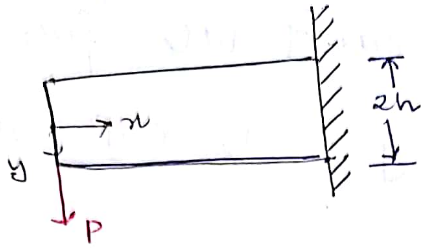
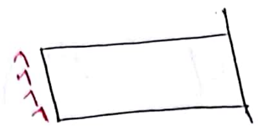
2D ELASTICITY ... CONTD.

- Bending of a cantilever under vertical end loading.





$$\int_{-h}^{h} \sigma_{xy} \cdot dy \cdot 1 = -P \quad \text{—— (1)}$$

$$\sigma_{xy}(x, \pm h) = 0 \quad \text{—— (2)}$$

considering $\varphi = A_{20}x^2 + A_{11}xy + A_{02}y^2$

set $A_{11} \neq 0$ others 0

$\sigma_{xy} = -A_{11}$

But (1) & (2) condition will never be simultaneously satisfied if we take $\varphi = A_{11}xy$

What is the way out?

$\nabla^4 \varphi = 0 \rightarrow$ Linear eqn

φ_I satisfies

φ_{II} satisfies.

$\varphi = \varphi_I + \varphi_{II}$ will also satisfy

Towards the end of finding a suitable φ_{II} which when added to $\varphi_I = A_{11}xy$ will hopefully satisfy the required BC.

- Consider a pure 4th degree polynomial form is a possible candidate for φ_{II}.

$$\varphi = A_{40}x^4 + A_{31}x^3y + A_{22}x^2y^2 + A_{13}xy^3 + A_{04}y^4$$

For the satisfaction of $\nabla^4\varphi = 0$ we must have,

$$3A_{40} + A_{22} + A_{04} = 0$$

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2} = 2A_{22}x^2 + 6A_{13}xy + 12A_{04}y^2$$

$$\sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2} = 2A_{22}y^2 + 6A_{31}xy + 12A_{40}x^2$$

$$\sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} = -3A_{31}x^2 - 4A_{22}xy - 3A_{13}y^2$$

Set everything $= 0$, except A_{13}

$$\sigma_{xx} = 6A_{13}\cdot xy$$

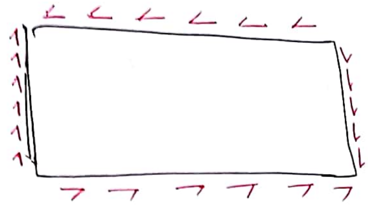
$$\sigma_{yy} = 0$$

$$\sigma_{xy} = -3A_{13}\cdot y^2$$

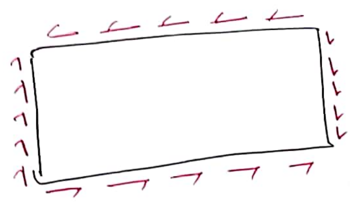
σ_{xy} on the horizontal edges i.e. $y = \pm h$

$$\sigma_{xy} = -3A_{13}h^2$$

σ_{xy} on the vertical edges, i.e. $x=0,\ x=l$



$\varphi_1 = \varphi = A_{13}xy^3$



$\varphi_{11} = \varphi = A_{11}\, x \cdot y$ $\sigma_{xx} = 0$ $\sigma_{yy} = 0$

$\sigma_{xy} = -A_{11}$

$$\varphi = \varphi_1 + \varphi_{11}$$
$$= A_{11} \cdot xy + A_{13}xy^3$$

The stresses resulting from $\varphi = \varphi_1 + \varphi_{11}$

$$\sigma_{xx} = 6A_{13}xy + 0$$
$$\sigma_{yy} = 0 + 0$$
$$\sigma_{xy} = -3A_{13}y^2 - A_{11}$$

We want that $\sigma_{xy}(x, \pm h) = 0$.

$$-3A_{13}h^2 - A_{11} = 0$$

$$\Rightarrow \boxed{A_{11} = -3A_{13}h^2}$$

~~$\sigma_{xy}(x, \pm h)$~~

$$\int_{-h}^{h} \sigma_{xy} \, dy = -P$$

$$+2A_{13}h^3 + 2A_{11}h = +P$$

$$(2A_{13} - 6A_{13})h^3 = P$$

$$\Rightarrow \quad A_{13} = \frac{-P}{4h^3}$$

$$A_{11} = \frac{3P}{4h^3}$$

$$\sigma_{xx} = -\frac{My}{I} = -\frac{P.x.y}{\frac{1}{12}8h^3}$$

$$= -\frac{3Pxy}{2h^3}$$

$$\sigma_{xx} = 6A_{13}xy = 6\left(\frac{-P}{4h^3}\right)xy$$

$$= -\frac{3P}{2h^3}xy$$

Deflection at end $\rightarrow \dfrac{PL^3}{3EI}$

Displacement

$u(x,y) = ?$

$v(x,y) = ?$

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu \cancel{\sigma_{yy}}^{\,0}\right)$$

$$= \frac{1}{E} \cdot \left(\frac{-3P}{2h^3} \cdot xy\right)$$

$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} = \frac{\nu}{E} \cdot \frac{3P}{2h^3} \cdot xy$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{-3P}{2E} \cdot \frac{xy}{h^3}$$

$$u = \frac{-3P}{2E} \cdot \frac{x^2 y}{2h^3} + f(y)$$

$$\frac{dv}{dy} = \frac{\nu}{E} \cdot \frac{3P}{2h^3} \cdot xy$$

$$v = \frac{\nu}{E} \cdot \frac{3P}{2} \cdot \frac{xy^2}{2h^3} + g(x)$$

$$\epsilon_{xy} = \frac{1}{G} \cdot \sigma_{xy} = \frac{1}{G}\left(\frac{-3Py^2}{4h^3} - \frac{3P}{4h}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1+\nu}{E}\left(\frac{-3Py^2}{4h^3} - \frac{3P}{4h}\right)$$

Scanned by CamScanner

$$\sigma_{xx} = -\frac{Pxy}{I} \qquad \sigma_{yy} = 0 \qquad \sigma_{xy} = -\frac{P}{2I}(h^2 - y^2)$$

$$u = -\frac{P}{2EI}x^2y + f(y) \cdot \frac{P}{2EI}$$

$$v = -\frac{\nu P}{2EI}xy^2 + g(x) \cdot \frac{P}{2EI}.$$

$$-\frac{P}{2EI} \cdot x^2 + f'(y) + \frac{\nu P}{2EI} \cdot y^2 + g'(x)$$

$$= -\frac{(1+\nu)}{E} \frac{2P}{2I}(h^2 - y^2)$$

$$-\frac{P}{2EI} \cdot x^2 + f'(y) + \frac{\nu P}{2EI} \cdot y^2 + g'(x)$$

$$= -\frac{2P}{2EI} \cdot h^2 + \frac{2P}{2EI} \cdot y^2 - \frac{2\nu P}{2EI} \cdot h^2 + \frac{2\nu P}{2EI} \cdot y^2$$

$$f'(y) + g'(x) = \frac{P}{2EI}\left[x^2 + 2y^2 + 3\nu y^2 - 2(1+\nu)h^2\right]$$

$$f_1'(y) + g_1'(x) = x^2 + (2+3\nu)y^2 - 2(1+\nu)h^2$$

$$\cancel{f_1'(y)} = \cancel{x^2 y}$$

$$f_1(y) = \frac{(2+3\nu)y^3}{3} - 2(\nu)h^2y + C_1$$

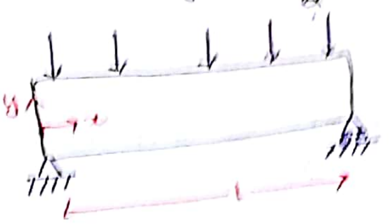
$$g_1(x) = \frac{x^3}{3} - 2(1+\cancel{\nu})h^2 x + C_2$$

$$u = \frac{-P}{2EI}\left(x^2 y + \frac{x^3}{3} - 2h^2 x + C_1\right)$$

$$v = \frac{-P}{2EI}\left((2+\nu)\left(\frac{-y^3}{3}\right) + 2\nu h^2 y + x y^2 + C_2\right)$$

Boundary conditions :-

Q. Find boundary conditions.



① $v(x=0)=0$

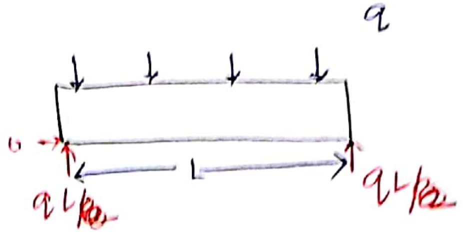
② $v(x=L)=0$, $\sigma_y(\ldots)=0$

Q. $\varphi = A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3 + A_{14}xy^4 + A_{05}y^5$

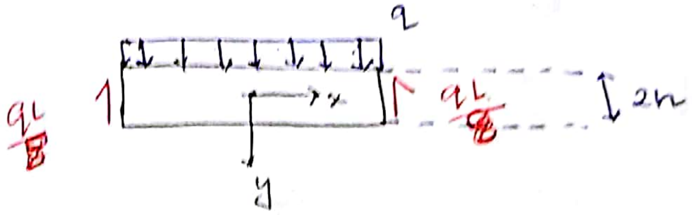
$\nabla^4\varphi = 0$

$$\Rightarrow 120A_{50}x + 24A_{41}y + 12A_{32}x + 12A_{23}y + 24A_{14}x + 120A_{05}y = 0$$

$$\Rightarrow (10A_{50} + A_{32} + 2A_{14})x + (10A_{05} + A_{23} + 2A_{41})y = 0$$



B.C.s in terms of $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$



$$\sigma_{xy}(x, \pm h) = 0$$

$$\sigma_{yy}(x, -h) = -q$$

$$\sigma_{yy}(x, h) = 0$$

$$\int_{-h}^{h} \sigma_{xy}(-l, y)\,.dy = ql$$

$$\int_{-h}^{h} \sigma_{xy}(l, y)\,.dy = -ql$$

$$\int_{-h}^{h} \sigma_{xx}(\pm l, y)\,.dy = 0$$

$$\int_{-h}^{h} \sigma_{xx}(\pm l, y).y\,.dy = 0$$

$$\varphi = A_{20}x^2 + A_{21}x^2y + A_{03}y^3 + A_{23}x^2y^3 + A_{05}y^5$$

check $\nabla^4\varphi = 0$ what conditions must hold?

why to satisfy the BCs.

$$A_{05} = \frac{-A_{23}}{5}$$

$A_{20} = \frac{-q}{4}$, $A_{21} = \frac{3q}{h}$, $A_{23} = \frac{-q}{5h^3}$

$A_{03} = -A_{23}\left(l^2 - \frac{2}{5}h^2\right) = \frac{q}{8h}\left(\frac{l^2}{h^2} - \frac{2}{5}\right)$

$\sigma_{xx} = \frac{q}{2I}(l^2 - x^2)y + \frac{q}{I}\left(\frac{y^3}{3} - \frac{h^2 y}{5}\right)$

$\sigma_{yy} = -\frac{q}{2I}\left(\frac{y^3}{3} - h^2 y + \frac{2}{3}h^3\right)$

$\sigma_{xy} = \frac{-q}{2I}x(h^2 - y^2)$

THEORIES

Some things t...
Engineered Sy...

- Excessive de...
- Undesirable d...
- Changes in ...
- Cutting tools ge...
- Creep deformati...
- σ_v, σ_{all} ...
- Maintenance
- Undesired vibr...
- Buckling -
- Fatigue
- Things just ...
- Environmenta...

* Excessive Def...
  - Inadequat...
  - Too much ...

Special typ...
Buckling
of loadin...

THEORIES OF FAILURE

Some things that can go wrong with Engineered Systems:

- Excessive deflection
- Undesirable deformation due to heating
- Changes in properties with time.
- Cutting tools getting blunt.
- Creep deformation
- σ_v, σ_{all}, F.S.
- Maintenance – taking care of corrosion, for example.
- Undesired vibrations.
- Buckling – wrinkling comes under this
- Fatigue
- Things just break apart
- Environmental factors.

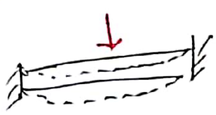
* Excessive Deflection

- Inadequate stiffness.
- Too much load.

Special type – buckling

Buckling doesn't take place in the direction of loading

buckling



bending.

· Vibrations.

* Fracture

Brittle

· Doesn't lengthen but suddenly fractures.

Ductile

· Yields plastic deformation before fracture.

· At high temp creep

· A flawed ductile material body presence of microcracks, notches can show brittle fracture.

· **Fatigue** (Progressive fracture)

· Abrupt changes in geometry – localised high stresses

· Microcracks – stress concentration.

