Substitution of these in (1) gives:

$I_o\ddot{\theta} + mgl_1 \sin\theta = 0$. ← The nonlinear DEOM

Linearizing by $\sin\theta \cong \theta$, we get the required DEOM:

$$I_o\ddot{\theta} + mgl_1\theta = 0$$

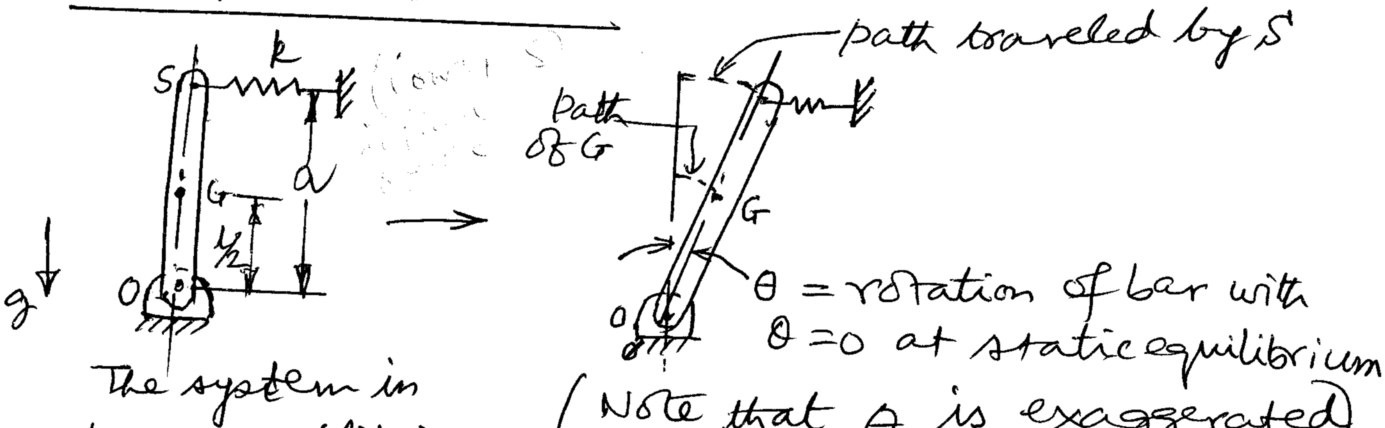
& so, $\omega_n = \sqrt{\dfrac{mgl_1}{I_o}}$ (Comparing with $I_o\ddot{\theta} + k_t\theta = 0$)

→ Thus, for a uniform bar of length l, $l_1 = l/2$, $I_G = \frac{1}{12}ml^2$ (Standard formula)

& $I_o = I_G + m(l/2)^2 = \frac{1}{3}ml^2$ (Remember)

Then, $\omega_n = \sqrt{\dfrac{mg\,l/2}{\frac{1}{3}ml^2}} = \sqrt{\dfrac{3g}{2l}}$

→ Example 3:~ The spring loaded inverted pendulum and its stability for small oscillations.

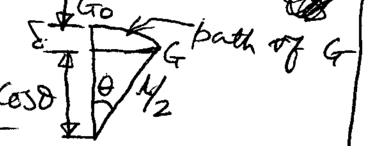


The system in Static equilibrium

(Note that θ is exaggerated very much for clarity)

If θ is small, compression in spring $\cong a\theta$, note. Also note that in static equilibrium the bar is vertical & so, the spring is neither compressed nor extended & hence has zero strain energy stored. Thus, $V_{spring} = \frac{1}{2}k(a\theta)^2 = \frac{1}{2}ka^2\theta^2$. But

Observe that the CG of the bar goes down in the gravitational field by an amount



$\delta = l/2 - l/2 \cos\theta = l/2(1-\cos\theta)$, assuming the bar is uniform and of length l and it is pivoted near its bottom end.

Then, $V_{gravity} = -mg\, l/2 (1-\cos\theta)$ and note the negative sign since ~~gravitational~~ gravitational potential energy has reduced, i.e. a negative change has occurred.

So, $V = V_{spring} + V_{gravity} = \frac{1}{2}ka^2\theta^2 - mg\frac{l}{2}(1-\cos\theta)$.

Also, Kinetic energy $= T = \frac{1}{2}I_0\dot{\theta}^2$ where $I_0 = \frac{1}{3}ml^2$ is the mass moment of inertia of bar about O, about an axis perpendicular to the plane of motion. [Note that the bar executes 'plane' motion while vibrating since every particle of it moves parallel to a single plane which you can visualize as the mid-plane of the bar (the plane of the paper).]

The Lagrange equation here is, once again,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{—— (1)}$$

Check that $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_0\ddot{\theta}$; $\frac{\partial T}{\partial \theta} = 0$ and

$\frac{\partial V}{\partial \theta} = ka^2\theta - mg\frac{l}{2}\sin\theta \simeq ka^2\theta - \frac{mgl}{2}\theta$ for small θ.

Substitution of these derivatives in (1) results in the DEOM:

$$I_0\ddot{\theta} + \left(ka^2 - \frac{mgl}{2}\right)\theta = 0$$

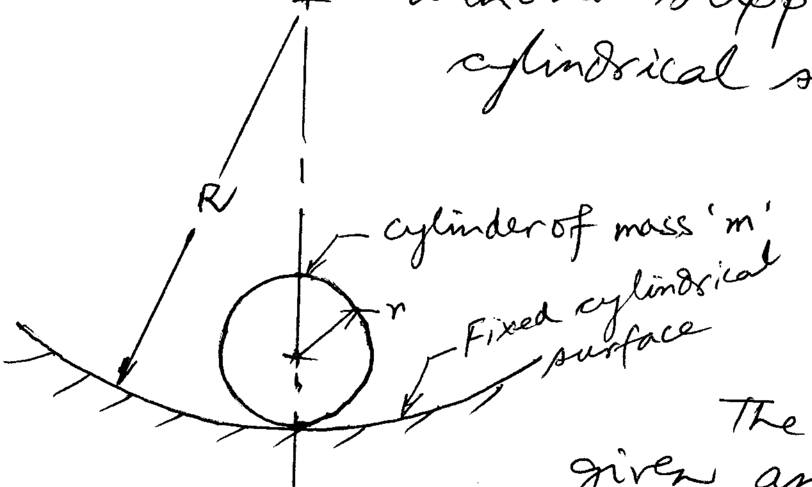
& $$\omega_n = \sqrt{\frac{ka^2 - \frac{mgl}{2}}{I_0}}.$$

→ Now note a very important thing about the above expression for ω_n. $\omega_n > 0$ if and only if $ka^2 - \frac{mgl}{2} > 0$.

