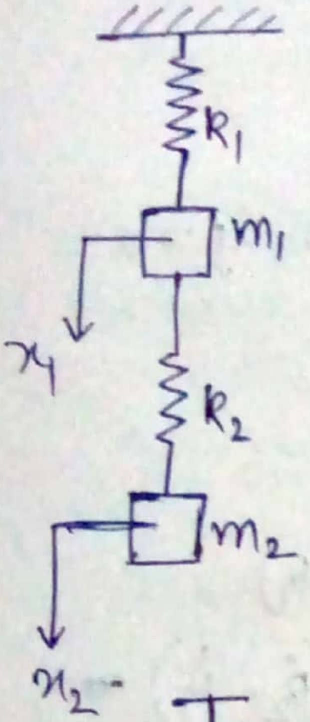
Two degrees of freedom (2DOF) systems

System A



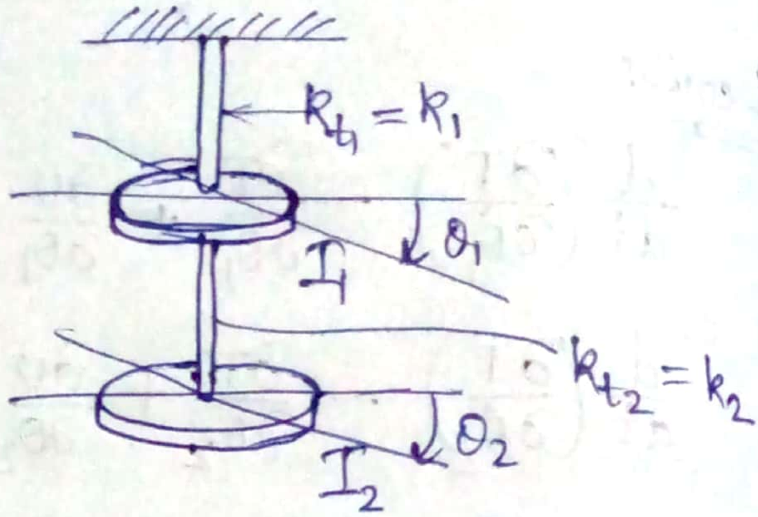
Translational model

$x_1(t), x_2(t) \longrightarrow$ generalised coordinates

Given $\begin{cases} x_1(0), \dot{x}_1(0) \\ x_2(0), \dot{x}_2(0) \end{cases}$

obtain $x_1(t)$ & $x_2(t)$

System B



Torsional oscillations (Rotational model)

Undamped free vibration

$$\left.\begin{array}{l} \theta_1(0), \dot{\theta}_1(0) \\ \theta_2(0), \dot{\theta}_2(0) \end{array}\right\} \text{Given}$$

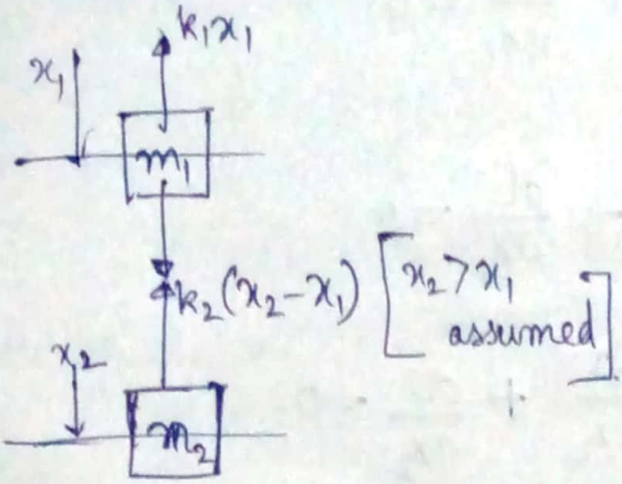
We want to find $\underbrace{\theta_1(t) \;\&\; \theta_2(t)}_{\text{generalised coordinates}}$

Step 1 :— Obtain the DEOM

System A

(i) Newton's method

~~So $m_1\ddot{x}_1$~~



So $m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1 \Rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$ —— (1)

& $m_2\ddot{x}_2 = -k_2(x_2-x_1) \Rightarrow m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$ —— (2)

reqd. DEOMs

In matrix form, we have

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

~~or, [m][~~ or $[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$ —— (3)

where $[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is the mass or inertia matrix

$[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ is the stiffness or elastic matrix

$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ = The vector of generalised coordinates (displacement vector)

$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$ = The acceleration vector.

2) Lagrange's Method

The Lagrange equ^{n}s are:-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0 \quad \text{---} (i)$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

& $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

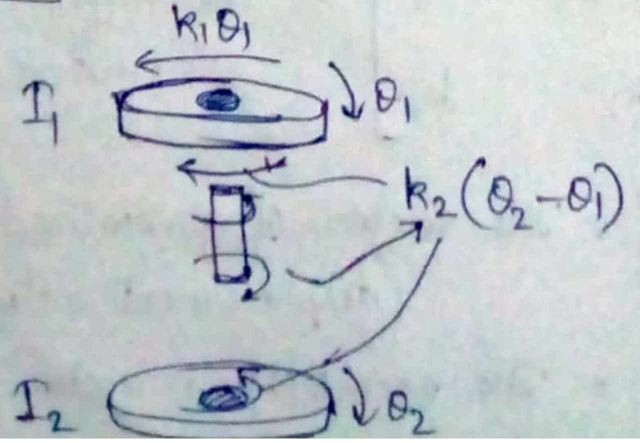
So, $$\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial U}{\partial x_1} = k_1 x_1 - k_2 (x_2 - x_1)$$
$$= (k_1 + k_2) x_1 - k_2 x_2$$

So (i) leads to $m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$

The other DEOM follows similarly.

