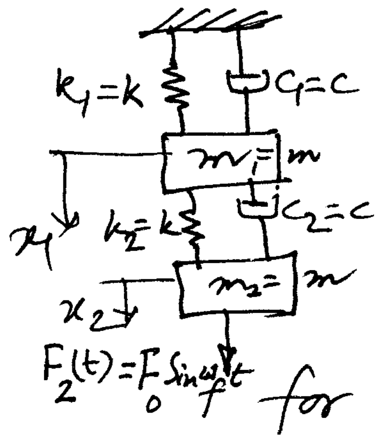
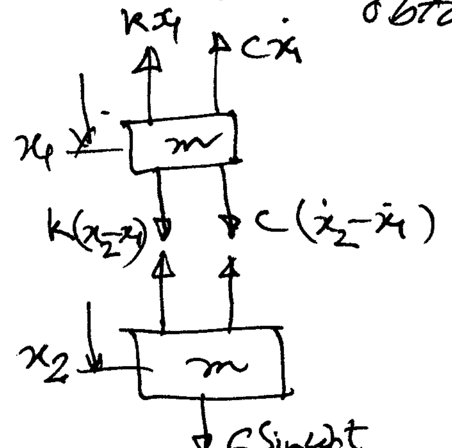
Forced vibration of damped 2-DOF systems



For simplicity, we take $k_1=k_2=k$, $c_1=c_2=c$, $m_1=m_2=m$. You will see that even this simplified system will lead to pretty complex expressions for response amplitudes. We aim at obtaining the steady-state response.



The DEOM are:

$$m\ddot{x}_1+2c\dot{x}_1-c\dot{x}_2+2kx_1-kx_2=0 \quad \text{--(1)}$$

$$m\ddot{x}_2-c\dot{x}_1+c\dot{x}_2-kx_1+kx_2=F_0\sin\omega_f t \quad \text{--(2)}$$

or, $$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+\begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix}+\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}$$

or, $$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\}=\begin{Bmatrix}0\\ F_0\sin\omega_f t\end{Bmatrix} = \{F\} \quad \text{--(3)}$$

(Check)

For the single DOF system described by $m\ddot{x}+c\dot{x}+kx=F_0\sin\omega_f t$, we had $x_{ss}=x(t)=X\sin(\omega_f t-\psi)$. We discussed that if we take the forcing function as $F_0e^{i\omega_f t}$, and use complex frequency response (after assuming $x(t)=\bar{X}e^{i\omega_f t}$ with $\bar{X}=Xe^{-i\psi}$), we can easily get the values (expressions) for X & ψ.

→ Drawing upon ~~this~~ experience with 1-DOF systems, we shall now assume that $x_1(t)=\bar{X}_1e^{i\omega_f t}$ & $x_2(t)=\bar{X}_2e^{i\omega_f t}$.

$\bar{X}_1 = X_1 e^{-i\psi_1}$

$\bar{X}_2 = X_2 e^{-i\psi_2}$

Then, $\dot{x}_1 = i\omega_f \bar{X}_1 e^{i\omega_f t}$, $\dot{x}_2 = i\omega_f \bar{X}_2 e^{i\omega_f t}$,

$\ddot{x}_1 = -\omega_f^2 \bar{X}_1 e^{i\omega_f t}$ & $\ddot{x}_2 = -\omega_f^2 \bar{X}_2 e^{i\omega_f t}$.

These lead to the following complex amplitude equations: (Note that $\bar{X}_1$ & $\bar{X}_2$ are the 'complex amplitudes'.)

$$\left[2K - m_1\omega_f^2 + i\cdot 2c\omega_f\right]\bar{X}_1 - \left[K + ic\omega_f\right]\bar{X}_2 = 0 \quad (4)$$

$$-\left[K + ic\omega_f\right]\bar{X}_1 + \left[K - m\omega_f^2 + ic\omega_f\right]\bar{X}_2 = F_0 \quad (5)$$

