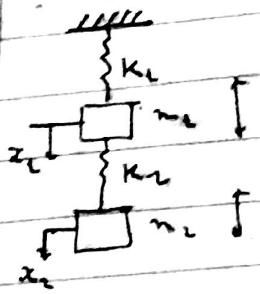
Two DOF systems

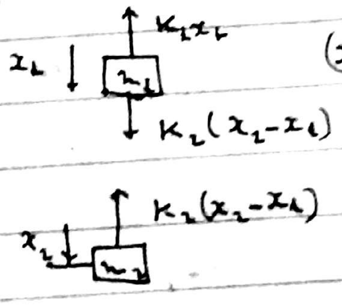
Undamped free vibrations of 2 DOF sys

$x_1, x_2 \rightarrow$ A set of generalized coordinates measured from static eqm position.



Derivation of DEOM

(1) Newton's Method



$(x_2 > x_1)$

So $m_1\ddot{x}_1 = -K_1x_1 + K_2(x_2-x_1)$

$\Rightarrow m_1\ddot{x}_1 + (K_2+K_1)x_1 - K_2x_2 = 0$ — ①

$m_2\ddot{x}_2 = -K_2(x_2-x_1)$

$\Rightarrow m_2\ddot{x}_2 + K_2x_2 - K_2x_1 = 0$ — ②

The reqd diff eqn of motion

① and ② can be put together in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (K_1+K_2) & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad — ③$$

$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is the mass / inertia matrix

$[K] = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$ is the stiffness or elastic matrix

$\{x\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$ is the displacement matrix

$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$ " " acceleration matrix

② Use of the Lagrange eqns

Here we have ② Lagrange eqns

These are :-

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0 \quad \text{——} ⓐ$$

$q_1 = x_1$

$q_2 = x_2$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0 \quad \text{——} ⓑ$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} = 0$$ for free vibration

a conservative system $j = 1, 2, \ldots n$

$n =$ no. of DOF

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0 \qquad \frac{\partial U}{\partial x_1} = K_1 x_1 - K_2 (x_2 - x_1)$$

— substitute in ⓐ to get ①.

$$\text{Similarly} \quad \frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = m_2 \ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0 \qquad \frac{\partial U}{\partial x_2} = K_2 (x_2 - x_1)$$

— Substitute in ⓑ to get ②

<u>The DEOM are coupled, also note this</u>

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So, we can't solve for $x_1(t)$ and $x_2(t)$ independently.

Here we take a heuristic approach; with our experience for the ~~simple~~ single DOF spring system, we try the following solution :-

$$x_1 = A_1 \sin(\omega t + \phi) \quad (4)$$

If (4) is true, then x_2 can only take the form

$$x_2 = A_2 \sin(\omega t + \phi) \quad (5)$$

$$\ddot{x}_1 = -A_1 \omega^2 \sin(\omega t + \phi) \quad (6)$$

Substitute (5) and (6) in (1)

$$x_2 = \frac{m_1 \ddot{x}_1}{K_2} + \frac{(K_1 + K_2) x_1}{K_2}$$

$$= \underbrace{\left[\frac{-m_1 \omega^2}{K_2} + \frac{(K_1 + K_2)}{K_2} \right]}_{A_2} A_1 \sin(\omega t + \phi)$$

The DEOM are :-

$$m_1 \ddot{x}_1 + (K_1 + K_2) x_1 - K_2 x_2 = 0 \quad (1)$$

$$m_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0 \quad (2)$$

Let $x_1 = A_1 \sin(\omega t + \phi)$ — (4) $\qquad \ddot{x}_1 = -A_1 \omega^2 \sin(\omega t + \phi)$

$x_2 = A_2 \sin(\omega t + \phi)$ — (5) $\qquad \ddot{x}_2 = -A_2 \omega^2 \sin(\omega t + \phi)$ (7)

$A_1, A_2, \omega, \phi \rightarrow$ to be determined

$$-m_1 A_1 \omega^2 \sin(\omega t + \phi) + (K_1 + K_2) A_1 \sin(\omega t + \phi) - K_2 A_2 \sin(\omega t + \phi) = 0$$

$$\Rightarrow (K_1 + K_2) A_1 - m_1 A_1 \omega^2 - K_2 A_2 = 0 \quad (8)$$

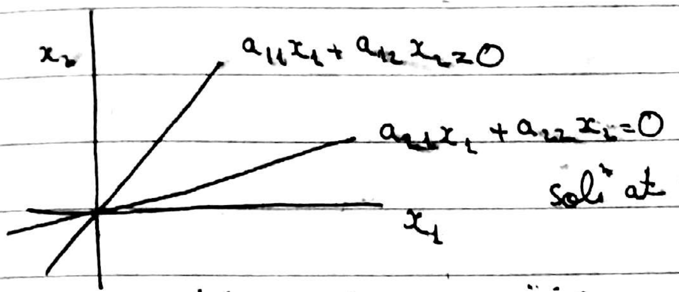
$$-K_2 A_1 + (K_2 - m_2 \omega^2) A_2 = 0 \quad (9)$$

since $\sin(\omega t + \phi) \neq 0$ at all times

We have to solve ⑧ and ⑨ for A_1 & A_2

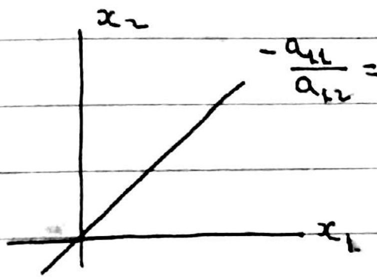
$(K_1+K_2-m_1\omega^2)A_1 - K_2A_2 = 0$ —— ⑧

$-K_2A_1 + (K_2-m_2\omega^2)A_2 = 0$ —— ⑨



soln at $x_1=0$ $x_2=0$

We want non-trivial solutions, thus



$-\frac{a_{11}}{a_{12}} = -\frac{a_{21}}{a_{22}}$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

$$\begin{vmatrix} (K_1+K_2-m_1\omega^2) & -K_2 \\ -K_2 & (K_2-m_2\omega^2) \end{vmatrix} = 0 \quad \text{—— ⑩}$$

which is called characteristic eqn. or frequency eqn. of our system

Let $m_1 = m_2 = m$

$K_1 = K_2 = K$

→ Obtain the frequency eqn.

→ " " natural frequencies

→ Obtain A_1 & A_2

The freq. eqn. is

$$\begin{vmatrix} 2K-m\omega^2 & -K \\ -K & K-m\omega^2 \end{vmatrix} = 0 \Rightarrow 2K^2-3Km\omega^2+m^2\omega^4-K^2=0$$

$$\Rightarrow m^2\omega^4 - 3Km\omega^2 + K^2 = 0$$

$$\Rightarrow \omega_1^2, \omega_2^2 = \frac{3km \pm \sqrt{5k^2m^2}}{2m^2}$$

$$\omega_1^2 = \frac{(3-\sqrt{5})}{2}\frac{k}{m} \Rightarrow \omega_1 = 0.618\sqrt{\frac{k}{m}}$$

$$\omega_2^2 = \frac{(3+\sqrt{5})}{2}\frac{k}{m} \Rightarrow \omega_2 = 1.618\sqrt{\frac{k}{m}}$$

Always designate the smaller of the nat. frequency as ω_1 and the bigger one as ω_2

The amplitudes A_1 & A_2 are obtained from

$$\left.\begin{aligned}(k_1+k_2-m_1\omega^2)A_1 - k_2 A_2 = 0\\ -k_2A_1 + (k_2 - m_2\omega^2)A_2 = 0\end{aligned}\right] \rightarrow$$

$$(2k - m\omega^2)A_1 - kA_2 = 0$$

$$-kA_1 + (k - m\omega^2)A_2 = 0$$

For $\omega = \omega_1$, $A_1 \rightarrow A_{11}$, $A_2 \rightarrow A_{21}$

A_{11}: the amplitude of m_1 for first principle mode of vibration

A_{21}: Amplitude of m_2 for first principal mode of vibration

For $\omega = \omega_1$, the free vibration response is

$$\left.\begin{aligned}x_1 &= A_{11}\sin(\omega_1 t + \phi_1)\\ x_2 &= A_{21}\sin(\omega_1 t + \phi_1)\end{aligned}\right\}$$ These represent the first principal or normal mode of vibration

For $\omega = \omega_2$

$$\left.\begin{aligned}x_1 &= A_{12}\sin(\omega_2 t + \phi_2)\\ x_2 &= A_{22}\sin(\omega_2 t + \phi_2)\end{aligned}\right\} \rightarrow$$ 2nd principal / normal mode of vibration

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$$(2k - m\omega_1^2) A_{11} - k A_{21} = 0 \quad \text{— (11)}$$

$$-k A_{11} + (k - m\omega_1^2) A_{21} = 0 \quad \text{— (12)}$$

$$\frac{A_{21}}{A_{11}} = \frac{2k - m\omega_1^2}{k}$$

$$\frac{A_{21}}{A_{11}} = \frac{k}{(k - m\omega_1^2)}$$

$$\frac{A_{21}}{A_{11}} = \frac{2k - (0.618)^2 k}{k} = 1.618 = \mu_1 = $$ Amplitude ratio case of 1st principal mode

$$\frac{A_{22}}{A_{12}} = \frac{2k - (1.618)^2 k}{k} = -0.618 = \mu_2 = $$ amplitude ratio case of 2nd principal mode

Here the first pr. mode is given by

$$x_1(t) = A_{11} \sin(\omega_1 t + \phi_1)$$

$$\& \; x_2(t) = 1.168 A_{11} \sin(\omega_1 t + \phi_1)$$

2nd principal mode is given by

$$x_1(t) = A_{12} \sin(\omega_2 t + \phi_2)$$

$$x_2(t) = \mu_2 A_{12} \sin(\omega_2 t + \phi_2)$$

$$= -0.168 A_{12} \sin(\omega_2 t + \phi_2)$$

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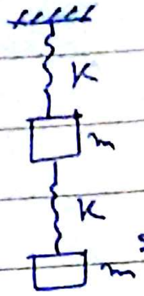


The principal modes of vibration are:

These satisfy the DEOM
$$\left.\begin{cases} x_1(t) = A_{11}\sin(\omega_1 t + \phi_1) \\ x_2(t) = \mu_1 A_{11}\sin(\omega_1 t + \phi_1) \end{cases}\right\} \text{1st principal mode}$$

satisfy the DEOM
$$\left.\begin{cases} x_1(t) = A_{12}\sin(\omega_2 t + \phi_2) \\ x_2(t) = \mu_2 A_{12}\sin(\omega_2 t + \phi_2) \end{cases}\right\} \text{2nd principal mode}$$

$$\left[\mu_1 = \frac{A_{21}}{A_{11}} \quad \mu_2 = \frac{A_{22}}{A_{12}}\right]$$

Note that $x_1(t)$ for first principal mode is linearly independent of $x_1(t)$ for 2nd principal mode

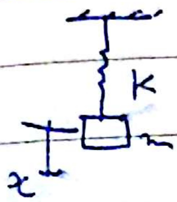
[This is so because
$$\frac{A_{11}\sin(\omega_1 t + \phi_1)}{A_{12}\sin(\omega_2 t + \phi_2)}$$ isn't a constant]

This ~~means~~ means that the two principal modes represent two linearly independent solutions of our DEOM. By the theory of differential eqns., the general solution (responses) of the masses would be the superposition of the principal modes.

Hence, the general free-vibration response of the system is given by
$$x_1(t) = A_{11}\sin(\omega_1 t + \phi_1) + A_{12}\sin(\omega_2 t + \phi_2)$$
$$\& \quad x_2(t) = \mu_1 A_{11}\sin(\omega_1 t + \phi_1) + \mu_2 A_{12}\sin(\omega_2 t + \phi_2)$$

Remember

The arbitrary constants of integration are $\phi_1, \phi_2, A_{11}, A_{12}$. These are to be determined for a given motion by using given initial conditions $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$ & $\dot{x}_2(0)$.



$$m\ddot{x} + Kx = 0$$

$$x_1 = A \sin \omega_n t \qquad x_1 + x_2 \rightarrow \text{gen sol}^n$$

$$x_2 = B \cos \omega_n t$$

(Q) What are the necessary and sufficient conditions for a principal mode of vibration?

~~This means that the two principal modes represent two linearly independent solutions of our DEOM.~~
~~By the~~

Part a :- the necessary conditions

Let the system is executing the first principal mode of vibration.

Then $x_1 = A_{11} \sin(\omega_1 t + \phi_1) \Rightarrow \dot{x}_1 = A_{11}\omega_1 \cos(\omega_1 t + \phi_1)$

$x_2 = \mu_1 A_{11} \sin(\omega_1 t + \phi_1) \Rightarrow \dot{x}_2 = \mu_1 A_{11}\omega_1 \cos(\omega_1 t + \phi_1)$

Here $x_2 = \mu_1 x_1 \Rightarrow \boxed{x_2(0) = \mu_1 x_1(0)}$ — (α)

Also $\dot{x}_2 = \mu_1 \dot{x}_1 \Rightarrow \boxed{\dot{x}_2(0) = \mu_1 \dot{x}_1(0)}$ — (β)

a necessary condition for first principal mode

(HW) Prove that conditions (α) and (β) are also sufficient conditions for first principal mode

Similarly, we can show that the necessary & conditions for the 2nd principal mode are

$$\left.\begin{aligned} x_2(0) &= \mu_2 x_1(0) \\ \dot{x}_2(0) &= \mu_2 \dot{x}_1(0) \end{aligned}\right\}$$

Hence for our example problem, to excite the first principal mode, we could do the following :-- [Here $\mu_1 = 1.618$, $\mu_2 = -0.618$]

Give m_1 an initial displacement of say 4 m

" m_2 " " " 1.618×4 m

Release the masses

The system will execute first pr. mode of vibration

linear independence of a set of n functions

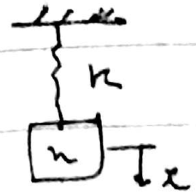
$f_1(x), f_2(x), f_3(x), \dots, f_n(x)$

The Wronskian of the set of functions is defined

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \cdots & \cdots & & \cdots \\ f_1^{(n-1)}(x) & f_2^{(n-2)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Here $f_i^{(n-1)}(x) = \dfrac{d^{(n-1)} f_i(x)}{dx^{(n-1)}}$

If $W(x)$ is identically zero over the interval of interest, then the functions are linearly dependent. Otherwise they are linearly independent.



$$W = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix}$$

$x_1 = A \sin(\omega_n t)$

$x_2 = B \cos(\omega_n t)$

$$= \begin{vmatrix} A \sin \omega_n t & B \cos \omega_n t \\ A\omega_n \cos \omega_n t & -B\omega_n \sin \omega_n t \end{vmatrix}$$

$$= -AB\omega_n \neq 0$$

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ \mu_1 A_{11} \end{Bmatrix} \qquad \{A_2\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$$

are the modal vectors for first and second pr. modes.

Note that A_{11} & A_{12} are arbitrary.
However, for some analytical advantage, we often normalize the modal vectors. There are several standard ways to normalize a modal vector.

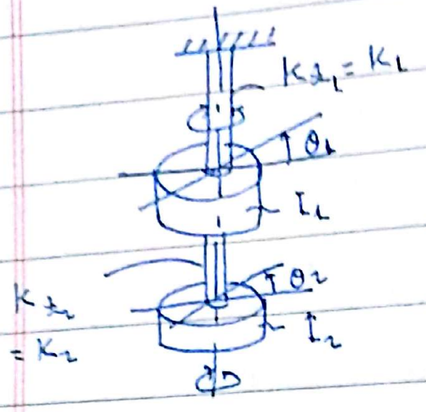
One way is to set $A_{11} = 1$, $A_{12} = 1$
Then the normalized modal vectors are

$$\{A_1\} = \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix}, \{A_2\} = \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$$

Definition of a modal matrix :- The matrix $[\mu] = [\{A_1\} \{A_2\}]$

$$= \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix}$$ is called a modal matrix of our system.

