Anisotropic Elasticity

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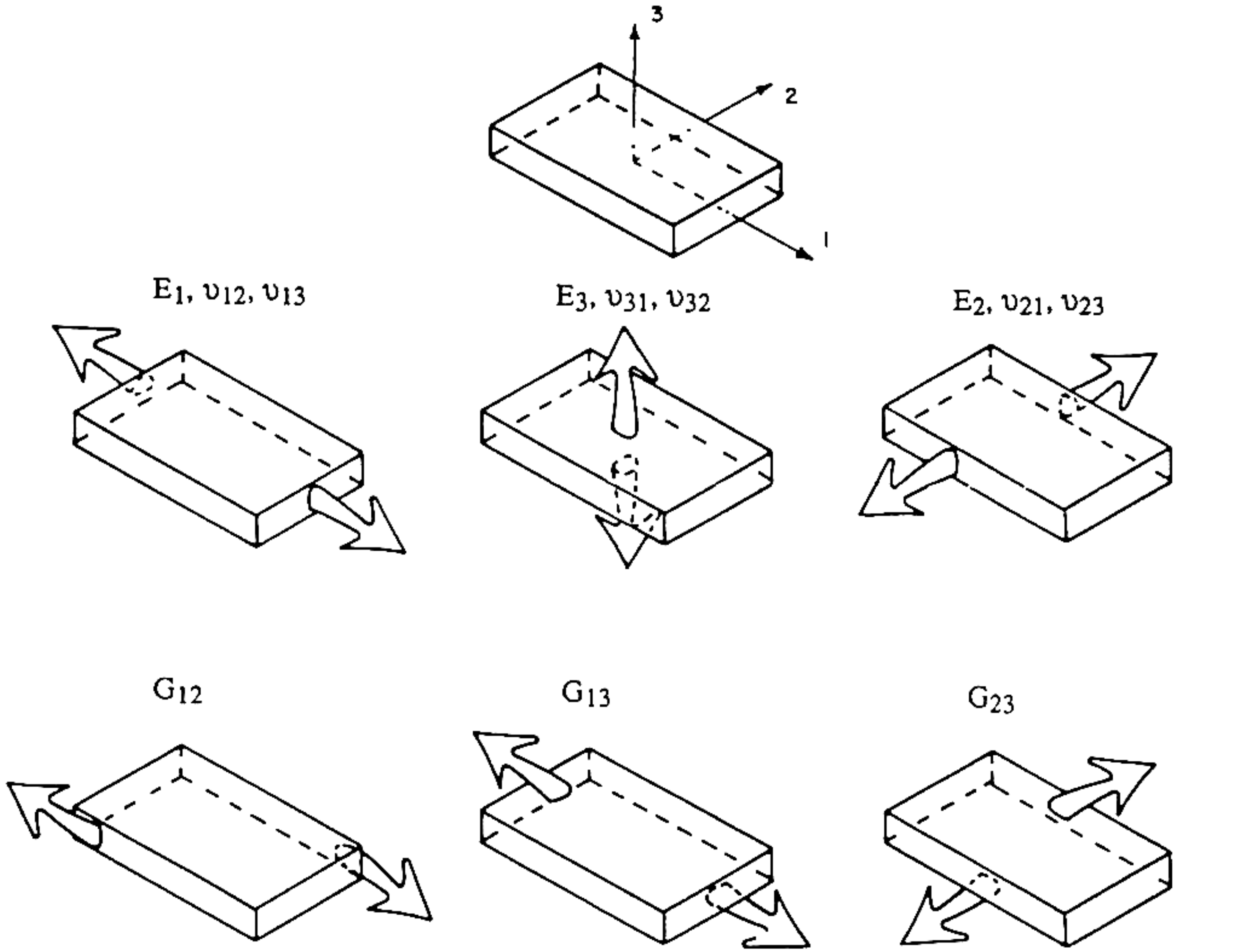
Matrix in terms of Elastic constants

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

$$\Delta = (1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13})/(E_1E_2E_3)$$

Graphic representation of the engineering constants

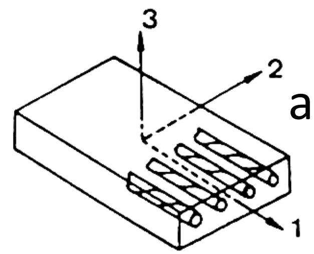


Orthotropic materials under plane stress

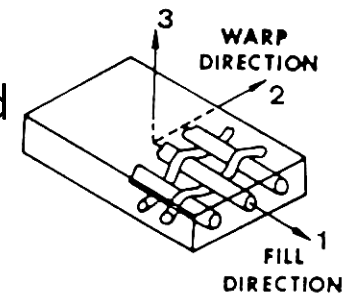
- Applies to thin orthotropic plies or laminae
- If (1-2) is orthotropy plane, state of plane stress means

$$\sigma_3 = \sigma_4 = \sigma_5 = 0$$

- Stress-strain relations reduce to



and



LAMINA WITH UNIDIRECTIONAL FIBERS

LAMINA WITH WOVEN FIBERS

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{pmatrix} = \begin{pmatrix}\frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & 0\\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix}$$

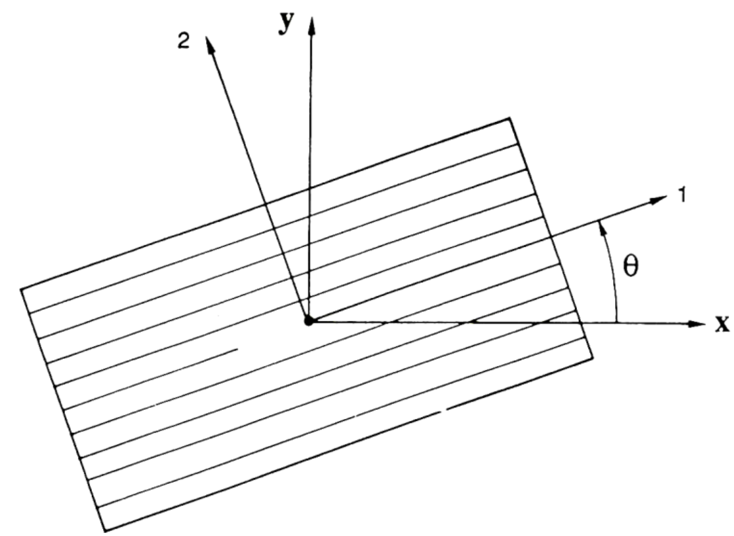
$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix} = \begin{pmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{pmatrix}$$

$$Q_{11} = \frac{E_1}{\left(1-\nu_{12}^2 \, E_2/E_1\right)} \qquad Q_{22} = \frac{E_2}{\left(1-\nu_{12}^2 \, E_2/E_1\right)}$$

$$Q_{12} = \frac{\nu_{12}E_2}{\left(1-\nu_{12}^2 \, E_2/E_1\right)} \qquad Q_{66} = G_{12}$$

