Assignment Sheet in Fluid Mechanics*
Kinematics

1. For the velocity field given by $\mathbf{v} = \frac{A}{x}\mathbf{i} + A\frac{y}{x^2}\mathbf{j}$, where $A = 2\ \mathrm{m^2 s^{-1}}$, determine the equation of the streamline through $(1\ \mathrm{m}, 3\ \mathrm{m})$. What is the time required for a fluid particle to move from $x = 1$ m to $x = 3$ m? [$y = 3x;\ t = 2$ s]

2. Determine the pathline equation of a particle that passes through the point $(1\ \mathrm{m}, 2\ \mathrm{m})$ at $t = 0$ in a velocity field given by $\mathbf{v} = ax\mathbf{i} - by\mathbf{j}$, where $a = b = 1\ \mathrm{s^{-1}}$. Verify that this equation of the pathline overlaps with that of the streamline passing through the same point. [$xy = 2$]

3. A velocity field is given by $\mathbf{v} = 4x\mathbf{i} + 2t\mathbf{j}\ \mathrm{ms^{-1}}$. Determine the equation of the streamline that passes through the point (2 m, 6 m) when $t = 1$ s. What is the equation of the pathline of a particle that passes through the same point at the same time instant? $\left[y = \left(\frac{1}{2}\ln\frac{x}{2} + 6\right)\mathrm{m};\ y = \left\{\left(\frac{1}{4}\ln\frac{x}{2} + 1\right)^2 + 5\right\}\mathrm{m}\right]$

4. A particle travels along the curve $y^3 = 8x - 12$. If its speed is $5\ \mathrm{ms^{-1}}$ when it is at $x = 1$ m, determine the two components of its velocity at this point. [Magnitude of velocity components: $v_x = 3.43\ \mathrm{ms^{-1}}$; $v_y = 3.63\ \mathrm{ms^{-1}}$]

5. Velocity is defined as $\frac{\mathrm{D}\mathbf{x}}{\mathrm{D}t}$, i.e. the material derivative of the position vector $\mathbf{x}$. Similarly, acceleration is defined as the material derivative of velocity.

   (a) Using this definition of acceleration and the relation $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$, express the acceleration in terms of the components of the velocity and appropriate derivatives in a rectangular Cartesian coordinate system.

   (b) The velocity field for a flow of water is given by $\mathbf{v} = 2x\mathbf{i} + 6tx\mathbf{j} + 3y\mathbf{k}\ \mathrm{ms^{-1}}$. Determine the position of a particle when $t = 0.5$ s if this particle is at (1 m, 0, 0) at $t = 0$. Also find the acceleration vector. [$2.72\mathbf{i} + 1.5\mathbf{j} + 0.634\mathbf{k}$ m; $10.9\mathbf{i} + 32.6\mathbf{j} + 24.5\mathbf{k}\ \mathrm{ms^{-2}}$]

6. Consider a flow field represented by the stream function $\psi = 10xy + 17$. Is this a possible two-dimensional, incompressible flow? Is the flow irrotational? [Yes; Yes]

7. Consider unsteady two-dimensional flow of a constant-density fluid, described by the velocity field, $\mathbf{v}(x, y, t) = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$, where x and y are Cartesian coordinates, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions respectively and t is the time.

   (a) Show that $\mathbf{v}\cdot\nabla\psi = 0$, where ψ is the stream function.

   (b) Show that $\nabla^2\psi = -\omega_z$, where ω_z is the z–component of the vorticity.

   (c) Show that if the flow is irrotational, $\nabla\phi\cdot\nabla\psi = 0$, where ϕ is the velocity potential.

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8. Consider a two-dimensional flow having the velcity field $\mathbf{v}(x, y, t) = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$, where x and y are Cartesian coordinates, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions respectively and t is the time.

   (a) Show that $\nabla \cdot \boldsymbol{\omega} = 0$.

   (b) If the fluid has a constant density so that the velocity field satisfies the condition $\nabla \cdot \mathbf{v} = 0$, then show that $\mathbf{v} = -\mathbf{k} \times \nabla\psi$, where $\mathbf{k}$ is the unit vector in the z–direction and ψ is the stream function.

   (c) If the flow is irrotational so that $\boldsymbol{\omega} = 0$, then by drawing a link between the velocity expressed in terms of the velocity potential, ϕ, and the same velocity expressed in terms of the stream function, ψ, show that $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$, where $\nabla^2() \equiv \frac{\partial^2()}{\partial x^2} + \frac{\partial^2()}{\partial y^2}$.

9. Consider the velocity field given by $\mathbf{v} = Axy\mathbf{i} + By^2\mathbf{j}$, where $A = 4\ \mathrm{m}^{-1}\mathrm{s}^{-1}$, $B = -2\ \mathrm{m}^{-1}\mathrm{s}^{-1}$, and coordinates are measured in meters. Determine the fluid rotation. Evaluate the circulation about the closed contour bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$. Obtain an expression for the stream function. [Rotation $= -2x\mathbf{k}\ \mathrm{s}^{-1}$; $\Gamma = -2\ \mathrm{m}^2\mathrm{s}^{-1}$; $\psi = 2xy^2 + c$]

10. The velocity field near the core of a tornado can be approximated as

$$\mathbf{v} = -\frac{q}{2\pi r}\mathbf{e}_r + \frac{K}{2\pi r}\mathbf{e}_\theta.$$

   Is this an irrotational flow field? Obtain the stream function for this flow. [Yes; $\psi = -\left(\frac{q\theta + K\ln r}{2\pi}\right) + c$]