$$u\frac{dT}{dx} - \alpha\frac{d^2T}{dx^2} + u\frac{(\Delta x)^2}{6}\frac{d^3T}{dx^3} - \alpha\frac{(\Delta x)^2}{12}\frac{d^4T}{dx^4} + \cdots$$

2nd order dispersion — second order dissipation

Higher order upwind scheme

$$u\frac{dT}{dx} - \alpha\frac{d^2T}{dx^2} = 0$$

How to know?? Is it because the term has velocity??

Centrale difference / upwind.

$\frac{dT}{dx} \rightarrow$ 1) 2nd order accurate central difference approximation

2) 2nd order accurate one sided upwind difference approximation

Take a weighted mean of the two. (NEW IDEA)

1) $$\boxed{\frac{dT}{dx} = \frac{T_{j+1} - T_{j-1}}{2\Delta x} + O(\Delta x^2)}$$ central difference 2nd order

2) $$T_{j-1} = T_j - \Delta x\left(\frac{dT}{dx}\right)_j + \frac{(\Delta x)^2}{2}\left(\frac{d^2T}{dx^2}\right)_j + O(\Delta x^3) \quad \times 4$$

$$T_{j-2} = T_j - 2\Delta x\left(\frac{dT}{dx}\right)_j + 4\frac{(\Delta x)^2}{2}\left(\frac{d^2T}{dx^2}\right) + O(\Delta x^3) \quad \times 1$$

using 2 upwind eq

$$T_{j-2} - 4T_{j-1} = T_j - 4T_j - 2\Delta x\left(\frac{dT}{dx}\right)_j + 4(\Delta x)\left(\frac{dT}{dx}\right) + O(\Delta x^3).$$

2nd order upwind eq's

$$T_{j-2} - 4T_{j-1} + 3T_j = 2\Delta x\left(\frac{dT}{dx}\right)_j + O(\Delta x^3)$$

$$\boxed{\left(\frac{dT}{dx}\right)_j = \frac{3T_j - 4T_{j-1} + T_{j-2}}{2\Delta x} + O(\Delta x^2)}$$

$$\left(\frac{dT}{dx}\right)_j = (1-a)\left(\frac{dT}{dx}\right)_j^{\text{central difference}} + a\left(\frac{dT}{dx}\right)_j^{\text{upwind approx}}$$

Why not central × upwind??

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$$= (1-a)\left(\frac{T_{j+1}-T_{j-1}}{2\Delta x}\right) + a\left(\frac{3T_j - 4T_{j-1} + T_{j-2}}{2\Delta x}\right)$$

$$= \frac{T_{j+1}-T_{j-1}}{2\Delta x} + a\left[\frac{-T_{j+1}+T_{j-1}}{2\Delta x} + \frac{3T_j - 4T_{j-1} + T_{j-2}}{2\Delta x}\right]$$

$$= \frac{T_{j+1}-T_{j-1}}{2\Delta x} + a\underbrace{\left[\frac{-T_{j+1}+3T_j - 3T_{j-1} + T_{j-2}}{2\Delta x}\right]}_{\text{only for } \frac{dT}{dx}}$$

$a = 0 \Rightarrow$ central diff app. $\Rightarrow q = 0$

$a = 1 \Rightarrow$ Upwind ~~appr~~ approx^m $\Rightarrow q = \frac{3}{2}$

In general,

$$\frac{a}{2} = \frac{q}{3}$$

$$\frac{dT}{dx} = \frac{T_{j+1}-T_{j-1}}{2\Delta x} - q\,\frac{T_{j+1} - 3T_j + 3T_{j-1} - T_{j-2}}{3\Delta x}$$

As $a \in [0,1]$ so $q = \frac{3}{2}a$, $q \to [0, 1.5]$

Numerical methods show that as q increases (weightage of upwind approx^m increases) oscillation decreases. For $q=0$, oscillation clearly seen for large Peclet. For $q = 0.5$ there is still some oscillation. For $q = 1$ and $q = 1.5$ oscillations are not observed.

$$u\left[\frac{T_{j+1}-T_{j-1}}{2\Delta x} - q\,\frac{T_{j+1} - 3T_j - 3T_{j-1} - T_{j-2}}{3\Delta x}\right] - \alpha\,\frac{T_{j+1} - 2T_j - T_{j-1}}{(\Delta x)^2} = 0$$