This means, we get sustained (stable) free-vibration of the system only if $k > \frac{mgl}{2a^2}$. If the spring has a stiffness which doesn't satisfy this relation, then the system doesn't oscillate but either stay at the displaced position or move away from equilibrium configuration.

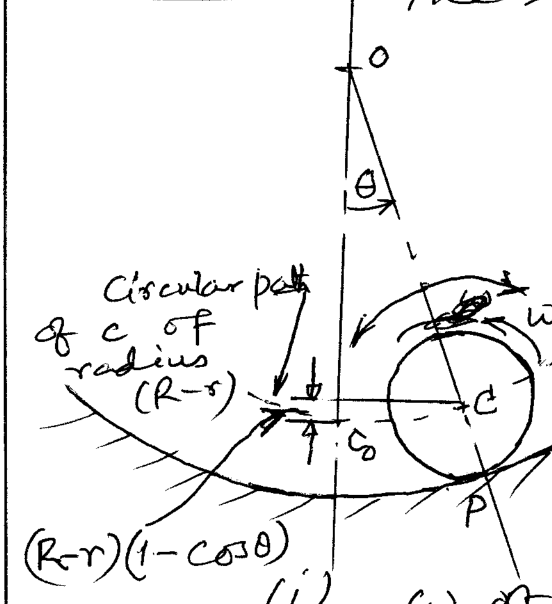
→ Example 4:- This is an important example worked out in many textbooks.

Statement:- A small cylinder rolls without slipping on a cylindrical surface of radius R.



The adjoining figure shows the system in static equilibrium. The cylinder is now given an initial angular displacement, and/or an angular velocity and left to itself. It will start rolling without slipping and climb up & down the contact surface on left and right and execute oscillations. Our aim is to obtain its DEOM and get ω_n (natural frequency) of the oscillations.

Solution:- The system at time 't' is shown here. The system has only one degree-of-freedom and the generalized coordinate θ is chosen as shown, positive counterclockwise.



There are several important points to note:-

(i) ω of cylinder is different from $\dot{\theta}$

(ii) Friction is essential for the block to have angular accelerations & decelerations while oscillating

(iii) Friction doesn't do any work here since in rolling without slipping, the point of contact (actually it is a line of contact here, the circle drawn shows the mid-plane of the circular cylinder) has zero velocity. Friction here is called a 'workless' or 'wattless' constraint force.

(iv) Thus, we have a conservative

system and ~~the~~ Lagrange's eqn. can be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=0 \quad \text{—} (1)$$

T = Kinetic energy $= \frac{1}{2} I_p \omega^2$, since the cylinder instantaneously rotates about the line of contact & I_p is its mass moment of inertia about the line of contact. By the parallel axes theorem, $I_p = I_c + mr^2 = \frac{1}{2}mr^2 + mr^2$ $= \frac{3}{2}mr^2$ where I_c is the moment of inertia of the cylinder about its own axis & $I_c = \frac{1}{2}mr^2$ (a well-known formula).

→ To obtain ω in terms of $\dot\theta$, we note that the centre of mass c of the cylinder moves in a ~~circle~~ circular path (shown dashed in the figure) of radius $(R-r)$. Clearly, its velocity is $v_c = (R-r)\dot\theta$. Looking at it from another point of view, that is, the pure instantaneous rotation of the cylinder about P, $v_c = r\omega$. Thus,

$$r\omega = (R-r)\dot\theta \quad \text{or,} \quad \boxed{\omega = \left(\frac{R-r}{r}\right)\dot\theta}$$

Hence, $T = \frac{1}{2} I_p \left(\frac{R-r}{r}\right)^2 \dot\theta^2 = \frac{1}{2}\times\frac{3}{2}m(R-r)^2\dot\theta^2$

Also, the centre of mass is raised by $(R-r)(1-\cos\theta)$ from the equilibrium level (when c was at c_0). Hence

$$V = mg(R-r)(1-\cos\theta) \quad \text{—} (3)$$

Thus, $\frac{\partial T}{\partial \dot\theta} = \frac{3m}{2}(R-r)^2\dot\theta$; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \frac{3}{2}(R-r)^2\ddot\theta$;

$\frac{\partial T}{\partial \theta} = 0$; $\frac{\partial V}{\partial \theta} = mg(R-r)\sin\theta \cong mg(R-r)\theta$ for small θ. Substitution of these in (1) gives the req'd DEOM as:

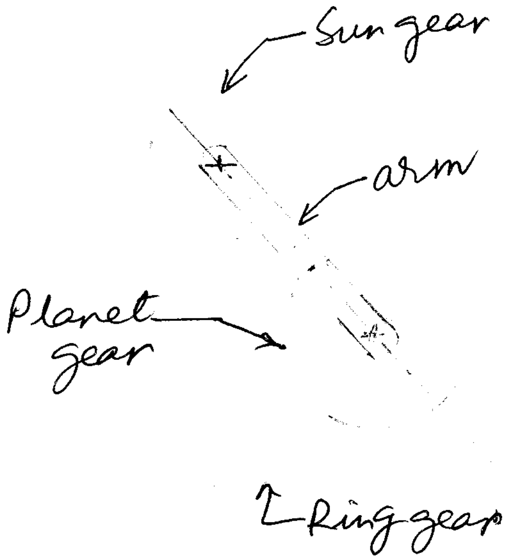
$$\frac{3}{2}m(R-r)^2\ddot{\theta} + mg(R-r)\theta = 0 \quad \text{--- (4)}$$

