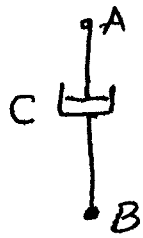
1-DOF damped systems

Every real ~~[illegible]~~ dynamic system has damping. Due to this, energy is dissipated in the form of heat.

We shall include this effect by incorporating a 'damper' into our spring-mass system. This damper will be a linear viscous damper as explained below:—



Schematic of a damper.

Properties of this damper:

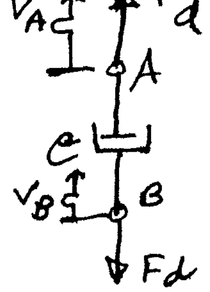
(1) It has negligible mass

(2) It can resist a force only if its two ends A & B have different speeds.

(3) The forces acting at its two ends are equal & opposite at every moment

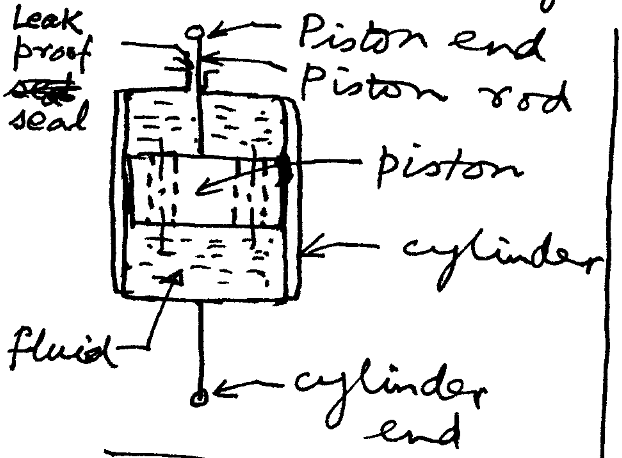
(4) It's constitutive law is:

$F_d = c(v_A \sim v_B)$, where F_d is the force in the damper, v_A & v_B are the velocities of the two ends, c is called the viscous damping constant, $\sim$ denotes 'difference'. The direction of F_d depends on the sign of the difference

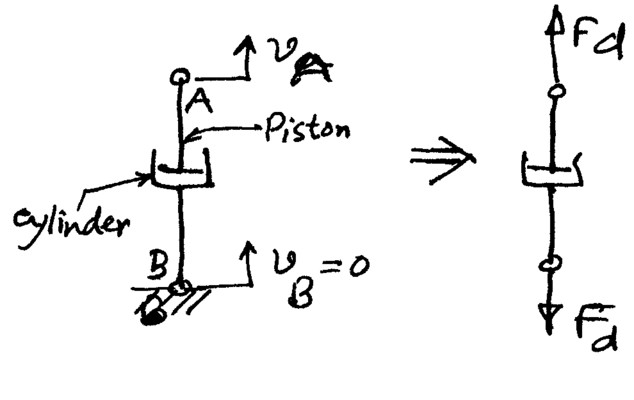


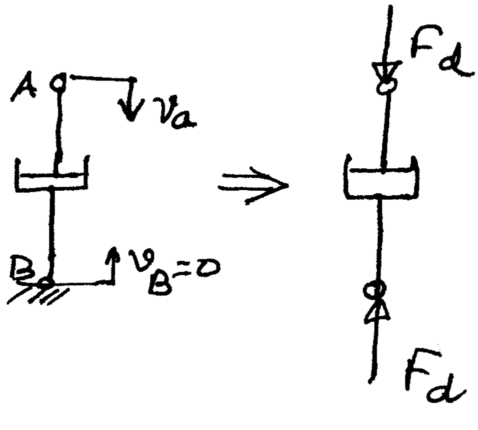
(5) Whatever energy is input to the damper, the whole amount is lost as heat.

→ Try to visualize the damper as follows:-



It has two parts, a cylinder and a piston. There are holes in the cylinder (shown by dashed lines in the figure) and it is partially filled with a viscous fluid. As the piston moves relative to the cylinder, fluid flows through these holes from one end of the cylinder to the other and damping force is provided. The direction of F_d is as indicated below.

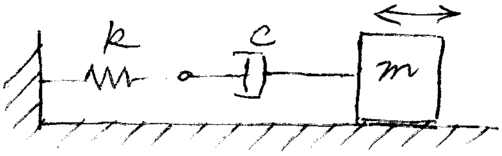




The next question is: How do we incorporate this damper in our spring-mass system?

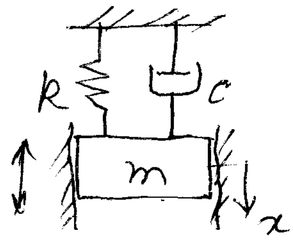
Apparently, there are several possibilities. The spring and the damper can be put in series.

What results is called the <u>Maxwell model</u> :-



Note that this model cannot be in static equilibrium if we make it <u>vertical</u> instead of horizontal as shown above. This is because the damper cannot support a load, say, the weight of the mass, unless it has a velocity difference between its two ends. Think a little and you should get the idea!

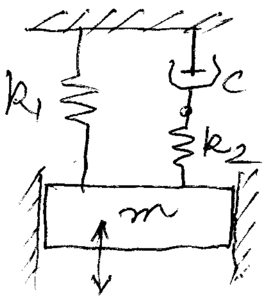
A more useful model for our vibration analysis is the so-called Kelvin-Voigt (K-V) model as shown below:



<u>The K-V model</u>

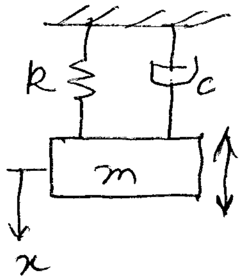
: A material whose viscoelastic properties can be described by coupling a damper with a spring in parallel, is called a Kelvin-Voigt material.