A rotational 2 DOF system



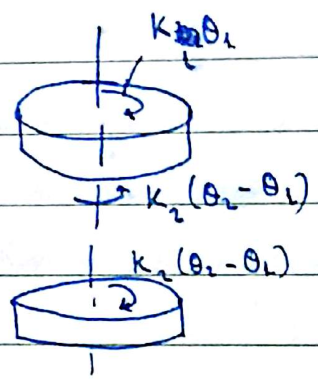
To obtain the DEOM

(I) By Newton's method

$$\left.\begin{aligned}\theta_1 &= \theta_1(t)\\ \theta_2 &= \theta_2(t)\end{aligned}\right\}$$ — A set of generalised coordinates measured from eqm, + ve as seen from above

Let $\theta_2 > \theta_1$

FBD :-



The MOM eqns for I_1 gives

$$I_1\ddot{\theta}_1 = -k_1\theta_1 + k_2(\theta_2 - \theta_1)$$

$$\boxed{I_1\ddot{\theta}_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0} \rightarrow \text{similar to } m_1\ddot{x}_1 + (k_1+k_2)x_1 - k$$

MOM eqn for I_2 gives

$$I_2\ddot{\theta}_2 = -k_2(\theta_2 - \theta_1) \Rightarrow \boxed{I_2\ddot{\theta}_2 + k_2\theta_2 - k_2\theta_1 = 0}$$

similar to $m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$

(II) By using the Lagrange eqns

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

$$U = \frac{1}{2} K_1 \theta_1^2 + \frac{1}{2} K_2 (\theta_2 - \theta_1)^2$$

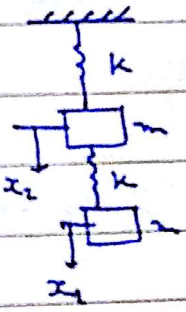
The L eqns are:-

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}_1}\right] - \frac{\partial T}{\partial \theta_1} + \frac{\partial U}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}_2}\right] - \frac{\partial T}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2} = 0$$

Modal Analysis

A modal matrix can be used as a coordinate transformation matrix to uncouple the DEOM. This leads to the modal analysis.



For this system, a modal matrix is

$$[\mu] = \begin{bmatrix} A_{11} & A_{12} \\ \mu_1 A_{11} & \mu_2 A_{12} \end{bmatrix}$$

or

$$[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$$ (a normalized modal matrix)

Sometimes it proves convenient to make each modal vector an orthogonal vector.

Let A_{11} be such that $|\{A_{11}\}| = 1$,

i.e $A_{11}^2 + \mu_1^2 A_{11}^2 = 1$

or $A_{11} = \frac{1}{\sqrt{1+\mu_1^2}}$

Similarly, make $|\{A_{12}\}| = 1$

where $\{A_{12}\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$

Then $A_{12}^2 + \mu_2^2 A_{12}^2 = 1 \Rightarrow A_{12} = \frac{1}{\sqrt{1+\mu_2^2}}$

Hence $[N] = \begin{bmatrix} \frac{1}{\sqrt{1+\mu_1^2}} & \frac{1}{\sqrt{1+\mu_2^2}} \\ \frac{\mu_1}{\sqrt{1+\mu_1^2}} & \frac{\mu_2}{\sqrt{1+\mu_2^2}} \end{bmatrix}$

is also a modal matrix where the modal vector $[A_{01}] = \begin{Bmatrix} \frac{1}{\sqrt{1+\mu_1^2}} \\ \frac{\mu_1}{\sqrt{1+\mu_1^2}} \end{Bmatrix}$ and

$[A_{02}] = \begin{Bmatrix} \frac{1}{\sqrt{1+\mu_2^2}} \\ \frac{\mu_2}{\sqrt{1+\mu_2^2}} \end{Bmatrix}$ are a set of orthogonal vectors.

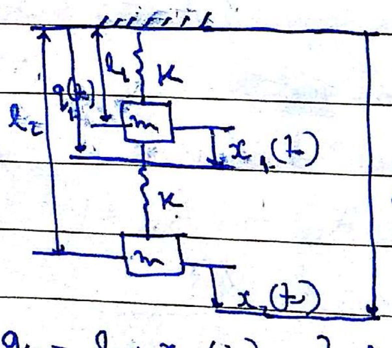
Modal Analysis :

Let the DEOM of the system for undamped free vibration be $[m]\{\ddot{x}\} + [K]\{x\} = \{0\}$ — (1)

Let $\{x(t)\} = [N]\{p(t)\}$ — (2)

defines a new set of coordinates $\{p(t)\}$ where $\{p(t)\} = [N]^{-1}\{x(t)\}$ — (3)

$x_1(t), x_2(t)$ are measured from the eqm positions



$q_1 = l_1 + x_1(t)$
$q_2 = l_2 + x_2(t)$ } Another set of generalised coordinates

$\{A_1\} = \begin{Bmatrix} A_{11} \\ 1.618A_{11} \end{Bmatrix}$, $\{A_2\} = \begin{Bmatrix} A_{12} \\ -0.618A_{12} \end{Bmatrix}$

$A_{11} = 1, A_{12} = 1$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Our DEOM are (for the example problem)

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \quad \text{— (i)}$$

where $[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ $\quad [k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \qquad [x] = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

If $\{x\} = [\mu]\{p\}$ — (ii)

$\Rightarrow \{\ddot{x}\} = [\mu]\{\ddot{p}\}$ — (iii)

then substitution of (ii) and (iii) in (i) leads to

$$[m][\mu][\ddot{p}] + [k][\mu]\{p\} = \{0\} \quad \text{— (iv)}$$

Premultiply both sides of (iv) by $[\mu]^T$

This gives

$$\underbrace{[\mu]^T [m][\mu]}\, [\ddot{p}] + [\mu]^T [k][\mu]\{p\} = [\mu]^T\{0\} = \{0\}$$

(v)

$$[\mu]^T[m][\mu] = \begin{bmatrix} 1 & 1.618 \\ 1 & -0.618 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} m & 1.618m \\ m & -0.618m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} m(1+1.618^2) & m(1-0.618 \times 1.618) \\ m(1-0.618\times 1.618) & m(1+0.618^2) \end{bmatrix}$$

$$= \begin{bmatrix} 3.62m & 7.6\times10^{-5}m \\ 7.6\times10^{-5}m & 1.382m \end{bmatrix} \approx \begin{bmatrix} 3.62m & 0 \\ 0 & 1.382m \end{bmatrix}$$

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$$[\mu]^T[K][\mu] = \begin{bmatrix} 1 & 1.618 \\ 1 & -0.618 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} k(2-1.618) & k(-1+1.618) \\ k(2+0.618) & k(-1-0.618) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} 0.382k & 0.618k \\ 2.618k & -1.618k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

$$= \begin{bmatrix} 1.382k & 0 \\ 0 & 3.618k \end{bmatrix}$$

$$\begin{bmatrix} 3.617m & 0 \\ 0 & 1.382m \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 1.382k & 0 \\ 0 & 3.618k \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{(VI)}$$

(VI) can be written as

$$3.617m\ddot{p}_1 + 1.382k\,p_1 = 0 \quad \text{— (vii)}$$
$$1.382m\ddot{p}_2 + 3.618k\,p_2 = 0 \quad \text{— (viii)}$$

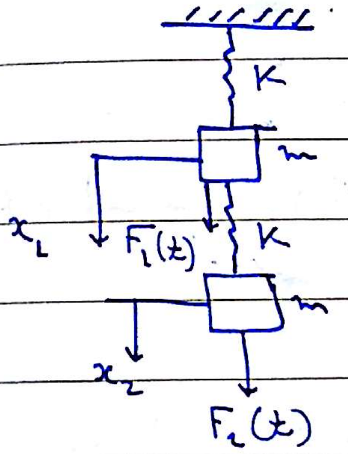
Eqns. (vii) and (viii) are reqd. uncoupled DEOM in terms of principal coordinates $p_1(t)$ & $p_2(t)$

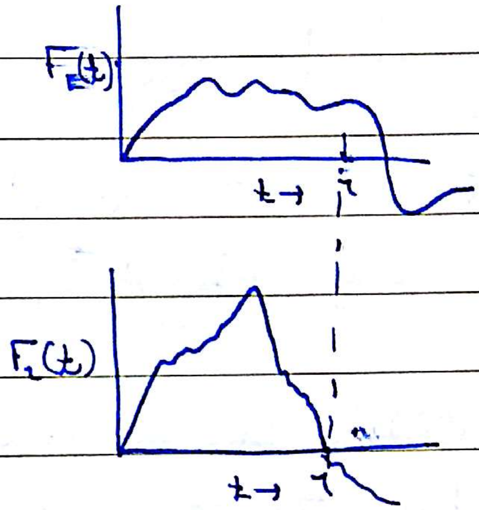
Note: There can be infinitely many sets of principal coordinates since A_{11} and A_{12} are arbitrary.

Since we normalized our modal matrix (setting $A_{11} = 1$, $A_{12} = 1$) we are getting only one particular set of principal coordinates.

$$\omega_1 = \sqrt{\frac{1.382K}{3.617m}} = 0.618\sqrt{K/m} \quad [\text{From (vii)}]$$

$$\omega_2 = \sqrt{\frac{3.618K}{1.382m}} = 1.618\sqrt{K/m} \quad [\text{From (viii)}]$$





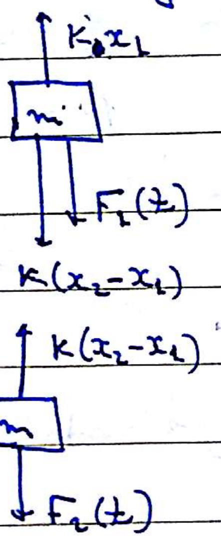
Here the DEOM will be

$$\left.\begin{aligned} m\ddot{x}_1 + 2Kx_1 - Kx_2 &= F_1(t) \\ m\ddot{x}_2 - Kx_1 + Kx_2 &= F_2(t) \end{aligned}\right]$$

$$[m]\{\ddot{x}\} + [K]\{x\} = \{F(t)\}$$

$$\{F(t)\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \quad \text{— (I)}$$

Aim: To obtain forced response of the system at any time t.



Let $\{x\} = [\mu]\{p\}$ — (II)

Substitute (II) in (I) & premultiply by $[\mu]^T$.

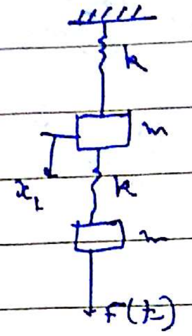
$$[\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[K][\mu][p] = [\mu]^T[F]$$

$$[\mu]^T\{F(t)\} = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = \{Q(t)\}$$

$$\Rightarrow M_{11}\ddot{p}_1 + K_{11}p_1 = Q_1(t) \quad \text{— (III)}$$

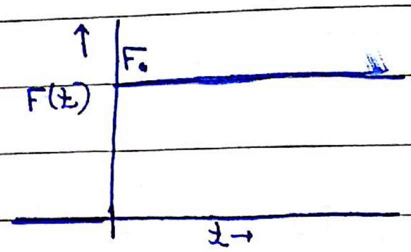
$$\Rightarrow M_{22}\ddot{p}_2 + K_{22}p_2 = Q_2(t) \quad \text{— (IV)}$$

Solve (III) and (IV) using Duhamel's integral for forced vibrations.



$F(t) = F_0 \, u(t)$

$u(t) \rightarrow$ unit step function



Obtain $x_1(t)$ & $x_2(t)$ for forced vibration using Modal Analysis

Step 1 - Obtain the DEOM in terms of x_1 & x_2.
Here these eqns. are

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \quad \text{--- (I)}$$

Step 2 - Obtain modal vector $\{A_1\} = \begin{Bmatrix} A_{11} \\ \mu_1 A_{11} \end{Bmatrix}$, $\{A_2\} = \begin{Bmatrix} A_{12} \\ \mu_2 A_{12} \end{Bmatrix}$

Step 3 - Use ~~to~~ the normalized modal matrix $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$

to uncouple the DEOM (I)

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix}$$

$$\Rightarrow [\mu]^T [m] [\mu] \{\ddot{p}\} + [\mu]^T [k] [\mu] \{p\} = [\mu]^T [\mu] \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix}$$

This results in the following uncoupled DEOM in terms of principal coordinates $\{p\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$, where

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [\mu] \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

$$M_{11}\ddot{p}_1 + K_{11}p_1 = Q_1(t) = \mu_1 F(t) \quad \text{——} \ (2)$$

$$M_{22}\ddot{p}_2 + K_{22}p_2 = Q_2(t) = \mu_2 F(t) \quad \text{——} \ (3)$$

M_{11} and M_{22} are called generalised masses.
K_{11} and K_{22} are " " stiffnesses.

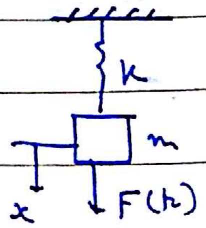
$$\omega_1 = \sqrt{\frac{K_{11}}{m_{11}}}, \quad \omega_2 = \sqrt{\frac{K_{22}}{m_{22}}}$$

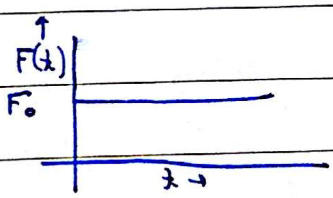
$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = [\mu]^{-1} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

(Q) Obtain a set of principal coordinates

$$(p_1)_{particular} = \frac{\mu_1 F_0}{K_{11}} \leftarrow M_{11}\ddot{p}_1 + K_{11}p_1 = \mu_1 F_0\, u(t)$$

$$(p_2)_{''} = \frac{\mu_2 F_0}{K_{22}} \leftarrow M_{22}\ddot{p}_2 + K_{22}p_2 = \mu_2 F_0\, u(t)$$





$$x_{forced} = x(t) = \frac{F_0}{k}[1 - \cos\omega_n t]$$

By Duhamel's Integral

$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau = \int_0^t \frac{F_0}{m\omega_n} \sin\omega_n(t-\tau)\, d\tau$$

With zero initial conditions $\boxed{x(0) = \dot{x}(0) = 0}$

$$= \frac{F_0}{m\omega_n^2} \Big| \cos\omega_n(t-\tau) \Big|_0^t$$

$$x = x_c + x_p = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0}{k}$$

$$x(0) = B + \frac{F_0}{k} \qquad B = -\frac{F_0}{k}$$

$$\dot{x} = A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t$$

$$\dot{x}(0) = 0$$

$$\Rightarrow A = 0$$

Hence

$$x = -\frac{F_0}{K}\cos\omega_n t + \frac{F_0}{K} = \frac{F_0}{K}(1-\cos\omega_n t)$$

$$\left\{\begin{array}{l} x_{1c} = A_{11}\sin(\omega_1 t+\phi_1) + A_{12}\sin(\omega_2 t+\phi_2) \\ x_{2c} = \mu_1 A_{11}\sin(\omega_1 t+\phi_1) + \mu_2 A_{12}\sin(\omega_2 t+\phi_2) \end{array}\right.$$

→ Free vibration part Use the I.C given and add the response obtained to the response from Duhamel's Integral

$$(x_1)_{forced} = p_1 + p_2$$

$$(x_2)_{forced} = \mu_1 p_1 + \mu_2 p_2$$

From DEOM (2) using Duhamel's integral, we get

$$p_1 = \int_0^t \frac{F(\tau)\sin\omega_1(t-\tau)}{M_{11}\omega_1}\, d\tau$$

$$m\ddot{x} + kx = F(t)$$

$$x = \int_0^t F(\tau)g(t-\tau)\, d\tau$$

$$g(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

$$g(t) = \frac{1}{M_{11}\omega_1}\sin\omega_1 t$$

$$p_1(t) = \frac{\mu_1 F_0}{M_{11}\omega_1^2}\Big|\cos\omega_1(t-\tau)\Big|_0^t \qquad p_1(t) = \frac{\mu_1 F_0}{K_{11}}[1-\cos\omega_1 t]$$

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$$p_2(t) = \frac{\mu_2 F_0}{K_{22}}\left[1-\cos(\omega_2 t)\right]$$

Hence, $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$ gives

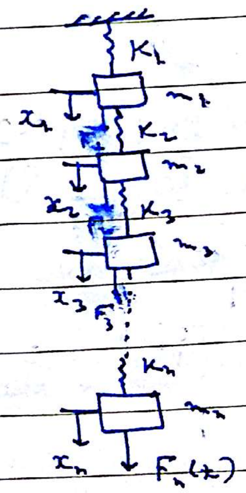
$$x_1(t) = p_1 + p_2 = \frac{\mu_1 F_0}{K_{11}}\left[1-\cos\omega_1 t\right] + \frac{\mu_2 F_0}{K_{22}}\left[1-\cos\omega_2 t\right]$$

$$\& \; x_2(t) = \mu_1 p_1 + \mu_2 p_2 = \frac{\mu_1^2 F_0}{K_{11}}\left[1-\cos\omega_1 t\right]$$

$$+ \frac{\mu_2^2 F_0}{K_{22}}\left[1-\cos\omega_2 t\right]$$

These are the reqd. Forced vibration responses.

