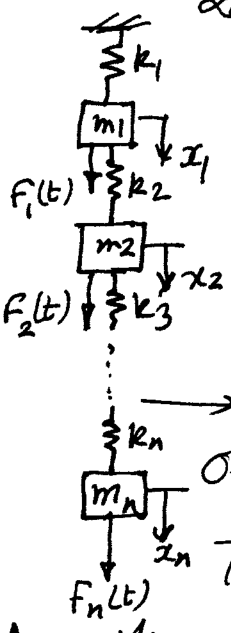
(5) Modal Analysis for an n-DOF system:



An example

Let the DEOM be $[m]\{\ddot{x}\}+[K]\{x\}=\{F(t)\}$ — (1)

Where $[m]=\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n1} & m_{n2} & \cdots & m_{nn}\end{bmatrix}$, $[K]=\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn}\end{bmatrix}$,

$\{x\}=\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{Bmatrix}$, $\{\ddot{x}\}=\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n\end{Bmatrix}$, $\{F(t)\}=\begin{Bmatrix} F_1(t) \\ \vdots \\ F_n(t)\end{Bmatrix}$.

→ Our aim is to obtain the forced response of the system (1) by uncoupling the DEOM. To this end, we first obtain the modal vectors and the associated natural frequencies. For a system having a large number of DOF, these can be obtained by a numerical technique such as the matrix iteration method. $(\omega_1<\omega_2\cdots<\omega_n)$

Let $[\mu]=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn}\end{bmatrix}$ be a normalized modal matrix. To uncouple (1), we introduce a new set of generalized coordinates

$\{p\}=\begin{Bmatrix} p_1(t) \\ \vdots \\ p_n(t)\end{Bmatrix}$ such that $\{x(t)\}=[\mu]\{p(t)\}$ — (2)

Then, $\{\ddot{x}\}=[\mu]\{\ddot{p}\}$ $\left(\{\ddot{p}\}=\{\ddot{p}_1\ \ddot{p}_2 \cdots \ddot{p}_n\}^T\right)$

& (1) transforms into:

$$[m][\mu]\{\ddot{p}\}+[K][\mu]\{p\}=\{F(t)\}$$

Premultiplying both sides by $[\mu]^T$, we get

$$[\mu]^T[m][\mu]\{\ddot{p}\}+[\mu]^T[K][\mu]\{p\}=\underset{n\times n}{[\mu]^T}\underset{n\times 1}{\{F(t)\}} \quad \text{— (3)}$$

But by virtue of the orthogonality principle (to be established soon), the off-diagonal terms

of $[\mu]^T[m]\{\mu\} = [M]$ would be all zeros.
Similarly, $[\mu]^T[k][\mu] = [K]$ also would be diagonal. Also, let $[\mu]^T\{F\} = \{Q\} = \begin{Bmatrix} Q_1(t) \\ \vdots \\ Q_n(t) \end{Bmatrix}$.
Then, (3) can be written as:

$$[M]\{\ddot{p}\} + [K]\{p\} = \{Q(t)\} \quad \text{--- (4)}$$

where $[M] = \begin{bmatrix} M_{11} & 0 & \cdots & 0 \\ 0 & M_{22} & \cdots & 0 \\ 0 & 0 & \cdots & M_{nn} \end{bmatrix}$, $[K] = \begin{bmatrix} K_{11} & 0 & \cdots & 0 \\ 0 & K_{22} & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & & K_{nn} \end{bmatrix}$.

→ M_{11}, M_{22} etc. are called <u>generalized masses</u> & K_{11}, K_{22} etc. are called generalized stiffnesses. Now (4) can be explicitly written as:

$$(5) \cdots \left\{ \begin{aligned} M_{11}\ddot{p}_1 + K_{11}p_1 &= Q_1(t) \\ M_{22}\ddot{p}_2 + K_{22}p_2 &= Q_2(t) \\ &\vdots \\ M_{nn}\ddot{p}_n + K_{nn}p_n &= Q_n(t) \end{aligned} \right\}$$

These are a set of n uncoupled (decoupled) DEOM for the system.

The coordinates $p_1(t), p_2(t), \cdots, p_n(t)$ are called <u>principal</u> coordinates. (Some authors call them <u>normal</u> coordinates or <u>natural</u> coordinates.)

[→ In passing, note that the number of sets of principal coordinates is theoretically infinite since the transformation $\{x(t)\} = [\mu]\{p(t)\}$ could also be invoked with

→

$$[\mu] = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ \mu_{21}X_{11} & \mu_{22}X_{12} & \cdots & \mu_{2n}X_{1n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1}X_{11} & \mu_{n2}X_{12} & \cdots & \mu_{nn}X_{1n} \end{bmatrix}$$ where $X_{11}, X_{12}, \ldots, X_{1n}$ are arbitrary.

Hence, $\{p(t)\} = [\mu]^{-1}\{x(t)\}$ will ~~have~~ be different for different $X_{11}, X_{12}, \ldots, X_{1n}$]

→ We now obtain $p_1(t), p_2(t), \ldots, p_n(t)$ from (5) by using Duhamel's integral.

