Algebraic Reconstruction Techniques and its Variants

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Problem Statement :

Aim of the project :

To implement, analyse and compare the different variants of the Algebraic Reconstruction Techniques :

- Additive ART
- Multiplicative ART
- Simultaneous Iterative RT

The Reconstruction Problem

CT Imaging as a linear system : $A x = b$

- ➔ x = unknown values of attenuation coefficients (vectorised Image)
- ➔ b = known data in Radon-transform domain (Vectorised Image)
- ➔ A = imaging/acquisition matrix (also known)
  - Models the imaging process
  - Each row corresponds to an integral along a line

Algebraic Reconstruction Techniques

Iterative algebraic techniques used to solve the Reconstruction Problem

Notations :

- Let the size of the image of attenuation coefficients (I) be [m x n]
- We denote the rows of A matrix as a_i^T (i.e. transpose of a_i).

Additive ART - Update Step

Initialisation : $x_0 = [0]_{mn \times 1}$

Iterative Step :

$$x^{k+1} = x^k + \lambda_k \frac{b_i - \langle a_i, x^k \rangle}{\|a_i\|^2} a_i$$

- λ_k is a free ("relaxation") parameter
- Typically $0 < \lambda_k <= 1$ ($\lambda_k < 2$ ensures convergence)

The specific case of lambda_k = 1 is called the Kaczmarz Method.

Multiplicative ART - Update Step

Initialisation : $x_0 = [1]_{mnx1}$

Iterative Step :

$$x_j^{k+1} = x_j^k \left(\frac{b_i}{\langle a_i, x^k \rangle} \right)^{\lambda_1 a_{ij}}$$

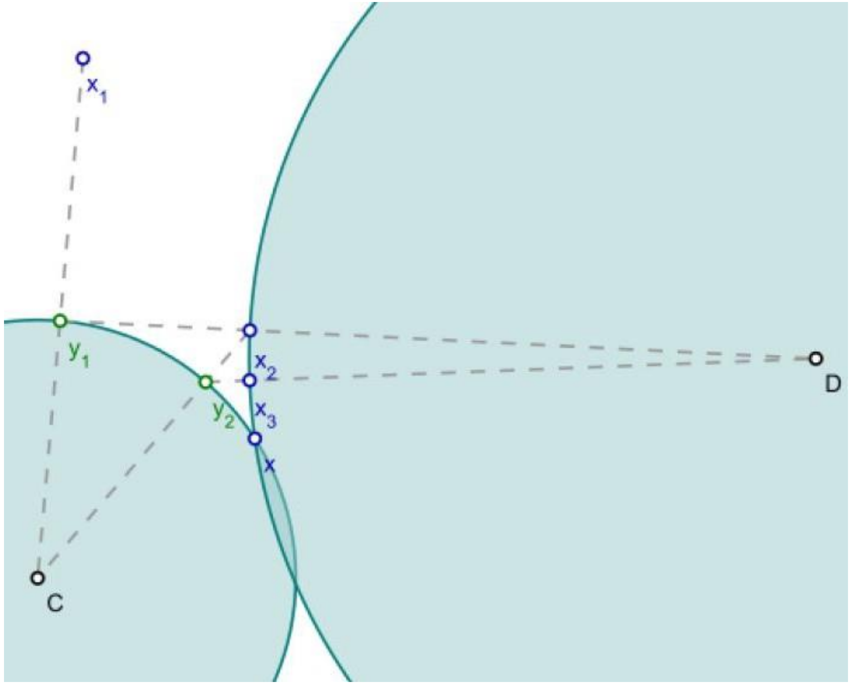
where λ_1 is a relaxation parameter

Simultaneous IRT - Update Step

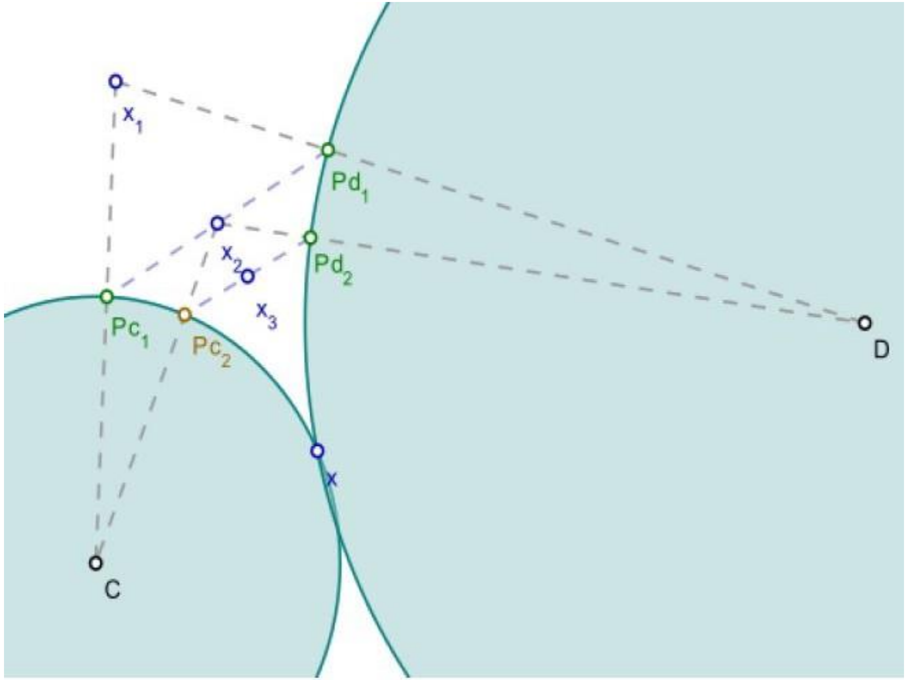
Initialisation : $x_0 = [0]_{mnx1}$

Iterative Step : It involves a cumulative update vector which accumulates the updates of the additive ART for all rows, without actually updating it and then takes a step in the average update direction.

AART vs SIRT - Update Step



Effective update - AART



Effective update - SIRT

Additive ART Solution Property

The Additive ART algorithm was proved to converge to the minimum norm solution, satisfying the constrained minimization problem:

$$\min_{x} \| x \|_2^2 \quad \text{such that } Ax = b.$$

Multiplicative ART Solution Property

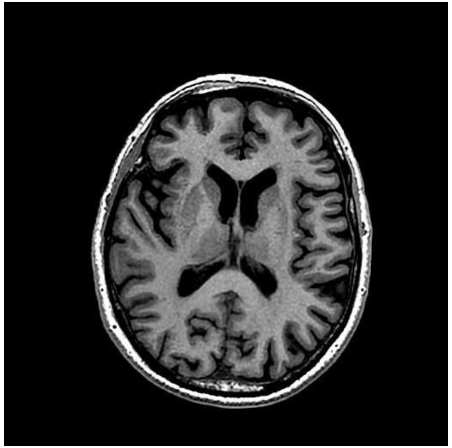
The Multiplicative ART algorithm was proved to converge to the maximum entropy solution, satisfying the constrained minimization problem:

$$\min_x \sum_i x_i \ln x_i$$ that Ax = b.

