Fuzzy Logic and Systems

Intelligent Systems Perspective

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Contents of the lectures

- Why are we talking of Fuzzy Systems?
- Fuzzy Logic Definitions
- Some basics of fuzzy logic
- Selected topics in fuzzy logic
- Application of Fuzzy Logic in Intelligent Systems

Intelligent Systems Evolution

- Traditional Design of products with fixed performance
- Design of products with Programmed (Programmable) Performance
- Intelligent Systems Design
  - Programmed + Adaptable performance
  - Interactive performance

New breed of products and systems

- House hold products
  - Intelligent Air Conditioners
  - Washing Machines
  - Microwave ovens, cooking etc.
  - Digital Cameras
  - Roboticc Toys
- Commercial systems
  - Intelligent Vehicles
  - Transport systems

Specifications for new generation products

- Qualitative description using Linguistic variables
- Less precision required instead of more accuracy
- Direct human interaction
- Minimal of training of skill sets required.

Computational (Artificial) Intelligence

- Design requirements have increasingly become more qualitative and linguistic
- Artificial Intelligence (traditional) approaches are being adopted.
- New form of the design is increasingly dependant on approximate reasoning
- Approximate Reasoning and Intelligence leads to Computational Intelligence.
- Fuzzy Logic and Systems is a major component

Components in an AI system

Philosophy	logic, methods of reasoning
	mind as physical system
	foundations of learning, language, rationality
Mathematics	formal representation and proof
	algorithms
	computation, (un)decidability, (in)tractability
	probability
Psychology	adaptation
	phenomena of perception and motor control
	experimental techniques (psychophysics, etc.)
Linguistics	knowledge representation
	grammar
Neuroscience	physical substrate for mental activity
Control theory	homeostatic systems, stability
	simple optimal agent designs

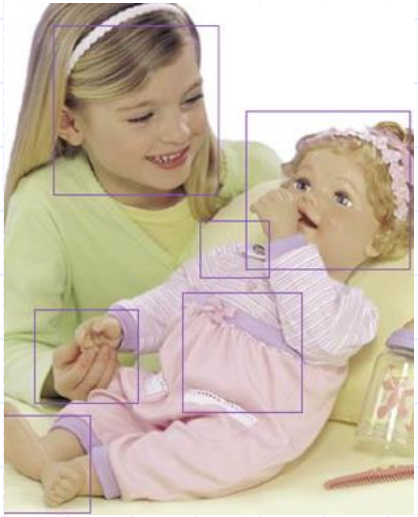
Furby

Furby's the **most** famous, **most** original interactive animatronic plush toy ever! It can be trained to dance, sing and play games. Each Furby has its own name -- and [is] capable of saying 800 different phrase combinations. … Far more than an electronic toy; **Furby is a friend**.

www.tigertoys.com

Fuzzy sound / Language / sentence processing

My Real Baby



(from Hasbro)

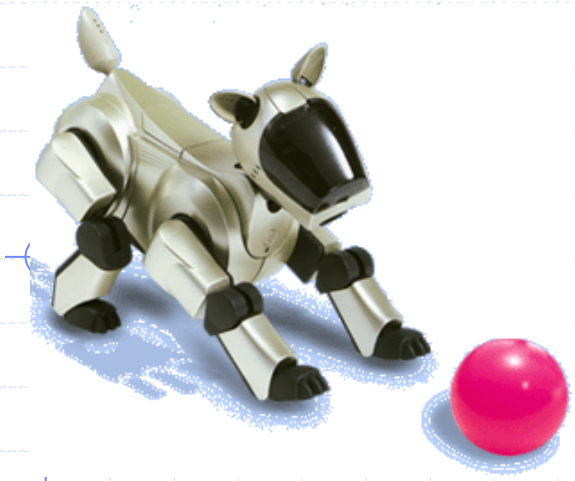
MRB has its own set of *emotions* and drives, and an incredibly expressive, completely animated, realistic face and voice. The child determines how she wants to play with her doll, and the doll responds – **naturally, emotionally, intelligently** – just like a real baby.

The MY REAL BABY … has hundreds of **facial** expressions and literally billions of different combinations of sounds and words ...

MRB knows when she's being hugged, rocked, fed, burped, bounced and more.

MRB uses a over 15 human-like emotions and levels of emotional intensity.

www.irobot.com



www.aibo.com

AIBO is not a toy! AIBO's a true companion with real emotions and instincts. With loving attention it can develop into a mature and fun-loving friend … The more interaction you have with AIBO, the faster it grows up. In short, AIBO is a friend for life.

AIBO has emotions and instincts programmed into its brain. AIBO *acts to fulfill the desires* created by its instincts.

Like any living being, AIBO learns how to get what it wants. Occasionally, AIBO will show signs of anger if it does not receive the kind of attention it requests from you. The way we respond to AIBO's emotional expressions greatly influences AIBO's personality and growth.

AIBO **learns from your praise or scolding**. Praise it for playing with its ball, and it will enjoy playing with the ball more than ever. But if you scold AIBO when it is only looking at the ball, it will soon ignore the ball altogether.

FUZZY LOGIC

LOGIC:

TRUE — TRUTH
FALSE — FALSEHOOD
} 2 STATES

BINARY LOGIC T & F
1 0

BIVALENT LOGIC — BIVALENCE

MULTIVALENCE → FUZZINESS

BIVALENCE: TRUE (1) AND FALSE (0)

CRISP → BLACK AND WHITE

IN BETWEEN — INDETERMINATE

MULTIVALENT

AMBIGUITY:

Ex. 1 DON'T TRUST ME! INDETERMINACY

Ex. 2. DOES THE LIAR FROM CRETE LIE WHEN HE SAYS ALL CRETANS ARE LIARS?

Ex. 3: CARD

SIDE 1: SENTENCE ON THE OTHER SIDE IS TRUE

SIDE 2: SENTENCE ON THE OHER SIDE IS FALSE

PARADOXES

PARADOXES

STATEMENT s

NEGATION OF STATEMENT NOT-s

TRUTH VALUE $t(s)$

$$t(s) = t(\text{not-}s)$$

Statement and its negation are same

BIVALENT LOGIC

$$\cancel{t(s)}\ t(\text{not-}s) = 1 - t(s)$$

$$t(s) = t(\text{not-}s) = 1 - t(s)$$

if s is <u>true</u> $t(s) = 1$ then $\underline{1 = 0}$

if s is <u>False</u> $t(s) = 0$ implies $\underline{0 = 1}$

CONTRADICTION

FUZZY OR MULTI VALUED INTERPRETETION

$$t(s) = 1 - t(s)$$

ACCEPTS $\quad 2\,t(s) = 1$

or $\quad t(s) = \underline{\frac{1}{2}}$ $\qquad \underline{0}$ or $\underline{1}$

HALF - TRUTHS $\longleftarrow$ PARADOXES

MULTIVALENCE $[0, 1]$

FUZZY LOGIC $\rightarrow$ L. ZADEH 1965

FUZZY <u>SETS</u>

SET THEORY $\longrightarrow$ FUZZY SETS

MEMBERSHIP FUNCTIONS

OR INDICATOR FUNCTIONS

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$\longrightarrow$ NON-FUZZY SETS (SUBSETS)

<u>MULTI-VALUED</u> INDICATOR / MEMBERSHIP FNS.

$$m_A : x \longrightarrow [0, 1]$$

$$m_A(x) = \text{Degree } (x \in A)$$

Degree of Belongingness / membership.

PROBABILITY → CHANCE OF OCCURENCE

FUZZINESS → DEGREE OF MEMBERSHIP

PURE WATER 1

PURE POISON 0

MIX VALUE [0, 1]

WATER
OR POISON } events / occurence.

FUZZINESS

INDETERMINACY

CRISP LOGIC (BIVALENT) T or F

MULTIVALENCE IN BETWEEN T & F

TEMPERATURE

COMFORTABLE (18-25) HOT → (30-38)

COLD (10-18) WARM → (28-32)

CHILLY (< 10)

AIR CONDITIONER

COMFORT 18 - 25 °C

VERY LOW LOW MEDIUM HIGH VERY HIGH

LINGUISTIC DESCRIPTIONS

NON - CRISP

VEHICLE SPEED

THROTTLE / BRAKE

LOW SPEED

MEDIUM

HIGH

LINGUISTIC DESCRIPTIONS

PERFORMANCE

EXCELLENT
VERY GOOD
GOOD
FAIR
⋮

} Ranges

NON - CRISP

CAMERA (PHOTOGRAPHY)

EXPOSURE (LIGHT) FILM SPEED

- DURATION - SHUTTER SPEED
- APERTURE → DEPTH OF FIELD

UNDER CORRECT OVER

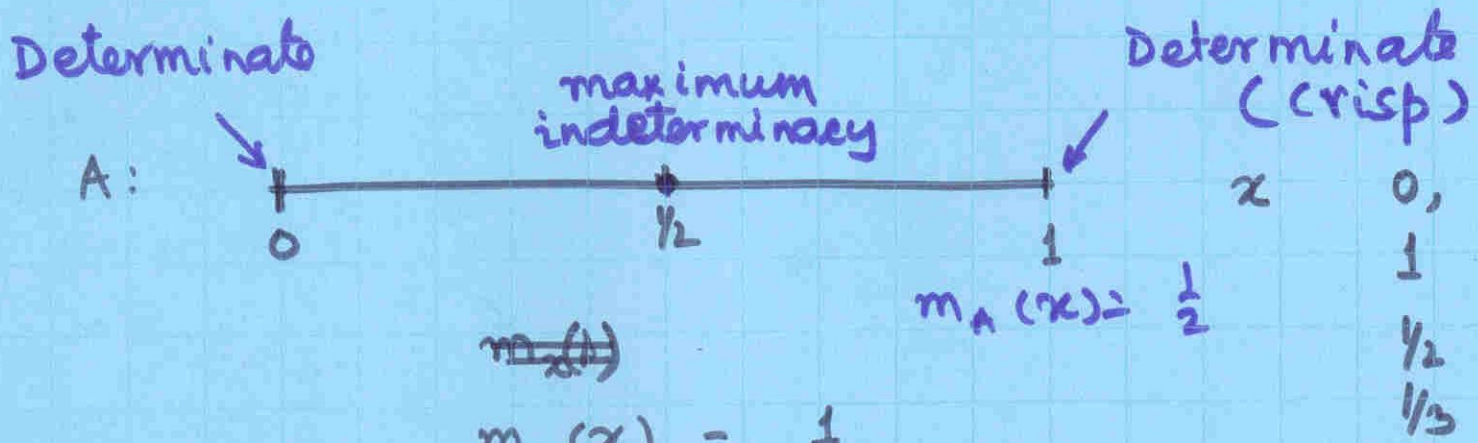
AUTOMATION -

S.S. : 1, 2, 4, 8, 15, 30, 60, 100, 200, - - -

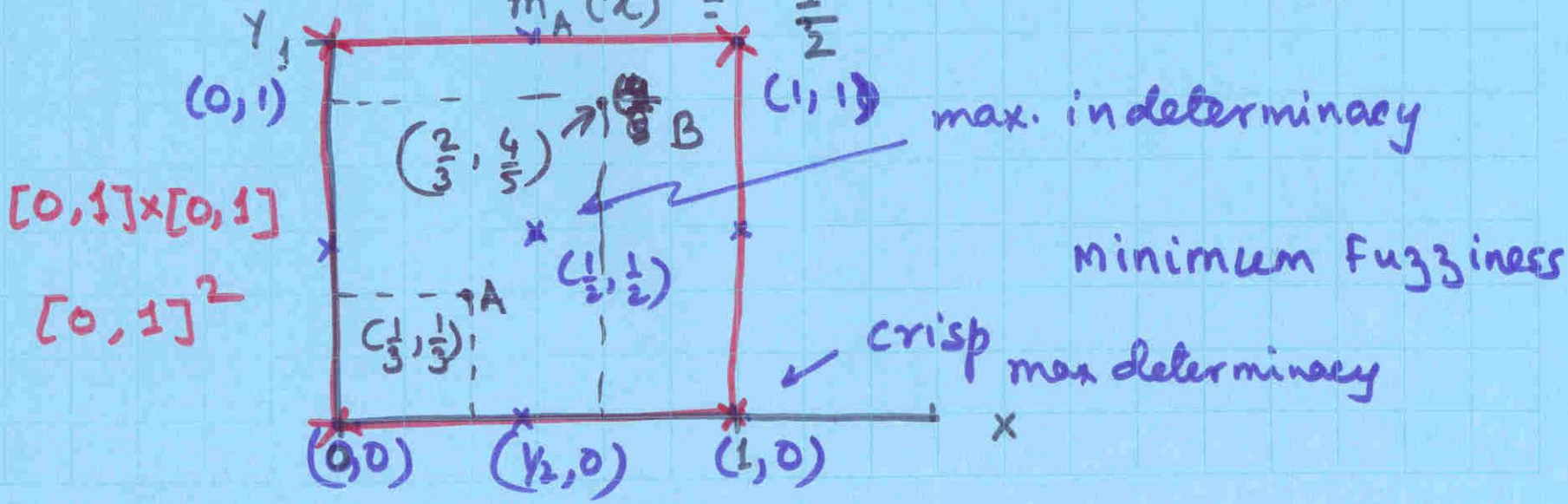
APERTURE: 22, 16, 8, 5.6, 4, - - -

INDETERMINACY , AMBIGUITY

$[0, 1]$

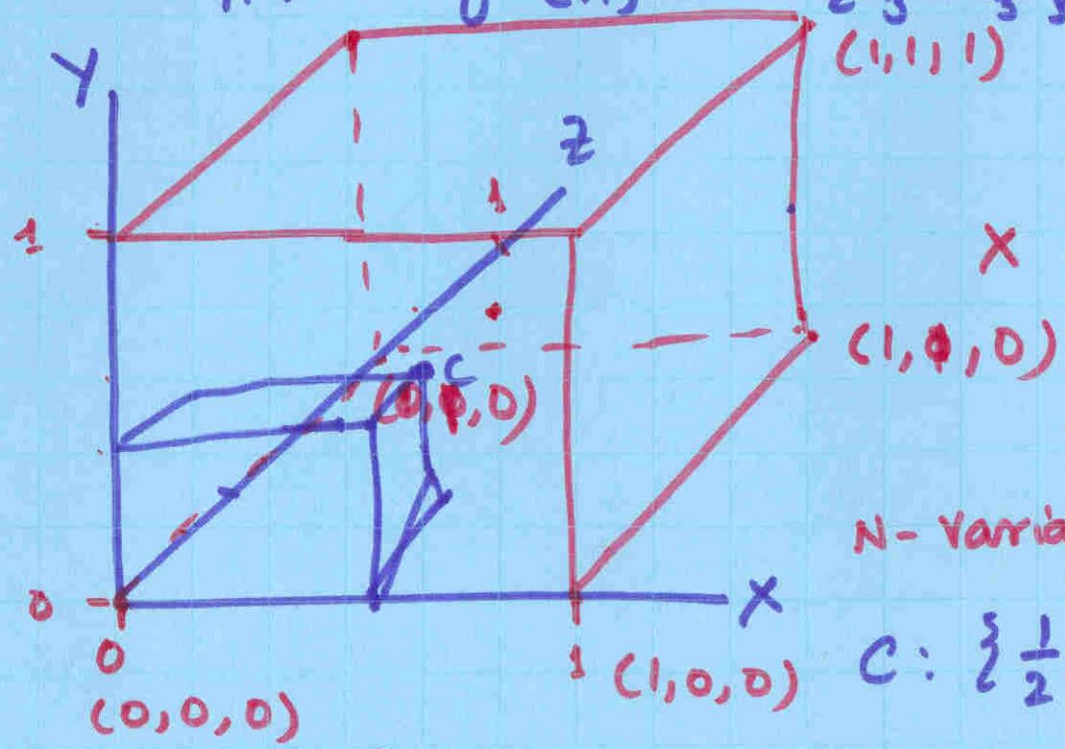


$m_A(x) = \frac{1}{2}$



B: $m_{B_U}(B) = \{\frac{2}{3}, \frac{4}{5}\}$ → membership values.

A: $m_U(A) = \{\frac{1}{3}, \frac{1}{3}\}$



$[0,1]\times[0,1]\times[0,1]$

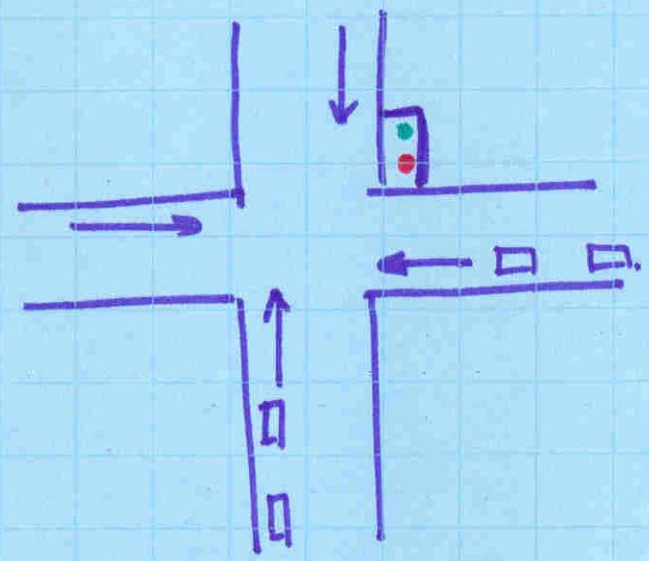
$[0,1]^3$

N-variable: $[0,1]^N$

C: $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{5}\}$

fitness value: $m_x(c) = \frac{1}{2}$, $m_y(c) = \frac{1}{3}$, $m_z(c) = \frac{2}{5}$

APPLICATIONS WITH RULES



Common Sense Rules

Rule:

If Traffic" is HEAVY
in this Direction
then
Keep the light GREEN LONGER

HEAVY → no of vehicles

LONGER → Duration

crisp
or use a function

"Linguistic" descriptors.
(HEAVY, LONGER) → Encoded Fuzzy Rule

FUZZY RULE BASED SYSTEM

(HEAVY, LONGER) → ASSOCIATION.

