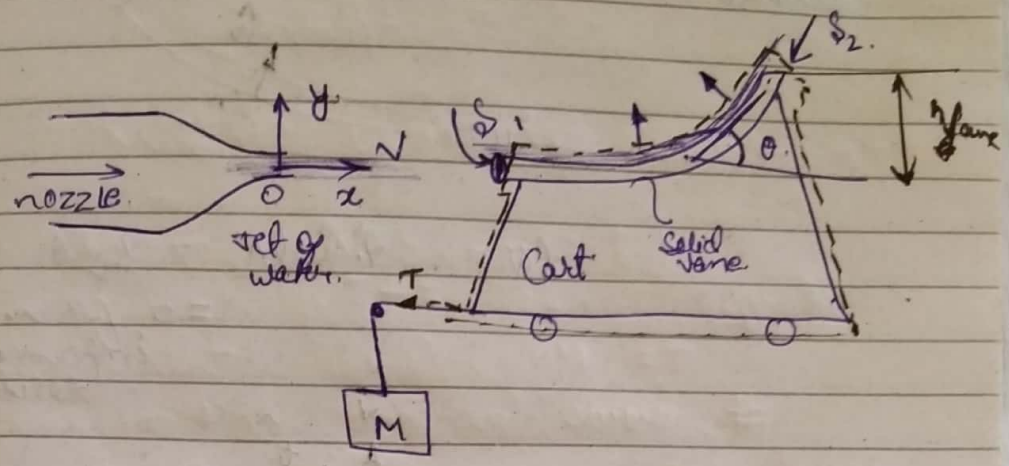
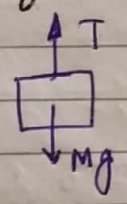
Turbomachines or Rotor-dynamics Machines



- find M for Cart to be stationary.

Assumptions -
- Inviscid flow.

F.B.D of Block:



$M(a_y)_{block} = T - Mg$

$\Rightarrow T = Mg \; (a_y = 0)$

Let $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

OR $= V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

using Reynolds Transport theorem -

Balancing of x - momentum -

$$\int_R \cancel{\frac{\partial}{\partial t}(\rho u)}^{\,0} dV + \int \rho u (V \cdot \hat{n}) dA = \Sigma F_x$$

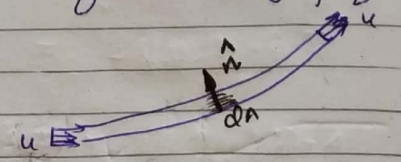
- steady state $\Rightarrow \frac{\partial(\quad)}{\partial t} = 0.$

Now,

$$\int_S \rho u (\vec{V} \cdot \hat{n}) dA = \int_{S_1} \rho u (\vec{V} \cdot \hat{n}) dA + \int_{S_2} \rho u (\vec{V} \cdot \hat{n}) dA$$

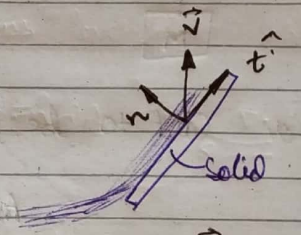
Scanned by CamScanner

• As assumed inviscid flow, velocity profile is constant

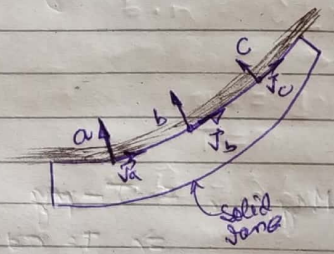


$$dQ = \vec{V}\cdot\hat{n}\,dA = v_n\,dA$$

$= 0$ for an impermeable solid



$$\vec{V} = v_t\hat{t} + \cancel{v_n}^{\,0}\hat{n}$$



Using Bernoulli's eqn.

$$p_e + \frac{1}{2}\rho V_e^2 + \rho g y_e = p_f + \frac{1}{2}\rho V_f^2 + \rho g y_f$$

$p_e = p_f = p_{atm}$

$V_e = V_{jet}$ (given)

$$\frac{1}{2}\rho\left\{(V_2)^2 - (V_{jet})^2\right\} = \rho g(y_e - y_f)$$

$$\frac{1}{2}\rho\left\{\left(\frac{V_2}{V_{jet}}\right)^2 - 1\right\} = -\frac{\rho g h_{vane}}{V_{jet}^2}$$

Scanned by CamScanner

$$\left(\frac{V_2}{V_{jet}}\right)^2 = 1 - \frac{2gh}{V_{jet}^2}$$

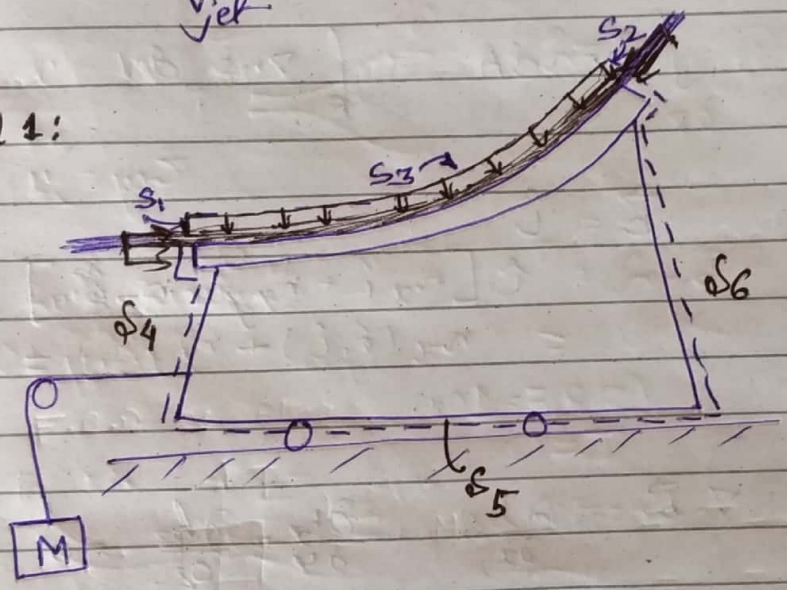
$$\frac{2gh}{V_{jet}^2} = \frac{2\times 10}{100} = \frac{1}{10} = 0.1$$

$$\frac{V_2}{V_{jet}} = \sqrt{1 - \frac{2gh}{V_{jet}^2}}$$

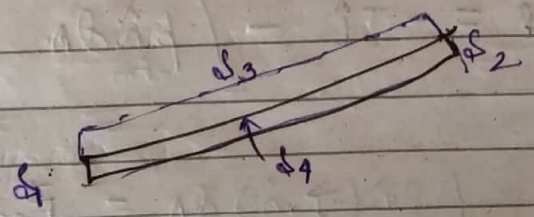
$$= 1 - \frac{1}{2}\times\frac{2gh}{V_{jet}^2} + \cdots$$

$$\frac{V_2}{V_{jet}} \simeq 1$$

Method 1:



Method 2 -

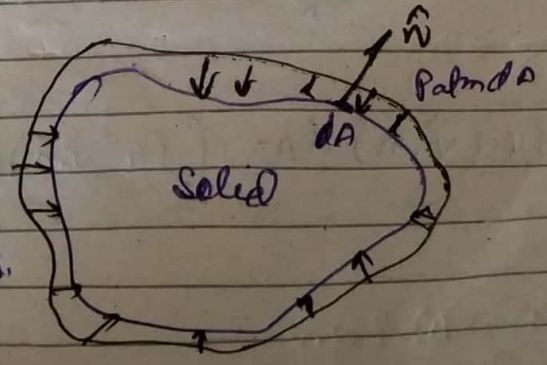


$p_1 = p_{atm}$

$p_2 \simeq p_{atm}$

$$d\vec{F}_{press} = -p\,dA\,\hat{n}$$

$$F_{press} = -\int_S p\,\hat{n}\,dA$$



Scanned by CamScanner

$$F_{press\,uniform} = -\int_S p_{atm}\,\hat{n}\,dA$$

$$= -p_{atm}\int_S \hat{n}\,dA$$

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}.$$

$$\int_S \hat{n} = \int_S n_x\,dA\,\hat{i} + \int_S n_y\,dA\,\hat{j} + \int_S n_z\,dA$$

Gauss divergence theorem.

$$\int_{\partial R} \vec{b}\cdot\hat{n}\,dA = \int_R \nabla\cdot\vec{b}\,dV$$

$$\vec{b} = \hat{i}$$

$$\vec{b}\cdot\hat{n} = \hat{i}\cdot[n_x\hat{i} + n_y\hat{j} + n_z\hat{k}]$$

$$= n_x(\hat{i}\cdot\hat{i}) + n_y(\hat{j}\cdot\hat{i}) + n_z(\hat{i}\cdot\hat{k})$$

$$= n_x.$$

$$\nabla\cdot\vec{b} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} = 0$$

$$b_x = 1,\ b_y = 0,\ b_z = 0$$

Now, $F_{net} = -T\hat{i} - \cancel{\int_S p_{atm}\hat{n}\,dA}^{\,0}$

$$\int_R \rho\,\underbrace{\cancel{\frac{\partial \vec{v}}{\partial t}}}_{0}\,dV + \int_{\partial R} \rho\vec{v}(\vec{v}\cdot\hat{n})\,dA = \vec{F}_{net}$$

$$\int_{S_1} \underset{+}{\rho u}\,\underset{-}{(\vec{v}\cdot\hat{n})}\,dA + \int_{S_2} \underset{+}{\rho u}\,\underset{-}{(\vec{v}\cdot\hat{n})}\,dA + \cancel{\int_{\partial R - S_1 - S_2} \rho u(\vec{v}\cdot\hat{n})\,dA}^{\,0} = -T$$

Scanned by CamScanner

$$-\int_{S_1} \rho u_1^2 \, dA + \int_{S_2} \rho u_2 (\vec{V}\cdot\hat{n})_2 \, dA = -T$$

$$p_1 + \frac{\rho v_1^2}{2} + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$p_1 = p_2 \simeq p_{atm}$$

$$\frac{1}{2}\rho (v_2^2 - v_{jet}^2) = \rho g (y_1 - y_2)$$

$$v_2^2 - v_{jet}^2 = -2g\, h_{vane}$$

Neglect change in potential energy,

$$v_2^2 - v_{jet}^2 \simeq 0$$

$$\Rightarrow v_2 \simeq v_{jet}$$

$$\frac{v_2}{v_{jet}} = \left\{1 - \frac{2gh_{vane}}{v_{jet}}\right\}^{1/2}$$

$$= 1 - \frac{1}{2} \times \frac{2gh_{vane}}{v_{jet}^2} + \cdots$$

$$\simeq 1 \text{ if } \frac{gh_{vane}}{v_{jet}^2} << 1$$

$$\frac{10 \times 0.1}{10^2} = 0.01$$

$$\hat{n}_2 = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$

$$\vec{V}_2 = V_2 \hat{n} \simeq v_{jet}\hat{n}_2$$

$$u_2 = (\vec{V}_2)_x \simeq v_j \cos\theta$$

$$Q = v_1 A_1 = v_2 A_2 = A_{jet} v_{jet}$$

$$x\text{-mom. out} = \int_{S_2} \rho u_2 (\vec{V}\cdot\hat{n})_2 \, dA$$

$$= \int_{S_2} \rho v_{jet} \cos\theta \, v_{jet} \, dA$$

Scanned by CamScanner

$= \rho V_{jet} \cos\theta \, V_{jet} A_2$

$= \rho Q V_{jet} \cos\theta$

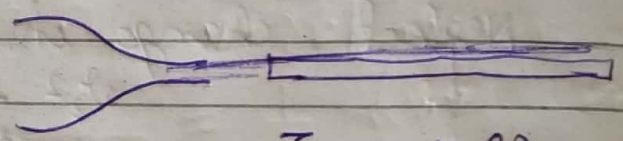
x-momentum eq^n in $= \rho Q V_{jet}$

$\rho Q V_{jet} (\cos\theta - 1) = -T$

$T = Mg = \rho Q V_{jet} (1 - \cos\theta)$

Special case :

① $\theta = 0$



$\tau_{xy} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$

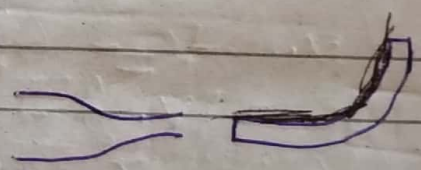
Inviscid fluid ; $\mu = 0$

$\tau_{xy} = 0.$

$\cos\theta = 1$

$T = 0$

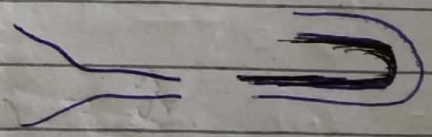
② $\theta = 90°$



$T = \rho Q . V_{jet}$

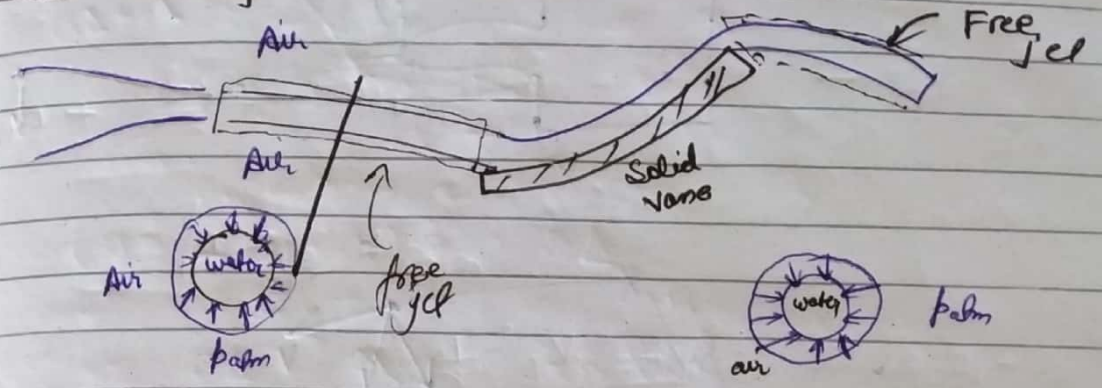
③ $\theta = 180°$

$T = 2\rho Q V_{jet}$

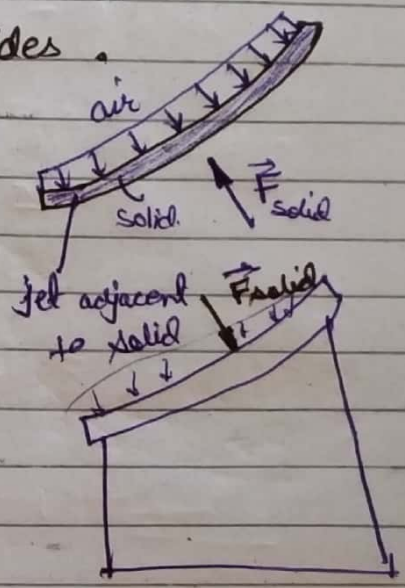


Scanned by CamScanner

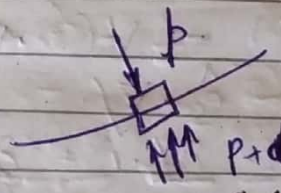
Inviscid Incompressible Model-



Free jet of water surrounded by air on all sides.



