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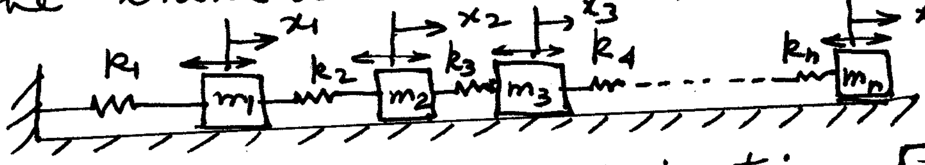
Vibration of discrete systems having (no. of) DOF > 2

[redacted]

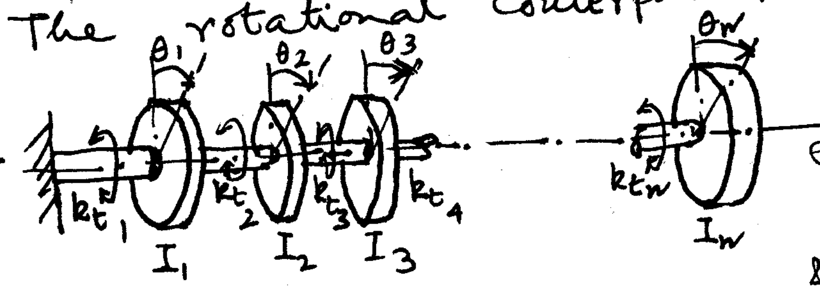
→ The translational undamped model (Free Vibration):



← Fig. 1

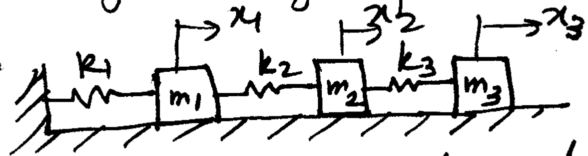
→ The rotational couterpart:

Fig. 2 →



[These are fixed-free systems. Other end (boundary) conditions such as fixed-fixed & free-free are possible]

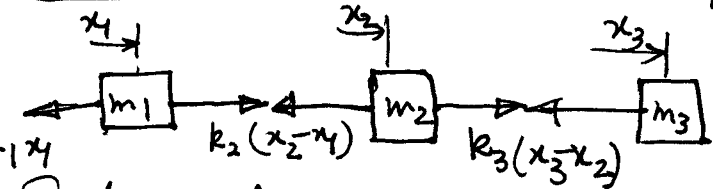
→ We shall first consider 3-DOF systems & then generalize for an n-DOF system.



For the translational system, x_1, x_2, x_3 are a set of generalized coordinates with static-equilibrium positions as reference.

At time t, the <u>relevant</u> FBDs are:—(Friction neglected)

(Let $x_3 > x_2 > x_1$)



By Newton's 2nd law of motion,

$$m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1 \quad \text{or,} \quad m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \quad \text{—— (1)}$$

$$m_2\ddot{x}_2 = k_3(x_3-x_2) - k_2(x_2-x_1) \quad \text{or,} \quad m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 - k_3x_3 = 0 \quad \text{—— (2)}$$

$$m_3\ddot{x}_3 = -k_3(x_3-x_2) \quad \text{or,} \quad m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0 \quad \text{—— (3)}$$

(1), (2) & (3) are the required DEOM for undamped free vibrations, obtained by Newton's Method.

In matrix form, the DEOM can be written as:

$$[m]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \text{—— (4).} \quad \text{Here } \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}$$

(Continued)

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}; \quad [k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \; \& \; \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

(Mass or inertia matrix) ; (Stiffness matrix)

For the corresponding n-DOF system, the DEOM will have the same form

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\},$$

where

$$\underset{n\times 1}{\{x\}} = \{x_1 \; x_2 \; \cdots \; x_n\}^T; \quad \underset{n\times 1}{\{\ddot{x}\}} = \{\ddot{x}_1 \; \ddot{x}_2 \; \cdots \; \ddot{x}_n\}^T;$$

(displacement vector) (Acceleration vector)

$$[m] = \underset{(n\times n)}{\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}}; \quad [k] = \underset{(n\times n)}{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}} \; \& \; \underset{n\times 1}{\{0\}} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}.$$

→ For the 3-DOF system, we now derive the DEOM using Lagrange's equations.

