The Matrix Iteration Method (For higher DOF systems) (The Power Method) (To get ω_n's & $\{X\}_n$'s numerically)

Part (A)

[In part (A), the procedure is illustrated using an example.
Part (B) is for those who want to go a little deeper into the theoretical aspects of the MI method]

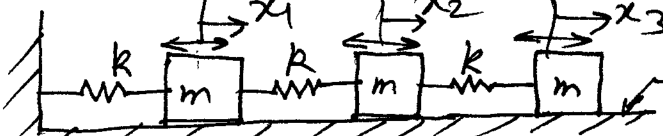
(I) If we want to obtain ω_1 & $\{X\}_1$ first, ω_2 & $\{X\}_2$ next & so on, our basis for the method is the relation

$\boxed{[D] = [k]^{-1}[m] = \text{The Dynamic matrix}}$

$$[D]\{X\}_r = \frac{1}{\omega_r^2}\{X\}_r \; ; \quad r = 1, 2, \dots, n \quad \text{(n-DOF system)}$$

→ The procedure is: start with an arbitrary trial vector, a 'guess' for $\{X\}_1$. Let this trial vector be $\{u\}_0$. Premultiply $\{u\}_0$ by $[D]$ to get a new, normalized estimate $\{u\}_1$. Premultiply $\{u\}_1$ by $[D]$ & get $\{u\}_2$ & so on, until a required level of convergence is achieved. [You are already familiar with this method applied to a 2-DOF system]

Example :- We continue with our earlier example of a 3-DOF system.



Friction neglected

→ Obtain ω_1, $\{X\}_1$ & ω_2 $\{X\}_2$ by the MI method. Achieve convergence upto 3 places after decimal for $\{X\}_1$, and upto 2nd place for $\{X\}_2$.

(PTO)

Solution:- From the analytical solution done before,

$$[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m[I]; \quad [k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$\therefore \det[k] = 2k(2k^2 - k^2) + k(-k^2) = k^3$$

$$\underset{(\text{adjoint of } [k])}{\mathrm{adj}[k]} = \begin{bmatrix} k^2 & k^2 & k^2 \\ k^2 & 2k^2 & 2k^2 \\ k^2 & 2k^2 & 3k^2 \end{bmatrix} \Rightarrow [k]^{-1} = \frac{\mathrm{adj}[k]}{\det[k]} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Hence, $$[D] = [k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

([D] must be obtained correctly. Otherwise everything computed subsequently goes wrong.)

→ The iterations begin now.

1st iteration:- Let $\{u\}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$\Rightarrow [D]\{u\}_0 = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m}{k}\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = \frac{3m}{k}\begin{Bmatrix} 1 \\ 1.6667 \\ 2 \end{Bmatrix}$$

(Retain 4 places after decimal for good accuracy while hand calculators are used)

→ We discard $\frac{3m}{k}$ & take $\begin{Bmatrix} 1 \\ 1.6667 \\ 2 \end{Bmatrix}$ as $\{u\}_1$, the approximation for $\{x\}_1$ after 1 (one) iteration.

2nd iteration:- $$[D]\{u\}_1 = \frac{4.6667m}{k}\underbrace{\begin{Bmatrix} 1 \\ 1.7857 \\ 2.2143 \end{Bmatrix}}_{\{u\}_2}$$

3rd iteration:- $$[D]\{u\}_2 = \frac{5m}{k}\underbrace{\begin{Bmatrix} 1 \\ 1.8000 \\ 2.2429 \end{Bmatrix}}_{\{u\}_3}$$

You carry out the details at each step.

Continuing, we get $\{u\}_4 = \begin{Bmatrix} 1 \\ 1.8017 \\ 2.2465 \end{Bmatrix}$,

$\{u\}_5 = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix}$. (Note, comparing $\{u\}_5$ with $\{u\}_4$, that convergence for the middle element is already achieved upto 3 places (1.802, rounded). However, for the bottom element, convergence upto 2 places is achieved (2.25, rounded).

→ Continuing iterations, we find that

$$\{u\}_6 = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \text{ \& now comparing } \{u\}_6$$

with $\{u\}_5$, we see that overall convergence upto 3 places has been achieved.

→ for classroom problems, we don't go beyond this accuracy for the time being.

