FUNDAMENTALS OF MECHANICAL VIBRATIONS

SECOND EDITION

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S. Graham Kelly
The University of Akron

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To Seala

S. Graham Kelly

ABOUT THE AUTHOR

S. Graham Kelly received an M.S. in 1977 and a Ph.D. in 1979, both in engineering mechanics, from Virginia Polytechnic Institute and State University. He served on the faculty at the University of Notre Dame from 1979 to 1982. Since 1982, Dr. Kelly has served on the faculty at The University of Akron where he has been active in teaching, research, and administration.

Besides vibrations, Dr. Kelly has taught undergraduate courses in statics, dynamics, mechanics of solids, fluid mechanics, compressible fluid mechanics, numerical analysis, and freshman engineering. Dr. Kelly's graduate teaching includes courses in vibrations of discrete systems, vibrations of continuous systems, nonlinear vibrations, continuum mechanics, advanced mathematics, and hydrodynamic stability. In 1994 Dr. Kelly received the Chemstress award for the Outstanding Teacher in the College of Engineering.

Dr. Kelly has served The University of Akron as Associate Provost and Interim Dean of Engineering.

Dr. Kelly is also the author of *Schaum's Outline in Mechanical Vibrations* and *Schaum's Electronic Tutor in Mechanical Vibrations*.

PREFACE

Engineers apply mathematics and science to solve problems. In a traditional undergraduate engineering curriculum, a student begins an academic career by taking courses in mathematics and basic sciences such as chemistry and physics. A student begins to develop problem-solving skills in basic engineering science courses. For a mechanical engineering student, these courses include statics, dynamics, mechanics of solids, fluid mechanics, and thermodynamics. In such courses, students learn to apply basic laws of nature, constitutive equations, and equations of state to develop solutions to abstract engineering problems.

Vibrations is one of the first courses where students learn to apply the knowledge obtained from mathematics and basic engineering science courses to solve practical problems. Indeed the problem-solving skills developed in a vibrations course are as valuable as the knowledge of the subject of vibrations. Solution of practical problems in vibrations requires modeling of physical systems. A system is abstracted from its surroundings. Assumptions appropriate to the system are made. Basic engineering science and mathematics are applied to derive a mathematical model. The resulting solution is used to learn about system behavior that could be used in applications such as design. The reader of this text will learn about vibrations by using such a problem-solving approach.

An application of vibration analysis is in engineering design. Design principles are developed using analysis of model one-degree-of-freedom systems. Design applications are presented for multi-degree-of-freedom systems and continuous systems. Many examples and homework problems have a design flavor.

This book is intended as a text in a junior or senior level undergraduate course in vibrations. It could be used in a course populated by both undergraduate and beginning graduate students. The prerequisites for such a course should include courses in statics, dynamics, mechanics of materials, and mathematics through differential equations. Some material usually covered in a fluid mechanics course is used, but this material can be omitted without loss of continuity.

An overview of the modeling procedure is presented in Chap. 1 (This material can be omitted if students have background in system dynamics). Two methods of dynamic analysis are presented and used throughout the book. The free-body diagram method is based on D'Alembert's principle. An energy method that includes the effect of nonconservative forces is presented as an alternative and is preferred in modeling multi-degree-of-freedom systems.

Chapters 2 through 4 focus on vibrations of linear one-degree-of-freedom systems while Chaps. 5 through 7 focus on vibrations of multi-degree-of-freedom systems. Chapter 8 presents methods of reducing unwanted vibrations of discrete linear systems. Chapter 9 provides a brief overview of continuous systems. Chapter 10 introduces the reader to the finite element method, while Chap. 11 focuses on nonlinear vibrations.

White the structure of the second edition is similar to that of the first edition, there is much new in the second edition. The use of complex algebra in the analysis of the forced response of discrete systems has been added. Free and forced vibrations of multi-degree-of-freedom systems with a general damping matrix are now presented. Many new examples are presented. Approximately one-third of the end-of-chapter problems are new for this edition. An appendix containing answers to selected problems has been added.

Chapter 10, "Finite-Element Method," is new. Engineering students are usually exposed to the finite-element method during undergraduate studies, but rarely are exposed to its application to vibrations problems. This chapter is a welcome addition for those who have previously studied the finite-element method, although it is self-contained in that the method is developed by using the assumed modes method and Lagrange's equations.

Examples throughout the book use MATLAB for numerical computation, symbolic computation, and visualization of results. MATLAB script files are consistent with the *Student Edition of MATLAB*, Version 5. The accompanying CD, titled *VIBES II*, contains all script files presented in the text as well as other script files used for a variety of vibrations applications. Problems using MATLAB are presented at the end of each chapter. Many problems require use of the VIBES II files to solve vibrations problems while others require the development of a MATLAB script file. Users are encouraged to explore the files and develop their own applications. A list of files summarizing their applications is available by printing the text file LIST.TXT.

MATLAB and similar software are used as tools in vibration analysis. Indeed, they are powerful tools, easy to use for computation and visualization. While it is important to understand how the mathematics used in solving a problem is performed, complex computations often obscure the use of the results. The use of MATLAB allows the focus to be on the modeling, analysis, and design aspects of a problem, rather than computational considerations.

