# Assignment 2

1. Given a function $f(x)$ and a set of uniformly spaced grid-points $x_i$, $i = 1, 2, \ldots,$ with $\Delta x = x_{i+1} - x_i$

   (a) obtain a finite-difference approximation for $(df/dx)_i$ which is second-order-accurate, using values of $f$ at $x_{i\pm1}$

   (b) obtain a finite-difference approximation for $(d^3f/dx^3)_i$ which is second-order-accurate using values of $f$ at $x_{i\pm1}$ and $x_{i\pm2}$. Using this approximation compute $d^3 \ln x/dx^3$ at $x = 1.5$ using $\Delta x = 0.1$ and compare with the exact value obtained by carrying out the differentiation analytically and evaluating the expression obtained, at $x = 1.5$. What is the error in the finite difference approximation compared to the exact value? Give quantitative justification for this error using the Taylor series.

2. Given a function $f(x)$ and a set of uniformly spaced grid-points $x_i$, $i = 1, 2, \ldots,$ with $\Delta x = x_{i+1} - x_i$, obtain a finite-difference approximation for $(df/dx)_i$, which is second-order-accurate, using values of $f$ at $x_{i\pm1}$. If you reduce $\Delta x$ by a factor of 2, by what factor would you expect the error, between the finite difference approximation and the exact value, to decrease? For the specific choice

$$f(x) = \sin 2x + 8 \sinh x$$

   where the argument of sin is in radians, compute $df/dx$ at $x = 0$ using the finite-difference approximation derived above, with $\Delta x = 0.2$ and $\Delta x = 0.4$. Using the analytical expression for the derivative compute the exact value and the error in the finite-difference approximation with the two different grid spacings. Compute the ratio of error for the two grid spacings and provide an explanation for your findings.

3. A function $f(x)$ is defined for $x \geqslant 0$. A set of uniformly spaced grid points is defined by $x_i = i \times \Delta x$, $i = 0, 1, 2, \cdots$. Write down finite-difference approximations for $df/dx$ at $x = 0$ which are first-order and second-order accurate, using values at grid points which have $x \geqslant 0$ only (one-sided difference approximations). First-order accurate finite-difference approximations for $(df/dx)_0$ can be written using $f_0$ and $f_2$ or using $f_1$ and $f_2$ instead of the usual $f_0$ and $f_1$. Write down such approximations. Is the approximation in terms of $f_0$ and $f_1$ superior? Give suitable justification for your answer.

4. Finite difference approximations for the derivatives of $T(x)$ are to be written using values at non-uniformly spaced grid points. The spacing between adjacent grid points are of comparable magnitude but all unequal. Express $T_{i-1}$ and $T_{i+1}$, values at $x_{i-1} = x_i - \Delta x_\ell$ and $x_{i+1} = x_i + \Delta x_r$, in terms of $T$ and its derivatives at $x_i$, using the Taylor series about the point $x_i$. Using these, write down finite difference approximations for $(dT/dx)_i$ and $(d^2T/dx^2)_i$, in terms of $T_{i-1}$, $T_i$ and $T_{i+1}$. What is the order of accuracy of these approximations? In each case choose the approximation in such a way that it has the highest order of accuracy possible using only these three points. For $T(x) = \ln x$, compute the value of $dT/dx$ and $d^2T/dx^2$ at $x = 2$, using values at $x = 1.9, 2.0$ and $2.2$ and the finite difference approximations you have derived. Values at how many points would you require in order to obtain a second-order-accurate approximation for $(d^2T/dx^2)_i$ and a fourth-order-accurate approximation for $(dT/dx)_i$, when the grid-spacing is non-uniform? Give reasoning for your answer.

5. (a) In cylindrical coordinates the one-dimensional diffusion equation, assuming no variation in the azimuthal and axial directions, is

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right)$$

   Explain how you will solve this equation using an explicit scheme, a fully implicit scheme and a Crank-Nicolson type scheme. What is the order of accuracy, in time and in space, of each of the three schemes?

