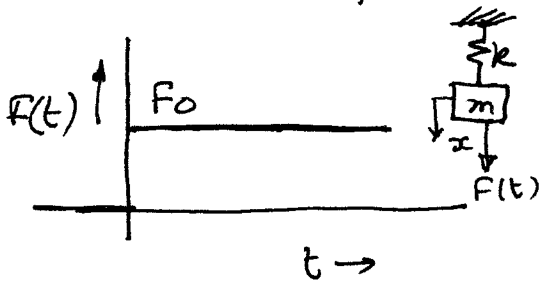
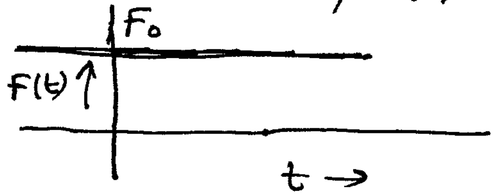
→ Note the following carefully :~



For the excitation shown in the figure, if you write down the DEOM as $m\ddot{x}+kx=F(t)=F_0$ & then write the ss solution as $x_{ss}=x(t)=$ the particular integral $=\frac{F_0}{k}$, you would be wrong. Here is the reason:—

The DEOM $m\ddot{x}+kx=F_0$ has the PI $x_p=x_{ss}=F_0/k$. But this analytical solution has the implicit assumption that the forcing function existed before $t=0$, i.e. the forcing function is like



& not like the one given above. In the above case, the forcing function is applied at $t=0$ with zero initial conditions ($x(0)=0$, $\dot{x}(0)=0$) present. The actual response, as obtained earlier by the use of Duhamel's integral is $x(t)=\frac{F_0}{k}[1-\cos\omega_n t]$.

→ To get this correct result using the ~~[illegible]~~ PI of $m\ddot{x}+kx=F_0$, you have to write the complete solution, viz.,

$$x(t)=\underbrace{A\sin\omega_n t+B\cos\omega_n t}_{\text{The free vibration part}}+\underbrace{\frac{F_0}{k}}_{\text{The forced vibration part}} \quad \text{---}(1)$$

[A, B arbitrary until initial conditions imposed]

and then impose zero initial conditions.

$x(0)=0$ & $\dot{x}(0)=0$ in (1).

$x(0)=0 \Rightarrow 0 = B+\frac{F_0}{K}$ or, $B=-\frac{F_0}{K}$

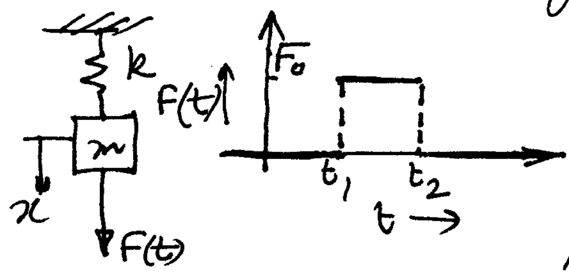
Also, $\dot{x} = A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t$

So, $\dot{x}(0)=0 \Rightarrow 0=A$

Hence, from (1), we get: $x(t)=\frac{F_0}{K}[1-\cos\omega_n t]$, the correct answer.

→ From all this, you shouldn't think that we can do without Duhamel's integral formula. In many practical situations, it is not possible to get a closed form solution such as (1) and in such situations, Duhamel's formula turns out to be very useful.

→ Next, carefully follow the following example:–



Obtain the response using Duhamel's integral formula.

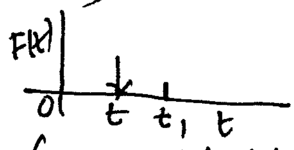
Solution:–

$$x(t)=\int_0^t F(\tau)g(t-\tau)\,d\tau$$

where $g(t)=\frac{1}{m\omega_n}\sin\omega_n t$.

