(8) Lagrange's Equations (The Lagrange equations of the 2nd kind)

For a holonomic dynamic system having n degrees-of-freedom, there are n number of Lagrange's equations.

These are usually written as:

$$(1) \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} = Q_j \; ; \; j = 1, 2, \cdots, n.$$

Remember

$q_1, q_2, \cdots, q_n$ are the n generalized coordinates used to describe the system configuration at any instant of time.

$\dot{q}_j = \frac{dq_j}{dt}$ is the j^{th} generalized velocity.

($q_1 = q_1(t)$, $q_2 = q_2(t)$ etc. Similarly, $\dot{q}_1 = \dot{q}_1(t)$ etc)

T = System kinetic energy

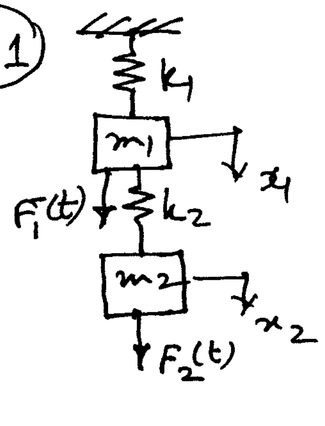
D = The Rayleigh dissipation function

V = System potential energy

Q_j = Generalized force corresponding to the generalized coordinate q_j.

[To arrive at the Lagrange's equations (1), one starts with the general equation of dynamics (See ~~Analy~~ "Lectures in Analytical Mechanics" by F. Gantmacher) which is also called generalized D'Alembert's principle. We are not going for the ~~da~~ derivations of equations 1 at this moment. We shall illustrate the use of 1 through several examples. (Single DOF cases have been ~~too~~ already takenup)

Ex. 1



Obtain the DEOM using Lagrange's equations.

Here we have a 2-DOF system. $x_1(t)$ & $x_2(t)$ are the generalized coordinates. The 2 Lagrange's eqns are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = Q_1 \quad \text{— (1)}$$

& $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = Q_2 \quad \text{— (2)}$$

Since $D = 0$ (no viscous damping). ** Contd on page 3.

[Note:- The Lagrangian function, L is defined as: $L = T - V$. Since V involves conservative forces only which are independent of generalized velocities, $\frac{\partial V}{\partial \dot{x}_1} = 0$ & $\frac{\partial V}{\partial \dot{x}_2} = 0$. Then, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{x}_1}\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}$,

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)$ & $-\frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = -\frac{\partial}{\partial x_j}(T-V)$

$= -\frac{\partial L}{\partial x_j}$ for $j = 1, 2$. Then (1) & (2) can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = Q_1$$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = Q_2$$

Thus, for free vibrations of above system, the Lagrange eqns can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

** Now, $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$,

$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$.

So, $\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$ --- (2)

$\frac{\partial T}{\partial x_1} = 0$ --- (2'), $\frac{\partial V}{\partial x_1} =$ ~~$m_1 x$~~ $k_1 x_1 - k_2(x_2 - x_1)$

$= (k_1 + k_2) x_1 - k_2 x_2$ --- (3)

→ To obtain Q_1, the generalized force associated with generalized coordinate x_1 :—

Here we keep x_2 fixed, give x_1 a virtual variation δx_1 & compute the virtual work δW_{x_1} done by $F_1(t)$.

You may denote δW_{x_1} as δW_1 also

Then, $Q_1 = \frac{\delta W_{x_1}}{\delta x_1}$, by definition.

So, here $\delta W_{x_1} = F_1(t)\, \delta x_1$ & $Q_1 = \frac{\delta W_{x_1}}{\delta x_1} = F_1(t)$. --- (4)

Putting (2), (3), (2') & (4) in (1), we get the DEOM of m_1 as:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t) \quad \text{--- (1')}$$

→ Again, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$, $\frac{\partial T}{\partial x_2} = 0$, $\frac{\partial V}{\partial x_2} = k_2(x_2 - x_1)$.