FUZZY ASSOCIATION

FUZZY ASSOCIATIVE MEMORY (FAM)

— MATRIX

EACH ENTRY IS A "FAM" RULE

MAPS INPUTS & OUTPUTS

FUZZY SYSTEMS

FUZZY LOGIC & SYSTEMS

FUZZY SETS

MEMBERSHIP , INDICATOR FUNCTION

$[0, 1]$

CRISP SETS

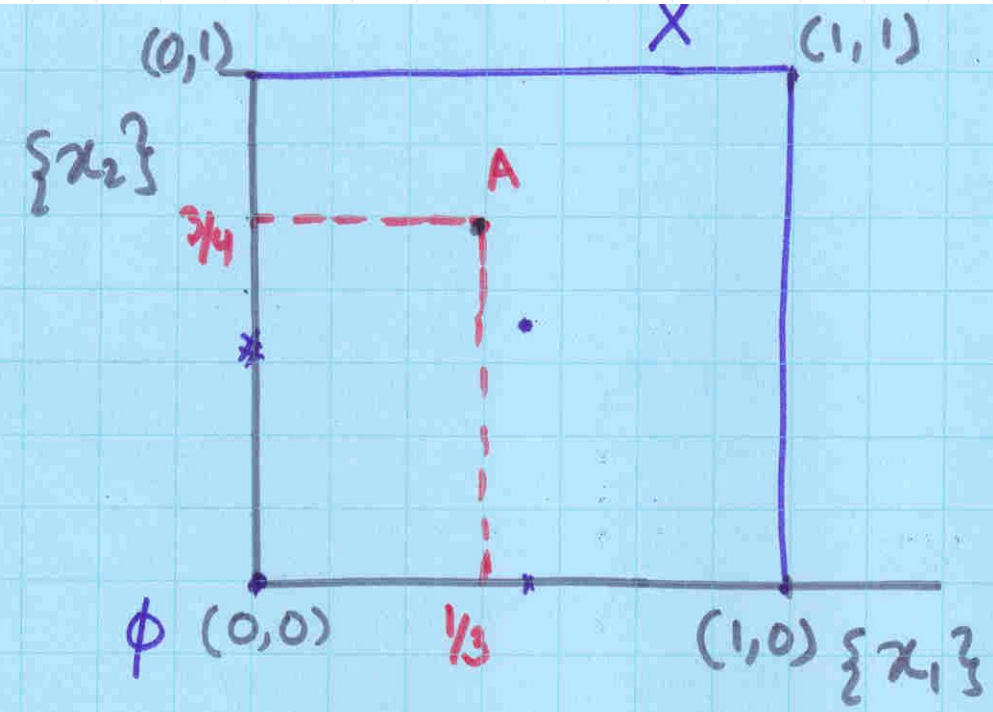
$[0,1] \times [0,1] \longrightarrow$ POWER SET FOR 2 VARIABLES

$I^n = [0,1]^n$

2^n vertices $\rightarrow$ "crisp" at vertices

"Least fuzzy"

determinate



Fuzzy Power set

$F(2^X)$ of X

$X = \{x_1, x_2\}$

A is a Fuzzy Subset

with fit vector $\{\frac{1}{3}, \frac{3}{4}\}$

fit value are membership values

$m_A(x_1) = 1/3$

$m_A(x_2) = 3/4$

encoding Fuzzy encoding

allows for indeterminacy

multivalence

parallel and distributed encoding

linguistic variables (low, medium, high)

"not crisp" mathematical descriptor

"approximations"

Approximate Reasoning → Intelligent Systems

FUZZY SET THEORY

MEMBERSHIP FUNTIONS / INDICATOR FNS.

$$I_A(x) : \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

NON-FUZZY SETS

$$m_A : x \rightarrow [0, 1]$$

degree of belongingness

$0 \rightarrow$ doesnot belong

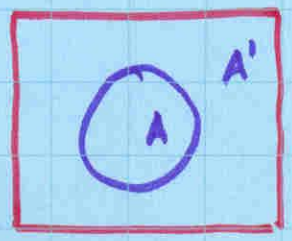
$\downarrow$

$1 \rightarrow$ completely belongs

$m_A(x) =$ Degree $(x \in A)$

"Crisp" Set Theory

Complement A $\quad A' = \{x \mid x \notin A\}$



Intersection A and B $\quad A \cap B$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Union $\quad A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$

Difference $\quad A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Symmetric Difference $\quad A + B = A - B \cup B - A$

Power Set of A $\quad P(A) = \{X \mid X \subseteq A\}$

Cartesian Product of A and B $\quad A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

Properties:

Involution: $(A')' = A$

Commutation $A \cup B = B \cup A$

$A \cap B = B \cap A$

Associativity $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Idempotence $A \cup A = A$

$A \cap A = A$

Adsorption: $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

Adsorption of complement

$$A \cup (A' \cap B) = A \cup B$$

$$A \cap (A' \cup B) = A \cap B$$

Adsorption by U and ϕ

$$A \cup U = U \quad \text{(Universe)}$$

$$A \cap \phi = \phi \quad \text{(Null set)}$$

Identity $\quad A \cup \phi = A$

$$A \cap U = A$$

Law of Contradiction $\quad A \cap A' = \phi$

Law of Excluded Middle $\quad A \cup A' = U$

De Morgan's Law $\quad (A \cap B)' = A' \cup B'$

$$(A \cup B)' = A' \cap B'$$

FUZZY SETS:

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$\mu_A : X \rightarrow [0, 1]$ degree of Membership (belongingness)

iff for all x $\mu_A(x) = \begin{cases} 1 & \text{when } x \in A \\ 0 & \text{when } x \notin A \end{cases}$

Complement of A A'

$$A' : \mu_{A'}(x) = 1 - \mu_A(x)$$

Intersection of A and B

$A \cap B$ $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

Union of A and B

$A \cup B$ $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$.

alternates

$$\mu_{A\cap B}(x) = \begin{cases} \mu_A(x)\cdot\mu_B(x) & \text{or} \\ \max(0, \mu_A(x)+\mu_B(x)-1) & \text{or} \\ \dfrac{\mu_A(x)\,\mu_B(x)}{\frac{3}{4}+\frac{1}{4}\left(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)\right)} & \end{cases}$$

Properties of fuzzy sets

$\cancel{\mu_A(x)}$ $\quad A\cap A' = \phi$

$\cancel{\mu_A(x)\cap}$

$$\mu_{A\cap A'}(x) \neq \phi$$

$$\mu_{A\cup A'}(x) \neq U$$

Subsethood & Probability

Subsethood: measures the degree to which set A belongs to set B.

the degree to which A is a subset of B

$$S(A, B) = \text{Degree}\ (A \subset B)$$

simplest case $A = \{x_i\}$ singleton set

$$S(\{x_i\}, B) = m_B(x_i)$$

Fuzzy Set Theory

$A = (1 \quad 0.8 \quad 0.4 \quad 0.5)$ $\quad M(A) = 2.7$

$B = (0.9 \quad 0.4 \quad 0 \quad 0.7)$ $\quad M(B) = 2.0$

$m_{A \cap B} = \min(m_A, m_B)$.

$m_{A \cup B} = \max(m_A, m_B)$.

$A \cap B = (0.9 \quad 0.4 \quad 0 \quad 0.5)$

$A \cup B = (1 \quad 0.8 \quad 0.4 \quad 0.7)$

$A' = A^c = (0 \quad 0.2 \quad 0.6 \quad 0.5)$

$A \cap A^c = (0 \quad 0.2 \quad 0.4 \quad 0.5)$ $\quad \cancel{M(A) = 1.1}$

$A \cup A^c = (1 \quad 0.8 \quad 0.6 \quad 0.5)$

A is Properly Fuzzy

iff $A \cap A^c \neq \phi$

iff $A \cup A^c \neq X$

How Fuzzy is a Fuzzy set ✓.

How big is a Fuzzy set

$$M(A) = \sum_{i=1}^{n} m_A(x_i)$$

Fuzziness.

Fuzzy Entropy

$$E(A) = \frac{m(A \cap A^c)}{M(A \cup A^c)} = \frac{1.1}{2.9}$$

Probability Theory says

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

probability of B occurring given that A occurs.

$$\boxed{S(A, B) = \frac{M(A \cap B)}{M(A)}}$$

Fuzzy Set Theory.

$M(A)$ = Fuzzy count of fit vector A

$$= a_1 + a_2 + \cdots + a_n$$

Fuzzy Entropy $E(A) = \dfrac{M(A \cap A^c)}{M(A \cup A^c)}$

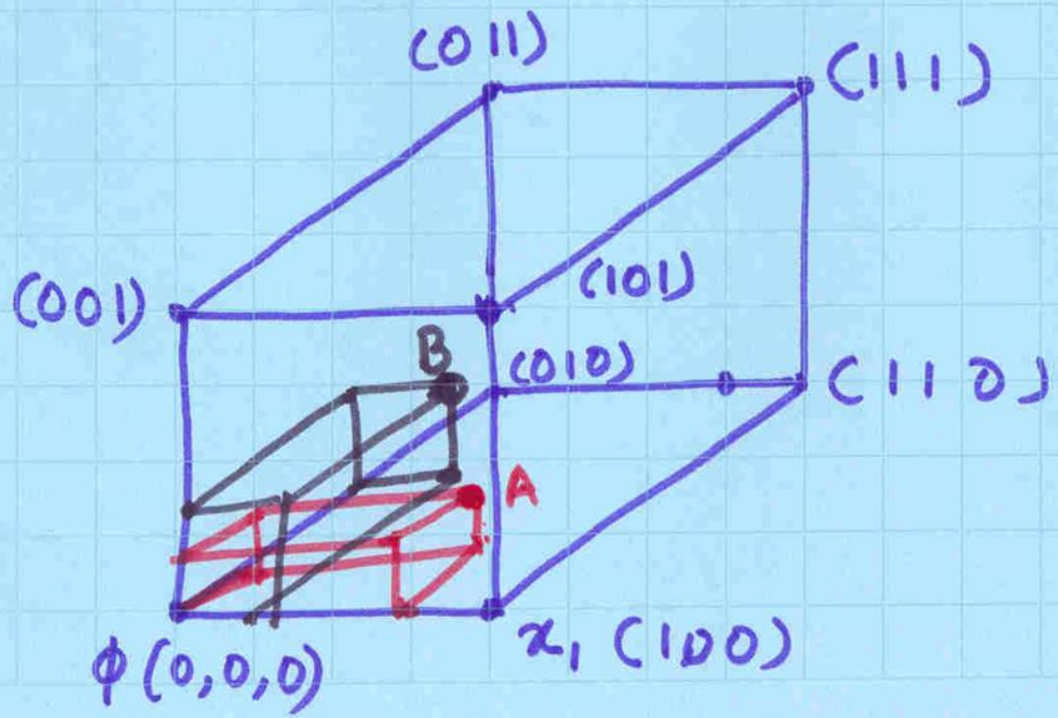
or $E(A) = S(A \cup A^c, A \cap A^c)$

$E(A)$ implies

degree to which

$\underline{A \cap A^c}$ belongs to its own superset $\underline{A \cup A^c}$

$$E(A) = \frac{M(A \cap A^c)}{M(A \cup A^c)} = S(A \cup A^c, A \cap A^c)$$



$X = (1\ 1\ 1) = \{x_1, x_2, x_3\}$

$A = \left(\frac{3}{4}\ \frac{1}{3}\ \frac{1}{6}\right)$

$B = \left(\frac{1}{4}\ \frac{1}{2}\ \frac{1}{3}\right)$

$$M(A) = \frac{3}{4} + \frac{1}{3} + \frac{1}{6} = \frac{9+4+2}{12} = \frac{15}{12} = \frac{5}{4}$$

$$M(A \cap B) = M\left(\frac{1}{4}\ \frac{1}{3}\ \frac{1}{6}\right) = \frac{1}{4} + \frac{1}{3} + \frac{1}{6} = \frac{3+4+2}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S(A, B) = \frac{(3/4)}{(5/4)} = \frac{3}{5} = 60\%.$$

if X defines the "sample space" of all elementary outcomes of an experiment

then X is a "sure event" since $P(X) = 1$

$$P(A) = P(A|X)$$

$\Longrightarrow$ X can partially belong to A

implies $P(A) = S(X, A)$

Probability in terms of relative frequency

$$\frac{n_A}{N}$$

References

- L. Zadeh, “Fuzzy sets as a basis of possibility” Fuzzy Sets Systems, Vol. 1, pp3-28, 1978.
- T. J. Ross, “Fuzzy Logic with Engineering Applications”, McGraw-Hill, 1995.
- K. M. Passino, S. Yurkovich, "Fuzzy Control" Addison Wesley, 1998.

Crisp (Traditional) Variables

- Crisp variables represent precise quantities:
  - $x = 3.1415296$
  - $A \in \{0,1\}$
- A proposition is either True or False
  - $\mathbf{A} \wedge \mathbf{B} \Rightarrow \mathbf{C}$
- $\text{King(Richard)} \wedge \text{Greedy(Richard)} \Rightarrow \text{Evil(Richard)}$
- Richard is either greedy or he isn't:
  - $\text{Greedy(Richard)} \in \{0,1\}$

Fuzzy Sets

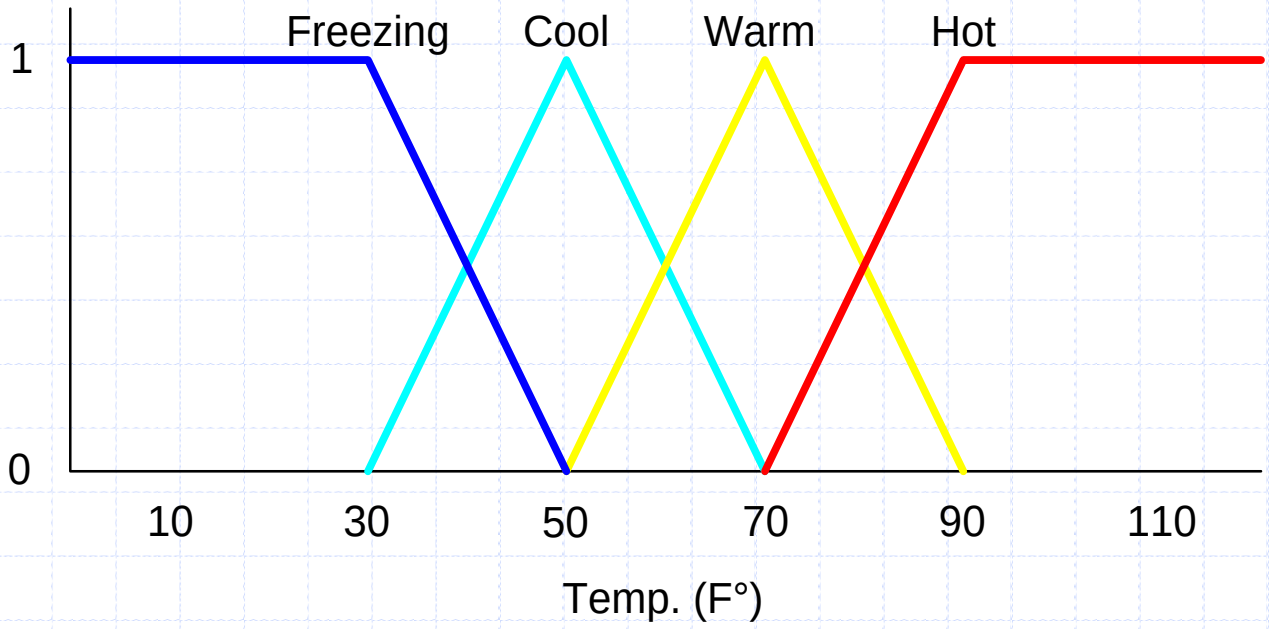
- What if Richard is only somewhat greedy?
- Fuzzy Sets can represent the degree to which a quality is possessed.
- Fuzzy Sets (Simple Fuzzy Variables) have values in the range of [0,1]
- Greedy(Richard) = 0.7
- Question: How evil is Richard?

Fuzzy Linguistic Variables

- Fuzzy Linguistic Variables are used to represent qualities spanning a particular spectrum
- Temp: **{Freezing, Cool, Warm, Hot}**
- Membership Function
- Question: What is the temperature?
- Answer: It is warm.
- Question: How warm is it?

