Chapter 5

Defuzzification Methods

Fuzzy rule based systems evaluate linguistic if-then rules using fuzzification, inference and composition procedures. They produce fuzzy results which usually have to be converted into crisp output. To transform the fuzzy results in to crisp, defuzzification is performed.

Defuzzification is the process of converting a fuzzified output into a single crisp value with respect to a fuzzy set. The defuzzified value in FLC (Fuzzy Logic Controller) represents the action to be taken in controlling the process.

Different Defuzzification Methods

The following are the known methods of defuzzification.

- Center of Sums Method (COS)
- Center of gravity (COG) / Centroid of Area (COA) Method
- Center of Area / Bisector of Area Method (BOA)
- Weighted Average Method
- Maxima Methods
  - First of Maxima Method (FOM)
  - Last of Maxima Method (LOM)
  - Mean of Maxima Method (MOM)

Center of Sums (COS) Method

This is the most commonly used defuzzification technique. In this method, the overlapping area is counted twice.

The defuzzified value x^* is defined as :

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{A_k}(x_i)}{\sum_{i=1}^{N} \quad \sum_{k=1}^{n} \mu_{A_k}(x_i)} ,$$

Here, n is the number of fuzzy sets, N is the number of fuzzy variables, $\mu_{A_k}(x_i)$ is the membership function for the k-th fuzzy set.

Example

The defuzzified value x^* is defined as :

$$x^* = \frac{\sum_{i=1}^{k} A_i \times \bar{x}_i}{\sum_{i=1}^{k} A_i},$$

Here, A_i represents the firing area of i^{th} rules and k is the total number of rules fired and $\bar{x}_i$ represents the center of area.

The aggregated fuzzy set of two fuzzy sets C_1 and C_2 is shown in Figure 1. Let the area of this two fuzzy sets are A_1 and A_2.

A_1 = ½ * [(8-1) + (7-3)] * 0.5 = ½ * 11 * 0.5 = 55/20=2.75

A_2 = ½ * [(9-3) + (8-4)] * 0.3 = ½ * 10 * 0.3 = 3/2 =1.5

Now the center of area of the fuzzy set C_1 is let say $\bar{x}_1$ = (7+3)/2= 5 and

the center of area of the fuzzy set C_2 is $\bar{x}_2$= (8+4)/2=6.

Now the defuzzified value $x^* = \frac{(A_1.\bar{x}_1 + A_2.\bar{x}_2)}{A_1 + A_2} = \frac{(2.75 * 5 + 1.5 * 6)}{(2.75 + 1.5)}$ = 22.75/4.25 = 5.35

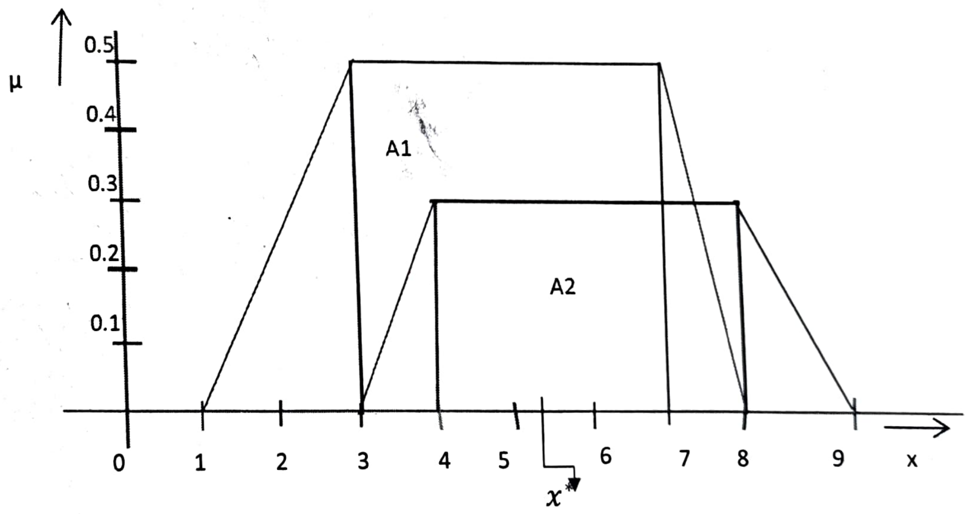


Figure 1 : Fuzzy sets $C_1$and C_2

Center of gravity (COG) / Centroid of Area (COA) Method

This method provides a crisp value based on the center of gravity of the fuzzy set. The total area of the membership function distribution used to represent the combined control action is divided into a number of sub-areas. The area and the center of gravity or centroid of each sub-area is calculated and then the summation of all these sub-areas is taken to find the defuzzified value for a discrete fuzzy set.