Free surface: Interface between liquid & ~~water~~ air



Euler's eqn of motion

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

$$\frac{kg}{m^3} \times \frac{m}{s^2}$$

$$\frac{N}{m^3}$$

$a = \frac{D\vec{V}}{Dt}$ - acceleration of fluid

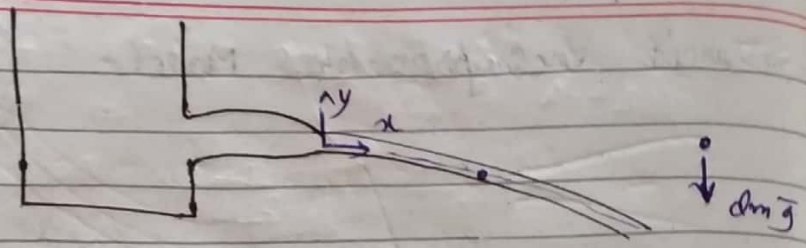
$$dm = \rho dv$$

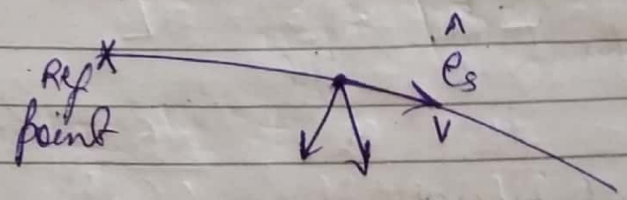
$$(\rho dv)\vec{a} = (\rho dv)\vec{g} - (\nabla p)\, dv$$

Free jet: $p_{liq} \simeq p_{atm}$

$$\nabla p_{liquid} = \frac{\partial p_{liq}}{\partial x}\hat{i} + \frac{\partial p_{liq}}{\partial y}\hat{j} + \frac{\partial p_{liq}}{\partial z}\hat{k}$$

Scanned by CamScanner





$$\vec{V} = V\hat{e}_s$$

$$V = |V| = \sqrt{V_x^2 + V_y^2 + V_z^2} = \sqrt{\vec{V}\cdot\vec{V}}$$

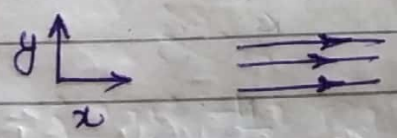
$$\vec{a} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{steady}} + \underbrace{(\vec{V}\cdot\nabla)\vec{V}}_{\text{convective acc}^n}$$

$$(\vec{V}\cdot\nabla) = V(\hat{e}_s\cdot\nabla)$$

$$\vec{V}\cdot\nabla f = \hat{e}_s\cdot\nabla f$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Special Cases:

1. 

$$\hat{e}_s = \hat{i}$$

$$\hat{i}\cdot\nabla f = \frac{\partial f}{\partial x}$$

2.

$$\hat{e}_s = -\hat{j}$$

$$-\hat{j}\cdot\nabla f = -\frac{\partial f}{\partial y}$$

3. $$\hat{e}_s = \hat{k}$$

$$\hat{e}_s\cdot\nabla f = \hat{k}\cdot\nabla f = \partial f/\partial z$$

Scanned by CamScanner

In steady flow,

$$\vec{a} = \vec{V}\cdot\nabla\vec{V} = V\frac{\partial \vec{V}}{\partial s}$$

$$\vec{V} = V\hat{e}_s$$

$$\frac{\partial \vec{V}}{\partial s} = \frac{\partial V}{\partial s}\hat{e}_s + V\frac{\partial \hat{e}_s}{\partial s}$$

$$V\frac{\partial \vec{V}}{\partial s} = V\frac{\partial V}{\partial s}\hat{e}_s + V^2\frac{\partial \hat{e}_s}{\partial s}$$

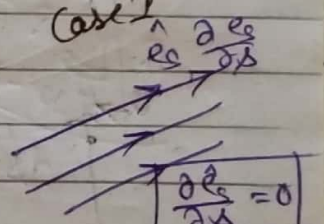
$\hat{e}_s\cdot\hat{e}_s = 1$ for all s along streamline.

$$\frac{\partial}{\partial s}(\hat{e}_s\cdot\hat{e}_s) = \frac{\partial}{\partial s}(1) = 0$$

$$2\,\hat{e}_s\cdot\frac{\partial \hat{e}_s}{\partial s} = 0$$

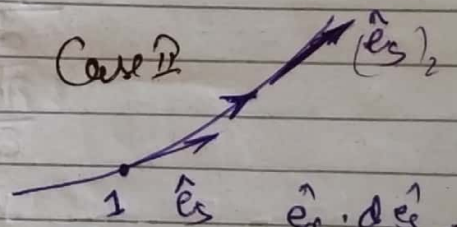
$$\hat{e}_s\cdot\frac{\partial \hat{e}_s}{\partial s} = 0.$$

Case I



$\boxed{\frac{\partial \hat{e}_s}{\partial s} = 0}$ for case I

Case II



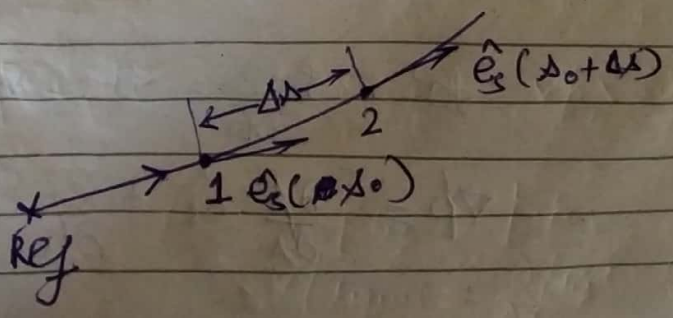
$$\hat{e}_s\cdot\frac{d\hat{e}_s}{ds} = |\hat{e}_s|\cdot\left|\frac{\partial \hat{e}_s}{\partial s}\right|\cdot\cos\phi$$

$$= 0$$

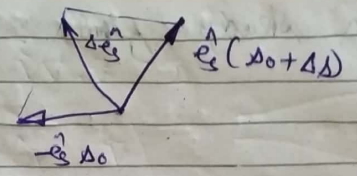
$$\therefore e\cos\phi = 0$$

$$\phi = 90^\circ$$

So $\frac{\partial \hat{e}_s}{\partial s} \perp \hat{e}_s$



Scanned by CamScanner

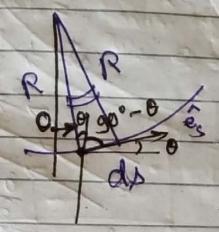


$$\Delta\hat{e}_s = \hat{e}_s(s_0+\Delta s) - \hat{e}_s(s_0)$$

$$\frac{\partial \hat{e}_s}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta \hat{e}_s}{\Delta s}$$

$$\left|\frac{\partial \hat{e}_s}{\partial s}\right| = \kappa = \frac{1}{R}$$

$$\frac{\partial \hat{e}_s}{\partial s} = \frac{1}{R}\hat{e}_n$$



$$\hat{e}_s = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\frac{d\hat{e}_s}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{\partial \hat{e}_s}{\partial s} = \frac{d\hat{e}_s}{d\theta} \times \frac{d\theta}{ds}$$

$$ds = R\,d\theta$$

$$\frac{d\theta}{ds} = \frac{1}{R}$$

$$\vec{a} = v\frac{\partial v}{\partial s}\hat{e}_s + v^2\frac{\partial \hat{e}_s}{\partial s}$$

$$= v\frac{\partial v}{\partial s}\hat{e}_s + \frac{v^2}{R}\hat{e}_n$$

$$\rho\vec{a} = \rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p$$

$$\nabla p = \frac{\partial p}{\partial s}\hat{e}_s + \frac{\partial p}{\partial n}\hat{e}_n + \frac{\partial p}{\partial b}\hat{e}_b$$

Scanned by CamScanner

Binomial vector: $\hat{e}_b = \hat{e}_s \times \hat{e}_n$

$$\hat{e}_n = \frac{\frac{\partial \hat{e}_s}{\partial s}}{\left|\frac{\partial \hat{e}_s}{\partial s}\right|}$$ — Principal normal vector.

$$\vec{a} = a_s \hat{e}_s + a_n \hat{e}_n + 0\,\hat{e}_b$$

$a_s = v\frac{\partial v}{\partial s}$ {tangential component}

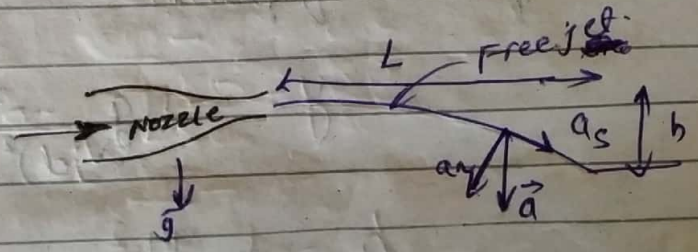
$a_n = \frac{V^2}{R}$ {centripetal acc^n component}

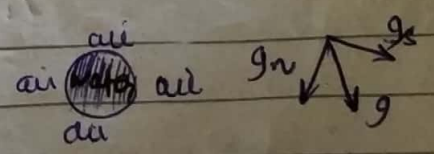
$$\rho(a_s \hat{e}_s + a_n \hat{e}_n) = \rho[g_s \hat{e}_s + g_n e_n + g_b \hat{e}_b] - \left\{\frac{\partial p}{\partial s}\hat{e}_s + \frac{\partial p}{\partial n}\hat{e}_n + \frac{\partial p}{\partial b}\hat{e}_b\right\}$$

$g_s = \hat{e}_s \cdot \vec{g}$, $g_n = \hat{e}_n \cdot \vec{g}$, $g_b = \hat{e}_b \cdot \vec{g}$

Euler's eq^n in curvilinear co-ordinates for steady flow.

$$\boxed{\begin{aligned} \rho a_s &= \rho v\frac{\partial v}{\partial s} = \rho g_s - \frac{\partial p}{\partial s} \\ \rho a_n &= \frac{\rho v^2}{R} = \rho g_n - \frac{\partial p}{\partial n} \\ 0 &= \rho g_b - \frac{\partial p}{\partial b} \end{aligned}}$$





Scanned by CamScanner

$$\vec{V} = V\hat{e}_s = v_x\hat{i} + v_y\hat{j}$$

$$\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

$$\vec{a} = a_s\hat{e}_s + a_n\hat{e}_n$$

$$\vec{a} = \vec{g}$$

$$\frac{d^2x}{dt^2} = 0 \Rightarrow \frac{dx}{dt} = \text{Const.}$$

$$\frac{d^2y}{dt^2} = -g \Rightarrow \frac{dy}{dt} = -gt + c_1$$

$$x_{point}(t) = V_{jet}\cdot t + \cancel{b_1}^{\to 0}$$

$$y_{point}(t) = -\frac{gt^2}{2} + \cancel{b_2}^{\to 0}$$

$$\frac{dx}{dt} = v_x = V_{jet}$$

$$\frac{dy}{dt} = v_y = -gt$$

$$t = \frac{x_{part}}{V_{jet}}$$

$$y_{part} = -\frac{1}{2}g\left(\frac{x_{part}}{V_{jet}}\right)^2$$

Equation of pathline -

$$y = -\frac{1}{2}\frac{g\,x^2}{(V_{jet})^2}$$

$$h = \frac{1}{2}\frac{gL^2}{V_{jet}^2}$$

$$\simeq \frac{1}{2}\times\frac{10\,L^2}{10^2}$$

$$= 0.05L^2$$

Scanned by CamScanner

$$\frac{\partial p}{\partial s} = \rho(g_s - a_s)$$

$$\frac{\partial p}{\partial n} = \rho\left(g_n - \frac{v^2}{R}\right)$$

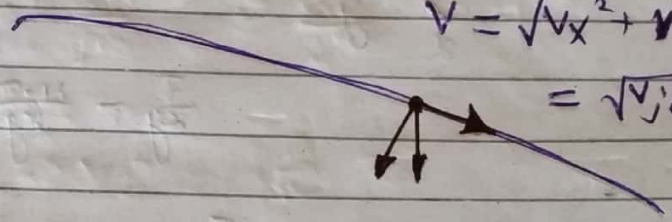
$$\nabla p = \rho(\bar{g} - \bar{a})$$

$$\frac{\partial p}{\partial n} dv = \rho\, dv.\, g_n + \underbrace{\rho\, dv.\left\{-\frac{v^2}{R}\right\}}_{\text{Centrifugal force.}}$$

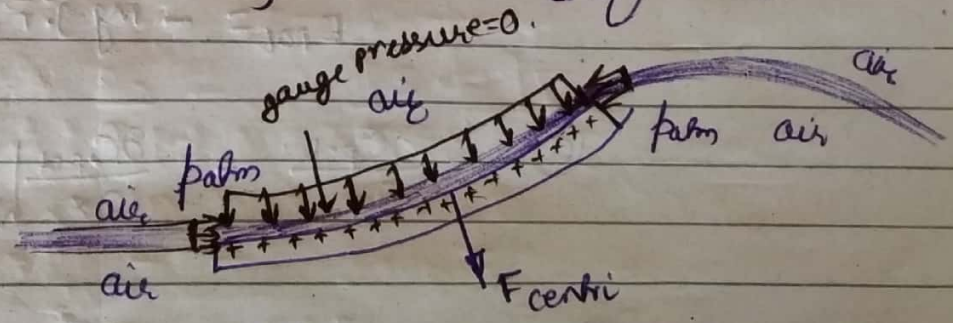
$$\nabla p = \frac{\partial p}{\partial s}\hat{e}_s + \frac{\partial p}{\partial n}\hat{e}_n$$

$$V = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{V_{jet}^2 + (\omega r)^2}$$



Now considering the case of vane



Why high pressure?

is it bcoz of liq. weight or due to motion of fluid.

$$p_{gauge} = p - p_{atm}.$$

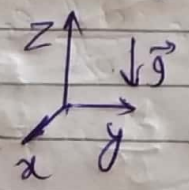
Euler's eq^n of motion-

$$\rho \vec{a} = \rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

$$e_{pot}(x,y,z) = -\vec{g}\cdot\vec{r}$$

$$g = g_x \hat{i} + g_y \hat{j} + g_z \hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{g}\cdot\vec{r} = g_x \cdot x + g_y \cdot y + g_z \cdot z$$

Ex. 1: $\vec{g} = -g\hat{k}$



$$e_{pot} = gz$$

$\frac{m}{s^2} \cdot m$

$$\frac{J}{kg} = \frac{N \cdot m}{kg} = \frac{\frac{kg\,m}{s^2} \cdot m}{kg} = \left(\frac{m}{s}\right)^2$$

$$E_{pot} = Mgz$$

kg, ms^{-2}, m

$$E_{pot} = -Mg\,\vec{g}\cdot\vec{r}$$

(a) $$\frac{\partial e_{pot}}{\partial x} = \left.\frac{\partial e_{pot}}{\partial x}\right|_{y,z} = -g_x$$

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial x}{\partial y} = \left.\frac{\partial x}{\partial y}\right|_{x,z} = 0$$

$$\frac{\partial x}{\partial z} = \left.\frac{\partial x}{\partial z}\right|_{x,y} = 0$$

(b) $$\frac{\partial e_{pot}}{\partial y} = -g_y$$

(c) $$\frac{\partial e_{pot}}{\partial z} = -g_z$$

Scanned by CamScanner

$$\nabla e_{pot} = \frac{\partial e_{pot}}{\partial x}\hat{i} + \frac{\partial e_{pot}}{\partial y}\hat{j} + \frac{\partial e_{pot}}{\partial z}\hat{k}$$

$$= -\{g_x\hat{i} + g_y\hat{j} + g_z\hat{k}\}$$

$$= -\vec{g}$$

$$\cancel{= -\rho\{}$$

$$\rho\vec{g} = -\rho\nabla e_{pot} = -\nabla(\rho e_{pot})$$

$$= -\rho\left\{\frac{\partial e_{pot}}{\partial x}\hat{i} + \frac{\partial e_{pot}}{\partial y}\hat{j} + \frac{\partial e_{pot}}{\partial z}\hat{k}\right\}$$

$$= -\left[\left\{\frac{\partial(\rho e_{pot})}{\partial x} - e_{pot}\frac{\partial\rho}{\partial x}\right\}\hat{i} + \left\{\frac{\partial(\rho e_{pot})}{\partial y} - e_{pot}\left(\frac{\partial\rho}{\partial y}\right)\right\}\hat{j} + \left\{\frac{\partial}{\partial z}(\rho e_{pot}) - e_{pot}\left(\frac{\partial\rho}{\partial z}\right)\right\}\hat{k}\right]$$

it only valid for incompressible flow

$$\rho\vec{g} - \nabla p = -\nabla(\rho e_{pot}) - \nabla p$$

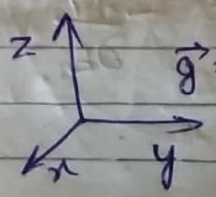
$$= -\nabla(p + \rho e_{pot}) = -\nabla P$$

$$\boxed{\rho\frac{D\vec{V}}{Dt} = -\nabla P}$$

Modified pressure (or piezometric pressure)

$$P = p + \rho e_{pot} = p - \rho\vec{g}\cdot\vec{r}.$$

special case :



$\vec{g} = -g\hat{k}$, $\vec{g}\cdot\vec{r} = -gz.$

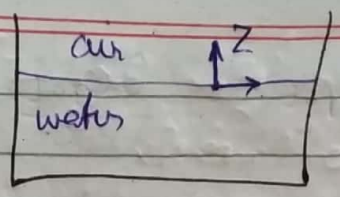
$$P = p + \rho g z.$$

Hydrostatic : $P = p + \rho g z$

$= \text{const}$

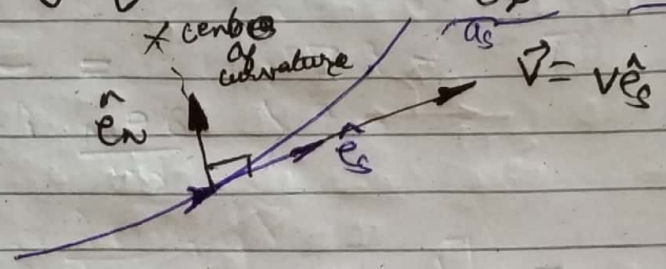
$$p = \text{const} - g\rho z.$$

Scanned by CamScanner



$$\rho \vec{a} = -\nabla p.$$

for steady flow, $\vec{a} = \underbrace{V\frac{\partial V}{\partial s}}_{a_s}\hat{e}_s + \underbrace{\frac{V^2}{R}}_{a_n}\hat{e}_n$



$$\nabla p = \frac{\partial p}{\partial s}\hat{e}_s + \frac{\partial p}{\partial n}\hat{e}_n + \frac{\partial p}{\partial b}\hat{e}_b$$

$$\hat{e}_b = \hat{e}_s \times \hat{e}_n$$

$$\vec{a} = a_s\hat{e}_s + a_n\hat{e}_n + 0\hat{e}_b$$

$$\nabla p = \frac{\partial p}{\partial s}\hat{e}_s + \frac{\partial p}{\partial n}\hat{e}_n + \frac{\partial p}{\partial b}\hat{e}_b$$

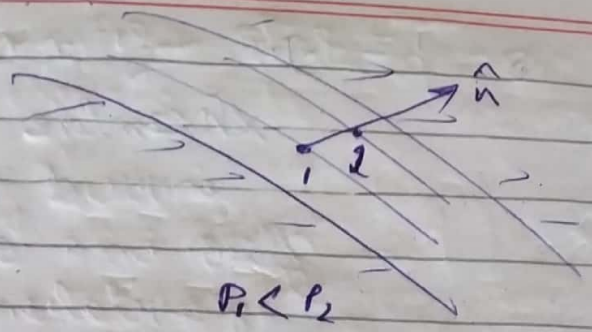
$$\vec{a} = a_s\hat{e}_s + a_n\hat{e}_n + 0\hat{e}_b$$

$$\rho a_s = \rho V\frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s}$$

$$\rho a_n = \rho\frac{V^2}{R} = -\frac{\partial p}{\partial n} = \frac{\partial p}{\partial n}$$

$$\rho a_b = 0 = -\frac{\partial p}{\partial b}$$

Scanned by CamScanner



$$\frac{\partial P}{\partial s} + \rho V \frac{\partial V}{\partial s} = 0$$

$$\frac{\partial}{\partial s}(V^2) = \frac{d}{dV}(V^2)\frac{\partial V}{\partial s} = 2V\frac{\partial V}{\partial s}$$

$$V\frac{\partial V}{\partial s} = \frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)$$

$$\frac{\partial P}{\partial s} + \rho V\frac{\partial V}{\partial s} = 0 \Leftrightarrow \frac{\partial P}{\partial s} + \rho\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right) = 0$$

$$\frac{\partial P}{\partial s} + \frac{\partial}{\partial s}\left(\frac{\rho V^2}{2}\right) = 0$$

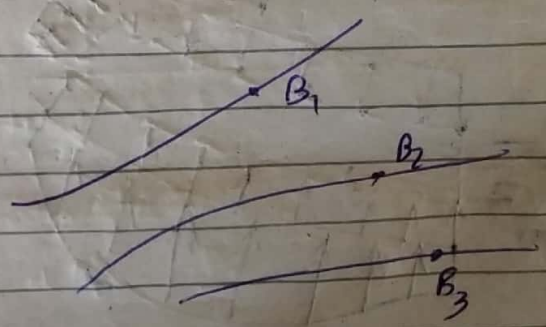
$$\frac{\partial}{\partial s}\left(P + \frac{\rho V^2}{2}\right) = 0$$

or $$\frac{\partial}{\partial s}\left(P - \rho\vec{g}\cdot\vec{r} + \frac{\rho V^2}{2}\right) = 0$$

Let $B(x,y,z) = P - \rho\vec{g}\cdot\vec{r} + \frac{\rho V^2}{2}$

$$\frac{\partial B}{\partial s} = 0 \Rightarrow B(x,y,z)$$

= const. along a streamline.