The author acknowledges the support and encouragement of Johnatan Plant, Senior Sponsoring Editor, during preparation of the second edition and the help of John Corrigan, formerly of McGraw-Hill, during preparation of the first edition. The help of former students Ken Kuhlmann, Mark Pixley, and Ashish Choski is greatly appreciated. Many valuable comments and suggestions were provided during preparation of the first and second editions by Donald Adams, University of Wyoming; Atila Ertas, Texas Tech University; Andrew Hansen, University of Wyoming; Eugene I. Rivin, Wayne State University; S. C. Sinha, Auburn University; Robert Steidel, University of California-Berkeley; J. Kim Vandiver, Massachusetts Institute of Technology; Dr. Aldo Ferri, Georgia Institute of Technology; Peter Philliou, Wentworth Institute of Technology; Richard Alexander, Texas A&M University; H. Nayeb-Hashemi, Northeastern University; Bala Balachandran, University of Maryland; William Webster, Kettering Institute; and Nester Sanchez, University of Texas at San Antonio. Finally the author expresses appreciation to his wife, Seala Fletcher-Kelly, and his son, Graham, for their patience and support.

S. Graham Kelly
July 30, 1999

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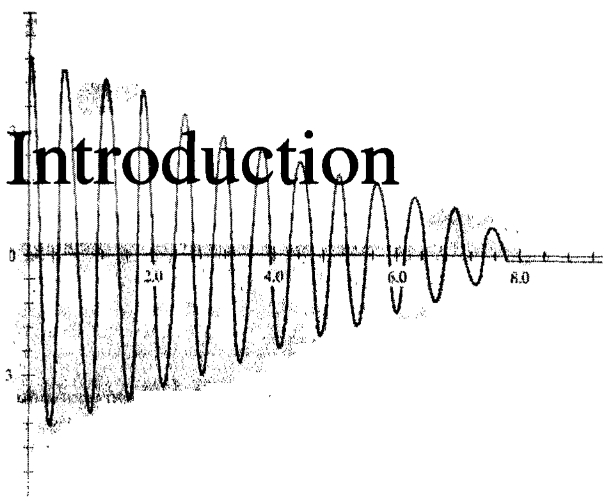
chapter 1

Introduction



1.1 THE STUDY OF VIBRATIONS

Vibrations are fluctuations of a mechanical or structural system about an equilibrium position. Vibrations are initiated when an inertia element is displaced from its equilibrium position due to an energy imparted to the system through an external source. A restoring force or moment pulls the element back toward equilibrium. When work is done on the block of Fig. 1.1*a* to displace it from its equilibrium position, potential energy is developed in the spring. When the block is released the spring force pulls the block toward equilibrium with the potential energy being converted to kinetic energy. In the absence of nonconservative forces, this transfer of energy is continual, causing the block to oscillate about its equilibrium position. When the pendulum of Fig. 1.1*b* is released from a position above its equilibrium position the moment of the gravity force pulls the particle, the pendulum bob, back toward equilibrium with potential energy being converted to kinetic energy. In the absence of nonconservative forces the pendulum will oscillate about the vertical equilibrium position.

We will develop mathematical models of the systems of Fig. 1.1. Solution of the mathematical problems leads to information about the vibrations. Using certain assumptions the oscillations of the systems of Fig. 1.1 are described as *simple harmonic motion*. The *time history* of the vibrations of a system undergoing simple harmonic motion is illustrated in Fig. 1.2. Simple harmonic motion is characterized by periodic oscillation about the equilibrium position. Each oscillation is one *cycle*. For simple harmonic motion the time it takes to execute one cycle, the *period*, is constant. The *frequency* of motion is the number of cycles executed in a fixed period of time, usually 1 second. The *amplitude*, the maximum displacement from equilibrium, is also constant in simple harmonic motion.

Vibrations occur in many mechanical and structural systems. If uncontrolled, vibration can lead to catastrophic situations. Vibrations of machine tools or machine tool chatter can lead to improper machining of parts. Structural failure can occur

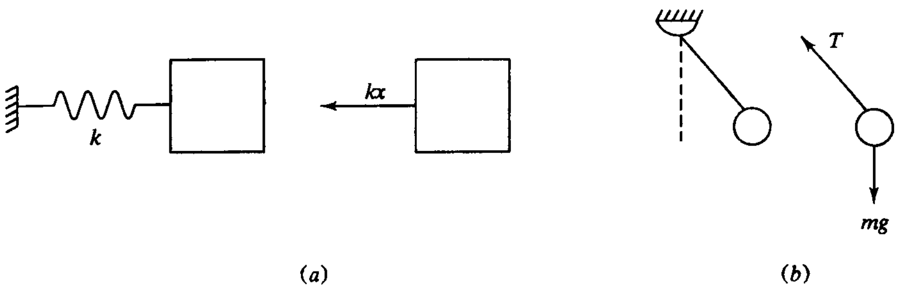


Figure 1.1 (*a*) When the block is displaced from equilibrium, the force developed in the spring as a result of stored potential energy pulls the block back toward its equilibrium position; (*b*) when the pendulum is rotated away from the vertical equilibrium position, the moment of the grovity force about the support pulls the pendulum back toward the equilibrium position.