System B



Using the MOM eqn

$$I_1 \ddot{\theta}_1 = -k_1 \theta_1 + k_2(\theta_2 - \theta_1)$$

& $$I_2 \ddot{\theta}_2 = -k_2(\theta_2 - \theta_1)$$

$$\left.\begin{array}{l} \Rightarrow I_1 \ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2 \theta_2 = 0 \\ \Rightarrow I_2 \ddot{\theta}_2 + k_2 \theta_2 - k_2 \theta_1 = 0 \end{array}\right\} \text{Reqd. DEOM}$$

The DEOMs are elastically coupled.

In matrix form,

$$[I]\{\ddot{\theta}\} + [k]\{\theta\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where $[I] = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$ & $[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix}$

Lagrange's method

The DEOM are :-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial U}{\partial \theta_i} = 0 \;; \quad i = 1, 2$$

$$\begin{cases} T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 \\ U = \frac{1}{2} k_1 \theta_1^2 + \frac{1}{2} k_2 (\theta_2 - \theta_1)^2 \end{cases}$$

These will lead to the same DEOM (HW)

We now want to ~~solve~~ solve the coupled set of DEOM

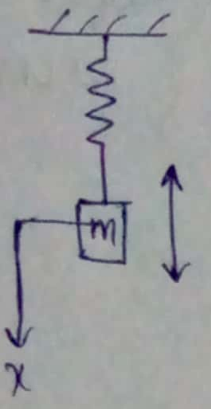
$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \quad \text{---} \quad (1)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \quad \text{---} \quad (2)$$

There are several ways for eg. — The method of undetermined coefficients / multipliers

We follow the heuristic approach

we draw from our earlier experience with the single DOF system.



$$x(t) = A\sin(\omega t + \phi)$$

We ask ~~ourls~~ ourselves ?
Could the 2DOF system have harmonic oscillations.

Let $x_1 = A_1 \sin(\omega t + \phi)$ ——— (3)

Substituting (3) in (1),

$$-A_1 m_1 \omega^2 \sin(\omega t+\phi) + (k_1+k_2) A_1 \sin(\omega t+\phi) - k_2 x_2 = 0$$

$$\Rightarrow x_2 = \frac{(k_1+k_2-m_1\omega^2) A_1 \sin(\omega t+\phi)}{k_2}$$

$$= A_2 \sin(\omega t + \phi) \quad \text{——— (4)}$$

$\boxed{\text{Aim:- To find } A_1; A_2, \phi, \omega}$

Substituting (3) & (4) in (1) & (2)

$$\cancel{m_1\omega^2} \; -m_1\omega^2 A_1 \sin(\omega t+\phi) + (k_1+k_2) A_1 \sin(\omega t+\phi) - k_2 A_2 \sin(\omega t+\phi) = 0$$

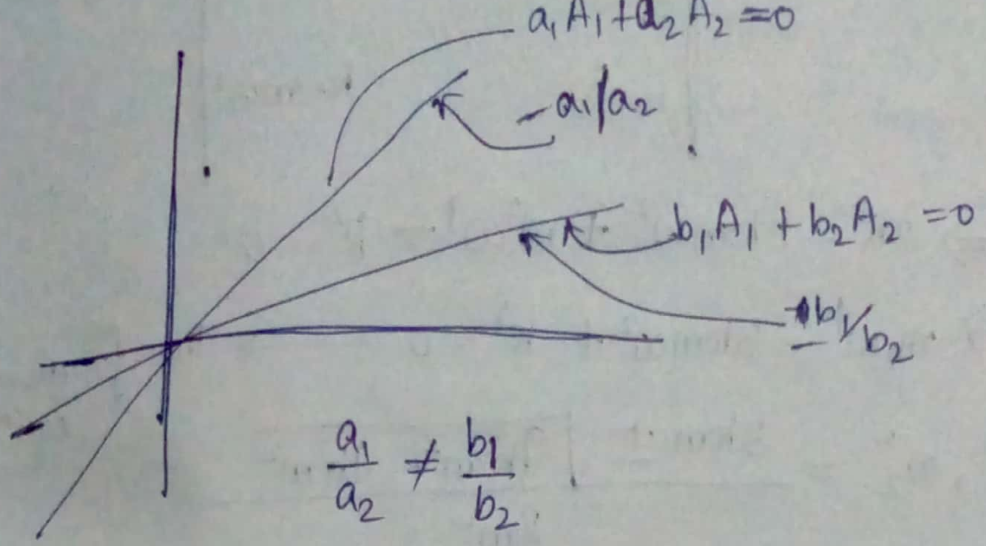
$$\Rightarrow (k_1+k_2-m_1\omega^2) A_1 - k_2 A_2 = 0 \quad \text{——— (5)}$$

$$-k_2 A_1 + (k_2 - m_2\omega^2) A_2 = 0 \quad \text{——— (6)}$$

For non-trivial A_1 & A_2

$$a_1 A_1 + a_2 A_2 = 0$$
$$b_1 A_1 + b_2 A_2 = 0$$

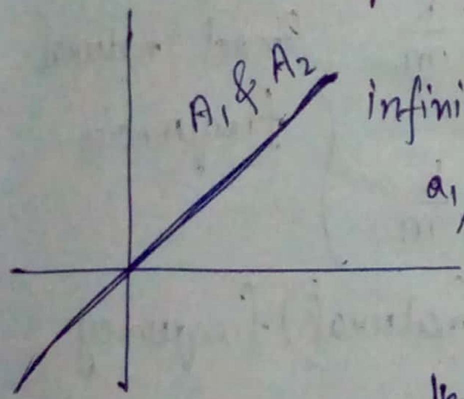
$A_1, A_2 \rightarrow$ Amplitudes of free vibration



$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

$$\Rightarrow a_1 b_2 - a_2 b_1 \neq 0$$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$$