→ Hence, we take $\{X\}_1 \simeq \{u\}_6 = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix}$

Now, you'd ~~easy~~ verify that

$$[D]\{u\}_5 = \frac{5.0489m}{k} \underbrace{\begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix}}_{\{u\}_6}$$

So, $\frac{1}{\omega_1^2} \simeq \frac{5.0489\,m}{k}$

$$\Rightarrow \omega_1 \simeq 0.4450\sqrt{\frac{k}{m}} \quad \text{(Check)}$$

(Note:- for the present problem, analytical (exact) solutions are available (Done before):-

$$(\omega_1)_{exact} = 0.4379\sqrt{\frac{k}{m}} \;\&\; \left(\{X\}_1\right)_{exact} = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix}$$

Thus, $(\omega_1)_{MI} > (\omega_1)_{exact}$, $\left(\%\text{ error} = \frac{(0.4450-0.4379)}{0.4379} \times 100 = 1.62\right)$

but $\left(\{X\}_1\right)_{MI} = \left(\{X\}\right)_{exact}$! (PTO)

→ We next try to get $\{x\}_2$ & ω_2 by this method.

→ This can be done in more than one ways

(1) The use of sweeping matrix

(2) " " " matrix deflation

(1) Sweeping matrix method:-

Any trial vector, $\{v\}_0$ for 2nd mode must be orthogonal to $\{x\}_1$ w.r.t. $[m]$, otherwise an arbitrary trial vector would lead to convergence towards $\{x\}_1$ & ω_1 only!

Hence, we must have $\left[\{v\}_0 = \begin{Bmatrix} v_{10} \\ v_{20} \\ v_{30} \end{Bmatrix}, \text{say}\right]$

$$\{v\}_0^T [m] \{x\}_1 = 0$$

or, $$\{v_{10} \;\; v_{20} \;\; v_{30}\} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix} = 0$$

or, $$v_{10} + 1.8019 v_{20} + 2.247 v_{30} = 0$$

or, $$v_{10} = -1.8019 v_{20} - 2.247 v_{30}$$

Thus, $$v_{10} = 0 \times v_{10} - 1.8019 v_{20} - 2.247 v_{30}$$

$$v_{20} = 0 \times v_{10} + 1 \times v_{20} + 0 \times v_{30}$$

& $$v_{30} = 0 \times v_{10} + 0 \times v_{20} + 1 \times v_{30}$$

or, $$\begin{Bmatrix} v_{10} \\ v_{20} \\ v_{30} \end{Bmatrix} = \begin{bmatrix} 0 & -1.8019 & -2.247 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{10} \\ v_{20} \\ v_{30} \end{Bmatrix}$$ non-symmetric

or, $$\{v\}_0 = [S]_2 \{v\}_0 \; ; \; [S]_2 = \begin{bmatrix} 0 & -1.8019 & -2.247 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ $[S]_2$ is called the sweeping matrix for the 2nd mode.

Thus, $\{v\}_0$ must be <u>premultiplied</u> by $[S]_2$ before the iterations for the 2nd mode begins.

Hence, $[D][S]_2\{v\}_0$ or $[D]_2\{v\}_0$ will give $\{v\}_1$; $[D]_2\{v\}_1$ will generate $\{v\}_2$ & so on.

Important $\rightarrow$ $[D]_2 = [D][S]_2$ is the dynamic matrix for the 2nd mode.

Here
$$[D]_2 = [D][S]_2 = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1.8019 & -2.247 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or,
$$[D]_2 = \frac{m}{k}\begin{bmatrix} 0 & -0.8019 & -1.2470 \\ 0 & 0.1981 & -0.2470 \\ 0 & 0.1981 & 0.7530 \end{bmatrix} \qquad \cancel{\frac{m}{k}\begin{bmatrix} 0 & -0.8019 & -2.247 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(non-symmetric) (Check)

Let $\{v\}_0 = \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}$ (Note that we have incorporated one sign change. We could also start with $\{1 \;\; 1 \;\; 1\}^T$

