Vibration of Continuous Systems (Part B)

In part A, we obtained DEOM for several systems. We shall now try to obtain the response.

⑤ Longitudinal vibration of bars :-

Let the bar be uniform in X-section.

The DEOM (free vibrations) is:

$$\boxed{C = \sqrt{\frac{AE}{m}} = \sqrt{E/\rho}} \qquad \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{—①} \quad \text{[See pg. 5, part A]}$$

① is a linear, 2nd order partial differential equation. [Why? → Done earlier]

E. Kreyszig, Advanced Engg. Mathematics, 9th Ed., Chapter 12

Note that ① involves derivatives w.r.t one variable (x or t) only, there is no mixed derivative term such as ($\propto \frac{\partial^2 u}{\partial t \partial x}$). In such a situation, the <u>method of separating variables</u>, or, <u>product method</u> works.

We assume $u(x,t) = U(x) f(t)$ —②

(You must have done it in a Maths or Physics course somewhere!)

Then, $\frac{\partial u}{\partial t} = U(x)\dot{f}(t)$; $\frac{\partial^2 u}{\partial t^2} = U(x)\ddot{f}(t)$ —③

where $\dot{f} = \frac{df}{dt}$ & $\ddot{f} = \frac{d^2 f}{dt^2}$.

Also, $\frac{\partial u}{\partial x} = U'(x) f(t)$; $\frac{\partial^2 u}{\partial x^2} = U'' f(t)$ —④

where $U' = \frac{dU}{dx}$ & $U'' = \frac{d^2 U}{dx^2}$

Substituting ③ + ④ in ①, we get

$$U(x)\ddot{f}(t) = c^2 U''(x) f(t) \Rightarrow \frac{\ddot{f}}{f} = \frac{c^2 U''}{U} \quad \text{—⑤}$$

Note that $\ddot{f}/f$ is a function of time t, or constant & $\frac{U''}{U}$ is a function of x, or constant only. Hence, here, each of these ratios must be equal to a constant which we represent as $-\omega^2$. Thus, ⑤ gives

$\frac{\ddot{f}}{f} = c^2\frac{U''}{U} = -\omega^2$ -- ⑥, implying the following two ordinary linear DEs of order 2 :-

$$\ddot{f} + \omega^2 f = 0 \text{ -- ⑦ } \& \ U'' + \left(\frac{\omega}{c}\right)^2 U = 0 \text{ -- ⑧}$$

[Look at ⑦ & see why in ⑥ we took $-\omega^2$ instead of ω^2. Had we taken $\frac{\ddot{f}}{f} = \omega^2$, then $\ddot{f} - \omega^2 f = 0$ would have unbounded solutions for f [Take $f = Ae^{st}$ to check this], which is not possible here.]

⑦ has the general solution

$$f(t) = A\sin\omega t + B\cos\omega t \text{ -- ⑨}$$

; A, B → arbitrary constants of integration

{You remember $m\ddot{x} + kx = 0$, or $\ddot{x} + \omega_n^2 x = 0$, which had the gen. soln. $x(t) = A\sin\omega_n t + B\cos\omega_n t = X_0\sin(\omega_n t + \phi)$?]

→ To find A & B for a particular motion, we need two initial conditions $f(0)$ & $\dot{f}(0)$.

Let us now turn to DE ⑧, i.e., $\frac{d^2U(x)}{dx^2} + \beta^2 U = 0$ -- ⑩

with $\beta = \frac{\omega}{c}$ -- ⑪. A comparison with $\frac{d^2x}{dt^2} + \omega_n^2 x = 0$ suggests the gen. soln. $U(x) = \overset{\text{Capital}}{C}\sin\beta x + D\cos\beta x$ -- ⑫

where C & D are arbitrary at this stage. For a particular motion, these are to be obtained using the Boundary Conditions at $x = 0$ & $x = l$.

→

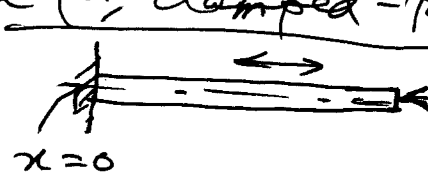
Thus, we have a boundary value problem here, like in many other branches of science and engineering.

Now remember the following:-

→ The DEOM is the same irrespective of the boundary conditions (BCs) the bar is subject to.

→ for each given configuration such as clamped-free, free-free(!), clamped-pinned, simply supported (pinned/pinned) etc, 2 boundary conditions, one at $x=0$ & the other at $x=l$ are required to be specified for our second order PDE. These 2 conditions, together with solution ⑫ will then determine the natural frequencies of our system.