~~$T_j = \gamma^j$~~ $T_j = \gamma^j$

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8/3/18
Thurs

$$\frac{\partial T}{dt} + \vec{V}.\nabla T = \alpha \nabla^2 T$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

one - dimensional Transport equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}$$

1. FTCS scheme

forward central — Explicit scheme — central

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} - \alpha\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{(\Delta x)^2} = 0$$

$$T_j^{n+1} - T_j^n + \frac{c}{2}\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right) = 0$$

Stability

$$G - 1 + \frac{c}{2}\left(e^{i\theta} - e^{-i\theta}\right) - s\left(e^{i\theta} - 2 + e^{-i\theta}\right) = 0$$

$$G - 1 + \frac{c}{2}\, 2i\sin\theta - s(2\cos\theta - 2) = 0$$

$$G = 1 - ic\sin\theta + 2s(\cos\theta - 1) = 1 - 2s(1-\cos\theta) - ic\sin\theta$$

$$|G|^2 = \left[1 - 2s(1-\cos\theta)\right]^2 + \left[c\sin\theta\right]^2$$

$$= \left[1 - 2s(1-\cos\theta)\right]^2 + c^2\sin^2\theta$$

$x = \cos\theta$
$\Rightarrow -1 \le x \le 1$

$$|G|^2 = \left[1 - 2s(1-x)\right]^2 + c^2(1-x^2)$$

$$= 1 - 4s(1-x) + 4s^2(1-x)^2 + c^2(1-x^2)$$

$$|G|^2 - 1 = -4s(1-x) + 4s^2(1-x)^2 + c^2(1-x^2)$$

$$= F(x)$$

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For stability we require,

$|g|^2 - 1 \le 0$ which is equivalent to

$F(x) \le 0$ for $-1 \le x \le 1$

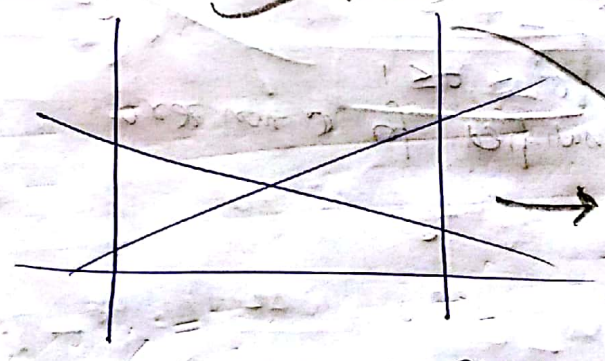
$F(x) = (1-x)\left[-4s + 4s^2(1-x) + c^2(1+x)\right] = (1-x)\,\hat{F}(x)$

For $-1 \le x \le 1$ we always have $1-x \ge 0$.

$\hat{F}(x) = -4s + 4s^2(1-x) + c^2(1+x)$

For stability we must have $\hat{F}(x) \le 0$ for $-1 \le x \le 1$

$\hat{F}(x) = -4s + 4s^2 + c^2 + \underbrace{(-4s^2 + c^2)}\,x$



$\rightarrow$ Linear function for stability for all values of θ.

$\hat{F}(x) \le 0$ in the range $-1 \le x \le 1$

provided $\hat{F}(x) \le 0$ for $x = -1$, $x = 1$.

$\hat{F}(1) = -4s + \cancel{4s^2} + c^2 + (\cancel{-4s^2} + c^2)$

$= -4s + 2c^2 = 2(-2s + c^2)$

$\hat{F}(-1) = -4s + 4s^2 + c^2 - (-4s^2 + c^2)$

$= -4s + 8s^2 = 4s(-1 + 2s)$

For stability we must have

1. $\boxed{2(-2s + c^2) \le 0} \Rightarrow c^2 \le 2s$

2. $\boxed{4s(-1 + 2s) \le 0} \Rightarrow 2s \le 1$

Finally for stability we require $\boxed{c^2 \le 2s \le 1}$

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When $u = 0 \Rightarrow$ ~~we consi~~ no convection term,

$$C = -\frac{u\Delta t}{\Delta x} = 0$$

the condition for stability reduces to

$$\sigma^2 \le 2s \le 1 \text{ or } 2s \le 1 \Rightarrow \boxed{s \le \frac{1}{2}}$$

(By default satisfies)

When $\alpha = 0 \Rightarrow$ no diffusion term

$$s = \frac{\alpha \Delta t}{(\Delta x)^2} = 0$$

the condition for stability reduces to

$$\boxed{c^2 \le 0 \le 1}$$

cannot be satisfied for c non zero.

