Mechanisms and Robot Kinematics

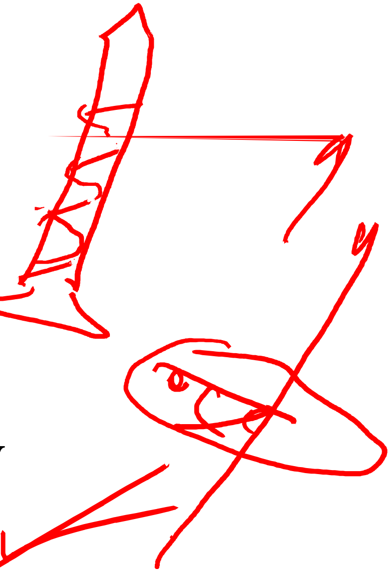
Rigid Body Motion

Screw Theory and Linear Algebra

Rigid Body Motion

axis of rot^n

translation axis



- Screw Theory developed in early 1800s by Chasles and Poinsot.
- Rigid Body can be moved from one position to another by a movement consisting of a rotation and a straight line followed by translation parallel to that line -> ***Screw Motion*** Gross Motion
- Infinitesimal version of screw motion is a ***twist*** – provides a description of the instantaneous velocity of a rigid body in terms of its linear and angular components.

Δtrans Drctn

Linear Force

Rotational ~~Force~~ + Force

Motion

Rigid Body Motion

scalar Power = effort x flow = Force x Velocity

- Any system of forces acting on a rigid body can be replaced by a single force applied along a line combined with a torque about the same line
  -> *Wrench*
- Wrenches are dual to twists

DUAL(ITY)

~~F, M~~ (v, ω)

- Many theorems applied to Screws can be applied to wrenches as well. F, M (F, Z)

duals

- Modern theory of screws is based on Linear Algebra and matrix groups

Flexible

Rigid Body Motion & Linear Algebra

Robotics

Computer graphics

CAD

- Basic Tools:
  - Use of Homogeneous coordinates to represent rigid motions and the matrix exponential which maps a twist to a corresponding screw motion.

Robotics

Gaming

Animation

- Advanced Tools:
  - Lie groups and Lie Algebra

Motion Planning

Control (Higher DOF)

- Advantages:
  - Allows a global description of rigid body motion which does not suffer from the singularities due to use of local coordinates
  - Provides a geometric description of rigid motion

R. John J Craig : Robotics

A review of vectors and matrix

- Vectors
  - Column vector and row vector

$$\boxed{v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} \qquad v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Eucledean Norm

  - Norm of a vector

$$|v| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

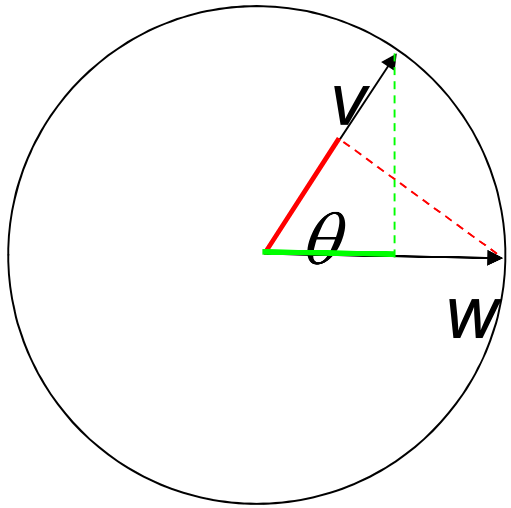
Dot product of two vectors

- Vector v and w

$$v \bullet w = |v||w|\cos\theta$$

- If $|v|=|w|=1$,

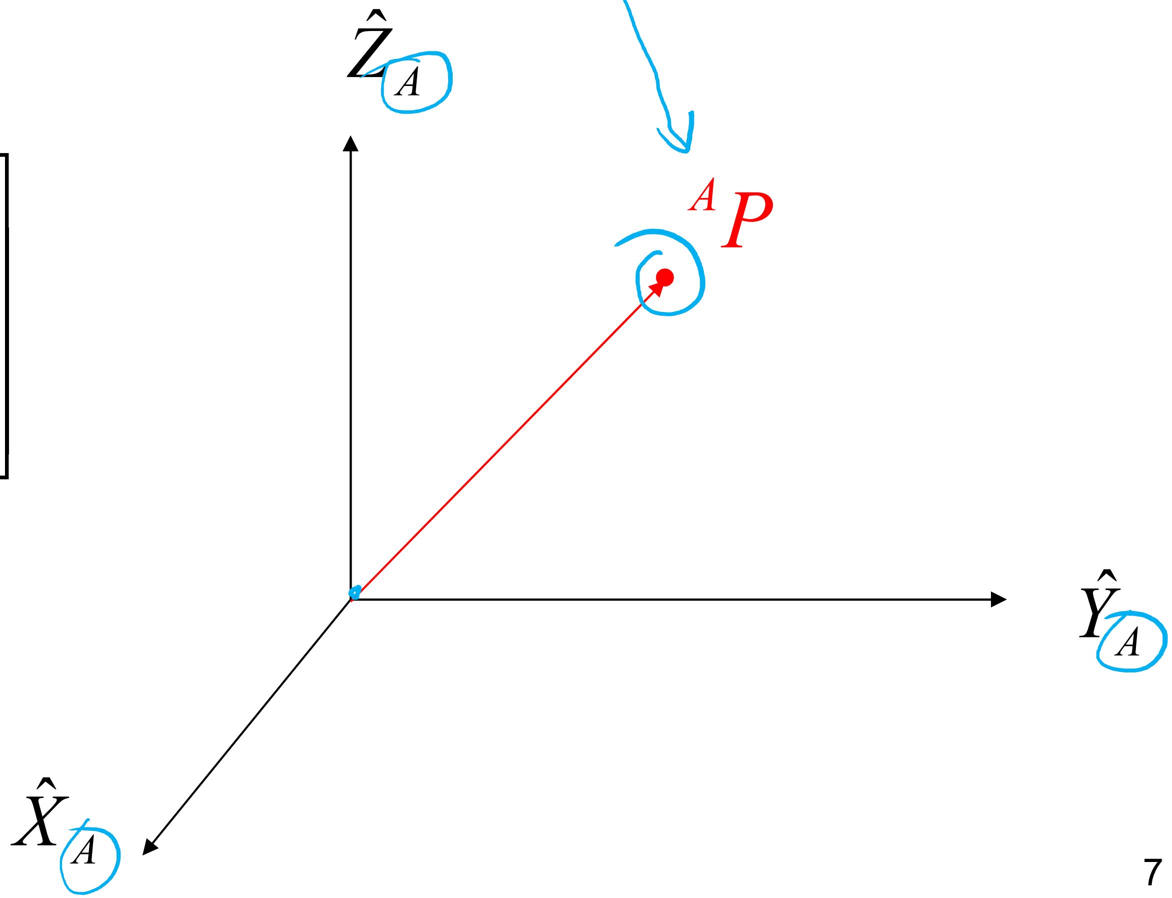
$$v \bullet w = \cos\theta$$



Position Description

- Coordinate System A

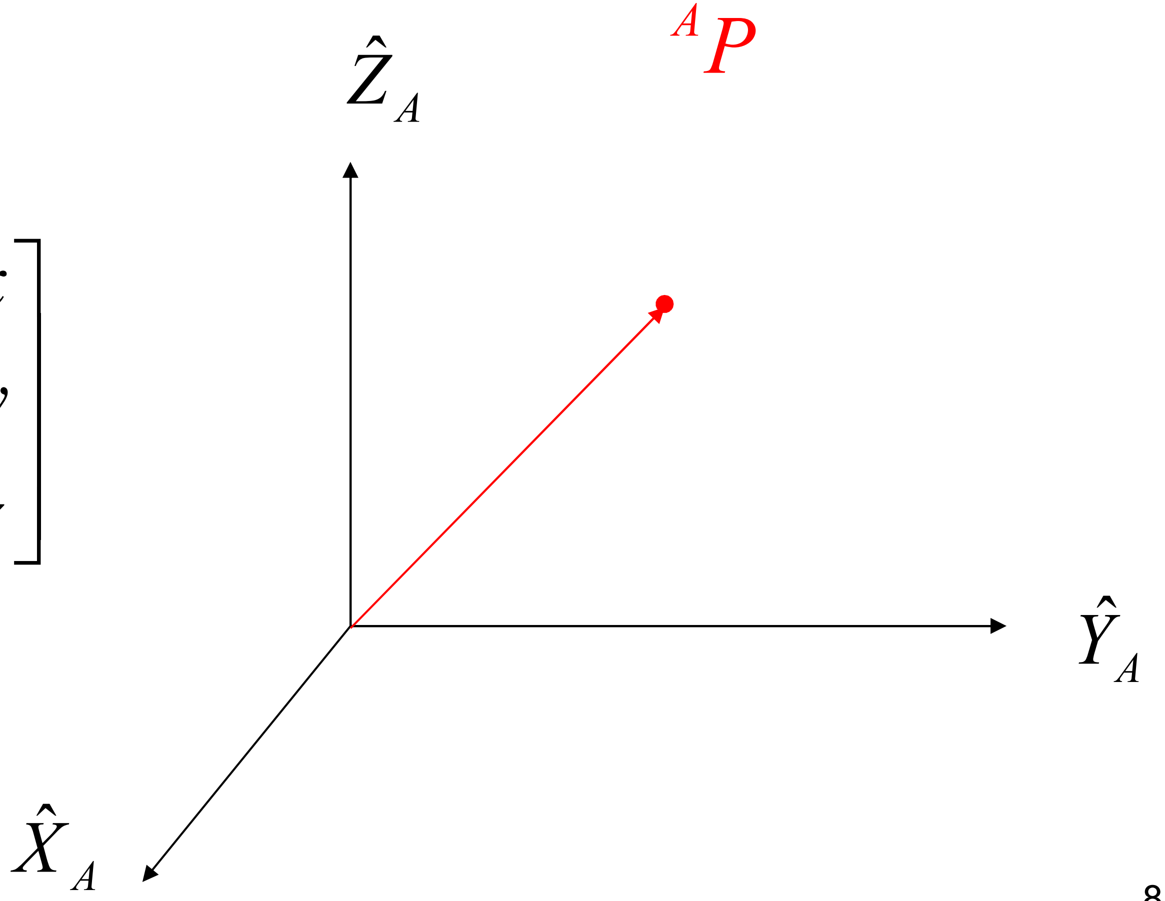
$$
{}^{A}P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$



Orientation Description

- Coordinate System A

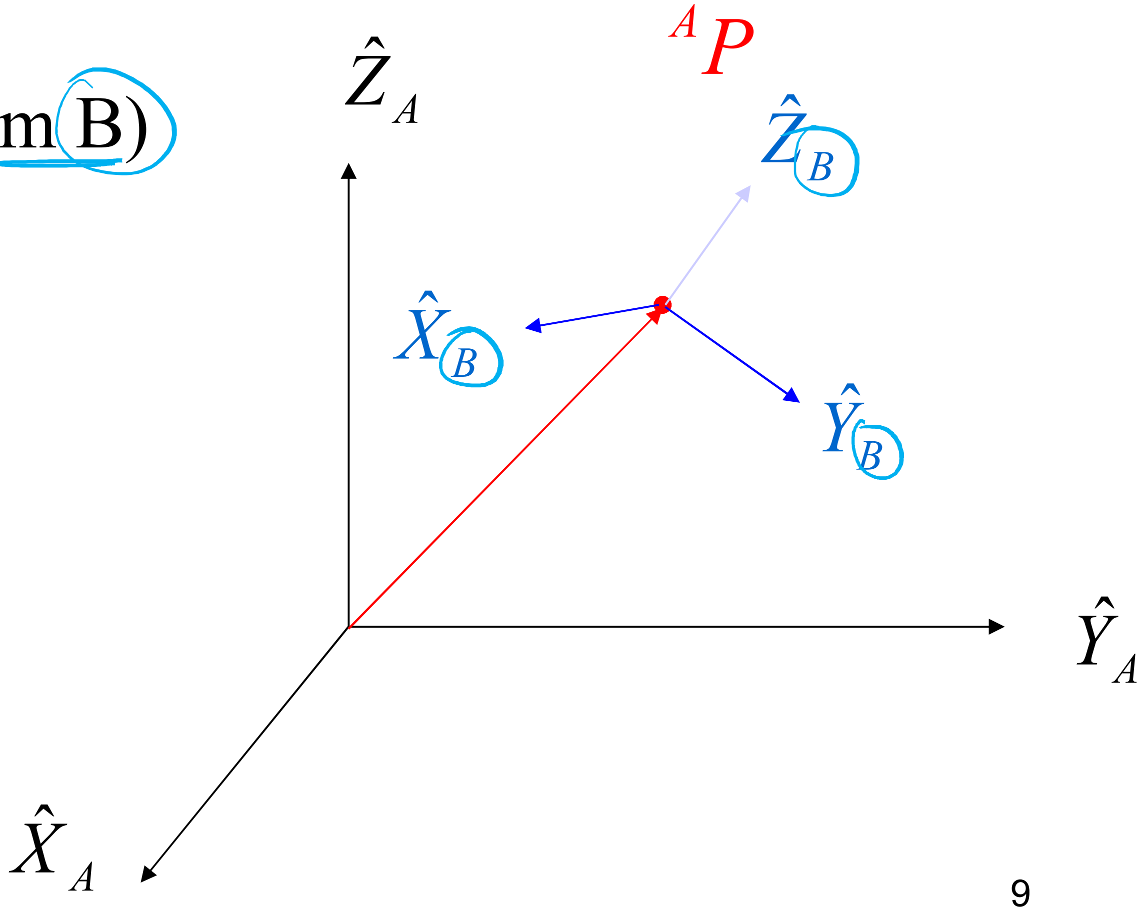
$$ {}^{A}P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} $$



