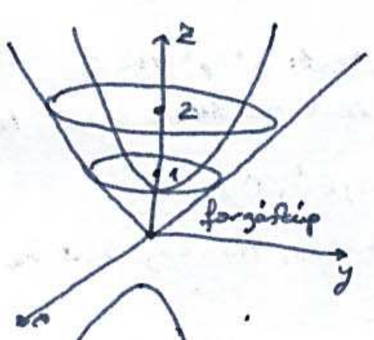


Pl: $\boxed{z = \sqrt{x^2+y^2}}$

$z=0 \quad \sqrt{x^2+y^2}=0$

$z=1 \quad \sqrt{x^2+y^2}=1$
$\qquad x^2+y^2=1^2$

$z=2 \quad \sqrt{x^2+y^2}=2$
$\qquad x^2+y^2=2^2$



$x=0: \; z=\sqrt{y^2} = |y|$

$x=\pm 1: \; z=\sqrt{1+y^2}$

$z^2-y^2=1$ hiperbola

– ha a fv x^2 és y^2 fv-e ⇒ mindig forgásfelület

(Pe) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+3}{x^2+y^2+1} = 3$

(Pe) $\lim\limits_{(x,y)\to(1,3)} \dfrac{9x^2-y^2}{3x-y} \underset{\frac{0}{0}}{=} \lim\limits_{(x,y)\to(1,3)} \dfrac{3x-y}{3x-y}(3x+y) = 6$

(Pe) Hol folytonos

$f(x,y) = \begin{cases} x^2 y \cdot \operatorname{arctg} \frac{1}{y}, & \text{ha } y \neq 0 \\ 0, & \text{ha } y = 0 \end{cases}$

$(a,0)$-ban kell megnézni, $a \in \mathbb{R}$

a tengely kivételével folytonos

$\lim\limits_{(x,y)\to(a,0)} \underset{\substack{\downarrow \\ a^2}}{x^2}\, \underset{\substack{\downarrow \\ 0}}{y}\, \underbrace{\operatorname{arctg}\frac{1}{y}}_{\text{korl}} = 0$

Tehát f mindenütt folytonos

(Pe) $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{ha } (x,y) \neq (0,0) \\ 0, & \text{ha } (x,y) = (0,0) \end{cases}$

Hol folyt? → origó kivételével folyt. fv-k összetétele folyt.
origót meg kell nézni.

$\lim\limits_{\substack{(x,y)\to(0,0) \\ \underline{x}\to\underline{0}}} \dfrac{2xy}{x^2+y^2} = ?$

1) átviteli elvvel

$P_n^{(1)} = \left(\frac{1}{n}, \frac{5}{n}\right) \xrightarrow[\infty]{n} (0,0)$

$f\left(\frac{1}{n}, \frac{5}{n}\right) = \dfrac{2\cdot\frac{1}{n}\cdot\frac{5}{n}}{\left(\frac{1}{n}\right)^2+\left(\frac{5}{n}\right)^2} = \frac{10}{26} \to \frac{10}{26}$

nem folytonos, $\neq 0$
de lehet még határértéke

$P_n^{(2)} = \left(\frac{-2}{n}, \frac{1}{n}\right) \xrightarrow[\infty]{n} (0,0)$

$f\left(-\frac{2}{n}, \frac{1}{n}\right) = \dfrac{2\cdot\left(-\frac{2}{n}\right)\cdot\frac{1}{n}}{\left(-\frac{2}{n}\right)^2+\left(\frac{1}{n}\right)^2} = \frac{-4}{5} \to \frac{-4}{5} \neq \frac{10}{26}$

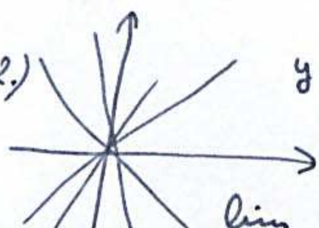
⇓

$\nexists$ határérték

⇓

f nem folytonos $(0,0)$-ban

2.)



$y = mx$

Ha a h.é függ az m-től ⇒ nem folytonos
Ha nem függ ⇒ semmit nem tudok.

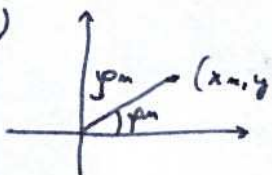
$\lim\limits_{x\to 0} f(x, mx) = \lim\limits_{x\to 0} \dfrac{2x\cdot mx}{x^2+m^2x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2}\cdot\dfrac{2m}{1+m^2} = \dfrac{2m}{1+m^2}$ függ m-től

⇓

$\nexists$ h.é.

Ezen egyenesek mentén más-más h.é-t kapok

3.)



$(x_n, y_n) = (\underset{\substack{\text{r} \\ \downarrow}}{\rho_n} \cdot \cos \underset{\substack{\text{fi} \\ \downarrow}}{\varphi_n}, \rho_n \cdot \sin \varphi_n) \to (0,0)$

$\rho_n \to 0 \quad (\rho_n \to +0)$

φ_n tetsz.

ha a φ benne marad, akkor függ az útvonaltól ⇒ $\nexists$ h.é.

$$\lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} f(\rho_n\cos\varphi_n, \rho_n \sin\varphi_n) = \lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} \frac{2\cdot \rho_n\cos\varphi_n \cdot \rho_n \sin\varphi_n}{\rho_n^2\cdot\cos^2\varphi_n + \rho_n \sin^2\varphi_n} =$$

$$= \lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} \frac{2\cos\varphi_n \sin\varphi_n}{1} = 2\cos\varphi_n\sin\varphi_n$$ függ a φ_n-től $\Rightarrow \nexists$ h.é.

marad ρ_n a számlálóban $\Rightarrow$ behúzza a 0-ba.

Pl $f(x,y) = \dfrac{2x^2y}{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ csak a 3 módszer!!!

$$\lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} \frac{2\rho_n^2\cos^2\varphi_n\, \rho_n\sin\varphi_n}{\rho_n^2\cos^2\varphi_n + \rho_n^2\sin^2\varphi_n} = \lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} \frac{2\cancel{\rho_n^2}}{\cancel{\rho_n^2}}\cdot \underset{0}{\underset{\downarrow}{\rho_n}} \underbrace{\cos^2\varphi_n \sin\varphi_n}_{\text{korl}} = 0,$$ tehát $\exists$ a h.é.

Pl $f(x,y) = \dfrac{x^2-y^2}{2x^2+3y^2}$

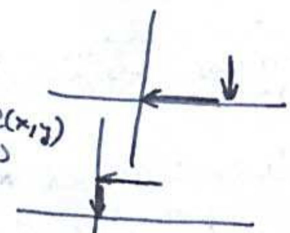
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

Behelyettesítve a φ_n bennmarad

$\exists$ h.é $\Rightarrow$ bm. módszer jó

iterált limesz:

$$\left.\begin{aligned} &\lim_{x\to 0}\lim_{y\to 0} \frac{x^2-y^2}{2x^2+3y^2} = \lim_{x\to 0} \frac{x^2-0}{2x^2+0} = \frac{1}{2} \\ &\lim_{y\to 0}\lim_{x\to 0} \frac{x^2-y^2}{2x^2+3y^2} = \lim_{y\to 0} \frac{0-y^2}{0+3y^2} = -\frac{1}{3} \neq \frac{1}{2} \end{aligned}\right\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$



ha a 2 útvonalon külön a h.é $\Rightarrow \nexists$ h.é.

Deriválhatóság

$\underline{x} = \underline{a}$ -ban

Egyváltozós: $f'(x_0) = \lim\limits_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

$\cancel{f'(\underline{a}) = \lim\limits_{\underline{h}\to 0} \dfrac{f(\underline{a}+\underline{h}) - f(\underline{a})}{\underline{h}}}$

Pl: $f(x,y) = x^2y + xe^{2xy}$

$y := 1$

$f(x,1) = x^2 + x e^{2x}$ egyvált. fv, így már deriválható

$f'_x(x,1) = 2x + 1\cdot e^{2x} + x\cdot e^{2x}\cdot 2$

$f'_x(5,1) = $ —"— $\Big|_{x=5}$ x szerinti parciális derivált.

$f'_y = \dfrac{\partial}{\partial y} f = \dfrac{\partial f}{\partial y} = x^2\cdot 1 + x\cdot e^{2xy}\cdot 2x\cdot 1$ (ez is egy kétváltozós fv lett)

x-et konstansnak tekintem

D $\underset{f(x,y_0)}{f'_x(x_0,y_0)} = \lim\limits_{h\to 0} \dfrac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$

$\underset{f(x_0,y)}{f'_y(x_0,y_0)} = \lim\limits_{k\to 0} \dfrac{f(x_0,y_0+k) - f(x_0,y_0)}{k}$

2008.04.03.
Kónya

(Pl)

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{2x^2+3y^2}$$ Iterált limesekkel most nem lehet dolgozni

$$\lim_{\substack{\rho_n\to 0\\ \varphi_n \text{ tetsz}}} \frac{\rho_n\cos\varphi_n\cdot\rho_n^3\sin^3\varphi_n}{2\rho_n^2\cos^2\varphi_n+3\rho_n^2\sin^2\varphi_n} = \frac{\overset{\to 0}{\rho_n^2}\cos\varphi_n\sin^3\varphi_n}{2\cos^2\varphi_n+3\sin^2\varphi_n} = 0$$

korl. mat $= \frac{\cos\varphi_n\cdot\sin^3\varphi_n}{2+\sin^2\varphi_n} \geq 2$

*! (Pl)

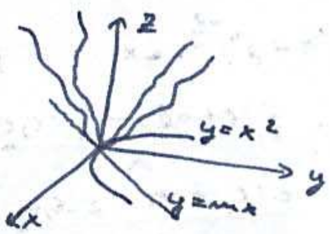
$$f(x,y) = \begin{cases} \frac{x^2y}{y^2+x^4}, & \text{ha } (x,y)\neq(0,0)\\ 0, & \text{ha } (x,y)=(0,0)\end{cases}$$

a) Mutassa meg, hogy a fv $\forall\cdot y=mx$ mentén folyt!
b) Folytonos-e a (0,0)-ban?

a.) $\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{mx+x^2}{m^2x^2+x^4} = \lim \frac{x^2}{x^2}\,\frac{m\overset{\to 0}{x}}{m^2+x^2} = 0 = f(0,0)$

y tengely mentén, $x=0$

$\lim_{y\to 0} f(0,y) = \lim_{y\to 0}\frac{y\cdot 0^2}{y^2+0^2} = 0 \to 0$



b.) $y=x^2$

$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \frac{1}{2} \neq f(0,0) \Rightarrow f$ nem folyt. a (0,0)-ban.

(Pl.) $f(x,y) = \begin{cases}\frac{xy}{x^2+y^2}+3x+2y, & \text{ha } (x,y)\neq(0,0)\\ 0, & \text{ha } (x,y)=0\end{cases}$

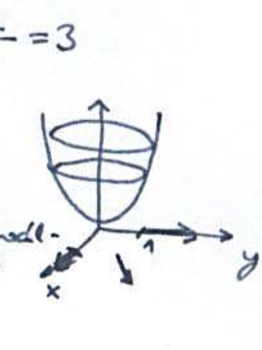
a). folytonos-e a (0,0)-ban? HF
b.) Írjuk fel $f'_x(x,y)$-t.

b.) $f'_x(x,y) = \begin{cases}\frac{y(x^2+y^2)-xy(2x)}{(x^2+y^2)^2}+3, & \text{ha } x^2+y^2\neq 0\\ 3, & \text{ha } (x,y)=(0,0)\end{cases}$

$f'_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h\cdot 0}{h^2+0}+3h+2\cdot 0}{h} = \lim_{h\to 0}\frac{3h}{h} = 3$

(Totális) deriválhatóság

(Pl) $f(x,y) = x^2+y^2$ Hol és deriválható? ($\equiv$ hol $\exists$ grad f)



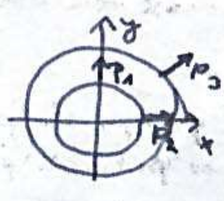
$f'_x = 2x+0$
$f'_y = 0+2y$ } mindenütt $\exists$ és folyt $\Rightarrow f$ mindenütt deriválható

grad $f = f'_x\,\underline{i} + f'_y\,\underline{j} = 2x\,\underline{i}+2y\,\underline{j}$ (vektor-vektor fv) pl: $\operatorname{grad} f(0,1) = f'_x(0,1)\,\underline{i} + f'_y(0,1)\,\underline{j} = 2\underline{j}$ (P_1)

$\operatorname{grad} f(1,0) = 2\underline{i}$ (P_2)

$\operatorname{grad} f(1,1) = 2\underline{i}+2\underline{j}$ (P_3)

Szintvonalak: $x^2+y^2=c$



A gradiensvektor az ért.tart.-hoz kötődik, merőleges a szintvonalakra, és a növekvő irányba mutat.

