15.07.2019

Applied Elasticity :-

Midsem :- 60 marks (30)

+ Teacher's assessment (20)

End sem - 100 marks (50)

2 assignments + Term project.

Textbook - Elasticity by Martin Saad.

1. MATHEMATICAL PRELIMINARIES :-

The rotation vector which does not obey the rules of vector addition - commutativity. - orientation differs with order of rotation.

Vectors :- $\vec{V}$, $\underset{\sim}{V}$: compact notations commonly used.

- no reference to any coordinate system.

Component form :- $\vec{V} \equiv \underset{\sim}{V} \equiv v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$

where $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are the unit vectors along the coordinate directions.

- the Basis vectors.

- v_1, v_2, v_3 are unique to the coordinate system.

The vector can be alternatively represented in another coordinate as -

$$\vec{V} \equiv v_1' \hat{e}_1' + v_2' \hat{e}_2' + v_3' \hat{e}_3' .$$

$$= \sum_{i=1}^{3} v_i \hat{e}_i$$ (short-hand component form)

Matrix representation of a vector -

$$[\vec{v}] \equiv [\underset{\sim}{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \boxed{\text{Column matrix}} \quad \text{or} \quad [v_1 \quad v_2 \quad v_3] \ \text{Row matrix.}$$

Basis vectors' representations in the component form :-

$$[\vec{e}_1] \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [\hat{e}_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\hat{e}_3] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[\vec{v}] = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The indicial (index) notation :-

- $\vec{v} \equiv \underset{\sim}{v} \equiv v_i \hat{e}_i$ (3 elements) $\qquad A_{ij} \rightarrow$ 9 elements.

Rules of the indicial notation -

1. No summation signs or unit vectors included.

2. A repeated index implies summation -

$$A_{ii} = A_{11} + A_{22} + A_{33} \quad (\text{Scalar result})$$

The repeated index can be replaced by any variable - hence it is the <u>dummy</u> index.

3. A repeated index is [illegible] dummy index can be replaced by another [illegible]

$$A_{iip} = A_{11p} + A_{22p} + A_{33p} \Rightarrow \text{a summation of vectors with free index } p$$

$$\equiv \begin{bmatrix} A_{111} \\ A_{112} \\ A_{113} \end{bmatrix} + \begin{bmatrix} A_{221} \\ A_{222} \\ A_{223} \end{bmatrix} + \begin{bmatrix} A_{331} \\ A_{332} \\ A_{333} \end{bmatrix}$$

$$= \begin{bmatrix} A_{111} + A_{221} + A_{331} \\ A_{112} + A_{222} + A_{332} \\ A_{113} + A_{223} + A_{333} \end{bmatrix}$$

: vector with 3 elements (1^{st} order tensor since 1 free index - p)

The order of tensor entity will be given by the number of free indices.

4. The number of non-repeated (free) indices in an index form gives the order of the tensor.

Eg. $A_{iip} \rightarrow$ here only p is repeated ("it is "free")

so that the order of A_{iip} is 1. That is, A_{iip} is a tensor of order 1

5. Any index repeated more than once is meaningless.

Dot Product :- $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$= a_i b_i$

$\vec{a}\cdot\vec{b} = [a_1 \quad a_2 \quad a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\vec{a}]^T[\vec{b}]$ which is the matrix form of the dot product.

Development of the expression for dot product in the index notation -

$\vec{a}\cdot\vec{b} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$

(the basis vectors belong to one coordinate system

$= (a_i\hat{e}_i)\cdot(b_i\hat{e}_i)$ represents an incorrect notation.

$= (a_i\hat{e}_i)\cdot(b_K\hat{e}_K)$

$= a_i b_K(\hat{e}_i\cdot\hat{e}_K)$

$\hat{e}_1\otimes\hat{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}[1 \quad 0 \quad 0] = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$\hat{e}_1\otimes\hat{e}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}[0 \quad 1 \quad 0] = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$\vec{a}\otimes\vec{b} = (a_i\hat{e}_i)\otimes(b_j\hat{e}_j)$

$= a_i b_j(\hat{e}_i\otimes\hat{e}_j)$

$= a_i b_1(\hat{e}_i\otimes\hat{e}_1) + a_i b_2(\hat{e}_i\otimes\hat{e}_2) + a_i b_3(\hat{e}_i\otimes\hat{e}_3)$

$= a_1 b_1(\hat{e}_1\otimes\hat{e}_1) + a_2 b_1(\hat{e}_2\otimes\hat{e}_1) + a_3 b_1(\hat{e}_3\otimes\hat{e}_1)$

$+ a_1 b_2(\hat{e}_1\otimes\hat{e}_2) + a_2 b_2(\hat{e}_2\otimes\hat{e}_2) + a_3 b_2(\hat{e}_3\otimes\hat{e}_2)$

$+ a_1 b_3(\hat{e}_1\otimes\hat{e}_3) + a_2 b_3(\hat{e}_2\otimes e_3) + a_3 b_3(\hat{e}_3\otimes\hat{e}_3)$

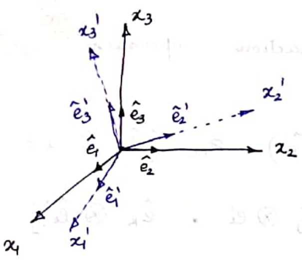
$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_3 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$

A second order tensor $\underset{\approx}{T}$ can be written as -

$\underset{\approx}{T} = T_{ij}\,\hat{e}_i\otimes\hat{e}_j$

$\underset{\approx}{T}^T = T_{ij}\,\hat{e}_j\otimes\hat{e}_i$ (prove). H.W.

Coordinate Transformation :-



The unit basis vectors in the unprimed coordinate. $\hat{e}_i$

Basis vectors in the primed coordinates : $\hat{e}_i'$

The direction cosines :- $\cos(x_i', x_j) = Q_{ij}$

which may be explicitly written as -

$\cos(x_1', x_1) = Q_{11}$ $\quad \cos(x_1', x_2) = Q_{12}$ $\quad \cos(x_1', x_3) = Q_{13}$

$$\hat{e}_1' = Q_{11}\hat{e}_1 + Q_{12}\hat{e}_2 + Q_{13}\hat{e}_3$$

$$\hat{e}_2' = Q_{21}\hat{e}_1 + Q_{22}\hat{e}_2 + Q_{23}\hat{e}_3$$

$$\hat{e}_3' = Q_{31}\hat{e}_1 + Q_{32}\hat{e}_2 + Q_{33}\hat{e}_3$$

$\therefore \hat{e}_i' = Q_{ij}\, e_j$. since $v' = Qv$

$$\begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3' \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$ and

$$\hat{e}_i = Q_{ji}\,\hat{e}_j'$$

22.07.2019. MATHEMATICAL PRELIMINARIES ... Contd.

Coordinate Transformations :-

$\hat{e}_i' = Q_{ij}\, e_j$ and $\hat{e}_i = Q_{ji}\,\hat{e}_j'$

where we know that $Q_{ij} = \cos(x_i', x_j)$.

A vector being a physical entity can be written as -

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

$$= v_1'\hat{e}_1' + v_2'\hat{e}_2' + v_3'\hat{e}_3'$$

which can be written in short as : $\vec{v} = v_i\hat{e}_i = v_j'\hat{e}_j'$

$\therefore v_i\hat{e}_i = v_j'\hat{e}_j' = v_i'\hat{e}_i'$

$\Rightarrow v_i\hat{e}_i = v_i'\hat{e}_i' = v_i' Q_{ij}\hat{e}_j$

$\Rightarrow v_j\hat{e}_j = v_i' Q_{ij}\hat{e}_j$

comparing both sides of the equation, or,

since $(v_j - v_i' Q_{ij})\hat{e}_j = 0$

$$\boxed{v_j = v_i' Q_{ij}}$$

$v_i\hat{e}_i = v_j'\hat{e}_j'$

$\Rightarrow v_i Q_{ji}\hat{e}_j' = v_j'\hat{e}_j'$ comparing or cancelling from the equations -

$\Rightarrow \boxed{v_j' = Q_{ji} v_i}$

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The equivalent matrix representation would be -

$$\vec{v}' = [Q]\vec{v}$$

$$\vec{v} = [Q]^T \vec{v}'$$

$\vec{a} \cdot \vec{b} = [a]^T [b]$ Q is a second order tensor.

we know that -

$$\vec{v}' = [Q]\vec{v}$$

$$\Rightarrow \vec{v}' = [Q][Q]^T \vec{v}' \quad \Rightarrow \boxed{[Q][Q]^T = I}$$

from which it can be gathered that Q is an orthogonal matrix.

which can be alternatively written in the indicial notation as -

$$\boxed{Q_{ji} \; Q_{ki} = \delta_{kj}}$$

H.W.

Prove that $\boxed{[\underset{\approx}{Q}]^T [\underset{\approx}{Q}]} = [\underset{\approx}{I}]$

~~KINEMATICS~~ :- Transformation of 2^{nd} order tensor components :-

Consider a tensor of order 2 :- $T_{ij} = [\underset{\approx}{T}]$

$T_{ij}' = a_i' \, b_j'$ (the tensor can be expressed as the dyadic product of 2 vectors)

$= Q_{ik} \, a_k \, Q_{jm} \, b_m$

$= Q_{ik} \, Q_{jm} \, a_k \, b_m$.

Hence we obtain the transformation relations for the tensor components as -

$$\boxed{T_{ij}' = Q_{ik} \; Q_{jm} \; T_{km}}$$

Transformation of a 3^{rd} order tensor components -

$$A_{ijk}' = Q_{il} \, Q_{jm} \, Q_{kn} \, A_{lmn}.$$

Equivalent matrix representation of the tensor transformations

$T_{ij}' = Q_{ik} \; Q_{jm} \; T_{km}$

$= Q_{ik} \; T_{km} \; Q_{jm}$

$= [\underset{\approx}{Q}][\underset{\approx}{T}][\underset{\approx}{Q}]^T$

$[\underset{\approx}{A}'] = [\underset{\approx}{Q}][\underset{\approx}{A}][\underset{\approx}{Q}]^T$

$= [Q][a][b]^T[Q]^T = [\underset{\approx}{Q}][\underset{\approx}{A}][\underset{\approx}{Q}^T]$

$A_{ij}' = \underbrace{Q_{im} \, a_m}_{u_i} \, \underbrace{Q_{jk} \, b_k}_{v_j}$

$= u_i \, v_j$

$= u_i \, e_i \otimes v_j \, e_j$

$= [\vec{u}][\vec{v}]^T$

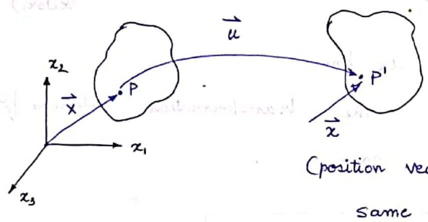
$= [Q][a]([Q][b])^T$

$= [Q][a][b]^T[Q]^T$

2. KINEMATICS :-

(i) Deformation and displacements -

- The Deformation map :



P and P' correspond to the same material point.

(position vector wrt the same coordinate system)

The final position vector must be a function of the initial undeformed position vector, so that -

$$\vec{x} \equiv \vec{x}(\vec{X}, t)$$ which can be explicitly written as -

$$x_1 \equiv x_1(X_1, X_2, X_3, t)$$
$$x_2 \equiv x_2(X_1, X_2, X_3, t)$$
$$x_3 \equiv x_3(X_1, X_2, X_3, t)$$

In the index notation,

$$x_i \equiv x_i(X_j, t)$$

- Displacements :-

Define the displacement vector :-

$$\underset{(\vec{X}, t)}{\vec{u}} := \vec{x}(\vec{X}, t) - \vec{X}$$

where the displacement $\vec{u}$ is a function of X

$\Rightarrow$ the original deformation.