2 Fully implicit scheme

T_j, T_{j+1}, T_{j-1}

$$\left(\frac{\partial T}{\partial t}\right)_j^{n+1} + u\left(\frac{\partial T}{\partial x}\right)_j^{n+1} - \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)_j^{n+1}$$

tridiagonal matrix (3 unknowns)

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\,\frac{T_{j+1}^{n+1} - T_{j-1}^{n+1}}{2\Delta x} - \alpha\,\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{(\Delta x)^2} = 0$$

Using the Von - Neumann method to determine whether the scheme is stable and is there a condition on stability.

Option 1 (for question framing)

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\,\boxed{\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x}}^{\text{explicit}} - \alpha\,\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{(\Delta x)^2} = 0$$

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explicit.

Option 2

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\,\frac{T_{j+1}^{n+1} - T_{j-1}^{n+1}}{2\Delta x} - \alpha\,\boxed{\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{(\Delta x)^2}} = 0$$

Derive the condition for these schemes to be stable

FTCS

$c^2 \le 2S \le 1$

Q. Upwind Scheme

only this change

$$\frac{T^{n+1} - T^n}{\Delta t} + u\,\boxed{\frac{T_j^n - T_{j-1}^n}{\Delta x}} - \alpha\,\frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{(\Delta x)^2} = 0$$

When is this stable? Find out

Fully implicit

Q

Final results given in Fletcher (in a table form)

→ Try both the schemes by the method that we used for FTCS and by the method in the reference given by Fletcher

Crank-Nicolson Scheme (at the time scale $n+\frac{1}{2}$)

$$\left(\frac{\partial T}{\partial t}\right)_j^{n+\frac{1}{2}} + u\left(\frac{\partial T}{\partial x}\right)_j^{n+\frac{1}{2}} - \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)_j^{n+\frac{1}{2}} = 0$$

What is the condition for stability?

12/3/18 Transport Equation

$$\frac{\partial t}{\partial t} + \vec{u}\cdot\nabla T = \alpha\nabla^2 T.$$

Two-dimensional transport equation.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \alpha_x\frac{\partial^2 T}{\partial x^2} - \alpha_y\frac{\partial^2 T}{\partial y^2} = 0.$$

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1. FTCS scheme

$$\frac{T_{j,k}^{n+1} - T_{j,k}^{n}}{\Delta t} + u\,\frac{T_{j+1,k}^{n} - T_{j-1,k}^{n}}{2\Delta x} + v\,\frac{T_{j,k+1}^{n} - T_{j,k-1}^{n}}{2\Delta y}$$

$$= \alpha_x \frac{T_{j+1,k}^{n} - 2T_{j,k}^{n} + T_{j-1,k}^{n}}{(\Delta x)^2} - \alpha_y \frac{T_{j,k+1}^{n} - 2T_{j,k}^{n} + T_{j,k-1}^{n}}{(\Delta y)^2} = 0$$

We assume the solution has the form –

$$e^{\sigma t} e^{i m_x \pi x} e^{i m_y \pi y}$$

$$\frac{G-1}{\Delta t} + u\,\frac{e^{i m_x \pi \Delta x} - e^{-i m_x \pi \Delta x}}{2\Delta x} + v\,\frac{e^{i m_y \pi \Delta y} - e^{-i m_y \pi \Delta y}}{2\Delta y}$$

$$- \alpha_x \left(\frac{e^{i m_x \pi \Delta x} - 2 + e^{-i m_x \pi \Delta x}}{(\Delta x)^2} \right) - \alpha_y \left(\frac{e^{i m_y \pi \Delta y} - 2 + e^{-i m_y \pi \Delta y}}{(\Delta y)^2} \right) =$$

We define,

$$\theta_x = m_x \pi \Delta x, \quad \Delta y = m_y \pi \Delta y$$

then

$$G - 1 + \frac{u \Delta t}{2\Delta x}\left(e^{i\theta_x} - e^{-i\theta_x}\right) + \frac{v \Delta t}{2\Delta y}\left(e^{i\theta_y} - e^{-i\theta_y}\right) - \frac{\alpha_x \Delta t}{(\Delta x)^2}$$

$$\left(e^{i\theta_x} - 2 + e^{-i\theta_x}\right) - \frac{\alpha_y \Delta t}{(\Delta y)^2}\left(e^{i\theta_y} - 2 + e^{-i\theta_y}\right) =$$