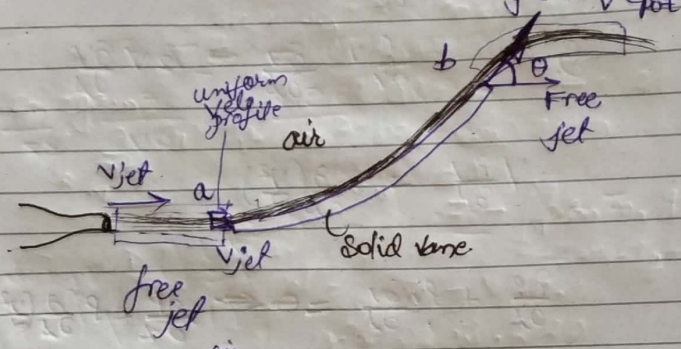


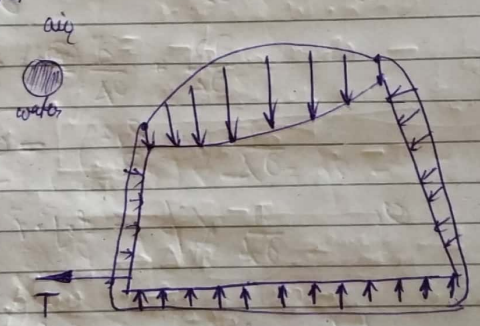
Scanned by CamScanner

$B = p - \rho \vec{g} \cdot \vec{r} + \frac{\rho V^2}{2} = \text{const.}$ along a streamline for steady, inviscid, const-density flow in a conservative gravitational force field.

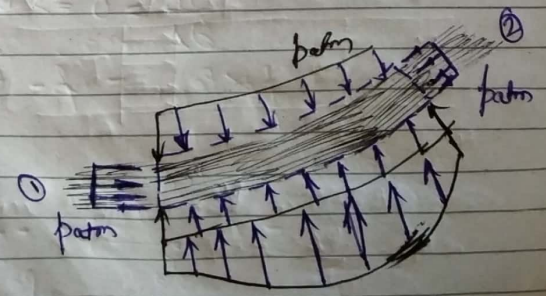
$\downarrow$

$\vec{g} = -\nabla e_{pot}$

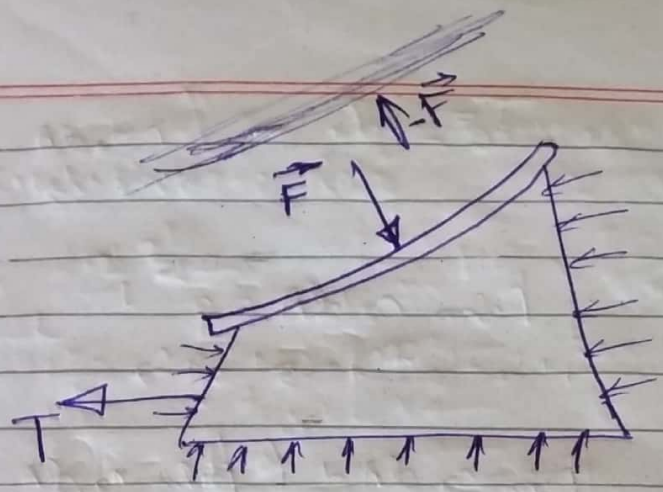




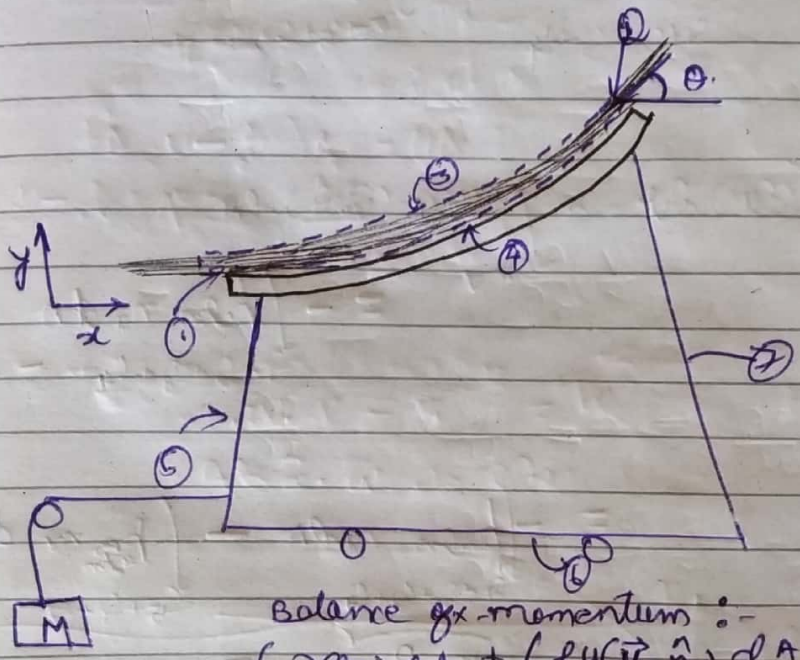
$(F_{net})_x = -T - \int p\, n_x\, dA$



Scanned by CamScanner

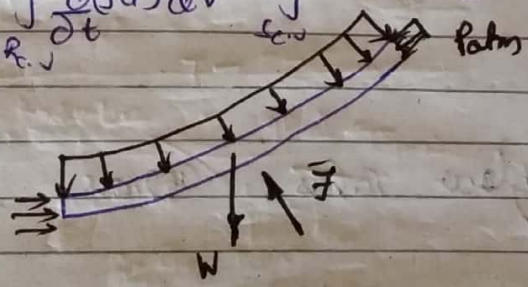


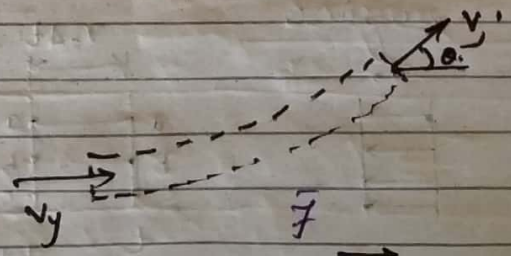
25/08/19

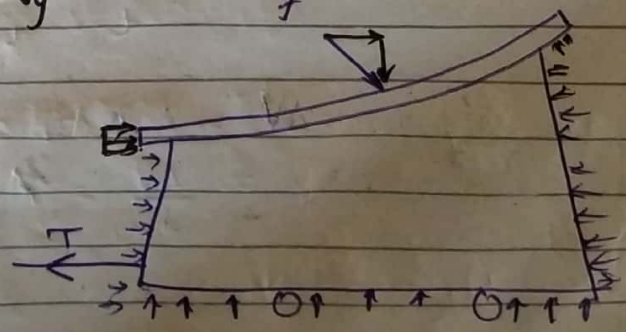


Balance of momentum :-

$$\int_{R.V} \frac{\partial(\rho u)}{\partial t} dv + \int_{C.S} \rho u(\vec{V}\cdot\hat{n})\, dA = ??$$







$$F_x - T - \int p_{atm} n_x \, dA = 0$$

Scanned by CamScanner

$$T = J_x - \int_{S_5+S_6+S_7} p\, n_x\, dA$$

$$\int_{S_1+S_2+S_3+S_4+S_5+S_6+S_7} p_{abs}\, n_x\, dA = 0$$

$$\int_{S_1+S_2+S_3+S_4} p_{abs}\, n_x\, dA + \int_{S_5+S_6+S_7} p_{abs}\, n_x\, dA = 0$$

So, again,

$$\int_{R_{c.v}} \frac{\partial}{\partial t}(\rho u)\, dV + \int_{S_{c.v}} \rho u\, (\vec{V}\cdot\hat{n})\, dA = -\int_{S_1+S_2+S_3+S_4} p_{abs}\, n_x\, dA - F_x$$

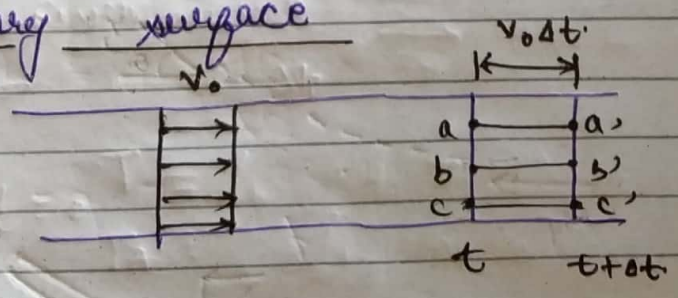
$$= \underbrace{-\int_{S_1+S_2+S_3+S_4} p_{abs}\, n_x\, dA - \int_{S_5+S_6+S_7} p_{abs}\, n_x\, dA}_{0 \quad -T}$$

$$= -T$$

Volume flow rates across surfaces -

① Stationary surface

Ex. ①

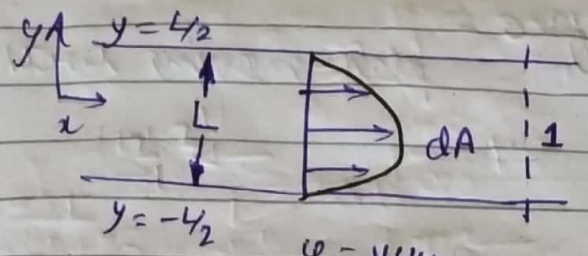


$$\Delta V = A V_0 \Delta t$$

$$Q = \frac{\Delta V}{\Delta t} = A V_0$$

Scanned by CamScanner

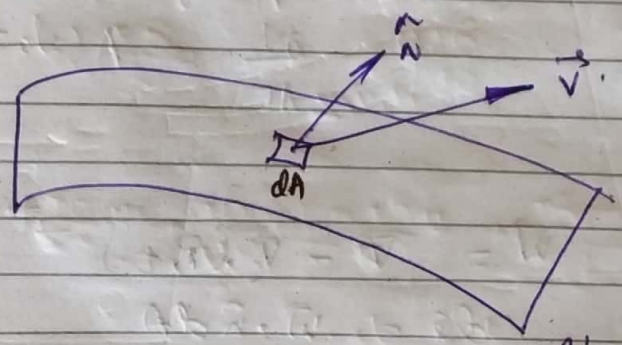
Ex. ②

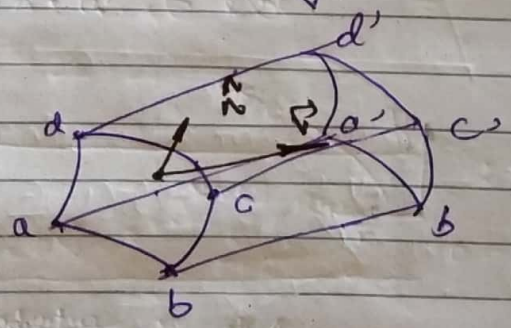


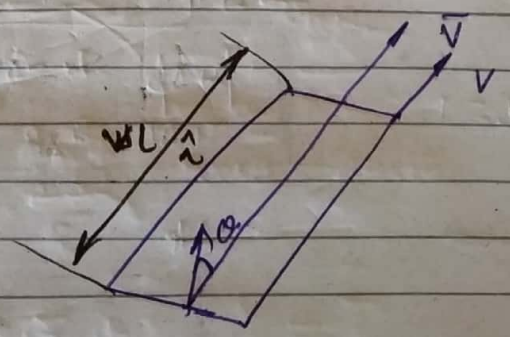
$u = u(y)$

$$u = u_{max}\left\{1 - \left(\frac{y}{L/2}\right)^2\right\}$$

$$= W \int_0^h v(y)\,dy$$







$$v_n = \vec{v}\cdot\hat{n} = |\vec{v}|\cdot|\hat{n}|\cos\theta$$
$$= V\cdot\cos\theta.$$

$$\Delta V = V\Delta t\cos\theta\, dA$$

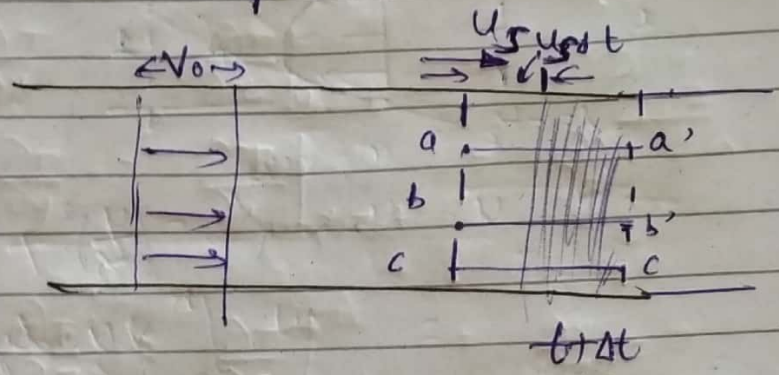
$$\frac{\Delta V}{\Delta t} = V\cos\theta\, dA = v_n\, dA$$

$$= \vec{v}\cdot\hat{n}\, dA$$

Scanned by CamScanner

(ii) Moving surface -

plate moving with u_s speed



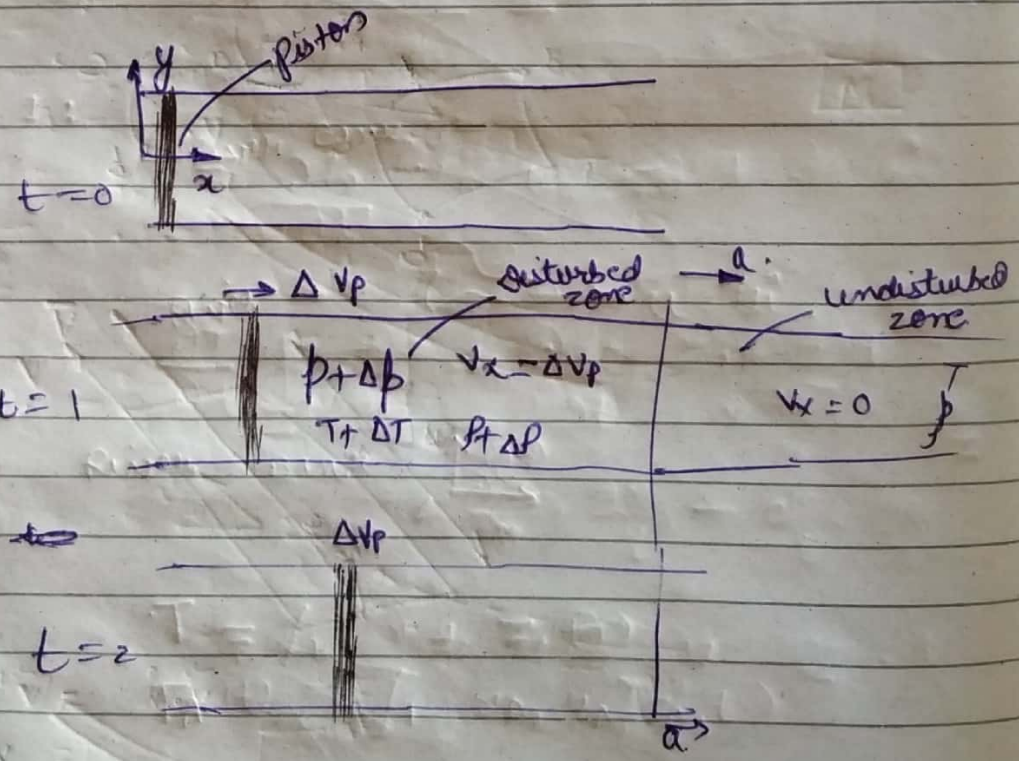
$$\Delta V = A(V_0 - u_s)\Delta t$$

$$Q = \frac{\Delta V}{\Delta t} = A(V_0 - u_s)$$

$$\vec{W} = \vec{V} - \vec{V}_s(\vec{r}, t)$$

$$dQ = \vec{W}\cdot\hat{n}\, dA$$

Ex 4



At an impermeable solid moving surface,

$$\vec{W}\cdot\hat{n} = 0.$$

$$\vec{W}\cdot\hat{n} = (\vec{V} - \vec{V}_s)\cdot\hat{n}$$

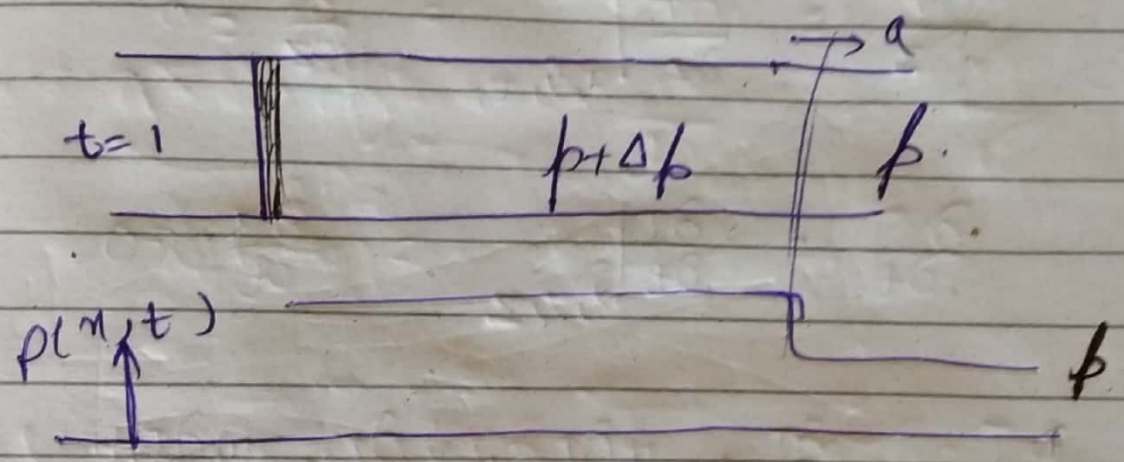
$$= \vec{V}\cdot\hat{n} - \vec{V}_s\cdot\hat{n} = 0$$