For instance, $M_{11}\ddot{p}_1 + K_{11}p_1 = Q_1(t)$ has the forced response $p_1(t) = \int_0^t Q_1(\tau)g_1(t-\tau)d\tau$ where

$$g_1(t) = \frac{1}{M_{11}\omega_1}\sin\omega_1 t.$$

[This $g_1(t)$ is obtained by comparison with

$$m\ddot{x} + kx = Q_1(t)$$ for which

$$x_{forced} = x(t) = \int_0^t Q_1(\tau)g(t-\tau)d\tau$$ with

$$g(t) = \frac{1}{m\omega_n}\sin\omega_n t.]$$

{Note ~~that~~, $\omega_1 = \sqrt{\frac{K_{11}}{M_{11}}}, \omega_2 = \sqrt{\frac{K_{22}}{M_{22}}}$ etc.

{So, after you obtain the $[M]$ & $[K]$ matrices, check whether $\sqrt{\frac{K_{11}}{M_{11}}}$ does indeed give ω_1 etc. If not, you made a mistake somewhere.}

So, in general, $p_r(t) = \int_0^t Q_r(\tau)g_r(t-\tau)d\tau$

where $g_r(t) = \frac{1}{M_{rr}\omega_r}\sin\omega_r t$; $r = 1, 2, \ldots, n$.

→

After obtaining $p_1(t), --, p_n(t)$ this way, the required forced responses in terms of $x_1(t), —, x_n(t)$ can be obtained from

$$\{x(t)\} = [\mu]\{p(t)\}, \quad \text{that is,}$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{bmatrix} \begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{Bmatrix}$$

OR

$$\left.\begin{aligned} x_1(t) &= p_1(t) + p_2(t) + \cdots + p_n(t) \\ x_2(t) &= \mu_{21} p_1(t) + \mu_{22} p_2(t) + \cdots + \mu_{2n} p_n(t) \\ &\vdots \\ x_n(t) &= \mu_{n1} p_1(t) + \mu_{n2} p_2(t) + \cdots + \mu_{nn} p_n(t) \end{aligned}\right\} \text{The reqd forced response.}$$

Note — If you are asked to obtain a set of principal coordinates for a given n-DOF system, you should use the relation $\{p(t)\} = [\mu]^{-1}\{x(t)\}$ & obtain $p_r(t)$ as a linear combination of $x_1(t), x_2(t), ---, x_n(t)$ for $r = 1, 2, --, n$.

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(5) We shall digress for a while before we come back to modal analysis.

→ We now prove the orthogonality principle (relations) $\{A\}_r^T[m]\{A\}_s = 0$ & $\{A\}_r^T[k]\{A\}_s = 0$ for $r \neq s$ $(\omega_r \neq \omega_s)$ when $[m] = [m]^T$ & $[k] = [k]^T$ (i.e., $[m]$, $[k]$ both symmetric).

The proof starts with the relations

$$\omega_r^2 [m]\{A\}_r = [K]\{A\}_r \quad \text{---(1)}$$

$$\omega_s^2 [m]\{A\}_s = [K]\{A\}_s \quad \text{---(2)}$$

$r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, n$

but $r \neq s$

(1) & (2) are obtained from $[m]\{\ddot{x}\} + [K]\{x\} = \{0\}$

with $\{x\} = \{A\}\sin(\omega t + \phi)$

$\{\ddot{x}\} = -\omega^2\{A\}\sin(\omega t + \phi)$

So, $-\omega^2[m]\{A\} + [K]\{A\} = 0$ since $\sin(\omega t + \phi) \neq 0$ at all times.

So, $\omega^2[m]\{A\} = [K]\{A\}$

For $\omega = \omega_r$, we have $\{A\} = \{A\}_r$

" $\omega = \omega_s$, $\{A\} = \{A\}_s$.

Thus, $\omega_r^2 [m]\{A\}_r = [K]\{A\}_r$ etc.

Premultiplying both sides of (1) by $\{A\}_s^T$, we get

$$\omega_r^2 \{A\}_s^T [m]\{A\}_r = \{A\}_s^T [K]\{A\}_r \quad \text{---(3)}$$

Premultiplying (2) by $\{A\}_r^T$, we get

$$\omega_s^2 \{A\}_r^T [m]\{A\}_s = \{A\}_r^T [K]\{A\}_s \quad \text{---(4)}.$$

Taking transpose of (3), we have

$$\omega_r^2 \left[\{A\}_s^T [m]\{A\}_r\right]^T = \left[\{A\}_s^T [K]\{A\}_r\right]^T$$

$$\Rightarrow \omega_r^2 \{A\}_r^T [m]^T (\{A_s\}^T)^T = \{A\}_r^T [K]^T (\{A\}_s^T)^T$$

$$\Rightarrow \omega_r^2 \{A\}_r^T [m]\{A\}_s = \{A\}_r^T [K]\{A\}_s \quad \text{---(5)}$$

(Since $([A]^T[B][C])^T = [C]^T[B]^T[A]^T$ & $([A]^T)^T = A$)

assuming $[m]$ & $[K]$ are both symmetric so that $[m]^T = [m]$ & $[K]^T = [K]$

Subtracting (5) from (4) we get

$$(\omega_s^2 - \omega_r^2)\{A\}_r^T [m]\{A\}_s = 0.$$

Hence, if $\omega_r \neq \omega_s$, then $\{A\}_r^T [m]\{A\}_s = 0$

This is the mass orthogonality relation.