Solving (4) & (5) by Cramer's rule, we get

(Check everything)

$$\text{(a)} \quad \bar{X}_1 = \frac{(K + ic\omega_f)F_0}{\Delta = (2K - m\omega_f^2 + 2ic\omega_f)(K - m\omega_f^2 + ic\omega_f) - (K + ic\omega_f)^2}$$

$$\& \quad \bar{X}_2 = \frac{(2K - m\omega_f^2 + 2ic\omega_f)F_0}{\Delta}$$

After simplification, $\bar{X}_1$ & $\bar{X}_2$ will be of the form:

$$\bar{X}_1 = \frac{a_1 + ib_1}{a_2 + ib_2} \quad \& \quad \bar{X}_2 = \frac{c_1 + id_1}{a_2 + ib_2}$$

$$\text{So, } X_1 = |\bar{X}_1| = \frac{|a_1 + ib_1|}{|a_2 + ib_2|} = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} \text{ etc.,}$$

$$-\psi_1 = \angle \bar{X}_1 = \angle a_1 + ib_1 - \angle a_2 + ib_2 \text{ etc.}$$

~~Thus,~~ $\cancel{X_1 = \frac{F_0\sqrt{K^2 + (c\omega_f)^2}}{}}$

$$\text{Then, } X_1 = \frac{F_0\sqrt{K^2 + (c\omega_f)^2}}{\Delta_1}, \quad X_2 = \frac{F_0\sqrt{(2K - m\omega_f^2)^2 + (2c\omega_f)^2}}{\Delta_1}$$

→

where $\Delta_1 = \sqrt{\left[m^2(\omega_f^2-\omega_1^2)(\omega_f^2-\omega_2^2)-c^2\omega_f^2\right]^2 + c^2\omega_f^2(2k-m\omega_f)^2}$

→ Never try to remember any of the above expressions. They have been presented here to make you aware of such expressions arising during our studies. Also, note that ω_1^2 & ω_2^2 (the squares of the undamped natural frequencies) have appeared in Δ_1. This is so because after putting Δ (see expression (a), pg. 2) in the form $A+iB$, we get $A = m^2\omega_f^4 - 3km\omega_f^2 + k^2$ & note that $A=0$ is the frequency equation if ω_f is replaced by ω & so, $m^2\omega_f^4 - 3km\omega_f^2+k^2$ can be written as: $m^2(\omega_f^2-\omega_1^2)(\omega_f^2-\omega_2^2)$.

[The frequency eqn is: $m^2\omega^4 - 3km\omega^2+k^2=0$

$\Rightarrow m^2(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)=0$.

So, $m^2\omega^4-3km\omega^2+k^2 = m^2(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)$, which is an identity. Thus, replacing ω by ω_f, we get

$m^2\omega_f^4 - 3km\omega_f^2+k^2 = m^2(\omega_f^2-\omega_1^2)(\omega_f^2-\omega_2^2)$ etc.]

So, to study the complicated expressions for X_1 & X_2 as ω_f is varied (i.e., to handle the frequency response of the system), it is found that introduction of the parameters r, X_0 & ζ helps where these parameters are defined as: $r = \frac{\omega_f}{\omega_1}$, $X_0 = F_0/K$, $\zeta = \frac{c}{m\omega_1}$ → This a new ζ with a new definition

note Also, $\omega_1^2 = 0.382 \frac{K}{m} = \gamma_1 \frac{K}{m}$ ($\gamma_1 = 0.382$) ← gamma

& $\omega_2^2 = 2.62 \frac{K}{m} = \gamma_2 \frac{K}{m}$ ($\gamma_2 = 2.62$)

[Applied Mechanical Vibrations - D.V. Hutton)

[Do you remember we had obtained the undamped natural frequencies of this system as
$\omega_1 = 0.618\sqrt{\frac{K}{m}}$ & $\omega_2 = 1.618\sqrt{\frac{K}{m}}$?]