Generalization for an n- DOF system :-



The DEOM are

$$\underset{n\times n}{[m]}\underset{n\times 1}{\{\ddot{x}\}} + \underset{n\times n}{[k]}\underset{n\times 1}{\{x\}} = \underset{n\times 1}{\{F\}} \quad \text{--- (1)}$$

where $\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}$, $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$

$$\{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{Bmatrix}$$

$$[m] = \begin{bmatrix} m_{11} & m_{12} & --- & m_{1n} \\ m_{21} & m_{22} & --- & m_{2n} \\ --- & --- & --- & --- \\ m_{n1} & m_{n2} & --- & m_{nn} \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_{11} & K_{12} & ---- & K_{1n} \\ K_{21} & K_{22} & --- & K_{2n} \\ \vdots & & & \\ K_{n1} & K_{n2} & --- & K_{nn} \end{bmatrix}$$

we get the following relation

$$\omega^2 [m]\{A\} = [K]\{A\} \quad (1)$$

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\{x\} = \{A\}\sin(\omega t + \phi)$$

$$-\omega^2 [m]\{A\} + [K]\{A\} = 0$$

$$\omega^2 [m]\{A\} = [K]\{A\} \quad (1)$$

when $\omega = \omega_r$, $\{A\} = \{A_r\}$

& when $\omega = \omega_s$, $\{A\} = \{A_s\}$

Then (1) becomes $\omega_r^2 [m]\{A_r\} = [K]\{A_r\}$ — (2)

$$\omega_s^2 [m]\{A_s\} = [K]\{A_s\} \quad (3)$$

Premultiply (2) by $\{A_s\}^T$

& " (3) by $\{A_r\}^T$

This gives

$$\omega_r^2 \{A_s\}^T [m]\{A_r\} = \{A_s\}^T [K]\{A_r\} \quad (4)$$

$$\omega_s^2 \{A_r\}^T [m]\{A_s\} = \{A_r\}^T [K]\{A_s\} \quad (5)$$

Transpose (4)

$$\Rightarrow \omega_r^2 \{A_r\}^T [m]^T \{A_s\} = \{A_r\}^T [K]^T \{A_s\}$$

let $[m]^T = [m]$ & $[K]^T = [K]$ (i.e. $[m]$ and $[K]$ are symmetric)

Then, $\omega_r^2 \{A_r\}^T [m]\{A_s\} = \{A_r\}^T [K]\{A_s\}$ — (6)

RHS of (5) and (6) are identical

Hence $(\omega_r^2 - \omega_s^2)\{A_r\}^T [m]\{A_s\} = 0$

But $\omega_r \neq \omega_s$

Hence, $\{A_r\}^T [m]\{A_s\} = 0$

From (6), $\{A_r\}^T [K]\{A_s\} = 0$

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} \quad \{B_1\} = \begin{Bmatrix} B_{11} \\ B_{21} \end{Bmatrix}$$

If $\bar{A}_1 \cdot \bar{B}_1 = 0$ i.e. if $\{A_1\}^T\{B_1\} = 0$ → Here $\{A_1\}$ & $\{B_1\}$ are orthogonal in an ordinary sense.

Then ↓

$$\{A_1\}^T[I]\{B_1\} = 0$$

→ weighting matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

↓

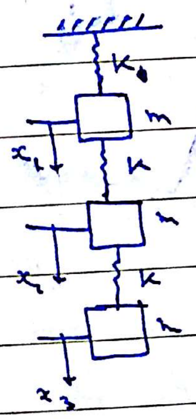
Identity matrix.

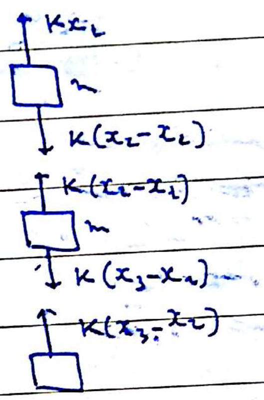
premultiply ② by $\{A_r\}^T$. This gives

$$\omega_r^2 = \frac{\{A_r\}^T[k]\{A_r\}}{\{A_r\}^T[m]\{A_r\}}$$

$r = 1, 2, \dots, n$

(Ex:-)





$$m\ddot{x}_1 = k(x_2 - x_1) - kx_1$$

$$m\ddot{x}_2 = k(x_3 - x_2) - k(x_2 - x_1)$$

$$m\ddot{x}_3 = -k(x_3 - x_2)$$

$$\Rightarrow m\ddot{x}_1 + 2kx_1 - kx_2 + 0x_3 = 0$$

$$\Rightarrow m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$

$$\Rightarrow m\ddot{x}_3 + 0x_1 - kx_2 + kx_3 = 0$$

$$[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad [k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

So the frequency eqn is

$$|[k] - \omega^2[m]| = 0$$

$$\Rightarrow \begin{vmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (2k - m\omega^2)\left[(2k - m\omega^2)(k - m\omega^2) - k^2\right] + k\left[-k(k - m\omega^2)\right] = 0$$

$$\Rightarrow (2k - m\omega^2)\left[k^2 + m^2\omega^4 - 3km\omega^2\right] - k^2(k - m\omega^2) = 0$$

$$\Rightarrow 2k^3 + 2km^2\omega^4 - 6k^2m\omega^2 - m\omega^2k^2 - m^3\omega^6 + 3km^2\omega^4 - k^3 + k^2m\omega^2 = 0$$

$$\Rightarrow -m^3\omega^6 + 5km^2\omega^4 - 6k^2m\omega^2 + k^3 = 0$$

$$\Rightarrow -\omega^6 + 5\left[\frac{k}{m}\right]\omega^4 - 6\left[\frac{k}{m}\right]^2\omega^2 + \left[\frac{k}{m}\right]^3 = 0$$

$$\omega^2 = 3.247\left(\frac{k}{m}\right),\ 1.555\left(\frac{k}{m}\right),\ 0.198\left(\frac{k}{m}\right)$$

$$\boxed{\begin{aligned} \omega_1 &= 0.445\sqrt{\frac{k}{m}} \\ \omega_2 &= 1.247\sqrt{\frac{k}{m}} \\ \omega_3 &= 1.802\sqrt{\frac{k}{m}} \end{aligned}}$$

The Rayleigh Method for multi DOF systems

For an n DOF system,

$$\omega_r^2 = \frac{\{A_r\}^T [K] \{A_r\}}{\{A_r\}^T [m] \{A_r\}} \quad \text{——} \; (1)$$

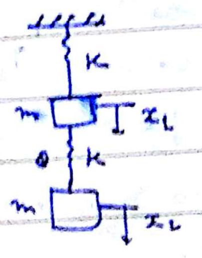
So we need to know $\{A_r\}$ & hence ω_r. However, we first obtain ω_r & then $\{A_r\}$ by the analytical method. Hence, formula (1) is apparently useless for computing any ω_r.

However, this is not so for $\omega = \omega_1$. Formula (1) can be used to obtain a pretty good estimate for ω_1, by making a (somewhat) guess for $\{A_1\}$.

That is the Rayleigh method for n DOF system.

It can be shown that a 100% error in the guess for $\{A_1\}$ results in only about 10% error in the estimate ω_1.

To illustrate, consider the system



We already know that

$\omega_1 = 0.618\sqrt{K/m}$

& $\{A_1\} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}$

To apply the Rayleigh method, we assume a trial vector $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ for $\{A_1\}$

Then, ω_R^2 = Square of Rayleigh frequency ω_R.

$$= \frac{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} [K] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} [m] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

or, $\omega_R^2 = \dfrac{\{1\ \ 1\}\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}}{\{1\ \ 1\}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}}$

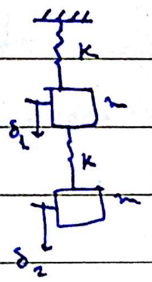
$= \dfrac{\{1\ \ 1\}\begin{Bmatrix}k\\0\end{Bmatrix}}{2m} = \dfrac{k}{2m}$

Hence, $\omega_R = \sqrt{\frac{1}{2}}\sqrt{\frac{k}{m}} = 0.707\sqrt{\frac{k}{m}}$

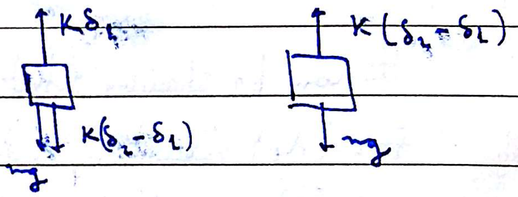
Hence % error $= \dfrac{(0.707-0.618)}{0.618} \times 100$

$\approx 14\%$

A better estimate for ω_1 can be obtained by taking the static deflection vector for $\{A_{11}\}$.



Then,



Hence,

$mg = k(\delta_2-\delta_1)$

& $mg + k(\delta_2-\delta_1) = k\delta_1$

$\Rightarrow mg + mg = k\delta_1 \Rightarrow \delta_1 = \dfrac{2mg}{k}$

$\Rightarrow \delta_2 = \dfrac{mg}{k} + \delta_1 = \dfrac{3mg}{k}$

Thus we take $\{A_1\} \approx \{\delta\} = \begin{Bmatrix}\delta_1\\ \delta_2\end{Bmatrix} = \begin{Bmatrix}2mg/k\\ 3mg/k\end{Bmatrix}$

or $\begin{Bmatrix}2\\3\end{Bmatrix}$ [normalizing $\{\delta\}$

Then, $\omega_R^2 = \dfrac{\{2\ \ 3\}\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}2\\3\end{Bmatrix}}{\{2\ \ 3\}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{Bmatrix}2\\3\end{Bmatrix}} = \dfrac{\{2\ \ 3\}\begin{Bmatrix}k\\k\end{Bmatrix}}{13m}$

$= \dfrac{5k}{13m}$

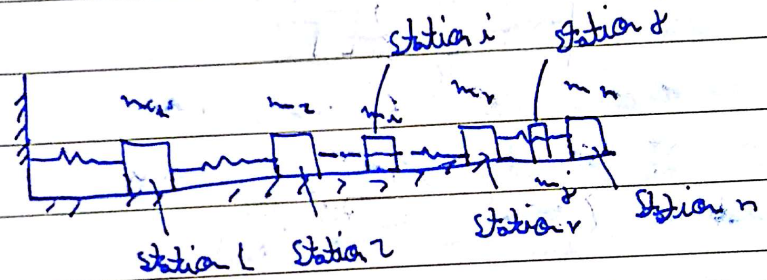
~~% error = 0.35~~ & so $\omega_1 = \omega_R = \sqrt{\frac{5k}{13m}} = 0.621\sqrt{\frac{k}{m}}$

$$\% \text{ error} = \frac{(0.621 - 0.618)}{(0.618)} \times 100 = 0.35\%$$

(6) The flexibility influence coefficients a_{ij} & flexibility matrix $\underset{n\times n}{[a]} = \underset{n\times n}{[a_{ij}]}$ for an n DOF system.

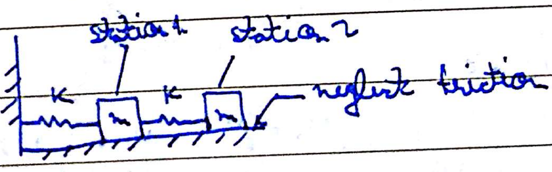
The flexibility influence coefficients a_{ij} are very useful for obtaining the stiffness matrix $[k]$ of a complex system by experimentation.

It can be shown that $[k] = [a]^{-1}$.

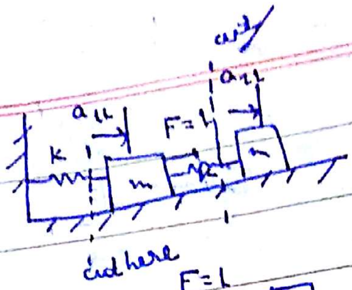


a_{ij} = deflection at station i due to unit force (at an appropriate direction) at station j with the other stations free of such forces.

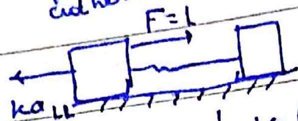
Ex : Obtain $[a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

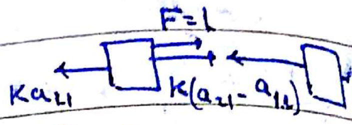


Step 1 :- Apply force $F = 1$ unit at station 1 & compute / measure deflection at stations 1 & 2. These will give a_{11} & a_{21}, i.e the first column of $[a]$



F.B.D





$ka_{11}=1$

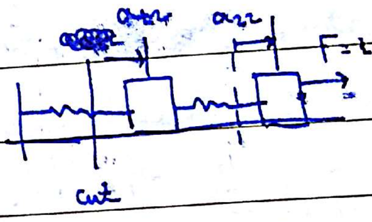
$\Rightarrow a_{11}=1/k$

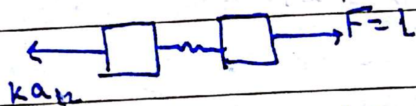
$k(a_{21}-a_{11})=0$

$a_{21}=a_{11}=1/k$

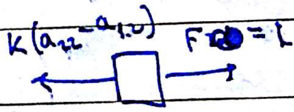
$$[a]=\begin{bmatrix} 1/k & \\ 1/k & \end{bmatrix}$$

Step 2: To obtain a_{12} and a_{22}, apply $F=1$ unit to station 2.





$ka_{12}=1 \Rightarrow a_{12}=1/k$



$1=k(a_{22}-a_{12})$

$\Rightarrow a_{22}=2/k$

$$[a]=\begin{bmatrix} 1/k & 1/k \\ 1/k & 2/k \end{bmatrix}$$

$$[a]^{-1}=k^2\begin{bmatrix} 2/k & -1/k \\ -1/k & 1/k \end{bmatrix}$$

$$=\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}=[k]$$

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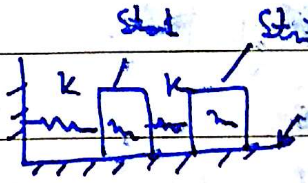
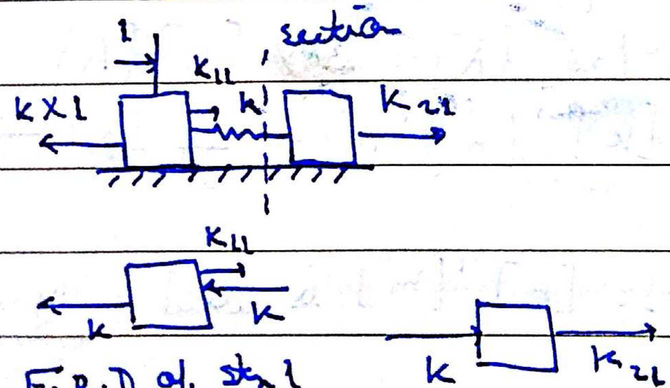
The stiffness influence coefficients k_{ij} and the stiffness matrix $[k] = [k_{ij}]$

k_{ij} = Force reqd. at station i to cause unit deflection at station j, all other stations being subject to forces simultaneously to arrest their movements.

Ex: Obtain $[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ using definition of k_{ij}

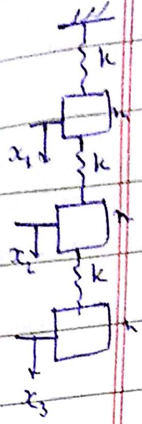


Step 1:- To get k_{11}, we apply a force on station 1 to cause a unit deflection of station 1 & no deflection at station 2. Clearly we must apply a force $= k_{12}$ to station 2 to accomplish this.