· Larger cracks – progression to fracture

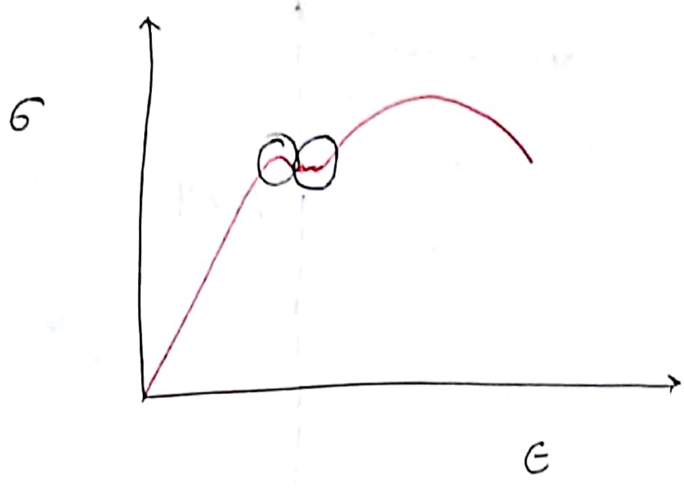
Under cyclic loading

* Permanent Deformation

Whenever a member yield plastically, we end up with some permanent deformation.

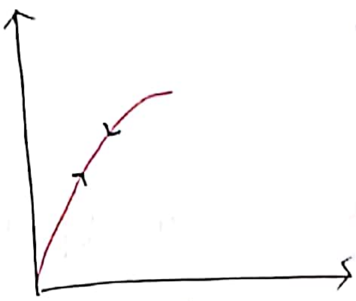
Plastic response of a material falls under the broader ambit of inelastic response

• Inelastic Response

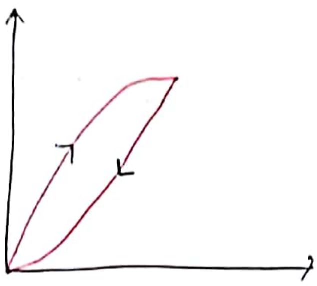


No man's land

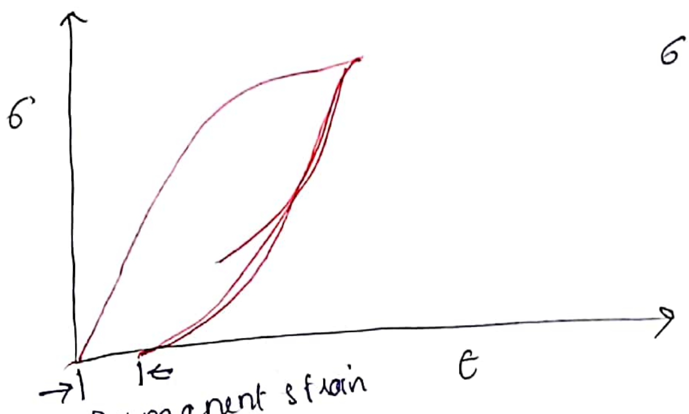
Non-elastic zone



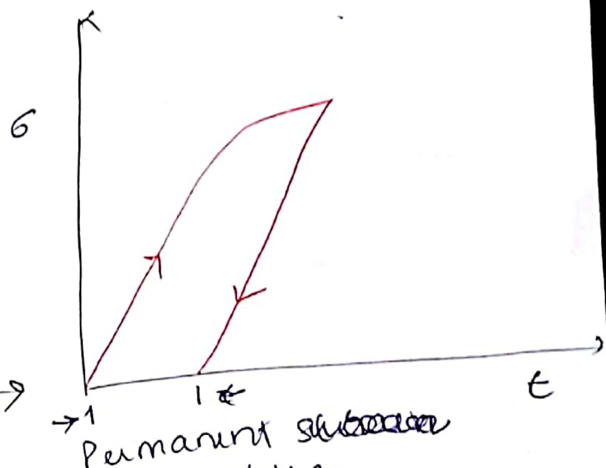
Elastic



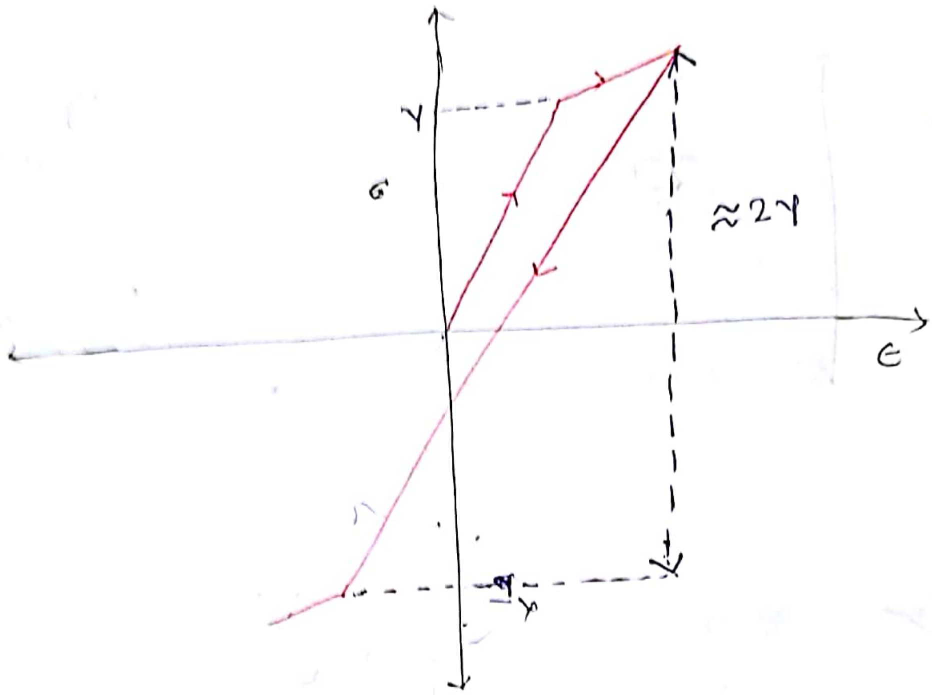
Visco elastic.



Visco plastic



Plastic



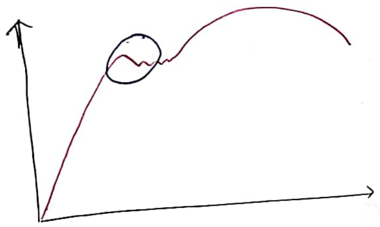
Barschinger effect

Inelastic Response ... Contd

Limitations of uniaxial test data:

- Temperature
- Rate of loading.
  high → ductile → brittle like behaviour.
- How to translate the data/info. from uniaxial mode to multiaxial mode?

Danger:



strictly $\sigma \geq Y$
↑
Yield stress value.

However, in multiaxial mode, even if none of the six stress components $\geq Y$, yielding (or inelastic response) can occur.

- How to identify yield in multiaxial mode? we define something called the effective stress which is a particular combination of the six stress components. There are various choices for defining effective stress, σ_{eff}

The σ_{eff} has to be compared to some limiting value to come up with various possible criteria for yielding in multiaxial mode.

Yielding Criterion (Mathematical form)

$$\underbrace{f(\sigma_{eff}, Y)}_{\text{yield func}} = 0$$

yield criterion is satisfied.

when $f(\sigma_{eff}, Y) < 0$, elastic stress state is present.

$$\sigma_{eff} = g(\sigma_{ij})$$

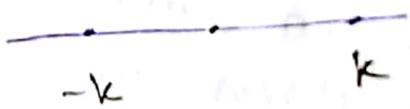
We wish to depict

$$f(\sigma_{eff}, Y) = 0$$

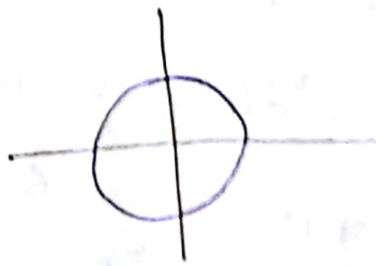
graphically. But there are issues to consider.

$$x - K = 0$$

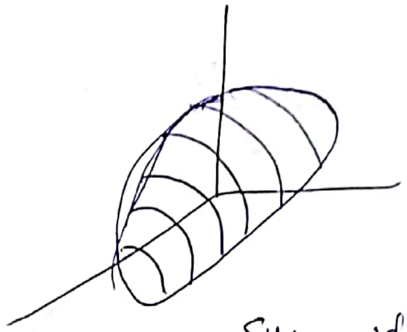
$$|x| - k = 0, \quad k > 0$$



$$x^2 + y^2 - r^2 = 0$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} = 1$$



Ellipsoid.

For a criterion involving six stress components, graphical depiction is impossible. However, if we transform the representation of the state of stress in terms of principal stresses, we would be dealing effectively with a 3D system whose graphical depiction is very much possible. This depiction in the principal stress space is referred to as the Haigh-Westergaard system (or space)

- Maximum principal stress criterion (Rankine)

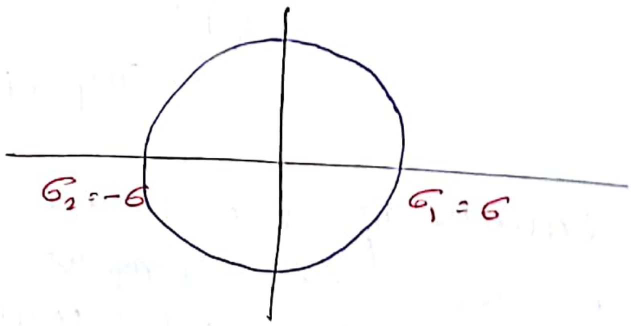
$$f = \max(|\sigma_1|, |\sigma_2|, |\sigma_3|) - Y$$

For uniaxial loading, we have σ_1 only and yielding is said to occurs when $|\sigma_1| = Y$

For biaxial loading, we have σ_1 & σ_2 and suppose $|\sigma_1| > |\sigma_2|$, then yielding is determined by σ_1 alone.

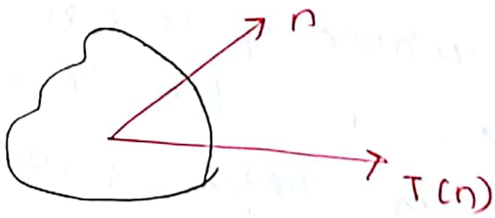
$$\sigma_1 = -\sigma_2 = \sigma$$

$$\tau = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma - (-\sigma)}{2} = \sigma$$



$$n = n_x\hat{i} + n_y\hat{j} + 0\hat{k}$$

$$\boxed{T^N = \sigma_{nn} = [\hat{n}]^T[\sigma][\hat{n}]}$$



For ductile materials, yielding occurs by the mechanism of slip. Slip is brought about by shear stresses.

Now we are obtaining from Rankine yield criterion that yielding occurs when $\sigma_1 = Y$.

But when $\sigma_1 = Y$, $\sigma_2 = -Y$ and $\tau = Y$, the problem is the uniaxial normal yield stress value. For ductile materials the shear stress yield value is actually less than Y.

So, τ reaching Y without yielding already is a physical impossibility.

From mixed bag of problems:

Principal strain directions are coincident with principal stress directions for a linear, elastic, isotropic, solid.

$$\varepsilon = \frac{\sigma - \lambda_{kk}}{2G}$$ → Reln between principal strain and principal stress

$$(\sigma_{ij} - \sigma \delta_{ij}) n_i = 0$$

$$[\underset{\approx}{\sigma} - \sigma \underset{\approx}{I}][\hat{n}] = 0 \quad \text{—— (1)}$$

$$(\varepsilon_{ij} - \varepsilon \delta_{ij}) n_i' = 0 \quad \text{—— } (*)$$

$$[\underset{\approx}{\varepsilon} - \varepsilon \underset{\approx}{I}][\hat{n}'] = 0 \quad \text{—— (2)}$$

$$\boxed{\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij}}$$ (linear, elastic, isotropic)

$$\Rightarrow (\lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij} - \sigma \delta_{ij}) \hat{n}_i = 0$$

$$\Rightarrow \left(\varepsilon_{ij} - \frac{\sigma - \lambda \varepsilon_{kk}}{2G} . \delta_{ij}\right) n_i = 0 \quad \text{—— } (**)$$

comparing $(*)$ and $(**)$

$n_i = n_i'$

$$\boxed{\varepsilon = \frac{\sigma - \lambda \varepsilon_{kk}}{2G}}$$ Reln between principal strain and principal stress.