A comparison with $I_d\ddot{\theta} + k_t\theta = 0$ gives

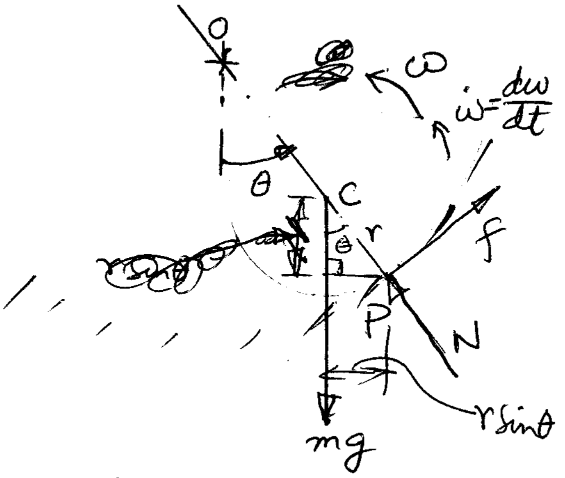
$$\omega_n = \sqrt{\frac{mg(R-r)}{\frac{3m}{2}(R-r)^2}} = \sqrt{\frac{2g}{3(R-r)}},$$

the req'd natural frequency.

→ The question now is – where do we find a set-up such as the one discussed above? You must have studied epicyclic (or, planetary) gear trains. So, if we put an arm and a sun-gear ~~and replace each gear~~ and replace each gear by its <u>pitch cylinder</u>, then the above set-up becomes part of an epicyclic gear train as shown below.



Thus, the above system can be considered to be a subsystem of the gear train shown here and a study of its vibrational characteristics could lead to a study of the same for the whole system shown here!



(Angles exaggerated for clarity)

You could also use the energy method (or, the power balance method) $\frac{d}{dt}(T+V)=0$ to get the same DEOM.

How could you get it using ~~the~~ Newton's method (the force balance method)?

For that, you must draw the FBD as shown in the figure here.

where f is the friction force & N the normal reaction. In order to eliminate the unknown forces f & N, we apply the moment balance method about point P which qualifies ~~for the~~ as the point about which the simple equation $I_P\dot{\omega}$ = Sum of moments of external forces applies, ~~Thus, taking~~ since the whole body (cylinder) instantaneously rotate about P.

Thus, taking moments about P, we get

$$I_P\dot{\omega} = -mgr\sin\theta \quad \left(\omega=\left(\frac{R-r}{r}\right)\dot{\theta} \Rightarrow \dot{\omega}=\left(\frac{R-r}{r}\right)\ddot{\theta}\right)$$

or, $$\frac{3}{2}mr^2\left(\frac{R-r}{r}\right)\ddot{\theta} + mgr\theta = 0 \quad (\text{Assuming } \sin\theta \cong \theta)$$

which is basically the same as DEOM (4) derived earlier.

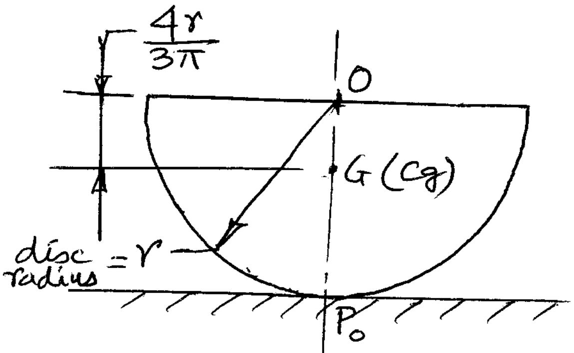
Notes:— (1) We have solved above problem in great details for the sake of explaining it properly. While ~~so~~lving a problem, <u>you</u> need to do the ~~basic~~ details only, without much of explanation of each step.

(2) You must have noticed that we are placing a lot of emphasis on Lagrange's equation as far as getting the DEOM is concerned. This turns out to be highly important. For some problems, this method may turn out to be lengthier that the Newton's method (see next example problem), however for complex problems the use of Lagrange's equations greatly simplies the analysis, especially in case of <u>multi-degree-of-freedom</u> systems.

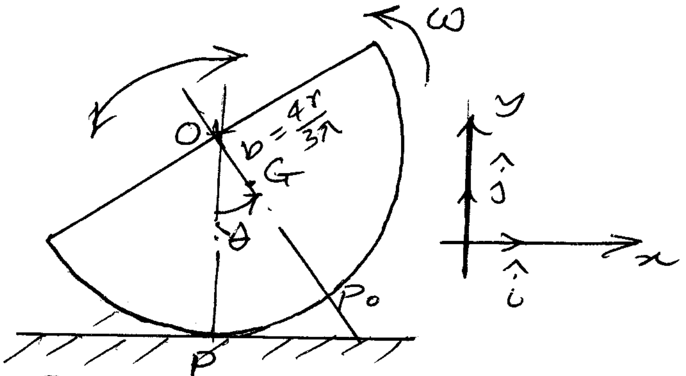
<u>Example 5:</u>— A semi-cylinder rolls without slipping on a rough horizontal surface. It actually performs small angular amplitude oscillations. Obtain its DEOM & find the expression for its natural frequency ω_n.

<u>Solution:</u>— For appling Lagrange's equation $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$, we need to evaluate

T for which an expression for the velocity of the centre of mass is required. This task is a little involved here as you will see. Direct application of ~~Newton's~~ moment balance method is easier & this will be done too. However, for getting a good grip on complex systems, you must solve this problem using Lagrange's equation.



Disc in static equilibrium



Disc in displaced configuration at time t while oscillating after being subjected to $\theta(0)$ (initial angular displacement) &/or $\dot{\theta}(0)$ (initial angular velocity, which can be generated by giving an angular ~~impt~~ impulse of very short duration, by tapping it, say.