F.B.D of stn 1

$k_{11} = 2k$ $\qquad$ $k + k_{21} = 0 \Rightarrow k_{21} = -k$



$\omega_1 = 0.44497\sqrt{\frac{k}{m}}$, $\{A_1\} = \begin{Bmatrix} 1.0 \\ 1.804 \\ 2.247 \end{Bmatrix}$

$\omega_2 = 1.247\sqrt{\frac{k}{m}}$, $\{A_2\} = \begin{Bmatrix} 1.0 \\ 0.447 \\ -0.802 \end{Bmatrix}$

$\omega_3 = 1.802\sqrt{\frac{k}{m}}$, $\{A_3\} = \begin{Bmatrix} 1.0 \\ -1.247 \\ 0.5605 \end{Bmatrix}$

⑤ The Matrix Iteration (MI) Method
(The Power Method etc)

This is a numerical method for obtaining ω_i & $\{A_i\}$ for an undamped n DOF system with $\omega_1 < \omega_2 < \dots < \omega_n$

Let the DEOM be

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

$$\{x\} = \{A\}\sin(\omega t + \phi)$$

$$\Rightarrow \omega^2 [m]\{A\} = [k]\{A\}$$

$$\Rightarrow [k]^{-1}[m]\{A\} = \frac{1}{\omega^2}\{A\} \quad \text{—— (1)}$$

$[D] = [k]^{-1}[m]$ is called a dynamic matrix

We also have

$$[m]^{-1}[k]\{A\} = \omega^2\{A\}$$

$$\Rightarrow [E]\{A\} = \omega^2\{A\} \quad \text{—— (2)}$$

where $[E] = [D]^{-1}$ is another dynamic matrix

If we use (1) for iteration, convergence will be to ω_1 & $\{A_1\}$ first, next to ω_2 & $\{A_2\}$ & so on

However, if ② is used for iteration, convergence will be to ω_n & $\{A_n\}$ & then to ω_{n-1} & $\{A_{n-1}\}$ etc.

$[D]$ (or E) must be correctly obtained.

We shall use ① for iteration since the first few natural frequencies are normally the most important ones.

Hence, $[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$, $[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$

$$[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

So, $[D] = [k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

We start iterating assuming a trial vector $\{u\}$ for $\{A_1\}$

If $\{u\} = \{A_1\}$, then $[D]\{u\} = \frac{1}{\omega_1^2}\{u\}$

Let the first trial vector be $\{u_1\}$

Let $\{u_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$\{u_1\} = \begin{Bmatrix} -1.0^3 \\ 1 \\ +1.0^3 \end{Bmatrix}$

$\{u_1\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$ etc.

$$[D]\{u_1\} = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m}{k}\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = \frac{3m}{k}\begin{Bmatrix} 1 \\ 5/3 \\ 2 \end{Bmatrix}$$

$$= \frac{3m}{k}\begin{Bmatrix} 1 \\ 1.667 \\ 2 \end{Bmatrix}$$

$\downarrow$

$\{u_2\}$

$\downarrow$

next trial vector

Page

$$[D]\{u_2\} = \frac{4.6667m}{k}\begin{Bmatrix}1\\1.7857\\2.2142\end{Bmatrix}$$

$$[D]\{u_3\} = \frac{4.9999m}{k}\begin{Bmatrix}1\\1.8000\\2.2428\end{Bmatrix} \rightarrow \{u_4\}$$

$$[D]\{u_4\} = \frac{m}{k}\begin{Bmatrix}5.0428\\9.0856\\11.3284\end{Bmatrix} = \frac{5.0428m}{k}\begin{Bmatrix}1\\1.8017\\2.2465\end{Bmatrix} \rightarrow \{u_5\}$$

$$[D]\{u_5\} = \cancel{\frac{m}{k}\begin{Bmatrix}25.4568\\45.8708\\57.1992\end{Bmatrix}} \quad \frac{5.0482m}{k}\begin{Bmatrix}1\\1.8019\\2.2469\end{Bmatrix} \rightarrow \{u_6\}$$

So, $\{A_1\} = \begin{Bmatrix}1\\1.8019\\2.2469\end{Bmatrix}$

$$\frac{1}{\omega_1^2} \approx \frac{5.04082m}{k} \Rightarrow \omega_1 = 0.4451\sqrt{\frac{K}{m}}$$

$$\{u\} = c_1\{A_1\} + c_2\{A_2\} + C_3\{A_3\}$$ [$\{A_1\}, \{A_2\}, \{A_3\}$ are linearly independent

$$\Rightarrow [D]\{u_1\} = c_1[D]\{A_1\} + c_2[D]\{A_2\} + C_3[D]\{A_3\}$$

$$\{u_2\} = \frac{c_1}{\omega_1^2}\{A_1\} + \frac{c_2}{\omega_2^2}\{A_2\} + \frac{c_3}{\omega_3^2}\{A_3\}$$

$$\{u_3\} = \frac{c_1}{\omega_1^4}\{A_1\} + \frac{c_2}{\omega_2^4}\{A_2\} + \frac{c_3}{\omega_3^4}\{A_3\}$$

$\omega_1 < \omega_2 < \omega_3$

$$\frac{1}{(\omega_1^2)^2} >>> \frac{1}{(\omega_2^2)^2} >>> \frac{1}{(\omega_3^2)^2}$$

For convergence to ω_2 & $\{A_2\}$
the trial vector $\{V\}$ must be orthogonal to $\{A_1\}$

$$\{V\} = \alpha\{A_1\} + \beta\{A_2\} + \gamma\{A_3\}$$

$$\{A_1\}^T[m]\{V\} = \alpha\underbrace{\{A_1\}^T[m]\{A_1\}}_{M_{11} \neq 0} + \beta\cancel{\{A_1\}^T[m]\{A_2\}}^{\,0} + \gamma\cancel{\{A_1\}^T[m]\{A_3\}}^{\,0}$$

If $\{A_1\}^T[m]\{V\} = 0$

then $\alpha = 0$

So $$\{A_{11} \quad A_{21} \quad A_{31}\}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = 0$$

$$\Rightarrow A_{11}v_1 + A_{21}v_2 + A_{31}v_3 = 0$$

$$\Rightarrow v_1 = 0\,v_1 - \frac{A_{21}v_2}{A_{11}} - \frac{A_{31}v_3}{A_{11}}$$

$$v_2 = 0\,v_1 + 1 \cdot v_2 + 0 \cdot v_3$$

$$v_3 = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3$$

or

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{A_{21}}{A_{11}} & -\frac{A_{31}}{A_{11}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{A Sweeping Matrix}} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$\{V\} = [S^{(2)}]\{V\}$ for convergence ω_2 & $\{A_2\}$
where $\{V\}$ is an arbitrary trial vector

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Instead of premultiplying $\{V\}$ by $[S^{(2)}]$ at every step we can make a new dynamic matrix $[D^{(2)}] = [D][S^{(2)}]$

The iteration should start with

$$[D^{(2)}]\{A_2\} = \frac{1}{\omega_2^2}\{A_2\}$$

For convergence to ω_3 & $\{A_3\}$, a trial vector $\{W\}$ must be orthogonal to $\{A_1\}$ & $\{A_2\}$ & hence,

$$\{A_1\}^T[m]\{W\} = 0 \longrightarrow W_1A_{11} + W_2A_{21} + W_3A_{31} = 0$$

$$\{A_2\}^T[m]\{W\} = 0 \longrightarrow W_1A_{12} + W_2A_{22} + W_3A_{32} = 0$$

2 constraint eqn

This leads to a new sweeping matrix $[S^{(3)}]$. The Dynamic matrix for ω_3 & $\{A_3\}$ will be

$$[D^{(3)}] = [D][S^{(3)}]$$

Solving for W_1 & W_2 in terms of W_3, we get eqns like

$$W_1 = 0W_1 + 0 \cdot W_2 + \alpha W_3$$

$$W_2 = 0 \cdot W_1 + 0 \cdot W_2 + \beta \cdot W_3$$

$$W_3 = 0 \cdot W_1 + 0 \cdot W_2 + 1 \cdot W_3$$

$$\Rightarrow \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 1 \end{bmatrix} = [S^{(3)}]$$

A good trial vector of $\{A_2\}$ should have 1 sign change

$$\begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}$$

A good " " $\{A_3\}$ " " 2 sign changes

$$\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

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Actually, after obtaining $\{A_1\}$ & $\{A_2\}$ it is no longer necessary to continue iteration, since $\{A_3\}$ can be obtained by invoking orthogonality, viz.

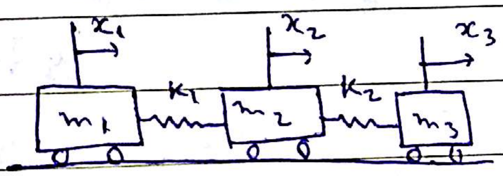
$$\left.\begin{aligned}\{A_1\}^T[m]\{A_3\} &= 0\\ \{A_2\}^T[m]\{A_3\} &= 0\end{aligned}\right\}$$ solve these two eqns for A_{32} & A_{33} in terms of A_{31}.

Let $A_{32} = \gamma A_{31}$ & $A_{33} = \delta A_{31}$

Then, $\{A\} = \begin{Bmatrix} A_{31} \\ \gamma A_{31} \\ \delta A_{31} \end{Bmatrix}$ or $\begin{Bmatrix} 1 \\ \gamma \\ \delta \end{Bmatrix}$

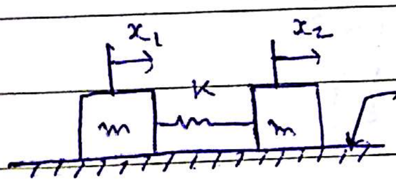
Then $\omega_3^2 = \dfrac{\{A_3\}^T[k]\{A_3\}}{\{A_3\}^T[m]\{A_3\}}$

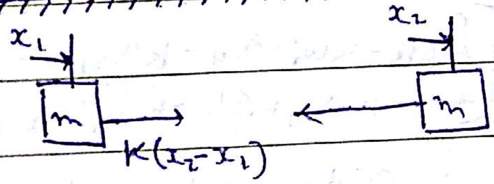
(5) Semi-definite Systems



(A free-free system)

(Ex:)





$m\ddot{x}_1 = k(x_2) - kx_1$ or $m\ddot{x}_1 + kx_1 - kx_2 = 0$

$m\ddot{x}_2 = -kx_2 + kx_1$ $\quad m\ddot{x}_2 - kx_1 + kx_2 = 0$

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$ det $[k] = 0$

$[k]^{-1}$ does not exist

Frequency eqn. is $\begin{vmatrix} k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$

$\Rightarrow k^2 - 2mk\omega^2 - k^2 + m^2\omega^4 = 0$

$\Rightarrow \omega^2(m^2\omega^2 - 2mk) = 0$

$\Rightarrow \omega_1 = 0$

$\omega_2 = \sqrt{\frac{2k}{m}}$

$\omega = 0 \Rightarrow$ Rigid body mode of motion is possible

CM moves with uniform velocity or remains at rest

$$\boxed{0 < \omega_1 < \omega_2 < \dots < \omega_{n-1} < \omega_n}$$

When $\omega_1 = 0$, we can reduce the DOF of the system by one (1) & then apply MI method as usual

Analytical Methods in Vibrations
by Leonard Meirovitch

For $\omega_1 = 0$, $\{A_1\} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ normalized

$\omega_1 = 0$, $\{A_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$(2k - m\omega_1^2)A_{11} - kA_{21} + 0 \cdot A_{31} = 0$

$-kA_{11} + (2k - m\omega_1^2)A_{21} - kA_{31} = 0$

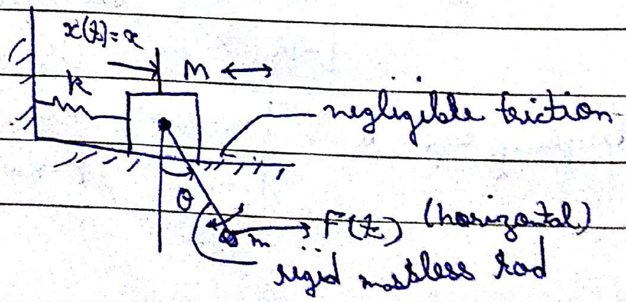
$0A_{11} - kA_{21} + (k - m\omega_1^2)A_{31} = 0$

These 3 eqns are not linearly independent. Take any 2 eqns, solve for A_{21} & A_{31} in terms of A_{11}

$\{A_1\} = \begin{Bmatrix} A_{11} \\ \alpha A_{11} \\ \beta A_{11} \end{Bmatrix}$

Lagrange's eqn. for 2 & 3 DOF systems

(Ex 1)



Let x, θ be a set of generalised coordinates

We have a 2 DOF system

Obtain the DEOM using Lagrange's eqns.

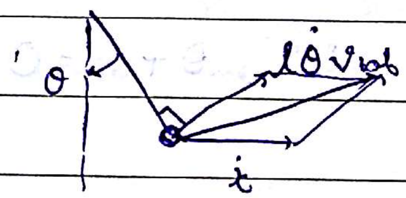
Linearize the DEOM

The Ls eqns are

① $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = Q_1$ ← generalized forces

② $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = Q_2$

No damping $\begin{cases} \frac{\partial D}{\partial \dot{x}} = 0 \\ \frac{\partial D}{\partial \dot{\theta}} = 0 \end{cases}$

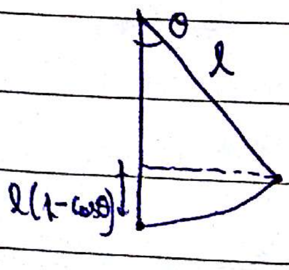


$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m v_{bob}^2$

$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right]$

Note that T is a function of θ, a gen coordinate

$\frac{\partial T}{\partial \theta} \neq 0$



$U = \frac{1}{2}Kx^2 + mgl(1-\cos\theta)$

$\delta W_x = F(t).\delta x \; [\delta\theta = 0]$

By def^n,

$Q_1 = \frac{\delta W_x}{\delta x} = F(t)$

x is kept const.

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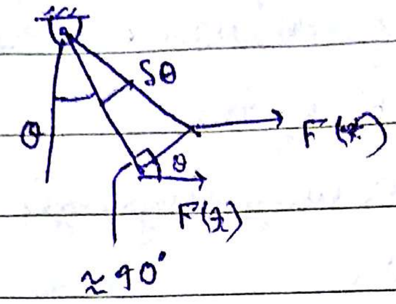

$(x \text{ const.}) \rightarrow \delta W_\theta = F(t)\cos\theta\,(l\cdot\delta\theta)$

$$\frac{\delta W_\theta}{\delta\theta} = F l \cos\theta = Q_2$$

$$\frac{\partial T}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial U}{\partial x} = kx$$

Hence DEOM (1) is

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + kx = 0 \quad \text{—— (A)}$$

(1) is a non linear DEOM. To linearize it, we must assume

$$\dot{\theta}^2\sin\theta \approx 0 \quad \& \quad \cos\theta \approx 1$$

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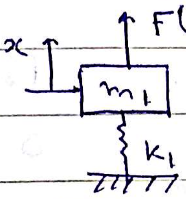
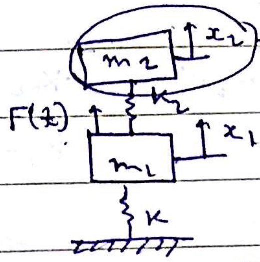
(8) The undamped dynamic vibration absorber (The tuned damper)

The basic problem :- We have a dynamic system approximately modeled as a simple spring-mass system. The mass is subjected to a force $F(t) = F_0 \sin(\omega_n t)$

It is reqd. to avoid resonance by using another spring-mass system



(original system)



→ the absorber

Given :- $\omega_n = \sqrt{\frac{k_1}{m_1}} = \omega_f$

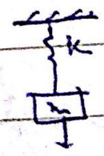
Aim :- To change m_2 & k_2 such that the amplitude of m_1 (i.e. x_1) is acceptable.