We define,

$$S_x = \frac{\alpha_x \Delta t}{(\Delta x)^2}, \quad S_y = \frac{\alpha_y \Delta t}{(\Delta y)^2}, \quad C_x = \frac{u \Delta t}{\Delta x}, \quad C_y = \frac{v \Delta t}{\Delta y}$$

then

$$G - 1 + \frac{C_x}{\cancel{2}} \times i \cancel{2} \sin\theta_x + \frac{C_y}{\cancel{2}} \cancel{2} i \sin\theta_y - S_x(2\cos\theta_x - 2) - S_y(2\cos\theta_y - 2) =$$

$$G = 1 - S_x(2 - 2\cos\theta_x) - S_y(2 - 2\cos\theta_y) - i(C_x \sin\theta_x + C_y \sin\theta_y)$$

$$G = 1 - 2S_x(1 - \cos\theta_x) - 2S_y(1 - \cos\theta_y) - i(C_x \sin\theta_x + C_y \sin\theta_y)$$

$$e^{i m_x \pi \Delta x} \rightarrow e^{i\theta_x} \qquad \theta_x = m_x \pi \Delta x$$

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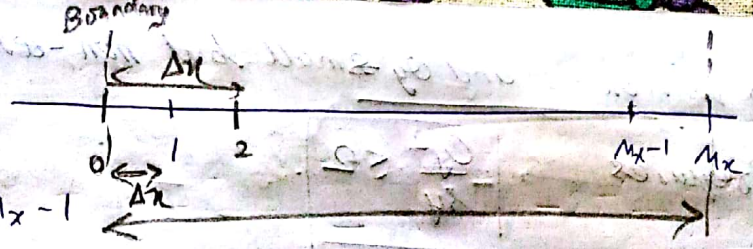
$\theta_x = m_x \pi \Delta x$

$\Delta x = \frac{1}{M_x}$

$M_x \to 1, \ldots M_x - 1$



$\theta_x = m_x \pi \Delta x$

$1 \times \pi \times \frac{1}{M_x} \ldots, (m_x - 1) \times \pi \times \frac{1}{M_x}$ i.e $\frac{\pi}{M_x}$ to $\frac{\pi(m_x-1)}{M_x}$

As $m_x \to \infty$

Considering one extreme case

Special case 1

θ_x ranges from 0 to π. Let us consider the mode with $\theta_x = \theta_y = \pi$ (largest)

$G = 1 - 2S_x \times 2 - 2S_y \times 2 = 1 - 4S_x - 4S_y$

For this mode to be stable we require

$-1 \le G \le 1 \Rightarrow -1 \le 1 - 4S_x - 4S_y \le 1$

The inequality on the right is always satisfied.

The inequality in the left requires $\boxed{S_x + S_y \le \frac{1}{2}}$

Considering other extreme

Now, let us consider mode for which θ_x and θ_y are small

$\sin\theta_x \approx \theta_x$, $\sin\theta_y \approx \theta_y$ $\quad 1 - \cos\theta_x \approx 1 - (1 - \frac{\theta_x^2}{2} + \ldots) \approx \frac{\theta_x^2}{2}$

$1 - \cos\theta_x = 1 -$ similarly, $1 - \cos\theta_y \approx \frac{\theta_y^2}{2}$

$|G|^2 = [1 - 2S_x(1 - \cos\theta_x) - 2S_y(1 - \cos\theta_y)]^2 + [C_x \sin\theta_x + C_y \sin\theta_y]^2$

$\approx [1 - 2S_x(\frac{\theta_x^2}{2}) - 2S_y \frac{\theta_y^2}{2}]^2 + [C_x\theta_x + C_y\theta_y]^2$

$|G|^2 \approx [1 - S_x\theta_x^2 - S_y\theta_y^2]^2 + [C_x\theta_x + C_y\theta_y]^2$ since θ_x, θ_y are small.

$\approx 1 - S_x\theta_x^2 - S_y\theta_y^2 + C_x^2\theta_x^2 + 2C_xC_y\theta_x\theta_y + C_y^2\theta_y^2$

dropped rest higher terms.

$= 1 - (2S_x - C_x^2)\theta_x^2 + 2C_xC_y\theta_x\theta_y - (2S_y - C_y^2)\theta_y^2$

Special case 2

For modes with $\theta_y = 0$ and θ_x small but non zero.