Datasets :

We used two images for our experiments, the standard phantom image and a more detailed Brain MRI image, and 2 real life CT images, downloaded from the internet. All the analyses have been performed on the phantom image, the Brain MRI image and the 2 CT images were used as a proof of concept

Phantom Image

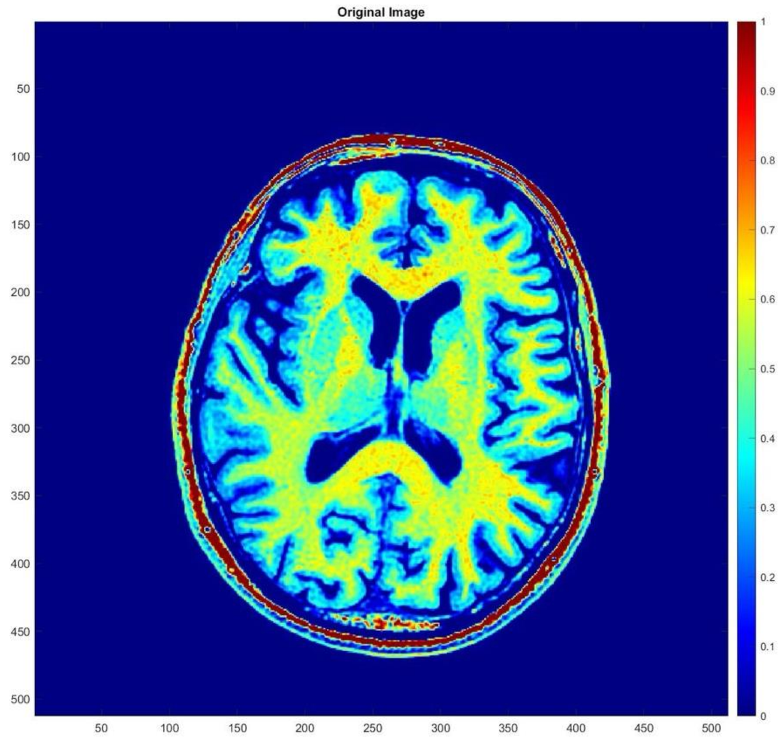


Brain MRI Image

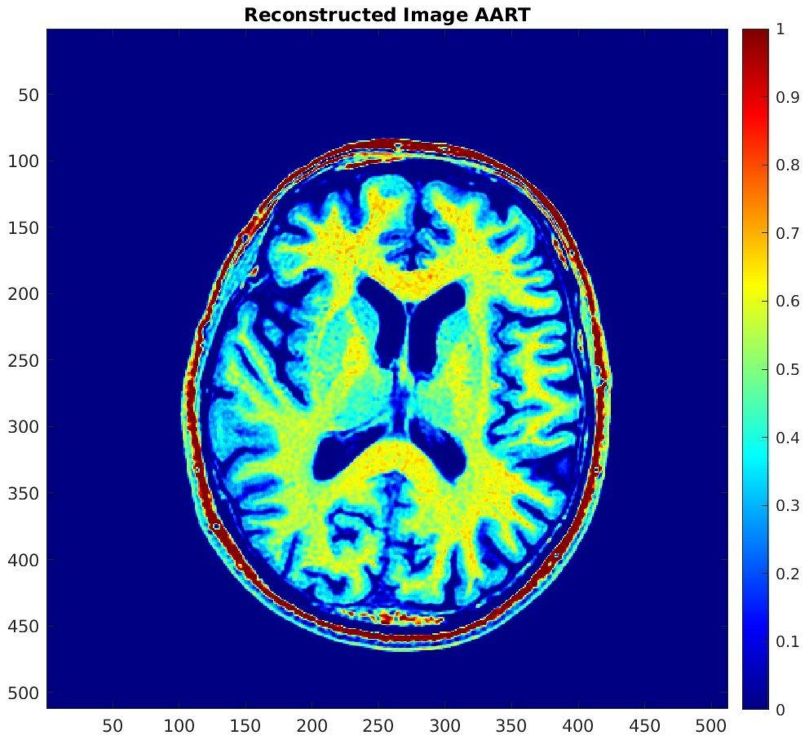
CT - 1

