Adott: $\int_0^3 x^2 f(x)^2\,dx = 1$

Kell: alsó becslés $I := \int_0^3 |f(x)|^3\,dx$ -re.

Megoldás

$$1 = \int_0^3 x^2 f(x)^2\,dx \le \int_0^3 |x^2 f(x)^2|\,dx$$

Ez legyen 3

$\hookrightarrow p = \frac{3}{2}$

$\hookrightarrow q = 3$

$$\overset{H_q}{\le} \left(\int_0^3 (x^2)^q\,dx\right)^{1/q} \left(\int_0^3 |f(x)^2|^p\,dx\right)^{1/p} = \left(\int_0^3 x^{2q}\,dx\right)^{1/q} \left(\int_0^3 |f(x)|^{2p}\,dx\right)^{1/p}$$

$$= \left(\int_0^3 x^6\,dx\right)^{1/3} \left(\int_0^3 |f(x)|^3\,dx\right)^{2/3} = \left(\frac{1}{7}3^7\right)^{1/3} \left(\int_0^3 |f(x)|^3\,dx\right)^{2/3}$$

Egyenlőség, ha $|f(x)|^3 = \lambda x^6$

$$\hookrightarrow 1 \le \left(\frac{1}{7}3^7\right)^{\frac{1}{2}} \int_0^3 |f(x)|^3\,dx \hookrightarrow I \ge \sqrt{\frac{7}{3^7}}$$

Ez optimális becslés, mert: $f(x) = c \cdot x^2$ esetén végig tud egyenlőség lenni.

Adott: $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ és $\int_{-\infty}^{\infty} f(t-s)\, e^{-|2s|}\, ds = \frac{1}{1+t^2}$

$$\mathcal{F}_\pm D_\lambda g = |\lambda|\, D_{1/\lambda} \mathcal{F}_\pm g$$

Kell: $I = \int_{-\infty}^{\infty} \frac{\sin(t)\, f(t)}{t}\, dt$

mert $\tilde{\mathcal{F}}_+$ unitér

Megoldás

$$I = \langle \operatorname{sinc}, f \rangle = \langle \tilde{\mathcal{F}}_+ \operatorname{sinc}, \mathcal{F}_+ f \rangle = \langle \sqrt{\tfrac{\pi}{2}}\, \chi_{[-1,1]}, \mathcal{F}_+ f \rangle = \sqrt{\tfrac{\pi}{2}} \int_{-1}^{1} \mathcal{F}_+ f(\omega)\, d\omega$$

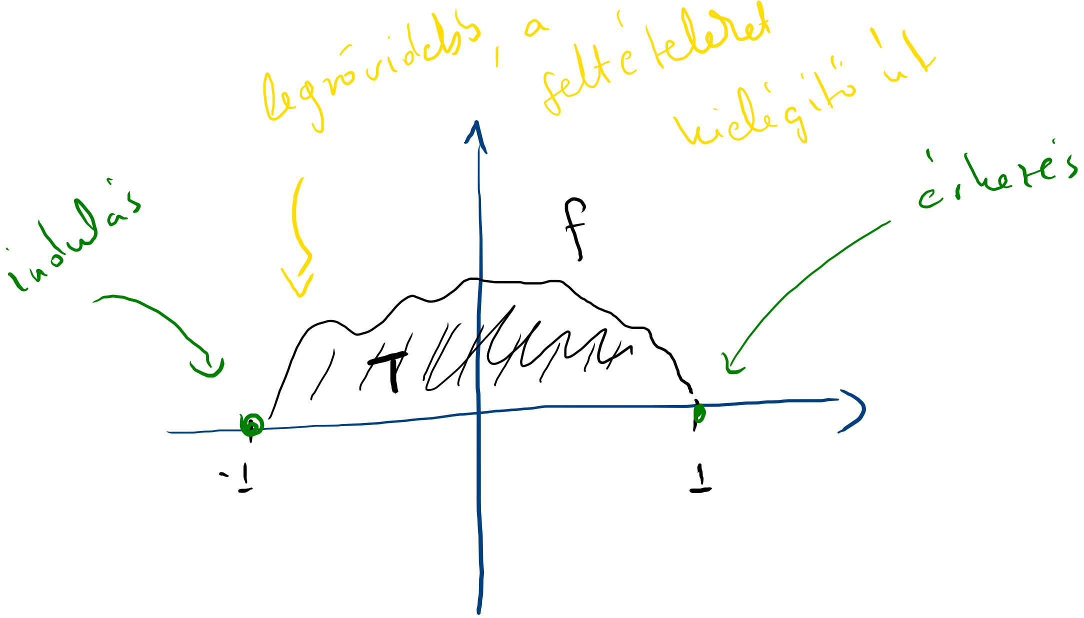
$$g(x) := e^{-|x|} \rightsquigarrow \int_{-\infty}^{\infty} f(t-s)\, e^{-|2s|}\, ds = (f * D_{1/2}\, g)(t), \quad h(x) := \frac{1}{1+t^2} \rightsquigarrow f * D_{1/2}\, g = h \quad \Big|\, \mathcal{F}_+$$

