Complex Analysis

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1 Complex Numbers and Properties

1.1 Definitions

A complex number is a number of the form $z = x + iy$, where x, y are real numbers and i is the imaginary unit, satisfying the equation $i^2 = -1$. The set of complex numbers is denoted by $\mathbb{C}$.

$$\boxed{\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}}$$

- For any complex number $z = x + iy$, x is called the real part of z and y is called the imaginary part of z. That is, $Re(z) = x$ and $Im(z) = y$.
- Since $i = \sqrt{-1}$, we have
  $$\Rightarrow i^2 = -1$$
  $$\Rightarrow i^4 = 1$$
  Hence $i^{4m} = 1$ for any $m \in \mathbb{Z}$.
- The conjugate of a complex number $z = x + iy$ is given by $\overline{z} = x - iy$.
- Modulus of a complex number z, $|z| = \sqrt{x^2 + y^2}$

1.2 Properties

If $z_1 = a + ib$ and $z_2 = c + id$ where $z_2 \neq 0$, then

- $z_1 + z_2 = (a + c) + i(b + d)$
- $z_1 - z_2 = (a - c) + i(b - d)$
- $z_1 z_2 = (ac - bd) + i(ad + bc)$
- $\frac{z_1}{z_2} = (\frac{ac+bd}{c^2+d^2}) + i(\frac{bc-ad}{c^2+d^2})$
- $arg(z_1) = \theta = tan^{-1}\frac{b}{a}$

- Polar form of a complex number $z = x + iy$ having $|z| = r$ is given by

$$z = r\cos\theta + ir\sin\theta$$
$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$
$$\Rightarrow z = re^{i\theta}$$

- $arg(z_1z_2) = arg(z_1) + arg(z_2)$
- $arg\frac{z_1}{z_2} = arg(z_1) - arg(z_2)$
- $arg(z^n) = n * arg(z)$
- $|z|^2 = z\overline{z}$

1.3 Examples

- Evaluate $\displaystyle\int_0^{2\pi} \cos^4\theta d\theta$.

<u>Solution.</u> We have that,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\Rightarrow cos^4\theta = \Big(\frac{e^{i\theta} + e^{-i\theta}}{2}\Big)^4$$
$$= \frac{1}{16}(e^{4i\theta} + 4e^{3i\theta}.e^{-i\theta} + 6e^{2i\theta}.e^{-2i\theta} + 4e^{i\theta}.e^{-3i\theta} + e^{-4i\theta})$$
$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6e^0 + 4e^{-2i\theta} + e^{-4i\theta})$$
$$= \frac{1}{16}[(e^{4i\theta} + e^{-4i\theta}) + 4(e^{2i\theta} + e^{-2i\theta}) + 6]$$
$$= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$$
$$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

$$\therefore \int_0^{2\pi} \cos^4\theta d\theta = \int_0^{2\pi} \frac{1}{8}(\cos^4\theta + 4\cos 2\theta + 3)d\theta \quad (1)$$
$$= \frac{1}{8}(\frac{1}{4}\sin 4\theta + \frac{4}{2}\sin 2\theta + 3\theta)\Big|_0^{2\pi} \quad (2)$$
$$= \frac{1}{8}(\frac{1}{4}\sin 8\pi + 2\sin 4\pi + 6\pi \ - \frac{1}{4}\sin 0 - 2\sin 0 - 0) \quad (3)$$
$$= \frac{1}{8}.6\pi \quad (4)$$
$$= \frac{3\pi}{4} \quad \square \quad (5)$$

2 Sequences of Complex Numbers

2.1 Convergence

A sequence $\{z_n\}$ of complex numbers is said to converge to a complex number w if for every $\epsilon > 0$ there exists a positive integer n_0 such that $|z_n - w| < \epsilon$ for all $n_{\geq} n_0$. **OR** a sequence of complex numbers is said to converge to $w \in \mathbb{C}$ if

$$\boxed{\lim_{n\to\infty} z_n = w}$$

.

2.2 Cauchy Sequence

A sequence $\{z_n\}$ of complex numbers is said to be Cauchy if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $m, n \geq n_0$.

For any $z_n = x_n + iy_n$, $\{z_n\}$ is Cauchy if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy.

2.3 Theorem: $\mathbb{C}$ is complete

Statement $\mathbb{C}$, the set of complex numbers, is complete.

Proof Let $z_n = x_n + y_n$ form a Cauchy sequence then $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since $\mathbb{R}$ is complete, $\{x_n\}$ and $\{y_n\}$ converge to x and y in $\mathbb{R}$ respectively. Then for any $\epsilon > 0$ there exist two natural numbers n_1 and n_2 such that $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$.