Orientation Description

- Coordinate System A
- Attach Frame B
  (Coordinate System B)

$$^{A}P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



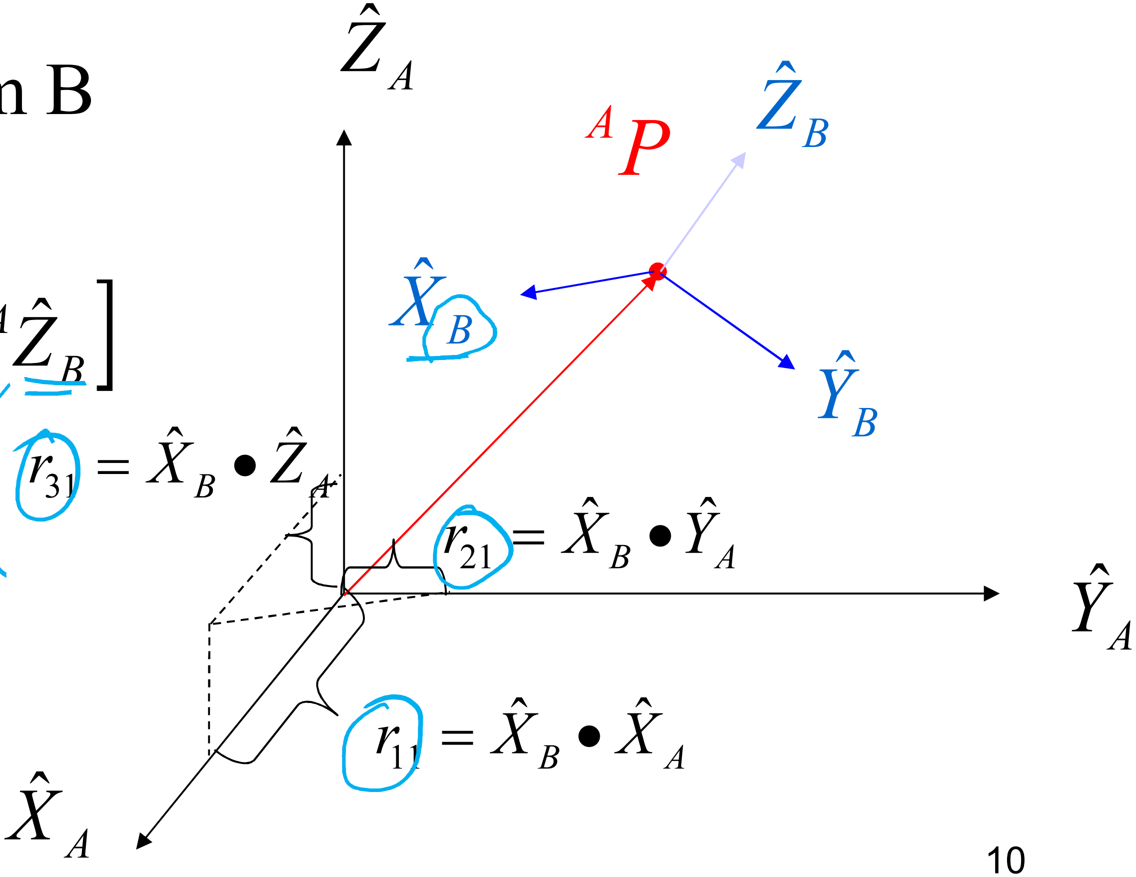
Orientation Description

- Coordinate System A
- Attach Frame
  Coordinate System B
- Rotation matrix

$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}
$$

$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

R



Rotation matrix

Directional Cosines

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}$$

$$= \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix}$$

${}^{B}\hat{X}_A^{T}$

Frame B

A

Directional Cosines

${}^{A}\hat{X}_B$

Frame A

$${}^{A}_{B}R$$

$$= \begin{bmatrix} {}^{B}\hat{X}_A^{T} \\ {}^{B}\hat{Y}_A^{T} \\ {}^{B}\hat{Z}_A^{T} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \end{bmatrix}^T = {}^{B}_{A}R^T$$

$${}^{A}_{B}R = \left({}^{B}_{A}R\right)^T$$

Rotation matrix

$$ {}^A_B R = {}^B_A R^T $$

$$ {}^A_B R \; {}^B_A R = I \Leftrightarrow {}^A_B R = {}^B_A R^{-1} $$

$$ {}^B_A R^T = {}^B_A R^{-1} $$

- For matrix M,
  - If $M^{-1} = M^T$, M is orthogonal matrix
  - ${}^A_B R$ is orthogonal!!

Orthogonal Matrix

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} $$

$$ = \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix} \quad 3 \times 3 $$

9 Parameters to describe orientation!

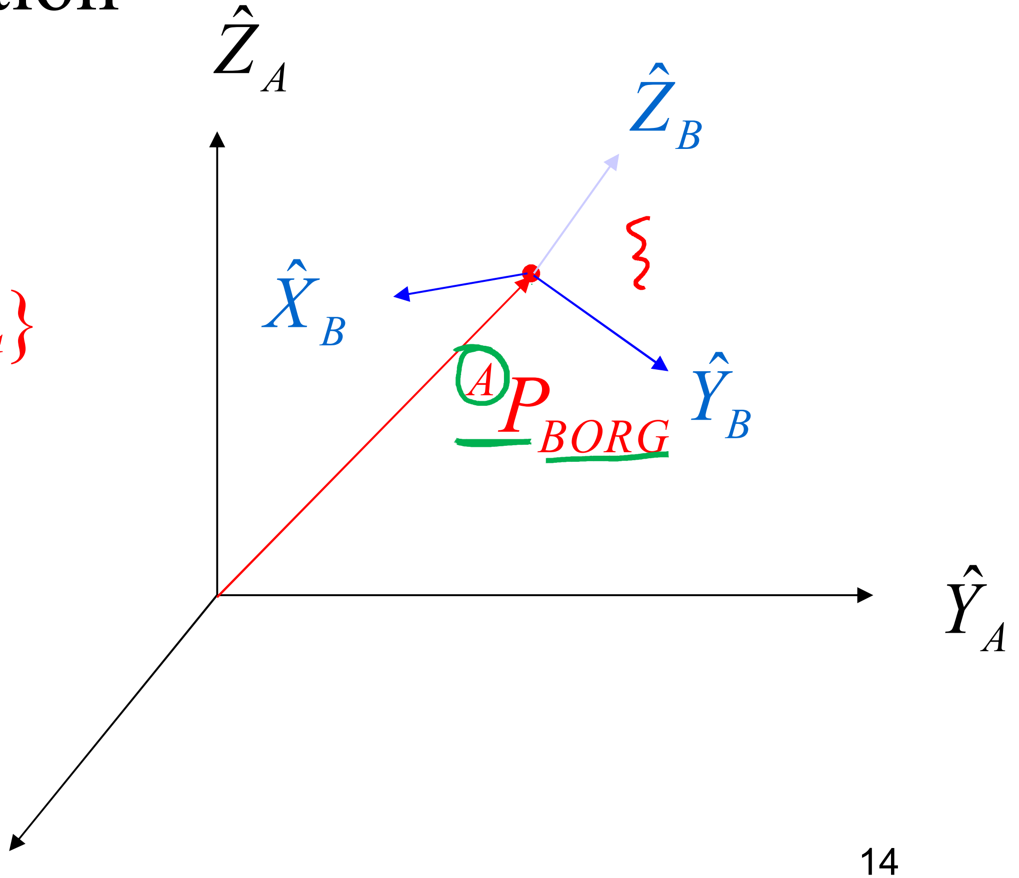
Description of a frame

$^A P_{BORG}$ = origin of frame B

- Position + orientation

$$\{B\} = \{ {}^A_B R, {}^A P_{BORG} \}$$

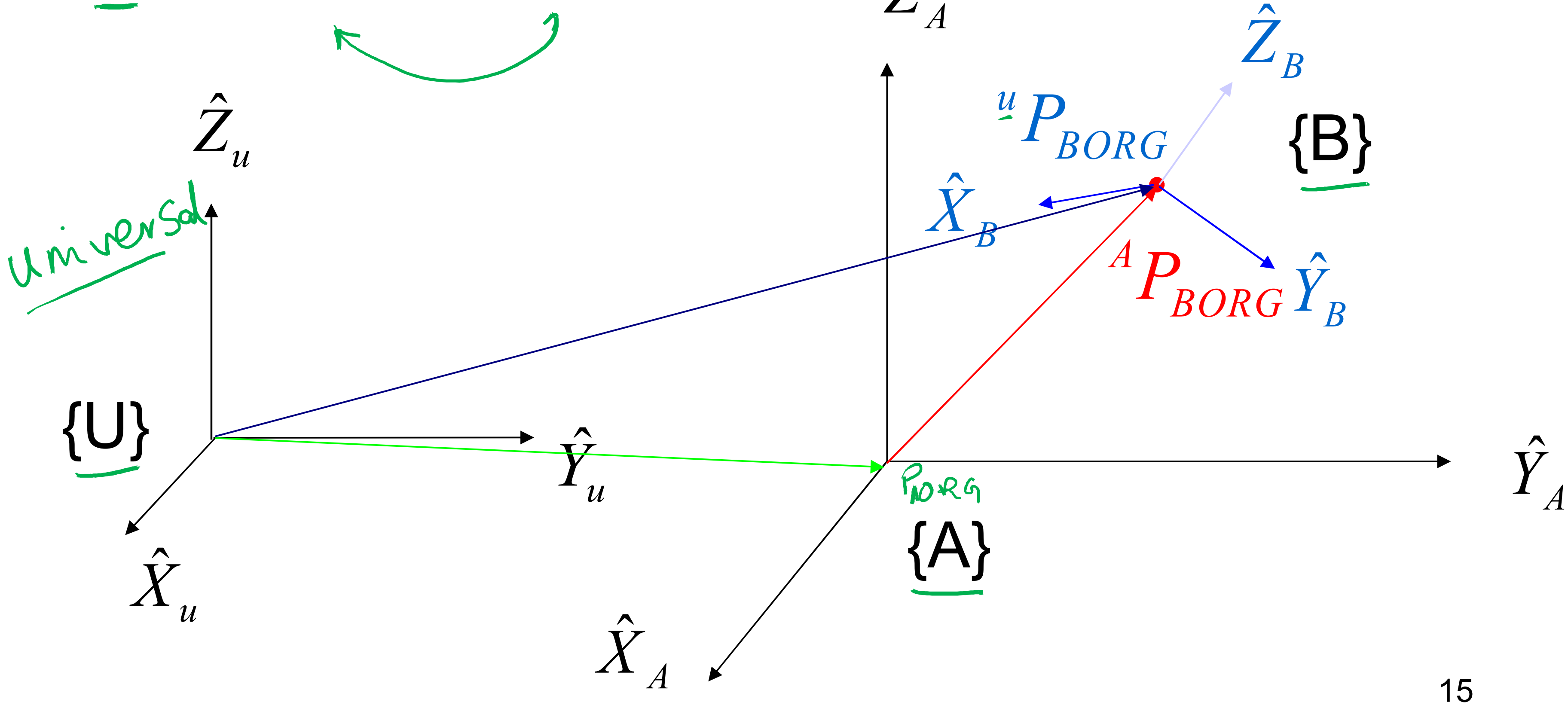
3x3 3x1



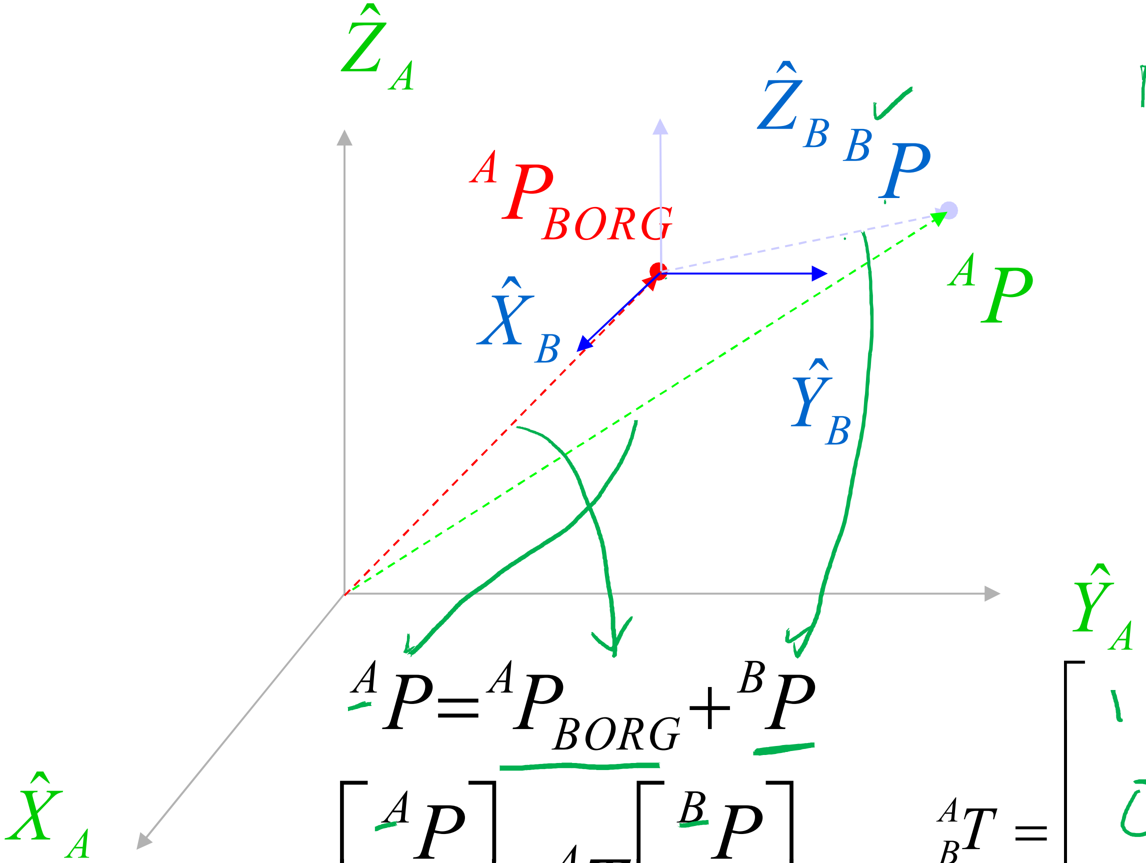
Graphical representation

$$\{A\} = \{{}^{u}_{A}R, {}^{u}P_{AORG}\}$$

$$\{B\} = \{{}^{u}_{B}R, {}^{u}P_{BORG}\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$



