EPR pardox $\longrightarrow$ Bell's inequality

* Dirac Notations

bra-ket notation

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \langle a| = (a_1 \; a_2 \; \ldots \; a_n)$$

ket $\qquad\qquad\qquad\qquad$ bra

$$|a\rangle + |b\rangle = |a+b\rangle$$

$$\alpha|a\rangle = |\alpha a\rangle$$

$$\langle a|b\rangle = \langle a_1^* \; a_2^* \; a_3^* \rangle \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}$$

$$\langle a|b\rangle = \langle b|a\rangle^*$$

$M^\dagger = \overline{M^T}$

if $M = M^*$ M is Hermitian

$$M|a\rangle = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

if $MM^* = I$ unitary matrix

H - Hermitian matrix

$H|a\rangle = \lambda_a |a\rangle$ → eigen

λ eigen value

H - measurable

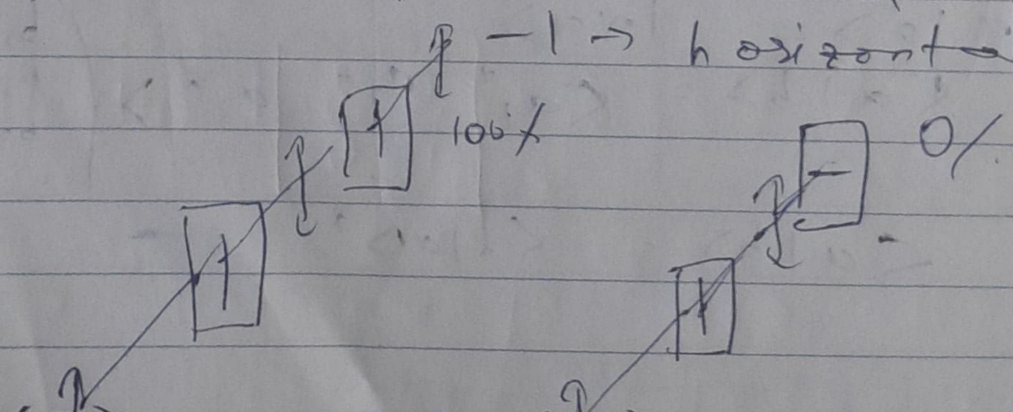
$|a\rangle$ - possible states of the system

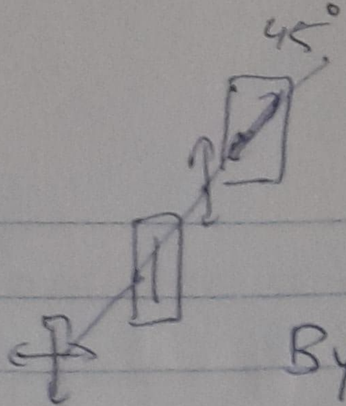
λ - actual values when measurement is made

E.g. H → Polarizer to detect the polarization of a light

$|a\rangle$ → vertically polarized
horizontally polarized

$\lambda = \pm 1$ +1 → vertical
−1 → horizontal





$$I = I_0 \cos^2\theta$$

By classical mechanics

$I \propto E^2$ $\qquad E = hf$

$\therefore I \propto f^2$

But this does not happen
Intensity is independent of frequency
Rather it depends on # of photons

Quantum mechanical explaination

Intensity reduces because the number of photons passing through polarizer change

$\leftrightarrow \Rightarrow$ horizontal polarization

$|x\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\updownarrow \Rightarrow$ vertical polarization

$|y\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\langle x|y\rangle$ - What is the chance that light in y state goes through x state?

$$\langle x|y\rangle = [1 \;\; 0]\begin{bmatrix}0\\1\end{bmatrix} = 0$$

Probability amplitude

$$|\langle x|y\rangle|^2 = \langle x|y\rangle\langle y|x\rangle = 0$$

$H_{\oplus} \rightarrow$ polarizer (Either vertical or horizontal)

$$H_{\oplus}|x\rangle = \lambda_x|x\rangle \;\&\; H_{\oplus}|y\rangle = \lambda_y|y\rangle$$

$\lambda_x = 1$ $\quad\quad$ $\lambda_y = -1$

$$H_{\oplus}\begin{bmatrix}1\\0\end{bmatrix} = +1\begin{bmatrix}1\\0\end{bmatrix} \quad H_{\oplus}\begin{bmatrix}0\\1\end{bmatrix} = -1\begin{bmatrix}0\\1\end{bmatrix}$$

