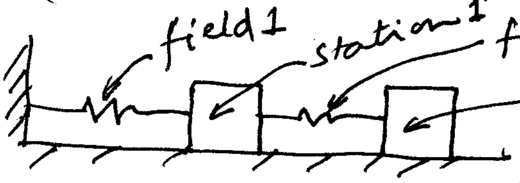
(5) Flexibility influence coefficients:~

Let us introduce the concepts of stations and fields in vibration studies.

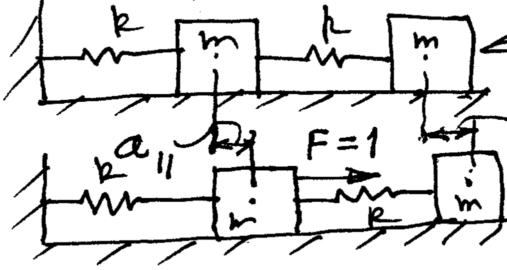


The word 'station' means (the position of) a body like the blocks in the above figure. A 'Field' means a spring which connects two bodies or a body & a wall. A field is assumed to be massless.

→ The flexibility influence coefficient a_{ij} is defined as the deflection at station i due to unit (static) force at station j.

Thus, a_{11} is the deflection at station 1 (left block) due to unit force at station 1 & a_{21} is the deflection at station 2 due to unit load at station 1.

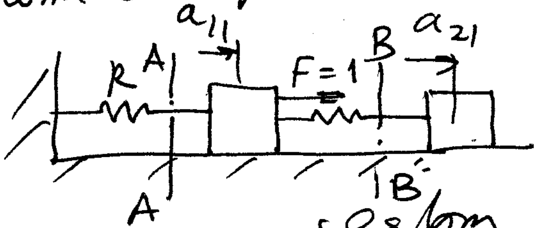
~~Let~~ us explain these further. (Forces & displacements have relevant directions only)



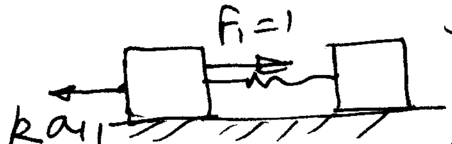
← System in static equilibrium

← System after a very small load is applied to left block & slowly increased to the unit value F=1 (One way of doing it). It is easy to compute a_{11} & a_{21}. We can use either the method of sections (where it is appropriate) or draw free-body diagrams for the blocks (this method is always applicable).

For the present system, method of sections will be quick & appropriate.



Cut (mentally) the system at A-A & consider the eq/bm of the portion to the right of the section. You can do this mentally w/o drawing a separate diagram, but here we draw a separate diagram to illustrate it.



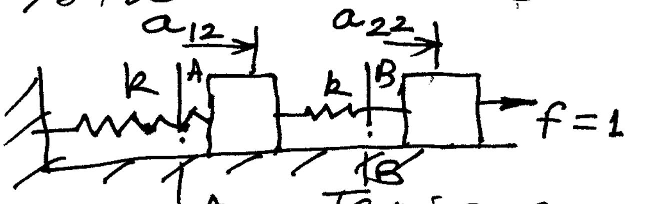
Hence, ka_{11} (the force in the left spring) balances the unit force.

So, $ka_{11} = 1$ or, $\boxed{a_{11} = \frac{1}{k}}$

Next, take a cut at B-B & consider the equilibrium of right block. The force in the right spring is $k(a_{21} - a_{11})$ which must be zero to keep the right block in equilibrium (Friction is neglected and there is no other horizontal force on this block).

Thus, $k(a_{21} - a_{11}) = 0$ or, $\boxed{a_{21} = a_{11} = \frac{1}{k}}$

→ Now, for a 2-DOF system, the number of stations is 2 and so, the ~~total~~ total number of influence coefficients $= 2 \times 2 = 4$ & these four coefficients would make a 2X2 matrix called the <u>flexibility matrix $[a] = [a_{ij}]$</u>. The two other elements come into picture when we apply unit load at station 2. This gives a_{12} & a_{22} since, by definition, a_{12} = deflection at station 1 due to unit load at station 2 & a_{22} = deflection at station 2 due to unit load at station 2. Let us obtain these.



