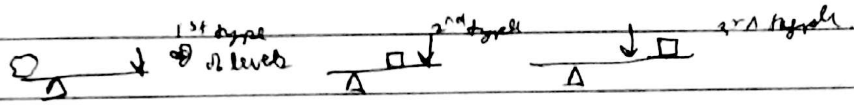
- Force analysis of mechanism
- Flywheels
- Gear Dynamics
- Governor
- Mechanical Vibration — 90% of syllabus.

Books
→ Kinematics & Dynamics of Machinery by
i) Ghosh Mallik
ii) Myszka
iii) Martin

Text books
→ Mechanical Vibration - S.S. Rao
→ 2) Theory of Vibrations with applications - W.T. Thomson et al.
3) Elements of vibration analysis - L. Meirovitch

Books for problems
1) Mechanical Vibration - Seto } Schaum's
2) " " → Kelly } outline series.

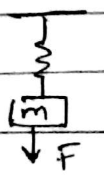
There are 6 simple M/cs
1) pulley
2) Levers



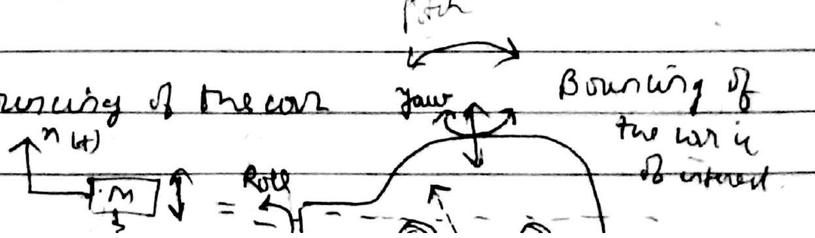

3) Inclined plane
4) Double inclined plane (wedge)
5) wheel & axle
6) screw

Definition of Machine - A m/c is a device which transmits energy/power from one place to another. A m/c may also transform ~~one~~ energy from one form to another.
↳ Example - An IC engine ↳

Forced
$F = F_0 \sin \omega t$

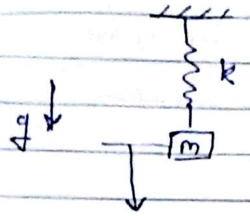



→ Bouncing of the car




Bouncing of the car is of interest

⑤ Free Vibration of an undamped 1DOF spring mass system




Assumptions:-

(i) The mass undergoes translation such that the centre of mass moves in a vertical line | Frictionless walls

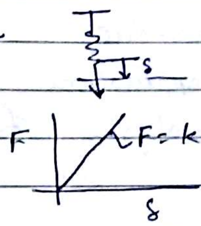
The block's rotational motion is arrested.

ii) All sorts of damping neglected (Frictionless walls eliminate damping by friction and the internal damping (hysteresis) in spring is also neglected.

iii) The mass is absolutely rigid

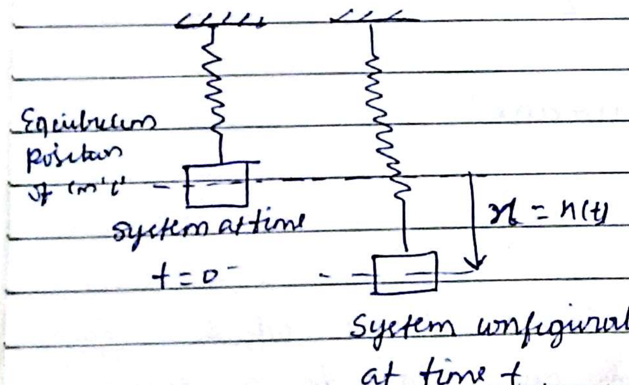
iv) The spring is linear

v) The spring is massless.




→ Free vibration means this:

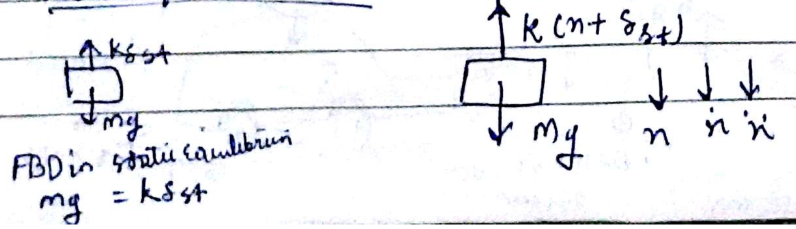
The mass would be given an initial displacement & /or an initial velocity & then the system will be left to itself. The subsequent motion is the free vibration.




Aim :- To obtain the differential Equation of Motion (DEOM) of our system.

Method I :- (Using Newton's 2nd law of motion)

FBD of the mass at time t



$x, \dot{x}, \ddot{x}$ → all are downwards

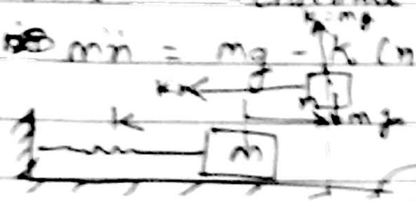
$m\ddot{x}$ , $\dot{x} = \frac{dx}{dt}$ , $\ddot{x} = \frac{d^2x}{dt^2}$

velocity

~~FBD of the mass~~

$m\ddot{x}$ = $\Sigma$ External force in the x-direction

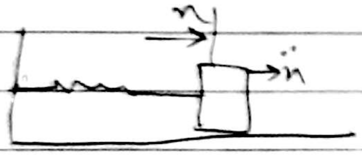
$m\ddot{x} = mg - k(x + \delta_{st}) = -kx$ [ $mg - k\delta_{st} = 0$ ]



frictionless

Method 2 – D'Alembert's method :-

D'Alembert said that if we introduce proper inertia terms, then a problem in dynamics can be reduced to a problem in statics



The inertia force will be $-m\ddot{x}$

then the FBD of block in horizontal

$\Sigma F_x = 0$

$\Rightarrow -m\ddot{x} - kx = 0$

$\Rightarrow m\ddot{x} + kx = 0$

Virtual displacement

$\delta x$

$\delta W$ = Virtual work = 0

$\delta W = (-kx - m\ddot{x})\,\delta x = 0$

$\delta x$ is arbitrary.

so we must have $-kx - m\ddot{x} = 0$

Method III The energy method :-

Ours is a conservative system & mechanical energy (KE + PE) is conserved.

T = KE = Kinetic energy of our system = $\frac{1}{2} m\dot{x}^2$

U or V = PE = Potential energy (includes strain energy) ?

Here PE (U) means the change in such energy over and above the equilibrium PE

So we shall take PE = ...

$dt$

$\frac{1}{2} 2m\left(\frac{d\dot{x}}{dt}\right)\dot{x} + \frac{1}{2}k\,2x\dot{x} = 0$

$\rightarrow m\dot{x}\ddot{x} + kx\dot{x} = 0$

Since $\dot{x} = 0$ at all times is not possible ... [illegible]

$(m\ddot{x} + kx = 0)$

Method IV - The use of Lagrange's equations (of ... [illegible])

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_j}\right) - \frac{\partial T}{\partial v_j} + \frac{\partial U}{\partial v_j} + \frac{\partial D}{\ldots}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j \text{ (for generalised ...)}$$

for a dynamic hollonomic n DOF system

$D \rightarrow$ Rayleigh. Dissipation function

11/11/17

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} + \frac{\partial D}{\partial q_j} = Q_j, \; j = 1,2,\dots,n$$

n = no. of D.O.F of holonomic system

For our system, $j=1$ & $q_1 = x$, $\dot{q}_1 = \dot{x}$

Also, $D=0$, $Q_j = 0$

(no damping) → ($\because$ only free vibration is considered)

Here $T = KE = \frac{1}{2}m\dot{x}^2$

$U = PE = \frac{1}{2}kx^2$

* Remember ~~at~~ that at this stage of derivation of the DEOM, $x$ & $\dot{x}$ are independent variables

The lagrange equation for our system is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0$

Now, $\Rightarrow \frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right) + \frac{dU}{dx} = 0$ — ①

[Now, $\frac{dT}{d\dot{x}} = \frac{d}{d\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}m\frac{d}{d\dot{x}}(\dot{x}^2) = m\dot{x}$

$\frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right) = m\ddot{x}$ — ② ], $\frac{dU}{dx} = \frac{d}{dx}\left(\frac{1}{2}kx^2\right) = kx$

Substituting ② & ③ in ①, we get $m\ddot{x} + kx = 0$

Method V :- The use of hamilton's principle

[Feynman Physics vol 2 ch 19 ]

$$\delta \int_{t_1}^{t_2} (T-U)\, dt = 0$$

→ lagrangian of the system

$\delta \left( \int_{t_1}^{t_2} L\, dt \right) = 0$

A variation $\delta x = dx$

$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$\int_{t_1}^{t_2} \delta L\, dt = 0$ $= \delta\left(\frac{1}{2}m\dot{x}^2\right) - \delta\left(\frac{1}{2}kx^2\right)$

$= \frac{m}{2}\delta(\dot{x}^2) - \frac{k}{2}\delta(x^2)$

$= \frac{m}{2} 2\dot{x}\,\delta(\dot{x}) - \frac{k}{2} \times 2x \times (\delta x)$

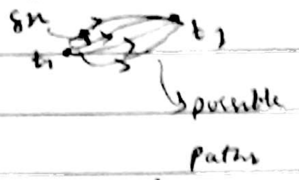
$= m\dot{x}\,\delta\dot{x} - kx\,\delta x$

So, $\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1} m\dot{x}(\delta\dot{x})\,dt - \int_{t_1} kx(\delta x)\,dt$

$$= \underbrace{\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x)\,dt}_{\text{Integration by parts.}} - \int_{t_1}^{t_2} kx(\delta x)\,dt$$

$$= \cancelto{0}{m\dot{x}\delta x\Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} m\ddot{x}\,\delta x\,dt - \int_{t_1}^{t_2} kx(\delta x)\,dt$$

$(\because \delta x = 0$ at $t_1$ & $t_2)$




$$= -\int_{t_1}^{t_2} (m\ddot{x} + kx)\,\delta x\,dt = 0, \text{ by H's principle}$$

Since $\delta x$ is arbitrary (Quite arbitrary)
for above relation to be true, we must have.

$$m\ddot{x} + kx = 0$$

which is the required ~~BO~~ DEOM

Method VI) Use of Schrodinger Wave equation.

The 1-D wave equation for our system can be shown to be

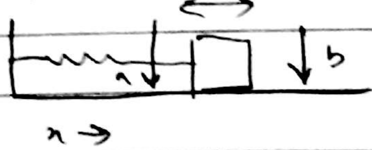
$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(\psi)$$

↑ A potential $\quad i = \sqrt{-1} \quad \hbar = \frac{h}{2\pi}$

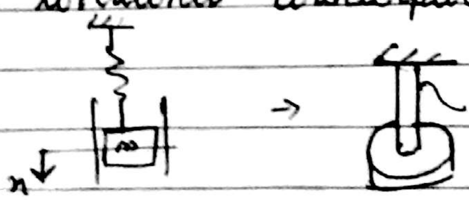
$\hbar = h/2\pi \qquad h$ = planck's constant.

$h$ = ~~Planck's constant~~

$\int_a^b |\psi|^2 dx$ gives the probability of finding the particle in $[a, b]$.

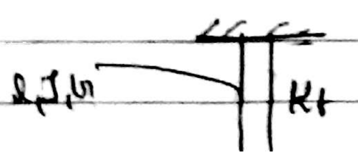



* The rotational counterpart of our model




massless shaft, which acts like a ~~torson~~ torsional spring

Rigid, circular shaft.

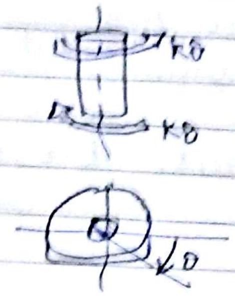



$T$ =

$T/_\theta$ = Torque per unit twist = $\boxed{k_t = \frac{GJ}{l}}$ and $\theta = \frac{TL}{GJ}$ from 1st year mechanics.

$J = \frac{\pi r^4}{2}$, $r$ = radius of the circular shaft. $J = I_p = \int r^2 dA$

$J = \frac{\pi r^4}{2}$




$I\ddot{\theta} + k_t\theta = 0$, which is the required DEOM

$I\ddot{\theta} = -k_t\theta$

$T = \frac{1}{2}I\dot{\theta}^2$ $\quad V = \frac{1}{2}k_t\theta^2$ $\quad \frac{d}{dt}(T+V) = 0$

$\Rightarrow \dot{\theta}(I\ddot{\theta} + k_t\theta) = 0$

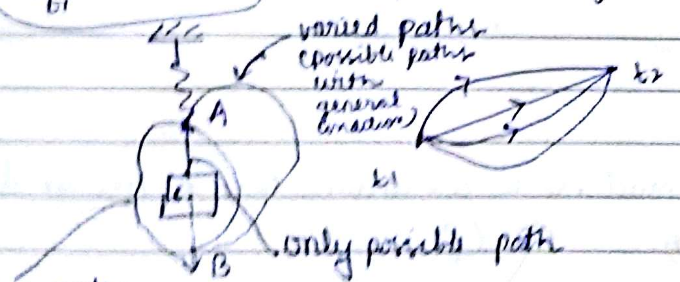
$\Rightarrow I\ddot{\theta} + k_t\theta = 0$

HW Obtain the DEOM using the Lagrange equation.

11/1/16

$\int_{t_1}^{t_2} L\,dt \rightarrow$ The action integral.

$\delta\int_{t_1}^{t_2} L\,dt = 0$ A principle of least action




Varied paths (possible paths with general conditions)

$m\ddot{x} + kx = 0$ --- (1)

Let $x = ce^{st}$ --- (2)

Substitute (2) in (1)

This leads to the auxiliary equation

$ms^2 + k = 0$

$\Rightarrow s_1, s_2 = \pm j\sqrt{\frac{k}{m}}$

$x = c_1e^{s_1t} + c_2e^{s_2t}$

$= c_1e^{-j\sqrt{\frac{k}{m}}t} + c_2e^{j\sqrt{\frac{k}{m}}t}$

$= A\sin\omega_nt + B\cos\omega_nt = x_0\sin(\omega_nt + \alpha)$

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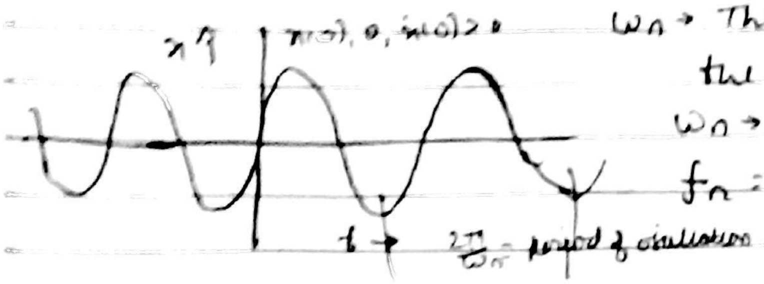
$x = A\sin\omega_n t + B\cos\omega_n t$

or $x = x_0 \sin(\omega_n t + \alpha)$

$A, B \rightarrow$ To be determined for given $x(0)$, $\dot{x}(0)$

$x_0, \alpha \rightarrow$ " " " " " " "




$\omega_n \rightarrow$ The natural frequency of the given system.