CT - 2

Additive ART Results



Original Image



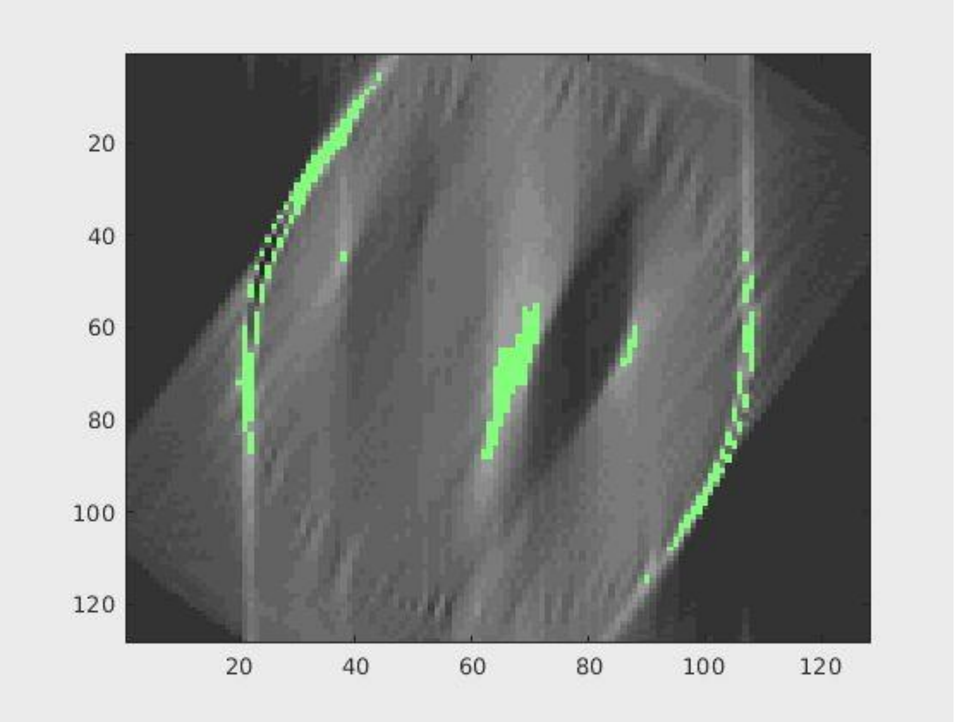
Reconstructed Image

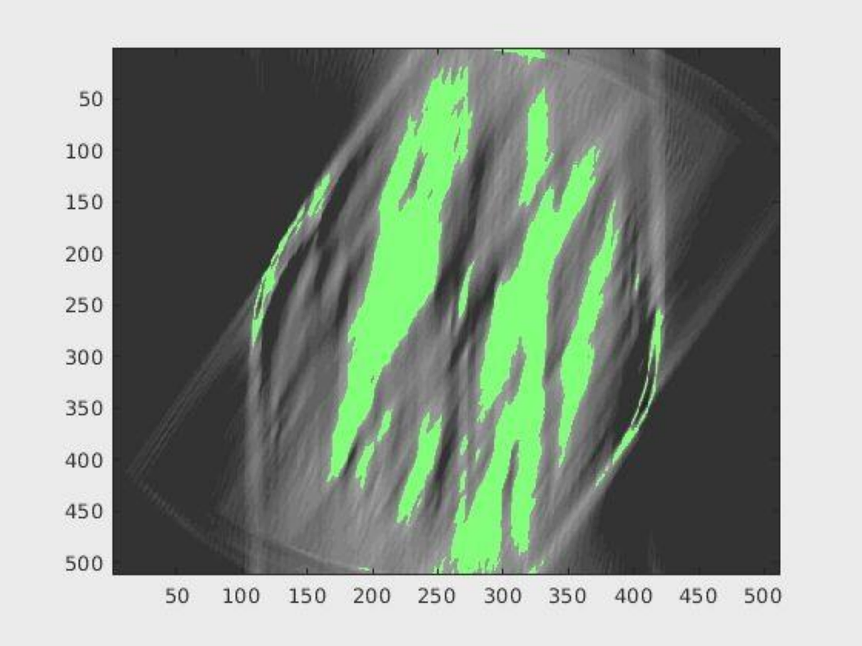
Additive ART Results

Original Image

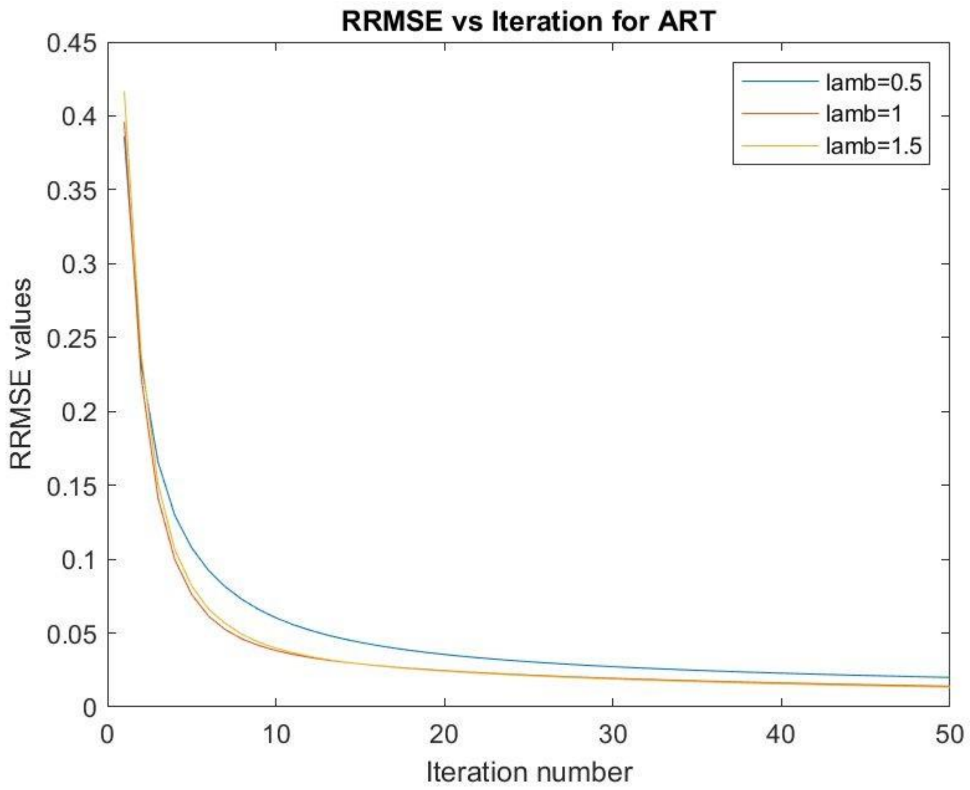
Reconstructed Image

Additive ART - Reconstruction over time

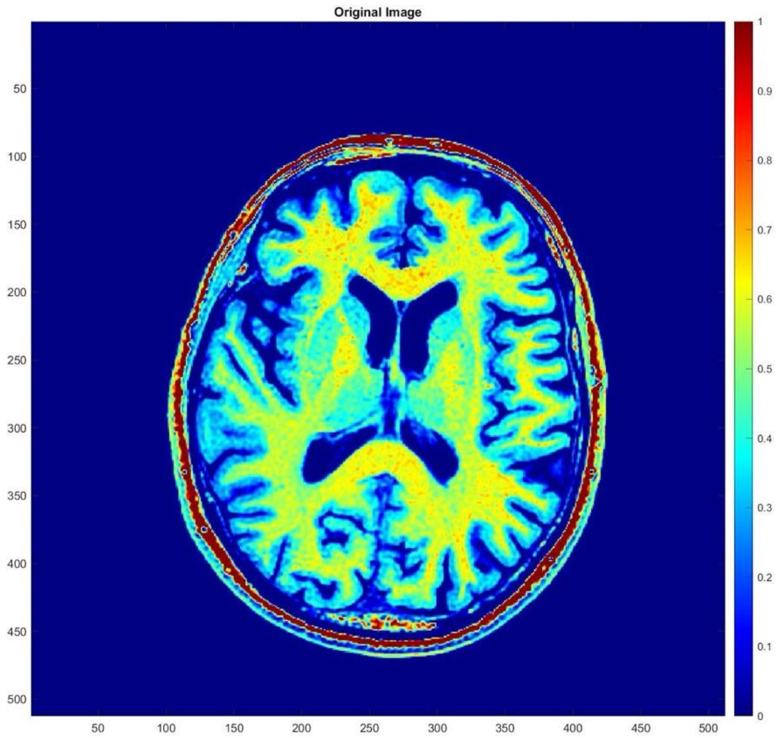




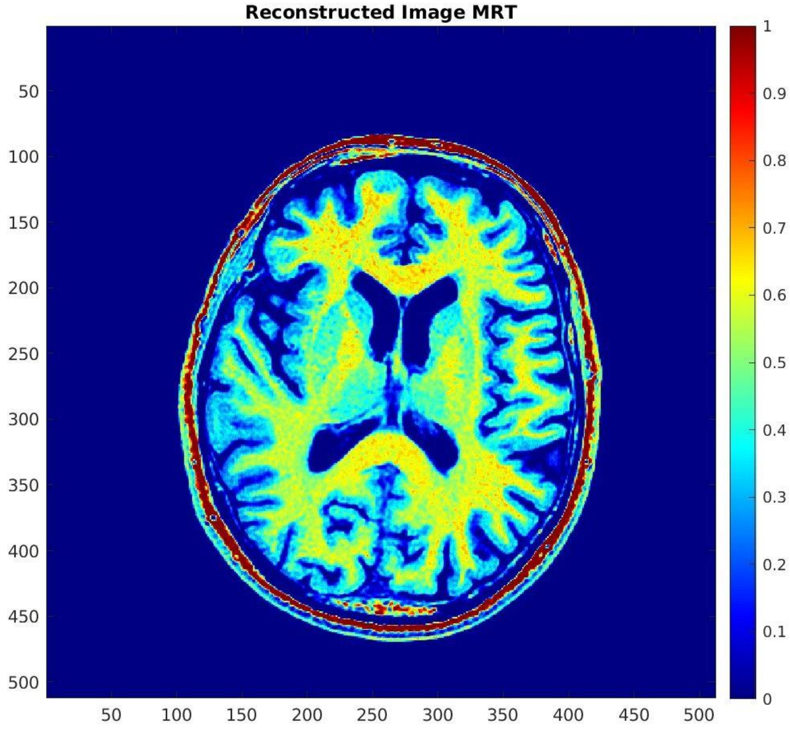
Additive ART - Effect of changing Lambda



