Design factor:

$$n_d = \frac{\text{loss of function parameter}}{\text{max. allowable parameter}}$$

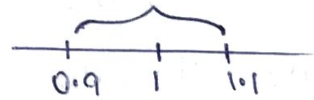
$$\text{Max. allowable parameter} = \frac{\text{loss of fu}^{n}\text{ parameter}}{n_d}.$$

Example: uncertainity in failure load $= \pm 10\%$

— " — max. load acting $= \pm 5\%$ on structure

Nominal failure load $= 10$ kN

ⓐ Assume uncertainity in failure load only



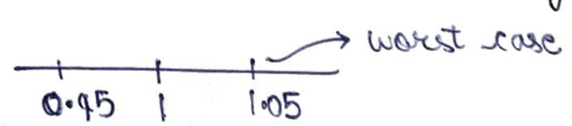
$$\text{max. allow parameter} = \frac{\text{failure load}}{n_d}$$

$$= 0.9 \times \text{Nominal failure load}$$

$$\Rightarrow \boxed{n_d = \frac{1}{0.9}}$$

ⓑ Assume uncertainity in max. load on structure only.

load:



$\Rightarrow$ Design based on: Max load $= 1.05 \times$ Nominal value.

$$\text{Nominal}^{\text{max.}}\text{ value} = \frac{1}{1.05} \times \text{ab. Max. value}$$

$$n_d = \frac{1}{\frac{1}{1.05}} = 1.05$$

Finally when both uncertainities are present, (assumed to be independent)

$$n_d = n_{d1} \times n_{d2} = \frac{1.05}{0.9} = \frac{7}{6}.$$

Max. allowable load $= \frac{\text{Failure load}}{n_d} = \frac{60}{7}$ kN.

Factor of safety

$n_f = n_d$ (after all dimensions are finalized)

Eg: A solid circular rod under torsion T = 100 N

Data available: Material shear strength = 85 MPa

Design factor $n_d = 2.5$

Design the rod. (diameter)

Unknown: diameter of the rod.

Max. shear stress acting on the rod

$$\tau_{max} = \frac{2T}{\pi r^3} \text{ or } \frac{Tr}{J} \qquad J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

(on top surface)

$$\leq \frac{\text{shear strength}}{n_d} = \frac{85 \text{ MPa}}{2.5}$$

$$\Rightarrow r^3 \geq \frac{2T}{\pi \times \frac{85}{2.5}} = \frac{2 \times 100 \times 2.5}{85 \times \pi}$$

$$\Rightarrow r \geq 12.3 \text{ mm}.$$

Diameter $= d = 24.6$ mm

Standard size: $d = 25$ mm (Table A-17 of Shigley's book)

Allowable shear stress (actual) $= \tau_{allow} = \frac{2T}{\pi r^3} = 32.59$ MPa

factor of safety $= n_f = \frac{85 \text{ MPa}}{32.59 \text{ MPa}} = 2.6$

Modes of Material Failure

1. Static yielding → plastic deformation in ductile materials

2. Brinelling — localized yielding in regions of contact betn curved surfaces.

3. Brittle fracture/ rupture

4. Ductile rupture

5. Fatigue

6. Corrosion

7. Wear

8. Spalling - small pieces get spontaneously removed from a part.

9. Buckling

Engineering Materials

1. Steel – predominantly used
   - alloy of iron and carbon
   - Cr, Mn, Ni, V, Mo

   Ni – increases strength without decreasing ductility

   Cr – increases wear resistance, hardness

   Mn – Acts as deoxidising and desulphurising agent (corrosion resistance)

2. Cast Iron

3. Non-ferrous metals
   - aluminium
   - copper based alloys

4. Non-metals
   - Glass
   - Plastic
   - Wood
   - Polycarbonate
   - Composites

Material Selection

- Need to pay attention to →
  - operational / functional aspects
  - potential failure modes
  - market related factors

Functional aspects

Property

1. High strength / volume ratio – yield / ultimate strength.
2. High strength to weight → yield / ultimate strength per unit mass.
3. Long term dimensional stability → Elastic modulus, thermal exp^n coefficient, corrosion resistance, creep resistance

⋮

Performance metric of a structure / part

- functional requirement
- geometry
- material properties

$$P = f(F, G, M)$$

functional, Geometry, man.

When the aspects are independent

$$P = f_1(F)\, f_2(G)\, f_3(M)$$

For optimum design, maximise or minimise P

Eg. Want to design light, stiff, axially loaded rod - of constant cross section.

Performance matrix $P = $ "m" is to be minimised.
↳ mass

Stiffness of the rod $K = \frac{F}{\delta} = \frac{EA}{l}$

$\Rightarrow A = \frac{Kl}{E}$

Mass of the rod $m = \rho A \cdot l = \rho \cdot \frac{Kl}{E} \cdot l$

$$= \underbrace{K}_{\text{functional}} \cdot \underbrace{l^2}_{\text{geometry}} \cdot \underbrace{\left(\frac{\rho}{E}\right)}_{\text{material}}$$

Minimise m:

Define Performance matrix $P = \frac{m}{K}$

∴ for optimum choice, minimise P

$$\boxed{\frac{m}{K} = l^2 \underbrace{\left(\frac{\rho}{E}\right)}_{= f_3(m)}}$$

↓
material efficiency coefficient

- For choosing material: maximize $\boxed{E/\rho}$

From Ashby chart, (2-11)

candidate materials - Al_2O_3

Technical ceramic

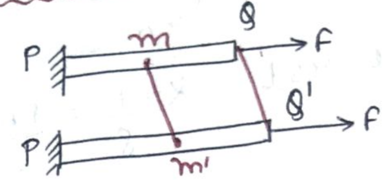
Si_3N_4

B_4C

→ Refer slides.

strain and deformation

1D :



(original)

(after deformation)

$Q \to Q'$, $m \to m'$

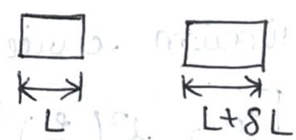
strain → 1. $e = \frac{L_f - L_0}{L_0} = \frac{\Delta u}{L_0}$ (engineering strain)

2. $\varepsilon = \frac{L_f - L_0}{L_f} = \frac{\Delta u}{L_f}$ (True strain)

3. $\varepsilon = \ln\left(\frac{L_f}{L_0}\right) = \ln(\lambda)$

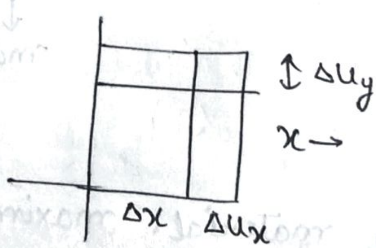
↳ stretch

(Logarithmic strain)



$\varepsilon = \int_{L_0}^{L_f} \frac{\delta L}{L}$ (true)

2D:

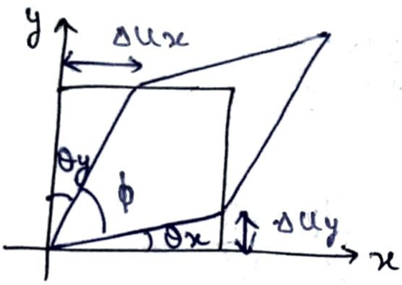


$\varepsilon_x = \frac{\Delta u_x}{\Delta x} = \frac{\partial u_x}{\partial x}$ (as $\Delta x \to 0$)

(strain along x- direction)

$\varepsilon_y = \frac{\Delta u_y}{\Delta y} = \frac{\partial u_y}{\partial y}$ (as $\Delta y \to 0$)

(along y- direction)



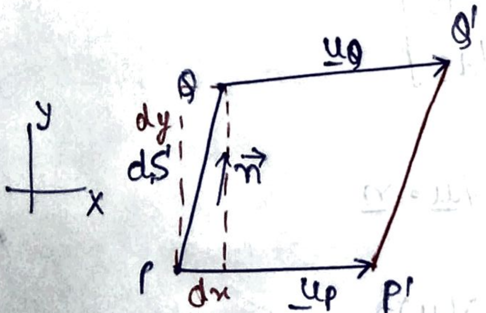
$$\gamma_{xy} = \frac{\pi}{2} - \phi = \theta_x + \theta_y$$

$$\cong \frac{\Delta u_x}{\Delta y} + \frac{\Delta u_y}{\Delta x} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

$$\Rightarrow \varepsilon_{xy} = \frac{1}{2}\gamma_{xy}.$$

Materials resist any relative displacement any two points.

Strain is a measure of quantifying such relative displacement.



PQ → small element

$\underline{u}_P$: displacement of point P

$\underline{u}_Q$: ——————— " ——————— Q

$\vec{n}$: unit vector along PQ

Relative displacement betn Q and P:

$$d\underline{u} = \underline{u}_Q - \underline{u}_P = \begin{bmatrix} du_x \\ du_y \end{bmatrix}$$ functions of X and Y.

unit relative displacement:

$$\frac{d\underline{u}}{ds} = \begin{bmatrix} \frac{du_x}{ds} \\ \frac{du_y}{ds} \end{bmatrix}$$

$$\frac{du_x}{ds} = \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{du_y}{ds} = \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial s}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{\partial u_x}{\partial s} \\ \frac{\partial u_y}{\partial s} \end{bmatrix}}_{\frac{d\underline{u}}{ds}} = \underbrace{\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix}}_{\nabla \underline{u}} \underbrace{\begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{bmatrix}}_{\underline{n}}$$

$$d\underline{s} = \underline{PQ} = dx\,\hat{i} + dy\,\hat{j}$$

$$\boxed{\underline{n} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}}$$

$\underline{n}$: unit vector along $\underline{PQ}$

$$\underline{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} dx/ds \\ dy/ds \end{bmatrix}$$

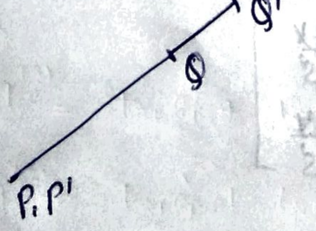
$$\Rightarrow \frac{d\underline{u}}{ds} = \nabla \underline{u} \cdot \underline{n}$$

strain : $\text{sym}(\nabla \underline{u})$
(E)
$$= [\nabla \underline{u} + (\nabla \underline{u})^T]/2$$

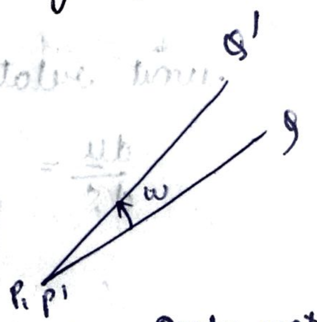
Rotation $\underline{\underline{\Omega}} = [\nabla \underline{u} - (\nabla \underline{u})^T]/2$

$\underline{\underline{E}} \cdot \underline{n}$ gives change in length of PQ

$\underline{\underline{\Omega}} \cdot n$ gives rotation of PQ.



1. Only stretch, no rotation



2. Only rotation

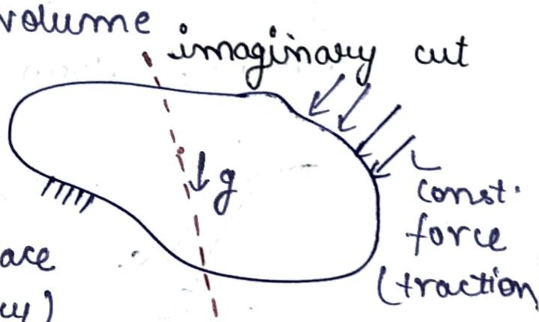
Verify that : $\underline{E}\cdot\underline{n} \neq 0$, $\underline{\Omega}\cdot\underline{n} = \underline{0}$ in 1

$\underline{E}\cdot\underline{n} = \underline{0}$, $\underline{\Omega}\cdot\underline{n} \neq 0$ in 2

★ In 3D the same relations hold.

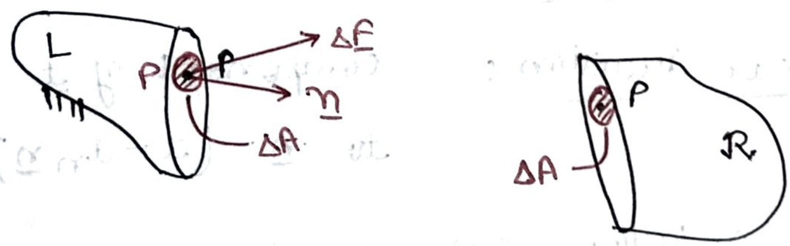
Stresses:

- Body force - acts on entire volume



- contact force (applied on surface or boundary)

Cauchy hypothesis: We can define a surface traction on any point inside the body.



$\underline{n}$: unit outward normal to ΔA at P

- Effect of material (in ΔA on R) on the ΔA in left half in a distributed force in ΔA around P →

Has a resultant $\Delta\underline{F}$

can think of an internal response at P.

