

APPENDIX A

Trigonometric identities

Formulas for rotation about the principal axes by θ :

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad (\text{A.1})$$

$$R_Y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad (\text{A.2})$$

$$R_Z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.3})$$

Identities having to do with the periodic nature of sine and cosine:

$$\begin{aligned} \sin \theta &= -\sin(-\theta) = -\cos(\theta + 90^\circ) = \cos(\theta - 90^\circ), \\ \cos \theta &= \cos(-\theta) = \sin(\theta + 90^\circ) = -\sin(\theta - 90^\circ). \end{aligned} \quad (\text{A.4})$$

The sine and cosine for the sum or difference of angles θ_1 and θ_2 :

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= c_{12} = c_1 c_2 - s_1 s_2, \\ \sin(\theta_1 + \theta_2) &= s_{12} = c_1 s_2 + s_1 c_2, \\ \cos(\theta_1 - \theta_2) &= c_1 c_2 + s_1 s_2, \\ \sin(\theta_1 - \theta_2) &= s_1 c_2 - c_1 s_2. \end{aligned} \quad (\text{A.5})$$

The sum of the squares of the sine and cosine of the same angle is unity:

$$c^2 \theta + s^2 \theta = 1. \quad (\text{A.6})$$

If a triangle's angles are labeled a , b , and c , where angle a is opposite side A , and so on, then the "law of cosines" is

$$A^2 = B^2 + C^2 - 2BC \cos a. \quad (\text{A.7})$$

The "tangent of the half angle" substitution:

$$\begin{aligned} u &= \tan \frac{\theta}{2}, \\ \cos \theta &= \frac{1 - u^2}{1 + u^2}, \\ \sin \theta &= \frac{2u}{1 + u^2}. \end{aligned} \quad (\text{A.8})$$

To rotate a vector \mathcal{Q} about a unit vector \hat{K} by θ , use **Rodrigues's formula**:

$$\mathcal{Q}' = \mathcal{Q} \cos \theta + \sin \theta (\hat{K} \times \mathcal{Q}) + (1 - \cos \theta) (\hat{K} \cdot \hat{\mathcal{Q}}) \hat{K}. \quad (\text{A.9})$$

See Appendix B for equivalent rotation matrices for the 24 angle-set conventions and Appendix C for some inverse-kinematic identities.

A P P E N D I X B

The 24 angle-set conventions

The 12 Euler angle sets are given by

$$R_{X'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{X'Z'Y'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{Y'Z'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Z'X'Y'}(\alpha, \beta, \gamma) = \begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$R_{X'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{X'Z'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Y'}(\alpha, \beta, \gamma) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$\begin{aligned}
 R_{Y'Z'Y'}(\alpha, \beta, \gamma) &= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
 R_{Z'X'Z'}(\alpha, \beta, \gamma) &= \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}, \\
 R_{Z'Y'Z'}(\alpha, \beta, \gamma) &= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.
 \end{aligned}$$

The 12 fixed angle sets are given by

$$\begin{aligned}
 R_{XYZ}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}, \\
 R_{XZY}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
 R_{YXZ}(\gamma, \beta, \alpha) &= \begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix}, \\
 R_{YZX}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
 R_{ZXY}(\gamma, \beta, \alpha) &= \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix}, \\
 R_{ZYX}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}, \\
 R_{XYX}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}, \\
 R_{XZX}(\gamma, \beta, \alpha) &= \begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
 R_{YXY}(\gamma, \beta, \alpha) &= \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},
 \end{aligned}$$

$$R_{YZY}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{ZXZ}(\gamma, \beta, \alpha) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix},$$

$$R_{ZYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

APPENDIX C

Some inverse-kinematic formulas

The single equation

$$\sin \theta = a \quad (\text{C.1})$$

has two solutions, given by

$$\theta = \pm \text{Atan2}(\sqrt{1 - a^2}, a). \quad (\text{C.2})$$

Likewise, given

$$\cos \theta = b, \quad (\text{C.3})$$

there are two solutions:

$$\theta = \text{Atan2}(b, \pm \sqrt{1 - b^2}). \quad (\text{C.4})$$

If both (C.1) and (C.3) are given, then there is a unique solution given by

$$\theta = \text{Atan2}(a, b). \quad (\text{C.5})$$

The transcendental equation

$$a \cos \theta + b \sin \theta = 0 \quad (\text{C.6})$$

has the two solutions

$$\theta = \text{Atan2}(a, -b) \quad (\text{C.7})$$

and

$$\theta = \text{Atan2}(-a, b). \quad (\text{C.8})$$

The equation

$$a \cos \theta + b \sin \theta = c, \quad (\text{C.9})$$

which we solved in Section 4.5 with the tangent-of-the-half-angle substitutions, is also solved by

$$\theta = \text{Atan2}(b, a) \pm \text{Atan2}(\sqrt{a^2 + b^2 - c^2}, c). \quad (\text{C.10})$$

The set of equations

$$\begin{aligned} a \cos \theta - b \sin \theta &= c, \\ a \sin \theta + b \cos \theta &= d, \end{aligned} \quad (\text{C.11})$$

which was solved in Section 4.4, also is solved by

$$\theta = \text{Atan2}(ad - bc, ac + bd). \quad (\text{C.12})$$