$\omega_n \rightarrow$ rad/s

$f_n = \frac{\omega_n}{2\pi}$ (cycles/s)

$k \rightarrow 2$ kN/m

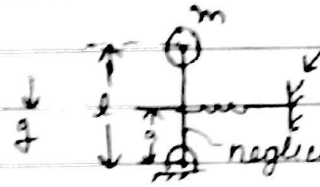
$m = 100$ g

$k \rightarrow$ N/m

$m \rightarrow$ kg

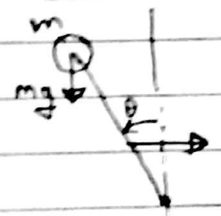
★ Example 1

Obtain the condition for stable oscillation for the following system




Step 1 To obtain the DEOM

Method 1 - Newton's method



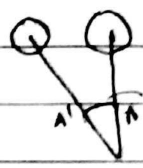

Let $\theta = \theta(t)$ be the generalized coordinate.

(Note that the given system has only one DOF, hence only one gen. coord. is req.

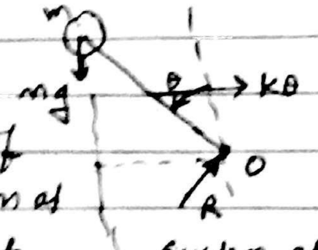
$\theta = +$ve CCW

$\dot{\theta}, \ddot{\theta} \rightarrow$ " "

We are assuming $\theta$ to be small, so we can assume that A' lies at the same horizontal level as A.




(The location of A and stiffness of spring are important to achieve stable oscillation




FBD of system at time $t$

System at time $t$.

If spring is very near to pivot then gravity will take the mass down and it will oscillate at complete rotation

$I_0 \ddot{\theta} = + mgl\sin\theta - Ka\theta \times a$

$\Rightarrow \boxed{ml^2\ddot{\theta} + (Ka^2 - mgl)\theta = 0}$.

$\omega_n$ (by comparision with $I\ddot{\theta} + K_t\theta = 0$ or $m\ddot{x} + kx = 0$)

$= \sqrt{\dfrac{Ka^2 - mgl}{ml^2}}$

Hence for stable oscillations, $Ka^2 - mgl > 0$

Energy method → H.W.

$\boxed{\dfrac{d}{dt}(T + U) = 0}$

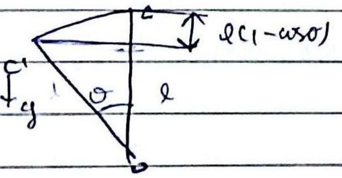
★ Method 3 – Using lagrange Equation

The lagrange equation here is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \quad \text{--- (1)}$$

$T = \frac{1}{2} I_0 \omega^2 = \frac{1}{2} m l^2 \dot{\theta}^2 \rightarrow v^2$

$U = -mgl(1-\cos\theta) + \frac{1}{2}k(a\theta)^2$ [→ Never linearize before differentiation]




$\dfrac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} ml^2 \times 2\dot{\theta} = ml^2\dot{\theta} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$

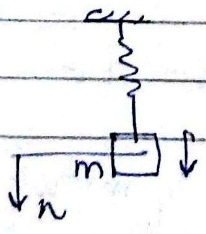
$\dfrac{\partial T}{\partial \theta} = 0$ [∵ $\dot{\theta}$ and $\theta$ are independent and expression for T does not contain $\theta$]

$\dfrac{\partial U}{\partial \theta} = -mgl\sin\theta + Ka^2\theta$

$\simeq -mgl\theta + Ka^2\theta$ (linearizing)

Substitution in (1) leads to.

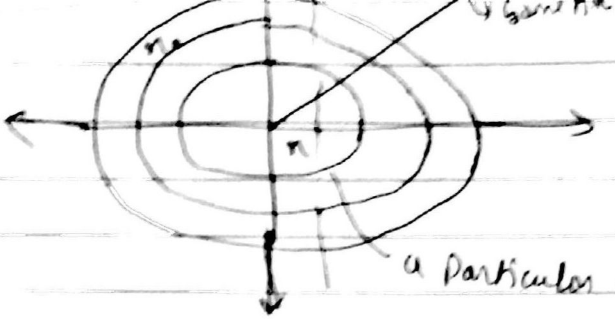
$\boxed{ml^2\ddot{\theta} + (Ka^2 - mgl)\theta = 0}$




$x = A\sin\omega_n t + B\cos\omega_n t$

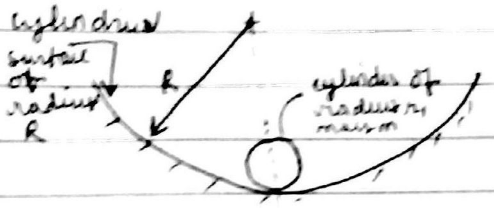
$\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$

$\left\{ \begin{array}{l} x = x_0\sin(\omega_n t + \phi) \\ \dot{x} = x_0\omega_n\cos(\omega_n t + \phi) \end{array} \right.$ , $\dfrac{x^2}{x_0^2} + \dfrac{\dot{x}^2}{(x_0\omega_n)^2} = 1$

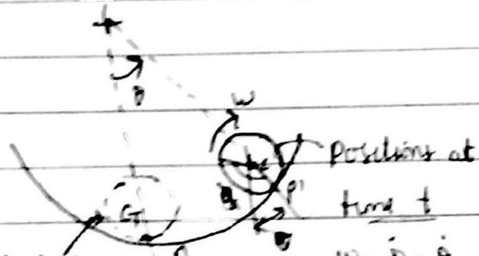



13/1/17

An important example




The cylinder rolls w/o slipping & oscillates back & forth. Obtain the DEOM & soln.




$\omega = \dot{\alpha} - \dot{\theta}$, for rolling w/o slipping $\widehat{PQ'} = \widehat{P'P''}$

$R\theta = r\alpha \Rightarrow \alpha = \frac{R}{r}\theta$

Is $\omega = \dot{\theta}$?

Ans → No.

To obtain $\omega$, we may consider the velocity of C' (the centre of cylinder at time)

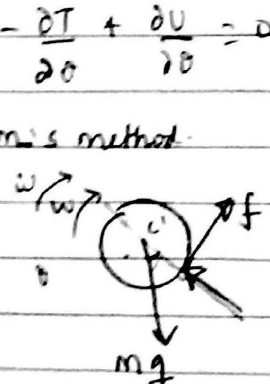
$v_{c'} = (R-r)\dot{\theta} = r\omega$

$\Rightarrow \quad \omega = \left(\frac{R-r}{r}\right)\dot{\theta}$

The lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \quad \text{--- (1)}$$

Using Newton's method




$I_{P'}\dot{\omega} = -mgr\sin\theta$

$I_{P'} = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$

For small oscillations, $\sin\theta \approx \theta$

$\frac{1}{2}I_P\omega^2 = T =$ K.E of cylinder $= \frac{1}{2}mV_{c'}^2 + \frac{1}{2}I_{c'}\omega^2$

$= \frac{1}{2}m(r\omega)^2 + \frac{1}{2}\,\frac{1}{2}mr^2\omega^2 = \frac{3}{4}mr^2\omega^2$

$= \frac{3}{4} m (R-r)^2 \dot{\theta}^2$

$\frac{\partial T}{\partial \dot{\theta}} = \frac{3}{2} m (R-r)^2 \dot{\theta}$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{3}{2} m (R-r)^2 \ddot{\theta}$ — (2) $\quad \frac{\partial U}{\partial \theta} = + mg(R-r)\sin\theta$ — (3)

Substitute (2) & (3) in

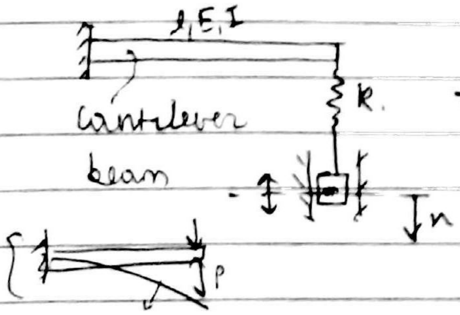
The required DEOM is

$$\frac{3}{2} m (R-r)^2 \ddot{\theta} + mg(R-r)\theta = 0$$

$$\& \quad \omega_n = \sqrt{\frac{2g}{3(R-r)}}$$

Using Newton's method

$I_P \cdot \dot{\omega} = -mgr\sin\theta \quad , \quad I_P = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$

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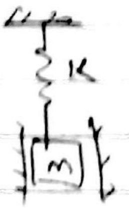
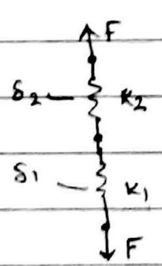



$\omega_n = \sqrt{\frac{k}{m}}$

The relastic line.

$\delta = \frac{PL^3}{3EI}$

Note:

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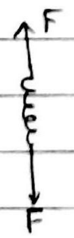



$F = k_2\delta_2 = k_1\delta_1 = k\delta$

$\delta_1 = \frac{k}{k_1}\delta$

$\delta_2 = \frac{k}{k_2}\delta$

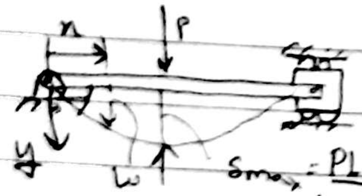
and $\delta = \delta_1 + \delta_2$

which gives $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

⇒ Getting back $k_1$ $k_2$ to our examples

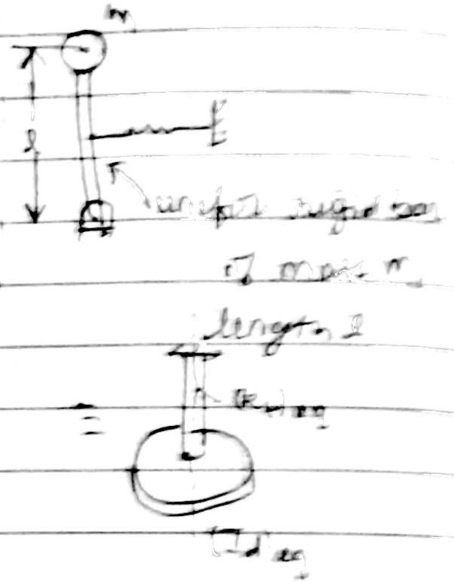
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

examples




$$M(x) = \pm EI\frac{d^2 w}{dx^2}$$

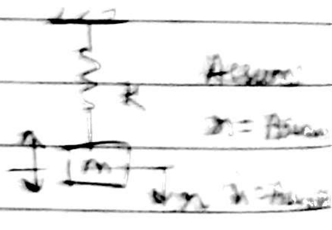
$$\delta_{max} = \frac{PL^3}{48EI}$$




A simply supported beam.

The Rayleigh Method

This method enables us to compute the fundamental natural frequency of a vibrating mechanical system




For a conservative system

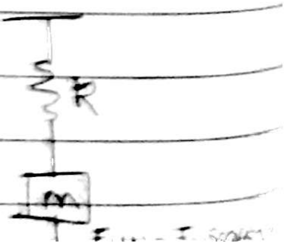
$T_{max} = U_{max}$

$\omega_n = ?$

For $x = A\sin\omega_n t$

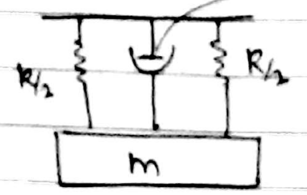
$\dot{x} = A\omega_n \cos\omega_n t$

$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_n^2\cos^2\omega_n t$

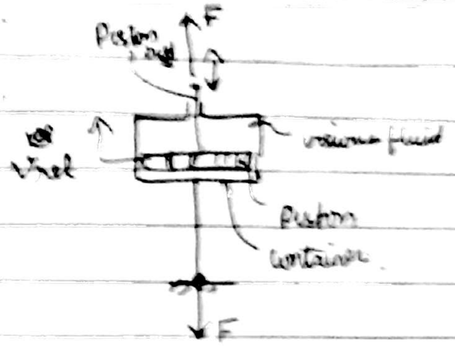



Scanned by CamScanner

Linearly viscous damper




viscous damper




$F_1(t) = F_0 \sin \omega t$

$F_2(t) =$

$F = C V_{rel}$ $\quad V_{rel}$ = relative velocity of end A w.r.t end B

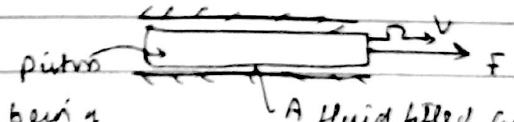
Properties of the damper

(i) It is massless

(ii) It is capable of resisting ~~a~~ a force only if there is a relative motion between its ends

$V_{rel}$ = relative velocity of end A w.r.t end B.

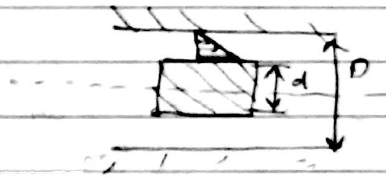
(iii) The constitutive law in $F = CV_{rel}$



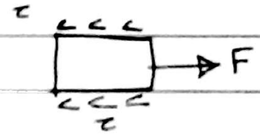

piston being pulled to the right at constant velocity v

Aim :- To show that for laminar flow conditions, $F \propto v$

We shall ignore forces on the flat ends of the cylindrical piston.




$$\tau = \mu \frac{du}{dy}$$




F = viscous Force on curved surface

$= \tau \times$ surface area

$= \pi d l \tau$

$$= \frac{2\pi d l \mu v}{(D-d)}$$

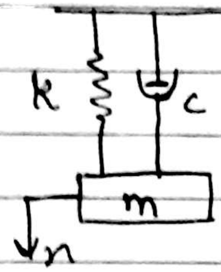
$= Cv$

$C = \frac{2\pi d l \mu}{(D-d)}$ = constant.