(! Nem az érintő sík normálvektora!)

Pl $f(x,y,z) = x^2z^3 + z\cdot e^{x-y}$

$$\left.\begin{array}{l} f'_x = 2x\cdot z^3 + z\cdot e^{x-y}\cdot 1 \\ f'_y = 0 + z\cdot e^{x-y}\cdot(-1) \\ f'_z = x^2\cdot 3z^2 + e^{x-y}\cdot 1 \end{array}\right\}$$ mindenütt $\exists$ és folyt $\Rightarrow f$ mindenütt deriválható

($\equiv$ grad f mindenütt $\exists$)

$\operatorname{grad} f = f'_x\underline{i} + f'_y\underline{j} + f'_z\underline{k} = \dots$

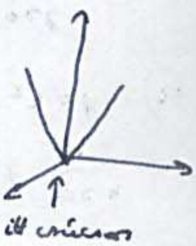
Pl $f(x,y) = \sqrt{3x^2+2y^4}$ $\quad f'_x=?\quad f'_y=?\quad \operatorname{grad} f=?$

$(x,y)\neq(0,0)$

$f'_x = \frac{1}{2}\cdot(3x^2+2y^4)^{-\frac{1}{2}}\cdot 6x \qquad f'_y = \frac{1}{2}(3x^2+2y^4)^{-\frac{1}{2}}\,8y^3 = \frac{1}{2\sqrt{3x^2+2y^4}}\cdot 8y^3$

$f'_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to0}\frac{\sqrt{3h^2}-0}{h} = \lim_{h\to0}\frac{\sqrt{3}\,|h|}{h} \not\exists$

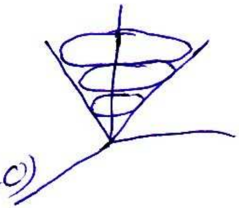
$f'_y(0,0) = \lim_{k\to0}\frac{f(0,k+0)-f(0,0)}{k} = \lim_{\to 0}\frac{\sqrt{+2k^4}}{k} = \frac{\sqrt{2}\,k^2}{k} = 0$



Ha $(x,y)\neq(0,0) \Rightarrow f'_x, f'_y$ $\exists$ és folyt $\Rightarrow \exists \operatorname{grad} f = f'_x\underline{i} + f'_y\underline{j} = \dots$

Ha $(x,y)=(0,0)$: $\nexists$ $\operatorname{grad} f(0,0)$, mert $\nexists$ $f'_x(0,0)$

IV.9 (!) $f(x,y)=\sqrt{x^2+y^2}$ (!)



Зад. $f(x,y)=\begin{cases}\dfrac{x^2y}{(x^2+y^2)^2}+6x & (x,y)\neq(0,0)\\ 0, & \text{ако } (x,y)=(0,0)\end{cases}$

а) f'_x -? f'_y -?

б) Къде $\exists$ grad f?

а) $f'_x=\begin{cases}\dfrac{2xy\,(x^2+y^2)^2-x^2y\,2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}+6, & \text{ако } (x,y)\neq(0,0)\\ 6, & \text{ако } (x,y)=(0,0)\end{cases}$

$$f'_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim\frac{\overbrace{\frac{0}{h^4}}^{=0}+6h-0}{h}=6$$

$f'_y=\begin{cases}\dfrac{x^2(x^2+y^2)^2-x^2y\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}, & \text{ако } (x,y)\neq(0,0)\\ 0, & \text{ако } (x,y)=(0,0)\end{cases}$

$$f_y'(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,k)}{k} = l\, \frac{0 = \frac{0}{k^4} + 0}{k} = 0$$

b) Ha $(x,y) \neq (0,0) \Rightarrow f_x'$ és f_y' $\exists$ és folytonosak $\Rightarrow \exists$ grad f

grad $f = f_x' \underline{i} + f_y' \underline{j}$

$(x,y) = (0,0)$ -ban $\nexists$ grad f, mert f nem folyt. $(0,0)$-ban, ugyanis:

(Pl) $f(x,y) = \dfrac{y \cdot e^{5x}}{y^2+4}$

a) $f_x' = ?$, $f_y' = ?$

b) grad $f\big|_{(0,2)} = ?$ $\quad df((0,2),(h,k)) = ?$

c) Írja fel a fv. $P_0(0,2)$ pontbeli érintősíkjának egyenletét!

a) $f(x,y) = \dfrac{y}{y^2+4}\, e^{5x}$

$$f_x' = \frac{y}{y^2+4}\, e^{5x} \cdot 5 \qquad f_y' = e^{5x} \cdot \frac{(y^2+4) - y(2y)}{(y^2+4)^2}$$

b) grad f $\forall$ütt $\exists$, mert az elégséges felt. teljesül:
f_x', f_y' $\forall$ütt $\exists$ és folyt.

$f_x'(0,2) = \frac{5}{4} \qquad f_y'(0,2) = 0$

grad $f(0,2) = \frac{5}{4}\underline{i} \; (+ 0\underline{j})$

$$df((0,2),(h,\ell)) = f'_x(0,2)\cdot h + f'_y(0,2)\cdot \ell =$$
$$= \frac{5}{4} h \,(+0\cdot \ell)$$

$$f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0$$

$x_0 = 0 \quad y_0 = 2$

$f(0,2) = \frac{1}{4}$

$$f'_x(0,2)(x-0) + f'_y(0,2)(y-2) - (z - f(0,2)) = 0$$

$$\boxed{\frac{5}{4}x\,(+\,0y) - \left(z - \frac{1}{4}\right) = 0}$$

általános lak.:

$\rightarrow = 0$ alakjára

Pl. $g(t)\big|_{t=x^2+xy} = g(x^2+xy) = h(x,y) \quad h'_x = ?\ h'_y = ?$

$h''_{xx} = ?\ h''_{xy} = ?\ h''_{yx} = ?$

$g \in C^2_{\mathbb{R}} \qquad h''_{yy} = ?$

$\Rightarrow$ 2-szer folyt. deriválható

belső fgv. x szerinti der.

$$h'_x = g'_{\cancel{x}}(x^2 + \cancel{x}y)\,\frac{\partial}{\partial x}(x^2+xy) = g'(x^2+xy)\cdot(2x+y)$$

NINCS! g egyváltozós!!!

$$h'_y = g'(x^2+xy)\cdot(x^2+xy)'_y = g'(x^2+xy)\cdot x$$

$$h''_{xx} = \left(\frac{\partial}{\partial x} g'(x^2+xy)\right)(2x+y) + (g'(x^2+xy))\left(\frac{\partial}{\partial x}(2x+y)\right) =$$

$$\left(g''(x^2+xy)\cdot(2x+y)\right)\cdot(2x+y) + g'(x^2+xy)\cdot 2$$

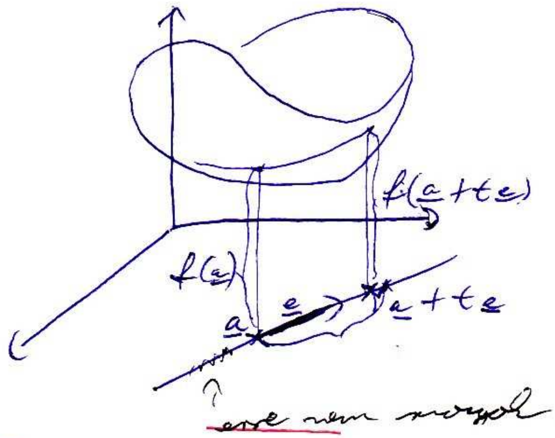
$h''_{xy} = h''_{yx} \ominus$, mivel folytonosak a leíró fgv-ek

$$\ominus \left(\frac{\partial}{\partial y} g'(x^2+xy)\right)(2x+y) + (g'(x^2+xy))\left(\frac{\partial}{\partial y}(2x+y)\right) =$$

$$= \left(g''(x^2+xy)\cdot x\right)(2x+y) + \left[g'(x^2+xy)\right]\cdot 1$$

$$h''_{xy} = x\left(\frac{\partial}{\partial y} g'(x^2+xy)\right) = x\cdot g''(x^2+xy)\cdot x$$

Iránymenti derivált (ez is egy részleges deriválhatóság)



$|\underline{e}| = 1$

$$\left.\frac{df}{d\underline{e}}\right|_{\underline{a}} = \left.\frac{\partial f}{\partial \underline{e}}\right|_{\underline{a}} = \lim_{t\to +0} \frac{f(\underline{a}+t\underline{e}) - f(\underline{a})}{t}$$

Összetett fv. deriválása

Pl $f(\underline{x}) = f(x,y) = e^{2x}\cdot \sin y$ $\quad x = x(t) = t^2-3$, $\; y = y(t) = (t+1)^2$ $\Rightarrow \underline{x} = \underline{x}(t) = \underline{\varphi}(t) = x(t)\underline{i} + y(t)\underline{j}$

$h(t) = f(\underline{x}(t)) = f(\underline{\varphi}(t)) = f(x(t), y(t))$ $\qquad \frac{d\underline{x}}{dt} = \frac{d\underline{\varphi}}{dt} = \dot{\underline{x}}(t) = \dot{\underline{\varphi}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j}$

$h'(t) = \dot{h}(t) = \underbrace{f'_x}_{\frac{\partial f}{\partial x}}\cdot \underbrace{\dot{x}(t)}_{\frac{dx}{dt}} + f'_y \cdot \dot{y}(t) =$

$= \underbrace{\text{grad} f}_{\text{grad} f|_{\underline{x}=\underline{x}(t)}} \cdot \dot{\underline{x}} = \text{grad} f(\underline{\varphi}(t)) \cdot \dot{\underline{\varphi}}(t)$

$\dot{h}(t) = 2e^{2x}\cdot \sin y\Big|_{\substack{x=x(t)\\ y=y(t)}} \cdot 2t + e^{2x}\cdot \cos y\Big|_{\vdots} \cdot 2(t+1)$

Pl. $f(u,v,w) = u\cdot v^2\cdot w^3$ $\qquad u = \ln(1+x^2y^2)$, $\; v = x+2y$, $\; w = x\cdot e^{2y}$

~~$h(x,y)$~~

$h(x,y) = f(u(x,y), v(x,y), w(x,y))$

$h'_x = f'_u\cdot u'_x + f'_v\cdot v'_x + f'_w\cdot w'_x$ $\qquad h'_x = v^2\cdot w^3\Big|_{\vdots} \cdot \frac{1}{1+x^2y^2}\cdot 2xy^2 +$

$h'_y = f'_u\cdot u'_y + f'_v\cdot v'_y + f'_w\cdot w'_y$ $\qquad + 2uvw^3\Big|\cdot 1 + 3uv^2w^2\Big|_{\vdots}\cdot e^{2y}$

M $f(x,y) \neq f(xy) = f(t)\Big|_{t=xy}$

Pl $f(x,y) = g(xy^2) = g(t)\Big|_{t=xy^2}$ $\qquad g\in C^2_{\mathbb{R}}$

!! Hat. meg. f ~~össz~~ első és másodrendű parc. deriváltjait.