(b) A finite difference approximation for

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} \tag{1}$$

can be written using three point central difference approximations for the first and second derivatives. What is the order of accuracy of this approximation? Using the Taylor series obtain the expression for the leading order term in the error. Alternatively the expression in (1) can be written as

$$\frac{1}{r}\frac{dF}{dr}$$

where

$$F = r\frac{dT}{dr}$$

These can be approximated by

$$\frac{1}{r_j}\left(\frac{F_{j+1/2} - F_{j-1/2}}{\Delta r}\right), \quad F_j = r_j\left(\frac{T_{j+1/2} - T_{j-1/2}}{\Delta r}\right)$$

Show how these approximations can be used to express $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$ at $r_j$ in terms of $T_j$ and $T_{j\pm1}$. What is the order of accuracy of this approximation? Give suitable justification for your answer. Logical reasoning, instead of explicit derivation of the error term, would be sufficient.

6. Explain how you will solve the one-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

with boundary conditions

$$T = T_A \quad \text{at } x = 0$$

$$T = T_B \quad \text{at } x = 1$$

where $T_A$ and $T_B$ are constants, and the initial condition

$$T = F(x) \quad \text{at } t = 0$$

using (i) the FTCS scheme, (ii) the fully implicit scheme and (iii) the Crank-Nicolson scheme. What is the order of the truncation error of these different schemes? Using the Taylor series provide justification for your statements. Use the von Neumann method and determine whether the Crank-Nicolson scheme is stable or not. State clearly the merits and demerits of the three different methods for solving this equation.

7. A transient one-dimensional conduction is governed by the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with boundary conditions

$$T = 0 \quad \text{at } x = 0, 1.5$$

and initial condition

$$T = x(1.5 - x) \quad \text{at } t = 0 \quad \text{for } 0 \le x \le 1.5$$

Compute $T$ at $x = 0.5$ and $t = 0.1$ using (a) the explicit method, (b) the fully implicit method and (c) the Crank-Nicolson method. Use $\Delta x = 0.5$ and $\Delta t = 0.1$.

8. Solve the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

with boundary conditions

$$T = 0 \quad \text{at} \quad x = 0$$

$$T = 1 \quad \text{at} \quad x = 1$$

and initial condition

$$T = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \le x < 1$$

using the hopscotch method. Assume $\alpha = 1$ and use $\Delta x = 0.25$, $\Delta t = 0.05$. Compute $T$ at all grid points, from $x = 0$ to $x = 1$, for $t = 0.05$ and $t = 0.10$. With grid points numbered from 0 to 4, in the first step, in the first sweep, use explicit method for odd-numbered nodes.

9. The partial differential equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + bu$$

where $a, b > 0$ are constants, is to be solved subject to boundary conditions

$$u = 0 \quad \text{at} \quad x = 0, 1$$

and initial condition

$$u = f(x) \quad \text{at} \quad t = 0$$

One way of obtaining a numerical solution is to use the explicit scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + bu_j^n$$

Using the von Neumann method carry out a stability analysis for this scheme and derive the condition for stability. Assume a form $e^{\sigma t}$ to avoid confusion with the coefficient $a$. Alternatively one can use fully implicit schemes

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} + bu_j^n$$

or

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} + bu_j^{n+1}$$

Carry out a stability analysis for both these schemes using the von Neumann method and explain when each scheme is stable. All the schemes given above have an error of $O(\Delta t, \Delta x^2)$. Suggest a scheme which has an error of $O(\Delta t^2, \Delta x^2)$.

10. How will you solve the equation

$$\frac{\partial T}{\partial t} = -\alpha \frac{\partial^4 T}{\partial x^4}$$

using an explicit method, alternatively called the Forward Time Central Space (FTCS) scheme? Use

$$\left(\frac{\partial^4 T}{\partial x^4}\right)_j = \frac{1}{\Delta x^2} \left[ \left(\frac{\partial^2 T}{\partial x^2}\right)_{j-1} - 2\left(\frac{\partial^2 T}{\partial x^2}\right)_j + \left(\frac{\partial^2 T}{\partial x^2}\right)_{j+1} \right]$$

Using the von Neumann method determine when this scheme will be stable. Assume $\alpha > 0$. The boundary and initial conditions are

$$T = \frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 0, 1$$

$$T = T_0(x) \quad \text{at} \quad t = 0$$

State clearly any assumptions that you make. How will you solve this equation using a Crank-Nicolson type scheme? Is this scheme stable? Now consider the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^4 T}{\partial x^4}$$

where again $\alpha > 0$. Using the von Neumann method determine the stability for the explicit and fully implicit method for solving this equation numerically. Is the exponential growth due to numerical instability?