Contact force on ΔA at P is $\Delta\underline{F}$

traction vector at P:

$$\underline{t} = \lim_{\Delta A \to 0} \frac{\Delta \underline{F}}{\Delta A}$$

(P is always inside ΔA during limit process)

- We always assume that limit exists

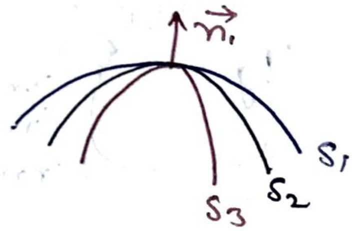
contact force per unit area $= \frac{\Delta \underline{F}}{\Delta A}$

- Traction vector $\underline{t}$ at P depends on the orientation of the imaginary cut plane.

$$\underline{t} = t(\underline{x}, \underline{n})$$

Hypothesis:

1. If different surfaces are chosen at P with same normal, then $\underline{t}$ remains the same.



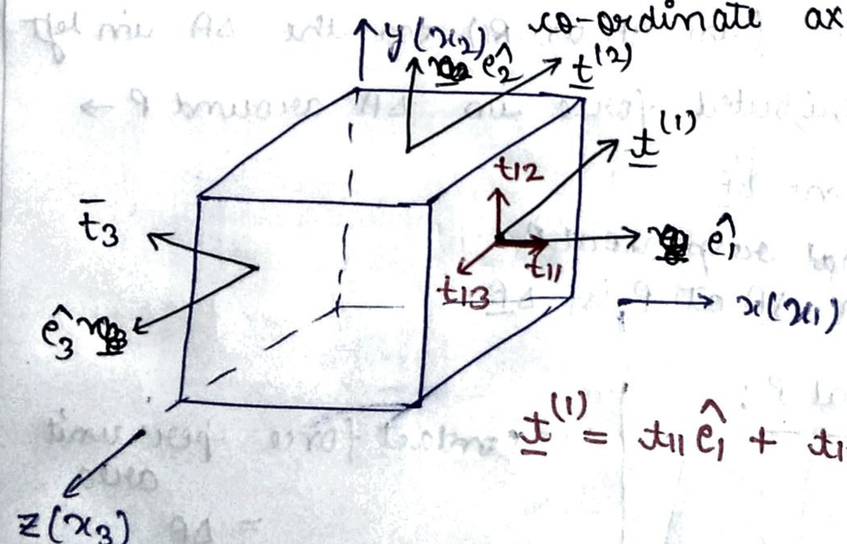
2. If two surfaces at P have different normal, then $\underline{t}$ is different.

$\Rightarrow$ There are infinitely many traction vector at P!!

- $t_n = \underline{t} \cdot \underline{n}$ is normal traction (stress) at P

- Shear traction: component of $\underline{t}$ perpendicular to $\underline{n} = (t - t_n \underline{n})$

In a 3D body: Define traction on 3 planes passing through P. Planes are perpendicular to co-ordinate axes.



$\hat{e}_i$ is unit vector along x_i axis.

$$\underline{t}^{(1)} = t_{11}\hat{e}_1 + t_{12}\hat{e}_2 + t_{13}\hat{e}_3 = t_{1i}\hat{e}_i$$

(using summation convention)

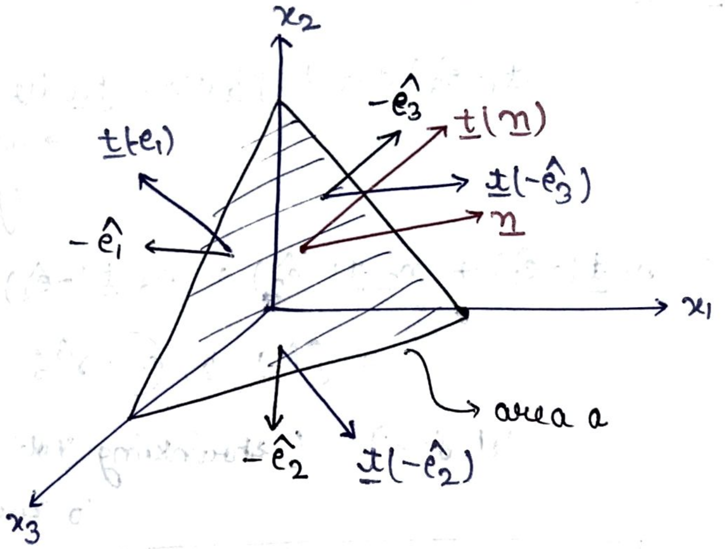
$$\underline{t}^{(2)} = t_{21}\hat{e}_1 + t_{22}\hat{e}_2 + t_{23}\hat{e}_3 = t_{2i}\hat{e}_i$$

$$\underline{t}^{(3)} = t_{31}\hat{e}_1 + t_{32}\hat{e}_2 + t_{33}\hat{e}_3 = t_{3i}\hat{e}_i$$

Normal stress components at P: t_{11}, t_{22}, t_{33}

Shear —— " —— : $t_{12}, t_{13}, t_{21}, t_{23}, t_{31}, t_{32}$.

Traction vector on an arbitrary plane



Inclined surface

unit normal $= \bar{n}$

traction vector $= \bar{t}(\bar{n})$

area $= a$.

On other surfaces:

OAC : normal: $-\hat{e}_1$, traction $\underline{t}(-\hat{e}_1)$

area $= a_1$

OAB : normal: $-\hat{e}_2$, traction: $\underline{t}(-\hat{e}_2)$, a_2

OAC : normal: $-\hat{e}_3$, traction: $\underline{t}(-\hat{e}_3)$, a_3

$$\underline{n} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$$

$da_1 =$ area of OAC $= da \cdot n_1 \sim \delta^2$

$da_2 =$ —— OAB $= da \cdot n_2 \sim \delta^2$

$da_3 =$ —— OBC $= da \cdot n_3 \sim \delta^2$

Volume of OABC $= dv$

$\delta =$ distance of inclined plane ABC from origin O.

$da \sim \delta^2$

$dv \sim \delta^3$

Linear momentum balance of OABC:

$\underline{t}(\underline{n})\, da + t(-\hat{e}_1)\, da_1 + t(-\hat{e}_2)\, da_2 + t(-\hat{e}_3)\, da_3 + \rho \underline{b}\, dv = \rho \underline{a}\, dv$ ↳ acceleration of OABC

$$\left\{ \underline{t}(\underline{n}) + n_1 \underline{t}(-\hat{e}_1) + n_2 \underline{t}(-\hat{e}_2) + n_3 \underline{t}(-\hat{e}_3) \right\} da\, \delta^2 = \rho(\underline{a} - \underline{b})\, dv\, \delta^3\, \delta$$

let $\delta \to 0$ (shrinking vol. OABC to zero)

$$\Rightarrow \boxed{\underline{t}(\underline{n}) + n_1 \underline{t}(-\hat{e}_1) + n_2 \underline{t}(-\hat{e}_2) + n_3 \underline{t}(-\hat{e}_3) = 0}$$

Let $\underline{n} = \hat{e}_1$, $\underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow n_1 = 1, \quad n_2 = n_3 = 0$

$$\Rightarrow \boxed{\underline{t}(\underline{n}) = n_1 \underline{t}(\hat{e}_1) + n_2 \underline{t}(\hat{e}_2) + n_3 \underline{t}(\hat{e}_3)}$$

$$\underline{t}_1 = \underline{t}(\hat{e}_1) = \sigma_{11}\hat{e}_1 + \sigma_{12}\hat{e}_2 + \sigma_{13}\hat{e}_3$$
$$= \sigma_{ij}\hat{e}_j$$

$\underline{t}(\underline{n}) = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, $\quad \underline{t}_1 = \underline{t}(\hat{e}_1) = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix}$

$\underline{t}_2 = \underline{t}(\hat{e}_2) = \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix}$ $\quad \underline{t}_3 = \underline{t}(\hat{e}_3) = \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix}$

$$\Rightarrow \quad \underline{t}(\underline{n}) = n_1\, \underline{t}(\hat{e}_1) + n_2\, \underline{t}(\hat{e}_2) + n_3\, \underline{t}(\hat{e}_3)$$

$$\begin{bmatrix} t_1(\underline{n}) \\ t_2(\underline{n}) \\ t_3(\underline{n}) \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} n_1 + \begin{bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{bmatrix} n_2 + \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix} n_3$$

$$\underline{t}(\underline{n}) = \underline{\underline{\sigma}} \cdot \underline{n} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\boxed{\underline{t}(\underline{n}) = \underline{\underline{\sigma}} \cdot \underline{n}}$$

Angular momentum balance:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

$\Rightarrow$ $\underline{\underline{\sigma}}$ is a symmetric ~~matrix~~ tensor

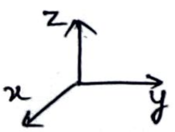
$\Rightarrow \sigma_{12} = \sigma_{21}$

$\sigma_{13} = \sigma_{31}$

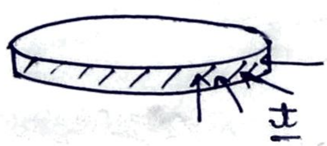
$\sigma_{23} = \sigma_{32}$

- $\underline{\underline{\sigma}}$ has six independent elements

Plane - stress



- Thin disk subject to lateral traction



- Traction on lateral surfaces
- The body is thin

Can assume:

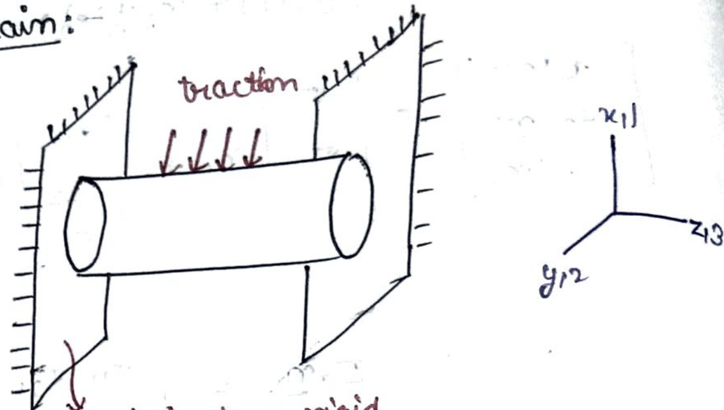
- Qualities of independent of z
- $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0.$

or $\sigma_{11} = \sigma_{13} = \sigma_{23} = 0.$

Unknown stresses: $\sigma_{11}, \sigma_{22}, \sigma_{12}$

or $\sigma_x, \sigma_y, \tau_{xz}$

Plane strain:



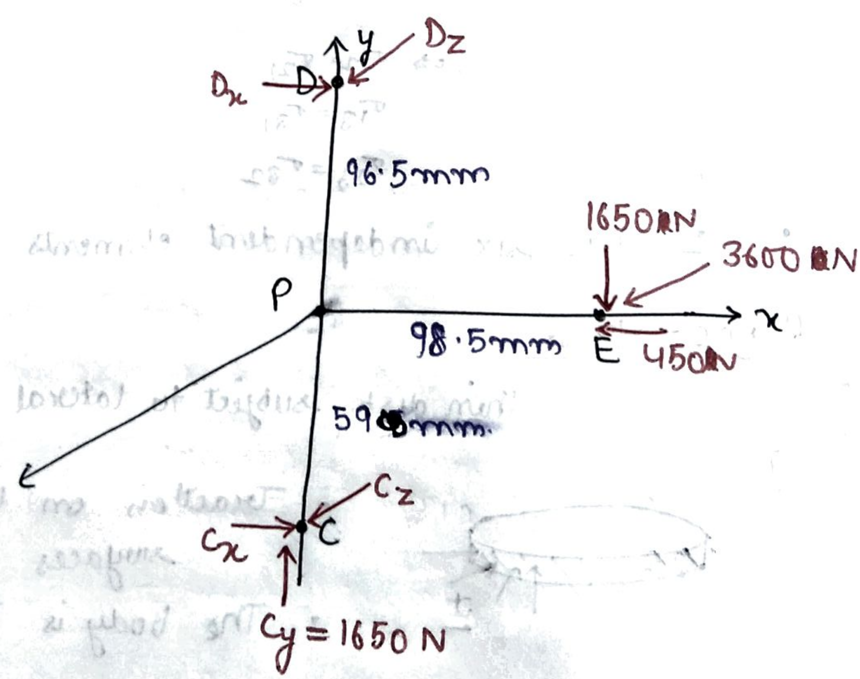
$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0.$

Non zero strain

$\varepsilon_x, \varepsilon_y$ and ε_{xy}

function of x and y only.