For discrete membership function, the defuzzified value denoted as x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$, Here x_i indicates the sample element, $\mu(x_i)$ is

the membership function, and n represents the number of elements in the sample.

For continuous membership function, x^* is defined as :

$$x^* = \frac{\int x\, \mu_A(x)\, dx}{\int \mu_A(x)\, dx}$$

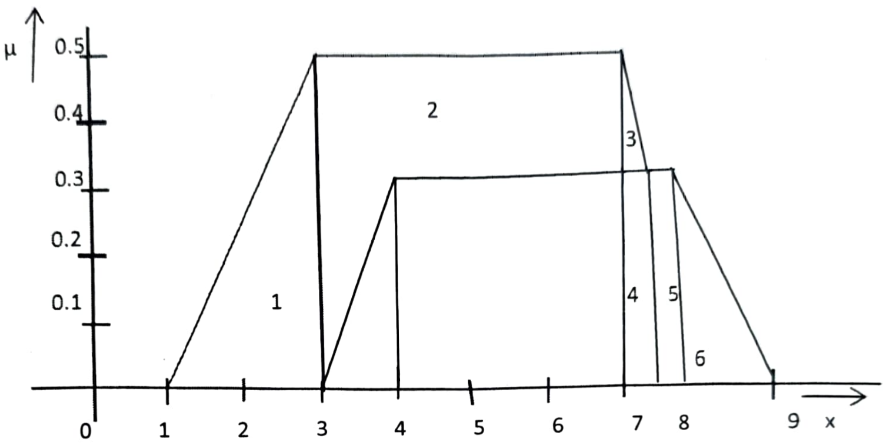


Figure 2 : Fuzzy sets C1 and C2

Example:

The defuzzified value x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{N} A_i \times \bar{x}_i}{\sum_{i=1}^{N} A_i}$$, Here N indicates the number of sub-areas, A_i and

$\bar{x}_i$ represents the area and centroid of area, respectively, of i^{th} sub-area.

In the aggregated fuzzy set as shown in figure 2. , the total area is divided into six sub-areas.
For COG method, we have to calculate the area and centroid of area of each sub-area.
These can be calculated as below.
The total area of the sub-area 1 is ½ * 2 * 0.5 = 0.5
The total area of the sub-area 2 is (7-3) * 0.5 = 4 * 0.5 = 2
The total area of the sub-area 3 is ½ * (7.5-7) * 0.2 = 0.5 * 0.5 *0.2 =.05
The total area of the sub-area 4 is 0.5* 0.3 = .15
The total area of the sub-area 5 is 0.5* 0.3 = .15
The total area of the sub-area 6 is ½ *1* 0.3 = .15
Now the centroid or center of gravity of these sub-areas can be calculated as

Centroid of sub-area1 will be (1+3+3)/3 = 7/3 =2.333
Centroid of sub-area2 will be (7+3)/2 = 10/2 = 5
Centroid of sub-area3 will be (7+7+7.5)/3 = 21.5/3 =7.166
Centroid of sub-area4 will be (7+7.5)/2 =14.5/2=7.25
Centroid of sub-are5 will be (7.5+8)/2 =15.5/2 = 7.75
Centroid of sub-area6 will be (8+8+9)/3 = 25/3 = 8.333
Now we can calculate $A_i \cdot \overline{x_i}$ and is shown in table 1.

Table 1

Sub-area number	Area(A_i)	Centroid of area($\overline{x_i}$)	$A_i \cdot \overline{x_i}$
1	0.5	2.333	1.1665
2	02	5	10
3	.05	7.166	0.3583
4	.15	7.25	1.0875
5	.15	7.75	1.1625
6	.15	8.333	1.2499

The defuzzified value x^* will be $\frac{\sum_{i=1}^{N} A_i \times \overline{x_i}}{\sum_{i=1}^{N} A_i}$

$$= \frac{(1.1665+10+0.3583+1.0875+1.1625+1.2499)}{(0.5+2+.05+.15+.15+.15)}$$

= (15.0247)/3 =5.008

$x^* = 5.008$

Center of Area / Bisector of Area Method (BOA)

This method calculates the position under the curve where the areas on both sides are equal. The BOA generates the action that partitions the area into two regions with the same area.