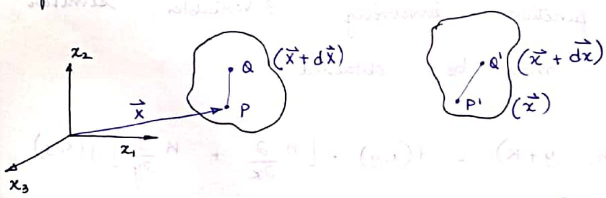
We will drop writing time 't' with the understanding that it is contained implicitly within the equations.

$$\therefore \vec{u}(\vec{X}) := \vec{x}(\vec{X}) - \vec{X}$$

In the index notation -

$$u_i := x_i(X_j) - X_i$$

Quantification of Deformation :-



Consider the displacement of the point Q, which is in the neighbourhood of P -

$$\vec{u}(\vec{X} + d\vec{X}) := (\vec{x} + d\vec{x}) - (\vec{X} + d\vec{X})$$
$$= (\vec{x} - \vec{X}) + d\vec{x} - d\vec{X}$$

From which we obtain that -

$$d\vec{x} = d\vec{X} + \vec{u}(\vec{X} + d\vec{X}) - \vec{u}(\vec{X})$$

Consider $u_1(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3) - u_1(X_1, X_2, X_3)$

Taylor series Expansions -

$$f(x+h) = f(x) + h \left.\frac{df}{dx}\right|_{h=0} + \frac{h^2}{2!} \left.\frac{d^2f}{dx^2}\right|_{h=0} + \cdots$$

if the function involves 2 variables -

$$f(x+h, y+K) = f(x,y) + \left[h \left.\frac{\partial f}{\partial x}\right|_{h=0} + K \left.\frac{\partial f}{\partial y}\right|_{K=0} \right]$$

$$+ \left[\frac{h^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{2hK}{2!} \frac{\partial^2 f}{\partial x \partial y} + \frac{K^2}{2!} \frac{\partial^2 f}{\partial y^2} \right] + \cdots$$

For a function involving 3 variables, similar expression may be obtained.

$$f(x+h, y+K) = f(x,y) + \left[h\frac{\partial}{\partial x} + K \frac{\partial}{\partial y} \right] f(x,y)$$

$$+ \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y} \right]^2 f(x,y)$$

where $\left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]^2 = \left(h^2 \frac{\partial^2}{\partial x^2} + 2hK \frac{\partial^2}{\partial x \partial y} + K^2 \frac{\partial^2}{\partial y^2} \right)$

Consider the expression :-

$$u_1(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3) - u_1(X_1, X_2, X_3)$$

We assume that dX_1, dX_2 and dX_3 are small.

$$= u_1(X_1, X_2, X_3) + \left[dX_1 \frac{\partial}{\partial X_1} + dX_2 \frac{\partial}{\partial X_2} + dX_3 \frac{\partial}{\partial X_3} \right] u_1(X_1, X_2, X_3) - u_1(X_1, X_2, X_3)$$

The higher order terms are neglected.

$$= \left(dX_1 \frac{\partial}{\partial X_1} + dX_2 \frac{\partial}{\partial X_2} + dX_3 \frac{\partial}{\partial X_3} \right) u_1(X_1, X_2, X_3)$$

Going back to the original equation -

$$d\vec{x} = d\vec{X} + \vec{u}(\vec{X} + d\vec{X}) - \vec{u}(\vec{X})$$

Considering the component with index 1 -

$$\Rightarrow dx_1 = dX_1 + u_1(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3) - u_1(X_1, X_2, X_3)$$

$$= dX_1 + \left(dX_1 \frac{\partial}{\partial X_1} + dX_2 \frac{\partial}{\partial X_2} + dX_3 \frac{\partial}{\partial X_3} \right) u_1(X_1, X_2, X_3)$$

$$dx_2 = dX_2 + \left(dX_1 \frac{\partial}{\partial X_1} + dX_2 \frac{\partial}{\partial X_2} + dX_3 \frac{\partial}{\partial X_3} \right) u_2$$

23.07.2019 **KINEMATICS** ... Contd.

Quantification of deformation :-

$$dx_i = dX_i + \left(dX_1 \frac{\partial}{\partial X_1} + dX_2 \frac{\partial}{\partial X_2} + dX_3 \frac{\partial}{\partial X_3} \right) u_i$$

otherwise, using the index notation -

$$\boxed{dx_i = dX_i + \left(dX_j \cdot \frac{\partial}{\partial X_j} \right) u_i}$$

which when written explicitly become -

$$dx_1 = dX_1 + \left(dX_j \frac{\partial}{\partial X_j} \right) u_1$$

$$dx_2 = dX_2 + \left(dX_j \frac{\partial}{\partial X_j} \right) u_2$$

$$dx_3 = dX_3 + \left(dX_j \frac{\partial}{\partial X_j} \right) u_3$$

Considering the equation -

$$dx_i = dX_i + dX_j \frac{\partial u_i}{\partial X_j}$$

A closer look at the term : $dX_j \dfrac{\partial u_i}{\partial X_j}$ -

Interpretation - (i):-

We can interpret the term as the scalar operator $\left(dX_j \dfrac{\partial}{\partial X_j} \right)$ operating on the vector u_i

[$dX_j \dfrac{\partial}{\partial X_j}$ is a scalar operation because it can be viewed as the dot product between $d\vec{X} \equiv dX_j$ and the gradient operator $\nabla \equiv \dfrac{\partial}{\partial X_j}$ $\Rightarrow (d\vec{X} \cdot \vec{\nabla})$]

where the gradient operator is :-

$$\nabla = \left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3} \right) \equiv \frac{\partial}{\partial X_1}\hat{e}_1 + \frac{\partial}{\partial X_2}\hat{e}_2 + \frac{\partial}{\partial X_3}\hat{e}_3$$

while the differential vector -

$$d\vec{X} = dX_1 \hat{e}_1 + dX_2 \hat{e}_2 + dX_3 \hat{e}_3$$

Interpretation (ii) :-

We can interpret the term as : - $dX_j \left(\dfrac{\partial u_i}{\partial X_j} \right)$ $\Leftarrow$

$\dfrac{\partial u_i}{\partial X_j}$ is a tensor of order 2 and we can view

$dX_j \dfrac{\partial u_i}{\partial X_j}$ as a contraction between the 2nd order tensor $\dfrac{\partial u_i}{\partial X_j}$ and the 1st order tensor dX_j.

$$\frac{\partial u_i}{\partial X_j} \equiv \nabla \vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Sometimes the partial derivatives are represented in short by a comma $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$

$$dX_j \left(\frac{\partial u_i}{\partial X_j} \right) \equiv \nabla \vec{u} \cdot d\vec{X} \equiv [\nabla \vec{u}][d\vec{X}]$$

$$= \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$

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Special note :-

$\frac{\partial u_i}{\partial x_j} \equiv \nabla \vec{u}$ which is a 3x3 matrix.

Since any matrix can be expressed as a sum of a symmetric and another anti symmetric matrix -

$$\nabla \vec{u} = \underbrace{\frac{1}{2}\left\{ (\nabla \vec{u}) + (\nabla \vec{u})^T \right\}}_{\text{Symmetric Part}} + \underbrace{\frac{1}{2}\left\{ (\nabla \vec{u}) - (\nabla u)^T \right\}}_{\text{Anti symmetric part}}$$

Getting back to our original equation -

$$dx_i = dX_i + dX_j \frac{\partial u_i}{\partial x_j}$$

we may write.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$\therefore dx_i = dX_i + dX_j \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right]$$

$$= dX_i + \frac{1}{2} dX_j \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{\text{Symmetric part}} + \frac{1}{2} dX_j \underbrace{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)}_{\text{anti symmetric part}}$$

The symmetric part of $\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ is defined to be the strain tensor ε_{ij}.

Strain :- $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ - symmetric part of the deformation gradient.

strictly, this is the small or infinitesimal strain tensors.

Written explicitly, the strain components are given by -

diagonal terms - normal strains.

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \; ; \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \; ; \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

off - diagonal terms.

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right)$$

and $\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right)$

the Strain - Displacement relations.

Sometimes the notation for shear strains is used as -

$$\gamma_{12} = \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = 2\varepsilon_{12} \quad \text{etc.}$$

H.W.

Reading assignment :-

Physical or geometric interpretation of the various strain terms.

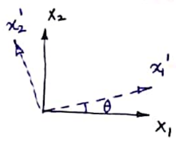
25.07.2019 KINEMATICS ... Contd.

Strain Transformation :-

A direct application of tensor transformations discussed before.

$$\varepsilon'_{ij} = Q_{ip}\, Q_{jq}\, \varepsilon_{pq}$$

$$\equiv [Q][\underset{\approx}{\varepsilon}][Q]^T$$



$$[Q]_{2\times 2} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

In 3 dimensions :- $[Q]_{3\times 3} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

H.W. Derive the plane stress conditions. (2D transformations)

Principal Strains :-

- Engineering strain -

$$\varepsilon_E := \frac{|\vec{dx}| - |\vec{dX}|}{|\vec{dX}|}$$

The expression for strain is a definition. In the case of a general 3D strain condition, the engineering strain varies in every direction.

$|\vec{dx}| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}$ ...

Because square roots are difficult to handle in a theoretical development, we will consider the squares :- $|\vec{dx}|^2$

$$|\vec{dx}|^2 = \vec{dx}\cdot\vec{dx}$$

$$\equiv [\vec{dx}]^T[\vec{dx}]$$

But we know that -

$$[\vec{dx}] = [\vec{dX}] + [\nabla\vec{u}][\vec{dX}]$$

$$\therefore |\vec{dx}|^2 = \{[\vec{dX}] + [\nabla\vec{u}][\vec{dX}]\}^T\{[\vec{dX}] + [\nabla\vec{u}][\vec{dX}]\}$$

$$= \{[\vec{dX}]^T + [\vec{dX}]^T[\nabla\vec{u}]^T\}\{[\vec{dX}] + [\nabla\vec{u}][\vec{dX}]\}$$

$$= [\vec{dX}]^T[\vec{dX}] + [\vec{dX}]^T[\nabla\vec{u}][\vec{dX}] + [\vec{dX}]^T[\nabla\vec{u}]^T[\nabla\vec{u}][\vec{dX}] + [\vec{dX}]^T[\nabla\vec{u}][\vec{dX}]$$

Alternatively we may write $[\vec{dX}] = |\vec{dX}|\,[\hat{N}]$

where $[\vec{N}]$ is the unit vector in the direction of $[dX]$

$$|\vec{dx}|^2 = |\vec{dX}|\{[\vec{N}] + [\nabla\vec{u}][\vec{N}]\}^T|\vec{dX}|\{[\vec{N}] + [\nabla\vec{u}][\vec{N}]\}$$

$$\Rightarrow \frac{|\vec{dx}|^2}{|\vec{dX}|^2} = \left([\vec{N}] + [\nabla\vec{u}][\vec{N}]\right)^T\left([N] + [\nabla\vec{u}][\vec{N}]\right)$$

$$= \left([\vec{N}]^T + [\vec{N}]^T[\nabla\vec{u}]^T\right)\left([\vec{N}] + [\nabla\vec{u}][\vec{N}]\right)$$

$$= [N]^T[N] + [\vec{N}]^T[\nabla\vec{u}][\vec{N}] + [\vec{N}]^T[\nabla\vec{u}]^T[\vec{N}] + [\vec{N}]^T[\nabla\vec{u}]^T[\nabla\vec{u}][\vec{N}]$$

$$= 1 + [\vec{N}^T]\left([\vec{\nabla u}] + [\vec{\nabla u}]^T\right)[\vec{N}]$$

The last term $[\vec{N}^T][\vec{\nabla u}][\vec{\nabla u}]^T[\vec{N}]$ will be neglected since it involves the nonlinear terms.

Also it is known that -

$$\varepsilon = \frac{1}{2}\left([\vec{\nabla u}] + [\vec{\nabla u}]^T\right)$$

$$\Rightarrow \left([\vec{\nabla u}] + [\vec{\nabla u}]^T\right) = 2\varepsilon.$$

$$\therefore \frac{|\vec{dx}|^2}{|\vec{dX}|^2} = 1 + 2[\vec{N}]^T[\underset{\approx}{\varepsilon}][\vec{N}]$$

But we have defined the engineering strain as -

$$\varepsilon_E = \frac{|\vec{dx}| - |\vec{dX}|}{|\vec{dX}|}$$

$$\Rightarrow 1 + \varepsilon_E = \frac{|\vec{dx}|}{|\vec{dX}|}$$

Squaring both sides -

$$\Rightarrow (1+\varepsilon_E)^2 = \frac{|\vec{dx}|^2}{|\vec{dX}|^2} = 1 + 2[\vec{N}]^T[\underset{\approx}{\varepsilon}][\vec{N}]$$

$$\Rightarrow 1 + 2\varepsilon_E + \varepsilon_E^2 = 1 + 2[\vec{N}]^T[\underset{\approx}{\varepsilon}][\vec{N}]$$

since ε_E^2 is a nonlinear term, we may neglect it -

So that $1 + 2\varepsilon_E = 1 + 2[N]^T[\underset{\approx}{\varepsilon}][\vec{N}]$

$$\Rightarrow \boxed{\varepsilon_E = [\vec{N}]^T[\underset{\approx}{\varepsilon}][\vec{N}]}$$

Engineering strain is a scalar ratio, which can be checked from the above equation.

$$\varepsilon_E = [\vec{N}]^T[\underset{\approx}{\varepsilon}][\vec{N}]$$

$$= N_i \, \varepsilon_{ij} \, N_j$$ (expressed in the index notation)

- engineering strain depends upon the direction $[\vec{N}]$.

Principal strain - at any point, the engineering strain takes a maximum value along one direction - this maximum value is the principal strain.

$$\frac{d(N_i \varepsilon_{ij} N_j)}{d N_K} = 0$$

since $[\vec{N}]$ is a unit vector, it must satisfy that $|\vec{N}| = 1$.

Hence the maximization problem is subject to the condition that $|\vec{N}| = 1$.

$$\frac{d(N_i \varepsilon_{ij} N_j)}{d N_K} = 0$$

has to be satisfied under the constraint that

$[\vec{N}]^T[\vec{N}] = 1$ or $N_K N_K - 1 = 0$

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This kind of constrained extremization can be done using the method of Lagrange multipliers.

So,

$$\frac{\partial L}{\partial N_K} = 0 \quad \text{where} \quad L = N_i \varepsilon_{ij} N_j + \lambda(1 - N_K N_K)$$

29.07.2019.