Stress-strain relations for orthotropic ply of arbitrary orientation

- Goal: write stress-strain relation in coordinate system (x-y) other than orthotropy axis (1-2)



- Angle between x and 1 is θ

- Tensor transformation laws yield

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=[T_\sigma]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix}\qquad \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ 2\varepsilon_{12}\end{pmatrix}=[T_\varepsilon]\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ 2\varepsilon_{xy}\end{pmatrix}$$

$$[T_\sigma]=\begin{pmatrix}m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & \left(m^2-n^2\right)\end{pmatrix}\quad [T_\varepsilon]=\begin{pmatrix}m^2 & n^2 & mn\\ n^2 & m^2 & -mn\\ -2mn & 2mn & \left(m^2-n^2\right)\end{pmatrix}$$

$$m=cos\theta \qquad n=sin\theta$$

Stress-strain relations arbitrary orientation

- In the (x-y) system, Hooke's law reads

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{xy} & S_{yy} & S_{ys} \\ S_{xs} & S_{ys} & S_{ss} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

Where $\left[\bar{\mathrm{S}}\right] = \left[\mathrm{T}_\varepsilon\right]^{-1}\left[\mathrm{S}\right]\left[\mathrm{T}_\sigma\right]$

- Algebra yields :

$$\begin{aligned}
S_{xx} &= m^4 S_{11} + m^2 n^2 \left(2S_{12} + S_{66}\right) + n^4 S_{22} \\
S_{xy} &= m^2 n^2 \left(S_{11} + S_{22} - S_{66}\right) + S_{12}\left(m^4 + n^4\right) \\
S_{yy} &= n^4 S_{11} + m^2 n^2 \left(2S_{12} + S_{66}\right) + m^4 S_{22} \\
S_{xs} &= 2m^3 n\left(S_{11} - S_{12}\right) + 2mn^3\left(S_{12} - S_{22}\right) - mn\left(m^2 - n^2\right)S_{66} \\
S_{ys} &= 2mn^3\left(S_{11} - S_{12}\right) + 2m^3 n\left(S_{12} - S_{22}\right) + mn\left(m^2 - n^2\right)S_{66} \\
S_{ss} &= 4m^2 n^2\left(S_{11} - S_{12}\right) - 4m^2 n^2\left(S_{12} - S_{22}\right) + \left(m^2 - n^2\right)^2 S_{66} \\
&with \quad m = \cos\theta, \quad n = \sin\theta
\end{aligned}$$

Contd.

- In the (x-y) system, Hooke's law reads

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix}$$

Where $\left[\overline{\mathrm{Q}}\right] = \left[\mathrm{T}_\sigma\right]^{-1}\left[\mathrm{Q}\right]\left[\mathrm{T}_\varepsilon\right]$

- Algebra yields :

$$Q_{xx} = m^4 Q_{11} + 2m^2n^2\left(Q_{12} + 2Q_{66}\right) + n^4 Q_{22}$$

$$Q_{xy} = m^2n^2\left(Q_{11} + Q_{22} - 4Q_{66}\right) + Q_{12}\left(m^4 + n^4\right)$$

$$Q_{yy} = n^4 Q_{11} + 2m^2n^2\left(Q_{12} + 2Q_{66}\right) + m^4 Q_{22}$$

$$Q_{xs} = m^3 n\left(Q_{11} - Q_{12}\right) + mn^3\left(Q_{12} - Q_{22}\right) - 2mn\left(m^2 - n^2\right)Q_{66}$$

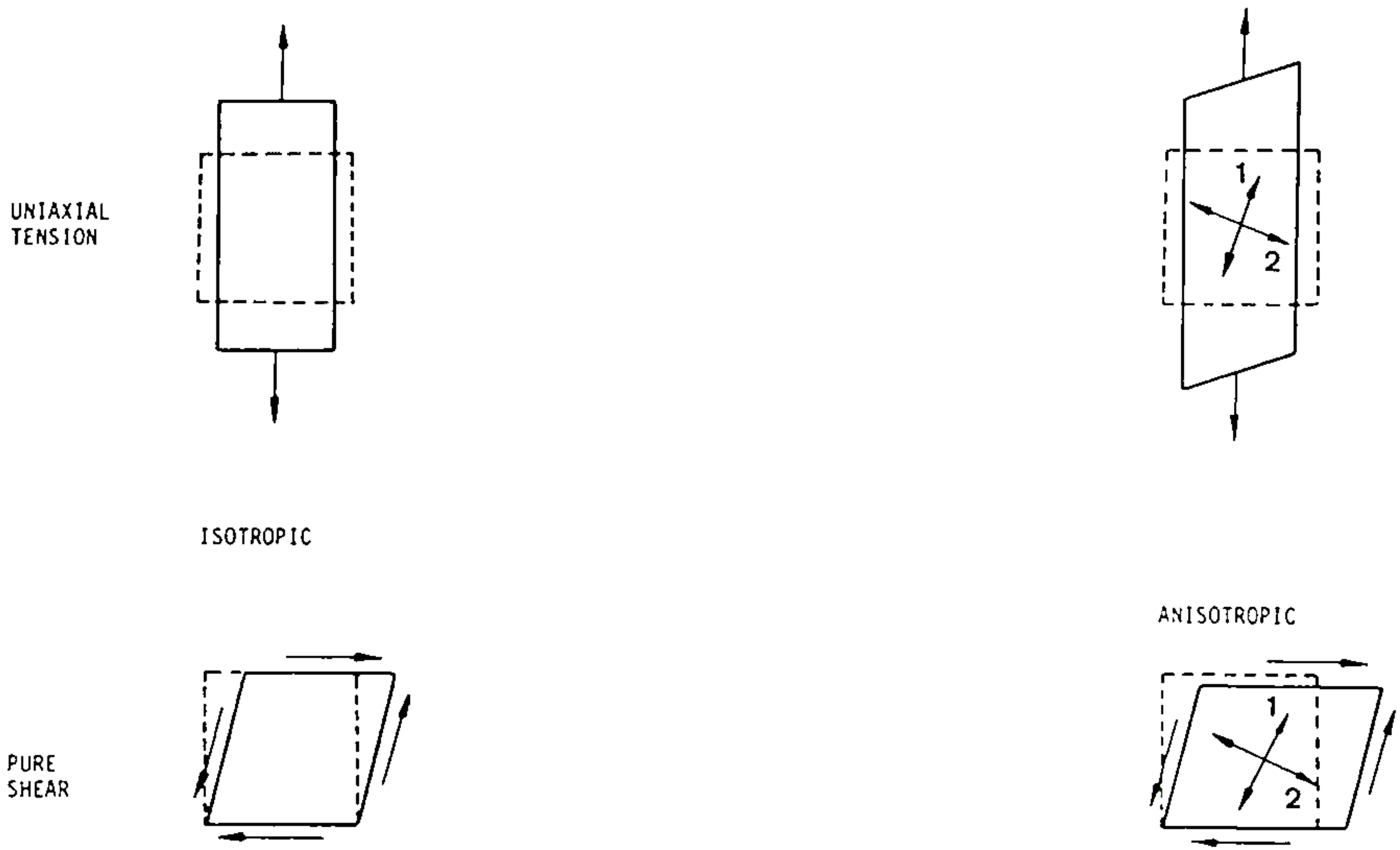
$$Q_{ys} = mn^3\left(Q_{11} - Q_{12}\right) + m^3 n\left(Q_{12} - Q_{22}\right) + 2mn\left(m^2 - n^2\right)Q_{66}$$