• Normal strains along principal stress direction.

$$\varepsilon_{(1)} = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$\varepsilon_{(2)} = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right]$$

$$\varepsilon_{(3)} = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]$$

We wish to show that $\varepsilon_1, \varepsilon_2, \varepsilon_3$, i.e. the principal strains obtained from $\varepsilon = \frac{\sigma - \lambda\varepsilon_{kk}}{2G}$ are exactly the same as $\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}$, i.e. the normal strains along the principal stress directions.

$$\varepsilon_1 = \frac{1}{2G}\left[\sigma_1 - \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\right]$$

$$\varepsilon_2 = \frac{1}{2G}\left[\sigma_2 - \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\right]$$

$$\varepsilon_3 = \frac{1}{2G}\left[\sigma_3 - \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\right]$$

[$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ – 1st strain invariant equal

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3$]

$$\Rightarrow \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3\lambda + 2G}$$

$$\varepsilon_1 = \frac{1}{2G}\left[\sigma_1 - \frac{\lambda}{3\lambda + 2G}(\sigma_1 + \sigma_2 + \sigma_3)\right]$$

$$= \frac{1}{2G}\left[\frac{2(\lambda + G)}{3\lambda + 2G}\sigma_1 - \frac{\lambda}{3\lambda + 2G}(\sigma_2 + \sigma_3)\right]$$

Check:

$$\frac{1}{2G}\,\frac{2(\lambda+G)}{(3\lambda+2G)} = \frac{1}{E}$$

$$\frac{1}{2G}\,\frac{\lambda}{3\lambda+2G} = \frac{\nu}{E}$$

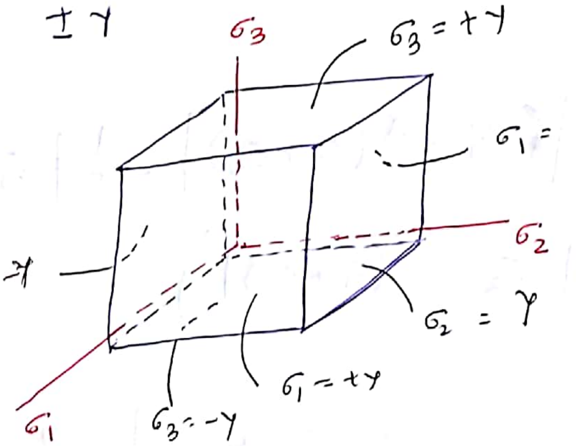
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)}$$

$$\boxed{f = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) - Y}$$

$\sigma_1 = \pm Y$

$\sigma_2 = \pm Y$

$\sigma_3 = \pm Y$



Maximum elastic strain criterion
OR
maximum principal strain criterion } St. Venant

Yielding will start when the max. elastic strain (or max. principal strain) equals the max. strain (at yield stress value) of a uniaxial tensile test.

$$\max\left(|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|\right) = \frac{Y}{E} \rightarrow$$ failure or yield criterion

$\sigma = Y$

$\varepsilon = \frac{Y}{E}$

In terms of stresses

$$|\varepsilon_1| = \frac{1}{E}\left|\sigma_1 - \nu(\sigma_2 + \sigma_3)\right|$$

$$|\varepsilon_2| =$$

$$|\varepsilon_3| =$$

$$\max\left(\frac{1}{E}\left|\sigma_1 - \nu(\sigma_2+\sigma_3)\right|, \frac{1}{E}\left|\sigma_2 - \nu(\sigma_3+\sigma_1)\right|, \frac{1}{E}\left|\sigma_3 - \nu(\sigma_1+\sigma_2)\right|\right)$$

$$\max\left|\sigma_i - \nu(\sigma_j + \sigma_k)\right| = Y \quad (i \neq j \neq k)$$

Special cases:

* Biaxial stress ($\sigma_3 = 0$) ; $\sigma_1 > 0$, $\sigma_2 > 0$

failure or yield criterion

$+\sigma_3)|$

$\frac{1}{E}|\sigma_2 - \nu(\sigma_3+\sigma_1)|$,

$= Y \quad (i \neq j \neq k)$

$\sigma_1 > 0, \ \sigma_2 > 0$

$\sigma_1 - \nu\sigma_2 = Y$, gives the yield condition

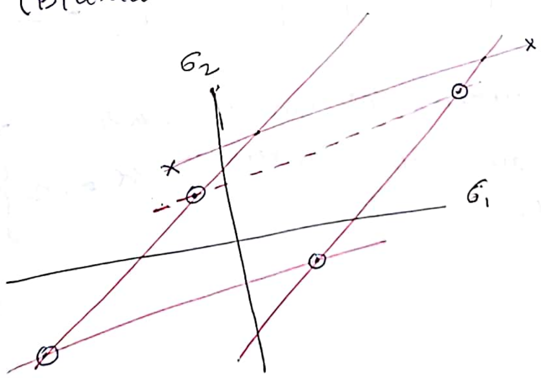
$\Rightarrow \sigma_1 = Y + \nu\sigma_2 < Y$, for a tensile case

* Biaxial stress $(\sigma_3 = 0)$, $\sigma_1 > 0$, $\sigma_2 > 0$

$\sigma_1 - \nu\sigma_2 = Y$ (again considering tensile case)

$\sigma_1 = Y + \nu\sigma_2 > Y$ Test for $|-\nu(\sigma_1+\sigma_2)|$ also for σ_{ff}

Yield surface (Biaxial stress)



Test for

$\sigma_1 - \nu\sigma_2 = Y$

$\sigma_1 - \nu\sigma_2 = -Y$

$\sigma_2 - \nu\sigma_1 = Y$

$\sigma_2 - \nu\sigma_1 = -Y$

- Drawbacks of max. principal stress criterion and max. principal strain criterion. → $\sigma_{ff} = \max(|\sigma_1|, \ldots \sigma_3|)$

$$\sigma_{ff} = \left| \sigma_i - \nu(\sigma_j + \sigma_k) \right| \quad , \quad i \neq j \neq k$$

Consider a state of hydrostatic stress.

$\sigma_1 = \sigma_2 = \sigma_3$ (compressive)

- Max shear stress criterion.

Given. $\sigma_1, \sigma_2, \sigma_3$.

τ_{max} ?

$$\max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right| \right) = \frac{Y}{2}$$

Yielding will start when the max. shear stress at a pt. equals the shear stress in a uniaxial tensile test.

Yield func.

$$f = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right| \right) - \frac{Y}{2}$$

$$f = \underbrace{\max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1| \right)}_{\sigma_{eff}} - Y$$

Special cases:

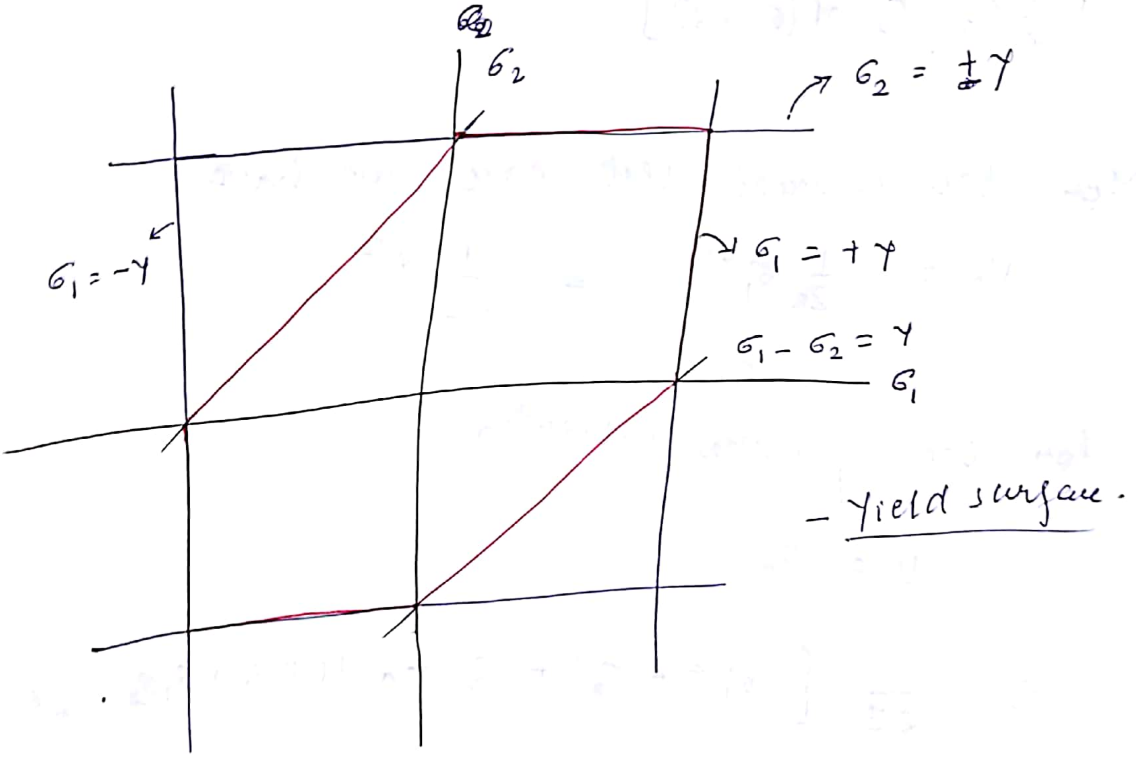
Biaxial stress:

$$\sigma_3 = 0$$

$$\sigma_1 - \sigma_2 = \pm Y$$

$$\sigma_2 = \pm Y$$

$$\sigma_1 = \pm Y$$



Strain energy criterion (not very useful ☹)

Yielding begins when the strain energy density at a pt equals the strain energy density corresponding to a uniaxial test case.

$$U = \frac{1}{2}\sigma_1 \varepsilon_1 + \frac{1}{2}\sigma_2 \varepsilon_2 + \frac{1}{2}\sigma_3 \varepsilon_3$$

$$= \frac{1}{2}\sigma_1\left[\sigma_1 - \nu(\sigma_2+\sigma_3)\right] + \frac{1}{2}\sigma_2\left[\sigma_2 - \nu(\sigma_3+\sigma_1)\right]$$

$$+ \frac{1}{2}\sigma_3\left[\sigma_3 - \nu(\sigma_2+\sigma_1)\right]$$

For the uniaxial test case, we have

$$U_0 = \frac{1}{2E}\sigma_1^2 \rightarrow Y^2 = \frac{1}{2E}Y^2$$

For the failure criterion,

$$U = U_0$$

$$\Rightarrow \frac{1}{2\cancel{E}}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$= \frac{1}{2\cancel{E}}Y^2$$

ery useful ☺)
in energy density
nergy density
test case.

$\frac{1}{2}\sigma_3 \varepsilon_3$

$- \nu(\sigma_3 + \sigma_1)]$

we have
γ^2

$-2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$

$\frac{1}{2E}\gamma^2$

Distortional Energy Criterion

We will be concerned with changes in shape so, consider the deviatoric part of the strain tensor.

First choose coordinate axis are aligned along the principal strain directions

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 - \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_2 - \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_3 - \varepsilon_m \end{bmatrix}$$

Total strain energy = volumetric strain energy + distortional energy.

$$[\sigma][\varepsilon] = [\sigma_m][\varepsilon_m] + E_{distortion}.$$

$$\sigma_1 = \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2G\varepsilon_1$$
$$\sigma_2 = \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2G\varepsilon_2$$
$$\sigma_3 = \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2G\varepsilon_3$$

$$(\sigma_1 + \sigma_2 + \sigma_3) = (3\lambda + 2G)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$
$$3\sigma_m = (3\lambda + 2G)\, 3\varepsilon_m$$

$$\sigma_m = \frac{E}{1-2\nu}\,\varepsilon_m$$

Now consider the mean strain tensor

$$\sigma = \lambda(\varepsilon_m + \varepsilon_m + \varepsilon_m) + 2G\,\varepsilon_m$$

$$= (3\lambda + 2G)\,\varepsilon_m$$

$$= \frac{E}{1-2\nu}\cdot\varepsilon_m$$

∴ Energy (per unit volume) stored as a result of pure volumetric change

$$U' = \frac{1}{2}\sigma_m\cdot\varepsilon_m + \frac{1}{2}\sigma_m\cdot\varepsilon_m + \frac{1}{2}\sigma_m\cdot\varepsilon_m$$

$$= \frac{3}{2}\cdot\sigma_m\,\varepsilon_m$$

$$= \frac{3}{2}\left(\frac{1-2\nu}{E}\right)\cdot\sigma_m^2$$

$$= \frac{(1-2\nu)}{6E}\cdot\underbrace{(\sigma_1+\sigma_2+\sigma_3)}_{I_1 \,\to\, \text{1st stress invariant}}{}^2$$

∴ Again consider the strain energy per unit volume

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

$$= \frac{1}{2E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right\}$$

$$= \frac{1}{2E}\left[(\sigma_1+\sigma_2+\sigma_3)^2 - 2(1+\nu)(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]$$

$$= \frac{1}{2E}\left[I_1^2 - 2(1+\nu)I_2\right]$$

The distortional energy per unit volume is

$$U^D = U - U'$$

$$= \frac{1}{6E}\left[3I_1^2 - 6(1+\nu)I_2 - (1-2\nu)I_1^2\right]$$

$$= \frac{1+\nu}{3E}\left[I_1^2 - 3I_2\right]$$

$$= \frac{2(1+\nu)}{6E}\left[(\sigma_1+\sigma_2+\sigma_3)^2 - 3(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]$$

$$= \frac{1}{6G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

The distortional energy (DE) failure criterion says that failure occurs when this DE equals the DE at yield in a uniaxial tensile test case

$\therefore U^D = U_0^P$

uniaxial tensile test case

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

$$= \frac{1}{12G}\left[2Y^2\right]$$

$$\Rightarrow \underbrace{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}_{\sigma_{eff}} = 2Y^2$$

$$f = (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2-2Y^2$$

$$\frac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2} = \frac{\sqrt{2}\,Y}{3}$$

