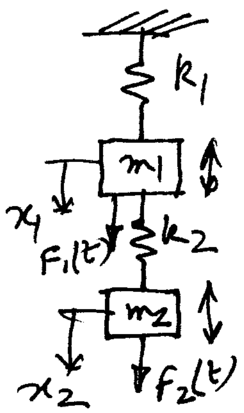
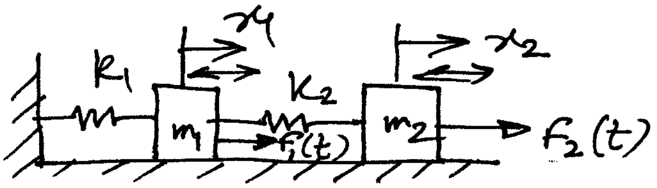
Two DOF systems :- (Undamped Systems)

The translational model we use is as follows :-



OR

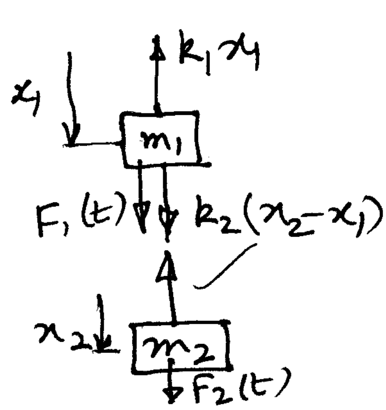


$x_1(t)$ & $x_2(t)$ are the generalized coordinates, measured from static equilibrium positions of the centres of mass of the blocks, +ive downward or, +ive towards the right. These senses could be reversed also.

→ Like the single DOF case, the static forces in the springs for the vertical system are, at all times, balanced by the forces of gravity & hence these are not shown in the subsequent FBDs. (Verify this)

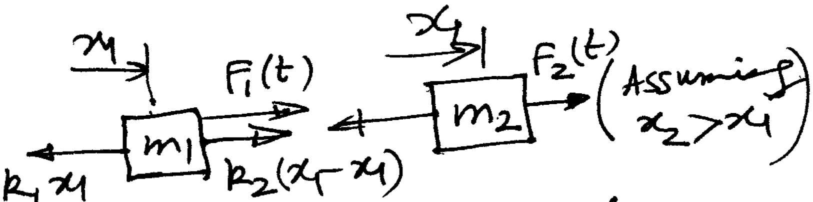
→ Derivation of the DEOM :-

(i) Newton's method :-



Relevant FBDs at time t

OR



So, by Newton's 2nd law,

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2-x_1) + F_1(t)$$

$$\& \quad m_2\ddot{x}_2 = -k_2(x_2-x_1) + F_2(t)$$

$$\Rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = F_1(t) \quad (1)$$

$$\& \quad m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F_2(t) \quad (2)$$

1 & 2 are the required DEOM. Thus, there are two DEOM for the 2 DOF system, each being a second order ordinary differential equation with constant coefficients.

→ Note that 1 & 2 can be written in the matrix form as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \quad --- (3)$$

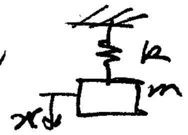
or, as: $$[m]\{\ddot{x}\} + [k]\{x\} = \{F(t)\} \quad --- (4)$$

where $[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is the mass matrix or the inertia matrix, $[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ is the stiffness matrix or elastic matrix, $\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$ is the acceleration vector, $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ is the displacement vector and $\{F(t)\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$ is the force vector. Note that $[m]$ & $[k]$ are symmetric, $[m]$ is diagonal.

(5) <u>Free-vibration of undamped 2-DOF systems:~</u> for free-vibration response, we have to consider the homogeneous DEOM (i.e., we set $F_1(t)=0$, $F_2(t)=0$)

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \quad -(i)$$
$$\& \quad m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \quad --(ii)$$

x_2 & x_1 both appear in each DEOM & so they must be solved simultaneously

There are formal ways of solving (i) & (ii) <u>simultaneously</u>, like the Laplace Transform method. However, we shall obtain $x_1(t)$ & $x_2(t)$ heuristically, that is, we shall use our experience with the single DOF undamped system 

whose free vibration response is given as $x = X \sin(\omega_n t + \phi)$.

$\rightarrow$ So, here also, we assume that m_1 executes harmonic oscillations & take

$x_1(t) = X_1 \sin(\omega t + \phi)$ --- (iii). Then, $\ddot{x}_1 = -X_1 \omega^2 \sin(\omega t + \phi)$.

Substituting these in (i), we get,

$$-m_1 X_1 \omega^2 \sin(\omega t + \phi) + (k_1 + k_2) X_1 \sin(\omega t + \phi) = k_2 x_2$$

or, $$x_2 = \left[\frac{(k_1 + k_2) - m_1\omega^2}{k_2}\right] X_1 \sin(\omega t + \phi)$$

$$= X_2 \sin(\omega t + \phi) \text{ --- (iv)}$$

From (iii) & (iv), we conclude that if m_1 executes simple harmonic free vibrations, m_2 does the same with same frequency & no difference in phase but with a different amplitude (unless $(k_1 + k_2 - m_1\omega^2)/k_2 = 1$).