Tutorial-3 - Problem 1



$C_y = 1650N$ as all the thrust is taken by bearing at C

$\Rightarrow D_y = 0N.$

$(\Sigma M_D)_z : -1650 \times 98.5 - 450 \times 96.5 + C_x \times (155.5) = 0$

$\Rightarrow C_x = 1324.437N$

$(\Sigma M_D)_x = 0 \Rightarrow -C_z \times 155.5 - 3600 \times 96.5 = 0$

$\Rightarrow C_z = -2234.084\,N.$

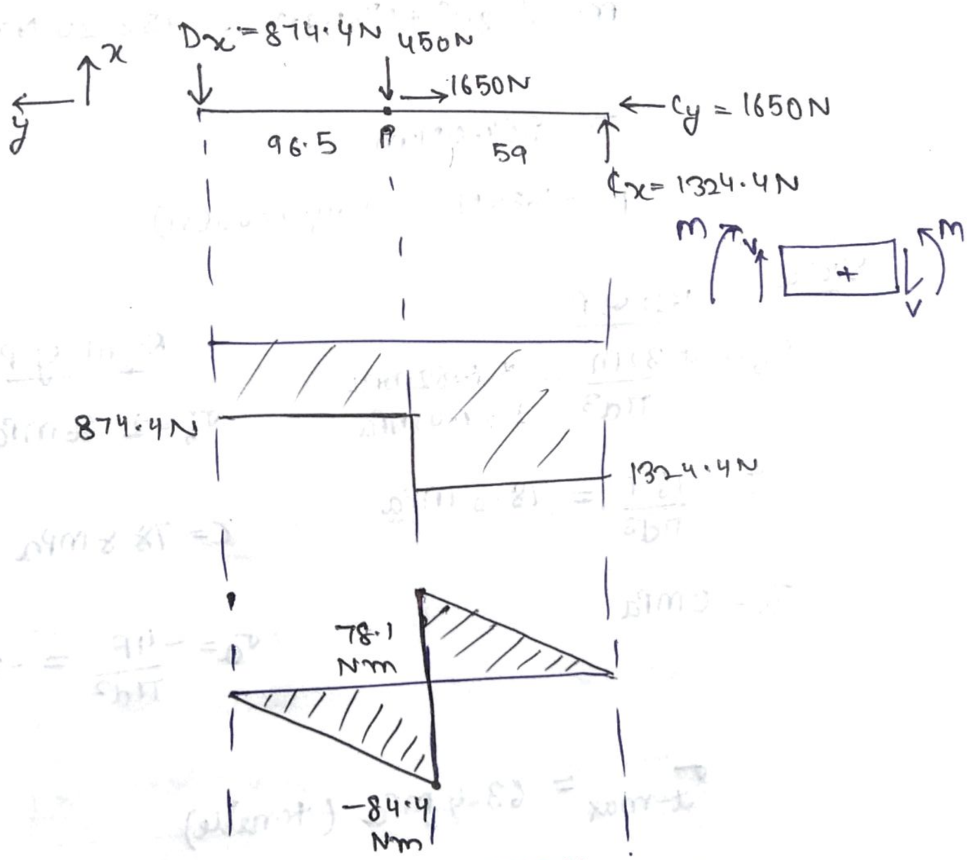
$\Sigma F_z = 0 \Rightarrow D_z + C_z = -3600\,N$

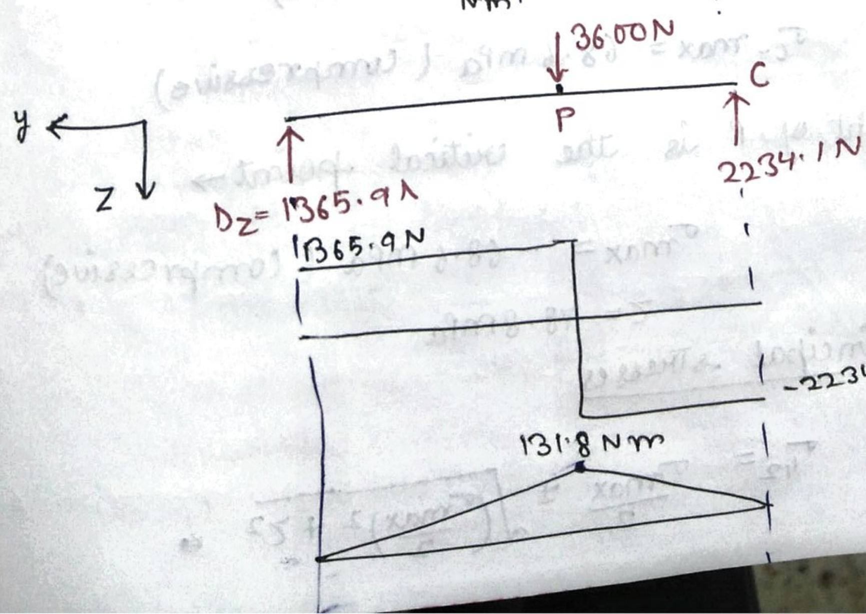
$\Rightarrow D_z = -1365.92\,N$

$\Sigma F_x = 0 \quad C_x + D_x = 450\,N$

$\Rightarrow D_x = -874.437\,N$

⑥ SFD and BMD





(c) Critical pt.

Left of P:

Bending moment $M = \sqrt{(84.4)^2 + (131.8)^2}$
$= 156.51\ Nm$

Torque: $T = 3600 \times 0.0985\ N\cdot m$
$= 354.6\ Nm$

Axial force ⇒ $F_a = 0N$

Right of P:

$m = \sqrt{78.1^2 + (131.8)^2} = 153.20\ Nm$

$T = 354.6\ Nm$

$A = -1650\ N$ (compression)

Stresses

Left of P	Right of P
$\sigma_b = \pm \frac{32M}{\pi d^3} = \cancel{\pm 0.62 MPa}$ $\pm 67.4\ MPa$	$\sigma_b = \pm 66\ MPa$
$\tau = \frac{16T}{\pi d^3} = 78.8\ MPa$	$\tau = 78.8\ MPa$
$\sigma_a = 0\ MPa$	$\sigma_a = \frac{-4F}{\pi d^2} = -2.6\ MPa$

$\sigma_{t-max} = 63.4\ MPa$ (tensile)

$\sigma_{c-max} = 68.6\ MPa$ (compressive)

Right of P is the critical point →

$\sigma_{max} = -68.6\ MPa$ (compressive)

$\tau = 78.8\ MPa$

(d) Principal stresses

$$\sigma_{1,2} = \frac{\sigma_{max}}{2} \pm \sqrt{\left(\frac{\sigma_{max}}{2}\right)^2 + \tau^2}$$

$$= -\frac{68.6}{2} \pm \sqrt{\left(\frac{68.6}{2}\right)^2 + (78.8)^2} = -120.2\,MPa,\ 51.6\,MPa$$

Maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{max}}{2}\right)^2 + \tau^2} = 85.94\,MPa.$$

Failure → A part is broken / functionality is ~~not~~ lost / not reliable.

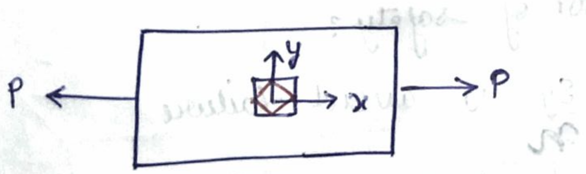
We will discuss - stress / strength based failure

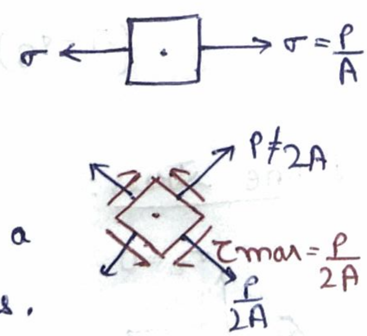
Under static Loading

- Maximum shear stress theory (Tresca)
  - Ductile material yielding

→ Failure or yielding happens when at a critical point, maximum shear stress exceeds a limit.

- → yield strength (in shear)
- → depends on material
- → experimentally obtained.





- Shear stress is max. on a plane at 45° with x-axis.
- $\tau_{max} = \frac{P}{2A} = \frac{\sigma}{2}$.

for failure : $\tau_{max} \geq S_{sy} \rightarrow$ yield strength in shear

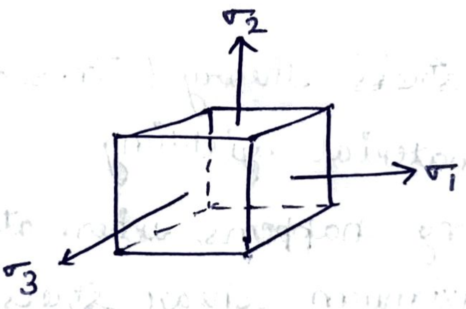
$$\Rightarrow \frac{\sigma}{2} \geq S_{sy} = \frac{S_y}{2}$$

$$\Rightarrow \boxed{\sigma \geq S_y}$$

$\boxed{S_y = 2S_{sy} = \text{yield strength}}$

3D state of stress :

In principal c.s.



Assume

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

Maximum shear stress

for failure, $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \cancel{\geq} \frac{S_y}{2}$

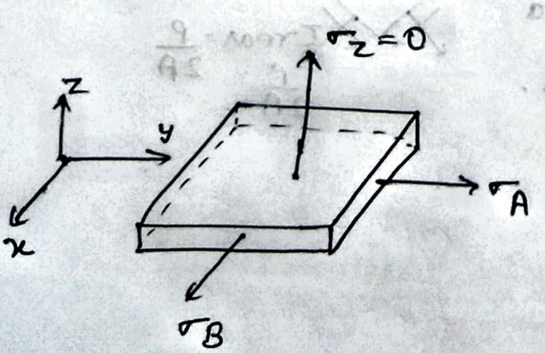
$$\Rightarrow (\sigma_1 - \sigma_3) \geq S_y$$

To include a factor of safety :

$(\sigma_1 - \sigma_3) < \frac{S_y}{n}$ to avoid failure

n → factor of safety

Plane stress :



- σ_A and σ_B are principal stresses in x-y plane.
- Plane stress is a 3D state of stress. (although approximate)

- Principal stresses are $\sigma_A, \sigma_B, 0$.

Case-1 →

$\sigma_A \geq \sigma_B \geq 0 \quad \Rightarrow \tau_{max} = \frac{\sigma_A}{2}$

For failure: $\frac{\sigma_A}{2} = \frac{S_y}{2}$ or $\boxed{\sigma_A = S_y}$

Case 2 →

$\sigma_B \geq \sigma_A \geq 0 \quad \tau_{max} = \frac{\sigma_B}{2}$

For failure $\boxed{\sigma_B = S_y}$

Case 3 →

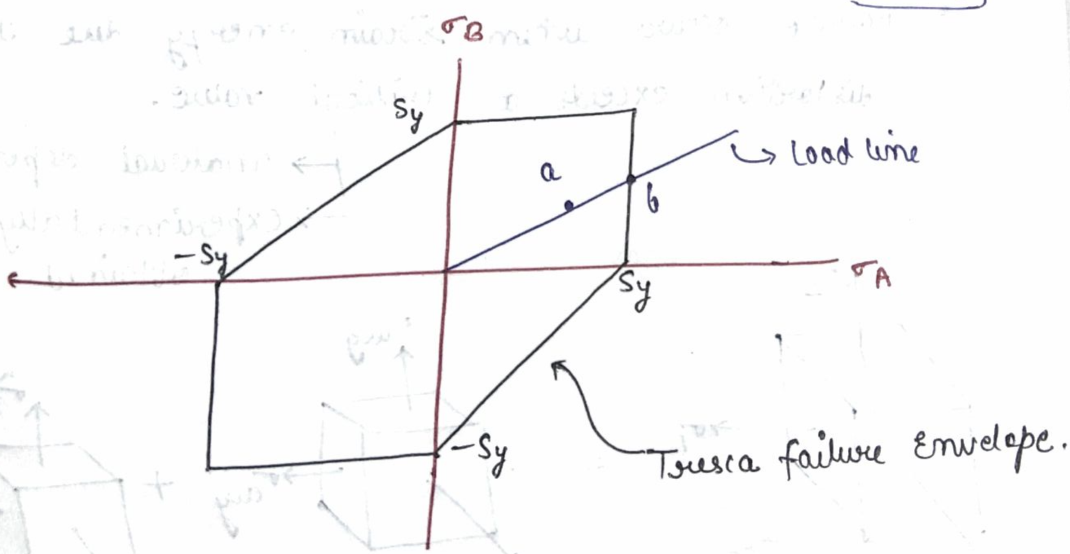
$0 \geq \sigma_B \geq \sigma_A \quad \Rightarrow \tau_{max} = -\frac{\sigma_A}{2}$

$\Rightarrow \boxed{\sigma_A = -S_y}$

Case 4 →

$0 \geq \sigma_A \geq \sigma_B \quad \Rightarrow \tau_{max} = -\frac{\sigma_B}{2}$

$\Rightarrow \boxed{\sigma_B = -S_y}$



Case 5 →

$\sigma_A \geq 0 \geq \sigma_B \quad \tau_{max} = \frac{\sigma_A - \sigma_B}{2}$

$\Rightarrow \boxed{\sigma_A - \sigma_B = S_y}$

Case 6 →

$\sigma_B \geq 0 \geq \sigma_A \quad \tau_{max} = \frac{\sigma_B - \sigma_A}{2}$

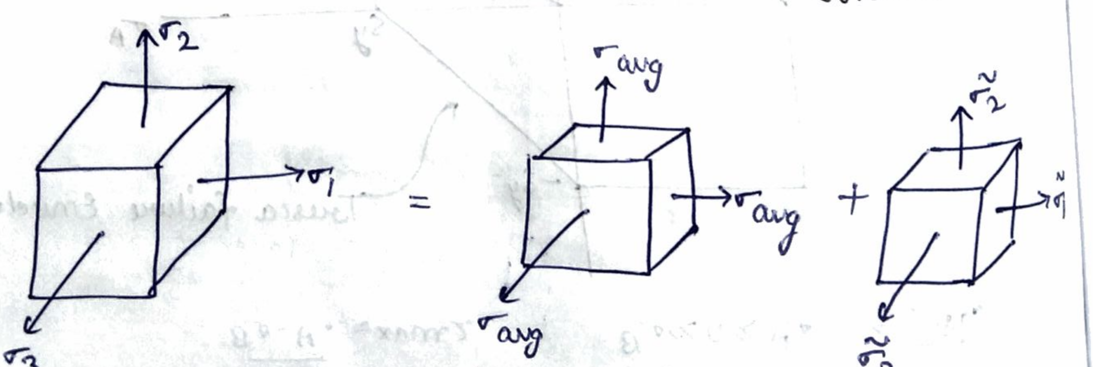
$\Rightarrow \boxed{\sigma_B - \sigma_A = S_y}$

- To avoid failure stress state (σ_A, σ_B) must lie inside this envelope.