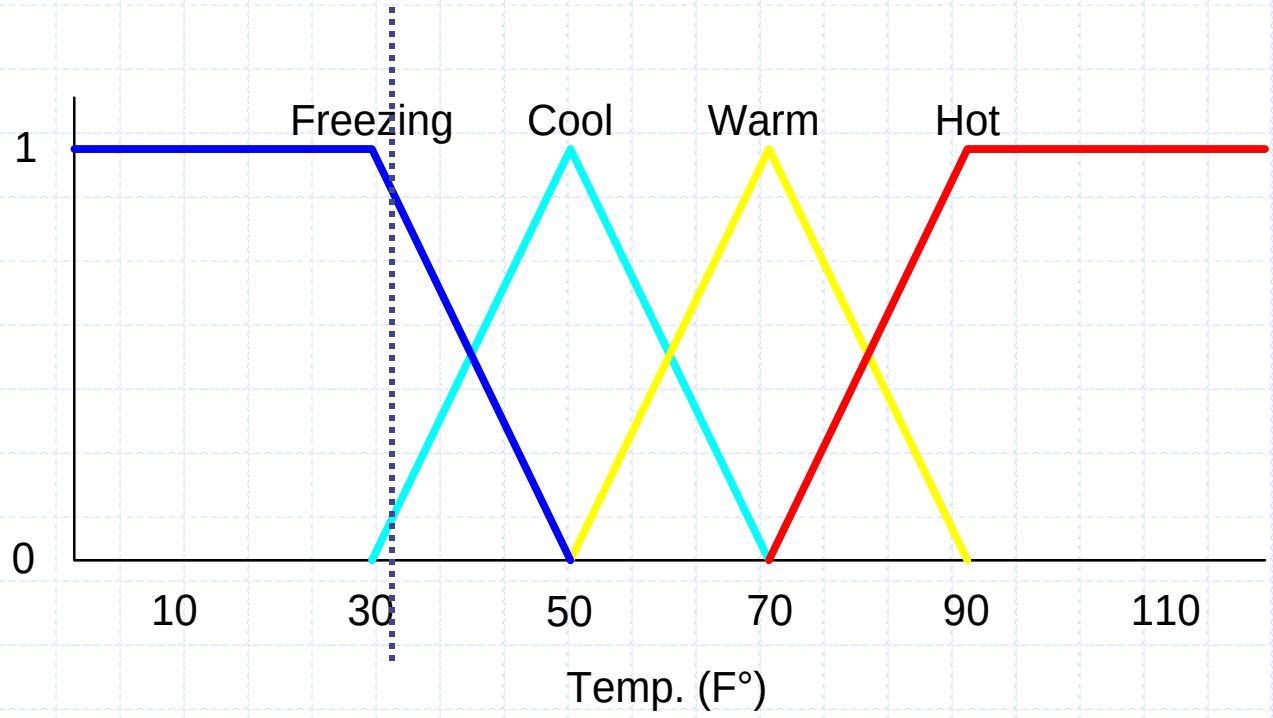
Membership Functions

- Temp: {Freezing, Cool, Warm, Hot}
- Degree of Truth or "Membership"



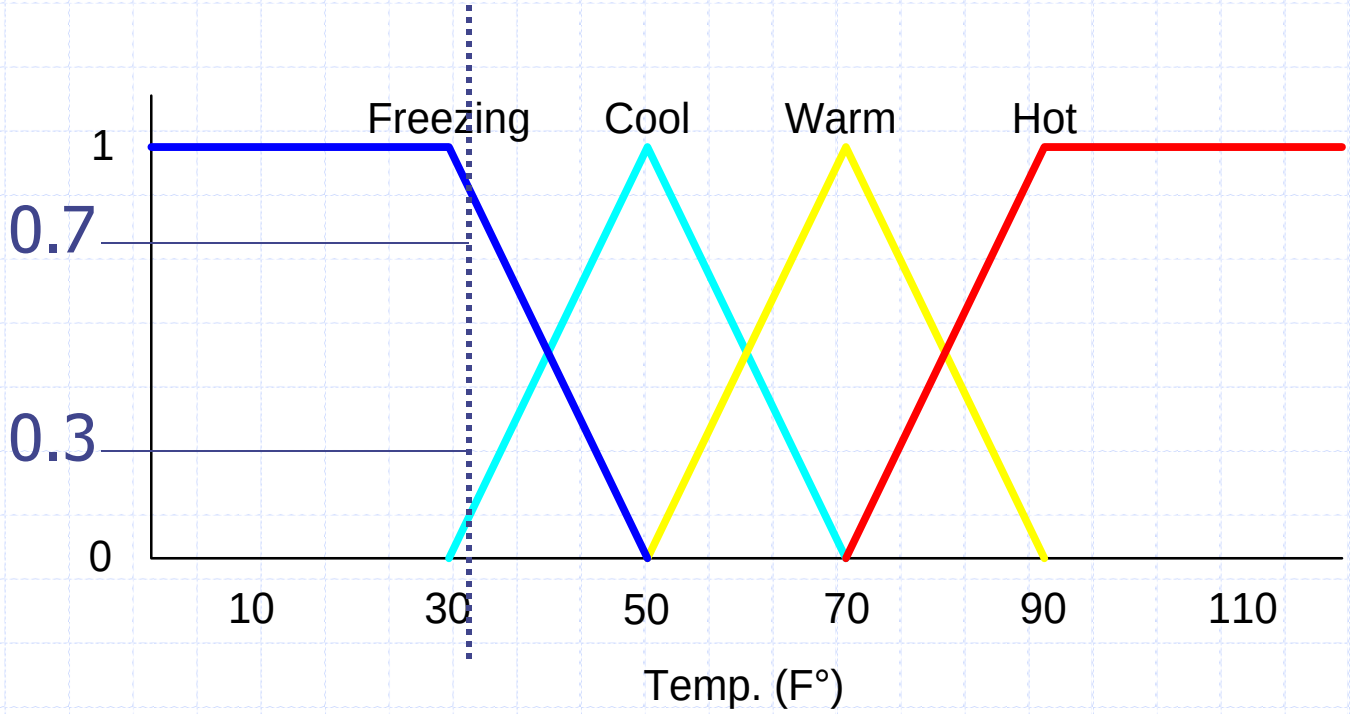
Membership Functions

- How cool is 36 F° ?



Membership Functions

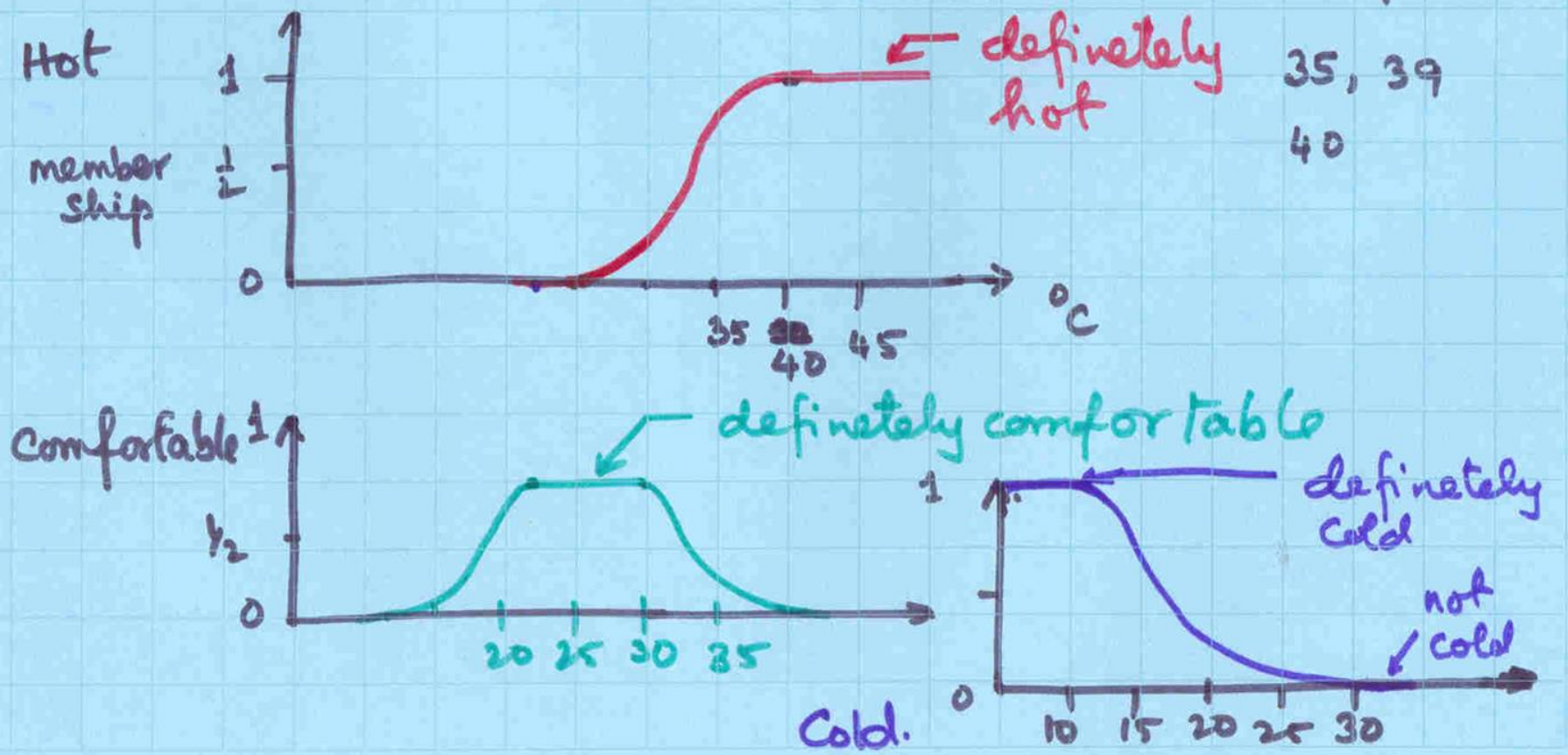
- How cool is 36 F° ?
- It is 30% Cool and 70% Freezing

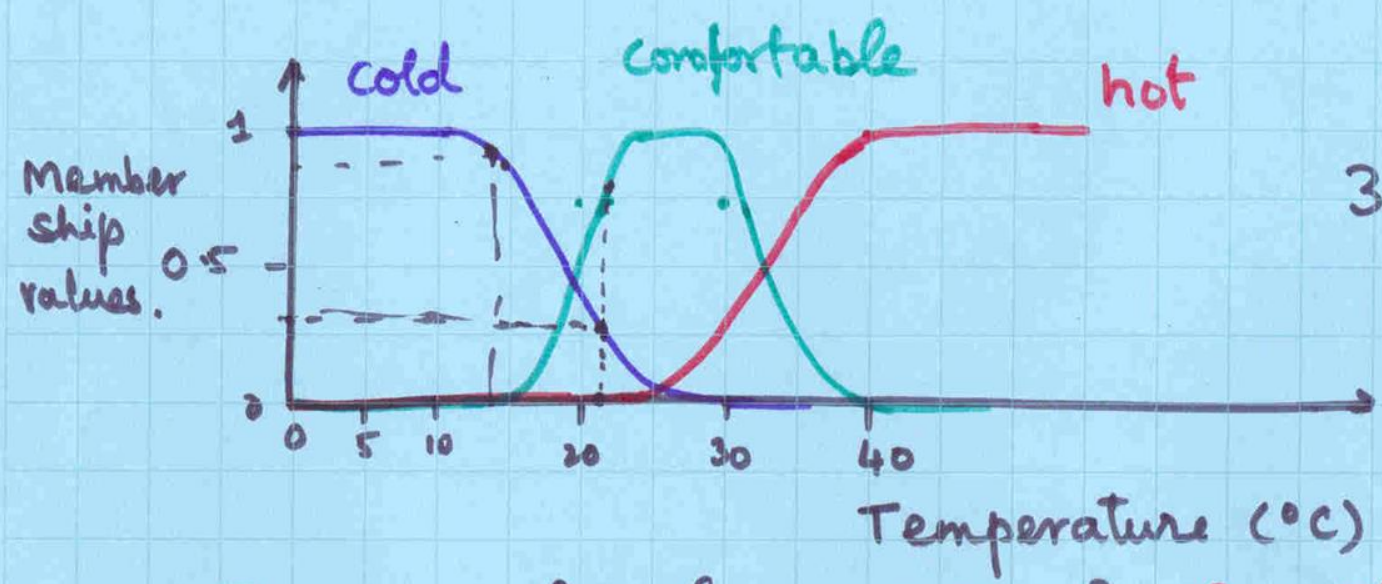


Fuzzy Associative Memory:

Temperature in a room with an airconditioner

Hot comfortable cold ⇒ Linguistic Description





3 variables

Temp. = 21°C : {0.3, 0.8, 0} Fuzzified Description of Temperature

$m_{cold}(21) = 0.3$

$m_{comfy}(21) = 0.8$

$m_{hot}(21°C) = 0.0$

$M(21) = 0.3 + 0.8 + 0.0$

$= 1.1$

Temp = ~~13°C : {0~~ 13 = {0.9, 0.0, 0.0}.

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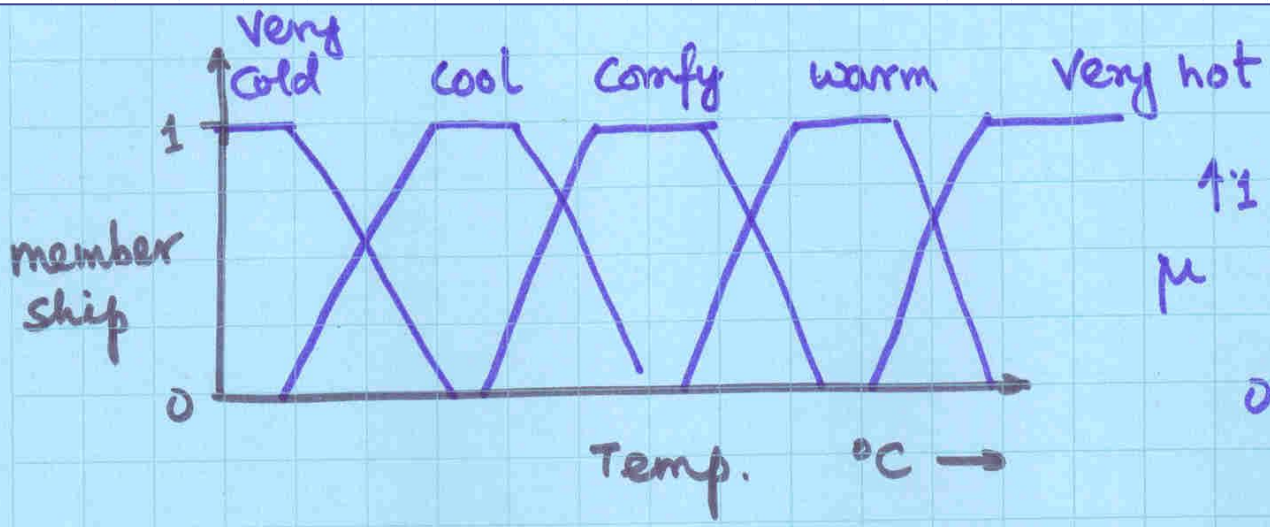
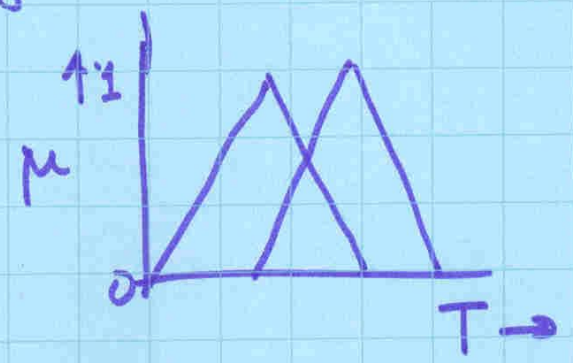
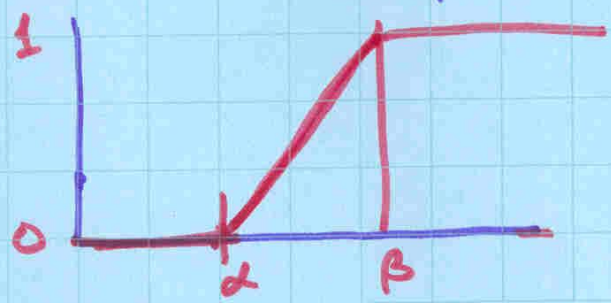




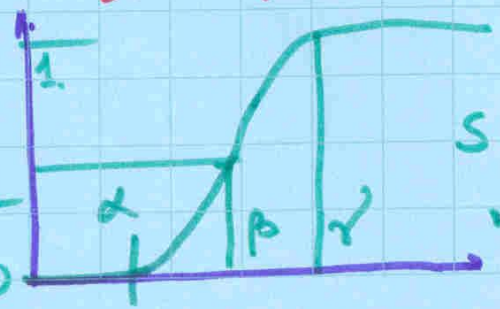
Trapezoidal membership functions



$$\Gamma(u,\alpha,\beta) = \begin{cases} 0 \\ (u-\alpha)/(\beta-\alpha) \\ 1 \end{cases}$$

Zadeh s function

$$S = \begin{cases} 0 & x \le \alpha \\ 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \alpha < x \le \beta \\ 1 - 2\left(\frac{x-\gamma}{\gamma-\alpha}\right) & \beta < x < \gamma \\ 1 & \gamma < x \end{cases}$$



$S(x,\alpha,\beta,\gamma)$

Properties of FUZZY SETS

Fuzzy vector-matrix Multiplication

a) Max-min composition

$$A \circ M = B \qquad M \longrightarrow I^{n \times p}$$

Inner Product

$$b_j = \max_{1 \le i \le n} \min (a_i, m_{ij})$$

$$A = \begin{pmatrix} 0.3 & 0.4 & 0.8 & 1 \end{pmatrix} \qquad n=4 \ , \ p=3$$

$$M = \begin{pmatrix} 0.2 & 0.8 & 0.7 \\ 0.7 & 0.6 & 0.6 \\ 0.8 & 0.1 & 0.5 \\ 0 & 0.2 & 0.3 \end{pmatrix} \qquad I^{n \times p}$$

$$b_1 = \max_{1 \le i \le n} \{ \min(a_1, m_{11}), \min(a_2, m_{21}), \cancel{\min} \min(a_3, m_{31}), \min(a_4, m_{41}) \}$$

$$= \max \{ \min(0.3, 0.2), \min(0.4, 0.7), \min(0.8, 0.8), \min(1, 0) \}$$

$$b_1 = \max \{ 0.2, \ 0.4 \ \ 0.8 \ \ 0 \} = 0.8$$

$$b_2 = \max \{ \min(0.3, 0.8), \min(0.4, 0.6), \min(0.8, 0.1), \min(1.0, 0.2) \}$$

$$= \max \{ 0.3, \ 0.4, \ \cancel{0.8} \ 0.1, \ 0.2 \} = 0.4$$

$$b_3 = \max \{ 0.3, 0.4, 0.5, 0.3 \} = 0.5$$

$$B = \{ 0.8 \ \ 0.4 \ \ 0.5 \}$$

Max-min composition (contd...)

$$(0.3 \quad 0.4 \quad 0.8 \quad 1) * \begin{pmatrix} 0.2 & 0.8 & 0.7 \\ 0.7 & 0.6 & 0.6 \\ 0.8 & 0.1 & 0.5 \\ 0 & 0.2 & 0.3 \end{pmatrix}$$

$$= (0.8 \quad 0.4 \quad 0.5)$$

b) Max product composition

$$b_j = \max_{1 \le i \le n} a_i m_{ij}$$

$$B = (\max\{0.06 \quad 0.28 \quad 0.64 \quad 0\} \quad \max\{0.24 \quad 0.24 \quad 0.08 \quad 0.2\}$$
$$\max\{0.21 \quad 0.24 \quad 0.4 \quad 0.3\})$$

$$= (0.64 \quad 0.24 \quad 0.4)$$

$B = A \circ M$ $\quad$ $\underline{M}$ encoded matrix

encodes a rule (A, B)

FUZZY RECALL

Encoding:

b) Correlation Product encoding – Fuzzy Rules.

$$M = A^T B$$

Inner Product)

$$m_{ij} = a_i b_j$$

$$M = A^T B = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.8 \\ 1.0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.4 & 0.5 \end{pmatrix}$$

$$M = A^T B = \begin{pmatrix} 0.24 & 0.12 & 0.15 \\ 0.32 & 0.16 & 0.20 \\ 0.64 & 0.32 & 0.40 \\ 0.8 & 0.4 & 0.5 \end{pmatrix}$$

II Fuzzy Hebbian FAM

given (X, Y)

encoding $M = X^T Y$ (outer product)

A) Correlation Minimum Encoding

$$m_{ij} = \min(a_i, b_j)$$

$$M = \underline{A}^T \circ \underline{B}$$

$$\underline{M} = A^T \circ B = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.8 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 0.8 & 0.4 & 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 0.5 \\ 0.8 & 0.4 & 0.5 \end{pmatrix}$$

rule:

(A, B) if A then B

Rule 1 (A_1, B_1) if A_1 then B_1

Rule 2 (A_2, B_2) if A_2 then B_2

$\vdots$

Rule m (A_m, B_m) if A_m then B_m

Superimposition of FAM rules

$$\underset{M_1}{(A_1, B_1)}, \underset{M_2}{(A_2, B_2)} \; - - - \; \underset{M_m}{(A_m, B_m)}$$

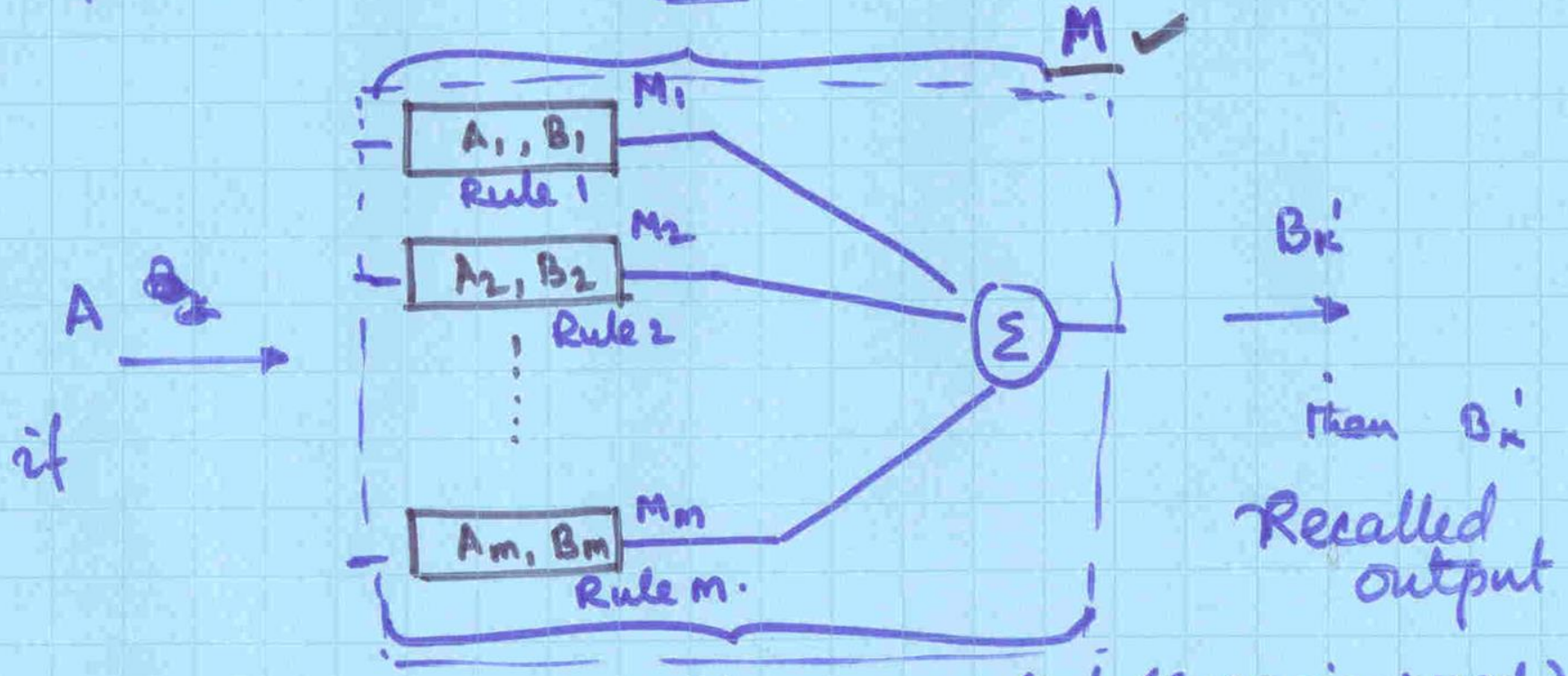
$$M = \max_{1 \le k \le m} \underline{M_k}$$ not effective in Hebbian Encoding

$$\max_{1 \le k \le m} \underline{\min(a_i^k, b_j^k)} \le \min\left(\max_{1 \le k \le m} a_i^k \;\; \max_{1 \le k \le m} b_j^k\right)$$

Fuzzy Approach to additively superimpose

$$M \circ M_k = A \circ (A_k^T \circ B_k)$$

if $\underline{A}$ then $\underline{B}$ $= \underline{B_k'}$ $\longrightarrow$ recalled output.



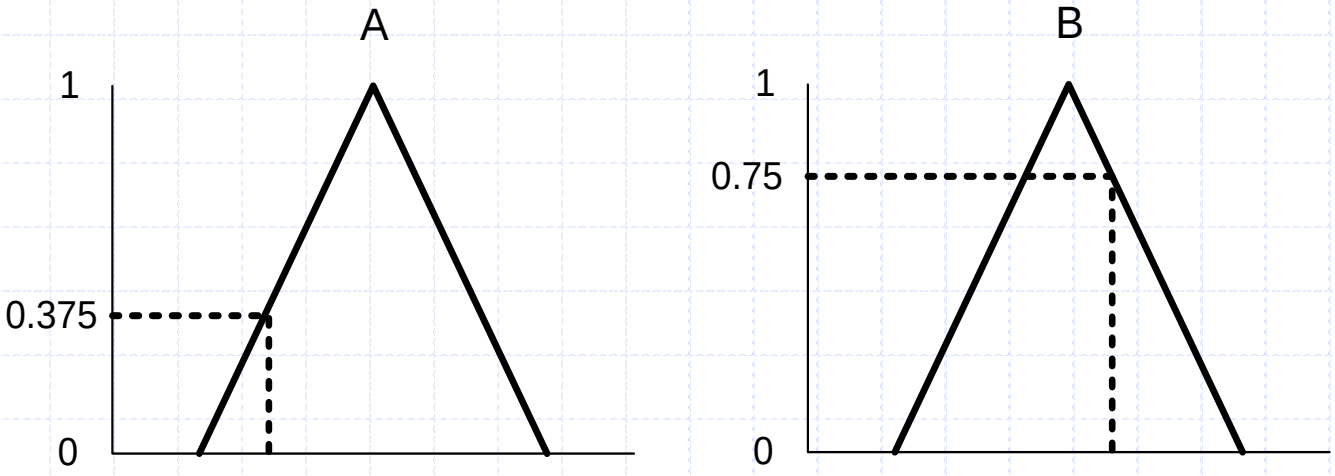
Approximate Reasoning

Fuzzy Logic

- How do we use fuzzy membership functions in predicate logic?
- Fuzzy logic Connectives:
  - Fuzzy Conjunction, $\wedge$
  - Fuzzy Disjunction, $\vee$
- Operate on degrees of membership in fuzzy sets

Fuzzy Disjunction

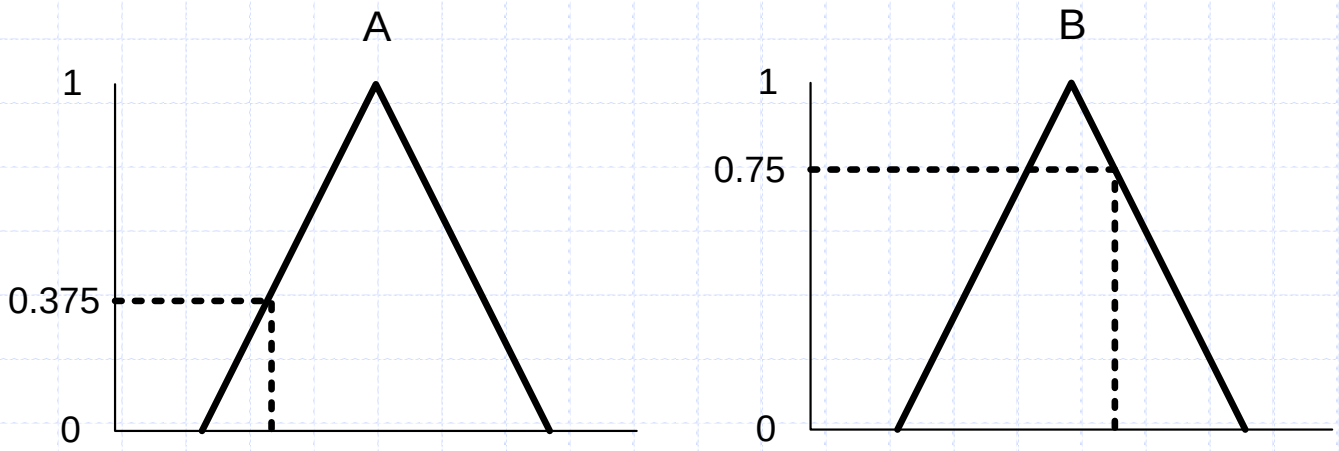
- $A \vee B \quad \max(A, B) \triangleq$
- $A \vee B = C$ "Quality C is the disjunction of Quality A and B"



$$(A \vee B = C) \Rightarrow (C = 0.75)$$

Fuzzy Conjunction

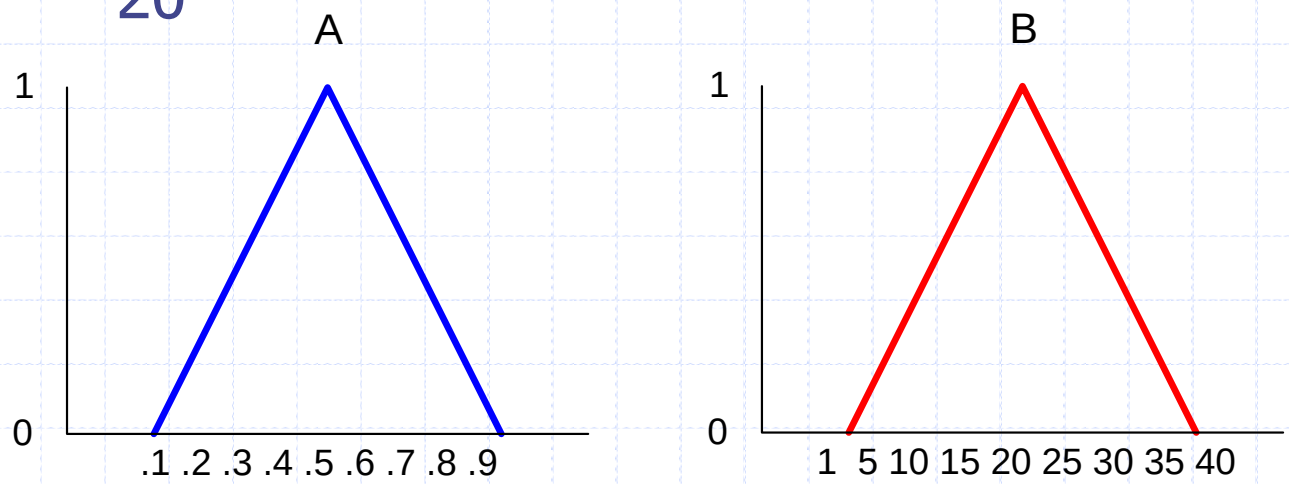
- $A \wedge B \triangleq \min(A, B)$
- $A \wedge B = C$ "Quality C is the conjunction of Quality A and B"



$$(A \wedge B = C) \Rightarrow (C = 0.375)$$

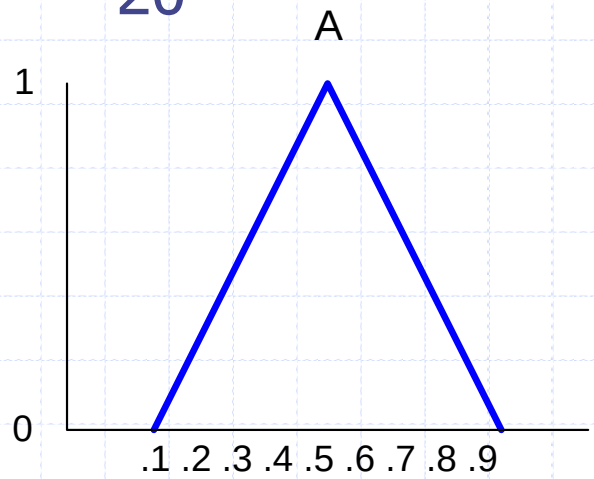
Example: Fuzzy Conjunction

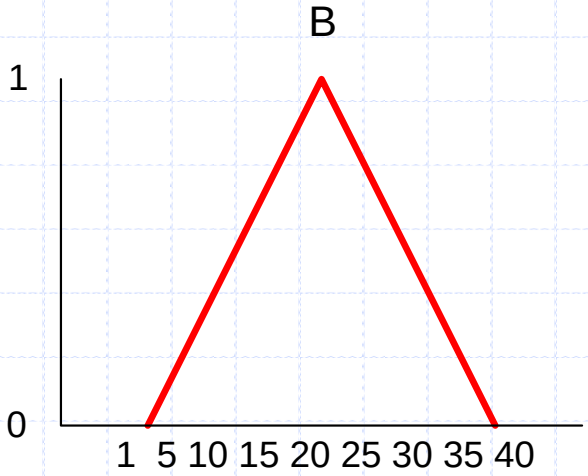
Calculate A∧B given that A is .4 and B is 20



Example: Fuzzy Conjunction

Calculate A∧B given that A is .4 and B is 20

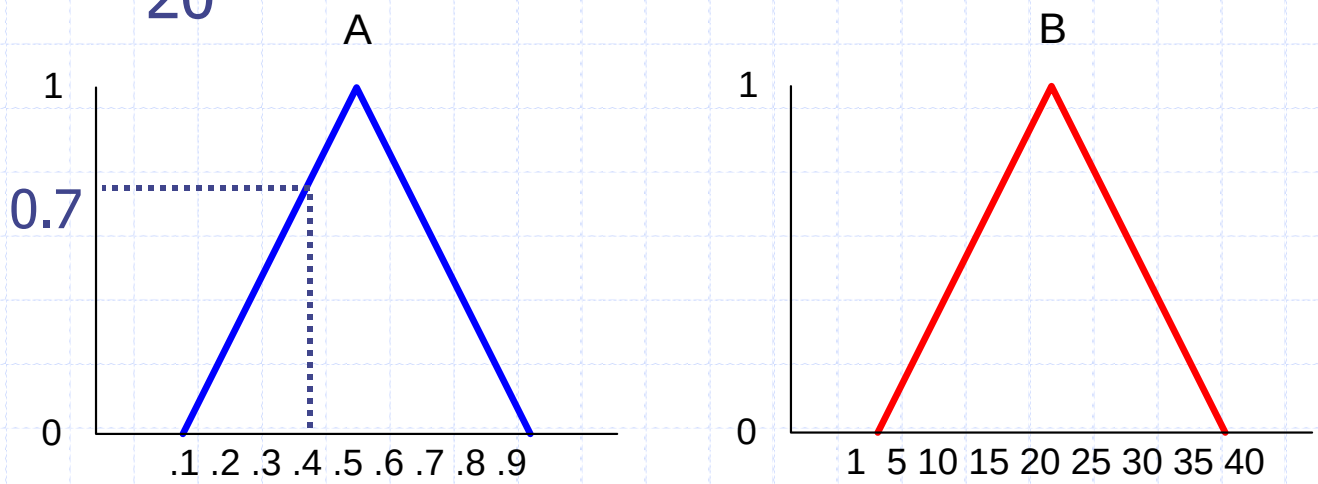




Determine degrees of membership:

Example: Fuzzy Conjunction

Calculate A∧B given that A is .4 and B is 20

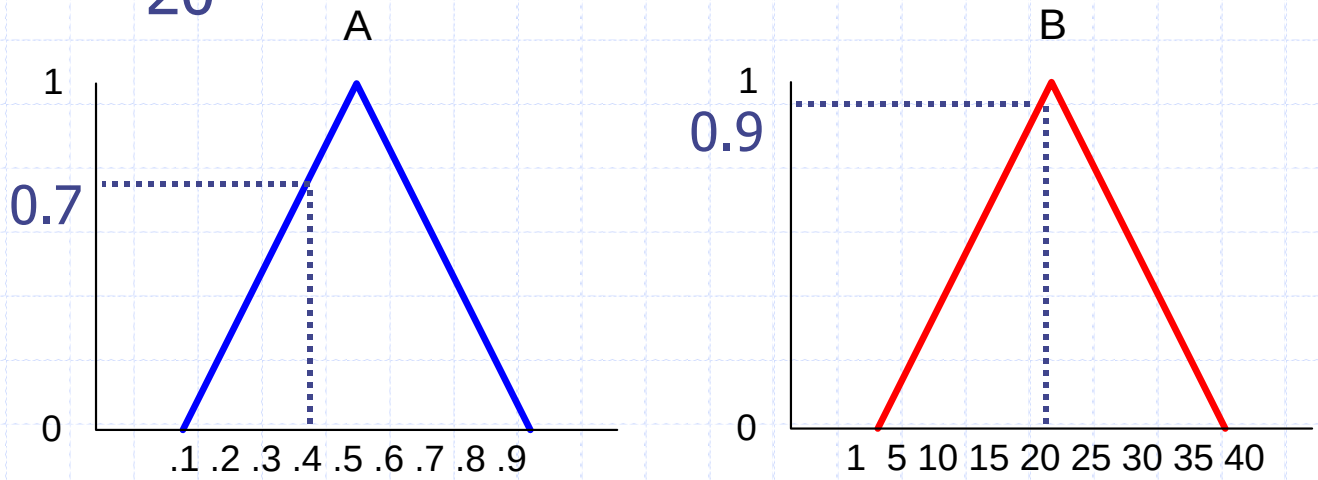


Determine degrees of membership:

A = 0.7

Example: Fuzzy Conjunction

Calculate A∧B given that A is .4 and B is 20

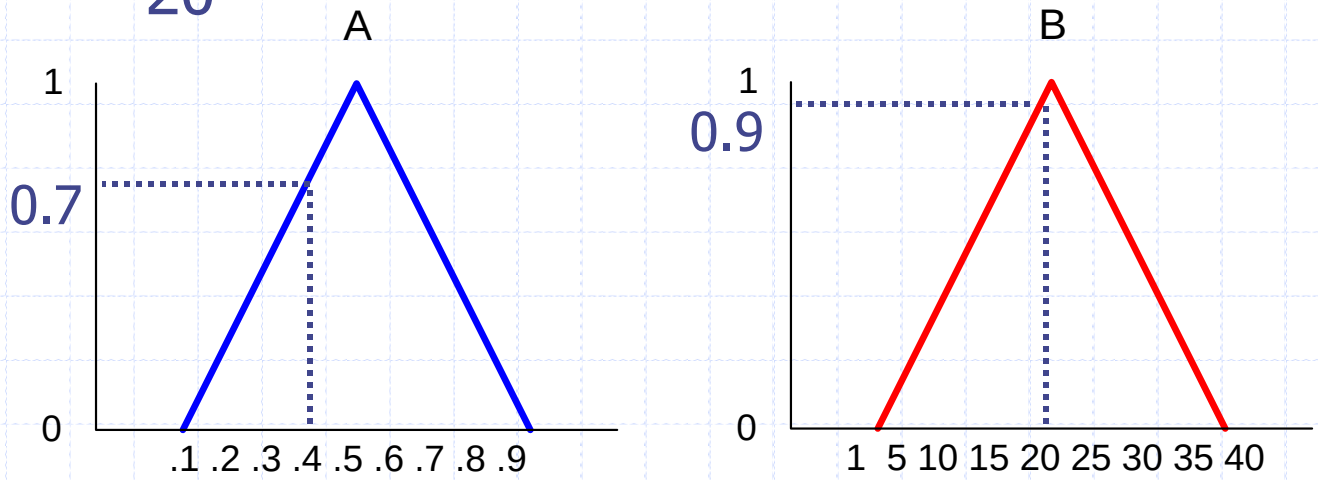


Determine degrees of membership:

A = 0.7 B = 0.9

Example: Fuzzy Conjunction

Calculate A∧B given that A is .4 and B is 20



Determine degrees of membership:

A = 0.7 B = 0.9

Apply Fuzzy AND

A∧B = min(A, B) = 0.7

FUZZY INFERENCE METHODS

use of Fuzzy Rules or Implication

R_1 : if u is A_1 and v is B_1 then w is C_1

I. Mamdani Method:

use minimum operator R_c

Rules: R_i : if u is A_i and v is B_i then w is C_i

$i = 1, 2, \ldots, n$

for $u \in U$, and $v \in V$, $w \in W$

$R_i = (A_i \text{ and } B_i) \longrightarrow C_i$ is defined by

$$\mu_{R_i} = \mu_{(A_i \text{ and } B_i \longrightarrow C_i)}(u, v, w)$$

Case 1: inputs are crisp and are treated as singleton.

$$\mu_{C_i'}(\omega) = \underbrace{[\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)]}_{\text{if (antecedent)}} \rightarrow \underbrace{\mu_{C_i}(\omega)}_{\text{then ...}}$$

Inference result (FUZZY)

E.g.: if temperature is high and humidity is high then fan speed is high

How to find the fan speed when temperature = 32°C and Humidity is 95%?