Then,
$$[D]_2\{v\}_0 = \frac{0.4451m}{k}\begin{Bmatrix} 1 \\ 1 \\ -1.2467 \end{Bmatrix} \rightarrow \{v\}_1 \quad \text{(Carry out the details)} \rightarrow \text{check computations}$$

$$\rightarrow [D]_2\{v\}_1 = \frac{0.7527m}{k}\begin{Bmatrix} 1 \\ 0.6723 \\ -0.9840 \end{Bmatrix} \rightarrow \{v\}_2$$

$$\rightarrow [D]_2\{v\}_2 = \frac{0.6879m}{k}\begin{Bmatrix} 1 \\ 0.5469 \\ -0.8835 \end{Bmatrix} \rightarrow \{v\}_3$$

$$\rightarrow [D]_2\{v\}_3 = \frac{0.6631m}{k}\begin{Bmatrix} 1 \\ 0.4925 \\ -0.8399 \end{Bmatrix} \rightarrow \{v\}_4$$

$$\rightarrow [D]_2\{v\}_4 = \frac{0.6524m}{k}\begin{Bmatrix} 1 \\ 0.4675 \\ -0.8190 \end{Bmatrix} \rightarrow \{v\}_5$$

$$\rightarrow [D]_2\{v\}_5 = \frac{0.6475m}{k}\begin{Bmatrix} 1 \\ 0.4558 \\ -0.8105 \end{Bmatrix} \rightarrow \{v\}_6$$

Comparing $\{v\}_6$ with $\{v\}_5$, convergence is seen to be upto 1 place only. Thus, convergence is slower compared with the first mode.

[The rate of convergence for 1st mode depends on the ratio $\frac{\omega_2}{\omega_1}$, for the 2nd mode, on $\frac{\omega_3}{\omega_2}$ & so on. The larger these ratios, the faster the ~~convergence~~ convergence. Also, a better guess for the initial trial vector results in quicker convergence.]

$$\rightarrow [D]_2\{v\}_6 = \frac{0.6452m}{k}\begin{Bmatrix}1\\0.4502\\-0.8060\end{Bmatrix} \rightarrow \{v\}_7$$

$$\rightarrow [D]_2\{v\}_7 = \frac{0.6441m}{k}\begin{Bmatrix}1\\0.4475\\-0.8038\end{Bmatrix} \rightarrow \{v\}_8$$

$$\rightarrow [D]_2\{v\}_8 = \frac{0.6435m}{k}\begin{Bmatrix}1\\0.4463\\-0.8028\end{Bmatrix} \rightarrow \{v\}_9$$

$$\rightarrow [D]_2\{v\}_9 = \frac{0.6432m}{k}\begin{Bmatrix}1\\0.4457\\-0.8024\end{Bmatrix} \rightarrow \{v\}_{10}$$

Comparing $\{v\}_{10}$ with $\{v\}_9$, it can be seen that convergence has been achieved upto the 2nd place.

Thus, $\{X\}_2 \simeq \{v\}_{10} = \begin{Bmatrix}1\\0.4457\\-0.8024\end{Bmatrix}$

[Home Work: Achieve convergence upto the 3rd place]

Also, $\frac{1}{\omega_2^2} \simeq \frac{0.6432m}{k} \Rightarrow \omega_2 \simeq 1.2469\sqrt{\frac{k}{m}}$

Note that $(\{X\}_2)_{exact} = \begin{Bmatrix}1\\0.4451\\-0.8021\end{Bmatrix}$

& $(\omega_2)_{exact} = 1.2469\sqrt{\frac{k}{m}}$

$\{X\}_3$ & ω_3 should be obtained next $\longrightarrow$

→ By now, it is clear that a trial vector

$\{W\}_0^T[m]\{X\}_1=0$ & $\{W\}_0^T[m]\{X\}_2=0$ | $\{W\}_0 = \{W_{10} \ W_{20} \ W_{30}\}^T$ must be mass orthogonal to both $\{X\}_1$ & $\{X\}_2$. This will lead to a new dynamic matrix $[D]_3=[D][S]_3$ for convergence to $\{x\}_3$ & ω_3.

→ <u>Home Work:-</u> obtain $[D]_3$ & then $\{X\}_3$ & ω_3 by MI method, starting with $\{W\}_0 = \begin{Bmatrix}1\\-1\\1\end{Bmatrix}$, incorporating <u>two</u> sign changes.