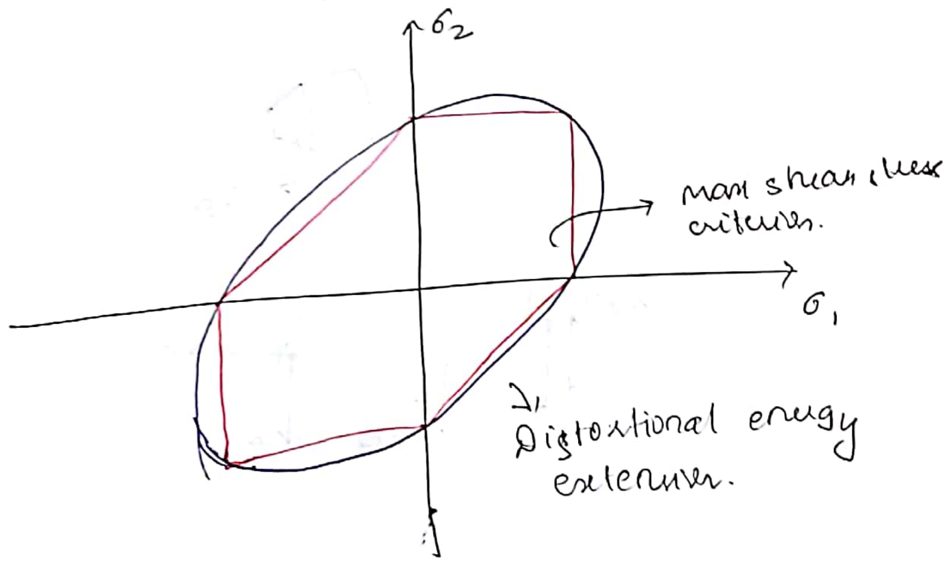
(Octahedral shear stress)

$$f = 2(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) - 2Y^2 = 0$$

To find the intersection points of $f = 0$ with σ_1 & σ_2 axes, set $\sigma_1 = 0$ & $\sigma_2 = 0$;

$\Rightarrow \sigma_1^2 = Y \quad \Rightarrow \sigma_1 = \pm Y$

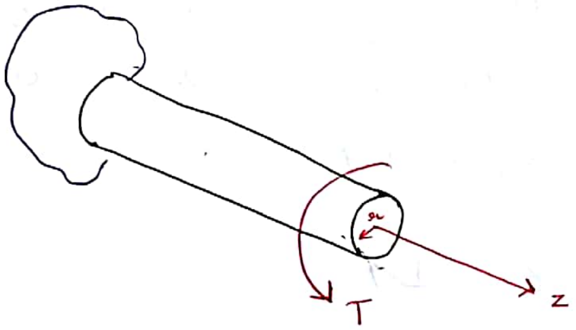
& $\Rightarrow \sigma_2^2 = Y \quad \Rightarrow \sigma_2 = \pm Y$.



The max shear stress criterion is a more conservative criterion for failure compared to distortional energy criterion.

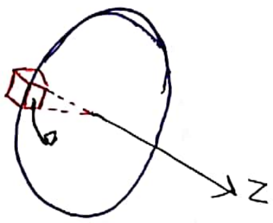
Different comparison basis for framing failure criteria. Instead of uniaxial tensile test, we may choose the torsion test, we may choose the torsion test as our basis of comparison.

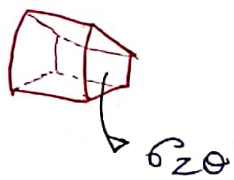
Torsion Test

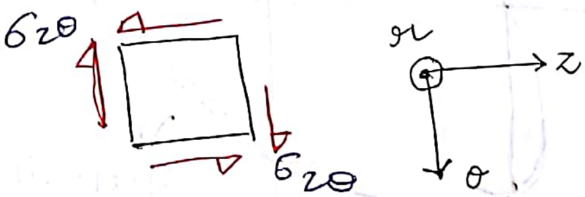


$$\tau(r) = \frac{Tr}{I_p}$$

$\sigma_{z\theta}$







$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0-\sigma_p & 0 & 0 \\ 0 & 0-\sigma_p & \tau \\ 0 & \tau & 0-\sigma_p \end{vmatrix} = 0$$

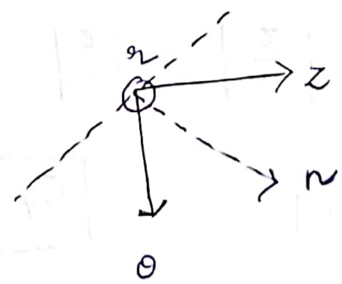
$$\sigma_p(\sigma_p^2 - \tau^2) = 0$$

$$\boxed{\sigma_p = 0, \quad \sigma_p = \tau, \quad \sigma_p = -\tau}$$

For $\sigma_p = \tau$, considering effectively biaxial state of stress

$$\begin{cases} \begin{bmatrix} -\tau & \tau \\ \tau & -\tau \end{bmatrix} \begin{bmatrix} n_\theta \\ n_z \end{bmatrix} = 0 \\ \text{alongwith} \quad n_\theta^2 + n_z^2 = 1 \end{cases}$$

$$n_\theta = n_z = \pm \frac{1}{\sqrt{2}}$$



For the torsion test, the principal stresses come out to be,

$\sigma_1 = \tau, \quad \sigma_2 = 0, \quad \sigma_3 = -\tau$

- Max. principal stress criterion in term of comparing with torsion test at yield

$$\max(|\sigma_1|, |\sigma_2|, |\sigma_3|) = \underbrace{\max(|\sigma_1|, |\sigma_2|, |\sigma_3|)}_{\text{For torsion test at yield.}}$$

$$\Rightarrow \max(|\sigma_1|, |\sigma_2|, |\sigma_3|) = \tau|_{\text{at yield}}$$

$$= \boxed{\tau_y}$$

- Max. shear stress criterion again comparing with torsion test at yield

$$\max\left(\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_3-\sigma_1}{2}\right|\right)$$

$$= \underbrace{\max\left(\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_3-\sigma_1}{2}\right|\right)}_{\text{For the torsion test at yield}}$$

$$LHS = \max\left(\left|\frac{\tau}{2}\right|, \left|\frac{\tau}{2}\right|, \left|\frac{-2\tau}{2}\right|\right)$$

$$= \tau|_{\text{at yield}} = \boxed{\tau_y}$$

- Distortional energy criterion comparing with torsion test at yield

$\frac{1}{12G}$

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

$$=\underbrace{\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}_{\text{for torsion at yield}}$$

$$\Rightarrow \text{LHS} = \frac{1}{12G}\left[\tau^2+\tau^2+4\tau^2\right]\Big|_{\text{at yield}}$$

$$\Rightarrow (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 6\tau^2\Big|_{\text{at yield}}$$

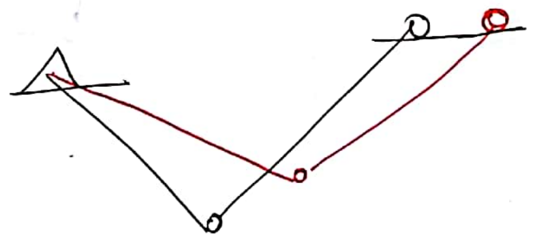
$$=\boxed{6\tau_y^2}$$

ENERGY METHODS

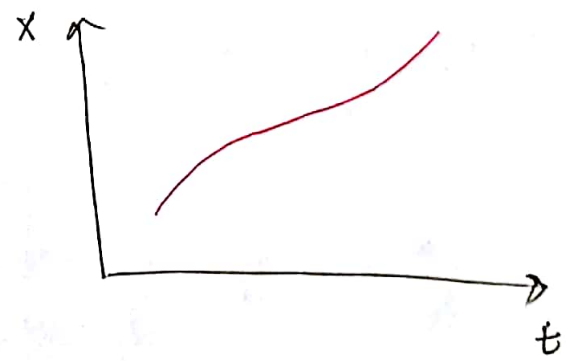
Configuration: The material pt. positions of a b[ody]
collectively referred to as the configuration of the
body

configuration space: The set of all configurations
of a body is referred to as the configuration
space of the body.

Distance in configuration space: Taken as the
max. magnitude of the displacements of
various material points.



Path in a configuration space:



consider that the configuration is a function of a scalar $\bar{t}$ as $\bar{t} \in [a, b]$
(not necessarily time)

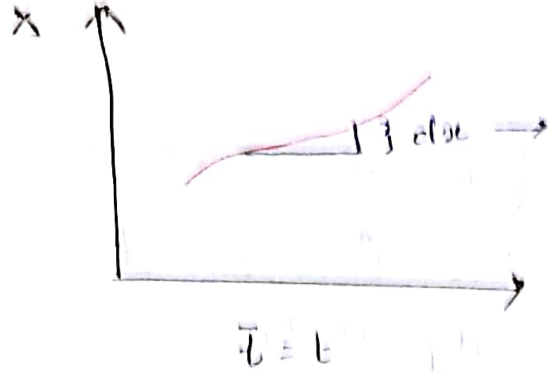
$$\vec{x} = \vec{x}(\bar{t})$$

not necessarily p.v. of a pt, in configuration space, just an identity of a pt.
i.e. just an identity of a particular config.

If $\bar{t}$ is indeed time, then the path in configuration space is referred to as the motion of the mechanical system.

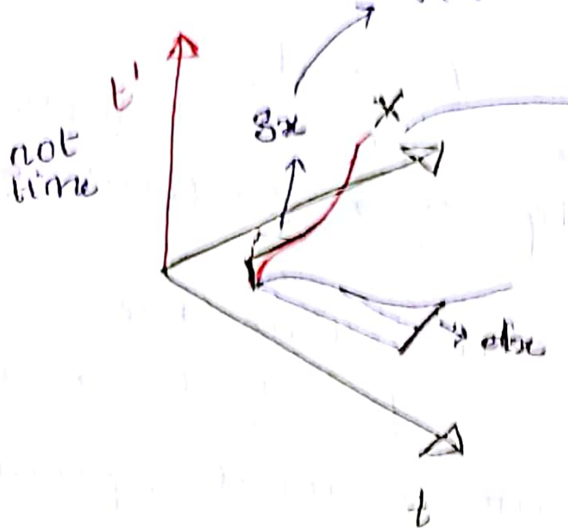
when going from one configuration to another configuration, we must respect the constraints of the mechanical system.

$\Rightarrow$ Not every path between two configurations is an admissible path.



→ real displacement corresponds to actual motion of the par

virtual displacement

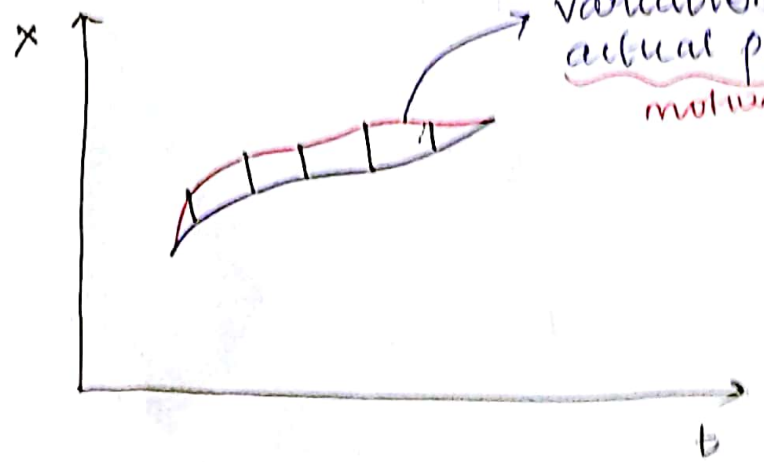


→ In a plane ∥ to t' - x plane

For virtual displacements:

∥ freeze time

Though the path is virtual displacement, but it should be admissible - should not violate mechanical constraints



variation on the actual path motion

$\delta x \rightarrow$ virtual increament in x without changing the ind. variable t.
(completely different from dx)

- Law of Kinetic Energy

In an inertial reference frame, the work of all forces (external and internal) on the mechanical system equals the change of kinetic energy.

$$W = \int_{t_0}^{t_1} \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F} \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\frac{D\vec{r}}{Dt}} \, dt = \int_{t_0}^{t_1} \vec{F} \cdot \vec{v} \, dt$$

$$= \int_{t_0}^{t_1} m \frac{d\vec{v}}{dt} \cdot v \, dt = \int_{t_0}^{t_1} m \, d\vec{v} \cdot \vec{v} = \frac{1}{2} \int_{t_0}^{t_1} m \, d(\vec{v} \cdot \vec{v})$$

$$W = \frac{1}{2} \int_{t_0}^{t_1} m \, d(\vec{v} \cdot \vec{v})$$

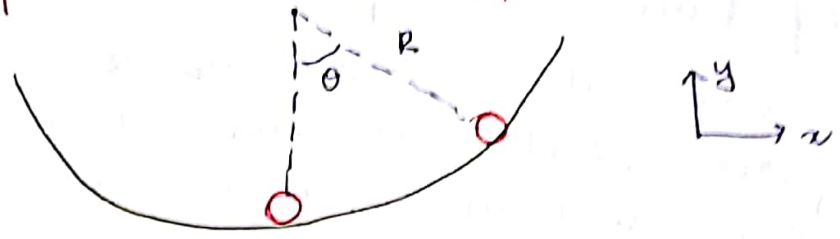
$$= \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2$$

- $= \Delta T \leftarrow$ kinetic energy

when a body starts to move, it is gaining KE.