There are 3 such equations & these are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = 0; \quad i = 1, 2, 3.$$

Here $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$

$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2$

Hence,

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1; \quad \frac{\partial T}{\partial x_1} = 0$

$\frac{\partial V}{\partial x_1} = k_1 x_1 + k_2 (x_2 - x_1)(-1)$
$= (k_1 + k_2) x_1 - k_2 x_2$

Thus, the first DEOM is:

$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$

These DEOM are the same as the ones obtained using Newton's method

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2; \quad \frac{\partial T}{\partial x_2} = 0$

$\frac{\partial V}{\partial x_2} = k_2 (x_2 - x_1) - k_3 (x_3 - x_2)$
$= -k_2 x_1 + (k_2 + k_3) x_2 - k_3 x_3$

& hence, the 2nd DEOM is:

$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 - k_3 x_3 = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m_3 \ddot{x}_3; \quad \frac{\partial T}{\partial x_3} = 0$

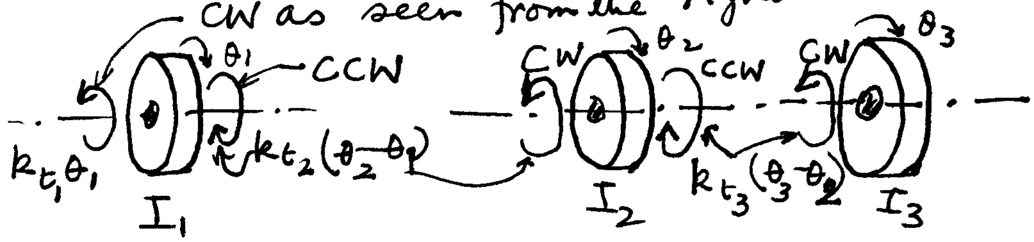
$\frac{\partial V}{\partial x_3} = k_3 (x_3 - x_2)$

& so, the 3rd DEOM is:

$m_3 \ddot{x}_3 - k_3 x_2 + k_3 x_3 = 0$

contd →

→ For fig. 2, if Newton-Euler (Moment of momentum) ~~equa~~ method is used, draw a relevant FBD for each disk by assuming, say, $\theta_3 > \theta_2 > \theta_1$.
Then: (θ_1, θ_2, θ_3 +ive CCW as seen from right side)



Then,
$$\left.\begin{aligned}
I_1\ddot{\theta}_1 &= k_{t_2}(\theta_2-\theta_1) - k_{t_1}\theta_1 \\
I_2\ddot{\theta}_2 &= k_{t_3}(\theta_3-\theta_2) - k_{t_2}(\theta_2-\theta_1) \\
\& \quad I_3\ddot{\theta}_3 &= -k_{t_3}(\theta_3-\theta_2)
\end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned}
&I_1\ddot{\theta}_1 + (k_{t_1}+k_{t_2})\theta_1 - k_{t_2}\theta_2 = 0 \\
&I_2\ddot{\theta}_2 - k_{t_2}\theta_1 + (k_{t_2}+k_{t_3})\theta_2 - k_{t_3}\theta_3 = 0 \\
\& \quad &I_3\ddot{\theta}_3 - k_{t_3}\theta_2 + k_{t_3}\theta_3 = 0
\end{aligned}\right\} \text{The reqd. DEOM}$$

OR

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} (k_{t_1}+k_{t_2}) & -k_{t_2} & 0 \\ -k_{t_2} & (k_{t_2}+k_{t_3}) & -k_{t_3} \\ 0 & -k_{t_3} & k_{t_3} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

OR

$$[I]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\}$$

where $[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$; $[K] = \begin{bmatrix} (k_{t_1}+k_{t_2}) & -k_{t_2} & 0 \\ -k_{t_2} & (k_{t_2}+k_{t_3}) & -k_{t_3} \\ 0 & -k_{t_3} & k_{t_3} \end{bmatrix}$;

$\{\ddot{\theta}\} = \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix}$ & $\{\theta\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$.

(PTO)

→ The Lagrange's Equns in this case are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = 0; \quad i = 1, 2, 3$$

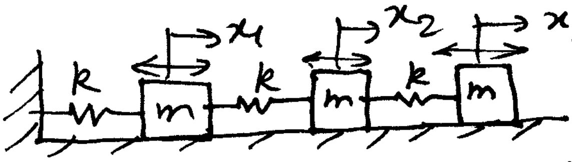
$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \dot{\theta}_3^2$$

$$V = \frac{1}{2} k_{t_1} \theta_1^2 + \frac{1}{2} k_{t_2} (\theta_2 - \theta_1)^2 + \frac{1}{2} k_{t_3} (\theta_3 - \theta_2)^2.$$

→ HW → Obtain the DEOM & compare with the ones already obtained by using Newton-Euler (moment-balance) method.

