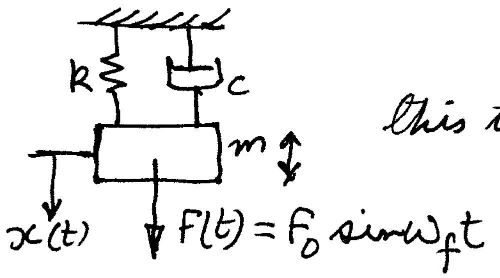
Forced vibration of 1-DOF systems:~

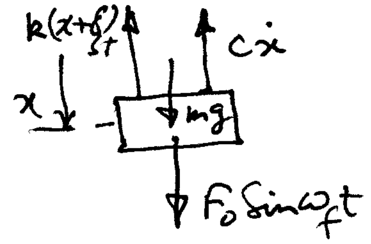


Every textbook starts this topic with a sinusoidal forcing function, why? This is because this forcing function can be generated quite easily. Also, based on the result obtained by this analysis, other complex forcing functions like periodic forcing of any kind may be easily handled.

→ We now obtain the DEOM.

(I) <u>Newton's method</u>:-

FBD:-



$$\therefore \; m\ddot{x} = F_0 \sin\omega_f t + mg - c\dot{x} - k(x+\delta_{st})$$

But $mg = k\delta_{st}$

Hence, $\boxed{m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t}$ ---- (1)

(Remember)

which is the required DEOM.

(II) Use of Lagrange's Equation:~

The Lagrange equation in this case shall be the following:~

$$\boxed{\text{Remember}} \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial V}{\partial x} = Q_x(t) \quad \text{---} (2)$$

→ The terms on the LHS are familiar. $Q_x(t)$ on the RHS is called the <u>generalized</u> <u>force</u> corresponding to the generalized coordinate x. →

Here is how to find $Q_x(t)$. You learn it mechanically for the time being. We shall justify the steps later.

→ To compute $Q_x(t)$, give the block a virtual displacement δx. Compute the virtual work done by the forcing function. Let δW_x be this virtual work.

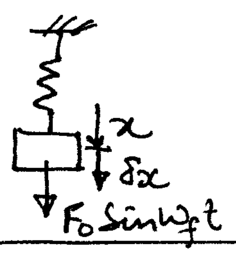
Then, $Q_x = \frac{\delta W_x}{\delta x}$ (Remember)

→ Somewhat funny, isn't it?

I hope you know what a virtual displacement means. If forgotten, here is a recapitulation. A virtual displacement is an infinitesimal displacement compatible with the constraints, when we assume that moving constraints, if any, are 'frozen', i.e., made stationary.

If all this sounds weird, don't worry. All sorts of clarifications will be provided soon. For the present problem, just remember that δx is nothing but 'dx', a real infinitesimal displacement of the block. Thus, $\delta x = dx$ here, but this is not so always, remember.

(No figure to scale)
Quantities exaggerated very much for clarity



$\therefore\ \delta W_x = F_0 \sin \omega_f t \times \delta x$

$\Rightarrow Q_x = \frac{\delta W_x}{\delta x} = F_0 \sin \omega_f t$!

So, it may seem a little silly to get the generalized force as the applied force only by spending so much of words about those virtual things! But wait — when you come to complex multi-DOF systems, you will realize, surely, the power of this method.

→ Let us complete the derivation of the DEOM.

We have $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$, $D = \frac{1}{2}c\dot{x}^2$

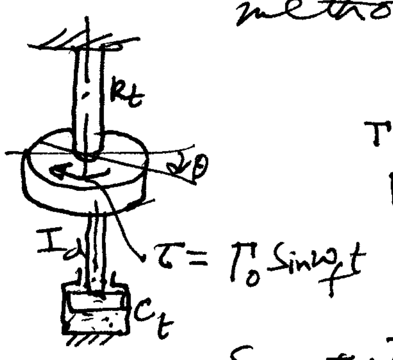
So, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$, $\frac{\partial T}{\partial x} = 0$, $\frac{\partial V}{\partial x} = kx$, $\frac{\partial D}{\partial \dot{x}} = c\dot{x}$

Also, $Q_x = F_0 \sin\omega_f t$. Putting these in the Lagrange eqn. (2) (page 1), we get

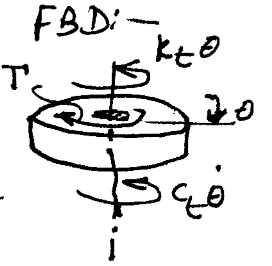
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t,$$ the reqd. DEOM.