- Net work on the body due to ext. & int. forces must be +ve.

• If a body is at rest then it will rem. only when for small movements from the rest position the net work on it is +ve.
↳ Only sufficient condition for equilibrium.



$$W = \int_0^\theta (-mg\hat{j}) \cdot d\vec{x}$$

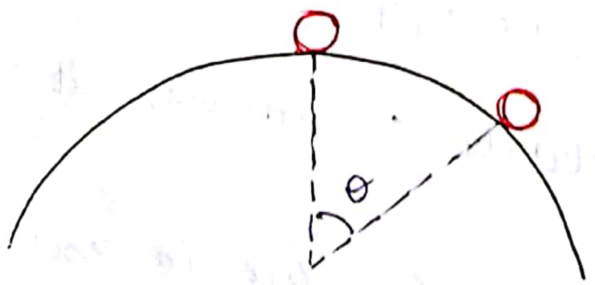
$$d\vec{x} = R\,d\theta\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$W = -\int_0^\theta mg\hat{j} \cdot R\,d\theta\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= -mgR \int_0^\theta \sin\theta\,d\theta = -mgR\,[-\cos\theta]_0^\theta$$

$$= mgR\,(\cos\theta - 1)$$

$$= -2mgR\,\sin^2\theta/2$$



$$W = \int_0^\theta (mg\,\hat{j}) \cdot d\vec{x}$$

$$\Rightarrow W = \int_0^\theta mg\,\hat{j} \cdot R\,d\theta\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= mgR(1 - \cos\theta) = +2mgR\sin^2\theta/2$$

Fourier's inequality $W \leq 0$ is only a sufficient condition for equilibrium, not necessary

If $W \leq 0 \Rightarrow$ Equilibrium

But, if equilibrium $\not\Rightarrow W \leq 0$

Q Does there exist a necessary and sufficient condition for equilibrium

YES

- Necessary and sufficient condition for equilibrium

$$\lim_{S \to 0} \frac{W}{S} = 0$$

where S represents the distance between two configurations between which the work done is calculated.

$W = -2mgR\sin^2\theta/2$

If θ is small

$W \simeq -\frac{1}{2} mgR\theta^2$

$s = R\theta$

$\lim_{s \to 0} \frac{W}{s} = 0$

$\Rightarrow$ Eqb.

$W = 2mgR\sin^2\theta/2$

$W \approx \frac{1}{2} mgR\theta^2$

$s = R\theta$

$\lim_{s \to 0} \frac{W}{s} = 0$

$\Rightarrow$ Eqb.

$$\boxed{\lim_{s \to 0} \frac{W}{s} = 0}$$

$\downarrow$

Principle of virtual work.

In the language of calculus of variations, $\lim_{s \to 0} \frac{W}{s} = 0$ can be written as $\boxed{\delta W = 0}$ (provided some very, very special conditions are not true)

Back to solid deformable bodies

For solid deformable bodies it is appropriate to identify the configuration of the body using the displacement field (same displacement field we have been discussing in pre mid-sem part, i.e. u, v, w)

If the solid deformable body is in eqb then according to principle of virtual work

$\delta W = 0$

Decompose $W = \underbrace{\underset{\text{ext force}}{W_e} + \underset{\text{int force}}{W_i}}_{\text{work done}}$

$$-\delta W_i = \delta W_e$$

$$\delta W_e = \int_V \rho \vec{b} \cdot \delta \vec{u} \, dV + \int_S \underset{\text{traction vector on surface}}{\vec{T}} \cdot \delta \vec{u} \, ds$$

Displacement field : $\vec{u} = \vec{u}(x, y, z)$

$$= \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix}$$

$$\delta \vec{u} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} ; \quad \delta \vec{u} = \delta u \hat{i} + \delta v \hat{j} + \delta$$

Considering only the surface integral part of δW_e

$$\int_S \vec{T} \cdot \delta\vec{u}\, ds = \int_S T_i\, \delta u_i\, ds$$

$$\vec{T} = \underset{\approx}{\sigma} \cdot \hat{n} \iff T_i = \sigma_{ji} n_j$$

$n_j \downarrow$ unit vector at a surface point

$$\int_S \sigma_{ji}\, n_j\, \delta u_i\, dS$$

$$= \int_V \frac{\partial}{\partial x_j} (\sigma_{ji}\, \delta u_i)\, dV \quad \text{(Gauss div. theorem)}$$

$$= \int_V \left(\frac{\partial \sigma_{ji}}{\partial x_j} \delta u_i + \sigma_{ji} \frac{\partial\, \delta u_i}{\partial x_j} \right) dV \quad \text{(simply expanding)}$$

$$\delta W_e = \int_V \rho b_i\, \delta u_i\, dV + \int_V \frac{\partial \sigma_{ji}}{\partial x_j} \delta u_i\, dV + \int_V \sigma_{ji} \frac{\partial\, \delta u_i}{\partial x_j} dV$$

$$= \int_V \underbrace{\left(\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right)}_{\rho\vec{b} + \nabla \cdot \sigma = 0} \delta u_i\, dV + \int_V \sigma_{ji} \frac{\partial\, \delta u_i}{\partial x_j}\, dV$$

(mechanical eqb. eqns)

$$\delta W_e = \int_V \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} dV$$

$$= \int_V \sigma_{ji} \, \delta \underbrace{\left(\frac{\partial u_i}{\partial x_j}\right)}_{\nabla \vec{u}} dV$$

we can write

$$\nabla \vec{u} = \frac{1}{2} \left\{ \nabla \vec{u} + (\nabla \vec{u})^T \right\} + \frac{1}{2} \left\{ \nabla \vec{u} - (\nabla \vec{u})^T \right\}$$

$$= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$\delta W_e = \int_V \sigma_{ji} \, \delta \Big(\underbrace{\varepsilon_{ij}}_{\mathrm{I}} + \underbrace{\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)}_{\mathrm{II}} \Big) . dV$$

$$\mathrm{II} = \frac{1}{2} \int \left(\sigma_{ji} \, \delta \frac{\partial u_i}{\partial x_j} - \sigma_{ij} \, \delta \frac{\partial u_j}{\partial x_i} \right) dV$$

(using $\sigma_{ji} = \sigma_{ij}$)

$$= 0$$

$$\delta W_e = \int_V \sigma_{ji} \, \delta \varepsilon_{ij} \, dV$$

$$-\delta W_i = \delta W_e = \int_V \sigma_{ji} \, \delta \varepsilon_{ij} \, dV$$

1st Law of Thermodynamics.

$$\delta W_e + \underset{\substack{\downarrow \\ \text{heat into} \\ \text{the body}}}{\delta Q} = \underset{\substack{\downarrow \\ \text{int. energy}}}{\delta U} + \underset{\substack{\downarrow \\ \text{KE}}}{\delta T}$$

Body is in equilibrium : $\boxed{\delta T = 0}$

Adiabatic condition : $\boxed{\delta Q = 0}$

$$\delta W_E = \delta U$$

$$\therefore \quad -\delta W_i = \delta W_e = \delta U.$$

$$= \delta \int U_0 \, dV$$

Potential Energy → Strain Energy
↓
P.C.E. ← complementary Energy
↓
Castigliano's theorem of deflection.

Potential energy

- If the virtual work associated with virtual disp. in a closed path is zero, then ~~and~~ mech. system is conservative
- W_e and W_i can be treated separately
- If ext. forces ~~are~~ are const. $W_e = -V_e(\vec{x}, \vec{x}_0)$

  $W_e = -V_e(\vec{x})$

- If ~~ext~~ int forces are const.

  $W_i = -V_i(\vec{x})$

- If both int. & ext. forces are const.

  $W = W_i + W_e = -(V_i(\vec{x}) + V_e(\vec{x}))$

  $= -\pi(\vec{x})$

Strain energy

- If the internal forces are conservative in the kinetic sense (i.e. virtual displacement with brought about not necessarily infinitesimal speed) then the mech. sys. is elastic in nature. The potential energy associated with these internal forces is referred to as strain energy.

- The strain energy density can be defined as

$$\underbrace{\vartheta_i}_{\text{strain energy}} = \int_V \underbrace{V_{io}}_{\text{strain energy density.}} \, dV$$

~~So, for a deformation process,~~

- For the kind of mech. systems we will be interested, the strain energy differs from the internal energy only by an additive const.

- So, for a deformation process,
$\vartheta_i = U$ and $V_{io} = U_0$

- Strain energy density is a func. of the strains (could be a func. of temp. also, but we disregard such dependence in this course)

$$V_{io} = V_{io}(\varepsilon_{ij}) \Rightarrow \delta V_{io} = \frac{\partial V_{io}}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}$$

But $V_{io} \equiv V_{io}$

$$\therefore \quad \delta U_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \qquad \text{—— (iv)}$$

From 1st law of Thermo D,

$$\delta W_e = \delta \int_V V_0 \, dV = \int_V \delta V_0 \, dV \quad \text{—} (*)$$

Further, from principle of virtual work

$$\delta W_e = \int_V \sigma_{ij} \delta \epsilon_{ij} \, dV \quad \text{—} (**) \quad \text{(last class)}$$

comparing $(*)$ and $(**)$

$$\int_V \delta V_0 \, dV = \int_V \sigma_{ij} \delta \epsilon_{ij} \, dV$$

$$\Rightarrow \quad \delta V_0 = \sigma_{ij} \delta \epsilon_{ij}$$

compare with (xx) to obtain

$$\frac{\partial V_0}{\partial \epsilon_{ij}} \delta \epsilon_{ij} = \sigma_{ij} \delta \epsilon_{ij}$$

$$\Rightarrow \quad \sigma_{ij} = \frac{\partial V_0}{\partial \epsilon_{ij}}$$

- Complementary Energy

We must to have something like $\varepsilon_{ij} = \dfrac{\partial \square}{\partial \sigma_{ij}}$

Complement energy

Define $V_0' = -V_0 + \sigma_{pq} \varepsilon_{pq}$

complementary energy density

To show: $\varepsilon_{ij} = \dfrac{\partial U_0'}{\partial \sigma_{ij}}$

$$\frac{\partial U_0'}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}} \left(-U_0 + \sigma_{pq} \cdot \varepsilon_{pq} \right)$$

$$= -\frac{\partial U_0}{\partial \sigma_{ij}} + \frac{\partial \sigma_{pq}}{\partial \sigma_{ij}} \varepsilon_{pq} + \sigma_{pq} \cdot \frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}$$

$$= -\frac{\partial U_0}{\partial \varepsilon_{pq}} \cdot \frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} + \frac{\partial}{\partial \sigma_{ij}} \left(\delta_{ip} \cdot \delta_{jq} \sigma_{ij} \right) \varepsilon_{pq}$$

$$+ \sigma_{pq} \frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}$$

$$= -\cancel{\sigma_{pq} \cdot \frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}} + \delta_{ip} \delta_{jq} \varepsilon_{pq} + \cancel{\sigma_{pq} \frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}}$$

$$\Rightarrow \boxed{\frac{\partial U_0}{\partial \sigma_{ij}} = \varepsilon_{ij}}$$

