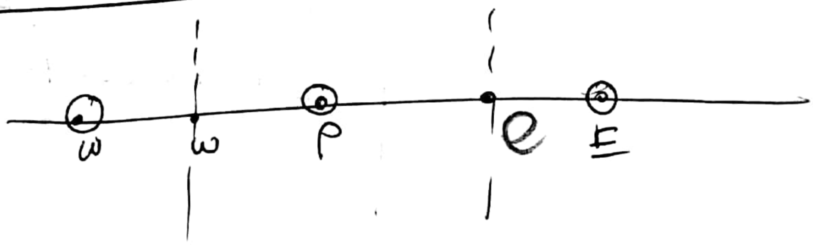
2/4/18 - Exam
Mon

4/4/18
Wed

Two dimensional Driven Cavity

Marker and cell (MAC) method stream function vorticity formulation . THE SIMPLE algorithm semi-implicit method for pressure-linked equations. — N.S equations.

One dimensional problem



$$\int_w^e -\frac{\partial p}{\partial x}\,dx = -p\Big|_w^e = p_w - p_e$$

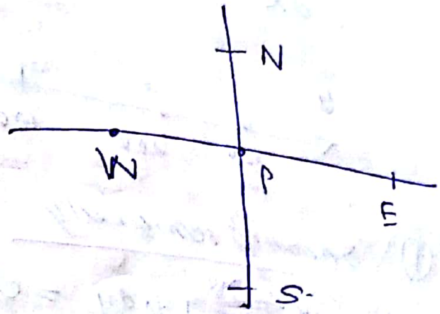
$$= \frac{p_W + p_P}{2} - \frac{p_P + p_E}{2}$$

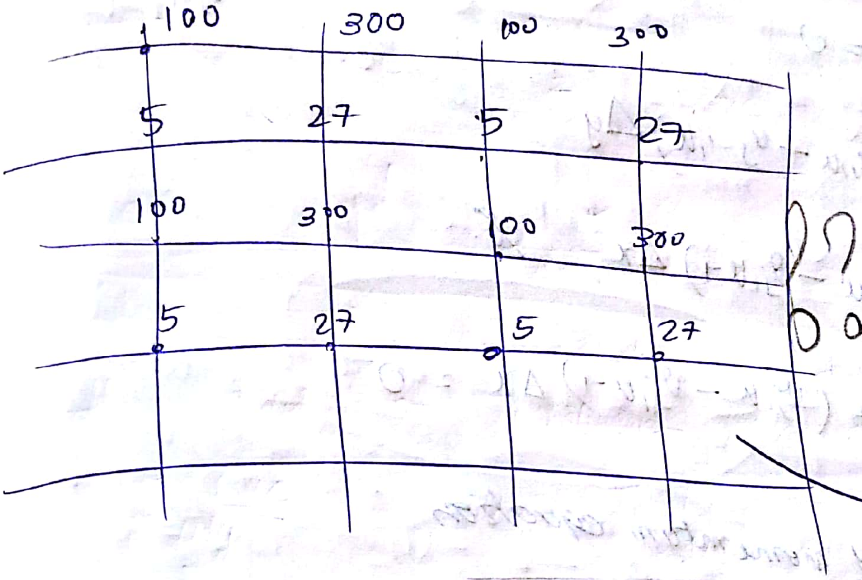
$$= \frac{p_W - p_E}{2}$$ two grid spaces apart (lower accuracy)

Two-dimensional problem

$$\iint -\frac{\partial p}{\partial x}\,\partial x\,\partial y = \left(\frac{P_W - P_E}{2}\right)\Delta y$$

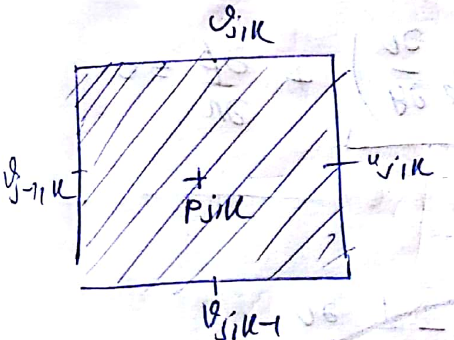
$$\iint -\frac{\partial p}{\partial y}\,\partial x\,\partial y = \left(\frac{P_S - P_N}{2}\right)\Delta x$$