Now choose $n_0 = max\{n_1, n_2\}$ and let $w = x + iy$.

$$\begin{aligned}
\therefore |z_n - w| &= |(x_n + iy_n) - (x + iy)| && (6)\\
&= |(x_n - x) + i(y_n - y)| && (7)\\
&\leq |x_n - x| + |y_n - y| && (8)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (9)\\
&< \epsilon && (10)
\end{aligned}$$

So for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|z_n - w| < \epsilon$ for all $n \geq n_0$
Thus $\{z_n\}$ converges to w, hence $\mathbb{C}$ is complete. $\square$

3 Sets in Complex Plane

3.1 Open Disc

$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$

3.2 Closed Disc

$\overline{D_r}(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$

3.3 Interior Point

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an interior point of Ω if there exists $r > 0$ such that
$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\} \subset \Omega$.

3.4 Open Set

A set $\Omega \subset \mathbb{C}$ is open if every point in Ω is an interior point of Ω.

3.5 Interior

The interior of Ω consists of all its interior points.

3.6 Limit Point

A point $z \in \mathbb{C}$ is said to be a limit point of a set $\Omega \subset \mathbb{C}$ if there exists a sequence points $z_n \in \Omega$ s.t. $z_n \neq z$ and

$$\lim_{n \to \infty} z_n = z$$

3.7 Closed Set

A set $\Omega \subset \mathbb{C}$ is closed if $\Omega^c = \mathbb{C} - \Omega$ is open.

3.8 Closure($\overline{\Omega}$)

The closure of any set Ω is the union of Ω and its limit points.

3.9 Diameter

If Ω is bounded, we define its diameter by

$$diam(\Omega) = Sup_{z,w \in \Omega} |z - w|$$

3.10 Compact Set

A set Ω is said to be compact if it is closed and bounded.

3.11 Open Covering

An open covering of a set Ω is the family of open sets $\{U_\alpha\}$ (not necessarily countable) such that $\Omega \subset \{U_\alpha\}$.

3.12 Connected Sets

An open set $\Omega \subset \mathbb{C}$ is said to be connected if it is not possible to find two disjoint non-empty open sets Ω_1 and Ω_2 s.t. $\Omega = \Omega_1 \cup \Omega_2$.

4 Complex-valued Functions

Let $\Omega \subset \mathbb{C}$, a function is defined on Ω is a rule which assigns every $z \in \Omega$ to a complex number w i.e. $f(z) = w$.

- For $w = f(z) = u(z) + iv(z)$, $Re(f) = u$ and $Im(f) = v$.

4.1 Examples

- Given $f(z) = \frac{z}{1+z}$, find real and imaginary part of f..

  Solution. Let $z = x + iy$, for $x, y \in \mathbb{R}$

$$
\begin{aligned}
f(z) &= \frac{x+iy}{1+(x+iy)} && (11)\\
&= \frac{x+iy}{(1+x)+iy} && (12)\\
&= \frac{x+iy}{(1+x)+iy} * \frac{(1+x)-iy}{(1+x)-iy} && (13)\\
&= \frac{x+x^2-ixy+iy+ixy-i^2y^2}{(1+x)^2-i^2y^2} && (14)\\
&= \frac{x+x^2+y^2+iy}{(1+x)^2+y^2} && (15)\\
&= \frac{x^2+y^2+x}{(1+x)^2+y^2} + i\frac{y}{(1+x)^2+y^2} && (16)
\end{aligned}
$$

  $Re(f) = \frac{x^2+y^2+x}{(1+x)^2+y^2}$ and $Im(f) = \frac{y}{(1+x)^2+y^2}$

- Find the domain of $f(z) = \frac{1}{z^2+4}$.

  Solution. $f(z)$ is not defined for $z^2 + 4 = 0$

$$\Rightarrow z^2 = -4$$

$$\Rightarrow z = \sqrt{-4}$$
$$\Rightarrow z = \pm 2i$$

Therefore, the domain of f is $\mathbb{C} - \{\pm 2i\}$.

4.2 Region

A non-empty open connected subset of $\mathbb{C}$ is called a region.

4.3 Domain

An open connected set is called a domain.

4.4 Limit of a Function

Consider a function $f(z)$ defined on D, we say that L is the limit of the function $f(z)$ as $z \to z_0$ if for every $\epsilon > 0$ there exists $\delta > 0$ s.t.

$|f(z) - L| < \epsilon$ whenever $|z - z_0| < \delta$.

- We say that $\lim_{z\to z_0} f(z) = L$.

4.5 Algebra of Limits

Let f and g be two complex valued functions defined in a neighbourhood of z_0 except possibly at z_0 and given that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then

- $\lim_{z\to z_0}[f(z) \pm g(z)] = L \pm M$
- $\lim_{z\to z_0} f(g(z)) = LM$
- $\lim_{z\to z_0} \frac{f}{g}(z) = \frac{L}{M}$ for $M \neq 0$.
- $\lim_{z\to z_0} \overline{f(z)} = \overline{L}$
- $\lim_{z\to z_0} |f(z)| = |L|$.