→ The rotational counterpart of the above model is considered now.

(I) Moment-balance Method (Moment of momentum methods)



FBD:—



So, $I_d\ddot{\theta} = -k_t\theta - c_t\dot{\theta} + T$

or, $I_d\ddot{\theta} + c_t\dot{\theta} + k_t\theta = T_0 \sin\omega_f t$, the DEOM reqd.

(II) Use of Lagrange's equation

Here, the Lagrange eqn. is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = Q_\theta$$

$$\delta W_\theta = T_0 \sin\omega_f t \times \delta\theta$$

$$\therefore Q_\theta = \frac{\delta W_\theta}{\delta\theta} = T_0 \sin\omega_f t$$

$T = \frac{1}{2}I_d\dot{\theta}^2$, $V = \frac{1}{2}k_t\theta^2$, $D = \frac{1}{2}c_t\dot{\theta}^2$

Hence, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_d\ddot{\theta}$, $\frac{\partial T}{\partial \theta} = 0$, $\frac{\partial V}{\partial \theta} = k_t\theta$, $\frac{\partial D}{\partial \dot{\theta}} = c_t\dot{\theta}$ & so,

$I_d\ddot{\theta} + c_t\dot{\theta} + k_t\theta = T_0 \sin\omega_f t$ is the reqd. DEOM

→

8 We take up the linear ~~s~~ translational system & consider its DEOM, viz.

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t \quad \text{— (1)}$$

The complete solution of (1) (as you all know) is $x(t) = x_c(t) + x_p(t)$ ← the particular integral

↓ Complementary function

$x_c(t)$ is the solution of $m\ddot{x} + c\dot{x} + kx = 0$ & it represents the free-vibration part. $x_p(t)$ gives the forced vibration response. $x_c(t)$, if $c \neq 0$, dies down after sometime and is often called the 'transient part' of the solution. $x_p(t)$, which lasts as long as the forcing function acts, is called the Steady-State (SS) or forced response.

→ $x_c(t)$ has already been obtained. We now turn to finding $x_p(t)$.

Note:- In books, $x_p(t)$ is ~~o~~ often denoted as $x(t)$ only, presuming the free-vibration isn't of much importance since it dies down after sometime (except when $c=0$, i.e. the system is undamped). So, we will do the same thing often while solving problems.

I'm sure you know not one, but several methods to solve (1), especially to obtain the particular integral $x_p(t)$. for instance, (i) the operator method, $D \equiv \frac{d}{dt}$ etc. (ii) Use of the Laplace transform method etc. However, another approach, based on simple observations and past experience, may prove to be quite interesting.

→ Let us first take $m\ddot{x} + kx = F_0 \sin\omega_f t$, --(i) the undamped case. A little observation reveals that, if we assume $x = X \sin\omega_f t$, (ii) we may be able to get X, since, $\ddot{x} = -X\omega_f^2 \sin\omega_f t$ ..(iii) & substitution of this x & $\ddot{x}$ in (i) will lead to 'cancellation' (!) of $\sin\omega_f t$ & enable evaluation of X. Then, $x = X\sin\omega_f t$ will be THE solution (particular integral) reqd., by the uniqueness theorem of solution of differential equations! Let us do this. Substituting (ii) & (iii) in (i), we get:

$$[-m\omega_f^2 + k]X \sin\omega_f t = F_0 \sin\omega_f t$$

$$\Rightarrow (k - m\omega^2)X = F_0 \Rightarrow X = \frac{F_0}{(k - m\omega^2)}$$

Hence, if $k - m\omega_f^2 \neq 0$, i.e., $\omega_f \neq \sqrt{\frac{k}{m}} = \omega_n$, then,

$$x_p(t) = X \sin\omega_f t = \frac{F_0}{(k - m\omega_f^2)} \sin\omega_f t$$

or, $$x_p(t) = \frac{F_0/k}{(1 - \omega_f^2/\omega_n^2)} \sin\omega_f t$$

$$= \frac{F_0/k}{(1 - r^2)} \sin\omega_f t \quad \text{--(iv)}$$

where $r = \frac{\omega_f}{\omega_n}$ = ratio of forcing frequency ω_f

to the undamped natural frequency ω_n is called the "frequency ratio".