Mamdani Method:

use of minimum operator ($\wedge$)

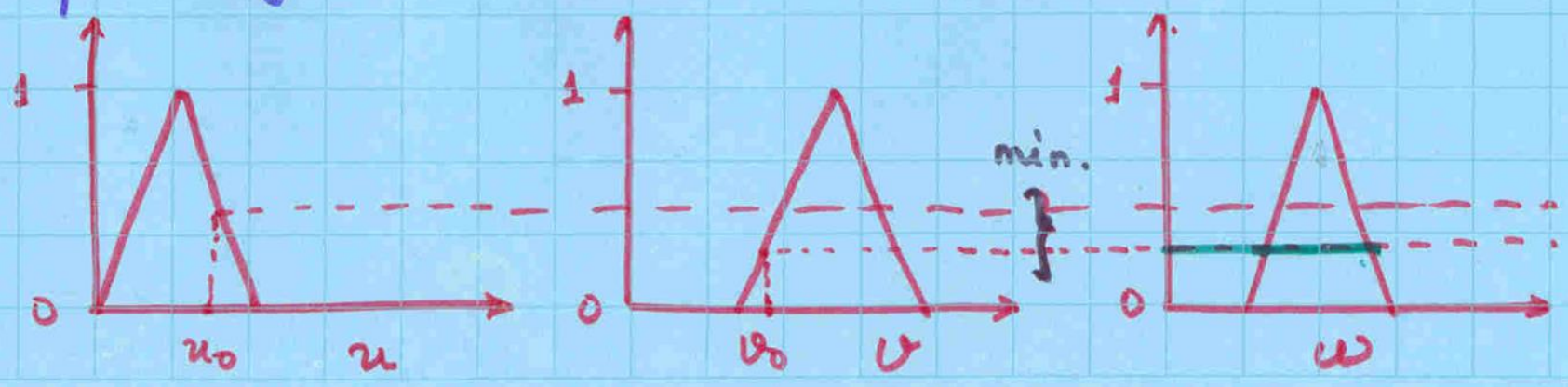
as a fuzzy implication function ($\longrightarrow$):

$$\mu_{C_i'}(w) = \alpha_i \wedge \mu_{C_i}(w) \qquad i = 1, 2, \ldots, n$$

where $\alpha_i = \mu_{A_i}(\underline{u_0}) \wedge \mu_{B_i}(\underline{v_0})$

α_i is called the "firing strength", "matching degree", "satisfaction degree".

Graphically. (one rule)



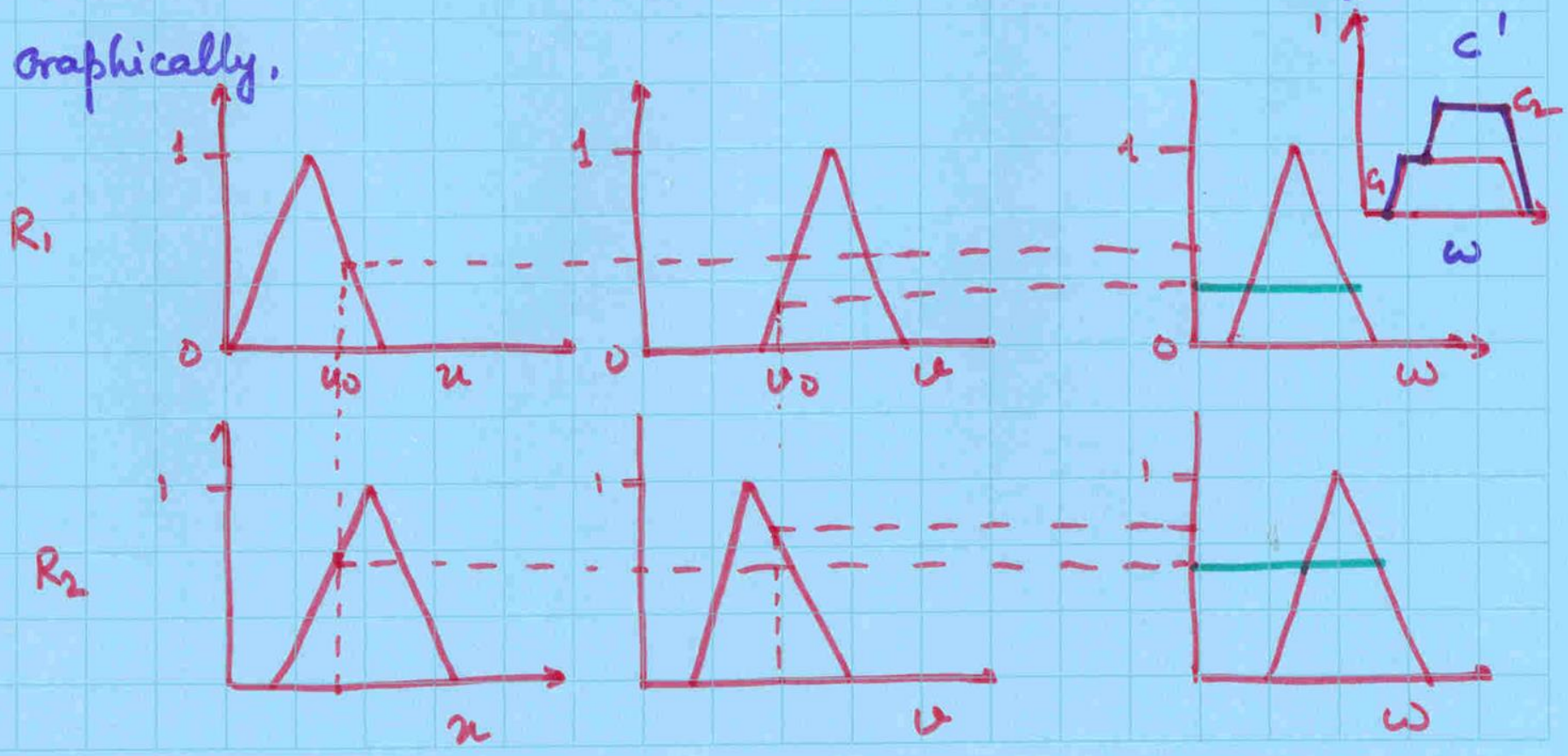
Mamdani Method – multiple rules

2 rules (e.g.) R_1 & R_2

$$\mu_{C'}(\omega) = \mu_{C_1'} \vee \mu_{C_2'}$$

$$= [\alpha_1 \wedge \mu_{C_1}(\omega)] \vee [\alpha_2 \wedge \mu_{C_2}(\omega)]$$

Graphically,



In general

$$\mu_{C'}(\omega) = \underbrace{\bigvee_{i=1}^{n}}_{\max} \underbrace{[\alpha_i \wedge \mu_{C_i}(\omega)]}_{\min} = \bigvee_{i=1}^{n} \mu_{C_i'}(\omega)$$

more than 2 rules.

Case 2: if inputs are Fuzzy Sets A' and B' (2 rules)

$$\mu_{C'}(\omega) = \alpha_i \wedge \mu_{C_i}(\omega)$$

$$\alpha_i = \min\left[\max\left(\mu_{A'}(u) \wedge \mu_A(u)\right), \max\left(\mu_{B'}(v) \wedge \mu_B(v)\right)\right]$$

$$\mu_{C_i'}(\omega) = \bigvee_{i=1}^{n} [\alpha_i \wedge \mu_{C_i}(\omega)] = \bigvee_{i=1}^{n} \mu_{C_i'}(\omega)$$

$$C' = \bigcup_{i=1}^{n} C_i'$$

Example:

Rule base is given by

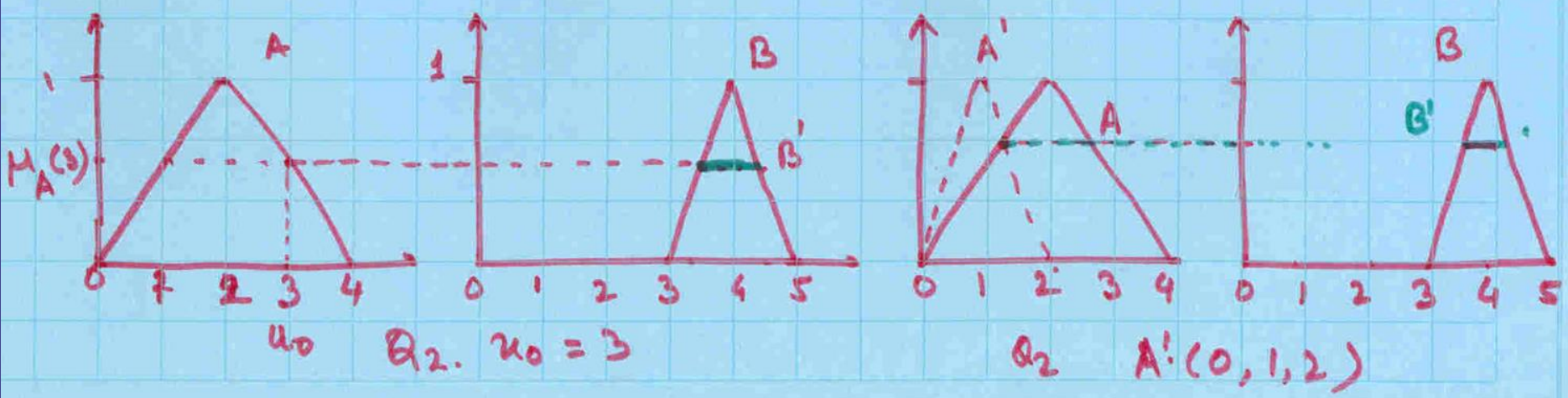
R: If u is A then v is B

A = (0, 2, 4) and B = (3, 4, 5)

triangular fuzzy sets.

Q_1: output B'? if the input is crisp value $u_0 = 3$

Q_2: output B'? if the input is fuzzy set A' = (0, 1, 2)



Q_2. $u_0 = 3$

Q_2 A' = (0, 1, 2)

Larsen Method :

Case 1: Inputs are crisp & Single tons

$$\mu_{C'_i}(\omega) = [\underline{\mu_{A_i}}(u_0) \text{ and } \underline{\mu_{B_i}}(v_0)] \longrightarrow \mu_{C_i}(\omega)$$

result.

if ... (A and B) then ... C

$\underline{u \text{ is } A}$ and $\underline{v \text{ is } B}$ $\underline{\omega \text{ is } C}$

$$= [\mu_{A_i}(u_0) \wedge \mu_{B_i}(v_0)] \cdot \mu_{C_i}(\omega)$$

$$= \alpha_i \cdot \mu_{C_i}(\omega) \quad \text{where } \alpha_i = [\mu_{A_i}(u_0) \wedge \mu_{B_i}(v_0)]$$

For multiple rules:

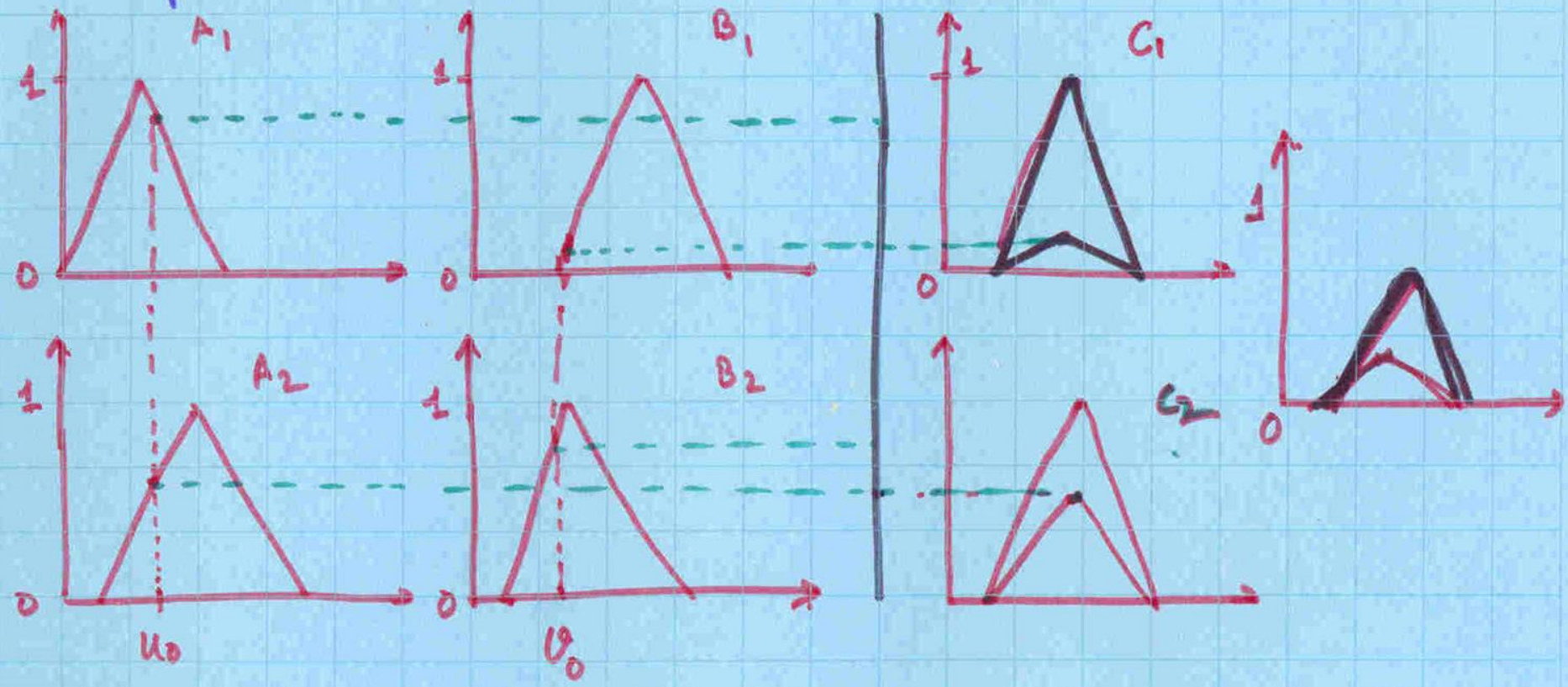
$$\mu_{C'}(\omega) = \underset{\substack{i=1 \\ \text{max}}}{\overset{n}{\bigvee}} [\alpha_i \cdot \underset{\text{product}}{\mu_{C_i}(\omega)}] = \bigvee_{i=1}^{n} \mu_{C'_i}(\omega)$$

$$C'_i = \bigcup_{i=1}^{n} C'_i$$

max product

Graphically

I.I.T. KGP



Case 2: inputs are fuzzy sets A', B'

$$\mu_{C'_i}(\omega) = \alpha_i \cdot \mu_{C_i}(\omega)$$

where $\alpha_i = \min\left[\max\left[\mu_{A'}(u) \wedge \mu_{A_i}(u)\right],\ \max\left[\mu_{B'}(v) \wedge \mu_{B_i}(v)\right]\right]$

multiple rules:

$$\mu_{C'}(\omega) = \bigvee_{i=1}^{n}\left[\alpha_i \cdot \mu_{C_i}(\omega)\right] = \bigvee_{i=1}^{n} \mu_{C'_i}(\omega)$$