Of course, a ~~gen~~ more general model could be like this:

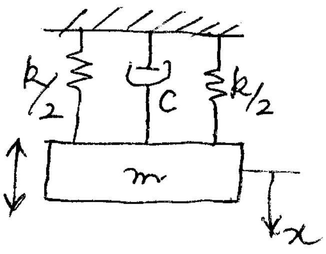


This model is important in vibration isolation where the Objective is to isolate the mass from surrounding vibration & vice-versa.

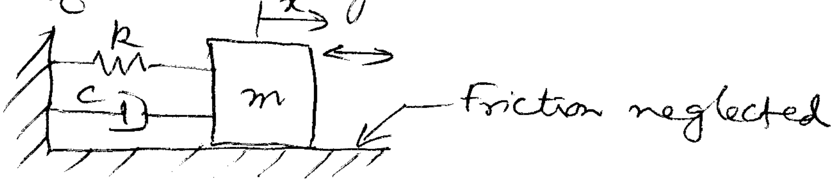
So, we next study the free-vibrational characteristics of the K-V model in



OR



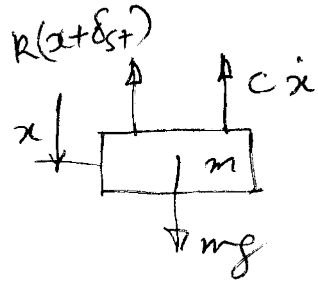
Here also, note that the weight of the mass is, at all times, balanced by the static spring force $k\delta_{st}$ & hence, $V = \frac{1}{2}kx^2$ is valid. This of course doesn't arise if the system moves horizontally:



→ We first obtain the DEOM.

(1) Newton's method (the force-balance method)

<u>The FBD</u>:- (x is measured from static equilibrium position, here positive downward)



Hence,
$$m\ddot{x} = mg - k(x+\delta_{st}) - c\dot{x}$$
$$= \underbrace{(mg - k\delta_{st})}_{\text{zero}} - kx - c\dot{x}$$

or, $$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{--- (1)}$$

which is the required DEOM.

(2) <u>Use of Lagrange's equation</u>

For our damped system free-vibration an additional term is to be added to the Lagrange equation you learned earlier. The new equation is :-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0 \quad — (2)$$

where $D = \frac{1}{2}c\dot{x}^2$ is called the Rayleigh dissipation energy. Learn it mechanically now & just rememb~~s~~ it. Its basis will be discussed much later.

So, here, $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$ as before.

Also, $D = \frac{1}{2}c\dot{x}^2$.

So, $\frac{\partial T}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{\cancel{2}}m \cdot \cancel{2}\dot{x} = m\dot{x}$

& $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$; $\frac{\partial T}{\partial x} = 0$ as before.

$\frac{\partial V}{\partial x} = \frac{1}{2}k\frac{\partial}{\partial x}(x^2) = kx$; $\frac{\partial D}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right) = c\dot{x}$.

Substituting these in (2) we get

$m\ddot{x} + c\dot{x} + kx = 0$, which is the required DEOM.

→ Note that the mechanical energy is <u>not</u> conserved for this system since ~~the~~ there is dissipation. Hence our present system is <u>not</u> conservative. For this non-conservative system, naturally, $\frac{d}{dt}(T+V) = 0$ ~~[illegible]~~ doesn't hold good & so, energy method is not used.

→ Our next aim is to solve $m\ddot{x} + c\dot{x} + kx = 0$. For this, like before, we assume $x = Ae^{st}$

Then, $\dot{x} = Ase^{st}$ & $\ddot{x} = As^2e^{st}$ leads to

$$mAs^2e^{st} + cAse^{st} + kAe^{st} = 0$$

or $$(ms^2 + cs + k)Ae^{st} = 0$$

But Ae^{st} can't be zero & so,

$ms^2 + cs + k = 0$, which is the auxiliary or characteristic equation.

Let its roots be s_1 & s_2.

Then, $s_1 = (-c - \sqrt{c^2 - 4km})/(2m)$

& $s_2 = (-c + \sqrt{c^2 - 4km})/(2m)$

& so, $x = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

Now, $$s_1 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$
$$= \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

where we define a new quantity ζ (Zeta), such that $c/m = 2\zeta\omega_n$ or, $\boxed{\zeta = \frac{c}{2m\omega_n}}$.

You can easily show that ζ is ~~dimen~~ dimensionless. It is called the damping factor for our system.

Then, $s_2 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$

→ Case 1:- $\zeta > 1$ (the case of overdamping)

In this case,

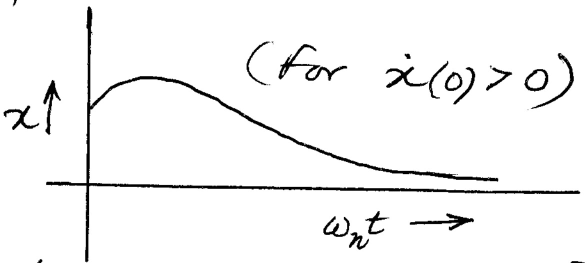
~~$x(t) = A_1 e$~~

(1) --- $$x(t) = A_1 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} + A_2 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

($A_1, A_2 \rightarrow$ Constants of integration) (Remember this formula for problem solving)

& the response curve ~~curve~~

is as follows:~



A mechanical system is seldom overdamped unless it is designed to be so.

