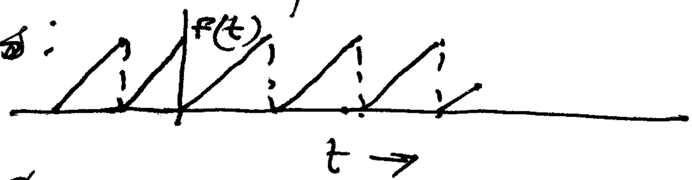
§ Periodic excitations:-

→ What happens to the forced response when we have a general periodic excitation such as:



?

Could we use our earlier experience with the special periodic excitation $F_0 \sin \omega_f t$? The answer is: Yes, we could. Here two things come into play, the Fourier series and the principle of superposition and I guess you know about both. Still, we shall recapitulate for the sake of those who might have forgotten!

① The fourier series:- for a general periodic function $f(t)$ of period τ, $f(t)$ can be represented by a fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_f t + b_n \sin n\omega_f t \right)$$

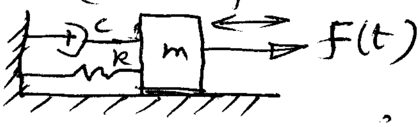
where $a_n = \frac{2}{\tau}\int_0^{\tau} f(t) \cos n\omega_f t \, dt$; $n = 0, 1, 2, \ldots$ ← note

& $b_n = \frac{2}{\tau}\int_0^{\tau} f(t) \sin(n\omega_f t)\, dt$; $n = 1, 2, 3, \ldots$

Of course, the function $f(t)$ has to satisfy a few conditions known as the Dirichlet conditions (check

from Engg. Maths books or the book 'Fourier Series & Integrals' by Carslaw.
Our engineering forcing period functions never fail to satisfy these conditions and hence we may proceed without further deliberations on this point.

You should note the relationship between τ & ω_f. Actually, $\tau = \frac{2\pi}{\omega_f}$ or, $\omega_f = \frac{2\pi}{\tau}$.

Hence, to obtain the forced response of the Kelvin-voigt model 

when $F(t+\tau) = F(t)$, i.e., $F(t)$ is periodic with period τ, we write

(i) the response due to the constant term $\frac{a_0}{2}$, which will be $\frac{a_0}{2k}$ only,

(ii) the response due to a general forcing term $a_n \cos n\omega_f t$ by comparison with the response due to $F_0 \cos\omega_f t$,

(iii) the response due to a general forcing term $b_n \sin n\omega_f t$, once again comparing with the response due to $F_0 \sin\omega_f t$ and then finally, we add up all these responses (valid due to the principle of superposition) to get the required forced response.

(2) The principle of superposition:–

Our ~~diff~~ DEOM $m\ddot{x}+c\dot{x}+kx=F(t)$ is a linear DEOM and hence, if it has the forced response $x_1(t)$ with forcing function $F_1(t)$ and response $x_2(t)$ when the forcing function is $F_2(t)$, then, when the forcing function $c_1F_1(t)+c_2F_2(t)$ acts, the response will simply be $c_1x_1+c_2x_2$. This can be shown pretty easily: (c_1, c_2 are constants)

We have,

$$\left.\begin{aligned} m\ddot{x}_1+c\dot{x}_1+kx_1&=F_1(t)\\ m\ddot{x}_2+c\dot{x}_2+kx_2&=F_2(t)\end{aligned}\right\}$$

So, $m(c_1\ddot{x}_1)+c(c_1\dot{x}_1)+k(c_1x_1)=c_1F_1(t)$

& $m(c_2\ddot{x}_2)+c(c_2\dot{x}_2)+k(c_2x_2)=c_2F_2(t)$

Adding these, we get

$$m(c_1\ddot{x}_1+c_2\ddot{x}_2)+c(c_1\dot{x}_1+c_2\dot{x}_2)+k(c_1x_1+c_2x_2)$$
$$=c_1F_1+c_2F_2$$

or, $m\ddot{x}_3+c\dot{x}_3+kx_3=F_3$

with $x_3=c_1x_1+c_2x_2$ as the particular integral (response, forced) due to forcing function $F_3=c_1F_1+c_2F_2$.

