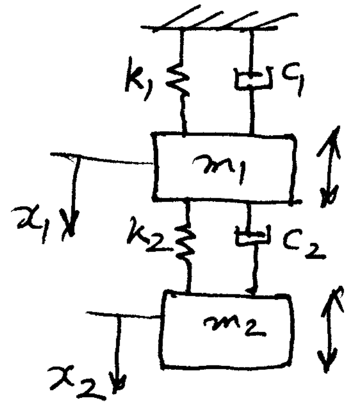
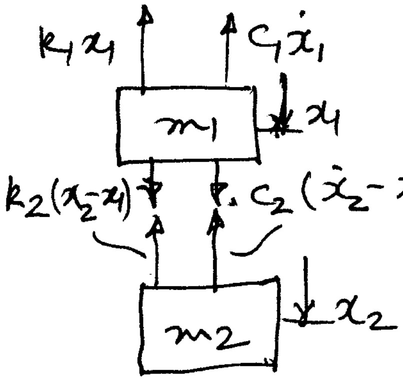
(§) Damped free vibration of 2-DOF Systems:



We obtain the DEOM using Force-balance (Newton's) method.



← Relevant FBDs (Assuming $x_2 > x_1$ & $\dot{x}_2 > \dot{x}_1$)

So, $m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1)$

& $m_2\ddot{x}_2 = -k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1)$

The DEOM are, thus,

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0 \quad \text{--- (1)}$$

& $$m_2\ddot{x}_2 - c_2\dot{x}_1 + c_2\dot{x}_2 - k_2x_1 + k_2x_2 = 0 \quad \text{--- (2)}$$

OR
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{--- (3)}$$

or, $$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{0\} \quad \text{--- (4)}$$

where $[c] = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$ is the damping matrix.

→ For finding the free vibration response, we assume $x_1(t) = C_1e^{st}$ & $x_2(t) = C_2e^{st}$ just the way we had assumed a solution for the single DOF case.

Substituting these in (1) & (2), (or, (3)) we get

$\because \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}e^{st}$, $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} sC_1 \\ sC_2 \end{Bmatrix}e^{st}$, $\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} s^2C_1 \\ s^2C_2 \end{Bmatrix}e^{st}$

→

two homogeneous linear algebraic equations in the unknowns $\bar{c}_1$ & $\bar{c}_2$. You should check that these two have the form

$$\left.\begin{array}{ll} (5) & a_{11}\bar{c}_1 + a_{12}\bar{c}_2 = 0 \\ (6) & a_{21}\bar{c}_1 + a_{22}\bar{c}_2 = 0 \end{array}\right\} \text{where } a_{11} = m_1 s^2 + (c_1+c_2)s + k_1 + k_2,\quad a_{12} = -c_2 s - k_2 = a_{21}\ \&\ a_{22} = m_2 s^2 + c_2 s + k_2$$

for non-trivial $\bar{c}_1$ & $\bar{c}_2$, we must have

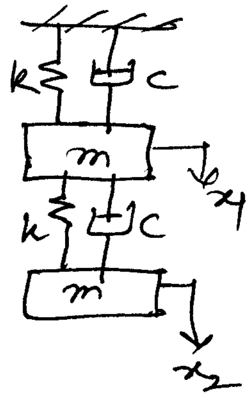
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

& this gives

$$m_1 m_2 s^4 + [m_1 c_2 + m_2(c_1+c_2)]s^3 + [m_1 k_2 + m_2(k_1+k_2) + c_1 c_2]s^2 + (k_1 c_2 + k_2 c_1)s + k_1 k_2 = 0 \quad (7)$$

[Don't mixup $\bar{c}_1, \bar{c}_2$ with c_1, c_2]

For simplicity, let us consider a system in which $m_1 = m_2 = m$, $c_1 = c_2 = c$ & $k_1 = k_2 = k$.