Principal Strains :-

$$\frac{d}{dN_K}\{\varepsilon_E + \lambda(1 - N_K N_K)\} = 0$$

$$\Rightarrow \frac{d}{dN_K}\{N_i \varepsilon_{ij} N_j + \lambda(1 - N_K N_K)\} = 0$$

But the strain ε_{ij} depends on a point and is independent of the direction -

$$\frac{dN_i}{dN_K}\varepsilon_{ij} N_j + N_i \varepsilon_{ij}\frac{dN_j}{dN_K} - 2\lambda . N_K = 0$$

$$\Rightarrow \frac{d}{dN_K}(N_K \delta_{iK})\varepsilon_{ij} N_j + N_i \varepsilon_{ij}\frac{d}{dN_K}(N_K \delta_{jK}) - 2\lambda N_K = 0$$

$$\Rightarrow \delta_{iK}\varepsilon_{ij} N_j + N_i \varepsilon_{ij}\delta_{jK} - 2\lambda N_K = 0$$

$$\varepsilon_{Kj} N_j + N_i \varepsilon_{iK} - 2\lambda N_K = 0$$

But the small strain tensor ε is a symmetric tensor

$\varepsilon_{pq} = \frac{1}{2}\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\right)$ so that $\varepsilon_{qp} = \frac{1}{2}\left(\frac{\partial u_q}{\partial x_p} + \frac{\partial u_p}{\partial x_q}\right)$

Hence $\varepsilon_{pq} = \varepsilon_{qp}$, or $\varepsilon_{iK} = \varepsilon_{Ki}$

$$\Rightarrow \varepsilon_{Kj} N_j + \varepsilon_{Ki} N_i - 2\lambda N_K = 0$$

$$\Rightarrow 2\varepsilon_{Kj} N_j - 2\lambda N_{jK} = 0$$

$$\Rightarrow \varepsilon_{Kj} N_j - \lambda N_K = 0$$

$$\Rightarrow \varepsilon_{Kj} N_j - \lambda . \delta_{jK} N_j = 0$$

$$\Rightarrow (\varepsilon_{Kj} - \lambda \delta_{jK}) N_j = 0$$

The equivalent matrix representation is :-

$$([\underset{\approx}{\varepsilon}] - \lambda[\underset{\approx}{I}])[\hat{N}] = 0$$

This represents an eigenvalue problem, where the eigenvalues are λ.

The eigenvalues are found from the characteristic equation which is given by -

$$\det([\underset{\approx}{\varepsilon}] - \lambda[\underset{\approx}{I}]) = 0$$

$$\Rightarrow \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where I_1, I_2 and I_3 are the strain invariants.

Ex. Take any symmetric tensor using Matlab / etc., find the eigenvalues.

Find the 3 eigenvalues from the solution of the algebraic characteristic equation

Find the λs using built in commands.

Strain Compatibility :-

Consider the six strain - displacement relations written earlier -

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

In the expanded form -

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

The strain - displacement equations are PDEs for evaluating u_1, u_2 and u_3.

An overconstrained problem with 3 unknowns & 6 equations.

Example of an incompatible strain field in 2D :-

$$\varepsilon_{11} = x_2^2 \qquad \varepsilon_{22} = 0 \qquad \varepsilon_{12} = 0$$

so that the strain matrix is :-

$$[\varepsilon] = \begin{bmatrix} x_2^2 & 0 \\ 0 & 0 \end{bmatrix}$$

using $\varepsilon_{11} = \partial u_1 / \partial x_1$

$$\frac{\partial u_1}{\partial x_1} = x_2^2 \Rightarrow u_1 = x_2^2 x_1 + f(x_2)$$

$$\therefore \frac{\partial u_1}{\partial x_2} = 2x_2 x_1 + \frac{df}{dx_2}$$

Next considering $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$

$\Rightarrow \frac{\partial u_2}{\partial x_2} = 0 \Rightarrow u_2 = g(x_1) \Rightarrow u_2$ is a function of x_1

The functions 'f' and 'g' may contain constant terms.

But $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$

$$\frac{\partial u_1}{\partial x_2} = 2x_2 x_1 + \frac{df}{dx_2} \qquad \frac{\partial u_2}{\partial x_1} = \frac{dg}{dx_1}$$

$$\therefore \varepsilon_{12} = \frac{1}{2}\left[2x_2 x_1 + \frac{df}{dx_2} + \frac{dg}{dx_1}\right] = 0$$

However, ε_{12} is given to be 0.

$$\frac{D}{Dt}\int_{\forall} \rho\vec{v}\, d\forall = \int_{\forall} \rho\vec{b}\, d\forall + \int_S \vec{T}\, dS.$$

Reynolds' Transport theorem - Leibnitz thm./rule

$$\frac{D}{Dt}\int_{\forall} \rho\vec{v}\, d\forall \equiv \int_{\forall} \frac{\partial(\rho\vec{v})}{\partial t} d\forall + \int_S (\rho\vec{v})\vec{v}\cdot\hat{n}\, dS$$

through RTT

Reynolds Transport Theorem (RTT)

$$\frac{D}{Dt}\int_{\forall} (\ )\, d\forall = \int_{\forall} \frac{\partial(\ )}{\partial t} d\forall + \int_S (\ )\vec{v}\cdot\hat{n}\, dS.$$

Gauss' Divergence Theorem :- $\int_S (\rho\vec{v})\vec{v}\cdot\hat{n}\, dS$

$$= \int_{\forall} \nabla\cdot\{(\rho\vec{v})\otimes\vec{v}\}\, d\forall$$

$$\therefore \frac{D}{Dt}\int_{\forall} \rho\vec{v}\, d\forall = \int_{\forall} \frac{\partial}{\partial t}(\rho\vec{v})\, d\forall + \int_{\forall} \nabla\cdot\{(\rho\vec{v})\otimes\vec{v}\}\, d\forall$$

$$= \int_{\forall}\left\{ \frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot[(\rho\vec{v})\otimes\vec{v}] \right\} d\forall$$

$$= \int_{\forall}\left\{ \frac{\partial\rho}{\partial t}\vec{v} + \rho\frac{\partial\vec{v}}{\partial t} + \{\nabla\cdot(\rho\vec{v})\}\vec{v} + (\rho\vec{v}\cdot\nabla\vec{v}) \right\} d\forall$$

$$= \int_{\forall}\left\{ \underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})}_{\text{Continuity equation}} \right\}\vec{v} + \rho\left\{ \underbrace{\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}}_{\text{LHS of the Navier Stokes eqn.}} \right\} d\forall$$

By conservation of mass -

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$\frac{\partial v}{\partial t} + \vec{v}\cdot\vec{\nabla v}$ is a representation of the acceleration.

$$= \int_{\forall} \rho\left\{ \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} \right\} d\forall$$

Getting back to the momentum equation -

$$\therefore \frac{D}{Dt}\int_{\forall} \rho\vec{v}\, d\forall = \int_{\forall} \rho\left\{ \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} \right\} d\forall$$

$$\text{or } \int_{\forall} \rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} \right) d\forall = \int_{\forall} \rho\vec{b}\, d\forall + \int_S \vec{T}\, dS.$$

Assuming that $\int_S \vec{T}\, dS = 0$, then the equation can be written as -

$$\int_{\forall} \rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} - \vec{b} \right] d\forall = 0$$

Derivation of the local (differential) form :-

$$\Rightarrow \rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} - \vec{b} \right] = 0$$

01.08.2019.

Law of Equilibrium of Tractions on Small volumes :-

$$\int_{\forall} \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} - \vec{b} \right) d\forall = \int_S \vec{T} \, dS$$

Let $\langle \ \rangle$ represent the average value of the arguments.

$$\sim \left\langle \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} - \vec{b} \right) \right\rangle l^3 \sim \int_S \vec{T} \, dS$$

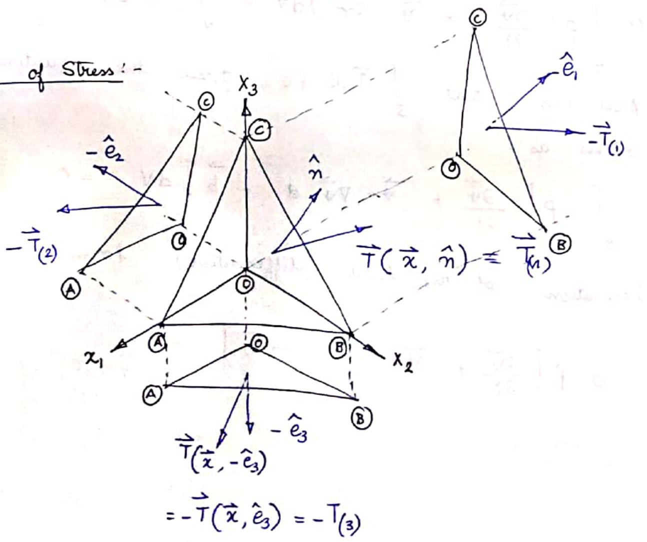
when the characteristic dimension $l \to 0$, LHS $\to 0$

$$\Rightarrow \int_S \vec{T} \, dS \to 0$$

$$\vec{T} := \lim_{A \to 0} \frac{\vec{F}}{A}$$

when the volume is small, the traction forces balance each other even when the body is not in static equilibrium.

State of Stress :-



Applying : $\int_S T \, ds = 0$

$$\vec{T}_{(n)} \, \Delta(ABC) + (-\vec{T}_{(1)}) \, \Delta(OCB) + (-\vec{T}_{(2)}) \, \Delta(OAC) + (-\vec{T}_{(3)}) \, \Delta(OAB) = 0$$

$$\Delta(OCB) = \Delta(ABC) \cdot (\hat{n} \cdot \hat{e}_1) = \Delta(ABC) \, n_1$$

$$\Delta(OAC) = \Delta(ABC) \, (\hat{n} \cdot \hat{e}_2) = \Delta(ABC) \, n_2$$

$$\Delta(OAB) = \Delta(ABC) \, (\hat{n} \cdot \hat{e}_3) = \Delta(ABC) \, n_3$$

$$\therefore \vec{T}_{(n)} \, \Delta(ABC) = \vec{T}_{(1)} \, \Delta(ABC) \, n_1 + \vec{T}_{(2)} \, \Delta(ABC) \, n_2 + \vec{T}_{(3)} \, \Delta ABC \, n_3 .$$

$$\Rightarrow \vec{T}_{(n)} = \vec{T}_{(1)} \, n_1 + \vec{T}_{(2)} \, n_2 + \vec{T}_{(3)} \, n_3 .$$

Considering the i^{th} component of $\vec{T}_{(n)}$:-

$$\vec{T}_{(n)i} = \vec{T}_{(1)i} \, n_1 + \vec{T}_{(2)i} \, n_2 + \vec{T}_{(3)i} \, n_3$$

$$= \vec{T}_{(j)i} \, n_j$$

The stress component $\boxed{\sigma_{ji} = \vec{T}_{(j)i}}$

$$T_{(n)i} = T_{(j)i} \, n_j \quad \text{(OR)} \quad T_{(n)i} = \sigma_{ji} \, n_j$$

$$T_{(n)1} = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \qquad [T] = [\sigma]^T [n]$$

$$T_{(n)2} = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$T_{(n)3} = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 .$$

In the tensor or compact form :-

$$\vec{T} = \underset{\approx}{\sigma}^T \cdot \hat{n}$$

In the matrix notation - $\boxed{[\vec{T}] = [\underset{\approx}{\sigma}]^T [\hat{n}]}$

Conserving the angular momentum gives the symmetry of the stress tensor - in the absence of body couples. $\Rightarrow \underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T$ OR, $\sigma_{ij} = \sigma_{ji}$

06.08.2019.

Balance of Angular momentum :- $\sigma_{ij} = \sigma_{ji}$

Getting back to the balance of linear momentum equation -

$$\int_{\forall} \rho \frac{D\vec{v}}{Dt} \, d\forall = \int_{\forall} \rho \vec{b} \, d\forall + \int_S \vec{T} \, ds$$

If the traction term were $= 0$ then -

$$\int_{\forall} \left(\rho \frac{D\vec{v}}{Dt} - \rho\vec{b}\right) d\forall = 0$$

Since $\forall$ is arbitrary, the integrand

$$\left(\rho \frac{D\vec{v}}{Dt} - \rho\vec{b}\right) = 0$$

For the case when the traction is non zero,

$$\int_{\forall} \left(\rho \frac{D\vec{v}}{Dt} - \rho\vec{b}\right) d\forall = \int_S \vec{T} \, ds$$

$$\Rightarrow \int_{\forall} \left(\rho \frac{D\vec{v}}{Dt} - \rho\vec{b}\right) d\forall = \int_{S} [\sigma]^T \hat{n} \, dS.$$

Gauss' Divergence Theorem :- $\int_S (\,) \cdot \hat{n} \, dS = \int_{\forall} \nabla \cdot (\,) \, d\forall$

$$\Rightarrow \int_{\forall} \left(\rho \frac{D\vec{v}}{Dt} - \rho\vec{b}\right) d\forall = \int_{\forall} \left(\nabla \cdot \underset{\approx}{\sigma}^T\right) d\forall$$

$$\Rightarrow \int_{\forall} \left[\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) - \nabla \cdot \underset{\approx}{\sigma}^T\right] d\forall = 0$$

$$\therefore \left[\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) - \nabla \cdot \sigma^T\right] = 0$$

Since the integral equation is true for all arbitrary volumes, we must have the integrand $= 0$.

or, $\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{b} + \nabla \cdot \sigma^T}$ Cauchy's equation of motion.

For bodies in equilibrium we have $\frac{D\vec{v}}{Dt} = 0$

$\therefore \boxed{\rho\vec{b} + \nabla \cdot \sigma^T = 0}$ Mechanical equilibrium equations. (local)

This equation is valid only for a particular material point within the continuum.