Multiplicative ART Results



Original Image



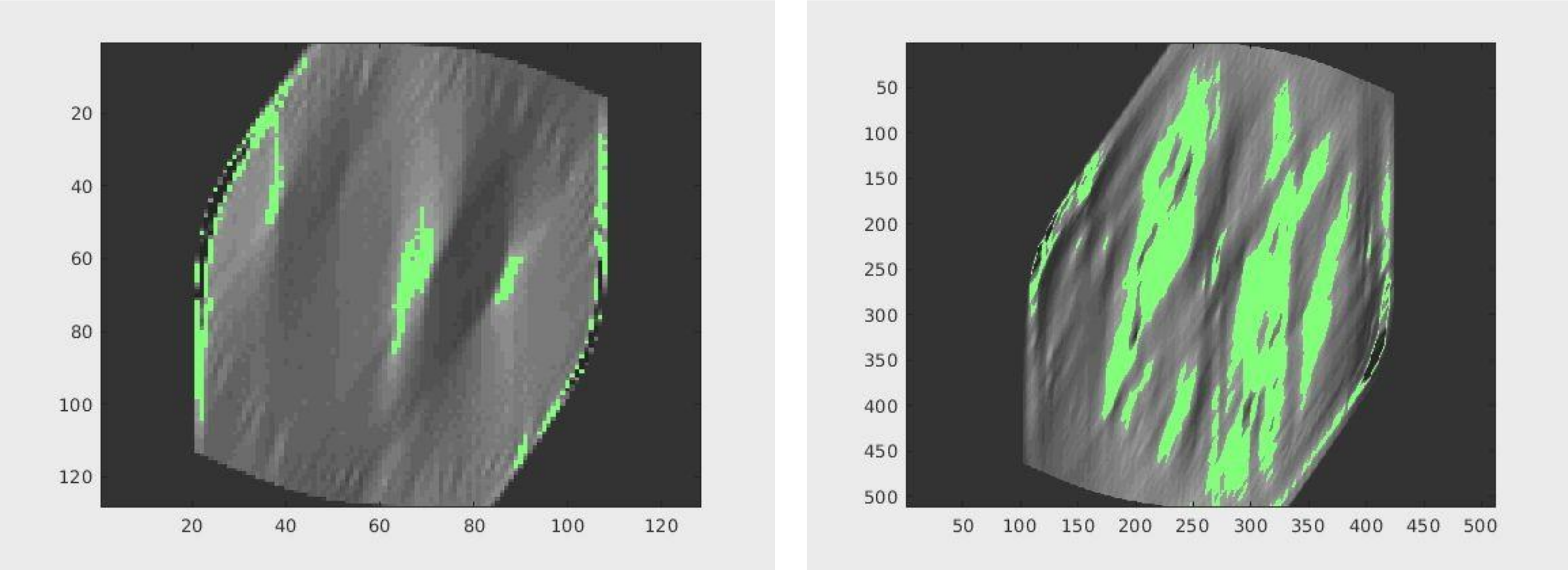
Reconstructed Image

Multiplicative ART Results

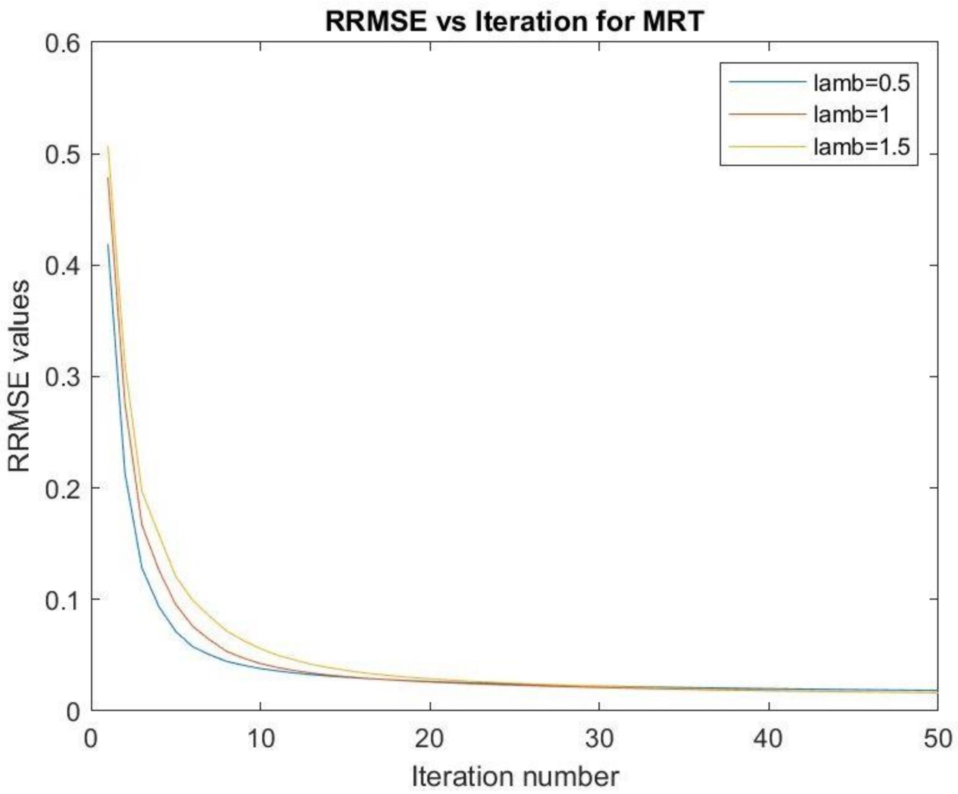
Original Image