4.6 Limit at Infinity

$$\lim_{z\to\infty} f(z) = L$$

if for given $\epsilon > 0$ there exists any $R > 0$ s.t.

$|f(z) - L| < \epsilon$ for $|z| > R$.

4.7 Infinite Limits

$\lim_{z \to z_0} f(z) = \infty$ if for given $R > 0$ there exists $\delta > 0$ s.t.

$|f(z) > R$ if $|z - z_0| < \delta$.

4.8 Continuous Functions

Let f be complex valued function defined on a set Ω of complex numbers. We say f is continuous at a point $z_0 \in \Omega$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

If f is continuous at z_0, then:

- $Re(f)$ is continuous at z_0
- $Im(f)$ is continuous at z_0
- $\overline{f}$ is continuous at z_0
- $|f|$ is continuous at z_0
- αf is continuous at z_0, where α is any scalar.

5 Differentiability, Holomorphic Functions or Analytic Functions

5.1 Definition

Let f be a complex-valued function in a domain Ω then f is said to be holomorphic at a point $z_0 \in \Omega$ if the limit:

$$\boxed{\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}}$$

exists and we denote it as $f'(z_0)$.

- If f is holomorphic at every point of Ω then f is called holomorphic on Ω.
- If Ω is a closed subset of $\mathbb{C}$ then we say that f is holomorphic on Ω if f is holomorphic on an open set containing Ω.

5.2 Entire Functions

A function f is said to be entire if f is holomorphic entirely on $\mathbb{C}$.

5.3 Analytic Functions

A function $f(z) = u(x, y) + iv(x, y)$ is analytic if

- f satisfies Cauchy-Riemann equations:

$$u_x = v_y \tag{17}$$
$$u_y = -v_x \tag{18}$$

- u_x, u_y, v_x and v_y exist and are continuous.

5.4 Theorem

Suppose $f(z) = u(x, y) + iv(x, y)$ is complex valued function on an open set Ω. If u and v are continuously differentiable and satisfy $C - R$ equations on Ω, then f is holomorphic on Ω and $f'(z) = \frac{\partial f}{\partial z}$.

Proof :- Given that (i) u and v are continuously differentiable and (ii) both satisfy $C - R$ equations on Ω.

To show that f is holomorphic, we have to prove that f is complex differentiable at every point of Ω. f is differentiable at a point $z \in \Omega$ if the limit:

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists, where h is a complex constant.

Let $z = x + iy$ and $h = \Delta x + i\Delta y$.
So, $f(z) = u(x, y) + iv(x, y)$
and

$$f(z + h) = f((x + iy) + (\Delta x + i\Delta y)) \tag{19}$$
$$= f((x + \Delta x) + i(y + \Delta y)) \tag{20}$$
$$= u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \tag{21}$$

Now,

$$\frac{f(z + h) - f(z)}{h} = \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \tag{22}$$

$$= \frac{[u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]}{\Delta x + i\Delta y} \tag{23}$$

Using Taylor's series expansion of a function $g(x, y)$,

$g(x + \Delta x, y + \Delta y) - g(x, y) = g_x \Delta x + g_y \Delta y + O(|h|)$

where $O(|h|) \to 0$ faster than $|h|$ when $h \to 0$. That is,

$$\boxed{\lim_{h \to 0} \frac{O(|h|)}{h} = 0}$$

Equation (13) becomes,

$$\frac{f(z+h) - f(z)}{h} = \frac{[u_x \Delta x + u_y \Delta y + O(|h|)] + i[v_x \Delta x + v_y \Delta y + O(|h|)]}{\Delta x + i\Delta y} \tag{24}$$

By $C - R$ equations, $u_x = v_y$ and $u_y = -v_x$, So

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{[u_x \Delta x - v_x \Delta y + O(|h|)] + i[v_x \Delta x + u_x \Delta y + O(|h|)]}{\Delta x + i\Delta y} && (25)\\
&= \frac{\Delta x(u_x + iv_x) + \Delta y(iu_x - v_x) + O(|h|)(1+i)}{\Delta x + i\Delta y} && (26)\\
&= \frac{\Delta x(u_x + iv_x) + i\Delta y(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{O(|h|)(1+i)}{h} && (27)\\
&= \frac{(\Delta x + i\Delta y)(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{O(|h|)(1+i)}{h} && (28)\\
&= (u_x + iv_x) + \frac{O(|h|)(1+i)}{h} && (29)
\end{aligned}$$

Taking $\lim_{h \to 0}$ both sides,

$$\begin{aligned}
f'(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} && (30)\\
&= \lim_{h \to 0} [(u_x + iv_x) + \frac{O(|h|)(1+i)}{h}] && (31)\\
&= u_x + iv_x && (32)
\end{aligned}$$

This shows that f is complex differentiable at any point of Ω, therefore f is holomorphic on Ω. $\square$

5.5 Conjugate Harmonic Functions

Let $f = u + iv$ be an analytic function then v is harmonic conjugate of u and u is harmonic conjugate of $-v$.