All these finally lead to

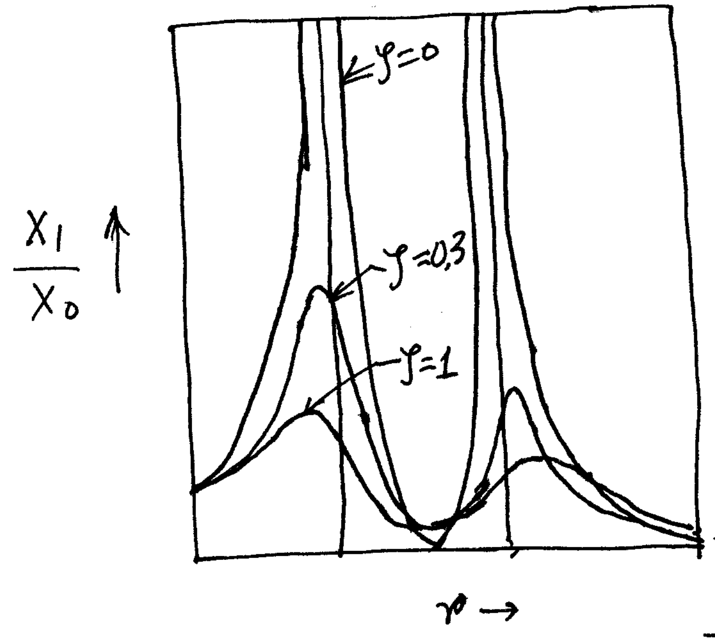
$$X_1 = \frac{X_0\sqrt{1+(\gamma_1 \zeta r)^2}}{\Delta_2} \quad \& \quad X_2 = \frac{X_0\sqrt{(2-\gamma_1 r^2)^2 + (\gamma_1 \zeta r)^2}}{\Delta_2}$$

where $\Delta_2 = \sqrt{\left[\gamma_1^2 (r^2-1)\left(r^2 - \frac{\gamma_2^2}{\gamma_1^2}\right) - (\gamma_1 \zeta r)^2\right]^2 + (\gamma_1 \zeta r)^2 (2-\gamma_1 r^2)^2}$.

(Donot try to remember these)

→ It is not difficult to see that the lower mass is more affected by the forced vibration since the external force acts directly on it. The upper mass is less affected because a lot of energy is dissipated by the the damper between the masses. A frequency response plot $\frac{X_2}{X_0}$ vs r would look like the following:
(See accurate plots from textbooks)

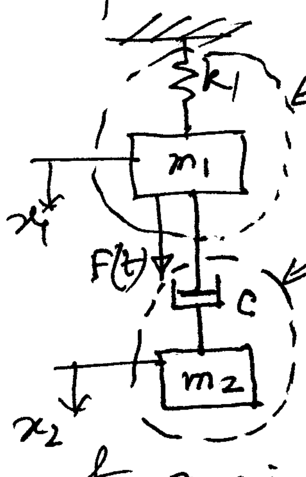
→



Note that these plots have similarities with the ones for 1-DOF systems. A remarkable difference, however, is that there is more than one resonant frequency. (Remember the general forms of these plots)

8 The damped vibration absorber (The untuned damper) (or, the viscous vibration absorber)

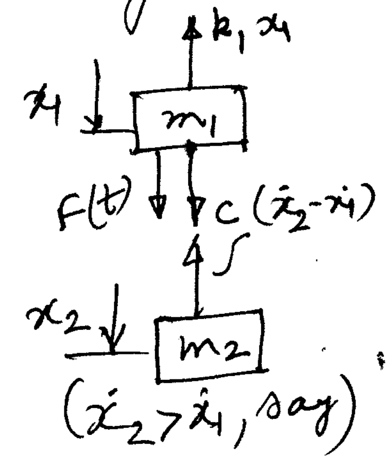
One way* to introduce a damper is as follows:- (* there are other ways too)



← The main system

← The absorber system.
(Assume there is a mechanism to keep the absorber system at rest while the absorber & main mass are not in motion. This is required because an ideal damper cannot resist any force unless its two ends have a difference in velocity.)

→ So, the problem is:- $F(t)=f_0 \sin \omega_f t$, where $\omega_f \approx \omega_n = \sqrt{\frac{k_1}{m_1}}$ & hence, large amplitude motions occur. To reduce this, the shown absorber system is added to the main system. We now have a 2-DOF system & the DEOM are:-



$$m_1 \ddot{x}_1 = F(t) + c(\dot{x}_2 - \dot{x}_1) - k_1 x_1$$

$$\& \quad m_2 \ddot{x}_2 = -c(\dot{x}_2 - \dot{x}_1)$$

OR

$$m_1 \ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + k_1 x_1 = F(t) \quad (1)$$

$$\& \quad m_2 \ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 = 0 \quad (2)$$