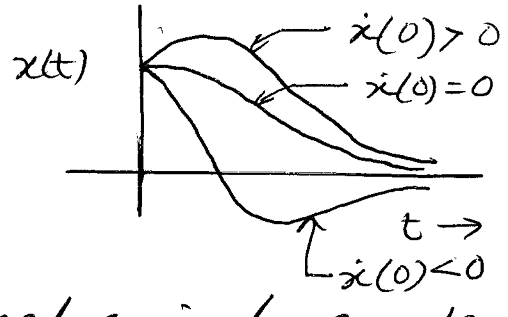
→ Case 2:~ ($\zeta = 1$) The critically damped system.

In this case, $s_1 = s_2 = -\zeta\omega_n$ & from theory of differential equations, we know that the solution $x(t)$ will be of the form:

$$x(t) = (A_1 + A_2 t)e^{-\zeta\omega_n t} \quad \text{--- (2)},$$

another formula to remember.

The response curves would look like:



Some mechanical systems are made critically damped by proper design. For instance, an automatic door closing mechanism is usually critically damped so that after the door is opened or closed and then left to itself, it closes without banging and without too much delay. Analog electrical meters such as an ammeter or voltmeter are also critically damped so that the pointer quickly gives the correct value without oscillations.

→ Case 3:- (The case of underdamping, $\zeta < 1$)

This is the most common case with mechanical systems. Now the response is given as:

$$x(t) = A_1 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} + A_2 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left[A_1 e^{-i\omega_d t} + A_2 e^{+i\omega_d t}\right]$$

where $\omega_d = (\sqrt{1-\zeta^2})\omega_n$ & $i = \sqrt{-1}$.

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the above can be finally expressed as: (Do it, HW)

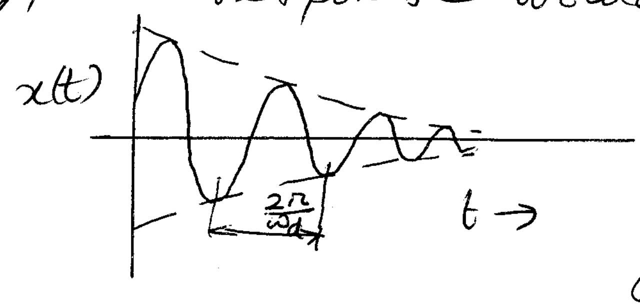
$$x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad \text{--- (3)}$$

where X_0 & ϕ are the constants of integration to be determined from given initial conditions $x(0)$ & $\dot{x}(0)$.

(3) is another important formula to remember.

$\omega_d = (\sqrt{1-\zeta^2})\omega_n$ is called the damped natural frequency.

A typical response would look like:



This is exponentially decaying sinusoidal (harmonic) oscillation.

The period of oscillation is $\frac{2\pi}{\omega_d}$, its frequency being ω_d (rad/s).

→ For solving numerical problems, note that the SI unit for damping constant c is N-s/m ($\because F_d = c\dot{x}$, $c = \frac{F_d}{\dot{x}}$, $\frac{N}{m/s} = \frac{N\text{-}s}{m}$) which is the same as kg/s (Remember).

So, first obtain ζ & depending upto its value, choose a formula:—

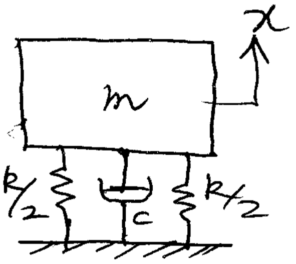
If $\zeta > 1$, $x(t) = A_1 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} + A_2 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$

If $\zeta = 1$, $x(t) = (A_1 + A_2 t) e^{-\zeta\omega_n t}$

If $\zeta < 1$, $x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$ or,
$x(t) = e^{-\zeta\omega_n t}(A \sin\omega_d t + B \cos\omega_d t)$ {A, B → constants of integration}

The constants A_1, A_2 or, X_0, ϕ are to be evaluated using given values of $x(0)$ & $\dot{x}(0)$, ~~ie~~ that is, using the given initial conditions.

Example:— The figure here represents a machine mounted on springs & damper for vibration isolation. If $m = 50$ kg, $k = 5$ kN/m & $c = 150$ N-s/m, obtain the free-vibration response if the mass is given an initial velocity of 20 cm/s. What is the displacement at $t = 0.5$ s?



Solution:— (For avoiding mistakes in numerical computations, it is advisable to convert each system parameter as well as other data in SI units. So, $m = 50$ kg is ok, but $k = 5$ kN/m should be converted to $k = 5\times10^3$ N/m, $c = 150$ N-s/m is ok. However, $\dot{x}(0) = 20$ cm/s should be ~~con~~ converted into $\dot{x}(0) = 20\times10^{-2}$ m/s)

Here $m = 50$ kg, $k = 5$ kN/m $= 5000$ N/m, $c = 150$ N-s/m, $\dot{x}(0) = 20\times10^{-2}$ m/s $= 0.2$ m/s
$x(0) = 0$. (Although x is now positive upward, basically this is no problem, you should realize)

→ The first step is to evaluate ζ.