→ For a 3-DOF system, however, $\{X\}_3$ & ω_3 can be obtained without further iterations. This can be seen as follows:-

$$\{W_{10} \ W_{20} \ W_{30}\}[m]\begin{Bmatrix}X_{11}\\X_{21}\\X_{31}\end{Bmatrix}=0,$$ for our example problem, gives: ($\because [m]=m[I]$)

$$\{W_{10} \ W_{20} \ W_{30}\}\begin{Bmatrix}1\\1.8019\\2.2470\end{Bmatrix}=0$$

$$\Rightarrow W_{10}+1.8019W_{20}+2.247W_{30}=0 \quad \text{— ⓐ}$$

& similarly, $\{W\}_0^T[m]\{X\}_2=0$ leads to

$$\{W_{10} \ W_{20} \ W_{30}\}\begin{Bmatrix}1\\0.4457\\-0.8024\end{Bmatrix}=0$$

or, $$W_{10}+0.4457W_{20}-0.8024W_{30}=0 \quad \text{— ⓑ}$$

In ⓐ & ⓑ, set $W_{10}=1$ & solve for W_{20} & W_{30} to obtain $\{W\}_0=\{X\}_3$ (<u>HW</u>) & compare this $\{X\}_3$ with $(\{X\}_3)_{exact}$.

→ Compute ω_3 from the formula:

$$\omega_3^2 = \frac{\{X\}_3^T[K]\{X\}_3}{\{X\}_3^T[m]\{X\}_3}$$

(PTO)

& Compare with $(\omega_3)_{exact}$.

In general, for an n-DOF system, after obtaining the first $(n-1)$ modal vectors & natural frequencies by MI, one may obtain ω_n & $\{X\}_n$ by invoking $(n-1)$ orthogonality relations & also using the formula $\omega_n^2 = \dfrac{\{X\}_n^T [k] \{X\}_n}{\{X\}_n^T [m] \{X\}_n}$

or, one may find $[D]_n$ & iterate to get $\{X\}_n$ & ω_n.

Important → One may also use the basic formula $[m]^{-1}[k]\{X\} = \omega^2\{X\}$ (i.e., $[D]^{-1}\{X\} = \omega^2\{X\}$) to start with an arbitrary trial vector & obtain convergence to $\{X\}_n$ & ω_n first. (See later)

(§) The use of 'deflated' matrices to obtain $\{X\}_2$ to $\{X\}_n$ & ω_2 to ω_n

→ For finding $\{X\}_2$ & ω_2, one may use a special dynamic matrix $[D]_{2d}$, called a deflated matrix corresponding to the second principal mode of vibration. It is defined as

REMEMBER → $$\boxed{\underset{n\times n}{[D]_{2d}} = \underset{n\times n}{[D]} - \frac{1}{\omega_1^2}\underset{n\times 1}{\{X\}_1}\underset{1\times n}{\{X\}_1^T}\underset{n\times n}{[m]}}$$

or, if $\dfrac{1}{\omega_1^2} = \lambda_1$, then $[D]_{2d} = [D] - \lambda_1\{X\}_1\{X\}_1^T[m]$

(In many books, $\dfrac{1}{\omega_r^2}$ is denoted as λ_r; $r = 1, 2, \ldots n$ for an n-DOF system.) (PTO)

Now, using an arbitrary trial vector & $[D]_{2d}$, one can obtain $\{X\}_2$ & ω_2 just the same way as these were obtained using an arbitrary trial vector & $[D]_2$.

→ Here there is one important difference of course. One has to normalize the already found $\{X\}_1 = \begin{Bmatrix} X_{11} \\ \mu_{2,1} X_{11} \\ \mu_{3,1} X_{11} \end{Bmatrix}$, X_{11} being arbitrary, such that

This is another way of normalizing a modal vector

$$\{X\}_1^T [m] \{X\}_1 = 1 \quad \left(X_{11} \text{ no longer arbitrary!}\right)$$

(See later (Part B) for details of derivation/proof)

→ For our example problem, we take as $\{X\}_1$ not $\begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix}$ (See page 3) but we take $\{X\}_1 = \begin{Bmatrix} X_{11} \\ 1.8019 X_{11} \\ 2.247 X_{11} \end{Bmatrix}$