$f'_x = g'(xy^2)\cdot y^2$ ~~$g(xy^2)\cdot 2xy$~~ $\qquad f'_y = g'(xy^2)\cdot 2xy$

$f''_{xx} = y^2\cdot g''(xy^2)\cdot y^2$ $\qquad f''_{xy} (= f''_{yx}) = (g''(xy^2)\cdot 2xy)\cdot y^2 + g'(xy^2)\cdot(2y) = f''_{yx}$

$f''_{yy} = (g''(xy^2)\cdot 2xy)\cdot 2xy + g'(xy^2)\cdot(\;2x\;)$

$\text{grad}\, \underline{x}^2 = 2\underline{x}$; $\quad f(\underline{x}) = \underline{x}^2$

$\underline{x}^2 = \underline{x}\cdot\underline{x} = (\underline{x}|\underline{x}) = x_1^2 + x_2^2 + \ldots + x_n^2 = |\underline{x}^2| = \|\underline{x}\|^2$

1. mo.: $f'_{x_1} = 2x_1$ $\qquad \text{grad} f = f'_{x_1}\underline{e}_1 + \ldots + f'_{x_n}\underline{e}_n = (f'_{x_1}, f'_{x_2}, \ldots, f'_{x_n}) =$

$= (2x_1, 2x_2, \ldots, 2x_n) = 2\underline{x}$

2. mo.: $\Delta f = f(\underline{x}+\underline{h}) - f(\underline{x}) = \underbrace{\underline{A}}_{\text{grad} f}\cdot \underline{h} + \underbrace{\underline{\varepsilon}(\underline{h})}_{\substack{h\to 0\\ \downarrow\\ \underline{0}}}\cdot \underline{h}$

fgt. $\underline{h}$-tól

Ez $\Delta f = (\underline{x}+\underline{h})^2 - \underline{x}^2 = \underbrace{2\underline{x}}_{\text{grad} f}\cdot\underline{h} + \underbrace{\underline{h}}_{\varepsilon}\cdot\underline{h}$

$x^2 + 2\underline{x}\cdot\underline{h} + h^2$

$\boxed{\text{grad } \underline{x}^n, \; n > 2 \text{ értelmetlen!}}$

$\underline{x}^3$ ilyen nincs (persze vegyesszorzat van: $\underline{a}\,\underline{b}\,\underline{c}$)

$$\text{grad } |\underline{x}| = \text{grad } \sqrt{\underline{x}^2} = \frac{1}{2\sqrt{y}}\Big|_{y=\underline{x}^2} \cdot \text{grad } \underline{x}^2 = \frac{1}{2\sqrt{\underline{x}^2}} \cdot 2\underline{x} = \frac{\underline{x}}{|\underline{x}|} \qquad \underline{x} \neq \underline{0}$$

$\left(f(x,y) = \sqrt{x^2+y^2}\right)$ $\quad |\underline{x}|^2 = \underline{x}^2$

$$\sqrt{\underline{x}^2} = \sqrt{y}\Big|_{y=\underline{x}^2}$$

— o —

$\boxed{\dfrac{df}{d\underline{e}}}$ iránymenti derivált

$$f(x,y) = \frac{e^{3x+2y^2}}{y^2+2} \qquad P_0(0,1)$$

a) Határozza meg az f fv gradiensét a P_0 pontban! Miért $\exists$ a gradiens?

b.) Írja fel az f fv P_0 pontbeli érintősíkjának egyenletét.

c) $\dfrac{df}{d\underline{e}} = ?$ $\qquad \underline{e} \parallel 3\underline{i} - 4\underline{j}$

d) $\max \dfrac{df}{d\underline{e}}\Big|_{P_0} = ?$ és $\underline{e} = ?$; $\min \dfrac{df}{d\underline{e}}\Big|_{P_0} = ?$ és $\underline{e} = ?$

a) $f(x,y) = e^{3x} \cdot \dfrac{e^{2y^2}}{y^2+2}$ $\qquad f'_x = 3e^{3x} \cdot \dfrac{e^{2y^2}}{y^2+2}$ $\qquad f'_y = e^{3x} \cdot \dfrac{4ye^{2y^2}(y^2+2) - e^{2y^2} \cdot 2y}{(y^2+2)^2}$

$$f'_x(0,1) = \frac{3e^2}{3} \qquad f'_y(0,1) = \frac{4e^2 \cdot 3 - e^2 \cdot 2}{3^2} = \frac{10}{9}e^2$$

$f'_x, f'_y \;\exists\; K_{P_0}$-ban és folytonosak $\Rightarrow \exists$ grad $f(P_0) = f'_x(P_0)\underline{i} + f'_y(P_0)\underline{j} = e^2\underline{i} + \frac{10}{9}e^2\underline{j}$

b) $f'_x(P_0)(x-x_0) + f'_y(P_0)(y-y_0) - (z - f(P_0)) = 0$ $\qquad f(0,1) = \dfrac{e^2}{3}$

$$e^2(x-0) + \frac{10}{9}e^2(y-1) - \left(z - \frac{e^2}{3}\right) = 0$$

c) (T) $\dfrac{df}{d\underline{e}}\Big|_{P_0} = \text{grad } f(P_0)\,\underline{e}$ $\qquad$ (grad f mindenütt létezik)

$$\underline{v} = 3\underline{i} - 4\underline{j}$$

$$|\underline{v}| = \sqrt{3^2+(-4)^2} = 5 \qquad \underline{e} = \frac{\underline{v}}{|\underline{v}|} = \left(\frac{3}{5}, -\frac{4}{5}\right)$$

$$\frac{df}{d\underline{e}}\Big|_{P_0} = \left(e^2, \frac{10}{9}e^2\right) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = e^2 \cdot \frac{3}{5} + \frac{10}{9}e^2\left(-\frac{4}{5}\right) = \ldots$$

d) $\max \dfrac{df}{d\underline{e}}\Big|_{P_0} = |\text{grad } f(P_0)|$ $\qquad \underline{e} \parallel \text{grad} f(P_0)$ $\qquad |\text{grad } f(P_0)| = \sqrt{(e^2)^2 + \frac{100}{81}(e^2)^2} =$

$$\underline{e} = \frac{\text{grad} f(P_0)}{|\text{grad} f(P_0)|} \qquad = e^2 \underbrace{\sqrt{1+\frac{100}{81}}}_{=c}$$

$\min \dfrac{df}{d\underline{e}}\Big|_{P_0} = -|\text{grad} f(P_0)|$

$$\underline{e} = -\frac{\text{grad} f(P_0)}{|\text{grad} f(P_0)|} \qquad \max \frac{df}{d\underline{e}}\Big|_{P_0} = c \cdot e^2 \qquad \underline{e} = \frac{(e^2, \frac{10}{9}e^2)}{ce^2} = \frac{1}{c}\left(1, \frac{10}{9}\right)$$

$\min \ldots = -ce^2$

[illegible] Kónya

(Pl) $f(x,y) = \frac{4}{x^2} - \frac{4}{x} + y^3 - 3y \qquad x \neq 0$

a) $df((1,-2),(h,k)) = ?$

b) $d^2f((1,-2),(h,k)) = ?$

c) Hol lehet a fv-nek lok. sz.é-e? Hol és milyen jellegű lok. sz.é-e van a fv-nek?

$f'_x = -\frac{8}{x^3} + \frac{4}{x^2} \qquad f'_y = 3y^2 - 3 \qquad f''_{xx} = \frac{24}{x^4} - \frac{8}{x^3} \qquad f''_{yy} = 6y, \quad f''_{xy} = f''_{yx} = 0$

$f'_x(1,-2) = -4 \qquad f'_y(1,-2) = 9 \qquad f''_{xx}(1,-2) = 16 \qquad f''_{yy}(1,-2) = -12$

a) $df((1,-2),(h,k)) = f'_x(1,-2)\cdot h + f'_y(1,-2)\,k = -4h + 9k$

b) $d^2f((1,-2),(h,k)) = [h \;\; k] \begin{bmatrix} f''_{xx}(1,-2) & f''_{xy}(1,-2) \\ f''_{yx}(1,-2) & f''_{yy}(1,-2) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = [h \;\; k] \begin{bmatrix} 16 & 0 \\ 0 & -12 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} =$

$= 16h^2 + (-12)k^2$

c) $f'_x = 0 \;\wedge\; f'_y = 0$

$f'_x = \frac{-8}{x^3} + \frac{4}{x^2} = \frac{-8+4x}{x^3} = 0 \Rightarrow x = 2$

$f'_y = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$

Tehát a $P_1(2,1)$ és $P_2(2,-1)$ pontokban lehet lok. szé.-e.

$D(x,y) = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix} \qquad D\big|_{P_1} \qquad D\big|_{P_2}$

ha $D\big|_{P_0} > 0 \Rightarrow \exists$ lok. sz.é.

ha $D\big|_{P_0} < 0 \Rightarrow \nexists$ lok. sz.é.

ha $D\big|_{P_0} = 0 \Rightarrow$ további vizsgálat (pl. a fv.érték előjele alapján)

$D(x,y) = \begin{vmatrix} \frac{24}{x^4} - \frac{8}{x^3} & 0 \\ 0 & 6y \end{vmatrix} \qquad P_1:\; D(2,1) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 6 \end{vmatrix} = 3 > 0 \Rightarrow \exists$ lok. szé.

$f_{xx}(P_1) > 0 \Rightarrow$ lok. min.
(bal felső elem)

$P_2:\; D(2,-1) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -6 \end{vmatrix} = -3 < 0 \Rightarrow \nexists$ lok. szé. P_2-ben

Pl. $f(x,y)=3x+y-\frac{9}{xy}$ Lok. szé?

$xy\neq 0$

$f'_x=3+\frac{9}{x^2y}=0$ $f'_y=1+\frac{9}{xy^2}=0$

$1+\frac{3}{x^2y}=0 \Rightarrow y=-\frac{3}{x^2}$ $1+\frac{9}{x\cdot\frac{9}{x^4}}=0 \Rightarrow x^3+1=0 \Rightarrow x=-1$

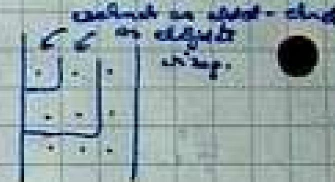
$y=-3$

Csak a $P_0(-1,-3)$-ban lehet lok. szé.

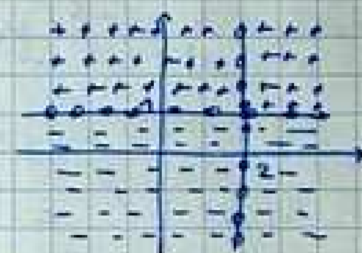
$D(x,y)=\begin{vmatrix} \frac{9}{y}\cdot\frac{-2}{x^3} & \frac{9}{x^2}\cdot\left(\frac{-1}{y^2}\right) \\ -\frac{9}{x^2y^2} & \frac{9}{x}\cdot\frac{-2}{y^3}\end{vmatrix}=\frac{18^2}{x^4y^4}-\frac{81}{x^4y^4}$ $D(x_0,y_0)>0 \Rightarrow \exists$ lok. szé. P_0-ban. $\forall\, x,y$-ra

$f''_{xx}(-1,-3)<0 \Rightarrow f$ lok. max. $f(-1,-3)$ értéke...

$D(x,y)=\begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy}\end{vmatrix}>0$ $D(x,y)=f''_{xx}f''_{yy}-\underbrace{f''_{xy}}_{<0}{}^2$



Pl. $f(x,y)=(y-1)^3(x-2)^4$ Lok. szé.



$f'_x=(y-1)^3\cdot 4(x-2)^3=0$

$f'_y=3(y-1)^2\cdot(x-2)^4=0$ $\Rightarrow$ $y=1$ ill. $x=2$

∞ sok pont

Tehát az $(x,1)$ ill. $(2,y)$ pontokban lehet lok. szé.

HF: $D(x,1)=0$ (mert minden szorzatban marad $(y-1)$ tényező

$D(2,y)=0$

Akkor most meg kell nézni a függvény előjelét.

$(x-2)^4\geq 0$

$(y-1)^3>0$, ha $y>1$

<0, ha $y<1$

Az $y=1$ pontjaiban $\nexists$ lok. szé. (fent >0, lent <0, akármilyen kis sugarú környezetben)

$x=2$, $y>1$: lok. min.

$x=2$, $y<1$: lok. max.

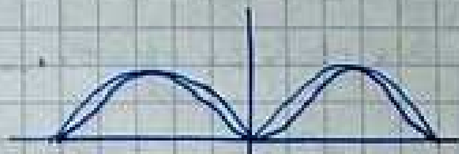
(T) $a_k^{f+g} = a_k^f + a_k^g$ (az együttható a függvények összegében

$b_k^{f+g} = b_k^f + b_k^g$ a függvények együtthatóinak összege, integrálás miatt)

$a_k^{c\cdot f} = c \cdot a_k^f$

$b_k^{c\cdot f} = c \cdot b_k^f$

(Pl) $f(x) = \sin^4 x$ F. sor?



$$f(x) = \frac{a_0}{2} + b_1 \sin x + a_1 \cos x + b_2 \sin 2x + a_2 \cos 2x + \dots$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^4 x \cos kx \, dx \; ; \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^4 x \sin kx \, dx$$

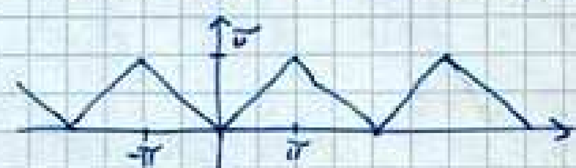
nem tudjuk megoldani.

$$f(x) = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1 - 2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}}\right) = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$\uparrow$ ez maga a Fourier-sor

$\frac{a_0}{2} = \frac{3}{8}$ $\quad a_2 = -\frac{1}{2}$ $\quad a_4 = \frac{1}{8}$ $\quad$ a többi 0.