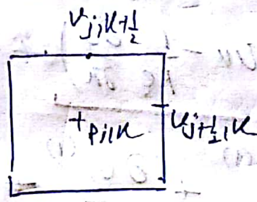




Checker board pressure field would have no effect on momentum equations

The staggerred Grid





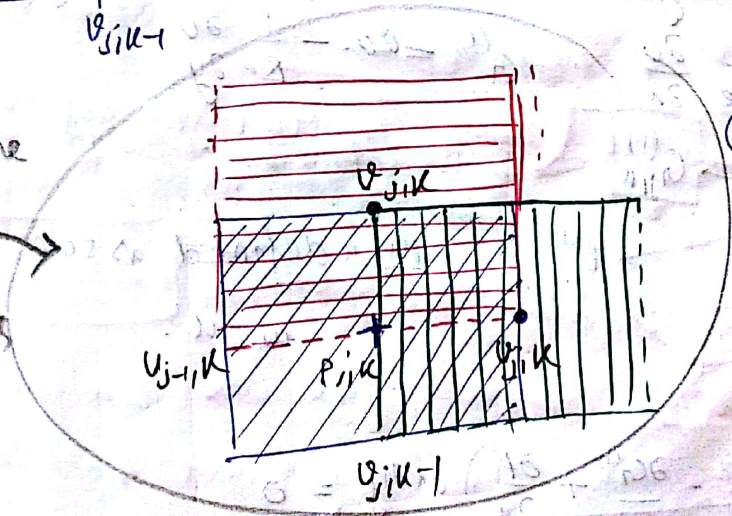
n.s eq^n

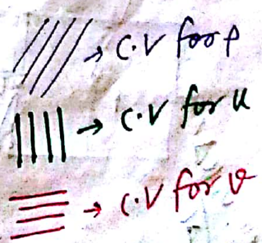
① for p

② for u

③ for v

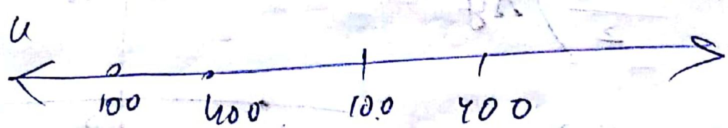
what is the use of this diff origins

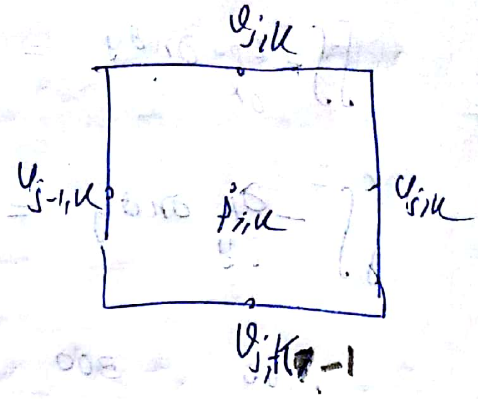




$$\frac{\partial p}{\partial x},\ \frac{\partial p}{\partial y},\ \frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y},$$

$$\int \frac{\partial u}{\partial x}\, \partial x = \frac{u_E - u_W}{2}$$





① incompressibility

$$\iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \partial x\, dy = 0$$

$$\iint \frac{\partial u}{\partial x}\, \partial x\, dy = (u_{j,k} - u_{j-1,k})\, \Delta y$$

$$\iint \frac{\partial v}{\partial y}\, dy\, dx = (v_{j,k} - v_{j,k-1})\, \Delta x \quad \cancel{v}$$