Example:- The fixed-free (or, clamped-free) configuration:



For this, BC at $x=0$ is, clearly, $u(x,t)=0$, at all times, i.e., $u(0,t)=0$ OR $u(x,t)\big|_{x=0}=0$ — ⑬

→ A little more difficult BC occurs at $x=l$. Note that for the free-vibration situation, axial force in bar at $x=l$ is zero. This axial force is $EA\frac{\partial u}{\partial x}$, $0 \le x \le l$ (See pg. 4, part A) Hence, the BC at $x=l$ is: $EA\frac{\partial u}{\partial x}\big|_{x=l}=0$ — ⑭

So, ⑬ & ⑭ are the two boundary conditions to be satisfied. To obtain C & D in ⑫, we need $U(0)$ & $U'(l)$, i.e., $\frac{dU(x)}{dx}\Big|_{x=L}$ which are obtained as follows:

$$u(x,t) = U(x)f(t) \Rightarrow u(0,t) = U(0)f(t)$$

But $u(0,t) = 0 \xrightarrow{\text{By } (13)}$ & so, $U(0)f(t) = 0$

$\Rightarrow \boxed{U(0)=0}$ —(13') [Since $f(t)=0$ is trivial, won't do]

we have kept EA since $EA\frac{du}{dx}$ has a physical meaning of a force

Also, $EA\frac{du}{dx}\Big|_{x=l} = 0 \xrightarrow{\text{By } (14)} \Rightarrow EA\frac{dU}{dx}f(t)\Big|_{x=l} = 0$

$\Rightarrow \boxed{EA\frac{dU}{dx}\Big|_{x=l} = 0}$ —(14')

Important Remember that use of BCs (13') & (14') in ⑫ will lead to the frequency equation whose solutions give the natural frequencies of the system.

Let us obtain the frequency equation.

We have: $U(x) = C\sin\beta x + D\cos\beta x$ —⑫

So, $\frac{dU}{dx} = C\beta\cos\beta x - D\beta\sin\beta x$ —(12')

BC (13'), viz., $U(0)=0$ gives (using ⑫)

$$0 = U(0) = C\sin 0 + D\cos 0 = D$$

So, $\boxed{D = U(0) = 0 \text{ ~~given~~}}$ —(15) $\Rightarrow \boxed{U(x) = C\sin\beta x}$ —⑯

Also, from (12'), BC $\frac{dU}{dx}\Big|_{x=l} = 0$ gives

$$0 = \frac{dU}{dx}\Big|_{x=l} = C\beta\cos\beta l - \overset{0}{\cancel{D}}\beta\sin\beta l = C\beta\cos\beta l$$

But $C \neq 0$ & β can't be zero. Hence,

$\boxed{\cos\beta l = 0}$ —⑰, which is the reqd. frequency equation.

Now, $\beta = \frac{\omega}{c}$ & so, the frequency eqn. can be written as:

$$\cos\left(\frac{\omega l}{c}\right) = 0 \quad \text{—— (17')}$$

(17') has an infinity of solutions, namely,

$$\frac{\omega l}{c} = (2n-1)\frac{\pi}{2}; \; n = 1, 2, 3, \ldots$$

So, to interpret things better, we replace ω by ω_n in above relation so ~~that~~

$$\frac{\omega_n l}{c} = (2n-1)\pi/2 \text{ or } \boxed{\omega_n = (2n-1)\frac{\pi c}{2l}} \text{ — (18)};$$

$$\boxed{c = \sqrt{\frac{EA}{m}} = \sqrt{\frac{E}{\rho}}}$$

$n = 1, 2, 3, \ldots$

Remember

(18) gives the (infinitely many) natural frequencies of the bar in axial vibration.

ω_1 is the fundamental or lowest natural frequency. Note that usually only first few natural frequencies are of importance.

Relation (16), pg. 4 gives $U(x) = C\sin\beta x$,

~~or, $U(x) = C\sin\omega$~~ ~~which~~ or, $U(x) = C\sin\frac{\omega x}{c}$.