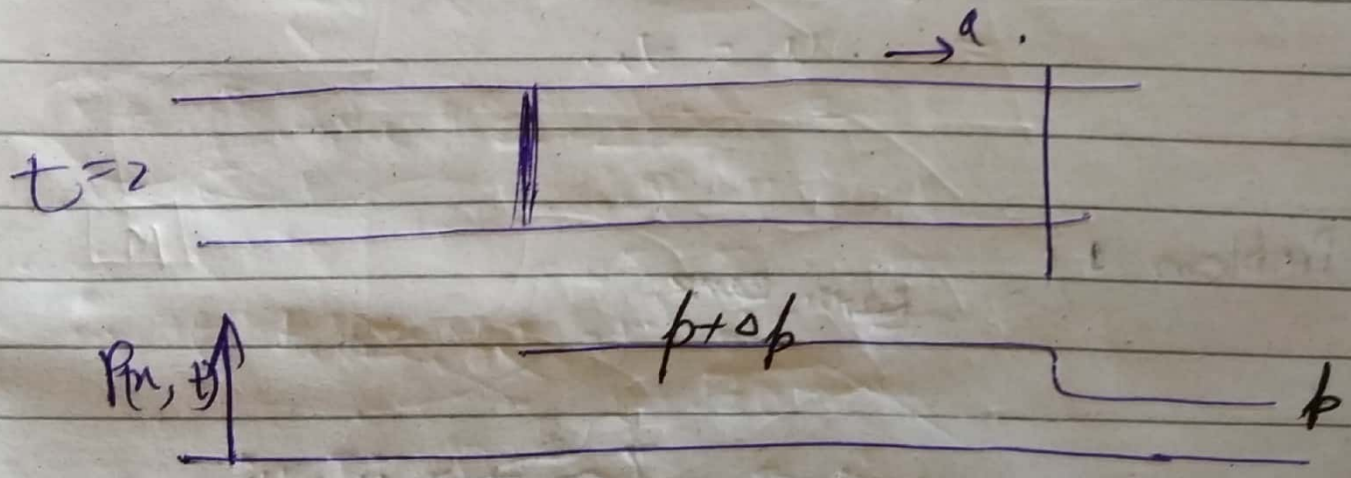
Scanned by CamScanner

$$\vec{v}\cdot\hat{n} = \vec{v}_s\cdot\hat{n}_s$$

$$v_n = (v_s)_n$$

Also, $a = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{\left(\frac{\partial \rho}{\partial p}\right)_s}$





Problem 4:

(a) $T(x,y,z,t) = x$

$$\frac{\partial T}{\partial x} = \left.\frac{\partial T}{\partial x}\right|_{y,z,t}$$

$$\frac{\partial T}{\partial y} = \left.\frac{\partial T}{\partial y}\right|_{x,z,t} = \frac{\partial x}{\partial y} = 0$$

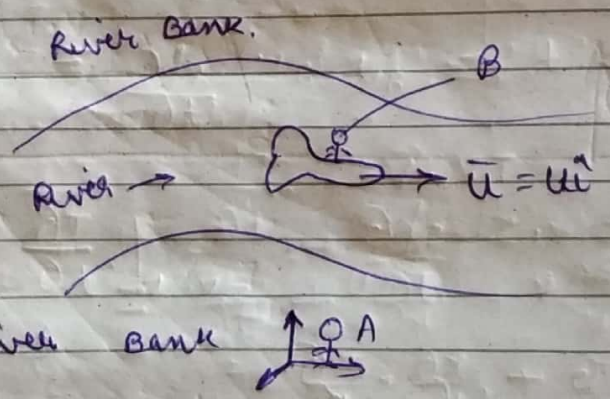
$$\frac{\partial T}{\partial z} = \left.\frac{\partial T}{\partial z}\right|_{x,y,t} = \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial T}{\partial t} = \left.\frac{\partial T}{\partial t}\right|_{x,y,z} = \frac{\partial x}{\partial t} = 0$$

$$\frac{DT}{Dt} = \cancel{\frac{\partial T}{\partial t}}^{0} + u\frac{\partial T}{\partial x} + \cancel{v\frac{\partial T}{\partial y}}^{0} + \cancel{w\frac{\partial T}{\partial z}}^{0}$$

$$= u = V_x.$$

Problem 1:



$$T = f_1(x, y, z, t)$$

$$T_A(t) = f_1(x_A, y_A, z_A, t)$$

$$\frac{dT_A(t)}{dt} = \lim_{\Delta t \to 0} \frac{T_A(t+\Delta t) - T_A(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f_1(x_A, y_A, z_A, t+\Delta t) - f(x_A, y_A, z_A, t)}{\Delta t}$$

$$= \frac{\partial f_1}{\partial t}(x_A, y_A, z_A, t)$$

Scanned by CamScanner

why not converting to some other function

$$\frac{dT_B}{dt} = \frac{\partial f_1}{\partial x}(x_B(t), y_B(t), z_B(t), t) \times \frac{\partial x_B}{\partial t}$$

$$+ \frac{\partial f_1}{\partial y}(x_B(t), y_B(t), z_B(t), t) \times \frac{\partial y_B}{\partial t}$$

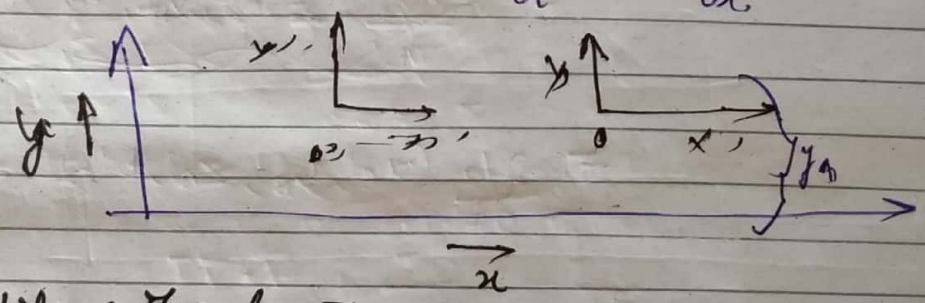
$$+ \frac{\partial f_1}{\partial z}(x_B(t), y_B(t), z_B(t), t) \frac{\partial z_B}{\partial t}$$

$$+ \frac{\partial f_1}{\partial t}(x_B(t), y_B(t), z_B(t), t)) \frac{\partial t}{\partial t}$$

$$= \frac{\partial f_1}{\partial t} + u_B \frac{\partial f_1}{\partial x} + v_B \frac{\partial f_1}{\partial y} + w_B \frac{\partial f_1}{\partial z}$$

$$= \frac{\partial f_1}{\partial t} + \vec{v}_B \nabla f_1$$

$$= \frac{\partial f_1}{\partial t} + U \frac{\partial f_1}{\partial x}$$



Galilean Transformation

$$\boxed{\begin{aligned} y &= y_{init} + y' \\ z &= z_{init} + z' \\ x &= x_{init} + x' + Ut' \\ t &= t' \end{aligned}}$$

$y' = y - y_{init}$

$z = z - z_{init}$

$z' = x - x_{init} - Ut$

$t' = t$

$T = f_1(x, y, z, t)$

$T = f_2(x', y', z', t')$

$$\frac{\partial T}{\partial t} = \left.\frac{\partial T}{\partial t}\right|_{x, y, z}$$

Scanned by CamScanner

$$\frac{\partial T}{\partial x'} = \left(\frac{\partial T}{\partial t}\right)_{x',y',z'} = \left(\frac{\partial f_2}{\partial t'}\right)$$

$$= \frac{\partial f_2}{\partial t}$$

$$\frac{\partial T_A}{\partial t} = \frac{\partial f_1}{\partial t}(\underbrace{x_A, y_A, z_A}_{\text{const. parameters}}, t)$$

$$\frac{\partial T_B}{\partial t} = \frac{\partial f_1}{\partial t} + u\frac{\partial f_1}{\partial x}$$

$$= \frac{\partial f_2}{\partial t'}(x_B', y_B', z_B', t')$$

$$\frac{\partial T}{\partial t'}\Big|_{x',y',z'} = \frac{\partial f_2}{\partial t}(x', y', z', t')$$

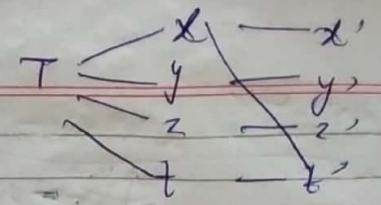
$$= \frac{\partial T}{\partial x} \times \frac{\partial x}{\partial t'} + \frac{\partial T}{\partial t} \times \frac{\partial t}{\partial t'}$$

$$= \frac{\partial T}{\partial x} u + \frac{\partial T}{\partial x} \cdot 1$$

$$\frac{dT_B}{dt} = \frac{\partial f_1}{\partial x} \cdot \frac{\partial x_B}{\partial t} + \frac{\partial f_1}{\partial t} \cdot \frac{dt}{dt}$$

$$= \frac{\partial f_1}{\partial t} + u\frac{\partial f_1}{\partial x}$$

$$= \frac{\partial f_2}{\partial t'}$$



$$\frac{\partial T}{\partial x'} = \frac{\partial T}{\partial x} \times \frac{\partial x}{\partial x'}$$

$$\frac{\partial T}{\partial y'} = \frac{\partial T}{\partial y} \times \frac{\partial y}{\partial y'} = \frac{\partial T}{\partial y}$$

$$\frac{\partial T}{\partial z'} = \frac{\partial T}{\partial z} \times \frac{\partial z}{\partial z'} = \frac{\partial T}{\partial z}$$

$$\frac{\partial T}{\partial t'} = \frac{\partial T}{\partial x} \times \frac{\partial x}{\partial t'} + \frac{\partial T}{\partial t} \times \frac{\partial t}{\partial t'}$$

$$= u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t'}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x'} \times \frac{\partial x'}{\partial x}$$

$$= \frac{\partial T}{\partial x'}$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial y'} \times \frac{\partial y'}{\partial y} = \frac{\partial T}{\partial y'}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial z} \times \frac{\partial z'}{\partial z} = \frac{\partial T}{\partial z'}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t'} \times \frac{\partial t'}{\partial t} + \frac{\partial T}{\partial x'} \times \frac{\partial x'}{\partial t}$$

$$= \frac{\partial T}{\partial t'} \times 1 + \frac{\partial T}{\partial x'}(-u)$$

$$\frac{\partial T}{\partial t'} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}$$

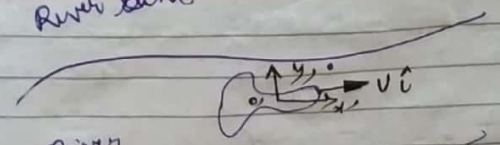
$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t'} - u\frac{\partial T}{\partial x}$$

Scanned by CamScanner

$(\nabla \times \bar{v})_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$

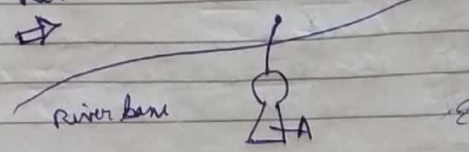
$\int \bar{r} \times p\, \hat{n}\, dA = -\int \bar{r} \times \nabla p\, dV$

River bank



$c = a \times b$

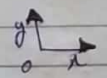
River

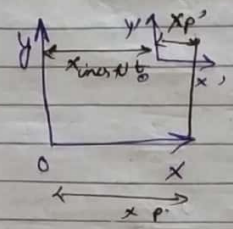


$c_i = \varepsilon_{ijk} a_j b_k$

River bank

$\varepsilon_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \\ 1 & \text{cyc} \\ -1 & \text{acyc} \end{cases}$





$\varepsilon_{123} = 1$

$\varepsilon_{231} = 0$

$\varepsilon_{321} = -1$

$j \sim k$

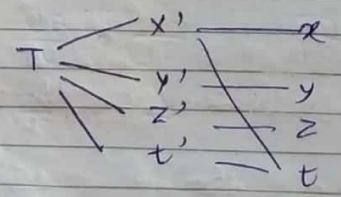
$\varepsilon_{111} = 0$

$\varepsilon_{112} = 0$

$x' = x - x_{iner} - Ut$	$x = x' + x_{iner} + Ut'$
$y' = y - y_{iner}$	$y = y' + y_{iner}$
$z' = z - z_{iner}$	$z' = z' + y_{iner}$
$t' = t$	$t = t'$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T$$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$



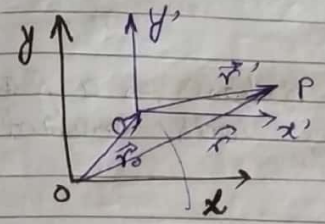
$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t'} \times \frac{\partial t'}{\partial t} + \frac{\partial T}{\partial x'} \times \frac{\partial x'}{\partial t}$$

$$= \frac{\partial T}{\partial t'} - \frac{u\partial T}{\partial x'}$$

what about $\frac{DT}{Dt'}$

$$= \frac{\partial T}{\partial t'} + u'\frac{\partial T}{\partial x'} + v'\frac{\partial T}{\partial y'} + \frac{\partial T}{\partial z'}$$

thus, $\frac{DT}{Dt} = \left(\frac{\partial T}{\partial t'} - U\frac{\partial T}{\partial x'}\right) + u\frac{\partial T}{\partial x'} + v\frac{\partial T}{\partial y'} + w\frac{\partial T}{\partial z'}$



$\vec{r} = \vec{r}' + \vec{r}_0$

$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{r}_0}{dt}$

$x' = x - x_{ine} - \int_0^t x(t')\,dt'$

$\vec{V} = \vec{w} + \vec{U}$

$\vec{w} = u'\hat{i}' + v'\hat{j}' + w'\hat{k}'$

$$\boxed{\begin{aligned} u' &= u - U \\ v' &= v \\ w' &= w \end{aligned}}$$

$\hat{i} \equiv \hat{i}'$

$\hat{j} = \hat{j}'$

$\hat{k} = \hat{k}'$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t'} + (u - U)\frac{\partial T}{\partial x'} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t'} + u'\frac{\partial T}{\partial x'} + v'\frac{\partial T}{\partial y} + w'\frac{\partial T}{\partial z'}$$

$T = f_1(x, y, z, t) = a(x + y + 3z - bt)$

$\frac{\partial T}{\partial t} = -ab \qquad \frac{\partial T}{\partial x} = a$

$$\frac{\partial T}{\partial t'} = \frac{\partial T}{\partial t} + U_0\frac{\partial T}{\partial x}$$

$$= -ab + U_0 a$$

$$= a(U_0 - b)$$

Scanned by CamScanner

Not steady in stationary reference system.
steady in moving " " only if $U_0 = 0$.

Euler's equation of motions in a frame of reference translating with constant velocity in x-dir^n.