So far, we have not reached any contradiction anywhere & so, we go ahead with the assumptions:

$$x_1 = X_1 \sin(\omega t + \phi) \text{ -- (iii)}$$
$$x_2 = X_2 \sin(\omega t + \phi) \text{ -- (iv)}$$
$$\Rightarrow \begin{cases} \ddot{x}_1 = -\omega^2 X_1 \sin(\omega t + \phi) \text{ -- (v)} \\ \ddot{x}_2 = -\omega^2 X_2 \sin(\omega t + \phi) \text{ -- (vi)} \end{cases}$$

Substituting these in (i) & (ii), we get:

$$\left[(k_1 + k_2 - m_1\omega^2) X_1 - k_2 X_2\right] \sin(\omega t + \phi) = 0 \text{ -- (vii)}$$

& $$\left[-k_2 X_1 + (k_2 - m_2\omega^2) X_2\right] \sin(\omega t + \phi) = 0 \text{ -- (viii)}$$

$\longrightarrow$

Since $\sin(\omega t+\phi) \neq 0$ at all times, for (vii) & (viii) to be true, we must have

$$(k_1+k_2-m_1\omega^2)X_1 - k_2X_2 = 0 \quad \text{--- (ix)}$$

& $$-k_2X_1 + (k_2-m_2\omega^2)X_2 = 0 \quad \text{--- (x)}$$

Solving (ix) & (x), we should be able to find X_1 & X_2, the amplitudes of m_1 & m_2 for free vibration.

Note that $X_1=0$ & $X_2=0$ satisfy (ix) & (x). But then $x_1=0$ & $x_2=0$ at all times and we are not interested in these. For (ix) & (x) to have non-zero solution, we must have

$$\begin{vmatrix} (k_1+k_2-m_1\omega^2) & -k_2 \\ -k_2 & (k_2-m_2\omega^2) \end{vmatrix} = 0$$

[(ix) & (x) are like
$a_{11}x_1 + a_{12}x_2 = 0$
& $a_{21}x_1 + a_{22}x_2 = 0$
So, $x_2/x_1 = -\frac{a_{11}}{a_{12}} = -\frac{a_{21}}{a_{22}}$
So, $a_{11}a_{22} - a_{21}a_{12} = 0$
or, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$]

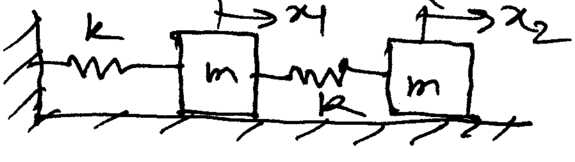
The determinant on the LHS is called the <u>characteristic determinant</u> of the system and the above equation, when expanded, gives the <u>characteristic eqn</u> or the <u>frequency eqn</u> of the system.

Hence, the frequency equation of our system is:

$$(k_1+k_2-m_1\omega^2)(k_2-m_2\omega^2) - k_2^2 = 0 \quad \text{--- (xi)}$$

It's solution gives the natural frequencies ω_1 & ω_2.

→ Let us take a specific example problem:



So here $k_1 = k_2 = k$
& $m_1 = m_2 = m$

Hence, the DEOM of this system is:
(You derive it separately)

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad \text{--- ⓐ}$$

$$\& \quad m\ddot{x}_2 - kx_1 + kx_2 = 0 \quad \text{--- ⓑ}$$

(For free-vibration)

Let $x_1 = X_1 \sin(\omega t + \phi)$ --- ⓒ

& $x_2 = X_2 \sin(\omega t + \phi)$ --- ⓓ

Then, from ⓐ & ⓑ, we get

$$(2k - m\omega^2)X_1 - kX_2 = 0 \quad \text{--- ⓔ}$$

$$\& \quad -kX_1 + (k - m\omega^2)X_2 = 0 \quad \text{--- ⓕ}$$

For non-trivial X_1 & X_2, we must have:

$$\begin{vmatrix} (2k - m\omega^2) & -k \\ -k & (k - m\omega^2) \end{vmatrix} = 0$$

$$\Rightarrow (2k - m\omega^2)(k - m\omega^2) - k^2 = 0$$

$$\Rightarrow 2k^2 - 3km\omega^2 + m^2\omega^4 - k^2 = 0$$

$$\Rightarrow m^2\omega^4 - 3km\omega^2 + k^2 = 0 \quad \text{--- ⓖ}$$

← A quadratic in ω^2, note

This is the frequency equation

Hence, $\omega^2 = \dfrac{3km \pm \sqrt{9k^2m^2 - 4k^2m^2}}{2m^2} = \dfrac{3km \pm \sqrt{5}\,km}{2m^2}$

Let $\omega_1^2 = \dfrac{(3 - \sqrt{5})km}{2m^2} = \left(\dfrac{3 - \sqrt{5}}{2}\right)\dfrac{k}{m}$

& $\omega_2^2 = \left(\dfrac{3 + \sqrt{5}}{2}\right)\dfrac{k}{m}$.