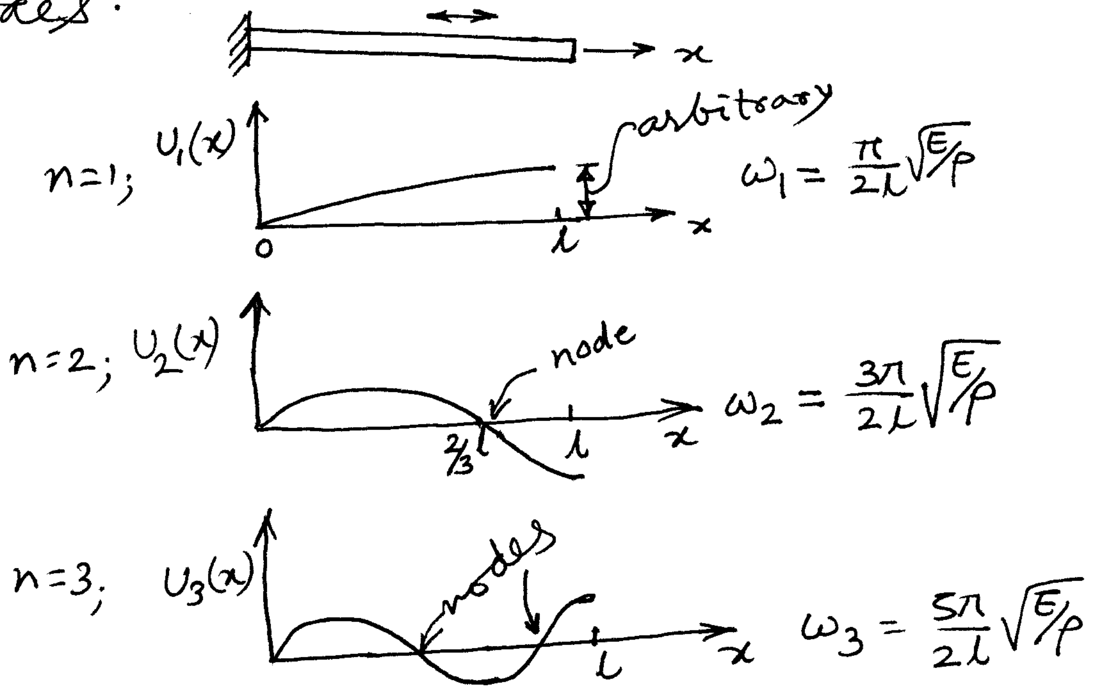
Capital — small

However, in view of the fact that ω can take on values such as $\omega_1, \omega_2, \omega_3 \ldots$, we should write above relation as

$$\text{(19)} \; \text{----} \; \boxed{U_n(x) = C_n \sin\left(\frac{\omega_n x}{c}\right)}; \quad n = 1, 2, 3, \ldots$$

→ $U_n(x)$ is called the nth eigenfunction or, the eigenfunction corresponding to the nth natural frequency ω_n, or, the eigenfunction corresponding to the nth mode of free vibration. Note that C_n is arbitrary!

→ The following figure shows the first three modes.



The higher frequencies ω_2, ω_3 ---, are called overtones. Overtones that are integral multiples of the fundamental frequency ω_1 are called higher harmonics.

→ Now go back to relation ⑨, page 2. Clearly, this relation should now be rewritten as:

$$f_n(t) = A_n \sin\omega_n t + B_n \cos\omega_n t \quad --(9')$$

$(n=1,2,3,--)$

Also, recall that $u(x,t) = U(x)f(t)$ & hence, $u_n(x,t) = U_n(x)f_n(t)$, $n=1,2,3---$

or, $u_n(x,t) = C_n \sin\left(\frac{\omega_n x}{c}\right)\left[A_n \sin\omega_n t + B_n \cos\omega_n t\right]$ -- ⑳, in which C_n, A_n, B_n are still arbitrary.

Writing (C_nA_n) as A_n' & (C_nB_n) as B_n', we finally →

get $$\boxed{u_n(x,t) = (A_n' \sin\omega_n t + B_n' \cos\omega_n t) \sin\left(\frac{\omega_n x}{c}\right)} \text{ --- } ⑴$$

$n = 1, 2, 3, \ldots$;

→ Next, understand the following clearly: for each n, ㉑ is a solution of the DEOM $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, which is a linear DEOM. Hence, by the principle of superposition, the general solution of the DEOM is:

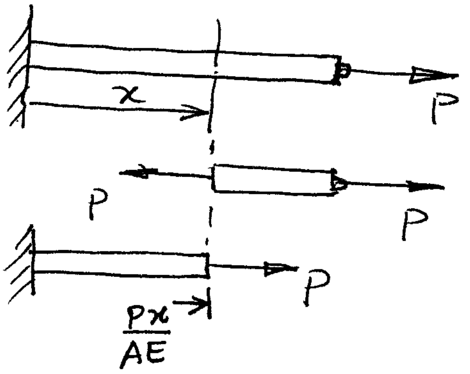
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (A_n' \sin\omega_n t + B_n' \cos\omega_n t) \sin\left(\frac{\omega_n x}{c}\right) \text{ --- } ㉒$$

& of course, ㉒ is a pretty complex relation, where, for a given set of initial conditions, corresponding to a particular motion, A_n' & B_n' are to be evaluated & these constants are infinite in number!