This is true even when $F(t)=\sum_{i=1}^{n}c_iF_i(t)$, n being any positive integer, the response would simply be

$x(t)=\sum_{i=1}^{n}c_i\,x_i(t)$. where $x_i(t)$ is the response due to $F_i(t)$ $(i=1,2,\to n)$.

→ Let us now sum things up:–

(i) When $F(t)=\frac{a_0}{2}$, the response is,

$$\frac{a_0}{2k} \text{ (by simple observation)}$$

(ii) When $F(t)=a_n\cos n\omega_f t$, the response is:

$$\left[\frac{a_n/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right]\cancel{\sin}\cos(n\omega_f t-\psi)$$

Compare with $\frac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t-\psi)$, $r=\frac{\omega_f}{\omega_n}$, $\psi=\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$

where $r=\frac{n\omega_f}{\omega_n}$ & hence, we represent it as $r_n=\frac{n\omega_f}{\omega_n}$ for convenience, since a simple r won't do. Similarly, $\psi=\psi_n=\tan^{-1}\left(\frac{2\zeta r_n}{1-r_n^2}\right)$

So, for $F(t)=a_n\cos n\omega_f t$, the response is:

$$\frac{a_n/k}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}\cancel{\sin}\cos(n\omega_f t-\psi_n)$$

(iii) When $F(t)=\sum_{n=1}^{\infty}a_n\cos n\omega_f t$, the forced response is:

$$\sum_{n=1}^{\infty}\frac{a_n/k}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}\cos(n\omega_f t-\psi_n),$$

by the principle of superposition.

(iv) When $F(t)=\sum_{n=1}^{\infty}b_n\sin n\omega_f t$, the forced response is, in a similar manner,

$$\sum_{n=1}^{\infty}\frac{b_n/k}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}\sin(n\omega_f t-\psi_n).$$

Hence, when the forcing function is

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_f t + \sum_{n=1}^{\infty} b_n \sin n\omega_f t,$$

the response is (~~again~~ applying principle of superposition again):

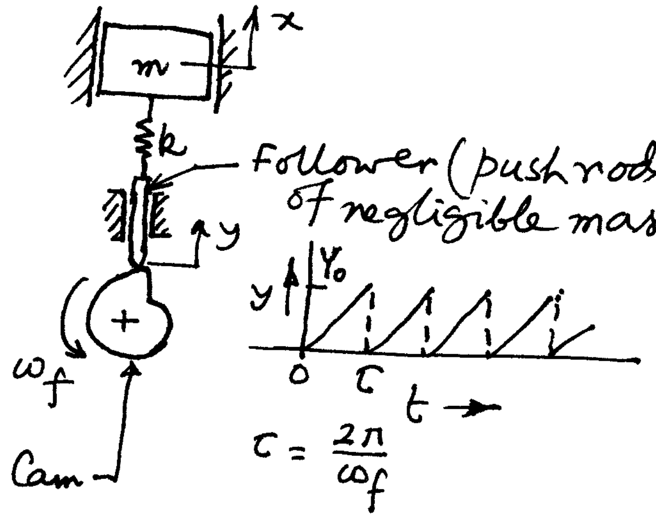
$$x(t) = \frac{a_0}{2K} + \sum_{n=1}^{\infty} \frac{a_n/k}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \cos(n\omega_f t - \psi_n)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n/k}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \sin(n\omega_f t - \psi_n)$$

or, $$x(t) = \frac{a_0}{2K} + \frac{1}{k}\sum_{n=1}^{\infty} \frac{1}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \Big[a_n \cos(n\omega_f t - \psi_n) + b_n \sin(n\omega_f t - \psi_n)\Big]$$

& you might remember this RHS expression if you feel like!

• An important example! [Also, see Ex. 1.12 (Cam follower mechanism) S.S. Rao, 6th Ed. Pg. 73]



← This example could be a simplified system corresponding to a mechanical press, say.

The DEOM is:

$$m\ddot{x} = -k(x-y)$$

or, $$m\ddot{x} + kx = ky \quad \text{--- (1)}$$

Relevant FBD:- (x>y) $\boxed{m}$ $\downarrow k(x-y)$

y is periodic with period $\tau = \frac{2\pi}{\omega_f}$. This is an example of a periodic base excitation. [See also ex. 1.12, (SS Rao 6th Ed, pg. 73) for another practical example of periodic excitation]

So, $y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_f t + b_n \sin n\omega_f t)$ --- (2)

$$a_n = \frac{2}{\tau}\int_0^{\tau} y(t) \cos n\omega_f t\, dt \; ; \quad n=0,1,2,3,\ldots$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} y(t) \sin n\omega_f t\, dt \; ; \quad n=1,2,3,\ldots$$

Note that $y(t) = \frac{Y_0}{\tau} t$ in $[0, \tau]$.

