VA-4, Part2

With the assumption of harmonic motion, we arrived at the two principal modes of vibration. The question is: Are these the general free-vibration response of our system? The answer is— No. You can easily show that the principal modes are linearly independent, that is,

$$x_1(t) = X_{11} \sin(\omega_1 t + \phi_1) \text{ (first pr. mode)}$$

$$\& \; x_1(t) = X_{12} \sin(\omega_2 t + \phi_2) \text{ (2nd. pr. mode)}$$

are linearly independent. Hence, from the theory of differential equations, we know that their superposition will also be a solution. Thus,

$$x_1(t) = X_{11} \sin(\omega_1 t + \phi_1) + X_{12} \sin(\omega_2 t + \phi_2) \text{ --- (i)}$$

is a more general solution. Actually, it is <u>the</u> general free-vibration response of m_1 containing <u>four</u> arbitrary constants of integration, namely, X_{11}, X_{12}, ϕ_1 & ϕ_2. Note that our system is of order $2+2=4$ (Two 2nd order DEOM) & hence the general solution can have 4 & only 4 arbitrary constants.

Very similarly,

$$x_2(t) = \mu_1 X_{11} \sin(\omega_1 t + \phi_1) + \mu_2 X_{12} \sin(\omega_2 t + \phi_2) \text{ -- (ii)}$$

is the general free-vibration response of m_2:

→ <u>An interesting question:-</u> How can the principal modes be excited, that is, under what kind of initial conditions would m_1 & m_2 execute simple harmonic oscillations? We now answer it:

→ Suppose our example system is executing the first principal mode. Then,

$$x_1 = X_{11}\sin(\omega_1 t+\phi)$$

& $$x_2 = \mu_1 X_{11}\sin(\omega_1 t+\phi)$$

Hence, $$x_1(0) = X_{11}\sin\phi$$

$$x_2(0) = \mu_1 x_{11}\sin\phi.$$

So, $$\boxed{x_2(0) = \mu_1 x_1(0)} \quad \text{---(iii)}$$

Now, $$\dot{x}_1 = X_{11}\omega_1\cos(\omega_1 t+\phi)$$

& $$\dot{x}_2 = \mu_1 X_{11}\omega_1\cos(\omega_1 t+\phi)$$

& so, $$\dot{x}_1(0) = X_{11}\omega_1\cos\phi$$

& $$\dot{x}_2(0) = \mu_1 X_{11}\omega_1\cos\phi$$

Thus, $$\boxed{\dot{x}_2(0) = \mu_1\dot{x}_1(0)} \quad \text{---(iv)}$$

We say that (iii) and (iv) are the <u>necessary conditions</u> for 1st pr. mode. That is, while the system is executing 1st pr. mode, these conditions are automatically satisfied.

Similarly, the necessary conditions for the 2nd pr. ϕ mode are:

$$x_2(0) = \mu_2 x_1(0) \quad \text{---(v)}$$

& $$\dot{x}_2(0) = \mu_2\dot{x}_1(0) \quad \text{---(vi)}$$

But remember we started with the aim of finding initial conditions to excite the principal modes. That is, we are actually looking for sufficient conditions which, when satisfied, would excite either principal mode. We shall now show that the necessary conditions obtained above are also the required sufficient conditions.

→ Suppose $x_2(0) = \mu_1 x_1(0)$ & $\dot{x}_2(0) = \mu_1 \dot{x}_1(0)$.

Now, the general responses are:

$$x_1(t) = X_{11} \sin(\omega_1 t + \phi_1) + X_{12} \sin(\omega_2 t + \phi_2)$$

& $$x_2(t) = \mu_1 X_{11} \sin(\omega_1 t + \phi_1) + \mu_2 X_{12} \sin(\omega_2 t + \phi_2).$$

So, ~~$x_2(0) = \mu_1 X_{11} \sin(\omega_1 t + \phi_1$~~

$$x_2(0) = \mu_1 X_{11} \sin\phi_1 + \mu_2 X_{12} \sin\phi_2$$

& $$\mu_1 x_1(0) = \mu_1 X_{11} \sin\phi_1 + \mu_1 X_{12} \sin\phi_2$$

So, $x_2(0) = \mu_1 x_1(0) \Rightarrow \mu_2 X_{12} \sin\phi_2 = \mu_1 X_{12} \sin\phi_2$

or, $(\mu_2 - \mu_1) X_{12} \sin\phi_2 = 0$ ---- ⓐ

Also, $$\dot{x}_1(t) = X_{11} \omega_1 \cos(\omega_1 t + \phi_1) + X_{12} \omega_2 \cos(\omega_2 t + \phi_2)$$