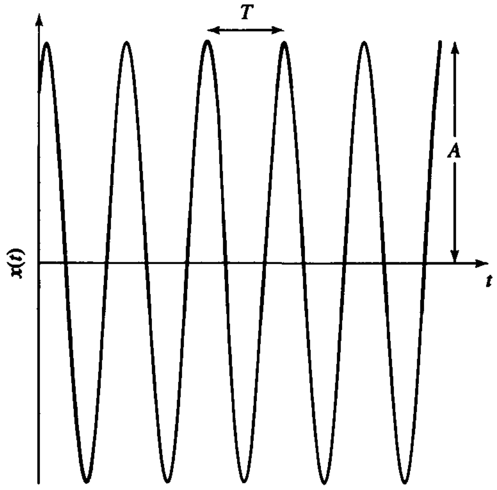


Figure 1.2 Simple hormonic motion with period T and amplitude A.

because of large dynamic stresses developed during earthquakes or even wind-induced vibration. Vibrations induced by an unbalanced helicopter blade while rotating at high speeds can lead to the blade's failure and catastrophe for the helicopter. Excessive vibrations of pumps, compressors, turbomachinery, and other industrial machines can induce vibrations of the surrounding structure, leading to inefficient operation of the machines while the noise produced can cause human discomfort.

Vibrations can be introduced, with beneficial effects, into systems in which they would not naturally occur. Vehicle suspension systems are designed to protect passengers from discomfort when traveling over rough terrain. Vibration isolators are used to protect structures from excessive forces developed in the operation of rotating machinery. Cushioning is used in packaging to protect fragile items from impulsive forces.

Our study of vibrations begins with the mathematical modeling of vibrating systems. Solutions to the resulting mathematical problems are obtained and analyzed. The solutions are used to answer basic questions about the vibrations of a system as well as to determine how unwanted vibrations can be reduced or how vibrations can be introduced into a system with beneficial effects. Mathematical modeling leads to the development of principles governing the behavior of vibrating systems.

1.2 MATHEMATICAL MODELING

Solution of an engineering problem often requires mathematical modeling of a physical system. The modeling procedure is the same for all engineering disciplines, although the details of the modeling vary between disciplines. The steps in the procedure are presented and the details are specialized for vibrations problems.

1.2.1 PROBLEM IDENTIFICATION

The system to be modeled is abstracted from its surroundings, and the effects of the surroundings are noted. The information to be obtained from the modeling is specified. Known constants and variable parameters are identified.

1.2.2 ASSUMPTIONS

Assumptions are made to simplify the modeling. If all effects are included in the modeling of a physical system, the resulting equations are usually so complex that a mathematical solution is impossible. When assumptions are used, an approximate physical system is modeled. An approximation should only be made if the solution to the resulting approximate problem is easier than the solution to the problem if the assumption were not made and that with the assumption the results of the modeling are accurate enough for the use they are intended.

Certain implicit assumptions are used in the modeling of most physical systems. These assumptions are taken for granted and rarely mentioned explicitly. Implicit assumptions used throughout this book include:

1. Physical properties are continuous functions of spatial variables. This *continuum assumption* implies that a system can be treated as a continuous piece of matter.
2. The earth is an inertial reference frame, thus allowing application of Newton's laws in a reference frame fixed to the earth.
3. Relativistic effects are ignored.
4. Gravity is the only external force field. The acceleration due to gravity is 9.81 m/s^2 (32.2 ft/s^2) on the surface of the earth.

5. The systems considered are not subject to nuclear reactions, chemical reactions, external heat transfer, or any other source of thermal energy.
6. All materials are linear, isotropic, and homogeneous.
7. The usual assumptions of mechanics of material apply (i.e., plane sections remain plane for beams in bending, circular sections under torsional loads do not warp).

Explicit assumptions are those specific to a particular problem. An explicit assumption is made to eliminate negligible effects from the analysis or to simplify the problem, while retaining appropriate accuracy. An explicit assumption should be verified, if possible, on completion of the modeling.

All physical systems are inherently nonlinear. Exact mathematical modeling of any physical system leads to nonlinear differential equations which often have no analytical solution. Since exact solutions of linear differential equations can usually be easily determined, assumptions are often made to *linearize* the problem. A linearizing assumption leads either to the removal of nonlinear terms in the governing equations or to the approximation of nonlinear terms by linear terms. The exact differential equation governing the oscillations of the pendulum of Fig. 1.1*b* contains several nonlinear terms. A *geometric nonlinearity* occurs as a result of the system's geometry. If the maximum angular displacement of the pendulum bob from its equilibrium position is small enough, the nonlinear term in the differential equation due to the geometric nonlinearity can be approximated by a linear term. As the pendulum oscillates it encounters friction in the form of aerodynamic drag. If included in the analysis, the drag force leads to a nonlinear term in the governing differential equation. In certain cases the drag may be neglected.