Mapping: Change Coordinates – Translation Difference



$${}^A_BT \ (4 \times 4)$$

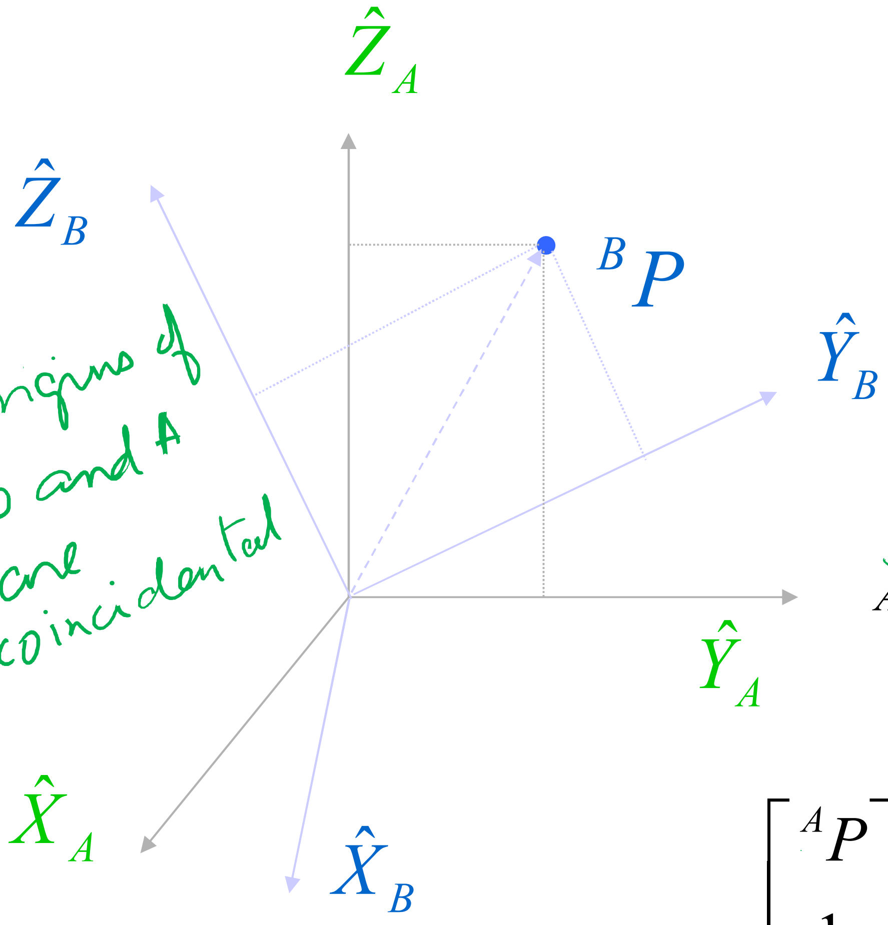
Translation (operator) Matrix

$${}^AP = {}^AP_{BORG} + {}^BP$$

$$\begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = {}^A_BT \begin{bmatrix} {}^BP \\ 1 \end{bmatrix}$$

$${}^A_BT = \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix}}_{3\times 3} & \begin{matrix} x \\ y \\ z \end{matrix}\ {}^AP_{BORG} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] \ 3\times 1$$

Mapping – rotation difference



origins of B and A are coincidented

$$^{B}P = {}^{B}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = P_x \hat{X}_B + P_y \hat{Y}_B + P_z \hat{Z}_B$$

3x1

$$^{A}P = P_x\, {}^{A}\hat{X}_B + P_y\, {}^{A}\hat{Y}_B + P_z\, {}^{A}\hat{Z}_B$$

$$= [\, {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B \,]\, {}^{B}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$= {}^{A}_{B}R\, {}^{B}P$$

$$^{A}P = {}^{A}_{B}R\, {}^{B}P$$

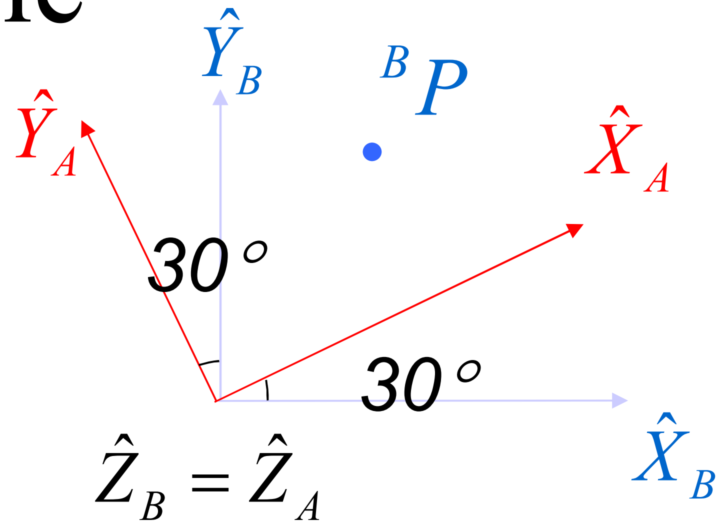
no Trans

$$^{A}_{B}T = \begin{bmatrix} & {}^{A}_{B}R & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

Example

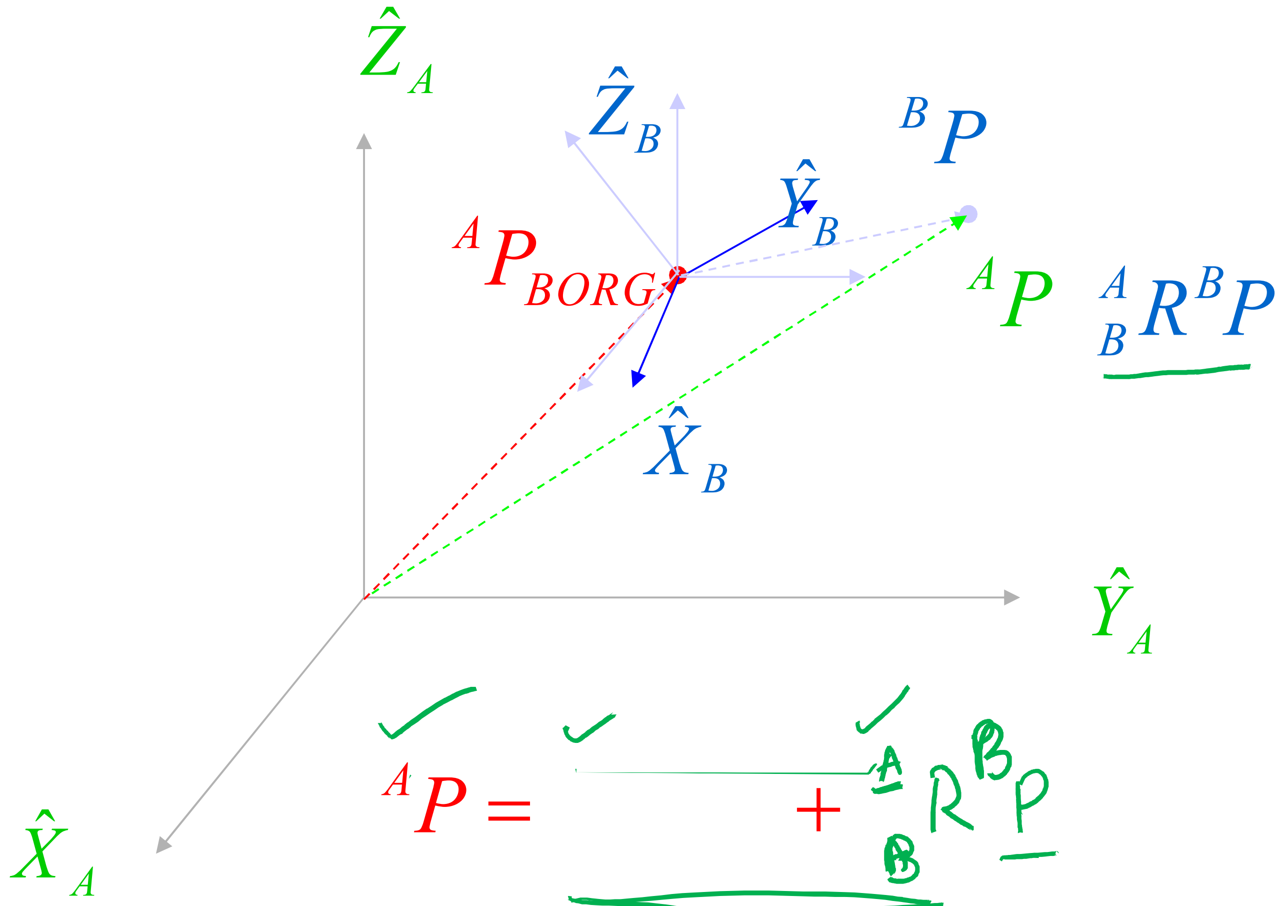
$$ {}^{B}P = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow {}^{A}P\,? $$



$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} $$

$$ = \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 120^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} $$

Mapping: Rotation + Translation Difference



$${}^{A}P = \underline{\qquad\qquad} + {}^{A}_{B}R\,{}^{B}P$$

Homogeneous Transformation for Mapping

$$^{A}P = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P$$

Translation

Rot^n matrix

$$\begin{bmatrix} ^{A}P_x \\ ^{A}P_y \\ ^{A}P_z \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} & & & ^{A}P_{BORG\_x} \\ & ^{A}_{B}R & & ^{A}P_{BORG\_y} \\ & & & ^{A}P_{BORG\_z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{^{A}_{B}T} \begin{bmatrix} ^{B}P_x \\ ^{B}P_y \\ ^{B}P_z \\ 1 \end{bmatrix}$$

H = Rotation + Translation included

A

B

$^{A}_{B}R$

4x4

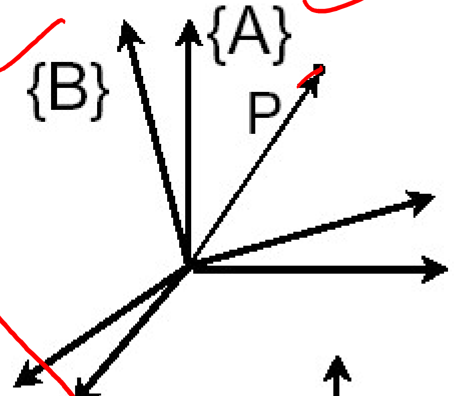
20/1/2021

Operators

Mapping: changing descriptions from frame to frame

Operators: moving points (within the same frame)

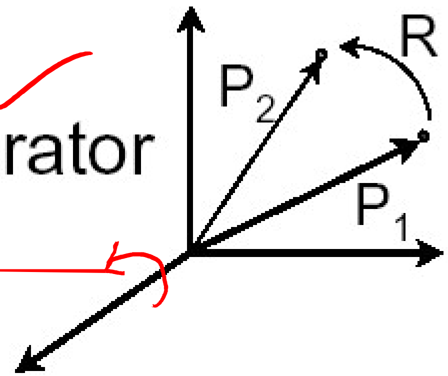
Mapping



A B

$${}^{A}P = {}^{A}_{B}R\, {}^{B}P$$

Rotational Operator



$$R: P_1 \longrightarrow P_2$$

$$\boxed{P_2 = R \ P_1}$$

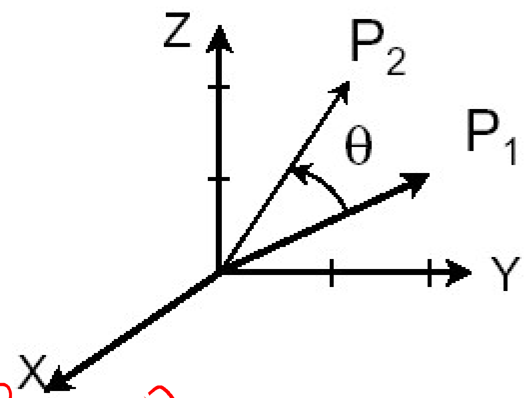
~~operator~~

Rotational Operators

$$R_K(\theta):\quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta)\, P_1$$