The DEOM of the main system + absorber are

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_0 \sin \omega_f t \quad (1)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \quad (2)$$

Keeping in mind what happens to the forced vibration of



$F = F_0 \sin \omega t$

, we assume $(x_1)_{forced} = x_1 = X_1 \sin \omega_f t$

& $(x_2)_{forced} = x_2 = X_2 \sin \omega_f t$

$\ddot{x}_1 = -X_1 \omega_f^2 \sin \omega_f t$

$\ddot{x}_2 = -X_2 \omega_f^2 \sin \omega_f t$

Substitute these in (1) & (2)

This gives $(k_1 + k_2 - m_1 \omega_f^2) X_1 - k_2 X_2 = F_0$

$$-k_2 x_1 + (k_2 - m_2\omega_f^2)\,x_2 = 0$$

So, if $\omega_f \neq \omega_1$ or ω_2 (ω_1, ω_2 are the nat frequencies of the 2DOF system)

So by Cramer rule

$$x_1 = \frac{\begin{vmatrix} F_0 & -k_2 \\ 0 & k_2 - m_2\omega_f^2 \end{vmatrix}}{\Delta} = \frac{F_0\,(k_2 - m_2\omega_f^2)}{\Delta}$$

$$x_2 = \frac{\begin{vmatrix} k_1 + k_2 - m_2\omega_f^2 & F_0 \\ -k_2 & 0 \end{vmatrix}}{\Delta}$$

where $\Delta = \begin{vmatrix} k_1 + k_2 - m_1\omega_f^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_f^2 \end{vmatrix}$

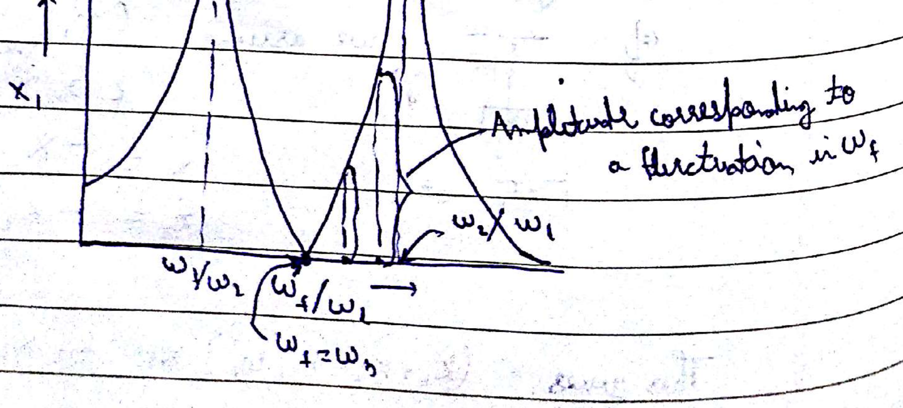
Here, $x_1 = 0$ if $k_2 - m_2\omega_f^2 = 0$

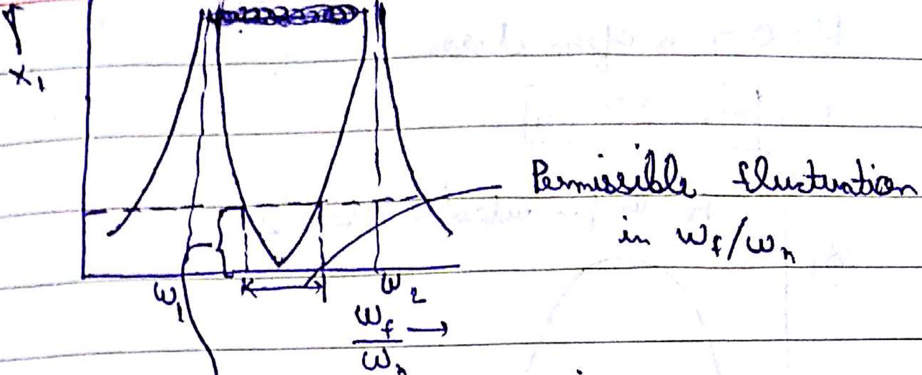
i.e if $\frac{k_2}{m_2} = \omega_f^2$

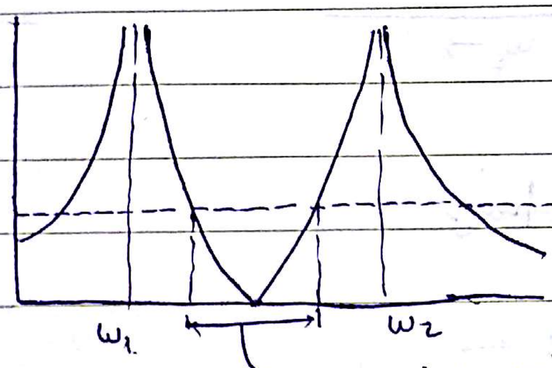
Since $\omega_f = \omega_n$, this implies

$$\boxed{\frac{k_2}{m_2} = \frac{k_1}{m_1}} \quad — (3)$$

(3) gives us a way of choosing k_2 & m_2. The forced response of amplitudes look like the following







Make $(\omega_2 - \omega_1)$ as large as possible

The frequency eqn. is

$$\begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (k_1+k_2)k_2 - \omega^2\left[k_2 m_1 + (k_1+k_2)m_2\right] + m_1 m_2 \omega^4 - k_2^2 = 0$$

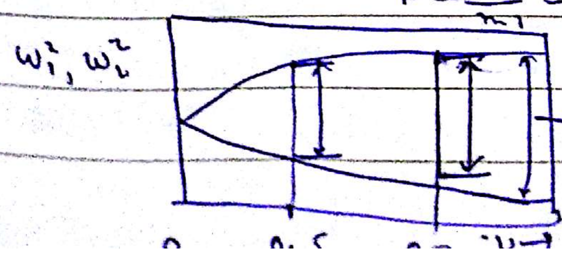
$$\Rightarrow m_1 m_2 \omega^4 - \omega^2\left[k_2 m_1 + (k_1+k_2)m_2\right] + k_1 k_2 = 0$$

$$\Rightarrow \omega^4 - \omega^2\left[\frac{k_2}{m_2} + \frac{(k_1+k_2)}{m_1}\right] + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\Rightarrow \omega^4 - \omega^2 \omega_n^2\left[2 + \frac{m_2}{m_1}\right] + \omega_n^4 = 0 \quad \left[\omega_n = \frac{k_2}{m_2} = \frac{k_1}{m_1}\right]$$

This gives ω_1 & ω_2

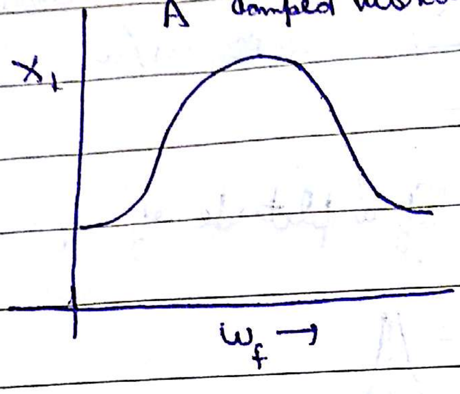
$\mu = \frac{m_2}{m_1}$ = The mass ratio



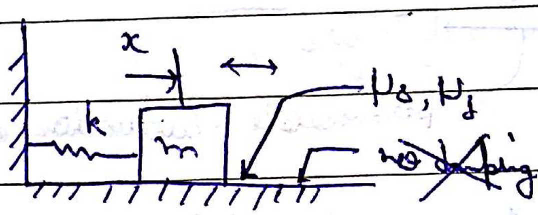
$\mu = 0.7$ is often chosen

i.e $\boxed{m_2 = 0.7\, m_1}$

A damped vibration Absorber



Coulomb Damping

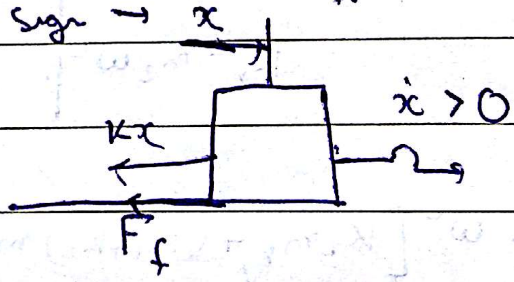


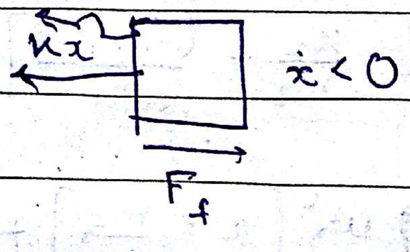
The DEOM is $m\ddot{x} + kx = \pm F_f$

where F_f = force of friction $= \mu_d mg$, while m is in motion

+ive sign → while the mass moves to the right.

− ve sign → " " " " left.





$$\rho = \frac{2.9}{\frac{\pi (36)(1)^2}{4}} = 0.103 \text{ lbf/in}^3$$

This is closest to the density of Aluminium which also satisfies the physical properties.

(2-23) Performing the visual magnetic and scratch tests ~~lead to~~ hint towards a non-ferrous copper based material.
Performing the weight test to find density gives

$$\rho = \frac{W}{Al} = \frac{9}{\frac{\pi(1)^2}{4}\times 36} = 0.318 \text{ lbf/in}^3$$

This is close to copper (0.322 lbf/in^3) and brass (0.31 lbf/in^3).

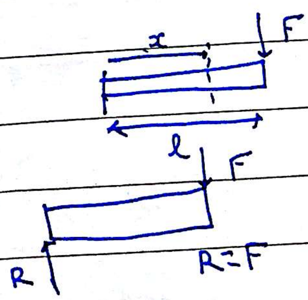
The deflection test of end-loaded cantilever beam is performed to find the Young's modulus.

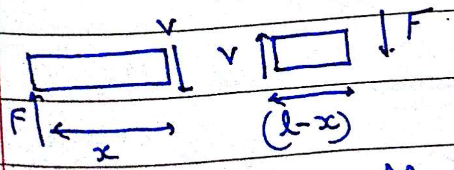
$$\delta = \frac{Fl^3}{3IE} \Rightarrow E = \frac{Fl^3}{3I\delta}$$

$$= 17.7\times 10^6 \text{ psi}$$

This is quite close to Cu (17.2 Mpsi).
Thus the material is copper.

(2-28) $\sigma = \frac{M_y z}{I_y}$





$V = F$ $\qquad M_y = Vx = Fx$ $\qquad$ Max at $x = l$

$$\sigma_{max} = \frac{Fl h}{2 I_y} = \frac{CFl}{h^3}$$

$$\sigma_{max} = \frac{CFl}{A^{3/2}} \quad (I_y \propto h^4)$$

To avoid failure $\sigma_{max} \leq S$

$$\Rightarrow \frac{CFl}{A^{3/2}} \leq S$$

$$\Rightarrow A \geq \left[\frac{CFl}{S}\right]^{2/3}$$

$$m = A\rho l \geq \left[\frac{CF}{S}\right]^{2/3} \rho \left[l\right]^{5/3}$$

$$m \geq (CF)^{2/3} \frac{\rho}{S^{2/3}} (l)^{5/3}$$

l, C, F are fixed

we need to check $\frac{\rho}{S^{2/3}}$

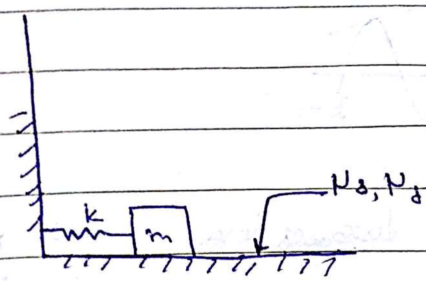
when $\frac{\rho}{S^{2/3}}$ is mi m will be minimum

Thus $S^{2/3}/\rho$ should be max. From the chart by drawing lines

$\frac{\rho}{S^{2/3}}$ ~~minimum for aluminium alloy~~

b|| to $S^{2/3}/\rho = C$ we see Al alloy is the best suited material since the value of $S^{2/3}/\rho$ for Al alloy is max.

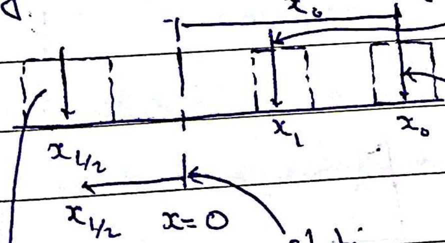
Coulomb Damping (contd.)



$$m\ddot{x} + Kx = -F_f \quad \text{while } \dot{x} > 0$$

Then $x = A\sin\omega_n t + B\cos\omega_n t - \frac{F_f}{K}$ is the complete response.

This is valid until t becomes t_1 such that $\dot{x}(0) = 0$. After $t = t_1$, the motion is governed by $m\ddot{x} + Kx = +F_f$, provided $x(t_1)$ is sufficient to overcome $\mu_s mg$. This clearly shows that the oscillations occur at the undamped natural frequency ω_n with diminishing amplitude.



position at the end of the cycle

position of mass m at the beginning of a cycle of motion

static eqm position

position of m after half cycle of motion

For movement b/w x_0 & $x_{1/2}$

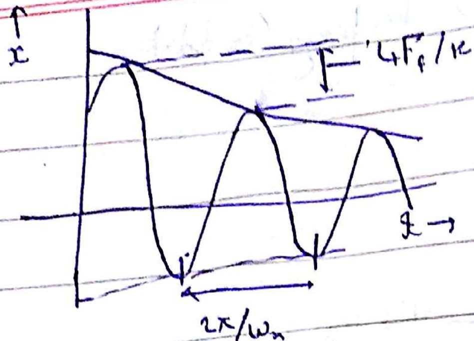
$$\frac{1}{2}k(x_0^2 - x_{1/2}^2) = F_f(x_0 + x_{1/2})$$

$$\Rightarrow x_0 - x_{1/2} = \frac{2F_f}{k}$$

Aim :- $x_0 - x_1 = ?$

Similarly, $x_{1/2} - x_1 = \frac{2F_f}{k}$

$\Rightarrow x_0 - x_1 = \frac{4F_f}{k}$ = reduction in amplitude of free vibration per cycle of motion

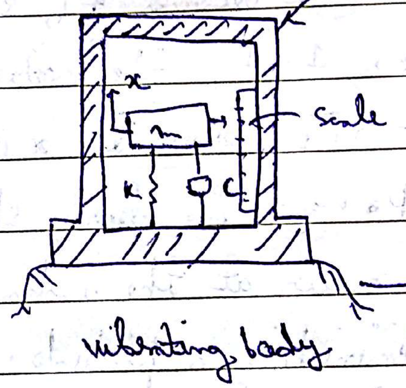


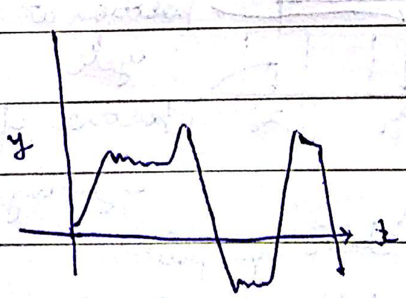
$x_0, x_{1/2}, x_1 \rightarrow$ distances (+ve) $\quad x(0), \dot{x}(0)$ $\downarrow$ given

§ Vibration measuring Instruments

Vibrometer – Basic setup

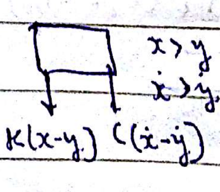
(Laser Doppler ~~vibrat~~ vibrometer)





Aim :- To measure y $(\dot{y}, \ddot{y})$

The DEOM is (for m)



$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

$$= mY_0\omega^2 \sin \omega t$$

where $z = x - y$

Hence for forced vibration $z = \dfrac{mY_0 \omega^2/k}{\sqrt{(1-r^2)^2 + (2\rho r)^2}} \sin(\omega t - \psi)$ — ②

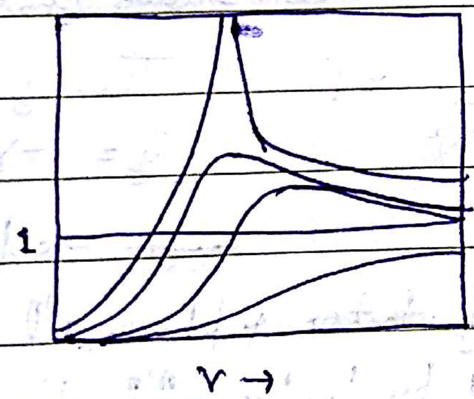
$$\tan \psi = \frac{2\rho r}{1-r^2}$$

Hence if $\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\rho r)^2}} = 1$ → can be satisfied only if r is large i.e ω_n is small ⇒ ?

then $z = Y_0 \sin(\omega t - \psi)$

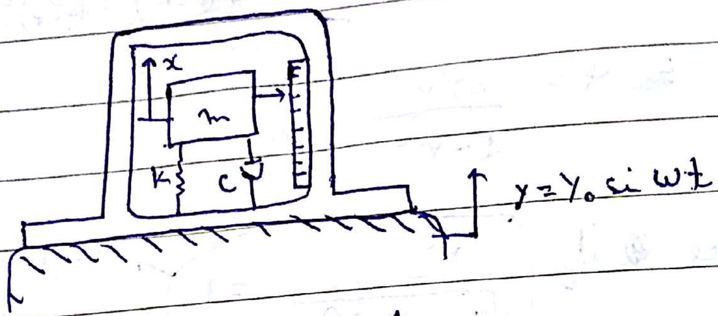
Then z faithfully reproduces y except for the phase lag ψ which is equivalent to a time lag ψ/ω & can be easily taken care of.