& $$\dot{x}_2(t) = \mu_1 X_{11} \omega_1 \cos(\omega_1 t + \phi_1) + \mu_2 X_{12} \omega_2 \cos(\omega_2 t + \phi_2)$$

$\therefore$ $$\dot{x}_2(0) = \mu_1 X_{11} \omega_1 \cos\phi_1 + \mu_2 X_{12} \omega_2 \cos\phi_2$$

& $$\mu_1 \dot{x}_1(0) = \mu_1 X_{11} \omega_1 \cos\phi_1 + \mu_1 X_{12} \omega_2 \cos\phi_2$$

Thus, $\dot{x}_2(0) = \mu_1 \dot{x}_1(0) \Rightarrow \mu_2 X_{12} \omega_2 \cos\phi_2 = \mu_1 X_{12} \omega_2 \cos\phi_2$

or, $(\mu_2 - \mu_1) X_{12} \omega_2 \cos\phi_2 = 0$ —— ⓑ

From ⓐ & ⓑ we can conclude that $X_{12} = 0$ (since $\mu_2 \neq \mu_1$, $\omega_2 \neq 0$, $\sin\phi_2$ & $\cos\phi_2$ are not simultaneously zero). Thus, $x_1(t) = X_{11} \sin(\omega_1 t + \phi_1)$ & $x_2(t) = \mu_1 X_{11} \sin(\omega_1 t + \phi_1)$

That is, only 1st pr. mode is excited.

Similarly, the sufficient conditions for exciting the 2nd pr. mode are:

$$x_2(0) = \mu_2 x_1(0)$$
$$\& \quad \dot{x}_2(0) = \mu_2 \dot{x}_1(0)$$

Remember

<u>Let us sum up:-</u> The <u>necessary and sufficient</u> conditions for the first principal mode of vibration are: $x_2(0) = \mu_1 x_1(0)$ & $\dot{x}_2(0) = \mu_1 \dot{x}_1(0)$. The same for the 2nd pr. mode are: $x_2(0) = \mu_2 x_1(0)$ & $\dot{x}_2(0) = \mu_2 \dot{x}_1(0)$.

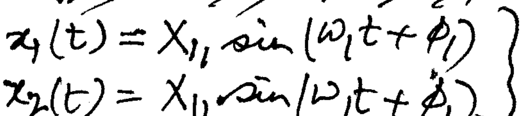
→ So, for example, if we pull down m_1 by 5 mm & m_2 by 1.618×5 mm & release the masses, the system would execute the first principal mode since, $\mu_1 = 1.618$.

So, we are applying the initial conditions

$$x_2(0) = \mu_1 x_1(0) \quad \& \quad \dot{x}_2(0) = \mu_1 \dot{x}_1(0) \text{ because,}$$
$$\dot{x}_2(0) = 0 \ \& \ \dot{x}_1(0) = 0 \text{ too, } \dot{x}_2(0) = \mu_1 \dot{x}_1(0) \text{ automatically.}$$

→ Similarly, if we pull down m_1 by 5 mm & push <u>up</u> m_2 by 0.618×5 mm & release, the system will execute the 2nd pr. mode.

→ <u>Homework Problem</u>:- Obtain the principal modes of response for the system



Derive DEOM, set up the frequency eqn. & amplitude eqns. etc.

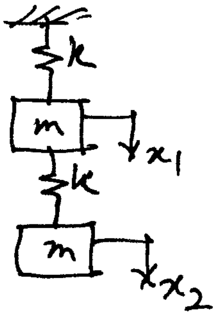
<u>ANSWERS</u>:~

$$\left.\begin{array}{l} x_1(t) = X_{11} \sin(\omega_1 t + \phi_1) \\ x_2(t) = X_{11} \sin(\omega_1 t + \phi_1) \end{array}\right\} \text{First Pr. mode}$$

→

$$\left.\begin{aligned} x_1(t) &= X_{12}\sin(\omega_2 t+\phi_2) \\ x_2(t) &= -X_{12}\sin(\omega_2 t+\phi_2) \end{aligned}\right\}\ \text{2nd Pr. mode} \quad \text{where } \omega_1=\sqrt{\frac{k}{m}}\ \&\ \omega_2=\sqrt{\frac{3k}{m}}$$

→ Let us get back to our example problem.