Example



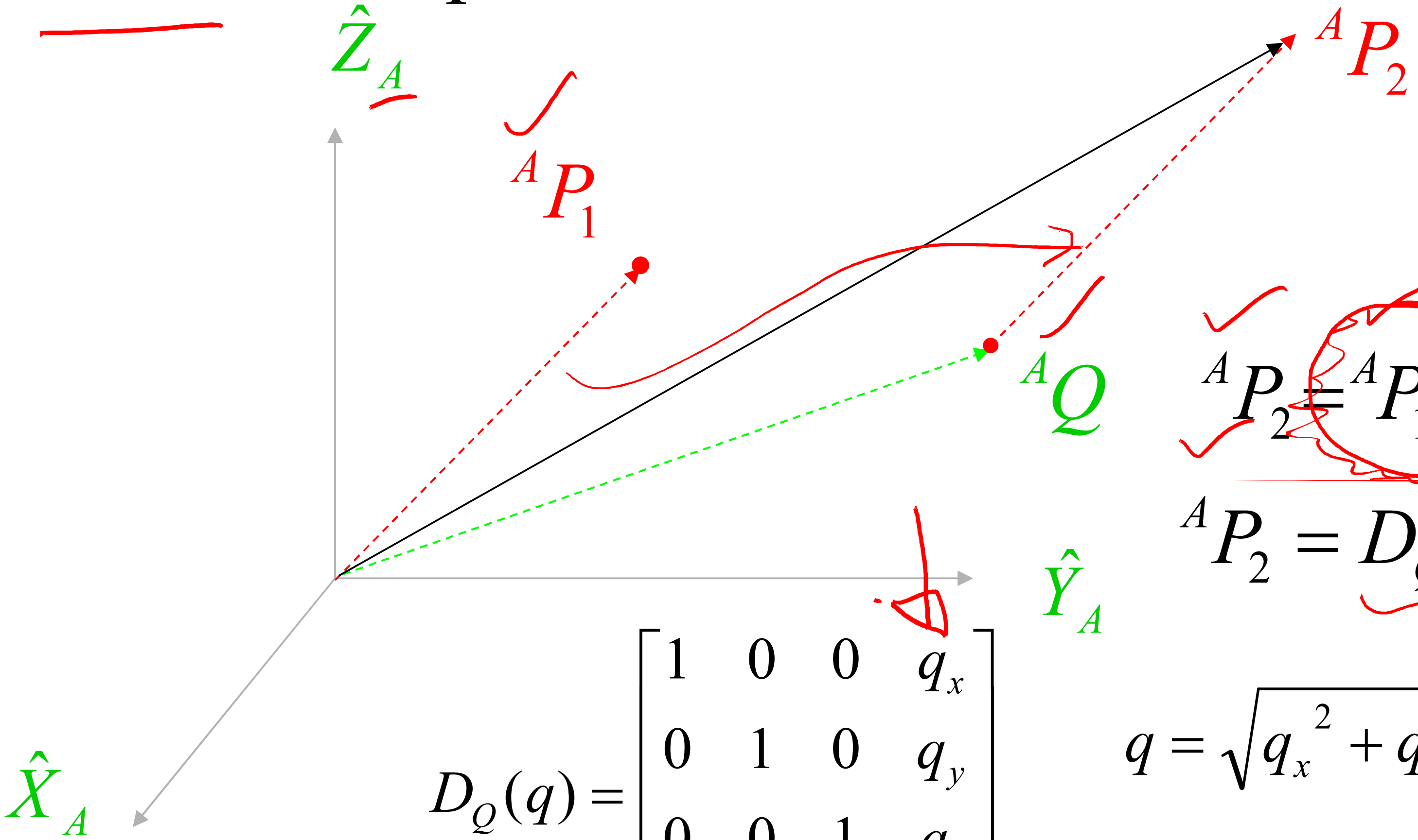
$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

P_1 P_2

$$P_2 = R_X(\theta)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translation Operator

- Translation operator



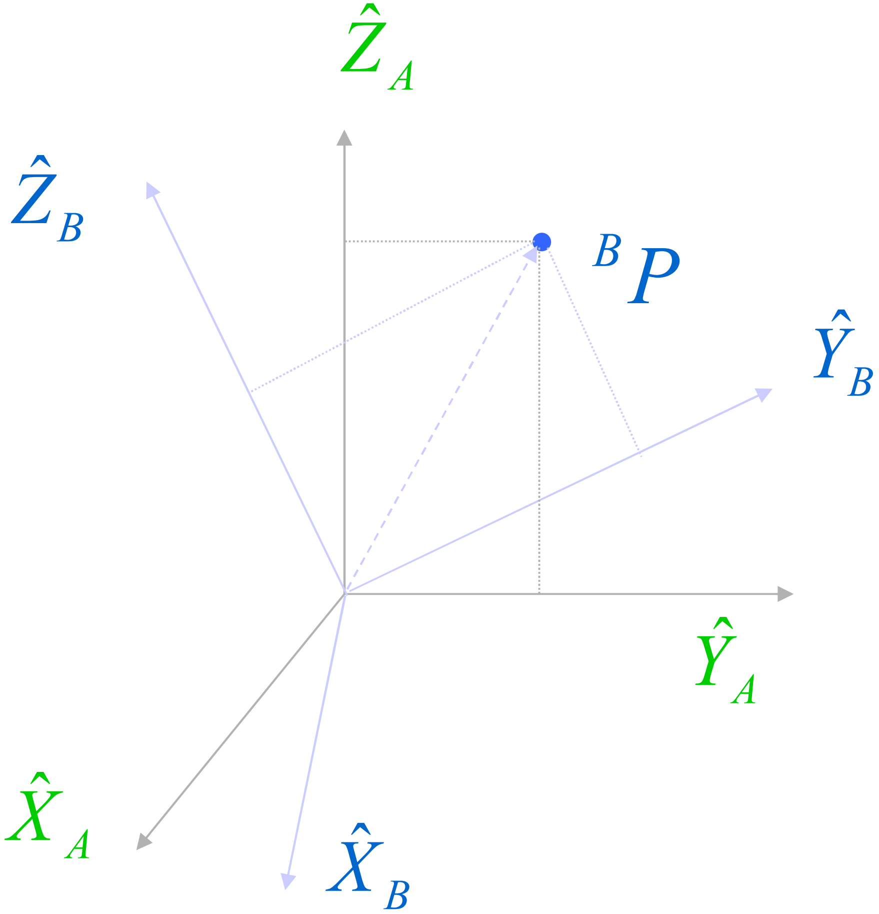
$${}^A P_2 = {}^A P_1 + {}^A Q$$

$${}^A P_2 = D_Q(q)\,{}^A P_1$$

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q = \sqrt{q_x^{\ 2} + q_y^{\ 2} + q_z^{\ 2}}$$

Recall: Mapping – rotation difference



$$ {}^A P = {}^A_B R \, {}^A P $$

A B

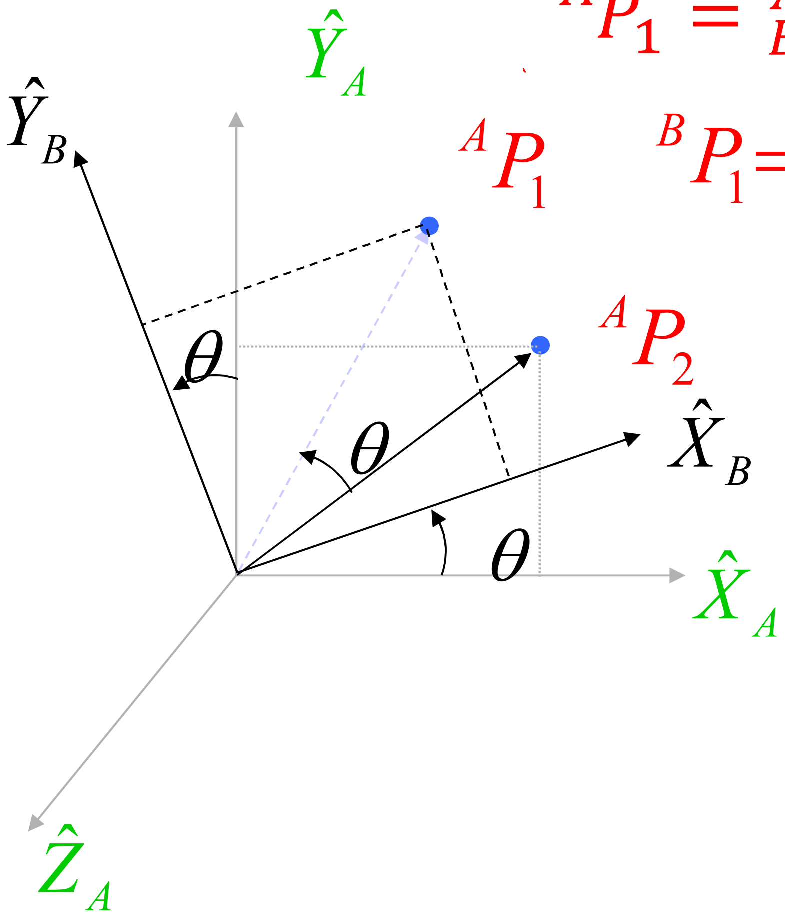
$$ \begin{bmatrix} {}^A P \\ 1 \end{bmatrix} = {}^A_B T \begin{bmatrix} {}^B P \\ 1 \end{bmatrix} $$

Rot^n matrix

$$ {}^A_B T = \left[\begin{array}{ccc|c} & {}^A_B R & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Relationship between Mapping with only Rotational Difference and Rotation Operator

$$^{A}P_1 = {}^{A}_{B}R\ {}^{B}P_1 = R_Z(\theta)\ {}^{A}P_2$$



$$^{B}P_1 = {}^{A}P_2 \qquad {}^{A}_{B}R = R_Z(\theta)$$

$$
\begin{aligned}
{}^{A}_{B}R &= \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} \\
&= \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90^\circ + \theta) & \cos 90^\circ \\ \cos(90^\circ - \theta) & \cos\theta & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Relationship between Mapping with only Rotational Difference and Rotation Operator

$$ {}^{A}_{B}R = R_{Z}(\theta) $$

- The rotation matrix that rotates vectors through some rotation, R, is the same as the rotation matrix that describes a frame rotated by R relative to the reference frame.

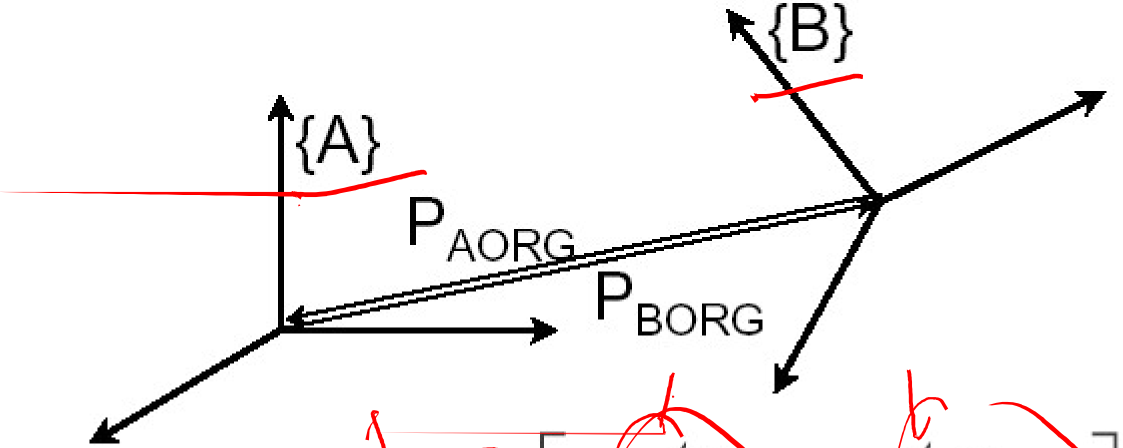
General Operators

$$P_2 = \left(\begin{array}{ccc|c} & R_K(\theta) & & Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right) P_1$$