But $\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T$

$$\therefore \rho\vec{b} + \nabla \cdot \underset{\approx}{\sigma} = 0$$

$$\rho b_1 + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

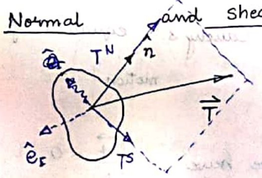
$$\rho b_2 + \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\rho b_3 + \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0$$

Expressed in the index notation -

$$\rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Normal and shear components of traction :-



T^N : normal component of traction (Component of $\vec{T}$ along $\hat{n}$)

T^S : shear component of traction - (Component of $\vec{T}$ which is $\perp$ to $\hat{n}$ AND coplanar with $\vec{T}$ and $\vec{n}$)

Normal Component T^N :-

$$\vec{T}^N = (\vec{T} \cdot \hat{n})\, \hat{n}$$

where the magnitude is given by -
$T^N = \vec{T} \cdot \vec{n}$

$$T^N = \vec{T} \cdot \hat{n}$$

$$= (\underset{\approx}{\sigma}^T \cdot \hat{n}) \cdot \hat{n}$$

$$\equiv [(\underset{\approx}{\sigma}^T \cdot \hat{n})]^T [\hat{n}]$$

$$\equiv ([\underset{\approx}{\sigma}]^T [\hat{n}])^T [\hat{n}]$$

$$\equiv [\hat{n}]^T [\underset{\approx}{\sigma}] [\hat{n}]$$

Shear Component T^S :-

$$(T^S)^2 = |\vec{T}|^2 - (T_N)^2$$

$$\cos\theta = \frac{\vec{T} \cdot \hat{n}}{|T^N|}$$

$$T_s\, \hat{e}_s = \vec{T} - T^N \hat{n}$$

$$= \vec{T} - (\vec{T} \cdot \hat{n})\, \hat{n}$$

Unit vector $\hat{e}_s$ along which T^S lies :-

$$\hat{e}_I = \frac{\vec{T} \times \hat{n}}{|\vec{T} \times \hat{n}|}$$

$$\hat{e}_S = \frac{\hat{n} \times \hat{e}_I}{|\hat{n} \times \hat{e}_I|}$$

For nontrivial solutions we must have

$$\begin{vmatrix} \sigma_{11} - T^N & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - T^N & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - T^N \end{vmatrix} = 0$$

The characteristic equation is given as –

$$(T^N)^3 - I_1 (T^N)^2 + I_2 (T^N) - I_3 = 0$$

the values of T^N serve as the eigenvalues.

where $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

I_1, I_2 and I_3 are known as the stress invariants.

- The eigenvalues of a real, symmetric matrix are always real. (Fact 1)
- Corresponding to 3 distinct eigenvalues, we obtain 3 mutually perpendicular eigenvectors. (Fact 2)

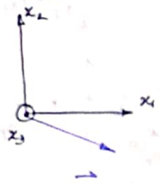
The stress tensor is symmetric and the stress components are always real. Therefore the eigenvalues, i.e., the principal stresses (the 3 roots of T^N) are all real.

When the eigenvalues are distinct then the eigenvectors are perpendicular to each other. Thus if the 3 roots of T^N are p_1, p_2, p_3 and the eigenvectors corresponding to p_1, p_2 and p_3 are respectively $\hat{n}_p^{(1)}$, $\hat{n}_p^{(2)}$, $\hat{n}_p^{(3)}$ then

$$\hat{n}_p^{(1)} \cdot \hat{n}_p^{(2)} = 0 \; ; \; \hat{n}_p^{(1)} \cdot \hat{n}_p^{(3)} = 0$$

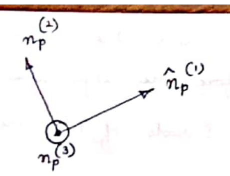
$$\hat{n}_p^{(2)} \cdot \hat{n}_p^{(3)} = 0$$

Following from Fact 2, we can choose to set up a coordinate frame such that the coordinate axes are along the 3 mutually perpendicular eigenvectors – $\hat{n}_p^{(1)}$, $\hat{n}_p^{(2)}$, $\hat{n}_p^{(3)}$



$\vec{T}_{(1)}$ –

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ – components of $\vec{T}_{(1)}$ along $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$



$\vec{T}_{(P_1)} \rightarrow \sigma_{11} = P_1$

$\sigma_{12} = 0$

$\sigma_{13} = 0$

$\vec{T}_{(P_2)} \rightarrow \sigma_{12} = 0$

$\sigma_{22} = P_2$

$\sigma_{23} = 0$

$\vec{T}_{(P_3)} \rightarrow \sigma_{13} = 0$

$\sigma_{23} = 0$

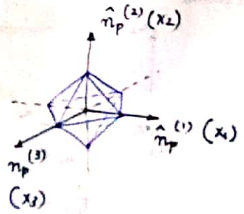
$\sigma_{33} = P_3$

so that the stress tensor becomes -

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}$$ Known as the State of Stress referred to the principal directions.

Octahedral Stresses :-

Referring to the state of stress corresponding to coordinate axes along the principal directions, consider a special plane which is equally inclined to the 3 axes (such that the 3 axes are along the principal directions)



The unit outward normal to this plane will have components (along the 3 axes) which have equal magnitudes.

$$\therefore |n_1| = |n_2| = |n_3|$$

where n_1, n_2, n_3 are the components of the unit outward normal to the special plane.

But as n_1, n_2 and n_3 are the components of a unit outward normal, $\Rightarrow n_1^2 + n_2^2 + n_3^2 = 1$

$$\therefore |n_1| = |n_2| = |n_3| = \frac{1}{\sqrt{3}}$$

For each such plane, consider the normal component of traction.

$$T^N = \vec{T} \cdot \hat{n} \equiv [\vec{T}]^T [\hat{n}] = [\hat{n}]^T [\sigma] [\hat{n}]$$

$$= [|n_1| \quad |n_2| \quad |n_3|] \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= [n_1 \quad n_2 \quad n_3] \begin{bmatrix} P_1 n_1 \\ P_2 n_2 \\ P_3 n_3 \end{bmatrix}$$

$$= P_1 n_1^2 + P_2 n_2^2 + P_3 n_3^2$$

$$= \underline{P_1 + P_2 + P_3}$$

For each such plane consider the shear component T^S :-

$$(T^S)^2 = |\vec{T}|^2 - (T^N)^2$$

$$\vec{T} = [\underset{\approx}{\sigma}][\hat{n}]$$

$$= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} p_1 n_1 \\ p_2 n_2 \\ p_3 n_3 \end{bmatrix}$$

$$(T^S)^2 = \left[(p_1 n_1)^2 + (p_2 n_2)^2 + (p_3 n_3)^2 \right] - \left[p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2 \right]^2$$

$$= (p_1 n_1)^2 + (p_2 n_2)^2 + (p_3 n_3)^2 - \left(p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2 \right)^2$$

13.08.2019

Mohr's Circle for 3D State of Stress :-

Referring to coordinate axes aligned along the principal directions -

$$|\vec{T}|^2 = (T^N)^2 + (T^S)^2$$

Then the state of stress is given by $\sigma = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$

$$|\vec{T}|^2 = (p_1 n_1)^2 + (p_2 n_2)^2 + (p_3 n_3)^2 \quad \text{--- (1)}$$

$$T^N = [\hat{n}]^T [\underset{\approx}{\sigma}][\hat{n}] \quad \text{--- (2)}$$

$$= p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2$$

since $\hat{n}$ is a unit vector :- — (3)

$$1 = n_1^2 + n_2^2 + n_3^2$$

Equations (1), (2) and (3) may be viewed as simultaneous equation involving the variables n_1^2, n_2^2, n_3^2.

Solving for n_1^2, n_2^2, n_3^2, we obtain :

$$n_1^2 = \frac{(T^N - p_2)(T^N - p_3) + (T^S)^2}{(p_1 - p_2)(p_1 - p_3)} \quad \text{--- (4)}$$

$$n_2^2 = \frac{(T^N - p_1)(T^N - p_3) + (T^S)^2}{(p_2 - p_1)(p_2 - p_3)} \quad \text{--- (5)}$$

$$n_3^2 = \frac{(T^N - p_1)(T^N - p_2) + (T^S)^2}{(p_3 - p_1)(p_3 - p_2)} \quad \text{——} ⑥$$

Let the principal values be arranged such that

$\boxed{p_1 > p_2 > p_3}$ (in descending order)

Now, n_1^2, n_2^2 and $n_3^2 \geq 0$

For ④ :- $(p_1 - p_2)(p_1 - p_3) \geq 0$

$$\Rightarrow (T^N - p_2)(T^N - p_3) + (T^S)^2 \geq 0$$

For ⑤ :- $(p_2 - p_1)(p_2 - p_3) \leq 0$

$$\Rightarrow (T^N - p_3)(T^N - p_1) + (T^S)^2 \leq 0$$

For ⑥ :- $(p_3 - p_1)(p_2 - p_1) > 0$

$$\Rightarrow (T^N - p_1)(T^N - p_2) + (T_S)^2 \geq 0$$

Considering the first inequality :

$$(T^N - p_2)(T^N - p_3) + (T^S)^2 \geq 0$$

$$\Rightarrow (T^N)^2 - (p_2 + p_3) T^N + p_2 p_3 + (T^S)^2 \geq 0$$

$$\Rightarrow (T^N)^2 - 2 \times \left(\frac{p_2 + p_3}{2}\right) T^N + \frac{(p_2 + p_3)^2}{4} - \frac{(p_2 + p_3)^2}{4} + p_2 p_3 + (T^S)^2 \geq 0$$

$$\Rightarrow \left[T^N - \frac{(p_2 + p_3)}{2}\right]^2 + (T^S)^2 \geq \frac{(p_2 + p_3)^2}{4} - p_2 p_3$$

$$\Rightarrow \left[T^N - \frac{(p_2 + p_3)}{2}\right]^2 + (T^S)^2 \geq \left(\frac{p_2 - p_3}{2}\right)^2$$

$\therefore$ From the 1^{st} inequality, we obtain -

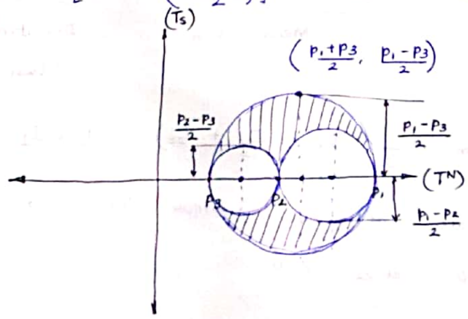
$$\left[T^N - \left(\frac{p_2 + p_3}{2}\right)\right]^2 + (T^S)^2 \geq \left(\frac{p_2 - p_3}{2}\right)^2$$

From 2^{nd} inequality.

$$\left[T^N - \left(\frac{p_1 + p_3}{2}\right)\right]^2 + (T^S)^2 \leq \left(\frac{p_1 - p_3}{2}\right)^2$$

From 3^{rd} inequality,

$$\left[T^N - \left(\frac{p_1 + p_2}{2}\right)\right]^2 + (T^S)^2 \geq \left(\frac{p_1 - p_2}{2}\right)^2$$



This is the maximum value of shear component of traction, T^S (provided that $p_1 > p_2 > p_3$)

State of Pure Shear :-

The stress tensor has the form $\sigma = \begin{bmatrix} 0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix}$

The stress invariants will be -

$I_1 = 0$ however, as I_1 is invariant, σ expressed in any coordinate frame must have $I_1 = 0$

So, if we consider on the octahedral plane, the normal component of the traction on that plane

$$\sigma_{oct}^{N} = \frac{1}{3}(p_1 + p_2 + p_3)$$

$$= 0 \quad (\text{for a pure state of shear})$$

Mean and Deviatoric Stress : Tensors.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{Mean Stress}} + \underbrace{\begin{bmatrix} \sigma_{xx}-\sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_m & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_m \end{bmatrix}}_{\text{Deviatoric Stress}}$$

where $\sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3} \times I_1$

For a state of pure shear, the mean stress $\sigma_m = 0$ since $I_1 = 0$

18.08.2019

MATERIAL BEHAVIOUR - STRESS : STRAIN RELATIONS :-

Unknown Quantities :-

- Density : ρ — ①
- Displacements : $\vec{u}$ — ③
- Strains : $[\varepsilon]$ — ⑥
- Stresses : $[\sigma]$ — ⑥

16.

Equations.

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{v}) = 0 \; ; \; \vec{v} = \frac{D\vec{u}}{Dt}$$

Strain - Displacement Relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - 6 \text{ eqns}$$

Mechanical Equilibrium equations in local form -

$$\nabla . \sigma = 0 \Rightarrow \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad - 3 \text{ eqns}$$

Total no. of available equations = 10.

The six missing equations are given by the Stress - strain relations.

Now, stress - strain relations are usually expressed in the form that -

$$\underset{\approx}{\sigma} = f(\underset{\approx}{\varepsilon})$$

A ssumption which we will use -

- Linearity : 1^{st} powers of ε are permitted on the RHS
- No rate or history dependence.
- Isotropy - properties are same in all directions.
- Uniformity - also known as homogeneity.

Using the assumption of linearity, we may write -

$$\boxed{\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}} \quad \text{—— (1)}$$

where C_{ijkl} is a 4^{th} order tensor, which consists of 81 constants.