→ We next illustrate the method of setting up the frequency equation, obtaining the natural frequencies ω_1, ω_2 & ω_3 & also obtain the general modal vectors, the general free-vibration response and the conditions for principal modes of vibration, by following what was done for a 2-DOF system in a similar situation.

→ Example:-



(Neglect friction)

Obtain the natural frequencies and modal vectors. Obtain the general free-vibration response & conditions for the principal modes. Obtain a normalized modal matrix

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Solution:- (PTO)

→ You can get the DEOM by either method.
Using $k_1 = k_2 = k_3 = k$ & $m_1 = m_2 = m_3 = m$ in the system in fig. 1 (page 1), we have:

$$\text{(I)} \cdots \left\{ \begin{aligned} & m\ddot{x}_1 + 2k x_1 - k x_2 = 0 \\ & m\ddot{x}_2 - k x_1 + 2k x_2 - k x_3 = 0 \\ & m\ddot{x}_3 - k x_2 + k x_3 = 0 \end{aligned} \right\} \xrightarrow{\text{The DEOM}} [m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

Here $[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$; $[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$,

$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}$; $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$; $\{0\} \equiv \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

Compare with the 2-DOF case all along

We assume :–
$$\left. \begin{aligned} x_1 &= X_1 \sin(\omega t + \phi) \\ x_2 &= X_2 \sin(\omega t + \phi) \\ x_3 &= X_3 \sin(\omega t + \phi) \end{aligned} \right\} \text{— (II)}$$

Substitution of (II) in (I) leads to

Equations for obtaining X_1, X_2, X_3
$$\left\{ \begin{aligned} & (2k - m\omega^2) X_1 - k X_2 = 0 \quad \text{——— (1)} \\ & -k X_1 + (2k - m\omega^2) X_2 - k X_3 = 0 \quad \text{— (2)} \\ & -k X_2 + (k - m\omega^2) X_3 = 0 \quad \text{— (3)} \end{aligned} \right.$$

We have no unique solution for X_1, X_2, X_3

For non-trivial X_1, X_2 & X_3, we must have

$$(4) \text{ ——— } \begin{vmatrix} (2k - m\omega^2) & -k & 0 \\ -k & (2k - m\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0 \quad \left(\begin{array}{c} \text{Reason} \\ \text{explained} \\ \text{earlier} \end{array} \right)$$

Relation (4) gives the frequency equation.

Also, from (1), $\dfrac{X_2}{X_1} = \dfrac{2k - m\omega^2}{k}$ — (5)

& from (3), $\dfrac{X_3}{X_2} = \dfrac{k}{k - m\omega^2}$ — (6) [We ignore (2) since it is a little complicated] (40)

Then $\frac{X_3}{X_1} = \frac{X_3}{X_2} * \frac{X_2}{X_1} = \frac{2k - m\omega^2}{k - m\omega^2}$ (Using (5) & (6)) --- (7)

Relations (5) & (7) will be used for computing $\frac{X_{21}}{X_{11}}, \frac{X_{31}}{X_{11}}$ etc. for all 3 modes.

X_{31} = amplitude of m_3 for 1st principal mode etc.

We now expand relation (4) to get the frequency equation:

$$m^3\omega^6 - 5km^2\omega^4 + 6k^2m\omega^2 - k^3 = 0$$

(check this)

OR

$$\left[\frac{\omega^2}{\left(\frac{k}{m}\right)}\right]^3 - 5\left[\frac{\omega^2}{\left(\frac{k}{m}\right)}\right]^2 + 6\left[\frac{\omega^2}{\left(\frac{k}{m}\right)}\right] - 1 = 0 \quad \text{--- (8)}$$

→ We now use a calculator to solve (8) for $\left[\frac{\omega^2}{\left(\frac{k}{m}\right)}\right]$. This leads to