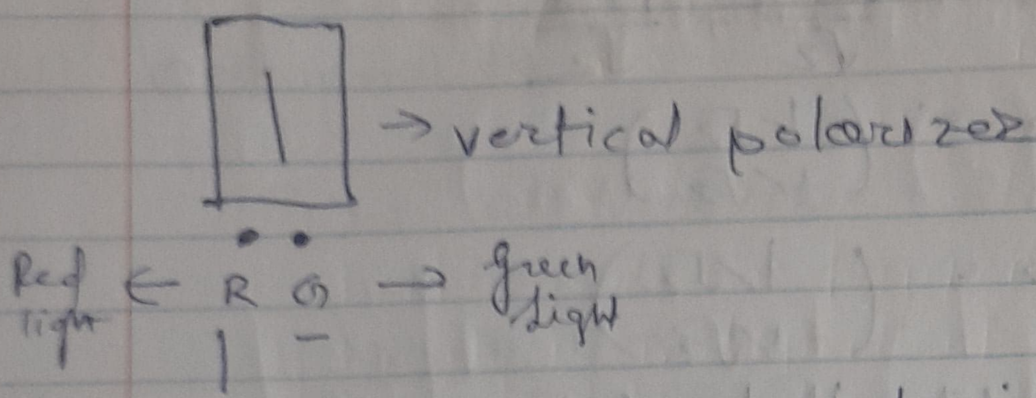
$$\begin{bmatrix}a & c\\b & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}a & c\\b & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

$\therefore a = 1 \quad c = 0$

$\quad b = 0 \quad d = -1$

$$\therefore H_{\oplus} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Consider a device



This measures polarizability
This equipment corresponds to H

Eigenvalues are result i.e. either red/green light passes

$|x\rangle = -$ $\quad |y\rangle = |$

$\therefore |/\rangle = |x\rangle + |y\rangle = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$

$\therefore |/\rangle = \begin{pmatrix}1\\1\end{pmatrix}$

$|\langle / | / \rangle|^2 = 100\%$ (prob. amplitude)

$|\langle / | / \rangle|^2 = \langle / | / \rangle \langle / | / \rangle = 2 \times 2 = 4$

$\langle / | / \rangle = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 2$

but $|\langle / | / \rangle| \leq 1$

$$\therefore |/\rangle = \frac{|x\rangle}{\sqrt{2}} + \frac{|y\rangle}{\sqrt{2}}$$

$$|/\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$|\backslash\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Probability that $|\backslash\rangle$ goes through $|/\rangle$

$$|\langle \backslash | / \rangle|^2 = \left(\tfrac{1}{\sqrt{2}} \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{2} \\ +\tfrac{1}{\sqrt{2}} \end{pmatrix} = 0$$

Probability that | goes through /

$$\langle / | 1 \rangle = \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

~~In~~ Intensity (or probability amplitude)

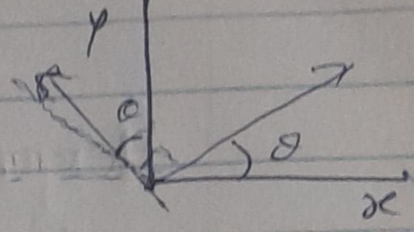
$$|\langle / | 1 \rangle|^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Here we have considered x, y as our basis vector

$|-\rangle = 1|x\rangle + 0|y\rangle$ $\qquad$ $|1\rangle = 0|x\rangle + 1|y\rangle$

$|/\rangle = \frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle$ $\qquad$ $|\backslash\rangle = \frac{1}{\sqrt{2}}|x\rangle - \frac{1}{\sqrt{2}}|y\rangle$

$\therefore |\theta\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle$



Orthogonal to θ

$|\theta_\perp\rangle = \cos(90+\theta)|x\rangle + \sin\theta(90+\theta)|y\rangle$

$= -\sin\theta|x\rangle + \cos\theta|y\rangle$

$|\theta_\perp\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Probability that $|\theta\rangle$ passes through x

$\langle x|\theta\rangle = (1 \quad 0)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta$

$\boxed{\therefore |\langle x|\theta\rangle|^2 = \cos^2\theta}$

Generalization

$\langle\beta|\alpha\rangle = (\cos\beta \quad \sin\beta)\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\langle\beta|\alpha\rangle = \cos(\alpha-\beta)$