11. A finite difference approximation for the one-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

at $x = x_j$ and $t = t_n$ which has error of $O(\Delta t^2, \Delta x^2)$ is to be obtained. For the spatial derivative on RHS use a second-order-accurate central difference approximation. To obtain a second-order-accurate approximation for the time derivative on LHS, use values at $t_n$, $t_{n-1}$ and $t_{n-2}$, all at $x_j$. Using these write down the finite difference approximation for the partial differential equation. Using the von Neumann method for stability analysis, obtain an expression for the amplification factor $G$. Considering this as a quadratic equation for $1/G$ prove that this scheme is unconditionally stable.

12. Using the Taylor series derive a central difference approximation for

$$\frac{\partial^2 T}{\partial x^2}$$

which is fourth-order-accurate. Explain how you will solve the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

by an explicit scheme, using a first-order-accurate forward difference approximation for the time derivative and a fourth-order-accurate central difference approximation for the spatial derivative. Using the von Neumann method determine when this scheme would be stable.

Assume that the initial condition is

$$T = 1.2\left(\frac{1}{2} - \left|\frac{1}{2} - x\right|\right)$$

and the boundary conditions are

$$T = 0 \quad \text{at} \quad x = 0, 1 \quad \text{for} \quad t > 0$$

Use the finite-difference scheme explained above with $\Delta x = 0.2$ and $\Delta t = 0.01$. Assume that $\alpha = 1$. What is the value of $T$ at $x = 0.2$ and $t = 0.02$? Compute only the values you require and not the values at all grid-points. In order to overcome the difficulty with nodes near the boundary, for a finite-difference approximation which is fourth-order-accurate in space, we extend the definition of $T$ in the region outside $0 \leqslant x \leqslant 1$. In the region $-1 \leqslant x \leqslant 0$ we define $T(x) = -T(-x)$. For example $T(-0.4) = -T(0.4)$. Now that we have $T$ defined in the interval $-1 \leqslant x \leqslant 1$, we assume that this repeats periodically so that $T(x + 2n) = T(x)$, where $n$ is any integer. For example $T(0.4 + 2)$, i.e. $T(2.4)$, as well as $T(0.4 - 2)$, i.e. $T(-1.6)$, are both equal to $T(0.4)$.

13. The system of equations

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 v}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 v}{\partial x^2}$$

where $a$, $b$, $c$ and $d$ are constants, $u$ and $v$ are specified on the boundaries at $x = 0, 1$ and initial conditions are specified for $u$ and $v$, is to be solved numerically. Give an algorithm which uses first-order-accurate forward difference approximation for time derivatives and second-order-accurate central difference approximation for the spatial derivatives. Carry out a stability analysis for this scheme using the von Neumann method. Assume that the errors in $u$ and $v$ have similar dependence on $x$ and $t$. Derive a quadratic equation for the amplification factor of a typical Fourier component. Is the amplification factor real or complex? Can you derive conditions for the stability of the scheme?

14. The equation governing transient one-dimensional conduction with a distributed heat source is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + q$$

This is to be solved in the region $0 \leqslant x \leqslant 1$, with boundary conditions

$$\frac{\partial T}{\partial x} = hT \quad \text{at} \quad x = 0$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 1$$

At $t = 0$ we assume the initial condition

$$T = 0 \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

This is reasonable if $q = 0$ for $t < 0$ and steady state has been attained. At $t = 0$, $q$ is switched on and we wish to compute the consequent change in $T$. Assume the data

$$\alpha = 1, \; h = 6 \; \text{ and } \; q = 2 \text{ for } t > 0$$

Use finite difference approximations with $\Delta x = 0.1$ and $\Delta t = 0.004$. Use the FTCS scheme to compute $T$ in the interior and the finite difference approximation of the boundary conditions to compute values on the boundary. In the boundary conditions use one-sided finite difference approximations so that no values outside the region $0 \leqslant x \leqslant 1$ are involved. Compute $T_0^2$, ie, $T$ at $x = 0$ and $t = 0.008$

(a) using first-order-accurate approximation for the boundary condition

(b) using second-order-accurate approximation for the boundary condition, for consistency with the second-order spatial accuracy of the FTCS scheme used in the interior.