Taking a section at A–A, we have

$$k a_{12} = 1 \Rightarrow a_{12} = 1/k. \text{ Note that } a_{21} = \frac{1}{k} \text{ too.}$$

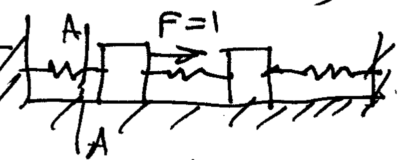
Taking section at BB, we have, $k(a_{22} - a_{12}) = 1$

$$\Rightarrow a_{22} = a_{12} + \frac{1}{k} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$

$$\text{Hence, } [a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1/k & 1/k \\ 1/k & 2/k \end{bmatrix}.$$ ← Symmetric

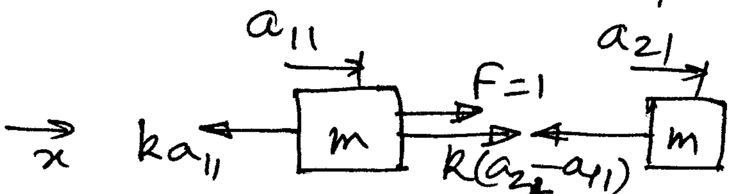
So, at each step, we get a whole column of the flexibility matrix, note.

→ Let us use the equilibrium of each block separately (because, method of sections may not work so well, if, for instance, we have a system like this –



Here, taking a section at A–A & consideration of equilibrium of the system on the right side of A–A won't give a_{11} directly since there is a third spring attached to the wall. However, FBD method will work) ⟶

To obtain a_{11} & a_{21} (i.e. ~~firs~~ elements of the first column of the flexibility matrix [a]).



(assuming $a_{22} > a_{11}$)

(Relevant FBDs)

Hence, for left block, we must have

$\Sigma F_x = 0 \Rightarrow 1 + k(a_{21}-a_{11}) - ka_{11} = 0$

or, $1 + ka_{21} - 2ka_{11} = 0$ — (1).

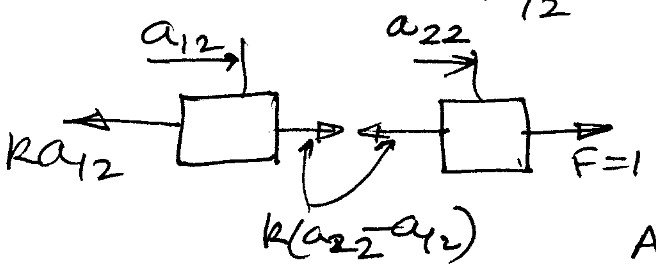
for right block, $-k(a_{21}-a_{11}) = 0$ ($\Sigma F_x = 0$ again)

$\Rightarrow a_{21} = a_{11}$ — (2) Then from (1),

$1 + ka_{11} - 2ka_{11} = 0 \Rightarrow \boxed{a_{11} = \frac{1}{k}}$

So, from (2), $\boxed{a_{21} = \frac{1}{k}}$

Next to obtain a_{12} & a_{22} by this method.



Hence, $ka_{22} - ka_{12} - ka_{12} = 0$

or, $a_{22} = 2a_{12}$ — (i)

Also, $1 - k(a_{22}-a_{12}) = 0$

$\Rightarrow 1 = k(2a_{12}-a_{12}) = ka_{12}$

$\Rightarrow \boxed{a_{12} = 1/k}$ ∴ from (i), $\boxed{a_{22} = 2/k}$

So, $[a] = \begin{bmatrix} 1/k & 1/k \\ 1/k & 2/k \end{bmatrix}$ as before.