$P_0 \rightarrow$ point of contact at the mid plane of semi-cylindrical disc at equilibrium.

$P \rightarrow$ point of contact at time t

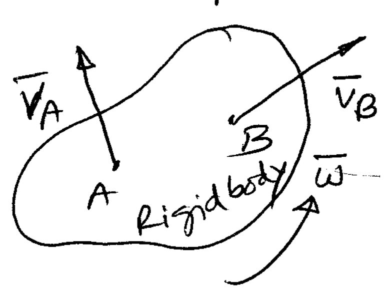
$\hat{i}, \hat{j} \rightarrow$ unit vectors

θ is taken as the generalized coordinate, measured from vertical direction, positive counterclockwise.

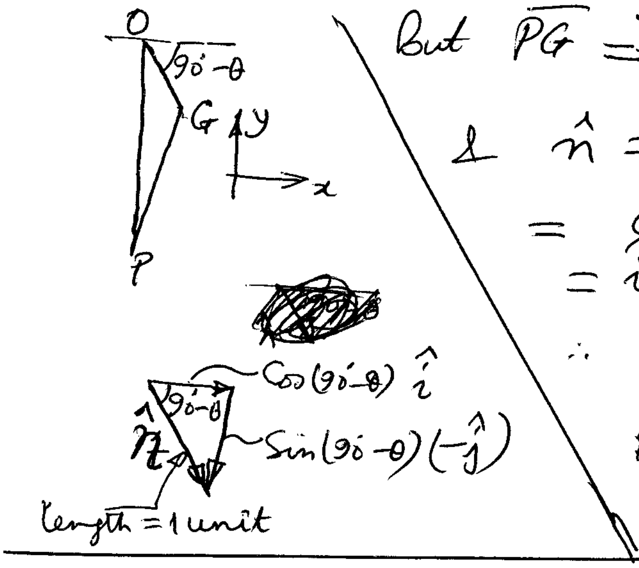
clearly, ω = angular velocity of disc $= \dot{\theta}$

We first compute $\overline{V_G}$ which is required for finding an expression for T. Note that $\overline{V_p}=0$ for rolling without slipping.

Now,
$$\overline{V_G} = \overline{V_p} + \overline{\omega} \times \overline{PG}$$
(We use the well-known result from dynamics for two points A & B on a rigid body having angular velocity $\overline{\omega}$:—
$$\overline{V_B} = \overline{V_A} + \overline{\omega} \times \overline{AB}\ )$$



Now, $\overline{V_P} = 0$ & so,
$$\overline{V_G} = \overline{\omega} \times \overline{PG} = \dot{\theta}\hat{k} \times \overline{PG}$$



But $\overline{PG} = \overline{PO} + \overline{OG}$, $OG = \frac{4r}{3\pi}$, $\overline{PO} = r\hat{j}$

& $\hat{n}$ = unit vector along OG
$$= \cos(90^\circ-\theta)\,\hat{i} - \sin(90^\circ-\theta)\,\hat{j}$$
$$= \hat{i}\sin\theta - \hat{j}\cos\theta$$
$$\therefore \overline{OG} = \frac{4r}{3\pi}\hat{n} = \frac{4r}{3\pi}(\hat{i}\sin\theta - \hat{j}\cos\theta)$$

~~Hence,~~ $\cancel{\overline{V_B} = \dot{\theta}\hat{k} \times \frac{4r}{3\pi}(\hat{i}\sin\theta - \hat{j}\cos\theta)}$

So,
$$\overline{PG} = r\hat{j} + \frac{4r}{3\pi}(\hat{i}\sin\theta - \hat{j}\cos\theta)$$
$$= r\hat{j} + b(\hat{i}\sin\theta - \hat{j}\cos\theta)$$
[Let $b = \frac{4r}{3\pi}$ for simplicity]

~~or,~~ $\cancel{\overline{PG} = b\hat{i} + (r - b\cos}$

or,
$$\overline{PG} = (b\sin\theta)\,\hat{i} + (r - b\cos\theta\,\hat{j})$$
$$\therefore \overline{V_G} = \dot{\theta}\hat{k} \times [(b\sin\theta)\hat{i} + (r - b\cos\theta)\hat{j}]$$
$$= -(r - b\cos\theta)\dot{\theta}\,\hat{i} + (b\sin\theta)\dot{\theta}\,\hat{j}$$

Hence,
$$T = \frac{1}{2}I_g\dot{\theta}^2 + \frac{1}{2}m(\overline{V_G}\cdot\overline{V_G}) = \frac{1}{2}I_C\dot{\theta}^2 + \frac{1}{2}m[r^2 + b^2 - 2br\cos\theta]\dot{\theta}^2$$

after simplification (Do it). ~~...~~

→ [This could be done without the vector approach as we shall show later. However, doing it vectorially better prepares you to tackle more involved, more complex problems, especially those involving 3-dimensional motion/vibration.]

For finding T, we have used the formula:

$$T = \underbrace{\frac{1}{2}m(V_{cg})^2}_{\text{Translational Kinetic Energy}} + \underbrace{\frac{1}{2}I_c\dot{\theta}^2}_{\text{Rotational Kinetic energy}}$$

Translational Kinetic Energy T_{tr}, assuming the whole mass is at the cg.

Rotational Kinetic energy T_{rot}, about the centroidal axis perpendicular to the plane of motion

Of course, we could also use

$T = \frac{1}{2}I_P\dot{\theta}^2$ where I_P = moment of inertia about an axis through P in z-direction, since a pure instantaneous rotation takes place about this axis. You should check that this gives the same kinetic energy as above.

Question: What is I_g & what is I_P?

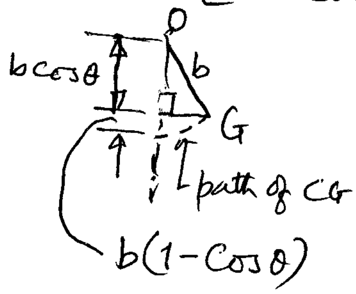
~~Note that~~, $I_O = \frac{1}{2} \times \frac{1}{2}mr^2$ (for 'half' cylinder)

Now, by the parallel axes theorem for moment of inertia, $I_O = I_C + mb^2 \Rightarrow I_C = \frac{1}{4}mr^2 - mb^2$

You should verify that both approaches give the same result.