Assume $F(t) = F_0 e^{i\omega_f t}$ — (3)

Then, proceeding as before, you can show that

$$X_1 = \frac{X_0 \sqrt{\mu^2 r^2 + 4\zeta^2}}{\sqrt{\mu^2(1-r^2)^2 + 4\zeta^2\left[\mu r^2 - (1-r^2)\right]^2}}$$

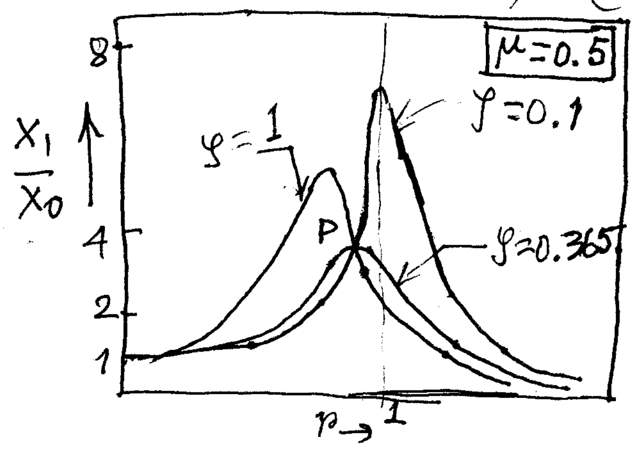
Amplitude of main mass

(No need to remember)

With $X_0 = \frac{F_0}{k_1}$, $\omega_n^2 = \frac{k_1}{m_1}$, $\zeta = \frac{c}{2m_1\omega_n}$,

$\mu = \frac{m_2}{m_1}$, $r = \frac{\omega_f}{\omega_n}$

Thus, $X_1 = X_1(\mu, r, \zeta)$ and it is observed that the most convenient way of studying the frequency response (i.e. variation of X_1 as ω_f varies) is by plotting curves X_1/X_0 vs r for various damping ratios ζ while μ (mass ratio) is kept constant.



An interesting point here is that all the curves pass through a common point P. (How to prove this? Take two arbitrary $\zeta = \zeta_1$ & $\zeta = \zeta_2$ & solve for the points of intersection near $r = 1$

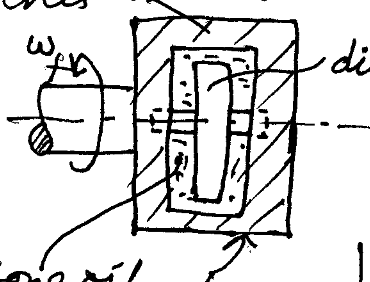
which would be independent of ζ_1 & ζ_2? Another way?) Hence, the value of ζ for which the tangent to the curve at P is horizontal, will give the minimum value for $\left(\frac{X_1}{X_0}\right)_{maxm}$. For a given μ, it can be shown that $\zeta_{Optimum} = \zeta_0$ is given by: $\zeta_0 = \dfrac{1}{\sqrt{2(1+\mu)(2+\mu)}}$

and the peak amplitude occurs at $r = \sqrt{\dfrac{2}{2+\mu}}$.

(Equate the values of X_1/X_0 for any two values of ζ & solve the resulting expression for r. Setting the slope of the curve at this r_1 to zero then gives ζ_0)

⟶ From the plots, it is apparent that this damper is effective over a wider range of variation of ω than the undamped vibration absorber.

The torsional counterpart of this damper is often used to reduce the torsional crankshaft oscillations which occur in IC engines. A device of this kind is shown schematically here.

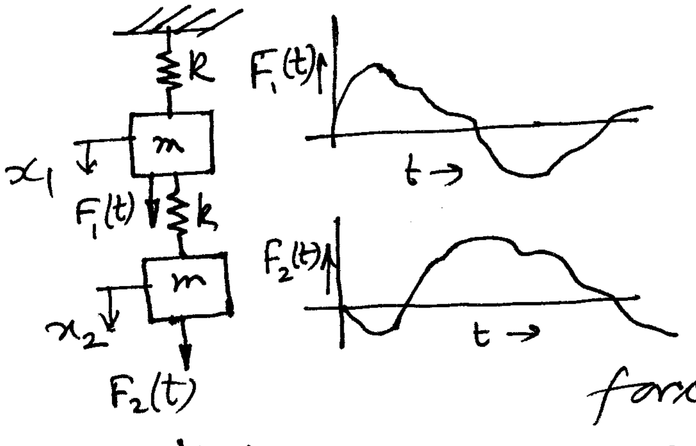


A disc is mounted inside a housing which is attached to the crankshaft. The space between disc & housing inside is filled with an oil. The damping torque is produced by the oil viscosity and is proportional to the relative angular velocity of disc w.r.t. the housing. This type of damper is known as a Houdaille damper or viscous Lanchester damper.