→ There is a systematic procedure for obtaining A_n' & B_n' using the orthogonality principle for those continuous systems which are self-adjoint. See 'Analytical Methods in Vibrations' by L. Meirovitch if you are interested in knowing more about it. →

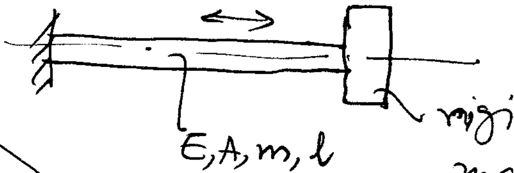
→ Example of an initial condition for our bar axial vibration problem :-

→ The bar could be subject to a load P at $x=l$ as shown. This causes a static deflection of $\frac{Px}{AE}$ at the section at location x. The load is removed at $t=0$. Then the initial conditions $u(x,0)=\frac{Px}{AE}$, $\left.\frac{\partial u}{\partial t}\right|_{t=0}=0$ are induced and the bar would execute axial free-vibrations.



→ In practical structures, the bar axial stiffness would be pretty high resulting in large values of natural frequencies & small amplitude oscillations. Such oscillations are more difficult to visualize compared with bending or beam (transverse) vibrations of the same bar.

→ Home Work Problem:-



rigid disc ~~of~~ mass M fixed at free-end (the end is no longer 'free'!)

(i) Write the BCs for this problem.
[Hint:- At $x=l$, $M\frac{\partial^2 u}{\partial t^2}=-AE\frac{\partial u}{\partial x}$?]

(ii) show that the frequency equation can be written as: $\frac{\rho A l}{M}=\frac{\omega l}{c}\tan\frac{\omega l}{c}$ or, $\boxed{\mu=\alpha\tan\alpha}$ where μ is the ratio of shaft mass to disc mass & $\alpha=\frac{\omega l}{c}$.

(PTO)

Note that we often represent ω^2 by λ and call the following an <u>eigenvalue</u> problem:

See pg. 2, eqn. ⑧ & pg. 4, relations ⑬ & ⑭

$$\left\{ \begin{array}{l} U'' + \dfrac{\lambda}{c^2} U = 0 \quad (\lambda = \omega^2; \text{ an eigenvalue}) \\ \text{with } U(0) = 0 \;\&\; EA \left.\dfrac{dU}{dx}\right|_{x=l} = 0 \\ \text{where } \lambda \text{ is the eigenvalue sought} \end{array} \right\}$$

This type of an eigenvalue problem is also called a Sturm-Liouville problem

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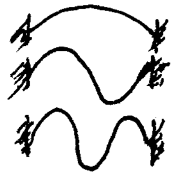
More HW problems:

① For the string vibration problem, obtain the frequency equation for the fixed-fixed case. [Hint – DEOM is similar to that of the bar in axial vibration (Pg. 6). The BC at $x = l$, however is different]

[Ans:- $\sin\frac{\omega l}{c} = 0$]

→ Show that $\omega_n = \frac{n\pi}{l}\sqrt{\frac{T}{m}}$

→ Plot the first 3 mode shapes [Hint:



② For the uniform E-B beam, the DEOM (pg. 10) is:

$$m\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = 0 \;..\; \text{or, } m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = 0$$

For the simply supported case, the 4 BCs are:

(Note that the DEOM is of order 4 & hence 2 BCs at each end are required to make a total of 4 BCs needed to solve the problem)

$$v(0,t) = 0 \;;\; v(l,t) = 0 \;;\; \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=l} = 0$$

(Since bending moment $= EI\frac{\partial^2 v}{\partial x^2}$ is zero at both ends of the beam when it is simply supported)

→ Show that the frequency eqn. can be written as: $\sin \lambda l = 0$ where $\lambda^2 = \omega/c$ & $\omega_n = n^2\pi^2\sqrt{\frac{EI}{ml^4}}$ →

$n = 1, 2, 3, ---$.

→ Vibration of continuous systems is huge in scope. We have barely touched the tip of the iceberg.

→ There is a class of continuous systems called the self-adjoint systems. For these, there is an expansion theorem like the one for discrete systems.

Analytical Methods in Vibrations — L. Meirovitch

→ There is a modal analysis for continuous systems too to mainly study forced vibration. Damping can be included.

→ Many approximate methods exist to deal with vibration of continuous systems. Some of these are: The Rayleigh Ritz Method, The Galerkin Method, The Collocation Method, The method of assumed modes.

→ After the basics of beam, plate & shell vibrations are mastered, one should go for studying the Finite Element Method to be able to apply this knowledge for obtaining useful information about vibrations of real life structures & machines. Thus, our scope of studies didn't permit us to do much, unfortunately.

[END OF PART B]