Then (7) becomes

$$m^2 s^4 + (3mc)s^3 + (3mk + c^2)s^2 + 2kcs + k^2 = 0 \quad (8)$$

(8) is an algebraic equation of fourth degree in s & has 4 roots, say, s_1, s_2, s_3 & s_4.

Just like the free vibration of undamped 2-DOF systems, corresponding to each s, we get one ratio $\frac{\bar{c}_2}{\bar{c}_1}$ from (5) or (6).

When $s = s_1$, we get $\frac{\bar{c}_{21}}{\bar{c}_{11}} = \alpha_1$ (like μ_1)

" $s = s_2$, we get $\frac{\bar{c}_{22}}{\bar{c}_{12}} = \alpha_2$

" $s = s_3$, " " $\frac{\bar{c}_{23}}{\bar{c}_{13}} = \alpha_3$

& " $s = s_4$, " " $\frac{\bar{c}_{24}}{\bar{c}_{14}} = \alpha_4$.

& thus, the general free vibration responses can be written as:

(7) —— $x_1(t) = C_{11} e^{s_1 t} + C_{12} e^{s_2 t} + C_{13} e^{s_3 t} + C_{14} e^{s_4 t}$

(8) — & $x_2(t) = \alpha_1 C_{11} e^{s_1 t} + \alpha_2 C_{12} e^{s_2 t} + \alpha_3 C_{13} e^{s_3 t} + \alpha_4 C_{14} e^{s_4 t}$

Now, we are dealing with a <u>stable system</u> and so, s_i $(i = 1, 2, 3, 4)$ can't have a positive real part. The various possibilities are as follows:—

(i) s_1, s_2, s_3, s_4 are all real & negative.
(ii) Two of these real, $-$ive & other two complex conjugates with $-$ive real parts.
(iii) Two pairs of complex conjugates with $-$ive real parts.

So, clearly, when possibility (a) occurs, we have exponentially decaying responses without oscillations like our <u>overdamped</u> 1-DOF system.

If, say, $s_3 = -a_3 + ib_3$, $s_4 = -a_3 - ib_3$,

then, $e^{s_3 t} = e^{-a_3 t} \cdot e^{ib_3 t}$, $e^{s_4 t} = e^{-a_3 t} \cdot e^{-ib_3 t}$

Also, let $s_1 = -a_1$, $s_2 = -a_2$ $(a_i > 0)$

Then, $x_1(t) = C_{11} e^{-a_1 t} + C_{12} e^{-a_2 t} + \left(C_{13} e^{ib_3 t} + C_{14} e^{-ib_3 t}\right) e^{-a_3 t}$.

Just like the 1-DOF case, the bracketed term will give a response like $A \sin(b_3 t + \phi)$

& thus, $x_1(t) = C_{11} e^{-a_1 t} + C_{12} e^{-a_2 t} + A e^{-a_3 t} \sin(b_3 t + \phi)$.

The four constants C_{11}, C_{12}, A & ϕ can be evaluated for given $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$ & $\dot{x}_2(0)$.

So, we have exponentially decaying harmonic oscillations.
Thus, $x_1(t) \to 0$ as time passes.
Similarly, $x_2(t) \to 0$ as time passes.

If we have two pairs of complex conjugate roots like $s_1 = -a_1 + ib_1$, $s_2 = -a_1 - ib_1$,

$$s_3 = -a_3 + ib_3, \quad s_4 = -a_3 - ib_3,$$

then
$$x_1(t) = B e^{-a_1 t} \sin(b_1 t + \phi_1) + A e^{-a_3 t} \sin(b_3 t + \phi_3)$$
& B, A, ϕ_1, ϕ_3 can be evaluated using given initial conditions.

- $x_2(t)$ will vary in a similar manner.

Hence, we now can obtain the free vibration response of our damped system. However, the algebra involved may be quite lengthy. In any case, the free vibration dies down to an insignificant value after some time.