So, the forced or ss (steady state) response of the undamped system is, indeed, $x_p = \frac{F_0/k}{(1-r^2)} \sin\omega_f t \quad (\omega_f \neq \omega_n)$.

(What happens when $\omega_f = \omega_n$ will be taken up soon. We get resonance then.)

Hence, the complete response is:

(V) --- $x(t) = x_c(t) + x_p(t) = X_0 \sin(\omega_n t + \phi) + \frac{F_0/k}{(1-r^2)} \sin\omega_f t$

Note that in this case, $x_c(t)$ remains at all times, unlike the damped case. But even in this ~~case~~ undamped case, the forced vibration part is still called the <u>steady-state</u> response.

→ We turn to the damped case & consider $m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t$ --- (VI)

If you observe keenly, you'd see here we can't assume $x_p = X \sin\omega_f t$ due to the presence of $c\dot{x}$ term, which now would be $c\omega_f X \cos\omega_f t$ & now substitution of these in (VI) would give something like $a \sin\omega_f t + b \cos\omega_f t = F_0 \sin\omega_f t$,

& so, either $b = c\omega_f X = 0$ or $\cos\omega_f t = 0$ at all times, none of which is true!

{ I hope you know that $\sin\omega_f t$ & $\cos\omega_f t$ are linearly independent & hence, if anywhere you find an expression like $a \sin\theta + b\cos\theta = 0$, with $\theta = \theta(t)$, to be true at all times, you must have $a=0$ & $b=0$. We shall be back to the linear independence of functions later.] Hence, the conclusion from all this is that for the damped system, we can't start with a solution like $x = X\sin\omega_f t$, instead, we must take $x = X_1 \sin\omega_f t + X_2 \cos\omega_f t$

or, $x = X\sin(\omega_f t - \psi)$ &

this would work. So, now there is a phase lag (the minus sign) ψ, see? This is compatible with the fact that in damped systems, the response (the forced response, to be precise) lags the forcing function. (An analogous situation exists in electrical circuits)

<u>Home Work:-</u> Substitute $x = X\sin(\omega_f t - \psi)$ and its derivatives in DEOM (VI) on page 6, obtain X & ψ (Equating coefficients of $\sin\omega_f t$ & $\cos\omega_f t$ on both sides etc.). What you'll get finally is one of the very important expressions in mechanical vibration analysis & you must remember it!

→

This is:

$$x_p(t) = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t - \psi) \quad \boxed{\text{Remember}} \quad \text{--- (VII)}$$

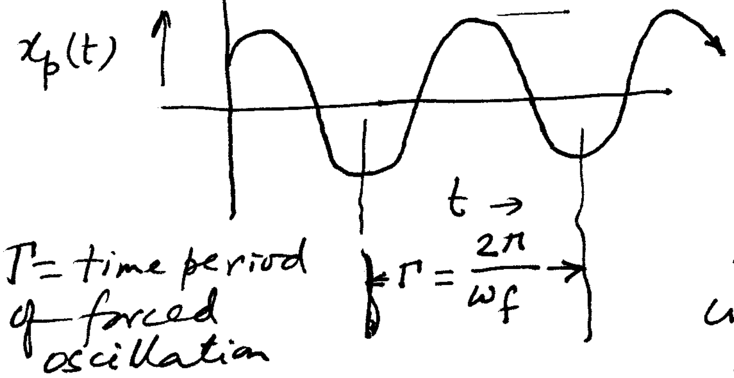
$$= \frac{F_0}{k} \times (MF)\sin(\omega_f t - \psi)$$

Where MF = the Magnification Factor (Also called 'Amplitude ratio' $\frac{x}{F_0/k}$)

or, $$\boxed{MF = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}} \quad \boxed{\text{Remember}} \quad \text{--- (VIII)}$$

$$\& \ \psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \quad \text{--- (ix)}$$

Note that F_0/k is the deflection created by a static force F_0, (which is equal to) the amplitude of the forcing function. This is sometimes called the 'equivalent static deflection'. Hence, the MF determines whether the amplitude of forced vibration is greater than, equal to or less than the equivalent static deflection. Relation (VII), (i.e., the forced response) when plotted, would look like:

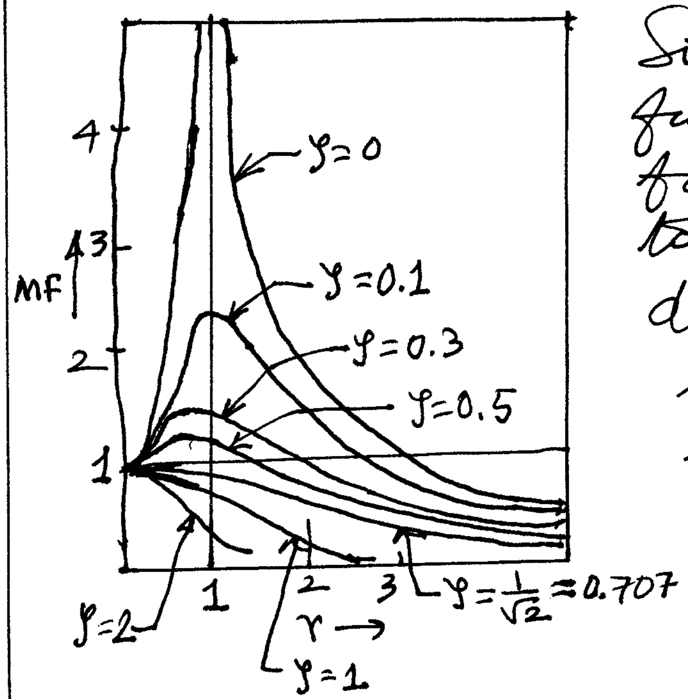


Γ = time period of forced oscillation

Note:- Γ was earlier a torque & now it is a time period! You shouldn't feel confused. The context should tell you what is what.

The expressions for MF & ψ are quite complicated and can be best understood in the graphical form which is taken up next. In passing, note that our approach here is often called a heuristic approach. (Heuristic teaching or education encourages you to learn by discovering things for yourself → Oxford Advanced Learner's Dictionary]

Study the following plots carefully in

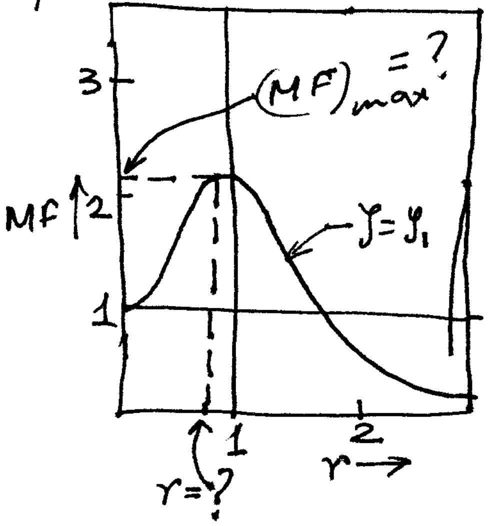


Since the MF is a function of ζ & r, it is found to be convenient to get MF ~~\$~~ r plots for different values of ζ such as those shown in this figure.

You should try to remember the form of the plots because this may come handy while solving some tricky problems. However, the plots here are hand drawn & quite arbitrary except for presenting the essential characteristics of such plots. ~~The~~ Take a close look at the accurate plots given in the textbook. You yourself could generate these plots using MATLAB, if you are familiar with it.

Note that: (1) For $\zeta < \frac{1}{\sqrt{2}}$, a max^m occurs near $r=1$. (2) $MF \to 0$ as $r \to \infty$ & hence to get a small MF, a high value of r is required for $\zeta < 1/\sqrt{2}$. These characteristics are often used for vibration isolation & control.