Step 1: write Euler's eq^n in a stationary co-ordinate system.

$$\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p$$

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x}$$

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y}$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z}$$

Step 2. Express derivative w.r.t x, y, z & t in terms of derivatives w.r.t x', y', z' & t'.

$$\frac{D}{Dt'} = \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'} + w'\frac{\partial}{\partial z'}$$

$$= \frac{\partial}{\partial t'} + \vec{w}_0 \cdot \nabla'$$

Step 3: Change the dependent variables.

$$\rho \frac{Du'}{Dt'} = \rho g_x - \frac{\partial p}{\partial x'}$$

$$\rho \frac{Dv'}{Dt'} = \rho g_y - \frac{\partial p}{\partial y'}$$

$$\rho \frac{Dw'}{Dt'} = \rho g_z - \frac{\partial p}{\partial z'}$$

$v' = v \Rightarrow \frac{Dv'}{Dt} = \frac{Dv}{Dt}$

$u = u' + U$

$$\frac{DU}{Dt} = \frac{Du'}{Dt} + \cancel{\frac{DU}{Dt}}$$

$$\frac{DU}{Dt} = \cancel{\frac{\partial u'}{\partial t'}} + \cancel{u'\frac{\partial u'}{\partial x'}} + \cancel{v'\frac{\partial u'}{\partial y'}} + \cancel{w'\frac{\partial u'}{\partial z'}}$$

$$\Rightarrow \rho \frac{D\vec{w}}{Dt'} = \rho\vec{g} - \nabla p'$$

$$\nabla' = \hat{i}'\frac{\partial}{\partial x'} + \hat{j}'\frac{\partial}{\partial y'} + \hat{k}'\frac{\partial}{\partial z'}$$

We can apply Euler's eq^n in moving frame with co-ordinate system transformed in that way.

Bernoulli's eq^n in a frame of reference translating with a const. velocity.

$\vec{U}$ = velocity of the origin of the translating frame.

$\vec{W} = \vec{v} - \vec{U}$ = velocity of fluid relative to translating frame.

$\vec{v}$ = absolute velocity of fluid (w.r.t stationary frame)

$$p + \frac{1}{2}\rho w^2 - \rho\vec{g}\cdot\vec{r}' = \text{const. along a "relative streamline"}$$

1. flow is steady in the translating frame.
2. Flow is inviscid.
3. density of fluid is const.
4. $\vec{g}$ is const.

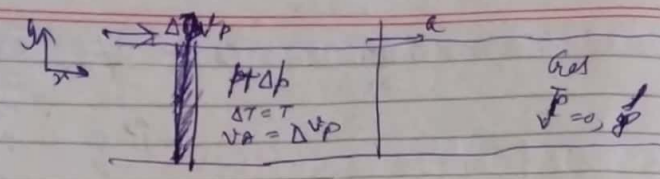
$$V^2 = u^2 + v^2 + w^2.$$

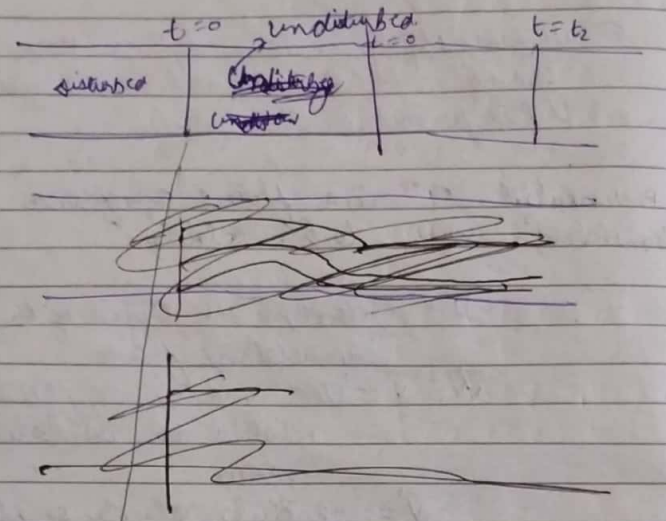
$$w^2 = u'^2 + v'^2 + w'^2.$$

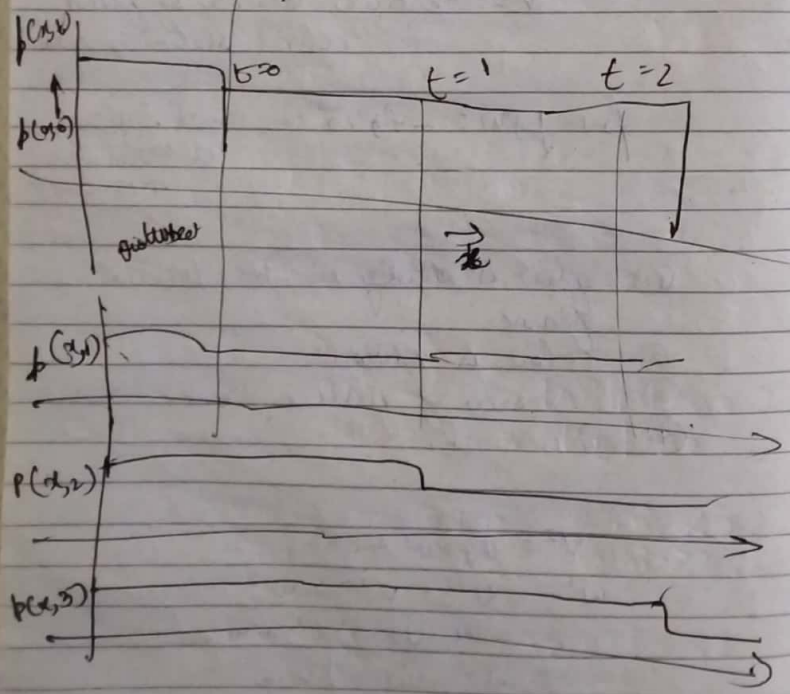
$$= (u-U)^2 + v^2 + w^2.$$

$$w = |\vec{w}| = \sqrt{\vec{w}\cdot\vec{w}}$$

Scanned by CamScanner







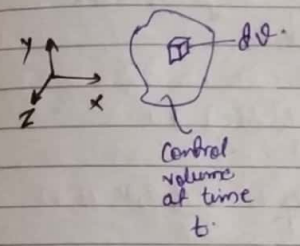
Moving Control Volumes - the extended Leibnitz rule.

Φ = extensive property.

ϕ = corresponding specific property

eg. $\Phi = M$

$\phi = M/M = 1$



$dm = \rho dv$

$d\Phi = \phi dm = \phi \rho dv$

$\Phi = L_x$ = x-component of linear Momentum

$\phi = v_x = u$

$dM = \rho dv$ $\quad (\phi = 1)$

$dL_x = \rho v_x dv$ $\quad (\phi = v_x = u)$

$dL_y = \rho v_y dv$ $\quad (\phi = v_y = v)$

$dL_z = \rho v_z dv$ $\quad (\phi = v_z = w)$

$d\bar{L} = \rho \bar{v} dv$ $\quad (\phi = \vec{v})$

$d\bar{H} = \rho (\vec{r} \times \vec{v}) dv$ $\quad (\phi = \vec{r} \times \vec{v})$

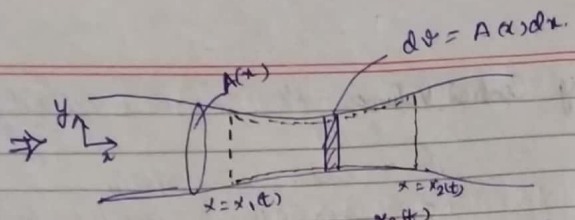
$$\Phi_{c.v} = \int_{R_{c.v}(t)} \phi \rho \, dv$$

Let. $\rho\phi = f_1(x, y, z, t)$

$$\Phi_{c.v} = \iiint_{R_{c.v}(t)} f_1(x, y, z, t) \, dx\,dy\,dz$$

$$\frac{d\Phi_{c.v}}{dt} = ?$$

Scanned by CamScanner



$$\Phi_{c.v}(t) = \int_{x_1(t)}^{x_2(t)} \phi(x,t)\, f(x,t) A(x)\, dx$$

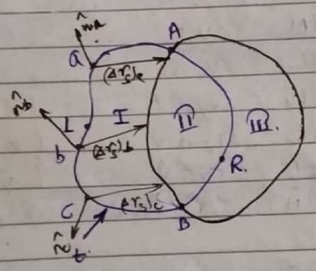
$$\frac{d}{dt}\Phi_{c.v}(t) = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} F(x,t)\, dx.$$

$$= \int_{x_1(t)}^{x_2(t)} \frac{\partial F}{\partial t}(x,t)\, dx.$$

$$+ \frac{dx_2(t)}{dt} \cdot F(x_2(t),t)$$

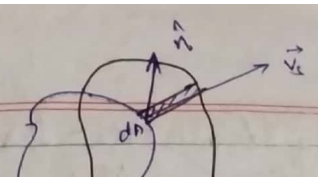
$$- \frac{dx_1(t)}{dt} \cdot F(x_1(t),t)$$

Now extending above to 3d -

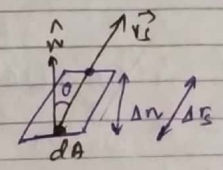


Now, $\Delta \vec{r}_s \cdot \hat{n} = |\Delta \vec{r}_s||\hat{n}| \cos\theta$
$= |\Delta \vec{r}_s| \cos\theta$

Note : on ALB, $\vec{V}_s \cdot \hat{n} < 0$
Similarly, on ARB, $\vec{V}_s \cdot \hat{n} > 0$



$\Delta V = dA\, \Delta n$
$\downarrow$
$\Delta \vec{r}_s \cos\theta$



So, now

$$\Phi_{c.v}(t) = \int_{R_I} f_1(x,y,z,t)\, dv + \int_{R_{II}} f_1(x,y,z,t)\, dv.$$

Also $\Phi_{c.v}(t+\Delta t) = \int_{R_{II}} f_1(x,y,z,t)\, dv + \int_{R_{III}} f_1(x,y,z,t)\, dv$

$$\Delta \Phi_{c.v}(t+\Delta t) = \Phi_{c.v}(t+\Delta t) - \Phi_{c.v}(t)$$

$$\Delta \Phi_{c.v} = \Phi_{c.v}(t+\Delta t) - \Phi_{c.v}(t)$$

$$= \int_{R_{II}} f_1(x,y,z,t+\Delta t) - f_1(x,y,z,t)\, dv$$

$$+ \int_{R_{III}} f_1(x,y,z,t+\Delta t)\, dv$$

$dv = dA\, \Delta \vec{r}_s \cdot \hat{n}$

$$= \int_{R_I} f_1(x,y,z,t)\, dv$$

$dv = -dA\, \Delta \vec{r}_s \cdot \hat{n}$

$dv = dA\, \Delta \vec{r}_s \cdot \hat{n}$

Scanned by CamScanner

$$\frac{\Delta \Phi_{C.V}}{\Delta t} = \frac{\Phi_{C.V}(t+\Delta t) - \Phi_{C.V}(t)}{\Delta t}$$

$$= \int_{R_{II}} \frac{f_1(x,y,z,t+\Delta t) - f_1(x,y,z,t)}{\Delta t} dv$$

$$+ \int_{R_{III}} f_1(x,y,z,t+\Delta t) \frac{\Delta \vec{r}_s}{\Delta t} \cdot \hat{n}\, dA$$

$$+ \int_{R_I} f_1(x,y,z,t) \frac{\Delta \vec{r}_s}{\Delta t} \cdot \hat{n}\, dA$$

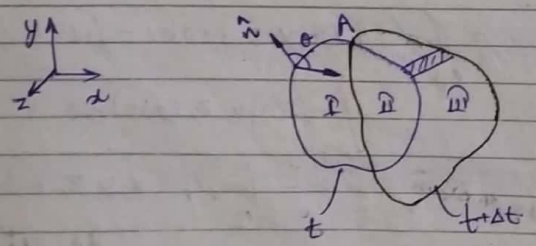
$$\frac{d\Phi_{C.V}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \Phi_{C.V}}{\Delta t} = \int_{R_{C.V}(t)} \frac{\partial f_1}{\partial t}(x,y,z,t)\, dv$$

$$+ \int_{\partial R_{C.V}(t)} f_1(x,y,z,t)\, \vec{v}_s \cdot \hat{n}\, dA$$

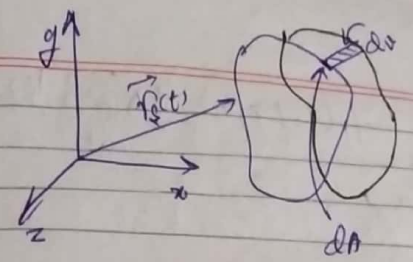
∂ means boundary of R.

Moving control volumes Contd....

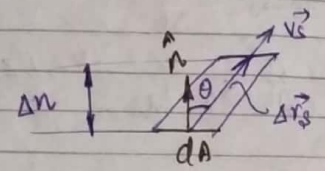
$$\Phi_{C.V}(t) = \int_{R_{C.V}(t)} \phi \rho\, dv$$



$$\frac{d}{dt}\Phi_{C.V} = \frac{d}{dt}\int_{R.C.V(t)} \underbrace{\phi(x,y,z,t)\rho(x,y,z,t)}_{f_1(x,y,z,t)} dv$$



$\Delta v = dA\,\Delta n$



$$\Phi_{C.V}(t) = \int_{R_{C.V}(t)} \phi \rho\, dv = \int_{R_I} f_1(x,y,z,t)\, dv + \int_{R_{II}} f_1(x,y,z,t)\, dv.$$

$$\Phi_{C.V(t)}(t+\Delta t) = \int_{R_{C.V}(t+\Delta t)} f_1(x,y,z,t+\Delta t)\, dv$$

$$= \int_{R_{II}} f_1(x,y,z,t)\, dv + \int_{R_{III}} f_1(x,y,z,t+\Delta t)\, dv$$

$$\Delta \Phi_{C.V} = \Phi_{C.V}(t+\Delta t) - \Phi_{C.V}(t)$$

$$= \int_{R_{II}} \{f_1(x,y,z,t+\Delta t) - f_1(x,y,z,t)\} dv$$

$$+ \int_{R_{III}} f_1(x,y,z,t+\Delta t)\, dv - \cancel{\int_{R_I} f_1(x,y,z,t)\, dv}$$

$$\frac{\Delta \Phi_{C.V}}{\Delta t} = \int_{R_{II}} \frac{f_1(x,y,z,t+\Delta t) - f_1(x,y,z,t)}{\Delta t} dv$$

$$+ \int_{R_{III} \text{ or } A R_B} f_1(x,y,z,t+\Delta t) \frac{|\Delta \vec{r}_s \cdot \hat{n}|\, dA}{\Delta t}$$

Scanned by CamScanner

$$- \int_{R_I \text{ or } ALB} f_1(x,y,z,t) \frac{|\Delta \vec{r}_s \cdot \hat{n}|}{\Delta t} dA$$

$$\frac{d\Phi_{c.v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \Phi_{c.v}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \int_{R_I} \frac{f_1(x,y,z,t+\Delta t) - f_1(x,y,z,t)}{\Delta t} dv$$

$$+ \lim_{\Delta t \to 0} \int_{ARB} f_1(x,y,z,t+\Delta t) \left| \frac{\Delta \vec{r}_s}{\Delta t} \cdot \hat{n} \right| dA$$

$$- \lim_{\Delta t \to 0} \int_{ALB} f_1(x,y,z,t) \left| \frac{\Delta \vec{r}_s}{\Delta t} \cdot \hat{n} \right| dA$$

$$= \int_{R_{c.v}(t)} \frac{\partial f_1(x,y,z,t)}{\partial t} dv + \int_{ARB} f_1(x,y,z,t) \vec{V}_s \cdot \hat{n} dA$$

$$- \int_{ALB} f_1(x,y,z,t) \cdot (-\vec{V}_s \cdot \hat{n}) dA$$

$$\boxed{\frac{d\Phi_{c.v}}{dt} = \iiint_{R_{c.v}(t)} \frac{\partial f_1}{\partial t}(x,y,z,t)\, dx\,dy\,dz + \iint_{\partial R_{c.v}(t)} f_1(x,y,z,t)\, \vec{V}_s(x,y,z,t) \cdot \hat{n}\, dA} \quad -(1)$$

~~for below~~ extended Leibnitz rule

Special Case -

1. $\vec{V}_s = \vec{0}$ (Stationary control volume)

$$\frac{d\Phi_{c.v}}{dt} = \int_{R_{fixed}} \frac{\partial}{\partial t}(\rho\phi)\, dv$$

2. $\vec{V}_s = \vec{v}$

$$\frac{d}{dt}\Phi_{sys} = \frac{d}{dt}\int_{R_I(t)} \rho\phi\, dv$$

$$\boxed{\frac{d\Phi_{sys}}{dt} = \int_{R_{c.v}} \frac{\partial}{\partial t}(\rho\phi)\, dv + \int_{\partial R_{c.v}} \rho\phi\, \vec{v} \cdot \hat{n}\, dA} \quad -(2)$$

Eq^n (2) − Eq^n (1)

$$\frac{d\Phi_{sys}}{dt} - \frac{d\Phi_{c.v}}{dt} = \int \rho\phi (\vec{v} - \vec{V}_s) \cdot \hat{n}\, dA$$

$\vec{w} = \vec{v} - \vec{V}_s$ — relative velocity

$$\frac{d}{dt}\Phi_{sys} = \frac{d}{dt}\Phi_{c.v} + \int_{\partial R_{c.v}} \rho\phi\, \vec{w} \cdot \hat{n}\, dA$$

