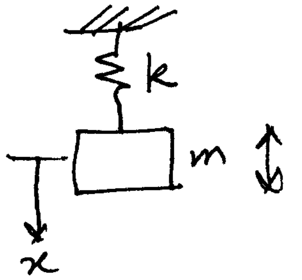
§ Rayleigh's Method (VA-6, Part 4) ①



Aim:– To estimate ω_n for the harmonic oscillator.

Assumption:– x is sinusoidally varying at frequency ω_n, i.e.,

let $x = A\sin(\omega_n t + \phi)$

Then $\dot{x} = A\omega_n \cos(\omega_n t + \phi)$

The system is conservative & so,

$\frac{d}{dt}(T+U) = 0$ or, $T_{max} = U_{max}$,

where $T = KE$ & $U = PE$ of system.

U_{max} occurs at the extreme mass positions & T_{max} occurs at static eq/bm position corresponding to $x = 0$.

Now $T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m A^2\omega_n^2 \cos^2(\omega_n t + \phi)$

& so, $T_{max} = \frac{1}{2} m A^2 \omega_n^2$ —— (1)

Also, $U = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \sin^2(\omega_n t + \phi)$

$\therefore U_{max} = \frac{1}{2} k A^2$ —— (2)

So, $T_{max} = U_{max} \Rightarrow \frac{1}{2} m A^2 \omega_n^2 = \frac{1}{2} k A^2$

or, $\omega_n^2 = \frac{k}{m}$ or, $\omega_n = \sqrt{\frac{k}{m}}$.

→ All this is school physics stuff.

However, Rayleigh's method can be applied to a multi DOF undamped system for a quick & reliable estimation →

of ω_1, the fundamental (natural) frequency of the system. Once again, for an n-DOF system, Rayleigh's method is still based upon the assumption that the system executes the first principal mode of vibration so that all masses pass through equilibrium positions simultaneously (T_{max} occurs) & also all masses reach extreme positions (U_{max} occurs) simultaneously so that $T_{max} = U_{max}$ holds good.

Actually, this holds good for all the principal modes, but the method is no good for estimating any higher natural frequency such as $\omega_2, \omega_3, -$ etc.

All this is because one needs to make a guess for a modal vector and it is (almost) impossible of make a good guess for $\{A\}_2$ & above.

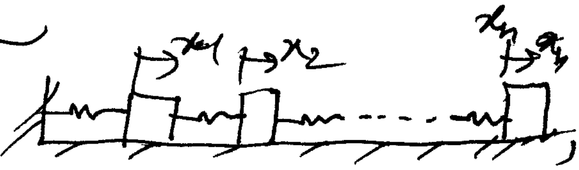
→ What about a guess for $\{A\}_1$?

For many systems, a <u>static deflection</u> vector provides a fairly good guess for $\{A\}_1$ as will be illustrated in the following example. It can be shown theoretically that even if there is 100% error in the guessed $\{A\}_1$, the error in estimated ω_1 will be of a smaller order of magnitude, maybe about 10%! If you are interested, see more about it from Meirovitch's 'Fundamentals of Vibrations', first edition § 7.13, page 331.

→ Here all you need to do is follow the following example based on the formula $\omega_r^2 = \dfrac{\{A\}_r^T [K] \{A\}_r}{\{A\}_r^T [m] \{A\}_r}$;

$r = 1, 2, ---, n$ for an n-DOF system.

→ If you have forgotten, here is a bit of recapitulation :~

for an n-DOF system such as: ,

the DEOM for free vibration are:

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \text{--- (1)}.$$

Let $\{x\} = \{A\}_r \sin(\omega_r t + \phi_r)$ for the r^{th} principal mode; $r = 1, 2, --, n$.

Then, (1) becomes $-\omega_r^2 [m]\{A\}_r + [K]\{A\}_r = 0$, ···(2)

since $\sin(\omega_r t + \phi_r)$ is not ~~zero~~ zero at all times. (2) can be written as:

$$\omega_r^2 [m]\{A\}_r = [K]\{A\}_r \Rightarrow \omega_r^2 \{A\}_r^T [m]\{A\}_r = \{A\}_r^T [K]\{A\}_r$$

$$\Rightarrow \omega_r^2 = \frac{\{A\}_r^T [K]\{A\}_r}{\{A\}_r^T [m]\{A\}_r}; \quad r = 1, 2, --, n.$$

For $r=1$, $\omega_1^2 = \dfrac{\{A\}_1^T [K] \{A\}_1}{\{A\}_1^T [m] \{A\}_1}$ --- (3)

