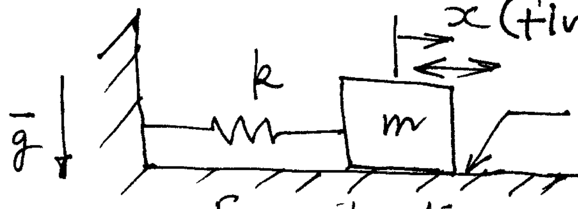
SOME SPECIAL TOPICS

(8) Coulomb Damping or Dry Friction Consideration



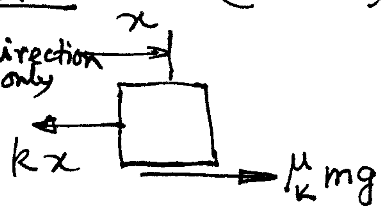
Coulomb damping (friction); μ_s, μ_K

Static (limiting) friction coefficient | Kinetic friction coefficient

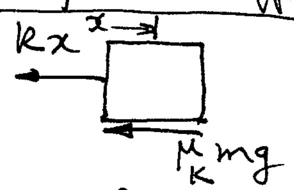
($\mu_K < \mu_s$)

Free vibration

Relevant FBDs:~ (in horizontal direction only)



while $\dot{x} < 0$



while $\dot{x} > 0$

While in motion: $\boxed{\mu_K mg = f_f = \text{friction force}}$

→ There are two DEOM :~

(i) for motion towards right ($\dot{x} > 0$),

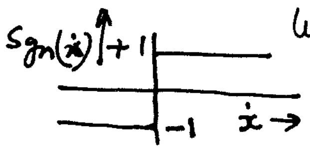
$$m\ddot{x} = -kx - \mu_K mg \quad \text{or,} \quad m\ddot{x} + kx = -\mu_K mg \quad \text{---(1)}$$

(ii) for motion towards left ($\dot{x} < 0$),

$$m\ddot{x} = -kx + \mu_K mg \quad \text{or,} \quad m\ddot{x} + kx = +\mu_K mg \quad \text{---(2)}$$

(1) & (2) can be combined into a single DEOM:

$$m\ddot{x} + \mu_K mg\, \mathrm{sgn}(\dot{x}) + kx = 0 \quad \text{---(3)}$$



where $\left.\mathrm{sgn}(\dot{x})\right\} \begin{array}{l} = +1 \text{ for } \dot{x} > 0 \\ = -1 \text{ for } \dot{x} < 0 \end{array}$ $\left[\mathrm{sgn}(\dot{x}) = \dfrac{\dot{x}}{|\dot{x}|}\right]$

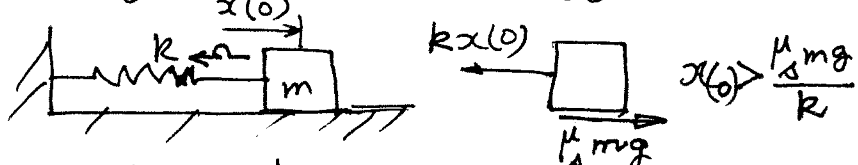
(The signum or sign of function)

Note that (1) & (2) seems to indicate forced vibration but since the damping force is passive in nature, through a simple coordinate change (transformation), the DEOM can be rendered homogeneous & hence represent free vibration.

→ So, what is happening is this:– The mass is initially pulled to right or pushed to left initially by an amount $x(0)$ such that the spring force generated is

(PTO)

sufficient to overcome the limiting friction force of $\mu_s mg$ & when released, vibration occurs. Of course, we could add some initial velocity $\dot{x}(0)$ to $x(0)$ but for now we assume a simple situation where $x(0) > 0$ & $\dot{x}(0) = 0$.



→ So, as the mass is released, it accelerates towards left & a velocity towards left ($\dot{x} < 0$) is generated. The mass goes past the equilibrium position & comes to standstill momentarily after half cycle of motion. If in that position, spring force can overcome limiting friction force, the mass starts moving towards right ($\dot{x} > 0$) until it comes to standstill momentarily again at the end of one full cycle of motion & the oscillations continue until, finally, at the end of a particular half-cycle, the displacement is insufficient so that spring force fails to ~~overcome~~ overcome friction force and the motion stops.

→ <u>We now</u> study this situation analytically.

For the first half cycle of motion, $\dot{x}(t)$ is < 0 (towards left). So, we use DEOM ②. For convenience, let $x(0) = x_0$.

So, $m\ddot{x} + kx = \mu_k mg \Rightarrow \ddot{x} + \omega_n^2 x = \frac{F_f}{m}$ ($F_f = \mu_k mg$ = friction force)

$\Rightarrow \ddot{x} + \omega_n^2 x = \frac{k}{m} \cdot \frac{F_f}{k} = \omega_n^2 X_e$ where $X_e = \frac{F_f}{k}$ = an equivalent displacement.

The motivation behind putting the DEOM in the form $\ddot{x} + \omega_n^2 x = \omega_n^2 X_e$ is that it results in a simplified, easily interpretable solution as will be seen soon.

→ The DEOM $\ddot{x} + \omega_n^2 x = \omega_n^2 X_e$ — (1) is subject to initial conditions $x(0) = x_0$ & $\dot{x}(0) = 0$ & so, its solution is

$$x(t) = (x_0 - X_e)\cos\omega_n t + X_e \quad \text{(\underline{Show this})} \quad \text{--- (2)}$$

which represents harmonic oscillations superposed on the average response X_e.

→ Equn (2) is valid for $0 \le t \le t_1$, where t_1 is the time at which the velocity becomes zero & the motion is about to reverse its direction.