- If $f = u + iv$ is an analytic function and u is harmonic conjugate of v as well as v is harmonic conjugate of u, then f is constant.
- If u is a harmonic function then $f(z) = u_x - iu_y$ is analytic.
- If u and v are conjugate harmonic functions, then uv is harmonic.

5.6 Problems on Analytic Functions

- Is f analytic, if $f(z) = (2x - 3y) + i(3x + 2y)$?
  Solution. Here

  $$u(x, y) = 2x - 3y \tag{33}$$
  $$v(x, y) = 3x + 2y \tag{34}$$

  $u_x = 2$,$u_y = -3$,$v_x = 3$ and $v_y = 2$
  $u_x = v_y$ and $u_y = -v_x$

  So f is analytic.

- If $f = u + iv$ is analytic and $u = v^2$ then prove that f is constant.
  Solution. Given $u = v^2$

  Differentiating w.r.t. x partially,

  $\frac{\partial u}{\partial x} = 2v\frac{\partial v}{\partial x}$

  Again differentiating w.r.t. y partially,

  $\frac{\partial u}{\partial y} = 2v\frac{\partial v}{\partial y}$

  From the above,

$$
\begin{align}
2v = \frac{u_x}{v_x} = \frac{u_y}{v_y} \tag{35}\\
\Rightarrow \frac{u_x}{v_x} = \frac{-v_x}{u_x} \tag{36}\\
\Rightarrow u_x^2 = -v_x^2 \tag{37}\\
\Rightarrow u_x^2 = -(-u_y^2) \tag{38}\\
\Rightarrow u_x^2 + u_y^2 = 0 \tag{39}\\
\tag{40}
\end{align}
$$

So u is constant $\Rightarrow f$ is constant. $\square$

- Compute the complex form of $C - R$ equations.

  Solution. The Cauchy-Riemann equations, for a complex function $f(z) = u(x, y) + iv(x, y)$ are:

  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  In terms of complex derivatives, using the Wirtinger derivatives:

  $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

  The Cauchy-Riemann equations are equivalent to the condition:

  $$\frac{\partial f}{\partial \bar{z}} = 0$$

  which ensures that $f(z)$ is holomorphic.

- Test whether the function $e^x(\cos y - i \sin y)$ is analytic.

  Solution. Let

  $$f(z) = e^x(\cos y - i \sin y)$$
  $$f(z) = e^x e^{-iy} = e^{x-iy}$$
  $$f(z) = e^x \cos y - ie^x \sin y$$

  Thus, we define:

  $$u(x, y) = e^x \cos y, \quad v(x, y) = -e^x \sin y$$

  Compute the Partial Derivatives:

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$
$$\frac{\partial v}{\partial x} = -e^x \sin y, \quad \frac{\partial v}{\partial y} = -e^x \cos y$$

Check the Cauchy-Riemann Equations:

The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Substituting:

$$e^x \cos y \neq -e^x \cos y$$

$\therefore$ Given function f is not analytic. $\square$

- Compute the polar form of $C - R$ equations.

  Solution. Let

$$\begin{aligned} z &= x + iy \quad (41) \\ &= r\cos\theta + ir\sin\theta \quad (42) \\ &= re^{i\theta} \quad (43) \end{aligned}$$

  So $f(z) = u(r,\theta) + iv(r,\theta)$

  Differentiating the above equation w.r.t. r, partially

$$f'(re^{i\theta})\frac{\partial}{\partial r}(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$
$$\Rightarrow e^{i\theta}f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

  Again differentiating f w.r.t. θ partially,

$$f'(re^{i\theta})\frac{\partial}{\partial \theta}(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$
$$\Rightarrow rie^{i\theta}f'(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$
$$\Rightarrow ir(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$
$$\Rightarrow ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$
$$\therefore \boxed{u_\theta = -rv_r}$$
$$\boxed{u_r = \frac{1}{r}v_\theta}$$

  □

- Prove that $u(x,y)$ and $u(x^2 - y^2, 2xy)$ are simultaneously harmonic.