$$(u_{j,k} - u_{j-1,k})\,\Delta y + (v_{j,k} - v_{j,k-1})\,\Delta x = 0$$

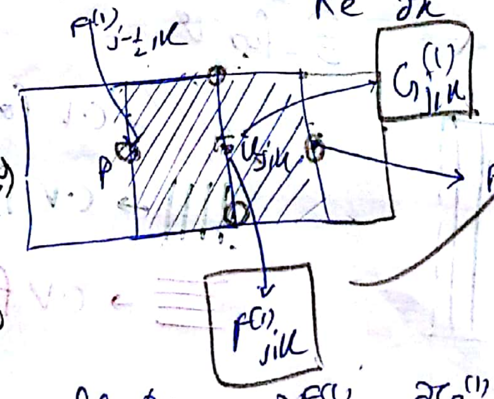
x component of momentum equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) + \frac{\partial p}{\partial x} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\underbrace{\left(uu - \frac{1}{Re}\frac{\partial u}{\partial x}\right)} + \frac{\partial}{\partial y}\underbrace{\left(vu - \frac{1}{Re}\frac{\partial u}{\partial y}\right)} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial F^{(1)}}{\partial x} + \frac{\partial G^{(1)}}{\partial y} + \frac{\partial p}{\partial x} = 0$$

$$F^{(1)} = uu - \frac{1}{Re}\frac{\partial u}{\partial x} \qquad G^{(1)} = vu - \frac{1}{Re}\frac{\partial u}{\partial y}$$



→ both defined at the pt. where u is defined

$F^{(1)}$ is defined as same point as u.

$$\frac{u^{n+1}_{j,k} - u^{n}_{j,k}}{\Delta t}\,\Delta x\,\Delta y$$

$$\iint \left(\frac{\partial u}{\partial t} + \frac{\partial F^{(1)}}{\partial x} + \frac{\partial G^{(1)}}{\partial y} + \frac{\partial p}{\partial x}\right) \partial x\, dy = 0$$

$$\left(\frac{\partial u}{\partial t}\right)_{j,k} \Delta x\,\Delta y + \left(F^{(1)}_{j+\frac{1}{2},k} - F^{(1)}_{j-\frac{1}{2},k}\right)\Delta y + \left(G_{j,k+\frac{1}{2}} - G_{j,k-\frac{1}{2}}\right)\Delta x$$

$$(p_{j+1,k} - p_{j,k})\,\Delta y$$

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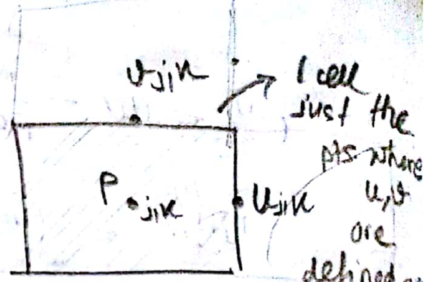


$$uu - \frac{1}{Re}\frac{\partial u}{\partial x^2} \qquad uv - \frac{1}{Re}\frac{\partial v}{\partial y^2}$$

$$\iint \left(\frac{\partial u}{\partial t} + \frac{\partial F^{(1)}}{\partial x} + \frac{\partial G^{(1)}}{\partial y} + \frac{\partial p}{\partial x} \right) \partial x \, \partial y = 0$$

$$\frac{u_{j,k}^{n+1} - u_{j,k}^{n}}{\Delta t} \Delta x \Delta y + \left(F^{(1)}_{j+\frac{1}{2},k} - F^{(1)}_{j-\frac{1}{2},k} \right) \Delta y + \left(G^{(1)}_{j,k+\frac{1}{2}} - G^{(1)}_{j,k-\frac{1}{2}} \right) \Delta x$$

$$+ \left(p_{j+1,k}^{n+1} - p_{j,k}^{n} \right) \Delta y = 0$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^{u}_{j,k} \right) u_{j,k}^{n+1} + \sum_{nb} a^{u}_{nb} u_{nb}^{n+1} + b^{u} + \left(p^{n+1}_{j+1,k} - p^{n+1}_{j,k} \right) \Delta y = 0 \qquad \text{——— (1)}$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^{v}_{j,k} \right) v_{j,k}^{n+1} + \sum_{nb} a_{nb} v_{nb}^{n+1} + b^{v} + \left(p^{n+1}_{j,k+1} - p^{n+1}_{j,k} \right) \Delta x = 0 \qquad \text{——— (2)}$$