Compute only those values which are needed in order to compute $T_0^2$. Derive the one-sided difference approximations that you use.

15. The one-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

is to be solved in the region $0 \leqslant x \leqslant 1$ with boundary conditions

$$\frac{\partial T}{\partial x} = hT \quad \text{at} \quad x = 0$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 1$$

and initial condition

$$T = 1 - 4(x - 0.5)^2 \quad \text{at} \quad t = 0$$

For $\alpha = 1$ and $h = 3$, computation is to be carried out using the hopscotch method with $\Delta x = 0.2$ and $\Delta t = 0.01$. Assume that the grid-points at $x = 0, 0.2, \ldots, 1$ are numbered as $j = 0, 1, \ldots, 5$. In the first step, in the first sweep, values at odd-numbered grid-points are computed using the explicit method, while in the second sweep, values at even-numbered grid-points are computed using the 'implicit' method. Compute values of $T$ at all grid-points $j = 0, 1, \ldots, 5$ at the end of the first time-step, i.e., at $t = 0.01$

(a) using first-order-accurate approximations for the boundary conditions

(b) using second-order-accurate approximations for the boundary conditions for consistency with the second-order spatial accuracy of the hopscotch method used in the interior

Derive the one-sided difference approximations that you use.

16. The equation

$$[A]\{x\} = \{b\}$$

where A is a pentadiagonal matrix can be solved using a generalized Thomas algorithm described in Sec. 6.2.4 in C A J Fletcher, Computational Techniques for Fluid Dynamics 1. What is the number of arithmetic operations (divisions, multiplications and additions/subtractions) required for elimination and for back substitution? Compare with the number of operations required for a tridiagonal matrix.

17. Explain how you will solve the two-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

using the alternating direction implicit (ADI) method as given by Peaceman and Rachford. Write the partial derivative over $x$ in terms of values at the intermediate time level, denoted by $*$, and the partial derivative over $y$ in terms of values at the beginning or end of the complete time step.

We now want to solve this equation in the square region $0 \leqslant x, y \leqslant 0.6$. Initially the temperature $T$ is zero in the entire region. So the initial condition is

$$T = 0 \quad \text{at} \quad t = 0$$

At $t = 0$ the temperature of the boundary at $x = 0$ is instantaneously increased to unity, so that the boundary conditions for $t > 0$ are

$$T = 1 \quad \text{for} \quad x = 0$$

$$T = 0 \quad \text{for} \quad x = 0.6$$

$$T = 0 \quad \text{for} \quad y = 0, 0.6$$

Define a grid with uniform spacing $\Delta x = \Delta y = 0.2$. Assume that $\alpha = 1$ and choose $\Delta t = 0.008$. Using the ADI method which you have explained above, compute the values of $T$ at all internal grid points for starred values at the intermediate time level and the values at the end of the full time step at $t = 0.008$.

In the ADI method for the two-dimensional diffusion equation as defined above, are each of the half-steps and the complete time-step unconditionally stable, conditionally stable or unconditionally unstable? If the stability of the half-steps and of the complete time-step are not the same, how are the two consistent? What is the order of accuracy of the ADI scheme? State your answers without giving the mathematical proof.

18. The diffusion equation in three dimensions, written in Cartesian coordinates, is

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

where $\alpha$ is a constant. Explain how you will solve this equation by an explicit scheme and by a fully implicit scheme. Assume a uniformly spaced grid in space with $\Delta x = \Delta y = \Delta z$ and use a finite difference approximation. What is the order of accuracy of these schemes? Assuming that the temperature is specified on the boundary (Dirichlet boundary condition), carry out a stability analysis of the two schemes, using the von Neumann method, to determine when the schemes will be stable.