- Let 'a' denote state of stress under some load
- Upon increase of load (1 parameter) – σ_A and σ_B will increase proportionally with load
  $\Rightarrow$ stress state will follow the load line
- Failure occurs when load line intersects with failure envelope.
- Factor of safety $\Rightarrow$ $\boxed{n = \frac{Ob}{Oa}}$

Maximum distortion Energy Theory (Von mises)

- Failure occurs when strain energy due to distortion exceeds a critical value.
  - → material dependent
  - → experimentally obtained



$$\underbrace{\quad}_{\text{Hydrostatic state of stress}}$$

Hydrostatic state of stress
- causes change in volume only

causes distortion only (no change in volume)

$\tilde{\sigma}_1 = \sigma_1 - \sigma_{avg}$

$\tilde{\sigma}_2 = \sigma_2 - \sigma_{avg}$

$\tilde{\sigma}_3 = \sigma_3 - \sigma_{avg}$

Total strain energy

$$w = \frac{1}{2}\sigma . \epsilon = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + 2\tau_{xy}\epsilon_{xy} + 2\tau_{xz}\epsilon_{xz} + 2\tau_{yz}\epsilon_{yz})$$

$$= \frac{1}{2}(\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3)$$

$$\epsilon_1 = \frac{1}{E}\{\sigma_1 - \nu(\sigma_2+\sigma_3)\}$$

$$\epsilon_2 = \frac{1}{E}\{\sigma_2 - \nu(\sigma_1+\sigma_3)\}$$

$$\epsilon_3 = \frac{1}{E}\{\sigma_3 - \nu(\sigma_1+\sigma_2)\}$$

$$w = \frac{1}{2E}\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\}$$

- strain energy due to volume change

$$(\epsilon_1+\epsilon_2+\epsilon_3) = \frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$

- Average strain. $\epsilon_{avg} = \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3} = \frac{1-2\nu}{E}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)$

- $w_{vol} = \frac{1}{2}\sigma_{avg}.\epsilon_{avg} = \frac{1}{2}\left(\frac{1-2\nu}{E}\right)\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)^2$

Take $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{avg}$

$$\Rightarrow w_v = \frac{1}{2}(\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3)$$

$$= \frac{1}{2}\sigma_{avg}(\epsilon_1+\epsilon_2+\epsilon_3)$$

$$= \frac{3}{2E}(1-2\nu)\sigma_{avg}^2$$

- strain energy due to distortion

$$w_d = w - w_v = \frac{1}{2E}\left\{(\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right\} - \frac{(1-2\nu)}{6E}(\sigma_1+\sigma_2+\sigma_3)^2$$

$$= \left(\frac{1+\nu}{3E}\right)\left[((\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2)/2\right]$$

Uniaxial Elongation

$\boxed{\sigma_1 \neq 0}$ $\sigma_2 = \sigma_3 = 0$

$$\Rightarrow \quad \omega_{vol} = \left(\frac{1+\nu}{3E}\right)\sigma_1^2 \geq \omega_{cri} = \left(\frac{1+\nu}{3E}\right)S_y^2$$

$$\boxed{\sigma_1 \geq S_y}$$

In general

$$\underbrace{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}_{(\sigma')^2} \geq S_y^2$$

or

$$\sigma' = \underbrace{\left\{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right\}^{1/2}}_{\text{Von mises stress}} \geq S_y$$

for failure

Von mises Failure Theory

Von mises stress:

$$\sigma' = \left\{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right\}^{1/2} \geq S_y$$

for failure

for general stress state:

$$\underline{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

$$\sigma' = \left\{\frac{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)}{2}\right\}$$

For Plane stress:

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0.$$

$$\sigma' = \left\{ \sigma_x^2 + \sigma_y^2 + 3\tau_{xy}^2 - \sigma_x\sigma_y \right\}^{1/2}$$

or when principal stresses in the plane are known

$$\sigma_1 = \sigma_A, \quad \sigma_2 = \sigma_B, \quad \sigma_3 = 0.$$

$$\Rightarrow \quad \sigma' = \sqrt{\sigma_A^2 + \sigma_B^2 - \sigma_A\sigma_B}$$

For failure,

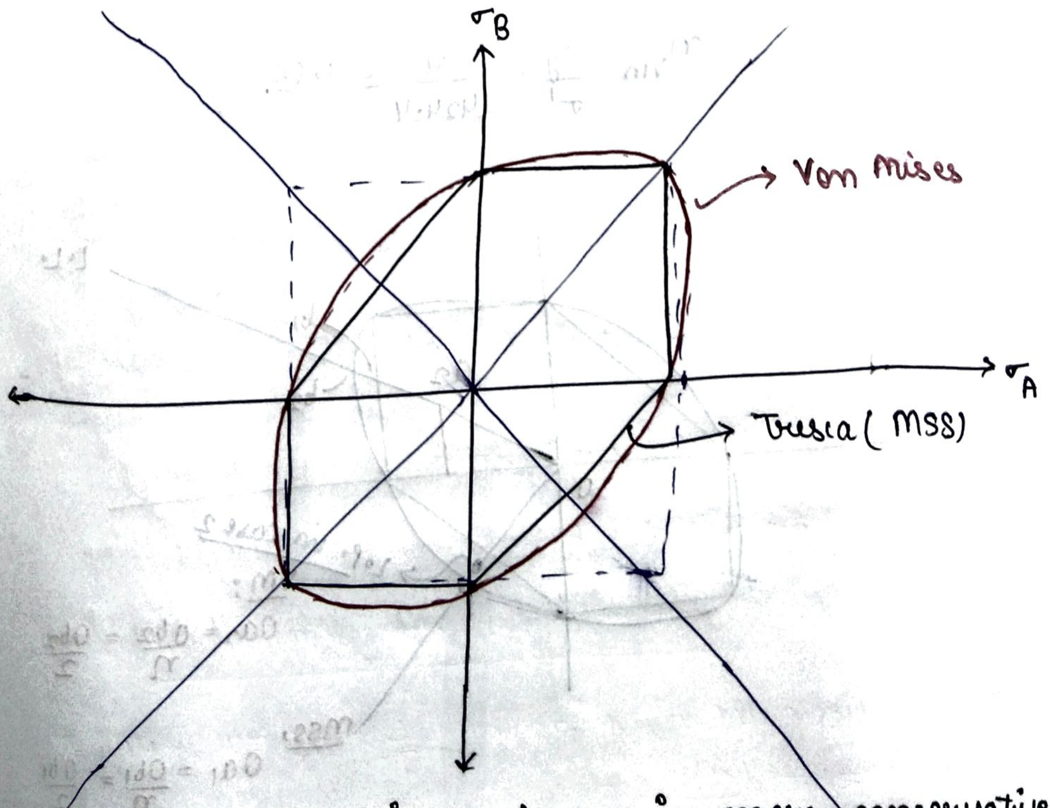
$$\sigma' = \sqrt{\sigma_A^2 + \sigma_B^2 - \sigma_A\sigma_B} \geq S_y \text{ or } \frac{S_y}{n}$$

Equation for failure Envelope:

$$\boxed{\sigma_A^2 + \sigma_B^2 - \sigma_A\sigma_B = S_y^2}$$

or

$$\boxed{\frac{1}{4}(\sigma_A + \sigma_B)^2 + \frac{3}{4}(\sigma_A - \sigma_B)^2 = S_y^2}$$



★ Tresca failure theory is more conservative.

Eg. Yield strength : $S_y = 700\,\text{MPa}$ (both in tension and compression)

Stresses due to loading:

$\sigma_1 = 490\,\text{MPa}$

$\sigma_2 = 250\,\text{MPa}$

$\sigma_3 = 0.$

Estimate the factor of safety using Von mises and Tresca criteria (Max. shear stress)

Sol. → $\tau_{max} = \dfrac{490}{2} = 245\,\text{MPa}.$

$$\sigma' = \left\{ \frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2} \right\}^{1/2}$$

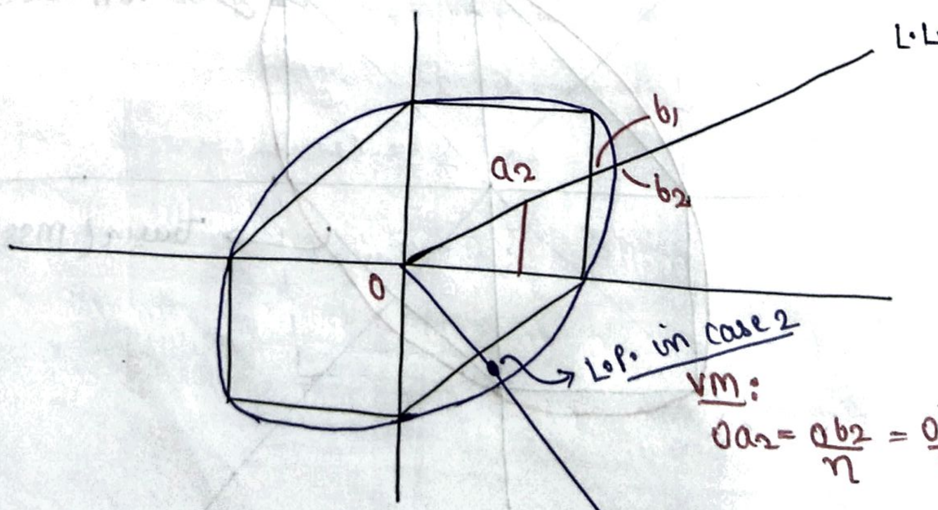
$\Rightarrow \sigma' = 424.4\,\text{MPa}.$

Factor of safety:

$$n_{mss} = \frac{700}{2\tau_{max}} \quad \left(\tau_{max} \geq \frac{S_y}{2n} \right. \text{ for failure}$$

$$n_{mss} = 1.43$$

$$n_{vm} = \frac{S_y}{\sigma'} = \frac{700}{424.4} = 1.65.$$



VM: $Oa_2 = \dfrac{Ob_2}{n} = \dfrac{Ob_2}{2}$

MSS: $Oa_1 = \dfrac{Ob_1}{n} = \dfrac{Ob_1}{2}$

$\Rightarrow$ For given n and S_y, v.m. permits larger load.