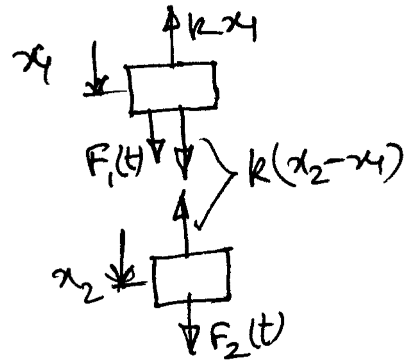
END

Also see §5.7, Pg. 140, Theory of Vibration, 3rd Edition, W.T. Thomson

9 The Modal Analysis :~



Let the system is subjected to general forcing functions $F_1(t)$ & $F_2(t)$ and we want to obtain the forced response.



The DEOM are:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1(t) \quad \text{— (1)}$$

$$m\ddot{x}_2 - kx_1 + kx_2 = F_2(t) \quad \text{— (2)}$$

or, $$[m]\{\ddot{x}\} + [k]\{x\} = \{F(t)\} \quad \text{— (3)}$$

where $[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $[k] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$, $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$.

The natural frequencies of the system are obtained ~~by solv~~ from $[m]\{\ddot{x}\}+[k]\{x\}=\{0\}$ & $\omega_1 = 0.618\sqrt{\frac{k}{m}}$, $\omega_2 = 1.618\sqrt{\frac{k}{m}}$, we already know.

→ For general forcing functions, it is difficult to solve (1) & (2) simultaneously. Think of a system having a large (number of) DOF. Solving for the forced response becomes a daunting task.

→ However, if we could uncouple the DEOM by using a different set of generalized coordinates, the task would become much simpler. Each such independent DEOM can be solved for forced response using Duhamel's integral, however complex the forcing functions may be.

→ It can be shown that ~~the~~ a modal matrix can be used as a coordinate transformation matrix to uncouple the DEOM. This is the theme of the <u>Modal Analysis</u>. (Note that some authors say that finding the modal vectors is modal analysis. Others say that finding the modal vectors together with using a modal matrix for uncoupling the DEOM is modal analysis.)

→ Basically, the orthogonality of modal vectors w.r.t. $[m]$ & $[K]$ is responsible for the uncoupling.

→ The normalized modal matrix $[\mu] = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$ is easier to handle & so we use this form. Even if you use $[\mu] = \begin{bmatrix} X_{11} & X_{12} \\ \mu_1 X_{11} & \mu_2 X_{12} \end{bmatrix}$, the results will be the same, you may check.

→ Note that a matrix product $[\mu]^T[m][\mu]$ $= \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}$ is a special case of a product like $[A]^T[B][A]$. ⟶

If $[A] = \left[\{A\}_1, \{A\}_2\right] = \begin{bmatrix} \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix} \end{bmatrix}$, $[B] = \begin{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \end{Bmatrix} \begin{Bmatrix} B_{12} \\ B_{22} \end{Bmatrix} \end{bmatrix}$, then

$$[A]^T[B][A] = \begin{bmatrix} \{A\}_1^T[B]\{A\}_1, & \{A_1\}^T[B]\{A\}_2 \\ \{A\}_2^T[B]\{A\}_1, & \{A\}_2^T[B]\{A\}_2 \end{bmatrix}$$

If the off-diagonal elements are zero, then the above matrix product is diagonal. This is precisely what happens if

$$[A] = [\mu], \quad [B] = [m] \text{ or } [k].$$

→ To bring this type of matrix product into picture, let us go for the following coordinate transformation:

$$④ \cdots \text{Remember} \rightarrow \boxed{\{x\} = [\mu]\{p\}}; \quad \{p\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \text{ being}$$

a new set of generalized coordinates.