$$C' = \bigcup_{i=1}^{n} C'_i$$

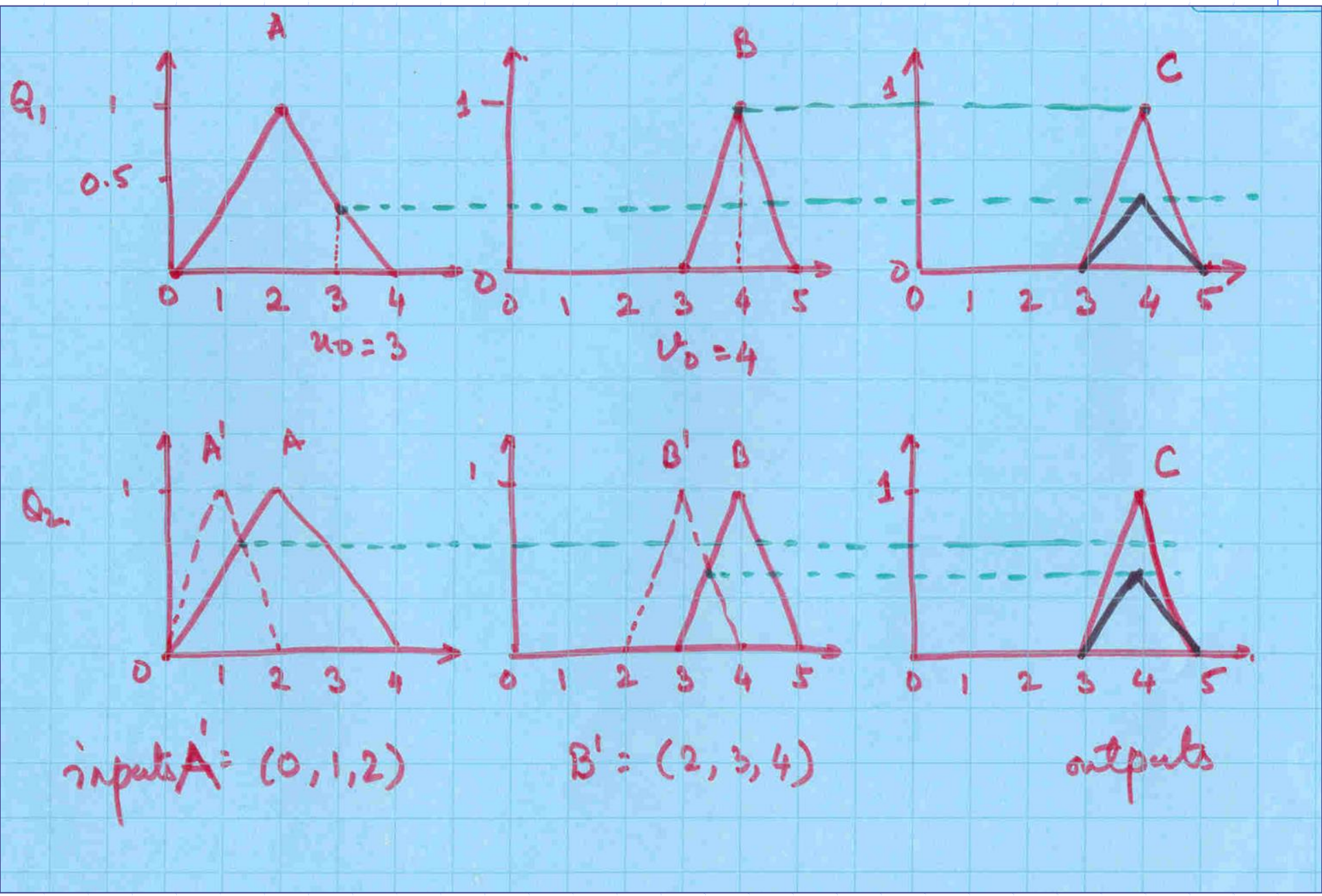
Example: Rule R: if u is A and v is B then w is C

$A = (0, 2, 4)$ $B = (3, 4, 5)$

and $C = (3, 4, 5)$ all are triangular sets

Q1: what is output C' if inputs are crisp values $u_0 = 3$. and $v_0 = 4$?

Q2: what is output C' if inputs are fuzzy sets $A' = (0, 1, 2)$ and $B' = (2, 3, 4)$?


A
B
C
Q_1
1
0.5
0
0 1 2 3 4
$u_0 = 3$
0 1 2 3 4 5
$v_0 = 4$
Q_2
A'
B'
inputs A' = (0, 1, 2)
B' = (2, 3, 4)
outputs

Inference → yields

a fuzzy set

input is crisp — output is fuzzy set.
(implication)

input is fuzzy set — output is fuzzy set

end result is often desired a crisp value!

Fuzzy set $\xrightarrow[\text{Defuzzification}]{}$ crisp value
(singleton)

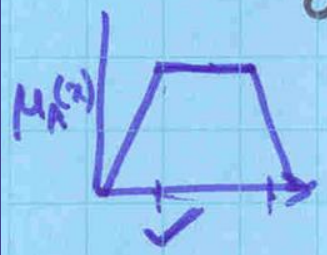
Defuzzification:

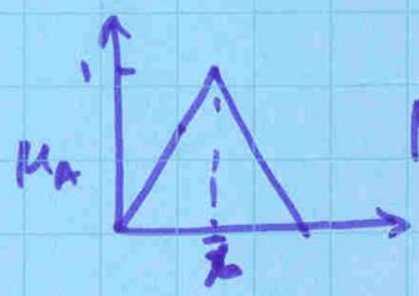
single membership function $\mu_A(x)$, set of membership functions $\mu_{A_i}(x)$

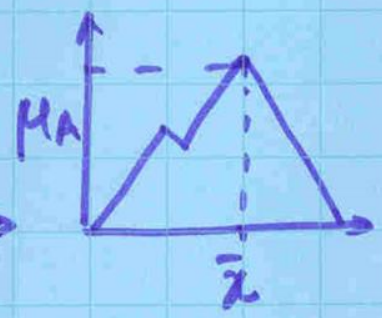
$\longrightarrow$ $\bar{x}$ crisp value.

1) Height or Maximum membership (single peak)

$\bar{x}$ is chosen such that $\mu_A(\bar{x})$ is maximum.



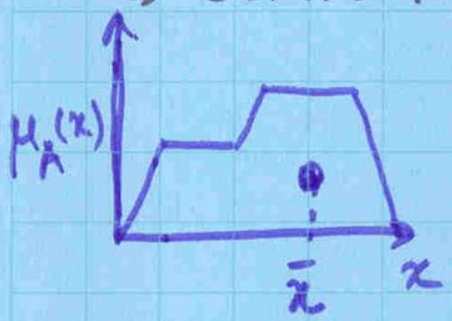




2) Mean-max or middle of maxima

more than one value of x with maximum membership then we take average of smallest & largest x

3) Centroid, or center of area / centre of gravity:



$$\bar{x} = \frac{\int_{x \in A} x \, \mu_A(x) \, dx}{\int_{x \in A} \mu_A(x) \, dx}$$

union is taken if there are a number of membership function.

4) Bisector of area: $\bar{x}$ divides the area into 2 equal parts

$$\int_{x < \bar{x}} \mu_A(\bar{x}) \, dx = \int_{x > \bar{x}} \mu_A(\bar{x}) \, dx$$

5) Weighted average: set of membership fns

$$\bar{x} = \frac{\sum x_m \mu_{A_i}(x_m)}{\sum \mu_{A_i}(x_m)}$$

x_m max value at $\mu_{A_i}(x_m)$

6) Centre of Sums: set of membership functions each of them are weighted

$$\bar{x} = \frac{\int_{x \in A} x \sum \mu_A(x)\, dx}{\int_{x \in A} \sum \mu_A(x)\, dx.}$$

similar to weighted average.

use integrals (area) instead of $x_{max}\, \mu_{A_i}(x_m)$

overall process.

Crisp value → fuzzy set → fuzzy rules → inference → crisp

fuzzification — defuzzification

fuzzy logic & sets.

↓

fuzzification

encoding

implication

inference, recall

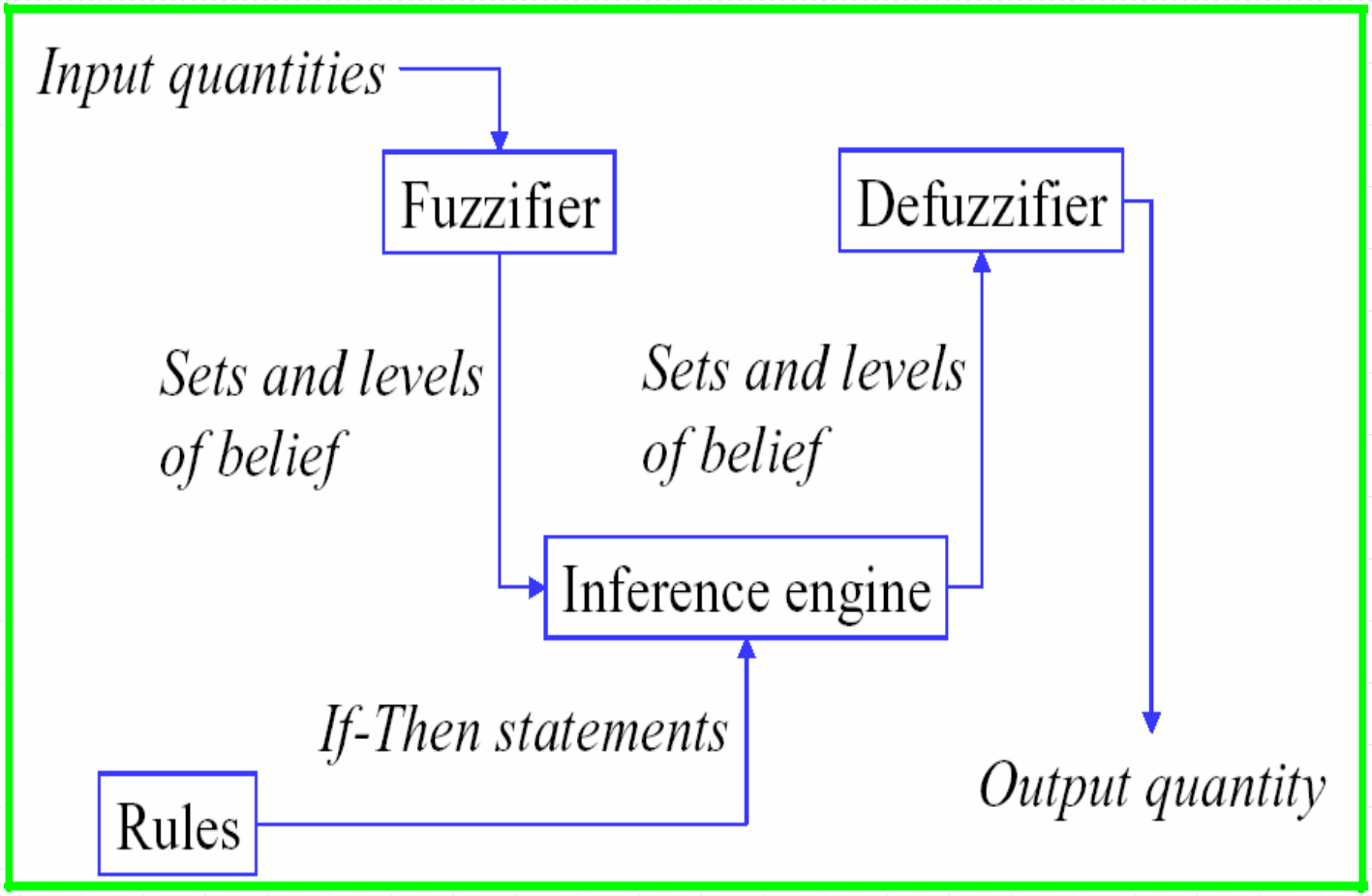
} linguistic commonsense. rules

defuzzification

Steps by Step Approach

- The activities covered are summarized into three main stages
  - Fuzzification
    - Membership functions used to graphically describe a situation
  - Evaluation of Rules
    - Application of the fuzzy logic rules
  - Diffuzification
    - Obtaining the crisp results

Steps by Step Approach



FUZZY LOGIC

RULES

if $\underline{x \text{ is } A}$ then $\underline{y \text{ is } B}$

A & B are linguistic values (Fuzzy Logic) defined by Fuzzy Sets of Discourse

universe of Discourse X and Y

RULE also called FUZZY IMPLICACTION

"x is A" is the <u>antecedant</u> or Premise

"y is B" is the <u>consequence</u> or Conclusion

Examples:

if <u>Pressure is High</u> then <u>Volume is Small</u>

if "Road is slippery" then "Driving is Dangerous"

if "an Apple is Red" then "it is ripe"

if "Speed is high" then "Apply brakes a little"

<u>Reasoning</u>

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REASONING

e.g. Play CRICKET on SATURDAY or SUNDAY and you dont want to wet when you play CRICKET

use of RULES:

if "it rains" then "you get wet"

if "you get wet" then "you can't play CRICKET"

if "it rains on SATURDAY" and "won't on SUNDAY"

- You play CRICKET on SUNDAY!

$\Longrightarrow$ REASONING.

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Forms of Reasoning:

I. Generalized Modus Ponens.

Premise	x is A'	Input
Implication	if x is A then y is B	RULE
Consequence	y is B'	output

A, A', B, B' are Fuzzy Sets and x and y are symbolic names for objects

II Generalized Modus Tolens:

Premise	y is B'
Implication	if x is A then y is B
Consequence	x is A'

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Fuzzy Rule as a Relation

if x is A then y is B

"x is A", "y is B" fuzzy Predicates $A(x)$, $B(y)$

if $A(x)$ then $B(y)$

can be described as a Relation

$$R(x, y) : A(x) \longrightarrow B(y)$$

$R(x, y)$ considered as a 2-Dimensional membership function (fuzzy set)

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y))$$

$\downarrow$

f is the Implication function (fuzzy Rule)

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MIN Fuzzy Implication

<u>Mamdani</u> Approach

$$R_c = A \times B$$

$$= \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) / (x, y)$$

where $\wedge$ is the "min" operator

PRODUCT FUZZY Implication

LARSEN Approach

$$R_p = A \times B$$

$$= \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) / (x, y)$$

where "$\cdot$" is the algebraic product

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Example

"If temperature is <u>high</u>, Then humidity is <u>fairly high</u>"

Let.

- T - universe of discourse for Temperature
- H - universe of discourse for Humidity
- $t \in T$, $h \in H$ - variables for Temp, Humidity

"high" Denoted by $A \subseteq T$

"fairly high" denoted by $B \subseteq H$

Rule becomes:

$R(t, h)$: if t is A then h is B or

$$R(t, h) : R(t) \longrightarrow R(h)$$

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If we know A and B

we can find $R(t,h) = A \times B$

t	20	30	40
$\mu_A(t)$	0.1	0.5	0.9

h	20	50	70	90
$\mu_B(h)$	0.2	0.6	0.7	1

Mamdani Approach
(implication)

$R_C(t,h) = A \times B$

$= \int \mu_A(t) \wedge \mu_B(h) / (t,h)$

Rule →

$t \backslash h$	20	50	70	90
20	0.1	0.1	0.1	0.1
30	0.2	0.5	0.5	0.5
40	0.2	0.6	0.7	0.9

Implication operators: Mamdani → min

Larsen → Algebraic Product

Composition operators Mamdani : max-min

Larsen : max product

If we know $R_c(t, h)$ for fuzzy rule

"If temp is high then humidity is fairly high"

when

what is the humidity when temp. is fairly high

or t is A', $A' \subseteq T$?

t	20	30	40	
$\mu_{A'}(t)$	0.01	0.25	0.81	$R(t)$

$$R(h) = R(t) \circ R_c(t, h)$$

h	20	50	70	90	
$\mu_{B'}(t)$	0.2	0.6	0.7	0.81	$R(h)$

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Inference

Graphically:

1. Build a cylindrical extension of A.
   $A(x, y)$
2. Determine intersection of $R(x,y)$ and $A(x,y)$
3. Build projection of $R(x,y) \wedge A(x,y)$

R_1 : if u is A_1 and v is B_1 then w is C_1

u & v

a) Crisp inputs: treated as fuzzy singletons

b) fuzzy sets A' and B'

Mamdani Method : $\underset{\text{Result}}{\mu_{C_i'}(w)} = \underbrace{[\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)]}_{\text{if} \rightarrow \text{min approach.}} \rightarrow \mu_{C_i}(w)$ — then

Larsen Method : $\mu_{C_i'}(w) = [\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)] \rightarrow \mu_{C_i}(w)$

product.

Inverted Pendulum

- Task:
  - To balance a pole on a mobile platform that can move in only two directions, either to the left or to the right.