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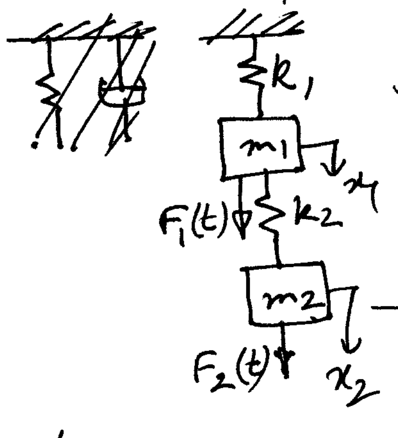
9 Undamped forced Vibration of 2-DOF systems



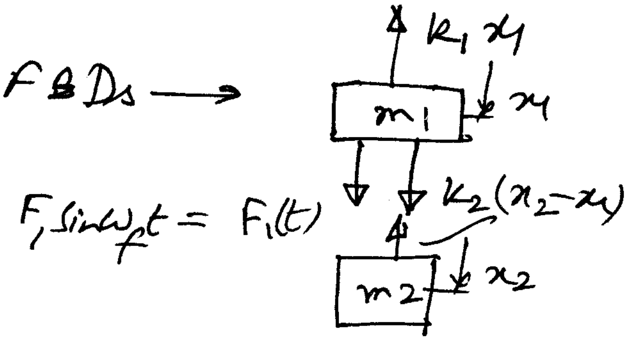
Suppose the system shown is subject to $F_1(t) = F_1 \sin \omega_{f1} t$ & $F_2(t) = F_2 \sin \omega_{f2} t$.

→ We can first apply $F_1(t)$ only & find $(x_1)_{forced}$ & $(x_2)_{forced}$.

→ We can next apply $F_2(t)$ only & obtain corresponding forced responses. By superposing these, we can get $x_1(t)$ & $x_2(t)$ for forced vibration when $F_1(t)$ & $F_2(t)$ act simultaneously. Hence, for simplicity,

We shall take $F_2=0$ & proceed to obtain
$(x_1)_{forced} = x_1(t)$, $(x_2)_{forced} = x_2(t)$.

Relevant FBDs $\longrightarrow$



$(x_2 > x_1, \text{ say})$

Hence, $m_1 \ddot{x}_1 = F_1(t) + k_2(x_2-x_1) - k_1 x_1$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$\Rightarrow \left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 &= F_1(t) = F_1 \sin\omega_f t \quad \text{— (1)} \\ m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 &= 0 \quad \text{— (2)} \end{aligned}\right\} \text{DEOM}$$

$\to$ We shall assume $x_1(t) = X_1 \sin\omega_f t$ — (3)
& $x_2(t) = X_2 \sin\omega_f t$ — (4)

in line with our experience with 1-DOF systems.

Then, $\ddot{x}_1(t) = -X_1\omega_f^2 \sin\omega_f t$

$$\ddot{x}_2(t) = -X_2\omega_f^2 \sin\omega_f t$$

& substitution in (1) & (2) results in
($\because \sin\omega_f t \neq 0$ at all times)

$$(k_1+k_2-m_1\omega_f^2)X_1 - k_2 X_2 = F_1 \quad \text{— (5)}$$

$$-k_2 X_1 + (k_2 - m_2\omega_f^2)X_2 = 0 \quad \text{— (6)}$$

$\to$ We assume $\omega_f \neq \omega_1$ or ω_2.

Then, $\Delta = \begin{vmatrix} (k_1+k_2-m_1\omega_f^2) & -k_2 \\ -k_2 & (k_2-m_2\omega_f^2) \end{vmatrix} \neq 0$ & the $\to$

(Note that $\begin{vmatrix} (k_1+k_2-m_1\omega^2) & -k_2 \\ -k_2 & (k_2-m_2\omega^2) \end{vmatrix} = 0$ is the frequency equation) system of equns 5 & 6 can be solved for X_1 & X_2 by ~~Cramergers~~ Cramer's rule.

