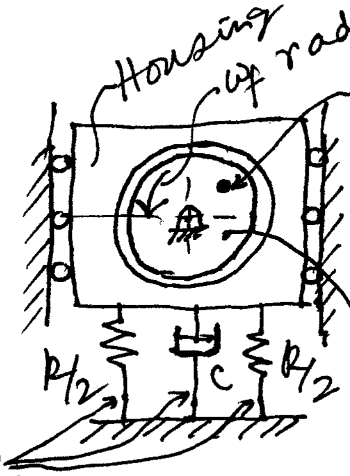
Forced Vibration of 1-DOF systems (contd)

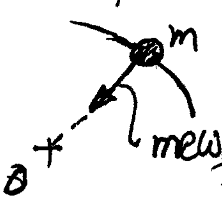
§ The rotating unbalance problem:-

All of you know how important it is to 'balance' a rotor properly. If there is an unbalance, it causes rotating forces/couple moments to act on the bearings & may give rise to bearing failure. We now consider a machine which has a rotor spinning at ω_f rad/s. Let this rotor has an unbalanced mass in its midplane. This mass is 'm' and is located at a distance 'e' from the centre of rotation. (The figure shows the mid plane of rotor with unbalanced mass 'm' with eccentricity 'e'. The machine housing is constrained to move between rollers for easy visualization of its vertical only motion.

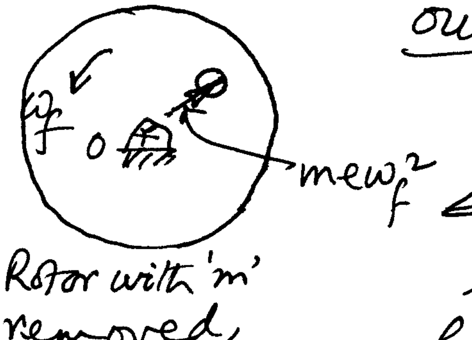


(There are two of these at the two ends of rotor.

You probably already know, from your study of the field balancing (P.T.O) of rotors, that m & e cannot be obtained separately. ~~a~~ We can only find the product me through experimentation. This product is called the 'unbalance'. For a thin unbalanced rotor, balancing can be done by putting/removing an appropriate amount of material at a proper radius on/from one of its end faces. For a long rotor, this is to be done at both end faces. Anyway, what is happening here is the following:

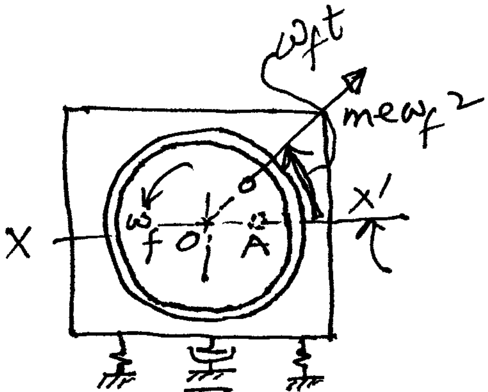


m is moving in a circle of radius e & hence is subjected to a centripetal force of mew_f^2. This force comes from its ~~surrounding~~ surrounding material & by Newton's third law, the rotor (minus the 'm') is subjected to an equal & opposite radially outward force as shown in this figure. This 'pull' on the rotor causes an equal pull on the two bearings. The horizontal component of this force is taken up by the rollers & walls. The vertical component causes oscillations.

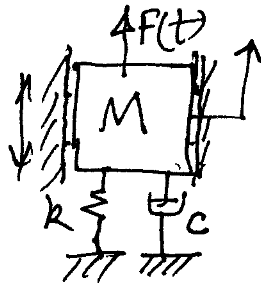


Rotor with 'm' removed, say

of the m/c housing. We now derive the DEOM for these vertical oscillations. We measure t from the instant the unbalanced mass passes through position A on the horizontal line XX' as shown in the figure. (i.e., $t=0$ corresponds to this position)



Then, at time t, $me\omega_f^2$ is inclined to the right hand horizontal at an angle of $\omega_f t$. So, its horizontal component is $(me\omega_f^2)\cos\omega_f t$ & is balanced by the reaction from the rollers (not shown here). The vertical component is $(me\omega_f^2)\sin\omega_f t$ & this component causes vibrations. Hence, our equivalent system for the vibration study can be shown as:-



where $F(t) = (me\omega_f^2)\sin\omega_f t$

So, the DEOM is:—

$$\mathbf{M}\ddot{x} + c\dot{x} + kx = (me\omega_f^2)\sin\omega_f t \quad \text{--- (1)}$$

