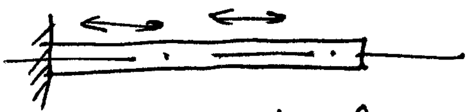
# Vibration of Continuous systems

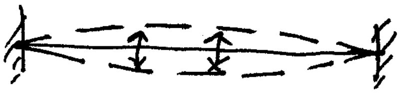
## PART-A

As the name 'Continuous system' implies, it has distributed inertia, storage & dissipation properties. This means an element of the system possesses mass, it can store elastic strain energy and it can dissipate energy, especially by an internal/material/structural/hysteresis damping.

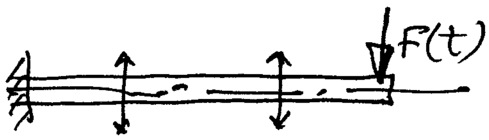
Examples:

1) 

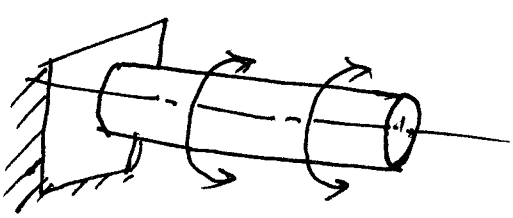
A clamped-free bar executing axial, free vibration.

2) 

A stretched string executing transverse, free vibration in the fundamental mode

3) 


An Euler-Bernoulli beam executing transverse, forced oscillations

4) 

A circular bar undergoing torsional free vibrations.

There can be much more complex Continuous systems consisting of a number of above elements as well as other types of elements such as plates & shells (curved plates of certain kinds).

→ Our scope of studies on vibrating continuous systems is limited to the free vibration of systems shown in examples 1 to 4 above.

~~Fo~~ Forced vibration of such systems as well as of systems containing plates & shells are studied in specialized courses on structural vibration and <u>finite element</u> method.

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→ A continuous system such as a vibrating bar is said to have infinitely many degrees-of-freedom (dof) because it has an infinity (!) of mass elements.

However, funnily enough, the DEOM of such a system is usually a small set of partial differential equations & <u>not</u> a set of infinitely many ordinary DEOM.

Leonard Meirovitch, a great author in this field of studies, has shown (See Vibration Analysis, 2nd Ed., L. Meirovitch, chapter 5, §5.2, page 205) how a system of ordinary DEOM for a a discrete system becomes a single partial DEOM as the number of vibrating mass elements is increased indefinitely.

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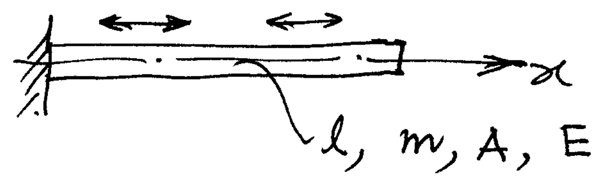
(8) Axial free vibration of a uniform straight bar :~

Aim:– To obtain the DEOM using Newton's method.

Let the given system be as shown in the figure:

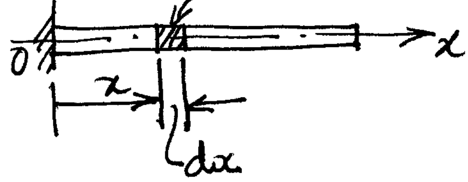


$l$ = (undeformed) bar length
$m$ = mass per unit length
$A$ = Gross sectional area
$E$ = Young's modulus

Assumptions:– (i) Amplitude of vibration is small & so, effective length of the bar remains '$l$' at all times. Similarly, '$A$' remains (almost) constant. (ii) Bar material is homogeneous, isotropic & linearly elastic.