$\int_{\alpha}^{x^*} \mu_A(x)\, dx = \int_{x^*}^{\beta} \mu_A(x) dx$, where α = min {x| x ∈ X} and β = max {x| x ∈ X}

Weighted Average Method

This method is valid for fuzzy sets with symmetrical output membership functions and produces results very close to the COA method. This method is less computationally intensive. Each membership function is weighted by its maximum membership value. The defuzzified value is defined as :

$$x^* = \frac{\sum \mu(x).x}{\sum \mu(x)}$$

Here $\sum$ denotes the algebraic summation and x is the element with maximum membership function.

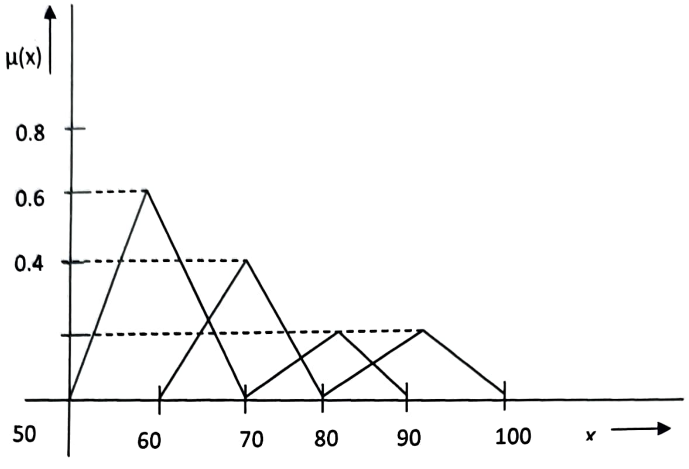


Figure 3: Fuzzy set A

Example:

Let A be a fuzzy set that tells about a student as shown in figure 3 and the elements with corresponding maximum membership values are also given.

A = {(P, 0.6), (F, 0.4),(G, 0.2),(VG, 0.2), (E, 0)}

Here, the linguistic variable P represents a Pass student, F stands for a Fair student, G represents a Good student, VG represents a Very Good student and E for an Excellent student.

Now the defuzzified value x^* for set A will be

$$x^* = \frac{(60*0.6+70*0.4+80*0.2+90*0.2+100*0)}{0.6+0.4+0.2+0.2+0}$$

= 98/1.4=70

The defuzzified value for the fuzzy set A with weighted average method represents a Fair student.

Maxima Methods

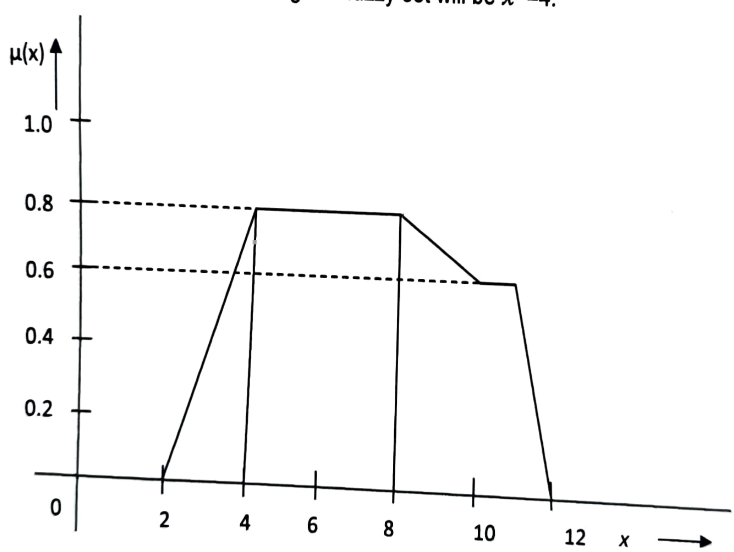
This method considers values with maximum membership. There are different maxima methods with different conflict resolution strategies for multiple maxima.

- First of Maxima Method (FOM)
- Last of Maxima Method (LOM)
- Mean of Maxima Method (MOM)

- **First of Maxima Method (FOM)**

This method determines the smallest value of the domain with maximum membership value.
Example:

The defuzzified value x^* of the given fuzzy set will be x^*=4.



- **Last of Maxima Method (LOM)**

Determine the largest value of the domain with maximum membership value.

In the example given for FOM, the defuzzified value for LOM method will be x^*= 8

- **Mean of Maxima Method (MOM)**

In this method, the defuzzified value is taken as the element with the highest membership values. When there are more than one element having maximum membership values, the mean value of the maxima is taken.
Let A be a fuzzy set with membership function $\mu_A(x)$ defined over $x \in X$, where X is a universe of discourse. The defuzzified value is let say x^* of a fuzzy set and is defined as,

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|},$$

Here, $M = \{x_i |\ \mu_A(x_i)$ is equal to the height of the fuzzy set A$\}$ and $|M|$ is the cardinality of the set M.