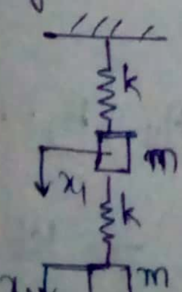


infinitely many solutions

$$a_1/a_2 = b_1/b_2 \Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = 0$$

We must have
$$\begin{vmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_2 - m\omega^2 \end{vmatrix} = 0$$

This gives the characteristic or frequency eqn



$$k_1 = k_2 = k$$
$$m_1 = m_2 = m$$

Frequency equⁿ is

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow 2k^2 - 3km\omega^2 + m^2\omega^4 - k^2 = 0$$

$$\Rightarrow m^2\omega^4 - 3km\omega^2 + k^2 = 0 \longrightarrow \text{The frequency equ}^n$$

$$\omega_1^2, \omega_2^2 = \frac{3km \pm \sqrt{9k^2m^2 - 4k^2m^2}}{2m^2}$$

$$= \left(\frac{3 \pm \sqrt{5}}{2}\right)\frac{k}{m}$$

The smaller one

$$\left.\begin{aligned} \omega_1 &= 0.6180\sqrt{\frac{k}{m}} \\ \omega_2 &= 1.6180\sqrt{\frac{k}{m}} \end{aligned}\right\} \text{Exact natural frequencies}$$

$\omega_1 \longrightarrow$ The fundamental (natural) frequency

$\omega_2 \longrightarrow$ The 2nd harmonic

<u>To obtain A_1 & A_2</u>

From equⁿ ⑤; $\dfrac{A_1}{A_2} = \dfrac{k_1 + k_2 - m_1\omega^2}{k_2}$

Corresponding to $\omega = \omega_1$

we shall have $A_1 = A_{11}$, $A_2 = A_{21}$

→ Amplitude ~~of~~ 1st mass for 1st

principal mode of oscillation

$A_{21} \rightarrow$ Amplitude of 2nd mass for Ist principal mod

$$\omega = \omega_2 \rightarrow \begin{cases} A_1 \rightarrow A_{12} \\ A_2 \rightarrow A_{22} \end{cases}$$

Thus,

$$\frac{A_{21}}{A_{11}} = \frac{k_1 + k_2 - m_1\omega^2}{k_2}$$

$$= \frac{2k - (0.618)^2 k}{k}$$

$$= 2 - (0.618)^2 = \cancel{\ \ } 1.618$$

So, $\boxed{A_{21} = 1.618 A_{11}}$

From equ (6),

$$\frac{A_2}{A_1} = \frac{k_2}{k_2 - m_2\omega^2}$$

$$\frac{A_{21}}{A_{11}} = \frac{k}{k - m\omega_1^2} = 1.618$$

Thus we can't obtain unique values for amplitudes.

For $\omega = \omega_2$,

$$\frac{A_{22}}{A_{12}} = \frac{k_1 + k_2 - m_1\omega_2^2}{k_2} = \frac{k_2}{k_2 - m_2\omega_2^2}$$

$$= \frac{2K_1 - m\omega_2^2}{K_2} = -0.618$$

Conclusions :- (i) For 1st principal mode of oscillation,

$$x_1(t) = A_{11} \sin(\omega_1 t + \phi_1)$$

$$x_2(t) = A_{21} \sin(\omega_1 t + \phi_1) = \mu_1 A_{11} \sin(\omega_1 t + \phi_1)$$

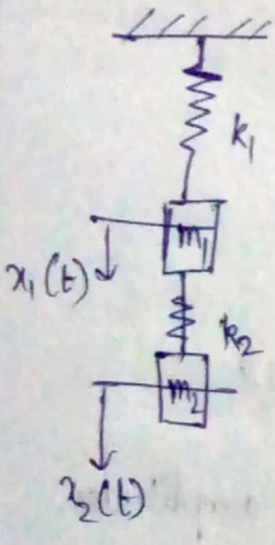
where $\mu_1 = \frac{A_{21}}{A_{11}} = 1.618$ for ex. problem

For the 2nd pr. mode,

$$x_1(t) = A_{12} \sin(\omega_2 t + \phi_2)$$

$$x_2(t) = A_{22} \sin(\omega_2 t + \phi_2) = \mu_2 A_{12} \sin(\omega$$

$$\mu_2 = \frac{A_{22}}{A_{12}} = -0.618$$



m_1 & m_2 can oscillate harmonically a two frequencies ω_1 & ω_2

$\omega_1 \longrightarrow$ fundamental (natural) frequency

$\omega_2 \longrightarrow$ 2nd harmonic

$x_1(t) = A_{11} \sin(\omega_1 t + \phi_1)$

$x_2(t) = \mu_1 A_{11} \sin(\omega_1 t + \phi_1)$,

For the 1st principal mode of vibration

&

$x_1(t) = A_{12} \sin(\omega_2 t + \phi_2)$

$x_2(t) = \mu_2 A_{12} \sin(\omega_2 t + \phi_2)$

for the 2nd principal mode.

ω_1 & ω_2 are obtained from the frequency / characteristic eq^n.-

$$\begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix} = 0$$

We can show that $A_{11}\sin(\omega_1 t+\phi_1)$ & $A_{12}\sin(\omega_2 t+\phi_2)$ are <u>linearly independent</u>

Let $x_1(t), x_2(t), \cdots, x_n(t)$ be a set of n continuously differentiable functions. They will be linearly independent if & only if

$$c_1x_1(t)+c_2x_2(t)+\cdots+c_nx_n(t)=0$$

is true when $c_1=c_2=\cdots=c_n=0$

$$c_1\dot{x}_1(t)+c_2\dot{x}_2(t)+\cdots+c_n\dot{x}_n(t)=0$$

$$c_1\ddot{x}_1(t)+c_2\ddot{x}_2(t)+\cdots+c_n\ddot{x}_n(t)=0$$

The Wronskian determinant $W(t)=\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} & \cdots & \frac{dx_n}{dt} \\ \vdots & & & \\ \frac{d^{n-1}x_1}{dt^{n-1}} & \frac{d^{n-1}x_2}{dt^{n-1}} & \cdots & \frac{d^{n-1}x_n}{dt^{n-1}} \end{vmatrix}$

For non-trivial $c_1, c_2, c_3, \cdots, c_n$ to satisfy above set of n eqn's we must have

<u>$W(t)=0$</u>

for eg.