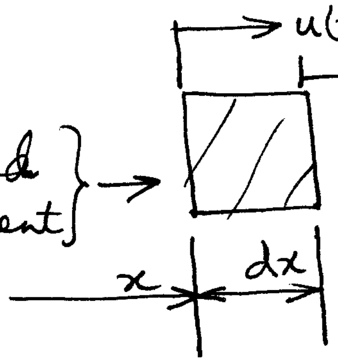
(Note:– The above is a clamped-free configuration/boundary condition.

Remember that other boundary conditions (BCs) are possible, such as pinned-pinned or simply supported, clamped-pinned, clamped-guided etc. In each such situation, the DEOM remains the same, only the BCs differ. This results in different sets of natural frequencies and eigenfunctions. (Eigenfunctions are the 'continuous' counterpart of Eigenvectors of discrete systems)




An 'element' of the bar at location $x$.

An enlarged view of element



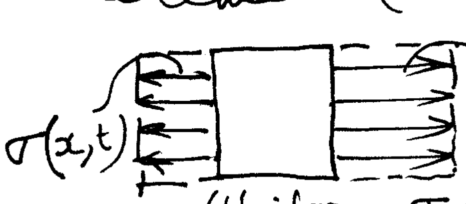

In the (element) figs, $dx$ is shown very much exaggerated

$$\boxed{\begin{aligned}\epsilon_x &= \frac{u+\frac{\partial u}{\partial x}dx-u}{dx}=\frac{\partial u}{\partial x}\\ &= \text{Axial strain} = \epsilon(x,t)\\ \therefore\ \sigma(x,t) &= E\epsilon(x,t)=E\frac{\partial u}{\partial x}\end{aligned}} \quad \cdots (\alpha)$$

$u(x,t)$ = Axial displacement of bar cross-section at original location '$x$' and at time '$t$' while vibrating;

$u(x,t)+\frac{\partial u(x,t)}{\partial x}dx$ = Axial displacement at original location '$x+dx$'

Hence, if $\frac{\partial u}{\partial x}$ is $>0$ at an instant, the element is stretched at that instant & vice-versa. This stretching and compression is due to a variation of axial stresses across the cross-section. The relevant FBD of the element (in the axial direction) is as follows:-

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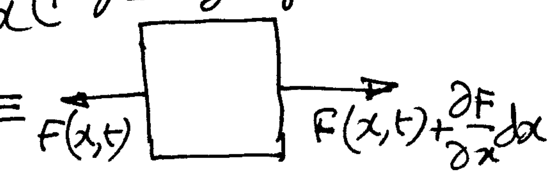



where $\sigma$ is the normal stress & $F$ is axial force at the x-section at $x$.

$\boxed{F=EA\frac{\partial u}{\partial x}, \text{ see } (\alpha) \text{ above}}$

$\therefore\ F=\sigma A, \quad F+\frac{\partial F}{\partial x}dx=\sigma A+A\frac{\partial \sigma}{\partial x}dx$

$\longrightarrow$ Axial acceleration of any (material) point in the element $\simeq \frac{\partial^2 u}{\partial t^2}$; mass of element $= m\,dx$

Hence, $m\,dx\times\frac{\partial^2 u}{\partial t^2}=F+\frac{\partial F}{\partial x}dx-F=\cancel{\sigma A}+A\frac{\partial \sigma}{\partial x}dx-\cancel{\sigma A}$

$\Rightarrow\ m\frac{\partial^2 u}{\partial t^2}=A\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right)=AE\frac{\partial^2 u}{\partial x^2}$, by Newton's 2nd law applied to the centre of mass of the element at time $t$, in the $x$-direction.

Hence, $m\dfrac{\partial^2 u}{\partial t^2} = AE\dfrac{\partial^2 u}{\partial x^2}$ is the reqd. DEOM of the bar for small, axial free-vibration.

Small

Note that $c = \sqrt{\dfrac{AE}{m}}$ = Speed of longitudinal ~~ve~~ waves in the bar.