Then,

$$X_1 = \frac{\begin{vmatrix} F_1 & -k_2 \\ 0 & (k_2 - m_2\omega_f^2) \end{vmatrix}}{\Delta} = \frac{F_1(k_2 - m_2\omega_f^2)}{\Delta}$$

&

$$X_2 = \frac{\begin{vmatrix} (k_1 + k_2 - m_1\omega_f^2) & F_1 \\ -k_2 & 0 \end{vmatrix}}{\Delta} = \frac{F_1 k_2}{\Delta}$$

Hence, the required forced response is:-

$$x_1(t) = \frac{F_1(k_2 - m_2\omega_f^2)}{\Delta} \sin\omega_f t$$

& $$x_2(t) = \frac{F_1 k_2}{\Delta} \sin\omega_f t$$

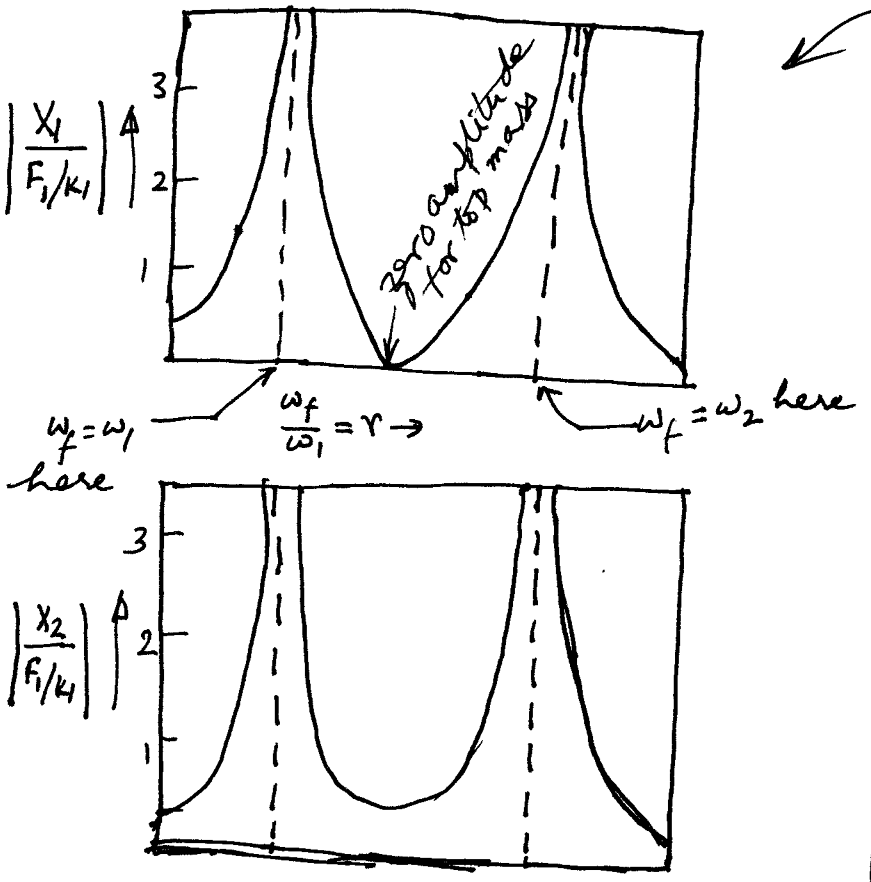
Check

$\boxed{\text{~~HW problem~~}}$

If $\Delta = 0$, $x_1(t), x_2(t) \to \infty$.