Then, $\{\ddot{x}\} = [\mu]\{\ddot{p}\}$ --- ⑤

Substitution of ④ & ⑤ in ③ gives

$$[m][\mu]\{\ddot{p}\} + [k][\mu]\{p\} = \{F(t)\} \text{ --- ⑥}$$

Premultiplying both sides of ⑥ by $[\mu]^T$, we get

$$\underbrace{[\mu]^T[m][\mu]}_{\text{Diagonal}}\{\ddot{p}\} + \underbrace{[\mu]^T[k][\mu]}_{\text{Diagonal}}\{p\} = [\mu]^T\{F(t)\}$$

$$\Rightarrow \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \end{bmatrix}\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$M_{11} = \{x\}_1^T[m]\{x\}_1$, etc. So, $M_{11} = \{1\ \mu_1\}[m]\begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix}$ & $M_{22} = \{1\ \mu_2\}[m]\begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$

$$\Rightarrow M_{11}\ddot{p}_1 + K_{11}p_1 = F_1 + \mu_1 F_2 = Q_1(t) \text{ --- ⑦}$$

$$\& \quad M_{22}\ddot{p}_2 + K_{22}p_2 = F_1 + \mu_2 F_2 = Q_2(t) \text{ --- ⑧}$$

$K_{11} = \{1\ \mu_1\}[k]\begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix}$

$K_{22} = \{1\ \mu_2\}[k]\begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}$

⑦ & ⑧ are the required uncoupled DEOM. Note that $\omega_1 = \sqrt{\dfrac{K_{11}}{M_{11}}}$, $\omega_2 = \sqrt{\dfrac{K_{22}}{M_{22}}}$

$\omega_1, \omega_2 \rightarrow$ natural frequencies of the system

→ M_{11} & M_{22} are called <u>Generalized masses</u>.

→ K_{11} & K_{22} are the <u>generalized stiffnesses</u>.

→ Each of (7) & (8) can be solved ~~for~~ as <u>follows</u>:– The forced vibration response of $m\ddot{x}+kx=F(t)$ is given by:

$$x(t)=\int_0^t F(\tau)g(t-\tau)\,d\tau \text{ where}$$

$$g(t)=\frac{1}{m\omega_n}\sin\omega_n t.$$

Natural frequencies can be obtained from (7) & (8) by setting $Q_1(t)=Q_2(t)=0$.
The free vibration DEOM are:–
$M_{11}\ddot{p}_1+K_{11}p_1=0$
&
$M_{22}\ddot{p}_2+K_{22}p_2=0$
Hence,
$\omega_1=\sqrt{\frac{K_{11}}{M_{11}}}$
& $\omega_2=\sqrt{\frac{K_{22}}{M_{22}}}$.
Note ~~that~~ same natural frequencies result whatever the ~~gen~~ generalized coordinates may be.

Hence, the forced response of

$$M_{11}\ddot{p}_1+K_{11}p_1=Q_1(t) \text{ is:}$$

$$p_1(t)=\int_0^t Q_1(\tau)g_1(t-\tau)\,d\tau \text{ where}$$

$$g_1(t)=\frac{1}{M_{11}\omega_1}\sin\omega_1 t.$$

So, $$p_1(t)=\int_0^t Q_1(\tau)\,\frac{1}{M_{11}\omega_1}\sin\omega_1(t-\tau)\,d\tau$$

(Check)

Similarly, $$p_2(t)=\int_0^t Q_2(\tau)\frac{1}{M_{22}\omega_2}\sin\omega_2(t-\tau)\,d\tau$$

with $g_2(t)=\frac{1}{M_{22}\omega_2}\sin\omega_2 t$.