Now, make a guess for $\{A\}_1$ & call it $\{A\}_R$ (Rayleigh $\to R$). The R.H.S. of (3) becomes:

$$\frac{\{A\}_R^T [K] \{A\}_R}{\{A\}_R^T [m] \{A\}_R}$$

& it is called a 'Rayleigh Quotient', giving ω_R^2, which is to be treated as an estimate for ω_1^2. ($\omega_R \to$ called Rayleigh frequency)

→ At this stage, are you baffled a little bit? See, the formula

$$\omega_r^2 = \frac{\{A\}_r^T [K] \{A\}_r}{\{A\}_r^T [m] \{A\}_r}$$

is no good for estimating ω_r unless $\{A\}_r$ is already known. But the analytical as well as MI methods studied before, gives ω_r first & then $\{A\}_r$ (Analytical method) or, $\{A\}_r$ & ω_r simultaneously (MI method).

Thus, if we want to use above formula for estimating ω_r when $\{A\}_r$ is <u>not</u> known, we have to make a guess for $\{A\}_r$. It is very difficult to make such a guess except when $r=1$.

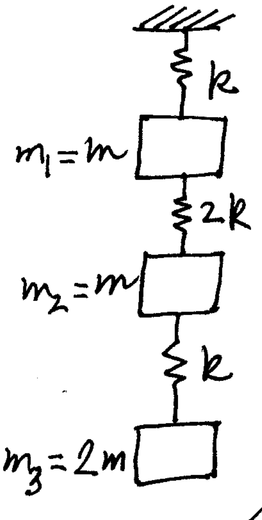
& thus, above formula is used to estimate ω_1 only. Also, as we have already stated, the error in estimated ω_1 shall be much less than error in the guess vector for $\{A\}_1$.

→ Another interesting point you should know about is that the estimated ω_1 shall never be less than the $(\omega_1)_{exact}$! More formally, this is stated as follows:- The Rayleigh Quotient provides an upper bound for ω_1.

→ In many situations, a so-called "static deflection vector" provides a good guess.

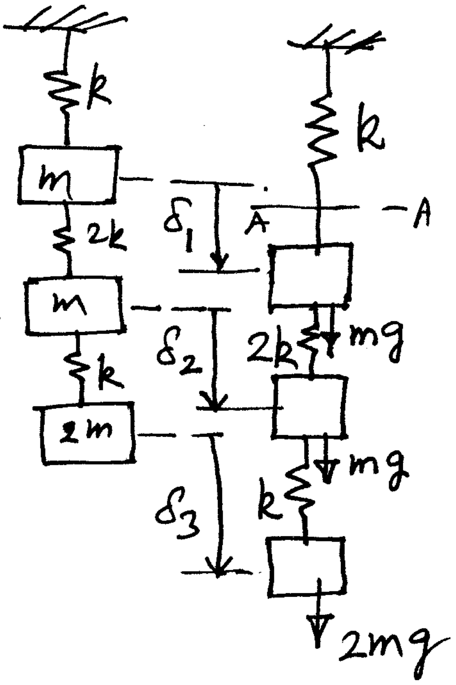
→ All this is now illustrated through an example.

Example:- For the system shown, obtain an estimate for ω_1 using the Rayleigh technique.



Solution:- For a guess of $\{A\}_1$, we take the 'static deflection' vector $\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}$ where δ_1, δ_2 & δ_3 are the deflections m_1, m_2 & m_3 undergo as the system is slowly brought to the static equilibrium position from the free length spring configuration, under the action of the weights of the masses. This is shown as follows:–

'Free length' Configuration. Assume the masses are held by hands

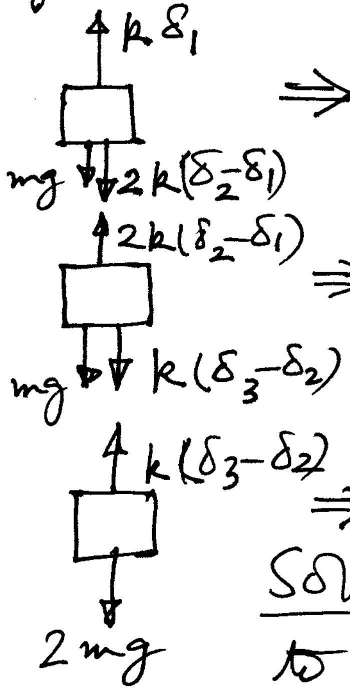


Static eqbm. Configuration after the masses are lowered by gravity & hand actions.

Clearly, all springs are extended

Hence, top spring is extended by δ_1,
middle " " " $(\delta_2-\delta_1)$
& bottom " " " $(\delta_3-\delta_2)$.