The vector $\{X\}_1 = \begin{Bmatrix} X_{11} \\ X_{21} \end{Bmatrix} = \begin{Bmatrix} X_{11} \\ \mu_1 X_{11} \end{Bmatrix}$ is called a modal vector or eigenvector or characteristic vector for the first principal mode. Note that X_{11} is arbitrary until specific initial conditions are given. Similarly, $\{X\}_2 = \begin{Bmatrix} X_{12} \\ X_{22} \end{Bmatrix} = \begin{Bmatrix} X_{12} \\ \mu_2 X_{12} \end{Bmatrix}$ is a modal vector for the 2nd principal mode, where X_{12} is arbitrary etc. $\{X\}_1$ & $\{X\}_2$ are nothing but amplitude vectors corresponding to the principal modes.

→ The matrix $[\overset{\text{mu}}{\mu}] = \left[\{x\}_1, \{x\}_2\right] = \begin{bmatrix} X_{11} & X_{12} \\ \mu_1 X_{11} & \mu_2 X_{12} \end{bmatrix}$ is called a modal matrix (& not the modal matrix, since X_{11} & X_{12} are arbitrary).

→ For simplifying computations, modal vectors are often normalized. This is called the 'Normalization of Modal Vectors'. This is done in several ways, two of which we mention here.

(i) We can set $X_{11} = 1$, $X_{12} = 1$ & then, $\{x\}_1 = \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix}$ & $\{x\}_2 = \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$ are the normalized modal vectors. $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$ is the normalized modal matrix. →

(ii) A secondway of normalizing is done by making the magnitudes of each modal vector (equal to) unity.

So, $\sqrt{X_{11}^2+(\mu_1 X_{11})^2}=1$ or, $X_{11}=\frac{1}{\sqrt{1+\mu_1^2}}$

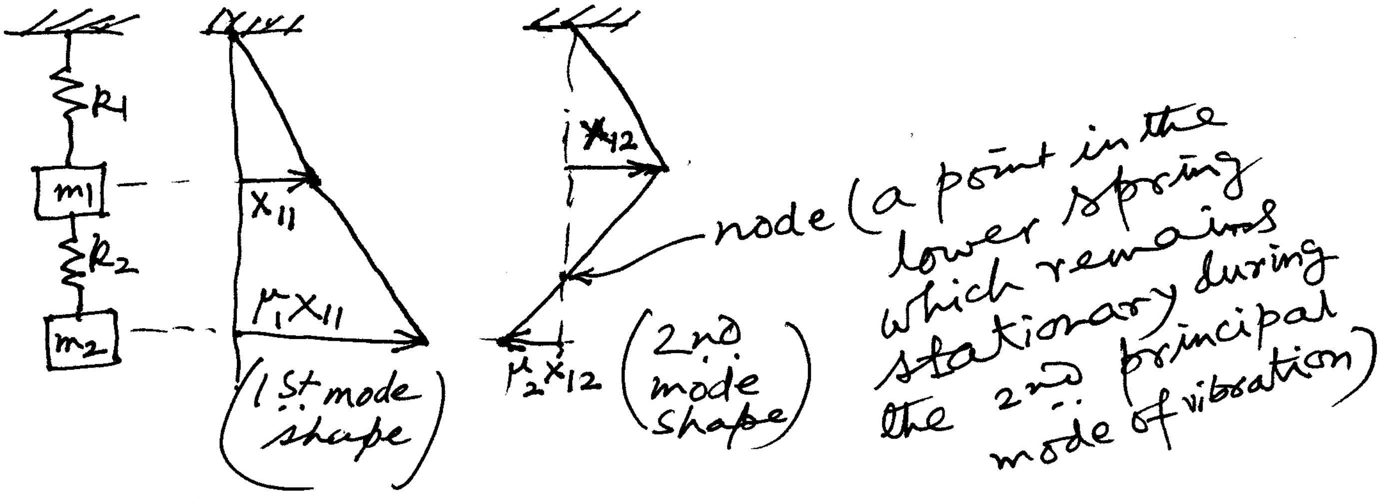
& $\sqrt{X_{12}^2+(\mu_2 X_{12})^2}=1$ or, $X_{12}=\frac{1}{\sqrt{1+\mu_2^2}}$

Hence, now $\{X\}_1 = \begin{Bmatrix} X_{11} \\ \mu_1 X_{11} \end{Bmatrix} = \frac{1}{\sqrt{1+\mu_1^2}} \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix}$

& $\{X\}_2 = \begin{Bmatrix} X_{12} \\ \mu_2 X_{12} \end{Bmatrix} = \frac{1}{\sqrt{1+\mu_2^2}} \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$.