Now, we normalize $\{X\}_1$ such that

$$\{X\}_1^T [m] \{X\}_1 = 1 \quad (\text{i.e. } M_{11} \text{ (generalized mass)} = 1)$$

or, $$\{X_{11},\ 1.8019 X_{11},\ 2.247 X_{11}\}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} X_{11} \\ 1.8019 X_{11} \\ 2.247 X_{11} \end{Bmatrix} = 1$$

or, $$m X_{11}^2 \left(1 + \overline{1.8019}^2 + \overline{2.247}^2\right) = 1$$

$$\Rightarrow X_{11} = \frac{0.328}{\sqrt{m}} \quad \text{(Check numerical computations.)}$$

→ Hence, for matrix deflation methods, we take

$$\{X\}_1 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.3280 \\ 0.5910 \\ 0.7370 \end{Bmatrix} \quad (PTO)$$

Also, $\frac{1}{\omega_1^2} = \frac{5.0489m}{k}$ (See page 3)

Then, $[D]_{2d} = [D] - \frac{1}{\omega_1^2}\{X\}_1 \{X\}_1^T [m]$

$$= \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \frac{5.0489m}{k} \times \frac{1}{m} \times m \begin{Bmatrix} 0.3280 \\ 0.5910 \\ 0.7370 \end{Bmatrix} \{0.328, 0.591, 0.737\} \times [I]$$

$(\because [m] = m[I])$

$$= \frac{m}{k}\begin{bmatrix} 0.4567 & 0.0210 & -0.2203 \\ 0.0210 & 0.2364 & -0.1993 \\ -0.2203 & -0.1993 & 0.2574 \end{bmatrix}$$

$\rightarrow$ Symmetric!

($[S]_2$ & $[D]_2$ were <u>not</u> symmetric)

Important

HOMEWORK Start with $\{v\}_0 = \{1, 1, -1\}^T$ & above $[D]_{2d}$ & obtain $\{X\}_2$ & ω_2 & compare with the ones obtained previously.

$\longrightarrow$ For $\{X\}_3$ & ω_3, the deflated matrix is

Remember $\rightarrow$ $\boxed{[D]_{3d} = [D]_{2d} - \frac{1}{\omega_2^2}\{X\}_2 \{X\}_2^T [m]}$

$\longrightarrow$ for $\{X\}_r$ & ω_r ($r = 4, 5, --, n$ for an n-DOF system),

the deflated matrix is

$$\boxed{[D]_{rd} = [D]_{(r-1)d} - \frac{1}{\omega_{(r-1)}^2}\{X\}_{(r-1)}\{X\}_{(r-1)}^T [m]}$$

HomeWork $\rightarrow$ Obtain $\{X\}_3$ & ω_3 by matrix deflation method.

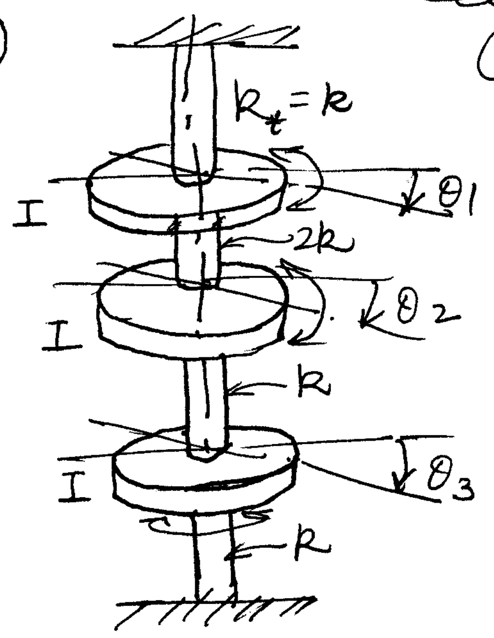
[Important:- You may use <u>either</u> Sweeping matrix based method <u>or</u> deflated matrices.

End of Part A

(See page 11 for HW problem)

HomeWork Problems on MI method

(1)



~~See~~ the system in the figure.

(A) Obtain the DEOM for undamped, free vibrations in the torsional mode.

(B) Find the natural frequencies & the associated modal vectors by the MI method. (Find ω_1, $\{X\}_1$ first) You should iterate till the last mode & finally check that the modal vectors are OK from the mass orthogonality point of view. For practice purpose, use both sweeping matrix & deflated matrix techniques.

(2) For the system in problem (1) above, obtain ω_3 & $\{X\}_3$ first by the MI method (i.e., use $[m]^{-1}[k]$ as the dynamic matrix etc.).