Thus, $T = \frac{1}{2} I_C \dot{\theta}^2 + \frac{1}{2} m [r^2 + b^2 - 2br\cos\theta]\dot{\theta}^2$ — (1)

[Kinetic energy a function of generalized coordinate θ!]



$V = mgb(1 - \cos\theta)$, — (2) since the CG is raised by an amount $b(1 - \cos\theta)$

The Lagrange eqn is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{— (3)}$$

~~From~~ From (1), $\frac{\partial T}{\partial \dot{\theta}} = I_c \dot{\theta} + m[r^2 + b^2 - 2br\cos\theta]\dot{\theta}$

So, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_c \frac{d}{dt}(\dot{\theta}) + m[r^2 - 2br\cos\theta]\frac{d}{dt}(\dot{\theta})$

$\cancel{+ m\, 2br(\sin\theta)\dot{\theta}}$

$+ (m\dot{\theta}) \frac{d}{dt}[r^2 + b^2 - 2br\cos\theta]$

Be patient and careful while you differentiate

$= I_c \ddot{\theta} + m[r^2 + b^2 - 2br\cos\theta]\ddot{\theta}$

$+ m\dot{\theta} \frac{d}{d\theta}(-2br\cos\theta) \cdot \frac{d\theta}{dt}$

$= [I_c + m(r^2 + b^2 - 2br\cos\theta)]\ddot{\theta}$

$+ 2mbr\dot{\theta}^2 \sin\theta$ — (4)

Note that $\frac{\partial T}{\partial \theta} \neq 0$ here. Actually,

$$\frac{\partial T}{\partial \theta} = \frac{1}{2} m \dot{\theta}^2 \frac{\partial}{\partial \theta}(-2br\cos\theta) = mbr\dot{\theta}^2 \sin\theta \quad \text{— (5)}$$

$$\frac{\partial V}{\partial \theta} = mgb\sin\theta \quad \text{— (6)}$$

Substitution of (4), (5) & (6) in (3) gives the required DEOM (nonlinear) as:—

$$\left[I_c + m(r^2+b^2-2br\cos\theta)\right]\ddot{\theta} + mbr\dot{\theta}^2\sin\theta + mgb\sin\theta = 0 \quad \text{--- (7)}$$

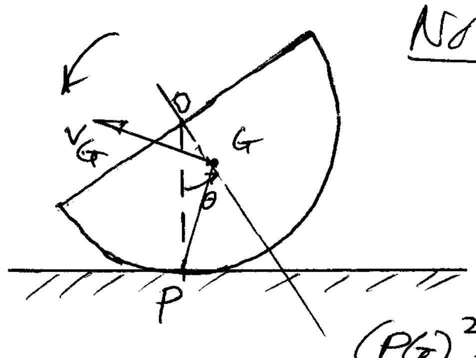
DEOM (7) is a pretty ~~comp~~ complex nonlinear differential equation whose analytical solutions cannot be found by elementary methods. However, assuming small oscillations so that $\cos\theta \simeq 1$ & $\sin\theta \simeq 0$ & neglecting the product $\dot{\theta}^2\sin\theta$, we get the <u>linearized</u> DEOM as:-

$$\left[I_c + m(r^2+b^2-2br)\right]\ddot{\theta} + mgb\theta = 0$$

or, $\left[I_c + m(r-b)^2\right]\ddot{\theta} + mgb\theta = 0$, which is our required DEOM.

Also, by comparison with $I_d\ddot{\theta} + k_t\theta = 0$ $\left\{\text{for which } \omega_n = \sqrt{\frac{k_t}{I_d}}\right\}$, $\omega_n = \sqrt{\dfrac{mgb}{\left[I_c + m(r-b)^2\right]}}$

You could further simplify it by obtaining the denominator under square root in terms of m & r. <u>Do it</u>.



<u>Note</u>:- V_G could be obtained ~~scalarly~~ scalarly as:-

$$v_G = |\bar{V}_G| = (PG)\,\dot{\theta}$$

where, from triangle OPG,

$$(PG)^2 = (OG)^2 + (OP)^2 - 2(OG)(OP)\cos\theta$$

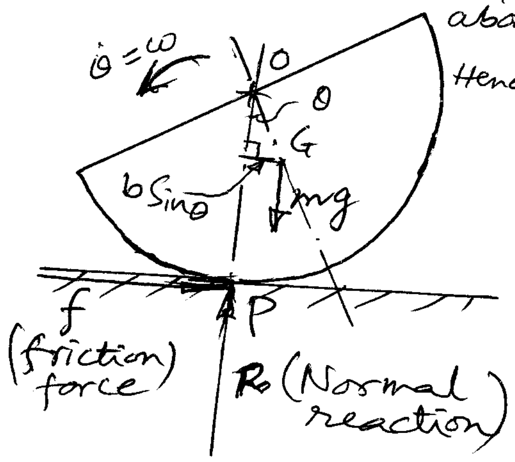
$$= b^2 + r^2 - 2br\cos\theta$$

Hence, $T_{tr} = \frac{1}{2}mV_G^2 = \frac{1}{2}m(PG)^2\dot{\theta}^2$

$= \frac{1}{2}m(b^2+r^2-2br\cos\theta)\dot{\theta}^2$, as before.

→ We now solve the problem by ~~Newton's~~ ~~or~~ the moment-balance method.

There is instantaneous pure rotation about P.



Hence, $I_P \ddot{\theta}$ = Sum of moments of external forces about P, +ive in CCW sense.

$= -mg \times b\sin\theta = -mgb\sin\theta.$

or, $I_P \ddot{\theta} + mgb\theta = 0$ (for small θ)

This is the reqd. DEOM.