Reconstructed Image

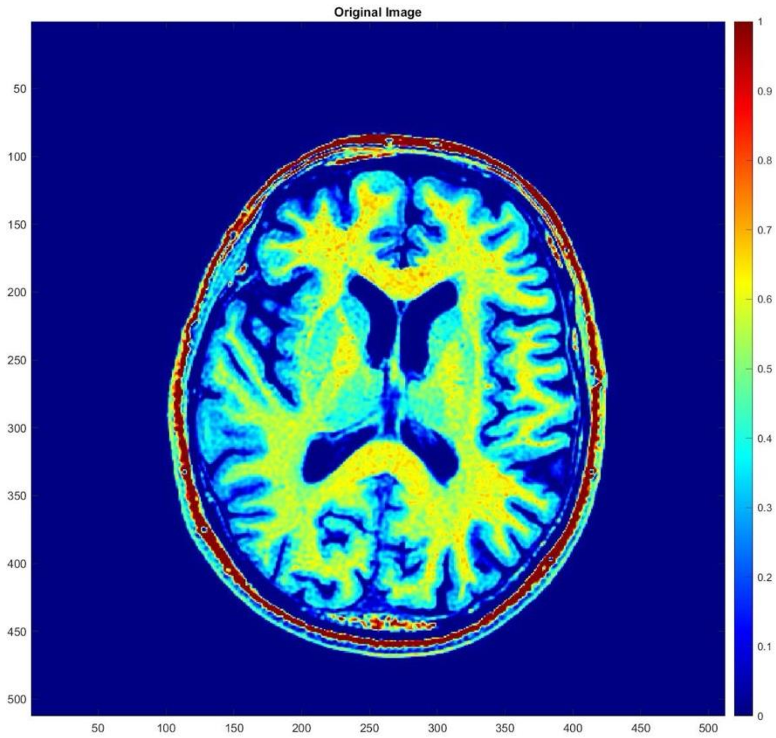
Multiplicative ART - Reconstruction over time



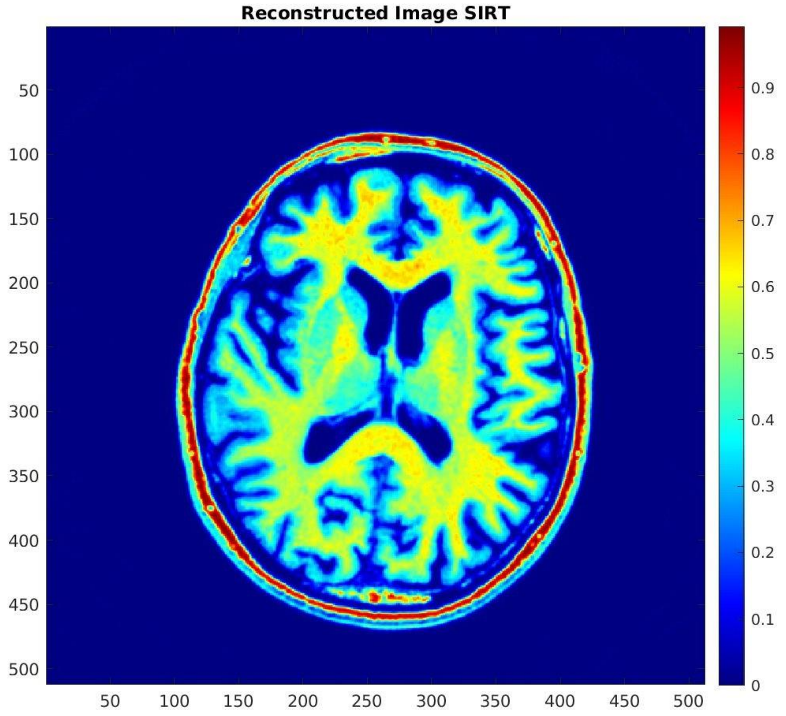
Multiplicative ART - Effect of changing Lambda