When analyzing the results of mathematical modeling, one has to keep in mind that the mathematical model is only an approximation to the true physical system. The actual system behavior may be somewhat different than that predicted using the mathematical model. When aerodynamic drag and all other forms of friction are neglected in a mathematical model of the pendulum of Fig. 1.1*b*, then perpetual motion is predicted for the situation when the pendulum is given an initial displacement and released from rest. Such perpetual motion is impossible. Even though neglecting aerodynamic drag leads to an incorrect time history of motion, the model is still useful in predicting the period, frequency, and amplitude of motion.

Once results have been obtained by using a mathematical model, the validity of all assumptions should be checked.

1.2.3 BASIC LAWS OF NATURE

A basic law of nature is a physical law that applies to all physical systems regardless of the material from which the system is constructed. These laws are observable, but cannot be derived from any more fundamental law. They are empirical. There exist only a few basic laws of nature: conservation of mass, conservation of momentum, conservation of energy, and the second and third laws of thermodynamics.

Conservation of momentum, both linear and angular, is usually the only physical law that is of significance in application to vibrating systems. Application of conservation of mass to vibrations problems is trivial. Applications of the second and third laws of thermodynamics do not yield any useful information. In the absence of thermal energy, the principle of conservation of energy reduces to the mechanical work-energy principle which is derived from Newton's laws.

1.2.4 CONSTITUTIVE EQUATIONS

Constitutive equations provide information about the materials of which a system is made. Different materials behave differently under different conditions. Steel and rubber behave differently because their constitutive equations have different forms. While the constitutive equations for steel and aluminum are of the same form, the constants involved in the equations are different. Constitutive equations are used to develop force-displacement relationships for mechanical components that are used in modeling vibrating systems.

1.2.5 GEOMETRIC CONSTRAINTS

Application of geometric constraints is often necessary to complete the mathematical modeling of an engineering system. Geometric constraints can be in the form of kinematic relationships between displacement, velocity, and acceleration. When application of basic laws of nature and constitutive equations lead to differential equations, the use of geometric constraints is often necessary to formulate the requisite boundary and initial conditions.

1.2.6 MATHEMATICAL SOLUTION

The mathematical modeling of a physical system results in the formulation of a mathematical problem. The modeling is not complete until the appropriate mathematics is applied and a solution obtained.

The type of mathematics required is different for different types of problems. Modeling of many statics, dynamics, and mechanics of solids problems leads only to algebraic equations. Mathematical modeling of vibrations problems leads to differential equations.

Exact analytical solutions, when they exist, are preferable to numerical or approximate solutions. Exact solutions are available for many linear problems, but for only a few nonlinear problems.

1.2.7 PHYSICAL INTERPRETATION OF RESULTS

After the mathematical solution is complete, the results are formulated. Physical interpretation of the results is an important final step in the modeling procedure. In certain situations this may involve drawing general conclusions from the mathematical solution, it may involve development of design curves, or it may require only simple arithmetic to arrive at a conclusion for the specific problem.

1.3 GENERALIZED COORDINATES

Mathematical modeling of a physical system requires the selection of a set of variables that describes the behavior of the system. *Dependent variables* are the variables that describe the physical behavior of the system. Examples of dependent variables are displacement of a particle in a dynamic system, the components of the velocity vector in a fluid flow problem, the temperature in a heat transfer problem, or the electric current in an AC circuit problem. *Independent variables* are the variables with which the dependent variables change. That is, the dependent variables are functions of the independent variables. An independent variable for most dynamic systems and electric circuit problems is time. The temperature distribution in a heat transfer problem may be a function of spatial position as well as time. The dependent variables in most vibrations problems are the displacements of specified particles from the system's equilibrium position while time is the independent variable.

The number of *degrees of freedom* for a system is the number of kinematically independent variables necessary to completely describe the motion of every particle in the system. Any set of n kinematically independent coordinates for a system with n degrees of freedom is called a set of *generalized coordinates*. The choice of generalized coordinates used to describe the motion of the system is not unique. The generalized coordinates are the dependent variables for a vibrations problem and are functions of the independent variable, time. If the time history of the generalized coordinates is known, the displacement, velocity, and acceleration of any particle in the system can be determined by using kinematics.

A single particle free to move in space has three degrees of freedom, and a suitable choice of generalized coordinates is the cartesian coordinates (x, y, z) of the particle with respect to a fixed reference frame. As the particle moves in space, its position is a function of time. An unrestrained rigid body has six degrees of freedom. A suitable choice for a system of generalized coordinates is the cartesian coordinates of the body's center of mass and the angular measure of an axis fixed to the body with respect to each of the cartesian coordinate axes fixed in space. The number of degrees of freedom is reduced if a particle or a rigid body is subject to constraints. A particle constrained to move in a plane has at most two degrees of freedom, while a rigid body undergoing planar motion has at most three degrees of freedom.

Each of the systems of Fig. 1.3 is in equilibrium in the position shown and undergoes planar motion. All bodies are rigid. Specify, for each system, the number of degrees of freedom and recommend a set of generalized coordinates.