(Pl) $f(x) = |x|$, ha $x \in [-\pi, \pi)$; $f(x + 2\pi) = f(x)$



$\Phi(x) = f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (b_k \sin kx + a_k \cos kx)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

$k = 0, 1, 2, \dots \qquad k = 1, 2, \dots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} |x|\, dx}\, (= \dots) = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \pi$$

p. fv: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{|x|}_{p.}\, \underbrace{\cos kx}_{p.}}_{p.}\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos kx \, dx =$$

$u = x \quad v' = \cos kx$
$u' = 1 \quad v = \frac{\sin kx}{k}$

$$= \frac{2}{\pi}\left(\underbrace{x \cdot \frac{\sin kx}{k}\Big|_0^{\pi}}_{0-0} - \int_0^{\pi} \frac{\sin kx}{k}\, dx\right) = -\frac{2}{\pi k}\cdot\frac{-\cos kx}{k}\Big|_0^{\pi} = \frac{2}{\pi k^2}\left(\underbrace{\cos k\pi}_{(-1)^k} - \cos 0\right) =$$

$$= \frac{2}{\pi}\cdot\frac{1}{k^2}\cdot((-1)^k - 1) = \begin{cases} 0, & \text{ha } k \text{ páros} \\ -\frac{4}{\pi}\cdot\frac{1}{k^2}, & \text{ha } k \text{ ptl.} \end{cases}$$

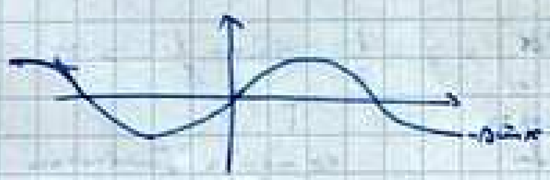
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{|x|}_{pe}\ \underbrace{\sin kx}_{pt}}_{pt}\, dx = 0$$

$$f(x) = \frac{a_0}{2} + a_1\cdot\cos x + a_2\cdot\cos 2x + a_3\cdot\cos 3x + \ldots =$$

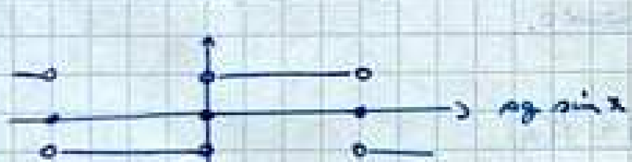
$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos[(2k-1)x]$$

$x=0$: $\quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\cdot\sum_{1}^{\infty}\frac{1}{(2k-1)^2} \Rightarrow \quad 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \sum_{1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$

(12) $f(x) = \text{sign}(\sin x)$ a) F sor
b) Egyenletesen konv.-e a F sor!
c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = ?$



$[0, 2\pi)$: $(0,\pi)$, $(\pi,2\pi)$ intervallumokban f monoton, végpontokban korlátos $\underset{\text{Dirichlet-t.}}{\Longrightarrow}$ F. sor mindenütt konv.



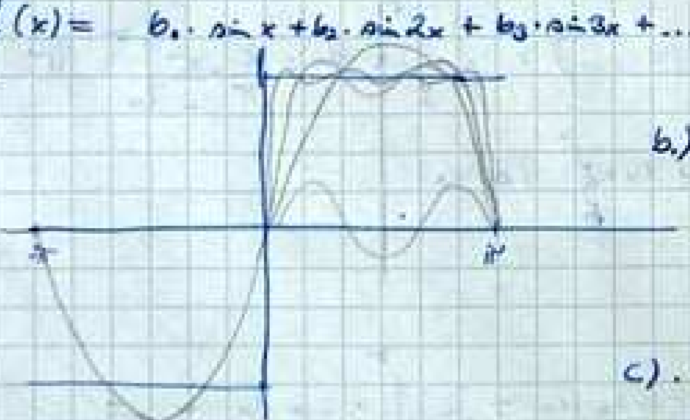
$f(x) = \Phi(x)$ mindenütt

a) f pt $\Rightarrow a_0 = 0$
$a_k = 0 \quad k = 1, 2, \ldots$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{f(x)}_{pt}\ \underbrace{\sin kx}_{pt}}_{pa}\, dx = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin kx = \frac{2}{\pi}\cdot\left.\frac{-\cos kx}{k}\right|_0^{\pi} =$$

$$= \frac{-2}{\pi k}\left(\underbrace{\cos k\pi}_{(-1)^k} - 1\right) = \begin{cases} 0, & k = k \text{ ps.} \\ \frac{4}{\pi}\cdot\frac{1}{k}, & k = k \text{ pt.} \end{cases}$$

$$f(x) = b_1\cdot\sin x + b_2\cdot\sin 2x + b_3\cdot\sin 3x + \ldots = \frac{4}{\pi}\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\cdot\sin 5x + \ldots\right)$$



b.) Nem egyenletes a konvergencia, mert az összegfv nem folytonos, bár az $f_n(x)$-ek azok voltak.

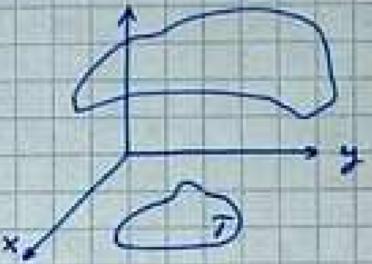
c). $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

$f(x) = \Phi(x)$ minden x-re

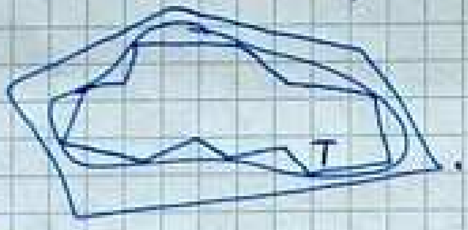
$$1 = f\left(\frac{\pi}{2}\right) = \Phi\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} + \ldots\right) \Rightarrow$$

Többes integrálok

Kettős integrálok



Jordan mérhetőség (T korlátos)



$t_b = \sup\{$ benne foglalt sokszögek területe $\}$
ha nincs ilyen $\rightarrow t_b = 0$

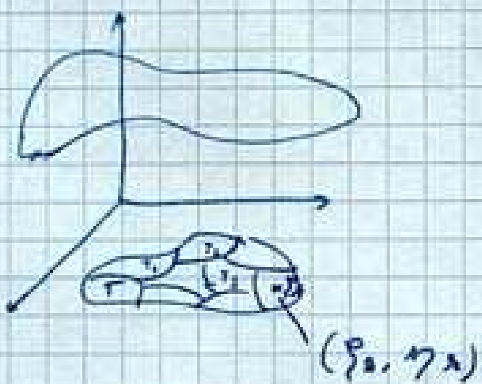
$t_k = \inf\{$ befoglaló sokszögek területe $\}$

(D) $T \subset \mathbb{R}^2$ Jordan mérhető, ha $t_b = t_k$

(Pl) nem mérhető halmaz (tartomány)

$$T = \{(x,y) : 0 \le x < 1;\ 0 \le y < 1,\ x \in \mathbb{Q},\ y \in \mathbb{Q}\}$$

$t_b = 0$
$t_k = 1$



$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot \text{ter}\ T_k$$

$$\iint_T f\ ,\ \iint_T f(x,y)\, dT\ ,\ \iint f(x,y)\, dx\, dy$$

Kettős integrál → kétszeres int

1.) φ

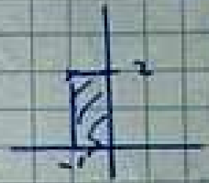
$f \in C^0_T$

$$\iint_T f(x,y)\, dx\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

$$\int_0^1 \int_{-1}^2 x e^{x^2} (y+1)^7\, dy\, dx$$

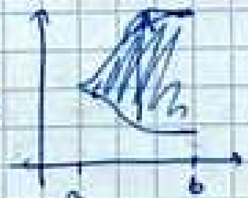
Anal elm — 2008.04.24. Kónya

Pl. $\iint_T x(xy+1)^2\,dT = ?$ T: $-1 \le x \le 0$, $0 \le y \le 2$



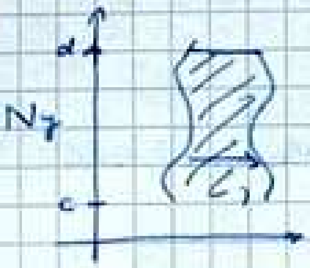
$\int_{x=-1}^{0}\int_{y=0}^{2}$

Normáltartomány:



N_x: $g_1(x) \le y \le g_2(x)$, $a \le x \le b$

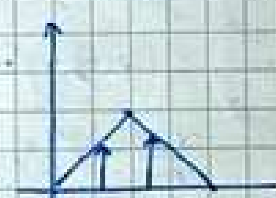
$$\iint_T f\,dT = \int_{x=a}^{b}\left(\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$$



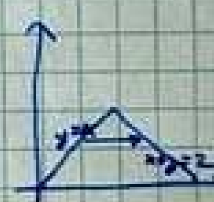
N_y: $h_1(y) \le x \le h_2(y)$, $c \le y \le d$

$$\iint_T f\,dT = \int_{y=c}^{d}\left(\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

Pl. $\iint_T (y+1)\,dT = ?$



Nem N_x (mert függvény-kettő-ig megy) (hanem 2 N_x összege)



N_y, mert fv-től fv-ig megy

$$J = \int_{y=0}^{1}\int_{x=y}^{x=2-y}(y+1)\,dx\,dy = \int_0^1 (y+1)x\Big|_{x=y}^{x=2-y}dy =$$

$$= \int_0^1\big((y+1)(2-y) - (y+1)y\big)\,dy = -2\frac{y^3}{3} + 2y\Big|_0^1 = -\frac{2}{3} + 2$$

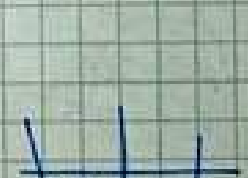
$2y - y^2 + 2 - y - y^2 - y$
$= -2y^2 + 2$

Pl. $J := \int_{-\frac{1}{2}}^{0}\int_{4x^2}^{1} 2x\cos y^2\,dy\,dx$ — nincs primitív fv-e, Taylor-sorba fejt!!!!

$4x^2 \le y \le 1$

$-\frac{1}{2} \le x \le 0$

$\left.\begin{array}{l} y = 4x^2 \\ x = -\frac{1}{2}\end{array}\right\} \Rightarrow y = 1$



$-\sqrt{\frac{y}{4}} \le x \le 0$

$0 \le y \le 1$

$$J = \int_{y=0}^{1}\int_{x=-\sqrt{\frac{y}{4}}}^{0} 2x\cos y^2\,dx\,dy = \int_0^1 x^2\Big|_{x=-\frac{\sqrt{y}}{2}}^{x=0}\cdot\cos y^2\,dy =$$

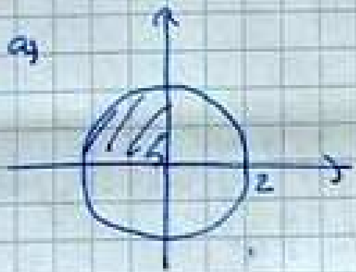
$$= \int_0^1 -\frac{y}{4}\cos y^2\, dy = -\frac{1}{4}\cdot\frac{\sin y^2}{2}\Big|_{y=0}^{1} = -\frac{1}{8}\cdot\sin 1 - 0$$

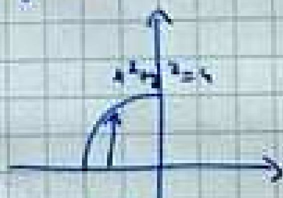
Pl. $\iint_T x^2\, dT = ?$ a) $T: x^2+y^2 \le 4$ $x\le 0,\ y\ge 0$

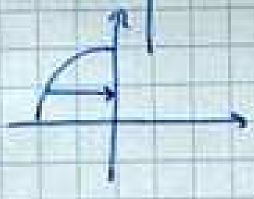
b) $T: 1\le x^2+y^2\le 4$ $x\le 0$

Írja fel a kétszeres integrált.

a)







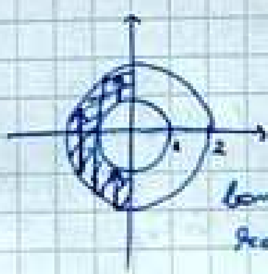
$$\int_{x=-2}^{0}\int_{y=0}^{\sqrt{4-x^2}} x^2\, dy\, dx$$

$$\int_{y=0}^{2}\int_{x=-\sqrt{4-y^2}}^{0} x^2\, dx\, dy = \int_{y=0}^{2} \frac{x^3}{3}\Big|_{-\sqrt{4-y^2}}^{0} dy =$$

$$= \frac{1}{3}\int_0^2 \left(0+\left(\sqrt{4-y^2}\right)^3\right) dy$$

$\frac{y}{2} = \sin t$ helyettesítéssel ... hosszú

b.)



bonyolult 3 tartomány kell...