Inverted Pendulum

- Classic fuzzy control application
- Task is to keep an inverted pendulum balanced on a mobile platform
- Platform can move left and right
  - moved by an electric motor
- Motor can operate at different speeds
  - controlled by current
  - Or speed controlled

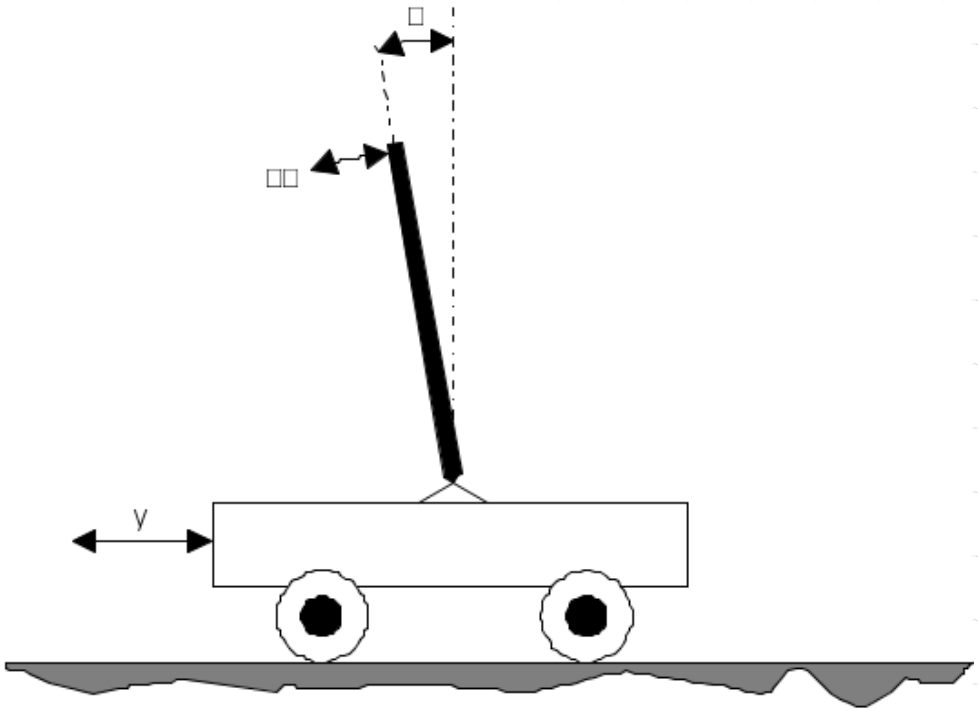
Inverted Pendulum

- The input and output relationships of the variables of the fuzzy system are then determined.
  - Inputs:
    - Angle between the platform and the pendulum
    - Angular velocity of this angle.
  - Outputs:
    - Speed of platform

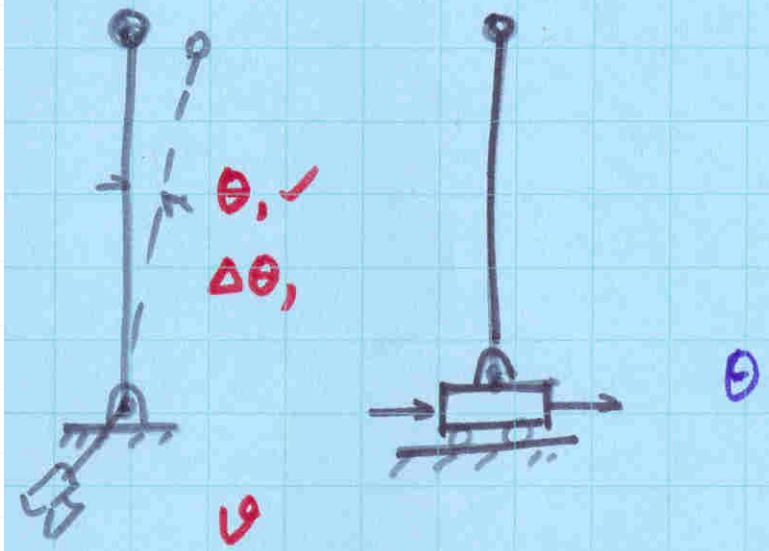
Inverted Pendulum

- Use membership functions to graphically describe the situation (Fuzzification)
- The output which is speed can be high speed, medium speed, low speed, etc. These different levels of output of the platform are defined by specifying the membership functions for the fuzzy-sets

Inverted Pendulum



CONTROL OF AN INVERTED PENDULUM



θ, Δθ and v are **Fuzzy Variables**

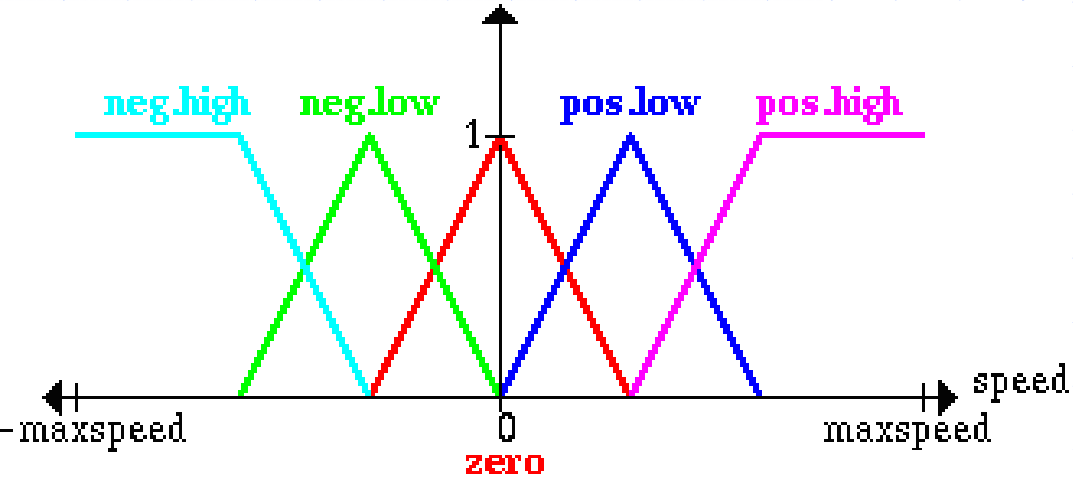
ZE	-2° to 2°
PS	1° - 5°
PM	3° - 8°
PL	6° - 15°

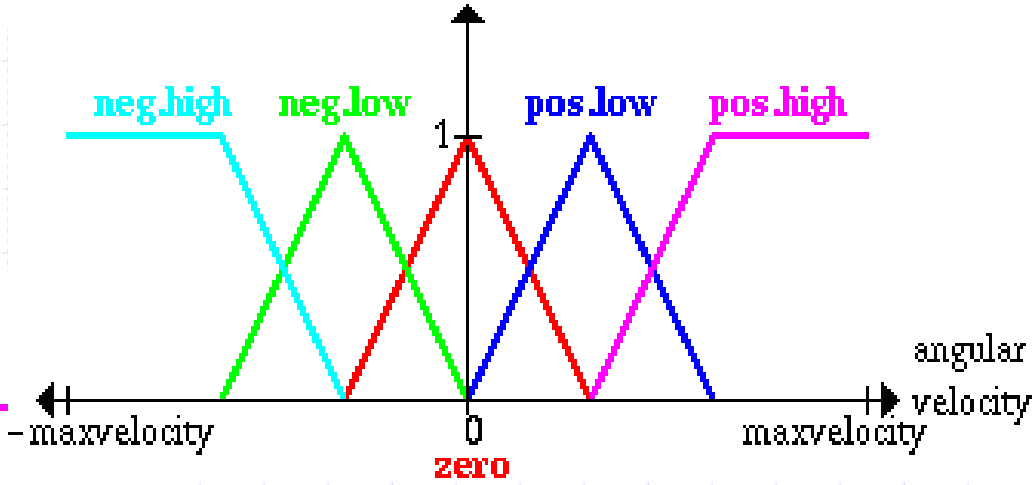
Fuzzy Variable Descriptions

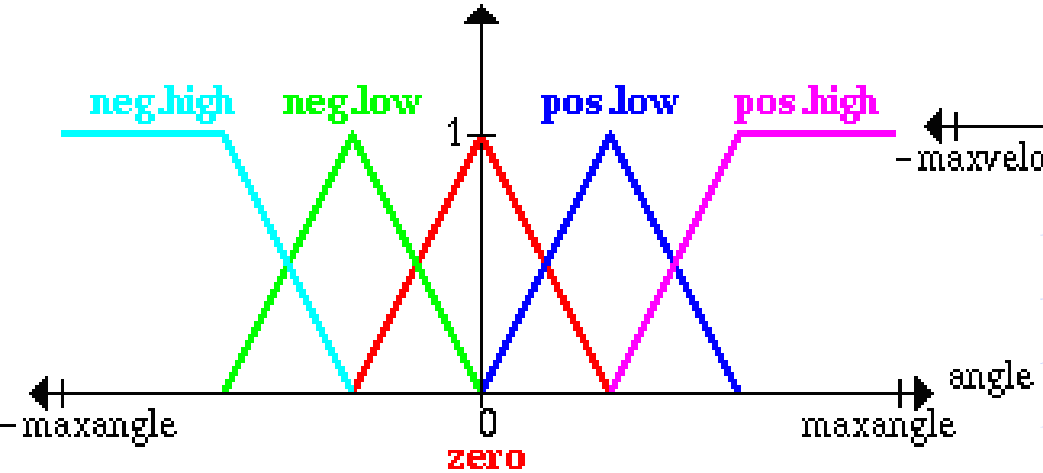
- NS, NM, NL

Δθ	→	NL, NM, NS, ZE, PS, PM, PL	(7)
v	→	NL, NM, NS, ZE, PS, PM, PL	(7)

Inverted Pendulum







Inverted Pendulum

- Define Fuzzy Rules
  - Examples
    - If angle is zero and angular velocity is zero, then speed is also zero
    - If angle is zero and angular velocity is negative low, the speed is negative low
    - If angle is positive low and angular velocity is zero, then speed is positive low
    - If angle is positive low and angular velocity is negative low, then speed is zero

FUZZY RULE BASE

$(\theta, \Delta\theta, v)$

$\downarrow \quad \downarrow \quad \downarrow$

(ZE, ZE, ZE) → COMMON SENSE DESCRIPTION RULE LINGUISTIC

(ZE, PS, NS) → ANOTHER RULE

⋮

(ZE, NS, PS) → ANOTHER RULE

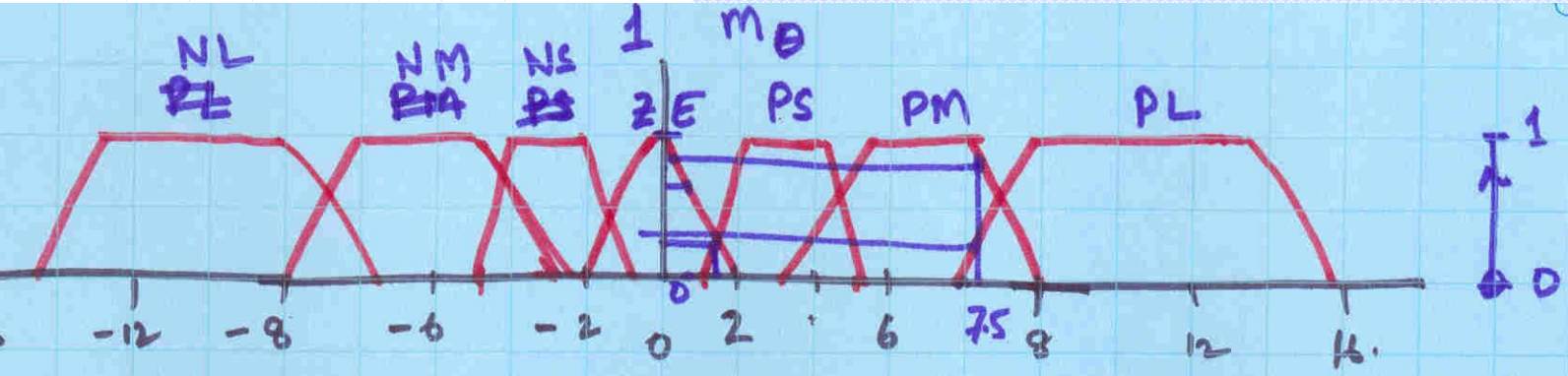
encoding

FUZZY ASSOCIATIVE MEMORY (FAM) $\Delta\theta$

Bank of Fuzzy Rules

$\Delta\theta$ ↓ / θ →	NL	NM	NS	ZE	PS	PM	PL (θ)
NL	PL	PL	PM	PM			PM
NM	PL	PL		PS			
NS			PS	PS			
ZE	PM	PS	PS	ZE	NS	NS	NM
PS				NS			
PM				NS			
PL				NM			NL

7×7 = 49 RULES



θ in ° (degrees)

no of fuzzy variables = 7

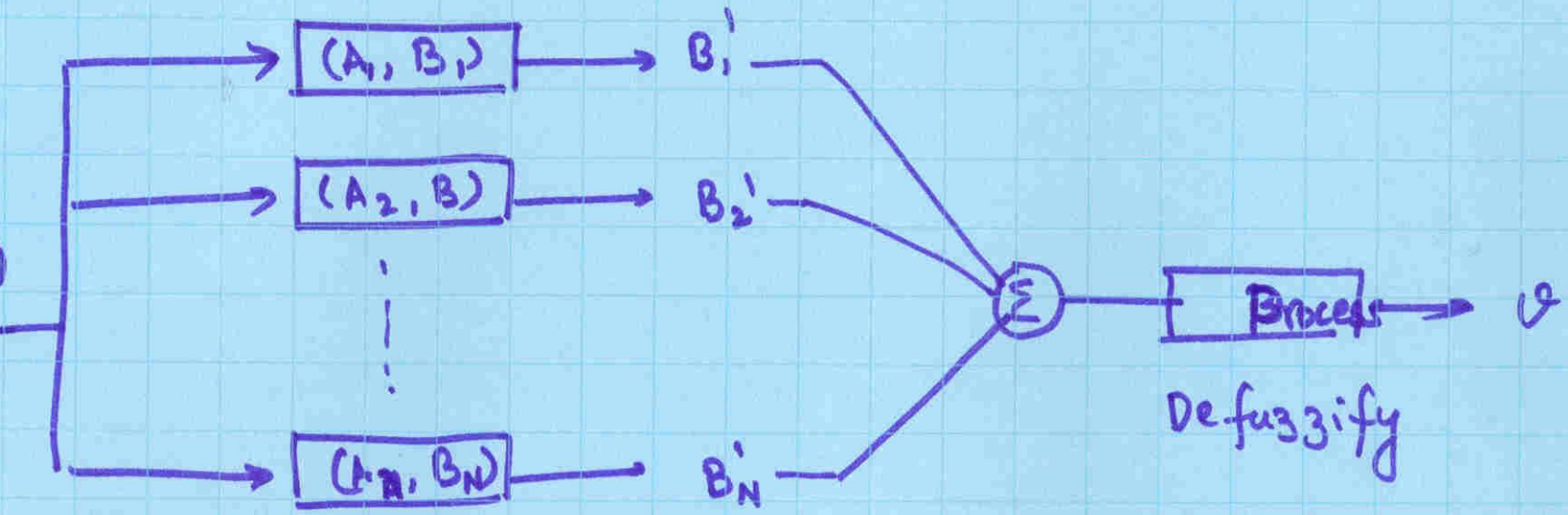
$\theta = 0.5°$

real value ~~$m_\theta(0.5)$~~ ↓ fuzzy encoded descriptor

$\theta = 0.5°$: $\{0, 0, 0, 0.75, 0, 0, 0\}$ fit vector

$\theta = 1.5°$: $\{0, 0, 0, 0.25, 0.25, 0, 0\}$

$\theta = \underline{7.5}°$: $\{0, 0, 0, 0, 0, 0.8, 0.3\}$



IAM RULES

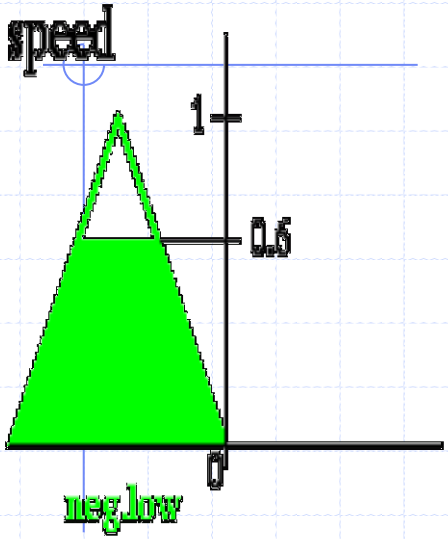
Fuzzified

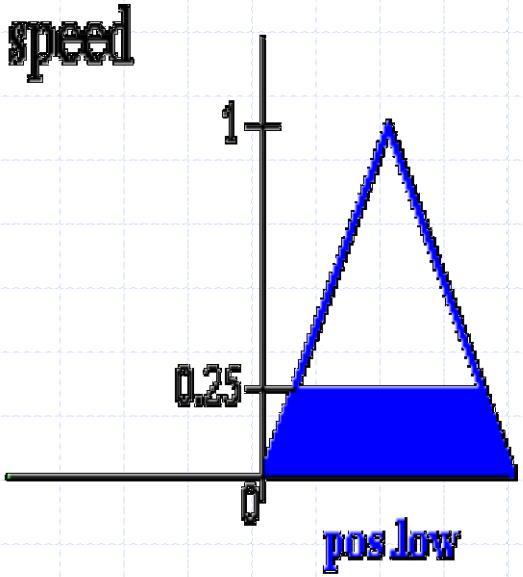
FUZZY SYSTEM

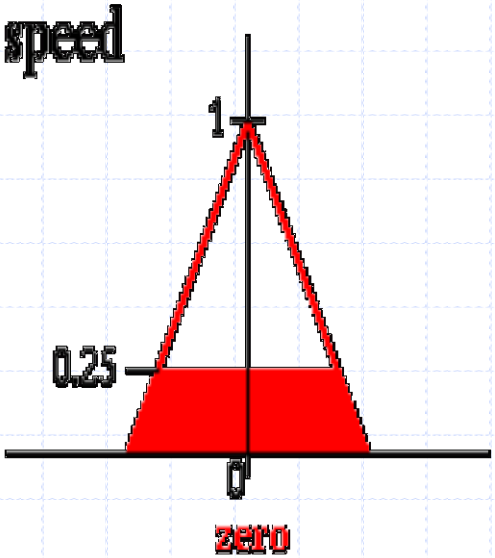
APPROXIMATE REASONING.

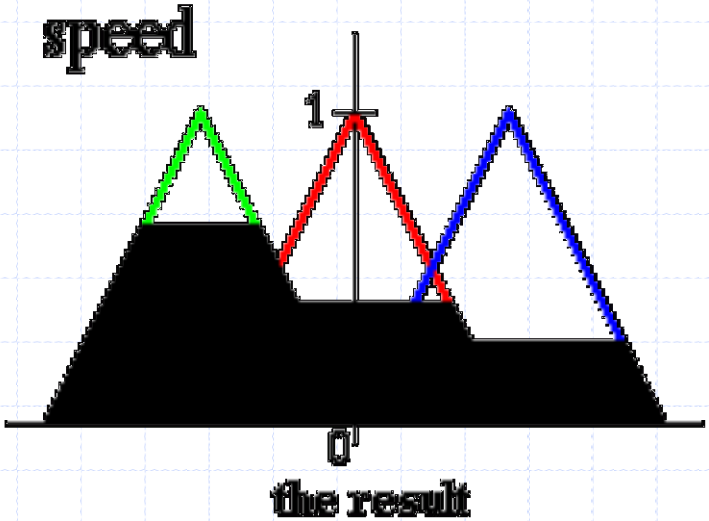
INTELLIGENT SYSTEMS.

Inverted Pendulum



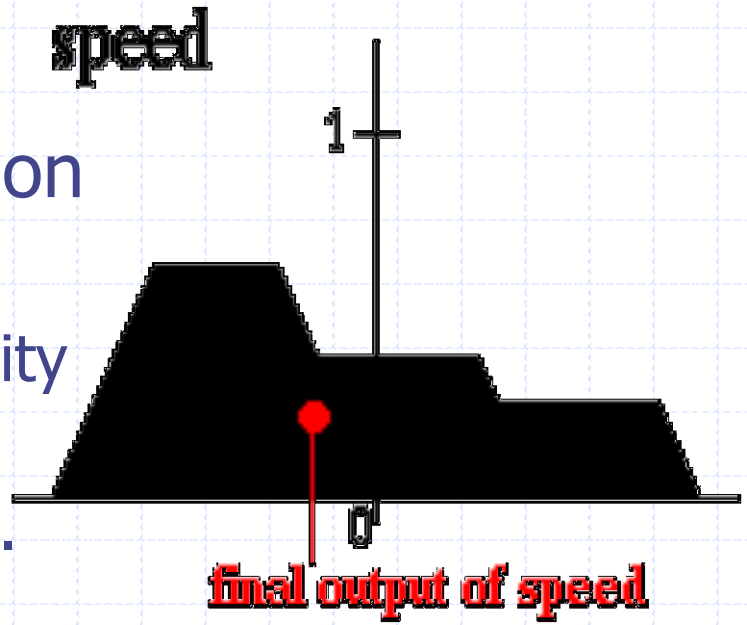






Inverted Pendulum

- Finally, the Defuzzification stage is implemented.
- Two ways of defuzzification is by
  - Finding the center of Gravity and
  - Finding the average mean.