$\to$ means $\omega_f = \omega_1$ or ω_2

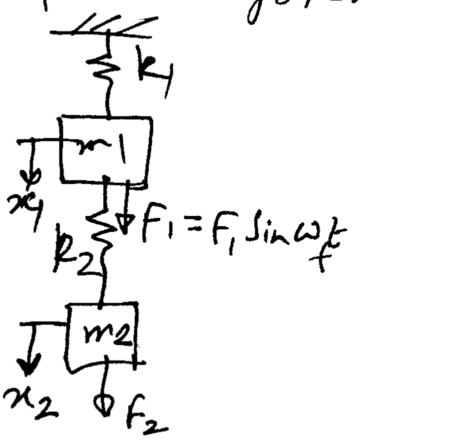
The frequency responses look like the following:-



See accurate plots from a textbook

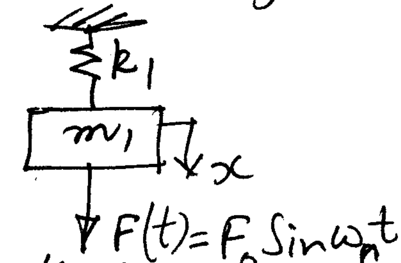
HW problem:-

When $m_1 = m_2 = m$, $k_1 = 3k$ & $k_2 = 2k$, $F_2 = 0$, obtain the steady state (forced) response of the system

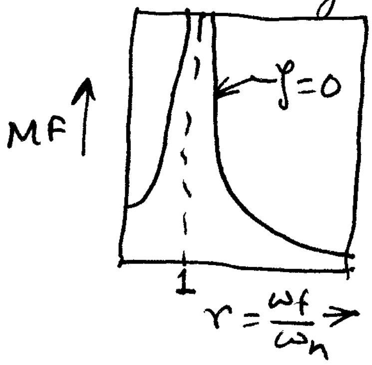


(8) The Undamped Vibration Absorber (The Tuned Damper):-

The basic problem:- We have a single DOF undamped system:



However, ω_f is very close to ω_n & very large amplitude motion (vibration) occurs.



← Remember this?

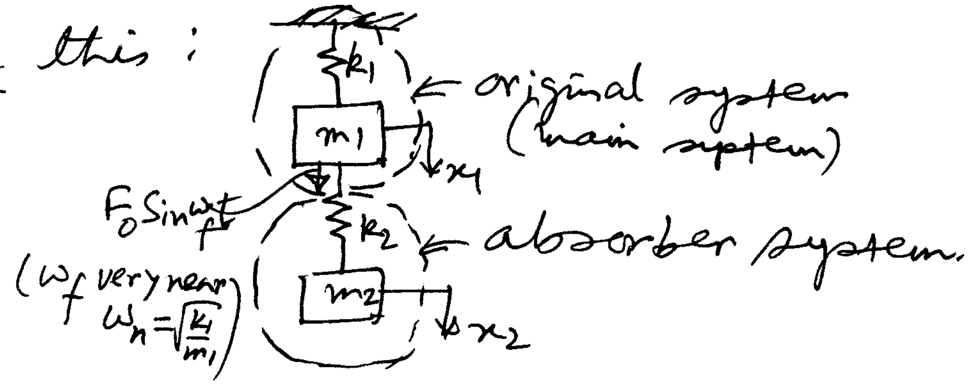
Note ~~that~~ $\omega_n = \sqrt{\frac{k_1}{m_1}}$ & $\omega_f \approx \omega_n$

How could we reduce the vibration level? We could, for instance, decrease or increase r so that r value is sufficiently away from unity & vibration levels drop down to acceptable values. To do this, we must change ω_n since ω_f is given and can't be changed. ω_n can be increased by increasing k_1 but this usually is a complicated affair. (k_1 may be due to springs which don't look like a spring at all, say rubber pads). Of course, we could add a dead weight to increase m & thus change ~~[illegible]~~ r. But all this type of measures are seen to produce limited result in practice.

→ So, instead of changing the spring or adding dead weight, how about putting/joining another spring-mass system to the original system so that we now get a

2-DOF system having two natural frequencies ω_1 & ω_2 ($\omega_1 < \omega_2$) & thus, the resulting 2-DOF would have large responses near $\omega_f = \omega_1$ or, $\omega_f = \omega_2$ but a small, acceptable response at $\omega_f = \omega_n$? If you take a look at the top frequency response plot on page 6, you should get the idea.