Case 2 $\rightarrow$

$\sigma_1 = 490\,MPa$

$\sigma_2 = -250\,MPa$

$\sigma_3 = 0\,MPa.$

$\tau_{max} = 370\,MPa$

$\sigma' = 652\,MPa.$

$n_{mss} = \frac{700}{2\tau_{max}} = 0.95$

$n_{vm} = \frac{S_y}{\sigma'} = 1.07$

Case 3 $\rightarrow$ $\sigma_x = 300\,MPa$

$\sigma_y = 210\,MPa$

$\tau_{xy} = 100\,MPa.$

$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 109.66\,MPa. \approx 110\,MPa.$

$\sigma' = 317.96\,MPa \approx 318\,MPa$

XX

$n_{mss} = \frac{700}{2 \times 110} = 3.18$

$n_{vm} = \frac{700}{318} = 2.20$

MSS: $\sigma_A > \sigma_B > 0 \Rightarrow \tau_{max} = \frac{\sigma_A}{2} = \frac{365}{2}\,MPa.$

$\Rightarrow n_{mss} = \frac{700}{365} =$

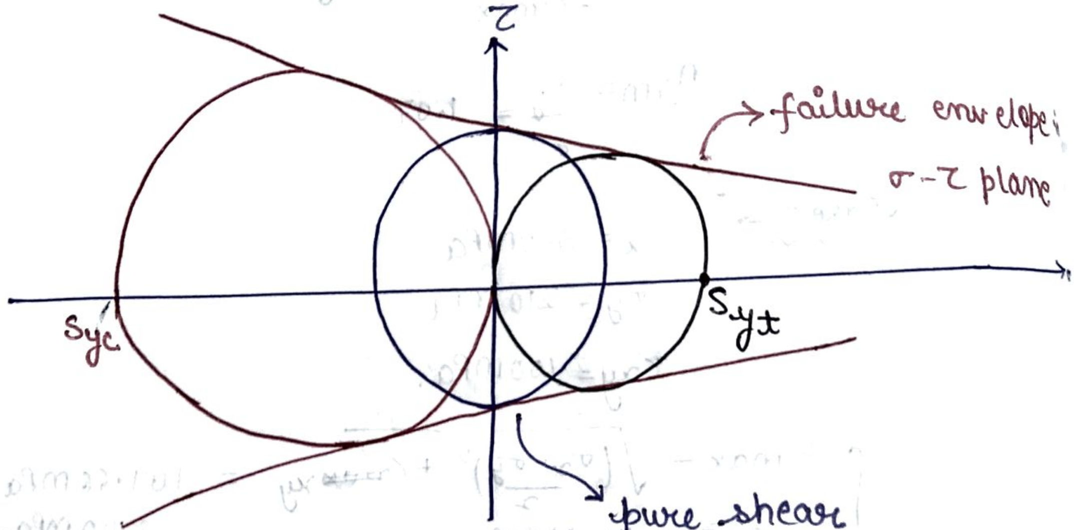
$n_{v.m.} = \frac{300}{318} =$

When yield strength in tension and compression are different

- $S_{yt} \rightarrow$ yield strength in tension
- $S_{yc} \rightarrow$ —— " —— compression

Mohr - criteria for ductile materials:

Uses test results for pure tension, pure compression (uniaxial) and pure shear



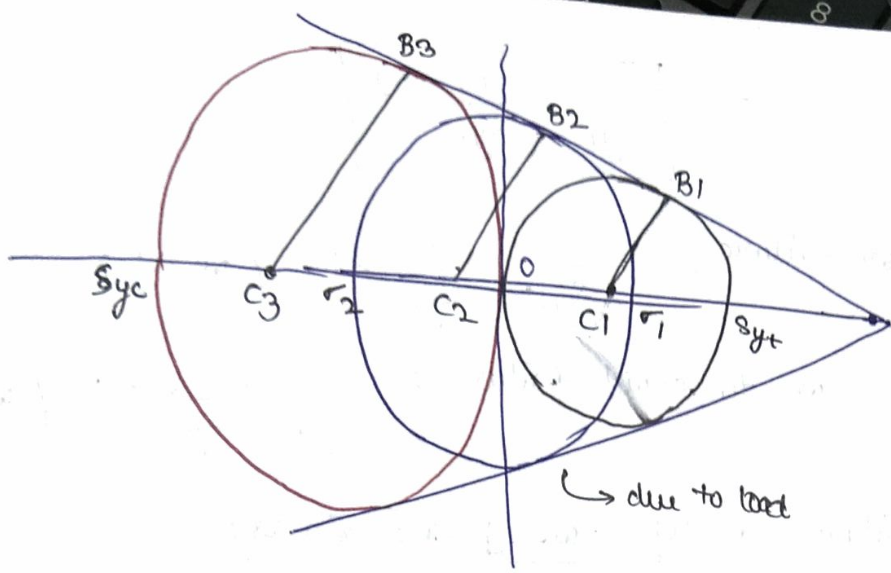
pressure - $p \Rightarrow \sigma_A = 2p$; $\sigma_B = -p \Rightarrow \sigma_{avg} = \frac{p}{2}$, $\tau_{max} = \frac{3p}{2}$

$\rightarrow 2p \Rightarrow \sigma_A = 4p$; $\sigma_B = -2p \Rightarrow \sigma_{avg} = p$, $\tau_{max} = \frac{3p}{}$

Mohr - Coulomb: (Internal friction theory)

- Assumes failure curve is a straight line

$\Rightarrow$ only two test data are sufficient.



$$B_1C_1 = \frac{S_{yt}}{2} \qquad B_2C_2 = \frac{(\sigma_1 - \sigma_3)}{2} \qquad B_3C_3 = \frac{S_{yc}}{2}$$

$$\left.\begin{aligned} C_3 &\rightarrow \left(-\frac{S_{yc}}{2}, 0\right) \\ C_2 &\rightarrow \left(\frac{\sigma_1+\sigma_3}{2}, 0\right) \\ C_1 &\rightarrow \left(\frac{S_{yt}}{2}, 0\right) \end{aligned}\right\} \text{co-ordinates}$$

ΔOB_1C_1, ΔOB_2C_2, ΔOB_3C_3 are similar

$$\Rightarrow \frac{B_3C_3}{OC_3} = \frac{B_2C_2}{OC_2} = \frac{B_1C_1}{OC_1} \Rightarrow \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1} = \frac{B_2C_2 - B_1C_1}{OC_2 - OC_1}$$

$$\Rightarrow \frac{B_3C_3 - B_1C_1}{C_1C_3} = \frac{B_2C_2 - B_1C_1}{C_1C_2}$$

$$\Rightarrow \frac{(S_{yc} - S_{yt})/2}{(S_{yc} + S_{yt})/2} = \frac{\{(\sigma_1 - \sigma_3) - S_{yt}\}/2}{\{S_{yt} - (\sigma_1 + \sigma_3)\}/2}$$

$$\Rightarrow \frac{S_{yc} - S_{yt}}{S_{yc} + S_{yt}} = \frac{(\sigma_1 - \sigma_3) - S_{yt}}{S_{yt} - (\sigma_1 + \sigma_3)}$$

$$\Rightarrow \boxed{\frac{\sigma_1}{S_{yt}} - \frac{\sigma_3}{S_{yc}} = 1}$$

For failure, $\frac{\sigma_1}{S_{yt}} - \frac{\sigma_3}{S_{yc}} \geq 1$ (or $\frac{1}{n}$)

or to avoid failure : $\frac{\sigma_1}{S_{yt}} - \frac{\sigma_3}{S_{yc}} < 1$ (or $\frac{1}{n}$)

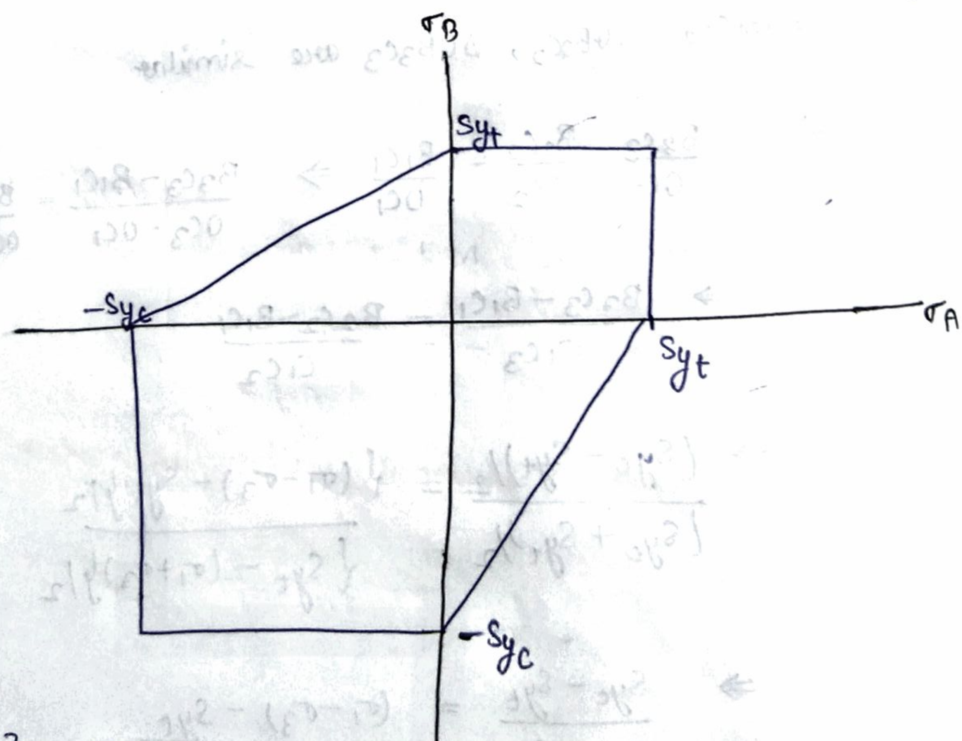
Failure envelope for plane stress:

Principal stresses are : $\sigma_A, \sigma_B, 0$

1. $\sigma_A \geq \sigma_B > 0$: $\sigma_1 = \sigma_A$, $\sigma_3 = 0$

$\Rightarrow \frac{\sigma_A}{S_{yt}} \geq 1$ for failure

2. $0 \geq \sigma_B \geq \sigma_A$ $\Rightarrow \sigma_1 = 0$, $\sigma_3 = \sigma_A$ $\Rightarrow \frac{-\sigma_A}{S_{yc}} \geq 1$ for failure



3. $\sigma_A \geq 0 \geq \sigma_B$ $\Rightarrow \sigma_1 = \sigma_A$, $\sigma_3 = \sigma_B$ $\Rightarrow \frac{\sigma_A}{S_{yt}} - \frac{\sigma_B}{S_{yc}} \geq 1$

4. $\sigma_B \geq 0 \geq \sigma_A$ $\Rightarrow \sigma_1 = \sigma_B$, $\sigma_3 = \sigma_A$ $\Rightarrow \frac{\sigma_B}{S_{yt}} - \frac{\sigma_A}{S_{yc}} \geq 1$

Load in pure shear:

$\Rightarrow \tau \geq S_{sy}$ for failure

or limit: $\tau = S_{sy} \rightarrow$ yield strength in shear

$\sigma_A = S_{sy}$ $\sigma_B = -S_{sy}$

$\Rightarrow$ Envelope, $\frac{\sigma_A}{S_{yt}} - \frac{\sigma_B}{S_{yc}} = 1$

$$\Rightarrow \frac{S_{sy}}{S_{yt}} + \frac{S_{sy}}{S_{yc}} = 1 \Rightarrow S_{sy} = \frac{S_{yt} S_{yc}}{(S_{yt} + S_{yc})}$$

Von mises

find σ for pure shear

$\sigma' < S_y$

for pure shear

$\sigma_A = S_{sy}$, $\sigma_B = -S_{sy}$ $\sigma' = \sqrt{3} S_{sy}$

$\Rightarrow \sqrt{3} S_{sy} = S_y$

or $S_{sy} = \frac{S_y}{\sqrt{3}}$ or $0.572 S_y$

Failure of ductile materials

1. If $S_{yt} = S_{yc}$, use Tresca or Von mises
2. If $S_{yt} \neq S_{yc}$ use Mohr-Coulomb.

Failure of brittle materials:

- Very less deformation before failure
- Failure due to fracture

Maximum Normal Stress Theory:

- Material fails when normal stress exceeds a limit experimentally obtained.
- $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses.

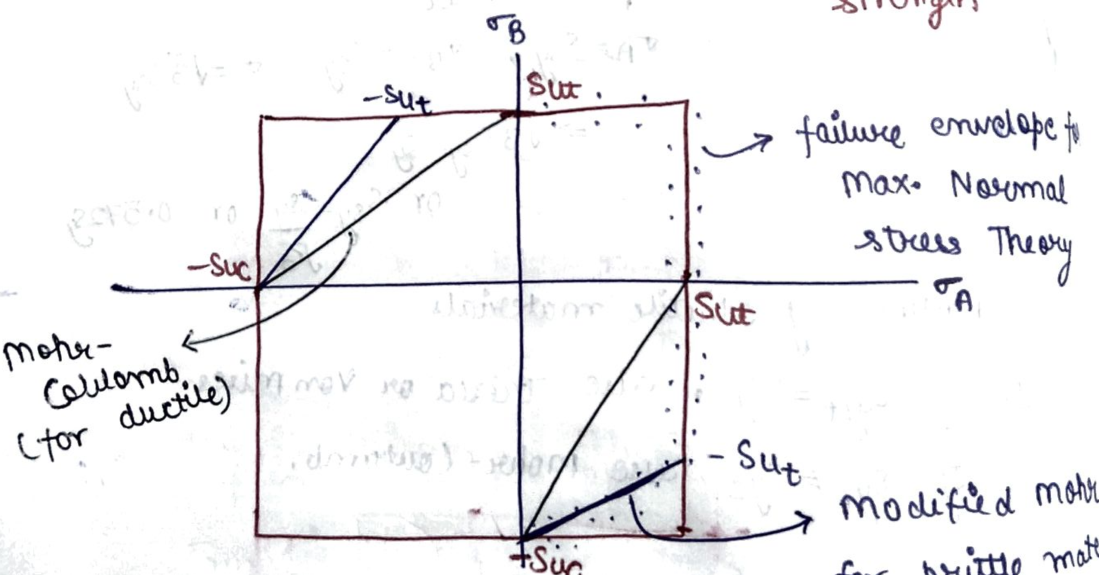
$\Rightarrow$ Failure condition:

$\sigma_1 \geq S_{ut}$ or $\left(\frac{S_{ut}}{n}\right)$

↳ ultimate tensile strength

or $\sigma_3 \leq -S_{uc}$ (S_{uc}/n)

ultimate compressive strength



Modified Mohr (Plane stress condition)

1. $\sigma_1 = S_{ut}$ or $\frac{S_{ut}}{n}$ if $\sigma_A \geq \sigma_B \geq 0$

$\sigma_A \geq 0 \geq \sigma_B$

and $\left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$

2. $$\frac{S_{uc}-S_{ut}}{S_{uc}\,S_{ut}}\cdot\sigma_A - \frac{\sigma_B}{S_{uc}} = 1 \left(\text{or } \frac{1}{n}\right) \quad \text{if}$$

$\sigma_A \geq 0 \geq \sigma_B$
and $\left|\frac{\sigma_B}{\sigma_A}\right| > 1$

3. $$\sigma_B = -S_{uc} \left(\text{or } -\frac{S_{uc}}{n}\right) \quad \text{if} \quad 0 \geq \sigma_A \geq \sigma_B$$

Failure by instability: (Column buckling)

- Column (member under compression) failure does not happen due to yielding or fracture. Rather deflection becomes large
- Critical buckling load depends on
  1. Length
  2. Young's Modulus
  3. Cross section.