$\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\rho r)^2}}$



Hence, vibrometers (which measures the displacement of a vibrating body) turn out to be bulky & these are not commonly used now-a-days

Accelerometers



We have seen, that

$$z_{ss} = z = x - y = \frac{Y_0 r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \varphi)$$

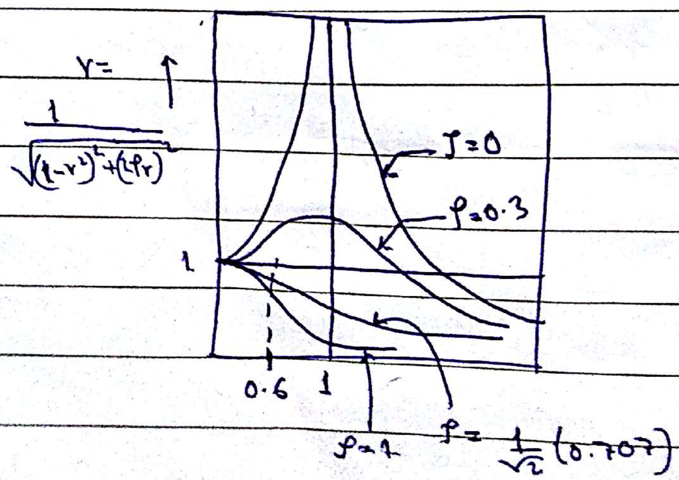
$$\left(r = \frac{\omega}{\omega_n}\right)$$

$$\Rightarrow -\omega_n^2 z = \frac{-Y_0 \omega_f^2 \sin(\omega t - \varphi)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

& this relation shows that if

$$\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \approx 1$$

Then z reproduces the accn. $\ddot{y} = -Y_0 \omega^2 \sin \omega t$ faithfully ~~excpt~~ except for the sign & a multiplying factor & phase all of which can be easily taken care of by proper calibration of the scale. Hence, for our spring-mass-damper system to act as an accelerometer, we must have $\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \approx 1$.



So, if $\zeta = 0.707$, $r < 0.6$, then Y will be close to unity.

Hence, an accelerometer should have a large ω_n, i.e a small m & large k, both of which can be easily achieved.

Let $y = \sum_{n=1}^{\infty} Y_n \sin n\omega t$ [i.e y is periodic with period $= \frac{2\pi}{\omega}$]

Hence $z = \sum_{n=1}^{\infty} \frac{Y_n r_n^2}{\sqrt{(1-r_n^2)^2 + (2\rho r_n)^2}} \sin(n\omega t - \psi_n)$

where $\tan\psi_n = \frac{2\rho r_n}{1-r_n^2}$, $r_n = \frac{n\omega}{\omega_n}$

Hence, for our accelerometer to work properly, we must have $\frac{1}{\sqrt{(1-r_n^2)^2 + (2\rho r_n)^2}} \approx 1$ & also $\frac{\psi_n}{n\omega} =$ constant

$\sin[n\omega t - \alpha]$

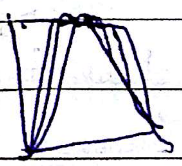
$\uparrow$

$\sin[n\omega(t - \frac{\psi_n}{n\omega})]$

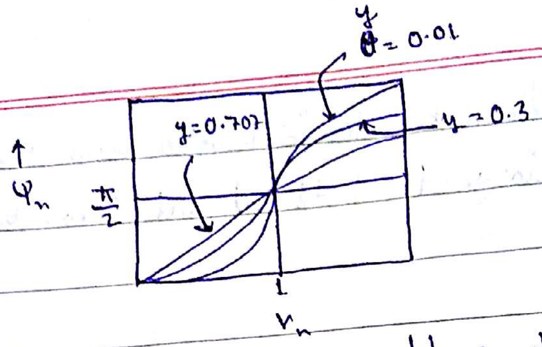
$$-\omega_n^2 z = \sum_{n=1}^{\infty} \frac{-Y_n (n\omega)^2}{\sqrt{(1-r_n^2)^2 + (2\rho r_n^2)^2}} \overbrace{\sin(n\omega t - \psi_n)}$$

$$\ddot{y} = \sum_{n=1}^{\infty} -Y_n (n\omega)^2 \sin(n\omega t)$$

Hence by making ω_n sufficiently high, we can easily make $r_n < 0.6$ for values of n upto, say, 5.
For $n > 5$, the contribution from higher harmonic would be quite negligible & hence the set up works.



$n = 1, 2, 3,$ etc.



For $y = 0.707$,

ψ_n v/s r_n is a straight line $\therefore$ Hence, $\psi_n = \beta r_n$

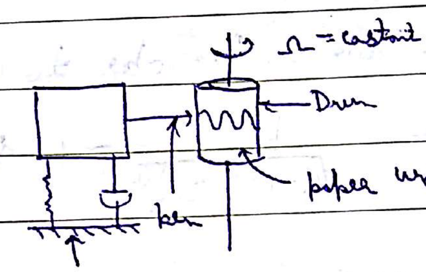
(β = const.)

$$\psi_n = \beta \frac{n\omega}{\omega_n}$$

$$\frac{\psi_n}{n\omega} = \frac{\beta}{\omega_n} = \text{const.} = \alpha$$

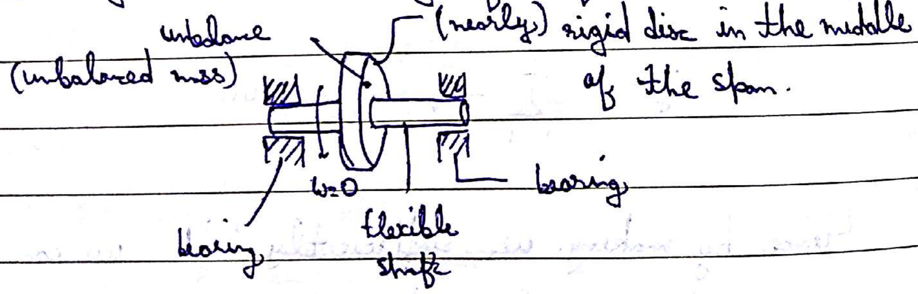
Here, the phase distortion error is taken care of.

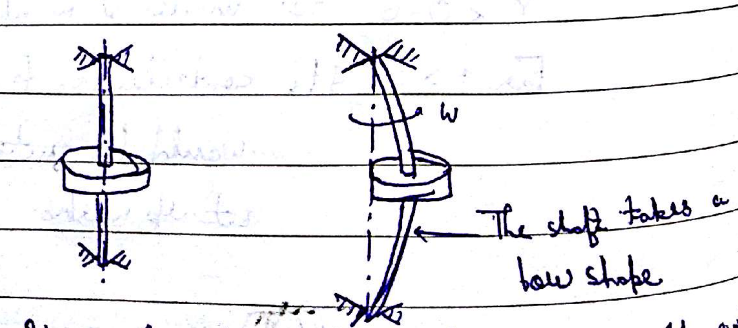
$$-\omega_n^2 z \approx \sum_{n=1}^{\infty} -Y_n (n\omega)^2 \sin(n\omega\tau) \text{ where, } \tau = t - \alpha$$



§ Rotor Dynamics

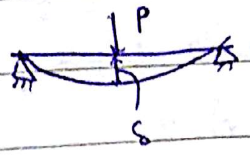
→ The synchronous whirling of shafts.





It can be shown that whenever ω = the natural frequency of transverse vibration, resonance occurs & large amplitude vibrations can destroy the s...

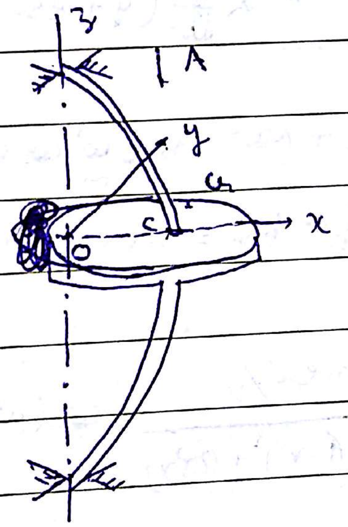
$$\delta = \frac{Pl^3}{48EI} \Rightarrow k = \frac{P}{\delta} = \frac{48EI}{l^3}$$

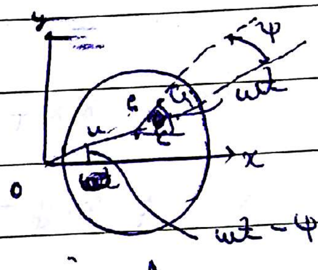


ω_n for beam or lateral or transverse vibration $\omega = \sqrt{k/m}$

Here the plane made by the bent axis of the shaft and the axis of the bearings rotate which is called whirling of the shaft.

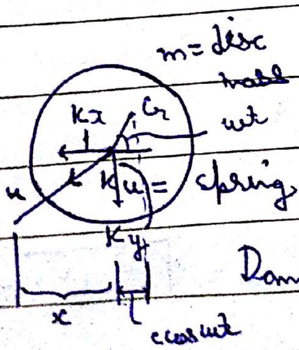
In asynchronous whirl, the shaft and disc have different angular velocities.





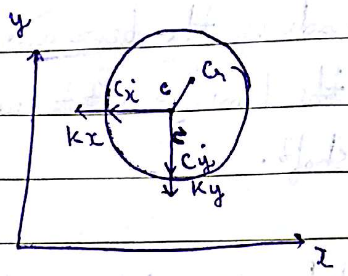
view - A

Let us consider a special situation when the geometric center C of the disc/rotor is different from its centre of gravity G. The shaft is connected at C. This situation gives rise to unbalance & whirling would occur.



= spring force applied by shaft and disc

Damping force on rotor (due to say air friction) is assumed to be proportional to the speed of the geometric centre. Hence, the damping force components would be $C\dot{x}$ & $C\dot{y}$ in -ve x & -ve y dir^n c being equivalent viscous damping constant.



FBD of disc at time t
(neglecting weight)

So, $m \times \frac{d^2}{dt^2}(x + e\cos\omega t) = -kx - C\dot{x}$

$m \times \frac{d^2}{dt^2}(y + e\sin\omega t) = -ky - C\dot{y}$

$$\left.\begin{array}{l} m\ddot{x} + C\dot{x} + kx = me\omega^2\cos\omega t \\ \& \quad m\ddot{y} + C\dot{y} + ky = me\omega^2\sin\omega t \end{array}\right]$$

$$X_{ss} = x = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \cos(\omega t - \phi) \qquad r = \frac{\omega}{\omega_n}$$

$$Y_{ss} = y = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega t - \phi)$$

$$x = \frac{er^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \cos(\omega t - \phi)$$

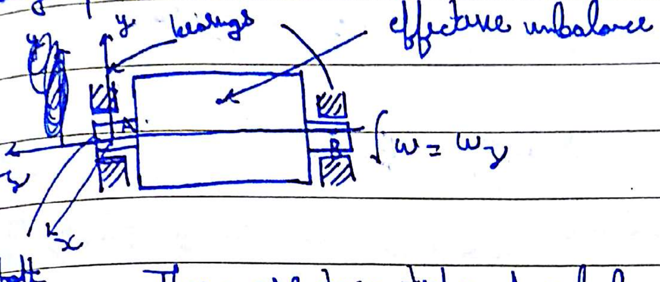
$$y = \frac{er^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega t - \phi)$$

Hence $u = \sqrt{x^2 + y^2} = \dfrac{er^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

(2) Balancing of m/cs

Effects of unbalance:

1) Large (unacceptable) levels of vibrations.
2) Fatigue / fracture failure of bearings
3) Noisy operation.



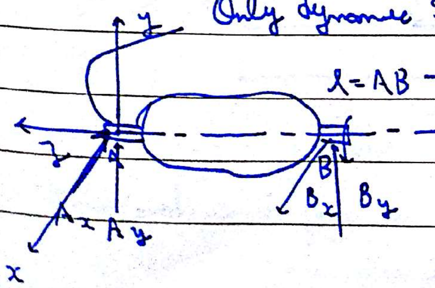
There are two types of unbalance:

(i) Static unbalance
(ii) Dynamic "

The system is said to be in static balance if its centre of mass lies on the axis of rotation. It is said to be in dynamic balance if it is in static balance & in addition, the axis of rotation is a principal axis at any point on the axis of rotation.

If the rotor-shaft system is in static balance, it will remain in equilibrium at any angular position. If the system is in dynamic balance, no unbalanced rotating forces will act on the bearings & hence vibration & failure of the setup may be avoided.

Only dynamic reactions are shown



$l = AB \rightarrow$ distance b/w midpoints of bearings A & B

A_x, A_y, B_x, B_y are components of bearing reactions at A & B.

$z \to$ axis of rotation

We have plane motion, since every particle of the system moves in a plane ||l to the xy-plane. Hence the MOM eqns. are :-

$$M_x = B_y l = I_{xz}\dot{\omega}_z - I_{yz}\omega_z$$

$$M_y = B_x l = -I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2$$

$$M_z = I_{zz}\dot{\omega}_z$$

where $[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{zy} & I_{zz} \end{bmatrix}$ is the inertia term for the shaft rotor system at A.

Here if $I_{xz} = I_{yz} = 0$, then $B_y = 0$ & $B_x = 0$

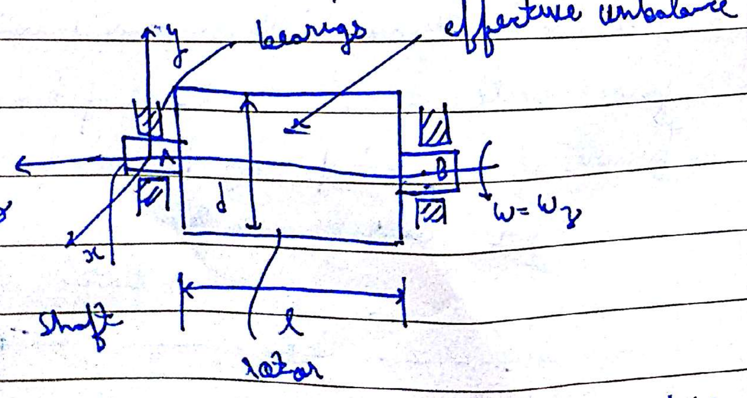
Here $A_x = 0$ & $A_y = 0$

They would balance the weight of the setup at all times. | Thus no rotating fatigue causing forces on the bearings.