$\boxed{\text{Your Calculator may not give solutions in proper order}}$

[Be careful here. Remember that $\omega_1 < \omega_2 < \omega_3$

$$\left.\begin{aligned} \omega_1^2 &= 0.1981\frac{k}{m} \\ \omega_2^2 &= 1.5549\frac{k}{m} \\ \omega_3^2 &= 3.2470\frac{k}{m} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \omega_1 &= 0.4379\sqrt{\frac{k}{m}} \\ \omega_2 &= 1.2469\sqrt{\frac{k}{m}} \\ \omega_3 &= 1.8019\sqrt{\frac{k}{m}} \end{aligned}\right.$$

↑ The reqd natural frequencies

Using these in (5) & (7), we get

$$\frac{X_{21}}{X_{11}} = 1.8019; \quad \frac{X_{31}}{X_{11}} = 2.2470$$

Thus, $\{X\}_1 = \begin{Bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{Bmatrix} = \begin{Bmatrix} X_{11} \\ 1.8019X_{11} \\ 2.2470X_{11} \end{Bmatrix}$, the first modal vector (eigenvector) corresponding to $\omega = \omega_1$

Similarly $\{X\}_2 = \begin{Bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{Bmatrix} = \begin{Bmatrix} X_{12} \\ 0.4451X_{12} \\ -0.8021X_{12} \end{Bmatrix}$ (PTO)

↳ 2nd modal vector

one sign change +, +, −

[X_{32} → amplitude of m_3 for 2nd pr. mode etc.]

$$\& \quad \{X\}_3 = \begin{Bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{Bmatrix} = \begin{Bmatrix} X_{13} \\ -1.2470\, X_{13} \\ 0.5550\, X_{13} \end{Bmatrix}$$ ← The 3rd modal vector corr. to $\omega = \omega_3$

Two sign changes, i.e. $+, -, +$ (plus to minus, minus to plus)

Where X_{11}, X_{12} & X_{13} are arbitrary, nonzero.

→ The other amplitudes are not arbitrary, though.

→ Remember ~~that~~

X_{11} = amplitude of left mass (corr. to x_1) or, mass 1 for first principal (normal) mode of vibration etc., i.e.,

X_{ij} = amplitude of the i^{th} mass for j^{th} mode of vibration

with $\begin{Bmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{Bmatrix}$ where the i^{th} mass is associated with the i^{th} generalized coordinate x_i.

Thus, for example, X_{23} = amplitude of the 2nd mass for 3rd principal mode of vibration.

Now, ~~Hence~~ the three principal modes of vibration are given as follows:-

First Principal mode
$$\left[\begin{aligned} x_1(t) &= X_{11} \sin(\omega_1 t + \phi_1) \\ x_2(t) &= X_{21} \sin(\omega_1 t + \phi_1) \\ x_3(t) &= X_{31} \sin(\omega_1 t + \phi_1) \end{aligned}\right\} \text{ OR } \begin{cases} x_1 = X_{11} \sin(\omega_1 t + \phi_1) \\ x_2 = \mu_{21} X_{11} \sin(\omega_1 t + \phi_1) \\ x_3 = \mu_{31} X_{11} \sin(\omega_1 t + \phi_1) \end{cases}$$

[[Note:- We adopt the following notations:- [See Page 9 for clarification]

$$\mu_{21} = \frac{X_{21}}{X_{11}}, \quad \mu_{31} = \frac{X_{31}}{X_{11}}$$

Similarly, $\mu_{22} = \frac{X_{22}}{X_{12}}$, $\mu_{32} = \frac{X_{32}}{X_{12}}$; $\boxed{\text{Read the clarification first \& then come back}}$

$\mu_{23} = \frac{X_{23}}{X_{13}}$, $\mu_{33} = \frac{X_{33}}{X_{13}}$. This leads to the

(PTO)

following modal matrix:

$$[\mu] = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \mu_{21}X_{11} & \mu_{22}X_{12} & \mu_{23}X_{13} \\ \mu_{31}X_{11} & \mu_{32}X_{12} & \mu_{33}X_{13} \end{bmatrix}$$

with X_{11}, X_{12} & X_{13} arbitrary (nonzero)

& a normalized modal matrix could now be written as

$$[\mu] = \begin{bmatrix} 1 & 1 & 1 \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}.$$

→ For a 2-DOF system, however, we had adopted a simpler notation scheme so that $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$ instead of $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_{21} & \mu_{22} \end{bmatrix}$