Capital letter, note

Comparing (1) with our old friend

$$m\ddot{x} + c\dot{x} + kx = f_0 \sin\omega_f t; \quad \text{--- (2)}$$

we see that $me\omega_f^2 \equiv f_0$

& so, the steady state response,

obtained by comparison, is simply:

$$x_{ss} = x(t) = \frac{\frac{me\omega_f^2}{k}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t - \psi) \quad \text{--- (3)}$$

where $r = \frac{\omega_f}{\omega_n}$; $\omega_n = \sqrt{\frac{k}{M}} \leftarrow$ note ;

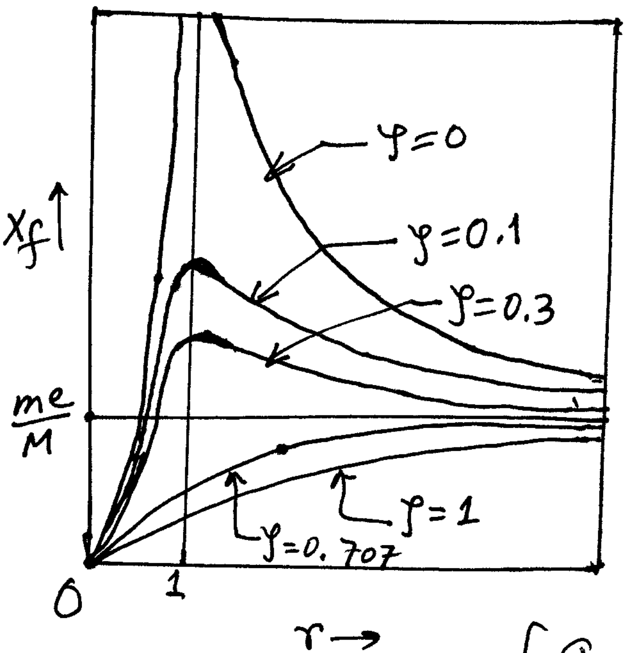
M = Total m/c mass; $\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ as before. So, amplitude of forced vibration is: $\underset{\text{forced}}{X_f} = \frac{me\omega_f^2/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

$$= \frac{\frac{\frac{me}{M}\omega_f^2}{k/M}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{\left(\frac{me}{M}\right)r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{\left(\frac{U}{M}\right)r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

($U = me$ = the 'unbalance') [me is usually very small]

So, $$\boxed{X_f = \frac{\left(\frac{me}{M}\right)r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$, a complicated expression. So, we resort to plots like these:—

Remember



← Remember the nature of these plots, it may help in solving some complicated problems! Note that

$$\lim_{r\to\infty} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 1.$$

for all ζ, $X_f \rightarrow \frac{me}{M}$ as $r \rightarrow \infty$.

[See accurate plots from books & study them]

Variation of ψ with r plots are the same as before.

Thus, once more: The steady-state response due to unbalance 'me' is given by $x(t) = X_f \sin(\omega_f t - \psi)$, where

$$X_f = \frac{\left(\frac{me}{M}\right) r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Example ①:- An industrial sewing m/c has a rotating unbalance of 0.15 kg-m. The m/c operates at 125 Hz and is mounted on a foundation of equivalent stiffness 2×10^6 N/m and damping ratio 0.12. Find the steady-state amplitude of the resulting vibration. [Kelly - Mechanical Vibrations]

Solution:- $me = 0.15$ kg-m, $\omega_f = 125\times2\pi$ rad/s

$M = 65$ kg. $\omega_n = \sqrt{\frac{2\times10^6}{65}} = 175.4$ rad/s $\Rightarrow r = \frac{\omega_f}{\omega_n} = 4.48$

$\therefore$ Reqd. ss amplitude $= \dfrac{\frac{me}{M} r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