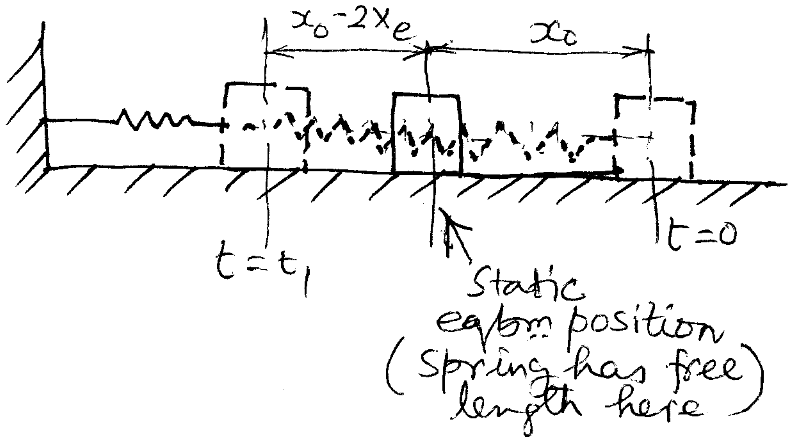
From (2)

$$\dot{x}(t) = -\omega_n (x_0 - X_e)\sin\omega_n t \quad \text{--- (3)}$$

$\therefore \dot{x}(t) = 0 \Rightarrow \sin\omega_n t = 0$ & the <u>least</u> value t_1 of t which satisfies this relation for $t>0$ is given by $t_1 = \frac{\pi}{\omega_n}$

[$\sin\omega_n t = 0 \Rightarrow \omega_n t = r\pi;\ r = 0, 1, 2, \ldots$] omit since it corresponds to $t=0$.
So, $\omega_n t_1 = 1\times\pi = \pi \Rightarrow t_1 = \frac{\pi}{\omega_n}$ etc.]

Then, $\underline{x(t_1) = -(x_0 - 2X_e)}$. [Value of $x(t)$ at the end of ½ cycle]



$$2X_e = \frac{2F_f}{k} = \frac{2\mu_k mg}{k}$$

Let us assume that $x(t_1)$ is large enough to initiate motion towards right. For this motion, $\dot{x}(t)$ is >0 & hence the DEOM to be used is $m\ddot{x}+kx=-\mu_k mg$, or,

(4) .... $\ddot{x}+\omega_n^2 x = -\omega_n^2 X_e$, subject to the (initial) conditions $x(t_1)=-(x_0-2X_e)$, $\dot{x}(t_1)=0$.

The solution of (4) is (show this)

$$x(t) = (x_0-3X_e)\cos\omega_n t - X_e \quad \text{--- (5)}$$

Response (5) is valid in $t_1 \le t \le t_2$ where t_2 is the time after t_1 when $\dot{x}(t)$ becomes zero again & a full cycle of vibration is completed. This value, obviously, is $t_2 = \frac{2\pi}{\omega_n}$ & at this time, motion is ready to reverse direction. Also, $x(t_2) = (x_0-3X_e)\cos 2\pi - X_e = x_0 - 4X_e$

→ Note that reduction in amplitude is $4X_e = \frac{4F_f}{k}$ for the first cycle of motion. You can easily prove that the same is true of subsequent cycles of motion provided they take place.

→ Let n = number of half cycles of motion just before motion stops.

By closely observing (2) & (5), we see that a pattern emerges and after n half cycles, we get

$$x(t) = \{x_0-(2n-1)X_e\}\cos\omega_n t \pm X_e$$

where $t = \frac{n\pi}{\omega_n}$ when $\dot{x}(t)$ becomes zero again!

→ After n half cycles, the mass reaches a position at a distance of $x_0 - 2nX_e$ from static eqbm position. So, if the force in

the spring, i.e., $k(x_0 - 2nX_e) <$ Max^m friction force, ($\mu_s mg$)
motion ceases.

Hence, n is the least integer satisfying the relation $x_0 - 2nX_e < \frac{\mu_s mg}{k}$.

Example:- (from 'Fundamentals of vibrations' by Meirovitch)

Let $m = 400$ kg, $k = 14\times10^4$ N/m, $\mu_s = 0.11$ & $\mu_k = 0.1$.

Given:- $x(0) = x_0 = 3$ cm, $\dot{x}(0) = 0$

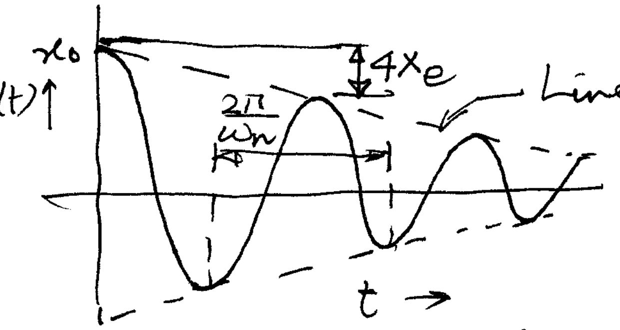
(i) What is the decay/cycle? Obtain
(ii) No. of half-cycles until oscillation stops
(iii) Position of mass after oscillation stops

Solution:-

Do the details

(i) Decay of amplitude per cycle $= 4X_e = 1.1212$ cm
(ii) $n = 5$
(iii) Reqd $x = -0.197$ cm.

→ The Response looks like this:



→ Most interestingly, free vibration under Coulomb damping occurs at the undamped natural frequency ω_n!

Home Work:- Using the work-energy theorem & considering half cycle at a time, show that the reduction in amplitude per cycle is $4F_f/k$ where $F_f = \mu_k mg$.