  Solution. A function $u(x,y)$ is harmonic if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

  Let $u(x,y)$ be harmonic, meaning

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Define new coordinates:

$$X = x^2 - y^2, \quad Y = 2xy.$$

Compute partial derivatives:

$$\frac{\partial X}{\partial x} = 2x, \quad \frac{\partial X}{\partial y} = -2y, \quad \frac{\partial Y}{\partial x} = 2y, \quad \frac{\partial Y}{\partial y} = 2x.$$

Apply the chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} = 2x\frac{\partial u}{\partial X} + 2y\frac{\partial u}{\partial Y}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} = -2y\frac{\partial u}{\partial X} + 2x\frac{\partial u}{\partial Y}.$$

Differentiate again:

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial X} + 4x^2\frac{\partial^2 u}{\partial X^2} + 8xy\frac{\partial^2 u}{\partial X \partial Y} + 4y^2\frac{\partial^2 u}{\partial Y^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial X} + 4y^2\frac{\partial^2 u}{\partial X^2} - 8xy\frac{\partial^2 u}{\partial X \partial Y} + 4x^2\frac{\partial^2 u}{\partial Y^2}.$$

Adding these:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right).$$

Since $u(x, y)$ is harmonic, the left-hand side is zero, implying

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = 0.$$

Thus, $u(X, Y) = u(x^2 - y^2, 2xy)$ is also harmonic. □

- Prove that $u(x, y)$ and $u(x^3 - y^3, 2xy)$ are simultaneously harmonic.

  <u>Solution.</u> A function $u(x, y)$ is harmonic if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  Given $u(x, y)$, we define new variables $\alpha = x^3 - y^3$ and $\beta = 2xy$, and consider $u(\alpha, \beta)$. To show that both $u(x, y)$ and $u(\alpha, \beta)$ are harmonic, we use the chain rule.

  First, compute the derivatives of α and β:

$$\frac{\partial \alpha}{\partial x} = 3x^2, \quad \frac{\partial \alpha}{\partial y} = -3y^2$$

$$\frac{\partial \beta}{\partial x} = 2y, \quad \frac{\partial \beta}{\partial y} = 2x$$

  The second derivatives are:

$$\frac{\partial^2 \alpha}{\partial x^2} = 6x, \quad \frac{\partial^2 \alpha}{\partial y^2} = -6y$$

$$\frac{\partial^2 \beta}{\partial x^2} = 0, \quad \frac{\partial^2 \beta}{\partial y^2} = 0$$

  Now, applying the chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \alpha \partial \beta}\frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial x} + \frac{\partial^2 u}{\partial \beta^2}\left(\frac{\partial \beta}{\partial x}\right)^2 + \frac{\partial u}{\partial \alpha}\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial u}{\partial \beta}\frac{\partial^2 \beta}{\partial x^2}$$

  Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \alpha \partial \beta}\frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial y} + \frac{\partial^2 u}{\partial \beta^2}\left(\frac{\partial \beta}{\partial y}\right)^2 + \frac{\partial u}{\partial \alpha}\frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial u}{\partial \beta}\frac{\partial^2 \beta}{\partial y^2}$$

  Summing these,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  Thus, $u(x, y)$ and $u(\alpha, \beta)$ are simultaneously harmonic.

- If $f(z)$ is analytic, show that

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2.$$

Solution.Let $f(z) = u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are real-valued functions.
Since $f(z)$ is analytic,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

$$|f(z)| = \sqrt{u^2 + v^2}.$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{uu_x + vv_x}{\sqrt{u^2 + v^2}},$$
$$\frac{\partial}{\partial y}|f(z)| = \frac{uu_y + vv_y}{\sqrt{u^2 + v^2}}.$$

Squaring bothsides,

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2$$
$$= \left(\frac{uu_x + vv_x}{\sqrt{u^2 + v^2}}\right)^2 + \left(\frac{uu_y + vv_y}{\sqrt{u^2 + v^2}}\right)^2.$$
$$= \frac{u^2u_x^2 + 2uvu_xv_x + v^2v_x^2}{u^2 + v^2} + \frac{u^2u_y^2 + 2uvu_yv_y + v^2v_y^2}{u^2 + v^2}.$$

Using Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$,

$$= \frac{u^2v_y^2 + 2uvv_yv_x + v^2v_x^2}{u^2 + v^2} + \frac{u^2v_x^2 - 2uvv_yv_x + v^2v_y^2}{u^2 + v^2}.$$

$$= \frac{u^2(v_y^2 + v_x^2) + v^2(v_x^2 + v_y^2)}{u^2 + v^2}.$$

$$= \frac{(u^2 + v^2)(v_x^2 + v_y^2)}{u^2 + v^2}.$$

$$= v_x^2 + v_y^2.$$

Since $v_x^2 + v_y^2 = |f(z)|^2$, we get:

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2.$$

□

6 Complex Integration

6.1 Definitions

6.1.1 Simple Closed Path

A closed arc that does not intersect or touch itself is called a simple closed path.

6.1.2 Simply Connected Domain

Every simple closed path in a domain D encloses only points of D then the domain D is called simply connected.