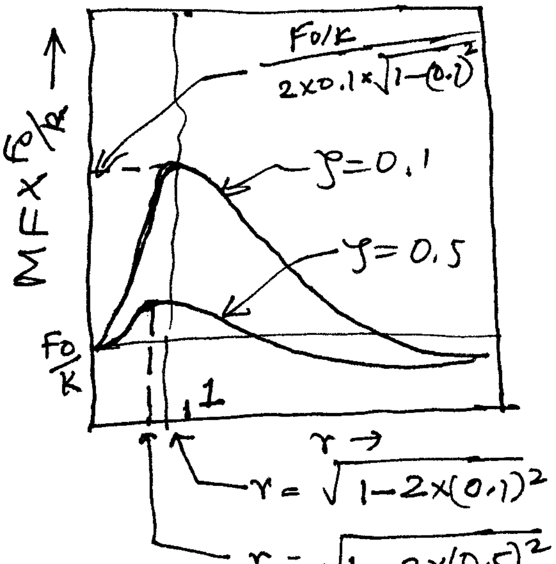
$$= \frac{(0.15/65)\times(4.48)^2}{\sqrt{[1-(4.48)^2]^2 + (2\times0.12\times4.48)^2}} = 2.43\times10^{-3}\,m$$
$$= 2.43 \text{ mm}$$

~~→Example ②~~

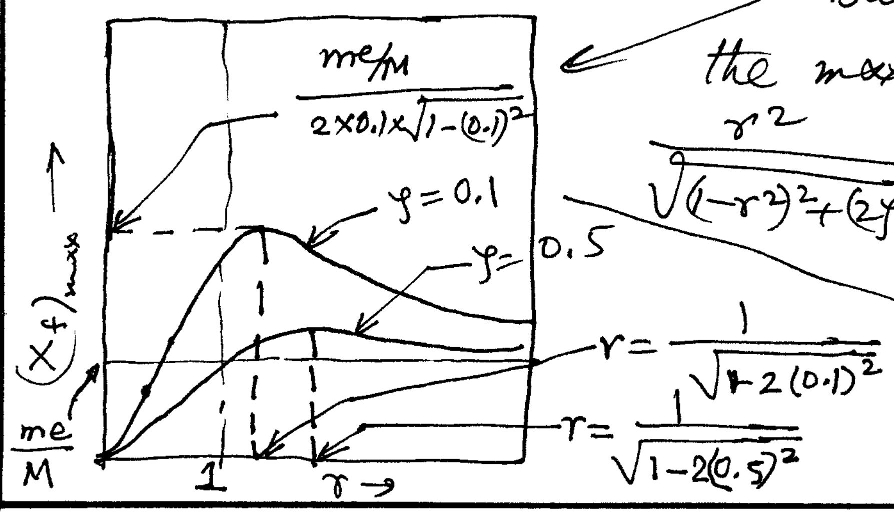
A question:- What is the value of $(X_f)_{max}$ & at what value of r does it occur? We have already seen that when the forcing function is

$F_0 \sin \omega_f t$, the maxm amplitude of forced or SS vibration is $\dfrac{F_0/k}{2\zeta\sqrt{1-\zeta^2}}$ & it occurs at $r = \sqrt{1-2\zeta^2}$. ($\zeta < \sqrt{2}$). Thus, as ζ increases, this value of r decreases & hence, the peak occurs at lower & lower value of r until ζ reaches the value $\frac{1}{\sqrt{2}}$.

This is shown below.



But what happens in the case of rotating unbalance? Here the opposite happens. You can easily show that $(X_f)_{max}$ now occurs at $r = \dfrac{1}{\sqrt{1-2\zeta^2}}$ $\left(\zeta < \frac{1}{\sqrt{2}}\right)$, i.e., as ζ increases, the $(X_f)_{max}$ value occurs for a value of $r > 1$ & this value of r increases as ζ increases as shown below:

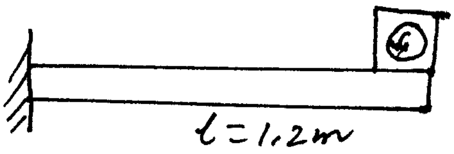


But funnily enough, the maxm value of $\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ is still $\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$!

So, check this by treating this as home-work.