So, $\omega_n = \sqrt{\dfrac{mgb}{I_P}}$

Question:- Does this tally with the previously obtained ω_n?

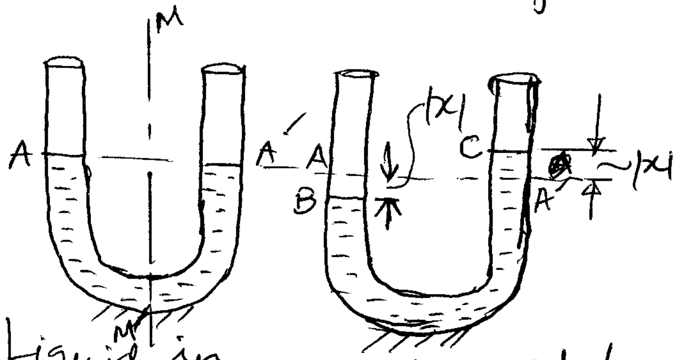
Note that for small θ, $\cos\theta \cong 1$

& $PG \cong r-b$. Then, $I_P = I_C + m(GP)^2 \cong I_C + m(r-b)^2$

& so, $\omega_n = \sqrt{\dfrac{mgb}{I_P}} = \sqrt{\dfrac{mgb}{[I_C + m(r-b)^2]}}$ as before.

Many (most ?) of you must have become extremely bored by now because, apparently, we have spent too much time discussing this problem when it could be solved so quickly by the moment balance method! However, we actually have wasted no time. A detailed, vector approach and the use of Lagrange equation to obtain the nonlinear DEOM is a very good exercise you won't regret!

Example 6 :- We now go over to another interesting problem, that of a liquid column oscillating in a thin U-tube! This problem might occur in some equipments such as a blood pressure measuring equipment (old type) using Hg column and various manometers, using water, say.



Liquid in static equilibrium.

Liquid levels at time t while oscillating

Say, x is +ive upwards.

We have an incompressible liquid in a fixed (say) ~~mass~~ U-tube at level AA. Imagine pushing down the liquid in the left arm using a pushrod of diameter almost same as that of the innertube diameter (just a little bit less) & then removing it quickly. The liquid column starts oscillating. Our aim is to obtain the DEOM for small oscillations & the natural frequency.

Solution :- Let at time t, the liquid level in left arm has gone down by x & so the level in the right arm has gone up by same amount x only. Let us use ~~the~~ Lagrange's equn. with $x = x(t)$ as the generalized coordinate:-

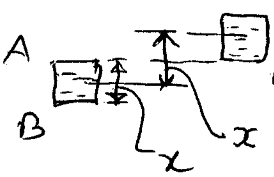
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \quad \text{--- (1)}$$

Let l = total length of liquid column (m)
A_0 = Area of cross-section of tube (not that of tube material) (m^2)
ρ = mass density (Kg/m^3) of liquid.

Then m = total liquid mass $= \rho l A_0$ (Kg)

$$\therefore \quad T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} \rho l A_0 \dot{x}^2 \quad \text{--- (2)}$$

To compute V, note that the liquid column that was AB, has become A'C & the rest of the liquid, as if, has not moved (for this computation only). Hence, its CG is raised by an amount x. It's mass $= \rho A_0 x$. Hence, $V = (\rho A_0 x) g x = \rho A_0 g x^2$ --- (3)



So, from (2), $\frac{\partial T}{\partial \dot{x}} = \rho l A_0 \dot{x}$;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \rho l A_0 \ddot{x}\,; \quad \frac{\partial T}{\partial x} = 0; \quad \frac{\partial V}{\partial x} = 2 \rho A_0 g x$$

Substitution in (1) gives:

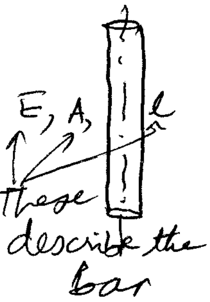
$$\rho l A_0 \ddot{x} + 2 \rho A_0 g x = 0 \quad \leftarrow \text{The required DEOM.}$$

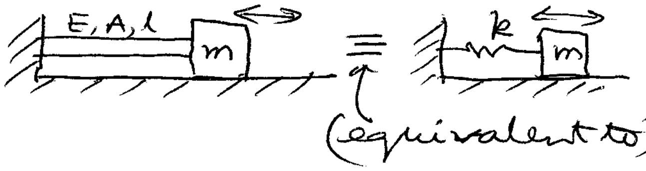
So, a comparison with $m\ddot{x} + kx = 0$ (for which the $\omega_n = \sqrt{\frac{K}{m}}$), we have our $\omega_n = \sqrt{\frac{2 \rho A_0 g}{\rho A_0 l}} = \sqrt{\frac{2g}{l}}$ & we have solved our problem.

Comments:- It would be interesting to obtain the DEOM by force-balance method - Try it!

→ We shall presently talk about various types of springs and their combinations. We shall see how to obtain the equivalent stiffness of such combinations.

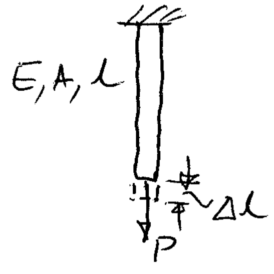
→ A rod of steel or aluminium may act as a spring. Let us consider a uniform bar of length l, cross-sectional area A and Young's modulus E. The question is: What is its stiffness k for axial vibrations? That is, suppose we have an oscillating system like:





What is k?

We find k as follows:- We take the bar & subject it to a load P as shown, Let it extends by Δl.