1. For simply supported ends:

$$P_{cr} = \frac{n^2\pi^2 EI}{l^2}$$

Lowest value is for $n=1$.

$\Rightarrow$ Critical buckling load : $\boxed{P_{cr} = \frac{\pi^2 EI}{l^2}}$

2. Fixed-fixed ends:

$$P_{cr} = \frac{4\pi^2 EI}{l^2}$$

$\boxed{c=4}$

3. Fixed-free

$P_{cr} = \frac{1}{4}\,\frac{\pi^2 EI}{l^2}$ $\boxed{c=\frac{1}{4}}$

4. Pinned-fixed:

$P_{cr} = 2\,\frac{\pi^2 EI}{l^2}$ $\boxed{c=2}$

In summary:

$$P_{cr} = c\cdot\frac{\pi^2 EI}{l^2}$$

↳ end condition constant

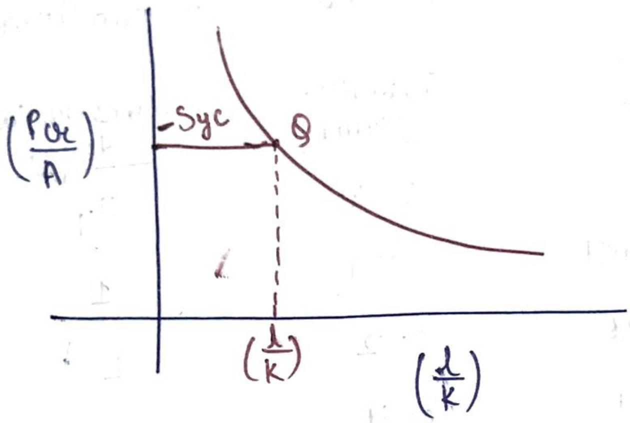
→ $P_{cr} = c\cdot\frac{\pi^2 EI}{l^2}$ $I = A\cdot K^2$ ↓ radius of gyration

$$= c\cdot\frac{A\pi^2 E}{(l/k)^2}$$

σ_{avg} (Average stress)

$$\boxed{\frac{P_{cr}}{A} = c\,\frac{\pi^2 E}{\left(\frac{l}{K}\right)^2}}$$

$\left(\frac{l}{K}\right)$: Slenderness ratio



- Critical buckling load depends more on $\left(\frac{l}{K}\right)$ than material
- Mild steel, High strength steel have same E, but significant diff. yield strength.
- If $\frac{P_{cr}}{A} \leq -S_{yc}$ or $-S_{uc}$ failure is due to yielding or fracture

when

$$\left(\frac{l}{K}\right) = \left(\frac{l}{K}\right)_Q \quad : \quad \left|\frac{P_{cr}}{A}\right| = S_{yc} \Rightarrow S_{yc} = c\frac{\pi^2 E}{(l/K)_Q^2}$$

$$\Rightarrow \boxed{\left(\frac{l}{K}\right)_Q = \sqrt{\frac{c\pi^2 E}{S_{yc}}}}$$

Euler-buckling:

$$\frac{P_{cr}}{A} = c\frac{\pi^2 E}{(l/K)^2} \quad \text{when } \left(\frac{l}{K}\right) > \left(\frac{l}{K}\right)_Q$$

$$= S_{yc} \quad \text{when } \left(\frac{l}{K}\right) \leq \left(\frac{l}{K}\right)_Q$$

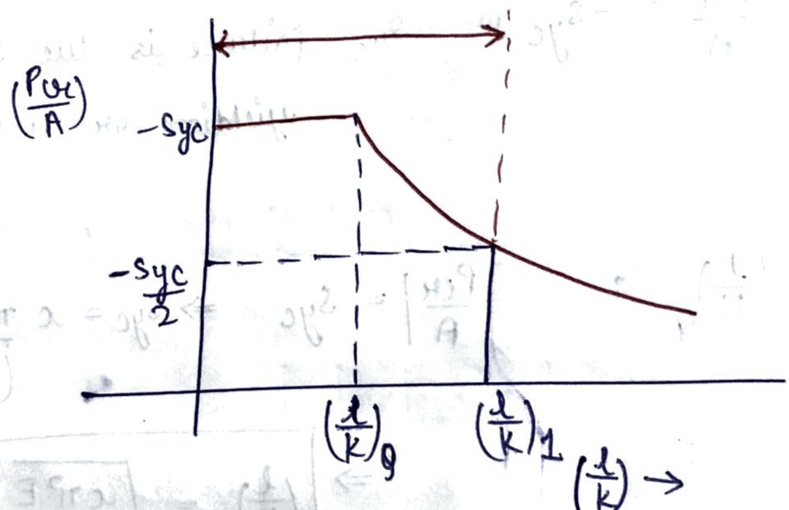
- Recommended values for end-condition consta[nts]

End cond^n	Theoritical Value	Conservative design	Recom[mended] val[ue]
Fixed-free	$C = 1/4$	$1/4$	$1/4$
Pinned-Pinned	$C=1$	1	1
Fixed-Pinned	$C=2$	1	$\boxed{1.2}$
Fixed-fixed	$C=4$	1	$\boxed{1.2}$

→ Euler-Column theory works well for large (l/k)

Near $\left(\frac{l}{k}\right) = \left(\frac{l}{k}\right)_Q$ not very accurate.

intermediate length column



- Rule of Thumb: Apply Euler-column formul[a]

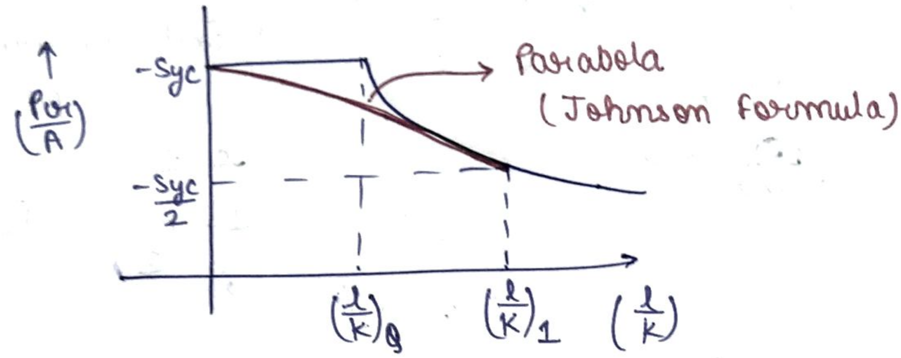
if $\left(\frac{l}{k}\right) > \left(\frac{l}{k}\right)_1$

Typically $\left(\frac{l}{k}\right)_1$ is chosen such that

$$\frac{P_{cr}}{A} = \frac{S_{yc}}{2} \quad \text{or} \quad \left(\frac{l}{k}\right)_1 = \sqrt{2C\frac{\pi^2 E}{S_{yc}}}$$

Intermediate length columns

$$0 \le \left(\frac{l}{k}\right) \le \left(\frac{l}{k}\right)_1$$



→ Johnson proposed a parabola for $\left(\frac{l}{k}\right) \le \left(\frac{l}{k}\right)_1$

$$\frac{P_{cr}}{A} = a - b\left(\frac{l}{k}\right)^2 \quad \text{for} \quad 0 \le \frac{l}{k} \le \left(\frac{l}{k}\right)_1$$

- $\frac{P_{cr}}{A} = S_y \quad \text{for } \frac{l}{k} = 0.$
- $\frac{P_{cr}}{A} = \frac{S_{yc}}{2} \quad \text{for } \left(\frac{l}{k}\right) = \left(\frac{l}{k}\right)_1 = \sqrt{\frac{2c\pi^2 E}{S_{yc}}}$

$$\Rightarrow a = S_{yc}, \quad b = \left(\frac{S_{yc}}{2\pi}\right)^2 \cdot \frac{1}{CE}$$

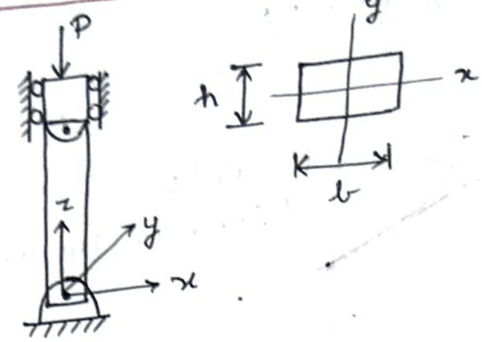
Johnson Column formula:

$$\frac{P_{cr}}{A} = S_{yc} - \left(\frac{S_{yc}}{2\pi}\frac{l}{K}\right)^2 \cdot \frac{1}{CE} \quad \text{for } 0 \le \frac{l}{K} \le \left(\frac{l}{K}\right)_1$$

$$= c\frac{\pi^2 E}{(l/K)^2} \quad \text{for } \left(\frac{l}{K}\right) > \left(\frac{l}{K}\right)_1$$

$$\left(\frac{l}{K}\right)_1 = \sqrt{\frac{2c\pi^2 E}{S_{yc}}} \quad \left(\text{at } \frac{l}{K} = \left(\frac{l}{K}\right)_1 : \frac{P_{cr}}{A} = \frac{S_{yc}}{2}\right)$$

A note on end condition:



Buckling in x-z pl…

- End condition – pinned-pi…

$c = 1$

$I = \frac{hb^3}{12}$ $\quad P_{cr} = \frac{\pi^2 E \ldots}{l^2}$

- Buckling in y-z plane (out of plane)

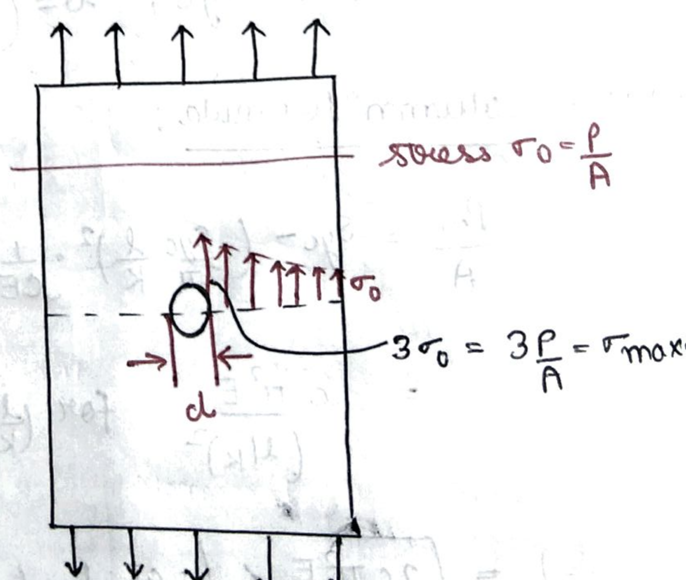
  → End condition – fixed-fixed

  $c = 4$ (theoretical)

  $= 1.2$ (recommended)

  $I = \frac{bh^3}{12}$

Stress concentration



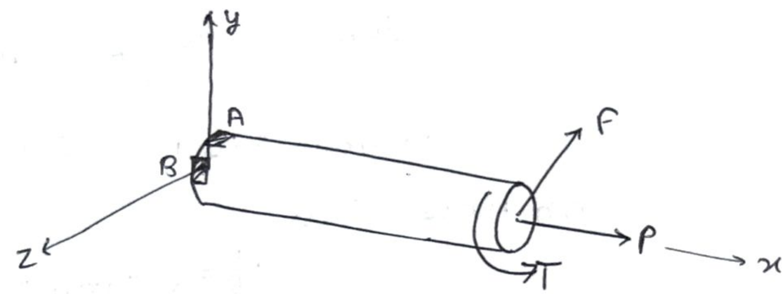
stress concentration factor ← $K_t = \frac{\sigma_{max}}{\sigma_0}$

$\Rightarrow \sigma_{max} = K_t \cdot \sigma_0$

Problem → Length = 100 mm

Dia = 15 mm

Material = AISI 1006 CD.



$F = 0.55\ \text{kN}$

$P = 4\ \text{kN}$

$T = 25\ \text{N·m}$

Compute stress at A & B.