Interchange i and j in (1) -

Symmetry of the Stress Tensor -

$$\sigma_{ji} = C_{jikl}\,\varepsilon_{kl}$$

$$\Rightarrow \sigma_{ij} = C_{jikl}\,\varepsilon_{kl}$$

$$\Rightarrow C_{ijkl}\,\varepsilon_{kl} = C_{jikl}\,\varepsilon_{kl}$$

$$\Rightarrow 0 = (C_{ijkl} - C_{jikl})\,\varepsilon_{kl}$$

From which we may conclude, $C_{ijkl} = C_{jikl}$

($\because \varepsilon_{kl} \neq 0$ in general)

The number of independent components reduce from $(3\times3\times3\times3 =)\ 81$ to $(3\times2\times3\times3) = 54$

Symmetry of the Strain Tensor -

Interchanging k and l in (1), we obtain -

$$\sigma_{ij} = C_{ijlk}\,\varepsilon_{lk}$$

But $\varepsilon_{lk} = \varepsilon_{kl}$,

$$\sigma_{ij} = C_{ijlk}\,\varepsilon_{kl} \quad \text{—— (3)}$$

$\therefore$ (1) − (3)

$$\Rightarrow 0 = (C_{ijkl} - C_{ijlk})\,\varepsilon_{kl}$$

since $\varepsilon_{kl} \neq 0$ we have -

$$C_{ijkl} = C_{ijlk}$$

$\therefore$ k and l can be interchanged without changing the value of C_{ijkl}, the number of independent components reduce from $(3\times2\times3\times3) = 54$ to $(3\times2\times3\times2 =)\ 36$.

Voigt Notation -

$\underset{\sim}{11} \rightarrow 1$; $\underset{\sim}{22} \rightarrow 2$; $\underset{\sim}{33} \rightarrow 3$;

$\underset{\sim}{23} \rightarrow 4$; $\underset{\sim}{13} \rightarrow 5$; $\underset{\sim}{12} \rightarrow 6$

so that

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \text{ and } \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

The $\{\sigma\}$ and $\{\varepsilon\}$ column matrices (6×1) can be constructed through a 6×6 matrix consisting of the 36 independent components of C_{ijkl}.

$\bar{C}$ represents the 4^{th} order C matrix expressed in the Voigt notation. so that

$$C_{1122} \longrightarrow \bar{C}_{12}$$
$$C_{1233} \longrightarrow \bar{C}_{63}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} & \bar{C}_{26} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} & \bar{C}_{36} \\ \bar{C}_{41} & \bar{C}_{42} & \bar{C}_{43} & \bar{C}_{44} & \bar{C}_{45} & \bar{C}_{46} \\ \bar{C}_{51} & \bar{C}_{52} & \bar{C}_{53} & \bar{C}_{54} & \bar{C}_{55} & \bar{C}_{56} \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{63} & \bar{C}_{64} & \bar{C}_{65} & \bar{C}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

Minor ~~Major~~ **symmetries :-** $C_{ijkl} = C_{jikl} = C_{ijkl} = C_{ijlk}$

$$u = \sigma_{ij}\,\varepsilon_{ij} = C_{ijkl}\,.\,\varepsilon_{kl}\,.\,\varepsilon_{ij} = (C:\varepsilon):\varepsilon$$

$$\therefore \frac{\partial u}{\partial \varepsilon_{ij}} = C_{ijkl}\,\varepsilon_{kl} \qquad \frac{\partial^2 u}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}} = C_{ijkl}.$$

But $$\frac{\partial^2 u}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}} = \frac{\partial^2 u}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}}. \qquad u = \sigma_{kl}\,\varepsilon_{kl}$$

$$= C_{klij}\,\varepsilon_{ij}\,\varepsilon_{kl}$$

$$\Rightarrow C_{ijkl} = C_{klij} \qquad \frac{\partial u}{\partial \varepsilon_i} = C_{klij} \qquad \frac{\partial^2 u}{} = C_{klij}$$

Major symmetry :-

Strain energy density is a function of strain invariants.

$$\sigma_{ij} = \frac{\partial u_0}{\partial \varepsilon_{ij}}$$

$$\Rightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 u_0}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}}$$

Alternatively, the 1^{st} expression can be written as -

$$\sigma_{kl} = \frac{\partial u_0}{\partial \varepsilon_{kl}}. \Rightarrow \frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} = \frac{\partial^2 u_0}{\partial \varepsilon_{ij}\,\partial \varepsilon_{kl}}$$

in case u_0 has second order continuity in strain, then the order of the differentiation is immaterial.

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$$

We know that, $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

$$\therefore \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl}$$

$$\sigma_{kl} = C_{klij}\,\varepsilon_{ij}$$

$$\frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} = C_{klij}$$

Hence we may conclude - $\boxed{C_{ijkl} = C_{klij}}$ or $\bar{C}_{pq} = \bar{C}_{qp}$

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which proves the symmetry of the whole constitutive tensor, and no. of independent constants reduces from 36 to 21.

A material with 21 independent components $\Rightarrow$ triclinic system.

<u>Result from Tensor Analysis</u> :-

For a general 4th order isotropic tensor, its only representation is -

$$C_{ijkl} = \alpha\,\delta_{ij}\,\delta_{kl} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk} \quad \text{— (4)}$$

From the above expression, it appears that 3 independent coefficients are present.

Interchanging i and j (using minor symmetry) in (4) -

$$C_{jikl} = \alpha\,\delta_{ji}\,\delta_{kl} + \beta\,\delta_{jk}\,\delta_{il} + \gamma\,\delta_{jl}\,\delta_{ik}$$

$$= \alpha\,\delta_{ij}\,\delta_{kl} + \gamma\,\delta_{jl}\,\delta_{ik} + \beta\,\delta_{jk}\,\delta_{il} \quad \text{— (5)}$$

$$[\text{since } \delta_{ij} = \delta_{ji}]$$

using (4) - (5) :-

$$\Rightarrow 0 = (\beta-\gamma)\,\delta_{jl}\,\delta_{ik} + (\gamma-\beta)\,\delta_{jk}\,\delta_{il}$$

$$= (\beta-\gamma)(\delta_{jl}\,\delta_{ik} - \delta_{jk}\,\delta_{il})$$

eg., $i = k = 1$ and $j = l = 2$

$\therefore (\beta-\gamma)[1-0] = 0 \Rightarrow \beta = \gamma$.

Exploiting the second major symmetry that - $C_{ijkl} = C_{ijlk}$, we again conclude that $\beta = \gamma$.

So, in reality there are only two independent coefficients in (4) :-

$$C_{ijkl} = \alpha\,\delta_{ij}\,\delta_{kl} + \beta(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

or, using the Lamé's parameters, λ and $G(\mu)$ -

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) \quad \text{— (6)}$$

General constitutive relation for a linear elastic isotropic solid where λ and G are referred to as the Lamé parameters.

Substituting the expression (6) in the stress-strain relation (1), we obtain -

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\Rightarrow \sigma_{ij} = [\lambda\,\delta_{ij}\,\delta_{kl} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})]\,\varepsilon_{kl}$$

$$= \lambda\,\delta_{ij}\,\delta_{kl}\,\varepsilon_{kl} + \mu(\delta_{ik}\,\delta_{jl}\,\varepsilon_{kl} + \delta_{il}\,\delta_{jk}\,\varepsilon_{kl})$$

$$= \lambda\,\varepsilon_{kk}\,\delta_{ij} + \mu(\varepsilon_{ij} + \varepsilon_{ji})$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + G(\varepsilon_{ij} + \varepsilon_{je})$$

But we know that the strain tensor is symmetric, that is -

$$\boxed{\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \delta \varepsilon_{ij}}$$

The constitutive relation for a linear, elastic, isotropic solid.

$$\therefore \sigma = \lambda(\mathrm{tr}\,\varepsilon) I + 2\mu \varepsilon$$

$$\mathrm{tr}\,\sigma = 3\lambda(\mathrm{tr}\,\varepsilon) + 2\mu \cdot (\mathrm{tr}\,\varepsilon) = (3\lambda + 2\mu)(\mathrm{tr}\,\varepsilon)$$

$$\therefore \sigma_0 = \text{avg stress} = \frac{\mathrm{tr}(\sigma)}{3} = \frac{(3\lambda+2\mu)}{3}(\mathrm{tr}\,\varepsilon)$$

$$\sigma^{dev} = \sigma - \sigma_0 I.$$

19.08.2019.

MATERIAL BEHAVIOUR : STRESS - STRAIN RELATIONS ... Contd.

We know that

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

Coefficient of G -

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

Let $i = k$ and $i \neq l$ or $j \neq l$ and $k \neq j$

$i = j$ and $k = l$; $i \neq k$ and $k \neq j$

So that $i = j$ and $\neq$ $l = k$.

Setting $i = j = 1$; $k = l = 2$

$$C_{1122} = \bar{C}_{12} = \lambda = \bar{C}_{21} = C_{2211} \text{ (by major symmetry)}$$

We can also have -

$$\lambda = C_{1133} = C_{3311} = C_{2233} = C_{3322}$$

$\bar{C}_{11} = \lambda + 2\mu$	$\bar{C}_{12} = \lambda$	$\bar{C}_{13} = \lambda$	$\bar{C}_{14} = 0$	$\bar{C}_{15} = 0$	$\bar{C}_{16} = 0$
$\bar{C}_{21} = \lambda$	$\bar{C}_{22} = \lambda + 2\mu$	$\bar{C}_{23} = \lambda$	$\bar{C}_{24} = 0$	$\bar{C}_{25} = 0$	$\bar{C}_{26} = 0$
$\bar{C}_{31} = \lambda$	$\bar{C}_{32} = \lambda$	$\bar{C}_{33} = \lambda + 2\mu$	$\bar{C}_{34} = 0$	$\bar{C}_{35} = 0$	$\bar{C}_{36} = 0$
$\bar{C}_{41} = 0$	$\bar{C}_{42} = 0$	$\bar{C}_{43} = 0$	$\bar{C}_{44} = \mu$	$\bar{C}_{45} = 0$	$\bar{C}_{55} = 0$
$\bar{C}_{51} = 0$	$\bar{C}_{52} = 0$	$\bar{C}_{53} = 0$	$\bar{C}_{54} = 0$	$\bar{C}_{55} = \mu$	$\bar{C}_{56} = 0$
$\bar{C}_{61} = 0$	$\bar{C}_{62} = 0$	$\bar{C}_{63} = 0$	$\bar{C}_{64} = 0$	$\bar{C}_{65} = 0$	$\bar{C}_{66} = \mu$

So that the entire constitutive matrix will be :-

$$C = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Next, setting $i=j=k=l=1$,

$$C_{1111} = \lambda + G(1+1)$$

$$\Rightarrow G = \frac{1}{2}(C_{1111} - \lambda)$$

$$= \frac{1}{2}(C_{1111} - C_{1122}) = \frac{1}{2}(\bar{C}_{11} - \bar{C}_{12})$$

But we could have alternatively arrived at -

$$G = \frac{1}{2}(\bar{C}_{22} - \bar{C}_{12}) = \frac{1}{2}(\bar{C}_{33} - \bar{C}_{12})$$

Furthermore, λ has various alternative expressions. So, overall, various alternatives for G will be possible.

Set $i=j=p$ (say)

$$\therefore \sigma_{pp} = \lambda\, \varepsilon_{kk}\, \delta_{pp} + 2G\, \varepsilon_{pp}.$$

But $\delta_{pp} = 3$.

$$\therefore \sigma_{pp} = 3\lambda\, \varepsilon_{kk} + 2G\, \varepsilon_{pp} = (3\lambda + 2G)\, \varepsilon_{kk}$$

But $\varepsilon_{kk} = \varepsilon_{pp}$.

$$\therefore \varepsilon_{kk} = \frac{\sigma_{pp}}{(3\lambda+2G)}$$

Substituting back into the expression of stress -

$$\sigma_{ij} = \lambda \cdot \frac{\sigma_{pp}}{(3\lambda+2G)}\, \delta_{ij} + 2G\, \varepsilon_{ij}$$

$$\therefore \varepsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\lambda\, \sigma_{pp}}{(3\lambda+2G)}\, \delta_{ij}\right]$$

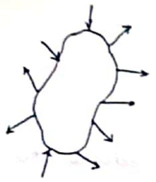
Derive relations between the material constants.

DISCUSSION ON BOUNDARY CONDITIONS :-

Overall, till now we have found the 16 equations corresponding to the 16 unknowns (ρ, $\vec{u}$, $\underset{\approx}{\varepsilon}$, $\underset{\approx}{\sigma}$). The system is closed. However, this system of equations can be applied to solve for $\vec{u}$, $\underset{\approx}{\varepsilon}$, $\underset{\approx}{\sigma}$ etc., only when proper boundary conditions are satisfied (specified).

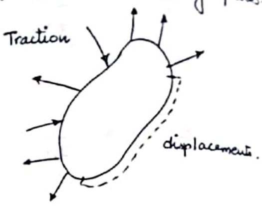
It is extremely important to note that while the equations remain the same for all problems/applications, it is the boundary conditions which vary from problem to problem and gives rise to various solutions.

Boundary conditions may be specified entirely in terms of traction or entirely in terms of displacements over the surface of the body, OR, tractions may be specified over a part (or parts) of the surface and displacements may be specified over the remaining parts.



BCs entirely in terms of tractions.

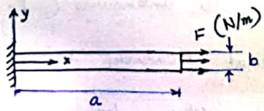
BCs in terms of displacements entirely.

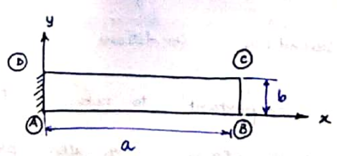


BCs in terms of both Tractions and displacements.