F = viscous Force on curved surface of cylinder

It is found that damping is better in ① than ②

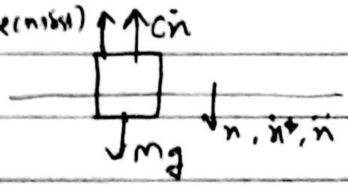
18/1/17

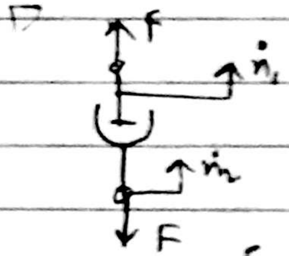



The K-V model

Aim :- To derive the DEOM of the K-V model for free vibration

(i) Newton's method




$$m\ddot{x} = \Sigma F_x = mg - k(x+\delta_{st}) - c\dot{x}$$

$F = c(\dot{x}_1 - \dot{x}_2)$ difference

$$m\ddot{x} + c\dot{x} + kx = 0 \quad — ①$$

$(\because mg = k\delta_{st})$ [linear homogeneous differential equation]

① is the reqd. DEOM

method② :- The use of Lagrange equation

(Note that $\frac{d}{dt}(T+U) = 0$ won't work)

The lagrange eqn here is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0;$$

where $D = \frac{1}{2}c\dot{x}^2$ = Rayleigh dissipation function.

$T = \frac{1}{2}m\dot{x}^2$ $\quad U = \frac{1}{2}kx^2$

We solve (eq. 1) for $x(t)$

let, $x = Ce^{st}$

$\dot{x} = Cse^{st}$

$\ddot{x} = Cs^2e^{st}$

Substitution in (1) gives $ms^2 + cs + k = 0$ — (3), the auxiliary equation ($\because C \neq 0$)

If $s_1$ and $s_2$ be the roots of (3)

$$s_1 = \frac{-c - \sqrt{c^2 - 4km}}{2m}$$

$$s_2 = \frac{-c + \sqrt{c^2 - 4km}}{2m}$$

involves all the system parameters, viz. $m, k$ & $c$.

Hence $x = C_1e^{s_1t} + C_2e^{s_2t}$ — (4) $C_1$ & $C_2$ being arbitrary constants of integration.

The advantage of non-dimensionalization

Aim - To non-dimensionalize

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{— (1)}$$

We introduce non-dimensional displacement & non-dim time as $\bar{x}$ & $\bar{t}$

Let $\bar{x} = \frac{x}{l}$, $l$ being some characteristic length associated with our system. It can be the free-length.

$\bar{t}$ = non-dim time $= \frac{t}{t_0}$, $t_0$ being some characteristic time associated with our system. $\omega_n = \frac{1}{t_0}$ $\boxed{\text{and } \omega_n = \sqrt{k/m}}$

$$x = \bar{x}l$$

$$\dot{x} = \frac{dx}{dt} = l\frac{d\bar{x}}{d\bar{t}}\frac{d\bar{t}}{dt} = l\omega_n\frac{d\bar{x}}{d\bar{t}}$$

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{d\bar{t}}\left(l\omega_n\frac{d\bar{x}}{d\bar{t}}\right)\frac{d\bar{t}}{dt} = l\omega_n^2\frac{d^2\bar{x}}{d\bar{t}^2}$$

Substitute in (1)

$$\therefore ml\omega_n^2\frac{d^2\bar{x}}{d\bar{t}^2} + cl\omega_n\frac{d\bar{x}}{d\bar{t}} + kl\bar{x} = 0$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \frac{c}{m\omega_n}\frac{d\bar{x}}{d\bar{t}} + \frac{k/m}{\omega_n^2}\bar{x} = 0$$

$$\Rightarrow \frac{d^2\bar{x}}{d\bar{t}^2} + \left(\frac{c}{m\omega_n}\right)\frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0 \quad \text{— (1')}$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2-4km}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

Remember $\frac{c}{2\sqrt{km}} = \xi$

Let, $\xi$ = damping factor $= \frac{c}{2m\omega_n} \Rightarrow \frac{c}{2m} = \xi\omega_n$.

If $s_1$ & $s_2$ be the roots of ③,

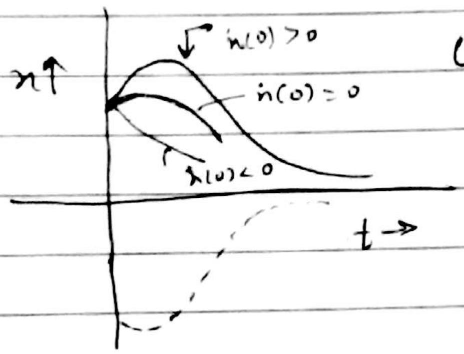
~~$s_1 = -c \pm \sqrt{c^2-4km}$~~

$$s_{1,2} = -\xi\omega_n \pm \left(\sqrt{\xi^2-1}\right)\omega_n \quad [\xi \neq 1]$$

Case (i) $\xi > 1$ ; The overdamped case
(case of overdamping)

$$x = Ae^{(-\xi-\sqrt{\xi^2-1})\omega_n t} + Be^{(-\xi+\sqrt{\xi^2-1})\omega_n t}$$

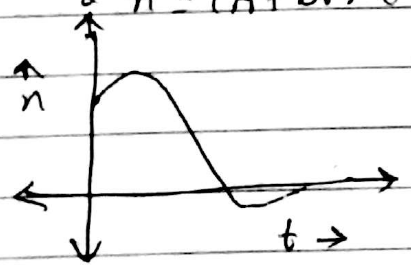
→ non oscillatory solution




Case (ii) $\xi = 1$. The ~~is~~ is the case of critical damping.
Here $s_1 = s_2 = -\xi\omega$
& $x = (A+Bt)e^{-\xi\omega_n t}$

Case (ii) $\xi = 1$, This is the case of critical damping.
Here $s_1 = s_2 = -\xi\omega_n$
& $x = (A+Bt)\, e^{-\xi\omega_n t}$

- Many measuring instruments are critically damped. So are the automatic door closing mechanism.



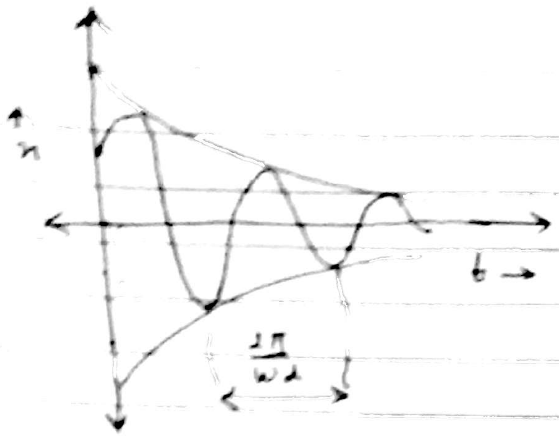

Case (iii) $\xi < 1$ (The case of underdamping)

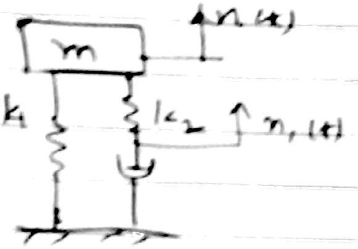
$$s_{1,2} = -\xi\omega_n \pm j\left(\sqrt{1-\xi^2}\right)\omega_n \; ; \quad j = \sqrt{-1}$$

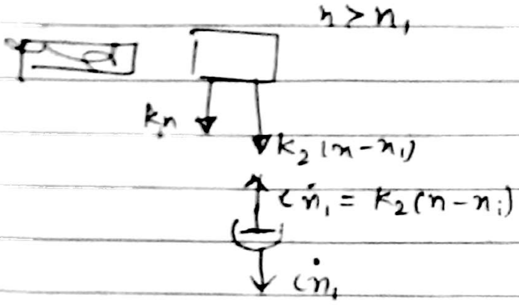
$$= -\xi\omega_n \pm j\omega_d \; ; \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$ = The damped natural frequency.

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

H.W. → $\boxed{x = X_0 e^{-\xi\omega_n t}\sin(\omega_d t + \phi)}$ Remember.




$m\ddot{x} = -k_1 x - k_2(x - x_1)$

$c\dot{x}_1 = k_2(x - x_1)$

(HW) Eliminate $x_1$ & obtain the 3rd or DE of m in $x$.

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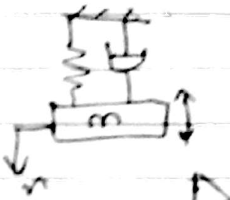
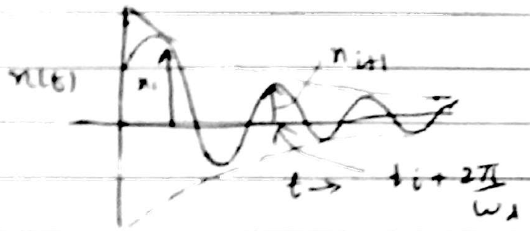
The logarithmic decrement




Let $\zeta < 1$.

The $x(t) = e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$ --- (1)




We use the logarithmic decrement $\delta$

where $\delta = \ln\left(\dfrac{x_i}{x_{i+1}}\right) = \ln\left[\dfrac{e^{-\zeta\omega_n t_i}\sin(\omega_d t_i + \phi)}{e^{-\zeta\omega_n\left(t_i + \frac{2\pi}{\omega_d}\right)}\sin\left(\omega_d\left(t_i + \frac{2\pi}{\omega_d}\right) + \phi\right)}\right]$

$\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ $= \ln e^{\left[\frac{2\pi\zeta\omega_n}{\omega_d}\right]} = \dfrac{2\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}$

For $\zeta \ll 1$ (which happens often)

$x_i$ will be quite close to $x_{i+1}$

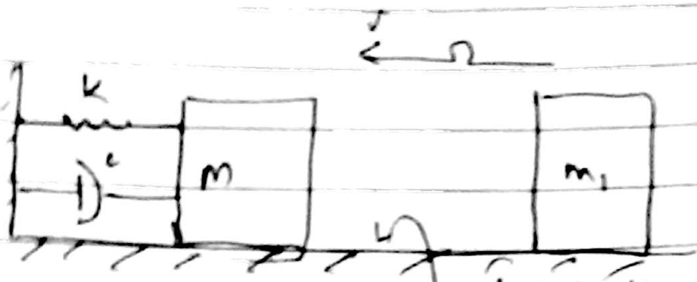
& experimental error occurs.

For better accuracy,

* obtain $\xi$ from the relation

$$\delta_n = \ln \frac{x_i}{x_{i+n}} = \frac{2\pi \xi n}{\sqrt{1-\xi^2}}$$

Ex 1




Step 1 - Find $\xi$

Let $\xi<1$

Immediately after the inelastic impact, the two blocks move with a common velocity $\dot{x}(0^+)$

$$x(0) = \int_0^{0^+} \dot{x}\,dt \approx 0.$$

$$m_1 v = (m+m_1)\,\dot{x}(0^+)$$

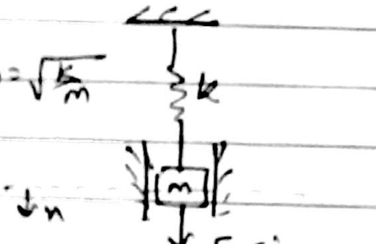
$$\dot{x}(0^+) = \frac{m_1 v}{m+m_1}$$

Let $\xi<1$, $\quad \xi = \frac{c}{2\sqrt{mk}}$ $\quad \xi = \frac{c}{2\sqrt{(m+m_1)k}}$ $\quad$ obtain $\xi$ and use the approximate formula

Find $x_{max}$ after impact.

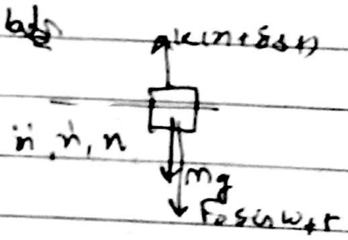
(5) Forced Vibration of 1DOF system

An undamped 1DOF system




Step 1 - obtain the DEOM

a) by Newton's method




$$m\ddot{x} = mg + F_0 \sin w_f t - k(x+\delta_{st})$$

$$\boxed{m\ddot{x} + kx = F_0 \sin w_f t} \sim ①$$

b) By using the lagrange system

The lagrange eqn here is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = Q_x - Ⓐ$$ where $Q_x$ is the generalized

$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x} \quad , \quad \frac{\partial T}{\partial x} = 0 \, , \quad \frac{\partial V}{\partial x} = kx$

For computing $Q_i$, obtain the virtual work done by the applied force over a virtual displacement $\delta x$ of the mass

If $\delta W$ is this virtual work, then $Q_1 = \frac{\delta W}{\delta x}$

$\delta W = F \delta x$

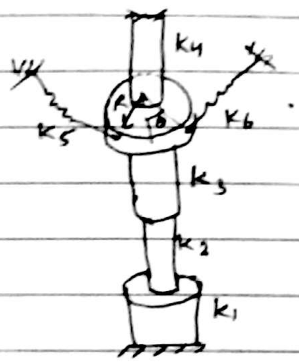
$Q_1 = \frac{\delta W}{\delta x} = F(t)$

Substitution in (A) gives $m\ddot{x} + kx = F_0 \sin \omega_f t$.

---

20/1/17 Tutorials [1.9, 1.12, 1.19, 1.30, 1.34, 2.10 (easy), 2.12, 2.15, 2.39 (easy)] $\boxed{2.97,\ 2.98,\ 2.89, \ 2.90,\ 2.91}$ H.W.

1.9




The springs 1, 2, 3 are in series, Equivalent spring constant of 1, 2, 3 is $\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$ [∵ Same resulting torque in all three of them.

$\frac{1}{k'} = \frac{k_2k_3 + k_1k_3 + k_1k_2}{k_1k_2k_3}$

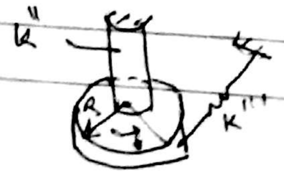
$\therefore k' = \frac{k_1k_2k_3}{k_2k_3 + k_1k_3 + k_1k_2}$

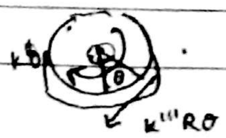
Now, $k'$ and $k_4$ are in parallel (∵ Same angular deflection $\theta$ is experienced in $k'$ and $k_4$)

so $k'' = k' + k_4$

and $k_5$ and $k_6$ translational springs are in parallel.