At A: Axial $\sigma_{axial} = \frac{4P}{\pi d^2} = 22.6\ \text{MPa}$

Shear stress: $\tau = \frac{16T}{\pi d^3} = 37.7\ \text{MPa}.$

Bending moment at $x=0$ $M = Fl = 55\ \text{Nm}$

Bending stress: $\sigma_{bending} = 0\ \text{MPa}.$

Shear stress due to transverse force F

maximum at A: $\tau_F = \frac{4F}{3A} = \frac{16F}{3\pi d^2}$

$= 4.14\ \text{MPa}.$

Total shear stress at A:

$$\tau = \frac{16T}{\pi d^3} - \frac{4F}{3A} = 33.56\ \text{MPa}.$$

Von Mises stress: $\sigma' = \sqrt{(\sigma_{bending} + \sigma_{axial})^2 + 3\tau^2}$ ✓

$= \sqrt{22.6^2 + \cancel{3}3\,33.56^2} = \cancel{40.46\ \text{MPa}}$

$= 62.4\ \text{MPa}$

At B: Axial stress $\sigma_{axial} = 22.6\ MPa$

Bending stress $\sigma_b = \frac{32M}{\pi d^3} = 165\ MPa$

Torsion $\tau = \frac{16T}{\pi d^3} = 37.7\ MPa$

Transverse force F: $\tau = 0\ MPa$

Von mises stress

$$\sigma'_B = \sqrt{(\sigma_{axial} + \sigma_b)^2 + 3\tau^2}$$

$$= \sqrt{(166+22.6)^2 + 3\times37.7^2}$$

$$= 199.6\ MPa.$$

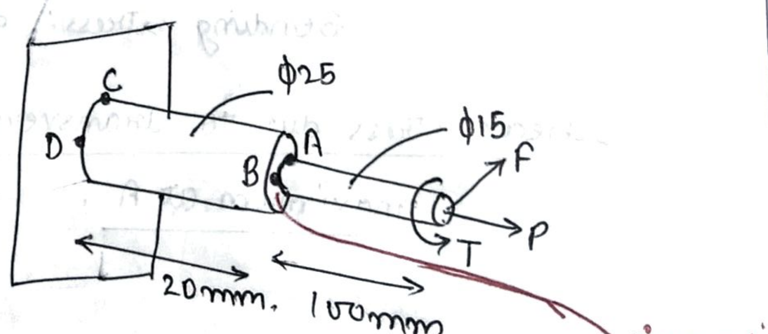
Find material prop of AISI 1006 CD

From table A20, $S_y = 280\ MPa$

Factor of safety $\Rightarrow$ $n_A = \frac{280}{62.4} = 4.48$

$n_B = \frac{280}{199.6} = 1.4$

$\rightarrow$ Added a step $\rightarrow$



$\Rightarrow$ B is still the critical point.

Stresses at B:

	Nominal	Stress con. factor	Actual stress
Axial	$\sigma_{a0} = 22.6\ MPa$	1.65 (Fig A15-7)	37.3 MPa
Bending	$\sigma_{b0} = 166\ MPa$	1.42 (A-15-9)	235.7 MPa
Torsion	$\tau_0 = 37.7\ MPa$	1.23 (A-15-8)	46.4 MPa
Transverse force	$\tau_0 = 0\ MPa$	—	—

$$\boxed{\begin{aligned} \frac{D}{d} &= \frac{25}{15} = 1.67 \\ \frac{r}{d} &= \frac{3}{15} = 0.2 \end{aligned}}$$

$$\sigma'_B = \sqrt{(137.3+235.7)^2 + 3 \times 46.4^2} = 286.6 \text{ MPa}.$$

$$n = \frac{280}{286.5} = 0.97$$

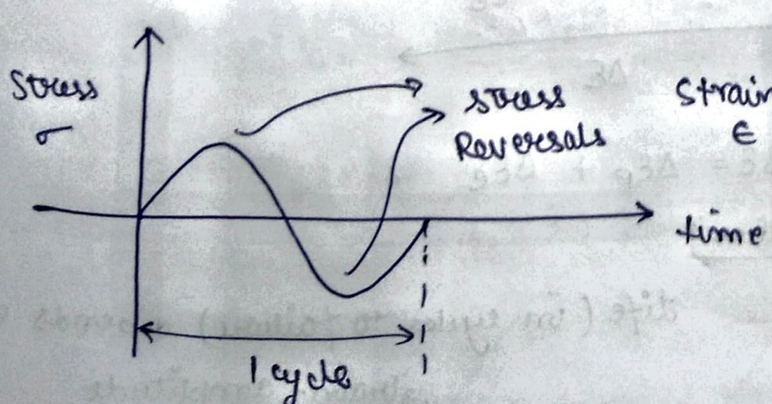
Fatigue Failure

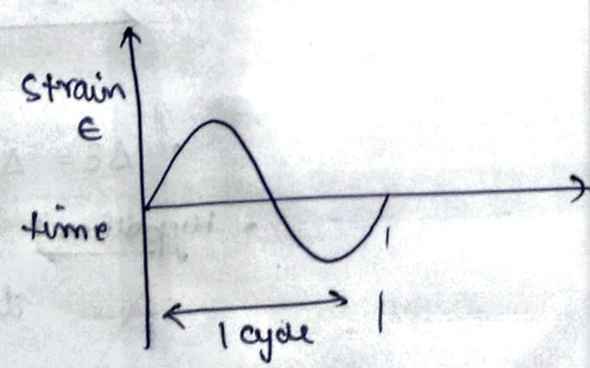
- Load varies with time
- Most of the time failure happens much before yield strength or ultimate strength is reached.
- Fails mainly due to crack formation and propagation
  (comes without warning)

Failure location →

- Rapid change in cross section or geometry.
- Surfaces under high contact pressure or relative motion.
- Microscopic defects due to processing.

Fatigue failure





Estimating fatigue life:

- strength at which a part fails depends on No. of cycles of operation

N = No. of cycles

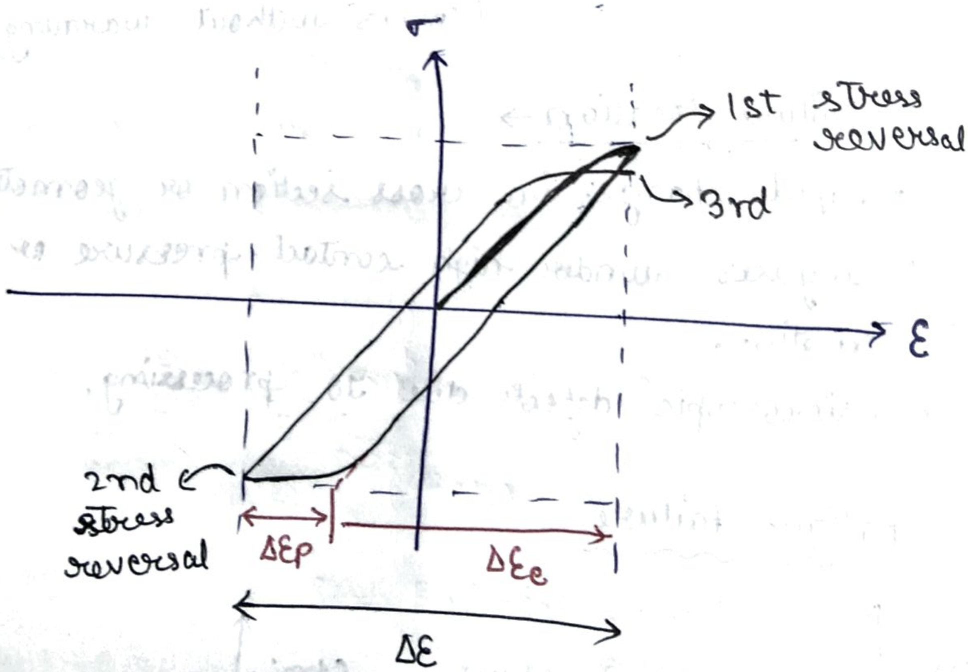
$\Rightarrow 2N$ = No of Reversals

Methods of estimating life

1. strain-life method
2. stress life method
} will discuss
3. Linear Elastic Fracture Mechanics

strain-life

For fatigue failure to occur there must be plastic strain accumulation.



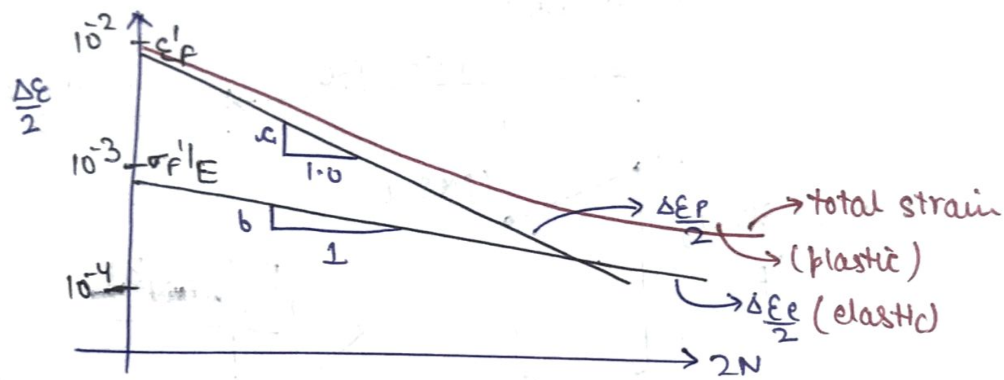
- $\Delta\varepsilon = \Delta\varepsilon_p + \Delta\varepsilon_e$

- Hypothesis

Life (in cycles to failure) depends on strain-amplitude

Mason-coffin relation

$$\frac{\Delta\varepsilon}{2} = \underbrace{\frac{\sigma_f'}{E}(2N)^b}_{\frac{\Delta\varepsilon_e}{2}} + \underbrace{\varepsilon_f'(2N)^c}_{\frac{\Delta\varepsilon_p}{2}}$$

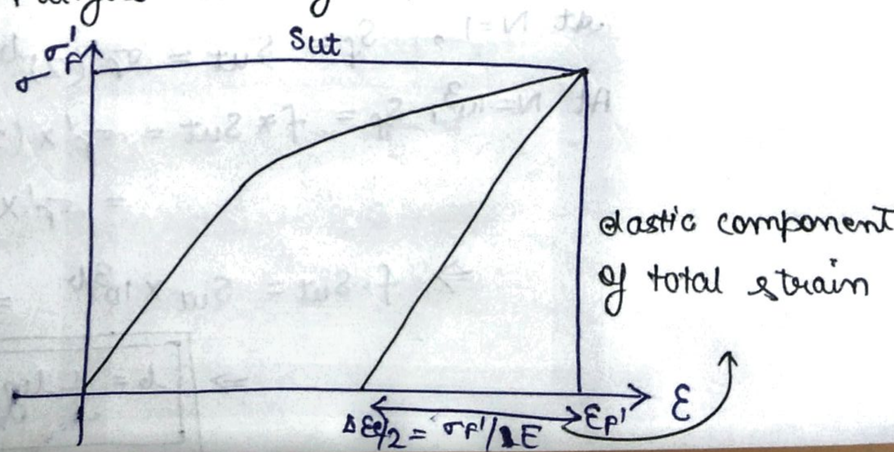


$$\frac{S_{ut}}{E} = \Delta\varepsilon_e = \frac{\sigma_f'}{E}$$

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'}{E}(2N)^b + \varepsilon_f'(2N)^c$$

- ε_f' = True strain at failure after one reversal (Fatigue ductility coefficient)
- σ_f' = True stress ——— " ——— (Fatigue strength coefficient)

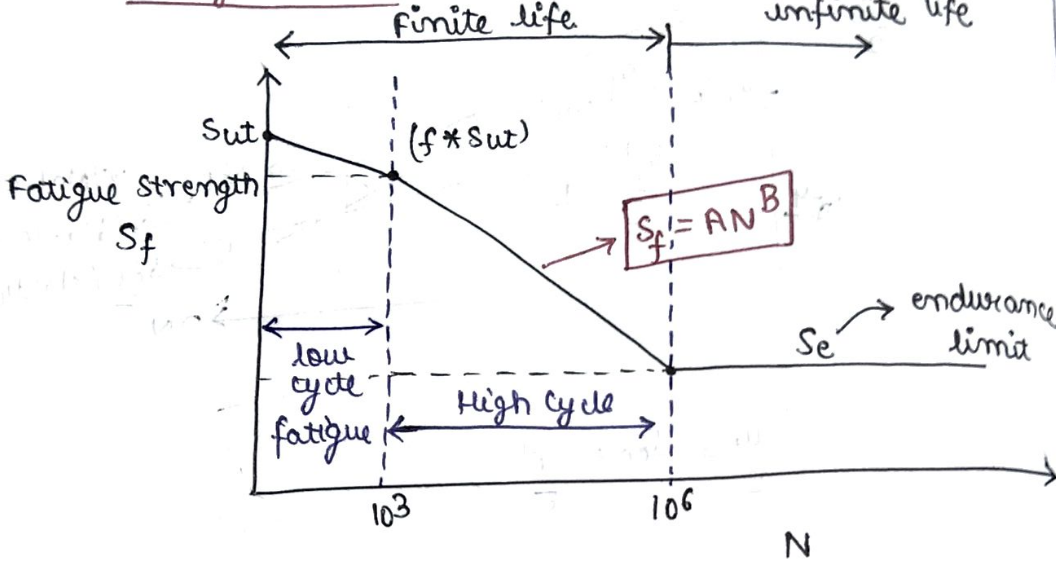
- b = fatigue strength exponent
- c = Fatigue ductility exponent



Limitations of strain-life method

1. Determination of local strain at notches, (corners) (stress-concentration location)
2. Accurate for low cycle fatigue ($1-10^3$ cycles)

Stress Life Method



- Accurate for high cycle ($10^3 - 10^6$)
- Easy to implement in design

For $1 < N < 10^3$ (from strain-life method)

$$\frac{\Delta \varepsilon}{2} = \underbrace{\frac{\sigma_f'}{E}(2N)^b}_{(\Delta \varepsilon_e/2)} + \varepsilon_f'(2N)^c$$

After N-cycles

$$S_f = E \cdot \left(\frac{\Delta \varepsilon_e}{2}\right) = \sigma_f'(2N)^b$$

At $N=1$, $S_f = S_{ut} = \sigma_f'(2)^b$

At $N = 10^3$, $S_f = f * S_{ut} = \sigma_f' \times (2 \times 10^3)^b$

$$= \sigma_f' \times 2^b \times 10^{3b}$$

$\Rightarrow f \cdot S_{ut} = S_{ut} \times 10^{3b} \Rightarrow f = 10^{3b}$

$$\Rightarrow \boxed{b = \frac{1}{3}\log_{10} f}$$

Also, $\boxed{f = \frac{\sigma_F' \times 2^b \times 10^{3b}}{S_{ut}}}$ (from table) ★ A-23 ★

At $N = 10^6$: $\boxed{S_f = S_e} \rightarrow$ endurance limit

★ Fig 6-18, figure of relⁿ b/w f and S_{ut}.