$$\sqrt{2\pi}\, \mathcal{F}_+ f \underbrace{\mathcal{F}_+ D_{1/2}\, g}_{\frac{1}{2} \cdot D_2 \mathcal{F}_+ g \,=\, \frac{1}{2} D_2 \sqrt{\frac{2}{\pi}}\, h} = \mathcal{F}_+ h = \sqrt{\tfrac{\pi}{2}}\, g$$

$$\Rightarrow \mathcal{F}_+ f \cdot D_2 h = \sqrt{\tfrac{\pi}{2}}\, g \quad \text{vagyis}$$

$$\mathcal{F}_+ f(\omega) \cdot \frac{1}{1+(\omega/2)^2} = \sqrt{\tfrac{\pi}{2}}\, e^{-|\omega|} \Rightarrow \mathcal{F}_+ f(\omega) = \left(1+\left(\tfrac{\omega}{2}\right)^2\right) \sqrt{\tfrac{\pi}{2}}\, e^{-|\omega|}$$

$$\Leftrightarrow I = \frac{\pi}{2} \int_{-1}^{1} \left(1+\left(\tfrac{\omega}{2}\right)^2\right) e^{-|\omega|}\, d\omega = \pi \int_{0}^{1} \left(1+\left(\tfrac{\omega}{2}\right)^2\right) e^{-\omega}\, d\omega = \text{(számológép megmondja)}$$



$$\int_{-1}^{1} f \doteq T \in (0, \pi/2)$$

<u>Átfogalmazás</u>: ? $f : [-1,1] \to \mathbb{R}$ (kellően szép), mely kielégíti a $\begin{cases} f(-1) = 0 \\ f(1) = 0 \end{cases}$ peremfeltételeket, az $\int_{-1}^{1} f = T$ integrál-feltételt és ezen feltételek mellett minimalizálja az

$$I = \int_{-1}^{1} \sqrt{1 + (f'(x))^2}\, dx$$

integrált.

Lagrange-multiplikátor

$$\left.\begin{aligned}\mathcal{L}(f, f') &= \sqrt{1+(f')^2} \\ \mathcal{G}(f, f') &= f\end{aligned}\right\} \qquad \tilde{\mathcal{L}}(f, f') = \sqrt{1+(f')^2} - \lambda f$$

E-L:

$$\underbrace{\frac{\partial \tilde{\mathcal{L}}}{\partial f}} - \underbrace{\left(\frac{\partial \tilde{\mathcal{L}}}{\partial f'}\right)'} = 0$$

$$-\lambda - \left(\frac{f'}{\sqrt{1+(f')^2}}\right)' = 0$$

$f'(x)$ és $Ax+B$ előjele azonos

$$\Rightarrow \frac{f'(x)}{\sqrt{1+(f'(x))^2}} = Ax+B$$

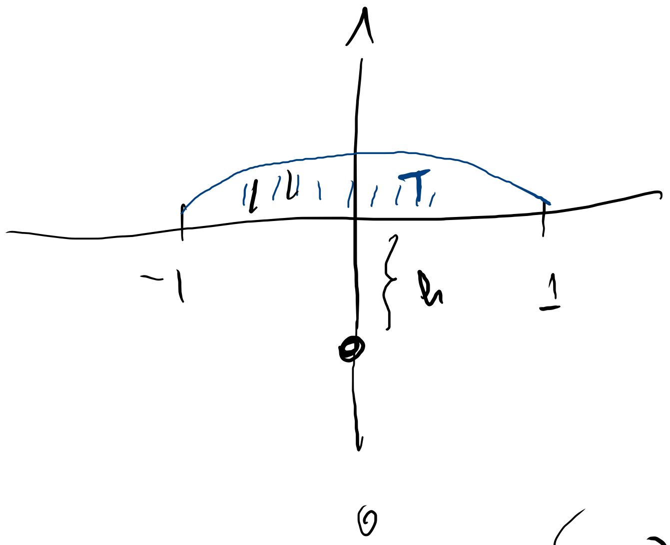
$$\Rightarrow f'(x)^2 = (Ax+B)^2\left(1+(f'(x))^2\right)$$

$$\Rightarrow (f'(x))^2\left(1-(Ax+B)^2\right) = (Ax+B)^2$$

$$\Rightarrow f'(x) = \frac{Ax+B}{\sqrt{1-(Ax+B)^2}}$$

$$f'(x) = \frac{Ax+B}{\sqrt{1-(Ax+B)^2}} \Rightarrow \text{mivel } f(-1)=0, \text{ ezért}$$

$$f(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} \frac{At+B}{\sqrt{1-(At+B)^2}}\,dt = \ldots \text{valami} \ldots (A, B, x)$$



$f \geq 0, \; h \geq 0$

$$(f(x)+h)^2 + x^2 = h^2+1$$

$$(f(x)+h)^2 = h^2+1-x^2$$

$$f(x) = \sqrt{h^2+1-x^2} - h$$

$\Leftrightarrow f'$ a kívánt alakú.