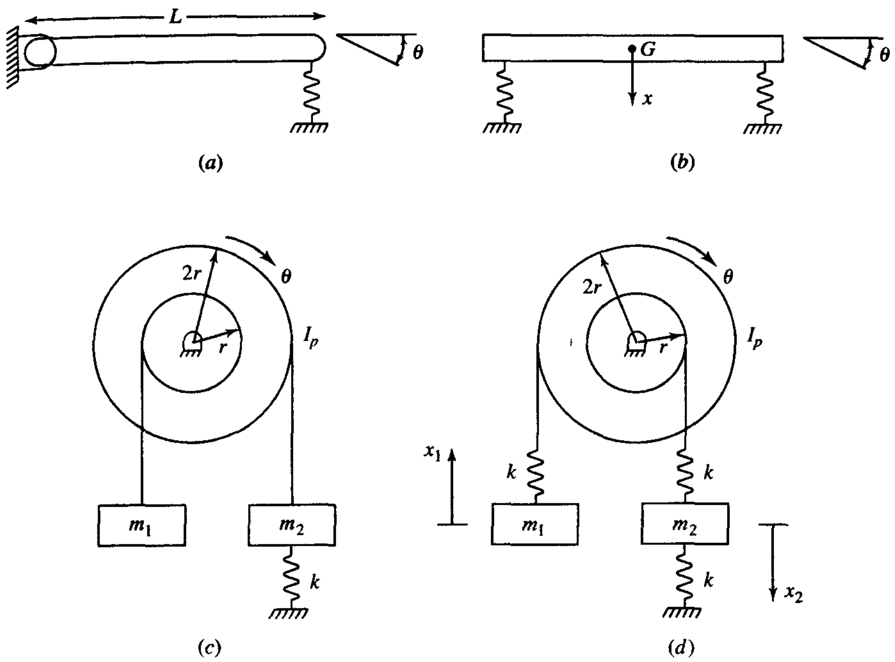


Figure 1.3 Systems for Example 1.1. One possible choice of a set of generalized coordinates is illustrated for each system.

Solution:

(a) The system has one degree of freedom. If θ, the clockwise angular displacement of the bar from the system's equilibrium position, is chosen as the generalized coordinate, then a particle initially a distance l from the fixed support has a horizontal position $l \cos \theta$ and a vertical displacement $l \sin \theta$.

(b) The system has two degrees of freedom, assuming it is constrained from side-to-side motion. If θ, the clockwise angular displacement of the bar measured from its equilibrium position, and x, the displacement of the bar's mass center measured from equilibrium, are chosen as generalized coordinates, then the displacement of a particle a distance d to the right of the mass center is $x + d \sin \theta$. Another choice for the generalized coordinates is x_1, the displacement of the right end of the bar, and x_2, the displacement of the left end of the bar, both measured from equilibrium.

(c) The system has one degree of freedom. If θ, the clockwise angular displacement of the pulley measured from the system's equilibrium position, is chosen as the generalized coordinate, then, assuming no slip between the pulley and the cables, the displacement of the block of mass m_1 is $r\theta$ upward and the displacement of the block of mass m_2 is $2r\theta$ downward.

(d) The system has three degrees of freedom. Since an elastic element is placed between the pulley and the blocks, no kinematic relationship exists between the displacement of either of the blocks and the angular rotation of the pulley. A suitable choice of generalized coordinates is θ, the clockwise angular rotation of the pulley, x_1, the upward displacement of the block of mass m_1, and x_2, the downward displacement of the block of mass m_2, all measured from the equilibrium position of the system.

The systems of Example 1.1 are assumed to be composed of rigid bodies. The relative displacement of two particles on a rigid body remains fixed as motion occurs. Particles in an elastic body may move relative to one another as motion occurs. Particles A and C lie along the neutral axis of the cantilever beam of Fig. 1.4 while particle B is in the cross section obtained by passing a perpendicular plane through the neutral axis at A. Because of the assumption that plane sections remain plane during displacement, the displacements of particles A and B are the same. However the displacement of particle C relative to particle A depends on the loading of the beam. Thus the displacements of A and C are kinematically independent. Since A and C represent arbitrary particles on the beam's neutral axis, it is inferred that there is no kinematic relationship between the displacements of any two particles along the neutral axis. Since there are an infinite number of particles along the neutral axis, the cantilever beam has an infinite number of degrees of freedom. In this case an independent spatial variable x, the distance along the neutral axis to a particle when the beam is in equilibrium, is defined. The dependent variable, displacement, is a function of the independent variables x and time.

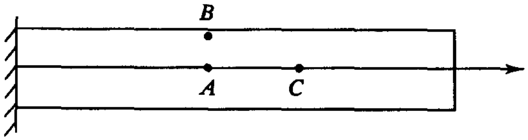


Figure 1.4 The displacements of particles A and B are related through the assumptions of elementary beam theory. However, no kinematic relationship exists between the displacements of particles A and C. Thus the cantilever beam is modeled as a continuous system.

1.4 REVIEW OF DYNAMICS

A brief review of rigid-body dynamics is presented to familiarize the reader with notation and methods. The reader is also encouraged to review the basic concepts of mechanics of solids and the solution of second-order ordinary differential equations.