19. A different form of the alternate direction implicit method for the two-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha_x \frac{\partial^2 T}{\partial x^2} + \alpha_y \frac{\partial^2 T}{\partial y^2}$$

uses the two successive approximations

$$\frac{T^* - T^n}{\Delta t} = \alpha_x L_{xx} \left( \frac{T^* + T^n}{2} \right) + \alpha_y L_{yy} T^n$$

$$\frac{T^{n+1} - T^n}{\Delta t} = \alpha_x L_{xx} \left( \frac{T^* + T^n}{2} \right) + \alpha_y L_{yy} \left( \frac{T^{n+1} + T^n}{2} \right)$$

Explain why this implicit scheme involves solving only tridiagonal systems. In the von Neumann stability analysis of this scheme show that the amplification factor of a Fourier mode of the form $\exp(i m_x \pi x + i m_y \pi y)$, after the two successive approximations, is

$$G = \frac{1 - a - b + ab}{(1+a)(1+b)}$$

where $a = s_x(1 - \cos\theta_x)$, $b = s_y(1 - \cos\theta_y)$, $\theta_x = m_x \pi \Delta x$, $\theta_y = m_y \pi \Delta y$, $s_x = \alpha_x \Delta t/(\Delta x)^2$, $s_y = \alpha_y \Delta t/(\Delta y)^2$. Hence show that the scheme is unconditionally stable. Taking the difference of the two successive approximations show that

$$T^* = T^{n+1} - \alpha_y \Delta t \, L_{yy} \left( \frac{T^{n+1} - T^n}{2} \right)$$

Substituting in the second approximation show that

$$\begin{aligned} \frac{T^{n+1} - T^n}{\Delta t} &= \alpha_x L_{xx} \left( \frac{T^{n+1} + T^n}{2} \right) + \alpha_y L_{yy} \left( \frac{T^{n+1} + T^n}{2} \right) \\ &\quad - \frac{1}{4} (\Delta t)^2 \alpha_x \alpha_y L_{xx} L_{yy} \left( \frac{T^{n+1} - T^n}{\Delta t} \right) \end{aligned}$$

Hence show that the truncation error of this scheme is $O(\Delta t^2, \Delta x^2, \Delta y^2)$.

The diffusion equation is to be solved in the region $0 \leqslant x, y \leqslant 1$ with $T$ specified on the entire boundary (Dirichlet boundary condition). If

$$T = b(y,t) \quad \text{at } x = 1$$

how will you impose this boundary condition on $T^*$, so that the global truncation error is $O(\Delta t^2, \Delta x^2, \Delta y^2)$. Can you impose a Neumann boundary condition so that global second order accuracy in time is obtained?

20. The scheme in Problem 19 can be generalized to solve the three-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha_x \frac{\partial^2 T}{\partial x^2} + \alpha_y \frac{\partial^2 T}{\partial y^2} + \alpha_z \frac{\partial^2 T}{\partial z^2}$$

using the successive approximations

$$\frac{T^* - T^n}{\Delta t} = \alpha_x L_{xx} \left( \frac{T^* + T^n}{2} \right) + \alpha_y L_{yy} T^n + \alpha_z L_{zz} T^n$$

$$\frac{T^{**} - T^n}{\Delta t} = \alpha_x L_{xx} \left( \frac{T^* + T^n}{2} \right) + \alpha_y L_{yy} \left( \frac{T^{**} + T^n}{2} \right) + \alpha_z L_{zz} T^n$$

$$\frac{T^{n+1} - T^n}{\Delta t} = \alpha_x L_{xx} \left( \frac{T^* + T^n}{2} \right) + \alpha_y L_{yy} \left( \frac{T^{**} + T^n}{2} \right) + \alpha_z L_{zz} \left( \frac{T^{n+1} + T^n}{2} \right)$$