Hence, $a_n = \frac{2}{\tau}\int_0^{\tau} \frac{Y_0}{\tau} t \cos n\omega_f t\, dt$

Home work | & use integration by parts to get a_n.
Similarly, obtain b_n

Now, the RHS of (1) becomes:

$$\frac{k a_0}{2} + k \sum (a_n \cos n\omega_f t + b_n \sin n\omega_f t).$$

→ SS response (Particular integral) due to $\frac{k a_0}{2}$ is $\frac{a_0}{2}$

→ SS response due to $k a_n \cos n\omega_f t$ is

$$\frac{k a_n / k}{\sqrt{(1-r_n^2)^2 + \underset{0}{2\zeta r_n}}} \cos(n\omega_f t - \psi_n) \; , \; (\psi_n = 0)$$

$$= \frac{a_n}{(1-r_n^2)} \cos n\omega_f t \qquad (r_n \neq 1) \quad \left\{ r_n = \frac{n\omega_f}{\omega_n} \right\}$$

→ SS response due to $k b_n \sin n\omega_f t$ is

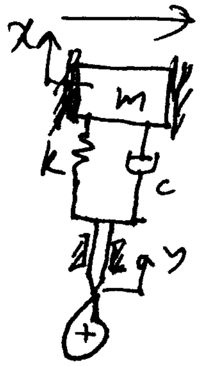
$\frac{b_n}{(1-r_n^2)} \sin n\omega_f t$.

Hence, the reqd. SS response is:

$$x(t) = \frac{a_0}{2} + \sum \frac{1}{(1-r_n^2)} [a_n \cos n\omega_f t + b_n \sin n\omega_f t]$$

etc.

<u>HW</u>:— Obtain $x(t)$ after substituting values of a_ns, b_ns in above expression.



→ Do a similar exercise when there is a damper parallel to the spring, y being the same.

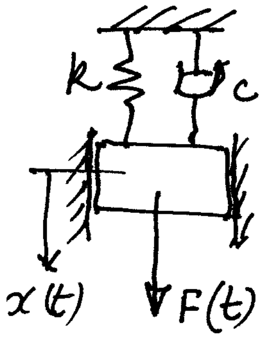
→ It is important to note that in practical computations, we take only first few a_n & b_n into account since the values of a_n & b_n for higher values of n become very small.

<u>Important</u>:— Do example problem 4.4 (SS Rao, 6th Ed., pg. 414).

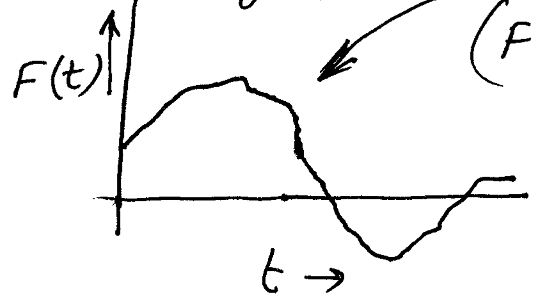
Also, study § 4.3 (pg. 418) to see what is to be done when an experimentally obtained periodic excitation cannot be put in an analytical form.

—x—

(8) Response to a general excitation



F(t) can now be any general forcing function such as:



(F(t) is applied at t=0 & was zero for t<0. This looks more practical)

We are interested in finding the steady-state (forced) response although it would be better to call it a transient response only because except for a few special F(t)s, there won't be any 'steady state'.

→ The basic idea is to imagine the forcing function to be made of impulses of short durations. Find the response due to an impulse of this type and then add-up (superpose) such responses for all the impulses over the interval [0, t] to get the response at time t.