$k''' = k_5 + k_6$




$\rightarrow I\ddot{\theta} = -k''\theta - k'''(R\theta).R$

$\rightarrow (I\ddot{\theta} + k''\theta + k'''R^2\theta) = 0$

Equivalent spring constant = $(k' + k'' R^2)$

$k_{eq} = \left( \dfrac{k_1 k_2}{k_1 + k_2} + k_3 \right) + [k_5 + k_6] R^2$

$l = \dfrac{l \cos\gamma}{\cos\beta} + x \sin\beta$

Extension in spring $= (x \cos\beta) + \dfrac{l\cos\gamma}{\cos\beta} - l$

assuming $\gamma \approx \beta$ extension in spring $= x_{\cos\gamma} + 0$

Ex. The 3 legs of the tripod constitute 3 springs in parallel.

$\left(\dfrac{3k x}{\cos\gamma}\right)\cos\gamma + m\ddot{x} = 0$

$k_{eq} = \left(\dfrac{3k}{\cos\gamma}\right) \quad k_{eq} = 3k$

$F = ax + bx^3 \qquad a = 20000\ N/m^2 \qquad b = 40\times10^6\ N/m^3$

Spring constant when $x = 0.01$ is $k = \left.\dfrac{dF}{dx}\right|_{x=0.01}$

$= (a + 3bx^2)$

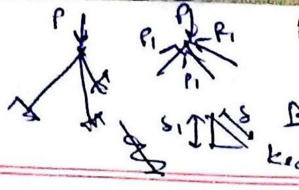
$= 20\times10^3 + 3\times40\times10^6\times10^{-4}$

$= 20\times10^3 + 12\times10^3$

$= 32\times10^3\ N/m$

Which this spring constant should be used for studying small amplitude vibration of the system about $x_1$

P1.12



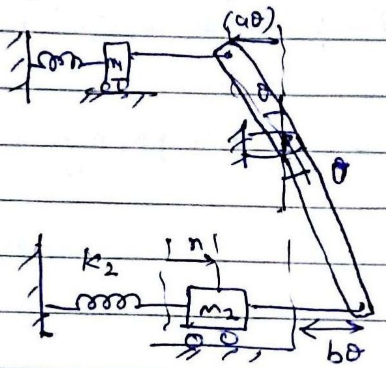
$P = 3P_1 \cos\theta$

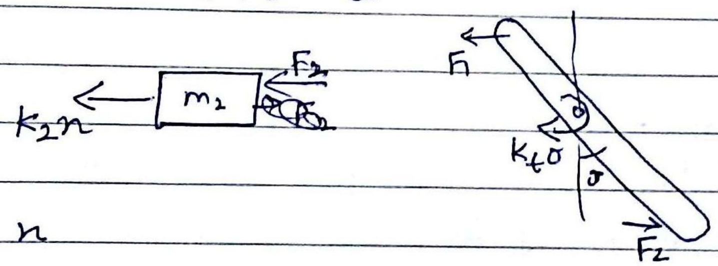
$P_1 = \frac{P}{3\cos\theta}$

$\delta = \frac{P_1 l}{AE}$

$k_{eq} = \frac{P}{\delta_1}$ $\quad \delta_1 = \delta\cos\theta$

1.30 Using Newton's method




$b\theta \simeq x$


For rocker arm $\quad F_2 b + F_1 a - K_t\theta = J_0\ddot{\theta}$

For mass $m_2$ $\quad \cancel{K_2x}\ F_2 + K_2 x = m_2\ddot{x}$

1.30

$$T = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_1 \frac{a^2 \dot{x}^2}{b^2} + \frac{1}{2} J_0 \frac{\dot{x}^2}{b^2}$$

$$V = \frac{1}{2} k_2 x^2 + \frac{1}{2} k_1 \frac{a^2 x^2}{b^2} + \frac{1}{2} k_t \frac{x^2}{b^2}$$

$b\theta = x$

Using lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0$$

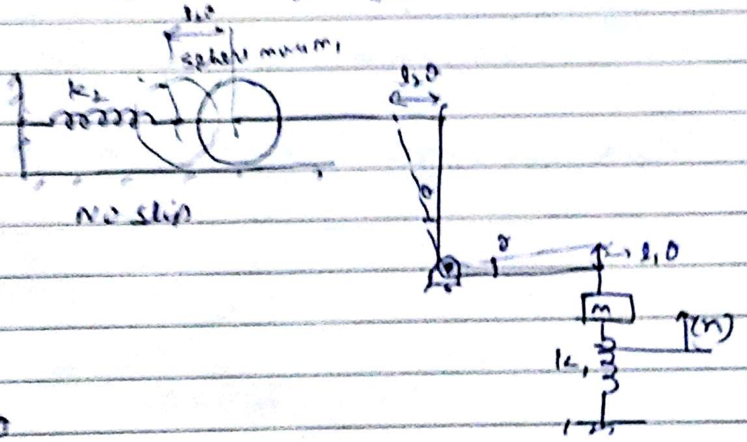
$$\frac{\partial T}{\partial \dot{x}} = m_2 \dot{x} + m_1 \frac{a^2}{b^2} \dot{x} + \frac{J_0 \dot{x}}{b^2}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m_2 \ddot{x} + m_1 \frac{a^2}{b^2} \ddot{x} + \frac{J_0 \ddot{x}}{b^2} \quad , \quad \frac{\partial T}{\partial x} = 0$$

~~and~~ and $\frac{\partial U}{\partial x} = k_2 x + k_1 \frac{a^2}{b^2} x + \frac{k_t x}{b^2}$

DEOM is $\left(m_2 + m_1 \frac{a^2}{b^2} + \frac{J_0}{b^2}\right)\ddot{x} + k_2 x + k$

1.34





$V_c = R\omega$

$\omega = \frac{V_c}{R}$

and $V_0 = l_2\dot{\theta}$

$= \dot{x}\frac{l_2}{l_1}$

$x = l_1\theta$

$l_1\theta = \left(x \frac{l_2}{l_1}\right)$

$$T = \underbrace{\frac{1}{2} m \dot{x}^2}_{T_m} + \underbrace{\frac{1}{2} J_0 \dot{\theta}^2}_{T_{pendulum}} + \underbrace{\frac{1}{2} m_s v_0^2 + \frac{1}{2} I_s \omega_s^2}_{T_{sphere}}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{l_1}\right)^2 + \frac{1}{2} m_s \left(\dot{x}\frac{l_2}{l_1}\right)^2 + \frac{1}{2} I_s \left(\frac{\dot{x} l_2}{R l_1}\right)^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \frac{\dot{x}^2}{l_1^2} + \frac{1}{2} m_s \frac{l_2^2}{l_1^2} \dot{x}^2 + \frac{1}{2}\left(\frac{2}{5} m_s R^2\right) \times \dot{x}^2 \frac{l_2^2}{R^2 l_1^2}$$

$$= \underbrace{\left(m + \frac{J_0}{l_1^2} + m_s \frac{l_2^2}{l_1^2} + \frac{2}{5}\frac{l_2^2}{l_1^2} m_s\right)}_{m_{eq}} \frac{1}{2}\dot{x}^2$$

2-14-7



2.87

2.87 The EOM for critically damped system $x=(A+Bt)e^{-\zeta\omega_n t}$

$\zeta$ = damping factor = $\frac{c}{2m\omega_n}$


Critically damped system will reach maximum value when $\frac{dx}{dt}=0$

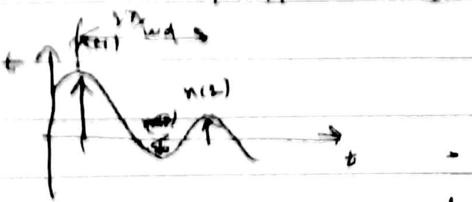
$\Rightarrow \frac{dx}{dt} = Be^{-\zeta\omega_n t} + (A+Bt)(-\zeta\omega_n)e^{-\zeta\omega_n t} = 0$

$B=(A+Bt)\zeta\omega_n$

$\left(\frac{B}{\zeta\omega_n} - A\right) = Bt$

$t = \frac{\cancel{B}}{\cancel{B}}\left(\frac{1}{\zeta\omega_n} - \frac{A}{B}\right)$

2.88.




K-V underdamped model

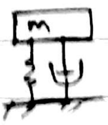
EOM is $x = x_0 e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$

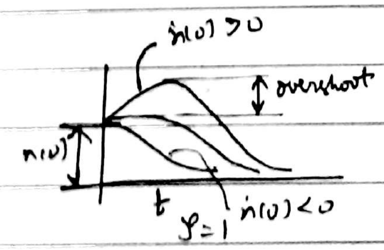
Damping factor $\zeta<1$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ = damped natural frequency.

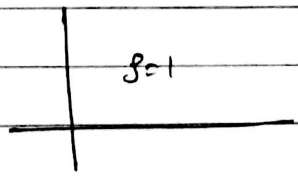
$\dot{x} = x_0(-\zeta\omega_n)e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) + x_0\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t+\phi)$

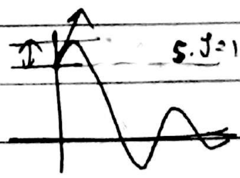
$= x_0 e^{-\zeta\omega_n t}\left[-\zeta\omega_n\sin(\omega_d t+\phi)+\omega_d\cos(\omega_d t+\phi)\right]$

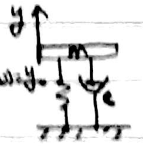
2.98)




Ans) Weight of motor = 800 N

(i) Nature of damping

a) Underdamped. b) Underdamped

(iii) $x = X_0 e^{-\zeta \omega_n t} \sin(\omega_d t + \alpha)$

$\frac{2\pi}{\omega_d} = 0.1$

$\omega_d = 20\pi = 20 \times 3.14 = 31.4 \times 2 = 62.8$ rad/s $= \omega_n\sqrt{1-\zeta^2}$

$\omega_n = \sqrt{\frac{k}{m}}$

For a) The logarithmic decrement

$\delta = \ln\left(\frac{x_1}{x_{n+1}}\right)$

$= \ln\left(\frac{4}{2}\right)$

$\delta = \ln 2 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

$\therefore \ln 2 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

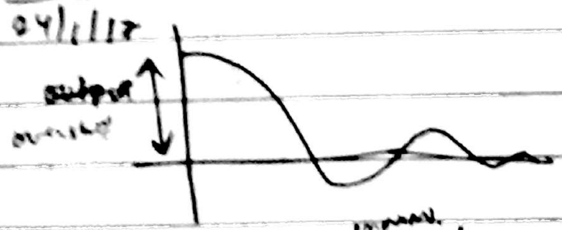
$(0.11)^2 = \frac{\zeta^2}{1-\zeta^2}$

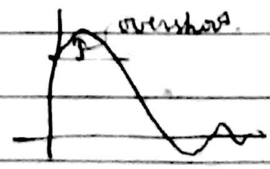
$(0.11)^2 = (1 + (0.11)^2)\zeta^2$

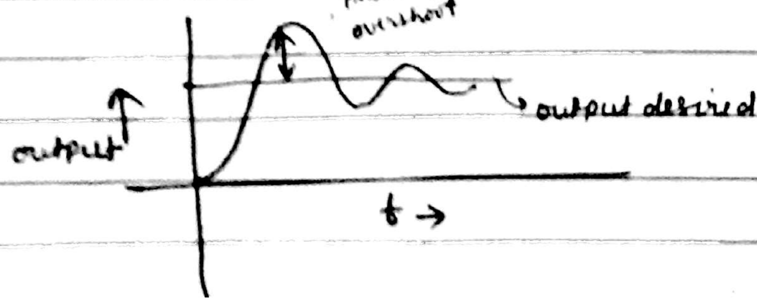
$(0.11)^2 = (1.0121)\zeta^2$

$\zeta = 0.109$

04/11/17

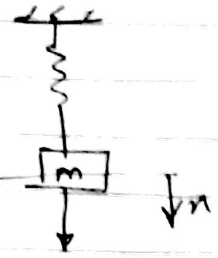


24/1/17 ①

Undamped forced vibration of 1 DOF system




$F(t) = F_0 \sin\omega_f t$

DEOM is

$m\ddot{x} + kx = F_0 \sin\omega_f t$ — ①

Here $x(t) = x(t) + x(t)$

compl ↓ Free Vibr part. forced / steady state

Let $\frac{d}{dt} = D$

Then ① becomes,

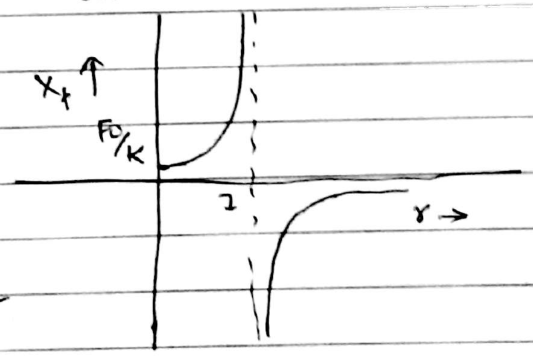
$(D^2 + \omega_n^2)x = \frac{F_0}{m}\sin\omega_f t$

$x_{particular} = \frac{F_0}{m}\frac{1}{(D^2+\omega_n^2)}[\sin\omega_f t]$

If $\omega_f \neq \omega_n$, then

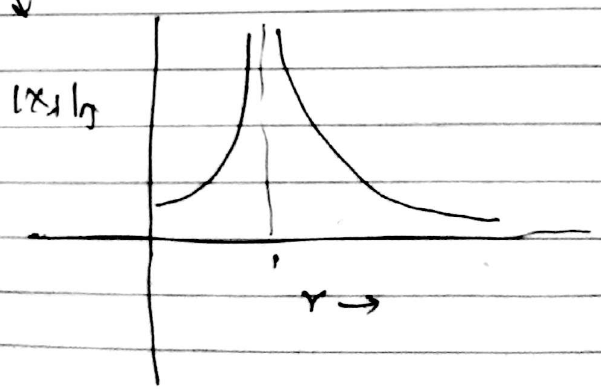
$x_p = \frac{F_0}{m}\frac{1}{(\omega_n^2-\omega_f^2)}\sin\omega_f t = \frac{F_0/k}{(1-r^2)}\sin\omega_f t = X_f \sin\omega_f t$

where $r = \frac{\omega_f}{\omega_n}$ = The frequency ratio




OR

(At resonance, theoretically the amplitude jumps to infinity)