$\boxed{|\langle\beta|\alpha\rangle|^2 = \cos^2(\alpha-\beta)}$

Hermitian operator for θ

$$\hat{H} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$|\langle\theta|\partial\rangle| = [\cos\theta \ \sin\theta]\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

$$= \frac{\cos\theta + i\sin\theta}{\sqrt{2}}$$

$$|\langle\theta|\partial\rangle|^2 = \langle\theta|\partial\rangle\langle\partial|\theta\rangle$$

$$= \left(\frac{\cos\theta + i\sin\theta}{\sqrt{2}}\right)\left(\frac{\cos\theta - i\sin\theta}{\sqrt{2}}\right)$$

$$= \frac{\cos^2\theta + \sin^2\theta}{2}$$

$$\boxed{|\langle\theta|\partial\rangle|^2 = \frac{1}{2}}$$

∂ - circular polarization

Circular polarized photon

$$|\partial\rangle = \alpha|x\rangle + \beta|y\rangle$$

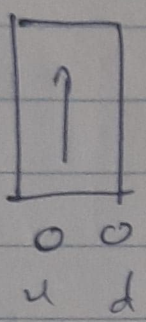
$$\alpha^2 + \beta^2 = 1$$

$$\alpha = \frac{1}{\sqrt{2}} \quad \beta = \frac{i}{\sqrt{2}}$$

$$|g\rangle = \frac{1}{\sqrt{2}}|x\rangle - \frac{i}{\sqrt{2}}|y\rangle$$

* Applying for an electron

$|\nearrow\rangle = \alpha|u\rangle + \beta|d\rangle$



$P_u = \alpha\alpha^* \quad P_d = \beta\beta^*$

$\alpha\alpha^* + \beta\beta^* = 1$

$|\uparrow\rangle = 1|u\rangle + 0|d\rangle$

$|\downarrow\rangle = 0|u\rangle + 1|d\rangle$

$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$

$P_u = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \quad P_d = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

Pauli Matrices for measuring spin along

x axis (r/l) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

y axis (i/o) $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

z axis (u/d) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

	Eigen vector	Corresponding Eigen Values	
σ_z	$\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$	$+1$	-1
σ_x	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ $\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$	$+1$	-1
σ_y	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\ i/\sqrt{2}\end{pmatrix}$ $\begin{pmatrix}i/\sqrt{2}\\-i/\sqrt{2}\end{pmatrix}$	$+1$	-1

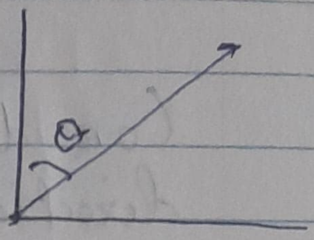
E.g $\langle r|u\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

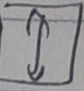
$|\langle r|u\rangle|^2 = \frac{1}{2}$

$\langle I|L\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix} = \frac{i+1}{2}$

$|\langle I|L\rangle|^2 = \frac{(1+i)}{2}\frac{(1-i)}{2} = \frac{1}{2}$

$\langle \uparrow | \rightarrow \rangle = \alpha |u\rangle + \beta |d\rangle$



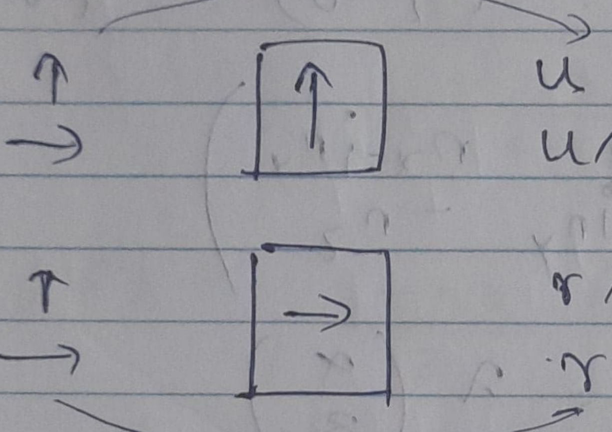


$\alpha - \beta = \cos \theta/2$

Prob. amplitude $= \cos^2 \frac{\theta}{2}$

$\therefore |\theta\rangle = \cos\frac{\theta}{2}|u\rangle + \sin\frac{\theta}{2}|d\rangle$