So, ϵ = axial strain = $\dfrac{\Delta l}{l}$

σ = " stress = $E\epsilon = E\dfrac{\Delta l}{l}$

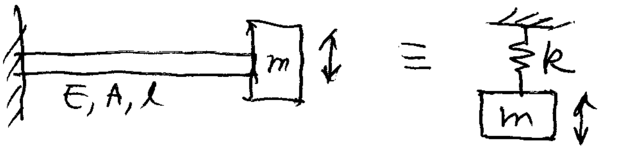
Hence, $P = \sigma A = \dfrac{EA}{l}\Delta l$

~~So, k = force per unit deflection~~

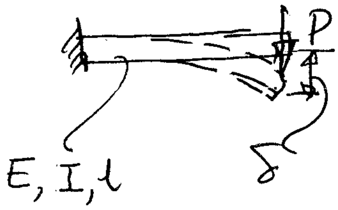
So, Axial stiffness k = Force per unit deflection

or, $k = \dfrac{P}{\Delta l} = \dfrac{EA}{l}$ (Remember)

→ The same bar will have a different stiffness for bending or lateral vibrations:-



What is k?

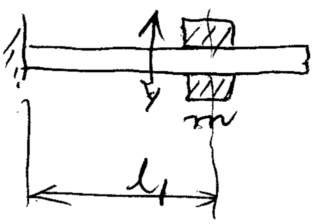


To obtain k, we take the bar as a cantilever beam, apply a load P in transverse direction at the free-end as shown & measure δ. Then, $k = \frac{P}{\delta}$.

Let the bar has length l and its area moment of inertia of a cross section about the neutral axis of the cross-section is I. (I is in m^4, note). Then, we know from elementary strength of materials that

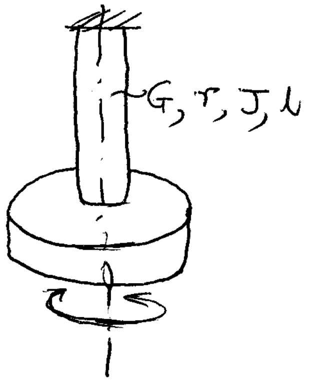
$\delta = \frac{Pl^3}{3EI}$. Hence, $k = \frac{P}{\delta} = \frac{3EI}{l^3}$ (Remember)

→ Note that the same bar can have a different lateral stiffness if the mass is attached at a different point as shown below:



In this case, for finding k, you have to apply the load P at the location of the mass, compute the transverse deflection using analytical or moment-area method & obtain the required k by dividing P by the deflection so obtained. See examples from the textbook.

→ The same bar, assuming it has a circular cross section, could be a part of a torsional vibration set-up:



G = Shear modulus
r = radius
l = length
J = Polar area moment of ~~ax~~ inertia
$= \frac{\pi r^4}{2}$

What is the torsional stiffness of the bar? We fix the bar at the top & apply a torque T at the bottom. Let ϕ be the twist angle at the bottom. Then, required torsional stiffness $= \frac{T}{\phi} = k_t$ (or, 'k' simply)

Now, you know the formula from strength of materials; $\frac{T}{J} = \frac{G\phi}{l}$. Hence $\frac{T}{\phi} = \frac{GJ}{l} = k_t$ (Remember)

~~Hence,~~ ~~$k_t =$~~

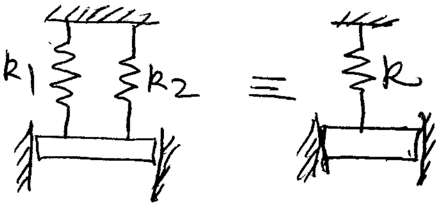
So, remember these 3 formulae & apply ~~them~~ whenever required :-

$\boxed{k = \frac{EA}{l}}$ $\boxed{k = \frac{3EI}{l^3}}$ & $\boxed{GJ/l}$

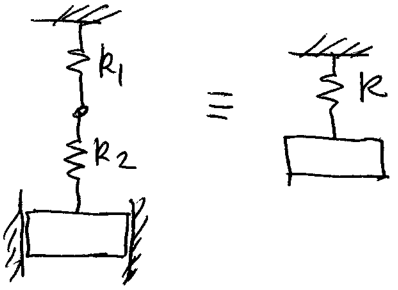
Axial stiffness — Bending stiffness — Torsional stiffness

A spring may also look like  & is called a <u>leaf spring</u> used in carts & automobiles.

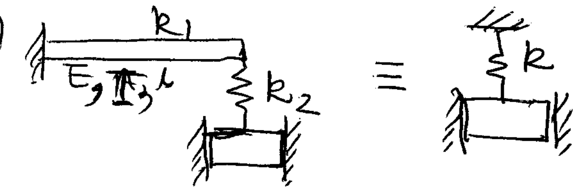
→ We can have series and parallel combinations of springs and you must be capable of finding the equivalent stiffness of such springs. You must have done it before. Here are some examples:

① 

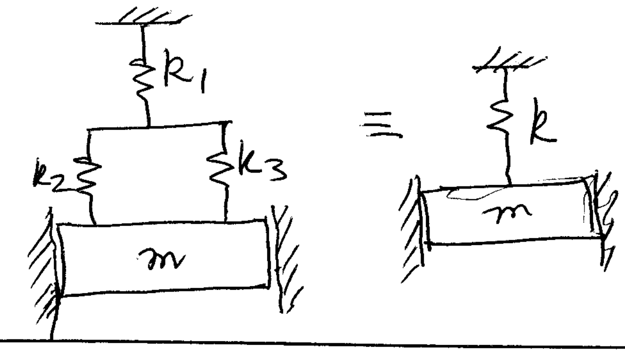
$k = k_1 + k_2$
(Springs in parallel)

② 

$k = \frac{k_1 k_2}{k_1 + k_2}$ (Springs in series)

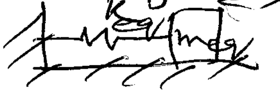
③ 

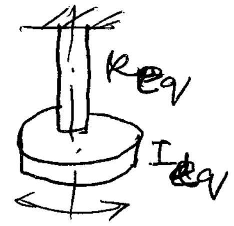
$k = \frac{k_1 k_2}{k_1 + k_2}$ where $k_1 = \frac{3EI}{l^3}$

④ 

Obtain k (HW)
(First combine k_2 & k_3 in parallel & then combine this with k_1 in series)