→ Now let us obtain $[a]^{-1}$. Note that

$\underbrace{\det[a]}_{\text{determinant of matrix }[a]} = \frac{2}{k^2} - \frac{1}{k^2} = 1/k^2$, $\underbrace{\text{adj}[a]}_{\text{adjoint of }[a]} = \begin{bmatrix} 2/k & -1/k \\ -1/k & 1/k \end{bmatrix}$

→

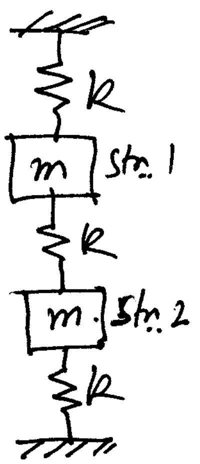
Hence, $(a)^{-1} = \dfrac{adj[a]}{det[a]} = k^2 \begin{bmatrix} 2/k & -1/k \\ -1/k & 1/k \end{bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$

& thus $[a]^{-1} = [k]$! ← Remember this.

→ Now you can see why [a] is important. It is quite easy to experimentally obtain the ~~a~~ matrix [a]. All you need to do is apply unit force at each station, in turn, and measure ~~s~~ displacement at all the stations. After obtaining [a], just invert it ~~to~~ get matrix [K]. We shall soon see that obtaining [K] ~~ex~~ directly experimentally is a much more complex affair, especially for a system with a higher DOF.

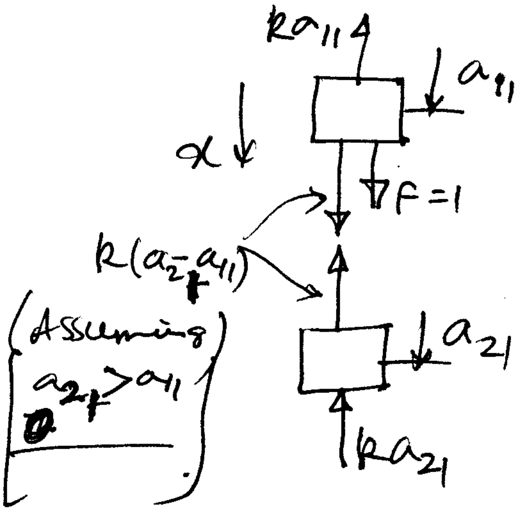
→ So, although we have spent several pages discussing the flexibility matrix, it is basically a small and ~~but~~ routine method that is required for getting [a].

Example: Obtain [a], using definition of a_{ij}, for the system shown in the figure. (Note that gravity & static spring forces are ignored, as usual. If the system executes vertical motion, you can assume it is doing so on a horizontal plane instead & proceed.) (To avoid worries about gravity & static spring forces!)



→

Solution:- Step 1:- To get a_{11} & a_{21}:- (the first column) (F applied to station 1)



For top block equilibrium, we have

$$1+k(a_{21}-a_{11})-ka_{11}=0$$

or, $\frac{1}{k}+a_{21}-2a_{11}=0$ — (1)

For bottom block,

$$-ka_{21}-k(a_{21}-a_{11})=0$$

or, $-2a_{21}+a_{11}=0$ or, $a_{11}=2a_{21}$ — (2)

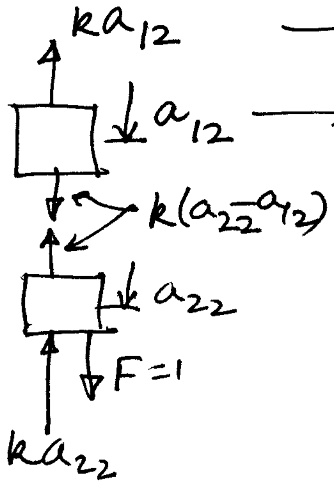
Try this problem using method of sections. Is it any good or different?