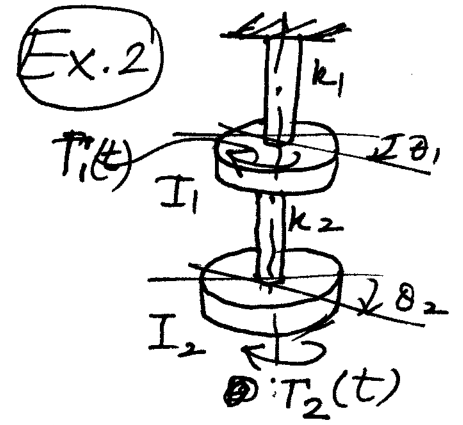
→ To get Q_2, keep x_1 fixed. give m_2 a Virtual displ $= \delta x_2$. Then $\delta W_{x_2} = \delta W_2 = F_2(t) . \delta x_2$ & so,

$Q_2 = \frac{\delta W_2}{\delta x_2} = F_2(t)$. Putting all this in equn ② we get the reqd DEOM as:

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = F_2(t) \quad \text{--- } (2')$$

(1') & (2') are the reqd DEOM.

Ex.2



$T_1(t), T_2(t) \rightarrow$ External torques

Obtain the DEOM using Lagrange's equations. θ_1 & θ_2 are the gen. coordinates.

Here Lagrange's equns are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = Q_1 \quad \text{--- } ①$$

&

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = Q_2 \quad \text{--- } ②$$

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$V = \frac{1}{2} k_1 \theta_1^2 + \frac{1}{2} k_2 (\theta_2 - \theta_1)^2$$

$\delta W_1 = T_1(t) \times \delta\theta_1$ (θ_2 kept constant)

$\Rightarrow Q_1 = \dfrac{\delta W_1}{\delta \theta_1} = T_1(t)$. Similarly, $Q_2 = \dfrac{\delta W_2}{\delta \theta_2} = T_2(t)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = I_1 \ddot{\theta}_1, \quad \frac{\partial T}{\partial \theta_1} = 0, \quad \frac{\partial V}{\partial \theta_1} = k_1 \theta_1 - k_2(\theta_2 - \theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = I_2 \ddot{\theta}_2, \quad \frac{\partial T}{\partial \theta_2} = 0, \quad \frac{\partial V}{\partial \theta_2} = k_2(\theta_2 - \theta_1).$$

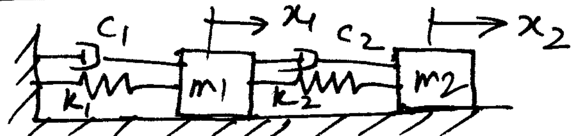
So, the DEOM are:-

$$I_1 \ddot{\theta}_1 + (k_2 + k_2)\theta_1 - k_2 \theta_2 = T_1(t)$$

&

$$I_2 \ddot{\theta}_2 - k_2 \theta_1 + k_2 \theta_2 = T_2(t).$$

Ex.3



Obtain the DEOM using Lagrange's equns

Here the Lagrange equns are:-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0 \quad \&$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = 0$$

(Damped free-vibration)

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$D = \frac{1}{2} c_1 \dot{x}_1^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2$$

Then, $\dfrac{\partial D}{\partial \dot{x}_1} = c_1 \dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) = (c_1 + c_2)\dot{x}_1 - c_2 \dot{x}_2$

& $\dfrac{\partial D}{\partial \dot{x}_2} = c_2(\dot{x}_2 - \dot{x}_1) = -c_2 \dot{x}_1 + c_2 \dot{x}_2$.

Other terms are as in Ex. 1.