⑤ The concepts of equivalent mass & stiffness :-

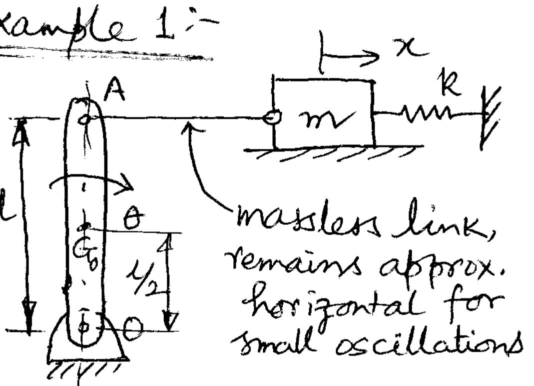
All single degree-of-freedom systems can be reduced to either 

or, 

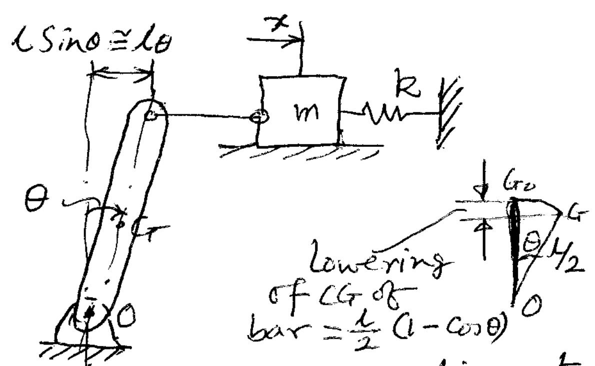
where k_{eq} is the equivalent stiffness; m_{eq} or I_{eq} is the equivalent inertia. This notion of equivalence can simplify analysis in many situations such as vibration analysis of gear trains.

Here we explain it with the help of a few examples.

Example 1:-



System in Static equilibrium



System at time t while oscillating freely

OA is a rigid bar whose moment of inertia about pivotal axis through O is I_0. The bar is vertical in static equilibrium and the spring is at its free length. x denotes the displacement of the block of mass m.

For small θ, clearly, $x \cong l\theta$, so we take $x = l\theta$.

→ Note that the system has only one DOF. We ~~to~~ can take either θ or x as the generalized coordinate.

(i) Let θ be the generalized coordinate chosen.

Then, Kinetic energy of the system = T = Kinetic energy of bar plus that of block

$= \frac{1}{2} I_0 \dot\theta^2 + \frac{1}{2} m \dot x^2$ —(1). But $x = l\theta$. So, $\dot x = l\dot\theta$

& $T = \frac{1}{2} I_0 \dot\theta^2 + \frac{1}{2} m l^2 \dot\theta^2 = \frac{1}{2}(I_0 + ml^2)\dot\theta^2$

$= \frac{1}{2} I_{eq} \dot\theta^2$ and the equivalent (rotary) inertia of the system is $I_{eq} = I_0 + ml^2$.

Also, Potential energy $= V = \frac{1}{2} k x^2 - Mg\frac{l}{2}(1-\cos\theta)$

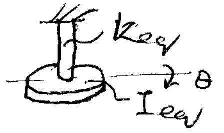
where M is the mass of bar, assumed uniform.

So, $V = \frac{1}{2} k l^2 \theta^2 - \frac{Mgl}{2}(1-\cos\theta)$ [Don't use $\cos\theta \cong 1$ here. Why?

Taking $\cos\theta \cong 1 - \frac{\theta^2}{2!}$ (Taylor's expansion, first two terms) we get $V = \frac{1}{2}kl^2\theta^2 - \frac{Mgl}{2}\cdot\frac{\theta^2}{2} = \frac{1}{2}\left(kl^2 - \frac{Mgl}{2}\right)\theta^2$

$= \frac{1}{2}(k_t)_{eq}\theta^2$, where $(k_t)_{eq}$ = equivalent torsional stiffness $= kl^2 - \frac{Mgl}{2}$

So now our system is equivalent to



& its $\omega_n = \sqrt{\frac{k_{eq}}{I_{eq}}} = \sqrt{\frac{kl^2 - \frac{Mgl}{2}}{I_0 + ml^2}}$ — (I)

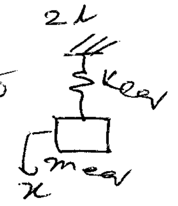
(ii) If we now choose x to be our generalized coordinate, then, $T = \frac{1}{2} I_0 \frac{\dot{x}^2}{l^2} + \frac{1}{2} m\dot{x}^2 = \frac{1}{2}\left(m + \frac{I_0}{l^2}\right)\dot{x}^2$

& so, the equivalent mass $= m_{eq} = m + \frac{I_0}{l^2}$.

Also, $V = \frac{1}{2}kx^2 - \frac{Mgl}{2}\cdot\frac{1}{2}\left(\frac{x}{l}\right)^2 = \frac{1}{2}\left[k - \frac{Mg}{2l}\right]x^2$

& so, equivalent stiffness $= k_{eq} = k - \frac{Mg}{2l}$

Now, the given system is equivalent to



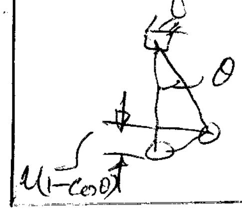
& $\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{k - \frac{Mg}{2l}}{m + I_0/l^2}}$ — (II)

(I) & (II) are the same, you may check.

It is hoped that the above example clearly illustrates the meanings of equivalent inertia & equivalent stiffness and you will be able to solve similar problems with ease.

Question:- Why should we never linearize before differentiating? (Remember this)

Because, if we do so, we loose vital information & land up with wrong result. Take the case of the simple pendulum. If $\cos\theta \cong 1$ is used in $V = mgl(1-\cos\theta)$, we get $V = 0$! However, after differentiation (in Lagrange equation, say), $\cos\theta$ becomes $-\sin\theta$ & then $\sin\theta \cong \theta$ is OK.



END OF VA-I