We use this in the next example. $\longrightarrow$

Example 2:– A 40 kg fan has a rotating unbalance of magnitude 0.1 kg-m. It is mounted on a beam as shown in the figure. The beam has been specially treated to add viscous damping. As the speed ω_f of the fan is varied, it is noted that its max^m ss amplitude is 20.3 mm. What is the fan's ss amplitude when it operates at 100 rpm? [Kelly] [From given data, we obtain ξ & ω_n & use these to get



$E = 200\,GPa$

$I = 1.3\times10^{-6}\,m^4$

Solution:– We know that $(X_f)_{max^m}$ occurs at $r = \dfrac{1}{\sqrt{1-2\xi^2}}$ & has a value of $\dfrac{\frac{me}{M}}{2\xi\sqrt{1-\xi^2}}$.

not reqd. here

~~Now, $(X_f)_{max^m}$~~ $me = 0.1$ kg-m, $M = 40$ kg,

$$(X_f)_{max^m} = 20.3\times10^{-3}\ m.$$

Hence, $$20.3\times10^{-3} = \frac{(0.1/40)}{2\xi\sqrt{1-\xi^2}} \Rightarrow \underline{\xi = 0.0617}.$$

The beam stiffness is: $k = \dfrac{3EI}{l^3} = 4.51\times10^5\ N/m$

$$\omega_n = \sqrt{\frac{K}{M}} = 106.2\ rad/s$$

When $\omega_f = 1000\ rpm = \dfrac{1000}{60}\times 2\pi\ rad/s.$

$$\therefore r = \frac{\omega_f}{\omega_n} = 0.986$$

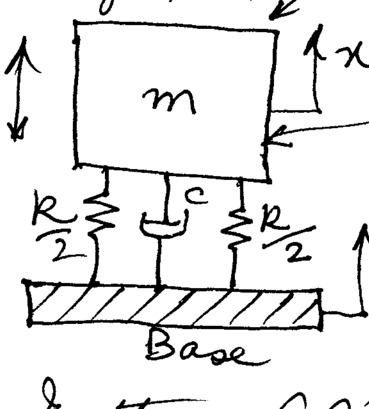
$\therefore$ The reqd. ss amplitude is: $\left[x_f = \dfrac{me}{M}\times\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \right.$

$$x_f = \frac{0.1}{40}\times\frac{(0.986)^2}{\sqrt{[1-(0.986)^2]^2+(2\times0.0617\times0.986)^2}}\ m$$

$$= 19.48\times10^{-3}\ m = \underline{19.48\ mm}$$

(8) **Base Excitation:-** (Support motion) Suppose the base of a m/c foundation is vibrating due to the vibration created by another machine on the same shop floor. We want to minimize the vibration of our m/c caused by this base excitation through a proper choice of the stiffness and damping characteristics of its foundation. This is how we approach this problem: (We assume a sinusoidal excitation of the base. For instance, such an excitation can be easily generated using a Scotch-yoke mechanism you had studied in your Kinematics course. You may check this from S.S. Rao's book also. Later on, we shall generalize the excitation)

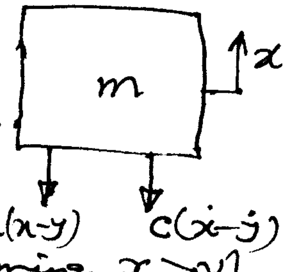
- The system



← our machine

$y = Y_0 \sin \omega t$ ← The base excitation

- The relevant FBD →



(Assuming $x > y$)

In the FBD, note that we assumed $x > y$ (& $\dot{x} > \dot{y}$). We could as well assume $x < y$, then the spring force would be $k(y-x)$

upwards & damping force $c(\dot{y}-\dot{x})$ upward. These would give rise to the <u>same</u> DEOM & hence you can use either in a FBD. Do not be confused about it.

So, by Newton's method,

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$$

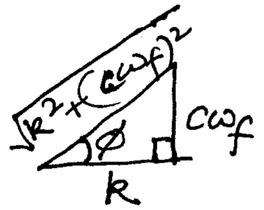
or, $m\ddot{x}+c\dot{x}+kx = ky+c\dot{y} = \underbrace{kY_0\sin\omega_f t + c\omega_f Y_0\cos\omega_f t}$

The RHS can be written as:

Aim is to put it in the form $\sin(\omega_f t+\phi)$

$$Y_0\left(k\sin\omega_f t + c\omega_f\cos\omega_f t\right)$$

$$= Y_0\sqrt{k^2+(c\omega_f)^2}\left[\frac{k}{\sqrt{k^2+(c\omega_f)^2}}\sin\omega_f t + \frac{c\omega_f}{\sqrt{k^2+(c\omega_f)^2}}\cos\omega_f t\right]$$



$$= Y_0 k\sqrt{1+\left(\frac{c\omega_f}{k}\right)^2}\left[\sin\omega_f t\cos\phi + \cos\omega_f t\sin\phi\right]$$

$$= Y_0 k\sqrt{1+(2\zeta r)^2}\,\sin(\omega_f t+\phi) \text{ where}$$

$$\boxed{\tan\phi = \frac{c\omega_f}{k} = 2\zeta r} \text{ (Remember)}$$

check:–
$\frac{c\omega_f}{k} = \frac{c}{2m\omega_n}\times 2m\omega_n\frac{\omega_f}{k}$
$= 2\zeta\omega_n\frac{\omega_f}{k/m}$
$= 2\zeta\omega_n\frac{\omega_f}{\omega_n^2}$
$= 2\zeta\frac{\omega_f}{\omega_n} = 2\zeta r$, OK

Hence, the DEOM is:

$$m\ddot{x}+c\dot{x}+kx = \underbrace{kY_0\sqrt{1+(2\zeta r)^2}}_{F_0}\sin(\omega_f t+\phi) \quad \text{---(i)}$$

Now a small but important point:

If $m\ddot{x}+c\dot{x}+kx = F_0\sin\omega_f t$ has the ss solution $x_p = x(t) = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t-\psi)$

then $m\ddot{x}+c\dot{x}+kx = F_0\sin(\omega_f t+\phi)$ has the ss solution $x(t) = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t+\phi-\psi)$.

You may verify it or you may simply remember it.

Hence, by comparison, ① has the following ss response:

$$x(t) = \frac{\left(k Y_0 \sqrt{1+(2\zeta r)^2}\right)/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t + \phi - \psi)$$

or, $$x(t) = Y_0 \left(\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \right) \sin(\omega_f t + \phi - \psi)$$

Remember

where $\psi = \tan^{-1} \frac{2\zeta r}{1-r^2}$

& $\phi = \tan^{-1}(2\zeta r)$

So, $x(t) = Y_0 \, \alpha_{MT} \sin(\omega_f t + \phi - \psi)$

where $\alpha_{MT} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ is called the Motion Transmissibility which is the ratio of the amplitude of the motion transmitted to the amplitude of the base excitation. It is also represented as TR or, $(TR)_{motion}$. Thus,

In some books

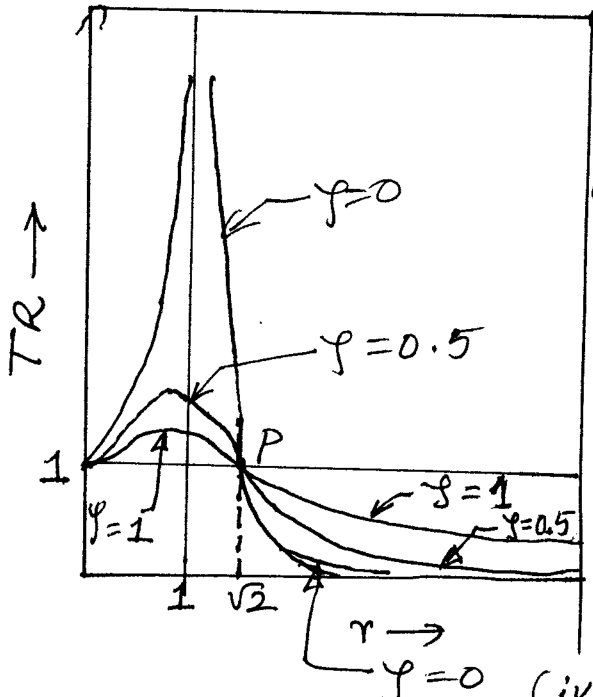
$$T = TR = (TR)_{motion} = \alpha_{MT} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

[Remember]

$$= \frac{Y_0 \sqrt{1+(2\zeta r)^2} / \sqrt{(1-r^2)^2 + (2\zeta r)^2}}{Y_0}$$

So, if TR (or, α_{MT}) is < 1, base motion is 'isolated', we say. If $TR > 1$, motion is 'multiplied'. This is so because with $TR < 1$, we have amplitude of mass m less than the amplitude Y_0 of base excitation. Similarly, if TR > 1, then amplitude of our m/c is more than the amplitude of the base excitation & so, motion (amplitude) is 'multiplied'.