4×4

$$\boxed{P_2 = T\,P_1}$$

Inverse Transform



$$\boxed{{}^{A}_{B}T} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

B → A

A → B

$$R^{-1} = R^{T} \qquad (T^{-1} \neq T^{T})$$

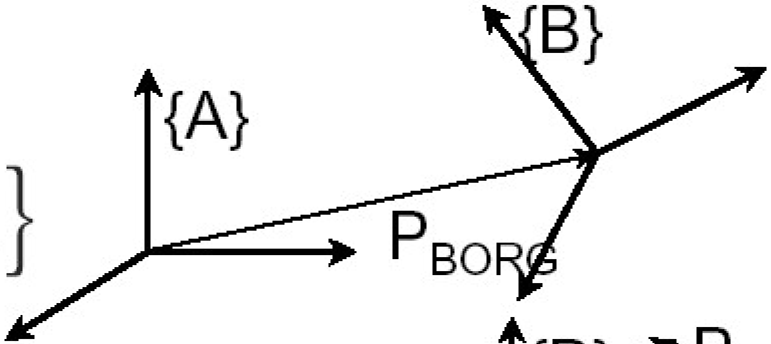
$$\boxed{{}^{B}P_{AORG}}$$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

Homogeneous Transform Interpretations

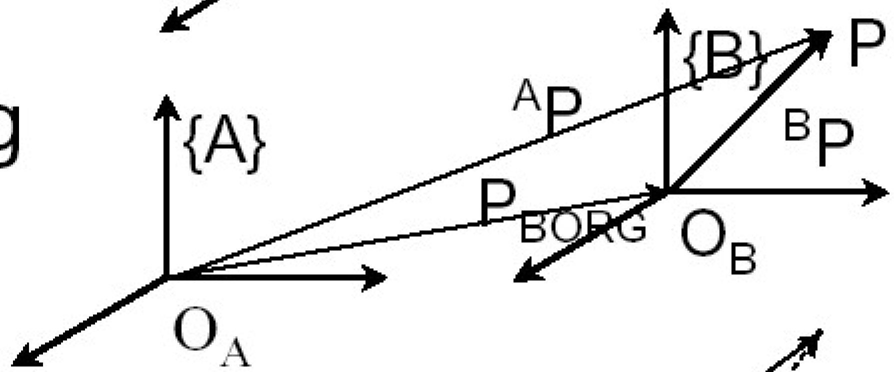
Description of a frame

$$ {}^{A}_{B}T: \quad \{B\} = \left\{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \right\} $$



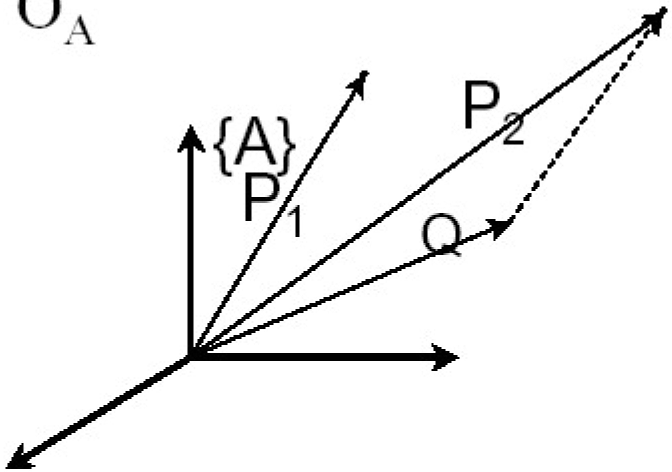
Transform mapping

$$ {}^{A}_{B}T: \quad {}^{B}P \rightarrow {}^{A}P $$

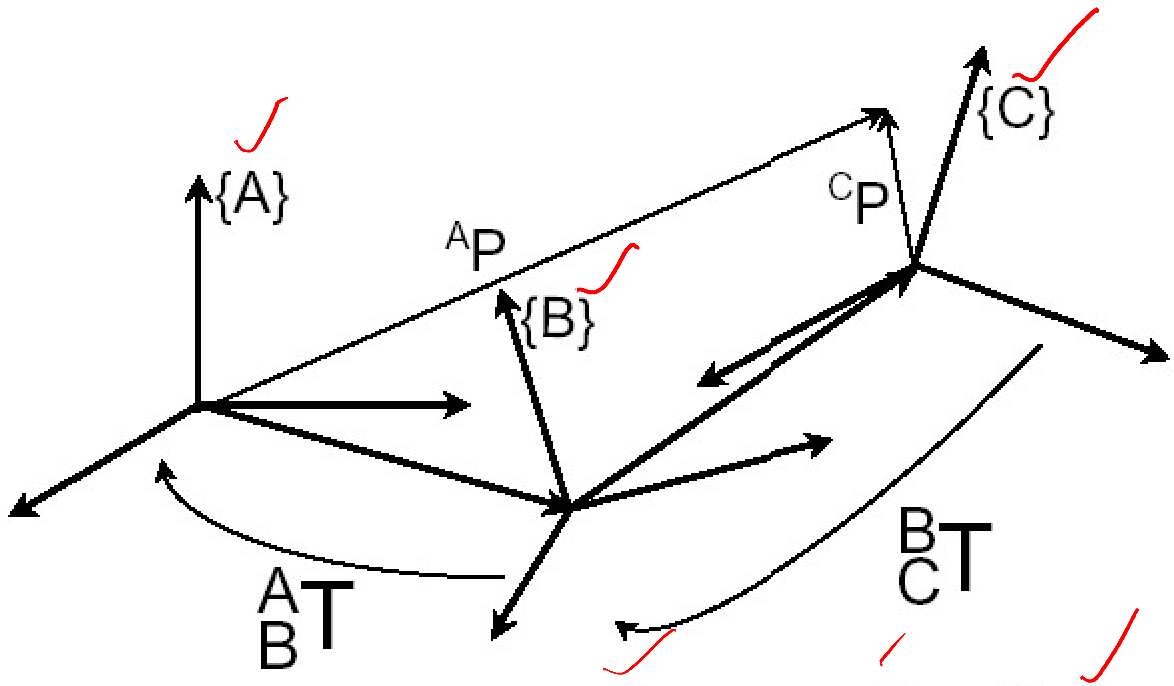


Transform operator

$$ T: \quad P_1 \rightarrow P_2 $$



Compound Transformations



$$^{B}P = {}^{B}_{C}T \, {}^{C}P$$

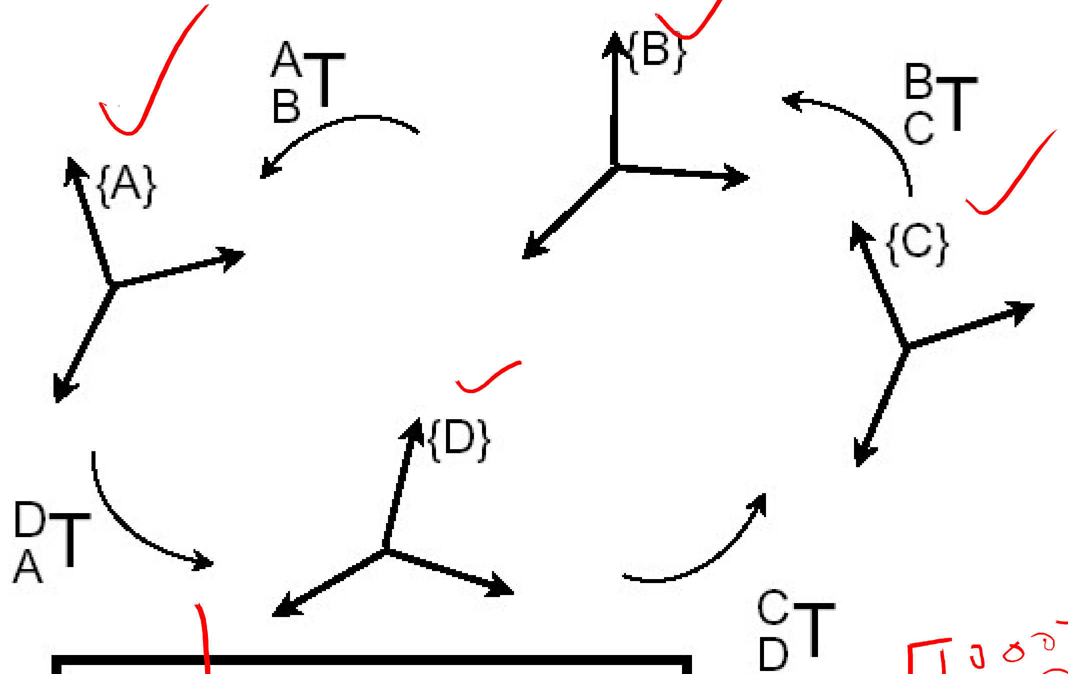
$$^{A}P = {}^{A}_{B}T \, {}^{B}P$$

$$^{A}P = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}P \quad \Rightarrow \quad \boxed{{}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T}$$

$$ {}^{A}_{C}T = {}^{A}_{B}T \; {}^{B}_{C}T $$

$$
{}^{A}_{C}T = \left[\begin{array}{cc} \overset{3\times 3}{\boxed{{}^{A}_{B}R \, {}^{B}_{C}R}} & \overset{4\times 1}{\boxed{{}^{A}_{B}R \, {}^{B}P_{Corg} + {}^{A}P_{Borg}}} \\ 0 \quad 0 \quad 0 & 1 \end{array} \right] \begin{array}{l} 3\times 1 \\ 1 \end{array}
$$

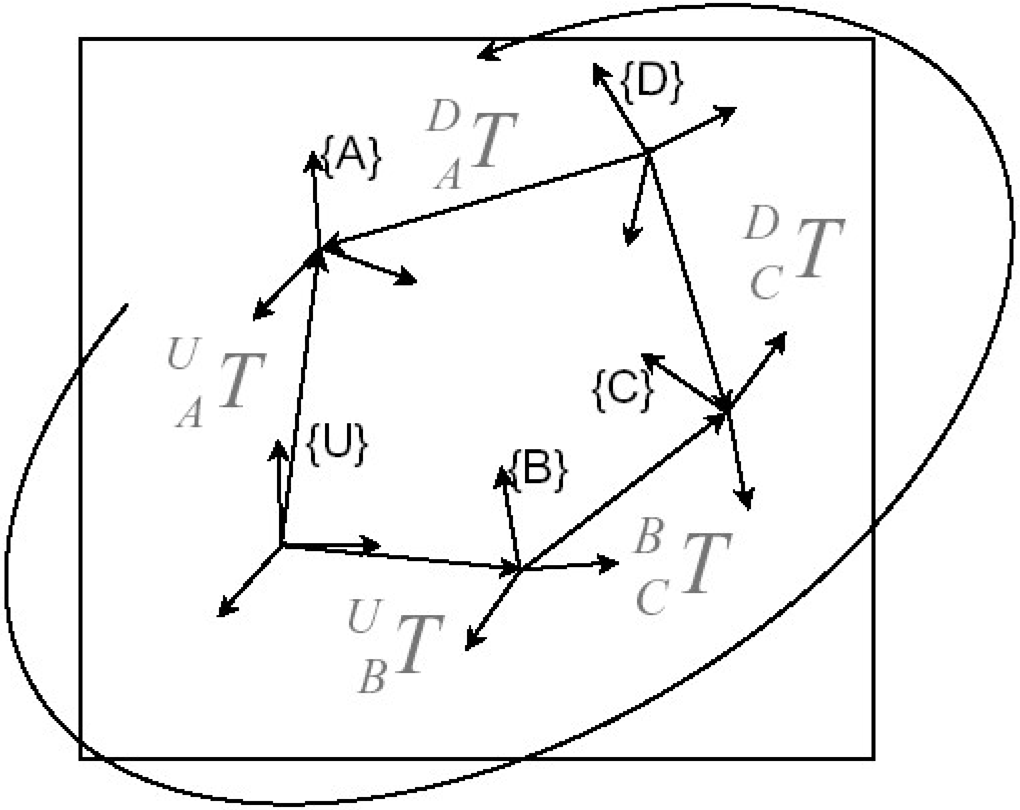
Transform Equation



$$\boxed{{}^A_B T\,{}^B_C T\,{}^C_D T\,{}^D_A T = I}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\ {}^A_B T^{-1} = {}^B_A T = {}^B_C T\,{}^C_D T\,{}^D_A T$$



$${}^D_A T^{-1} \cdot {}^D_C T \cdot {}^B_C T^{-1} \cdot {}^U_B T^{-1} \cdot {}^U_A T \equiv I$$

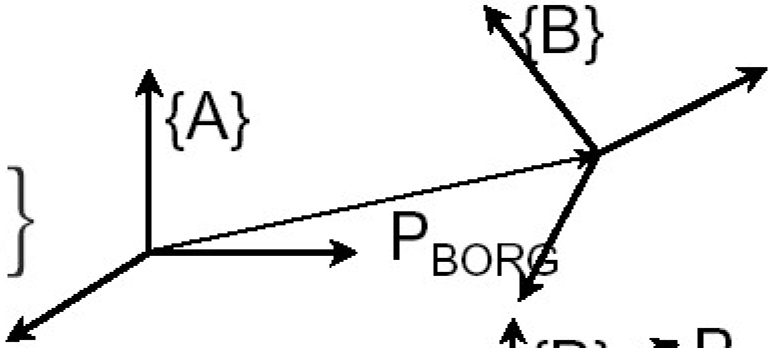
$${}^U_A T = {}^U_B T \cdot {}^B_C T \cdot {}^D_C T^{-1} \cdot {}^D_A T$$