Step 1:- To obtain the response due to a Dirac's delta function or the unit impulse function $\delta(t)$:-

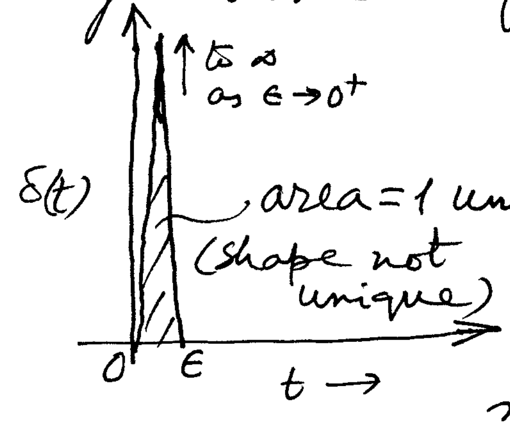
$\delta(t)$ is defined as follows:-

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\int_0^\infty \delta(t)\,dt = 1$$

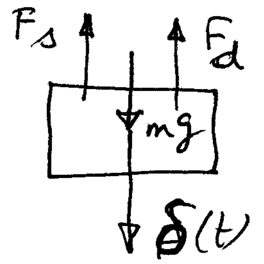
$$\int_0^\infty \delta(t) f(t)\,dt = f(0).$$

To visualize $\delta(t)$, you can't go strictly by above definition. Assume that



$\delta(t)$ occurs over a an interval of time $[0, \epsilon]$ where $\epsilon \to 0^+$, i.e. $\epsilon \to 0$ from above ($\epsilon > 0$) and as $\epsilon \to 0^+$, the maximum value of $\delta(t) \to \infty$ such that the area under ~~the~~ the $\delta(t)$ ~~vs~~ t plot $\to 1$ unit.

$\to$ Let us find the response of the system at time $t = \epsilon$ (as $\epsilon \to 0^+$).



$F_s \to$ spring force
$F_d \to$ damping force

Since ϵ is very small, the impulses $\int_0^\epsilon F_s\,dt$ & $\int_0^\epsilon F_d\,dt$ & $\int_0^\epsilon mg\,dt$ are negligible because mg, F_s & F_d are finite.
Impulse of $\delta(t)$ is $\int_0^\epsilon \delta(t)\,dt = 1$ as $\epsilon \to 0^+$ & is not negligible.

So, we now apply the impulse-momentum theorem to the block in the vertical direction (Let downward direction is positive) to get (change in momentum = force impulse)

$$mv(\epsilon) - mv(0) = 1 \text{ as } \epsilon \to 0, \text{ i.e., } m(v(0^+)) = 1$$
(since $v(0) = 0$

where $v(t)$ is the velocity of block.

Thus, $\boxed{v(0^+) = \frac{1}{m}}$ or, $\left(\dot{x}(0^+) = \frac{1}{m}\right)$

→ What is $x(0^+)$? (x = displacement of block)

Now, $x(\epsilon) = \int_0^\epsilon v(t)\,dt = \epsilon\, v(\epsilon_1) \to 0$ as $\epsilon \to 0$ by mean value theorem of integral calculus, where $0 < \epsilon_1 < \epsilon$

Hence, $\boxed{x(0^+) = 0}$

So, physically, when a unit impulse of very short duration is applied to the block at $t=0$ (say, by a proper blow of a hammer), all it does is causes a velocity $= \frac{1}{m}$ without any appreciable displacement.

What happens subsequently is just free-vibration with initial conditions $x(0^+) = 0$ & $\dot{x}(0^+) = \frac{1}{m}$.

→ This response for the case of an underdamped system has a special name. It is called the '<u>impulsive response function</u>' or '<u>impulse response function</u>' which is denoted by $g(t)$ in many text books. Let us obtain this $g(t)$.

→ For an underdamped system,

$$x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad \text{--- (I)}$$

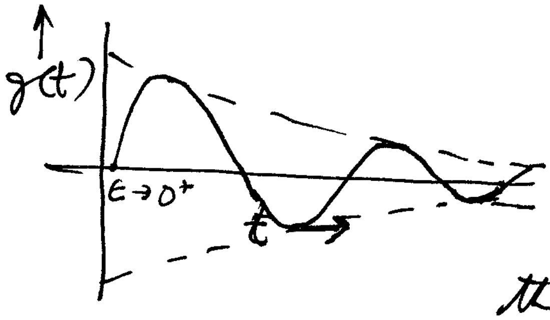
So, $x(0^+) = 0 \Rightarrow 0 = \sin\phi$ or, $\underline{\phi = 0}$ →