→ Once $p_1(t)$ & $p_2(t)$ are obtained, our required forced response in terms of $x_1(t)$ & $x_2(t)$ are obtained from:

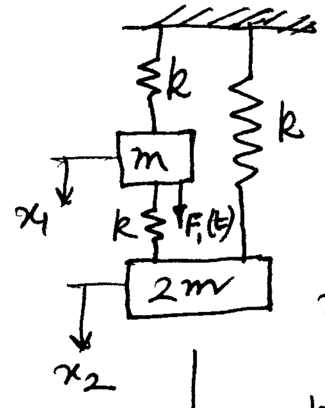
$\{x\}=[\mu]\{p\}$ or, $\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\begin{bmatrix}1 & 1\\ \mu_1 & \mu_2\end{bmatrix}\begin{Bmatrix}p_1\\p_2\end{Bmatrix}$ as

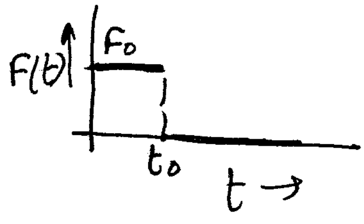
$$x_1=p_1+p_2$$

$$\& \quad x_2=\mu_1 p_1+\mu_2 p_2$$

Remember:– $p_1(t)$, $p_2(t)$ are called a set of principal coordinates. They are called normal coordinates by some authors. The name 'natural coordinates' is also there.

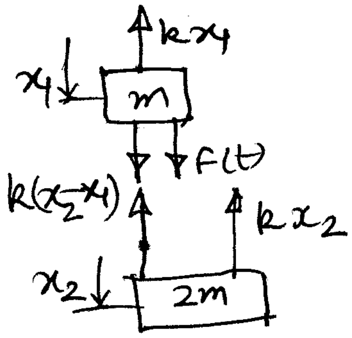
Example :- For the system shown, obtain the forced response using Modal Analysis & Duhamel's integral.





Solution:-

$x_2 > x_1$ (say)



$$m\ddot{x}_1 = k(x_2-x_1)+F(t)-kx_1$$

$$2m\ddot{x}_2 = -k(x_2-x_1)-kx_2$$

OR

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F(t) \quad \text{--- (1)}$$

$$\& \quad 2m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \quad \text{--- (2)}$$

or, $$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$$

or, $$[m]\{\ddot{x}\} + [k]\{x\} = \{F\} \quad \text{--- (3)}$$

for free-vibration, (to get $\omega_1, \omega_2, [\mu]$)

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad \text{--- (4)}$$

$$2m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \quad \text{--- (5)}$$

$\boxed{\text{Check everything}}$

(6) --- $x_1 = \cancel{X_1}\sin(\omega t+\phi) \Rightarrow \ddot{x}_1 = -X_1\omega^2 \sin(\omega t+\phi)$

(7) --- $x_2 = X_2\sin(\omega t+\phi) \Rightarrow \ddot{x}_2 = -X_2\omega^2\sin(\omega t+\phi)$

$$\Rightarrow (2k - m\omega^2)X_1 - kX_2 = 0 \quad \text{--- (8)}$$

$$-kX_1 + (2k - 2m\omega^2)X_2 = 0 \quad \text{--- (9)}$$

Hence, frequency equation is:

$$\begin{vmatrix} (2k-m\omega^2) & -k \\ -k & (2k-2m\omega^2) \end{vmatrix} = 0 \Rightarrow 2m^2\omega^4 - 6km\omega^2 + 3k^2 = 0$$

$$\Rightarrow \omega_1 = 0.7962\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.5382\sqrt{\frac{k}{m}}$$

Then, from (8) & (9), we get

$$\mu_1 = \frac{A_{21}}{A_{11}} = 1.3661 \quad \& \quad \mu_2 = \frac{A_{22}}{A_{12}} = -0.3661$$

Hence, $$[\mu] = \begin{bmatrix} 1 & 1 \\ 1.3661 & -0.3661 \end{bmatrix}$$

→ We next use the coordinate transformation

(10) --- $\{x\} = [\mu]\{p\}$; $\{p\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$

Then (3) is transformed to

$$[m][\mu]\{\ddot{p}\} + [k][\mu]\{p\} = \{F\}$$

$$\Rightarrow [\mu]^T[m][\mu]\{\ddot{p}\} + [\mu]^T[k][\mu]\{p\} = [\mu]^T\{F\} \quad \text{--- (11)}$$

$$[\mu]^T[m][\mu] = m\begin{bmatrix} 1 & 1.3661 \\ 1 & -0.3661 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1.3661 & -0.3661 \end{bmatrix}$$

will come close to zero

$$= \begin{bmatrix} 4.7324m & 0 \\ 0 & 1.268m \end{bmatrix} \quad \text{--- (12)}$$

$$\& \quad [\mu]^T[k][\mu] = \begin{bmatrix} 3k & 0 \\ 0 & 3k \end{bmatrix} \quad \text{--- (13)}$$

$$[\mu]^T\{F\} = \begin{bmatrix} 1 & 1.3661 \\ 1 & -0.3661 \end{bmatrix}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 \end{Bmatrix} \quad \text{--- (14)}$$