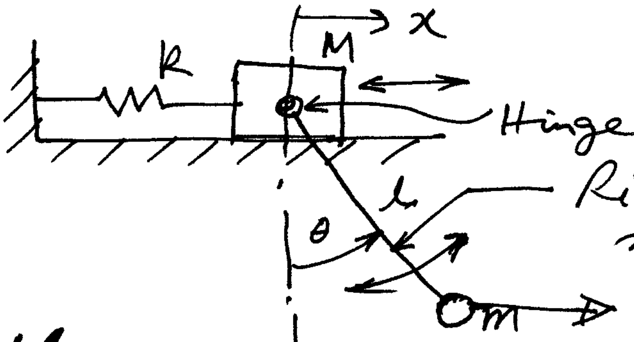
The reqd DEOM are:

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

& $$m_2 \ddot{x}_2 - c_2 \dot{x}_1 + c_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

or, $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Ex. 4 (Important)



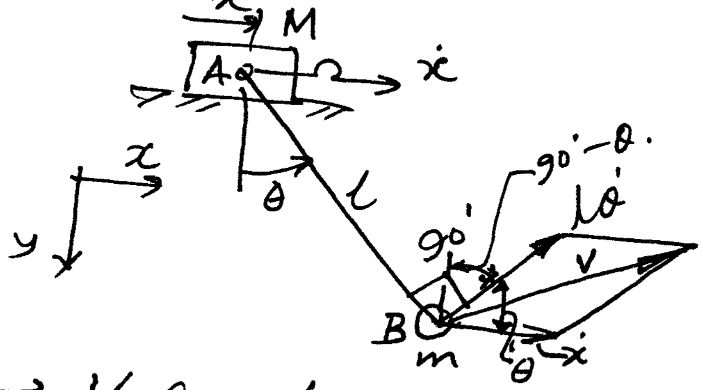
The block translates & the pendulum oscillates. Obtain the nonlinear DEOM using Lagrange's equations. A horizontal force $F(t)$ acts on the bob.

Solution:- Let $x(t)$ & $\theta(t)$ be the generalized coordinates (This is a 2-DOF system). θ, is measured from the vertical, +ve CCW.

The Lagrange equations here are:—

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)-\frac{\partial T}{\partial x}+\frac{\partial V}{\partial x}=Q_1 \quad \text{—— (1)}$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=Q_2 \quad \text{—— (2)}$$

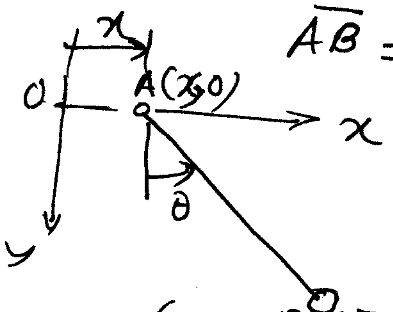


$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}mV^2$
where V = velocity of bob w.r.t. ground reference.

→ V can be obtained in several ways. One of the ways is shown above.

Then, $V^2=\dot{x}^2+(l\dot{\theta})^2+2\dot{x}l\dot{\theta}\cos\theta$

[You can also use $\overline{V_B}=\overline{V_A}+\overline{\omega}_{AB}\times\overline{AB}$
where $\overline{V_A}=\dot{x}\hat{i}$, $\overline{\omega}_{AB}=\dot{\theta}\hat{k}$,
$\overline{AB}=(l\sin\theta)\hat{i}+(l\cos\theta)\hat{j}$ etc.

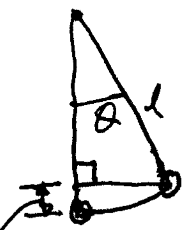


(x_B, y_B) or, $B(x+l\sin\theta,\ l\cos\theta)$

You can also find $V=|\overline{V_B}|$ as follows:— $\overline{V_B}=\dot{x}_B\hat{i}+\dot{y}_B\hat{j}$
where $x_B=x+l\sin\theta$
& $y_B=l\cos\theta$.

So, $V^2=|\overline{V_B}|^2=\dot{x}_B^2+\dot{y}_B^2$ etc.]