The FBDs of the masses are as follows:–



$\Rightarrow mg+2k(\delta_2-\delta_1)-k\delta_1=0$ —— (1)

$\Rightarrow mg+k(\delta_3-\delta_2)-2k(\delta_2-\delta_1)=0$ —— (2)

$\boxed{\text{You may use method of sections (A-A etc.) \& get } \delta_i \text{ more quickly}}$

$\Rightarrow 2mg-k(\delta_3-\delta_2)=0$ —— (3)

Solve (do it) (1), (2) & (3) simultaneously to get $\delta_1=\dfrac{4mg}{k}$, $\delta_2=\dfrac{11mg}{2k}$, $\delta_3=\dfrac{15mg}{2k}$

Hence, $\{\delta\}$ = guess for $\{A\}_1 = \frac{mg}{K}\begin{Bmatrix} 4 \\ 11/2 \\ 15/2 \end{Bmatrix}$

→ You may discard $\frac{mg}{K}$ & take

$$\{\delta\} = \begin{Bmatrix} 4 \\ 11/2 \\ 15/2 \end{Bmatrix}.$$

(Why?) (because a modal vector can be multiplied by an arbitrary constant)

→ Next obtain [m] & [K] by obtaining the DEOM. Here $[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$

& $[K] = \begin{bmatrix} 3K & -2K & 0 \\ -2K & 3K & -K \\ 0 & -K & K \end{bmatrix}$ (Verify all this)

Then, $$\omega_R^2 = \frac{\{\delta\}^T[K]\{\delta\}}{\{\delta\}^T[m]\{\delta\}} = \frac{1.531K}{9.922m}$$

$$\Rightarrow \omega_1 \cong \omega_R = 0.393\sqrt{\frac{K}{m}}.$$

Hence, by Rayleigh's method,

$$(\omega_1)_{approx.} = 0.393\sqrt{\frac{K}{m}}$$

HW → Obtain $(\omega_1)_{exact}$ by analytical method. You should get $(\omega_1)_{exact} = 0.391\sqrt{\frac{K}{m}}$, accurate upto 3 places after decimal.

Note that $(\omega_1)_{approx} > (\omega_1)_{exact}$.

$$\% \text{ error} = \frac{(\omega)_{approx} - (\omega_1)_{exact}}{(\omega_1)_{exact}} \times 100 = 0.51 \;!$$

Hence, $(\omega_1)_{approx}$ is quite accurate!

→ You now do the following :- ~~[illegible]~~ Get $\{A\}_1$

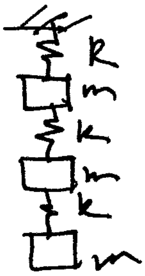
by analytical method & normalize it.

Compare it with $\{\delta\}_{\text{normalized}} = \begin{Bmatrix} 1 \\ 11/8 \\ 15/8 \end{Bmatrix}$

& see the error in $\{\delta\}$. ~~You should be able~~

You will observe ~~that~~ error in $\{\delta\}$ is much greater than error in the estimated ω_1.

(HW) for the system

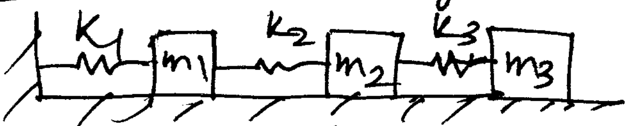


, ω_1 & $\{A\}_1$ was obtained earlier. Now, using Rayleigh's method, obtain $(\omega_1)_{approx}$ & estimate the % error.

Imp. Notes:-

(Note 1:-) To obtain $\{\delta\}$, you may use the method of sections where applicable. This method was discussed in the class while discussing flexibility influence coefficients.

(Note 2:-) for a horizontal system such as



, apply imaginary horizontal forces $m_1 g$, $m_2 g$ & $m_3 g$ on the masses (left to right) & estimate $\{\delta\}$.

(Note 3:-) Rayleigh's method was once widely used for estimating the lowest critical speed of whirling shaft-rotor systems. See optional

§ 5.3, page 195, example 3 from 'Mechanical Vibrations' by Tse, Morse, Hinkle, 2nd Edition. Also see Martin's book on Kinematics & Dynamics of Machines, chapter on critical speeds.

Note 4:- for theoretical aspects of Rayleigh's

optional

method, see 'Theory of Vibration' (3rd Ed.) by W.T. Thomson, § 11.1, Pg. 292.

Note 5:- There are ~~many~~ interesting

optional

applications of Rayleigh's quotient. For instance, the famous Rayleigh-Ritz method is based upon it. See 'Analytical Methods in Vibrations' by L. Meirovitch.

END OF VA-6 — Part 4