$$Q_{ss} = m^2n^2\left(Q_{11} + Q_{22} - 2Q_{12}\right) + \left(m^2 - n^2\right)^2 Q_{66}$$

$$with \quad m = cos\theta, \quad n = sin\theta$$

- *Shear-extension coupling* occurs if (x-y) is different from (1-2)

Coefficient of shear coupling coefficient



Engineering constants for orthotropic ply of arbitrary orientation

- Simple shear σxy yields

$$\varepsilon_x = S_{xs}\sigma_{xy} \quad \varepsilon_y = S_{ys}\sigma_{xy} \quad 2\varepsilon_{xy} = S_{ss}\sigma_{xy}$$

- This defines the different apparent engineering constants :

$$G_{xy} = \frac{\sigma_{xy}}{2\varepsilon_{xy}} = \frac{1}{S_{ss}}$$

$$\eta_{x,xy} = \frac{\varepsilon_x}{2\varepsilon_{xy}} = \frac{S_{xs}}{S_{ss}} \quad \eta_{y,xy} = \frac{\varepsilon_y}{2\varepsilon_{xy}} = \frac{S_{ys}}{S_{ss}}$$

- The coupling coefficients satisfy the following relations :

$$\frac{\eta_{xy,x}}{E_x} = \frac{\eta_{x,xy}}{G_{xy}} \quad \frac{\eta_{xy,y}}{E_y} = \frac{\eta_{y,xy}}{G_{xy}}$$

Engineering constants for orthotropic ply of arbitrary orientation

- In terms of apparent engineering constants, Hooke's law reads

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_x} & \frac{\eta_{xy,x}}{E_x} \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{E_y} \\ \frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

Engineering constants for orthotropic ply of arbitrary orientation

- Directional dependence of apparent engineering constants :

$$\frac{1}{E_x} = m^4 \frac{1}{E_1} + m^2 n^2 \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) + n^4 \frac{1}{E_2}$$

$$\nu_{xy} = E_x \left[\frac{\nu_{12}}{E_1} \left(m^4 + n^4 \right) - m^2 n^2 \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) \right]$$

$$\frac{1}{E_y} = n^4 \frac{1}{E_1} + m^2 n^2 \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) + m^4 \frac{1}{E_2}$$

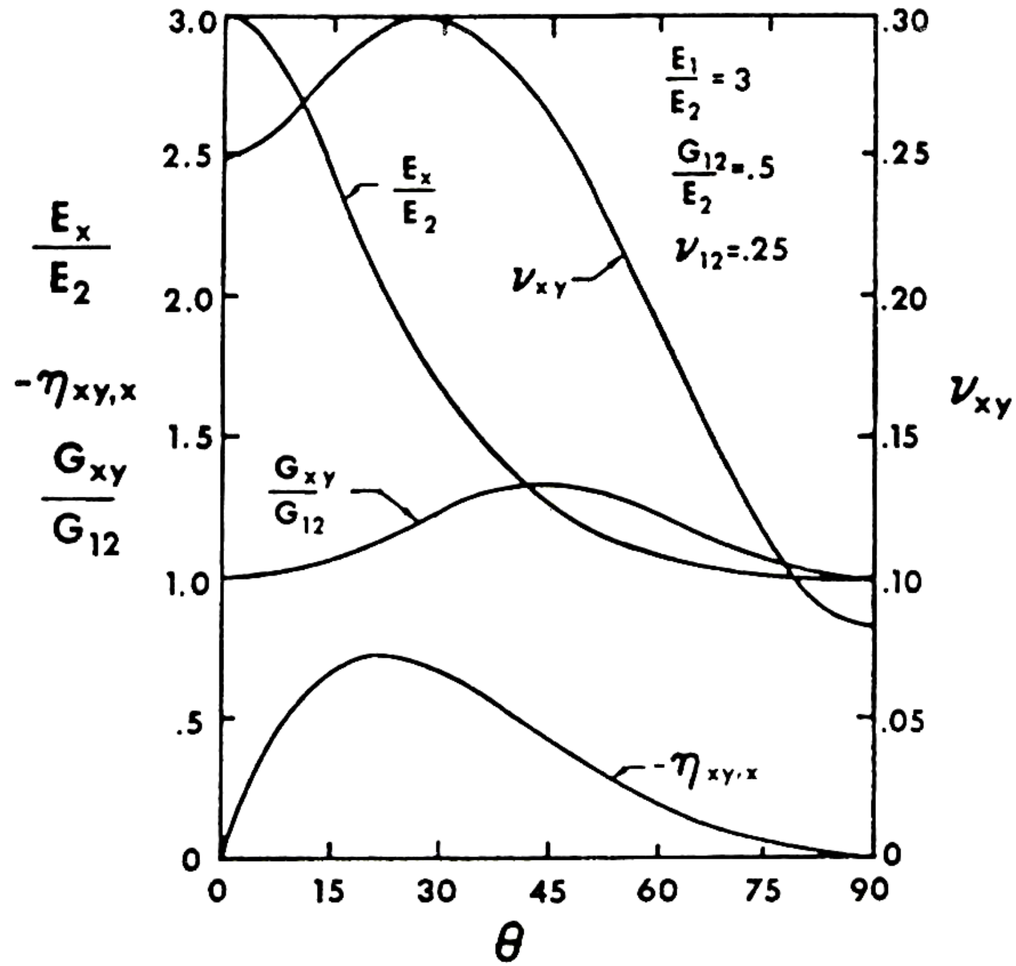
$$\eta_{xy,x} = E_x \left[m^3 n \left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) - m n^3 \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) \right]$$