Q. What happens if $\omega_f = \omega_n$ !

Now the DEOM becomes

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega_n t \quad \text{— (2)}$$

$\frac{D^2 x +}{D+2}$

We start with a forcing fn. $\frac{F_0}{m} e^{j\omega_n t}$ — (3), so that our desired steady state soln. will be the imaginary part of the particular integral of $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{j\omega_n t}$ — (4)

$$x_p = \frac{F_0}{m} \left[\frac{1}{D^2+\omega_n^2}\right] e^{j\omega_n t}$$

$$= \frac{F_0}{m} \left[\frac{1}{(D-j\omega_n)(D+j\omega_n)}\right] e^{j\omega_n t}$$

$$= \frac{F_0}{2mj\omega_n} \left[\frac{1}{D-j\omega_n}\right] e^{j\omega_n t}$$

$$= \frac{F_0}{2mj\omega_n} e^{j\omega_n t} \int e^{-j\omega_n t} e^{j\omega_n t}\, dt$$

$$= \frac{F_0 t}{2mj\omega_n} e^{j\omega_n t}$$

[Note: $y = \left(\frac{1}{D+P}\right) g$

$(D+P) y = g$

$\frac{dy}{dx} + P(x) y = g(x)$

I.F. $= e^{\int P dx}$

$\int d[y e^{\int P dx}] = \int e^{\int P dx} g$

$y = e^{-\int P dx} \int e^{\int P dx} g$ ]

$$x_p = -\frac{j F_0 t}{2m\omega_n} (\cos\omega_n t + j \sin\omega_n t)$$

of DE (4)

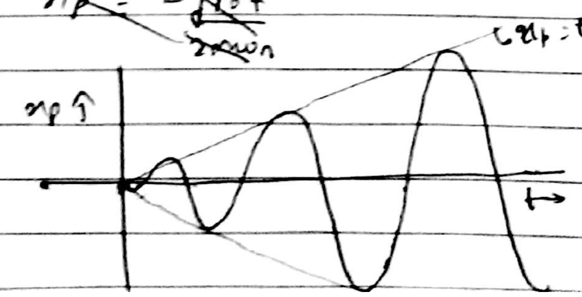
So, reqd. $x_p = \text{Im}(x_p \text{ of DE } 4)$

$$= -\frac{F_0 t}{2m\omega_n} \cos\omega_n t$$

So, at $\omega_f = \omega_n$, the forced response is given as

$$x_p = x_{forced} = x_{ss} = -\frac{F_0 t}{2m\omega_n} \cos\omega_n t$$

~~$x_p = -\frac{jF_0 t}{2m\omega_n}$~~



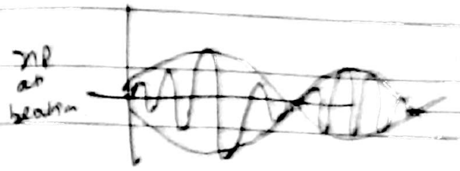

$x(t) = X_0 \sin(\omega_n t + \psi) + \text{SS response}$

$= (A \sin \omega_n t + B \cos \omega_n t)$

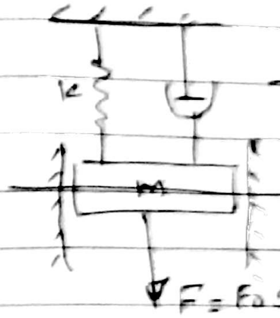
$+ \frac{F_0/k}{1-r^2} \sin \omega_f t$

Note that the free ~~vibration part~~ doesn't die down

What happens when $\omega_f \approx \omega_n$ ?
The phenomenon of beating occurs
Obtain the forced response ... when $\omega_f \approx \omega_n$




★ ⇒




→ The GEOM is
① - $m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t$
⇒ $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega_f t$

Essay in the History of Mechanics - Truesdell et al

$x_t = X_0 e^{-\xi\omega_n t}\sin(\omega_d t + \phi)$

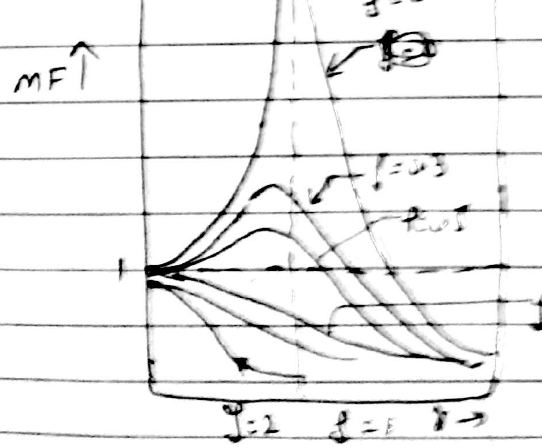
Equivalent static ~~deflection~~

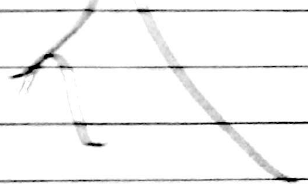
Remember

$$\boxed{x_p = \frac{F_0/K}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin(\omega_f t - \phi) \qquad \phi = \tan^{-1}\frac{2\xi r}{1-r^2}}$$

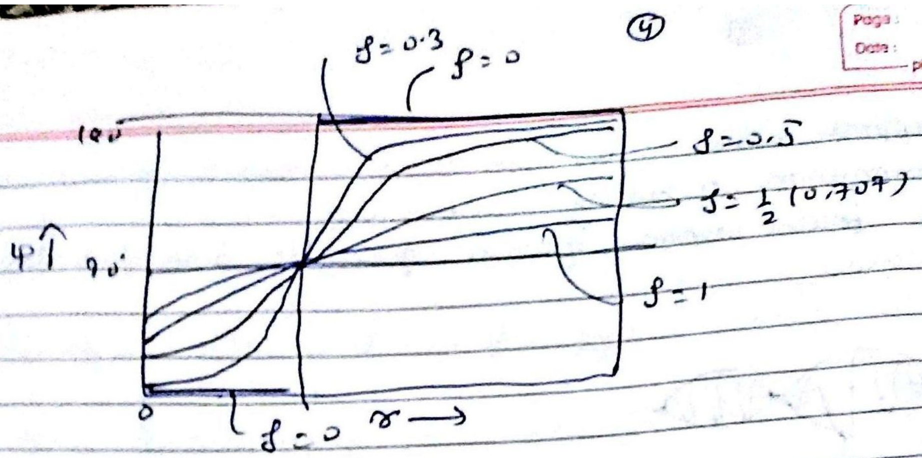
or $x_{ss}$

$\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$ → The Magnification factor



④




Tutorial 1

~~1.31)~~

1.32) The co-efficient of $\frac{1}{2}\dot{\theta}_1^2$ in the expression of K.E. as the $I\ddot{\theta}_1 + k_r\theta_1 = 0$ & in the OEUM.

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2$$

and $N_1\theta_1 = N_2\theta_2$ (as $\because$ gear 1 & 2 are in mesh)

$$\Rightarrow \theta_2 = \frac{N_1\theta_1}{N_2}$$

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\left(\frac{N_1\dot{\theta}_1}{N_2}\right)^2$$

$$= \frac{1}{2}J_1 + \frac{1}{2}J_2\left(\frac{N_1}{N_2}\right)^2\dot{\theta}_1^2$$

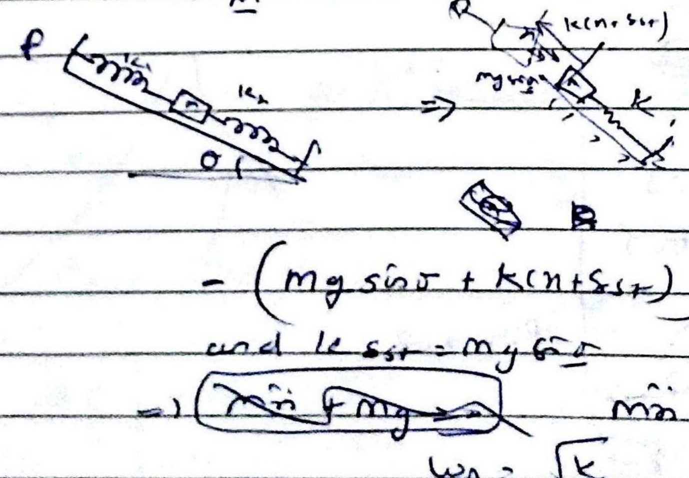
$$I = \left(J_1 + J_2\left(\frac{N_1}{N_2}\right)^2\right)\frac{\dot{\theta}_1^2}{2}$$

2.08. $M = 2000\,kg$ $\quad k\delta_{st} = mg$

$k = 2000 \times 9.81/0.02$

$\omega_n = \sqrt{\frac{k}{m}}$

2.09)




$$-\left(mg\sin\theta + k(x+\delta_{st})\right) = m\ddot{x}$$

and $k\delta_{st} = mg\sin\theta$

$\Rightarrow$ $\cancel{m\ddot{x} + mg}$ $\quad m\ddot{x} + kx = 0$

$$\omega_n = \sqrt{\frac{k}{m}}$$

here $k = k_{eq} = k_1$ & $k_2$ in parallel

2.4(a)

$$T = \tfrac{1}{2} m (l\dot\theta)^2 + \tfrac{1}{2} k_1 (a\theta)^2 + \tfrac{1}{2} k_2 (b\theta)^2$$

$$U = \tfrac{1}{2} k_1 (a\theta)^2 + \tfrac{1}{2} k_2 (b\theta)^2 - mgl\sin\theta$$

$$\frac{du}{d\theta} = k_1 a\theta + k_2 b\theta - mgl\cos\theta$$

2.72

$$\delta = \frac{PL^3}{48EI}$$

$\sigma = E\varepsilon$

$\left(\frac{F}{A_0}\right) = E\frac{\Delta l}{l}$

$\frac{F}{(EA/l)} = \Delta l$ $\qquad F \propto x$

$k = \left(\frac{EA}{l}\right)$

$\left(\frac{EA}{l}\right)$

$mg = 9810$

$m = 1000\ kg$

$x(0) = 0 \qquad x = A\sin(\omega_n t + \alpha)$

$\dot x(0) = 2\ m/s \qquad x = A\sin\omega_n t$

$A\omega_n = 2$

$A = \frac{2}{\omega_n} = \left(\frac{2}{\sqrt{\frac{k}{m}}}\right)$

$f_n = 0.5\ Hz$ $\qquad$ ~~$x(t) = A_n \sin($~~

$f_d = 0.45$

$\omega_d = \sqrt{1-\xi^2}\,\omega_n$

$\frac{0.45}{0.5} = \sqrt{1-\xi^2}$

$\left(\frac{9}{10}\right)^2 = 1-\xi^2$

$\xi^2 = 1 - \frac{81}{100}$

$\xi^2 = \frac{19}{100}$

$\boxed{\xi = \frac{\sqrt{19}}{10}} = \boxed{\frac{c}{2m\omega_n}}$

2.87) $x = (A+Bt)e^{-\omega_n t}$ critically damped system

$\dot{x} = Be^{-\omega_n t} - \omega_n(A+Bt)e^{-\omega_n t}$

$\dot{x} = 0 \Rightarrow (A+Bt)\omega_n = B$

$t = \dots$

$t = \left(\frac{B}{\omega_n} - A\right)\frac{1}{B}$

2.88)

$x = Ae^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t}$

$x(0) = (A+B)$

$\dot{x}(0) =$

$\dot{x} = -\omega_n A(\zeta+\sqrt{\zeta^2-1})e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} + (-\zeta+\sqrt{\zeta^2-1})\omega_n B e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t}$

$\dot{x}(t) = 0$

when $A(\zeta+\sqrt{\zeta^2-1}) = (\sqrt{\zeta^2-1}-\zeta)Be^{2\sqrt{\zeta^2-1}\omega_n t}$

$A(\zeta+\sqrt{\zeta^2-1})$ ...

2.89) a) Underdamped $x = Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$

$\frac{2\pi}{\omega_d} = 0.1\,s$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{4}{2}\right)$

$\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln 2$

$\zeta^2 = (0.11)^2\times(1-\zeta^2)$

$\zeta^2 = 0.0121\times(1-\zeta^2)$ $\qquad \zeta^2\times 0.9879 = 0.0121$

$\zeta = \frac{c}{2m\omega_n}$ $\qquad \zeta = 0.11$

$\omega_d = 62.8 = \sqrt{1-(0.11)^2}\,\omega_n$

$\Rightarrow \omega_n = 63.183 = \sqrt{\frac{k}{m}}$

$m = 500/9.81 = 50.96$ kg

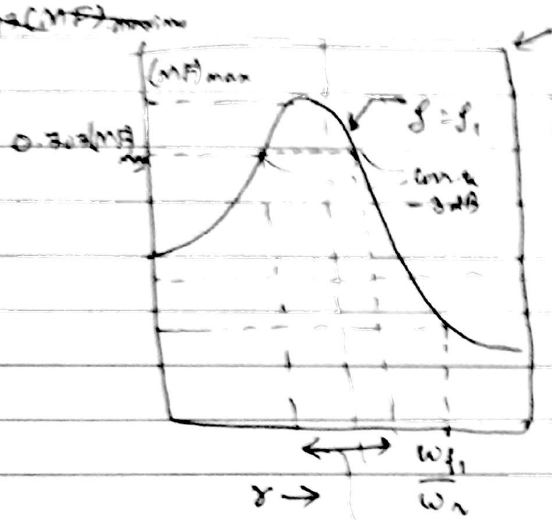
$k = 63.183^2\times 50.96$ N/m

$= 203.43$ kN

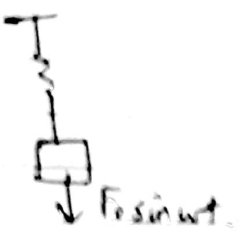
$\zeta = 44.922$ overdamped

$x(0) = 0 \Rightarrow A + B = 0$

$\dot{x}(0)$

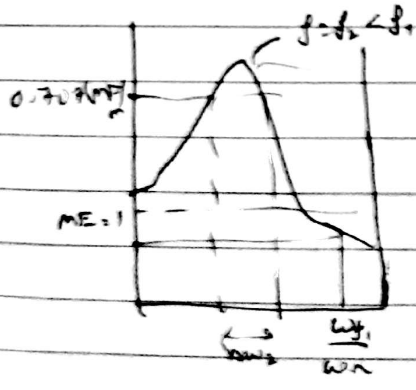



This fig shows that the spring mass system can act as a mechanical filter.