We can learn more about these plots as follows:– Let us take one plot as follows:–



We want to find the resonance amplitude $(MF)_{max}$ of forced vibration which is $\frac{F_0}{K} \times (MF)_{max}$. We also want to know at what value of r this occurs. (From the plots on page (9), doesn't it appear that $(MF)_{max}$ occurs at a value of r different from 1 except for the undamped case ($\zeta = 0$)? Note that $r = 1$ corresponds to $\omega_f = \omega_n$, i.e. when the forcing frequency is equal to the undamped natural frequency.) To find $(MF)_{max}$, we set

$$\left.\frac{\partial(MF)}{\partial r}\right|_{\zeta=\zeta_1} = 0 \quad \text{or,} \quad \frac{\partial}{\partial r}\left[(1-r^2)^2 + (2\zeta r)^2\right]^{-1/2} = 0$$

$$\text{or,} \quad -\frac{1}{2}\cdot\left[(1-r^2)^2 + (2\zeta r)^2\right]^{-3/2} \times \frac{\partial}{\partial r}\left[(1-r^2)^2 + 4\zeta^2 r^2\right] = 0$$

HW | You must simplify above expression & see for yourself that $\left.\frac{\partial(MF)}{\partial r}\right|_{\zeta=\zeta_1} = 0$ gives 3 values of r, these are:-

$r = 0$, $r = \sqrt{1-2\zeta^2}$ & $r = \infty$ (! $r = \infty$ may be an invalid, a crazy relation to a mathematician & he may insist we should write 'as $r \to \infty$,' $\left.\frac{\partial(MF)}{\partial r}\right|_{\zeta=\zeta_1} \to 0$.

But we engineers often do it like this & get away with it. →

Buy studying the sign of $\left.\frac{\partial^2(MF)}{\partial r^2}\right|_{\zeta=\zeta_1}$ at these values of r, you can establish that a minimum occurs at r=0. Same thing happens as $r\to\infty$. However, at $r=\sqrt{1-2\zeta^2}$, a maximum occurs.

So, $$(MF)_{maxm} = \left.\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right|_{r=\sqrt{1-2\zeta^2}}$$

$r^2 = 1-2\zeta^2$
$1-r^2 = 2\zeta^2$

$$= \frac{1}{\sqrt{4\zeta^4+4\zeta^2(1-2\zeta^2)}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}.$$

Thus, rember these:

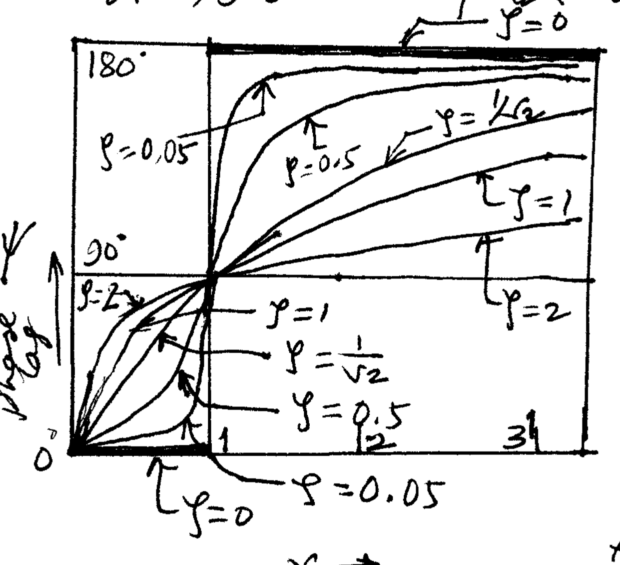
$(MF)_{maxm} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ & this occurs at $r=\sqrt{1-2\zeta^2}$.

We say that amplitude resonance occurs at $r=\sqrt{1-2\zeta^2}$

Since $r>0$, $\sqrt{1-2\zeta^2}>0$, i.e., $\zeta<\frac{1}{\sqrt{2}}\cong 0.707$
So, ζ must be less than $\frac{1}{\sqrt{2}}$ for a resonance peak to occur. For $\zeta \geq \frac{1}{\sqrt{2}}$, no such peaks as the plots on page 9 reveals.

→ Note the following:– In books on AUTOMATIC CONTROL, the scales along the MF axis (y-axis) are usually in decibels (dB) & the r axis (x-axis) is logarithmic. The plots are called Bode plots or a Bode diagram and are extensively used to design automatic control systems using conventional control techniques. See, for instance 'Modern Control Engineering' by K. Ogata.