Thus, $T=T_{block}+T_{bob}=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m[\dot{x}^2+l^2\dot{\theta}^2+2l\dot{x}\dot{\theta}\cos\theta]$



$l-l\cos\theta$
$=l(1-\cos\theta)$

$V=\frac{1}{2}Kx^2+mgl(1-\cos\theta)$

Thus, $\frac{\partial T}{\partial \dot{x}}=M\dot{x}+\frac{1}{2}m[2\dot{x}+2l\dot{\theta}\cos\theta]$

& $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)=M\ddot{x}+m[\ddot{x}+l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta]$

(Be careful while differentiating - Note that the generalized coordinates & gen. velocities are all independent of each other at this stage of obtaining the DEOM) ($\frac{d}{dt}(\cos\theta) = \frac{d}{d\theta}(\cos\theta) \cdot \frac{d\theta}{dt} = -\dot{\theta}\sin\theta$)

---

or, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\rightarrow \frac{\partial T}{\partial x} = 0, \quad \frac{\partial V}{\partial x} = kx$$

$$\rightarrow \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{\cancel{2}} m\left[\cancel{2}l^2\dot{\theta} + \cancel{2}l\dot{x}\cos\theta\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\left[l^2\ddot{\theta} + l\ddot{x}\cos\theta - l\dot{x}\dot{\theta}\sin\theta\right]$$

[Now comes the interesting term. Till now, the derivatives $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial x_1}$, $\frac{\partial T}{\partial x_2}$, $\frac{\partial T}{\partial \theta}$ used to be zero. Here it is not so, note. This is because T is a function of θ (or, $\cos\theta$)]

$$\rightarrow \frac{\partial T}{\partial \theta} = \frac{1}{2} m\left[\frac{\partial}{\partial \theta}(2l\dot{x}\dot{\theta}\cos\theta)\right] = -ml\dot{x}\dot{\theta}\sin\theta$$

$$\rightarrow \frac{\partial V}{\partial \theta} = mgl\sin\theta$$

Now to obtain Q_1 & Q_2.

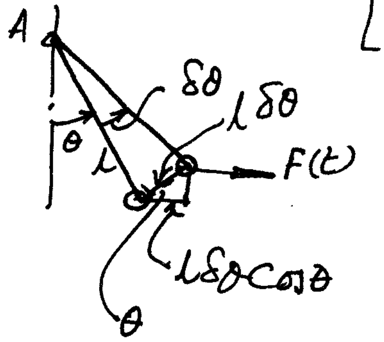
For obtaining Q_1, θ is to be kept constant. The block is given a virtual displacement δx. Then, $\delta W_x = F(t)\delta x$ & so, $\underline{Q_1} = \frac{\delta W_x}{\delta x} = \underline{F(t)}$.

(Note that while the block undergoes the virtual displacement, the bar of the pendulum moves & the bob also undergoes a displacement δx & hence, $F(t)$ does virtual work $= F(t)\,\delta x$.

Have you noticed we take the force to remain constant at $F(t)$ while δx is given? This is so because time 'congeals', that is remains unchanged while the virtual displacement is given. This is a little deep & you need to study Analytical ~~Dynamics~~ Mechanics a bit to properly understand it)

→ To get Q_2, keep x ~~and~~ unchanged, i.e. the block remains fixed at location x and θ is changed by $\delta\theta$ (see fig.)

[Actually $\delta x, \delta\theta$ are infinitesimals. $\delta\theta$ is exaggerated for clarity]



Note that due to the virtual displacement $\delta\theta$, $F(t)$ moves an amount $l\delta\theta\cos\theta$ in the horizontal direction. Hence, $\delta W_\theta = F(t) \times l\cos\theta\,\delta\theta$

So, $$Q_2 = \frac{\delta W_\theta}{\delta\theta} = F(t)\,l\cos\theta.$$

Thus, Q_2 is not a force but the moment of a force. Hence, the name 'generalized force'.

→ We now have obtained everything needed to get the DEOM.

Substitutions in (1) gives:

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + kx = F(t) \quad \text{--(a)}$$

& (2) gives:

$$ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - \cancel{ml\dot{x}\dot{\theta}\sin\theta} + \cancel{ml\dot{x}\dot{\theta}\sin\theta} + mgl\sin\theta = F(t)\,l\cos\theta \quad \text{--(b)}$$

ⓐ & ⓑ are the required nonlinear DEOM and there is no way you can get a closed form analytical solution for these.