Corresponds to the bandwidth of the system ($\omega_b$)

$$20 \log \frac{(MF)_{min}}{(MF)_{max}}$$




- Sharper resonance

$r \to$ at smaller value of $\xi$

H.W. $\left.\dfrac{\partial MF}{\partial r}\right|_{\xi\cdot\xi} = 0$ Find where MF is a maximum or a minimum

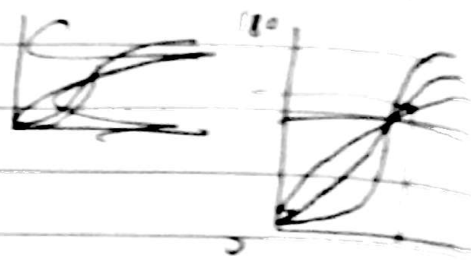
If $\xi << 1$

$(MF)_{max} = \dfrac{1}{2\xi} \to$ The Q factor

★ Phase resonance occurs at $r=1$ $(\Psi = 90°)$

Find $\Psi$ for $r = 1.1$ $\xi = 0.3$

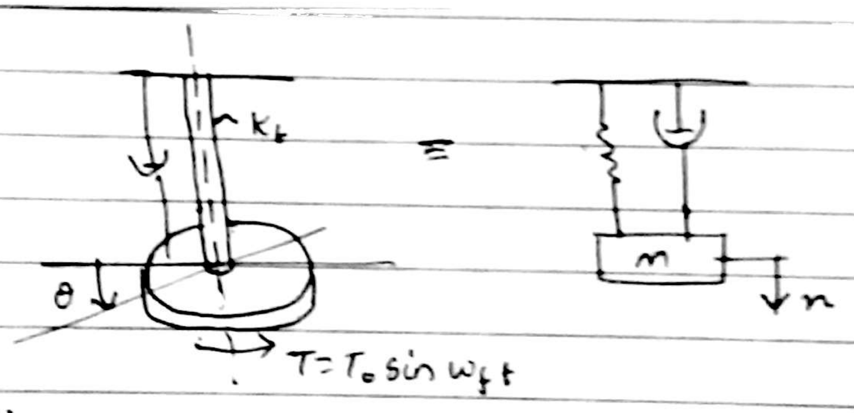
$\tan^{-1}\left(\dfrac{2\times\xi\times r}{1-r^2}\right) = -72°.08$



Now $\tan(180+\theta) = \tan\theta$

$180 - 72.08 = \boxed{108}$ ✓ Right answer

⇛




$I_d\ddot{\theta} + C_t\dot{\theta} + K_t\theta = T_0 \sin w_f t$

Transient Response

$\theta_c(t) = H_1 \sin w_n t + H_2 \cos w_n t$

$[\, x_c = A\sin w_n t + B\cos w_n t \,]$

dies down after few cycles

$\boxed{\theta = \theta_c + \theta_{ss}}$ , $\theta_{ss} = \dfrac{T_0/k_t \text{ (rad)}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(w_f t - \Psi)$

$\xi = \dfrac{c}{c_c} = \dfrac{c}{2mw_n}$

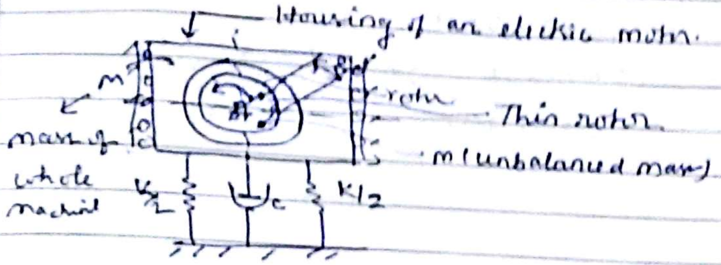
$\xi = \dfrac{c_t}{2I_d w_n}$

$C_c \to$ Critical damping constant
corresponds to $\xi = 1$

Here, $\zeta = \frac{c_c}{2 J_1 \omega_n}$

$(MF)_{max} = \left(\frac{1}{2\zeta}\right) \rightarrow$ The Q factor.

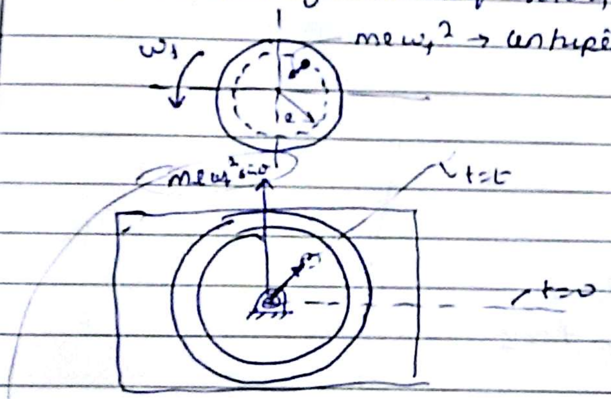
(5) Rotating Unbalance [Harris Shock & Vibration book]




Q) How do we get m for a real thin rotor?

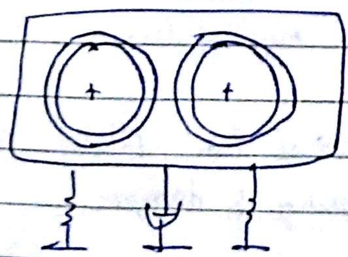
A. We never get m experimentally, however, we can get obtain the product em, which is called the unbalance U [That is U = me = The unbalance
~~me = 10^-3 gm-cm~~

Let θ is measured from the right hand horizontal line through centre of rotor, +ve ccw




$\theta = \omega_f t$

This is the vertical force on the m/c housing caused by the unbalanced mass.



Vertical force = $m e \omega_f^2 \sin(\omega_f t)$

Assume $F_0 = m e \omega_f^2$

$\omega_n = \sqrt{\frac{k}{M}}$

$x = x_h + x_{ss}$

(The equivalent system)

$$x_{ss} = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$$

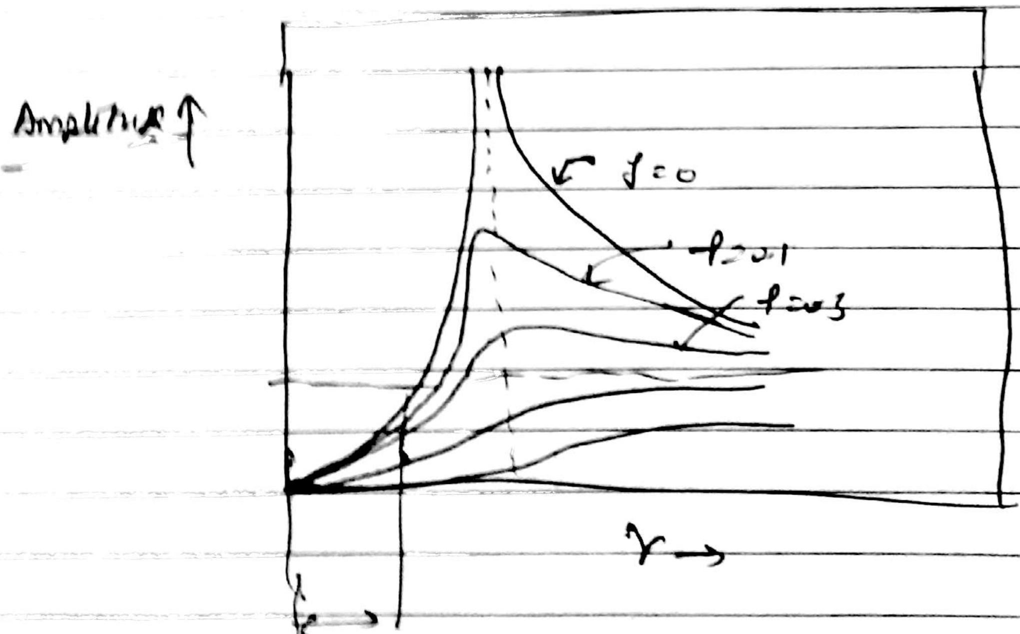
$$= \frac{m e \omega_f^2/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$$

$$r = \frac{\omega_f}{\omega_n}$$

So $\dfrac{\left(\frac{me}{M}\right)\omega_f^2}{k/M} = \left(\dfrac{U}{M}\right)\dfrac{\omega_f^2}{\omega_n^2} = \dfrac{U}{M} r^2$

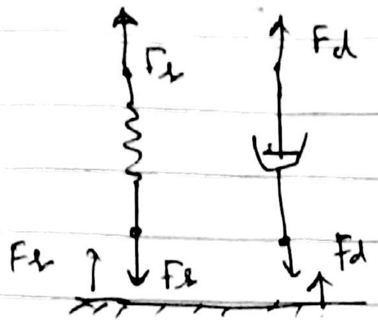
So $x_{ss} = \dfrac{\frac{U}{M} r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$

Amplitude of forced vibration $= \dfrac{\frac{U r^2}{M}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$




In this range amplitude due to unbalance mass is less

→ We are interested in finding the characteristics of the force transmitted to the rule foundation by the spring & damper combination.




$F_s = kx$ $\quad$ $F_T$ = Force transmitted to foundation

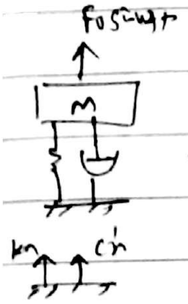
$F_d = c\dot{x}$ $\quad$ $= F_s + F_d = kx + c\dot{x}$

$F_T$ = Dynamic Force $= F_s + F_d = kx + c\dot{x}$

$x = \dfrac{F_0/k}{\sqrt{(1-r^2)^2}}$ ⇝ Assuming $F(t) = F_0 \sin \omega_f(t)$

$F_0$ independent of $r$

$$\dot{x} = \frac{F_0/k \; \omega_f}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \cos(\omega_f t - \psi)$$




$$F_T = kx + c\dot{x} = \frac{F_0}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega_f t + \psi) + \frac{F_0 \left(\frac{c\omega_f}{k}\right)}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \cos(\omega_f t - \psi)$$

$(2\xi r)$

$$F_T = kx + c\dot{x} = \frac{F_0}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega_f t + \psi) + \frac{F_0 \left(\frac{c\omega_f}{k}\right)}{\sqrt{\phantom{x}}} \cos(\omega_f t + \psi)$$

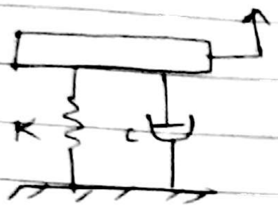
$$= f_T \sin(\omega_f t - \psi + \beta) \quad \text{(say)}$$

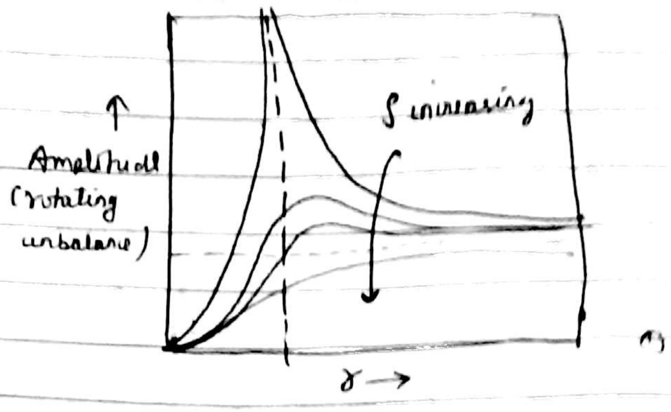
$$\frac{c\omega_f}{k} = \frac{c\omega_f}{m\,k/m}$$

$$= \frac{c}{m\omega_n}\,\frac{\omega_f}{\omega_n} = 2\xi r$$

$$F_T = F_0 \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega_f t - \psi + \beta)$$

$$\beta = \tan^{-1}(2\xi r)$$





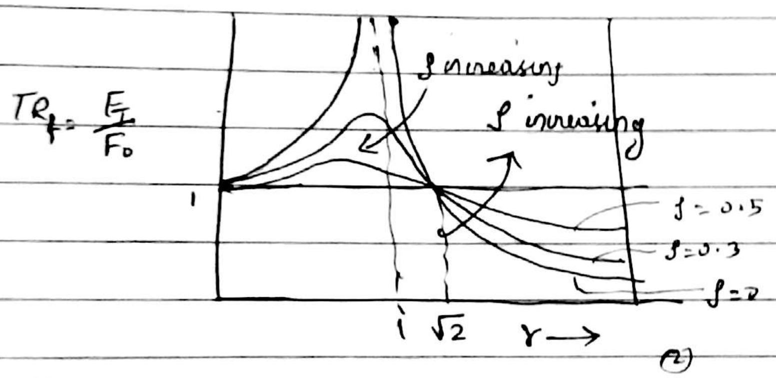

Force transmitted to the foundation

$$= \frac{F_0\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin(\omega_f t - \psi + \beta)$$

$\beta = \tan^{-1}(2\xi r)$

Definition: Force transmissibility $TR_f$ = Amplitude of force transmitted / " " applied force.

$$= \frac{F_T}{F_0} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$




We have force multiplication for $0 < r < 1$

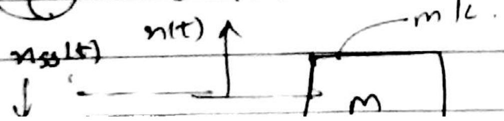
We have force isolation ($\Rightarrow TR_f < 1$) only for $r > 1$

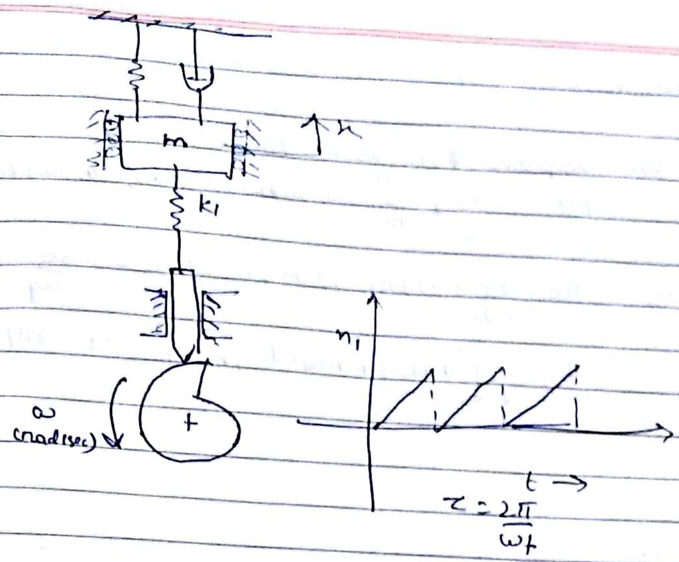
[From plots 1 & 2, in 1 as ξ increases amplitude is increased ~~whereas~~ whereas in 2 as ξ increases $\frac{F_T}{F_0}$ decreases.