Explain why again only tridiagonal systems need to be solved. In the von Neumann stability analysis, the amplification factor of a Fourier mode of the form $\exp(im_x\pi x + im_y\pi y + im_z\pi z)$ is now given by

$$G = \frac{1 - a - b - c + ab + bc + ca + abc}{(1+a)(1+b)(1+c)}$$

where $c = s_z(1 - \cos\theta_z)$, $\theta_z = m_z\pi\Delta z$, $s_z = \alpha_z\Delta t/(\Delta z)^2$, with $a$ and $b$ as defined in problem 1. Hence show that the scheme is unconditionally stable. By eliminating $T^{**}$ and $T^*$ show that

$$\begin{aligned}\frac{T^{n+1} - T^n}{\Delta t} &= (\alpha_x L_{xx} + \alpha_y L_{yy} + \alpha_z L_{zz})\left(\frac{T^{n+1} + T^n}{2}\right)\\ &\quad -(\Delta t)^2(\alpha_x\alpha_y L_{xx}L_{yy} + \alpha_y\alpha_z L_{yy}L_{zz} + \alpha_z\alpha_x L_{zz}L_{xx})\left(\frac{T^{n+1} - T^n}{4\Delta t}\right)\\ &\quad +(\Delta t)^3\alpha_x\alpha_y\alpha_z L_{xx}L_{yy}L_{zz}\left(\frac{T^{n+1} - T^n}{8\Delta t}\right)\end{aligned}$$

Hence show that the scheme has a truncation error of $O(\Delta t^2, \Delta x^2, \Delta y^2, \Delta z^2)$.

The diffusion equation is to be solved in the region $0 \leqslant x, y, z \leqslant 1$, with $T$ specified on the entire boundary (Dirichlet boundary condition). If

$$T = b(y, z, t) \quad \text{at } x = 1$$

how will you impose this boundary condition on $T^*$ and on $T^{**}$ so that the global truncation error is $O(\Delta t^2, \Delta x^2, \Delta y^2, \Delta z^2)$

21. The equation governing three-dimensional diffusion

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)T$$

is to be solved by the algorithm

$$\frac{T^* - T^n}{\Delta t/3} = \alpha(L_{xx}T^* + L_{yy}T^n + L_{zz}T^n)$$

$$\frac{T^{**} - T^*}{\Delta t/3} = \alpha(L_{xx}T^* + L_{yy}T^{**} + L_{zz}T^*)$$

$$\frac{T^{n+1} - T^{**}}{\Delta t/3} = \alpha(L_{xx}T^{**} + L_{yy}T^{**} + L_{zz}T^{n+1})$$

where

$$L_{xx}T_{i,j,k} = \frac{T_{i-1,j,k} - 2T_{i,j,k} + T_{i+1,j,k}}{(\Delta x)^2}$$

and $L_{yy}$ and $L_{zz}$ are defined similarly. In this scheme the first fractional step is implicit in $x$, the second in $y$ and the third in $z$. For the explicit terms the updated values from the previous fractional step are used. Using the von Neumann method for stability, calculate the amplification factors $G'$, $G''$ and $G'''$ for the three fractional steps. Assuming that the most stringent condition on stability comes from the highest Fourier modes in $x$, $y$ and $z$, and that $\Delta x = \Delta y = \Delta z$, what is the criterion for this scheme to be stable?

22. The equation governing three-dimensional diffusion

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

is to be solved in three iterative steps by the algorithm

$$\frac{T^* - T^n}{\Delta t} = \alpha L_{xx}T^* + \alpha L_{yy}T^n + \alpha L_{zz}T^n$$

$$\frac{T^{**} - T^{*}}{\Delta t} = \alpha L_{yy}(T^{**} - T^{n})$$

$$\frac{T^{n+1} - T^{**}}{\Delta t} = \alpha L_{zz}(T^{n+1} - T^{n})$$

where

$$L_{xx}T_{i,j,k} = \frac{T_{i-1,j,k} - 2T_{i,j,k} + T_{i+1,j,k}}{(\Delta x)^2}$$

and $L_{yy}$ and $L_{zz}$ are defined in a similar manner. Using the von Neumann method for stability, calculate the amplification factors $G'$, $G''$ and $G$, up to the first and second iteration and for the complete step. Is the scheme stable or unstable and is there a condition for stability?