3. $\vec{V}_s = \vec{U}$ = const. (control vol. translating with const. velocity $\vec{U}$)

$$\Phi_{c.v}(t) = \iiint_{R_{c.v}(t)} \phi\rho\, dx\,dy\,dz$$

$$= \iiint_{R_{c.v}(t)} f_1(x,y,z,t)\, dx\,dy\,dz$$

Scanned by CamScanner

$$(x, y, z, t) \longrightarrow (x', y', z', t')$$

$$\rho\phi \equiv f_1(x, y, z, t) \equiv f_2(x', y', z', t')$$

$$\equiv \iiint f_2(x', y', z', t')\, dx'\, dy'\, dz'$$

$$\frac{d\Phi_{sys}}{dt} = \frac{d}{dt}\int_{R_{sys}(t)} \rho\phi\, dv$$

$$= \int_{R_{c.v}(t)} \left\{\frac{\partial}{\partial t}(\rho\phi)\right\} dv + \int_{\partial R_{c.v}(t)} \rho\phi\, \vec{v}\cdot\hat{n}\, dA$$

$$\frac{d\Phi_{sys}}{dt} = \int_{R'} \left[\frac{\partial}{\partial t'}(\rho\phi)\right]_{x', y', z'} dv$$

$$\frac{\partial f_2}{\partial t'} + \int_{\partial R_{c.v}} \rho\phi\, \vec{w}\cdot\hat{n}\, dA$$

$$w = \vec{v} - \vec{U}$$

?? what about time derivative in other frame

$$\frac{\partial(\rho\phi)}{\partial t} = \frac{\partial(\rho\phi)}{\partial t'} - U\frac{\partial(\rho\phi)}{\partial x'} = \frac{\partial(\rho\phi)}{\partial t'} - \frac{\partial(\rho\phi U)}{\partial x'}$$

$$\left(\frac{\partial f_1}{\partial t}\right)_{x,y,z} \qquad \left(\frac{\partial f_2}{\partial t'}\right)_{x', y', z', t'}$$

$$\int \frac{\partial}{\partial t}(\rho\phi)\, dx\, dy\, dz = \int \frac{\partial}{\partial t}(\rho\phi)\, dx'\, dy'\, dz' - \int \frac{\partial}{\partial x}(\rho\phi U)\, dv$$

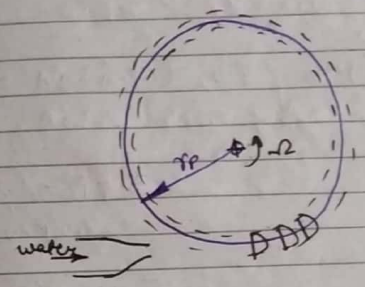
$$= \int \frac{\partial}{\partial t'}(\rho\phi)\, dv' - \int \nabla\cdot\vec{F}\, dv$$

$$= \int \frac{\partial}{\partial t'}(\rho\phi)\, dv - \int_{\partial R} (\vec{F}\cdot\hat{n})\, dA$$

steady in moving frame

$$\Rightarrow \int_{R_{c.v}} \frac{\partial}{\partial t}(\rho\phi)\, dv = -\int_{\partial R} \rho\phi\, \vec{U}\cdot\hat{n}\, dA$$

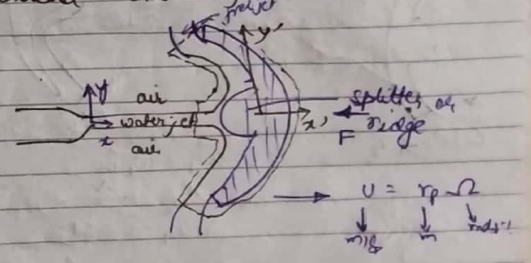
Pelton turbine or Pelton wheel



r_p = pitch radius
r_i = inner radius
r_o = outer radius
$h = r_o - r_i$
= height of buckets

- $\frac{h}{r_p} \ll 1$

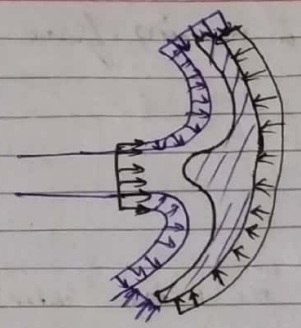
Bucket Geometry



$$\vec{v}_{rel} = \vec{w} = \vec{v} - \vec{U}$$

$$\vec{U} = U\hat{i} = r_p\Omega\hat{i}$$

Scanned by CamScanner



How do we know?

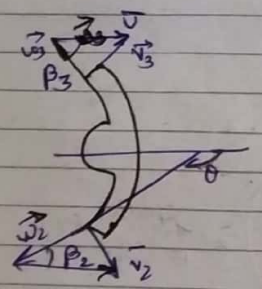
$$\int_{R_{c.v}'} \frac{\partial}{\partial t}(\rho w_x)\, dV' + \int_{\partial R_{c.v}'} \rho w_x (\vec{w}\cdot\hat{n})\, dA' = \Sigma F_x'$$

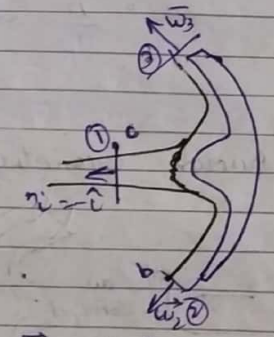
$\to 0$

(Flow is steady in travelling frame)

$$= -F - \int_{\partial R_{c.v}'} p_{atm}\, n_x\, dA'$$

$$= -F - p_{atm} \int_{\partial R_{c.v}} \cancel{n_x}^{\,0}\, dA'$$





$\vec{w}_1 = w_1 \hat{i}$

$\hat{n}_{c.s} = -\hat{i}$

$w_1 \cdot \hat{n} = 1$

$\vec{w}_3 = w_3(-\cos\beta_3\, \hat{i} + \sin\beta_3\, \hat{j})$

$\vec{w}_2 = w_2(-\cos\beta_2\, \hat{i} - \sin\beta_2\, \hat{j})$

$\vec{U} = r_p \Omega\, \hat{i}$

fixed ref. frame

$\to p_a + \frac{1}{2}\rho v_a^2 + \rho g y_a = p_b + \frac{1}{2}\rho v_b^2 + \rho g y_b$

moving ref. frame $\to$ $p_a + \frac{1}{2}\rho w_a^2 + \rho g y_a' = p_b + \frac{1}{2}\rho w_b^2 + \rho g y_b'$

$$p_a = p_b = p_{atm}$$

$$w_a = w_1 = \cancel{w_{jet}} \; V_{jet} - V$$

$$\frac{1}{2}\rho w_2^2 = \frac{1}{2}\rho w_1^2 + \rho g (y_a' - y_b')$$

$$\boxed{w_2 = w_1}$$

$$\int_{\partial R_1'} \rho w_x (\vec{w}\cdot\hat{n})\, dA + \int_{\partial R_2'} \rho w_x (\vec{w}\cdot\hat{n})\, dA$$

$$+ \int_{\partial R_3'} \rho w_x (\vec{w}\cdot\hat{n})\, dA$$

$$+ \int_{\partial R_{c.v} - \partial R_1 - \partial R_2 - \partial R_3} \rho w_x \cancel{(\vec{w}\cdot\hat{n})}^{\,0}\, dA = -F$$

$$-\int_{\partial R_1'} \rho w_1^2\, dA' - \int_{\partial R_2'} \rho w_2^2\, dA' - \int_{\partial R_3} \rho w_3^2\, dA = -F$$

$$-\rho w_1^2 A_1 - \rho w_2^2 A_2 \cos\beta_2 - \rho w_3^2 A_3 \cos\beta_3 = -F$$

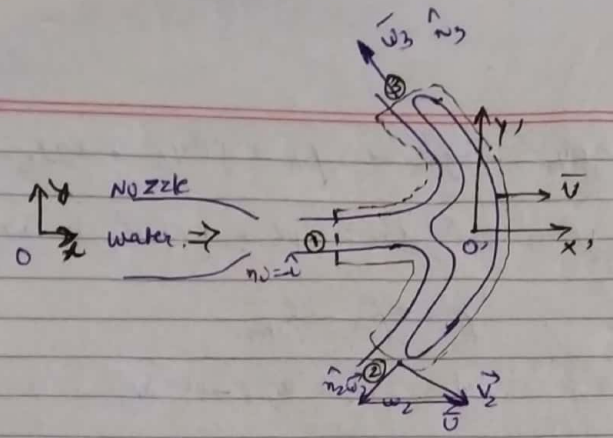
$$F = \rho w_1^2 (A_1 + A_2 + A_3)$$

$$F = \rho w_1 (w_1 A_1) + \rho w_2 \cos\beta_2 (w_2 A_2) + \rho w_3 \cos\beta_3 (w_3 A_3)$$

$$Q_1 = w_1 A_1 + w_2 A_2 + w_3 A_3$$

$$F = \rho (V_j - U) Q_1 + \rho (V_j - U) \cos\beta_2 (w_2 A_2) + \rho (V_j - U) \cos\beta_2\, w_3 A_3$$

Scanned by CamScanner

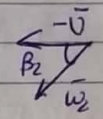


$$\int_{\partial R_{c.v}} \rho w_x (\vec{w}\cdot\hat{n})\, dA' = -F$$

$$\int_{\partial R_1} \rho w_x \vec{w}\cdot\hat{n}\, dA = \int_{\partial R_1} \rho w_1 (-w_1)\, dA$$

$$= -\rho w_1 (w_1 A_1)$$

$$= -\rho Q_1 w_1$$



$$\bar{w} = \bar{V} - \bar{U}$$

$$\bar{V} = \bar{w} + \bar{U}$$

$$n_2 = -\cos\beta_2 \hat{i} - \sin\beta_2 \hat{j}$$

$$\vec{w}_2 = w_2 \hat{n}_2$$

$$\int_{\partial R_2} \rho w_x \vec{w}\cdot\hat{n}\, dA = -\int_{\partial R_2} \rho w_2 \cos\beta_2\, w_2\, dA$$

$$= -\rho w_2 \cos\beta_2\, w_2 A_2$$

$$\int_{\partial R_3} \rho w_x \vec{w}\cdot\hat{n}\, dA = -\int_{\partial R_3} \rho w_3 \cos\beta_3\, w_3\, dA$$

$$= -\rho w_3 \cos\beta_3 (w_3 A_3)$$

$$\Rightarrow -\rho Q_1 w_1 - \rho w_2 \cos\beta_2 Q_2 - \rho w_3 \cos\beta_3 Q_3 = -F$$

$$w_1 = w_2 = w_3 = V_{jet} - U$$

$$F = \rho (V_{jet} - U)\{Q_1 + Q_2\cos\beta_2 + Q_3\cos\beta_3\}$$

Global Cons. of mass

$$Q_1 = Q_2 + Q_3$$

& $(\beta_2 = \beta_3)$

$$= \rho (V_{jet} - U)\{Q_1 + (Q_2 + Q_3)\cos\beta_2\}$$

$$F = \rho (V_{jet} - U) Q_1 (1 + \cos\beta_2)$$

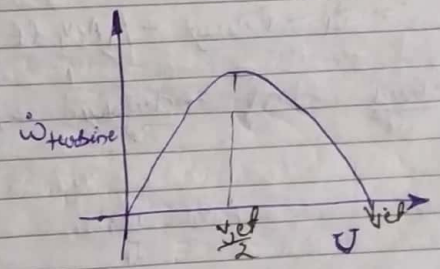
Power developed by one blade or bucket

$$F_b = F \cdot U$$

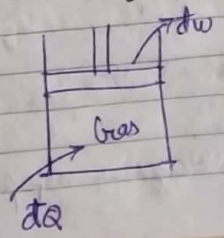
$$= \rho Q_1 (V_{jet} - U)\, U (1 + \cos\beta_2)$$

Power developed by the turbine –

$$\dot{W}_{turbine} = \rho Q (V_{jet} - U)\, U (1 + \cos\beta_2)$$



1st law of thermodynamics –



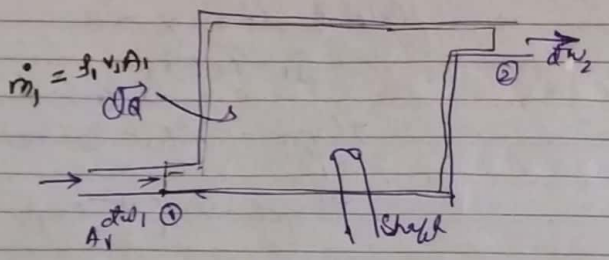
$$dE_{int} = dQ - dW$$

→ Gas. (+ve)

$$d(E_{int} + E_{kin}) = dQ - dW_{bdy} - dW_{surf} \quad (1)$$

what business??

$$d(E_{int} + E_{kin} + E_{pot}) = dQ - dW_{surf} \quad (2)$$



$$E_{sys} = E_{int} + E_{kin} + E_{pot}$$

$$dE_{sys} = dQ - dW_{shaft} + dW_1 - dW_2$$

$$dW_1 = (p_1 A_1)(v_1\, dt)$$

$$dW_2 = (p_2 A_2)\, v_2\, dt$$

$$dE_{sys} = dQ - dW_{shaft} + p_1 A_1 v_1\, dt - p_2 A_2 v_2\, dt$$

$$\frac{dE_{sys}}{dt} = \frac{dQ}{dt} - \frac{dW_{shaft}}{dt} + p_1 A_1 v_1 - p_2 A_2 v_2$$

$$A_1 v_1 = \frac{\dot{m}_1}{\rho_1}$$

$$A_2 v_2 = \frac{\dot{m}_2}{\rho_2}$$

$$= \frac{dQ}{dt} - \frac{dW_{shaft}}{dt} + \dot{m}_1 \frac{p_1}{\rho_1} - \dot{m}_2 \frac{p_2}{\rho_2}$$

Specific vol, $v = \frac{1}{\rho}$

$$\frac{dE_{sys}}{dt} = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1 p_1 v_1 - \dot{m}_2 p_2 v_2$$

$$E_{int} = \int e\, dm = \int_{R_m(t)} e \rho\, dV$$

$$E_{kin} = \int \frac{1}{2} \vec{V}\cdot\vec{V}\, dm = \int_{R_m(t)} \frac{1}{2} \vec{V}\cdot\vec{V} \rho\, dV$$

$$E_{pot} = \int dm\, g z = -\int dm\, \vec{g}\cdot\vec{r}$$

$$= -\int \vec{g}\cdot\vec{r} \rho\, dV$$

$$E_{sys} = \int_{R_m(t)} \underbrace{\left\{ e + \frac{1}{2}\vec{V}\cdot\vec{V} - \vec{g}\cdot\vec{r} \right\}}_{\phi} \rho\, dV$$

$$\frac{dE_{sys}}{dt} = \frac{d}{dt}\int_{R_m(t)} \rho\phi\, dV \quad \text{where } \phi = e + \frac{1}{2}\vec{V}\cdot\vec{V} - \vec{g}\cdot\vec{r}$$

How to distinguish the systems & C.V

$$\int_{R_{C.V}} \frac{\partial}{\partial t}(\rho\phi)\, dV + \int_{\partial R_{C.V}} \rho\phi\, \vec{V}\cdot\hat{n}\, dA = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1 \phi_1 - \dot{m}_2 \phi_2$$

$$\int_{\partial R_1} \rho\phi\, \vec{V}\cdot\hat{n}\, dA = -\rho_1 \phi_1 V_1 A_1$$

$$\int_{\partial R_2} \rho\phi\, \vec{V}\cdot\hat{n}\, dA = \rho_2 \phi_2 V_2 A_2$$

$$\int_{R_{C.V}} \frac{\partial}{\partial t}(\rho\phi)\, dV + \dot{m}_2 \phi_2 - \dot{m}_1 \phi_1 = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1 p_1 v_1 - p_2 \dot{m}_2 v_2$$

$$\int_{R_{C.V}} \frac{\partial}{\partial t}(\rho\phi\, dV) + \dot{m}_2 \phi_2 - \dot{m}_1 \phi_1 = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1 p_1 v_1 - \dot{m}_2 p_2 v_2$$

Scanned by CamScanner

• potential, K.E & pressure energy are high grade energy

$$\frac{dE_{C.V}}{dt} = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1\left(e_1 + \frac{\vec{V}_1\cdot\vec{V}_1}{2} - \vec{g}\cdot\vec{r} + p_1 v_1\right)$$

$$- \dot{m}_2\left(e_2 + \frac{1}{2}\vec{V}_2\cdot\vec{V}_2 - \vec{g}\cdot\vec{r} + p_2 v_2\right)$$

$$\frac{dE_{C.V}}{dt} = \dot{Q} - \dot{W}_{shaft} + \dot{m}_1\left(h_1 + \frac{1}{2}\vec{V}_1\cdot\vec{V}_1 - \vec{g}\cdot\vec{r}_1\right)$$

$$- m_2\left(h_2 + \frac{1}{2}\vec{V}_2\cdot\vec{V}_2 - \vec{g}\cdot\vec{r}_2\right)$$

Head-

$$B = p + \frac{1}{2}\rho V^2 - \rho\vec{g}\cdot\vec{r}$$ - Bernoulli func^n.

$$= B(x, y, z, t)$$

In steady, inviscid flow, $B(x,y,z)$ = const. along a streamline.

$$H = \frac{B}{\rho g} = \frac{p}{\rho g} + \frac{V^2}{2g} - \frac{\vec{g}\cdot\vec{r}}{g}$$

$$= \frac{p}{\rho g} + \frac{1}{2}\frac{\vec{V}\cdot\vec{V}}{g} - \frac{\vec{g}\cdot\vec{r}}{g}$$

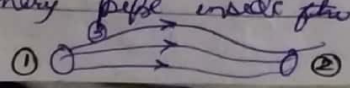
for viscous flow,

$$\tau_{yy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

$$\sigma_{xx} = -p + 2\mu\frac{\partial v_x}{\partial x}$$

Stream - surface - Collection of stream lines

Stream tube - imaginary pipe inside the which fluid flow



Steady flow Energy equation (SFEE) with 1 inlet & 1 outlet.