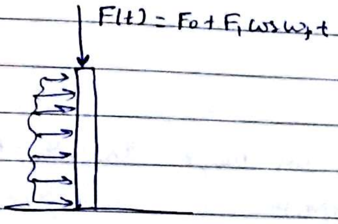
To keep the vibration amplitude of the machine to a reqd. minimum & also to obtain a better force isolation, a design has to strike a compromise

## (E) Base excitation

$y(t) \rightarrow$ displacement of the m/c foundation






Hence the DEOM is $m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$

$\Rightarrow \quad m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$ —— (1)

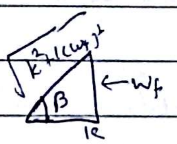
Let $y(t) = Y_0 \sin\omega_f t$

Eq. (1) becomes $m\ddot{x} + c\dot{x} + kx = Y_0(k\sin\omega_f t + c\omega\cos\omega_f t)$

$$= Y_0\sqrt{k^2+(c\omega)^2}\left(\frac{k}{\sqrt{k^2+(c\omega_f)^2}}\sin\omega_f t + \frac{c\omega_f}{\sqrt{k^2+(c\omega_f)^2}}\cos\omega_f t\right)$$

$$= \underbrace{Y_0 k\sqrt{(1+(2\xi r)^2)}}_{F_0}\sin(\omega_f t + \beta), \quad \beta = \tan^{-1}(2\xi r)$$

$$\frac{2c\omega_f}{(2m\omega_n^2)\cancel{k}} = 2\xi r$$




$$\xi = \frac{c}{c_c}$$

Amplitude of forced vib. of our system due to the base excitation.

$(F_0/k)$

$$x_{ss} = x(t) = \frac{Y_0\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin(\omega_f t - \psi + \beta),$$

$$\psi = \tan^{-1}\frac{2\xi r}{(1-r^2)}$$

$TR_m$ = motion transmissibility = $\dfrac{\text{amplitude of force vibr.}}{\text{amplitude of base vibr.}} = \dfrac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$

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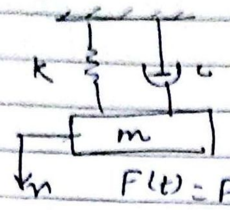
(8) Periodic Excitation



We express $F(t)$ as a fourier series.

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_f t + b_n \sin n\omega_f t).$$

$F(t) = F(t+\tau)$

$$a_n = \frac{2}{\tau}\int_0^{\tau} F(t)\cos n\omega_f t\,dt,\ n = 0,1,2,3,\ldots \quad \tau = \frac{2\pi}{\omega_f}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} F(t)\sin n\omega_f t\,dt,\ n = 0,1,2,3,\ldots \quad \omega_f = 2\pi/\tau$$

~~$x_{ss}$ due to $\frac{a_0}{2}$ is~~

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$

$x_{ss}$ due to $\frac{a_0}{2}$ is $\frac{a_0}{2k}$

$x_{ss}$ due to $a_n \cos\omega_f t$ is $\dfrac{a_n/k}{\sqrt{(1-r_n)^2 + (2\zeta r_n)^2}} \cos(n\omega_f t - \psi_n)$, $\psi_n = \tan^{-1}\dfrac{2\zeta r_n}{1-r_n^2}$

$r = \dfrac{\omega_f}{\omega_n}$

$r = \dfrac{n\omega_f}{\omega_n}$

$r_n$

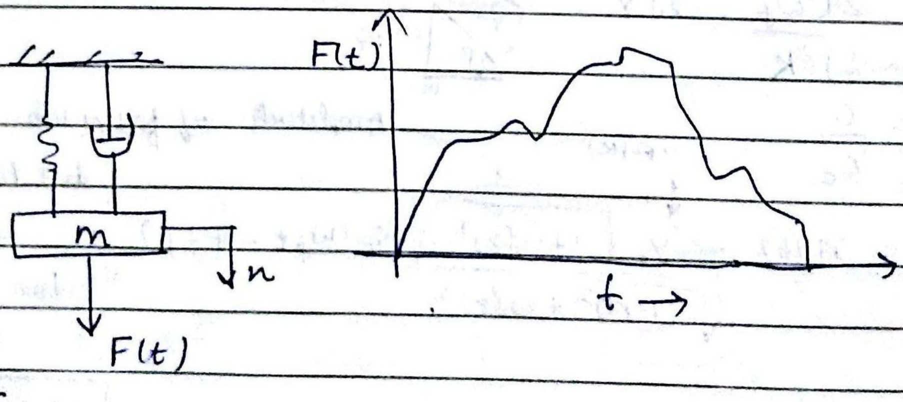
$x_{ss}$ due to $b_n \sin\omega_f t \longrightarrow \dfrac{b_n/k}{\sqrt{(1-r_n)^2 + (2\zeta r_n)^2}} \sin(n\omega_f t - \psi_n)$

By the principle superposition

$$x_{ss} = \frac{a_0}{2k} + \sum_{n=1}^{\infty} \frac{1}{k\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}}\left[a_n \cos(n\omega_f t - \psi_n) + b_n \sin(n\omega_f t - \psi_n)\right]$$

~~xxxx~~

2/1/17

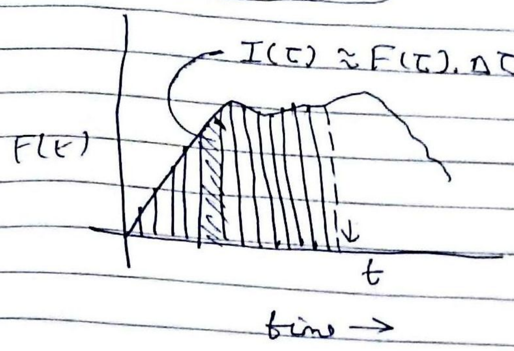



$x(t)$ = Forced response of the K-V model under the action of a force $F(t)$, which can have any complex form.

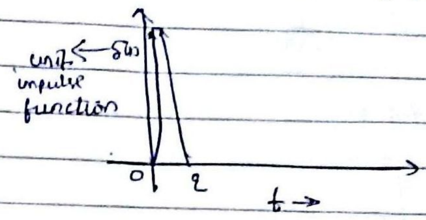
The force acts for $t \geq 0$. & we have zero initial conditions.

Brüel & Kjaer → company.

$$\boxed{x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau} \rightarrow \text{The Duhamel integral / convolution integral}$$




The dirac delta function (Dirac's delta fn)




$$I = \int_{t_1}^{t_2} F(t)\, dt$$

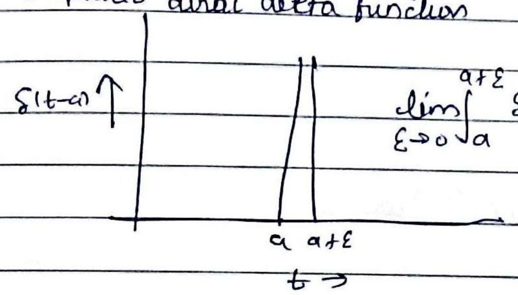
As $\epsilon \rightarrow 0^+$, we get $\delta(t)$, provided

$$\underbrace{\int_0^\infty \delta(t)\, dt}_{\text{Impulse of } \delta(t)} = 1$$

$\delta(t) = 0$ for $t \neq 0$

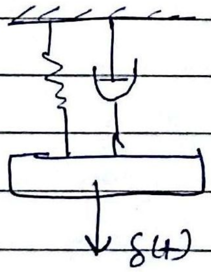
$$\int_0^\infty f(t)\,\delta(t)\, dt = f(0)$$

Displaced dirac delta function




$$\lim_{\epsilon \to 0} \int_a^{a+\epsilon} \delta(t-a)\, dt = 1$$

so $$\int_0^\infty f(t)\,\delta(t-a)\, dt = f(a)$$




$\delta(t)$ will produce, over $[0, \epsilon]$ $(\epsilon \rightarrow 0^+)$ a finite velocity of the mass but it won't produce any considerable displacement. In other words, after $\delta(t)$ is applied $x(0^+) = 0$, $\dot{x}(0^+) \rightarrow$ finite.

Over the interval $[0, t]$ several forces act on the body, however the impulses of spring, damper & gravity forces are negligible.

Hence $mv(0^+) - mv(0) = 1$ → (Impulse of dirac delta function is unity)

$$v(0^+) = \dot{x}(0^+) = \frac{1}{m}$$

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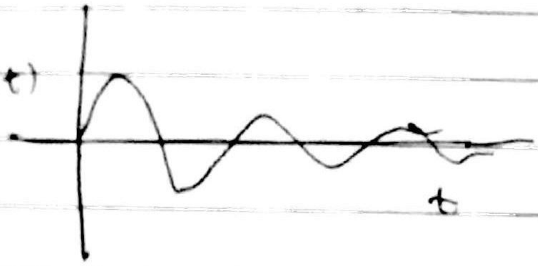
$$\left\{\begin{array}{l} x(0^+) = 0 \\ \dot{x}(0^+) = \dfrac{1}{m} \end{array}\right.$$

If $\zeta < 1$




$$\underbrace{m\ddot{x} + c\dot{x} + kx}_{} = F(t)$$

$x_{ss}$ $\qquad\qquad\qquad \downarrow$

$\alpha x_{ss}$ $\qquad\qquad\qquad \alpha F(t)$

Let $\zeta < 1$

Then $x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$

$x(0^+) = 0 \Rightarrow \phi = 0$

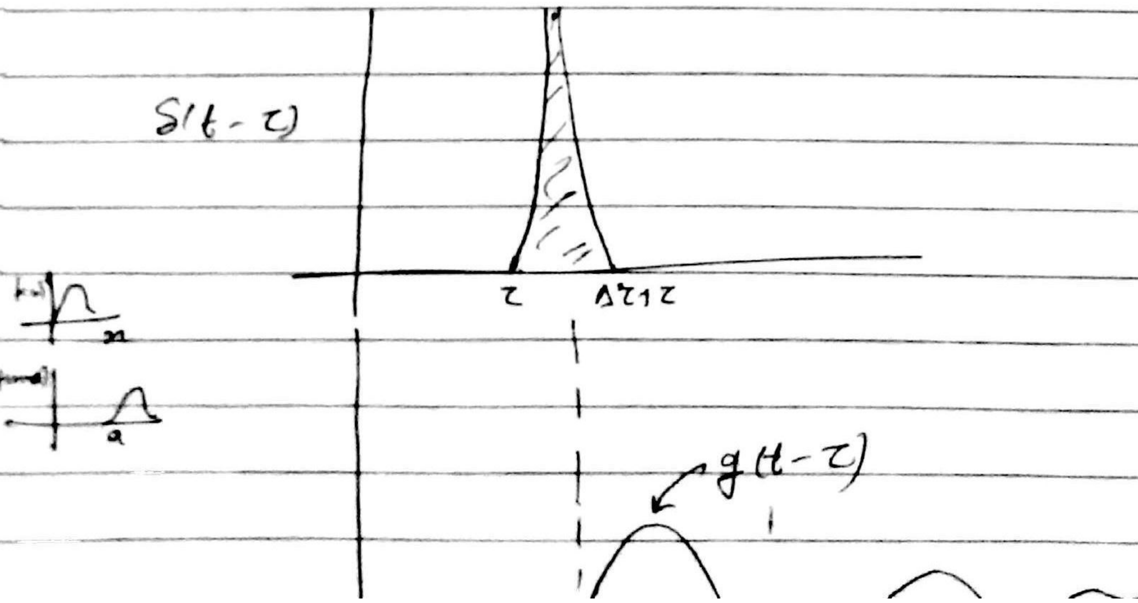
$\dot{x}(t) = -X_0 \zeta\omega_n e^{-\zeta\omega_n t} \sin\omega_d t + X_0 \omega_d e^{-\zeta\omega_n t} \cos\omega_d t$

$\dot{x}(0^+) = \dfrac{1}{m} \Rightarrow X_0\omega_d = \dfrac{1}{m} \quad X_0 = \dfrac{1}{m\omega_d}$

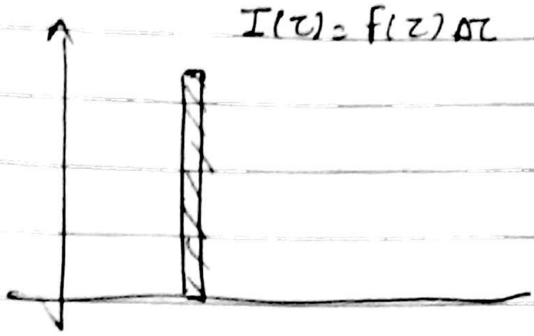
Hence, $\boxed{x(t) = \dfrac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t}$

$g(t)$, the impulse response function.

Response of our system due to dirac ~~delta~~ delta excitation.



$g(t-\tau)$ → The response of the system due to unit impulse excitation, applied at $t=\tau$



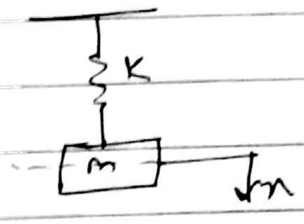

This $I(\tau)$ generates a response of $I(\tau)\,g(t-\tau) = F(\tau)\,g(t-\tau)\,\Delta\tau$

The total response of time $t$, can be obtained by the principle of superposition & will be obviously given by

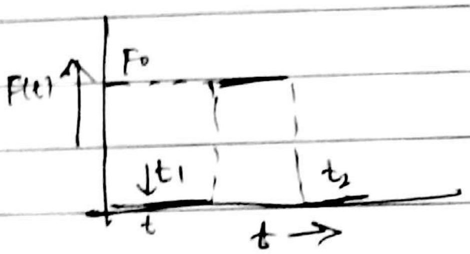
$$x(t) = \int_0^t F(\tau)\,g(t-\tau)\,d\tau$$

$$= \int_0^t F(t-\tau)\,g(\tau)\,d\tau.$$

Example




$\xi = 0$




Find $x(t)$ for $t>0$.
Using Duhamel's integral.