→ S_f is linear with N in log-log plot

$\Rightarrow \boxed{S_f = AN^B} \rightarrow$ eqⁿ of line

$\Rightarrow f * S_{ut} = A(10^3)^B = A \times 10^{3B}$

$S_e = A \times 10^{6B}$

$\Rightarrow \frac{f \cdot S_{ut}}{S_e} = \frac{1}{10^{3B}}$

or $\boxed{B = -\frac{1}{3} \log\left(\frac{f S_{ut}}{S_e}\right)}$

$\Rightarrow A = \frac{(f \cdot S_{ut})^2}{S_e}$

Summary: 1. $S_f = S_{ut} \, N^{(\log f/3)}$

for $N < 10^3$.

2. $S_f = \frac{(f \cdot S_{ut})^2}{S_e} \times N^{-\frac{1}{3}\log\left(f\frac{S_{ut}}{S_e}\right)}$

for $10^3 < N < 10^6$.

Two parameters :→

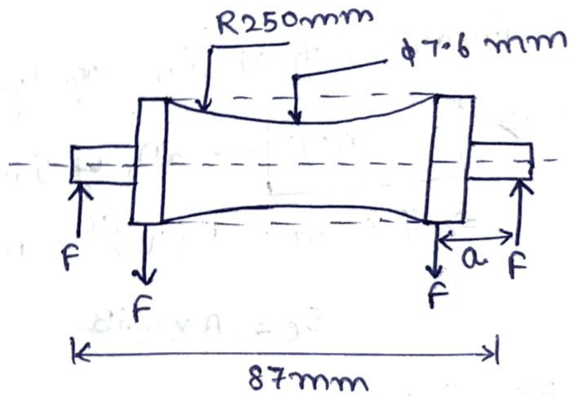
$f \rightarrow$ obtain from Fig 6-18 (or from values of σ_F' and b in A-23)

2. S_e the endurance limit

Endurance limit:

From R.R. Moore specimen:

$$S_e' = \begin{cases} 0.5\, S_{ut} & \text{for } S_{ut} \leq 1400\,\text{MPa} \\ 700\,\text{MPa} & \text{for } S_{ut} > 1400\,\text{MPa.} \end{cases}$$



Bending moment

$$\boxed{M = F \cdot a}$$

Note:

1. Al alloys do not have endurance limit
2. Actual endurance limit $Se \neq \underbrace{Se'}_{R.R.\,Moore}$

   Typically, $\boxed{Se \leq Se'}$

- Modification is due to - shape, size, loading scenario, manufacturing processes.

For actual Scenario:

$$Se \nless Se'$$

Marin Equation:

$$Se = k_a\, k_b\, k_c\, k_d\, k_e\, k_f\, Se'$$

1. $k_a = a S_{ut}^{b}$: surface modification factor (ground, cold rolled, ...)

   Table 6.2 for a and b.

2. K_b = size factor (if the size is different from R.R. Moore specimen)

Bending and torsion

$$K_b = \begin{cases} 1.24\, d^{-0.107} & \text{for } 2.79 \leq d \leq 51 \text{ mm} \\ 1.51\, d^{-0.157} & \text{for } 51 \leq d \leq 254 \text{ mm} \end{cases}$$

For axial loading $K_b = 1$.

For non-rotating and non-circular cross-section use an equivalent diameter (d_e) in place of d

(Table 6.3)

3. K_c = load factor

$$= \begin{cases} 1 & \text{for pure bending} \\ 0.59 & \text{for pure torsion} \\ 0.85 & \text{for pure axial loading} \end{cases}$$

For combined loading: take $K_c = 1$.

4. K_d = temperature factor

$\boxed{K_d = 1}$ always (for the course)

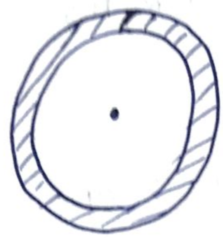
5. K_e = Reliability factor (due to scatter in data)

(Table 6.5)

6. K_f = miscellaneous effect factor

$\boxed{K_f = 1}$ (for the course)

Rotating

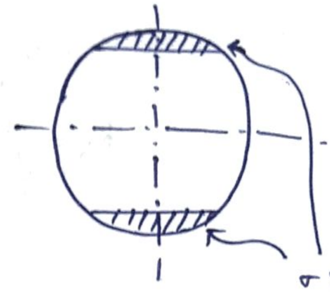


$\sigma \geq 0.95\,\sigma_{max}$

$$A_{0.95\sigma_{max}} = \frac{\pi}{4}\left[d^2 - (0.95d)^2\right]$$

$$= 0.0766\,d^2$$

Non rotating



$\sigma \geq 0.95\sigma_{m}$

$$A_{0.95\sigma} = 0.0104\,d^2$$

$$0.0104\,d^2 = 0.0766\,d_e^2$$

$$\Rightarrow \boxed{d_e = 0.370d}$$

AISI 1020CD:

$S_e' = 235$ MPa $\quad (S_{ut} = 470$ MPa$)$ < 1400 MPa

AISI 4340 heat treated to get $S_{ut} = 1760$ MPa

$S_e' = 700$ MPa

2024 T3 Al : (Al- alloy) – no endurance limit

Stress concentration and Notch sensitivity:

- For fluctuating load effect of stress concentration gets reduced
- A reduced value of stress concentration factor is used.
- For fatigue : K_F, K_{FS} (in shear)

Define notch sensitivity

$$q = \frac{K_f - 1}{K_t - 1},$$

K_f → fatigue SCF

K_t → static SCF

$$q_{shear} = \frac{K_{fs} - 1}{K_{ts} - 1}$$

K_{fs} → fatigue SCF

K_{ts} → static SCF in shear

$$\Rightarrow \boxed{\begin{aligned} K_f &= 1 + q(K_t - 1) \\ K_{fs} &= 1 + q_{shear}(K_{ts} - 1) \end{aligned}}$$

$0 \le q \le 1$ and $0 \le q_{shear} \le 1$

- $q = 1$ or $q_{shear} = 1$ $\Rightarrow$ $K_f = K_t$, $K_{fs} = K_{ts}$ → fully sensitive notch

- $q = 0$ or $q_{shear} = 0$ $\Rightarrow$ $K_f = 1$, $K_{fs} = 1$ → not sensitive

Fig 6-20 for q }
Fig 6-21 for q_{shear} } wrt notch radius

Eg → steel rotating beam

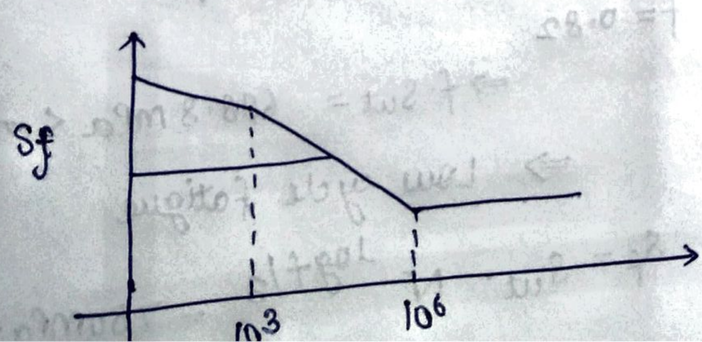
$S_{ut} = 840$ MPa (122 KSi)

Stress amplitude: $\sigma_{rev} = 490$ MPa (completely reversible)

<u>Estimate life</u>

$S_e' = S_e = 420$ MPa. < 490 MPa

$\Rightarrow$ finite life



Find f: (From 6-18)

$f = 0.82$

$\Rightarrow f \cdot S_{ut} = 0.82 \times 840 = 688.8\,MPa. > \sigma_{rev}$

$\Rightarrow$ High cycle Fatigue

$S_f = A N^B$, $\quad B = \frac{1}{3}\log\left(\frac{f \cdot S_{ut}}{S_e}\right)$

$A = \frac{(f \cdot S_{ut})^2}{S_e}$

$\Rightarrow$ ~~A = 968.26 MPa~~ $\quad B = -0.0716$

$A = 1129.632\,MPa$

For life $\quad S_f = \sigma_{rev} = 490\,MPa$

$\Rightarrow 490 = A \cdot N^B \Rightarrow N = \left(\frac{490}{A}\right)^{1/B}$

$\Rightarrow N = 1.16 \times 10^5$ cycles.

Same question but $\quad S_{ut} = 1610\,MPa$

$\Rightarrow S_e' = 700\,MPa > \sigma_{rev}.$

$\Rightarrow$ infinite life

Now, $\quad S_{ut} = 840\,MPa$

$\sigma_{rev} = 750\,MPa.$

$S_e' = 420\,MPa < \sigma_{rev}$

$\Rightarrow$ finite life

$f = 0.82$

$\Rightarrow f \cdot S_{ut} = 688.8\,MPa < \sigma_{rev}$

$\Rightarrow$ Low cycle fatigue

$S_f = S_{ut} \cdot N^{\log f/3} = 750\,MPa = \sigma_{rev}$

Eg $\rightarrow$

$$\Rightarrow N = \left(\frac{S_{ut}}{750}\right)^{-3/\log f}$$

$$N = 51 \text{ cycles.}$$

Eg → 32 mm dia rod

- Material: AISI 1040 HR Steel
- Machined finish
- $S_{ut} = 710$ MPa (after heat treatment)

(a) Estimate endurance strength

$$S_e' = 0.5 S_{ut} = 355 \text{ MPa}$$

$$S_e = k_a k_b k_c k_d k_e k_f S_e'$$

1. <u>Surface finish factor</u>

$$k_a = a S_{ut}^{b} \qquad \left(\begin{matrix} a = 4.51 \\ b = -0.265 \end{matrix} \right\} \text{Table 6.2}$$

$$= 0.792$$

2. <u>Size factor</u>

$$k_b = 1.24 d^{-0.107}$$

$$= 0.86$$

3. <u>Load factor</u> $k_c = 1$

4. $k_d = k_e = k_f = 1.$

$$S_e = 0.792 \times 0.86 \times 1 \times 1 \times 1 \times 1 \times 355$$

$$= 241.8 \text{ MPa}$$

(b) Assume $\sigma_{rev} = 500$ MPa Estimate life

$\sigma_{rev} > S_e \Rightarrow$ finite life

Find f $\quad S_{ut} = 710 \text{ MPa} = 103.11$ KSi

$f = 0.842$ (fig 6-18)

$\Rightarrow f \cdot S_{ut} = 597.82 \text{ MPa} > \sigma_{rev}$

$\Rightarrow$ high cycle fatigue.

$\Rightarrow S_f = A N^B \qquad A = \dfrac{(f \cdot S_{ut})^2}{S_e} \qquad B = -\dfrac{1}{3}\log\left(\dfrac{f \cdot S_{ut}}{S_e}\right)$

$\Rightarrow N = 4501$ cycles $\qquad = 1474.97$ MPa $\qquad = -0.131$

Combined Loading:

Take load factor $K_c = 1$.

Bending → $\sigma_b \rightarrow K_f \cdot \sigma_b$ → fatigue SCF

Axial → $\sigma_{axial} \rightarrow K_f \cdot \sigma_{axial}$

Torsion/shear : $\tau \rightarrow K_{fs} \cdot \tau$

→ fatigue SCF in shear

Calculate von mises stress

$$\sigma' = \left\{ \left(K_f \sigma_{bending} \pm \frac{K_f \cdot \sigma_{axial}}{0.85} \right)^2 + 3 (K_{fs} \cdot \tau)^2 \right\}^{1/2}$$

takes into account $K_c = 0.59$ implicitly