Transzformáció (helyettesítés)

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt$$

$x=\varphi(t)$ $\frac{dx}{dt}=\varphi'(t)$ $\varphi'(t)\ne 0$

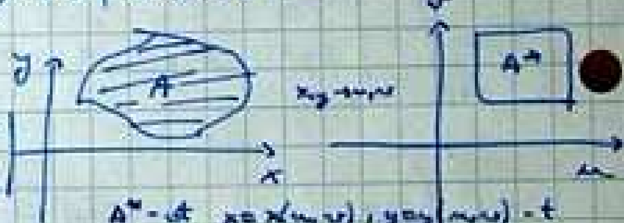
$$\iint_A f(x,y)\,dx\,dy = \iint_{A^*} f(x(u,v),y(u,v))\cdot \overset{\text{abszolút érték}}{|J|}\,du\,dv$$

$x = x(u,v)$

$y = y(u,v)$

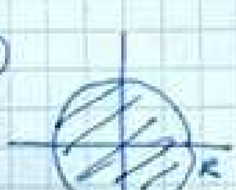
$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\text{determináns}} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

$J\ne 0$ A^*-on.

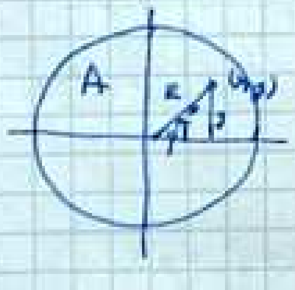


A^*-ot $x=x(u,v)$, $y=y(u,v)$-t kölcsönösen egyértelműen leképezi A-ra

Pl.



$x^2+y^2=R^2$



$x = r\cdot\cos\varphi = x(r,\varphi)$

$y = r\cdot\sin\varphi = y(r,\varphi)$

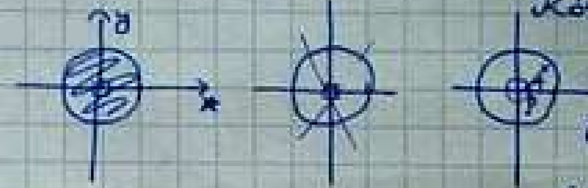
$x,y \to r,\varphi$

$$\left.\begin{array}{l} 0\le r\le R \\ 0\le\varphi\le 2\pi \end{array}\right\}\ A^* \text{ téglalap lesz.}$$

Anal. elm.

2008.04.30.
Kónya

$$\iint_T \frac{1}{(x^2+y^2)^3}\,dT \qquad T:\ x^2+y^2\le 1$$



$x = r\cos\varphi$
$y = r\sin\varphi$

$\rho \le r \le 1$

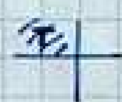
$0 \le \varphi \le 2\pi$

$$I=\lim_{\rho\to 0} I_\rho = \lim_{\rho\to 0}\int_{r=\rho}^{1}\int_{\varphi=0}^{2\pi}\frac{1}{(r^2)^3}\underbrace{\frac{r}{|J|}}_{r^{-5}}\cdot d\varphi\, dr =$$

$$= (2\pi-0)\lim_{\rho\to 0}\frac{r^{-4}}{-4}\Big|_{r=\rho}^{1} = -\frac{\pi}{2}\lim_{\rho\to 0}\left(1-\frac{1}{\rho^4}\right)=\infty$$

A tart. nem kor.

Pl a) $\iint_T e^{6x-8y}\,dT=?$



b) $\iint_T e^{-2x^2-2y^2}\,dT=?$

vagy

vagy

$$\lim_{R\to\infty}\int_{y=0}^{R}\int_{x=-R}^{0} e^{6x}\cdot e^{-8y}\,dx\,dy = \lim_{R\to\infty}\int_0^R e^{-8y}\,\frac{e^{6x}}{6}\Big|_{x=-R}^{0}dy = \frac{1}{6}\lim_{R\to\infty}\left(1-e^{-6R}\right)\int_0^R e^{-8y}\,dy =$$

$$= \frac{1}{6}\cdot\lim_{R\to\infty}\left(1-\underbrace{e^{-6R}}_{\downarrow\,0}\right)\cdot\underbrace{\frac{e^{-8y}}{-8}\Big|_{y=0}^{R}}_{-\frac{1}{8}(e^{-8R}-1),\ e^{-8R}\to 0} = \frac{1}{48} \Rightarrow \exists \text{ az improprius integrál.}$$

b)

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$
$|J| = r$

$0 \le r \le R$
$\frac{\pi}{2} \le \pi$

$$I_R=\int_{r=0}^{R}\int_{\varphi=\frac{\pi}{2}}^{\pi} e^{-2r^2}\, r\, d\varphi\, dr =$$

$$=\left(\pi-\frac{\pi}{2}\right)\left(-\frac{1}{4}\right)e^{-2r^2}\Big|_0^R = -\frac{\pi}{8}\left(e^{-2R^2}-1\right)$$

$$I=\lim_{R\to\infty} I_R = \lim_{R\to\infty}\left(-\frac{\pi}{8}\right)\left(e^{-2R^2}-1\right)=\frac{\pi}{8}$$

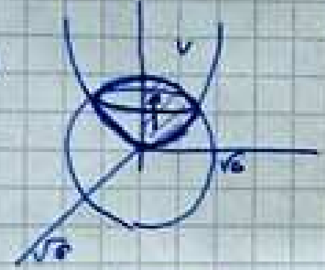
(22) $\iiint_V x\,dV$

a) $V: x^2+y^2+z^2 \le 6$; $z \ge x^2+y^2$; $x \ge 0$

b) $V: x^2+y^2+z^2 \le 6$; $z \ge \sqrt{x^2+y^2}$

c) $V: x^2+y^2+z^2 \le 6$, $x^2+y^2 \ge 2$; $z \ge 0$

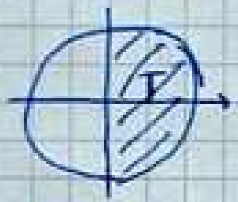
a)



$x^2+y^2+z^2 = 6$

$z = x^2+y^2$

$z^2+z-6=0 \Rightarrow z=2$: $x^2+y^2=2$



$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$
$z = z$
$|J| = r$

$$\mathcal{Y}_a = \iiint_{\substack{r=0\ \varphi=-\frac{\pi}{2}\ z=r^2}}^{\sqrt{2}\ \frac{\pi}{2}\ \sqrt{6-r^2}} r\cdot\cos\varphi\cdot \underbrace{r}_{|J|}\, dz = \int_0^{\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos\varphi\cdot r^2 z\Big|_{z=r^2}^{\sqrt{6-r^2}}}_{\cos\varphi\cdot r^2(\sqrt{6-r^2}-r^2)}\, d\varphi\, dr = \ldots$$

$x \ge 0$

$r^2 \le z \le \sqrt{6-r^2}$

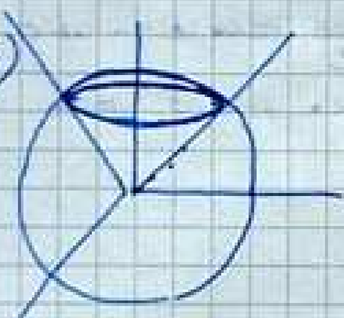
$f_1(x,y) = x^2+y^2$

$f_2(x,y) = \sqrt{6-x^2-y^2}$

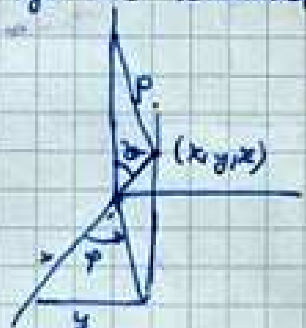
$0 \le r \le \sqrt{2}$

? ← csak hogy ki tudjuk integrálni

b)



gömbi transzformáció



$x = r\cdot\sin\vartheta\cos\varphi$

$y = r\sin\vartheta\sin\varphi$

$z = r\cdot\cos\vartheta$

$|J| = r^2\sin\vartheta$

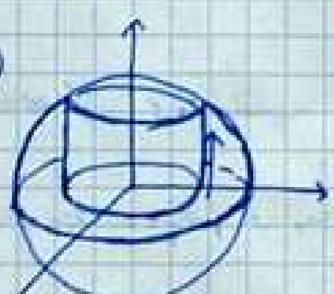
$0 \le \vartheta \le \frac{\pi}{4}$

$0 \le \varphi \le 2\pi$

$0 \le r \le \sqrt{6}$

$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{6}} r\sin\vartheta\cdot\cos\varphi\; r\cos\vartheta\; \underbrace{r\cdot\sin\vartheta}_{|J|}\, dr \overset{?}{=} \ldots$$

c.)

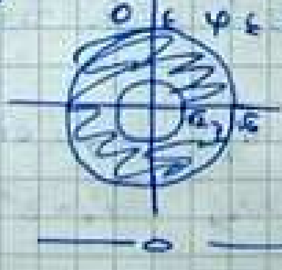


Hengerkoordinátás transzformáció

$0 \le z \le +\sqrt{6-r^2}$

$\sqrt{2} \le r \le \sqrt{6}$

$0 \le \varphi \le 2\pi$



---o---

Komplex függvény:

$w = f(z)$

$\varphi = \arg z \ (\in [-\pi, \pi])$

$w = u + jv = \rho e^{j\theta}$

$z = x + jy = r\cdot e^{j\varphi}$

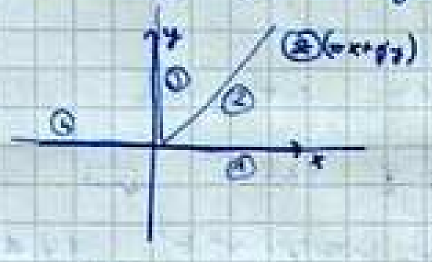
Pach Zs. Pálné: villamosmérnöki jegyzet
a jegyzetben ~~több~~ található még

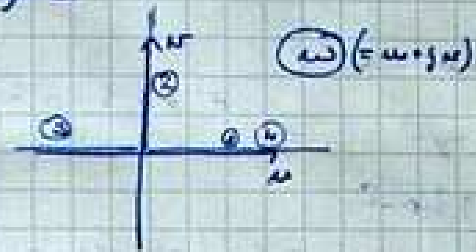
$$w = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + j\,\underbrace{2xy}_{v(x,y)} = r^2 \cdot e^{2j\varphi}$$

$$|w| = r^2 = |z|^2$$

$$\operatorname{arc} w = 2\varphi = 2 \operatorname{arc} z$$

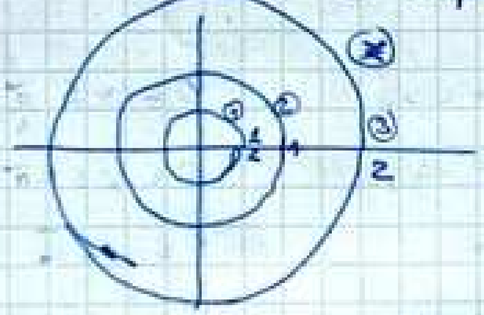
Ábrázolás: két sík segítségével

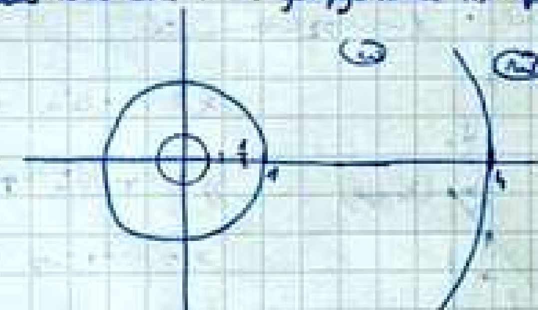




1. $\operatorname{arc} z = 0 \Rightarrow \operatorname{arc} w = 0$
2. $\operatorname{arc} z = \frac{\pi}{4} \Rightarrow \operatorname{arc} w = \frac{\pi}{2}$
3. $\operatorname{arc} z = \frac{\pi}{2} \Rightarrow \operatorname{arc} w = \pi$
4. $\operatorname{arc} z = \pi \Rightarrow \operatorname{arc} w = 2\pi$

a leképezés különleges: a félegyenes 1x fordulás körbe (z)-n az 0 körül





(w)-n 2x fordul körbe

1. $|z| = \frac{1}{2}$ $\quad |w| = |z|^2 = \frac{1}{4}$
2. $|z| = 1$ $\quad |w| = 1$
3. $|z| = 2$ $\quad |w| = 4$

(22) $f(z) = e^x \cdot \cos y + j e^x \sin y$ Hol differenciálható és hol reguláris? $f'(z) = ?$

$$\left.\begin{aligned} u(x,y) &= e^x \cdot \cos y \\ v(x,y) &= e^x \cdot \sin y \end{aligned}\right\} \forall$$ parc. der. $\exists$ és folyt. ($\Rightarrow$ u, v folyt. deriválható)

C–R ? $\quad u'_x = v'_y \qquad u'_x = e^x \cos y \qquad v'_y = e^x \cdot \cos y \qquad u'_x \equiv v'_y$

$\qquad u'_y = -v'_x \qquad u'_y = -e^x \sin y \qquad v'_x = e^x \sin y \qquad u'_y \equiv -v'_x$

$\Rightarrow$ f mindenütt deriválható $\Rightarrow$ f mindenütt reguláris.