You may use either

2nd Principal mode

$$\left.\begin{aligned} x_1(t) &= X_{12}\sin(\omega_2 t+\phi_2) \\ x_2(t) &= X_{22}\sin(\omega_2 t+\phi_2) \\ x_3(t) &= X_{32}\sin(\omega_2 t+\phi_2) \end{aligned}\right\} \text{OR} \left\{\begin{aligned} x_1(t) &= X_{12}\sin(\omega_2 t+\phi_2) \\ x_2(t) &= \mu_{22}X_{12}\sin(\omega_2 t+\phi_2) \\ x_3(t) &= \mu_{32}X_{12}\sin(\omega_2 t+\phi_2) \end{aligned}\right.$$

3rd pr. mode

$$\left.\begin{aligned} x_1(t) &= X_{13}\sin(\omega_3 t+\phi_3) \\ x_2(t) &= X_{23}\sin(\omega_3 t+\phi_3) \\ x_3(t) &= X_{33}\sin(\omega_3 t+\phi_3) \end{aligned}\right\} \text{OR} \left\{\begin{aligned} x_1(t) &= X_{13}\sin(\omega_3 t+\phi_3) \\ x_2(t) &= \mu_{23}X_{13}\sin(\omega_3 t+\phi_3) \\ x_3(t) &= \mu_{33}X_{13}\sin(\omega_3 t+\phi_3) \end{aligned}\right.$$

→ For our problem here,

$\mu_{21} = 1.8019$, $\mu_{31} = 2.2470$

$\mu_{22} = 0.4451$, $\mu_{32} = -0.8021$

$\mu_{23} = -1.2470$, $\mu_{33} = 0.5550$

(PTO)

Hence, the required general free-vibration response is:-

$$x_1(t) = X_{11}\sin(\omega_1 t+\phi_1) + X_{12}\sin(\omega_2 t+\phi_2) + X_{13}\sin(\omega_3 t+\phi_3)$$

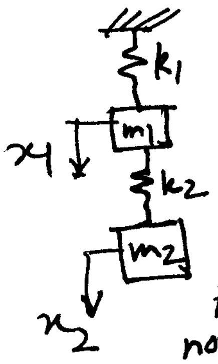
$$x_2(t) = \mu_{21}X_{11}\sin(\omega_1 t+\phi_1) + \mu_{22}X_{12}\sin(\omega_2 t+\phi_2) + \mu_{23}X_{13}\sin(\omega_3 t+\phi_3)$$

$$x_3(t) = \mu_{31}X_{11}\sin(\omega_1 t+\phi_1) + \mu_{32}X_{12}\sin(\omega_2 t+\phi_2) + \mu_{33}X_{13}\sin(\omega_3 t+\phi_3)$$

where X_{11}, X_{12}, X_{13} are arbitrary (non-zero) & μ_{21} etc. have the values mentioned on pg. 8.

→ ~~Some~~ CLARIFICATIONS regarding notation for amplitude ratios:~

→ This may be reqd for some students



← for this 2-DOF system, we used

Many books use these notations

$$\mu_1 = \frac{X_{21}}{X_{11}} = \frac{\text{Amplitude of 2nd mass for 1st pr. mode}}{\text{'' '' 1st '' '' '' ''}}$$

$$\mu_2 = \frac{X_{22}}{X_{12}} = \frac{\text{Ampl. of 2nd mass for 2nd pr. mode}}{\text{'' '' 1st '' '' '' ''}}$$

& the modal matrix was:

$$[\mu] = \begin{bmatrix} X_{11} & X_{12} \\ \mu_1 X_{11} & \mu_2 X_{12} \end{bmatrix} \text{ \& normalized}$$

$$[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}.$$ This was done for simplicity.

μ_1 = Amplitude ratio for 1st pr. mode

& μ_2 = '' '' '' 2nd '' ''

It worked nicely with a 2-DOF system. However, for an n-DOF system ($n > 2$), (Pto) this scheme should be modified. For a 2-DOF system, we should replace μ_1 by μ_{21} & μ_2 by μ_{22}

so that, $[\mu]_{\text{(normalized)}} = \begin{bmatrix} 1 & 1 \\ \mu_{21} & \mu_{22} \end{bmatrix}$ & μ_{21} is the element at (2,1) position (2nd row, first column) & μ_{22} is the element at (2,2) position of $[\mu]$ normalized.