Remember:- $\zeta = \dfrac{c}{2m\omega_n}$ or, $\zeta = \dfrac{c}{2\sqrt{km}}$

$$\therefore \zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{\frac{k}{m}}} = \frac{c}{2\sqrt{km}}$$

$$= \frac{150}{2\sqrt{5000\times50}} = 0.15 < 1.$$ Hence, we have the case of an underdamped system. So,

Use either of these formulae, it is your choice

$$x(t) = X_0 e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$

or,

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t)$$

Now, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{5000}{50}} = 10$ rad/s

So, $\omega_d = (\sqrt{1-\zeta^2})\omega_n = [\sqrt{1-(0.15)^2}]\times10$ ~~= 9.887 rad/s~~ $= 9.8868$ & $\zeta\omega_n = 0.15\times10 = 1.5$ rad/s.

So, $x(t) = e^{-1.5t}(A\sin\omega_d t + B\cos\omega_d t)$

$x(0) = 0 \Rightarrow 0 = B$. So, $x(t) = Ae^{-1.5t}\sin\omega_d t$

Hence, $\dot{x}(t) = \frac{dx}{dt} = A\left[-1.5e^{-1.5t}\sin\omega_d t + \omega_d e^{-1.5t}\cos\omega_d t\right]$

So, $\dot{x}(0) = A\omega_d$ ~~$\cos\omega_d t$~~ or, $A = \frac{\dot{x}(0)}{\omega_d} = \frac{20\times10^{-2}}{\cancel{0.989}\ 9.8868}\ m$ ~~$= 0.202\ m$~~ $= 0.0202\ m$

So, $x(t) = 0.0202\,e^{-1.5t}\sin(0.989t)\ m$ is the required response.

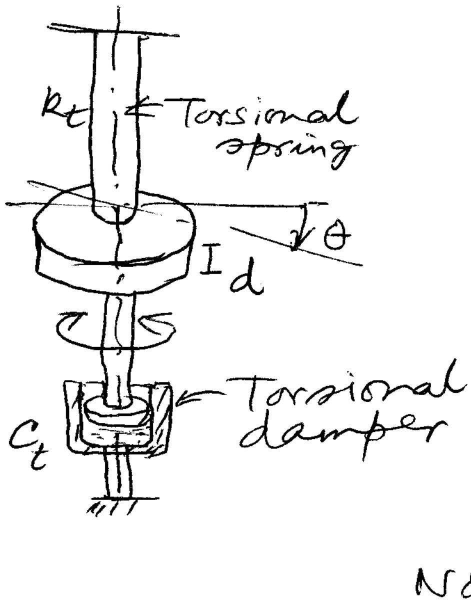
Also, at $t = 0.5\ s$,

$x(0.5) = 0.0202\,e^{(-1.5\times0.5)}\sin(0.989\times0.5)\ m$ In radian. Be careful

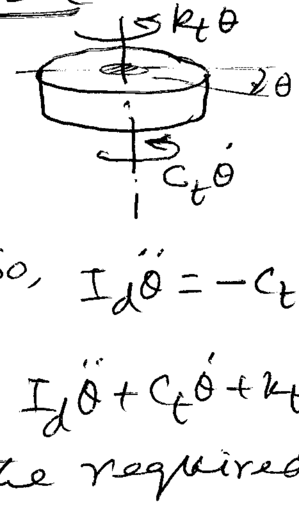
~~$= 0.0453\ m$~~ $= 0.00453\ m$ Ans.

---

(8) Free-vibration of a damped torsional system:-



FBD ~~of~~ Disc



So, $I_d\ddot{\theta} = -c_t\dot{\theta} - k_t\theta$

or, $I_d\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$. This is the required DEOM.

Note that SI unit of c_t is $\frac{N\text{-}m}{rad/s}$ or, N-m-s. The constitutive equation for the torsional damper is:

$\tau_d = c_t\dot{\theta}$ where τ_d is the torque in damper.

Hence, c_t is the damping torque per unit relative angular velocity between piston & cylinder like relative velocity between two ends of a damper for linear motion.

So, if τ_d is in N-m & $\dot{\theta}$ is in rad/s,

c_t will be in $\frac{N-m}{rad/s}$ or N-m-s as stated earlier.

~~→ for simplicity, we shall replace k_t by k & c_t by c from now on~~

Now, we have ω_d & ζ for torsional damped oscillations too. To find an expression for ζ in this case, just compare the two DEOM:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{—} (1)$$

$$\& \quad I_d\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \quad \text{—} (2)$$

For (1), $\zeta = \frac{c}{2\sqrt{km}}$. So, for (2), $\zeta = \frac{c_t}{2\sqrt{I_d k_t}}$

You should check that this ζ is also ~~non dimensional~~, dimensionless.

Also, for $\zeta > 1$, the torsional system is overdamped & $\theta(t) = A_1 e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t}$

as before. [$\omega_n = \sqrt{\frac{k_t}{I_d}}$

For $\zeta = 1$, the system is critically damped

& $\theta(t) = (A_1 + A_2 t)e^{-\zeta\omega_n t}$

For $\zeta < 1$, the system is underdamped

& $\theta(t) = A_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$

OR

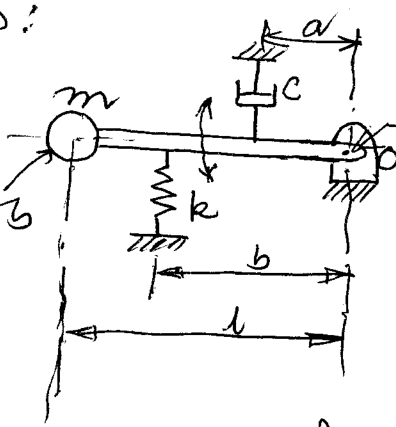
$$\theta(t) = e^{-\zeta\omega_n t}\left[A \sin(\omega_d t) + B \cos(\omega_d t)\right].$$