$\boxed{g(t) = \frac{1}{m\omega_n}\sin\omega_n t}$ → ($\xi = 0$ for undamped system)

For $0 \le t \le t_1$, $F(t) = 0$

Hence, $x(t) = \int_0^t F(\tau)\,g(t-\tau)\,d\tau$

$= 0$

For $t_1 \le t \le t_2$

For $t > t_2$

$F(t) = 0$

$x(t) = \int_0^t \text{———}$

$\int_0^{t_1} \nearrow + \int_{t_1}^{t_2} F(\tau)\, g(t-\tau)\, d\tau$

$+ \int_{t_2}^{t} \nearrow^{0}$

~~3.7~~ 3/1/17

Tutorial 3

3.7 $F(t) = 25 \cos\omega t$ , $k = 2000$, $m = \frac{100}{9.8} = 10.2\,kg$

$= 25 \sin(\omega t + 90^\circ)$

$\ddot{x} + \omega_n^2 x = F(t)/m$

$\ddot{x} + \omega_n^2 x = \frac{25 \cos\omega t}{m}$ 3.8)

$\omega_n = \sqrt{\frac{k}{m}}$ $x = x_c + x_{ss}$

The required steady state solution will be the real part of Particular solution of $\ddot{x} + \omega_n^2 x = \frac{25}{m} e^{j\omega t}$

$x_p = \frac{25}{m} \frac{e^{j\omega t}}{(D^2 + \omega_n^2)}$

$x_p = \frac{25}{m(D - j\omega_n)(D + j\omega_n)} e^{j\omega t}$

$x_p = \frac{25}{2mj\omega_n} \frac{e^{j\omega t}}{(D - j\omega)}$

$x_p = \frac{25}{2mj\omega_n} e^{j\omega t} \int e^{-j\omega_n t} e^{j\omega_n t}\, dt$

$x_p = \frac{25}{2mj\omega_n} e^{j\omega t}\, t$

$= \frac{25t}{2m\omega_n} (-j) (\cos\omega t + j\sin\omega t) \rightarrow$

Real part of $x_p$ is $\frac{25t}{2m\omega_n} \sin\omega_n t$

$x_p = \frac{25t}{2m\omega_n} \sin\omega_n t$ $x_p = x_{force} = \frac{25}{2m\omega_n} t \sin\omega_n t$

~~at t~~ and $T = \frac{2\pi}{\omega_n}$

at $\frac{1}{4}^{th}$ of the cycle ~~amplitude~~ : $t = \frac{T}{4} = \frac{\pi}{2\omega_n}$

$$x_p = \frac{25}{2m\omega_n}\left(\frac{\pi}{2\omega_n}\right)\sin\left(\frac{\pi}{2}\right), \quad t = \frac{5T}{2} = \frac{5}{2}\left(\frac{2\pi}{\omega}\right) = \left(\frac{5\pi}{\omega}\right)$$

$$= \frac{25\pi}{4m\omega_n^2}$$

$$= \frac{25\pi}{4m \times \frac{k}{m}} \qquad x_p = \frac{25t}{2m\omega_n}\sin(5\pi)$$

$$= \frac{25 \times \pi}{4 \times 2000} \qquad = 0$$

$$= 9.8125 \times 10^{-3}\ m$$

$$= 9.81\ mm$$

3.8) $K = 4000\ N/m$ $\quad F = F_0 \sin\omega_f t$ $\quad F_0 = 100,\ \omega_f = 5Hz$

Amplitude of forced motion $= \frac{F_0/k}{\cancel{\sqrt{1-r^2}}\ (1-r^2)}$

$$\frac{100/4000}{(1-r^2)} = |20 \times 10^{-3}|$$

$$\frac{1}{\cancel{(1-r^2)}} \qquad (1-r^2) = \frac{10 \times 10^3}{800}$$

$$1 - r^2 = |1.25|$$

~~Taking~~ $\cancel{x_p} = \frac{F_0/k}{(1-r^2)}$

$$1 - r^2 = -1.25$$

$$r^2 = 2.25$$

$$\boxed{r = 1.5}$$

$$\frac{\omega_f}{\omega_n} = 1.5$$

$$\frac{2\pi \times 5}{1.5} = \omega_n$$

$$\omega_n = 20.933$$

$$\sqrt{\frac{k}{m}} = 20.933$$

$$\frac{k}{m} = 438.19$$

$$m = \frac{4000}{438.19} = 9.128$$

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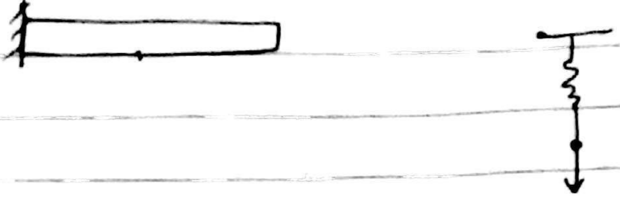
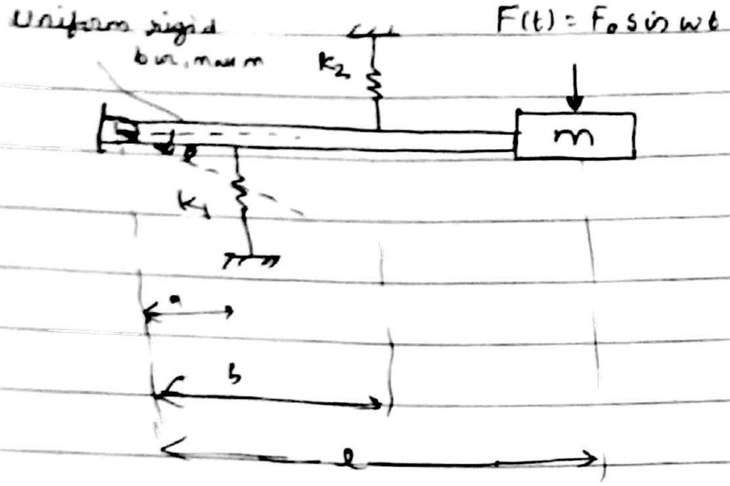
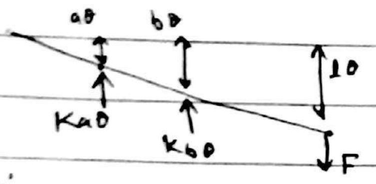
3.11)



3.A)




$I\ddot{\theta} = -(k_1 a\theta)a - (k_2 b\theta)b + F(l)$

$I\ddot{\theta} + (k_1 a^2 + k_2 b^2)\theta = F \times l$

$I\ddot{\theta} + (k_1 a^2 + k_2 b^2)\theta = (F_0 l)\sin\omega t$

$\theta_c = \theta_0 \sin\omega_n t$ $\qquad \omega_n = \sqrt{\dfrac{ka^2 + kb^2}{I}}$

$\omega_n = \sqrt{\dfrac{5000 \times ((0.25)^2 + (0.5)^2)}{ML^2 + \frac{1}{2} m_{rod} L^2}}$

$= \sqrt{\dfrac{5000 \times 0.3125}{50 \times 1 + \frac{1}{2} \times 10 \times 1}}$

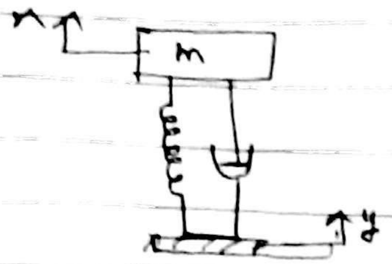
$= \sqrt{\dfrac{1562.5}{55}}$

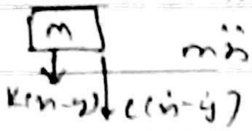
$\omega_n = 5.33$

$\omega_f = 1000\, rpm = \dfrac{1000 \times 2\pi}{60} = 104.67$ $\qquad \dfrac{\omega_f}{\omega_n} = \dfrac{104.67}{5.33} = 19.63$

$r = 19.63$

3.35) $y(t) = y_0 \cos \omega t$




$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$

$m\ddot{x} + k(x-y) + c(\dot{x}-\dot{y}) = 0$

$m\ddot{x} + kx + c\dot{x} = ky + c\dot{y}$

$y(t) = y_0 \cos \omega t = y_0 \sin(\omega t + 90)$

$$x_{ss} = \frac{x(t)}{y_0} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t + 90 + \psi - \beta)$$

$\beta = \tan^{-1}(2\zeta r)$

$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$

$\zeta = \frac{c}{2m\omega_n}$ , $\zeta = 0.8$

$\cancel{\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\ }}$

$$\frac{x(t)}{y_0} = \frac{0.1}{0.2} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\frac{1}{4} = \frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}$$

$(1-r^2)^2 + (2\zeta r)^2 = 4 + 4\times(2\zeta r)^2$

$(1-r^2)^2 - 4 = 3\times(2\zeta r)^2$

$(1-t)^2 - 4 = 3\times(2\times0.8)^2\times t$

$1+t^2-2t-4 = 7.68\times t$

$t^2 - 9.68t - 3 = 0$

$$t = \frac{9.68 \pm \sqrt{(9.68)^2+4\times3\times1}}{2(1)}$$

$$= \frac{9.68+10.28}{2}$$

$t = 9.98$

$r^2 = 9.98$

$r = 3.159 = \frac{\omega_f}{\omega_n}$

$\omega_f = 157.08$

$3.159 = \frac{157.08}{\omega_n}$

$y = \left(\frac{3l}{4}\theta\right)$

$I\ddot{\theta} = -k\left(\frac{l}{4}\theta\right)\frac{l}{4} + k\left(x - \frac{3L}{4}\theta\right)\left(\frac{3L}{4}\right)$

$I\ddot{\theta} + \left(k\left(\frac{l}{4}\right)^4 + k\left(\frac{3L}{4}\right)^2\right)\theta = kx\frac{3L}{4}$

$I\ddot{\theta} + \left(k\left(\frac{l}{4}\right)^2 + k\left(\frac{3l}{4}\right)^2\right)\theta = \cancel{(kx_0)\sin\omega t}_{\to F_0} \quad \underbrace{\left(kx_0\frac{3L}{4}\right)}_{F_0}\sin\omega t$

$\theta_{ss} = \frac{F_0/k_{eq}\ \sin\omega_f t}{(1-r^2)}$

$k_{eq} = k\left(\frac{l}{4}\right)^2 + k\left(\frac{3L}{4}\right)^2$

$r = \frac{\omega_f}{\omega_n}, \quad \omega_n = \sqrt{\frac{k_{eq}}{I}}$

$\cancel{I = \frac{4\,ML^2}{64}}$ $\boxed{I = \frac{7ML^2}{48}}$

$I = \int_{-l/4}^{3l/4} \cancel{(\ldots)}$

$I = \int_{-l/4}^{3l/4} \frac{M}{L} x^2\,dx$

$= \frac{m}{3L}\left(\left(\frac{3L}{4}\right)^3 + \left(\frac{L}{4}\right)^3\right)$

$= \frac{m}{3}\left(\frac{27+1}{64}\right)$

$= \frac{4\cancel{ML^2}}{\cancel{64}}$

$= \frac{m}{3}\frac{28}{64}$

$= \frac{7ML^2}{48}$

$F_0 = 1000 \times 10^{-2} \times \frac{3}{4} \times 1$

$F_0 = 7.5$

$k_{eq} = (1000) \times \left((0.25)^2 + (0.75)^2\right)$

$k_{eq} = 625$

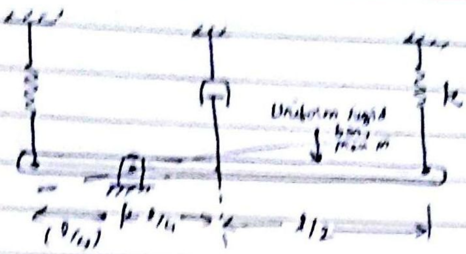
$\omega_n = \sqrt{\frac{625}{\frac{7}{48} \times 10 \times 1}}$

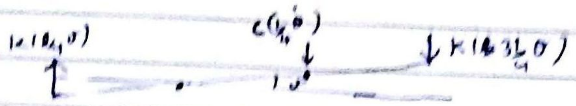
$\boxed{\omega_n = 20.70}$

$\omega_f = 10$ rad/s

$r = \frac{10}{\ldots} = 0.483 \qquad \theta_{ss} = \frac{7.5}{625} = \underline{0.0156}$

3.46)





$$I\ddot{\theta} = -k\left(\frac{3L}{4}\theta\right)\frac{3L}{4} - k\left(\frac{L}{4}\theta\right)\frac{L}{4} - c\left(\frac{3L}{4}\dot{\theta}\right)\frac{3L}{4} + M_0 \sin\omega t$$

$$I\ddot{\theta} + c\left(\frac{3L}{4}\right)^2\dot{\theta} + \left(k\left(\frac{3L}{4}\right)^2 + k\left(\frac{L}{4}\right)^2\right)\theta = M_0 \sin\omega t$$

$$\theta_{ss} = \frac{M_0/k_{eq}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega t + 90^\circ - \psi)$$

$$\xi = \frac{C_{eq}}{2m\omega_n}$$

$$C_e = c\left(\frac{L}{4}\right)^2 = 1000 \times (0.25)^2$$
$$= 62.5$$

$$I = \frac{7mL^2}{48} = \frac{7\times10\times1}{48} = 1.458$$

$$\omega_n = \sqrt{\frac{k_{eq}}{I}} = \sqrt{\frac{3125}{1.458}} = 46.29 \text{ rad/s}$$

$$\xi = \frac{62.5}{2\times1.458\times46.292}$$
$$= 0.463$$

Underdamped $\xi < 1$

$$\theta_c = \theta_0 e^{-\xi\omega_n t}\sin(\omega_d t + \phi)$$

$$\theta_{ss} = \frac{M_0/k_{eq}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \sin(\omega t + 90^\circ - \psi)$$

3.68)

$$\frac{\cancel{x m}}{\cancel{r^2}} \quad \frac{\frac{u r^2}{m}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

b) $r = \dfrac{\omega_f}{\omega_n}$ $\quad \omega_f = \dfrac{2\pi N}{60}$

$$\omega_n = \sqrt{\frac{k}{m}}$$