1.4.1 KINEMATICS

The location of a particle on a rigid body at any instant of time can be referenced to a fixed cartesian reference frame, as shown in Fig. 1.5. The particle's position vector is given by

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{1.1}$$

from which the particle's velocity and acceleration are determined

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k} \tag{1.2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k} \tag{1.3}$$

where a dot above a quantity represents differentiation of that quantity with respect to time.

In general, a rigid body is rotating and translating. Assume that at an arbitrary instant of time the rigid body is rotating about an axis defined by a unit vector $\mathbf{e}$ with an angular speed ω. The angular velocity vector is

$$\boldsymbol{\omega} = \omega\mathbf{e} \tag{1.4}$$

from which the angular acceleration vector is calculated

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \tag{1.5}$$

Consider two particles, A and B, fixed to the same rigid body. Let $\mathbf{r}_{B/A}$ be the position vector of B relative to A. The velocity of B relative to A is

$$\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{1.6}$$

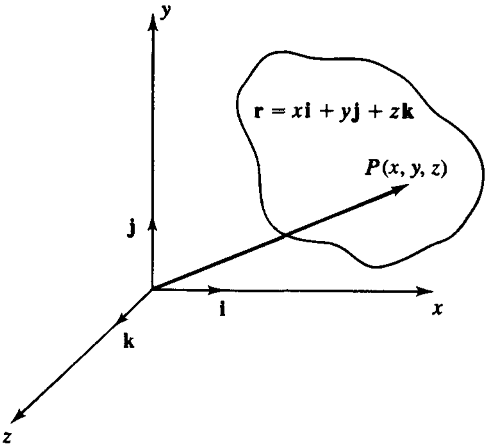


Figure 1.5 Position vector of a particle, P, on a rigid body, in a cartesian reference frame.

The acceleration of B relative to A is

$$\mathbf{a}_{B/A} = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \tag{1.7}$$

Consider a particle on a rigid body rotating about a fixed axis with an angular displacement θ, measured in a plane normal to the axis of rotation. Every particle on the rigid body travels on a circle centered on the axis of rotation. The velocity of a point of the rigid body, a distance r from the axis of rotation, is

$$\mathbf{v} = r\dot{\theta}\mathbf{i}_t \tag{1.8}$$

where $\mathbf{i}_t$ is a unit vector instantaneously tangent to the circle. The particle's acceleration is given by

$$\mathbf{a} = r\ddot{\theta}\mathbf{i}_t - r\dot{\theta}^2\mathbf{i}_n \tag{1.9}$$

where $\mathbf{i}_n$ is a unit vector instantaneously normal to the circle directed away from the axis of rotation, as shown in Fig. 1.6.

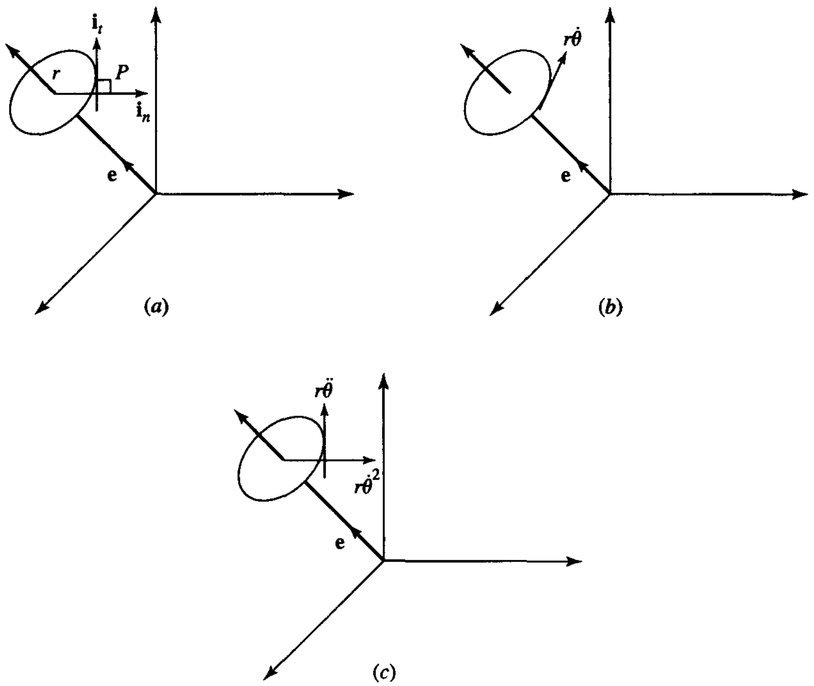


Figure 1.6 (a) The particle P is on a rigid body rotating about a constant axis defined by the unit vector, $\mathbf{e}$. P moves in a circle of radius r about the axis. $\mathbf{i}_n$ is instantaneously normal to the circle, directed away from the axis of rotation. $\mathbf{i}_t$ is instantaneously tangent to the circle, in the direction of rotation; (b) $\mathbf{v}_p = r\dot{\theta}\mathbf{i}_t$; (c) $\mathbf{a}_P = r\ddot{\theta}\mathbf{i}_t - r\dot{\theta}^2\,\mathbf{i}_n$.