Experiment (Bell's inequality)



getting u here

or getting l here

tells us nothing about the spin of e^-

↑↓ $\boxed{\uparrow}$ → definite proof

$|\uparrow\rangle = 1|u\rangle + 0|d\rangle$

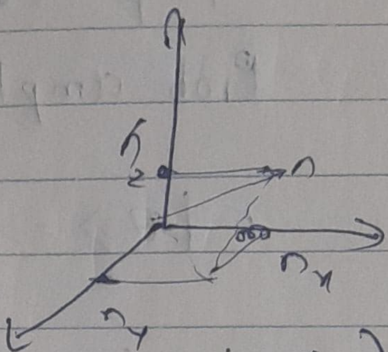
$|L\rangle = 0|u\rangle + 1|d\rangle$

$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$

$|I\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle$ $\quad |O\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

Consider an electron with spin in direction of a vector $\hat{n}$ (unit vector)



$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z = \hat{\sigma}_n$$

$$\hat{\sigma}_n = n_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \hat{\sigma}_n = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\therefore \hat{\sigma}_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$

$$\therefore \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = +1 \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$$

$$n_z + \gamma n_x - i \gamma n_y = 1$$

$$n_x + i n_y - \gamma n_z = \gamma$$

$$\therefore \gamma = \frac{1-n_z}{n_x - in_y}$$

$$|\psi\rangle = \alpha \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \qquad \psi\psi^* = 1$$

$$\alpha (1 \;\; \gamma^*) \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = 1$$

$$\therefore \alpha^2 (1 + \gamma^* \gamma) = 1$$

$$\therefore \alpha = \sqrt{\frac{1+n_z}{2}}$$

$$|\psi\rangle = \sqrt{\frac{1+n_z}{2}} \begin{pmatrix} 1 - n_z \\ n_x - in_y \end{pmatrix} \qquad \lambda = \pm 1$$

For a photon

$$|?\rangle = \frac{1}{\sqrt{2}} |x\rangle + \frac{1}{\sqrt{2}} |y\rangle$$

$$|\psi\rangle = a_1 |x_1\rangle + a_2 |x_2\rangle + \cdots + a_n |x_n\rangle$$

$$\therefore |\psi\rangle = \sum_{i=1}^{n} a_i |x_i\rangle$$

$$|\phi\rangle = \sum_{j=1} b_j |x_j\rangle$$

$\langle \phi | = \sum_j \langle x_j | b_j^*$

$\langle \phi | \psi \rangle = \sum \langle x_j | b_j^* \sum a_i | x_i \rangle$

$\langle \phi | \psi \rangle = \sum_{j,i} b_j^* a_i \langle x_j | x_i \rangle$

if $j \neq i$ $\quad \langle x_j | x_i \rangle = 0$

else $\quad = 1$.

$\langle \phi | \psi \rangle = \sum_j b_j^* a_j$

For electron

$|r\rangle = \frac{1}{\sqrt{2}} |u\rangle + \frac{1}{\sqrt{2}} |d\rangle$

$\cancel{\langle u | r \rangle = 1}$

$|\psi\rangle = \sum_i a_i |x_i\rangle$

$\langle x_j | \psi \rangle = \langle x_j | \sum a_i | x_i \rangle$

$= \sum a_i \langle x_j | x_i \rangle$.

$= \sum_j a_j$

$\langle x_j | \psi \rangle = \sum_j \psi_j$

$$\sum |x_i\rangle\langle x_i| = I$$

$$\sum |x_i\rangle\langle x_i|\psi\rangle = \sum |x_i\rangle\langle x_i| \sum_j a_j |x_j\rangle$$

$$= \sum_{ij} a_j |x_j\rangle\langle x_i|x_j\rangle$$

$$= \sum_i a_j |x_j\rangle$$

$$= \sum_i \psi_i$$

$$\boxed{\therefore I|\psi\rangle = |\psi\rangle}$$

* To check $\hat{H}$ is hermitian

(1) $\hat{H}|\psi\rangle = \lambda|\psi\rangle$

$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\lambda|\psi\rangle$

$= \lambda\langle\psi|\psi\rangle$

$$\boxed{\langle\psi|\hat{H}|\psi\rangle = \lambda}$$

(2) $\langle\phi|\hat{H}|\psi\rangle$

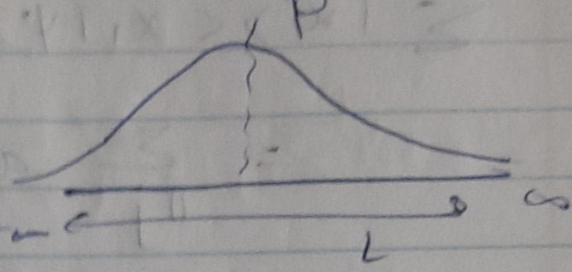
$\langle\phi|\hat{H}|\psi\rangle^* = \langle\psi|H^\dagger|\phi\rangle$