So, $x(t) = X_0 e^{-\zeta\omega_n t} \sin\omega_d t$

$\therefore \dot{x}(t) = -X_0 \zeta\omega_n e^{-\zeta\omega_n t} \sin\omega_d t + X_0 \omega_d e^{-\zeta\omega_n t} \cos\omega_d t$

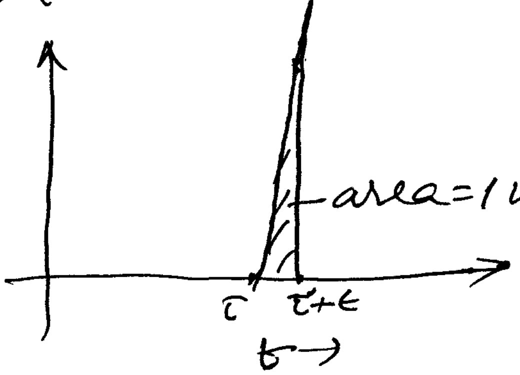
$\therefore \dot{x}(0^+) = \frac{1}{m} \Rightarrow \frac{1}{m} = X_0 \omega_d$ or, $\underline{\underline{X_0 = \frac{1}{m\omega_d}}}$

Hence, $\underline{\underline{g(t)}} = x(t) = \underline{\underline{\frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t}}$ (Remember)



Also note that in case the impulse has magnitude I, the response would be simply $g(t) \times I$, i.e., $I g(t)$ since our system is linear.

→ Let us now see what happens when the unit impulse function is applied at $t = \tau$ instead of at $t = 0$.



So, what we are applying to the system is the unit impulse function $\delta(t-\tau)$ defined as:

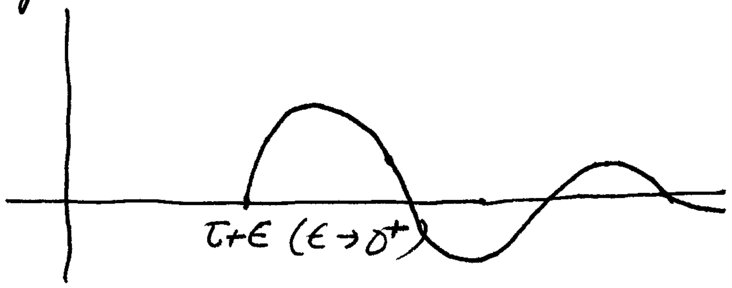
$$\delta(t-\tau) = 0 \text{ for } t \neq \tau$$

$$\int_0^\infty \delta(t-\tau)\,dt = 1$$

$$\int_0^\infty \delta(t-\tau) f(t)\,dt = f(\tau).$$

with $0 < \tau < \infty$.

Clearly, the system response now will simply be $g(t-\tau)$ which looks like:



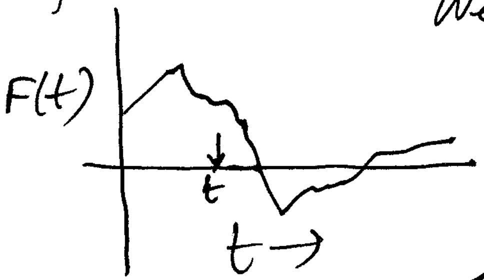
So, $g(t-\tau) = 0$ for $t < \tau$

$$= \frac{1}{m\omega_d} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)$$

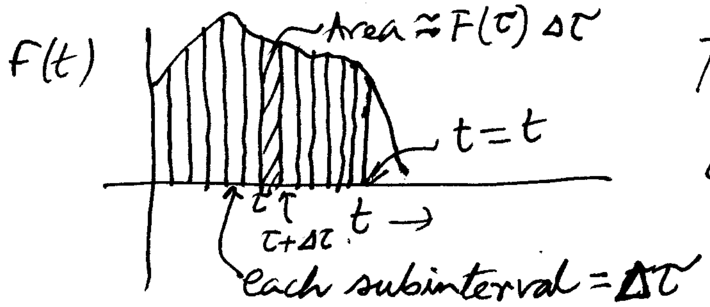
If the impulse has magnitude I, the response would be $I\,g(t-\tau)$.

→ After these preliminaries, let us go for the general forcing function $F(t)$:

We want $x(t)$, response at time t.



For response at time t, we need to take $F(t)$ over the interval $[0, t]$ only.