${}^U_C T$

Homogeneous Transform Interpretations

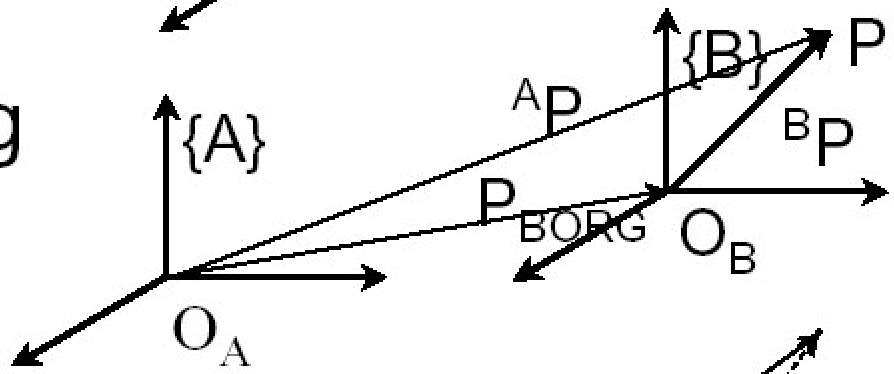
Description of a frame

${}^A_BT: \quad \{B\} = \{ {}^A_BR \quad {}^AP_{Borg} \}$



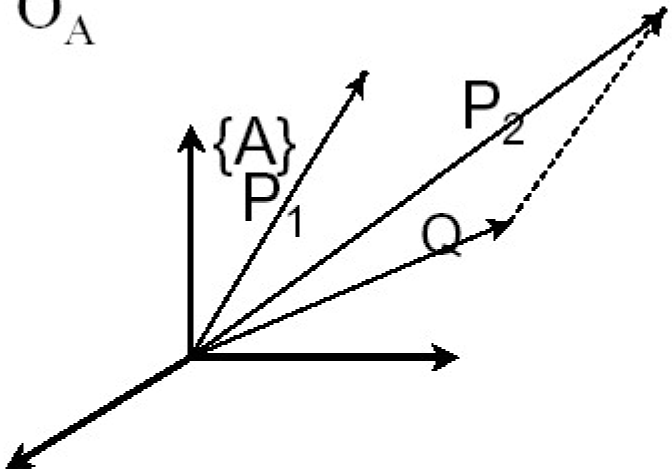
Transform mapping

${}^A_BT : \quad {}^BP \rightarrow {}^AP$



Transform operator

$T : \quad P_1 \rightarrow P_2$

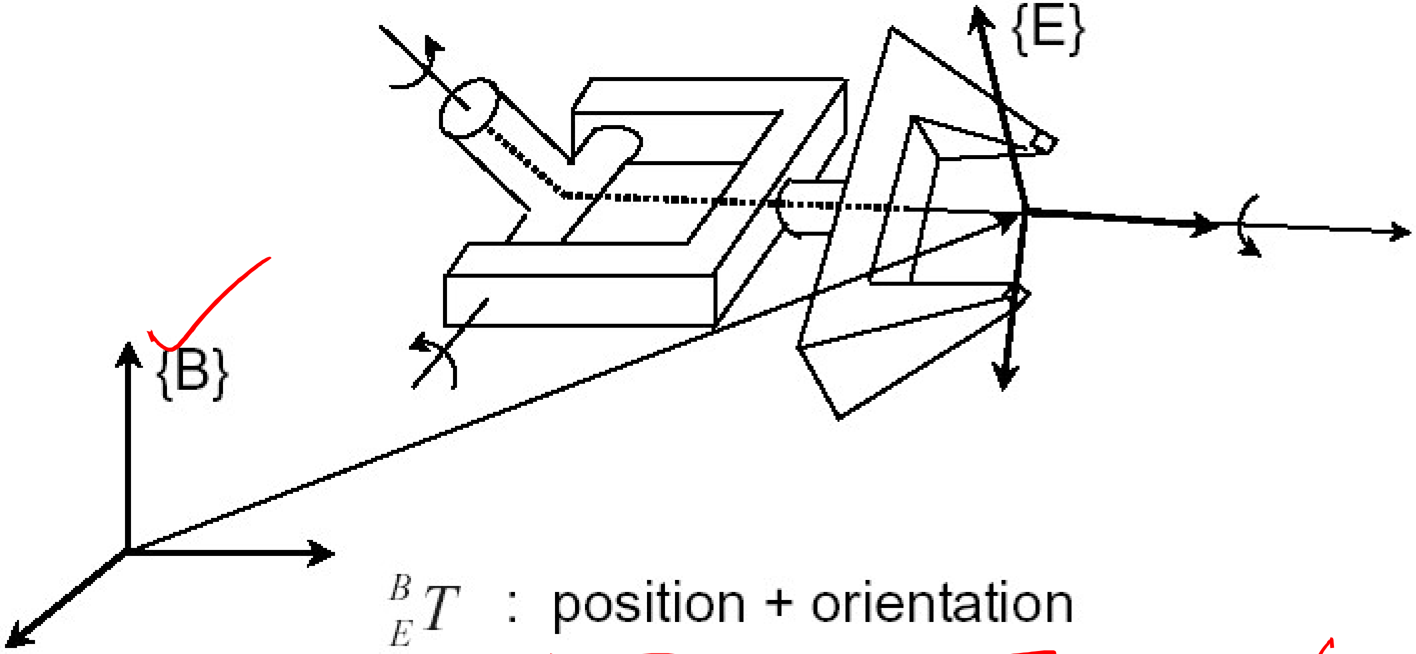


Spatial Descriptions

- Transformations
  - Mapping
  - Operator
- Representations

Example

End-Effector Configuration



$^{B}_{E}T$: position + orientation

6x1

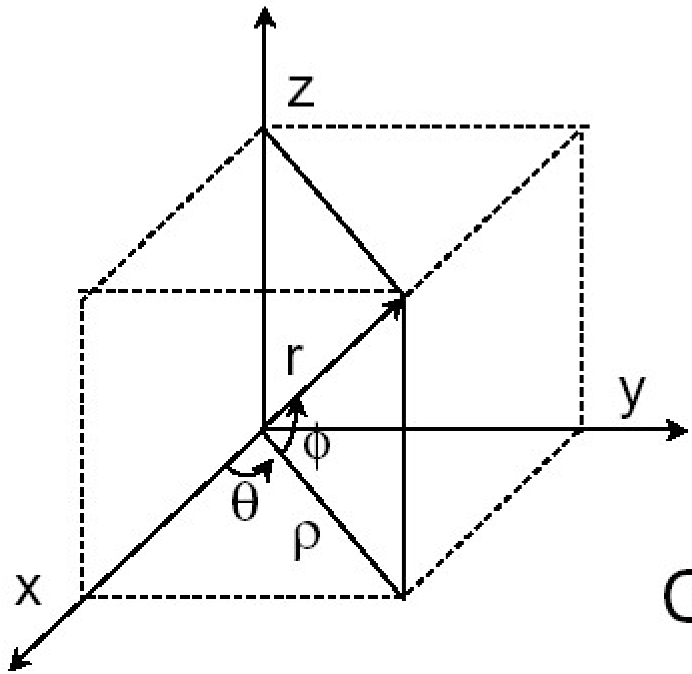
End-Effector Configuration Parameters

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

X_P ← position 3x1

X_R ← orientation 3x1

Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, ϕ)

Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

3 x 1

3x3

(3x1)

Direction Cosines

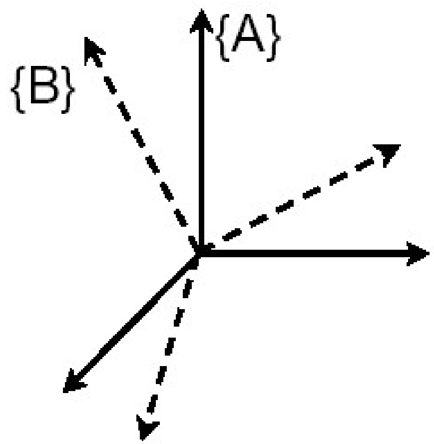
$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

Constraints

$$|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = 1$$

$$\mathbf{r}_1 . \mathbf{r}_2 = \mathbf{r}_1 . \mathbf{r}_3 = \mathbf{r}_2 . \mathbf{r}_3 = 0$$

Three Angle Representations



X - Y - Z
Y - X - Z
Z - X - Y . . . 12 sets

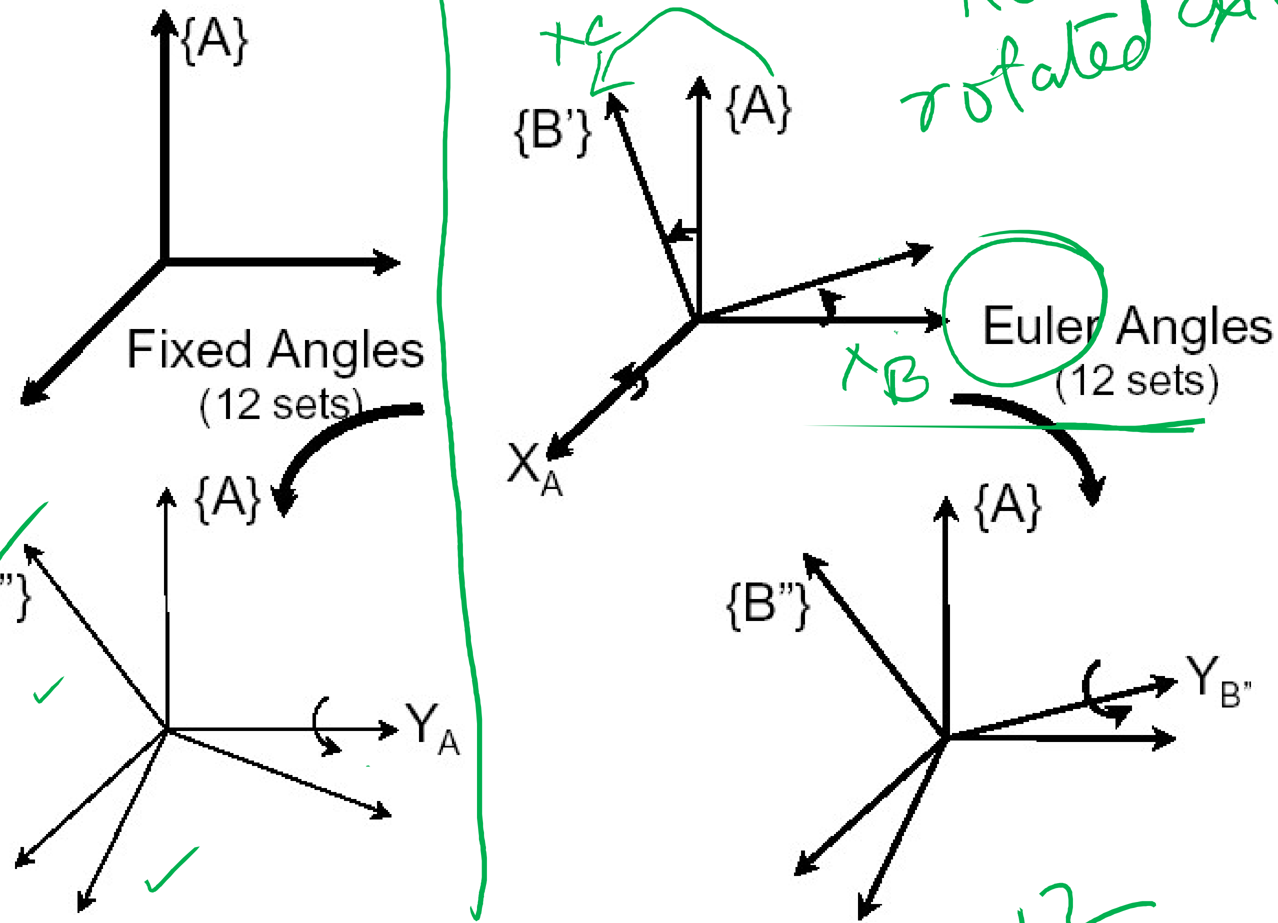
Compendium
Craig

Three Angle Representations

Rotation about rotated axis.