From (1), using (2), we get

$\frac{1}{k}+a_{21}-4a_{21}=0 \Rightarrow 3a_{21}=\frac{1}{k}$

or, $\boxed{a_{21}=\frac{1}{3k}}$ So, from (2), $\boxed{a_{11}=\frac{2}{3k}}$

Step 2:- To get a_{12} & a_{22} (the 2nd column) (F applied to station 2)



$\longrightarrow$ Hence, $k(a_{22}-a_{12})-ka_{12}=0$ (Eqlbm. of top block)

$$\Rightarrow \boxed{a_{22}=2a_{12}}$$

Also, $1=ka_{22}+ka_{22}-ka_{12}$

$$\Rightarrow 2a_{22}-\frac{a_{22}}{2}=\frac{1}{k}$$

$$\Rightarrow \boxed{a_{22}=\frac{2}{3k}}$$

& $a_{12}=\frac{1}{2}a_{22}=\frac{1}{3k}$

So, $[a]=\begin{bmatrix} 2/3k & 1/3k \\ 1/3k & 2/3k \end{bmatrix}$ ∴ $\det[a] = \frac{4}{9k^2}-\frac{1}{9k^2}=\frac{1}{3k^2}$.

$\text{Adj}[a]=\begin{bmatrix} \frac{2}{3k} & -1/3k \\ -1/3k & 2/3k \end{bmatrix}$ So, $[a]^{-1}=3k^2\begin{bmatrix} \frac{2}{3k} & -\frac{1}{3k} \\ -\frac{1}{3k} & \frac{2}{3k} \end{bmatrix}$

$$\therefore [k]=[a]^{-1}=\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}.$$

Obtain the DEOM of the system & check if this [k] is ok.

We shall see later that the flexibility matrix has usage in some approximate methods for estimating the lowest natural frequency of the system.

→ You may note in passing that the DEOM for free vibration of a 2-DOF system can be written as

$$x_1 = a_{11}(-m_1\ddot{x}_1) + a_{12}(-m_2\ddot{x}_2) \quad (1)$$

$$\& \quad x_2 = a_{21}(-m_1\ddot{x}_1) + a_{22}(-m_2\ddot{x}_2) \quad (2)$$

These may look strange at first sight. However, a closer look will reveal ~~that~~ there is nothing unusual. Since $-m_1\ddot{x}_1$ & $-m_2\ddot{x}_2$ are the inertia forces involved, the displacement at station 1 is given by the sum of the displacements caused by inertia forces $-m_1\ddot{x}_1$ & $-m_2\ddot{x}_2$ & from the definitions of a_{11} & a_{12}, ~~it follows that~~ DEOM (1) follows. Similarly, DEOM (2) is obtained.

→ (1) & (2) can be obtained in another way.

$[m]\{\ddot{x}\}+[K]\{x\}=\{0\}$ is the matrix DEOM of our system for undamped free vibration. Premultiplying both sides by $[K]^{-1}$, we get

$[K]^{-1}[m]\{\ddot{x}\}+\{x\}=\{0\}$.. But $[K]^{-1}=[a]$

So, $[a][m]\{\ddot{x}\}+\{x\}=\{0\}$ or $\{x\}=-[a][m]\{\ddot{x}\}$.

$$RHS = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} -a_{11}m_1\ddot{x}_1 - a_{12}m_2\ddot{x}_2 \\ -a_{21}m_1\ddot{x}_1 - a_{22}m_2\ddot{x}_2 \end{Bmatrix}$$

$LHS = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$. ~~Hence, $x_1 = -a_{11}m_1\ddot{x}_1 - a_{12}m_2\ddot{x}_2$~~

Hence, $x_1 = -a_{11} m_1 \ddot{x}_1 - a_{12} m_2 \ddot{x}_2$

$x_2 = -a_{21} m_1 \ddot{x}_1 - a_{22} m_2 \ddot{x}_2$

which are the same as (1) & (2).

→ We shall Comeback to the flexibility matrix a little later.

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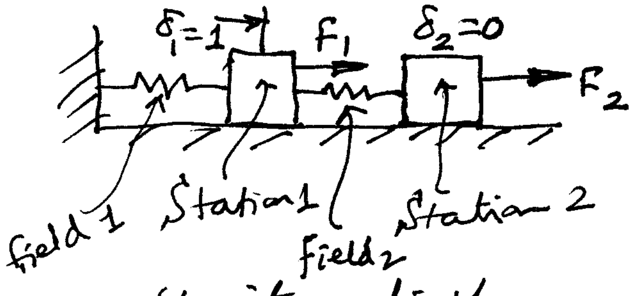
(8) The Stiffness influence coefficients:-

The stiffness influence coefficient k_{ij} is defined as the force required at station i to produce unit displacement at station j such that under appropriate forces applied to all stations, j th station is the only one to undergo a displacement. All other stations shouldn't move. Let us illustrate.