We get the following interesting plots for TR. (See accurate plots from textbook)



* The interesting features of these plots are:-

(i) They all pass through point P at which $r=\sqrt{2}$ and TR = 1.

(ii) For $r<\sqrt{2}$, we have motion multiplication for all ζ.

(iii) ~~Mo~~tion isolation is possible only if $r>\sqrt{2}$

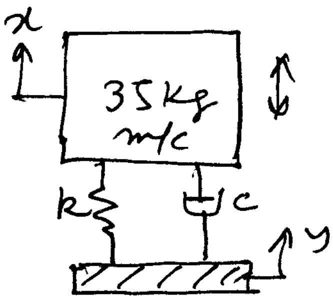
(iv) For $r>\sqrt{2}$, ~~the~~ a smaller ζ gives a better isolation (a lesser TR)! That is, lightly damped systems give better isolation than heavily damped ones! This is a little paradoxical, isn't it? Hence, intuition may lead to error, one must follow a logical analysis.

(v) The higher the value of r above $\sqrt{2}$, the better is the isolation for a given ζ.

→ Now here is another interesting thing: By comparing the above plots with those for rotating unbalance (page 4), you would observe →

that for the same system in the absence of base excitation, a higher ξ gives a lesser amplitude. Hence in case both rotating unbalance and base excitation are present and we have to minimize the amplitude of vibration, we must meet a contradictory requirement, that is ~~choosing~~ a proper ξ to minimize overall vibration due to these two effects is not very straightforward. (Here a proper ξ means a proper choice of c & k for mounting m/c).

Example:- For the set-up in the figure, obtain the steady-state absolute displacement of the m/c block. $k = 1.4$ MN/m, $c = 1.8 \times 10^3$ N-s/m. y is harmonic with amplitude 10 mm and frequency 35 Hz. [Kelly]



Solution:- $Y_0 = \cancel{10 \times 10^{-3}} = 10 \times 10^{-3}$ m,

$\omega_f = 35 \times 2\pi$ rad/s. To find r:-

$$\omega_n = \sqrt{\frac{k}{m}} = 200 \text{ rad/s}, \quad \xi = \frac{c}{2m\omega_n} = 0.129$$

$$r = \frac{\omega_f}{\omega_n} = \frac{35 \times 2\pi}{200} = 1.1$$

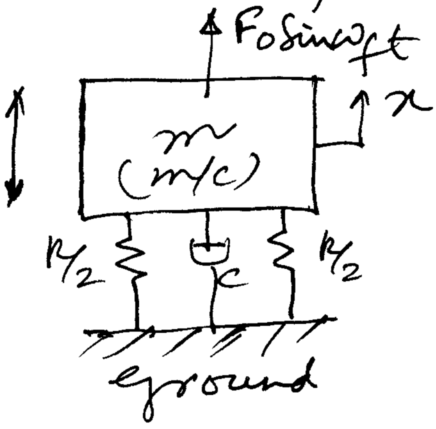
Hence reqd. amplitude $= Y_0 \times TR$

$$= 10 \times 10^{-3} \times \sqrt{\frac{1 + (2 \times 0.129 \times 1.1)^2}{[1-(1.1)^2]^2 + (2 \times 0.129 \times 1.1)^2}}$$

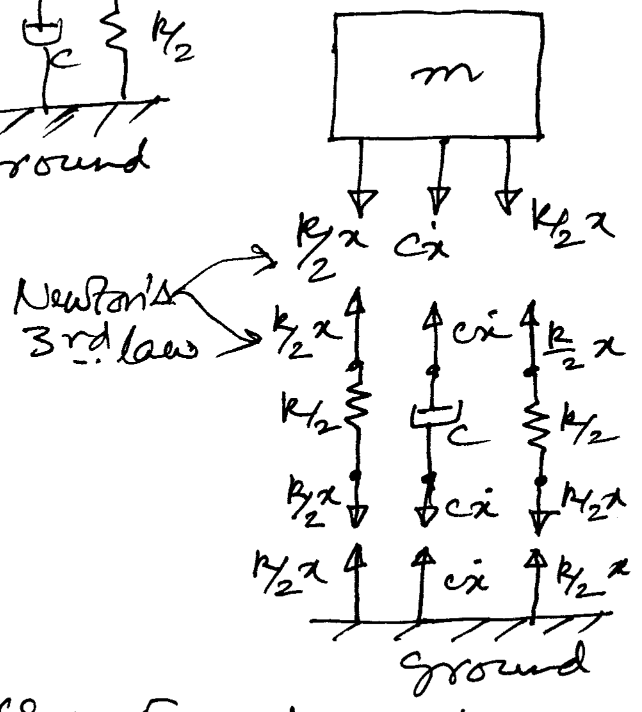
$$= 29.4 \times 10^{-3} \text{ m} = 29.4 \text{ mm}.$$