Let $\left.\begin{array}{l} x_1=\sin\omega t \\ x_2=\cos\omega t \end{array}\right\}\omega\neq 0$

$$W(t)=\begin{vmatrix} \sin\omega t & \cos\omega t \\ \omega\cos\omega t & -\omega\sin\omega t \end{vmatrix}$$

$= -\omega(\sin^2\omega t) - \omega\cos^2\omega t = -\omega \neq 0$

So $\sin\omega t$ & $\cos\omega t$ are linearly independent

HW show that $[x_1(t)]_1 = A_{11}\sin(\omega_1 t + \phi_1)$

& $[x_1(t)]_2 = A_{12}\sin(\omega_2 t + \phi_2)$

are linearly independent

Then, by the theory of linear differential eqn's,

$x_1(t) = [x_1(t)]_1 + [x_1(t)]_2$ will be the general solution or general motion of m_1.

Thus the superposition of the principal modes gives the general free vibration response

Hence, in general,

$$x_1(t) = A_{11}\sin(\omega_1 t + \phi_1) + A_{12}\sin(\omega_2 t + \phi_2)$$
$$x_2(t) = \mu_1 A_{11}\sin(\omega_1 t + \phi_1) + \mu_2 A_{12}\sin(\omega_2 t + \phi_2)$$

Here A_{11}, A_{12}, ϕ_1, ϕ_2 are arbitrary. These can be obtained using given initial conditions $x_1(0)$, $\dot{x}_1(0)$; $x_2(0)$, $\dot{x}_2(0)$.

Aim:- To find under what kind of initial conditions can either principal mode be excited.

Let the system is executing the first principal mode or normal mode.

Then, $x_1(t) = A_{11}\sin(\omega_1 t + \phi_1)$

$x_2(t) = \mu_1 A_{11}\sin(\omega_1 t + \phi_1)$

$$\Rightarrow \left.\begin{aligned} x_1(0) &= A_{11}\sin\phi_1 \\ x_2(0) &= \mu_1 A_{11}\sin\phi_1 \end{aligned}\right\} \Rightarrow \boxed{x_2(0) = \mu_1 x_1(0)}$$

REMEMBER

$\dot{x}_1(t) = \omega_1 A_{11} \cos(\omega_1 t + \phi_1)$

$\dot{x}_2(t) = \mu_1 \omega_1 A_{11} \cos(\omega_1 t + \phi_1)$

Hence $\boxed{\dot{x}_2(0) = \mu_1 \dot{x}_1(0)}$ (REMEMBER)

~~Let the system is executing the first principal or (normal) mode~~

HW → show that these conditions are sufficient too

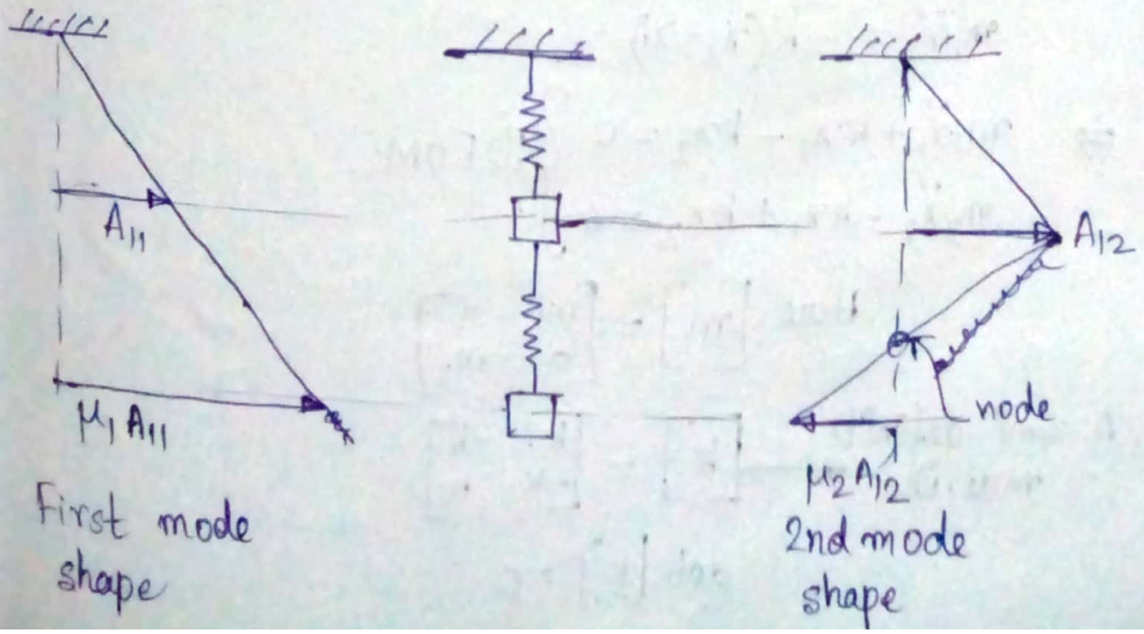
For 2nd mode (principal), the necessary & sufficient initial conditions are