$f'(z) = u'_x + j v'_x = e^x \cos y + j e^x \sin y = f(z)$ $\searrow$ ez lesz az e^z definíciója

Egyszeresen összefüggő tartomány (e.ö.t.)

bármely 2 pont a határon, tartománybeli vonallal összekötöm és szétesik 2 részre.

Nem e.ö.t.:

Szinguláris pont: ahol f nem reguláris

pl: nem értelmezett: $f(z)=\frac{1}{z}$ $z=0$ szing. pont.

Izolált szing. pont z_0, ha z_0-ban f nem reg, de $\exists\ \dot{K}_{z_0,\delta}$-ban igaz.

Elemi fv-k:

$$e^z = e^{x+jy} = e^x\cdot e^{jy} \overset{\text{Euler}}{=} e^x\cdot(\cos y + j\sin y)$$

$$\boxed{\begin{aligned} e^{j\varphi} &= \cos\varphi + j\sin\varphi \\ e^{-j\varphi} &= \cos\varphi - j\sin\varphi \end{aligned}} = \cos(-\varphi) + j\sin(-\varphi)$$

$$\downarrow$$

$$\cos\varphi = \frac{e^{j\varphi}+e^{-j\varphi}}{2} \qquad \sin\varphi = \frac{e^{j\varphi}-e^{-j\varphi}}{2j} \qquad \Rightarrow \varphi \to z$$

$\mathrm{sh}\, jz = j\sin z$

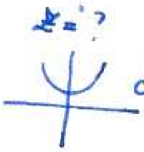
$\mathrm{ch}\, jz = \cos z$

$\sin jz = j\,\mathrm{sh}\, z$

$\cos jz = \mathrm{ch}\, z$

Anal elm

(Pl) $\operatorname{ch} z = 0$ $\quad z = ?$

(Valóstan  $\operatorname{ch} x > 0$)

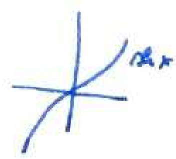
$\operatorname{ch}(x+jy) = \operatorname{ch} x \cdot \operatorname{ch} jy + \operatorname{sh} x \cdot \operatorname{sh} jy = \operatorname{ch} x \cos y + j \operatorname{sh} x \cdot \sin y$

(Hf : Mut. meg, hogy $(\operatorname{ch} z)' = \operatorname{sh} z$ (C.R))

$$\begin{cases} u(x,y) = \operatorname{ch} x \cdot \cos y = 0 & (1) \\ v(x,y) = \operatorname{sh} x \cdot \sin y = 0 & (2) \end{cases}$$

(1) $\operatorname{ch} x \neq 0 \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + k\pi$

(2) $\sin y \neq 0 \Rightarrow \operatorname{sh} x = 0 \Rightarrow x = 0$



$\operatorname{ch} z = 0$, ha $z = j\left(\frac{\pi}{2} + k\pi\right)$

HF : ell:
$\sin z = 0 \quad z = k\pi$
$\cos z = 0 \quad z = \frac{\pi}{2} + k\pi$
$\operatorname{sh} z = 0 \quad z = jk\pi$

(Pl) $f(z) = \operatorname{sh} 2\bar{z}$

a) Hol diff-ható és hol reguláris

b) $\operatorname{sh} 2\bar{z} = 0 \quad z = ?$

$f(z) = \operatorname{sh}(2(x-jy)) = \operatorname{sh} 2x \operatorname{ch} 2jy - \operatorname{ch} 2x \operatorname{sh} j2y = \operatorname{sh} 2x \cdot \cos 2y - j \operatorname{ch} 2x \cdot \sin 2y$

$$\begin{cases} u(x,y) = \operatorname{sh} 2x \cos 2y \\ v(x,y) = -\operatorname{ch} 2x \cdot \sin 2y \end{cases}$$
Totálisan deriválható, mert a parciálisok minden léteznek és folytonosak

C-R : $u'_x = v'_y$, $u'_y = -v'_x$

$u'_x = 2\operatorname{ch} x \cdot \cos 2y \quad v'_y = -2\operatorname{ch} 2x \cos 2y \Rightarrow \operatorname{ch} 2x \cdot \cos 2y =$

$\Downarrow$

$\operatorname{ch} 2x \neq 0 \Rightarrow \cos 2y$

$2y = \frac{\pi}{2} + k\pi$

$-v'_x = -(2 \operatorname{sh} 2x \sin 2y) \Rightarrow \operatorname{sh} 2x \underbrace{\sin 2y}_{\neq 0} = 0$

$u'_y = -v'_x$

$\Downarrow$

$x = 0$

$z = j\left(\frac{\pi}{4} + k \cdot \frac{\pi}{2}\right)$ pontokban deriválható, és sehol sem reguláris

b) $\operatorname{sh} z = 0 \quad z = jk\pi$

$\operatorname{sh} 2\bar{z} = 0 \Rightarrow 2\bar{z} = jk\pi$

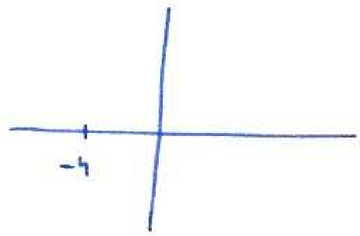
$\bar{z} = jk\frac{\pi}{2}$

$z = -jk\frac{\pi}{2}$

Pl $e^{j2\bar{z}} + 4 = 0$ $\quad z = ?$

$e^{j2\bar{z}} = -4$

$j2\bar{z} = \mathrm{Ln}\,-4$



$-4 = r \cdot e^{j\varphi} = -4 \cdot e^{j(-\pi)}$

$\ln(-4) = \ln 4 + j(-\pi)$

$\ln(z) = \ln \underbrace{|z|}_{r} + j \underbrace{\arg z}_{\varphi}$

$\mathrm{Ln}\,(-4) = \ln 4 + j(-\pi + 2k\pi)$

$j2\bar{z} = \ln 4 + j(-\pi + 2k\pi)$

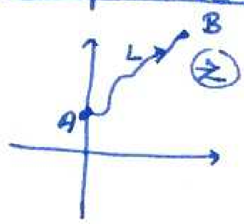
$\bar{z} = \frac{1}{2j} \ln 4 + \frac{1}{2}(-\pi + 2k\pi) = \frac{1}{2}(-\pi + 2k\pi) + j\left(-\frac{1}{2}\ln 4\right)$

$\frac{1}{j} = -j$

$\Downarrow$

$z = \frac{1}{2}(-\pi + 2k\pi) + j\,\frac{1}{2}\ln 8$

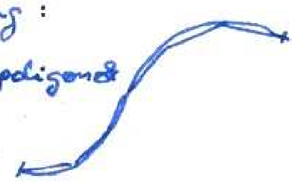
Komplex vonalintegrál



$L_{\overrightarrow{AB}}$

mérhető ívhosszúság:

sup{ leleírt húrpoligonok hossza }



Anal elm

2008.05.15.
Kónya

(Pl) Hol differenciálható, hol reguláris?

$f(z) = |z|^2$

$f(z) = x^2+y^2+j\cdot 0$ ~~$u(x^2+y^2)$~~ $u(x,y) = x^2+y^2$

$v(x,y) = 0$ totálisan deriválható mert...

C-R $u'_x = 2x$ $v'_y = 0$ $u'_x = v'_y \Rightarrow 2x = 0 \Rightarrow x = 0$

$u'_y = 2y$ $v'_x = 0$ $u'_y = -v'_x \Rightarrow 2y = 0 \Rightarrow y = 0$

metszete: $z = 0 (= 0 + j0)$

Tehát f $z=0$-ban deriválható, és sehol sem reguláris.

(Pl) Írja fel az összes olyan reguláris ffv-t, amelyre $u(x,y) (= \mathrm{Re}\, f) = \mathrm{sh}\, 2x \cdot \cos 2y + 3x$

$f = u + jv$, $\Delta u \equiv 0$, $\Delta v \equiv 0$ $f'(\pi j) = ?$

C-R: $u'_x = v'_y$

$u'_y = -v'_x$

$v'_y = 2\,\mathrm{ch}\, 2x \cos 2y + 3$ (1)

$v'_x = +2\,\mathrm{sh}\, 2x \sin 2y$ (2)

(1) $v(x,y) = \int (2\,\mathrm{ch}\, 2x \cos 2y + 3)\, dy = \mathrm{ch}\, 2x \sin 2y + 3y + C(x)$

(2) $2\,\mathrm{sh}\, 2x \sin 2y + C'(x) = 2\,\mathrm{sh}\, 2x \sin 2y$ $C'(x) = 0 \Rightarrow C(x) = k$

$f(z) = \mathrm{sh}\, 2x \cos 2y + 3x + j(\mathrm{ch}\, 2x \sin 2y + 3y + k)$ $k \in \mathbb{R}$

$f'(j\pi) = u'_x(0,\pi) + j\, v'_x(0,\pi) = \ldots$

(Pl) $\mathrm{Re}\, z_i = ?$ $\mathrm{Im}\, z_i = ?$

$z_1 = e^{\frac{\pi}{4} + j\frac{\pi}{4}} = e^{\frac{\pi}{4}} \left(\cos\frac{\pi}{4} + j\underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\right)$ $\mathrm{Re}\, z_1 = e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$ $\mathrm{Im}\, z_1 = e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$

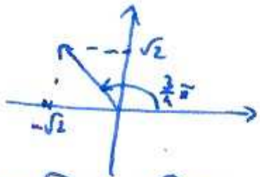
$e^{x+jy} = e^x(\cos y + j \sin y)$

$z_2 = \sin\left(\frac{\pi}{2} + j\pi\right) = \sin\frac{\pi}{2} \cos j\pi + \cos\frac{\pi}{2} \cdot \sin j\pi = \sin\frac{\pi}{2}\,\mathrm{ch}\,\pi + j\cos\frac{\pi}{2}\,\mathrm{sh}\,\pi$

$\mathrm{Re}\, z_2 = \mathrm{ch}\,\pi$ $\mathrm{Im}\, z_2 = 0$

HF: $z_3 = \mathrm{sh}(1+6j) = \ldots$

$z_4 = \ln(-\sqrt{2} + j\sqrt{2})$



$\ln z = \ln|z| + j \arg z$

$z = -\sqrt{2} + j\sqrt{2} = r \cdot e^{j\varphi} = 2 e^{j\frac{3}{4}\pi}$

$\sqrt{2+2} = 2$

$\ln(-\sqrt{2} + j\sqrt{2}) = \underbrace{\ln 2}_{\mathrm{Re}\, z_4} + j \underbrace{\tfrac{3}{4}\pi}_{\mathrm{Im}\, z_4}$

$z_5 = \mathrm{Ln}(-\sqrt{2} + j\sqrt{2}) = \ln 2 + j\left(\tfrac{3}{4}\pi + 2k\pi\right)$

$\mathrm{Ln}\, z = \ln|z| + j(\arg z + 2k\pi)$

$z_6 = (-\sqrt{2} + j\sqrt{2})^j = ? = e^{j \ln(-\sqrt{2} + j\sqrt{2})} = e^{j(\ln 2 + j\cdot\frac{3}{4}\pi)} = e^{-\frac{3}{4}\pi + j\ln 2} = e^{-\frac{3}{4}\pi}(\cos \ln 2 + j \sin \ln 2)$

$e^{\ln \ldots}$

$\mathrm{Re}\, z_6 = e^{-\frac{3}{4}\pi} \cdot \cos \ln 2$

$\mathrm{Im}\, z_6 = e^{-\frac{3\pi}{4}} \cdot \sin \ln 2$

Pl $\sin z = 2 \qquad z = ?$

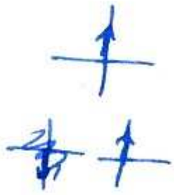
$\sin z = \dfrac{e^{jz} - e^{-jz}}{2j} = 2 \Rightarrow \underbrace{e^{jz}}_{=:u} - \dfrac{1}{e^{jz}} = 4j \quad / \cdot u$

$\sqrt{12} = \sqrt{12}\cdot\sqrt{-1} \qquad u^2 - 4ju - 1 = 0$

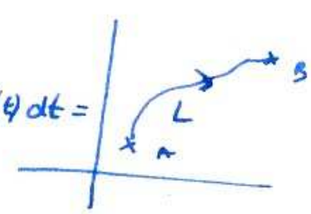
$u_{1,2} = \dfrac{4j + \sqrt{-16+4}}{2} = (2 \pm \sqrt{3})\, j$

$e^{jz} = (2+\sqrt{3})\, j \Rightarrow jz = \mathrm{Ln}(2+\sqrt{3})j = \ln(2+\sqrt{3}) + j\left(\tfrac{\pi}{2} + 2k\pi\right) \ldots$