5 Force transmission & force transmissibility :~

The problem here is this :- A m/c is subjected to a harmonic excitation & is executing a steady state vibration. We want to find the force transmitted to the ground on which the spring & the damper are mounted. This force may be important because it may induce unwanted vibration in other equipments nearby on the same floor.



At time t, the FBD of the springs & damper are shown below.



Springs & damper have negligible inertia & hence forces at the two ends must be equal & opposite at all times

Hence, F_T = force transmitted to the ground $= 2\times\frac{k}{2}x + c\dot{x} = kx + c\dot{x}$. —— (1)

~~But~~ $x_{ss} = x = \dfrac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t-\psi)$ —— (2)

or, $x = X \sin(\omega_f t - \psi)$ — (3) $\left[X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \right)$

$\therefore \dot{x} = X\omega_f \cos(\omega_f t - \psi)$ — (4)

So, from (1) $\rightarrow$ $F_T = kX \sin(\omega_f t - \psi) + cX\omega_f \cos(\omega_f t - \psi)$

or, $F_T = X\left[k \sin(\omega_f t - \psi) + c\omega_f \cos(\omega_f t - \psi)\right]$

$$= X\sqrt{k^2 + (c\omega_f)^2}\left[\frac{k}{\sqrt{k^2+(c\omega_f)^2}}\sin(\omega_f t - \psi) + \frac{c\omega_f}{\sqrt{k^2+(c\omega_f)^2}}\cos(\omega_f t - \psi)\right]$$

$$= kX\sqrt{1+(2\zeta r)^2}\,\sin(\omega_f t - \psi + \phi)$$ (Like before!)

[where $\phi = \tan^{-1}(2\zeta r)$]

or, $$F_T = \frac{F_0\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega_f t - \psi + \phi)$$

(Remember).

Hence, amplitude of transmitted force

$$= \frac{F_0\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}.$$

Definition: The force transmissibility $(TR)_{force}$

$$= \frac{\text{Amplitude of transmitted force}}{\text{Amplitude of forcing function } (F_0)}$$

Remember

$$= \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$ [the same expression as motion transmissibility!]

→ A note on forced vibration of rotational systems :— Let the DEOM of a rotational/torsional vibratory system be like : $I_d \ddot{\theta} + c_t \dot{\theta} + k_t \theta = T_0 \sin\omega_f t$ — (1)

Then, the steady-state response shall be:

$$\theta_{ss}(t) = \theta(t) = \frac{T_0/k_t}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi) \quad \text{--- (2)}$$

& follows from direct comparison with

$$m\ddot{x} + c\dot{x} + kx = f_0 \sin\omega_f t \quad \&$$

$$x_{ss}(t) = x(t) = \frac{f_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi).$$

In (2), $\zeta = \frac{c_t}{2 I_d \omega_n} = \frac{c_t}{2\sqrt{I_d k_t}}$.

Even if someone writes the DEOM as:

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = T_1 \sin\omega_f t,$$

θ_{ss} is still given by [by comparison with (1)]

$$\theta(t) = \frac{T_1/\beta}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi)$$

with $\zeta = \frac{\alpha}{2\sqrt{1\times\beta}} = \frac{\alpha}{2\sqrt{\beta}}$ etc.

(Because, $\alpha = \frac{c_t}{I_d}$, $\beta = \frac{k_t}{I_d}$

& so, $\frac{\alpha}{2\sqrt{\beta}} = \frac{c_t}{I_d \times 2 \times \sqrt{\frac{k_t}{I_d}}} = \frac{c_t}{2\sqrt{I_d k_t}}$

as before). (Similarly, $T_1/\beta = \frac{T_0}{K}$)

X

END OF VA-3, part 2