Thus, the DEOM can be written as:

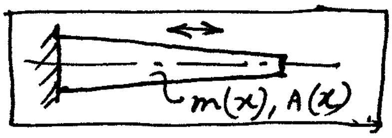
[Remember]

(I) - - - $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}}$ which is a

one-dimensional wave equation in its standard form with $c = \sqrt{\dfrac{AE}{m}} = \sqrt{E/\rho}$ = wave (longitudinal) velocity (speed) $\left[\rho = \dfrac{m}{A}\right.$ = mass/unit vol. = density

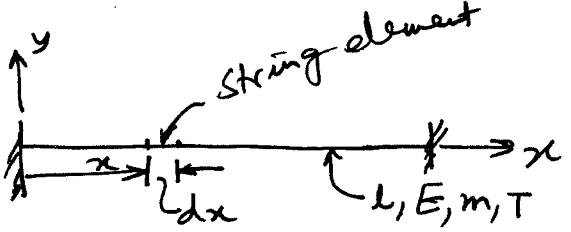
** [Home Work] If the bar is of variable x-section (say, a ~~tapered~~/conical bar), then $m = m(x)$ & $A = A(x)$. Show that the DEOM in this case is

$$m(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]$$

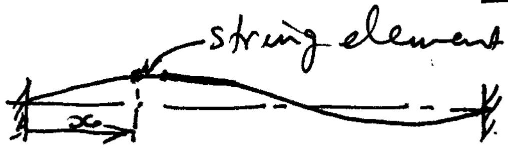



(5) Obtain the DEOM of a transverse vibration of a stretched, uniform string. Assume small amplitude vibration. The initial tension $T$ is so high that the small displacement vibration hardly causes any variation in $T$.

(PTO)



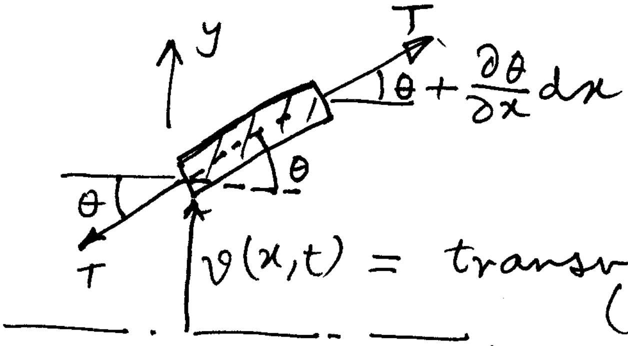

(At $t=0$)




(At time $t$)

$l$ = Length
$E$ = Young's Mod.
$m$ = mass/unit length

FBD of string element (weight neglected, why?)




$\theta = \theta(x,t)$ = angle made by the tangent to string centre-line with $x$-axis

$v(x,t)$ = transverse displacement of element (in $y$-direction)

Assumption:- $\theta$ is small so that

slope $= \frac{\partial v}{\partial x} = \tan\theta \simeq \theta$.

Net force on element in $y$-direction

$$= T \sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T \sin\theta$$

$$\simeq T\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T\theta$$

[For small $\theta$, $\sin\theta \simeq \theta$ etc.

$$= T \frac{\partial \theta}{\partial x} dx$$

Hence, applying Newton's 2nd law to the element centre of mass in $y$-direction,

$$(m\,dx) \cdot \frac{\partial^2 v(x,t)}{\partial t^2} = T \frac{\partial \theta}{\partial x} dx$$

$$\Rightarrow \quad m \frac{\partial^2 v}{\partial t^2} = T \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = T \frac{\partial^2 v}{\partial x^2}$$

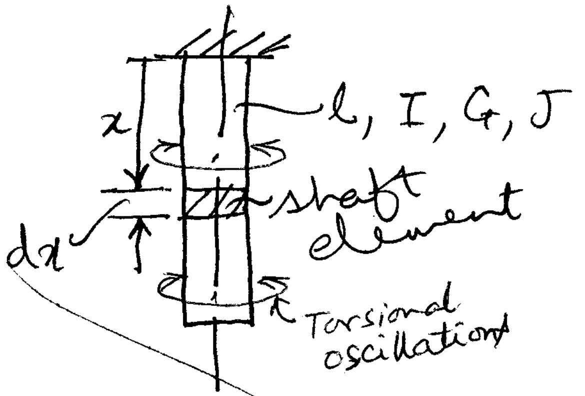
So, reqd. DEOM is:

(Remember) $\longrightarrow$ $$\boxed{\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}}$$ $\longrightarrow$ Reqd. DEOM

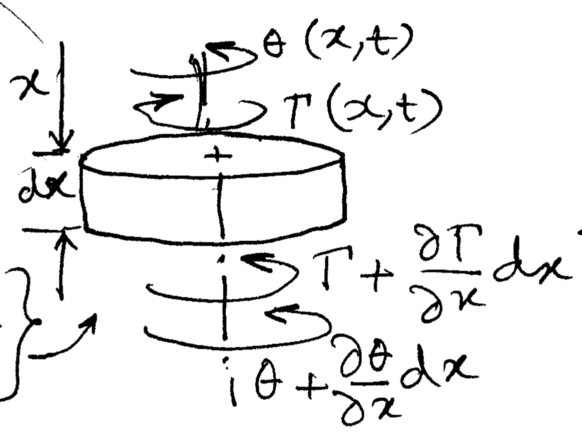
where $c = \sqrt{\frac{T}{m}}$ = wave speed.

This is a 1-D wave equation, once again! $\longrightarrow$

## (5) Torsional free-oscillations of a shaft/circular bar




$\left\{\begin{array}{l} l = \text{length of bar} \\ I = \text{moment of inertia (MI) } \underline{\text{per unit length}} \text{ about axis of rotation, } G = \text{shear modulus of shaft material, } J = \text{polar area MI of a x-section} \end{array}\right.$




FBD of enlarged element

$\theta(x,t)$ = rotation of a x-section at $x$ at time $t$

$\Gamma(x,t)$ = Torque on same x-section at time $t$

$\theta \rightarrow$ +ive CCW as seen from above

Applying MOM eqn.:–

$$\underbrace{(I\,dx)}_{\text{MI of element}} \underbrace{\frac{\partial^2 \theta(x,t)}{\partial t^2}}_{\text{angular acceleration, +ive CCW}} = \underbrace{\Gamma + \frac{\partial \Gamma}{\partial x} dx - \Gamma}_{\text{Net torque in CCW sense}}$$

$$\Rightarrow \quad I \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial \Gamma}{\partial x} \quad \text{—— (i)}$$

From elementary strength of materials studies, we know that $\frac{\Gamma}{J} = G \frac{\partial \theta}{\partial x}$, or, $\Gamma = GJ \frac{\partial \theta}{\partial x}$

( $\frac{\Gamma}{J} = \frac{G\phi}{L}$; Here $\phi = d\theta$, $L = dx$, $\frac{\phi}{L} = \frac{\partial \theta}{\partial x}$, partial derivative )

So, from (i), $I \frac{\partial^2 \theta(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(GJ \frac{\partial \theta}{\partial x}\right)$

Hence, for a uniform shaft ($J$ = Constant),

$\longrightarrow$

$$I \frac{\partial^2 \theta(x,t)}{\partial t^2} = GJ \frac{\partial^2 \theta}{\partial x^2} \rightarrow \text{the reqd. DEOM}$$

OR

Remember $\rightarrow$ $$\boxed{\frac{\partial^2 \theta(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \theta(x,t)}{\partial x^2}}\ ; \quad c = \sqrt{\frac{GJ}{I}} = \sqrt{\frac{G}{\rho}} = \text{speed of shear waves}$$

wave equation, once again!