6.1.3 Contour

It is a single point or a finite sequence of directed smooth curves γ_1,γ_2,.....,γ_n such that the initial point of γ_k coincides with the terminal point of γ_{k-1} for $k = 1, 2, ..., n$.

$$\boxed{\gamma = \gamma_1 + \gamma_2 + ... + \gamma_n}$$

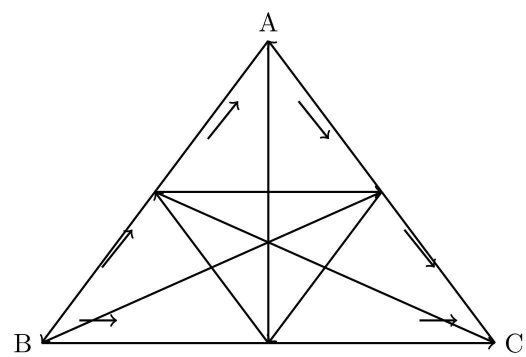
6.2 Goursat's Theorem

(a). If Ω is an open set in $\mathbb{C}$ and $T \subset \Omega$, a triangle whose interior is also contained in Ω then

$$\int_T f(z)dz = 0$$

$\forall z \in \Omega$, where f is holomorphic on Ω.

Proof



Let us consider the first original triangle $T^{(0)}$ with a positive orientation. Let $d^{(0)}$ and $p^{(0)}$ denote diameter and perimeter of $T^{(0)}$, respectively.

Now, bisecting the sides of triangle $T^{(0)}$, which yields four triangles, say $T_1^{(0)}$,

$T_2^{(0)}$, $T_3^{(0)}$, $T_4^{(0)}$ Hence,

$$\int_{T^{(0)}} f(z)dz = \sum_{j=1}^{4} \int_{T_j^{(1)}} f(z)dz$$

By canceling the integration over the sides with opposite direction, we get

$$\Big|\int_{T^{(1)}} f(z)dz\Big| \geq \Big|\int_{T_j^{(1)}} f(z)dz\Big|$$

$$\Rightarrow \Big|\int_{T^{(0)}} f(z)dz\Big| \leq 4\Big|\int_{T^{(1)}} f(z)dz\Big|$$

where $T^{(1)}$ is one among the four triangles with $d^{(1)} = \frac{1}{2}d^{(0)}$ and $p^{(1)} = \frac{1}{2}p^{(0)}$, where $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of triangle $T^{(1)}$, respectively.

Again, repeating the same process n-times, we obtain

$$\Big|\int_{T^{(0)}} f(z)dz\Big| \leq 4^n\Big|\int_{T^{(n)}} f(z)dz\Big|$$

with $d^{(n)} = \frac{1}{2^n}d^{(0)}$ and $p^{(n)} = \frac{1}{2^n}p^{(0)}$, where $d^{(n)}$ and $p^{(n)}$ are diameter and perimeter of triangle $T^{(n)}$ respectively.

Since f is holomorphic on Ω, so it is holomorphic at $z_0 \in \Omega$ and $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$, where $\psi(z) \to 0$ as $z \to z_0$.

As $f(z_0)$ and $f'(z_0)$ are constants, the first two terms have primitives, thus,

$$\int_{T^{(n)}} [f(z_0) + f'(z_0)(z - z_0)]dz = 0$$

Thus,

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz$$

Now, z_0 belongs to the closure of solid triangle Δ^n, and z belongs to the boundary of $T^{(n)}$. Thus,

$$|z - z_0| < d^{(n)}$$

$$\Big|\int_{T^{(n)}} f(z)dz\Big| \leq \epsilon_n d^{(n)}p^{(n)}$$

where $\epsilon_n = Sup_{z\in T^{(n)}}|\psi(z)| \to 0$ as $n \to \infty$.

$$\Rightarrow \Big|\int_{T^{(n)}} f(z)dz\Big| \leq \frac{\epsilon_n d^{(n)}p^{(n)}}{4^n}$$

$$\Rightarrow \left| \int_{T^{(0)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right| \leq \epsilon_n d^{(0)} p^{(0)}$$

For $n \to \infty$, $\epsilon_n \to 0$. Therefore,

$$\boxed{\int_T f(z)dz = 0.}$$

$\square$

(b). If f is holomorphic in an open set Ω that contains a rectangle R and its interior then

$$\int_R f(z)dz = 0.$$

6.3 Local Existence of Primitives

Theorem. A holomorphic function in an open disc has a primitive in that disc.