Using (12) – (14), (11) becomes

$$4.7324m\ddot{p}_1 + 3kp_1 = F_1(t) \quad \text{--- (15)}$$

$$\& \quad 1.268m\ddot{p}_2 + 3kp_2 = F_1(t) \quad \text{--- (16)}$$

(15) & (16) are the reqd uncoupled DEOM, each to be solved using Duhamel's integral. [Note:- At this stage you could check whether $\omega_1 = \sqrt{\frac{3k}{4.7324m}}$ & $\omega_2 = \sqrt{\frac{3k}{1.268m}}$. If these won't check, you've (I've!) made a mistake somewhere.]

→ We take up (15) first:–

$$4.7324m\ddot{p}_1 + 3kp_1 = F_1(t)$$

(i) for $0 \le t \le t_0$,

$$p_1(t) = \int_0^t F_0 \frac{1}{4.7324m\omega_1} \sin\omega_1(t-\tau)\,d\tau$$

$$= \frac{F_0}{3k}\left[1 - \cos\omega_1 t\right] \quad \text{--- (A)}$$

(ii) for $t > t_0$, $P_1(t) = \int_0^{t_0} \frac{F_0}{4.7324m\omega_1} \sin\omega_1(t-\tau)\,d\tau$

$$= \frac{F_0}{3k}\left[-\cos\omega_1 t + \cos\omega_1 (t-t_0)\right] \quad \text{---} \quad (B)$$

→ We now take up $1.268\, m\ddot{p}_2 + 3kp_2 = F_1(t)$

(i) For $0 \le t \le t_0$, $p_2(t) = \frac{F_0}{1.268\, m\, \omega_2^2}\left[1 - \cos\omega_2 t\right]$

or, $p_2(t) = \frac{F_0}{3k}\left[1 - \cos\omega_2 t\right] \quad \text{---} \quad (C)$

(ii) For $t > t_0$, $p_2(t) = \frac{F_0}{3k}\left[-\cos\omega_2 t + \cos\omega_2 (t-t_0)\right] \quad \text{---} \quad (D)$

→ Finally, $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.3661 & -0.3661 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$

So, for $0 \le t \le t_0$,

$$x_1(t) = p_1(t) + p_2(t) = \frac{F_0}{3k}\left[2 - \cos\omega_1 t - \cos\omega_2 t\right]$$

For $t > t_0$,

$$\cancel{x_1(t) = \frac{F_0}{3k}\left\{2 - \cos\omega_1 (t-t_0) - \cos\omega_2 (t-t_0)\right\}}$$

check everything. Mistakes are likely.

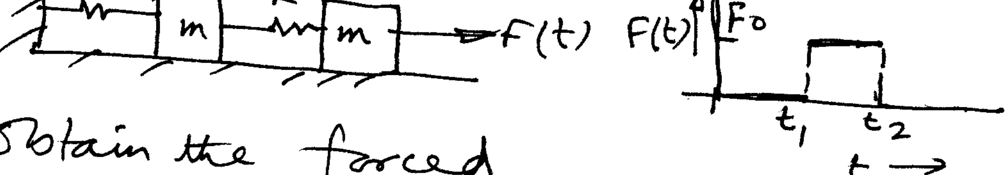
$$x_1(t) = p_1(t) + p_2(t) = (B) + (D)$$

$$= \frac{F_0}{3k}\left[\text{etc.}\right] \text{ (Complete it)}$$

→ Similarly, $x_2(t) = 1.3661\, p_1(t) - 0.3661\, p_2(t)$ & proceeding as above, complete the solution.

(HW)

8.



Obtain the forced response $x_1(t)$ & $x_2(t)$ using modal analysis and Duhamel's integral.

END OF VA-4, Part 5