1.4.2 BASIC PRINCIPLES OF RIGID-BODY KINETICS FOR PLANAR MOTION

A rigid body undergoes planar motion when its mass center moves on a plane and the body rotates about a fixed axis. The principles governing rigid-body kinetics of a body undergoing planar motion are obtained by applying the basic laws of particle kinetics to a system of particles and taking the limit as the number of particles in the system grows large. Applying Newton's second law for a particle and using the limiting process, it can be shown that for a rigid body in plane motion

$$\sum \mathbf{F} = m\bar{\mathbf{a}} \tag{1.10}$$

and

$$\sum \mathbf{M}_G = \bar{I}\alpha \tag{1.11}$$

where $\bar{I}$ is the moment of inertia of the body about an axis through its mass center and parallel to the axis of rotation. In general, a bar above a quantity refers to the quantity being evaluated for the body's mass center, G.

Recall that a system of forces and moments acting on a rigid body can be replaced by a force equal to the resultant of the force system applied at any point on the body and a moment equal to the resultant moment of the system about the point where the resultant force is applied. The resultant force and moment act equivalently to the original system of forces and moments. Thus Eqs. (1.10) and (1.11) imply that the system of external forces and moments acting on a rigid body is equivalent to a force equal to $m\bar{\mathbf{a}}$ applied at the body's mass center and a resultant moment equal to $\bar{I}\alpha$. This latter resultant system is called the system of effective forces. The equivalence of the external forces and the effective forces is illustrated in Fig. 1.7.

The previous discussion suggests the solution procedure for rigid-body kinetics problems that is used throughout this book. Two free-body diagrams are drawn for a rigid body. One free-body diagram shows all external forces and moments acting on the rigid body. The second free-body diagram shows the effective forces. If

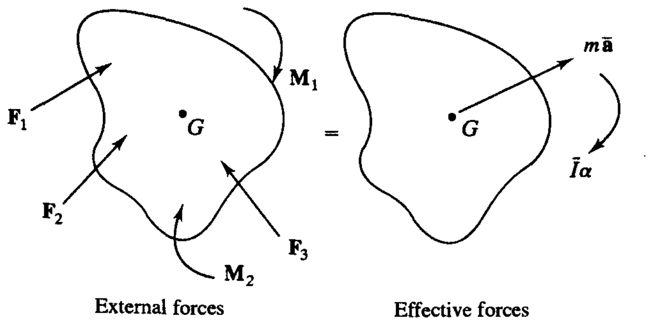


Figure 1.7 The system of external forces and moments acting on a rigid body in plane motion is equivalent to a force $m\bar{\mathbf{a}}$ applied at the body's mass center and a moment $\bar{\mathbf{I}}\alpha$.

the problem involves a system of rigid bodies, it may be possible to draw a single free-body diagram showing the external forces acting on the system of rigid bodies and one free-body diagram showing the effective forces of all of the rigid bodies. Equations (1.10) and (1.11) are equivalent to

$$\sum \mathbf{F}_{\text{ext}} = \sum \mathbf{F}_{\text{eff}} \qquad [1.12]$$

and

$$\sum \mathbf{M}_{O_{\text{ext}}} = \sum \mathbf{M}_{O_{\text{eff}}} \qquad [1.13]$$

taken about any point O on the rigid body.

---

2 The slender rod ($\bar{I} = \frac{1}{12}mL^2$) AC of Fig. 1.8 of mass m is pinned at B and held horizontally by a cable at C. Determine the angular acceleration of the bar immediately after the cable is cut.

Solution:

Immediately after the cable is cut, the angular velocity of the bar is zero. Equation (1.9) is used to determine the acceleration of the mass center in terms of the bar's angular acceleration, α, $\bar{a} = (L/4)\alpha$.

Summing moments about B, using the free-body diagrams of Fig. 1.8b, gives

$$\sum \overset{\curvearrowright +}{M}_{B_{\text{ext}}} = \sum \overset{\curvearrowright +}{M}_{B_{\text{eff}}}$$

$$mg\frac{L}{4} = \left(m\frac{L}{4}\alpha\right)\left(\frac{L}{4}\right) + \frac{1}{12}mL^2\alpha$$

$$\alpha = \frac{12g}{7L}$$

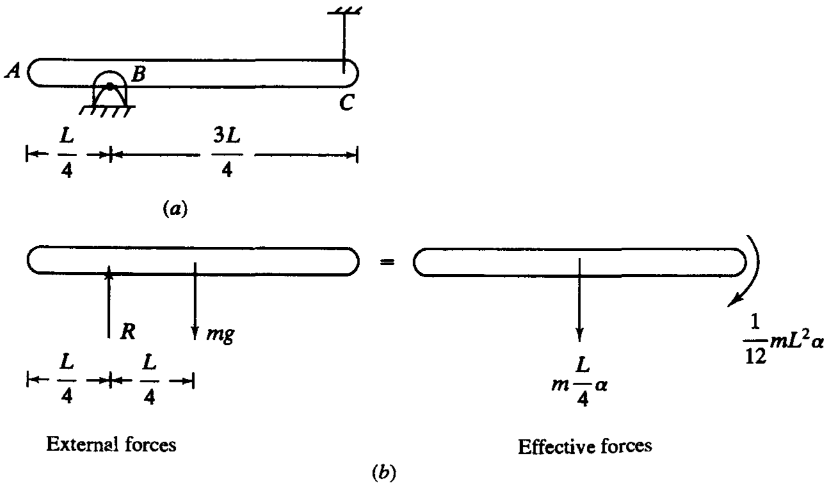


Figure 1.8 (a) Slender rod of Example 1.2 is pinned at B and held by cable at C; (b) free-body diagrams immediately after cable is cut.