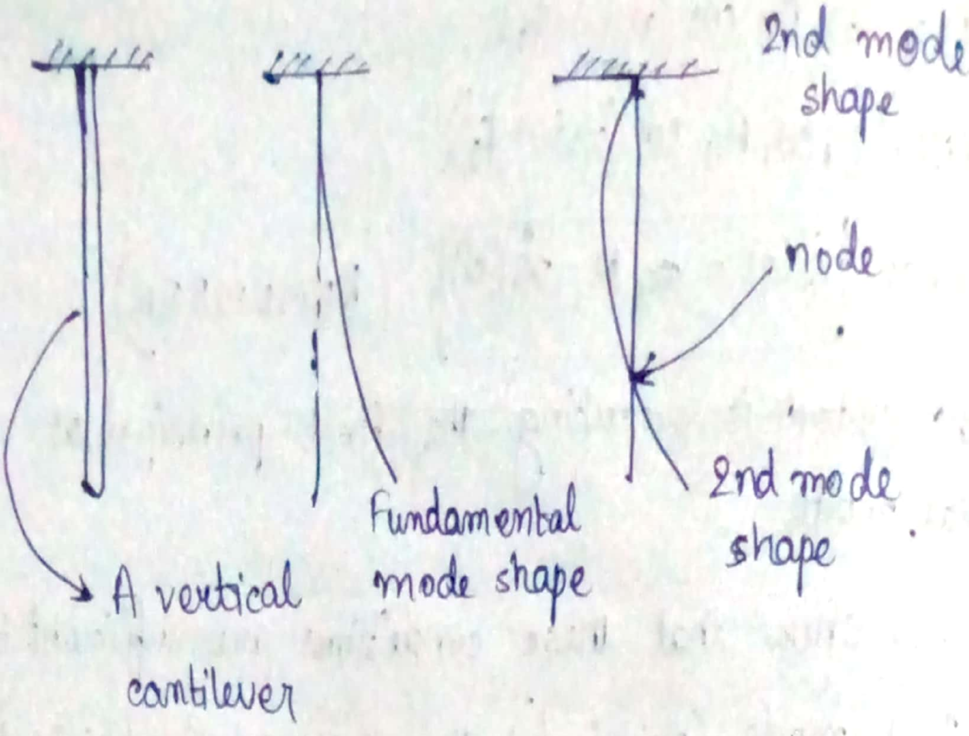
$$\boxed{\begin{aligned} x_2(0) &= \mu_2 x_1(0) \\ \dot{x}_2(0) &= \mu_2 \dot{x}_1(0) \end{aligned}}$$

Mode Shapes :-

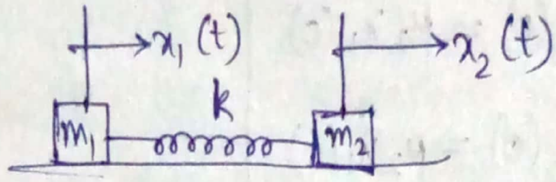
A mode shape is an approximate plot of amplitudes of various points in the system for a particular principal mode of vibration.

For our system, there will be two mode shapes



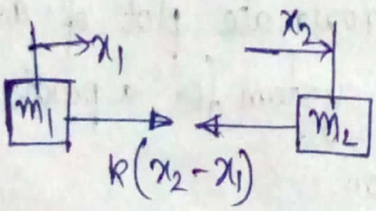


(8) A semi-definite system



Find ω_1, ω_2 for undamped free vibration

$x_2 > x_1$



FBD in the horiz. direction

$$m_1\ddot{x}_1 = k(x_2-x_1)$$

$$m_2\ddot{x}_2 = -k(x_2-x_1)$$

$$\Rightarrow \left.\begin{aligned} m_1\ddot{x}_1 + kx_1 - kx_2 = 0 \\ m_2\ddot{x}_2 - kx_1 + kx_2 = 0 \end{aligned}\right\} \text{DEOM}$$

Here $[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

A semi-definite matrix $\leftarrow [k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

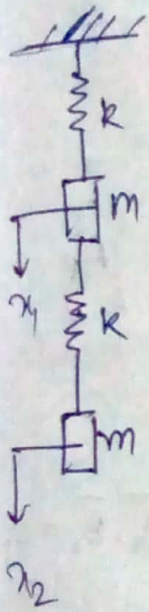
$\det[k] = 0$

Frequency eqn

$$\begin{bmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{bmatrix} = 0.$$

$$\omega_1 = 0, \quad \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$$

$\longrightarrow$ Rigid body motion is possible



We have seen that the principal mode vibrations are:-

$$\left.\begin{array}{l} \begin{cases} x_1(t) = A_{11}\sin(\omega_1 t + \phi_1) \\ x_2(t) = A_{11}\mu_1 \sin(\omega_1 t + \phi_1) \end{cases} \\ \& \\ \begin{cases} x_1(t) = A_{12}\sin(\omega_2 t + \phi_2) \\ x_2(t) = \mu_2 A_{12}\sin(\omega_2 t + \phi_2) \end{cases} \end{array}\right\}.$$

The vectors $\begin{Bmatrix} A_{11} \\ \mu_1 A_{11} \end{Bmatrix} = \{A_1\}$

&

$\begin{Bmatrix} A_{12} \\ A_{12}\mu_2 \end{Bmatrix} = \{A_2\}$

$$\{A_1\}^T [m] \{A_2\}$$

$$= \begin{Bmatrix} A_{11} \\ 1.618 A_{11} \end{Bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} A_{12} \\ -0.618 A_{12} \end{Bmatrix}$$