Example

In the example as shown in Fig. , x = 4, 6, 8 have maximum membership values and hence |M| = 3

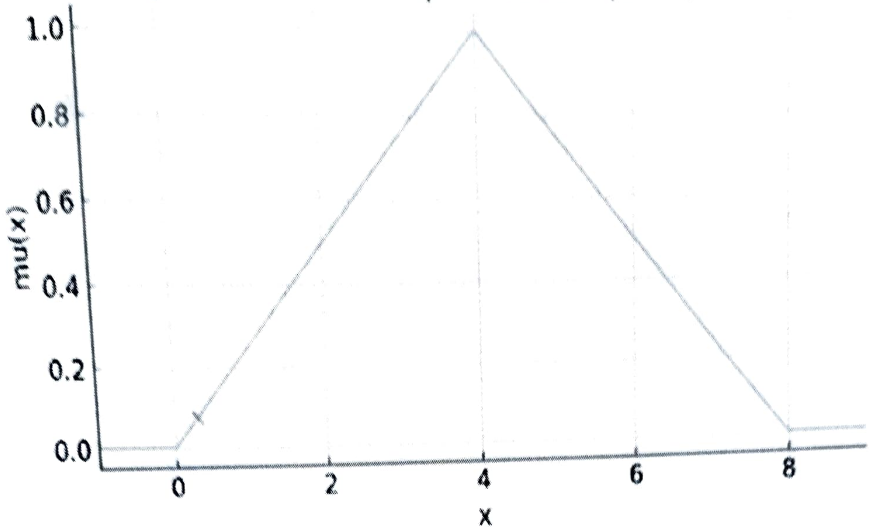
According to MOM method, $x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$

Now the defuzzified value x^* will be $x^* = \frac{4+6+8}{3} = \frac{18}{3} = 6.$

References:

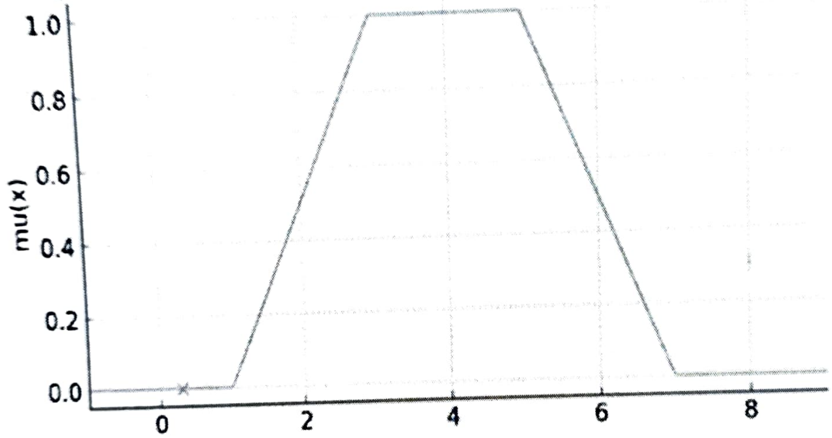
1. N. Mogharreban and L. F. DiLalla "Comparison of Defuzzification Techniques for Analysis of Non-interval Data", IEEE, 06.
2. Jean J. Saade and Hassan B. Diab. "Defuzzification Methods and New Techniques for Fuzzy Controllers", Iranian Journal of Electrical and Computer Engineering, 2004.
3. Aarthi Chandramohan, M. V. C. Rao and M. Senthil Arumugam: "Two new and useful defuzzification methods based on root mean square value", Soft Computing, 2006.
4. Soft Computing by D.K. Pratihar, Narosa Publication.