The positive square roots of these are:

$\omega_1 = 0.618\sqrt{\dfrac{k}{m}}$ → The fundamental or first natural frequency

& $\omega_2 = 1.618\sqrt{\dfrac{k}{m}}$ → The second natural frequency

<u>IMPORTANT</u>:~ If you are using your calculator

to get ω_1^2 & ω_2^2, it may so happen ~~that~~ at it gives ω_2^2 first & then ω_1^2 and so, you might mistakenly designate as ω_2 what is actually ω_1! Don't do this. Remember that the smaller of the natural frequencies here is ω_1 & the larger one is ω_2.

So, Never do this:– $\omega_1 = 1.618\sqrt{\frac{k}{m}}$, $\omega_2 = 0.618\sqrt{\frac{k}{m}}$ | This is wrong!

Designating these wrongly create problems which may not be obvious to you at this moment

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→ You can now see that our 2-DOF system has <u>two</u> natural frequencies, which are unequal.

Some special systems <u>may</u> have $\omega_1 = \omega_2$. Some systems may even have $\omega_1 = 0$, $\omega_2 > 0$. We shall illustrate these later.

→ After ω_1 & ω_2 are obtained, we get back to the <u>amplitude equations</u> [(ix) & (x) on page 4 and (e) & (f), page 5] to get information about the amplitudes X_1 & X_2. Note that these equations are valid for both ω_1 & ω_2. We take $\omega = \omega_1$ first.

Equation (e) becomes: $(2k - m\omega_1^2)X_1 - kX_2 = 0$

" (f) " $-kX_1 + (k - m\omega_1^2)X_2 = 0$

→ We now introduce a special notation for the amplitudes. We shall designate X_1 as X_{11} & X_2 as X_{21} where the second subscripts 1 means these are amplitudes corresponding to ω_1.

So, $(2k - m\omega_1^2) X_{11} - k X_{21} = 0$ --- (h)

& $-k X_{11} + (k - m\omega_1^2) X_{21} = 0$ -- (i)

Note that you cannot find any unique values of X_{11} & X_{21} from (h) & (i). There is infinitely many solutions & all we can do is obtain the ratio $\frac{X_{21}}{X_{11}}$. We can use either (h) or (i) because both will give the same ratio.

Using (h), $\frac{X_{21}}{X_{11}} = \frac{2k - m\omega_1^2}{k} = \frac{2k - 0.3819k}{k} = 1.618$

" (i), $\frac{X_{21}}{X_{11}} = \frac{k}{k - m\omega_1^2} = \frac{k}{k - 0.3819k} = 1.618$

So, you use either of the amplitude equations find $\frac{X_{21}}{X_{11}}$.

This amplitude ratio is often denoted as μ_1, that is, $\mu_1 = \frac{X_{21}}{X_{11}} = 1.618$

→ So, when $\omega = \omega_1$, $X_{21} = \mu_1 X_{11} = 1.618 X_1$ and the masses have the following response:-

$$\left.\begin{aligned} x_1 &= X_{11} \sin(\omega_1 t + \phi_1) \\ \& \quad x_2 &= X_{21} \sin(\omega_1 t + \phi_1) \end{aligned}\right\} \phi_1 = \text{value of } \phi \text{ for } \omega = \omega_1$$

OR

$$x_1 = X_{11} \sin(\omega_1 t + \phi_1)$$

$$\& \quad x_2 = \mu_1 X_{11} \sin(\omega_1 t + \phi_1) = 1.618 X_{11} \sin(\omega_1 t + \phi_1)$$

When this motion occurs, we say that the system is executing the first principal mode or first normal mode of vibration.

→ We now take $\omega = \omega_2$. With this, we have

$$(2K - m\omega_2^2)X_{12} - KX_{22} = 0$$

& $$-KX_{12} + (K - m\omega_1^2)X_{22} = 0$$

← Carefully note the notations for amplitudes.

& proceeding as before, we shall get

$$\mu_2 = \frac{X_{22}}{X_{12}} = -0.618,$$

where μ_2 is the amplitude ratio corresponding to the second principal mode of vibration.

Here X_{22} = Amplitude of m_2 corresponding to the 2nd principal mode &

X_{12} = Amplitude of m_1 corresponding to the 2nd principal mode.

→ The motion corresponding to the 2nd principal mode are:

$$x_1(t) = X_{12}\sin(\omega_2 t + \phi_2)$$

& $$x_2(t) = X_{22}\sin(\omega_2 t + \phi_2) = \mu_2 X_{12}\sin(\omega_2 t + \phi_2)$$

where ϕ becomes ϕ_2 for the 2nd pr. mode.

→ So, remember the following:-

X_{11} = Amplitude of m_1 corr. to 1st pr. mode.

X_{21} = " " m_2 " " " " "

X_{12} = Amplitude of m_1 corr. to 2nd pr. mode

X_{22} = " " m_2 " " " " "