→ After we add k_2 & m_2, the new system looks like this:



The question is: How do we choose ~~proper~~ proper k_2 & m_2?

To answer this, we must consider the forced vibration of the above system.

The DEOM are, (check this)

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = F_0 \sin\omega_f t \quad \text{—— (1)}$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \quad \text{—— (2)}$$

Let $x_1 = X_1 \sin\omega_f t$ & $x_2 = X_2 \sin\omega_f t$.

Then, from (1) & (2), we get, as before,

$$(k_1+k_2-m_1\omega_f^2)X_1 - k_2X_2 = F_0 \quad \text{—— (3)}$$

&

$$-k_2X_1 + (k_2-m_2\omega_f^2)X_2 = 0 \quad \text{—— (4)}$$

Now, ω_f is close to $\omega_n = \sqrt{\frac{k_1}{m_1}}$ & so, in general, ω_n will not be equal to either ω_1 or ω_2 where ω_1 & ω_2 are obtained from the frequency equation

$$\begin{vmatrix} (k_1+k_2-m_1\omega^2) & -k_2 \\ -k_2 & (k_2-m_2\omega^2) \end{vmatrix} = 0.$$

Hence $\Delta = \begin{vmatrix} (k_1+k_2-m_1\omega_f^2) & -k_2 \\ -k_2 & (k_2-m_2\omega_f^2) \end{vmatrix} \neq 0$

& so, by Cramer's rule,

$$X_1 = \frac{\begin{vmatrix} F_0 & -k_2 \\ 0 & (k_2-m_2\omega_f^2) \end{vmatrix}}{\Delta} = \frac{F_0(k_2-m_2\omega_f^2)}{\Delta}.$$

$$\& \quad X_2 = \frac{\begin{vmatrix} (k_1+k_2-m_1\omega_f^2) & F_0 \\ -k_2 & 0 \end{vmatrix}}{\Delta} = \frac{F_0 k_2}{\Delta}.$$

The above expression for X_1 indicates ~~that~~ if we choose k_2 & m_2 such ~~that~~ $k_2 - m_2\omega_f^2 = 0$ or, $\frac{k_2}{m_2} = \omega_f^2 \approx \omega_n^2 = \frac{k_1}{m_1}$, then, $X_1 = 0$ & the original system doesn't vibrate at all! So, the purpose of the absorber is served.

→ Hence, for the absorber system, k_2 & m_2 must be so chosen that $\frac{k_2}{m_2} = \frac{k_1}{m_1}$. Note that this is only one condition for choosing two parameters k_2 & m_2. Thus, k_2 & m_2 are <u>not</u> unique. These values should be such that above relation is satisfied.

The ss responses $x_1(t)$ & $x_2(t)$ would be

governed by amplitudes X_1 & X_2 of which $X_1 = 0$.
In practice, ω_f would ~~...~~ vary a little and then X_1 would no longer be zero.
This is why this undamped vibration absorber is called a 'tuned' damper because it is tuned to only one frequency $\omega_f = \omega_n$.
Researchers have tried to improve the performance of the tuned damper.