are called modal vectors or eigen vectors

For our eg., $\mu_1 = 1.618$

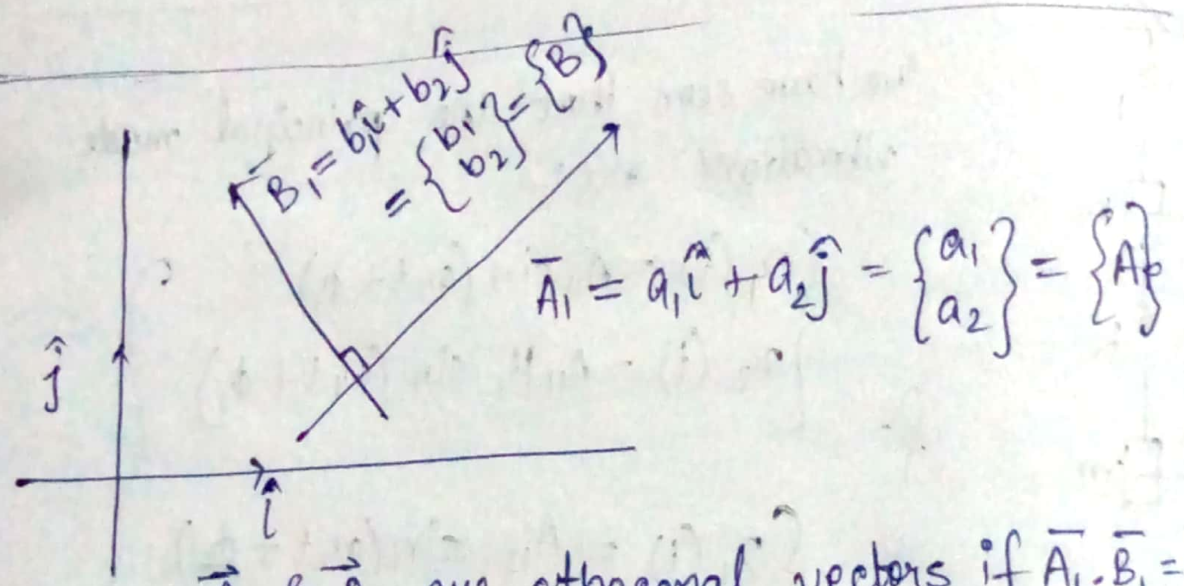
$\mu_2 = -0.618$

Verify that $\{A_1\}$ & $\{A_2\}$ are orthogonal w.r.t $[m]$ & $[k]$

$\Rightarrow$ Show that

$$\left.\begin{array}{l}\{A_1\}^T[m]\{A_2\}=0 \\ \& \quad \{A_1\}^T[k]\{A_2\}=0\end{array}\right]$$

$$\{A_1\}^T[k]\{A_2\} = \begin{Bmatrix} A_{11} \\ 1.618A_{11}\end{Bmatrix}\begin{bmatrix} 2k & -k \\ -k & k\end{bmatrix}\begin{Bmatrix} A_2 \\ -0.618A_{12}\end{Bmatrix}$$



$\vec{A}_1$ & $\vec{B}_1$ are othogonal vectors if $\bar{A}_1 \cdot \bar{B}_1 = 0$

ie, if $a_1b_1 + a_2b_2 = 0$ ie, $\boxed{\{A\}^T\{B\} = 0}$

$$\{A\}^T[I]\{B\}$$

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \rightarrow \text{a weighting matrix}$$

We say that $\{A\}$ & $[B]$ are othogonal w.r.t a weighting matrix $[c]$ if $\{A\}^T[c]\{B\} = 0$.

Verify that the modal matrix

$$[\mu] = \left[\{A_1\}\{A_2\}\right] = \begin{Bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{Bmatrix}$$ can

uncouple the DEOM.

Our DEOM in matrix form are :—

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \quad \text{——— (1)}$$

Let $\{x\} = [\mu]\{p\}$ (2), $\{p\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$, a new set of generalised coordinates.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$\Rightarrow x_1 = A_{11} p_1 + A_{12} p_2$$

$$x_2 = \mu_1 A_{11} p_1 + \mu_2 A_{12} p_2$$

Substitute (2) in (1)

$$[m][\mu]\{\ddot{p}\} + [k][\mu]\{p\} = \{0\} \quad \text{——— (3)}$$

where $\{\ddot{p}\} = \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix}$

Premultiply (3) by $[\mu]^T$

This gives $[\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[k][\mu]\{p\}$

$$= [\mu]^T\{0\} = \{0\} \quad \text{——— (4)}$$

Verify that $[\mu]^T[k][\mu]$ & $[\mu]^T[m][\mu]$ are both diagonal

$$[\mu]^T [m] [\mu] = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

&

$$[\mu]^T [k] [\mu] = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

where M_{11} & M_{22} are called generalised masses & K_{11} & K_{22} are the generalised stiffnesses

$M_{11} =$ $\quad$ $K_{11} =$

$M_{22} =$ $\quad$ $K_{22} =$

Let $A_{11} = 1$, Then $\{A_1\} = \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix} \longrightarrow$ called a normalised modal ~~matrix~~ vector

Similarly,

$\{A_2\} = \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$ is normalised modal vector.

$[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$, a normalised modal matrix

$\mu_1 = 1.618 \qquad \mu_2 = -0.618$

$$[\mu]^T [m] [\mu] = \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$

$$= m \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$

$$= m \begin{bmatrix} 1+\mu_1^2 & 1+\mu_1\mu_2 \\ 1+\mu_1\mu_2 & 1+\mu_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3.618m & \approx 0 \\ \approx 0 & 1.382m \end{bmatrix} \longrightarrow \begin{matrix} M_{11} = 3.618m \\ M_{22} = 1.382m \end{matrix}$$

$$[M]^T[k][M] = \begin{bmatrix} 1.382k & 0 \\ 0 & 3.618k \end{bmatrix}$$

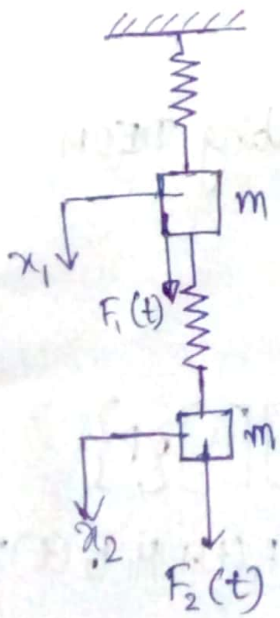
$$\omega_1 = \sqrt{\frac{1.382k}{3.618m}} = 0.618\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3.618k}{1.382m}} = 1.618\sqrt{\frac{k}{m}}$$