(Note that a single expression for $F(\tau)$ is not available for every $t>0$. Hence, we must consider various subintervals as follows)

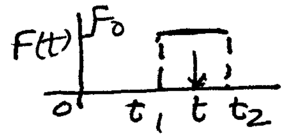
(i) $F(t)=0$ for $0\le t\le t_1$



Hence, $x(t)=\int_0^t 0\cdot g(t-\tau)\,d\tau=0$ for $0\le t\le t_1$.

(This is obvious too. No forcing function, no response, that's all.)
(with zero initial conditions)

(ii) For $t_1 < t \le t_2$, $F(t) = F_0$



So, $x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$

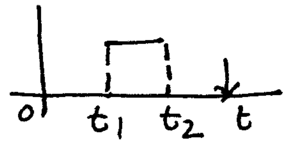
$$= \int_0^{t_1} F(\tau)\, g(t-\tau)\, d\tau + \int_{t_1}^{t} F(\tau)\, g(t-\tau)\, d\tau$$

$$= 0 + \int_{t_1}^{t} F_0 \cdot \frac{1}{m\omega_n} \sin\omega_n (t-\tau)\, d\tau \quad \text{--- (i)}$$

$$= \frac{F_0}{m\omega_n^2} \Big| \cos\omega_n (t-\tau) \Big|_{t_1}^{t} \quad \text{check}$$

$$= \frac{F_0}{k}\left[1 - \cos\omega_n (t - t_1)\right] \quad \text{--- (ii)} \quad \text{Ans.}$$

(iii) For $t > t_2$, $F(t) = 0$



Again $x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$

$$= \int_0^{t_1} \overset{0}{\cancel{F(\tau)}}\, g(t-\tau)\, d\tau + \int_{t_1}^{t_2} F(\tau)\, g(t-\tau)\, d\tau + \int_{t_2}^{t} 0 \cdot g(t-\tau)\, d\tau$$

$$= \int_{t_1}^{t_2} F_0 \cdot \frac{1}{m\omega_n} \sin\omega_n (t-\tau)\, d\tau \quad \text{--- (iii)}$$

(* At this step, be a little careful. Don't think the above integral $= \frac{F_0}{k}\left[1 - \cos(t_2 - t_1)\right]$ (from (ii)) just because the upper limit in (ii) is t_2 instead of upper limit t in (i) & so, all you need to do is put t_2 in place of t in (ii). Well, you may think you'd never

τ is the variable of integration, not t & so, you can't touch the t in the $\int$ integrand) do it yourself, but many guys are seen to commit this mistake. From another point of view also this can be seen.

$$x(t) = \frac{F_0}{k}\left[1 - \cos(t_2 - t_1)\right] = \text{a constant.}$$

But at $t = t_2$, we have $x(t_2)$ & $\dot{x}(t_2)$ non-zero & after $t = t_2$, a 'free-vibration' response occurs. Hence, it must be a function of time, not a constant.)

So, for $t > t_2$, $x(t) = \frac{F_0}{m\omega_n^2}\Big|\cos\omega_n(t-\tau)\Big|_{t_1}^{t_2}$

check

$$= \frac{F_0}{k}\left[\cos\omega_n(t-t_2) - \cos\omega_n(t-t_1)\right]$$

Ans.

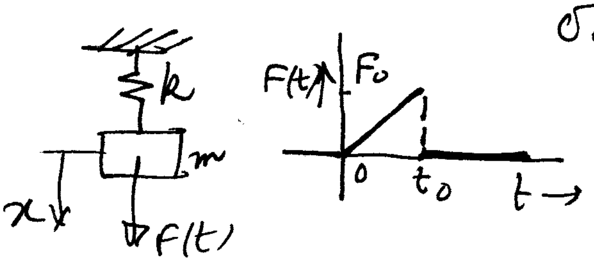
→ To sum up:

$x(t) = 0$ for $0 \le t \le t_1$

$x(t) = \frac{F_0}{k}\left[1 - \cos\omega_n(t-t_1)\right]$ for $t_1 \le t \le t_2$

& $x(t) = \frac{F_0}{k}\left[\cos\omega_n(t-t_2) - \cos\omega_n(t-t_1)\right]$ for $t > t_2$

Example:-



Obtain $x(t)$ by using the convolution integral formula.