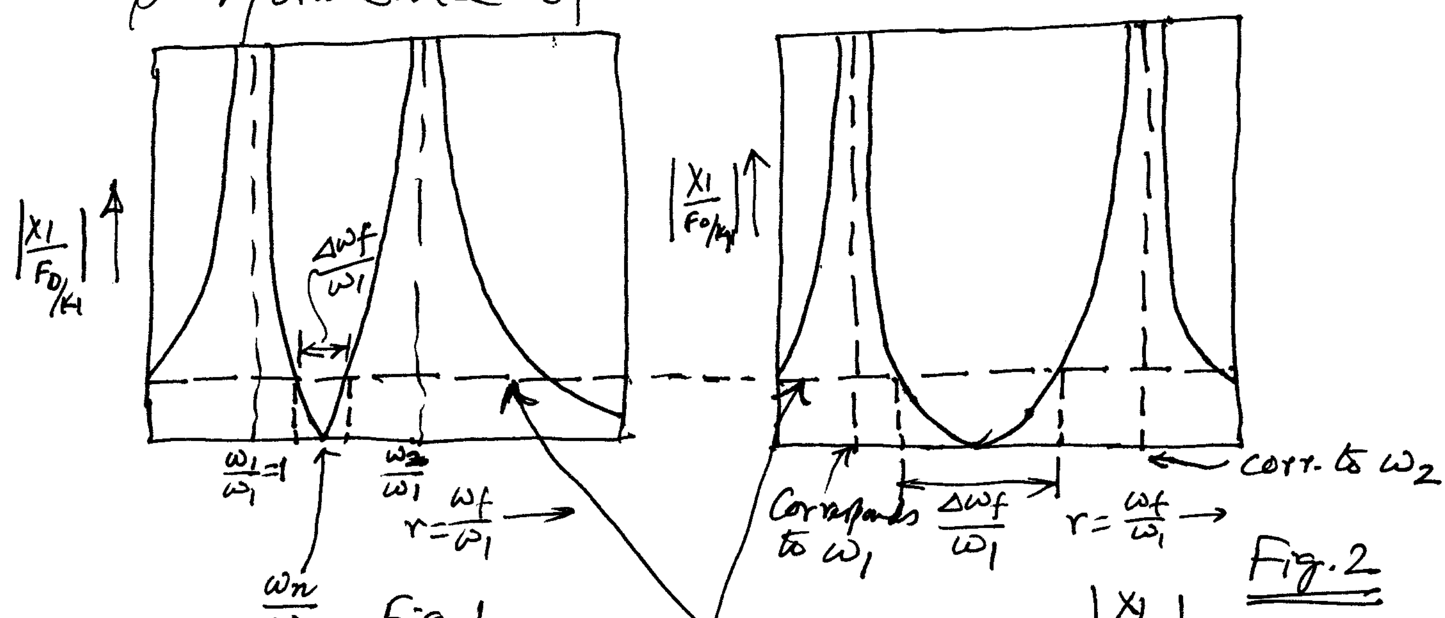


Fig. 1 Fig. 2

Maxm tolerable level of $\left|\frac{X_1}{F_0/K_1}\right|$

Study above figs. 1 & 2. Suppose ω_f fluctuates. Which of the above two figures gives a better vibration isolation for our main system? It is obviously fig. 2 because, the acceptable level of X_1 (F_0/K_1 is a constant) (below the dashed line) occurs over a larger $\Delta\omega_f$ compared to the $\Delta\omega_f$ in fig. 1.
This happens because the difference $(\omega_2 - \omega_1)$ is larger in fig. 2 than in fig. 1. Thus,

the aim should be to choose the absorber parameters k_2 & m_2 such that $(\omega_2-\omega_1)$ is as large as possible. This possibility can be studied as follows:-

The frequency eqn. is $\begin{vmatrix} (k_1+k_2-m_1\omega^2) & -k_2 \\ -k_2 & (k_2-m_2\omega^2) \end{vmatrix}=0$

or, $$m_1m_2\omega^4-\left[m_1k_2+m_2(k_1+k_2)\right]\omega^2+k_1k_2=0$$

$$\Rightarrow \frac{\omega^4}{\left(\frac{k_1k_2}{m_1m_2}\right)}-\left[\frac{m_1}{k_1}+\frac{m_2}{k_2}\left(1+\frac{k_2}{k_1}\right)\right]\omega^2+1=0$$

$$\Rightarrow \frac{\omega^4}{\left(\frac{k_1}{m_1}\right)^2}-\left[\frac{m_1}{k_1}+\frac{m_1}{k_1}\left(1+\frac{m_2}{m_1}\right)\right]\omega^2+1=0$$