$\left.\begin{matrix} a^{u}_{j,k},\ a^{u}_{nb},\ b^{u} \\ a^{v}_{j,k},\ a^{v}_{nb},\ b^{v} \end{matrix}\right\}$ are functions of u^n, v^n

$$\left(u_{j,k}^{n+1} - u_{j-1,k}^{n+1} \right) \Delta y + \left(v_{j,k}^{n+1} - v_{j,k-1}^{n+1} \right) \Delta x = 0 \qquad \text{——— (3)}$$

Use p^n instead of p^{n+1} in Eqns (1) and (2). The two equations can be solved by a procedure similar to ADI. The solution will give us u^* and v^* and not u^{n+1}, v^{n+1}. u^* and v^* will not satisfy eq (3). A ~~corre~~ correction in the pressure has to be introduced which will give rise to a change in velocity and comp the requirement that this corrected velocity must satisfy. the cond? of incompressibility

$$p^{n+1} = p^n + \delta p$$

$$u^{n+1} = u^* + u^c$$

$$v^{n+1} = v^* + v^c$$

These remain constant

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{j,i,k}\right) u^*_{j,i,k} + \sum_{nb} a^u_{nb} u^*_{nb} + \boxed{b^u} + (p^n_{j+1,i,k} - p^n_{j,i,k}) \cdot \Delta y = 0$$

———— (4)

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{j,i,k}\right) v^*_{j,i,k} + \sum_{nb} a^v_{nb} v^*_{nb} + \boxed{b^v} + (p^n_{j,i,k+1} - p^n_{j,i,k}) \Delta x = 0$$

———— (5)

Eq (4) − Eq (1)

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{j,i,k}\right) u^c_{j,i,k} + \sum_{nb} a^u_{nb} u^c_{nb} + (\delta p_{j+1,i,k} - \delta p_{j,i,k}) \Delta y = 0$$

———— (6)

Eq (5) − Eq (2)

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{j,i,k}\right) v^c_{j,i,k} + \sum_{nb} a^v_{nb} v^c_{nb} + (\delta p_{j,i,k+1} - \delta p_{j,i,k}) \Delta x = 0$$

———— (7)

incompressibility eqn

$$u^{n+1} = u^* + u^c, \quad v^{n+1} = v^* + v^c$$

$$(u^*_{j,i,k} + u^c_{j,i,k} - u^*_{j-1,i,k} - u^c_{j-1,i,k}) \Delta y + (v^*_{j,i,k} + v^c_{j,i,k} - v^*_{j,i,k-1} - v^c_{j,i,k-1}) \Delta x = 0$$

$$(u^c_{j,i,k} - u^c_{j-1,i,k}) \Delta y + (v^c_{j,i,k} - v^c_{j,i,k-1}) \Delta x = -(u^*_{j,i,k} - u^*_{j-1,i,k}) \Delta y - (v^*_{j,i,k} - v^*_{j,i,k-1}) \Delta x$$

In the simple Algorithm (SIMPLE) the effect of the neighbouring points is dropped.

??

$$\sum a^u_{nb} u^c_{nb} \approx 0$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{j,i,k}\right) u^c_{j,i,k} = -(\delta p_{j+1,i,k} - \delta p_{j,i,k}) \Delta y$$

$$\Rightarrow \left(1 + \underbrace{\frac{\Delta t}{\Delta x \Delta y}}_{E} a^u_{j,i,k}\right) u^c_{j,i,k} = -\underbrace{\frac{\Delta t}{\Delta x \Delta y}}_{E} (\delta p_{j+1,i,k} - \delta p_{j,i,k}) \Delta y$$

$\Rightarrow$

$$\Rightarrow u^c_{j,i,k} = \frac{-E^u \Delta y}{(1+E^u)\, a^u_{j,i,k}} \left(\cancel{\delta_{P_{j+1,i,k}}} - \delta_{P_{j,i,k}} \right)$$

$$= \frac{E^u \Delta y}{(1+E^u)\, a^u_{j,i,k}} \left(\delta_{P_{j,i,k}} - \delta_{P_{j+1,i,k}} \right)$$

$$\boxed{U^c_{j,i,k} = d^u_{j,i,k} \left(\delta_{P_{j,i,k}} - \delta_{P_{j+1,i,k}} \right)}$$