$$\eta_{xy,y} = E_y \left[m n^3 \left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) - m^3 n \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) \right]$$

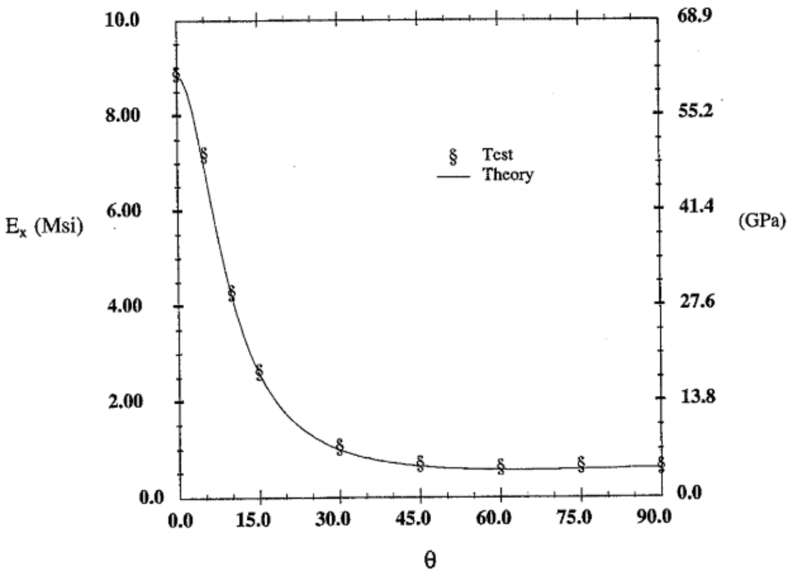
$$\frac{1}{G_{xy}} = 4 m^2 n^2 \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} \right) + \left(m^2 - n^2 \right)^2 \frac{1}{G_{12}}$$

Engineering constants for orthotropic ply of arbitrary orientation

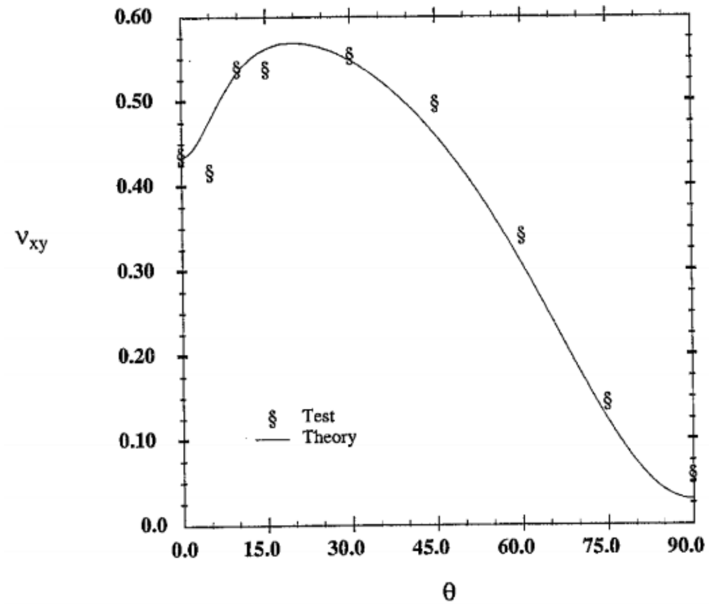
- Illustration : Variation of engineering constants as a function of the loading angle θ
  - For glass-epoxy



Comparison with experiments

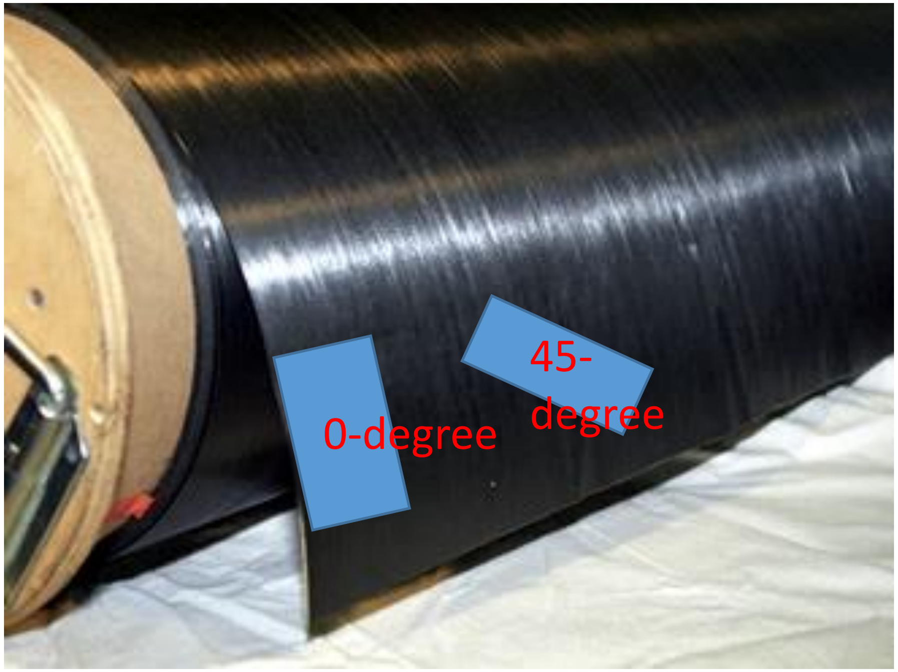


Aramid/Epoxy Off-Axis Modulus: Theory and Experiment



Aramid/Epoxy Off-Axis Poisson's Ratio: Theory and Experiment

Cutting lamina at different angle



Problem 1

- Calculate the engineering constants of the 0° and 90° orientations of the glass fabric composite, v_f = 40.9