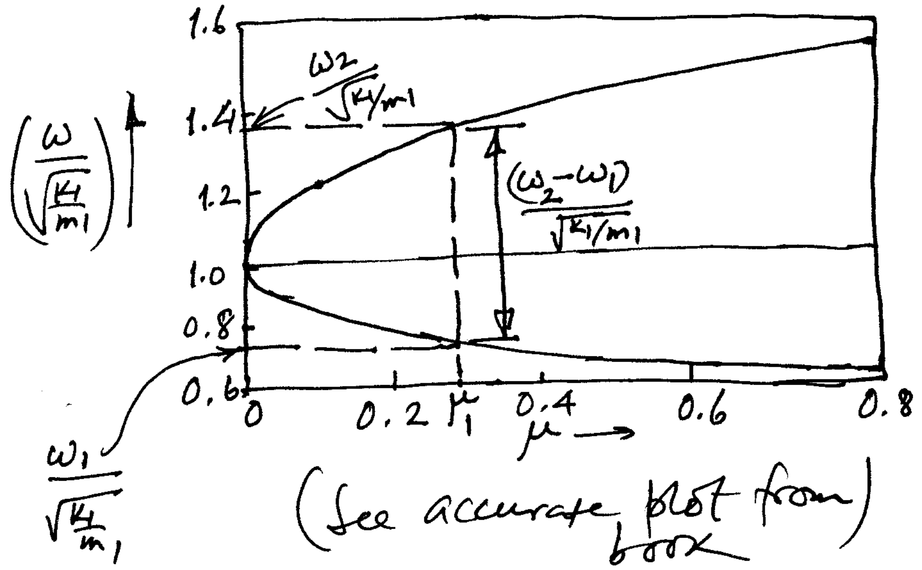
(Since, $\frac{k_1}{m_1}=\frac{k_2}{m_2}$, $\frac{k_2}{k_1}=\frac{m_2}{m_1}$ and also $\frac{m_2}{k_2}=\frac{m_1}{k_1}$)

$$\Rightarrow \left(\frac{\omega}{\sqrt{\frac{k_1}{m_1}}}\right)^4-\left[1+1+\frac{m_2}{m_1}\right]\left(\frac{\omega}{\sqrt{\frac{k_1}{m_1}}}\right)^2+1=0$$

$$\Rightarrow \left[\frac{\omega}{\sqrt{\frac{k_1}{m_1}}}\right]^4-(2+\mu)\left(\frac{\omega}{\sqrt{\frac{k_1}{m_1}}}\right)^2+1=0$$

where $\underline{\underline{\mu}}=\frac{m_2}{m_1}=\underline{\underline{\text{mass ratio}}}=\frac{\text{Absorber mass}}{\text{main mass}}$

A plot of $\left(\frac{\omega}{\sqrt{\frac{k_1}{m_1}}}\right)$ Vs μ looks like the following:



(See accurate plot from book)

So, as μ increases, $(\omega_2-\omega_1)$ increases. However, this increase rate reduces (the plot flattens) as μ increases. So, we normally don't go beyond $\mu=0.7$. After all, if a one tonne machine has a 800 Kg absorber, it doesn't look good, isn't it?

→ So, now we could arrive at another criterion for selecting the absorber parameter. We could, for instance, take $\mu = \frac{m_2}{m_1} = 0.65$?

Then, $m_2 = 0.65 m_1$ & $\frac{K_2}{m_2} = \frac{K_1}{m_1}$ gives

$K_2 = m_2 \cdot \frac{K_1}{m_1} = \mu K_1$. Thus the absorber has a unique mass & spring and it can work well over a moderate variation in ω_f.

→ (See 'Mechanical Vibration' by J.-P. DenHartog for a more detailed discussion on vibration absorbers.)

→ There are numerous research papers on Dynamic Vibration Absorbers using passive as well as active control elements.