Triangular Membership Function (a=0, b=4, c=8)



a=0, b=4,
c=8

Trapezoidal Membership Function (a=1, b=3, c=5, d=7)



Triangular vs Trapezoidal membership (with examples)

Definitions / formulas

- Triangular membership $\mu_{tri}(x; a, b, c)$ (support $[a, c]$, peak at b, $a \le b \le c$):

$$\mu_{tri}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ \dfrac{c-x}{c-b}, & b < x < c \\ 0, & x \ge c \end{cases}$$

- Trapezoidal membership $\mu_{trap}(x; a, b, c, d)$ (rises on $[a, b]$, plateau on $[b, c]$, falls on $[c, d]$):

$$\mu_{trap}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & b < x < c \\ \dfrac{d-x}{d-c}, & c \le x < d \\ 0, & x \ge d \end{cases}$$

Two concrete examples I used and computed for you

1. Triangular with $a = 0,\ b = 4,\ c = 8$ (peak at 4, base from 0 to 8).
   - This matches the earlier discrete points (μ(2)=0.5, μ(4)=1, μ(6)=0.5).
   - For $x = 0.3$:

take @ $x = 0.3$

μ 0.5
2
9

$$\mu_{tri}(0.3) = \frac{0.3 - 0}{4 - 0} = 0.075$$

2. Trapezoidal with $a = 1,\ b = 3,\ c = 5,\ d = 7$ (rises from 1→3, plateau 3→5, falls 5→7).
   - For $x = 0.3$: $x < a$ so $\mu_{trap}(0.3) = 0$.

I ran a short Python snippet that computed these values and plotted both membership functions. Output from the run:

- Triangular $\mu(0.3) = 0.075000$
- Trapezoidal $\mu(0.3) = 0.000000$

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1) Triangular membership $\mu_{tri}(x; a, b, c)$

Definition (peak at b, support $[a, c]$):

$$\mu_{tri}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x - a}{b - a}, & a < x \le b \\ \dfrac{c - x}{c - b}, & b < x < c \\ 0, & x \ge c \end{cases}$$

Area (A) (height = 1):

$$A_{tri} = \frac{c - a}{2}$$

Centroid (COG) C (first moment / area):

$$C_{tri} = \frac{a + b + c}{3}$$

Derivation note: integrate $x\mu(x)$ over $[a, c]$ and divide by area; simplifies to $(a + b + c)/3$.

Example (numeric): $a = 0,\ b = 4,\ c = 8$

- Area: $A = (8 - 0)/2 = 4$
- Centroid: $C = (0 + 4 + 8)/3 = 4.0$

2) Trapezoidal membership $\mu_{trap}(x; a, b, c, d)$

Definition (rising on $[a, b]$, plateau $[b, c]$, falling $[c, d]$):

$$\mu_{trap}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x - a}{b - a}, & a < x \le b \\ 1, & b < x < c \\ \dfrac{d - x}{d - c}, & c \le x < d \\ 0, & x \ge d \end{cases}$$

Area (A):

Compute as sum of rising triangle, plateau rectangle, falling triangle:

$$A_{trap} = \frac{b - a}{2} + (c - b) + \frac{d - c}{2} = \frac{-a - b + c + d}{2}$$

Centroid (COG) C **(closed form):**

$$C_{trap} = \frac{c^2 + cd + d^2 - a^2 - ab - b^2}{3\,(c + d - a - b)}$$

(You can equivalently use the negative-signed variant depending on algebraic rearrangement; this form uses the positive denominator $c + d - a - b$.)

(You can equivalently use the negative-signed variant depending on algebraic rearrangement; this form uses the positive denominator $c + d - a - b$.)

Example (numeric): $a = 1,\ b = 3,\ c = 5,\ d = 7$

- Area: $A = (-1 - 3 + 5 + 7)/2 = 4.0$
- Centroid: $C = 4.0$ (symmetric trapezoid centered at 4)

3) Center-of-Sum (CoS) — combining multiple fuzzy sets

If you have n fuzzy output sets (e.g. outputs of different rules), each with area A_i and centroid C_i, the Center-of-Sum (CoS) defuzzified value is the area-weighted centroid:

$$x^* = \frac{\sum_{i=1}^{n} A_i C_i}{\sum_{i=1}^{n} A_i}$$

Example A (two triangles):

- Set 1: triangular with $(a,b,c) = (2,3,4)$ → area $A_1 = (4-2)/2 = 1$, centroid $C_1 = (2+3+4)/3 = 3$.
- Set 2: triangular with $(a,b,c) = (6,7,8)$ → area $A_2 = 1$, centroid $C_2 = 7$.

CoS:

$$x^* = \frac{1 \cdot 3 + 1 \cdot 7}{1+1} = \frac{10}{2} = 5.0$$

Example B (triangle + trapezoid):

- Triangle (a=0,b=4,c=8): $A_{tri} = 4, C_{tri} = 4$.
- Trapezoid (a=1,b=3,c=5,d=7): $A_{trap} = 4, C_{trap} = 4$.

CoS:

$$x^* = \frac{4 \cdot 4 + 4 \cdot 4}{4+4} = 4.0$$