Finally, the necessary & sufficient condition for the dynamic balance of a rotor system

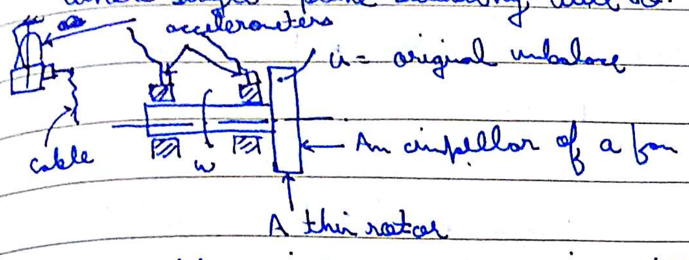
(1.) The CM must lie ~~of~~ on axis of rotation

(2.) This axis must be a principal axis at any pt. on the axis of rotation.



If $l/d \sim 1$ we call it a long rotor which requires 2 plane balancing.

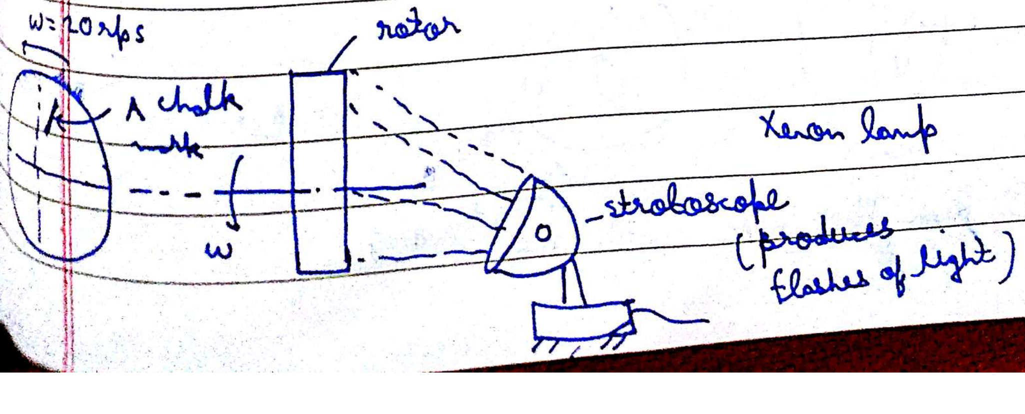
If $l \approx 0.5d$ or less, we may call it a thin rotor, where single-plane balancing will do.

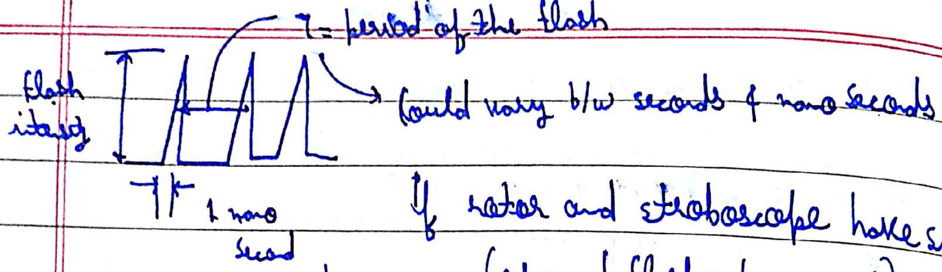


Here single plane balancing will do.

(3) Single-plane field balancing of rotors. This method is applicable to thin rotors only.

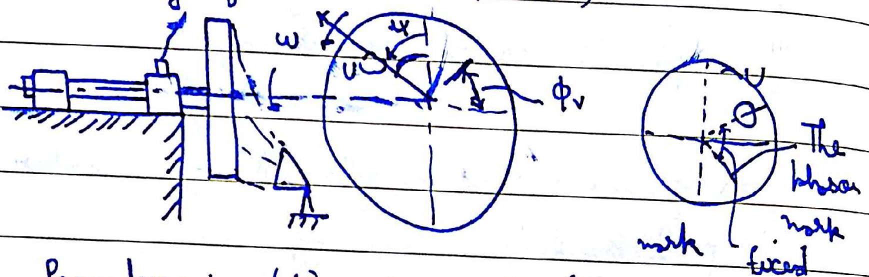
A General Purpose Vibration Analyzer contains mechanical & electrical / electronic components.
The electronic / electrical unit converts the mechanical / electrical output of the vibration transducers (accelerometers) to suitable form using frequency filters.
There is a stroboscopic unit for measuring phase angles.
(Modern GPVA, have phase meters too.)
The rotor is run & an a unacceptable amplitude of vibration is observed in the transverse dir^n at the rotational frequency (ω) of the shaft.
It is very likely caused by an unbalance U in the rotor.
Aim :- To measure U & take corrective action to eliminate unbalance.





If rotor and stroboscope have same frequency (rps and flashes per second) chalk mark will appear stationary. If 2 rps frequency and 1 rps frequency then 2 chalk marks will be seen

Field Balancing of thin rotors (contd.)

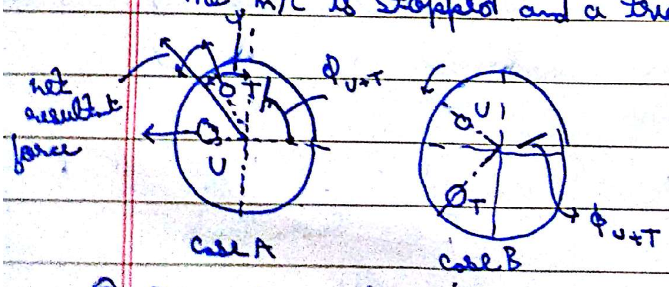


Procedure :- (1.) Make a valid (phaser) on the rotor

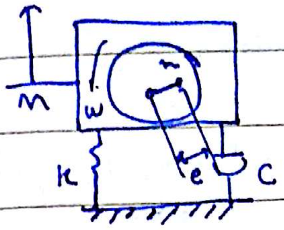
(2.) Adjust the GPVA so that it locks on frequency ω & the stroboscope flashes at the moment when the vibration pick-up A undergoes maxm. deflection in the ~~vertical~~ vertical direction.

(3.) The m/c is run with original unbalance U & the position of the phase mark from RH horizontal is measured to be ϕ_v. The amplitude A_u is also measured.

(4.) The m/c is stopped and a trial unbalance T is put.



(5) Run the m/c with trial unbalance



$$x_{forced} = x(t) = \left[\frac{\frac{U - (m e r^2)}{M}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \right] \sin(\omega t - \psi)$$

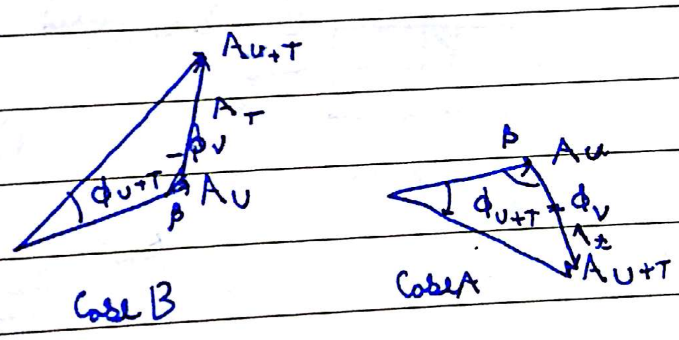
Amplitude

$$\psi = \tan^{-1} \left[\frac{2\zeta r}{1-r^2} \right]$$

A_T = amplitude of vibration with trial unbalance

$$\frac{A_T}{A_U} = \frac{T}{U}$$

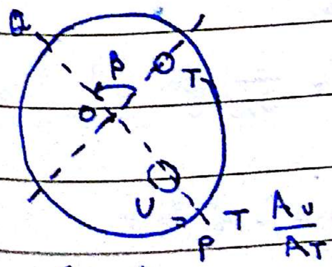
$$\boxed{A_U = A_T \frac{U}{T}}$$



$$\boxed{A_T^2 = A_{U+T}^2 + A_U^2 - 2A_{U+T}A_U \cos(\phi_{U+T} - \phi)}$$

Case A

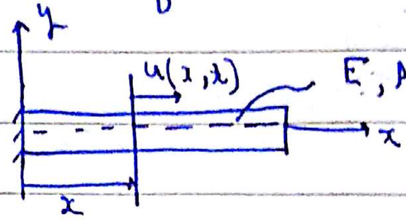
To balance the m/c, either some mass is removed at a convenient location on radius OP or, a mass is added (welded / glued) at a convenient location on OQ.



Case A

$$\boxed{m_c r_c = U}$$

* Vibration of Continuous Systems



$E, A(x), l, m(x)$

$m(x) \to$ mass per unit length at x

$A(x) \to$ x-sectional area at location x.

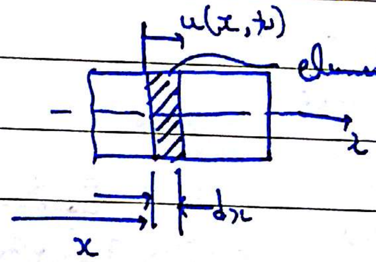
A continuous system has infinitely many natural frequency

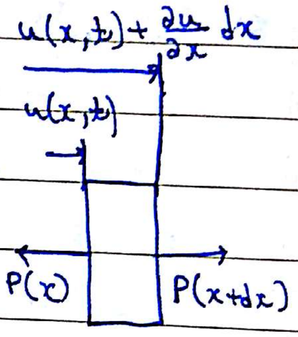
An example:-

Let us consider the axial free-vibration of a clamped-free bar.

Let $u(x,t)$ = axial deflection due to free vibration at location x & time t.

Aim:- To obtain the DEOM.





$$\epsilon_x = \frac{\frac{\partial u}{\partial x}dx + u(x,t) - u(x,t)}{dx}$$

Axial force at x : $\frac{\partial u}{\partial x} = \epsilon_x$

$$= \sigma_x A(x) = EA(x)\frac{\partial u}{\partial x} = P(x)$$

So axial force at $x+dx$ is

$$P(x+dx) = EA(x)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]dx$$

Hence, by Newton's 2nd law,

$$\underbrace{m(x)\,dx}_{\text{mass of element}} \times \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration of element}} = P(x+dx) - P(x)$$

$$= \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]dx$$

Hence

$$m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]$$ is the reqd.

DEOM

The boundary conditions are :-

$u(x,t) = 0$ at $x = 0$ at all times

$EA\left.\frac{\partial u}{\partial x}\right|_{x=l} = 0$ at all times.

12th Apr

The DEOM of a bar in axial vibration was seen to be

$$m(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] \quad \text{—— (1)}$$

For a uniform bar, it reduces to

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{EA}{m}\frac{\partial^2 u}{\partial x^2}}$$

or $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{—— (2)}$$

where $c = \sqrt{\frac{EA}{m}}$ = wave speed

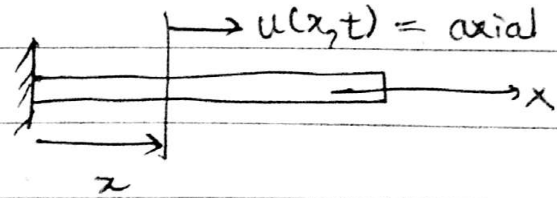
(2) is a 1D wave equation

To solve it, we use the method of variable separation

Let $u(x,t) = U(x)\, f(t)$ —— (3)

where $U(x)$ is an eigen function

and $f(t)$ is a generalized coordinate



$u(x,t)$ = axial displacement at location x and time t.

Substituting of (3) in (2) gives.

$$U(x)\ddot{f}(t) = c^2 U'' f \quad \text{or} \quad \frac{\ddot{f}(t)}{f(t)} = \frac{c^2 U''}{U} \quad \text{—— (4)}$$

where $\ddot{f}(t) = \dfrac{d^2 f}{dt^2}$

$$U'' = \frac{d^2 U(x)}{dx^2}$$

$\cancel{f''(t) - \omega^2 f(t) = 0}$

$\cancel{f(t) = ce^{st}}$

$\cancel{s^2 - \omega^2 = 0 \quad s = +\omega \quad s_1 = -\omega}$

Each ratio in (4) must be a constant

Hence $\dfrac{\ddot{f}(t)}{f} = \dfrac{c^2 U''}{U} = -\omega^2$, a -ve constant

(Otherwise f(t) would increase exponentially which isn't possible for our system executing stable oscillations

So then,

$$\ddot{f}(t) + \omega^2 f(t) = 0 \quad \text{—— (5)}$$

and $$\frac{d^2 U}{dx^2} + \left(\frac{\omega}{c}\right)^2 U = 0 \quad \text{—— (6)}$$

from (5), we get

$$f(t) = A \sin \omega t + B \cos \omega t \quad \text{—— (7)}$$

From (6).

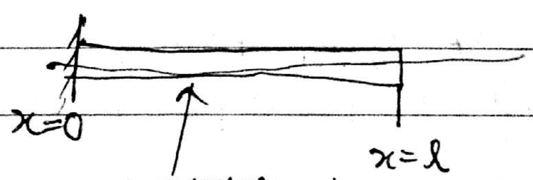
$$U = C \sin \beta x + D \cos \beta x \quad \text{—— (8)}$$

where $\beta = \dfrac{\omega}{c}$

A and B are to be obtained from given initial conditions $u(x,0)$ and $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}$

To get C and D, we need the BCs

Let our bar be clamped-free



clamped free bar in axial free vibrations

The BCs are

$$u(0,t) = 0, \quad \left. EA \frac{\partial u}{\partial x} \right|_{x=l} = 0$$

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$u(0,t) = U(0)\, f(t) = 0 \Rightarrow U(0) = 0$ — (9)

$\left.\frac{\partial u}{\partial x}\right|_{x=l} = \left.\frac{dU}{dx}\right|_{x=l} \times f(t) = 0 \Rightarrow \left.\frac{dU}{dx}\right|_{x=l} = 0$ — (10)

From (8),

$U(0) = D = 0$

Then $U(x) = A \sin \beta x$.

$\left.\frac{dU}{dx}\right|_{x=l} = A\beta\cos\beta x\Big|_{x=l} = A\beta\cos\beta l = 0$

$\Rightarrow \boxed{\cos\beta l = 0}$ — (11) the frequency eqn

$\Rightarrow \beta_n l = (2n-1)\frac{\pi}{2}$; $n = 1, 2, 3, \ldots \infty$

$\Rightarrow \omega_n = (2n-1)\frac{\pi c}{2l}$

$\boxed{\beta = \frac{\omega}{c}}$

$\boxed{\beta_n = \frac{\omega_n}{c}}$

$\Rightarrow \boxed{\omega_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{EA}{ml^2}}}$ $n = 1, 2, 3, \ldots$

Hence, the bar has ∞ly many natural frequencies of which the first few matter in practise.

Also, $U_n(x) = C_n \sin\frac{\omega_n x}{c} + D_n \cos\frac{\omega_n x}{c}$

Finally, $f(t) = A_n \sin\omega_n t + B_n \cos\omega_n t$

So, $u_n(x,t) = U_n(x)\, f_n(t)$ is a soln of eqn (2).

Thus, by the principle of superposition, (since ans is a linear system) the general free vibration response is given as

$$u(x,t) = \sum_{n=1}^{\infty} U_n(x)\, f_n(t) = \sum_{n=1}^{\infty} \left(A_n' \sin\omega_n t + B_n' \cos\omega_n t\right) \sin\frac{\omega_n x}{c}$$

A_n' and B_n' are evaluated using the given initial conditions.

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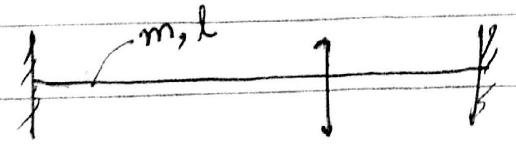
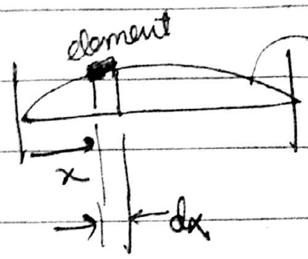
(5) Transverse free vibrations of a stretched string with a high initial tension T.