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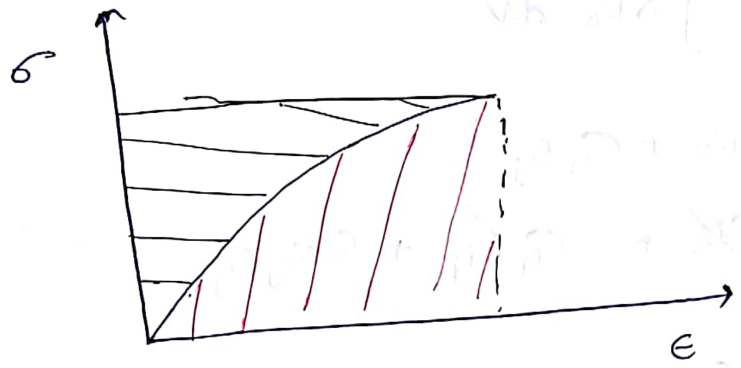
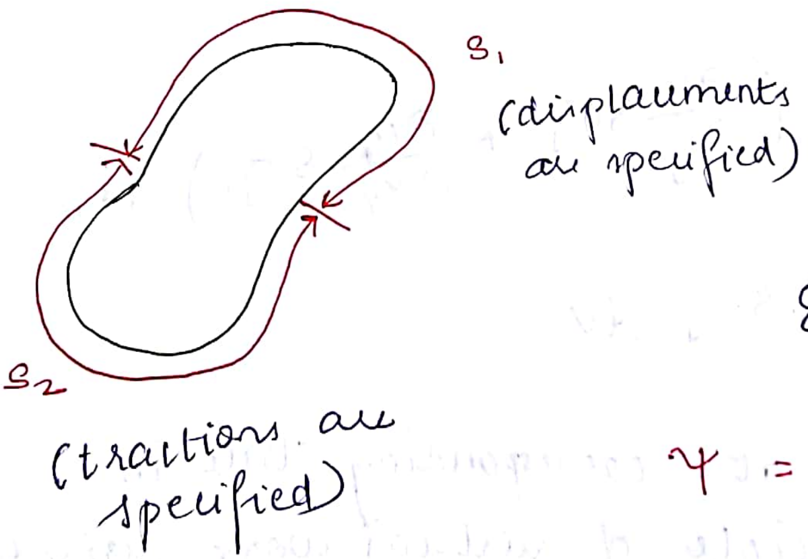
For physical interpretation of U_0 and U_0'

$$U = \int_0^{\varepsilon} \sigma \, \partial \varepsilon$$

$$U' = \int_0^{\sigma} \varepsilon \, \partial \sigma$$



- Generalisation of Castigliano's theorem of least work



$$\delta \psi = 0$$

$$\psi = U' - \int_{S_2} u_i T_i \, ds$$

$$\delta U' = \delta \int_V U_0' \cdot dV$$

$\delta U'$ → complementary energy

U_0' → complementary energy density

$$= \int \delta U_0' \cdot dV$$

$$U_0' = -U_0 + \sigma_{ij}\epsilon_{ij}$$

$$\delta U_0' = -\cancel{\delta U_0} + \cancel{\sigma_{ij}\delta\epsilon_{ij}} + \delta\sigma_{ij}\epsilon_{ij}$$

$$\delta U' = \int_V \epsilon_{ij}\,\delta\sigma_{ij} \cdot dV$$

$$= \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta\sigma_{ij}\, dV$$

$$= \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}\cdot\delta\sigma_{ij} + \frac{\partial u_j}{\partial x_i}\cdot\delta\sigma_{ji}\right)\cdot dV$$

$$= \int_V \frac{\partial u_i}{\partial x_j}\cdot\delta\sigma_{ij}\cdot dV$$

(look at corresponding state in principle of virtual work derivation)

$$= \underbrace{\int_V \frac{\partial}{\partial x_j}(u_i \delta\sigma_{ij})\, dV}_{\nabla\cdot(\vec{u}\,\underset{\sim}{\sigma})} - \underbrace{\int_V u_i \frac{\partial \delta\sigma_{ij}}{\partial x_j}\, dV}_{=0}$$

(from mech. eq.)

from mech equilibrium,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$$

Taking first variation

$$\frac{\partial \delta\sigma_{ij}}{\partial x_j} + \delta(\cancel{\rho b_i}^{\,0}) = 0$$

when something is specified to us, the first variation of it will be zero.

$$= \underbrace{\int_S u_i \delta\sigma_{ij} . n_j \, ds}_{(\ \ ).\vec{n}}$$

(using Gauss' divergence theorem)

$\sigma_{ij} n_j = T_i$

or

$\underset{\sim}{\sigma} . \hat{n} = \vec{T}$

$$\delta U' = \int_S u_i \delta\sigma_{ij} n_j \, ds$$

$$= \int_S u_i \,\delta(\sigma_{ij} n_j)\, ds$$

Here are assuming there is no significant change in geometry hence the 'n' vector will remain constant.

$$= \int_S u_i \, \delta T_i \, ds$$

$$= \int_{S_1} u_i \, \cancel{\delta T_i}^{\,0} \, ds + \int_{S_2} u_i \, \delta T_i \, ds$$

(because traction is specified)

$$\delta U' = \int_{S_2} u_i \, \delta T_i \, ds$$

Define : $\Psi = U' - \int_{S_2} u_i T_i ds$

$$\delta \Psi = \delta U' - \int_{S_2} \cancel{\delta u_i} T_i ds - \int_{S_2} u_i \, \delta T_i ds$$

(because displacement is specified)

$$\delta \Psi = 0$$

* Case

u

Deflection in the direction of the point force F

crucial of deflection

For linear as in

U :

We

* Castigliano's theorem of deflection *

$$u_F = \frac{\partial U'}{\partial F}$$

u_F → deflection in the direction of the point force F

F → point force

deflection f

Crucial thing to use Castigliano's theorem is to find U' (in terms of F) of deflection.

For linear elastic materials (referred to as Hookean materials),

$$U' = U$$

U' → compl. energy

U → strain energy

$$U = \int U_0 \, dV$$

U_0 → strain energy density

We know,

$$\frac{\partial U_0}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

For a uniaxial stress situation

$$\frac{\partial U_0}{\partial \epsilon} = \sigma$$

$$dU_0 = \frac{\partial U_0}{\partial \epsilon} \cdot d\epsilon$$

$$\Rightarrow \quad U_0 = \int_0^{\epsilon} \frac{\partial U_0}{\partial \epsilon} \cdot d\epsilon = \int_0^{\epsilon} \sigma \cdot d\epsilon$$

But for linear elastic materials

$$\sigma = E\epsilon$$

$$U_0 = \int_0^{\epsilon} \sigma \, d\epsilon$$

$$= \int_0^{\epsilon} E\epsilon \, d\epsilon$$

$$\boxed{U_0 = \frac{1}{2} E \epsilon^2 = \frac{\sigma^2}{2E}}$$

For a body subjected to bending (like problems from 1st year Mechanics)

We have,

$$\sigma = \frac{My}{I}$$

$$U_0 = \frac{1}{2E} \frac{M^2 y^2}{I^2}$$

$$\therefore \quad U = \int_V U_0 \, dV$$

$$= \int_V \frac{M^2 y^2}{2EI^2} \, dV$$

$$= \int_0^L \int_A \frac{M^2 y^2}{2EI^2} \, dA \, dx$$

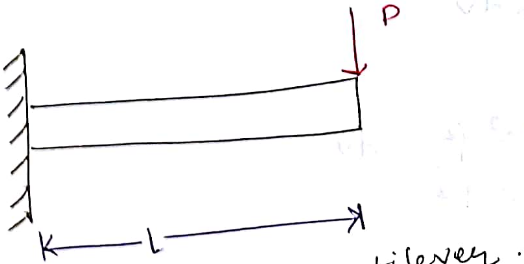
$$= \int_0^L \frac{M^2}{2EI^2} \underbrace{\left(\int_A y^2 \, dA \right)}_{\text{second moment of area}} dx$$

$$\therefore \quad U = \int_0^L \frac{M^2}{2EI} \, dx$$

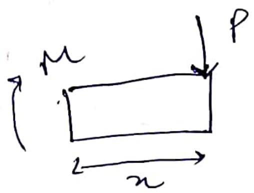
$$u_F = \frac{\partial U}{\partial F} = \frac{\partial}{\partial F} \int_0^L \frac{M^2}{2EI} \, dx$$

$$\boxed{\therefore \quad u_F = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial F} \, dx}$$

Q



Uniform CA of cantilever.
Deflection at tip ?



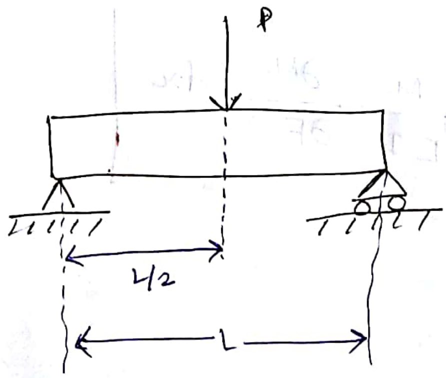
$$M = -Px$$

$$u_p = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial P} . dx$$

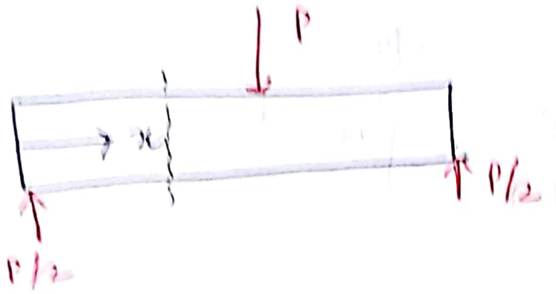
$$= \int_0^L \frac{P.x}{EI} . x . dx$$

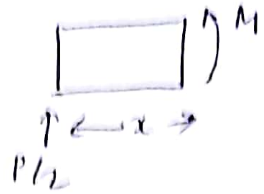
$$= \frac{P}{EI} \frac{L^3}{3} = \frac{PL^3}{3EI}$$

Q



c/s area uniform along length of simply-supported beam. Deflection at mid-point (along vertical dirn) ?





$$M = \frac{P}{2} x \quad , \quad 0 \le x < L/2$$

$$\cancel{M = \frac{P}{2}(L - x)} \quad L/2 \le x \le L$$

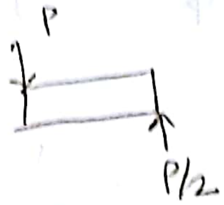
$$M = \frac{P}{2} x - P\left(x - L/2\right)$$

$$= \frac{P}{2}(L - x)$$

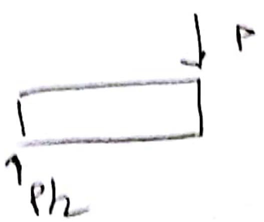
alternatively, exploiting symmetry we write

$$u_p = 2 \int_0^{L/2} \frac{M}{EI} \frac{\partial M}{\partial P} \, dx$$

=

2 

or



$$M = \frac{P}{2}x$$

$$U_p = \cancel{2}\int_0^{L/2} \frac{Px}{2EI} \cdot \frac{x}{\cancel{2}} \cdot dx$$

$$= \frac{P}{2EI}\int_0^{L/2} x^2 \cdot dx$$

$$= \frac{P}{2EI} \cdot \frac{L^3}{3 \cdot 8}$$

$$\boxed{U_p = \frac{PL^3}{48EI}}$$

* More general method without exploiting symmetry.

$$M = \frac{P}{2}x - \left(P\left(x - L/2\right)\right) H\left(x - L/2\right)$$

$$0 \le x \le L$$

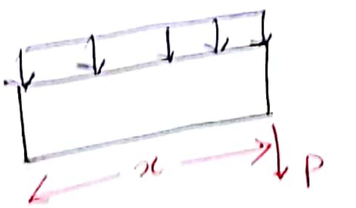
$$U_p = \int_0^{L} \frac{M}{EI} \cdot \frac{\partial M}{\partial P} \cdot dx$$

$$= \frac{1}{EI}$$

Deflection at free end?

Introduce a fictitious force P at the free end.

consider a cut-section at a free distance x from the free end. Then,



$$M + Px + \frac{wx^2}{2} = 0$$

$$\Rightarrow d = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial P} \cdot dx \Bigg|_{P=0}$$

$$\Rightarrow d = \int_0^L \frac{\left(-Px - w\frac{x^2}{2}\right)(-x) \cdot dx}{EI} \Bigg|_{P=0}$$

$$= \frac{1}{EI} \int_0^L \frac{wx^3}{2} \cdot dx$$

$$= \frac{wL^4}{8EI}$$

Corollary to Castigliano's theorem of Deflection (without Proof)

Just as we have, $d = \frac{\partial U'}{\partial P}$

so also we have,

$$\boxed{\theta = \frac{\partial U'}{\partial M_0}}$$

For linear elastic materials, i.e. Hookean materials

$$\boxed{U' = U}$$

so that,

$$d = \frac{\partial U}{\partial P}$$ (already used in previous problems)

and $$\theta = \frac{\partial U}{\partial M_0}$$

For typical problems on beams,

$$\theta = \frac{\partial}{\partial M_0} \int_0^L \frac{M^2}{2EI} \cdot dx$$

$$\boxed{\theta = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial M_0} \cdot dx}$$

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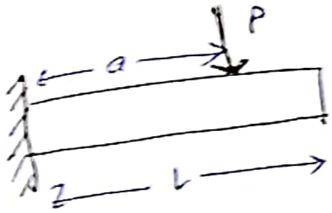
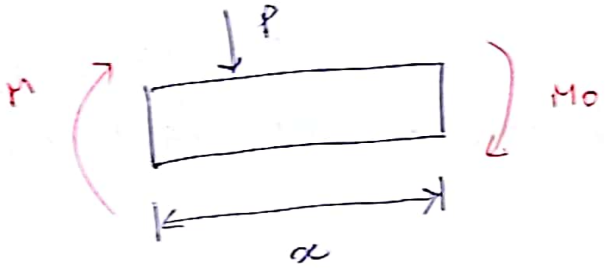


Find slope at free end.