Simultaneous IRT Results



Original Image



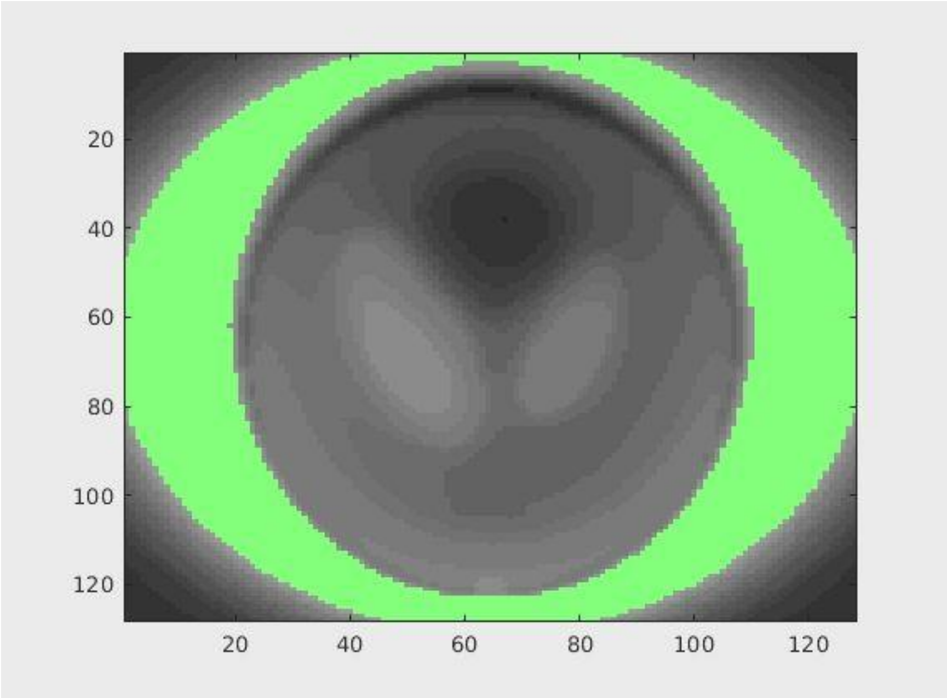
Reconstructed Image

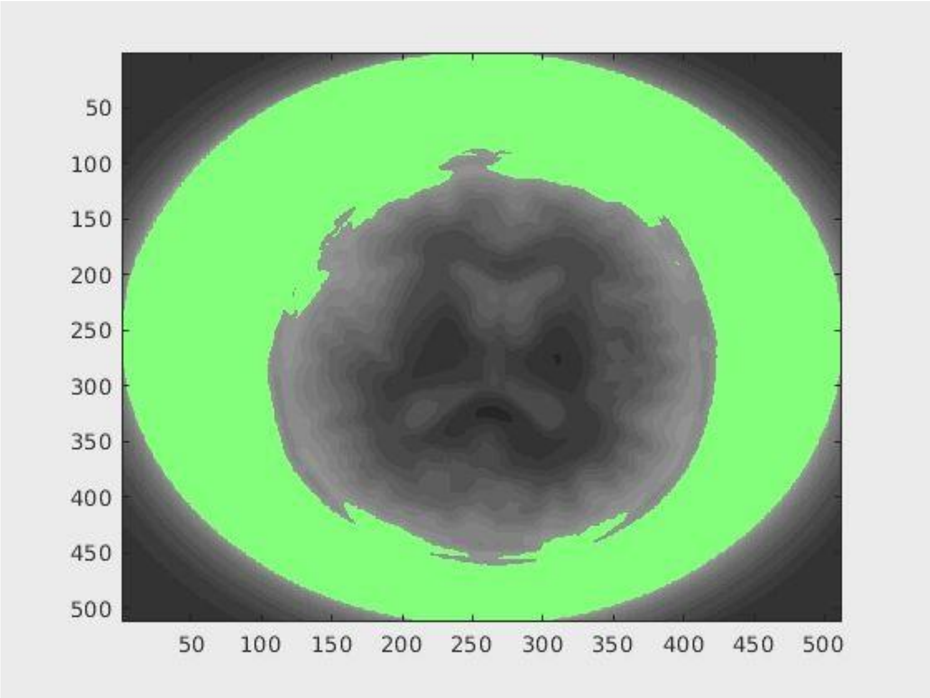
Simultaneous IRT Results

Original Image

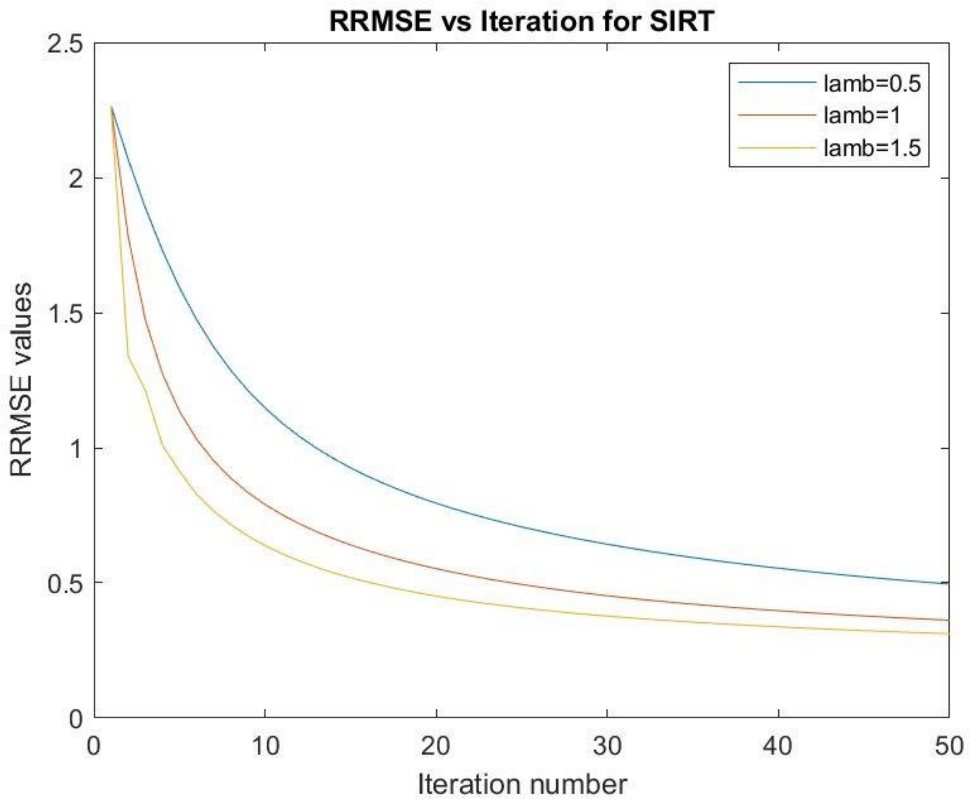
Reconstructed Image

Simultaneous IRT - Reconstruction over time

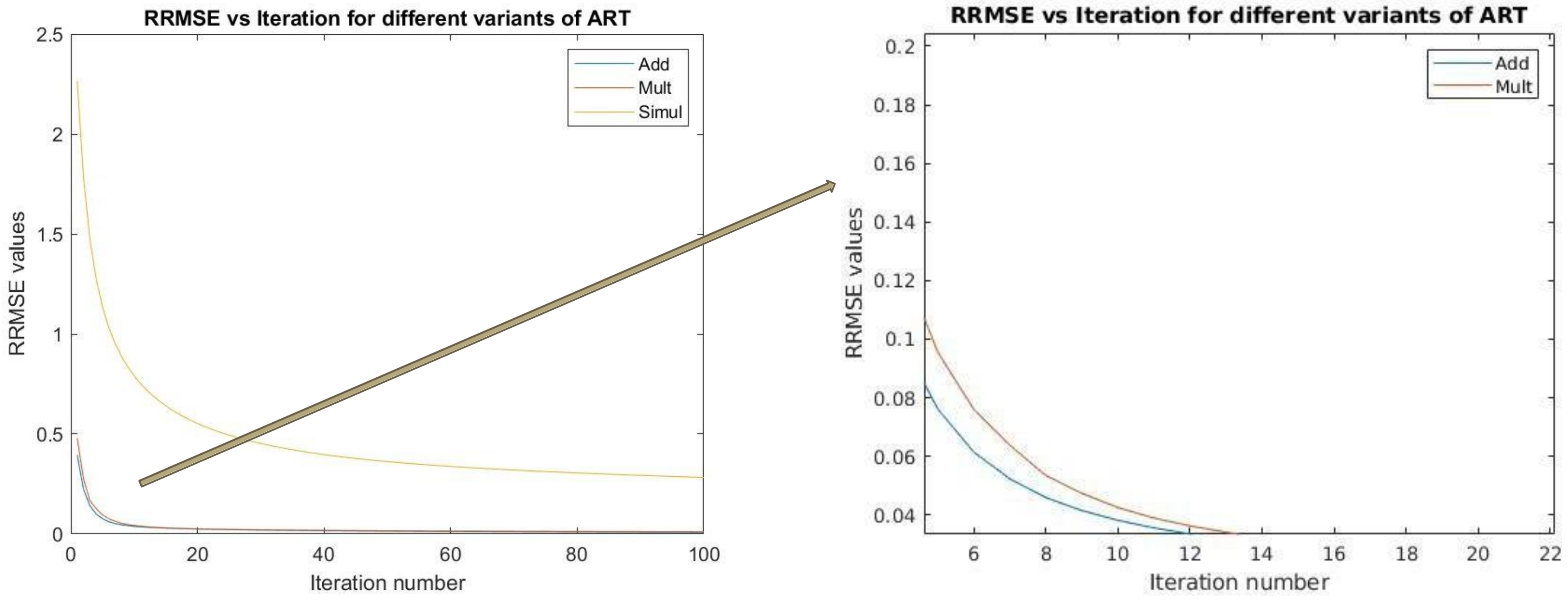