6.4 Cauchy's Integral in a Simply Connected Domain

Theorem. If f is holomorphic on a simply connected domain, then for every closed contour C,

$$\int_C f(z)dz = 0.$$

<u>Proof</u> Let $f(z) = u(x,y) + iv(x,y)$ be analytic on the domain D where $z = x+iy$. Then,

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx + iudy + ivdx + i^2 vdy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy)$$

By **Green's Theorem**,

$$\int_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

where R is the region enclosed by C.

$$\therefore \int_C (udx - vdy) = \iint_R (\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy$$

and

$$\int_C (vdx + udy) = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy$$

Since f is analytic, it satisfies the $C - R$ equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\therefore \int_C (udx - vdy) = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})dxdy = 0$$

and

$$\int_C (vdx + udy) = \iint_R (\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y})dxdy = 0$$

Hence,

$$\boxed{\int_C f(z)dz = 0}$$

□

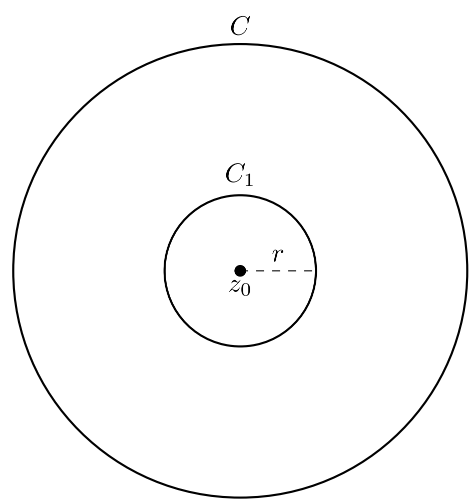
6.5 Cauchy's Integral Formula

Theorem. If f is an analytic function inside and on a closed contour C and if z_0 be any point inside C then

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

The integration being taken in counter clockwise.

<u>Proof</u> Given that f is analytic inside and on a simple closed contour C and z_0 is any point inside C. Consider a disk C_1 centered at z_0 with radius r.



So,

$$|z - z_0| = r$$

$$z - z_0 = re^{i\theta}$$

$$\Rightarrow dz = rie^{i\theta} d\theta$$

Clearly, $\frac{f(z)}{z-z_0}$ is analytic in $C - C_1$, So

$$\int_{C-C_1} \frac{f(z)}{z - z_0} dz = 0$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = \int_{C_1} \frac{f(z)}{z - z_0} dz$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = \int_{C_1} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = i \int_{C_1} f(z_0 + re^{i\theta}) d\theta$$

When $r \to 0$, then $f(z) \to f(z_0)$.

$$\therefore \int_C \frac{f(z)}{z - z_0} = i \int_{C_1} f(z_0) d\theta$$

$$= if(z_0) \int_0^{2\pi} d\theta$$

$$= 2\pi i f(z_0)$$

Therefore

$$\boxed{\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)}$$

□

6.6 Problems on Cauchy's Integral Formula

1. Evaluate $\displaystyle\int_C \frac{dz}{z - 3i}$ where C is the circle $|z| = \pi$ counter clockwise.

Solution. The singularities of $F(z) = \frac{1}{z-3i}$ is given by

$$z - 3i = 0$$

$$\Rightarrow z = 3i$$

which lies inside the circle $|z| = \pi = 3.141$ and here $f(z) = 1$.

Using Cauchy's Integral Formula,

$$\int_C \frac{dz}{z - 3i} = 2\pi i f(3i)$$

$$= 2\pi i(1)$$

$$= 2\pi i$$

$\square$

2. Evaluate $\displaystyle\int_C \frac{z}{(9-z^2)(z+1)}dz$ where C is the circle $|z| = 2$ counter clockwise.

Solution. The singularities of $F(z) = \frac{z}{(9-z^2)(z+1)}$ are given by

$$(9-z^2)(z+1) = 0$$

$$\Rightarrow 9 - z^2 = 0, z + 1 = 0$$

$$\Rightarrow z = \pm 3, z = -1$$

But the pole $z = \pm 3$ lies outside the circle $|z| = 2$, so

$$\int_C \frac{z}{(9-z^2)(z+1)}dz = \int_C \frac{\frac{z}{9-z^2}}{z-(-1)}dz$$

Here $f(z) = \frac{z}{9-z^2}$

$$\int_C \frac{z}{(9-z^2)(z+1)}dz = 2\pi\ i f(-1)$$

$$= 2\pi i(\frac{-1}{9-1})$$

$$= 2\pi i(\frac{-1}{8})$$

$$= \frac{-1}{4}\pi i$$

$\square$

3. Evaluate $\displaystyle\int_C \frac{3z-1}{z^3-z}dz$ where C is the circle

$$a.|z| = \frac{1}{2}$$

$$b.|z| = 2$$

Solution The singularities of $F(z) = \frac{3z-1}{z^3-z}$ are given by

$$z^3 - z = 0$$

$$\Rightarrow z(z^2-1) = 0$$

$$\Rightarrow z = 0, z^2 - 1 = 0$$

$$\Rightarrow z = 0, z = \pm 1$$

a. For the circle $|z| = \frac{1}{2}$, the poles $z = \pm 1$ do not lie inside. So,

$$\int_C \frac{3z-1}{z^3 - z} dz = \int_C \frac{\frac{3z-1}{z^2-1}}{z} dz$$

Here $f(z) = \frac{3z-1}{z^2-1}$.