→ We come to $\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ now and see how it varies with variations in r & ζ. r varies, for a given system, as ω_f is varied. ζ can be varied by changing the system parameters m, K & c [$\zeta = \frac{c}{2\sqrt{mK}}$, remember?], all or some of them. The plots are as follows:—



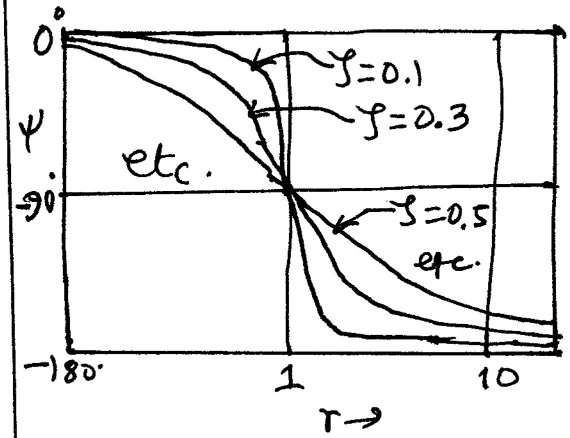
(See accurate plots from books)

Note that for $r=1$, ψ is 90° for all ζ. We say that the <u>phase</u> <u>resonance</u> always ~~occ~~ occurs at $\psi = 90°$ (or, $\psi = \frac{\pi}{2}$)

Hence, amplitude resonance occurs at $r = \sqrt{1-2\zeta^2}$ (for $\zeta < \frac{1}{\sqrt{2}}$) but phase resonance occurs at $r=1$ for all ζ. So, now you know that there are two kinds of resonance for our system, the amplitude resonance & the phase resonance.

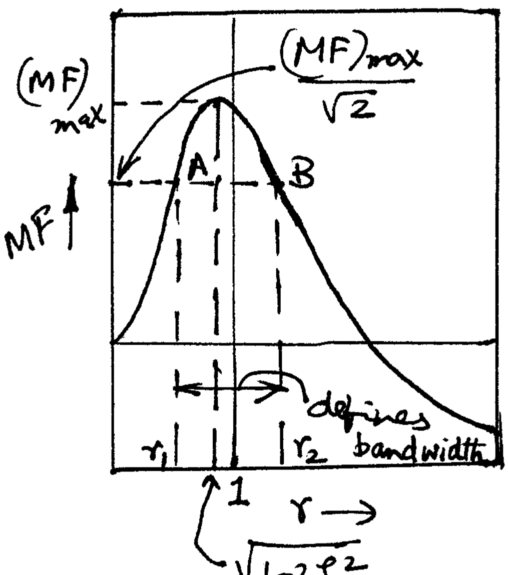
→ Once again, in books on AUTOMATIC CONTROL, the r-axis is logarithmic, the response is written as $x = X\sin(\omega_f t + \psi)$ +ive sign here

and the graphs are shown as follows:—



← This is another Bode plot, also used for control systems design.

Bandwidth and Damping Factor ζ :-



Let us consider a particular MF vs r plot as shown here (ζ < $\frac{1}{\sqrt{2}}$, of course).

Now, $(MF)_{max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ & it occurs at $r = \sqrt{1-2\zeta^2}$

If $\zeta \le 0.1$ (some authors put it at $\zeta \le 0.05$), we ignore ζ^2 & $2\zeta^2$ compared with unity & say that $(MF)_{max} = \dfrac{1}{2\zeta}$ and it occurs at $r = 1$! $(MF)_{max}$ has a special name. It is called the Q factor or Quality factor of the system. This is done in analogy with electrical circuits (an RLC circuit, say) having an analogous behaviour. Now take a look at the points A & B. These two points correspond to the value of $MF = \dfrac{(MF)_{max}}{\sqrt{2}}$. This value of MF occurs at frequencies corresponding to $r = r_1$ & r_2 as shown in the figure.

If $r_1 = \frac{\omega_{f_1}}{\omega_n}$ & $r_2 = \frac{\omega_{f_2}}{\omega_n}$, then $\Delta\omega = \omega_{f_2} - \omega_{f_1}$ is called the '<u>bandwidth</u>' of the system.
It can be measured experimentally with good accuracy & is related to ζ, the damping factor, as will be shown now.