→ On dropping the coupling with neighbouring cells/grids

$$u^c_{j,i,k} = \frac{E^u \Delta y}{(1+E^u)\, a^u_{j,i,k}} \left(\delta p_{j,i,k} - \delta p_{j+1,i,k} \right) = d^u_{j,i,k} \left(\delta_{P_{j,i,k}} - \delta_{P_{j+1,i,k}} \right)$$

where

$$d^u_{j,i,k} = \frac{E^u}{1+E^u} \frac{\Delta y}{a^u_{j,i,k}}$$

Similarly,

$$\boxed{v^c_{j,i,k} = d^v_{j,i,k} \left(\delta_{P_{j,i,k}} - \delta_{P_{j,i,k+1}} \right)}$$

$$\left[d^u_{j,i,k} (\delta p_{j,i,k} - \delta p_{j+1,i,k}) - d^u_{j-1,i,k} (\delta p_{j-1,i,k} - \delta p_{j,i,k}) \right] \Delta y$$

$$+ \left[d^v_{j,i,k} (\delta p_{j,i,k} - \delta p_{j,i,k+1}) - d^v_{j,i,k-1} (\delta p_{j,i,k-1} - \delta p_{j,i,k}) \right] \Delta x = 0$$

$$= -(u^*_{j,i,k} - u^*_{j-1,i,k}) \cdot \Delta y - (v^*_{j,i,k} - v^*_{j-1,i,k}) \, \Delta x \quad \text{—— (8)}$$

This is a Poisson equation for δp

Algorithm for solving Poisson equation

1) u^*, v^* are obtained from Eq (4) and Eq (5)

2) δp is obtained from eq (8)

3) u^c and v^c are obtained from Eq (6) and (7)

4) p^{n+1} is obtained from $p^{n+1} = p^n + \alpha_p \delta p$

α_p is a relaxation parameter

Wed 11/4/18

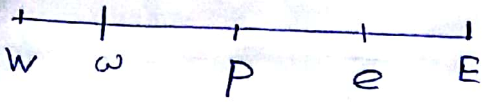
Conservative vs Non-conservative formulation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(u\phi) - \frac{\partial}{\partial x}\left(T\frac{\partial \phi}{\partial x}\right) = 0$$

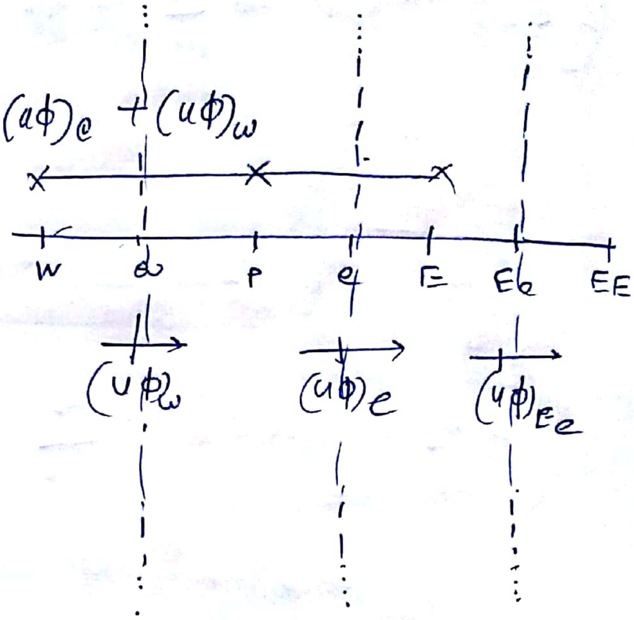
velocity diffusivity

from this equation

$$\int_w^e \frac{\partial \phi}{\partial t}dx = -\int \frac{\partial}{\partial x}(u\phi)\,dx$$



$$-\int_w^e \frac{\partial}{\partial x}(u\phi)\,dx = -(u\phi)\Big|_w^e = -(u\phi)_e + (u\phi)_w$$



Convection for cell around P

$$= -(u\phi)_e + (u\phi)_w$$

Convection for cell around E

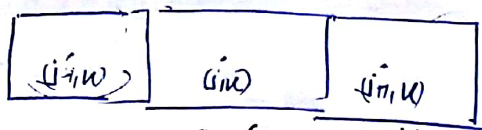
$$= -(u\phi)_{eE} + (u\phi)_e$$

The momentum equation

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y}$$

Finite difference approximation at (j,k).