Note:- In some numerical problems, there maynot be any torsional springs & dampers but only ~~linear~~ linear springs & dampers with an angle (say, θ) as the generalized coordinate, the DEOM would be of the form

$$\alpha\ddot{\theta}+\beta\dot{\theta}+\gamma\theta=0 \quad (\alpha, \beta, \gamma \text{ being constants}).$$

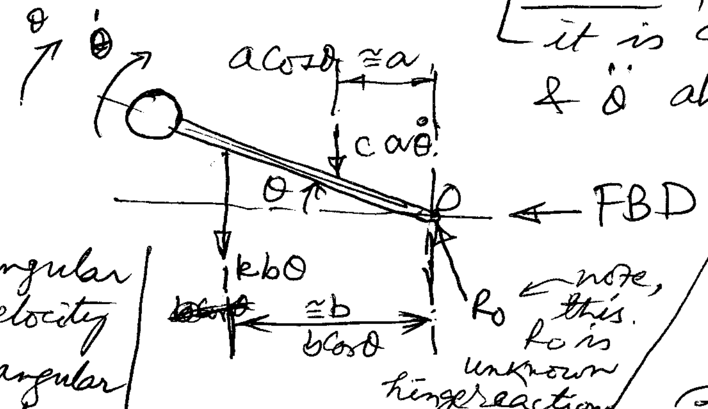
For ζ, $\theta(t)$ etc., you would follow same procedure just by comparing this DEOM with $I_d\ddot{\theta}+c_t\dot{\theta}+k_t\theta=0$.

An example of such a system is shown below:



Neglecting mass of the bar, obtain the DEOM of the system, ~~shown~~ shown in the figure, for small oscillations. In equilibrium, the bar is horizontal. Take θ as the generalized coordinate where $\theta(t)$ is the inclination of the bar with the horizontal, taken positive clockwise.

[Note:- If θ is +ive clockwise, it is customary to take $\dot{\theta}$ & $\ddot{\theta}$ also +ive CW]



$\dot{\theta}$ → angular velocity
$\ddot{\theta}$ → angular acceleration

To avoid confusion regarding weight of the bob and static deflection of spring, unless stated otherwise, you may assume that the motion takes place in a horizontal plane. The pin at O is vertical.

[Homework - If the motion is in a vertical plane, show that the moment of the weight of bob about O keeps balancing the moment of the static force in the spring about O at all times. Hence you need not consider these in the formulation]

(1) **Moment Balance Method:-**

$I_0\ddot{\theta}$ = Sum of moments of the spring & damper forces about O, CW moments being taken as positive. I_0 is the moment of inertia of the (bar+bob) system about the axis of rotation at O. Since bar inertia is neglected, $I_0 = ml^2$, simply. If bar mass is considered, you have to compute its moment of inertia about the same axis of rotation & add ~~that~~ to ml^2. If the bob size is small and the bar is pivoted very near its right end, then $(I_0)_{bar} = \frac{1}{3}Ml^2$, where M is the mass of the bar, assumed to be of uniform cross-section.]

So, $ml^2\ddot{\theta} = -kb\theta\times b - ca\dot{\theta}\times a$

or, $ml^2\ddot{\theta} + ca^2\dot{\theta} + kb^2\theta = 0$ — (1), which is the required DEOM.

Comparing with $m\ddot{x}+c\dot{x}+kx=0$, we can say that $\zeta = \frac{ca^2}{2\sqrt{kb^2\times ml^2}} = \frac{ca^2}{2bl\sqrt{km}}$. Check that this expression is dimensionless. For given values* ~~After this~~ After this, depending on the numerical value of ζ, ($<$, $>$ or $=1$), write the response $\theta(t)$.

2 Use of Lagrange's equation:–

The Lagrange equation in this case is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} + \frac{\partial D}{\partial \dot\theta} = 0 \quad \text{--- (1)}$$

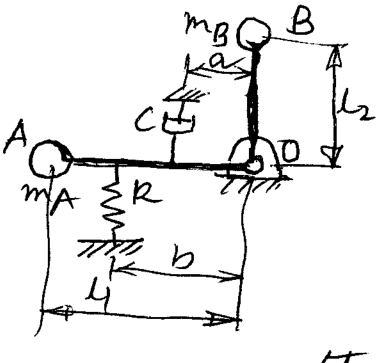
$$T = \tfrac{1}{2} m (l\dot\theta)^2, \quad V = \tfrac{1}{2} k (b\theta)^2, \quad D = \tfrac{1}{2} c (a\dot\theta)^2$$

So, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = ml^2\ddot\theta$, $\frac{\partial T}{\partial \theta} = 0$, $\frac{\partial V}{\partial \theta} = kb^2\theta$,

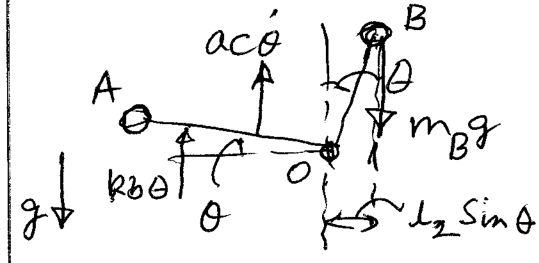
$\frac{\partial D}{\partial \dot\theta} = ca^2\dot\theta$ & substitution in (1) gives

$ml^2\ddot\theta + ca^2\dot\theta + kb^2\theta = 0$ as the required DEOM, as before.