To get $\Delta\omega$, you <u>can</u> ~~use~~ the accurate relation, viz,

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{1}{\sqrt{2}} \times \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$, solve for r $(r>0)$ to get r_1 & r_2. Do this, if you feel like. Here we shall do what the textbooks do: take the approximate value of $(MF)_{max}$ to be $\frac{1}{2\zeta}$ & so,

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{1}{\sqrt{2}} \times \frac{1}{2\zeta}$$

Squaring both sides & transposing, we get

$$r^4 - r^2(2-4\zeta^2) + (1-8\zeta^2) = 0, \text{ giving}$$

$$r_1^2 = 1-2\zeta^2-2\zeta\sqrt{1+\zeta^2} \quad \& \quad r_2^2 = 1-2\zeta^2+2\zeta\sqrt{1+\zeta^2}$$

Ignoring ζ^2 again, $r_1^2 \simeq 1-2\zeta$, $r_2^2 \simeq 1+2\zeta$.

So, $\frac{\omega_{f_1}^2}{\omega_n^2} \simeq 1-2\zeta$, $\frac{\omega_{f_2}^2}{\omega_n^2} \simeq 1+2\zeta$

$$\Rightarrow \frac{\omega_{f_2}^2 - \omega_{f_1}^2}{\omega_n^2} \simeq 4\zeta \Rightarrow \frac{(\omega_{f_2}+\omega_{f_1})\Delta\omega}{\omega_n^2} = 4\zeta \quad \text{—(i)}$$

For a lightly damped system, r_1 & r_2 are pretty close to unity & so, $\omega_{f_2}+\omega_{f_1} \simeq 2\omega_n$ & thus, (i) becomes: $\frac{2\omega_n \Delta\omega}{\omega_n^2} = 4\zeta$ or $\boxed{\Delta\omega \simeq 2\zeta\omega_n}$

So, if $\Delta\omega$ is measured experimentally, ζ can be obtained. Now, the Quality factor $Q = \frac{1}{2\zeta}$. Thus, $Q = \frac{\omega_n}{\Delta\omega}$

Q denotes the sharpness of resonance. The higher its value, the better our system acts as a mechanical filter. This means $\Delta\omega$ would be small and the range of frequencies over which the system forced response to a harmonic excitation force is significant would be small indeed.

→ The main use of the bandwidth (also called the half-power bandwidth is in the experimental modal analysis or modal testing. (See chapter 10, Mechanical vibrations by S.S. Rao, 6th edition)

Question:- Why the bandwidth is also called the half-power bandwidth?

→ We shall show later that the damper dissipates, per cycle of motion, an amount of energy proportional to X^2 where X is the amplitude of forced vibration under a harmonic force. So, when the amplitude is $\frac{X}{\sqrt{2}}$, the energy dissipated per cycle would be proportional to $\frac{X^2}{2}$, which is half the previous value. The same is true of the power absorbed by the damper and hence the name.

Example:- (from Mechanical vibrations by S.G. Kelly)
An 82 kg m/c tool is mounted on an elastic foundation. An experiment is performed to determine the stiffness and damping properties of the foundation. The tool is excited with a harmonic force of magnitude 8000 N at a variety of frequencies. The maxm. steady-state amplitude is obtained as 4.1 mm at a frequency of 40 Hz. Using this information, estimate the stiffness and the damping factor.

Solution:- $F_0 = 8000N$, Maxm. SS amplitude

$$= \frac{F_0}{K} \times \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 0.0041 \quad \text{—— (1)}$$

Also, $r_{res} = \sqrt{1-2\zeta^2}$, or, $\frac{(\omega_f)_{res}}{\omega_n} = \sqrt{1-2\zeta^2}$ —— (2)

where $(\omega_f)_{res} = 2\pi \times 40$ rad/s.

Finally, $\omega_n = \sqrt{\frac{k}{82}}$ (k in N/m) —— (3)

Using (1), (2) & (3), find ζ.

Actually, $\zeta^4 - \zeta^2 + 0.03107 = 0$

$\Rightarrow \zeta = 0.179$ & 0.984.

We must take the value of $\zeta < \frac{1}{\sqrt{2}}$.

Hence, $\zeta = 0.179$ Ans.

From (2), Compute $\omega_n = \frac{(\omega_f)_{res}}{\sqrt{1-2\zeta^2}} = 255.5$ rad/s.

So, $k = 82\omega_n^2 = 5.35 \times 10^6$ N/m $= 5.35$ MN/m Ans.

(Check calculations).

END OF PART 1, VA-3