Simultaneous IRT - Effect of changing Lambda

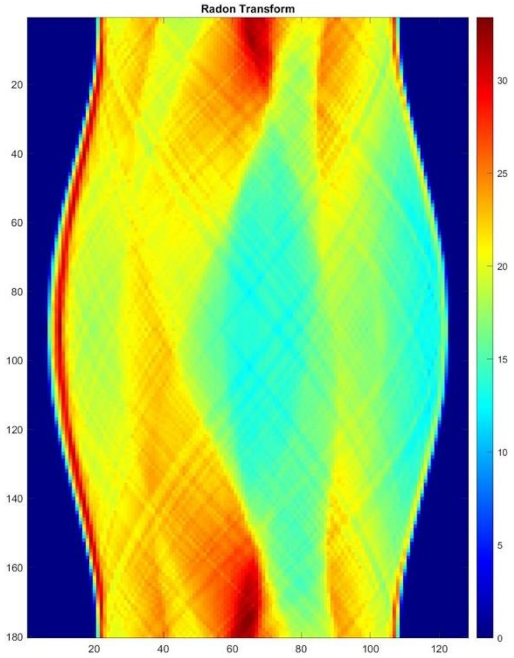


Comparisons between the algorithm

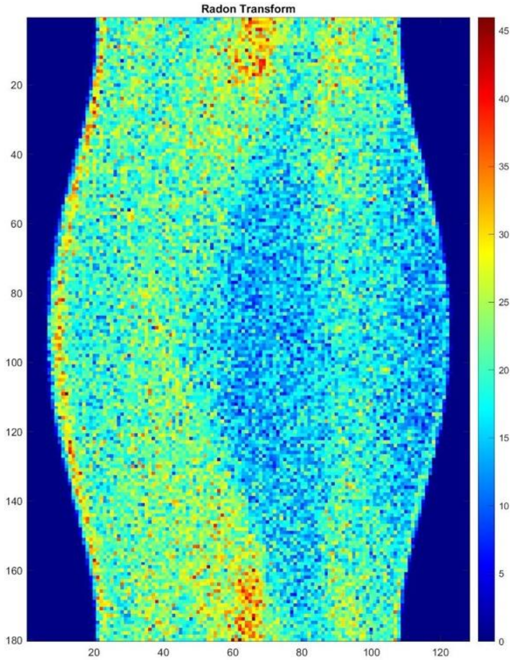


And then god said, let there be noise

Poisson Noise on the Radon Transform



Noiseless Radon Transform of Image



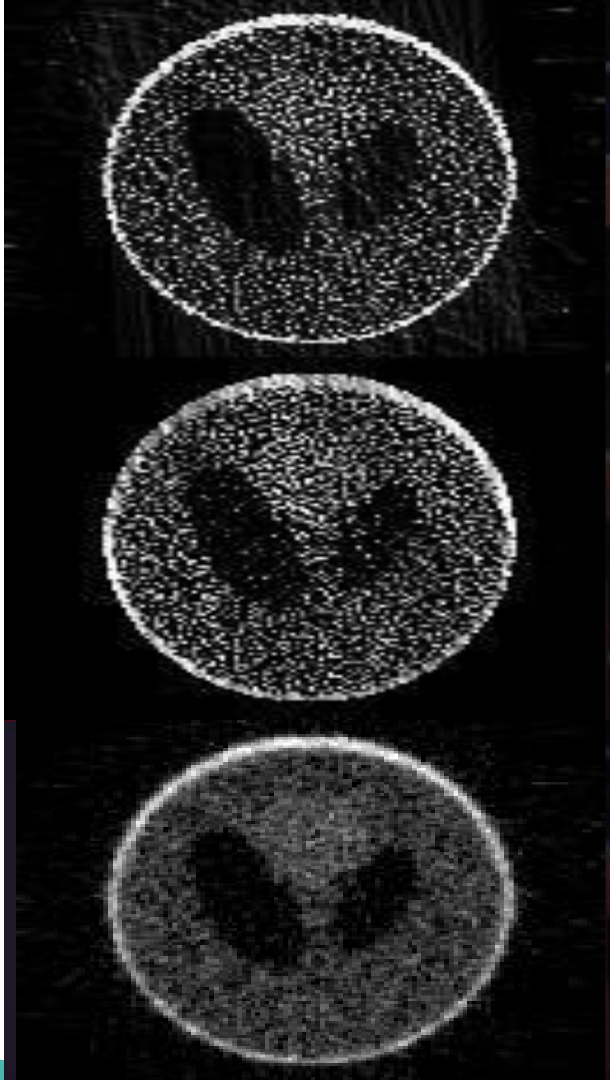
Noisy Radon Transform of Image

Reconstructions:

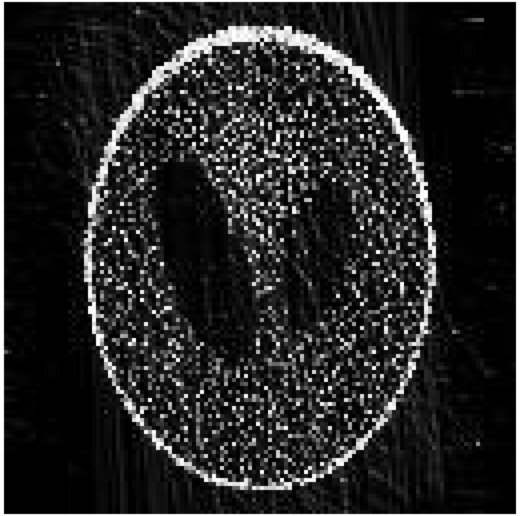
Additive

Multiplicative

Simultaneous

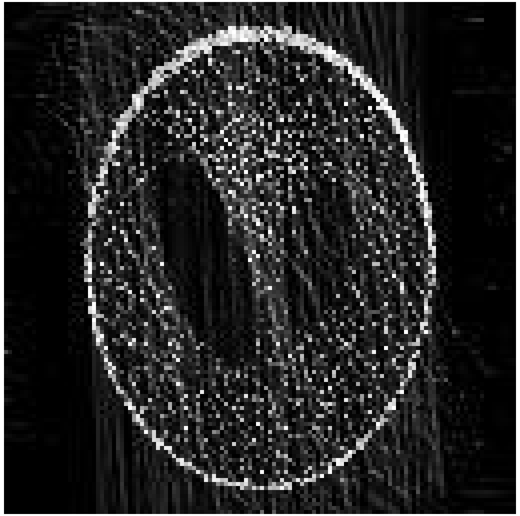


Variation with lambda : Additive ART



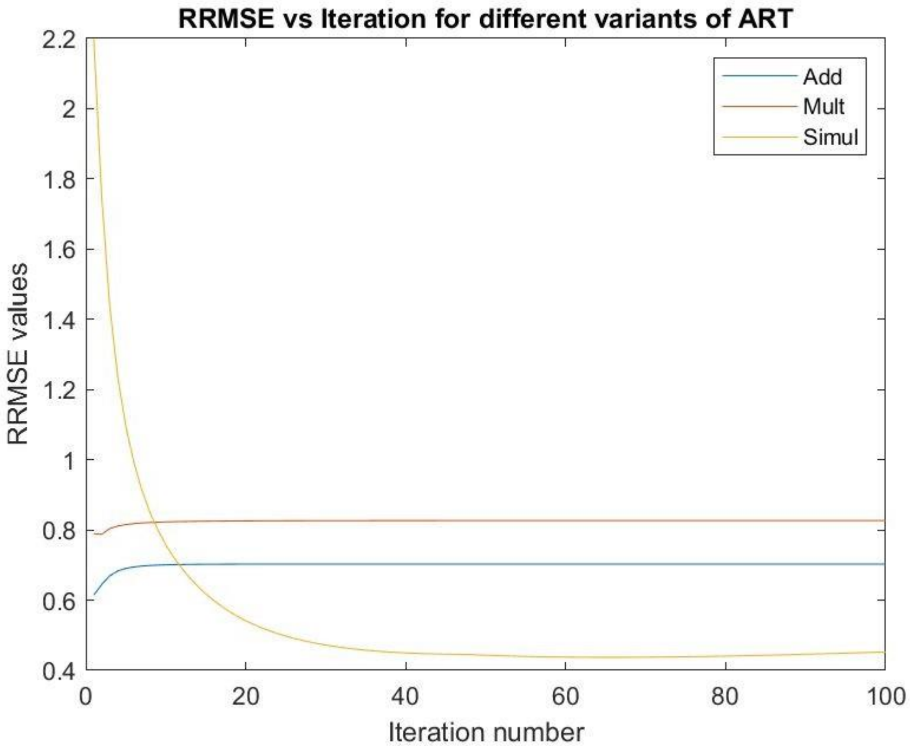
Lambda = 0.5

Lambda = 1



Lambda = 1.5

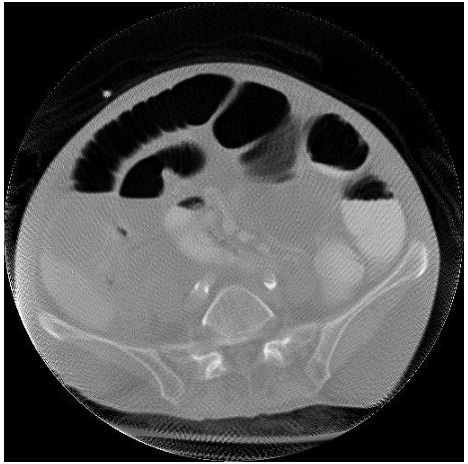
Comparison among the variants



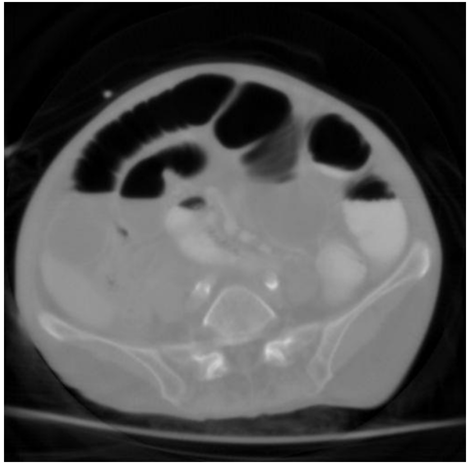
Real Life CT image slices - Reconstructions

Original CT image slice

Additive ART



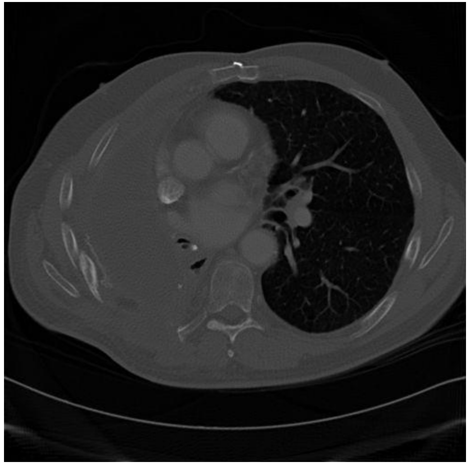
Multiplicative ART



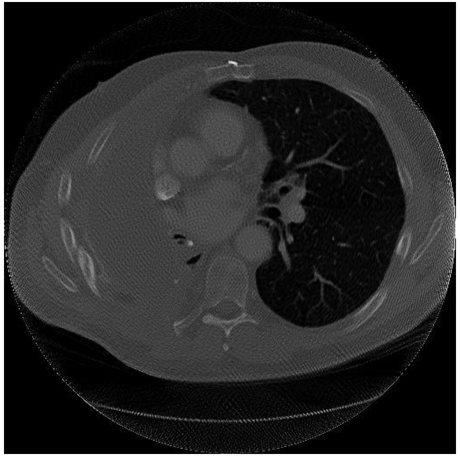
Simultaneous IRT

Real Life CT image slices - Reconstructions

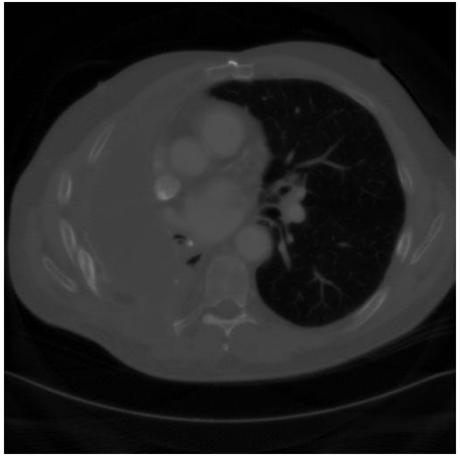
Original CT image slice



Additive ART



Multiplicative ART



Simultaneous IRT

Conclusions

1) Since the simultaneous IRT takes a step in the average direction of updates without completely projecting on any one of the convex sets, it converges slower than the other 2 variants in general.
2) However, the average direction does help in case of noisy data, where a complete projection onto any of the convex sets means getting more influenced by noise. As a result, SIRT performs better on noisy data.
3) Decreasing the value of lambda in case of AART leads to slower convergence, because of incomplete projections onto the convex sets.
4) Decreasing the value of lambda in case of MART leads to slightly faster convergence, because the intensity values are between 0 and 1 and lower power implies higher magnitude of updates.
5) AART and MART behave similarly in the cases we experimented on.

Contribution :

We have worked on all the parts of the project together, including coding and report, just as we did for our assignments!

References :

1 : https://crazybiocomputing.blogspot.com/2012/08/learning-tomography-toc.html

2 : http://www.mers.byu.edu/docs/thesis/msthesis_willism_lib.pdf

Thank You