→ A word of caution :– For some systems oscillating in a vertical plane, you should be careful to check the effect of weights, their moments about axis of rotation, static forces in springs & their moments as may be appropriate. Study the following example carefully.



Let the rigid bars OB & OA have negligible mass. The bobs are heavy with masses m_A & m_B respectively. If the motion occurs in a horizontal plane, we need not worry about the effects of weights, static spring forces and their moments about O. The spring would have free length in equilibrium with zero force. However, suppose motion occurs in the vertical plane & OA is horizontal but OB vertical at static equilibrium. In this case, the moment of the weight of A about O would be balanced by the ~~force in~~ moment of the force in spring at equilibrium. Note that the spring would be ~~compres~~ compressed by δ_{st} so that it applies an upward force of $k\delta_{st}$ on bar OA. So, in equilibrium, $mgl_1 = k\delta_{st} b_1$ & this relation would stay valid all along during small oscillations. However, the weight of bob B doesn't contribute to this balance. Hence, as links OA & OB rotate by an amount θ, the moment of the weight of B about O would come into the picture and contribute to the DEOM etc.



Complete the solution of this problem. Refer to the adjoining figure. Weight of A & static spring force are not shown, note.

However, in problem, if you are in doubt, do the complete analysis by taking all weights & static spring forces etc. into account, use the equilibrium force/moment relations, eliminate some of these & get the correct result.

8 The concept of <u>Logarithmic Decrement</u> & <u>its usage</u>:~

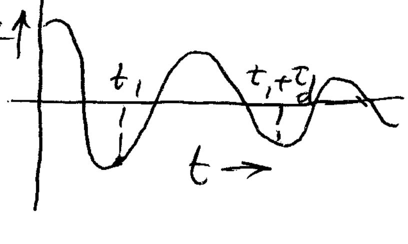
Suppose we have an Single DOF system & it is underdamped. (How do know a priori that it is underdamped? Give the system some initial displacement or velocity & see if it oscillates a few times. If it so does, it is underdamped)

Using proper vibration measuring equipment, we can measure ω_d, the frequency of damped vibration. So, $\tau_d = \frac{2\pi}{\omega_d}$, which is the period of such oscillations, can also be measured. Suppose now we measure the response x at times t_1 & $t_1+\tau_d$.

Then, $x(t_1) = X_0 e^{-\zeta\omega_n t_1} \sin(\omega_d t_1+\phi)$

& $x(t_1+\tau_d) = X_0 e^{-\zeta\omega_n(t_1+\tau_d)} \sin[\omega_d(t_1+\tau_d)+\phi]$

$= X_0 e^{-\zeta\omega_n t_1} . e^{-\zeta\omega_n \tau_d} \sin(\omega_d t+\phi)$ (Since $\omega_d(t+\tau_d) = \omega_d + 2\pi$ & $\sin(2\pi+\theta) = \sin\theta$ etc.)



So, $\dfrac{x(t_1)}{x(t_1+\tau_d)} = e^{+\zeta\omega_n\tau_d}$

& $\ln\left[\dfrac{x(t_1)}{x(t_1+\tau_d)}\right] = \zeta\cancel{\omega_n} \times \dfrac{2\pi}{\sqrt{1-\zeta^2}\,\cancel{\omega_n}}$ [$\because \tau_d = \frac{2\pi}{\omega_d}$ & $\omega_d = \omega_n\sqrt{1-\zeta^2}$]

The quantity on the left side of above relation is called the <u>Logarithmic decrement</u> usually ~~described by~~ denoted by 'δ' in textbooks. Thus, $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ (Remember).

So, solving for ζ, we get: $\zeta = \dfrac{\delta}{\sqrt{\delta^2+4\pi^2}}$ (Remember)

So, after obtaining δ experimentally, we can find ζ using above formula.

Hence, the <u>Logarithmic decrement</u> gives us an experimental value of ζ which is difficult to obtain otherwise.

→ Often, the ratio $\frac{x(t_1)}{x(t_1+\tau_d)}$ is difficult to measure accurately. So, what is done to improve accuracy of measurement is to measure $\frac{x(t_1)}{x(t_1+n\tau_d)}$, where n is a positive integer. $n=5$ is a reasonable value, say.

Now, $\ln\left[\frac{x(t_1)}{x(t_1+n\tau_d)}\right] = \zeta\omega_n n\tau_d$ (~~check this~~) $= n\delta$

or, $\delta = \frac{1}{n}\ln\left[\frac{x(t_1)}{x(t_1+n\tau_d)}\right]$.