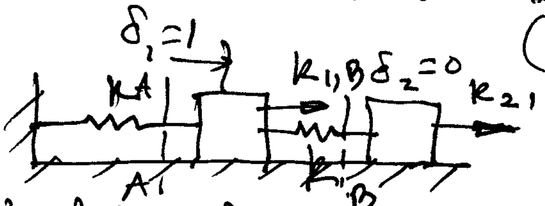


For our 2-DOF system shown, let F_1 & F_2 be forces applied at station 1 & station 2 respectively to produce unit displacement $\delta_1 = 1$ at station 1 & zero displacement at station 2. Then, by definition, $F_1 = k_{11}$ and $F_2 = k_{21}$

force at stn 1 to produce unit displ. at stn. 1

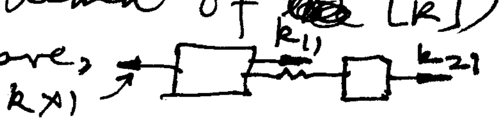
force at stn 2 to produce unit displ. at stn 1

(With no movement of stn 2)

So, 

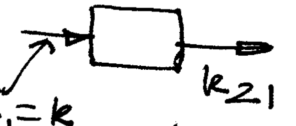
← This should be the fig. for obtaining k_{11} & k_{21} (The first column of [K]).

(Showing k_{11} & k_{21} as forces is OK, why?)

Using section at A-A, we have, 

So, $K_{11} + K_{21} - k = 0$ or, $K_{11} + k_{21} = k$ --- (1)

For section B–B,

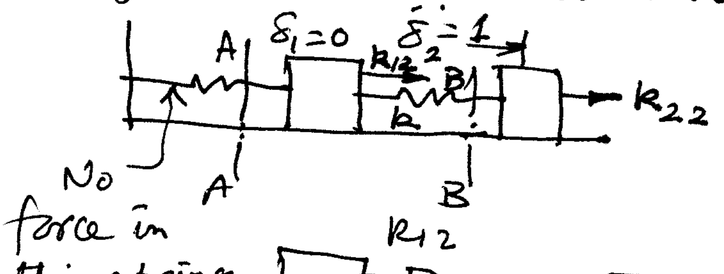


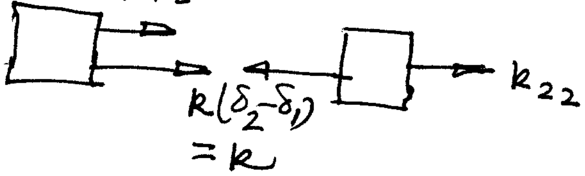
$\Rightarrow k_{21} + k = 0 \Rightarrow \underline{\underline{k_{21} = -k}}$

(Since field to is compressed by $\delta_1 = 1$)

So, from (1), $K_{11} = K - K_{21} = k - (-k) = 2k$.

→ To get the 2nd column $\begin{Bmatrix} K_{12} \\ K_{22} \end{Bmatrix}$ of $[k]$, we use:





So, $k_{12} + k = 0$ & $k_{22} - k = 0$

$\Rightarrow k_{12} = -k, \quad k_{22} = k$.

Thus, $[k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$, hence, <u>OK</u>.

→ Note that obtaining $[k]$ directly experimentally is a more involved affair compared with obtaining $[a]$. To obtain a_{11} & a_{21} for instance, all we need to do is apply unit force at station 1 & measure the displacements at stations 1 & 2.

For obtaining K_{11} & K_{21}, on the other hand, we have to apply simultaneous forces at both stations, keep adjusting these forces until deflection at station 1 is unity but that at station 2 is zero. This is a much more involved task. So we experimentally <u>obtain $[a]$</u> usually & not $[k]$.

—x—