$$\begin{aligned} \Rightarrow \int_C \frac{3z-1}{z^3-z} dz &= 2\pi i f(0) \\ &= 2\pi i \Big(\frac{-1}{-1}\Big) \\ & 2\pi i \end{aligned}$$

□

b. For the circle $|z| = 2$, all the singularities lie inside. So,

$$F(z) = \frac{3z-1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$\begin{aligned} \Rightarrow \frac{3z-1}{z(z-1)(z+1)} &= \frac{A(z^2-1) + Bz(z+1) + Cz(z-1)}{z(z-1)(z+1)} \\ \Rightarrow 3z - 1 &= Az^2 - A + Bz^2 + Bz + Cz^2 - Cz \\ \Rightarrow 3z - 1 &= (A + B + C)z^2 + (B - C)z - A \end{aligned}$$

By comparing the coefficients of the powers of z,

$$\boxed{A = 1}$$

$$A + B + C = 0 \Rightarrow B + C = -1$$

$$B - C = 3$$

$$\Rightarrow 2B = 2 \Rightarrow \boxed{B = 1}$$

$$\Rightarrow 1 - C = 3$$

$$\Rightarrow -C = 2 \Rightarrow \boxed{C = -2}$$

Therefore,

$$\begin{aligned} \int_C \frac{3z-1}{z^3-z} dz &= \int_C \frac{dz}{z} + \int_C \frac{dz}{z-1} + \int_C \frac{-2dz}{z-(-1)} \\ &= 2\pi i f(0) + 2\pi i f(1) + (-2) 2\pi i f(-1) \\ &= 2\pi i(1) + 2\pi i(1) - 4\pi i(1) = 0 \end{aligned}$$

□

4. Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle $|z+1-i|=2$.

Solution The given circle is,

$$|z+1-i|=2$$
$$\Rightarrow |z-(i-1)|=2$$

The singularities of $F(z)=\frac{z+4}{z^2+2z+5}$ are given by

$$z^2+2z+5=0$$
$$\Rightarrow z=\frac{-2\pm\sqrt{2^2-4.1.5}}{2.1}$$
$$\Rightarrow z=\frac{-2\pm\sqrt{-16}}{2}$$
$$\Rightarrow z=\frac{-2\pm 4i}{2}$$
$$\Rightarrow z=-1\pm 2i \Rightarrow z=-1+2i,-1-2i$$

Now,

$$|(-1+2i)-(i-1)|$$
$$=|-1+2i-i+1|$$
$$=|i|=1<2$$

So, $-1+2i$ lies in the circle $|z+1-i|=2$.

$$|(-1-2i)-(i-1)|$$
$$|-1-2i-i+1|=|-3i|=3>2$$

So, $-1-2i$ lies outside the circle. Then,

$$\int_C \frac{z+4}{z^2+2z+5}dz=\int_C \frac{\frac{z+4}{z+1+2i}}{z-(-1+2i)}dz$$

Here $f(z)=\frac{z+4}{z+1+2i}$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5}dz=2\pi i f(-1+2i)$$
$$=2\pi i\Big(\frac{-1+2i+4}{-1+2i+1+2i}\Big)$$
$$=2\pi i\Big(\frac{3+2i}{4i}\Big)=\frac{\pi}{2}(3+2i)$$

□

5. Evaluate $\int_C \frac{zdz}{z^4-1}$ where C is the circle $|z-2|=2$.

Solution The singularities of $F(z)=\frac{z}{z^4-1}$ are given by

$$z^4-1=0$$

$$\Rightarrow (z^2+1)(z^2-1)=0$$

$$\Rightarrow z^2+1=0, z^2-1=0$$

$$\Rightarrow z=\pm i, z=\pm 1$$

For $z=\pm i$, $|\pm i-2|=\sqrt{5}>2$, so $z=\pm i$ lie outside the circle.
For $z=1$, $|1-2|=1<2$, so it lies inside the circle, for $z=-1$, it lies outside the circle. Then,

$$\int_C \frac{z}{z^4-1}dz=\int_C \frac{\frac{z}{(z^2+1)(z+1)}}{z-1}dz$$

Here $f(z)=\frac{z}{(z^2+1)(z+1)}$,

$$\int_C \frac{z}{z^4-1}dz=2\pi i f(1)$$

$$=2\pi i\frac{1}{(1+1)(1+1)}$$

$$=2\pi i\frac{1}{4}$$

$$=\frac{\pi}{2}i$$

□