We shall mostly use the first type of normalization.

§ **Mode shapes:-** A mode shape is a geometric way of representing the amplitudes of various points of the system in either principal mode. It is drawn to an arbitrary scale:



[Note:- Some authors call the modal vectors 'mode shapes'. Hence, $\begin{Bmatrix} X_{11} \\ \mu_1 X_{11} \end{Bmatrix}$ is a mode shape for 1st mode & $\begin{Bmatrix} X_{12} \\ \mu_2 X_{12} \end{Bmatrix}$ is the one for 2nd pr. mode]

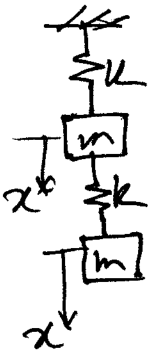
⑤ Some interesting properties of modal vectors:-

(i) Let $\omega_1 \neq \omega_2$, as in our example problem.

Then, $\{X\}_1$ & $\{X\}_2$ (the modal vectors) are orthogonal to each other w.r.t. weighting matrices $[m]$ & $[k]$. That is,

$$\{x\}_1^T [m] \{X\}_2 = 0 \quad \& \quad \{x\}_1^T [K] \{X\}_2 = 0.$$

Let us verify these: (We take normalized modal vectors)



$$\{X\}_1 = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix} \; \& \; \{X\}_2 = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix};$$

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}; \; [k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$ for the example problem. [Even with general modal vectors, these relations are true]

So, $\{X\}_1^T [m] \{X\}_2 = \{1 \;\; 1.618\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$

~~= 1.16~~ $= \{m \;\; 1.618m\} \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix} = m - 1.618 \times 0.618m$

$= 7.6 \times 10^{-5} m \simeq 0.$

{ Due to numerical approximations in the modal vectors (only three places after decimal were retained), we donot get exactly zero, but get a very small +ive number, $7.6 \times 10^{-5} m$]

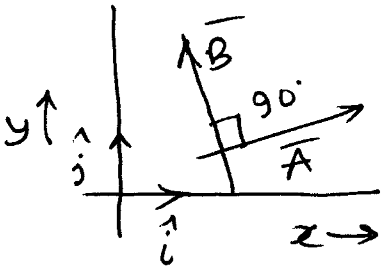
Also, $\{X\}_1^T [K] \{X\}_2 = \{1 \;\; 1.618\} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$

$= \{0.382k \;\; 0.618k\} \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix} = 7.6 \times 10^{-5} k \simeq 0.$

→ But what is the use of all this? You will see subsequently that the

Validity of the orthogonality relations shown above will enable us to uncouple (or, 'decouple') the DEOM which facilitates the handling of forced vibrations problems with complex forcing functions.

[**NOTE** :- Two vectors $\bar{A} = A_x\hat{i} + A_y\hat{j} = \begin{Bmatrix} A_x \\ A_y \end{Bmatrix} = \{A\}$ & $\bar{B} = B_x\hat{i} + B_y\hat{j} = \begin{Bmatrix} B_x \\ B_y \end{Bmatrix}$ ~~are~~ $= \{B\}$ are orthogonal in ordinary sense if $\bar{A} \cdot \bar{B} = 0$, i.e., if $\{A\}^T\{B\} = 0$. ($\bar{A} \cdot \bar{B}$ is same as $\{A\}^T\{B\}$, check)



Here, the identity ~~matrix~~ $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the weighting matrix since $\{A\}^T\{B\} = 0$ can also be written as $\{A\}^T[I]\{B\} = 0$. (check) ~~So, $\{A\}$ & $\{B\}$~~

~~are Our $[m]$ & modal [illegible] vectors~~

Our modal vectors $\{X\}_1$ & $\{X\}_2$ are however orthogonal not in the ordinary sense but in the generalized sense where the weighting matrices are $[m]$ & $[k]$, i.e., here

$$\{X\}_1^T[m]\{X\}_2 = 0 \quad \& \quad \{X\}_1^T[k]\{X\}_2 = 0.$$

We shall prove the validity of these later]

———x———

END OF PART 2