Examples :-

A Cantilever Beam :





Boundary conditions :-

(i) At $x=0$, along AD :- $u=0;\ v=0$

(ii) At $x=a$, along BC :- $t_x = \bar{F}$

We know that :-

$$\vec{T} = \underset{\approx}{\sigma}^T \cdot \hat{n} \equiv \underset{\approx}{\sigma} \cdot \hat{n}$$

where $\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$ and $[\hat{n}] = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \vec{T} = \underset{\approx}{\sigma} \cdot \hat{n} \Rightarrow T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$

$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y$

or

$T_x = \sigma_{xx}$ we know that $T_x = F$

$T_y = \sigma_{xy}$

$$\boxed{\begin{matrix} \therefore \sigma_{xx} = F \\ \sigma_{xy} = 0 \end{matrix}}$$ Required boundary condition.

(iii) At $y=0$, along line AB - (traction-free boundary)

$\vec{T} = \sigma \cdot n$ so that $T_x = \sigma_{xx} n_x + \sigma_{xy} n_y = -\sigma_{xy}$

$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y = -\sigma_{yy}$

But for this surface, $n_x = 0$ and $n_y = -1$

$$\therefore \boxed{\begin{matrix} -\sigma_{xy} = 0 \\ -\sigma_{yy} = 0 \end{matrix}}$$

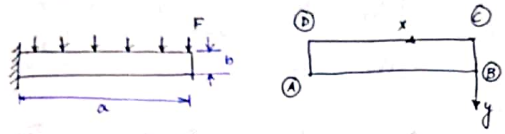
(iv) At $y=b$, along the line CD -

$\vec{T} = \sigma \cdot n \Rightarrow$ $T_x = \sigma_{xx} n_x + \sigma_{xy} n_y = \sigma_{xy}$

$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y = \sigma_{yy}$

$\therefore$ $\boxed{\begin{matrix} \sigma_{xy} = 0 \\ \sigma_{yy} = 0 \end{matrix}}$ are the required boundary conditions.

#



(i) At $x = a$; that is, along AD : $u = 0$; $v = 0$

(ii) At $y = 0$, i.e, along CD -

$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$

$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y$

$\vec{T} = \begin{pmatrix} 0 \\ +F \end{pmatrix}$ and $\hat{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\therefore$ $T_x = -\sigma_{xy}$

$T_y = -\sigma_{yy}$

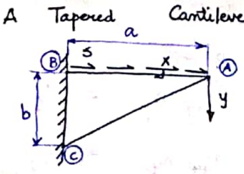
$\therefore$ $\boxed{\begin{matrix} -\sigma_{xy} = 0 \\ -\sigma_{yy} = +F \end{matrix}}$ OR $\begin{matrix} \sigma_{xy} = 0 \\ \sigma_{yy} = -F \end{matrix}$

20.08.2019.

BOUNDARY CONDITIONS ... Contd.

Examples -

A Tapered Cantilever Beam :-



where S is the uniformly distributed shear along AB -

(i) At $x = a$, that is, along BC - $u = 0$, $v = 0$

(ii) At $y = 0$, that is, along AB -

$\vec{T} = \underset{\approx}{\sigma}^T \cdot \hat{n} \equiv \underset{\approx}{\sigma} \cdot \hat{n}$ and

$[\hat{n}] = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

We know that -

$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$

$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y$

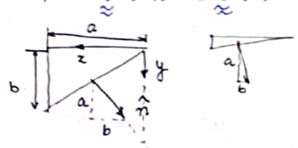
$\therefore$ $-S = -\sigma_{xy}$

$0 = -\sigma_{yy}$

$\Rightarrow$ $\boxed{\begin{matrix} -\sigma_{xy} = -S \\ -\sigma_{yy} = 0 \end{matrix}}$

(iii) Along AC, that is :- along $y = \frac{b}{a} x$:

Traction along AC $= 0$ then $\vec{T} = \underset{\approx}{\sigma}^T \cdot \hat{n} \equiv \underset{\approx}{\sigma} \cdot \hat{n}$



$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$

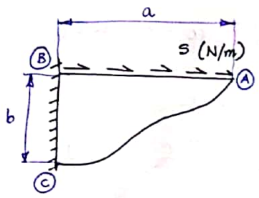
$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y$

We know that $\hat{n} = \begin{pmatrix} \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$

So that $\vec{n} = -b\hat{i} + a\hat{j}$

$\therefore$ $\hat{n} = -\frac{b}{\sqrt{a^2+b^2}}\hat{i} + \frac{a}{\sqrt{a^2+b^2}}\hat{j}$

Then the boundary conditions can be assessed accordingly. $= \begin{pmatrix} -b/\sqrt{a^2+b^2} \\ a/\sqrt{a^2+b^2} \end{pmatrix}$



Along AC, the values of n_x and n_y will keep changing.

So even though both the equations -

$$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$$

$$T_y = \sigma_{xy} n_x + \sigma_{yy} n_y$$

are applicable, but we have to consider the values at each point along AC locally.

Boundary conditions within a body :-

(A) Perfectly Bonded Interface -

$(u^n)^{(1)} = (u^n)^{(2)}$ [displacements in the normal direction]

$(u^s)^{(1)} = (u^s)^{(2)}$ [displacements in the shear directions]

$(T^n)^{(1)} = (T^n)^{(2)}$ [normal local component of traction]

$(T^s)^{(1)} = (T^s)^{(2)}$ [shear local component of traction]

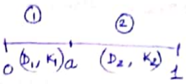
Let us consider a differential equation -

$$\frac{d^2u}{dx^2} = K \quad \text{where} \quad x \in [0,1]$$

We have 2 boundary conditions :- $u(0) = u_L$, $u(1) = u_R$

If the equation has the form :-

$$D\frac{d^2u}{dx^2} = K,$$



In $0 \le x \le a$ (within the domain)

$$D_1 \frac{d^2u_1}{dx^2} = K_1$$

and within $a \le x \le 1$: $D_2 \dfrac{d^2u_2}{dx^2} = K_2$

At $x = 0$, $u_1 = u_L$ At $x = 1$, $u_2 = u_R$

At $x = a$, $u_1 = u_2$ and $D_1 \left.\dfrac{du_1}{dx}\right|_{x=a} = D_2 \left.\dfrac{du_2}{dx}\right|_{x=a}$

Displacement Formulation :-

- Mechanical equilibrium equations $\longrightarrow$ 3

These involve the various stress components

- Stress - Strain relations $\longrightarrow$ 6 $\quad \underset{\approx}{\sigma} := f(\underset{\approx}{\varepsilon})$
- Strain displacement relations $\longrightarrow$ 6 $\quad \underset{\approx}{\varepsilon} := g(\vec{u})$

strain disp. $\longrightarrow$ stress strain relations $\Rightarrow$ $\underset{\approx}{\sigma} := h(\vec{u})$

stress - strain $\longrightarrow$ mechanical eqm $:\Rightarrow$ The mechanical equilibrium equations expressed in terms of $\vec{u}$ components.

(The entire problem can be expressed in terms of disps)

Stress Formulation :-

- Mechanical equilibrium equations expressed in terms of stresses $\longrightarrow$ 3

+

stress - strain relations

+

strain displacement relations.

The 3 compatibility equations are imposed over the mechanical equilibrium equations.

22.08.2019.

2D ELASTICITY :-

Assumptions of Two dimensional elasticity :-

Domain confined between two parallel plates.nes

Distance between the parallel planes is very small compared to the other dimensions of the domain.

Referring to a rectangular Cartesian coordinate system, the mid-surface between the parallel planes is taken to be (conventionally) the xy plane. The top surface is at $z = +h$ and the bottom surface is at $z = -h$.

The top and bottom surfaces are taken to be stress free. Thus,

$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. or $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$

Because the thickness is very small, the stresses σ_{zz}, σ_{zy}, σ_{zx} are assumed not to deviate from 0 value inside the domain.

Again, because the thickness is very small, the stress components : σ_{xx}, σ_{xy}, σ_{yy} are assumed to not vary with z. Thus,

$$\sigma_{xx} \equiv \sigma_{xx}(x,y)$$

$$\sigma_{xy} \equiv \sigma_{xy}(x,y)$$

$$\sigma_{yy} \equiv \sigma_{yy}(x,y)$$

Stress - Strain Relations :-

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \quad \text{--- } ①$$

But $\sigma_{zz} = 0$ so $\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] = \varepsilon_{xx}(x,y)$

Similarly -

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \quad \text{--- } ② \quad = \varepsilon_{yy}(x,y)$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad \text{--- } ③$$

$$= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

[so that even if $\sigma_{zz} = 0$, $\varepsilon_{zz} \neq 0$]

$$\varepsilon_{xy} = \frac{1}{2G} \times \sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} \equiv \varepsilon_{xy}(x,y) \quad \text{— (4)}$$

$\varepsilon_{yz} = 0$ — (5) $\quad \varepsilon_{zx} = 0$ — (6) $\quad$ since $\sigma_{yz} = \sigma_{zx} = 0$

Compatibility Equations :-

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \text{— (7)}$$

$$\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z^2}}^{\,0} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\cancel{\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}}^{\,0} \quad \text{— (8)}$$

$$\Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 0$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \quad \text{— (9)}$$

$$\Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 0$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \quad \text{— (10)}$$

But $\frac{\partial \varepsilon_{xx}}{\partial z} = 0 \Rightarrow \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = 0$

$\Rightarrow 0 = 0$ This compatibility equation yields redundant information.

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \Rightarrow 0 = 0 \quad \text{— (11)}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right) \quad \text{— (12)}$$

$$\Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = 0$$

Hence the important equations are -

- $\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$
- $\frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 0$
- $\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 0$
- $\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = 0$

29.08.2019. 2D ELASTICITY ... Continued.

PLANE STRESS :-

Consider the compatibility equation -

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \text{--- (7)}$$

Substituting in the above -

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}) \quad \text{--- (1)}$$

$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}] \quad \text{--- (2)}$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad \text{--- (4)}$$

$$\frac{1}{E}\left(\frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2}\right) + \frac{1}{E}\left(\frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2}\right) = 2\frac{(1+\nu)}{E}\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$

$$\Rightarrow \frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial^2\sigma_{xx}}{\partial x^2} - \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \frac{\nu\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} - \frac{\partial^2\sigma_{yy}}{\partial y^2} = 2(1+\nu)\cdot\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$

$$\Rightarrow \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} - \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2} - \frac{\partial^2\sigma_{yy}}{\partial y^2} = 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$

$$\Rightarrow \nabla^2\sigma_{xx} - (1+\nu)\cdot\frac{\partial^2\sigma_{xx}}{\partial x^2} - (1+\nu)\cdot\frac{\partial^2\sigma_{yy}}{\partial y^2} + \nabla^2\sigma_{yy} = 2(1+\nu)\cdot\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$

$$\Rightarrow \nabla^2(\sigma_{xx} + \sigma_{yy}) = (1+\nu)\left[\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}\right]$$

$$= (1+\nu)\left[\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial\sigma_{xy}}{\partial x\partial y}\right]$$

$$= (1+\nu)\left[\frac{\partial}{\partial x}\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x}\right)\right] \quad \text{--- (11)}$$

Now, considering the mechanical equilibrium equation

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + f_x = 0 \quad \text{--- 12(A)}$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + f_y = 0 \quad \text{--- 12(B)}$$

Taking $\frac{\partial}{\partial x}$ 12(A)

$$\Rightarrow \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial f_x}{\partial x} = 0$$

and $\frac{\partial (12)}{\partial y}$ (B)

$$\Rightarrow \frac{\partial^2 \sigma_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial f_y}{\partial y} = 0$$

Adding these 2 equations -

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = 0$$

$$\Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -(\nabla\cdot\vec{f}) \quad \text{—— (13)}$$

Substituting equation (13) in (11), we obtain -

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)(\nabla\cdot\vec{f}) \quad \text{—— (14)}$$

Consider the mechanical equilibrium equations - once again

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

$$\therefore \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -f_x = \frac{\partial v}{\partial x}$$ where v is a scalar potential.

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = -f_y = \frac{\partial v}{\partial y}$$

A conservative force field can be expressed in terms of a scalar potential -

Next, consider that $\vec{f}$ is conservative, so that $\vec{f}$ can be expressed as the gradient of a scalar potential -

$$\vec{f} = -\nabla \vartheta \longrightarrow \text{scalar potential.}$$

$$f_x = -\frac{\partial v}{\partial x}, \quad f_y = -\frac{\partial v}{\partial y}$$

$\nabla\cdot\vec{v} = 0$ where $\vec{v}$ is the velocity.

$$\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Considering the stream function, $v_x = \frac{\partial \psi}{\partial y}$

$$v_y = -\frac{\partial \psi}{\partial x}$$

If $\sigma_{xx} = \frac{\partial \phi}{\partial y}$ and $\sigma_{xy} = -\frac{\partial \phi}{\partial x}$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \text{(identically)}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

The equilibrium equations can be written as -

$$\frac{\partial}{\partial x}(\sigma_{xx} - V) + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - V) = 0$$

These two equations will be satisfied identically if we take -

$$\sigma_{xx} - V = \frac{\partial^2 \phi}{\partial y^2} \quad \text{— (15a)}; \quad \sigma_{yy} - V = \frac{\partial^2 \phi}{\partial x^2} \quad \text{and — (15b)}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad \text{— (15c)}$$

Substituting 15 (a), (b) and (c) and $\vec{f} = -\nabla V$ in (14) -

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\ \nabla \cdot \vec{f}$$

$$\Rightarrow \nabla^2\left[V + \frac{\partial^2 \phi}{\partial y^2} + V + \frac{\partial^2 \phi}{\partial x^2}\right] = +(1+\nu)\ \nabla^2 V$$

$$\Rightarrow 2\nabla^2 V + \nabla^2(\nabla^2 \phi) = +(1+\nu) \cdot \nabla^2 V .$$

$$\Rightarrow \nabla^2(\nabla^2 \phi) = +(\nu - 1)\ \nabla^2 V .$$

$$= -(1-\nu)\ \nabla^2 V$$

$$\Rightarrow \boxed{\nabla^4 \phi = -(1-\nu)\nabla^2 V}$$

which is a PDE in terms of only one scalar unknown.