$J = \frac{1}{2}\pi r^4$ ; $r$ = radius of shaft

$\underbrace{I\,dx = \frac{1}{2}(dm)r^2}_{\text{Using } I=\frac{1}{2}mr^2 \text{ for a circular cylinder}} = \frac{1}{2}(\rho\, dv)r^2$ ($\rho$ = density of shaft material)

$= \frac{1}{2}(\rho . \pi r^2 dx) r^2$

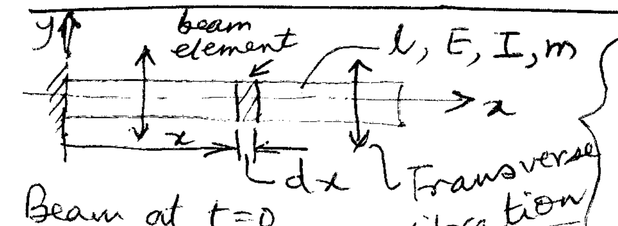
$\Rightarrow I = \left(\frac{1}{2}\pi r^4\right)\rho$. Thus, $\frac{J}{I} = \frac{1}{\rho}$

## ⑤ Vibration of an Euler-Bernoulli (E-B) beam

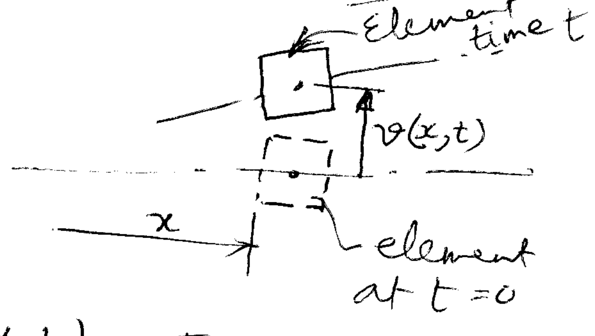
Note that by 'beam vibration', we usually mean 'transverse' or 'bending vibration', neglecting axial vibration.

$\rightarrow$ Also, we consider here free vibration only.

$\rightarrow$ To obtain the DEOM




Beam at t=0




$m$ = mass per unit length,
$l$ = length of beam,
$E$ = Young's modulus of beam material
$I$ = Area MI of a x-section about neutral axis of the x-section.
Note that the meaning of $I$ here is different from that of the previous topic. Unfortunately, most books follow this notation & so, we are using the same!

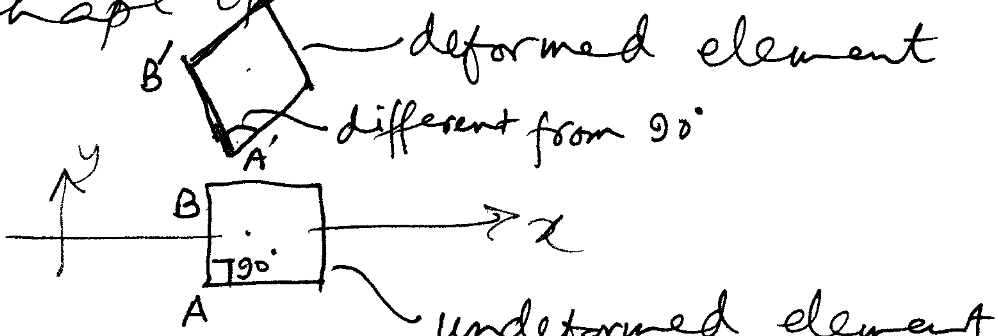
$v(x,t)$ = Transverse deflection of beam at $x$ (which is approx. same as tr. deflection at $(x+\frac{dx}{2})$ in fig. above) $\rightarrow$

Now note the following important assumptions:—

① Plane x-sections remain plane during vibration

② Deformations are linearly elastic & hence small. Beam slope is small too!