if H is hermitian $H = H^\dagger$

$$\boxed{\langle\phi|H|\psi\rangle^* = \langle\psi|H|\phi\rangle}$$

$\hat{x}|\psi\rangle = x|\psi\rangle$ $\hat{x}$ - position operator

What is probability of finding e^- from 0 to x



$$|\psi\rangle = \sum_i a_i|x_i\rangle = \sum_{-\infty}^{\infty} \psi_i \, dx$$

$$\langle\phi|\psi\rangle = \sum_j \phi_j^* \psi_j = \int_{-\infty}^{\infty} \psi(x)\,\phi\,\psi^*(x)\,dx$$

$-i\frac{d}{dx}|\psi\rangle$ - Hermitian (proof)

$$\langle\phi|\hat{H}|\psi\rangle^* = \langle\psi|\hat{H}|\phi\rangle$$

$$\left\langle\phi\middle| -i\frac{d\psi}{dx}\right\rangle^* = \left\langle\psi\middle|-i\frac{d\phi}{dx}\right\rangle$$

$$-i\int dx\,\psi^*\frac{d\phi}{dx} = \left[-i\int dx\,\phi^*\frac{d\psi}{dx}\right]^*$$

$$-i\int dx\,\psi^*\frac{d\phi}{dx} = \left[i\int dx\,\frac{d\phi^*}{dx}\psi\right]^*$$

Momentum operator

$$-i \frac{d|\psi\rangle}{dx} = k|\psi\rangle$$

$$\frac{d}{dx}|\psi\rangle = ik|\psi\rangle \qquad |\psi\rangle = e^{ipx/\hbar}$$

$$\frac{d|\psi\rangle}{dx} = \frac{ip}{\hbar}|\psi\rangle$$

$$\therefore -i\hbar \frac{d|\psi\rangle}{dx} = p|\psi\rangle$$

$$\hat{p}|\psi\rangle = p|\psi\rangle \qquad \therefore \hat{p} = -i\hbar\frac{d}{dx}$$

(momentum operator)

* Heisenberg's Uncertainity Principle

To ~~measurent~~ measure any two quantities at the same time commutator of their respective operator must be zero

i.e $\hat{M}_1|a\rangle = \alpha|a\rangle$

$\hat{M}_2|a\rangle = \beta|a\rangle$

$$M_1 M_2|a\rangle = M_1 \beta|a\rangle = \beta M_1|a\rangle = \beta\alpha|a\rangle$$

$$M_2 M_1|a\rangle = M_2 \alpha|a\rangle = \alpha M_2|a\rangle = \alpha\beta|a\rangle$$

$\therefore \langle (M_1 M_2 - M_2 M_1) | a \rangle = 0$

$\therefore \langle [M_1, M_2] | a \rangle = 0$

$[M_1, M_2] = 0$ since $|a\rangle \neq 0$

if we want to measure M_1 M_2 should have two state common eigen vector at the same time

to measure position and momentum at the same time

$[\hat{x}, \hat{p}] = 0$

$$\hat{x}\hat{p} - \hat{p}\hat{x} = x\left(-i\hbar \frac{d}{dx} |\psi\rangle\right) - \left(-i\hbar \frac{d}{dx} (x|\psi\rangle\right)$$

$$\therefore [\hat{x}, \hat{p}] = -i\hbar x \frac{d|\psi\rangle}{dx} + i\hbar \left(|\psi\rangle + x\frac{d|\psi\rangle}{dx}\right)$$

$$= \cancel{-i\hbar x \frac{d|\psi\rangle}{dx}} + \cancel{i\hbar x \frac{d|\psi\rangle}{dx}} + i\hbar |\psi\rangle$$

$\therefore [\hat{x}, \hat{p}] = i\hbar \neq 0$

$\therefore$ $\hat{x}$ and $\hat{p}$ cannot be measured at the same time

$$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$$

Can we measure along x & z simultaneously?

$$[\hat{\sigma}_x \, \hat{\sigma}_z] = \sigma_z \sigma_x - \sigma_x \sigma_z$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$= \frac{2}{-i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[\hat{\sigma}_x, \hat{\sigma}_z] = 2i\sigma_y \neq 0$$

Similarly for $[\sigma_x, \sigma_y]$, $[\sigma_y, \sigma_z]$