Fixed Angle Rotn

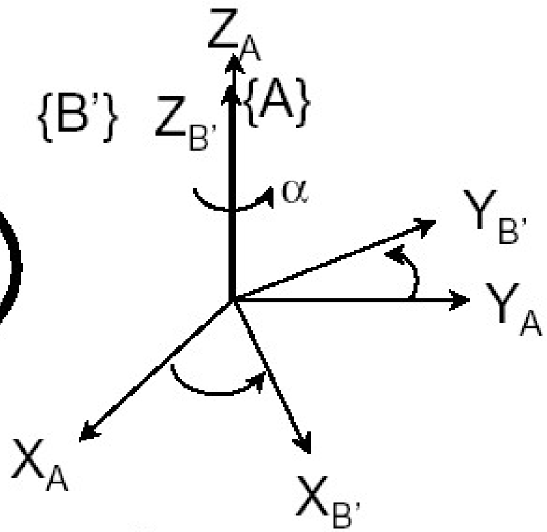
$X_A Y_A Z_A$ original axes



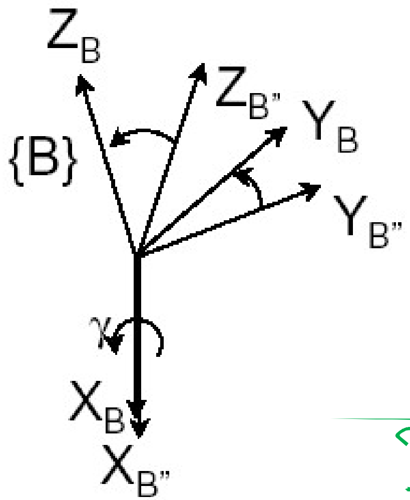
12

12

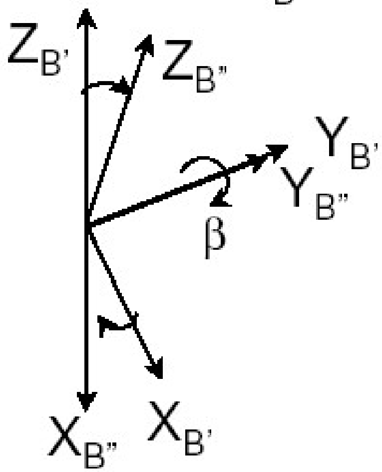
Euler Angles (Z-Y-X)



${}^{A}_{B'}R$



${}^{B''}_{B}R$



${}^{B'}_{B''}R$

3 rot^{n}s

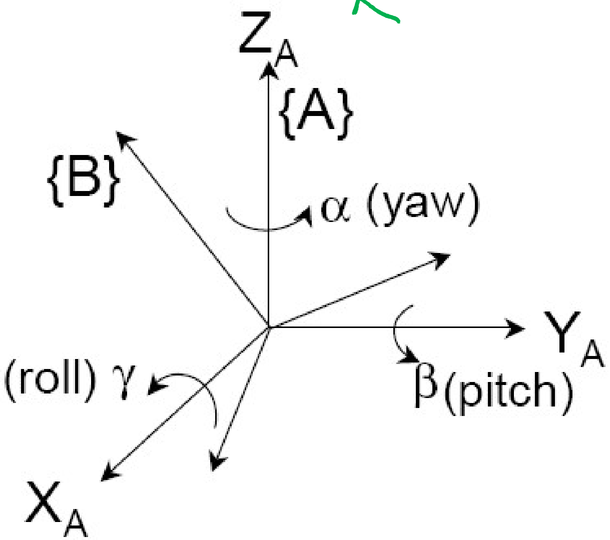
$${}^{A}_{B}R = {}^{A}_{B'}R \,.\, {}^{B'}_{B''}R \,.\, {}^{B''}_{B}R$$

$$\boxed{{}^{A}_{B}R = \underbrace{R_Z(\alpha) \,.\, R_Y(\beta) \,.\, R_X(\gamma)}}$$

X-Y-Z Fixed Angles

axis rotn

F



rotn about F

$$R_X(\gamma):\ \ v \rightarrow R_X(\gamma).v$$

rotn about F

$$R_Y(\beta):\ (R_X(\gamma).v) \rightarrow R_Y(\beta).(R_X(\gamma).v)$$

$$R_Z(\alpha):\ (R_Y(\beta).R_X(\gamma).v) \rightarrow R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$$

$$\boxed{{}^A_B R = {}^A_B R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

2D

Z-Y-X Euler Angles

$$ {}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{X'}(\gamma) $$

$$ \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

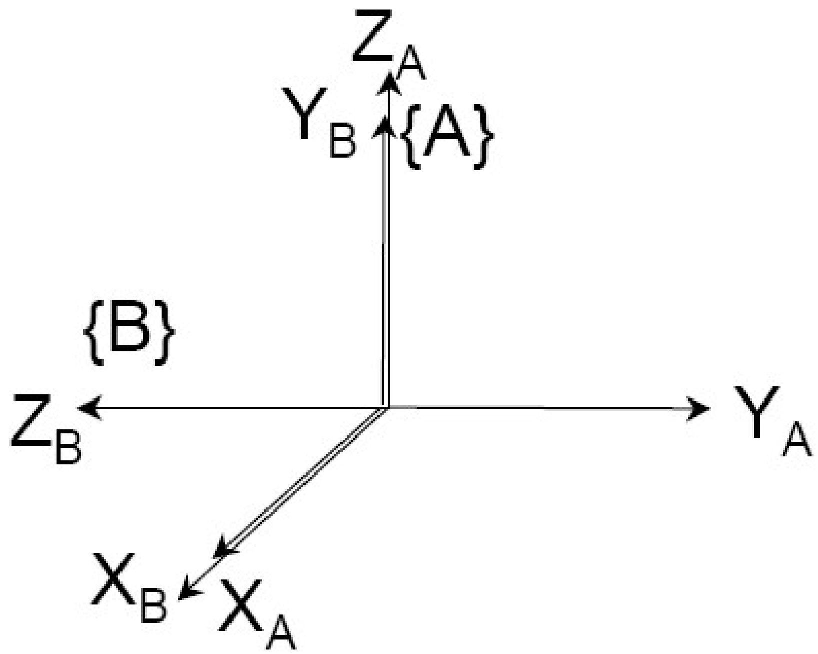
$$ {}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha.c\beta & X & X \\ s\alpha.c\beta & X & X \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix} $$

Z-Y-Z Euler Angles

$$ {}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma) $$

$$ {}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} X & X & c\alpha.s\beta \\ X & X & s\alpha.s\beta \\ -s\beta.c\gamma & s\beta.s\gamma & c\beta \end{bmatrix} $$

Example



$$R_{Z'Y'X'}(\alpha, \beta, \gamma):$$

$$\alpha = 0$$

$$\beta = 0$$

$$\gamma = 90^o$$

Fixed & Euler Angles

X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Inverse Problem

Given ${}^{A}_{B}R$ find (α, β, γ)

$\rightarrow R_{Z'Y'X'}$

$$
{}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}
$$

3×3

$$
\left.\begin{aligned} \cos\beta = c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta = -r_{31} \end{aligned}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})
$$

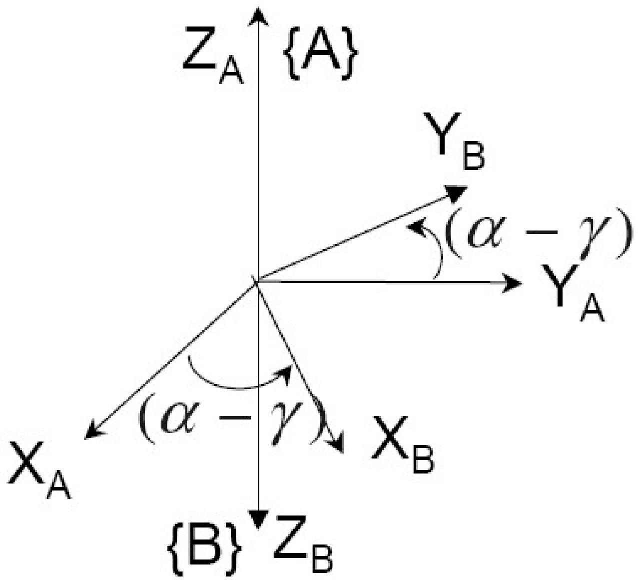
if $c\beta = 0$ $(\beta = \pm 90^{\circ})$ ⇨ Singularity of the representation

⇨ Only $(\alpha + \gamma)$ or $(\alpha - \gamma)$ is defined

Singularities - Example

$$\underline{c\beta = 0,\ s\beta = +1}$$

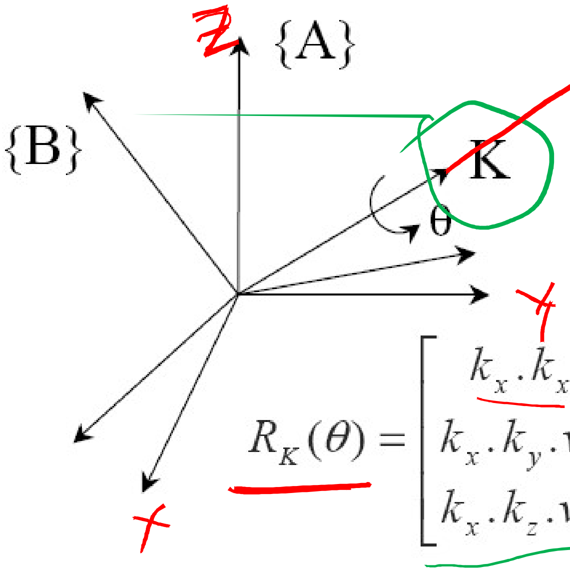
$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix}$$



$$\underline{c\beta = 0,\ s\beta = -1}$$

$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix}$$

Equivalent angle-axis representation, $R_K(\theta)$



$$X_r = \theta.K = \begin{bmatrix} \theta.k_X \\ \theta.k_Y \\ \theta.k_Z \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} k_x.k_x.v\theta + c\theta & k_x.k_y.v\theta - k_x.s\theta & k_x.k_z.v\theta + k_y.s\theta \\ k_x.k_y.v\theta + k_z.s\theta & k_y.k_y.v\theta + c\theta & k_y.k_z.v\theta - k_x.s\theta \\ k_x.k_z.v\theta - k_y.s\theta & k_y.k_z.v\theta + k_x.s\theta & k_z.k_z.v\theta + c\theta \end{bmatrix}$$

3x3

with $v\theta = 1 - c\theta$

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = Ar\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

3x1

$${}^{A}K = \frac{1}{2.\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

singularity for $\sin\theta = 0$

quaternian

Euler Parameters

4x1 vector

$$\varepsilon_1 = W_x . \sin\frac{\theta}{2}$$

$$\varepsilon_2 = W_y . \sin\frac{\theta}{2}$$

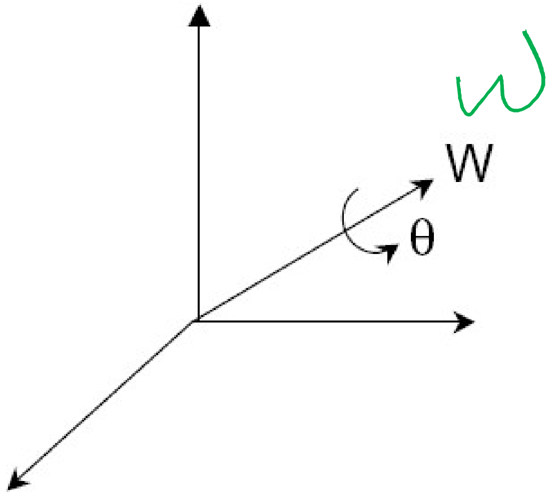
$$\varepsilon_3 = W_z . \sin\frac{\theta}{2}$$

3

+

$$\varepsilon_4 = \cos\frac{\theta}{2}$$

1



Normality Condition

$$|W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

ε : point on a unit hypershpere in four-dimensional space

Inverse Problem

Given ${}^{A}_{B}R$ find ε

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

Quaternion

$$r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)$$

$$(1-\varepsilon_4^2)$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4}$$

$\underline{\underline{\varepsilon_4 = 0?}}$

Vectors

Vector space: $\mathbb{V}$ (set closed under vector addition and scalar multiplication)

Normed linear space: vector space endowed with norm (magnitude of vector)

Inner product space: normed linear space endowed with inner product (like dot product, allows projection and concept of orthogonality)

We will consider 3-D Euclidean space, $\mathbb{E}^3$, which is an inner product space and has the additional operation of cross product.

Richard Murray, Shankar Sastry

(Murray, Li & Sastry)

Screw Theory

$\mathbb{E}^3$

Linear Transformation

$\mathcal{L} : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ is a linear transformation if it satisfies superposition: $\mathcal{L}(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 \mathcal{L}(\vec{v}_1) + \alpha_2 \mathcal{L}(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2$ in $\mathbb{V}_1$.

Example:

$\mathcal{L} : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$: $\mathcal{L}(\vec{w}) = \vec{v} \times \vec{w}$, $\vec{v}$ is a fixed vector.

$\mathcal{L} : \mathbb{E}^3 \longrightarrow \mathbb{R}$: $\mathcal{L}(\vec{w}) = \vec{v} \cdot \vec{w}$, $\vec{v}$ is a fixed vector.

$(\vec{v} \times)\vec{\omega}$ — vector

$\mathcal{L}(\vec{v} \cdot)\vec{\omega} \to$ scalar

Property preserving transformations ($\mathcal{L} : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$):

- length preserving $\|\mathcal{L}(\vec{v})\| = \|\vec{v}\|$
- cross product preserving $\mathcal{L}(\vec{v} \times \vec{w}) = \mathcal{L}(\vec{v}) \times \mathcal{L}(\vec{w})$

Euclidean norm

$\| \quad \|$

Orthonormal Frame

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal frame if

1. $||\vec{e}_i|| = 1$ (normality)
2. $\vec{e}_i \cdot \vec{e}_j = 0$ if $i \neq j$ (orthogonality)
3. $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$ (right hand rule)

By writing

$$\mathcal{E} = \left[\begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \right]$$

we can regard $\mathcal{E} : \mathbb{R}^3 \rightarrow \mathbb{E}^3$ as a linear transformation.

Coordinate Representation

Given $\vec{v} \in \mathbb{E}^3$, $\vec{v}$ *represented* in $\mathcal{E}$ is $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ where $\vec{v} = \mathcal{E} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

The *adjoint* of $\mathcal{E}$ is $\mathcal{E}^* : \mathbb{E}^3 \to \mathbb{R}^3$:

$$\mathcal{E}^* = \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \\ \vec{e}_3 \cdot \end{bmatrix}.$$

(adjoint is just like transpose, note that $\vec{w} \cdot (\mathcal{E} v) = (\mathcal{E}^* \vec{w})^T v$).