→ We can, however, linearize these DEOM. However, to linearize, only $\cos\theta \approx 1$ & $\sin\theta \approx \theta$ (for 'small' θ) won't do! Because, substitution of $\cos\theta = 1$ & $\sin\theta = \theta$ in ⓐ & ⓑ gives:

$$(M+m)\ddot{x} + ml\ddot{\theta} - ml\dot{\theta}^2\theta + kx = F(t) \quad \text{— ⓒ}$$

&

$$ml^2\ddot{\theta} + ml\ddot{x} + mgl\theta = F(t)l \quad \text{— ⓓ}$$

~~Although~~ ⓓ is linear, ⓒ is <u>not</u> due to the term $-ml\dot{\theta}^2\theta$. Hence we must additionally assume that $\dot{\theta}^2\theta$ is negligible & this seems a bit strange, isn't it?

So, the final linearized equation of motion are:

$$(M+m)\ddot{x} + ml\ddot{\theta} + kx = F(t)$$

&

$$\text{ⓔ} \ldots\ldots ml\ddot{x} + ml^2\ddot{\theta} + mgl\theta = F(t)l$$

Check everything

<u>OR</u>

$$\begin{bmatrix} (M+m) & ml \\ ml & ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mgl \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ F(t)l \end{Bmatrix}$$

& note that the inertia matrix is symmetric but non-diagonal but the stiffness matrix is diagonal (& automatically symmetric). These linearized eqns can be solved as before for free vibration & forced vibration response.

→ A WORD OF CAUTION

See DEOM ⓔ, pg. 9:

$$ml\ddot{x} + ml^2\ddot{\theta} + mgl\theta = F(t)l.$$

You might be tempted to reduce this to:

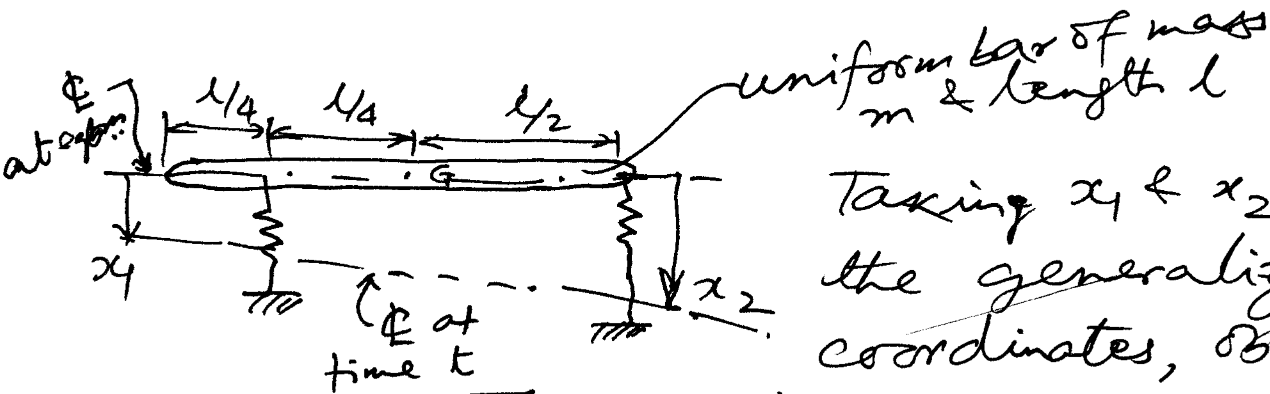
$$m\ddot{x} + ml\ddot{\theta} + mg\theta = F(t).$$

DONOT do this, because, although mathematically it seems ok, doing this would destroy the symmetric nature of the mass matrix & pose some analysis problem subsequently. Also, doing so will obliterate the nature of excitation force (it is the moment of a force) corresponding to gen. coordinate θ.

→ We have spent about 5 pages to do this problem. This is because quite few explanations were in order. You should not take more than 2 pages to complete it.

Example-5:— This example will illustrate how a new set of generalized coordinates result in a non-diagonal mass matrix. →

(11)



Taking x_1 & x_2 as the generalized coordinates, obtain the DEOM. Comment on the nature of [m] & [k].