$|G|^2 = 1 - (2S_x - C_x^2)\theta_x^2$

Now, the mode to be stable we require

$(2S_x - C_x^2)\theta_x^2 \ge 0 \Rightarrow 2S_x - C_x^2 \ge 0 \Rightarrow \boxed{\frac{C_x^2}{S_x} \le 2}$

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Case 3 For modes with $\theta_x = 0$ and θ_y small but non-zero, stability requires $\boxed{\frac{C_y^2}{S_y} \le 2}$.

- Can we find out the stability condition for θ_x, θ_y small but non zero

We define,

$a = 2S_x - C_x^2$, $b = C_x C_y$, $c = 2S_y - C_y^2$, $x = \theta_x$, $y = \theta_y$

$|G|^2 = 1 - ax^2 + 2bxy - cy^2$ $\quad ax^2 - 2bxy + cy^2$

Now $ax^2 - 2bxy + cy^2$ can be written as

$$= a\left(x^2 - 2\frac{b}{a}xy\right) + cy^2 = a\left(x - \frac{b}{a}y\right)^2 - a\frac{b^2}{a^2}y^2 + cy^2$$

So, $ax^2 - 2bxy + cy^2 = a\left(x - \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2$

For stability this quantity must be non-negative. This quantity will be non-negative of for all values of x and y provided —

$$a > 0 \quad \text{and} \quad c - \frac{b^2}{a} > 0 \Rightarrow \boxed{ac - b^2 > 0}$$

$\Rightarrow 2S_x - C_x^2 \ge 0$ &

$(2S_x - C_x^2)(2S_y - C_y^2) - C_x^2 C_y^2 \ge 0$

$\Rightarrow 4S_x S_y - 2S_x C_y^2 - 2S_y C_x^2 \ge 0$ $\quad$ Dividing by $2S_x S_y$

$\Rightarrow 2 - \frac{C_y^2}{S_y} - \frac{C_x^2}{S_x} \ge 0 \Rightarrow 2 \ge \frac{C_x^2}{S_x} + \frac{C_y^2}{S_y}$ $\quad$ Conditions $\Rightarrow \boxed{\frac{C_x^2}{S_x} + \frac{C_y^2}{S_y} \le 2}$

This condition is more stringent than the previous two conditions.

☆ Certain modes require $(\theta_x = \theta_y = \pi)$ require $S_x + S_y = \frac{1}{2}$ for stability.

Certain other modes $(\theta_x, \theta_y \ll 1)$ require

$$\frac{C_x^2}{S_x} + \frac{C_y^2}{S_y} \le 2 \text{ for stability}$$

- Can we show stability condition for all cases (without any assumptions)?

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14/3/18 Two dimensional transport equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \alpha_x \frac{\partial^2 T}{\partial x^2} - \alpha_y \frac{\partial^2 T}{\partial y^2} = 0$$

• FTCS scheme

$$G = 1 - 2\delta_x(1-\cos\theta_x) - 2\delta_y(1-\cos\theta_y) - i(C_x \sin\theta_x + C_y \sin\theta_y)$$

Conditions required for stability

For $\theta_x = \theta_y = \pi$.

$\delta_x + \delta_y \leq \frac{1}{2}$ if this condition is violated the mode with $\theta_x = \theta_y = \pi$. becomes unstable.

For θ_x, θ_y small

$\frac{C_x^2}{\delta_x} + \frac{C_y^2}{\delta_y} \leq 2$. if this condition is ~~to~~ violated then certain modes with θ_x, θ_y small will become unstable.

Q The question is if $\delta_x + \delta_y \leq \frac{1}{2}$ and $\frac{C_x^2}{\delta_x} + \frac{C_y^2}{\delta_y^2} \leq 2$ are both satisfied Q are modes with $\theta_x, \theta_y \neq \pi$ and θ_x, θ_y not small also stable?

→ If the conditions $\delta_x + \delta_y \leq \frac{1}{2}$, $\frac{C_x^2}{\delta_x} + \frac{C_y^2}{\delta_y} \leq 2$ are both satisfied can we prove that the scheme will be stable for all possible values of θ_x and θ_y?

Acc. to Cauchy – Schwartz inequality

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2 + \cdots)(b_1^2 + b_2^2 + \cdots) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \cdots)^2$$

So, $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1b_1 + a_2b_2)^2$.

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2$$

$$= (a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1b_1a_2b_2)$$ after cancelling terms

$$= (a_1b_2 - a_2b_1)^2 \geq 0$$ [Algebraic proof].