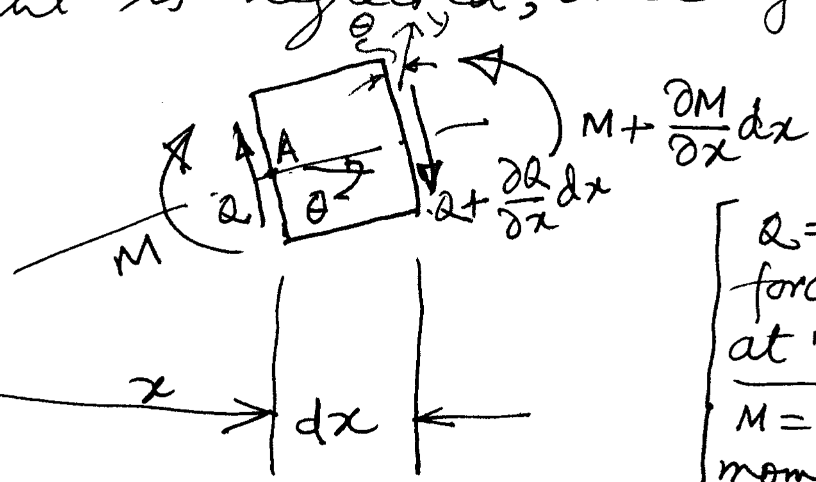
③ A beam element moves in transverse direction and it also 'turns' a bit. Also, there is 'shear deformation' of the shape of element:

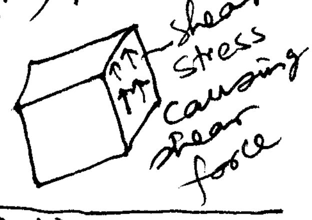



→ It has been experimentally seen that for long beams (Euler-Bernoulli type), the effects of rotational motion, or, 'rotary inertia' can be neglected. Similarly, the deformation in shape, called 'shear deformation' can also be neglected. These effects are prominent for short beams (called Rayleigh - Timoshenko beams) and these are neglected for our E-B beams.

→ Note also that a <u>net</u> force is required in the transverse direction to cause acceleration/deceleration of the element in the transverse direction. This is provided by <u>shearforces</u> on the left & right (edges / faces) of the →

element. Also, bending moments act on these faces. Hence, the FBD of the element is as follows (gravity force or weight is neglected, once again!):-




$Q = Q(x,t)$ = shear force on x-section at '$x$' at time $t$

$M = M(x,t)$ = bending moment on the same x-section

Here $\theta \simeq \tan\theta = \frac{\partial v}{\partial x}$ is the slope of the centre-line (℄) of the beam.

Using Newton's 2nd law in the $y$-direction, (since $\cos\theta \simeq 1$ & $\sin\theta \simeq \theta = \frac{\partial v}{\partial x}$) we get

$$\underbrace{(m\,dx)}_{\text{Element mass}}\underbrace{\frac{\partial^2 v(x,t)}{\partial t^2}}_{\text{element acceleration}} = Q\cos\theta - \left(Q+\frac{\partial Q}{\partial x}dx\right)\cos\theta \simeq Q - \left(Q+\frac{\partial Q}{\partial x}dx\right)$$
$$= -\frac{\partial Q}{\partial x}dx$$

Hence, $m\frac{\partial^2 v}{\partial t^2} = -\frac{\partial Q}{\partial x}$ —— ⓐ

Taking moments about A (see fig. above),

we get $\cancel{M} + \frac{\partial M}{\partial x}dx - \cancel{M} - \left(Q+\frac{\partial Q}{\partial x}dx\right)dx = 0$

$\Rightarrow \frac{\partial M}{\partial x} - Q = 0$ (neglecting $(dx)^2$, an infinitesimal of higher order) (Since angular acceleration $\simeq 0$ as per our assumption, net moment about any point $= 0$. A is chosen for convenience)

$\Rightarrow Q = \frac{\partial M}{\partial x}$ But $M = EI\frac{\partial^2 v}{\partial x^2}$, from elementary str. of materials.

So, ⓐ becomes:- $m\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = 0$ which is the reqd DEOM (continued in Part B)

END OF PART 'A'