This equation represents the Governing Equation for plane stress.

We have defined $f_x = -\frac{\partial V}{\partial x}$

$f_y = -\frac{\partial V}{\partial y}$

If there are no body forces present or if $\nabla^2 V = 0$, then

$$\boxed{\nabla^4 \phi = 0}$$

which is known as the Biharmonic equation.

∇^4 : the biharmonic operator.

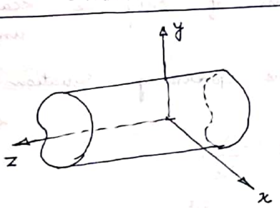
$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

$$= \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} .$$

31.08.2019 2D ELASTICITY ... Continued -

PLANE STRAIN FORMULATIONS : -



\# Infinitely long cylindrical (prismatic) domain.

\# Body forces and tractions on lateral surface will not have any z component along the longitudinal axis

\# $w = 0$ (that is, the displacement in the z direction)

\# $u \equiv u(x,y)$ and $v \equiv v(x,y)$

\# We consider the strain displacement equations

$\rightarrow \varepsilon_{xx} = \frac{\partial u}{\partial x} \equiv \varepsilon_{xx}(x,y)$ $\qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \equiv \varepsilon_{xy}(x,y)$

$\rightarrow \varepsilon_{yy} = \frac{\partial v}{\partial y} \equiv \varepsilon_{yy}(x,y)$ $\qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \equiv 0$

$\rightarrow \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$ $\qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$

Consider the stress - strain relations : -

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \quad \text{—— ①}$$

(in this case, $\sigma_{zz} \neq 0$)

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \quad \text{—— ②}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad \text{—— ③}$$

But we know that $\varepsilon_{zz} = 0$

$$\therefore \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \quad \text{—— ④} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} = 0$$

$$\Rightarrow \boxed{\sigma_{yz} = 0} \quad \text{—— ⑤}$$

$$\varepsilon_{zx} = \frac{(1+\nu)}{E}\sigma_{zx} = 0 \Rightarrow \boxed{\sigma_{zx} = 0} \quad \text{—— ⑥}$$

Substituting equation ③ in ① and ② : -

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left[\sigma_{yy} + \nu(\sigma_{xx} + \sigma_{yy})\right]\right]$$

$$= \frac{1}{E}\left[(1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}\right]$$

$$= \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \quad \text{—— ⑦}$$

Similarly,

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left[\sigma_{xx} + \nu(\sigma_{xx} + \sigma_{yy})\right]\right]$$

$$= \frac{1}{E}\left[(1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}\right]$$

$$= \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \quad \text{—— } (8)$$

We next consider the compatibility equations and implement the same kind of simplifications as we had done back in the PLANE STRESS FORMULATIONS. Ultimately, we end up with only one of those equations which is not $0 = 0$.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \text{—— } (9)$$

Substituting (7), (8), (4) in eqn (9) - we obtain

$$(1-\nu)\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2 \sigma_{yy}}{\partial y^2} + (1-\nu)\frac{\partial^2 \sigma_{xx}}{\partial x^2} - (1-\nu)\frac{\partial^2 \sigma_{xx}}{\partial x^2}$$

$$+ (1-\nu)\frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2 \sigma_{xx}}{\partial x^2} + (1-\nu)\frac{\partial^2 \sigma_{yy}}{\partial y^2} - (1-\nu)\frac{\partial^2 \sigma_{yy}}{\partial y^2}$$

$$= 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow (1-\nu)\nabla^2\sigma_{xx} + (1-\nu)\nabla^2\sigma_{yy} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial^2 \sigma_{xx}}{\partial x^2}$$

$$= 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow (1-\nu)(\nabla^2\sigma_{xx} + \nabla^2\sigma_{yy}) = \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad \text{—— } (10)$$

Now, consider the mechanical equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

Taking derivative of 1^{st} wrt x and 2^{nd} wrt y -

$$\Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial f_x}{\partial x} = 0$$

$$\frac{\partial^2 \sigma_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial f_y}{\partial y} = 0$$

Adding the two above equations, we obtain :-

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = 0$$

$$\Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \quad \text{—— } (11)$$

Substituting (11) in (10), we obtain -

$$(1-\nu)\ \nabla^2(\sigma_{xx} + \sigma_{yy}) = -\nabla\cdot\vec{f} \quad \text{— (12)}$$

02.09.2019. **2D ELASTICITY** ... Continued.

PLANE STRAIN FORMULATION —

$$(1-\nu)\ \nabla^2(\sigma_{xx} + \sigma_{yy}) = -\nabla\cdot\vec{f}$$

~~It~~ Just like the plane stress formulation, $\vec{f}$ is considered conservative so that -

$$\vec{f} = -\nabla v$$

$$\therefore (1-\nu)\ \nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2 v \quad \text{— (13)}$$

Again, consider the mechanical equilibrium equations,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -f_x = \frac{\partial V}{\partial x}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = -f_y = \frac{\partial v}{\partial y}$$

These equations can be identically satisfied by taking

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} + v \quad \text{— (14a)} \qquad \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} + v \quad \text{— (14b)} \qquad \sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \quad \text{— (14c)}$$

Substituting equations 14a, 14b and 14c in (13)

$$\Rightarrow (1-\nu)\cdot \nabla^2\left(\frac{\partial^2 \varphi}{\partial y^2} + v + \frac{\partial^2 \varphi}{\partial x^2} + v\right) = \nabla^2 v.$$

$$\Rightarrow (1-\nu)\cdot \nabla^2(\nabla^2 \varphi + 2v) = \nabla^2 v$$

$$\Rightarrow (1-\nu)\ \nabla^4 \varphi + (2-2\nu)\ v = \nabla^2 v$$

$$\therefore \boxed{\nabla^4 \varphi = -\frac{(1-2\nu)}{(1-\nu)} \nabla^2 v} \quad \text{— (15)}$$

If there are no body forces or they are such that $\nabla^2 v = 0$, then from (15), we obtain -

$$\boxed{\nabla^4 \varphi = 0.}$$

which is the biharmonic equation (exactly the same as in PLANE STRESS)

The function φ is known as the Airy stress function.

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SOLUTION OF THE BIHARMONIC EQUATION :-

$$\nabla^4 \varphi = 0$$

$$\Rightarrow \nabla^2(\nabla^2 \varphi) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = 0$$

$$\Rightarrow \frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0$$

We will seek solutions in the form -

$$\varphi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n$$

$$= A_{00} + A_{10} x + A_{01} y + A_{20} x^2 + A_{11} xy + A_{02} y^2$$

$$+ A_{30} x^3 + A_{31} x^2 y + A_{32} xy^2 + A_{03} y^3 + \cdots$$

For $m+n = 2$:-

$$\varphi = A_{20} x^2 + A_{11} xy + A_{02} y^2$$

which satisfies $\nabla^2 \varphi = 0$ identically.

Then, $\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} = 2A_{02}$ $\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} = -A_{11}$

$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} = 2A_{20}$

For $m+n = 3$:-

$$\varphi = A_{30} x^3 + A_{21} x^2 y + A_{12} xy^2 + A_{03} y^3$$

which satisfies the biharmonic equation identically.

Then $\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} = A_{12} x + 3A_{03} y^2$

$\sigma_{yy} = +\frac{\partial^2 \varphi}{\partial x^2} = +(3A_{30} x^2 + 2A_{21} xy)$

$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} = -(2A_{21} x + A_{12} y)$

For $m+n = 4$:-

$$\varphi = A_{40} x^4 + A_{31} x^3 y + A_{22} x^2 y^2 + A_{13} xy^3 + A_{04} y^4$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 12 A_{40} x^2 + 6A_{31} xy + 2A_{22} y^2$$

$$\frac{\partial^2 \varphi}{\partial y^2} = 2A_{22} x^2 + 6A_{13} xy + 12 A_{04} y^2$$

Then $\psi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}$

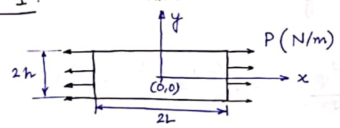
$$= 12 A_{40} x^2 + 6A_{31} xy + 2A_{22} y^2 + 2A_{22} x^2 + 6A_{13} xy + 12 A_{04} y^2.$$

$\frac{\partial^2 \psi}{\partial x^2} = 24 A_{40} + 4A_{22}$ $\frac{\partial^2 \psi}{\partial y^2} = 4A_{22} + 24 A_{04}$

$$\therefore \boxed{24A_{40} + 8A_{22} + 24A_{04} = 0.}$$

The coefficients are constrained by the above equation.

Example 1 :- Uniaxial tension - the thickness along $z = 1$ (assumed)



$\therefore$ the governing equation is : $\nabla^4 \varphi = 0$.

Boundary conditions -

1. along x at $y = \pm h$:-

$$\sigma \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \sigma_{xy} \\ \sigma_{yy} \end{pmatrix} = 0 \qquad \sigma_{xy}(x, \pm h) = 0, \quad \sigma_{yy}(x, \pm h) = 0$$

2. along any y at $x = -L$ and $x = +L$,

$$\sigma_{xx} = P \quad \text{or} \quad \sigma \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix}$$

$$\Rightarrow \sigma_{xx}(\pm l, y) = P \quad \text{and} \quad \sigma_{xy}(\pm l, y) = 0$$

Our motivation in selecting a particular form for φ is that it should give rise to $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$, which satisfy the boundary conditions.

So if we take $\varphi = A_{02} y^2$

then $\sigma_{xx} = 2A_{02}$, $\sigma_{xy} = 0$ and $\sigma_{yy} = 0$

These expressions of σ_{xx}, σ_{xy} and σ_{yy} satisfy the boundary conditions provided that

$$2A_{02} = P$$

$$\Rightarrow A_{02} = \frac{P}{2}$$

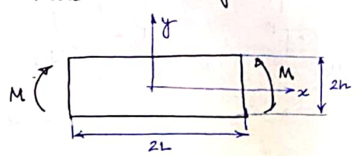
But since $\varphi = A_{02} y^2 = \frac{P}{2} y^2$ also satisfies the biharmonic equation, the expressions of $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ are valid.

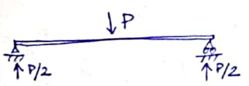
Solution throughout the entire domain -

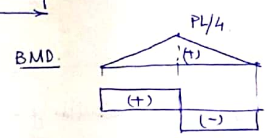
$\therefore$ Throughout the domain, we have

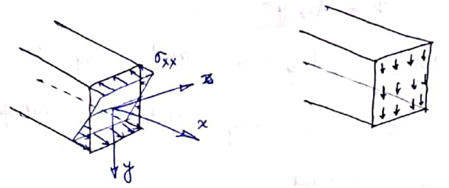
$$\sigma_{xx} = P \; ; \; \sigma_{xy} = 0 \quad \text{and} \quad \sigma_{yy} = 0$$

Example 2 :- Pure Bending -





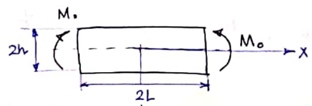




$dF = \tau_{xy}\, dA$.

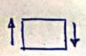
03.09.2019. **2D ELASTICITY** .... Continued.

Example 2 :- Pure Bending ... Contd.

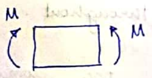


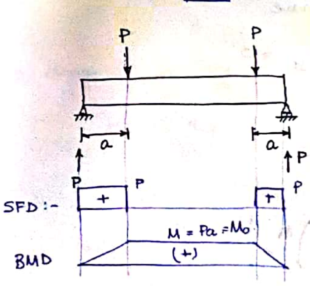
Sign conventions followed -

+ve shear



+ve BM :





Boundary conditions -

1. at $y = +h$, we have -

$$\vec{T} = \begin{bmatrix} \sigma_{xy} \\ \sigma_{yy} \end{bmatrix} = 0$$

$$\Rightarrow \sigma_{xy} = 0$$

$$\sigma_{yy} = 0$$

2. at $y = -h$, similarly - $\vec{T} = \begin{bmatrix} -\sigma_{xy} \\ -\sigma_{yy} \end{bmatrix} = 0 \Rightarrow \begin{matrix} \sigma_{xy} = 0 \\ \sigma_y = 0 \end{matrix}$

$\therefore \sigma_{yy}(x, \pm h) = 0$ and $\sigma_{xy}(x, \pm h) = 0$

$$M_0 = \int_{-h}^{h} \sigma_{xx}\, y\, dA \qquad \text{but} \quad dA = 1 \times dy.$$

The moment arm is y.

$$\therefore M_0 = \int_{-h}^{+h} \sigma_{xx}\, y\, dy$$

$$\int_{-h}^{+h} \sigma_{xx}(\pm l, y)\,(dy \times 1) = 0$$

Let us assume that $\varphi = A_{03}\, y^3$.