$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2$

$\vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2 \cdots$

$|\vec{a}|^2 = a_1^2 + a_2^2 + \cdots$

$|\vec{b}|^2 = b_1^2 + b_2^2 + \cdots$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$

$|\vec{a}||\vec{b}|\cos\theta = |\vec{a}\cdot\vec{b}|$

$(|\vec{a}\cdot\vec{b}|)^2 \leq |\vec{a}|^2 |\vec{b}|^2$

$\cos\theta \leq 1$.

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$$G = 1 - 2\delta_x(1-\cos\theta_x) - 2\delta_y(1-\cos\theta_y) - i\left(C_x \sin\theta_x + C_y \sin\theta_y\right)$$

$$|G|^2 = \underbrace{\left[1 - 2\delta_x(1-\cos\theta_x) - 2\delta_y(1-\cos\theta_y)\right]^2 + \left[C_x \sin\theta_x + C_y \sin\theta_y\right]^2}$$

we need to use Cauchy-Schwartz ineq, cond 1 and cond 2 to be able to show $|G|^2 \leq 1$ for stability at all pts.

So, $$|G|^2 = \left[\frac{C_x}{\sqrt{\delta_x}}\sqrt{\delta_x}\sin\theta_x + \frac{C_y}{\sqrt{\delta_y}}\sqrt{\delta_y}\sin\theta_y\right]^2 + \left[\quad\right]^2$$

now, $$\left[\frac{C_x}{\sqrt{\delta_x}}\sqrt{\delta_x}\sin\theta_x + \frac{C_y}{\sqrt{\delta_y}}\sqrt{\delta_y}\sin\theta_y\right]^2 \leq \left(\frac{C_x^2}{\delta_x} + \frac{C_y^2}{\delta_y}\right)\left(\delta_x \sin + \delta_y\right.$$

$$\leq 2\left[\delta_x \sin^2\theta_x + \delta_y \sin\right.$$

now $$|G|^2 = \underbrace{\left[1 - 2\delta_x(1-\cos\theta_x) - 2\delta_y(1-\cos\theta_y)\right]^2} + 2\left[\delta_x \sin\theta_x^2 + \delta_y \sin^2\theta_y\right]$$

We define, $z_x = 1-\cos\theta_x$; $z_y = 1-\cos\theta_y$

then $\sin^2\theta_x = 1-\cos^2\theta_x = (1-(1-z_x)^2) = 1-(1-2z_x+z_x^2)$

$$= 2z_x - z_x^2$$

$$\sin^2\theta_y = 2z_y - z_y^2$$

$$|G|^2 \leq |1 - 2\delta_x z_x - 2\delta_y z_y|^2 + 2\left[\delta_x(2z_x - z_x^2) + \delta_y(2z_y - z_y^2)\right]$$

$$= 1 - 4\delta_x z_x - 4\delta_y z_y + 4(\delta_x z_x + \delta_y z_y)^2 + 4\delta_x z_x - 2\delta_x z_x^2$$
$$+ 4\delta_y z_y - 2\delta_y z_y^2$$

$$= 1 + 4\underbrace{(\delta_x z_x + \delta_y z_y)^2} - 2(\delta_x z_x^2 + \delta_y z_y^2)$$

$(\delta_x z_x + \delta_y z_y)^2 = (\sqrt{\delta_x}\sqrt{\delta_x} z_x + \sqrt{\delta_y}\sqrt{\delta_y} z_y)^2$ Cauchy inequality

$$\leq (\delta_x + \delta_y)(\delta_x z_x^2 + \delta_y z_y^2) \leq \frac{1}{2}(\delta_x z_x^2 + \delta_y z_y^2)$$

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So $|G|^2 \le 1 + 4 \times \frac{1}{2}(S_x \xi_x^2 + S_y \xi_y^2) - 2(S_x \xi_x^2 + S_y \xi_y^2)$

$$\boxed{|G|^2 \le 1}$$ Hence proved

using only Cauchy-Schwartz inequality

& $S_x + S_y \le \frac{1}{2}$, $\frac{C_x^2}{S_x} + \frac{C_y^2}{S_y^2} \le 2$

So These are the necessary and sufficient conditions for the schemes to be stable.

2. ADI method

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \alpha_x \frac{\partial^2 T}{\partial x^2} - \alpha_y \frac{\partial^2 T}{\partial y^2} = 0$$

→ Homework

Peacemann - Rachford scheme

Q Is the scheme stable ?

Q Does it have second order accuracy in time ?

Q What about Douglas - Rachford scheme - Same qres ?

Class Test - 2 April

5:10 pM

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