Inverted Pendulum

- Online Demo

http://www.aptronix.com/fuzzynet/java/pend/pendjava.htm

More Applications

- Washing Machine
- Vehicle Control

- Additional Set of Slides

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Fuzzy knowledge based Control

Controller Design

Fuzzy Knowledge based Controller (FKBC)

Control Problem:

System: $f(\bar{x}, \dot{\bar{x}}, \bar{u}, \bar{z}) = 0$

$\bar{x}$ is the state vector $\dot{\bar{x}}$ derivative

$\bar{u}$ is the <u>control input</u> vector

$\bar{z}$ is the noise vector

output vector $\bar{y} = \bar{g}(\bar{x}, \bar{u})$

control $\bar{u} = \bar{h}(\bar{\omega}, \bar{y})$

$\bar{\omega}$ is the desired input.

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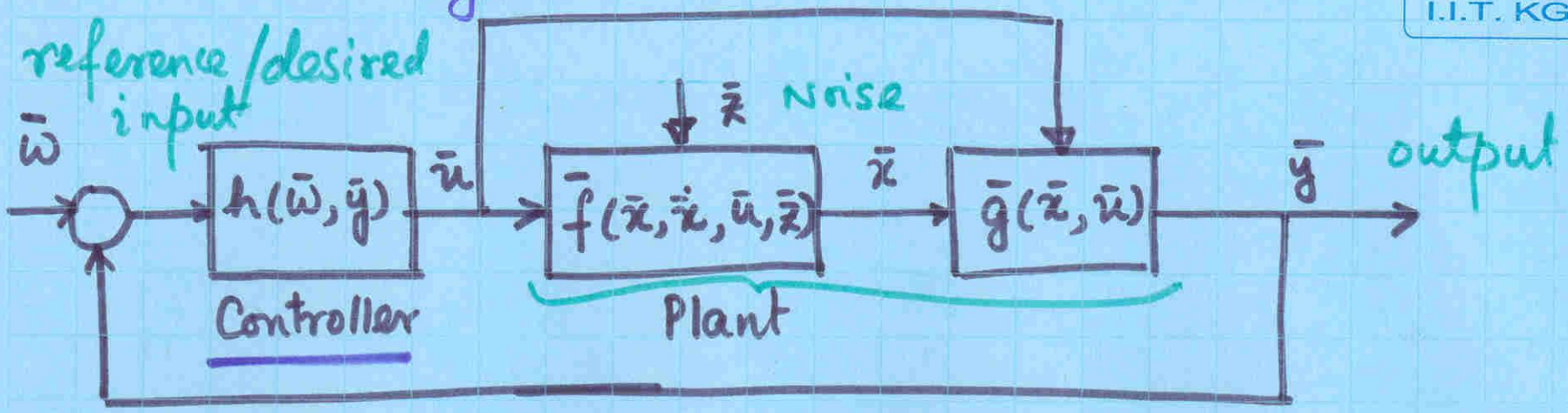
Control System



typically

$$\dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{B}\,\bar{u} + \bar{z}$$

$$\bar{y} = \bar{C}\,\bar{x} + \bar{D}\,\bar{u}$$

$$\underline{\bar{u} = \bar{h}(\bar{w}, \bar{y})}$$

Control input

$\bar{B}, \bar{C}, \bar{D}$ constant matrices

$\bar{f}(\bar{x})$ may be linear / non-linear fn

Conventional Approach

Crisp approach

PID Control

Phase-lead / Phase-lag Control

Full state feedback

LQG, LQR, Optimal Control

Fuzzy Knowledge Based Approach

Fuzzy Rules. - Approximate Reasoning

Noise

Unmodelled / unknown dynamics

FKBC: Fuzzy knowledge based Controller

following structural stage:

1. input scaling - normalization (0 to 1)
2. Fuzzification Crisp → Fuzzy sets or membership values
3. → use Rules - Inference or Rule firing
4. Defuzzification → crisp data
5. Denormalize → output scaling.

1, 5 → linear operation

* 2, 3, and 4 non-linear & work fuzziness.

PID Controller

Crisp Control vs FKB Control

$\bar{x} = f(\bar{x}, \dot{\bar{x}}, t, u)$ state eqn.

$\bar{y} = g(\bar{x})$ output eqn.

$e(t) = \bar{x}(t) - x_d(t)$ (error, ref input x_d)

Crisp Controller

error

↳ Proportional Integral Derivative Control.

Controller $u(t) = K_p e(t) + K_D \dot{e}(t) + \frac{K_I}{T} \int_0^t e(t)\,dt$

$K_p, K_D, K_I \rightarrow$ Proportional, Derivative and Integral Gain Term.

Fuzzy knowledge based control

Crisp

$$u_{PID} = k_p e + k_D \dot{e} + \frac{k_I}{T}\int_0^t e\,dt$$

$$u(t) = k_p \underline{e}(t) + k_D \underline{\dot{e}}(t) + k_I \underline{\delta}(t) \qquad \delta = \frac{1}{T}\int_0^t e\,dt$$

Cases:

$k_p e + k_D \dot{e} + k_I \delta > 0$ +ve output

or $k_p e + k_D \dot{e} + k_I \delta < 0$ -ve output

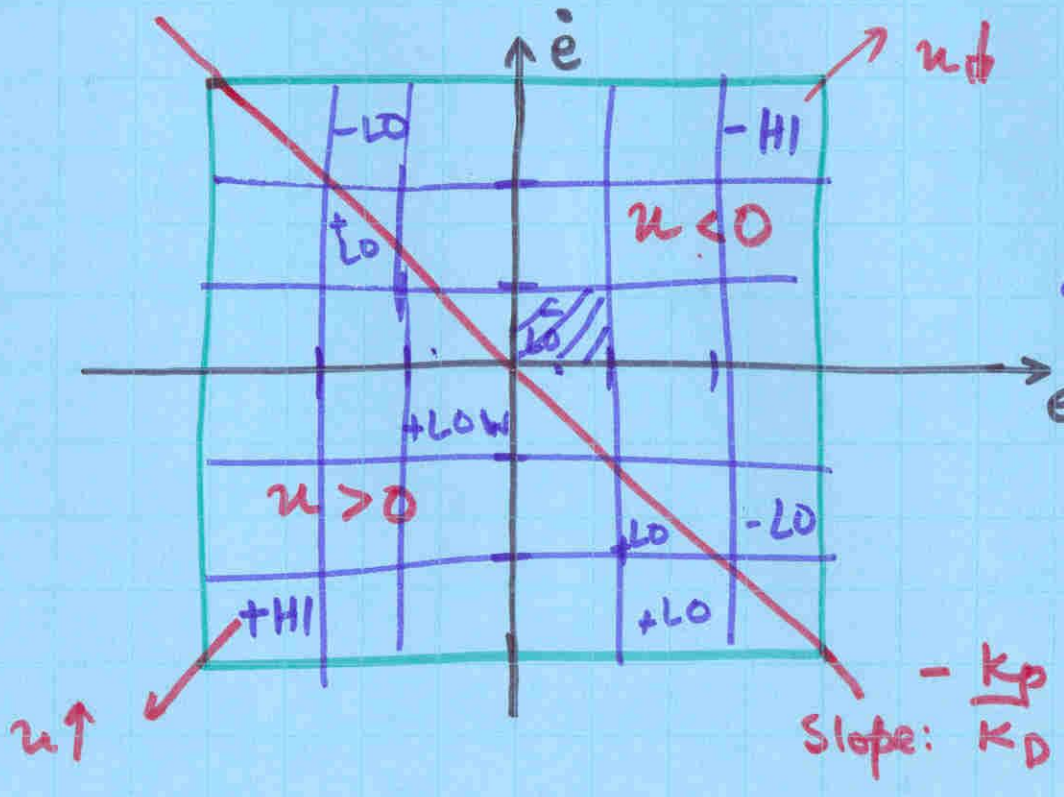
s : switching surface : $\dot{e} + \left(\frac{k_p}{k_D}\right) e + \left(\frac{k_I}{k_D}\right)\delta = 0$

s_N : e_N $\dot{e}_N$ δ_N

PD : Proportional Derivative Control

$$u_{PD} = k_p e + k_D \dot{e}$$

s : switching surface $\quad \dot{e} + \frac{k_p}{k_D} e = 0$



Line!

if $\underset{F.V}{e}$ is PL $\quad$ F.S.

and $\dot{e}$ is PL

then u is NL

Fuzzy Rule (Implication)

$e, \dot{e}, u$ - variables

PL, NL, PH, NH $\rightarrow$ Fuzzy sets.

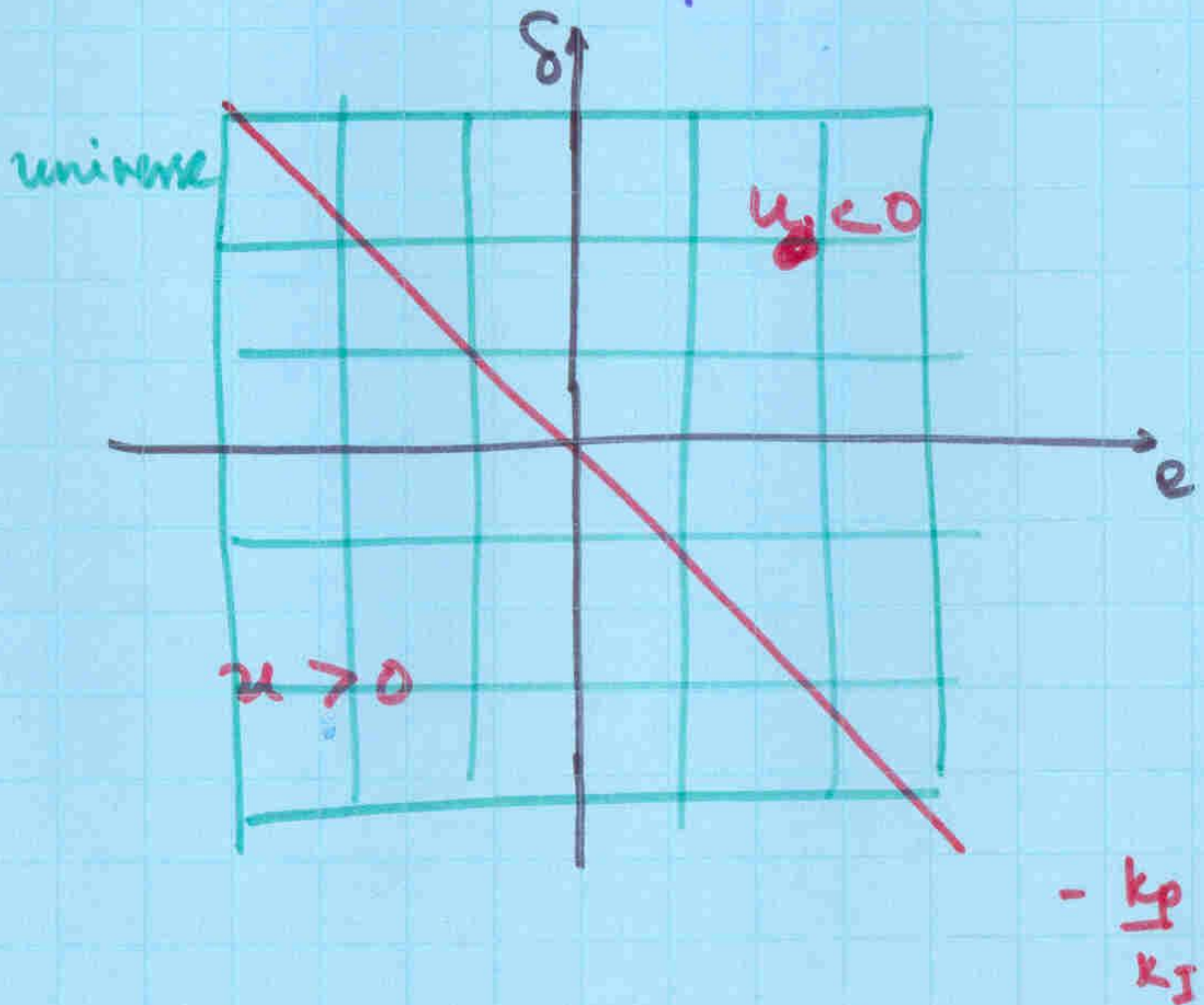
PI Controller

$$u_{PI} = k_p e + k_I \cdot \frac{1}{T} \int e\,dt$$

$$= k_p e + k_I \delta$$

$$k_p e + k_I \delta = 0$$

$$\delta + \frac{k_p}{k_I} e = 0$$



Rules can be set (Linguistic)

if $s: \quad \dot{e} + \left(\frac{K_P}{K_D}\right) e + \left(\frac{K_I}{K_D}\right) \delta = 0$

if ((e is PL) and ($\dot{e}$ is PL) and (δ is PL))

then (u is NL)

$e, \dot{e}, \delta \rightarrow$ PH, PM, PL, NL, NM, NH

No. of rules reqd. $6 \times 6 \times 6 = 216$.

More number fuzzy membership function

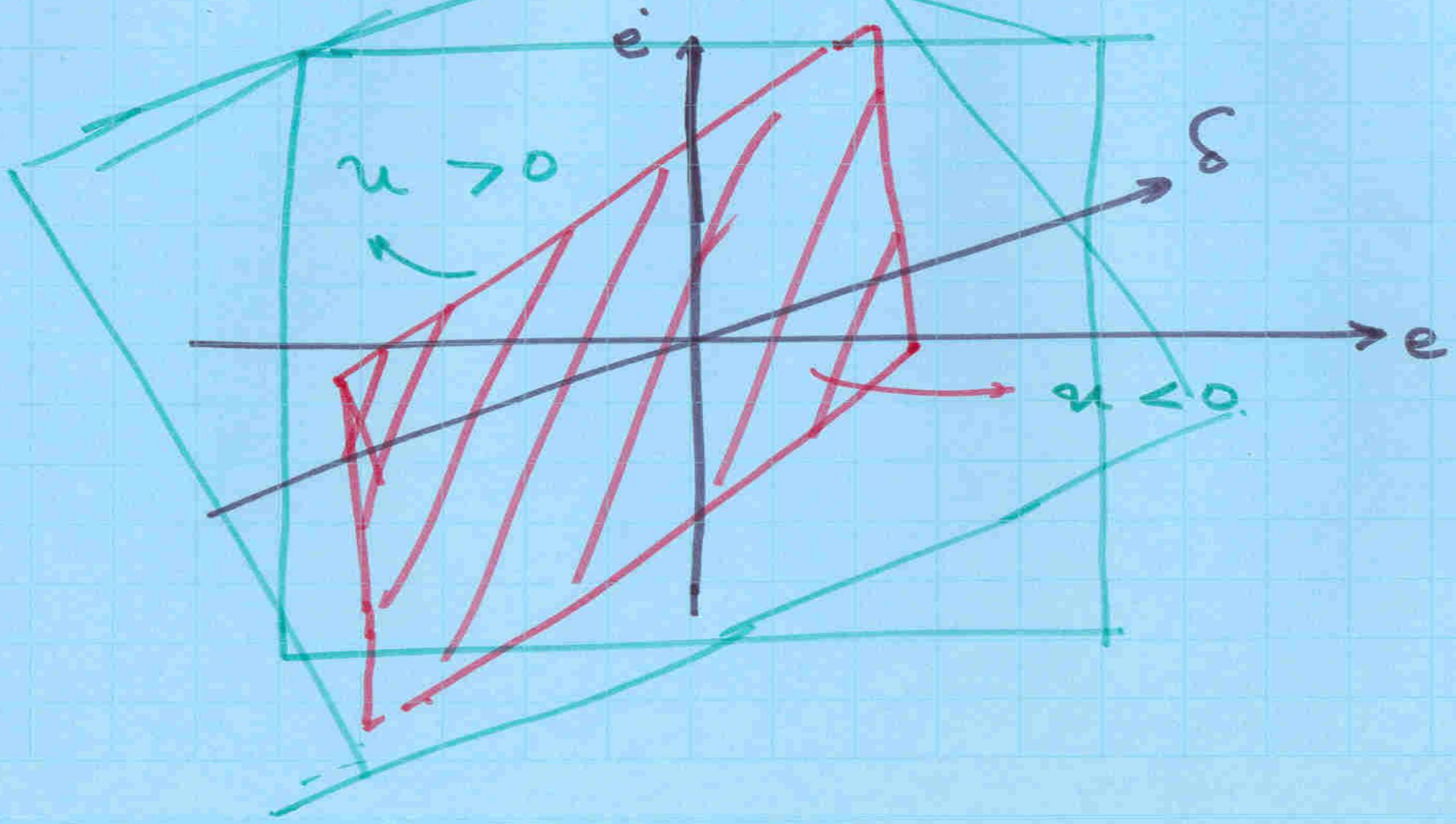
more rules reqd.

Memory requirement goes up.

$\rightarrow$ FUZZY ASSOCIATIVE MEMORY (FAM)

PID Control:

$$s: \quad \dot{e} + \left(\frac{K_P}{K_D}\right) e + \left(\frac{K_I}{K_D}\right) \delta = 0$$



- Large Number of rules required
  More memory

- Sequential application of rules/implications
  (rule firing) → may be parallel.
  is not mandatory

- missing / lost or rules not defined
  → some solution will be obtained
  graceful degradation. Intelligence!
  Robustness
  → approximate result exists & can be obtained

Examples / Applications

- Intelligent Automated Subways.
- Auto focus cameras, camcorders
- CTV tuning
- Control of Automobile transmission → CVT
- cruise controllers, emergency braking systems
  defrost refrigerators, a/c control, washing m/c.
- Robotic Guidance – autonomous vehicles
  underwater,
  Areal vehicles
- Finance stock markets:
  banking sector: Risk Assessment.

Thank you

For further details please contact

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Questions?