For example, if the amplitude of free vibration decreases by 75% after 5 cycles, then $n=5$,

$$x(t_1+n\tau_d) = 25\% \text{ of } x(t_1)$$

or, $$\frac{x(t_1)}{x(t_1+n\tau_d)} = 1/\left(\frac{25}{100}\right) = 4$$

So, $$\ln\left[\frac{x(t_1)}{x(t_1+n\tau_d)}\right] = \ln 4$$

or, $$\delta = \frac{1}{n}\ln\left[\frac{x(t_1)}{x(t_1+n\tau_d)}\right] = \frac{1}{5}\ln 4 = 0.277$$

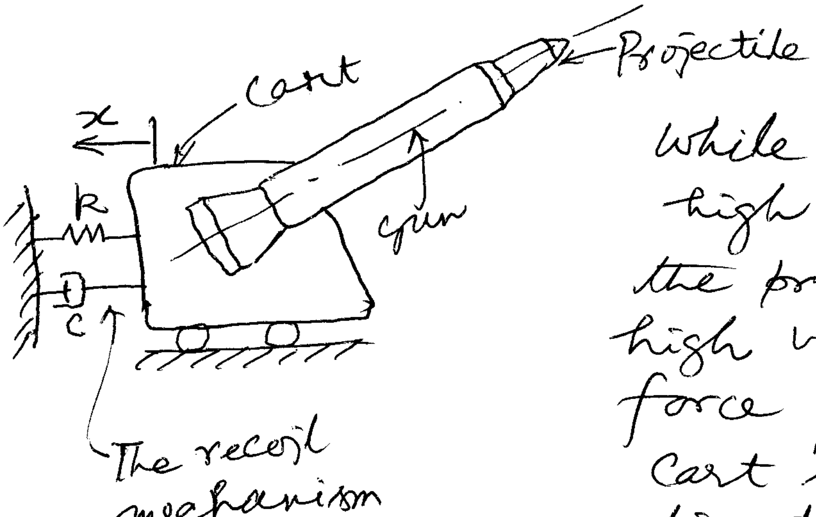
So, $$\zeta = \frac{\delta}{\sqrt{\delta^2+4\pi^2}} = \frac{0.277}{\sqrt{(0.277)^2+4\pi^2}} = 0.044.$$

→ If $\zeta \ll 1$, then, from $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, we have

$\delta \cong 2\pi\zeta$ or, $\zeta = \frac{\delta}{2\pi}$; note.

We next discuss several important problems.

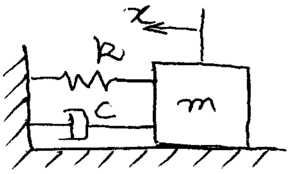
(1) The Problem of Recoil of a Gun (Cannon):



While the gun is fired, high pressure gas pushes the projectile with a very high velocity. The reaction force pushes the gun & cart in the opposite direction. It is desirable to bring the gun to rest as soon as possible for firing the next projectile. So, the recoil mechanism is critically damped.

In a particular case, the mass of the set-up is 1000 kg. $k = 20$ kN/m. After a firing, the amount of horizontal recoil is 0.35 m. (The vertical changes in reaction etc. are taken care of by an internal vibration absorber, not shown). Note that the firing ~~causes~~ causes only an appreciable initial velocity & negligible initial displacement & so, we take $x(0) = 0$. We have to find c, the damping constant, the initial recoil velocity $\dot{x}(0)$ and the time the gun takes to recoil 0.2 m.

Solution: We consider the following model:



$m = 1000\,kg$, $k = 20{,}000\ N/m$

For critical damping, $\zeta = 1 \Rightarrow \frac{c}{2\sqrt{km}} = 1$

$\Rightarrow c = 2\sqrt{20\times10^3 \times 10^3} = 8944.27\ N\text{-}s/m$ (check)

2nd part:- Since we have critical damping,

$x(t) = (A_1 + A_2 t)e^{-\omega_n t}$. So, $x(0) = 0 \Rightarrow 0 = A_1$

Hence, $x(t) = A_2 t e^{-\omega_n t}$ -- ①, $\dot{x}(t) = A_2 e^{-\omega_n t} - A_2\omega_n t e^{-\omega_n t}$ -- ②

So, $\dot{x}(0) = A_2$ & we have, from ①,

$x(t) = \dot{x}(0)\, t e^{-\omega_n t}$ -- ③

To find $\dot{x}(0)$, we use the fact that when

$x = x_{maxm} = 0.35\,m$, we have $\dot{x}(t_{max}) = 0$.

~~Let this happens at $t = t_{max}$. ... from ② with $A_2 = \dot{x}(0)$, $\dot{x}(t_{max}) = 0$~~

Using ② & the fact that $A_2 = \dot{x}(0)$, we have,

$$\dot{x}(0)e^{-\omega_n t_{max}}\left[1 - \omega_n t_{max}\right] = 0 \Rightarrow 1 - \omega_n t_{max} = 0 \quad \text{-- ④}$$

or, $t_{max} = \frac{1}{\omega_n} = \sqrt{\frac{m}{k}} = \sqrt{\frac{1000}{20{,}000}} = 0.224\,s$

At this time, $x = 0.35\,m$.

So, from ①, $0.35 = \dot{x}(0) \times 0.224 \times e^{-1}$ $\{\because -\omega_n t_{max} = -1$, from (4)

$\Rightarrow \dot{x}(0) = \underline{4.2473\ m/s}$.

3rd part:- Let $t = t_1$ when $x = 0.2\,m$.

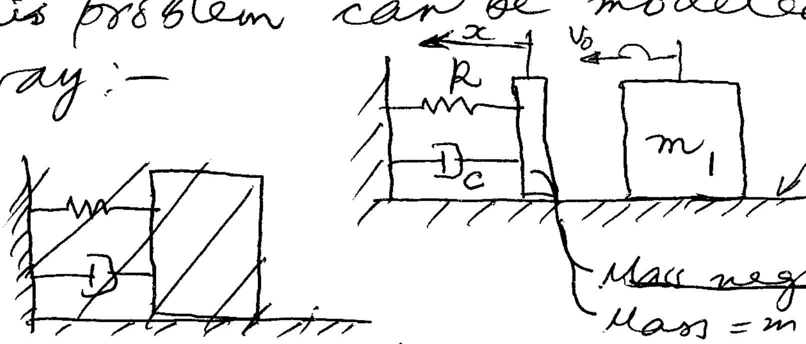
So, using ③, we have $0.2 = 4.2473 t_1 e^{-\sqrt{\frac{k}{m}} t_1}$

or, $t_1 e^{-4.4721 t_1} = 0.0471$.