$e^{jz} = (2-\sqrt{3})\, j \Rightarrow jz = \mathrm{Ln}(2+\sqrt{3})j = \ldots$



Integrálok

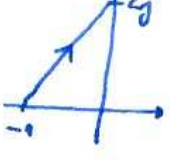
$\displaystyle\int_L f(z)\, dz = \int_\alpha^\beta f(z(t))\, z'(t)\, dt =$  $= \left(F(z)\Big|_A^B\right)$

$z = z(t)$

$A = z(\alpha)$

$B = z(\beta)$

Pl $\displaystyle I = \int_L (\mathrm{Im}\, 2\bar{z} + \mathrm{sh}\, 5z)\, dz = ?$



$y = 2x + 2$

$\displaystyle I = \int_L \underbrace{\mathrm{Im}\, 2\bar{z}}_{\mathrm{Im}(2x - 2jy) = -2y}\, dz + \int_L \mathrm{sh}\, 5z\, dz = I_1 + I_2$

$I_2 = \dfrac{\mathrm{ch}\, 5z}{5}\Big|_{z=-1}^{z=2j} = 5\big(\underbrace{\mathrm{ch}(10j)}_{\cos 10} - \underbrace{\mathrm{ch}(-5)}_{\mathrm{ch}\, 5}\big) = \frac{1}{5}(\cos 10 \ldots$

$x := t \qquad y := 2t + 2$

$z(t) = t + (2t+2)j$

$z'(t) = x'(t) + j y'(t) = 1 + 2j$

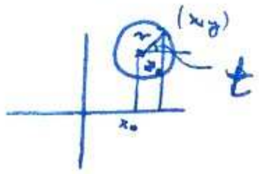
$t: -1 \to 0$

$\displaystyle I_1 = \int_{-1}^{0} \overbrace{-2(2t+2)}^{f(z(t))} \overbrace{(1+2j)}^{z'(t)}\, dt = -2(1+2j)\int_{-1}^{0}(2t+2)\, dt = (-2)(1+2 \ldots$

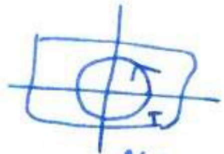
$\cdot (0 - (1-2))$

$$I=\oint_{|z|=2}\left(\frac{1}{\bar{z}}+z\cos z\right)dz=?=I_1+I_2$$

$|z-z_0|=r$



$$I_2=\oint_{|z|=2} z\cos z\,dz=0 \quad \text{C.f.a.t}$$



$f(z)=z\cdot\cos z$ reg

$z_0=x_0+jy_0$

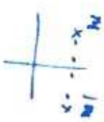
$$x=x_0+r\cdot\cos t=x(t)$$
$$y=y_0+r\cdot\sin t=y(t)$$

$$z(t)=x_0+r\cos t+(y_0+r\cdot\sin t)j=x_0+jy_0+r(\cos t+j\sin t)=z_0+r\cdot e^{jt}$$

$$z(t)=2\cos t+j2\sin t=2e^{jt}$$
$$t: 0\to 2\pi$$

$$z'(t)=2je^{jt}$$

$$I_2=\int_0^{2\pi}\frac{1}{-j\cdot 2e^{-jt}}\cdot\underbrace{2je^{jt}}_{z'(t)}dt=-\int_0^{2\pi}e^{j2t}dt=-\frac{e^{j2t}}{j2}\Bigg|_{t=0}^{2\pi}=j\frac{1}{2}\cdot\Big(\underbrace{e^{j4\pi}}_{=1}-\underbrace{e^{j0}}_{=1}\Big)=0$$

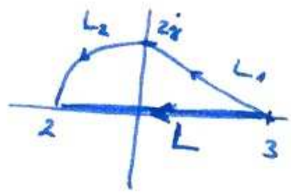


$\bar{j}=-j$

$\bar{z}(t)=2e^{-jt}$

(Nem csak reg. fv-nek lehet az integrálja 0 zárt görbére)

Pl $\displaystyle\int_{L_1\cup L_2}\underbrace{\frac{z-1}{e^{2z}}}_{reg}dz=?$



$$\int_{L_1}\ldots+\int_{L_2}\ldots$$

De egyszerűbb: Cauchy f.a.t.

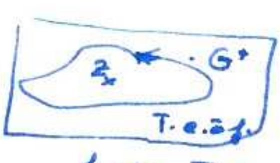
$$\int_{L_1\cup L_2}\ldots=\int_L\ldots=\int_3^{-2}\frac{t-1}{e^{2t}}\cdot 1\,dt=\int_3^{-2}(t-1)e^{-2t}dt=\ldots$$

parc. int.

$z(t)=t+0\cdot j$

$z'(t)=1$

$t: 3\to -2$



f reg. T-n

Cauchy-féle integrálformula:

$$f(z)=\frac{1}{2\pi j}\oint_{G^+}\frac{f(\zeta)}{\zeta-z}d\zeta$$

Általánosított Cauchy-féle integrálformula:

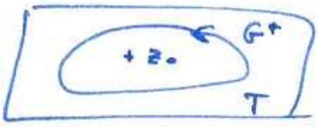
$$f^{(n)}(z)=\frac{n!}{2\pi j}\oint_{G^+}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta \qquad n=1,2,\ldots$$

z helyett z_0

ζ helyett z

$$f(z_0)=\frac{1}{2\pi j}\oint_{G^+}\frac{f(z)}{z-z_0}\,dz$$

$$f^{(n)}(z_0)=\frac{n!}{2\pi j}\oint_{G^+}\frac{f(z)}{(z-z_0)^{n+1}}\,dz$$



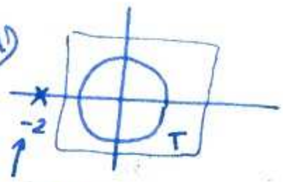
$$\oint_{G^+}\frac{f(z)}{z-z_0}\,dz=2\pi j\,f(z_0)$$

$$\oint_{G^+}\frac{f(z)}{(z-z_0)^{n+1}}\,dz=\frac{2\pi j}{n!}\,f^{(n)}(z_0)$$

f reg T-n

$$\oint_L\frac{e^{2zj}}{z+2}\,dz=?\qquad L:\ \text{a)}\ |z|=1\quad \text{b)}\ |z-6|=3\quad \text{c)}\ |z-j|=5$$

a)



Szinguláris pont (Serenset)

a) ~~f~~el tudok venni T-t, hogy g reg T-n $\Rightarrow I=0$

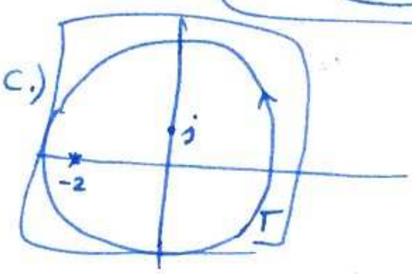
C.f.a.t

Cauchy-Goursat

b.)

g reg T-n $\Rightarrow I=\oint g(z)\,dz=0$

c.)



f reg T-n

$$\oint_L\frac{e^{2jz}}{z+2}\,dz=2\pi j\,e^{2jz}\Big|_{z=-2}=2\pi j\,e^{j(-4)}=$$

$z_0=-2$

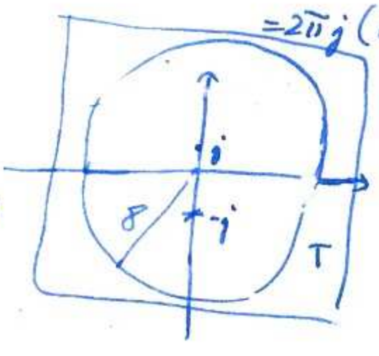
$$=2\pi j\left(\cos(-4)+j\sin(-4)\right)=\dots$$

Pl

$$\oint_{|z-j|=8}\frac{\sin 2z}{(z+j)^3}\,dz=?=\frac{2\pi j}{2!}(\sin 2z)''\Big|_{z=-j}=$$

$2\sin 2z$

$-4\cos 2z$



$z_0=-j$

$n+1=3$

$n=2$

$$=\frac{2\pi j}{2!}(-4\sin 2z)\Big|_{z=-j}=\dots$$

$$\sin(-2j)=j\,\mathrm{sh}(-2)=j\,\mathrm{sh}\,2.$$

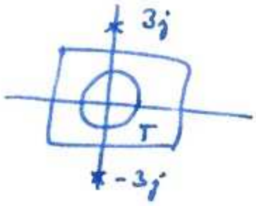
Pl $\displaystyle\oint_L \frac{z^3 e^{j\pi z}}{z^2+9}\,dz = ?$

L: a) $|z|=1$

b) $|z+2j|=1$

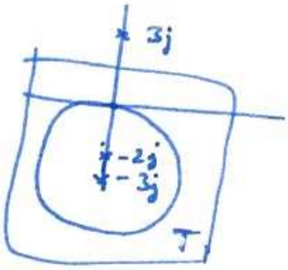
c) $|z|=5$

a) $z^2 = \pm 3j$



$I_a = 0$ (C f.a.t)

b)

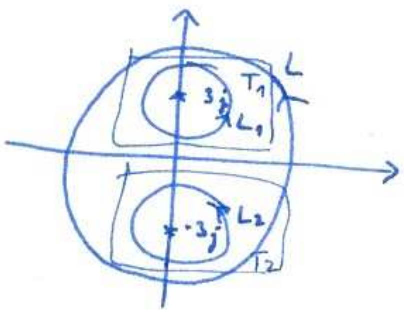


$$\oint \frac{\dfrac{z^3 \cdot e^{j\pi z}}{z-3j}}{z-(-3j)} = 2\pi j \left.\frac{z^3 \cdot e^{j\pi \cdot z}}{z-3j}\right|_{z=-3j} = \dots$$

$f(z)$ reg T-n

$z_0=-3j$

c.)



tétel: $$\oint_L = \oint_{L_1} + \oint_{L_2} = \oint_{L_1} \frac{\dfrac{z^3 \cdot e^{j\pi z}}{z+3j}}{z-3j}\,dz + \oint_{L_2} \frac{\dfrac{z^3 \cdot e^{j\pi z}}{z-3j}}{z+3j}\,dz$$

$f(z)$ reg T_1-n $f(z)$ reg T_2-n

$z_0=3j$

$$= 2\pi j \left.\frac{z^3 \cdot e^{j\pi z}}{z+3j}\right|_{z=3j} + 2\pi j \left.\frac{z^3 e^{j\pi z}}{z-3j}\right|_{z=-3j}$$

HF: $$\oint_{|z|=2} \left(\frac{\operatorname{sh} 5z}{z-j} + \operatorname{ch} 3z + \frac{z}{(z-1)^3(z+1)} \right) dz \,.$$

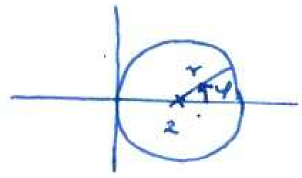
$$\oint_{|z|=4} \frac{\cos 3z}{z^2(z+2j)^5}\,dz$$

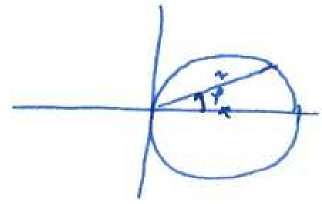
(E1) $\iint_T (x^2-4x+y^2)^5\, dT = ?$

$T:\ x^2-4x+y^2 \le 0$

$y \ge 0$

$(x-2)^2+y^2 \le 2$





$\alpha)\ \begin{cases} x = 2 + r\cdot\cos\varphi \\ y = r\cdot\sin\varphi \end{cases}$

$|J| = r$

$0 \le \varphi \le \pi$

$0 \le r \le 2$

$\beta)\ x = r\cdot\cos\varphi$

$y = r\cdot\sin\varphi$

$|J| = r$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le r \le 4\cos\varphi$

ez most nem jó

$\int_0^2\int_0^\pi (r^2-4)^5 \cdot r\, d\varphi\, dr = (\pi-0)\,\frac{1}{2}\int_0^2 2r(r^2-4)^5\, dr = \ldots$

$((x-2)^2+y^2-4)^5$

$r^2\cos^2\varphi + r^2\sin^2\varphi$

(P2) $f(x) = (1+3x)^{1/9}$

$g(x) = (1+x^2)^{1/7}$

T.sor $x_0 = 0$

$\lim_{x\to 0} \frac{3f(x) - 7g(x) + 4}{x^4} = ?$

$(1+u)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} u^n \quad R=1$

$f(x) = \sum_{n=0}^\infty \binom{1/9}{n}(3x^2)^n = \sum_{n=0}^\infty \binom{1/9}{n} 3^n \cdot x^{2n}$