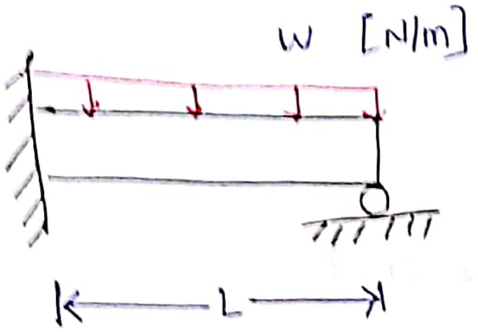


$$M + M_o + P(x-a)H(x-a) = 0$$

$$\theta = \int_0^L \frac{\{-M_o - P(x-a)H(x-a)\}(-1)\,dx}{EI}\Bigg|_{M_o=0}$$

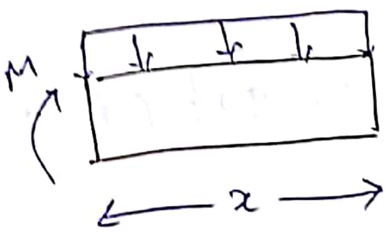
$$= \int_a^L \frac{P(x-a)}{EI}\,dx$$

$$= \frac{P}{2EI}(L-a)^2$$



Find the reaction force at the right end

The key idea to solve this problem is that the deflection at the right end must be 0.



R (unknown reaction force)

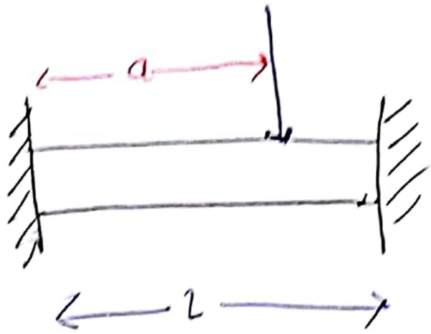
$$M - Rx + w\frac{x^2}{2} = 0$$

$$d = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial R} \cdot dx$$

$$\Rightarrow \quad 0 = \int_0^L \frac{(Rx - wx^2/2)}{EI} (x) \cdot dx$$

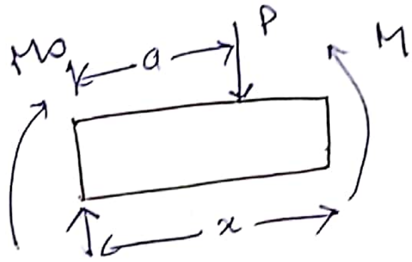
$$\Rightarrow \quad 0 = \frac{1}{EI}\left(R\frac{L^3}{3} - \frac{w}{2}\frac{L^4}{4}\right)$$

$$\Rightarrow \quad \boxed{R = \frac{3wL}{8}}$$



Find the reactions at end A
(forces and moments)

The key idea to solve this problem is that the deflection at the right end must be 0.



$$M - R_o x - M_o + P(x-a)H(x-a) = 0$$

$$\left.\begin{aligned} d &= \int_0^l \frac{M}{EI} \cdot \frac{\partial M}{\partial R_o}\, dx \\ \theta_A &= \int_0^l \frac{M}{EI}\, \frac{\partial M}{\partial M_o} \cdot dx \end{aligned}\right\}$$

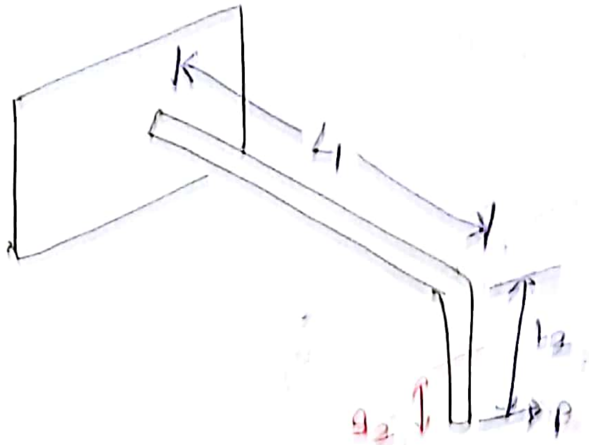
Linear simultaneous equations in M_o & R_o

$$M_o = \frac{-P(L-a)^2 a}{L^2}$$

$$R_o = \frac{P(L-a)^2(L+2a)}{L^3}$$

NOTES MISSING

5.35



We can use two body-fitted coordinate variables to track the length of the structure

For CB: s_2 and for BA: s_1

For CB, take a cut cut-section at C.

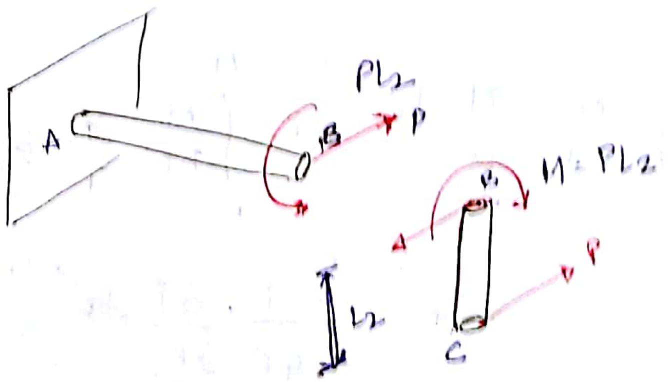
$M - Ps_2 = 0$

no axial force;

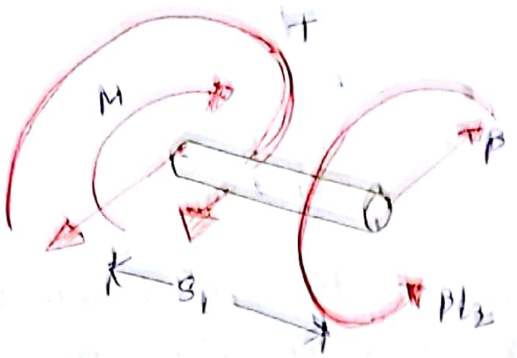
no torsion.

For BA,

Step 1: Make a cut at B and consider only BA.



Step 2: Take a cut-section at a distance s_1 from B.



$$\boxed{\begin{aligned} M - Ps_1 &= 0 \\ T - PL_2 &= 0 \end{aligned}}$$

For the deflection(d) at C:

$U = U_M + U_T$

$U = U_{CB} + U_{BA}$

$$= \int_0^{L_2} \frac{M^2}{2EI}\,dx + \underbrace{\int_0^{L_1} \frac{M^2}{2EI}\,dx + \int_0^{L_1} \frac{T^2}{2GJ}\,dx}_{BA}$$

$$d = \frac{\partial U}{\partial P}$$

$$= \int_0^{L_2} \frac{M}{EI}\frac{\partial M}{\partial P}\,\cancel{dx}^{\,ds_2} + \int_0^{L_1} \frac{M}{EI}\frac{\partial M}{\partial P}\,\cancel{dx}^{\,ds_1}$$

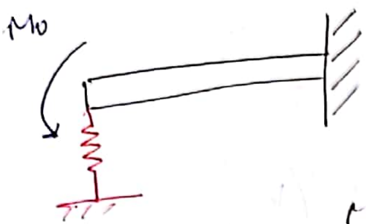
$$+ \int_0^{L_1} \frac{T}{GJ}\frac{\partial T}{\partial P}\,\cancel{dx}^{\,ds_1}$$

$$= \int_0^{L_2} \frac{Ps_2}{EI} \cdot s_2 \, ds_2 + \int_0^{L_1} \frac{Ps_1}{EI} \cdot s_1 \, ds_1 + \int_0^{L_1} \frac{PL_2}{GJ} L_2 \, ds_1$$

$$= \frac{PL_2^3}{3EI} + \frac{PL_1^3}{3EI} + \frac{PL_2^2 L_1}{GJ}$$

Strain energy due to spring

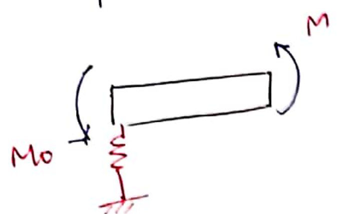
B/S 5.66



Find the force on the beam from the spring.

Make a cut section at a distance s from the left end

$M + M_0 - Rs = 0$



$\Delta_{el} \nearrow^0 = \frac{\partial U}{\partial R}$

$$U = \frac{\partial}{\partial R}\left(\int_0^L \frac{M^2}{2EI} dx + \frac{1}{2} k \Delta^2 \right)$$

$$= \int_0^L \left\{ \frac{-M_0 + Rs}{EI} \frac{\partial}{\partial R}(-M_0 + Rs) \right\} ds + \frac{1}{2k}(2R)$$

$$0 = \int_0^L \frac{-M_0 + RS}{EI} s\,ds + \frac{R}{k}$$

$$\Rightarrow \quad 0 = \frac{1}{EI}\left(-M_0\frac{L^2}{2} + \frac{RL^3}{3}\right) + \frac{R}{k}$$

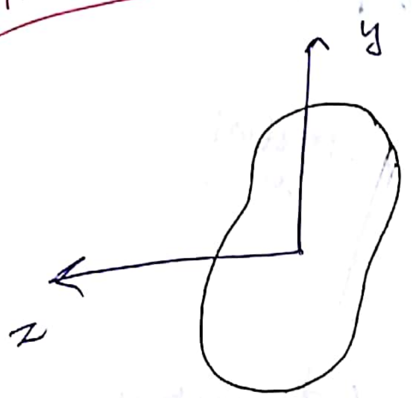
$$\Rightarrow \quad R\left(\frac{L^3}{3EI} + \frac{1}{k}\right) = M_0\frac{L^2}{2EI}$$

$$\Rightarrow \quad \boxed{R = \frac{M_0 L^2/2EI}{L^3/3EI + 1/k}}$$

~~Asymmetrical~~

ASYMMETRICAL BENDING OF ~~B~~ BEAMS

Preliminary Concepts



$$A\bar{y} = \int_A y\,dA$$

$$A\bar{z} = \int_A z.dA$$

Shifting the origin to the centroid:

$$0 = \int y\,dA$$

$$0 = \int z\,dA$$

Centroidal axes are the axes which pass through the centroid

OR,

centroidal axes are the ones about which the first moments of area are zero.

If the axes are so chosen that $I_{YZ} = 0$ then the axes are referred to as principal axes.

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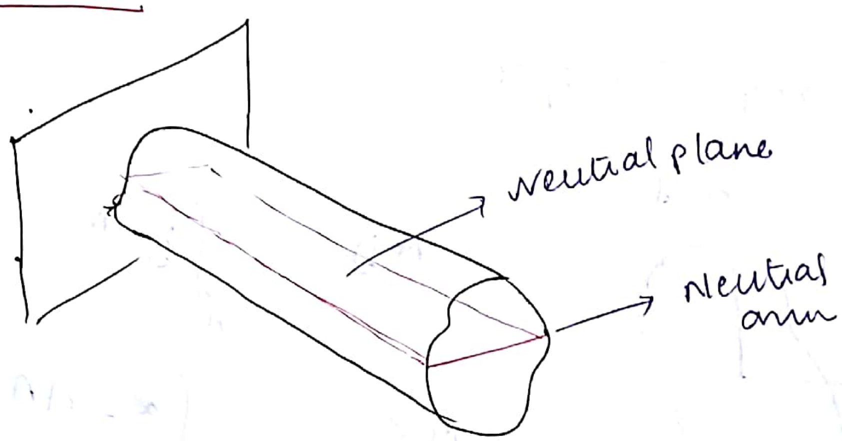
Neutral axis



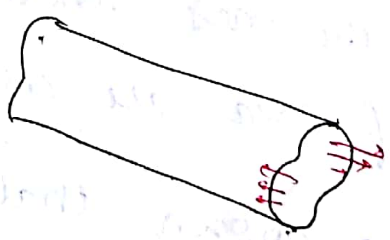
The axis in a c/s along which the bending stress is 0.

For pure bending, the neutral axis passes through the centroid.

For pure bending

$$\int_A \sigma_{xx}\, dA = 0$$

→ i.e. the axial force over a c/s is 0.



If we use $\sigma_{xx} = \dfrac{My}{I}$

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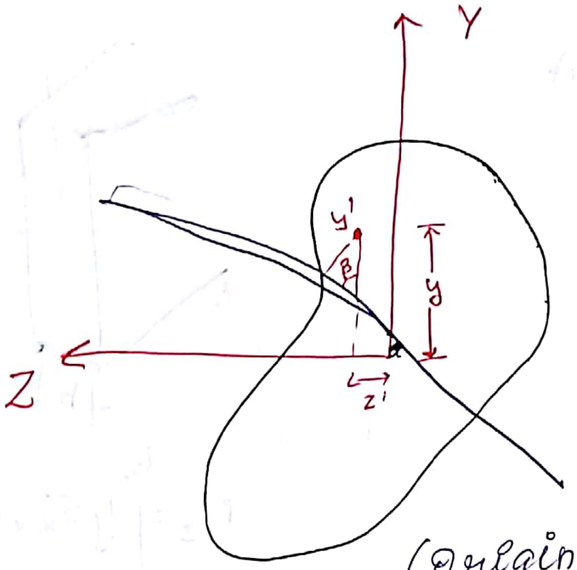
$$\int_A \frac{My}{I} \cdot dA = 0$$

$$\Rightarrow \quad \int y \, dA = 0$$



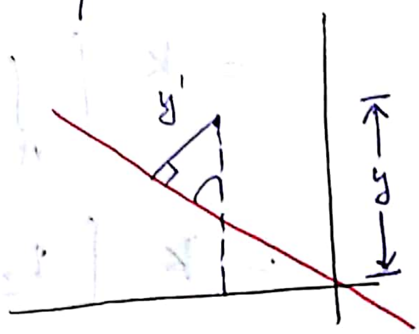
(Origin is at the centroid of the c/s)

$\sigma_{xx} = k y'$

Represent y' in terms of y and z.