Inlet - section 1

Outlet - section 2.

$$\dot{m}_1 = \dot{m}_2 = \dot{m}$$

$$\dot{m}\left[h_1 + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1\right] + \dot{Q}_{in} = \dot{m}\left[h_2 + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2\right] + \dot{W}_{shaft}$$

$$h = e + pv = e + \frac{p}{\rho}$$

$$\Rightarrow e_1 + \frac{p_1}{\rho_1} + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1 + \frac{\dot{Q}_{in}}{\dot{m}} = e_2 + \frac{p_2}{\rho_2} + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2$$

$$\left(\frac{p_1}{\rho_1} + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1\right) = \left(\frac{p_2}{\rho_2} + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2\right) + e_2 - e_1 - \frac{\dot{Q}_{in}}{\dot{m}}$$

$$gH_1 = gH_2 + \left[(e_2 - e_1) - \frac{\dot{Q}_{in}}{\dot{m}}\right]$$

$$H_1 = H_2 + \frac{1}{g}\left[(e_2 - e_1) - \frac{\dot{Q}_{in}}{\dot{m}}\right]$$

$$(\Delta H_{loss})_{1-2} = H_1 - H_2$$

$$= \left(\frac{p_1}{\rho_1} + \frac{V_1^2}{2g} - \vec{g}\cdot\vec{r}_1\right) - \left(\frac{p_2}{\rho_2} + \frac{V_2^2}{2g} - \vec{g}\cdot\vec{r}_2\right)$$

$$= \frac{1}{g}\left\{(e_2 - e_1) - \frac{\dot{Q}_{in}}{\dot{m}}\right\}$$

Scanned by CamScanner

Engineering Bernoulli eqⁿ –

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} - \frac{\vec{g}\cdot\vec{r}}{g}\right)_{inlet} = \left(\frac{p}{\rho g} + \frac{V^2}{2g} - \frac{\vec{g}\cdot\vec{r}}{g}\right)_{outlet} + (H_{loss})_{1-2}.$$

Now some analysis with $\dot{W}_{shaft}$ –

$$\left(p_1 + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1 + \frac{\dot{Q}_{in}}{\dot{m}}\right) = \left(p_2 + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2\right) + \frac{\dot{W}_{shaft}}{\dot{m}}$$

$$\left(e_1 + \frac{p_1}{\rho_1} + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1\right) + \frac{\dot{Q}_{in}}{\dot{m}} = e_2 + \frac{p_2}{\rho_2} + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2 + \frac{\dot{W}_{shaft}}{\dot{m}}$$

$$\left(\frac{p_1}{\rho_1} + \frac{V_1^2}{2} - \vec{g}\cdot\vec{r}_1\right) = \left(p_2 + \frac{V_2^2}{2} - \vec{g}\cdot\vec{r}_2\right) + \underbrace{\left[(e_2 - e_1) - \frac{\dot{Q}_{in}}{\dot{m}}\right]}_{g H_{loss}} + \frac{\dot{W}_{shaft}}{\dot{m}}$$

$$H_1 = H_2 + H_{loss} + \frac{\dot{W}_{shaft}}{\dot{m} g}$$

$$\dot{W}_{shaft} = \dot{m} g \{(H_1 - H_2) - H_{loss}\}$$

$$= \rho Q_{flow} \{H_{inlet} - H_{outlet} - H_{loss}\}$$

2nd Law of thermo $\Rightarrow$ $H_{loss} \geq 0$

How H_{loss} can be measured.

Ideal fluid : $H_{loss} = 0$

Thus, $\dot{W}_{shaft} = \rho Q_{flow} \{H_{inlet} - H_{outlet}\}$

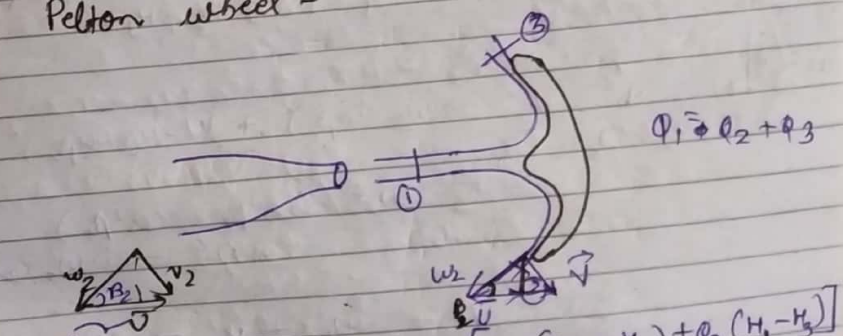
Pump : $\dot{W}_{shaft} < 0$.

$$\dot{W}_{pump} = |\dot{W}_{shaft}|$$
$$= -\dot{W}_{shaft}$$
$$= \rho Q_{flow} g [H_{outlet} - H_{inlet}]$$
$$= \rho Q_{flow} g H_{pump}.$$

Turbine : $\dot{W}_{shaft} > 0$

$$\dot{W}_{turbine} = |\dot{W}_{shaft}|$$
$$= \dot{W}_{shaft}$$
$$= \rho Q_{flow} g (H_{inlet} - H_{outlet})$$
$$= \rho Q_{flow} g H_{turbine}$$

Pelton wheel –



$$\dot{W}_{blade} = \rho g [Q_2 (H_1 - H_2) + Q_3 (H_1 - H_3)]$$

Now, lets see what happen by using momentum eqⁿ,

$$F_{blade} = \rho Q_1 (V_{jet} - U)(1 + \cos\beta_2)$$

Scanned by CamScanner

$T_{blade} = F_{blade}\, r_{pitch}$

$P_{blade} = T_{blade}\, \Omega$

$= F_{blade}(r_{blade}\, \Omega) = F_{blade}\, U$

P_{pelton} ~~P_{blade}~~ $= \rho Q (V_{jet} - U)\, U (1 + \cos\beta_2)$

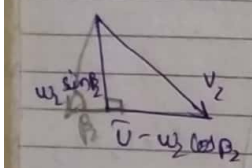
$Q = A_{jet} V_{jet} = Q_{nozzle}$ (negligible)

$$H_1 - H_2 = \cancel{\frac{p_1 - p_2}{\rho g}} + \frac{v_1^2 - v_2^2}{2g} - \cancel{\left(\frac{g \cdot \vec{r}_1 - g \cdot \vec{r}_2}{g}\right)}$$

0 as it is impulse turbine

$$H_1 - H_3 = \frac{V_1^2 - V_3^2}{2g} = \frac{V_1^2 - V_2^2}{2g} \quad \langle \text{symmetry} \rangle$$

$$= H_2 - H_3$$



$$V_2 = (w_2 \sin\beta_2)^2 + (U - w_2 \cos\beta_2)^2$$

$$= w_2^2 (\sin^2\beta_2 + \cos^2\beta_2) + U^2 - 2U w_2 \cos\beta_2$$

$$\vec{V}_2 = w_2^2 + U^2 - 2U w_2 \cos\beta_2$$

$$\vec{V}_1 = (w_1 + U)^2$$

$$= w_1^2 + U^2 + 2U w_1$$

$$V_1^2 - V_2^2 = 2U w_1 + 2U w_1 \cos\beta_2$$

$$= 2U w_1 (1 + \cos\beta_2)$$

$$g H_{turbine} = \frac{V_1^2 - V_2^2}{2}$$

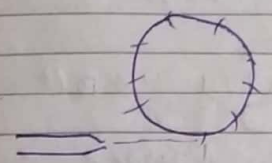
• Total fluid head ; $K = 0$

$$= U (V_{jet} - U)(1 + \cos\beta_2)$$

$$w_{blade} = \frac{\rho g Q_1 U (V_{jet} - U)(1 + \cos\beta_2)}{g}$$

$$= \rho Q_1 U (V_{jet} - U)(1 + \cos\beta_2)$$

Pelton turbine -



$F_{blade} = \rho Q_1 (V_{jet} - U)(1 + \cos\beta_2)$

$P_{blade} = \rho Q_1 (V_{jet} - U)(1 + \cos\beta_2)$

$$\dot{m}\left\{\frac{p_1}{\rho_1} + e_1 + \frac{v_1^2}{2} - \vec{g} \cdot \vec{r}_1\right\} + \dot{Q}_{in}$$

$$= \dot{m}\left\{\frac{p_2}{\rho_2} + e_2 + \frac{1}{2} v_2^2 - \vec{g} \cdot \vec{r}_2\right\} + \dot{W}$$

$$\dot{W} = \dot{m}(H_1 - H_2)$$

$$= \dot{m}\left(\frac{V_1^2}{2} - \frac{V_2^2}{2}\right)$$

Blade efficiency,

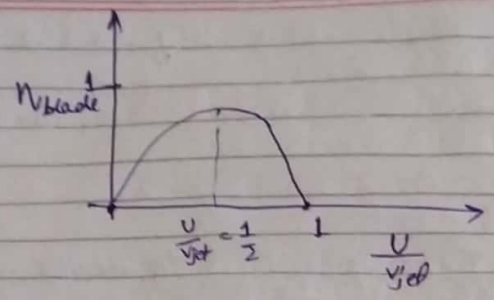
$$\eta_{blade} = \frac{P_{blade}}{P_{blade}|_{max}}$$

$$= \frac{\rho Q_1 U (V_{jet} - U)(1 + \cos\beta_2)}{\rho Q_1 \frac{1}{2} V_{jet}^2}$$

$$= \frac{2U}{V_{jet}}\left(1 - \frac{U}{V_{jet}}\right)(1 + \cos\beta_2)$$

$U = r_p \Omega$.

Scanned by CamScanner



"Runaway condition : $\frac{U}{V_{jet}} = 1$"

$$\eta_{blade}|_{max} = 2 \times \frac{1}{2} \times \frac{1}{2}(1+\cos\beta_2)$$
$$= \frac{(1+\cos\beta_2)}{2}$$

$P_{turbine} = ??$

$Q = A_{jet} V_{jet}$

flow rate going into moving C.V $\quad Q_1 = A_{jet}(V_{jet} - U) < Q$

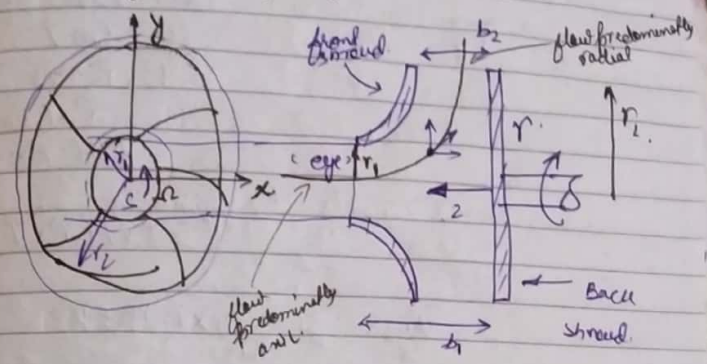
$$P_{turbine} = \rho Q U (V_{jet} - U)(1+\cos\beta_2)$$

$$P_{available} = \rho Q \frac{V_{jet}^2}{2}$$

$$A_{jet} = \frac{\pi}{4} d_{jet}^2$$

$$\eta_{wheel} = \frac{P_{turbine}}{P_{avail}} = \eta_{blade}$$

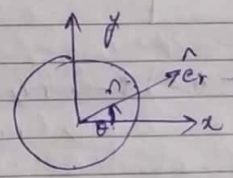
Centrifugal pump-



If blades are curved in same direction of rotation are called forward curved blade

If blades are curved in opposite direction of rotation, then its called ~~for~~ backward curved blade.

- Generally backward curved blades are used.



$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

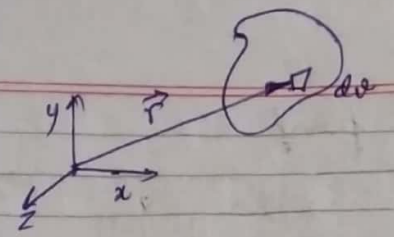
$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \vec{r} = r\hat{e}_r + z\hat{e}_z$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\hat{e}_z = \hat{k}$

Scanned by CamScanner



$$dm = \rho\, dv$$
$$d\vec{L} = dm\,\vec{v}$$
$$d\vec{H} = \vec{r} \times d\vec{L}$$
$$= \vec{r} \times (dm\,\vec{v})$$
$$= dm(\vec{r} \times \vec{v})$$

$$d\vec{H} = \rho\, dv\,(\vec{r} \times \vec{v})$$

$$(\vec{H}_{sys}) = \int_{R_m(t)} (\vec{r} \times \vec{v})\,\rho\, dv$$

$$\frac{d\vec{H}_{sys}}{dt} = \Sigma \vec{M}$$

Now, $\vec{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z$

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ r & 0 & z \\ v_r & v_\theta & v_z \end{vmatrix}$$

$$= \hat{e}_r(-v_\theta z) - \hat{e}_\theta(r v_z - v_r z) + \hat{e}_z(r v_\theta)$$

$$dH_z = dm\, r\, v_\theta$$

$$H_z|_{sys} = \int_{R_m(t)} \rho\, r v_\theta\, dv$$

$$= \int_{R_m(t)} \rho\, \phi\, dv \quad \text{where } \phi = r v_\theta$$

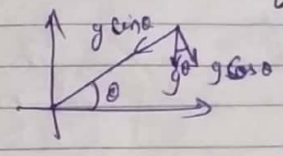
$$\frac{d(H_z)_{sys}}{dt} = M_z^{ext} + M_z^{body} + M_z^{surf}$$

$$d\vec{M}_{body} = r \times d\vec{F}_{body}$$
$$= \vec{r} \times dm\,\vec{g}$$
$$= dm(\vec{r} \times \vec{g})$$
$$= \vec{r} \times \vec{g}\,\rho\, dv$$

$$\vec{M}_{body} = \int_{R_m(t)} \vec{r} \times \vec{g}\,\rho\, dv$$

$$\vec{r} \times \vec{g} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ r & 0 & z \\ g_r & g_\theta & 0 \end{vmatrix}$$

(Since most pumps have horizontal shaft)

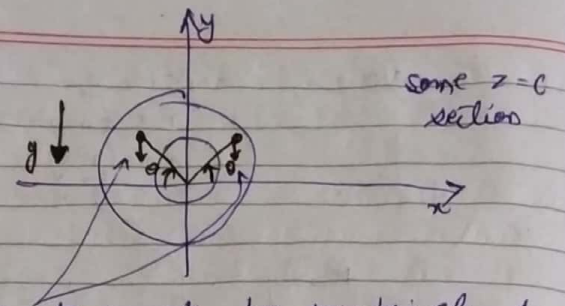


$$g_r = -g\sin\theta$$
$$g_\theta = -g\cos\theta$$

$$\therefore (\vec{r} \times \vec{g})_z = \begin{vmatrix} r & 0 \\ g_r & g_\theta \end{vmatrix} = r g_\theta$$
$$= -r g \cos\theta$$

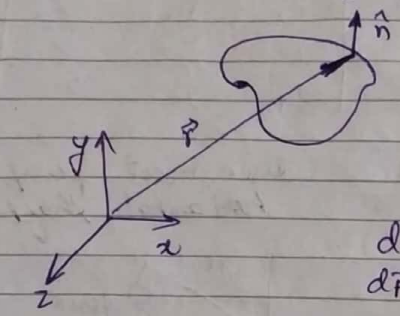
$$M_z^{body} = \int -r g \cos\theta\, \rho\, dv$$
$$= -r g \iiint \cos\theta\, \rho\, r\, d\theta\, dr\, dz$$
$$= \int_0^{2\pi} \cos\theta\, d\theta \left(\iint \rho r^2\, dr\, dz \right)$$
$$= 0$$

Scanned by CamScanner



some z = c section

Counter moments by symmetric element
thus $M_z^{body} = 0$



$$d\vec{M}^{surf} = \vec{r} \times d\vec{F}^{surf}$$

$$d\vec{F}^{surf} = p\,dA(-\hat{n})$$

$$dM^{surf} = -p(\vec{r} \times \hat{n})\,dA$$

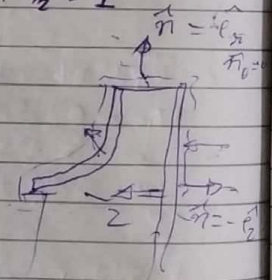
$$\hat{n} = n_r \hat{e}_r + n_\theta \hat{e}_\theta + n_z \hat{e}_z$$

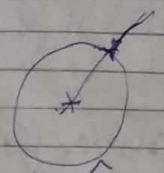
$\hat{n} = n_r \hat{e}_r + n_\theta \hat{e}_\theta + n_z \hat{e}_z$

$$n_r^2 + n_\theta^2 + n_z^2 = 1$$

$$(\vec{r} \times \hat{n})_z = r n_\theta$$

$$M_z^{surf} = 0$$





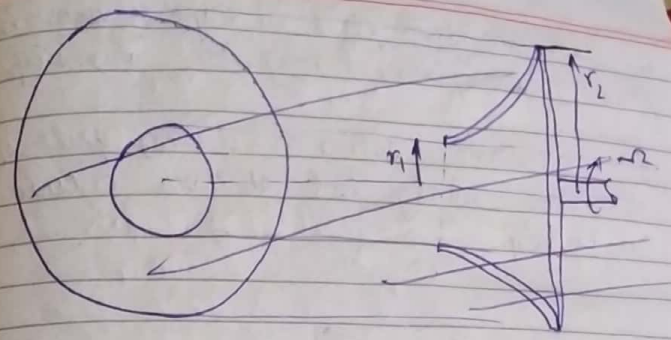
$$z_r = z_s(r)$$

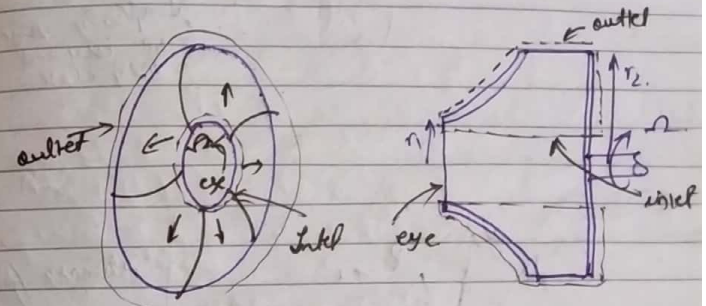
$$f(r, z) = z - z_s(r) = 0$$

$$\nabla f = \hat{e}_z - \frac{dz_s}{dr}\hat{e}_r$$

$$\hat{n} = \frac{\hat{e}_z - \frac{dz_s}{dr}\hat{e}_r}{\sqrt{1 + \left(\frac{dz_s}{dr}\right)^2}}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$





Balance of moment of momentum in z-dir^n about the axis of symmetry.

$$\frac{dH_z}{dt} = M_z^{ext} + \underbrace{M_z^{body}}_{0} + \underbrace{M_z^{surf}}_{0}$$

$$H_z = \int_{R_{ref}(t)} r v_\theta \rho\, dv, \quad \phi = r v_\theta$$

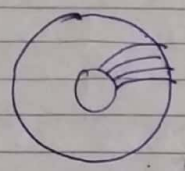
$$\frac{dH_z}{dt} = \frac{d}{dt}\int_{R_{ref}(t)} \rho\phi\, dv$$

$$= \int_{R_{c.v}} \frac{\partial}{\partial t}(\rho\phi)\, dv + \int_{\partial R_{c.v}} \rho\phi\, \vec{v}\cdot\hat{n}\, dA$$

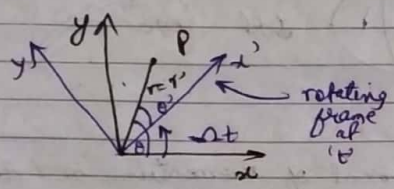
Scanned by CamScanner

Near vanes the fluid flows in shape of vanes but far away from vanes it may take any shape.

for an observer sitting on rotating part sees streamline steady, but observer outside sees it unsteady



for sake of simplicity vanes are assumed very close.