Solve this equation numerically to get t_1

② The problem of a railway carriage or engine hitting a buffer:-

This problem can be modeled in the following way:-



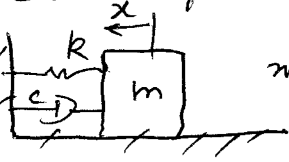
Friction neglected

~~Mass neglected (for now)~~

Mass = m_2

A mass m_1 moving with velocity v_0, strikes a plank of ~~negligible~~ mass, m_2 which is supported by a spring & damper as shown. With m_1, m_2 v_0, k & c given, find the maximum distance the mass plus plank move before coming to rest momentarily.

To solve this problem, assume completely inelastic collision ~~impact~~ & apply conservation of linear momentum to obtain the velocity v_f immediately after impact. This corresponds to our $t=0$. After this, the system model becomes



$m = m_1 + m_2$

with $x(0) = 0$, $\dot{x}(0) = v_f$

$\left[(m_1+m_2)v_f = m_1 v_0 \text{ etc.}\right]$
The masses move together after impact

→ First obtain ζ & depending on whether $\zeta >, =$ or < 1, use the appropriate formula for $x(t)$. For instance, if $\zeta < 1$,

$$x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \text{ or}$$

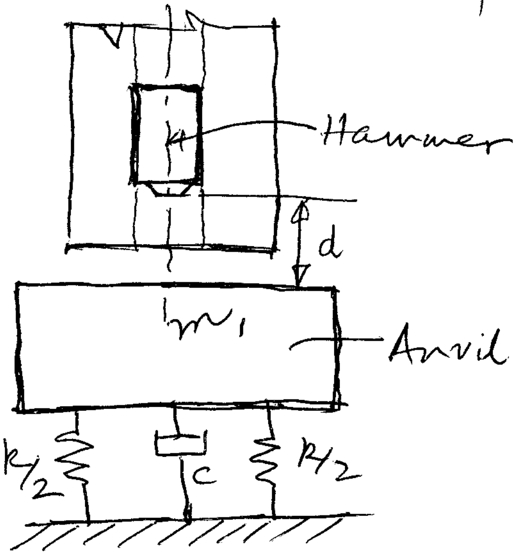
$$x(t) = e^{-\zeta\omega_n t}(A \sin\omega_d t + B \cos\omega_d t),$$ you can use either, as per your preference.

Obtain A & B using $x(0)$ & $\dot{x}(0)$ values.

Then set $\dot{x}(t)$ to zero to find t_{max} at which

$x(t)$ is maximum. Find $x(t_{max})$. You will find ~~two~~ tutorial problems related to this. Solve those problems.

Another similar problem could be this:-



See fig. This is a schematic of a drop forge set-up. Numerical values of m_1, m_2, d, k & c would be given. The hammer falls through distance 'd' from rest onto the anvil. The coefficient of restitution ϵ would also be given.

$\left(\epsilon = -\frac{\text{(velocity of separation)}}{\text{(" " approach)}}\right.$, in case you have forgotten. We're having a case of imperfectly elastic collision) ~~in which~~

You have to determine (i) the velocity of anvil just after collision, (ii) the maximum displacement of anvil.

(2) ~~Task~~ Home work:- Solve the above problem by taking $m_1 = 1450$ kg, $m_2 = 300$ kg, $d = 0.5$ m, $k = 0.5$ MN/m, $c = 30$ kN-s/m, $\epsilon = 0.4$

(3) In another category of problems, you are asked to identify the system characteristics from the given response.

<u>HomeWork problem</u>:- The response of a single DOF system that was subject to an initial displacement only (i.e., $\dot{x}(0)=0$) was seen to be given by:

$$x(t) = 0.03e^{-5t}\sin(4.5t+1.4)\ m.$$

<u>Obtain</u> ζ, ω_n, ω_d & $x(0)$.

[Hint: Compare with standard form:

$$x(t) = X_0 e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)]$$

→ <u>A few important things to note:~</u>

① Thus far, We have always taken the damping to be linearly viscous. You may wonder why? In real world, the damping mechanism is seldom linearly viscous. For instance, we encounter coulomb damping or dry friction quite often. Fluid friction could be velocity squared or, velocity raised to another power. We also always have material or hysteresis damping due to internal friction in a material. So, why almost always a linear viscous damper? The reason is that we can introduce the notion of an <u>equivalent viscous damping constant</u> for all these cases & use the linear model. This is done by considering energy dissipation per cycle of motion under a standard forcing function & this study will be taken up later.

② With these two chapters, we have covered approximately 200 pages of Prof. S. S. Rao's textbook. There are numerous practical and important problems given in that fantastic book, both ~~as~~ an exercise as well as worked out examples. Take a look at these. You should ~~s~~ solve the worked out examples yourselves.

③ Solved the ~~p~~ problems given in the tutorial-cum-homework problem sheets, as far as you can.

END OF VA-2