Then from (4) or (5), $\{A\}_r^T [K]\{A\}_s = 0$ & this is the stiffness orthogonality relation. Remember these. Note that

the symmetry of $[m]$ & $[k]$ are (6)
necessary for the orthogonality relations to be true. In certain formulations, $[m]$ &/or $[k]$ may not be symmetric. So, be a little careful before you presume orthogonality relations are valid for your problem. Another important but not so obvious point is that $\{A\}_r$ & $\{A\}_s$ must correspond to two different natural frequencies, i.e. $\omega_r \neq \omega_s$. There are systems with <u>repeated</u> natural frequencies, ~~that~~ is, there can be, for instance, a double root of the frequency equation giving rise to, say, $\omega_r = \omega_s$. The funny thing is, we can obtain linearly independent modal vectors $\{A\}_r$ & $\{A\}_s$ corresponding to ~~these~~ this repeated natural frequency. These may not satisfy the orthogonality relations.

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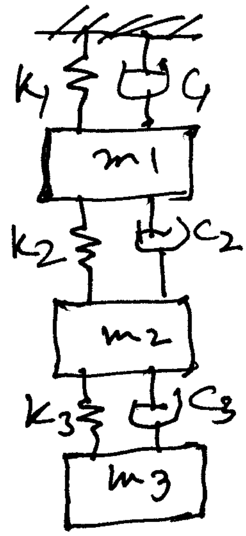
(5) → We now go back to modal analysis. Presently, we shall consider a damped system.

With linear viscous damping, the DEOM will be of the form: →

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\} \quad \text{---(1)}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

For example, for the 3-DOF system shown,



$$[c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

(Check this)

It can be shown that the coordinate transformation $\{x\} = [\mu]\{p\}$ won't uncouple the DEOM (1), where $[\mu]$ is a modal matrix for the corresponding undamped system $[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$.

This is because after using above transformation and premultiplication by $[\mu]^T$ would result in the following:

$$(2) \text{----} [\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[c][\mu]\{\dot{p}\} + [\mu]^T[k][\mu]\{p\} = [\mu]^T\{F\}$$

Although $[\mu]^T[m][\mu] = [M]$ & $[\mu]^T[k][\mu] = [K]$ would be diagonal, $[\mu]^T[c][\mu]$ won't be so. Thus, we don't get uncoupled DEOM.

→ In practice, however, often the off-diagonal terms of $[\mu]^T[c][\mu]$ are small & can be neglected. Then the uncoupled DEOM would be:

$$M_{11}\ddot{p}_1 + C_{11}\dot{p}_1 + K_{11}p_1 = Q_1(t)$$

$$M_{22}\ddot{p}_2 + C_{22}\dot{p}_2 + K_{22}p_2 = Q_2(t)$$

& so on.

→ Each of the above differential equations can be solved for steady-state solution using Duhamel's integral. This will involve the damped impulse response function $g_r(t) = \dfrac{1}{M_{rr}\,\omega_{dr}} e^{-\zeta_r \omega_r t} \sin(\omega_{dr} t)$

[Remember $g(t) = \dfrac{1}{m\,\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$ for $m\ddot{x} + c\dot{x} + kx = f(t)$?]

Here $\omega_{dr} = \omega_r\sqrt{1-\zeta_r^2}$, a damped natural frequency corresponding to

$$M_{rr}\ddot{p}_r + C_{rr}\dot{p}_r + K_{rr}p_r = Q_r(t);\ r = 1, 2, \ldots, n.$$

Also, $\zeta_r = \dfrac{C_{rr}}{2\sqrt{M_{rr}K_{rr}}}$.

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(5) Proportional Damping (Rayleigh Damping) –

If $[C] = \alpha[m]$ or $[C] = \beta[k]$, then it is a case of proportional damping. α & β are constants

$$[M]^T[C][M] = \alpha[M]^T[m][M] \rightarrow \text{Diagonal} \rightarrow$$

& $[M]^T[C][M] = \alpha[M]^T[K][M]$ is also diagonal.

The DEOM thus can be uncoupled and then solved for steady state response using Duhamel's integral.

→ A special type of proportional damping is Rayleigh Damping.

In this case, it is assumed that $[C]$ can be expressed as follows:—

$$[C] = \alpha[m] + \beta[K]. \quad \text{--- (1)}$$

$\boxed{\text{HW problem}}$ Starting with the DEOM

$$[m]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \quad \text{--- (2)}$$

show that when $[C]$ is of the form (1), the DEOM can be uncoupled.

END OF VA-6, Part 3