$$3.618m\ddot{p}_1 + 1.382k\,p_1 = 0$$
$$1.382m\ddot{p}_2 + 3.618k\,p_2 = 0$$

The DEOM in terms of generalised coordinates p_1 & p_2

⑨ Forced vibrations & Modal Analysis



The DEOM are:-

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1(t) \quad (1)$$

$$m\ddot{x}_2 - kx_1 + kx_2 = F_2(t) \quad (2)$$

$F_1(t)$ & $F_2(t) = 0$ for $t<0$

and $\left.\begin{matrix} F_1(t) = \text{- - -} \\ F_2(t) = \text{- - -} \end{matrix}\right\}$ for $t>0$

Aim:- To obtain the forced response using the Modal Analysis.

When a system response is obtained by uncoupling the DEOM using a modal matrix, we call it <u>modal analysis</u>.

<u>Step 1</u>: Consider the free-vibration first

Solve :- $m\ddot{x}_1 + 2kx_1 - kx_2 = 0$

$m\ddot{x}_2 - kx_1 + kx_2 = 0$

& obtain ω_1, ω_2

& then $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$, a normalised modal matrix.

<u>Step 2</u>:- Uncouple the DEOM using the coordinate transformation

$$\{x\} = [\mu]\{p\}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\} \quad \text{———} \quad (1)$$

where $\{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$

Substitute (2) in (1)

Premultiply both sides of the resulting DEOM by $[\mu]^T$

This gives

$$[\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[k][\mu]\{p\}$$

$$= [\mu]^T\{F(t)\} = \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \end{bmatrix}\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} F_1(t) + \mu_1 F_2(t) \\ F_1(t) + \mu_2 F_2(t) \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{P}_1 \\ \ddot{P}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix}$$

where $\left.\begin{aligned} Q_1(t) &= F_1(t) + \mu_1 F_2(t) \\ Q_2(t) &= F_1(t) + \mu_2 F_2(t) \end{aligned}\right\}$ are the generalised forces.

Hence, the uncoupled DEOM in terms of the principal coordinates p_1 & p_2 are :-

$$M_{11}\ddot{p}_1 + K_{11}p_1 = F_1(t) + \mu_1 F_2(t) \quad \text{—— (I)}$$

$$M_{22}\ddot{p}_2 + K_{22}p_2 = F_1(t) + \mu_2 F_2(t) \quad \text{—— (II)}$$

Step 3 :- obtain $p_1(t)$ & $p_2(t)$ using (I) & (II) & Duhamel's Integral

Step 4 :- Obtain $x_1(t)$ & $x_2(t)$ using

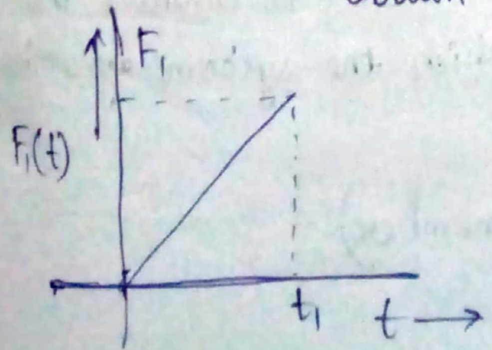
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

& hence forced responses $x_1(t)$ & $x_2(t)$ are obtained

Tu Problem :- Set $F_2(t) = 0$

$F_1(t) \longrightarrow$ as shown

obtain $x_1(t)$ & $x_2(t)$



A numerical method for obtaining the natural frequencies and associated modal vectors — The Matrix Iteration Method

For a system with a large no. of DOF, exact analysis is cumbersome. So we go for approximate methods

Let the DEOM be $[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$ —— ①

Let $\{x\} = \{A\}\sin(\omega t + \phi)$ —— ②

ie, $x_1 = A_1 \sin(\omega t + \phi)$

$x_2 = A_2 \sin(\omega t + \phi)$

for a 2 DOF system.

Substitute ② in ①

This ⇒

$$-\omega^2[m]\{A\}\sin(\omega t+\phi) + [k]\{A\}\sin(\omega t+\phi) = \{0\}$$

$$\Rightarrow [k]\{A\} = \omega^2 [m]\{A\}$$

$$\Rightarrow [k]^{-1}[m]\{A\} = 1/\omega^2 \{A\}$$

$$\Rightarrow [D]\{A\} = \frac{1}{\omega^2}\{A\}$$

where $[D] = [k]^{-1}[m] \doteq$ A Dynamic matrix (Since it contains the parameters upon which the system dynamics depend)

$$\boxed{[D]\{A\} = \frac{1}{\omega^2}\{A\}}$$ (Remember)

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So we have

$$[D]\{A_1\} = \frac{1}{\omega_1^2}\{A_1\}$$

$$[D]\{A_2\} = \frac{1}{\omega_2^2}\{A_2\}$$

$\{A_1\}$ = First eigen vector $= \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}$

$\{A_2\}$ = Second eigen vector $= \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix}$