(T does not change appreciably during vibration)

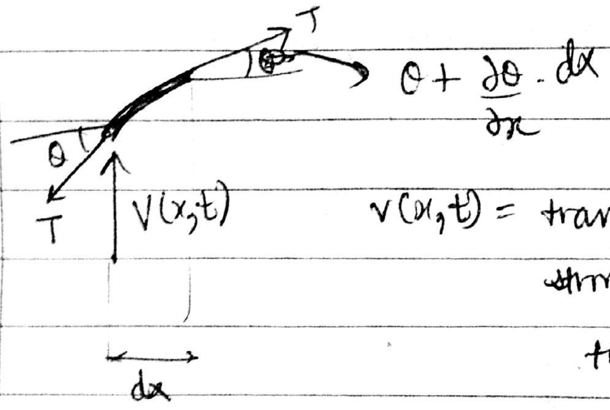


$m = \frac{\text{mass}}{\text{length}}$ = linear density

Initial configuration



at time t



$v(x,t)$ = transverse displacement of string at location x and time t

So, by newton's 2nd law,

$$m\,dx \cdot \frac{\partial^2 v}{\partial t^2} = \text{net component of } T \text{ in the } v \text{ direction}$$

Transverse acc^n of element

$$= T\sin\left(\theta + \frac{\partial\theta}{\partial x}dx\right) - T\sin\theta$$

Since θ is small

$$\sin\left(\theta + \frac{\partial\theta}{\partial x}dx\right) \simeq \theta + \frac{\partial\theta}{\partial x}dx$$

This gives

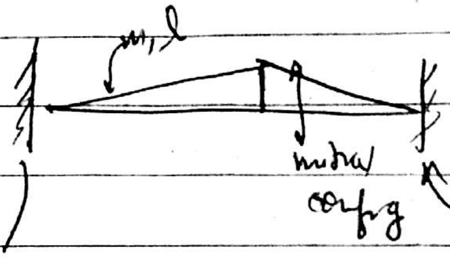
$$\boxed{m\frac{\partial^2 v}{\partial t^2} = T\frac{\partial \theta}{\partial x}}$$

Now $\theta \approx \frac{\partial v}{\partial x} =$ slope of the string/cable at location x and time t

Thus, we have, $\frac{\partial^2 v}{\partial t^2} = \frac{T}{m}\frac{\partial^2 v}{\partial x^2}$

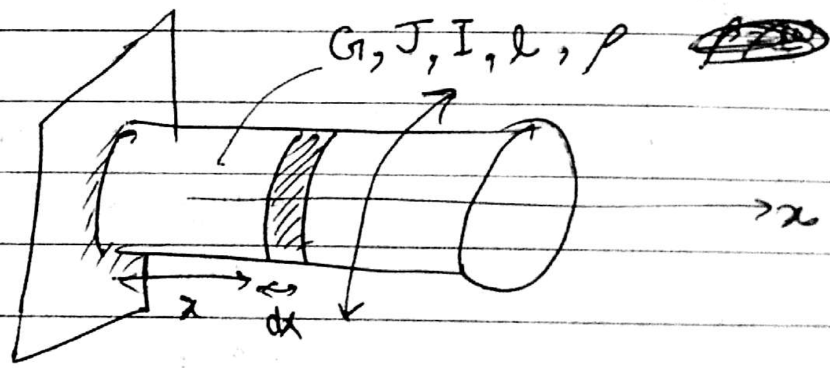
$$\Rightarrow \boxed{\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}}$$

$c = \sqrt{\frac{T}{m}}$ → wave speed in string



BC $v(0,t)=0$

(5) Torsional free oscillations of a circular bar



G — shear modulus

J → Polar moment of inertia about x axis

$J = \frac{\pi r^4}{2}$

I → moment of inertia / unit length about its own axis i.e x axis

$I_{total} = \frac{Mr^2}{2}$

l → length of shaft

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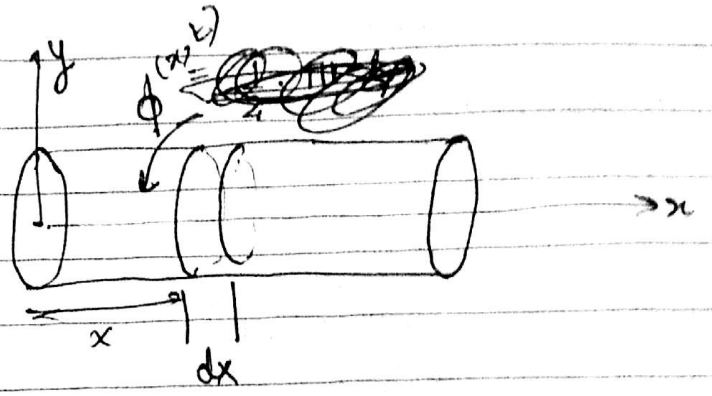
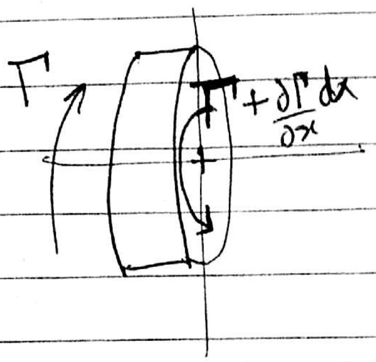
$$I \text{ (per unit length)} = \frac{I_{total}}{l} = \frac{1}{2}\frac{Mr^2}{l}$$

$M = \pi r^2 l \rho$





$\Gamma(x,t)$ = torque acting on section at location x, at time t

$$\frac{\Gamma}{J} = G\frac{\partial \phi}{\partial x} = \frac{G\,\phi_{max}}{t} \rightarrow \frac{\partial \Gamma}{\partial x} = \frac{\partial}{\partial x}\left[GJ\frac{\partial \phi}{\partial x}\right]$$

MOM eq^n for our element.

$$I dx \frac{\partial^2 \phi}{\partial t^2} = \cancel{\Gamma} + \frac{\partial \Gamma}{\partial x}dx - \cancel{\Gamma}$$

$$I\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[GJ\frac{\partial \phi}{\partial x}\right]$$

$$I(x)\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[G\,J(x)\frac{\partial \phi}{\partial x}\right]$$

If the shaft is uniform, then

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2}}$$

is the DEOM where

$$c = \sqrt{\frac{GJ}{I}}$$

One D wave eq^n.

= speed of shear waves in bar

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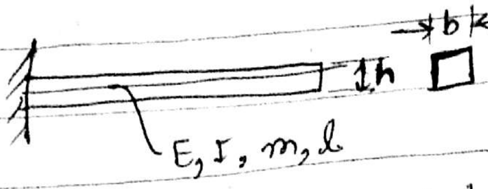
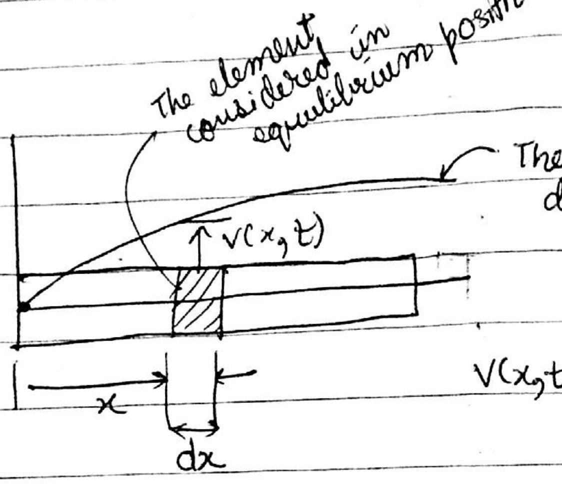
Transverse vibration of an Euler-Bernoulli Beam



A clamped-free (cantilever) EB beam

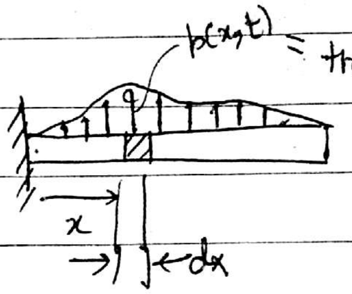
If the aspect ratio $\frac{l}{h}$ of a beam is ≥ 10, we may call it an EB beam.

Also, b should be comparable with h.

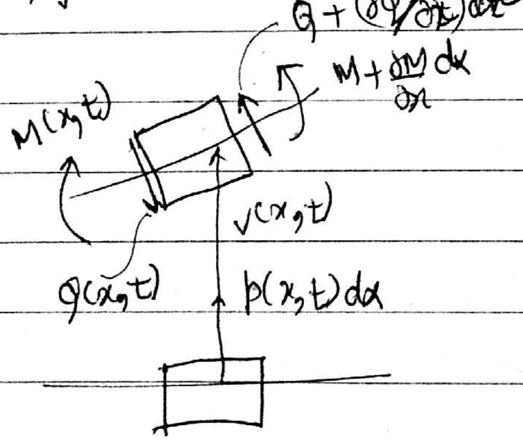


$V(x,t) \rightarrow$ lateral deflection of an element at location x at time t.

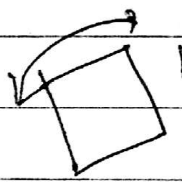
<u>Weight of beam is neglected</u>



$p(x,t)$ = transverse loading intensity = load/unit length at x at time t



If shear deformation is considered, we get a Rayleigh beam



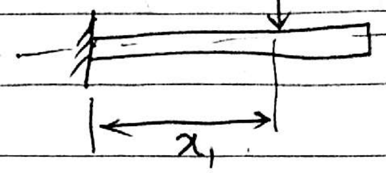
Rotary (Rotational) inertia of beam is neglected.

If it is taken into account, we get

a Timoshenko beam. These effects are important for short beams only.

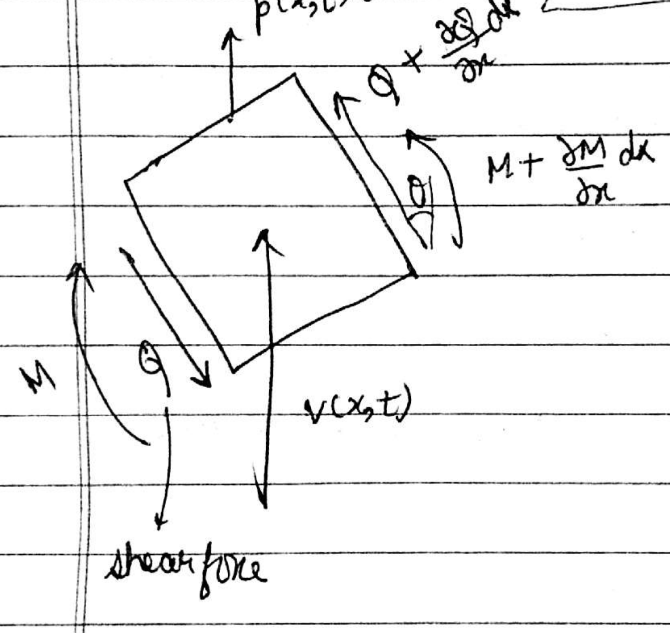
$\underline{\sin\theta \approx \theta \quad \cos\theta \approx 1}$. Also, slope of beam $= \frac{\partial v}{\partial x} = \tan\theta \approx \theta$

Assumptions: deflections as well as slope of beam remains small



$\delta(x-x_i) \rightarrow$ the spatial Dirac's Delta function

$\boxed{\cos\theta \approx 1}$



Using NLM (2^{nd}) in y direction we have

$$\underbrace{m\,dx}_{\text{mass of element}} \frac{\partial^2 v(x,t)}{\partial t^2} = p(x,t)\,dx - Q + Q + \frac{\partial Q}{\partial x}dx$$

$$\Rightarrow m\frac{\partial^2 v}{\partial t^2} = p(x,t) + \frac{\partial Q}{\partial x} \qquad \text{— (1)}$$

Hence, (i) reduces to

$$m\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 v}{\partial x^2}\right] = p \qquad \text{— (ii)}$$

which is the required DEOM

Taking moment about A,

$$Q\,dx + \frac{\partial M}{\partial x}dx = 0 \Rightarrow \boxed{Q = -\frac{\partial M}{\partial x}}$$

we know that

$$M = \text{bending moment} = EI \frac{\partial^2 v}{\partial x^2}$$

Also, $q = -\frac{\partial M}{\partial x}$

If $m \neq m(x)$

$I \neq I(x)$

i.e, if the beam is uniform (2) reduces to

$$\boxed{m \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^4 v}{\partial x^2} = p(x,t)}$$

which is the DEOM of an EB beam for forced vibration under a continuous loading.

(3) can be solved using the method of separation of variables for free vibrations, when $p(x,t) \equiv 0$

Hence, for free vibrations,

$$m \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^4 v}{\partial x^4} = 0$$

$$v(x,t) = V(x) \, f(t)$$

$V(x)$ → Eigen funn

$f(t)$ → generalized coordinate.

$$m \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^4 v}{\partial x^4} = 0 \quad \text{--- (i)}$$

$$v = V(x) \, f(t) \quad \text{--- (ii)}$$

$$\left.\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= V(x) \, \ddot{f}(t) \\ \frac{\partial^4 v}{\partial x^4} &= V^{iv}(x) \, f(t) \end{aligned}\right\} \text{Substitution in (i) results in}$$

$$m \, V(x) \, \ddot{f} = -EI \, V(x) \, f(t)$$

$$\Rightarrow \frac{\ddot{f}}{f} = -\frac{EI}{m}\frac{V_x^{(iv)}}{V(x)} = -\omega^2$$

$$\Rightarrow \ddot{f} + \omega^2 f = 0 \quad \text{—— (iii)}$$

$$\frac{d^4V}{dx^4} - \left(\frac{m\omega^2}{EI}\right)V = 0 \quad \text{—— (iv)}$$

(iv) represents an eigenvalue problem subject to proper BCs.

$$V = ce^{sx}$$

$$\Rightarrow s^4 - \beta^4 = 0$$

$$\Rightarrow s_1 = \beta \qquad s_3 = j\beta$$

$$s_2 = -\beta \qquad s_4 = -j\beta \qquad j = \sqrt{-1}$$

$$\frac{d^4V}{dx^4} - \beta^4 V = 0 \quad \text{—— (v)}$$

where $\beta^4 = \frac{m\omega^2}{EI}$

The general soln of (v) is

$V(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$.

A, B, C, D $\rightarrow$ to be determined using the 4 BCs.

Configuration	BC's
(1) Clamped-free (CF) 	$V = 0$, $\frac{\partial V}{\partial x} = \text{slope} = 0$ at $x=0$
	$EI\frac{\partial^2 V}{\partial x^2} = BM = 0$
	$-EI\frac{\partial^3 V}{\partial x^3} = \text{Transverse shear force} = 0$ at $x = L$

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② Simply supported (Pinned Pinned) (SSB)

$v=0, \; EI\frac{\partial^2 v}{\partial x^2}=0$ at $x=0$

$v=0, \; EI\frac{\partial^2 v}{\partial x^2}=0$ @ $x=l$ at all times

③ clamped-pinned CP

④ clamped-clamped

For simply supported Beam

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{ml^4}}$$

- The orthogonality of the eigen functions

For a uniform bar in axial, free vibration, the eigenfunctions are given by

$$U_r(x) = A_r \sin\beta_r x$$

We can easily show that

$$\int_0^l U_r(x)\, U_s(x)\, dx = 0 \quad \text{if } r\neq s.$$

$\beta_r = \frac{\omega_r}{c}$

$c = \sqrt{\frac{EA}{m}}$

We can also show that,

$$\int_0^l U_r(x)\, \frac{d^2U_s(x)}{dx^2}\, dx = 0 \quad \text{if } r\neq s.$$

These two relations are the orthogonality principle for the uniform bar in axial vibration.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

- The Expansion theorem for a continuous system.

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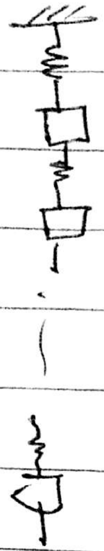

$a_1 < a_2 < \cdots < a_n$

$\{A_1\}, \{A_2\} \cdots \{A_n\}$ eigen vectors

Any n-dimensional vector $\{u\} = \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix}$

can be expressed as.

$\{u\} = c_1\{A_1\} + c_2\{A_2\} + \cdots + c_n\{A_n\}$.

$c_i \rightarrow$ constants, not all zero.

Any continuous bounded function of x can be expressed as a linear combination of the eigenfunctions,

i.e, $f(x) = \sum_{i=1}^{\infty} c_i u_i(x)$