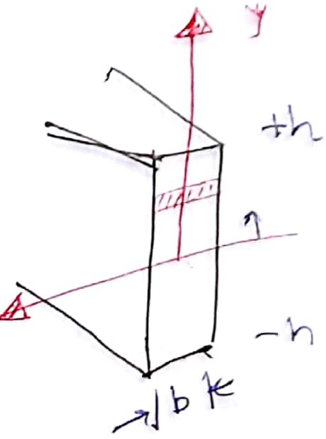
$$y = \frac{y'}{\sin\beta} + \frac{z}{\tan\beta}$$

$$\Rightarrow \quad y' = y \sin\beta - z\cos\beta$$

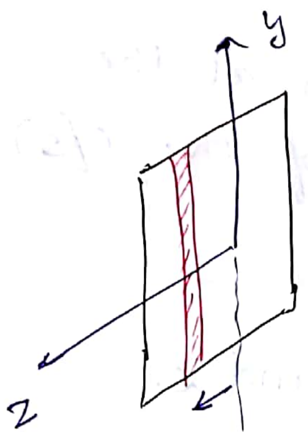


$M_z = -\int \sigma_{xx}\, y\, dA$

$M_y = \int \sigma_{xx}\, z\, dA$



$$M_z = \int_{-h}^{h} y\, \sigma_{xx}\, b\, dy$$



$$-M_z = \int_A k\,(y\sin\beta - z\cos\beta)\, y\, dA$$

$$= k\left[\int_A y^2 \sin\beta \, dA - \int_A yz\cos\beta\, dA\right]$$

$$= k\left[I_z \sin\beta - I_{yz}\cos\beta\right]$$

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similary,

$$M_Y = \int k(y\sin\beta - z\cos\beta)\, z\, dA$$

$$= k\left[I_{YZ}\sin\beta - I_Y\cos\beta \right]$$

$$-M_Z = k\left[I_Z\sin\beta - I_{YZ}\cos\beta \right]$$

$$M_Y = k\left[I_{YZ}\sin\beta - I_Y\cos\beta \right]$$

$$\frac{-M_Y}{M_Z} = \frac{I_{YZ}\sin\beta - I_Y\cos\beta}{I_Z\sin\beta - I_{YZ}\cos\beta}$$

$$= \frac{I_{YZ}\tan\beta - I_Y}{I_Z\tan\beta - I_{YZ}}$$

$\tan\beta =$

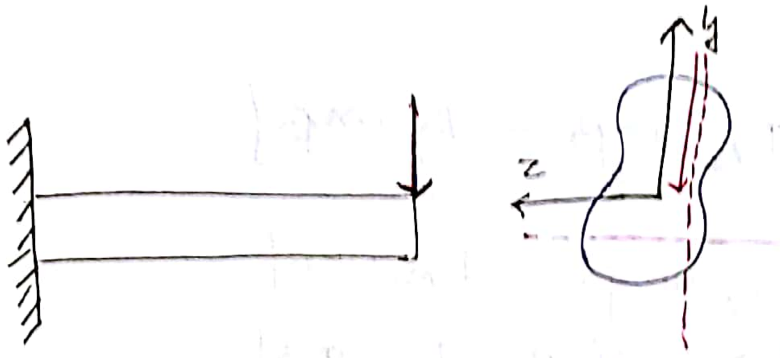
Then find k using either M_z or M_y

Then use $\sigma_{xx} = ky'$ to obtain the general expression.
flexure formula.

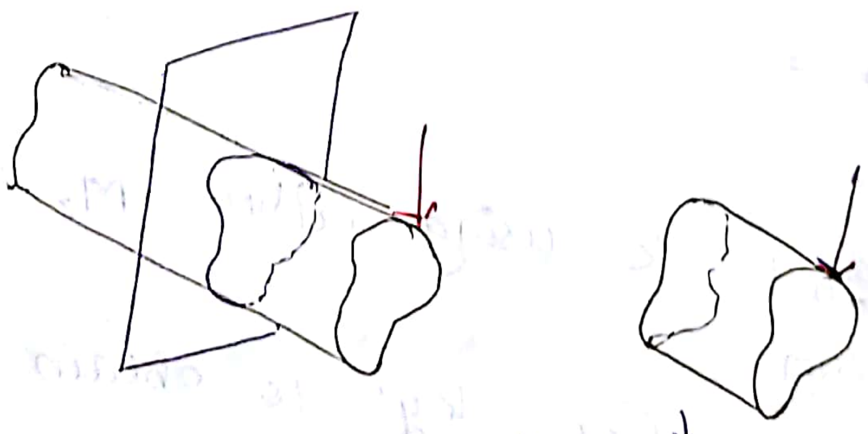
$$\boxed{\sigma_{xx} = \frac{M_Y(yI_{YZ} - zI_Z) - M_Z(zI_{YZ} - yI_Y)}{I_{YZ}^2 - I_Y I_Z}}$$

~~BENDIN~~

Bending, Twisting and Shear Centre

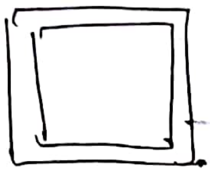


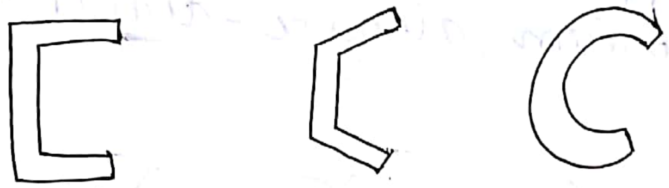
It is possible to find a line of application for each of the horizontal and vertical components of a general oblique load such that twisting induced is zero. The point of intersection of these two lines of application is called the shear centre.

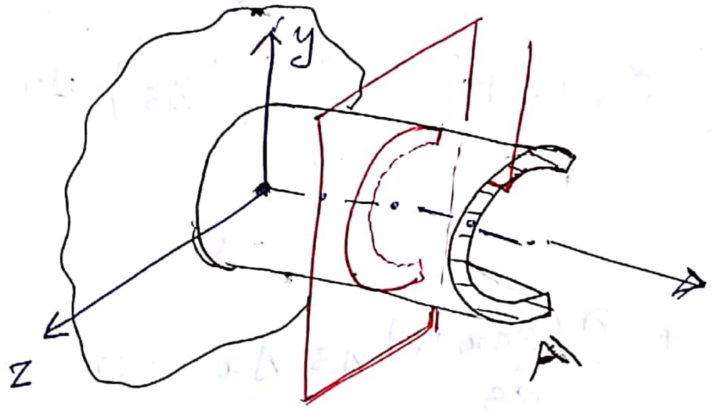


very difficult to find stress distribution
⇒ very difficult to find shear centre

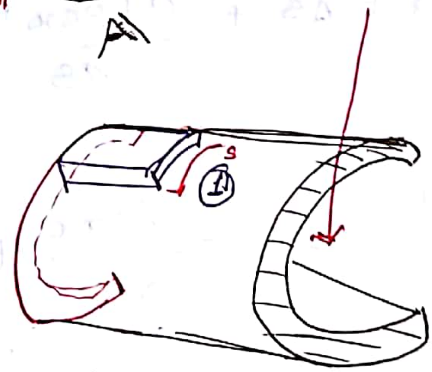
Relatively easy to find shear centre for thin-walled, open sections.



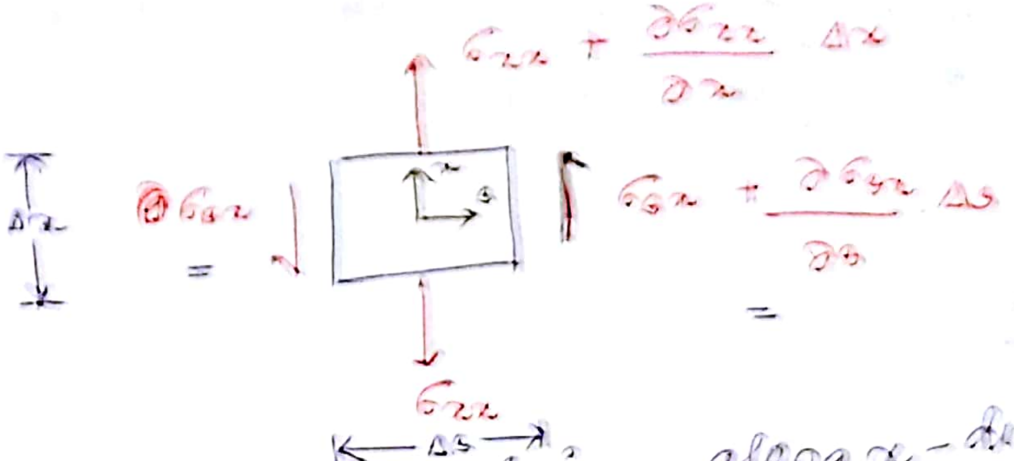




The wall is not necessarily of uniform thickness



II



Force equilibrium along x-direction

$$- \sigma_{xx}\, t\, \Delta s + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \Delta x \right) t\, \Delta s$$

$$- \sigma_{sx} t_s \Delta x + \left(\sigma_{sx} t_s + \frac{\partial (\sigma_{sx} t_s)}{\partial s} \Delta s \right) \Delta x = 0$$

$$\Rightarrow \frac{\partial \sigma_{xx}}{\partial x} \Delta x\, t \cdot \Delta s + \frac{\partial (\sigma_{sx} t_s)}{\partial s} \cdot \Delta s\, \Delta x = 0$$

$$\Rightarrow \boxed{\frac{\partial (\sigma_{sx} t_s)}{\partial s} = - \frac{\partial \sigma_{xx}}{\partial x} t}$$

From flexure formula,

$$\sigma_{xx} = - M_z \frac{(z I_{yz} - y I_y)}{I_{yz}^2 + I_y I_z}$$

$$\frac{\partial (\sigma_{sx} t_s)}{\partial s} = \frac{\partial M_z}{\partial x} \cdot \frac{z I_{yz} - y I_y}{I_{yz}^2 - I_y I_z}$$

Integrating w.r.t. s from 0 to a generic s:

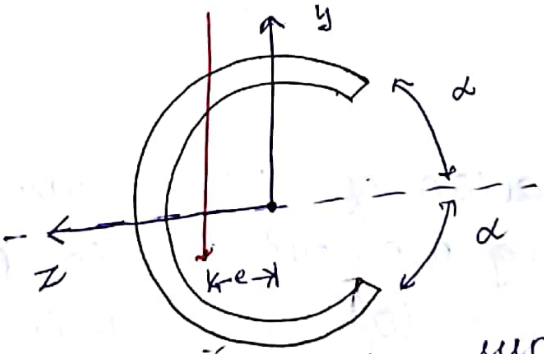
$$\sigma_{sx} t_s = \int \frac{\partial M_z}{\partial x} \cdot \frac{(z I_{yz} - y I_y)}{I_{yz}^2 - I_y I_z} \cdot t\, ds$$

$$= - \int_0^s V_y \frac{(z I_{yz} - y I_y)}{I_{yz}^2 - I_y I_z} \cdot t\, ds$$

$$= \frac{P}{I_{yz}^2 - I_y I_z} \int_0^s (z t I_{yz} - y t I_y)\, ds$$

$$\sigma_{sx} t_s = \frac{P}{I_{yz}^2 - I_y I_z} (I_{yz} Q_y - I_y Q_z)$$

where $Q_y = \int_0^s z t\, ds$ & $Q_z = \int_0^s y t\, ds$

Q 

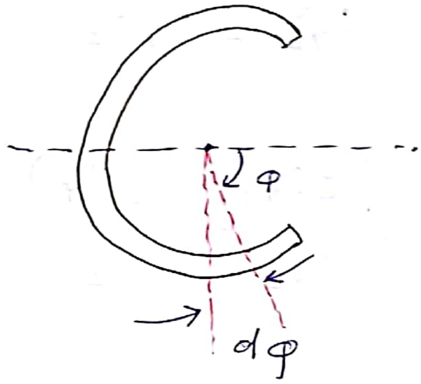
Find the shear centre

$I_{yz} = 0$ due to symm. as $\int yz \cdot dA = 0$

$$\sigma_{sx} = \frac{P}{+ I_y I_z t_s} (+ I_y Q_z)$$

$$= \frac{P Q_z}{I_z t_s}$$

$$Q_z = \int_0^s y t \, ds$$



$$Q_z = \int_\alpha^\varphi (t \underbrace{R\,d\varphi}_{ds}) R \sin\varphi$$

$$I_z = \int_\alpha^{2\pi-\alpha} \underbrace{(t R\,d\varphi)}_{dA} \underbrace{(R \sin\varphi)^2}_{y}$$

To find the location of the shear centre balance the twisting moment due to P by the twisting moment due to the distribution of σ

$$Pe = \int_\alpha^{2\pi-\alpha} \underbrace{\sigma_{sx} \underbrace{t\,R\,d\varphi}_{dA}}_{dF} \times R$$