1. Fibre properties		
Ef//	71	GPa
Ef^	71	GPa
ν_{12}	0.25	
Gf	28.50	GPa

2. matrix properties		
E	3.1	GPa
ν	0.2	
G	1.29	GPa

$$E_{//} = E_f V_f + E_m V_m$$

$$E_{\perp} = \frac{E_f E_m}{E_f V_m + E_m V_f}$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

$$G_{12} = \frac{G_f G_m}{G_f V_m + G_m V_f}$$

Solution

- Use rules of mixtures to calculate engineering constants

3. properties of the UD-ply (ply coordinate system)		
Vf	40,9%	
E11	30,87	GPa
E22	5,09	GPa
ν12	0,22	
G12	2,12	GPa

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{pmatrix} = \begin{pmatrix}\frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & 0\\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{pmatrix} \cdot \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix}$$

S-matrix 0° orientation		
0,0324	-0,0071	0,0000
-0,0071	0,1964	0,0000
0,0000	0,0000	0,4719

Q-matrix 0° orientation		
31,12	1,13	0,00
1,13	5,13	0,00
0,00	0,00	2,12

Q-matrix orientation		90
5,133	1,131	0,000
1,131	31,121	0,000
0,000	0,000	2,119

S-matrix orientatie		90
0,20	-0,01	0,00
-0,01	0,03	0,00
0,00	0,00	0,47

Exx	30,87	Gxy	2,12
Eyy	5,09	$\eta_{xy,x}$	0,00
ν_{xy}	0,22	$\eta_{xy,x}$	0,00

Exx	5,09	Gxy	2,12
Eyy	30,87	$\eta_{xy,x}$	0,00
ν_{xy}	0,04	$\eta_{xy,x}$	0,00

Expansional (Hygrothermal) Strains

All materials are sensitive to the environment in which they operate.

Expansional strains (volume changes in the absence of surface tractions) in composites are caused by:

1. Changes in temperature
2. Absorption of swelling agents (e.g. water, solvents)
3. Expansion of dissolved gases

In addition to inducing residual stresses (internal self-equilibrating stresses which exist in the absence of surface tractions), these factors may also affect the gross response of a structure, such as natural vibrational frequencies and buckling loads.

The Generalised Hooke's Law must therefore be amended to take expansional strains into account:

$$[\varepsilon] = [S][\sigma] + [\varepsilon]^T + [\varepsilon]^H + \ldots\ldots$$

where $[\varepsilon]^T$ and $[\varepsilon]^H$ represent thermal and hygroscopic strains, respectively.

Expansional (Hygrothermal) Strains (cont.)

In Eqn. (2.25), the thermal and hygroscopic strains are given by:

$$\varepsilon_i{}^T = \alpha_i \, \Delta T$$
$$\varepsilon_i{}^H = \beta_i \, \Delta C$$

Where α_i and ΔT denote thermal expansion coefficients and change in temperature, and where β_i and ΔC denote hygroscopic expansion coefficients and change in concentration of a swelling agent.

The inverted (stiffness) form yields:

$$[\sigma] = [C]\{[\varepsilon] - [\varepsilon]^T - [\varepsilon]^H\}$$

For orthotropic materials, free expansion produces only normal strains *(i.e.* $\varepsilon_4{}^{T,H} = \varepsilon_5{}^{T,H} = \varepsilon_6{}^{T,H} = 0$*).*

For a material which is transversely isotropic in the 1-2 plane, $\varepsilon_2{}^{T,H} = \varepsilon_3{}^{T,H}$.

Plane Stress Constit. Eqn. w. Expansional Strains

For plane stress in the 1-2 plane, become:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} + \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} + \Delta C \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 - \alpha_1 \Delta T - \beta_1 \Delta C \\ \varepsilon_2 - \alpha_2 \Delta T - \beta_2 \Delta C \\ \varepsilon_6 \end{bmatrix}$$

For rotations, expansional strains rotate, resulting in:

$$\begin{aligned} {\varepsilon_1}^T &= m^2\, {\varepsilon_x}^T + n^2\, {\varepsilon_y}^T \\ {\varepsilon_2}^T &= n^2\, {\varepsilon_x}^T + m^2\, {\varepsilon_y}^T \\ 0 &= 2mn\left({\varepsilon_y}^T - {\varepsilon_x}^T \right) + (m^2 - n^2)\, {\gamma_{xy}}^T \end{aligned}$$

Rotated Plane Stress Equations

For the rotated plane stress equation, with expansional strains:

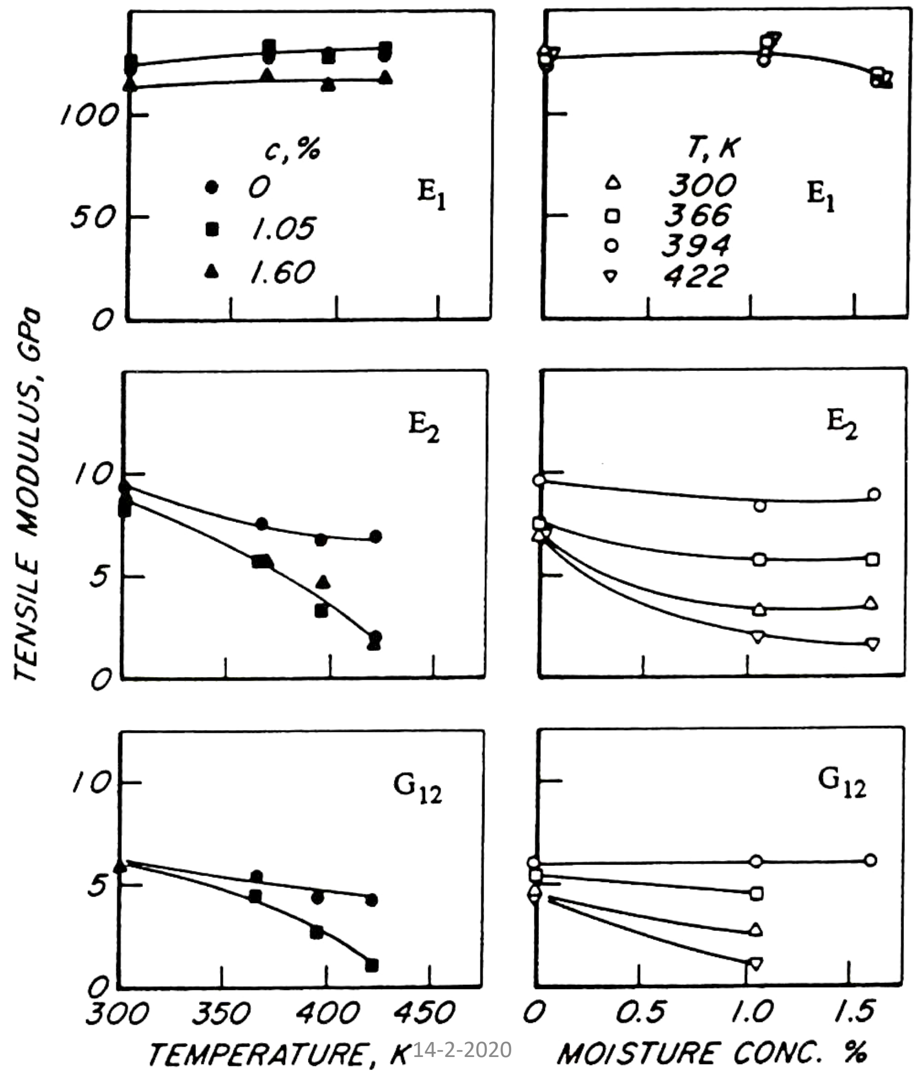
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x - \Delta T\, \alpha_x - \Delta C\, \beta_x \\ \varepsilon_y - \Delta T\, \alpha_y - \Delta C\, \beta_y \\ \gamma_{xy} - \Delta T\, \alpha_{xy} - \Delta C\, \beta_{xy} \end{bmatrix}$$

It can also be shown that:

$$\alpha_x = {\varepsilon_x}^T / \Delta T = m^2 \alpha_1 + n^2 \alpha_2$$
$$\alpha_y = {\varepsilon_y}^T / \Delta T = n^2 \alpha_1 + m^2 \alpha_2$$
$$\alpha_{xy} = {\gamma_{xy}}^T / \Delta T = 2(\alpha_1 - \alpha_2) mn$$

Temperature and moisture dependence

- Stiffness coefficients must be determined at the correct T and c



How much reinforcement?

Weight fraction

Used in manufacture.

May refer to fibre or resin - 'GRP' manufacturers will specify a ***glass*** content of (e.g.) 25 wt%; a prepreg supplier might give a ***resin*** content of 34 wt%.

Volume fraction

Used in design to calculate composite properties.

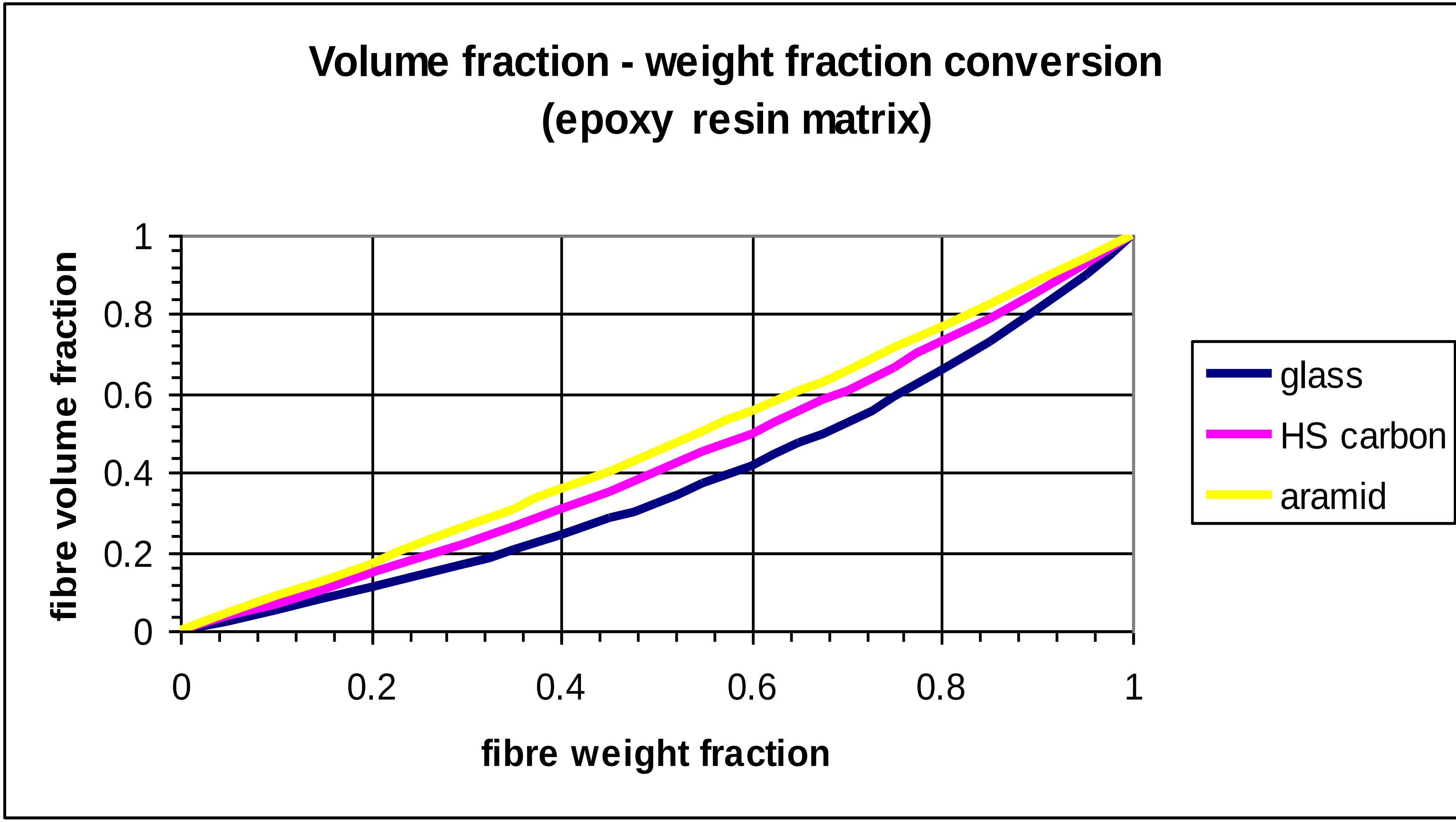
Almost always refers to *fibre* content.