Here $F(t) = \frac{F_0}{t_0}t$ for $0 \le t \le t_0$
$= 0$ for $t > t_0$

i) For $0 \le t \le t_0$,

$$x(t) = \int_0^t F(\tau)\,g(t-\tau)\,d\tau = \int_0^t \frac{F_0}{t_0}\tau \cdot \frac{1}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{t_0 m\omega_n}\int_0^t \tau \sin\omega_n(t-\tau)\,d\tau \quad \text{(integrating by parts)}$$

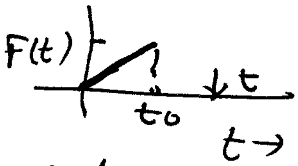
$$= \frac{F_0}{t_0 m\omega_n}\left[\frac{1}{\omega_n}\tau\cos\omega_n(t-\tau)\Big|_0^t - \frac{1}{\omega_n}\int_0^t 1.\cos\omega_n(t-\tau)\,d\tau\right]$$

$$= \frac{F_0}{t_0 m\omega_n}\left[\frac{t}{\omega_n} + \frac{1}{\omega_n^2}\sin\omega_n(t-\tau)\Big|_0^t\right]$$

$$= \frac{F_0}{t_0 m\omega_n^2}\left[t - \frac{1}{\omega_n}\sin\omega_n t\right] = \frac{F_0}{k t_0}\left[t - \frac{\sin\omega_n t}{\omega_n}\right]$$

Check

(ii) <u>For $t > t_0$</u>



$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\,d\tau$$

$$= \int_0^{t_0} F(\tau)\, g(t-\tau)\,d\tau + \int_{t_0}^t \overset{0}{\cancel{F(\tau)}}\, g(t-\tau)\,d\tau$$

$$= \frac{F_0}{t_0 m\omega_n}\left[\frac{t}{\omega_n} + \frac{1}{\omega_n^2}\sin\omega_n(t-\tau)\Big|_0^{t_0}\right]$$

(Using previous integration)

$$= \frac{F_0}{k t_0}\left[t + \frac{1}{\omega_n}\left\{\sin\omega_n(t-t_0) - \sin\omega_n t\right\}\right]$$

Check

<u>Important:-</u> For the sake of simplicity, we have taken an undamped system in the above examples. In the next example, damping is considered & integrations become more complicated. So, you may find it convenient to remember the following results

instead of using integration by parts to obtain the values of integrals like $\int_0^{t_1} e^{at} \sin bt \, dt$ etc.

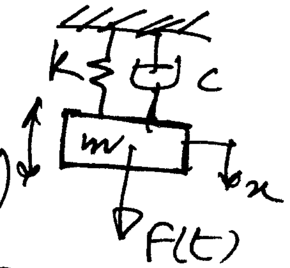
Remember:— check from a Calculus book.

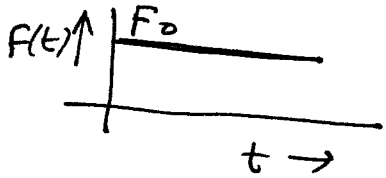
$$\int e^{at} \sin bt \, dt = \left(\frac{e^{at}}{a^2+b^2}\right)(a \sin bt - b \cos bt)$$

$$\int e^{at} \cos bt \, dt = \left(\frac{e^{at}}{a^2+b^2}\right)(b \sin bt + a \cos bt)$$

Ex.

$(\zeta < 1)$





Find $x(t)$ using Duhamel's integral.

Solution:— $x(t) = \int_0^t F(\tau) g(t-\tau) \, d\tau$

where $g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$.

Hence, $x(t) = \int_0^t \frac{F_o}{m\omega_d} e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) \, d\tau$

etc.

→ Complete the solution

Question:— for above example, will it be more convenient to use $x(t) = \int_0^t F(t-\tau) g(\tau) \, d\tau$?

check.