The effect of $F(t)$ over $[0, t]$ is the effect created by the impulses shown in the figure.

The impulse over $[\tau, \tau+\Delta\tau)$ gives the approximate response

$$\Delta x(t) \approx \underbrace{F(\tau)\Delta\tau}_{I}\, g(t-\tau)$$ as per our earlier discussion. So, total response $x(t)$ at time t will be

$$x(t) \simeq \overset{\Sigma \Delta x(t)}{\cancel{\ldots}} = \sum F(\tau)\, g(t-\tau)\, \Delta\tau$$

where the summation is carried over all the impulses shown in the figure. (We are using the principle of superposition, see?)

~~...~~ As $\Delta\tau \to d\tau$ & number of impulses $\to \infty$, the above summation gives way to integration & we get exact $x(t)$ as:

$$\underline{\underline{x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau}} \quad \text{(Remember)}$$

This is the famous <u>Duhamel's integral</u> or <u>Convolution integral</u> formula.

[It has an interesting geometrical interpretation. See Analytical Methods in Vibrations by L. Meirovitch]

→ Try to see what has been achieved.

If the above integral exists, then for any irregular $F(t)$, the integration can be performed numerically provided $F(t)$ is measured for a sufficiently large number of points on the t-axis in the interval $[0, t]$. Using proper instrumentation, this can be quite easily done & $x(t)$ can be obtained.

→ We shall now consider some simple

examples to illustrate the use of above formula. In passing, note that

$$\int_0^t F(\tau)\,g(t-\tau)\,d\tau = \int_0^t F(t-\tau)\,g(\tau)\,d\tau$$

which can be established by a change of variable of integration (say, substitute $z = t-\tau$ and use the fact that $\int_0^t F(t-z)\,g(z)\,dz$

$$= \int_0^t F(t-\tau)\,g(\tau)\,d\tau \text{ etc.})$$

So, you can use either

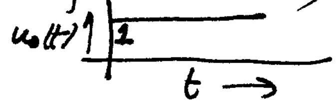
$$x(t) = \int_0^t F(\tau)\,g(t-\tau)\,d\tau$$

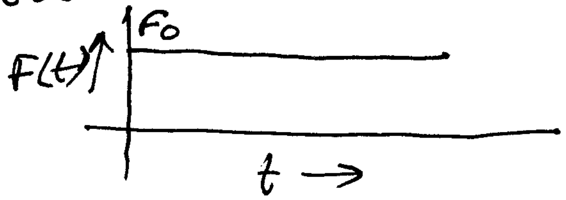
OR

$$x(t) = \int_0^t F(t-\tau)\,g(\tau)\,d\tau,$$

whichever is convenient.

Example 1:- For an undamped 1-DOF spring-mass system, obtain $x(t)$*
(* by Duhamel's method)
when $F(t) = F_0 u_0(t)$ ($u_0(t)$ → unit step function)
as shown:-





So, $m\ddot{x} + kx = F_0$ $(t>0)$ is the DEOM.

Since $\zeta = 0$, $\boxed{g(t) = \frac{1}{m\omega_n}\sin\omega_n t}$ $\longrightarrow$

∴ Required response is:

$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$$

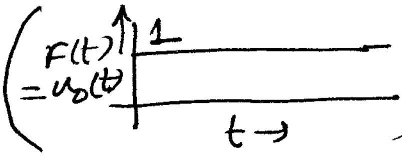
$$= \int_0^t F_0 \cdot \frac{1}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0}{m\omega_n} \times \frac{1}{\omega_n} \Big| \cos \omega_n (t-\tau) \Big|_0^t$$

$$= \frac{F_0}{m\omega_n^2} \left[1 - \cos \omega_n t \right]$$

$$= \frac{F_0}{k} \left[1 - \cos \omega_n t \right] \quad \underline{\underline{\text{Ans.}}}$$

[Note that in some books, $g(t)$ is denoted as $h(t)$ and the response to $u_0(t)$ (



) is denoted as $g(t)$.

We have used the notation found in most of the textbooks.

The response to $F(t) = u_0(t)$ has the special name 'the indicial response'.] It can be shown that

$$g(t) = \frac{dh(t)}{dt} + h(0)\,\delta(t)$$

(Meirovitch, Analytical ....)

(To be continued)

onto VA-3, part 6