Stationary frame of reference $Oxyz$
Rotating frame of reference $Ox'y'z'$.

$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r' = \sqrt{x'^2+y'^2}$
$\theta' = \theta - \Omega t$
$z' = z$
$t' = t$

$$\rho\phi = f_1(r,\theta,z,t) = f_2(r',\theta',z',t')$$

$$\rho\phi \begin{cases} r' & \text{—— } r \\ \theta' & \text{—— } \theta \\ z' & \text{—— } z \\ t' & \text{—— } t \end{cases}$$

$r' = r$
$\theta' = \theta - \Omega t$
$z' = z$
$t' = t$

$$\frac{\partial(\rho\phi)}{\partial t} = \frac{\partial}{\partial\theta'}(\rho\phi) \times \frac{\partial\theta'}{\partial t} + \frac{\partial}{\partial t'}(\rho\phi)\underbrace{\frac{\partial t'}{\partial t}}_{1}$$

$$\frac{\partial\theta'}{\partial t} = -\Omega$$

$$= -\Omega\frac{\partial}{\partial\theta'}(\rho\phi) + \frac{\partial}{\partial t'}(\rho\phi)$$

$$\frac{\partial(\rho\phi)}{\partial t} = \frac{\partial}{\partial t'}(\rho\phi) - \Omega\frac{\partial(\rho\phi)}{\partial\theta'}$$

$$\frac{\partial(\rho\phi)}{\partial r} = \frac{\partial}{\partial r'}(\rho\phi) \times \frac{\partial r'}{\partial r} = \frac{\partial(\rho\phi)}{\partial r'}$$

$$\frac{\partial(\rho\phi)}{\partial\theta} = \frac{\partial(\rho\phi)}{\partial\theta'} \times \frac{\partial\theta'}{\partial\theta} = \frac{\partial(\rho\phi)}{\partial\theta'}$$

$$\frac{\partial}{\partial z}(\rho\phi) = \frac{\partial}{\partial z'}(\rho\phi) \times \frac{\partial z'}{\partial z} = \frac{\partial(\rho\phi)}{\partial z'}$$

$$\int_{R.C.V}\frac{\partial(\rho\phi)}{\partial t}\,dv = \int_{R.C.V}\left\{\frac{\partial}{\partial t'}(\rho\phi) - \Omega\frac{\partial(\rho\phi)}{\partial\theta'}\right\}dv$$

$$= \int_{R.C.V}\frac{\partial}{\partial t'}(\rho\phi)\,dv - \int_{R.C.V}\frac{\partial}{\partial\theta'}(\Omega\rho\phi)\,dv$$

$$\vec{b} = b_r\hat{e}_r + b_\theta\hat{e}_\theta + \hat{e}_z b_z$$

$$\nabla\cdot\vec{b} = \frac{1}{r}\times\frac{\partial}{\partial r}(r b_r) + \frac{1}{r}\frac{\partial b_\theta}{\partial\theta} + \frac{\partial b_z}{\partial z}$$

If $b_r = 0$, $b_z = 0$

$$\vec{\nabla}\cdot\vec{b} = \frac{1}{r}\frac{\partial b_\theta}{\partial\theta}$$

So, if $b_\theta = (\text{const.}\ \Omega\rho\phi)$

Scanned by CamScanner

$$\int_{R.C.V} \frac{\partial}{\partial t}(\rho\phi)\,d\forall = \int_{R.C.V} \frac{\partial}{\partial t'}(\rho\phi)\,d\forall - \int_{R.C.V} \nabla\cdot\vec{b}\,d\forall$$

where $\vec{b} = -\Omega r(\rho\phi)\,\hat{e}_\theta$

$$= \int_{R.C.V} \frac{\partial}{\partial t'}(\rho\phi)\,d\forall - \int_{\partial R.C.V} \vec{b}\cdot\hat{n}\,dA$$

$$U(\vec{r}) = -\Omega r\,\hat{e}_\theta = \vec{\Omega}\times\vec{r}. \quad (\vec{\Omega} = -\Omega\hat{k})$$

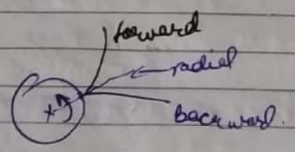
$$\vec{b} = -\Omega r(\rho\phi)\,\hat{e}_\theta$$
$$= \rho\phi\,\vec{U}(\vec{r})$$

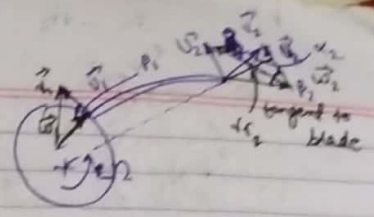
$$\int_{R.C.V} \frac{\partial}{\partial t}(\rho\phi)\,d\forall = \int_{R.C.V} \frac{\partial}{\partial t'}(\rho\phi)\,d\forall - \int_{\partial R.C.V} \rho\phi\,\vec{U}(\vec{r})\cdot\hat{n}\,dA$$

So, $$\frac{dH_z}{dt} = \int_{R.C.V} \frac{\partial}{\partial t'}(\rho\phi)\,d\forall - \int_{\partial R.C.V} \rho\phi\,\vec{U}\cdot\hat{n}\,dA + \int_{\partial R.C.V} \rho\phi\,\vec{V}\cdot\hat{n}\,dA$$

$$\frac{d}{dt}\int_{R.C.V} \rho\phi\,d\forall = \frac{dH_z}{dt} = \int_{R.C.V} \frac{\partial}{\partial t'}(\rho\phi)\,d\forall + \int_{\partial R.C.V} \rho\phi\,(\vec{V}-\vec{U})\cdot n\,dA$$

$$\vec{V}-\vec{U} = \vec{W} = \vec{V} - \Omega r\,\hat{e}_\theta$$

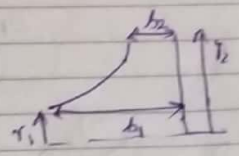




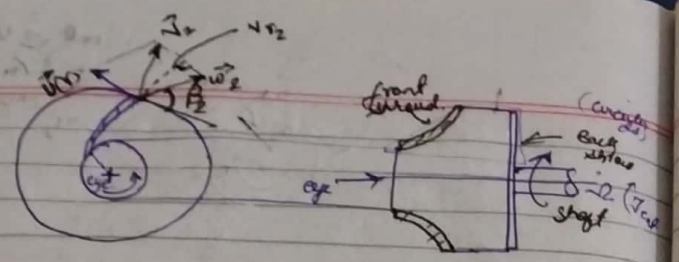
$$\vec{V}_2 = \vec{W}_2 + \vec{U}_2$$
$$\vec{V}_1 = \vec{W}_1 + \vec{U}_1$$

Volume flow rate, $Q = 2\pi r_1 b_1 V_{r_1}$

$$Q = 2\pi r_2 b_2 V_{r_2}$$



Scanned by CamScanner



Vol. flow rate through pump

(i) From conditions at inlet or suction pipe

D_p = diameter of inlet pipe (or suction pipe)

$A_p = \frac{\pi D_p^2}{4}$ = c.s area of pipe

$(V_z)_p$ = axial velocity in pipe (avg. velo.)

Q = Vol. flow rate $= A_p (V_z)_{pipe}$

(ii) From conditions at $r = r_1$ (inlet to passage b/w blades)

$Q = 2\pi r_1 b_1 V_{r1}$ (V_{r1} is assumed to be const.)

(iii) From Cond^n at $r = r_2$

$Q = 2\pi r_2 b_2 V_{r2}$

$\vec{w} = \vec{V} - \vec{U}(r)$

ϕ = is angle b/w tangents to blade & circle
= angle b/w $-U_2$ & $\vec{w}_2$

at impeller exit.

$V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z = w_r \hat{e}_r + w_\theta \hat{e}_\theta + w_z \hat{e}_z + U \hat{e}_\theta$

$\Rightarrow V_r = w_r$, $V_\theta = w_\theta + U$, $V_z = 0$, $w_z = 0$

$U_2 = r_2 \Omega$

$U_1 = r_1 \Omega$

Thus, $Q = 2\pi r_2 b_2 w_{r_2}$

Also $Q = 2\pi r_1 b_1 w_{r_1}$

Global balance of moment about z-axis,

$$\frac{d(H_z)_{sys}}{dt} = M_z^{ext} + \cancel{M_z^{body}}^{0} + \cancel{M_z^{surf}}^{0}$$

$$\frac{d}{dt}\int_{R_m(t)} \rho \phi \, dv = \frac{dH_z}{dt} = \int_{R_c \cdot V} \frac{\partial}{\partial t}(\rho\phi)\, dv^{\,=0 \text{ (steady in relative ref. frame)}} + \int_{\partial R_c \cdot V} \rho\phi \underbrace{(\vec{V} - \vec{U})}_{\vec{w}} \cdot \hat{n}\, dA$$

$$\int_{\partial R_c \cdot V} \rho r V_\theta \, \vec{w} \cdot \hat{n}\, dA = \int_{\partial R_1} \rho r V_\theta \underbrace{(\vec{w} \cdot \hat{n})}_{-} dA + \int_{\partial R_2} \rho r V_\theta \underbrace{\vec{w} \cdot \hat{n}}_{+} dA + \int_{\partial R_{c.v} - \partial R_1 - \partial R_2} \rho r V_\theta \underbrace{(\vec{w} \cdot \hat{n})}_{0} dA$$

$$= \int_{\partial R_1} \rho r_1 V_{\theta_1} (-w_{r_1}) dA + \int_{\partial R_2} \rho r_2 V_{\theta_2} (w_{r_2}) dA$$

$$= -\rho r_1 V_{\theta_1} w_{r_1} A_1 + \rho r_2 V_{\theta_2} w_{r_2} A_2$$

$$T^{ext} = \cancel{-\rho r (w_{r_2} A_2 - w_{r_1} A_1)}$$

$$T^{ext} = \rho Q (V_{\theta_2} r_2 - V_{\theta_1} r_1)$$

Hydraulic power, $P_{hydro} = T^{ext} \Omega$

$= \rho Q (V_{\theta_2} r_2 - V_{\theta_1} r_1)\Omega$

$= \rho Q (V_{\theta_2} U_2 - V_{\theta_1} U_1)$

$= \rho Q (U_2 V_{\theta_2} - U_1 V_{\theta_1})$

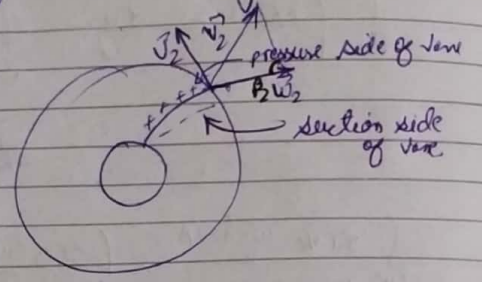
Scanned by CamScanner

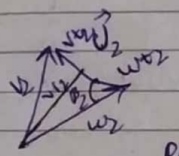
$P_{ext} = \rho Q g H_{pump}$

Head, $H = \dfrac{U_2 V_{t_2} - U_1 V_{t_1}}{g}$

"usual condition" : No pre-whirl.
At inlet to eye impeller, $V_\theta = V_t = 0$

The fluid acquires swirl only when it crosses r_1 & r_2.





Hydro Power $= \rho Q U_2 V_{t_2}$ $\langle V_{\theta_1} = V_{t_1} = 0 \rangle$

$\tan\beta_2 = \dfrac{V_{n_2}}{\omega_{t_2}}$ (ω_{n_2}) $\quad \therefore H = \dfrac{U_2 V_{t_2}}{g}$

$\omega_{t_2} = V_{n_2} \cot\beta_2$ $\quad H = \dfrac{U_2}{g}(U_2 - V_{n_2}\cot\beta_2)$

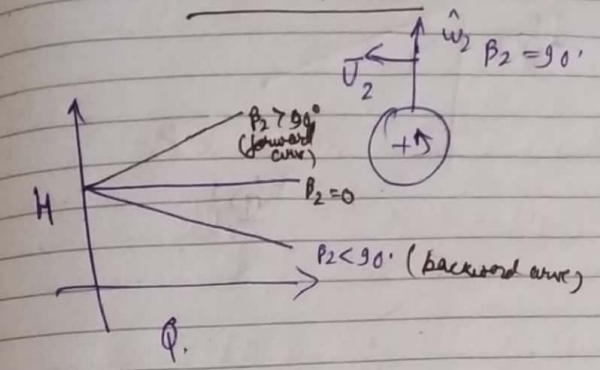
$$H = \frac{U_2^2}{g} - U_2 \frac{Q}{2\pi r_2 b_2} \cot\beta_2$$

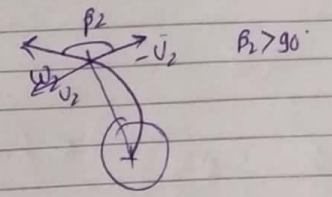
$$H = a - bQ$$

$$a = \frac{U_2^2}{g} = H_{shaft}$$

$$b = +\frac{U_2 \, a}{2\pi r_2 b_2} \cot\beta_2$$

Radial blades





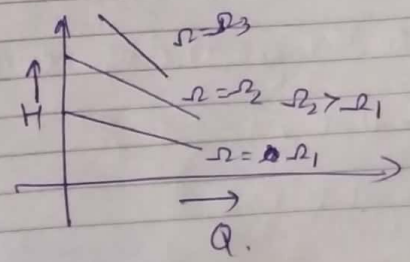
$$H = \frac{\Omega^2 r_2^2}{g} - \frac{\Omega r_2 \cot\beta_2 \, Q}{2\pi r_2 b_2 g}$$

$$gH = \Omega^2 r_2^2 - \frac{\Omega \cot\beta_2 Q}{2\pi b_2}$$

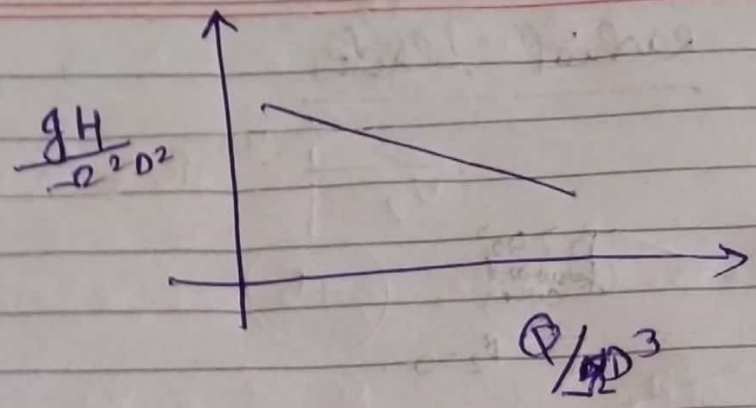
$\boxed{D = 2r_2}$

$$\underbrace{\frac{gH}{\Omega^2 D^2}}_{C_H} = \left(\frac{r_2}{D}\right)^2 - \frac{\cot\beta_2}{2\pi\left(\frac{b_2}{D}\right)} \cdot \left(\frac{Q}{\Omega D^3}\right)$$

C_H Head co-efficient



Scanned by CamScanner



Scanned by CamScanner