$|3x^2| = 3|x|^2 < 1$

$|x| < \frac{1}{\sqrt{3}} = R_f$

$g(x) = \sum_{n=0}^\infty \binom{1/7}{n} x^{2n}$

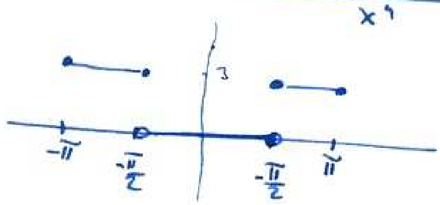
$|x^2| = |x|^2 < 1$

$|x| < 1 = R_g$

$\lim_{x\to 0} \frac{3\left(1 + \frac{1/9}{1}\cdot 3\cdot x^2 + \underbrace{\frac{(1/9)(-8/9)}{1\cdot 2}\cdot 3^2}_{=:C_1} x^4 + \ldots\right) - 7\left(1 + \frac{1/7}{1}\cdot x^2 + \underbrace{\frac{(1/7)(-6/7)}{2}}_{=:C_2} x^4 + \ldots\right) + 4}{x^4} =$

$= \frac{(3C_1 - 7C_2)x^4 + C_3x^6 + \ldots}{x^4} \xrightarrow[x\to 0]{} 3C_1 - 7C_2$

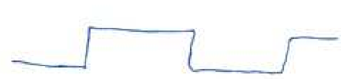
(F2)



$f(x) = \begin{cases} 3, \text{ ha } x \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi] \\ 0, \text{ ha } |x| < \frac{\pi}{2} \end{cases}$

2π szerint periodikus

Fourier sor? $\sim \phi(x) = !$



$\phi(x) = \begin{cases} f(x) & x \ne \frac{\pi}{2} + k2\pi \\ \frac{3}{2} & \text{ha } x = \frac{\pi}{2} + k\cdot 2\pi \end{cases}$

$\phi(x) = \frac{a_0}{2} + a_1\cdot\cos x + b_1\cdot\sin x + a_2\cos 2x + b_2\sin 2x + \ldots = \frac{a_0}{2} + \sum_{k=0}^\infty (a_k\cos kx + b_k\cdot\sin kx)$

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx$

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx$

f páros $\Rightarrow b_k = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 3\,dx = \frac{2}{\pi}\cdot 3\left(\pi - \frac{\pi}{2}\right) = 3$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos kx}_{ps}\,dx = \frac{1}{\pi}\cdot 2\cdot\int_0^{\pi} f(x)\cos kx\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} 3\cos kx\,dx = \frac{6}{\pi}\cdot\frac{\sin k\pi}{k}\Bigg|_{\pi/2}^{\pi} =$$

$$= \frac{6}{\pi k}\left(\sin k\pi - \sin k\frac{\pi}{2}\right) = \frac{-6}{\pi k}\sin k\frac{\pi}{2} = -\frac{6}{\pi}\cdot\frac{1}{k}\cdot\sin k\frac{\pi}{2}$$

$$\phi(x) = \frac{3}{2} - \frac{6}{\pi}\left(\frac{1}{1}\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cdot\cos 5x - \frac{1}{7}\cos 7x + \dots\right)$$

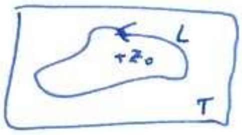
Pl

$$\int_L f(z)\,dz = \int_{\alpha}^{\beta} f(z(t))\cdot z'(t)\,dt$$

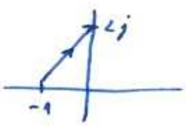
$$\int_L f(z)\,dz = F(z)\Big|_A^B$$

C. f. a. t is lehet

$$\oint_L \frac{f(z)^{\text{reg } T\text{-n}}}{(z-z_0)^{n+1}}\,dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$



$$\int_L \left(\sin 2\bar{z} - \frac{1}{z^2}\right) dz$$



konjugált, abszolút érték
⇓
paraméterezés

$I = I_1 - I_2$

$$I_2 = \int_L z^{-2}\,dz = \frac{z^{-1}}{-1}\Bigg|_{-1}^{2j} = -\left(\frac{1}{2j} - \frac{1}{(-1)}\right) = \dots$$

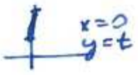
$y = 2x+2 \qquad x := t$

$z(t) = t + j(2t+2)$, ahol $t = x(t)$, $2t+2 = y(t)$

$z'(t) = 1 + j\cdot 2$

$x := t: \ -1 \to 0$

Körnél: $x(t) = r\cos t$, $y(t) = r\cdot\sin t$



$x=0$, $y=t$

$\sin 2\bar{z} = \sin 2\overline{(x+jy)} = \sin 2(x-jy)$

$$I_1 = \int_{-1}^{0} \sin\left(2t - j(4t+4)\right)\cdot(1+j\cdot 2)\,dt = (1+j\cdot 2)\int_{-1}^{0} \sin\left(2t - j(4t+4)\right)dt = (1+j\cdot 2)\frac{-\cos(2t-j(\dots}{2-j4}$$

$$= \frac{1+j\cdot 2}{2-4j}(-1)\cdot\left(\cos(-4j) - \cos(-2)\right)$$

konjugálttal bővítünk; $\operatorname{ch}(-4) = \operatorname{ch} 4$

Pl

$$\oint_{|z-j|=3} \frac{e^{jz}-1^{\text{ reg}}}{z^4}\,dz = \frac{2\pi j}{3!}\left(e^{jz}-1\right)'''\Big|_{z=0} = \frac{2\pi j}{3!}(-j)\underbrace{e^0}_{1}$$

$z_0 = 0$

$n+1 = 4 \Rightarrow n = 3$

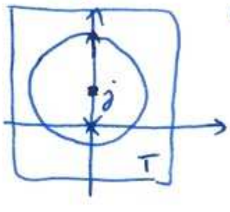
$j\cdot e^{jz}$

$j^2\cdot e^{jz}$

$j^3\cdot e^{jz}$

$|z-z_0| = r$

z_0 kp-ú r sugarú kör

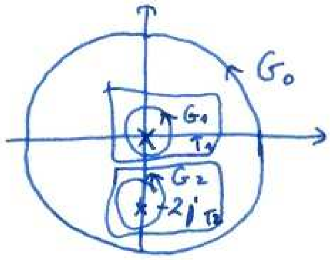


Anal konzultáció

2008.05.28.
Kónya

Pl. $\oint_{|z|=4} \frac{\cos 3z}{z^2(z+2j)^5}\,dz = \oint_{G_1} \frac{\frac{\cos 3z}{(z+2j)^5}}{(z-0)^2}\,dz + \oint_{G_2} \frac{\frac{\cos 3z}{z^2}}{(z-(-2j))^5}\,dz = \frac{2\pi j}{1!}\cdot\left(\frac{\cos 3z}{(z+2j)^5}\right)'\Big|_{z=0} +$

f reg T_1-en   f reg T_2-n

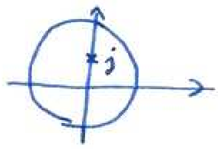


$\oint_{G_0} = \oint_{G_1} + \oint_{G_2}$

$+ \frac{2\pi j}{4!}\left(\frac{\cos 3z}{z^2}\right)^{IV}\Big|_{z=-2j}$

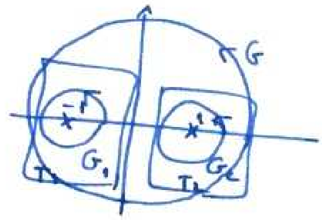
$\frac{-3\sin 3z\,(z+2j)^5 - \cos 3z\,(z+2j)^4}{(z+2j)^{10}}$

Pl. $\oint_{|z|=2}\left(\frac{sh\, 5z}{z-j} + ch\, 3z + \frac{3}{(z-1)^3\cdot(z+1)}\right)dz$



$I = I_1 + I_2 + I_3$

$I_2 = \oint_{|z|=2} \underbrace{ch\, 3z}_{reg}\, dz = 0 \quad (C\text{-}f.a.t)$



$I_1 = \oint_{|z|=2} \frac{sh\, 5z}{z-j}\,dz = 2\pi j\, \underbrace{sh\, 5z}_{sh\, 5j = j\sin 5}\Big|_{z=j} = -2\pi \sin 5$

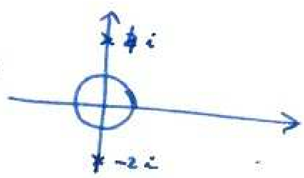
$I_3 = \oint_{G} \frac{3}{(z-1)^3(z+1)}\,dz = \oint_{G_1} \frac{\frac{3}{(z-1)^3}}{z+1}\,dz + \oint_{G_2} \frac{\frac{3}{z+1}}{(z-1)^3}\,dz = 2\pi j\,\frac{3}{(z-1)^3}\Big|_{z=-1} + \frac{2\pi j}{2!}\left(\frac{3}{z+1}\right.$

f reg T_1-en   f reg T_2-n

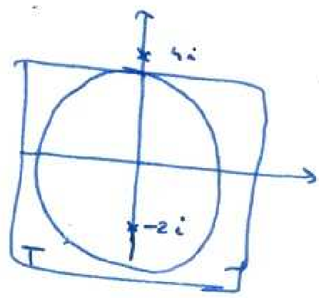
$-3(z+1)^3$

$6(z+1)^3$

Pl. a.) $\oint_{|z|=1} \underbrace{\frac{z}{(z+2i)(z-4i)}}_{reg}\,dz = 0 \quad$ C.f.a.t.



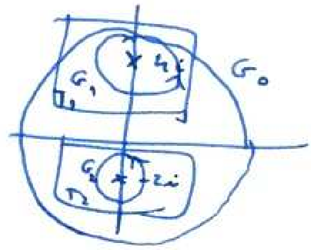
b.) $\oint_{|z|=3} \frac{z}{(z+2i)(z-4j)}\,dz = ?$ Szing: $z=-2i$, $z=4j$



$\oint \frac{\frac{z}{z-4i}}{z+2i}\,dz = 2\pi j\,\frac{z}{z-4i}\Big|_{z=-2i} = \dots$

reg T-n

c.) $\oint_{|z|=5} \frac{z}{(z+2i)(z-4i)}\,dz =$

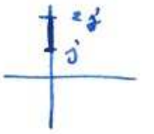


$= \oint_{G_2} \frac{\frac{z}{z-4i}}{z+2i}\,dz + \oint_{G_1} \frac{\frac{z}{z+2i}}{z-4i} = 2\pi j\,\frac{z}{z-4i}\Big|_{z=-2i} + 2\pi j\,\frac{z}{z+2i}\Big|_{z=4i}$

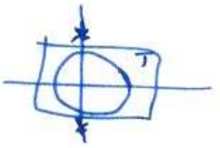
f reg T_2-n   f reg T_1-n

(Pe) a) $\oint\limits_{|z|=1} \frac{e^{2z}}{z^2+4}\,dz = ?$

b.) $\int\limits_L \bar{z}\, z\, dz = ?$



a) $z^2+4=0$
$z^2=-4$
$\sqrt{-4} = \pm 2j$



$I_a = 0$
f reg. T - m. (C. f.c.t)

b) $x=0$
$y:=t$
$z(t) = 0+jt.$
$z'(t) = j$

$$\int\limits_L \bar{z}\, z\, dz = \int\limits_\alpha^\beta f(z(t)) \cdot z'(t)\, dt = \int\limits_1^2 t^2 j\, dt = j\, \frac{t^3}{3}\Big|_1^2 = \dots$$

$\bar{z}\cdot z = |z|^2 = x^2+y^2$

(Pe) $z_1 = e^{1-\frac{\pi}{4}j}$

$e^z = e^{x+jy} = e^x \cdot e^{jy} = e^x(\cos y + j\sin y)$

$x=1 \quad y = -\frac{\pi}{4}$

$z = e\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right) = e\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$\operatorname{Re} z = \frac{e}{\sqrt{2}}$

$\operatorname{Im} z = -\frac{e}{\sqrt{2}}$

(Pe) $z_2 = \ln(-3) = ?$

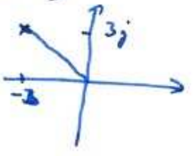
$z_3 = \operatorname{Ln}(-3) = ?$

$\ln z = \ln|z| + j \arg z \qquad \arg z \in [-\pi, \pi)$

$\operatorname{Ln} z = \ln|z| + j(\arg z + 2k\pi)$

$\ln(-3) = \ln 3 + j(-\pi)$
$\operatorname{Ln}(-3) = \ln 3 + j(-\pi + 2k\pi)$

$z_4 = \ln(-3+3j)$



$|-3+3j| = \sqrt{9+9} = \sqrt{18}$

$z_4 = \ln\sqrt{18} + j\left(3\frac{\pi}{4}\right)$

$z_5 = \ln 0 \ \nexists$