Equⁿ (I) shows that if $\{A\}$ is an eigenvector of the system, then premultiplication of $\{A\}$ by $[D]$ results in the same vector $\{A\}$ multiplied by a factor $1/\omega^2$, which is the natural frequency corresponding to $\{A\}$

$$\{A_1\}_{normalised} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix} \quad \{A_1\} = \begin{Bmatrix} A_{11} \\ 1.618\, A_{11} \end{Bmatrix}$$

$$[\mu] = \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix} \qquad [\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$

$$\omega_1 = 0.618\sqrt{\frac{k}{m}}$$

Start with any trial vector $\{a_1\} = \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$[D] = [k]^{-1}[m] \qquad \det[k] = 2k^2 - k^2 = k^2$$

$$[k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

$$[k]^{-1} = \frac{\operatorname{adj}[k]}{\det[k]} = \frac{\begin{bmatrix} k & k \\ k & 2k \end{bmatrix}}{k^2} = \frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[D] = \frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\{A_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$[D][A_1] = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{m}{k}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$= \frac{2m}{k}\begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix}$$

$$\{A_2\} = \begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix}$$

$$[D]\{A_2\} = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix} = \frac{m}{k}\begin{Bmatrix} 2.5 \\ 4 \end{Bmatrix}$$

$$= \frac{2.5m}{k}\begin{Bmatrix} 1 \\ 1.6 \end{Bmatrix}$$

$$\{A_3\} = \begin{Bmatrix} 1 \\ 1.6 \end{Bmatrix}, \quad [D]\{A_3\} = \frac{2.6m}{k}\begin{Bmatrix} 1 \\ 1.6154 \end{Bmatrix}$$

$$\{A_4\} = \begin{Bmatrix} 1 \\ 1.6154 \end{Bmatrix}$$

$$[D][A_4] = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.6154 \end{Bmatrix} = \frac{2.6154m}{k}\begin{Bmatrix} 1 \\ 1.6176 \end{Bmatrix}$$

So $\{A_1\} = \begin{Bmatrix} 1 \\ 1.6176 \end{Bmatrix}$

$$\frac{1}{\omega_1^2} \simeq \frac{2.6154m}{k} \text{ or, } \omega_1 \approx \sqrt{\frac{1}{2.615}}\sqrt{\frac{k}{m}}$$

$$= 0.6183\sqrt{\frac{k}{m}}$$

The Expansion Theorem

For a 2DOF system: Any 2-D vector $\begin{Bmatrix} u \\ v \end{Bmatrix}$ can be expressed as a linear combination of modal vectors $\{A_1\}$ & $\{A_2\}$

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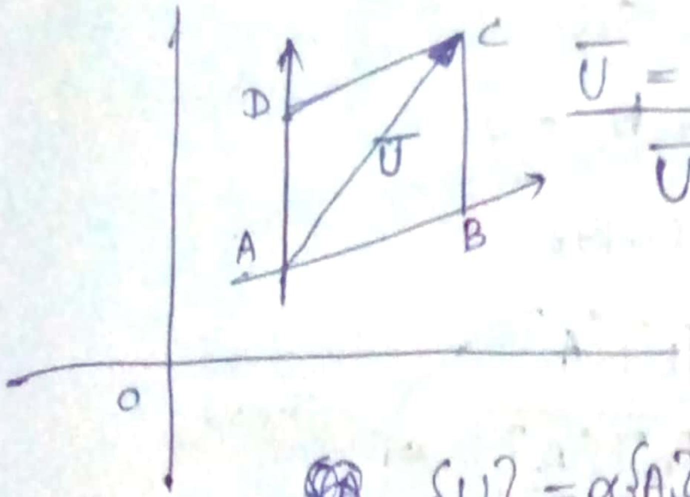
$\bar{A}_2 \neq \alpha \bar{A}_1$



$\bar{U} = \alpha_1 \bar{A}_1 + \alpha_2 \bar{A}_2$

$\bar{U} = \overline{AB} + \overline{BC}$

$= \alpha_1 \bar{A}_1 + \alpha_2 \bar{A}_2$

$\{U\} = \alpha_1 \{A_1\} + \alpha_2 \{A_2\}$

$$\{U_2\} = [D]\{U\} = \alpha_1 [D]\{A_1\} + \alpha_2 [D]\{A_2\}$$

$$= \frac{\alpha_1}{\omega_1^2}\{A_1\} + \frac{\alpha_2}{\omega_2^2}\{A_2\}$$

$$[D]\{U_2\} = \frac{\alpha_1}{(\omega_1^2)^2}\{A_1\} + \frac{\alpha_2}{(\omega_2^2)^2}\{A_2\}$$

$$[D]\{U_p\} = \frac{\alpha_1}{(\omega_1^2)^p}\{A_1\} + \frac{\alpha_2}{(\omega_2^2)^p}\{A_2\} \rightarrow \text{negligible}$$

$\underline{\omega_2 > \omega_1}$ $\quad \frac{1}{\omega_2^2} << \frac{1}{\omega_1^2}$

$$\Rightarrow \frac{1}{(\omega_2^2)^p} <<<< \frac{1}{(\omega_1^2)^p}$$

$\omega_1 = 0.6183\sqrt{\frac{k}{m}}$ $\quad \{A_1\} = \begin{Bmatrix} 1 \\ 1.6176 \end{Bmatrix}$

$\{A_1\}^T [m] \{A_2\} = 0$ (By mass orthogonality)

$$\begin{Bmatrix} 1 & 1.6176 \end{Bmatrix} m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix} = 0$$

$$\Rightarrow 1 + 1.6176\mu_2 = 0$$

$$\Rightarrow \mu_2 = -0.6178$$

$$\omega_2^2 = \frac{\{A_2\}^T [k] \{A_2\}}{\{A_2\}^T [m] \{A_2\}}$$ show

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

$$\Rightarrow -\omega^2 [m]\{A\} + [k]\{A\} = \{0\}$$

$$-\omega_1^2 [m]\{A_1\} + [k]\{A_1\} = \{0\}$$

$$\{A_1\}^T \omega_1^2 [m]\{A_1\} = \{A_1\}^T [k]\{A_1\}$$

$$\omega_1^2 = \frac{\{A_1\}^T [k]\{A_1\}}{\{A_1\}^T [m]\{A_1\}}$$

Similarly, $\omega_2^2 = \dfrac{\{A_2\}^T [k]\{A_2\}}{\{A_2\}^T [m]\{A_2\}}$