1. $$\left(\rho u\frac{\partial u}{\partial x}\right)_{j,k} = \rho u_{j,k}\frac{u_{j+1,k} - u_{j-1,k}}{2\Delta x}$$



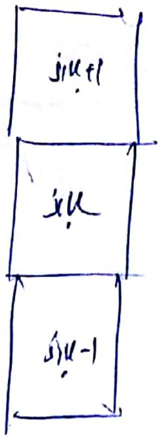
So, Conservative form

$u_{j-1,k} \times (\cancel{u_{j,k}} - u_{j-2,k})$ $u_{j,k} \times (\cancel{u_{j+1,k}} - \cancel{u_{j-1,k}})$ $u_{j+1,k}(u_{j+2,k} - \cancel{u_{j,k}})$

If we add up terms cancel

2. $$\left(\rho v\frac{\partial u}{\partial y}\right)_{j,k} = \rho . v_{j,k}\frac{u_{j+1,k} - u_{j-1,k}}{2\Delta y}$$

So, Non-conservative form



~~$v_{j,k+1}(u_{j+1,k} - u_{j-1,k})$~~ $v_{j,k+1} \times (u_{j,k+2} - u_{j,k})$

$v_{j,k} \times (u_{j,k+1} - u_{j,k-1})$

$v_{j,k-1} \times (u_{j,k} - u_{j,k-2})$

here there is no chance of terms cancelling

Similarly, $\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y}$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u)$$

$$\left(\frac{\partial}{\partial x}(\rho u u)_{j,k}\right) = \rho \frac{u_{j+1,k}\, u_{j+1,k} - u_{j-1,k}\, u_{j-1,k}}{2\Delta x}$$

$$\left(\frac{\partial}{\partial y}(\rho v u)_{j,k}\right) = \rho \frac{v_{j,k+1} u_{j,k+1} - v_{j,k-1} u_{j,k-1}}{2\Delta y}$$

j-2,k	j-1,k	j,k	j+1,k	j+2,k	

$$\cancel{u_{j,k}}\; u_{j-1,k}\, \cancel{u_{j-1,k}} + \cancel{u_{j,k} u_{j,k}} + \cancel{u_{j+1,k} u_{j+1,k}} + u_{j+2,k} u_{j+2} + u_{j+3,k} u_{j+3,k}$$

$$- u_{j-3,k} u_{j-3,k} - u_{j-2,k} u_{j-2,k} - \cancel{u_{j-1,k} u_{j-1,k}} - \cancel{u_{j,k} u_{j,k}} - \cancel{u_{j+1,k} u_{j+1,k}}$$

Now, $\left(\frac{\partial}{\partial y}(\rho v u)_{j,k}\right) = \rho \frac{v_{j,k+1} \cdot u_{j,k+1} - v_{j,k-1} u_{j,k-1}}{2\Delta y}$

j,k+1
j,k
j,k-1
j,k-2

$v_{j,k+3} \cdot u_{j,k+3} - \cancel{v_{j,k+1} u_{j,k+1}}$

$v_{j,k+2}\, u_{j,k+2} - v_{j,k} u_{j,k}$

$\cancel{v_{j,k+1} u_{j,k+1}} - \cancel{v_{j,k-1} u_{j,k-1}}$

$v_{j,k} u_{j,k} - v_{j,k-2} u_{j,k-2}$

$\cancel{v_{j,k-1} u_{j,k-1}} - v_{j,k-3} u_{j,k-3}$.

non conservative

$$\rho \frac{\partial \phi}{\partial t} + \rho \vec{u} \cdot \nabla \phi$$

Conservative

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho \vec{u} \phi)$$