Extending it to a 3-DOF system, we get

$$[\mu] = \begin{bmatrix} 1 & 1 & 1 \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \text{(normalized)}$$

or

$$[\mu] = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \mu_{21}X_{11} & \mu_{22}X_{12} & \mu_{23}X_{13} \\ \mu_{31}X_{11} & \mu_{32}X_{12} & \mu_{33}X_{13} \end{bmatrix}$$

with $\left.\begin{matrix} X_{11} \\ X_{12} \\ X_{13} \end{matrix}\right\}$ arbitrary (non-zero)

Here

$$\mu_{21} = \frac{X_{21}}{X_{11}}$$

$$\mu_{31} = \frac{X_{31}}{X_{11}}$$

$$\mu_{22} = \frac{X_{22}}{X_{12}}$$

etc.

For a general n-DOF system,

$$[\mu] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{bmatrix} \text{(normalized)}$$

OR $$[\mu] = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ \mu_{21}X_{11} & \mu_{22}X_{12} & \cdots & \mu_{2n}X_{1n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1}X_{11} & \mu_{n2}X_{12} & \cdots & \mu_{nn}X_{1n} \end{bmatrix}$$

$(X_{11}, X_{12}, \ldots, X_{1n} \rightarrow$ arbitrary (non-zero))

(PTO)

where $\mu_{21} = \frac{X_{21}}{X_{11}}$, $\mu_{n1} = \frac{X_{n1}}{X_{11}}$

$\mu_{22} = \frac{X_{22}}{X_{12}}$, $\mu_{n2} = \frac{X_{n2}}{X_{12}}$ & so on. → GO back TO page 7 now

→ We get back to our example problem (pg. 4)

→ It can be shown that for the n^{th} principal mode of vibration, there are specific relations to be satisfied among the given initial displacements $x_1(0), -, x_n(0)$ and the initial velocities $\dot{x}_1(0), --, \dot{x}_n(0)$.

- For a 2-DOF system, these were:

→ for first pr. mode, $x_2(0) = \mu_1 x_1(0)$ or, $\mu_{21} x_1(0)$
& $\dot{x}_2(0) = \mu_1 \dot{x}_1(0)$ or, $\mu_{21} \dot{x}_1(0)$

→ for 2^{nd} pr. mode, $x_2(0) = \mu_2 x_1(0)$ or, $\mu_{22} x_1(0)$
& $\dot{x}_2(0) = \mu_2 \dot{x}_1(0)$ or, $\mu_{22} \dot{x}_1(0)$

- For a 3-DOF system, these would be:

Required Conditions to excite the various principal modes of vibration

→ For first mode, $\left.\begin{matrix} x_2(0) = \mu_{21} x_1(0) \\ x_3(0) = \mu_{31} x_1(0) \end{matrix}\right\}$ & $\left\{\begin{matrix} \dot{x}_2(0) = \mu_{21} \dot{x}_1(0) \\ \dot{x}_3(0) = \mu_{31} \dot{x}_1(0) \end{matrix}\right.$

→ for 2^{nd} mode, $\left.\begin{matrix} x_2(0) = \mu_{22} x_1(0) \\ x_3(0) = \mu_{32} x_1(0) \end{matrix}\right\}$ & $\left\{\begin{matrix} \dot{x}_2(0) = \mu_{22} \dot{x}_1(0) \\ \dot{x}_3(0) = \mu_{32} \dot{x}_1(0) \end{matrix}\right.$

→ for 3^{rd} mode, $\left.\begin{matrix} x_2(0) = \mu_{23} x_1(0) \\ x_3(0) = \mu_{33} x_1(0) \end{matrix}\right\}$ & $\left\{\begin{matrix} \dot{x}_2(0) = \mu_{23} \dot{x}_1(0) \\ \dot{x}_3(0) = \mu_{33} \dot{x}_1(0) \end{matrix}\right.$

(PTO)

→ (Ex. problem continued) → See pg. 4

A normalized modal matrix $[\mu]$ is

(See pg. 8)

$$[\mu] = \begin{bmatrix} 1 & 1 & 1 \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}$$

(This completes the solution)

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1.8019 & 0.4451 & -1.2470 \\ 2.2470 & -0.8021 & 0.5550 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\{X\}_1 \qquad \{X\}_2 \qquad \{X\}_3$$

(Note:-

The first modal vector $\{X\}_1$ does not have any sign change as its elements are explored from top to bottom.

$\{X\}_2$ involves one sign change $\left(\begin{smallmatrix}+\\+\\-\end{smallmatrix}\right)$

$\{X\}_3$ " two " changes $\left(\begin{smallmatrix}+\\-\\+\end{smallmatrix}\right)$

plus to minus
minus to plus)