Weight fraction $\leftrightarrow$ volume fraction conversion

For the special case of a two-component composite (eg fibre and matrix):

$$V_f = \frac{W_f / \rho_f}{W_f / \rho_f + (1 - W_f) / \rho_m}$$

$$W_f = \frac{\rho_f V_f}{\rho_f V_f + \rho_m (1 - V_f)}$$


Volume fraction - weight fraction conversion
(epoxy resin matrix)
fibre volume fraction
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
fibre weight fraction
glass
HS carbon
aramid

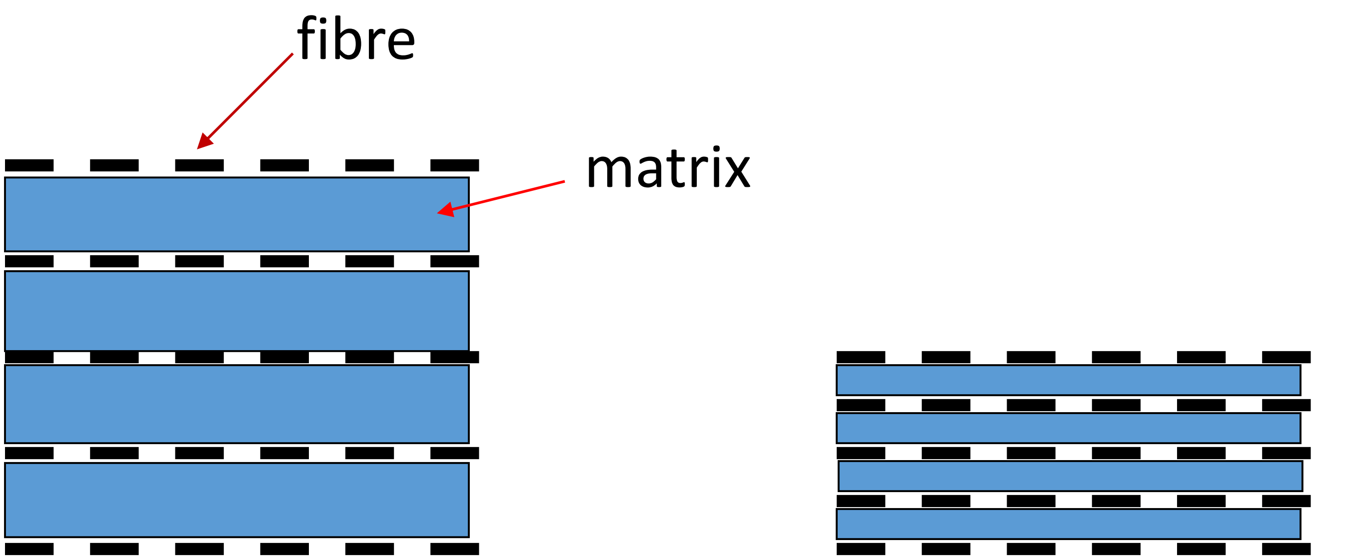
How much fibre?

Commercial reinforcements are characterised by their ***areal weight (A_w).*** This is simply the weight (usually given in g) of 1 m^2 of the reinforcement. A_w depends on many factors - fibre density, tow or bundle size, weave style, etc.

A_w may range from 50 g/m^2 or less (for lightweight surfacing tissues), up to more than 2000 g/m^2 for some heavyweight non-crimp fabrics.

Laminate thickness

Two laminates, both containing 5 plies of reinforcement:



high matrix content

low fibre content

= thick laminate

low matrix content

high fibre content

= thin laminate

Laminate thickness

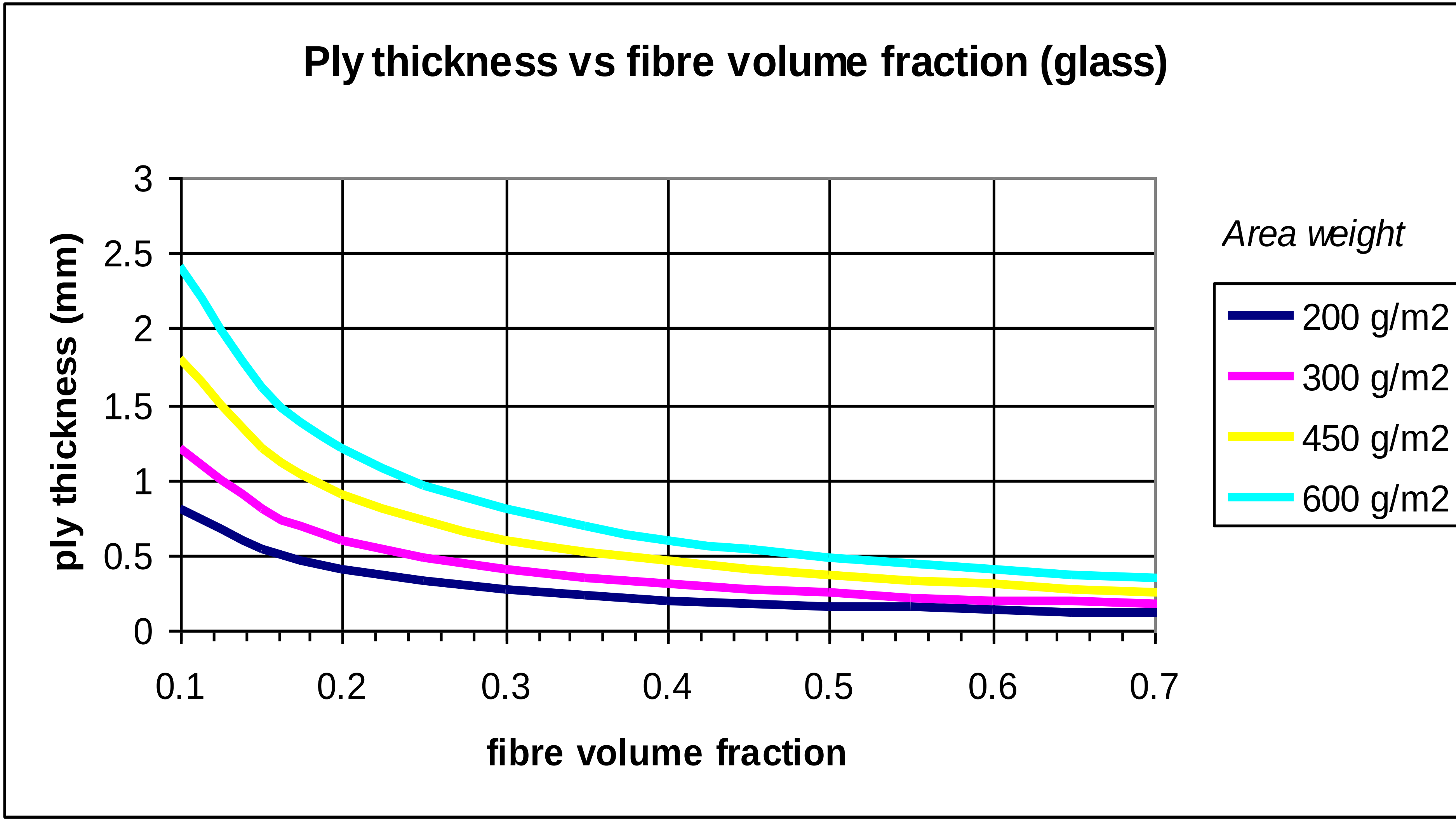
Fibre volume fraction is thus inversely proportional to laminate thickness.