Then $\mathcal{E}^* \mathcal{E} = I_3$ and

$$v = \mathcal{E}^* \vec{v}$$

$\mathcal{E}^*$ is the projection onto the coordinate axes, providing the coordinate readout.

Coordinate Transformation

Let $\mathcal{E}_a = \begin{bmatrix} \vec{e}_{a_1} & \vec{e}_{a_2} & \vec{e}_{a_1} \end{bmatrix}$ and $\mathcal{E}_b = \begin{bmatrix} \vec{e}_{b_1} & \vec{e}_{b_2} & \vec{e}_{b_1} \end{bmatrix}$ be two coordinate frames. Then

$$\vec{v} = \mathcal{E}_a v_a = \mathcal{E}_b v_b$$

where v_a is $\vec{v}$ represented in $\mathcal{E}_a$ and v_b is $\vec{v}$ represented in $\mathcal{E}_b$. It follows

$$v_a = \underbrace{\mathcal{E}_a^* \mathcal{E}_b}_{R_{ab}} v_b$$

where R_{ab} is a 3×3 matrix.

Show

$$R_{ab} = R_{ba}^{-1} = R_{ba}^T, \quad v_b = R_{ba} v_a$$

If we attach a frame $\mathcal{E}_a$ to a rigid body and $\mathcal{E}_b$ is the reference frame. Then R_{ab} (or R_{ba}) provides the *relative rotation* between the two frames. R_{ab} is therefore called the orientation (or attitude) matrix, representing the orientation of the body with respect to the reference.

Representation of vectors and transforms

Given an (orthonormal) *frame* $\mathcal{E}_a$ in $\mathbb{E}^3$. A vector $\vec{v} \in \mathbb{E}^3$ represented in $\mathcal{E}_a$ is $v_a = \mathcal{E}_a^* \vec{v}$.

Consider a linear transform $L : \mathbb{E}^3 \to \mathbb{E}^3$ (with corresponding frames $\mathcal{E}_a$ and $\mathcal{E}_b$). Representation of $L(\vec{v})$ in $\mathcal{E}_b$ is

$$\mathcal{E}_b^* L(\vec{v}) = \mathcal{E}_b^* L(\mathcal{E}_a v) = (\mathcal{E}_b^* L \mathcal{E}_a) v.$$

Therefore, the representation of L wrt $\mathcal{E}_a$ and $\mathcal{E}_b$ is $L_{a,b} = \mathcal{E}_b^* L \mathcal{E}_a$. If $\mathcal{E}_a = \mathcal{E}_b$, then $L_a = \mathcal{E}_a^* L \mathcal{E}_a$.

Summary:

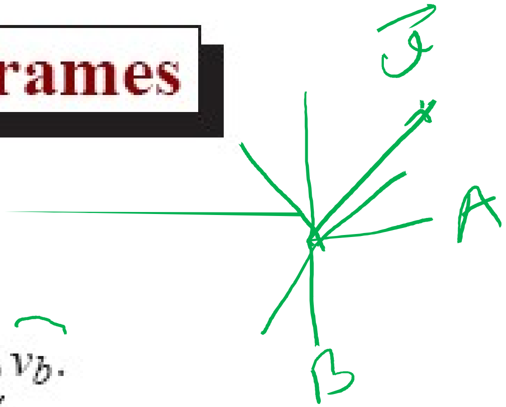
$\mathcal{E}_a^* \vec{v} = v_a \longleftrightarrow \vec{v} = \mathcal{E}_a v_a$

$\mathcal{E}_a^* L \mathcal{E}_a = L_a \longleftrightarrow L = \mathcal{E}_a L_a \mathcal{E}_a^*$

(Note: $(e_{a_i})_a$ is the ith unit vector and $(\mathcal{E}_a)_a = I_3$.)

$\mathcal{E}_a$ — $\mathcal{E}_b$ $\mathcal{E}_b$

Transformation between frames



Now consider two frames $\mathcal{E}_a$, $\mathcal{E}_b$ in $\mathbb{E}^3$. Then,

$$\vec{v} = \mathcal{E}_a v_a = \mathcal{E}_b v_b, \qquad v_a = \underbrace{\mathcal{E}_a^* \mathcal{E}_b}_{R_{ab}} v_b.$$

$\rightarrow$ Rot^n matrix

Similarly, for $L : \mathbb{E}^3 \to \mathbb{E}^3$

$$L = \mathcal{E}_a L_a \mathcal{E}_a^* = \mathcal{E}_b L_b \mathcal{E}_b^* \qquad L_a = \mathcal{E}_a^* \mathcal{E}_b L_b \mathcal{E}_b^* \mathcal{E}_a = R_{ab} L_b R_{ba}.$$

Again, $R_{ba} = (R_{ab})^T = R_{ab}^{-1}$.

$$R_{ba} = R_{ab}^T = R_{ab}^{-1} \qquad {}^A_B R = {}^B_A R^T$$

Examples

$(\vec{\omega}_a)$

- **Dot product**

r_x r_y r_z

$$\vec{w}\cdot\vec{v} = \vec{w}\cdot\mathcal{E}_a v_a = \begin{bmatrix} \underbrace{\vec{w}\cdot\vec{e}_{a_1}}_{\hat{x}} & \underbrace{\vec{w}\cdot\vec{e}_{a_2}}_{y} & \underbrace{\vec{w}\cdot\vec{e}_{a_3}}_{\hat{z}} \end{bmatrix} = w_a^T v_a = w_b^T v_b.$$

Therefore, $(\vec{w}\cdot)_a = w_a^T$**, and dot product is invariant under coordinate transform.**

- **Cross product**

$$\vec{w}\times\vec{v} = \vec{w}\times\mathcal{E}_a v_a = \begin{bmatrix} \vec{w}\times\vec{e}_{a_1} & \vec{w}\times\vec{e}_{a_2} & \vec{w}\times\vec{e}_{a_3} \end{bmatrix} v_a.$$

Therefore, $w\times$ **represented in** $\mathcal{E}_a$ **is**

$$\boxed{(\vec{w}\times)_a} = \mathcal{E}_a^* \begin{bmatrix} \vec{w}\times\vec{e}_{a_1} & \vec{w}\times\vec{e}_{a_2} & \vec{w}\times\vec{e}_{a_3} \end{bmatrix} = \begin{bmatrix} 0 & -w_{a_3} & w_{a_2} \\ w_{a_3} & 0 & -w_{a_1} \\ -w_{a_2} & w_{a_1} & 0 \end{bmatrix}.$$

operator

$\Rightarrow$ anti Symmetric Matrix (skew)

Notation

$\hat{\omega}$

matrix

vector

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \longrightarrow \hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \longrightarrow \underline{\underline{A^\vee}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

(hat)

Note $\underline{(\hat{w})^\vee = w.}$

$\vec{\omega} \to (\hat{\omega})^\vee = \vec{\omega}$

$\vec{\omega}$

vector

$(\vec{\omega}\times) = \hat{\omega}$ (ulta hat)

Matrix $\overset{\vee}{A}$ vector

A matrix

Skew Symmetric matrix

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

3×3

Identities Involving $\hat{v}$

- $\hat{v}v = 0$
- $\hat{v} = -\hat{v}^T$ ($\widehat{v}$ **is skew-symmetric)**
- $\hat{v}w = -\hat{w}v$
- $\hat{v}\hat{w} = wv^T - (v^T w)I_3$ $(\vec{v} \times (\vec{w} \times \cdot))$
- $\widehat{\hat{v}w} = wv^T - vw^T$ $((\vec{v} \times \vec{w})\times)$
- $\widehat{v}^3 = \widehat{v}\widehat{v}\widehat{v} = -\|v\|^2\,\widehat{v}$
- $\widehat{v}^4 = \widehat{v}\widehat{v}\widehat{v}\widehat{v} = -\|v\|^2\,\widehat{v}^2$
- **If** $\|v\| = 1$**, then** $\widehat{v}^{4k} = -\widehat{v}^2$, $\widehat{v}^{4k-1} = -\widehat{v}$, $\widehat{v}^{4k-2} = \widehat{v}^2$, $\widehat{v}^{4k-3} = \widehat{v}$, $k = 1, 2, \ldots$ **(Alternate derivation: Caley-Hamilton Theorem)**

Interpretation of R_{ab}

1. R_{ab} is the representation of $\mathcal{E}_b$ in $\mathcal{E}_a$.

2. R_{ab} is the direction cosine matrix.

3. R_{ab} changes the representation of the same vector from $\mathcal{E}_b$ to $\mathcal{E}_a$.

4. R_{ab} rotates $\mathcal{E}_a$ to $\mathcal{E}_b$.

Example

$\mathcal{E}_b = \mathcal{E}_a$ **rotated about** e_{a_3} **over** θ**:**

$$\vec{e}_{b_1} = \cos\theta \vec{e}_{a_1} + \sin\theta \vec{e}_{a_2}, \quad \vec{e}_{b_2} = -\sin\theta \vec{e}_{a_1} + \cos\theta \vec{e}_{a_2}.$$

Therefore, $R_{ab} = [(\vec{e}_{b_1})_a, (\vec{e}_{b_2})_a, (\vec{e}_{b_3})_a] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$**. To verify,** $R_{ab}(\vec{e}_{a_1})_b = (\vec{e}_{b_1})_b$**.**

Similarly, rotation about e_{a_1}**:** $R_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$ $(c = \cos\theta$**,** $s = \sin\theta)$

rotation about e_{a_2}**:** $R_{ab} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$**.**

R_{ob}

origin

Convention

Robotics literature is inertial-centric: orientation is usually in the form R_{ob} where o is the inertial frame and b is the body frame, e.g., tool frame $[n, o, a]$ (normal-orthogonal-approach axes represented in the inertial frame).

Spacecraft literature is body-centric: orientation is usually in the form R_{bo} where b is the spacecraft body frame and o is the inertial frame (inertial axes represented in the body frame).

R_{bo}

Representation of Rotation Operator

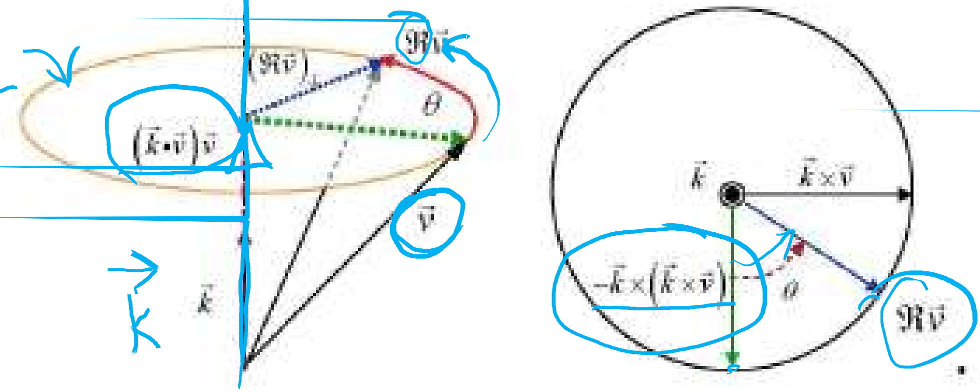
Consider a general rotation $\mathcal{R}$ about a unit vector $\vec{k}$ over an angle θ.

$R\vec{v}$

$(\vec{k}\cdot\vec{v})\vec{v}$

$\perp$

$\vec{k}$



projection

Consider $\mathcal{R}\vec{v}$. Decompose $\mathcal{R}\vec{v}$ into a component along $\vec{k}$ and a component perpendicular to $\vec{k}$:

$\vec{k}$ $\perp$

$$\mathcal{R}\vec{v} = (\vec{k}\cdot\vec{v})\vec{k} + (\mathcal{R}\vec{v})_\perp.$$

Consider the disk formed by the tip of $\vec{v}$ rotating about $\vec{k}$. Then $(\mathcal{R}\vec{v})_\perp$ may be decomposed as

$$(\mathcal{R}\vec{v})_\perp = \sin\theta\vec{k}\times\vec{v} - \cos\theta\vec{k}\times(\vec{k}\times\vec{v}).$$

in the vertical plane

Euler-Rodrigues Formula

$= (\vec{k}\cdot\vec{v})\vec{k} + \sin\theta\,\vec{k}\times\vec{v} \boxed{- \cos\theta\,\vec{k}\times(\vec{k}\times\vec{v})}$

Hence

$$\mathcal{R}\,\vec{v} = (\vec{k}\vec{k}\cdot + \sin\theta\vec{k}\times - \cos\theta\vec{k}\times(\vec{k}\times))\vec{v}.$$

1 2 3

Representing $\vec{v}$ and $\vec{k}$ with respect to a frame, we have $\mathcal{R}$ as a 3×3 matrix:

$$R = I_3 + \sin\theta\hat{k} + (1 - cos\theta)\hat{k}^2.$$

$\vec{K}$ axis of rotn & θ $\angle$ of rotn

This is called the $\boxed{\text{Euler-Rodrigues Formula.}}$

In spacecraft literature, you'll see

$$R^T = I_3 - \sin\theta\hat{k} + (1 - cos\theta)\hat{k}^2.$$

compact

$$\hat{K} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$