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} = 6 A_{03}\, y \qquad \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} = 0$$

$$\int_{-h}^{+h} 6 A_{03}\, y^2\, dy = \frac{6 A_{03}}{3}\, y^3 \Big|_{-h}^{+h} = \frac{6 A_{03}}{3} \times 2h^3 = 4 A_{03} h^3$$

$$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} = 0.$$

But we know that -

$$\int_{-h}^{+h} 6 A_{03}\, y^2\, dy = 4 A_{03} h^3 = M_0.$$

(here we find that as φ is independent of x, σ_{xx} is also independent of x).

$$\boxed{A_{03} = \frac{M_0}{4h^3}.}$$

$\therefore$ the stress function is given by -

$$\varphi = \frac{M_0}{4h^3} \cdot y^3$$

From the above, we know that -

$\sigma_{xx} = 6A_{03}\, y \qquad \sigma_{yy} = 0 \qquad \sigma_{xy} = 0$

To estimate the displacement fields from the stress values -

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy})\right] = \frac{6A_{03}y}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\{0 - \nu\sigma_{xx}) = -\frac{\nu}{E} \times 6A_{03}\,y = -\frac{6\nu A_{03}y}{E}$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy} = 0$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{6A_{03}y}{E}$$

Then $u = \frac{6A_{03}xy}{E} + f(y)$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -\frac{6A_{03}\nu}{E} \cdot y$$

$$\Rightarrow v = -\frac{6A_{03}\nu}{E}\frac{y^2}{2} + g(x) = -\frac{3A_{03}\nu}{E}y^2 + g(x)$$

since $\varepsilon_{xy} = 0 \Rightarrow \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$

$$\frac{\partial u}{\partial y} = \frac{6A_{03}}{E}x + \frac{df}{dy}$$

$$\frac{\partial v}{\partial x} = \frac{dg}{dx}$$

$$\therefore \frac{dg}{dx} + \frac{df}{dy} = -\frac{6A_{03}}{E}x$$

$$\Rightarrow \frac{df}{dy} = -\frac{dg}{dx} - \frac{6A_{03}}{E}x$$

LHS is a pure function of y and RHS = function of x.

$$\therefore \frac{df}{dy} = -\frac{dg}{dx} - \frac{6A_{03}}{E}x = \text{constant}.$$

$$\therefore \frac{df}{dy} = K \Rightarrow f(y) = Ky + C_1.$$

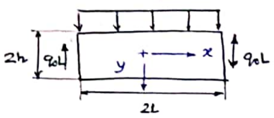
$$\frac{dg}{dx} + \frac{6A_{03}}{E}x = -K.$$

$$\Rightarrow g(x) = -\frac{6A_{03}}{E}.\frac{x^2}{2} - Kx + C_2$$

$$= -\frac{3A_{03}}{E}x^2 - Kx + C_2$$

09.09.2019. **2D ELASTICITY ... Continued.**

Example 3:- Uniform transverse loading on a beam ... continued.



We want to have a 5^{th} degree polynomial

$$\varphi = A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3 + A_{14}xy^4 + A_{05}y^5$$

We remember that

$$A_{50} = -\frac{1}{5}(A_{32} + A_{14})$$

$$A_{05} = -\frac{1}{5}(A_{23} + A_{41})$$

Set $A_{23} \neq 0$. Therefore, $A_{05} = -\frac{1}{5}A_{23}$ (while the rest $=0$)

$$\therefore \varphi = A_{23}x^2y^3 - \frac{1}{5}A_{23}y^5$$

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2} = 6A_{23}x^2y - 4A_{23}y^3$$

$$\sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2} = 2A_{23}y^3$$

$$\sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} = -6A_{23}xy^2$$

Boundary conditions :-

$$\sigma_{yy}(x,+h) = 0 \quad , \quad \sigma_{yy}(x,-h) = -q_0$$

$$\sigma_{xy}(x,\pm h) = 0$$

$$\int_{-h}^{+h}\sigma_{xy}(+l,y)\,dy = -q_0l \quad ; \int_{-h}^{+h}\sigma_{xy}(-l,y)\,dy = q_0l.$$

$$\int_{-h}^{+h}\sigma_{xx}(\pm l,y)\,dy = 0$$

$$\int_{-h}^{+h}y\,\sigma_{xx}(\pm l,y)\,dy = 0$$

(I) We wish to correct the non-zero value of σ_{yy} at $y=+h$ because $\sigma_{yy}(x,+h)=0$ whereas,

$$\sigma_{yy}\Big|_{y=h} = 2A_{23}y^3\Big|_{+h} \neq 0.$$

Superpose on the existing φ another $\varphi = A_{20}x^2$

so that when $\varphi = A_{20}x^2 \Rightarrow \begin{cases}\sigma_{xx} = 0\\ \sigma_{yy} = 2A_{20}\\ \sigma_{xy} = 0\end{cases}$

(II) We wish to correct the non zero value of σ_{xy} at $y=\pm h$ because $\sigma_{xy}(x,\pm h)=0$ whereas $\sigma_{xy} = -6A_{23}xy^2\Big|_{\pm h}$

(which is linear in x) $\rightarrow$ $= -6A_{23}h^2x$

Suppose $\varphi = A_{21} x^2 y$ on the existing φ -

$$\varphi = A_{21} x^2 y \Rightarrow \begin{cases} \sigma_{xx} = 0 \\ \sigma_{yy} = 2A_{21} y \\ \sigma_{xy} = -2A_{21} x \end{cases}$$

(III) We wish to correct the non-zero value of

$\int_{-h}^{+h} y\, \sigma_{xx}(\pm L, y)\, dy$ because our BC gives a 0 value

whereas, $\sigma_{xx}\big|_{\pm L} = 6A_{23} L^2 y - 4A_{23} y^3$ which does not

produce a 0 value for $\int_{-h}^{+h} y\, \sigma_{xx}(\pm L, y)\, dy$

Suppose that $\varphi = A_{03} y^3$ on the existing φ

$$\varphi = A_{03} y^3 \Rightarrow \begin{cases} \sigma_{xx} = 6A_{03} y \\ \sigma_{yy} = 0 \\ \sigma_{xy} = 0 \end{cases}$$

Original φ :-

$$\varphi = A_{23} x^2 y^3 - \frac{1}{5} A_{23} y^5$$

(I) Adding φ : $\varphi = A_{20} x^2$.

(II) Adding φ : $\varphi = A_{21} x^2 y$

(III) Adding φ : $\varphi = A_{03} y^3$

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The final corrections are -

I :- $\varphi = A_{20}x^2$; $\sigma_{yy} = 2A_{20}$

II :- $\varphi = A_{21}x^2 y$; $\sigma_{xy} = -2A_{21}x$; $\sigma_{yy} = 2A_{21}y$

III :- $\varphi = A_{03}y^3$; $\sigma_{xx} = 6A_{03}y$.

The final expression for φ will be -

$$\varphi = A_{23}x^2y^3 - \frac{1}{5}A_{23}y^5 + A_{20}x^2 + A_{21}x^2y + A_{03}y^3.$$

H.W Check with all the boundary conditions -

Corresponding to the final form of φ, we find σ_{xx}, σ_{yy}, σ_{xy}. Then use the boundary conditions to find the values of A_{mn} in terms of q_0, l, h.

Sometimes the final expression for σ_{xx} is expressed in a form including the second moment of area (or, the moment of inertia).

Here. $I = \frac{1}{12} \times 1 \times (2h)^3 = \frac{2h^3}{3}$

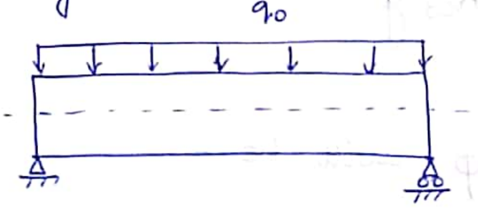
Finally, we obtain

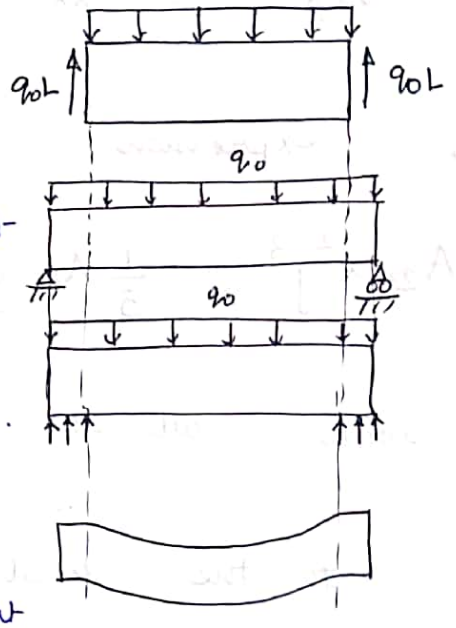
$$\sigma_{xx} = \frac{q_0}{2I}(l^2 - x^2)y + \frac{q_0}{I}\left(\frac{1}{3}y^3 - \frac{1}{5}h^2y\right)$$

where $\frac{q_0(l^2-x^2)}{2}y$ has the form of My

$$\therefore \sigma_{xx} = \frac{My}{I} + \frac{q_0}{I}\left(\frac{1}{3}y^3 - \frac{1}{5}h^2 y\right)$$

where the second term is a correction introduced to the strength of material formula. But since 2h is very small, its contribution is negligible.





stress concentrations appear at the support points A and B. which are usually neglected.

The calculations are not performed at the support. but at a distance slightly away from the supports.

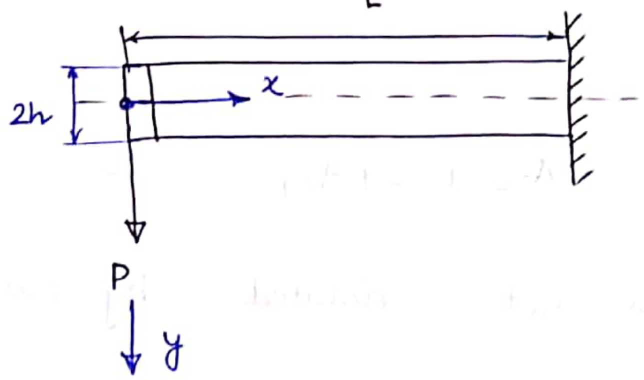
St. Venant approximation - to avoid stress conc. and we do not lose any information about the macroscopic quantities.

Mid Sem Question Paper - 60 marks.

- 2D elasticity will have highest weightage.
-

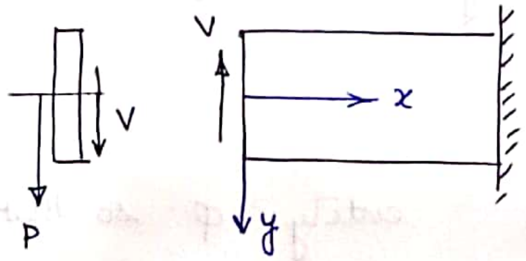
Example 4 : Cantilever Beam under end point loading.

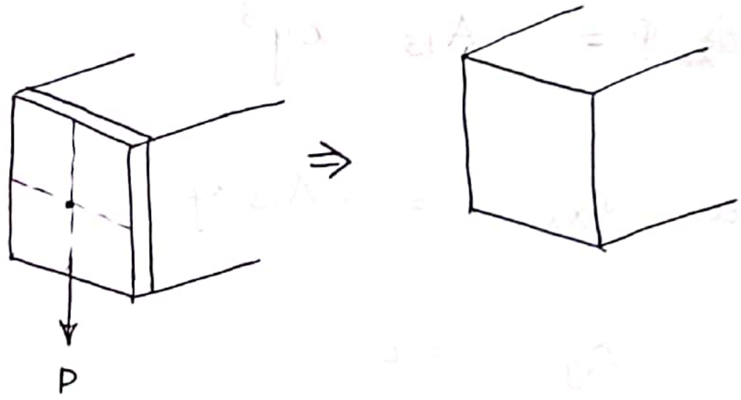
Deflection at the free end $= \dfrac{PL^3}{3EI}$.



Boundary conditions :- $\sigma_{yy}(x, \pm h) = 0$.

$\sigma_{yx}(x, \pm h) = 0$





$$\int_{-h}^{+h} \sigma_{xy}(0, y)\, dy = V = -P. \quad \text{(shear equation)}$$

$$\int_{-h}^{+h} y\, \sigma_{xx}(0, y)\, dy = M. \quad \text{(moment equation)}$$

Taking a 4th degree polynomial, for the Airy stress function φ -

$$\varphi = A_{40} x^4 + A_{31} x^3 y + A_{22} x^2 y^2 + A_{13} x y^3 + A_{04} y^4$$

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We set $A_{13} \neq 0$ while all the rest $= 0$.

But the coefficients follow a relation derived previously as -

$$24 A_{40} + 8 A_{22} + 24 A_{04} = 0$$

This condition is not violated by our specification that $A_{13} \neq 0$ and the rest $= 0$.

$\therefore \varphi = A_{13} \, xy^3$.

Then $\sigma_{xx} = 6 A_{13} xy$. $\sigma_{xy} = -3 A_{13} y^2$

$\sigma_{yy} = 0$

Superpose $\varphi = A_{11} xy$ to the entity φ so that the non-zero value of σ_{xy} at $y = \pm h$ gets corrected.

$$\varphi = A_{11} xy \Rightarrow \begin{cases} \sigma_{xx} = 0 \\ \sigma_{yy} = 0 \\ \sigma_{xy} = -A_{11} \end{cases}$$

$\therefore$ Overall, $\varphi = A_{13} xy^3 + A_{11} xy$

$\therefore \sigma_{xy} = -3 A_{13} y^2 - A_{11}$