If the fibre content and laminate thickness are defined, we can calculate the fibre volume fraction:

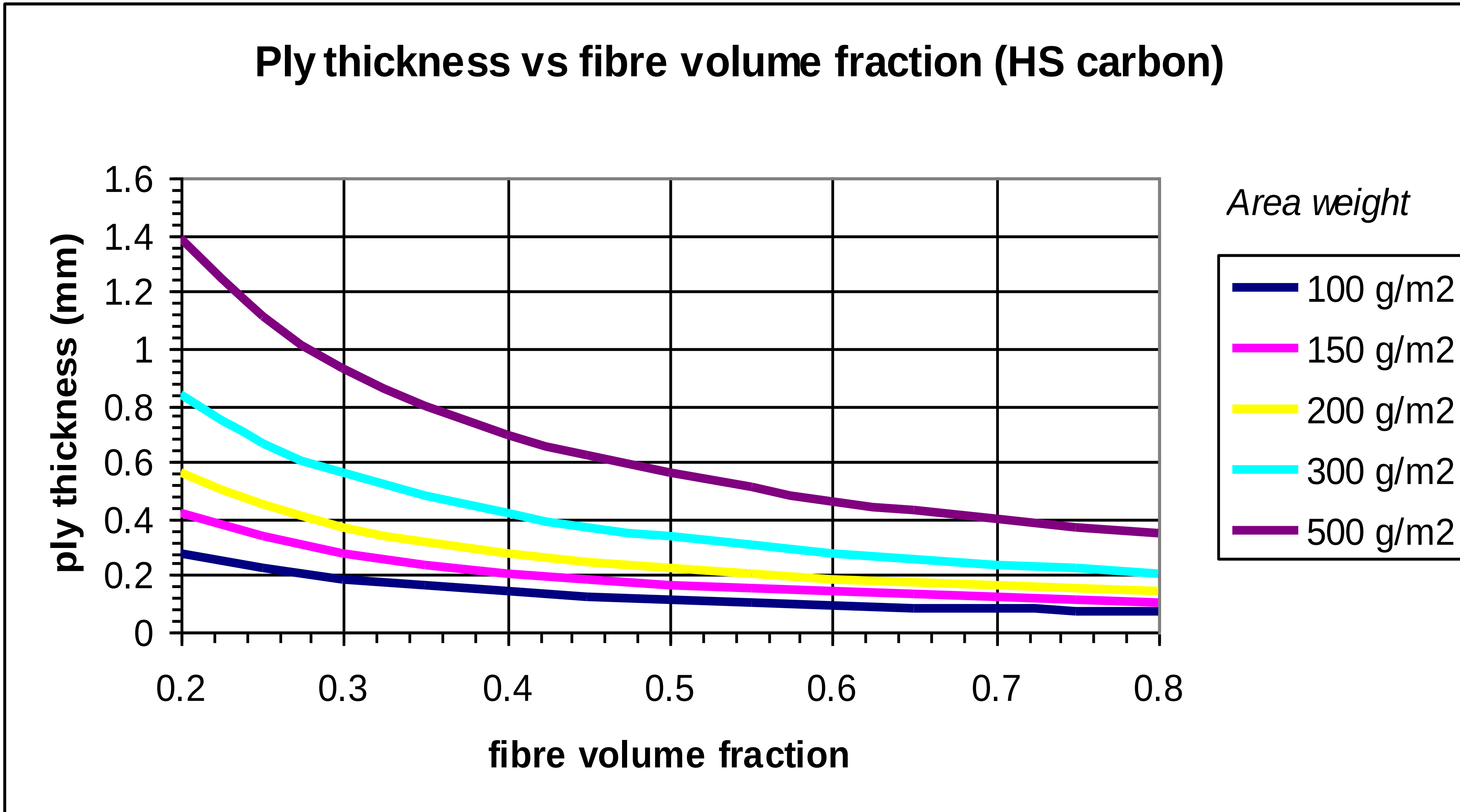
$$V_f = \frac{nA_w}{\rho_f t}$$

If the fibre content and volume fraction are defined, we can calculate the laminate thickness:

$$t = \frac{nA_w}{\rho_f V_f}$$



Ply thickness vs fibre volume fraction (glass)
ply thickness (mm)
3
2.5
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
fibre volume fraction
Area weight
200 g/m2
300 g/m2
450 g/m2
600 g/m2


Ply thickness vs fibre volume fraction (HS carbon)
ply thickness (mm)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
fibre volume fraction
Area weight
100 g/m2
150 g/m2
200 g/m2
300 g/m2
500 g/m2

Example calculations

1. What will be the thickness of a laminate consisting of 2 layers of 450 g/m^2 chopped strand mat if a resin to glass ratio (by weight) of 2:1 is used? Density of fibre and matrix is 2 g/cm3 and 1gm/cm3 respectively.

2. What fibre volume fraction is achieved if 3 layers of 800 g/m^2 glass woven roving are compression-moulded to a thickness of 2 mm?