Example

Determine the angular acceleration of the pulley of Fig. 1.9.

Solution:

Consider the system of rigid bodies composed of the pulley and the two blocks. If α is the counterclockwise angular acceleration of the pulley, then, assuming no slip between the pulley and the cables, block A has a downward acceleration of $r_A\alpha$ and block B has an upward acceleration of $r_B\alpha$.

Summing moments about the center of the pulley, neglecting axle friction in the pulley, and using the free-body diagrams of Fig. 1.9b yields

$$\sum \overset{\curvearrowleft +}{M}_{O_{\text{ext}}} = \sum \overset{\curvearrowleft +}{M}_{O_{\text{eff}}}$$

$$m_A g r_A - m_B g r_B = I_P\alpha + m_A r_A^2\alpha + m_B r_B^2\alpha$$

Substituting given values leads to $\alpha = 7.55\ \text{rad/s}^2$.

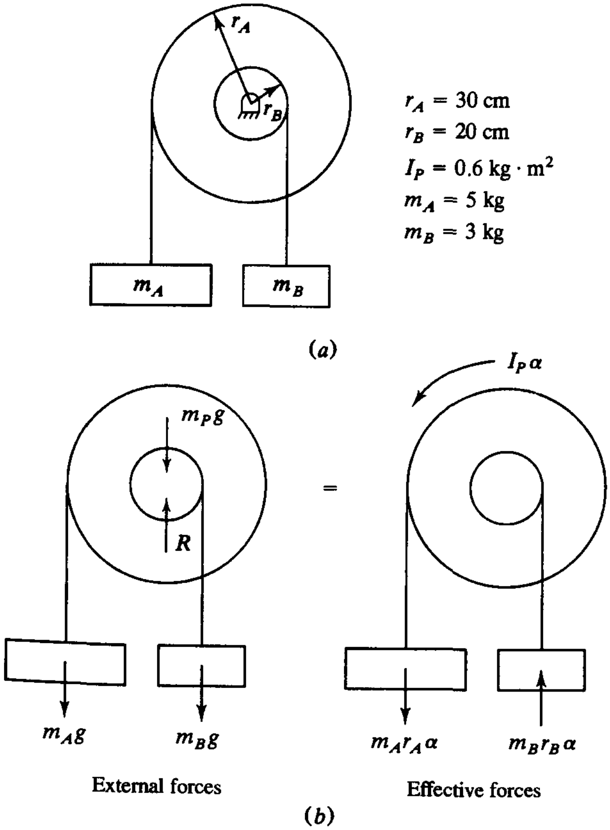


Figure 1.9 (a) System of Example 1.3; (b) free-body diagrams of pulley and blocks at an arbitrary time.

1.4.3 PRINCIPLE OF WORK-ENERGY

The kinetic energy of a rigid body is

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 \tag{1.14}$$

The work done by a force, F, acting on a rigid body as the point of application of the force travels between two points described by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ is

$$U_{A\to B} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F}\cdot d\mathbf{r} \tag{1.15}$$

where $d\mathbf{r}$ is a differential position vector in the direction of motion. The work done by a moment acting on a rigid body in planar motion is

$$U_{A\to B} = \int_{\theta_A}^{\theta_B} M\,d\theta \tag{1.16}$$

If the work of a force is independent of the path taken from A to B, the force is called *conservative*. Examples of conservative forces are spring forces, gravity forces, and normal forces. A potential energy function, $V(\mathbf{r})$, can be defined for conservative forces. The work done by a conservative force can be expressed as a difference in potential energies

$$U_{A\to B} = V_A - V_B \tag{1.17}$$

Since the system of external forces is equivalent to the system of effective forces, the total work done on a rigid body in planar motion is

$$U_{A\to B} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} m\bar{\mathbf{a}}\cdot d\mathbf{r} + \int_{\theta_A}^{\theta_B} \bar{I}\alpha\,d\theta \tag{1.18}$$

When integrated, the right-hand side of Eq. (1.18) is equal to the difference in the kinetic energy of the rigid body between A and B. Thus Eq. (1.18) yields the principle of work-energy,

$$T_B - T_A = U_{A\to B} \tag{1.19}$$

If all forces are conservative, Eq. (1.17) is used in Eq. (1.19) and the result is the principle of conservation of energy

$$T_A + V_A = T_B + V_B \tag{1.20}$$

Express the kinetic energy of each of the systems of Fig. 1.3 in terms of the specified generalized coordinates. The slender bars of Fig. 1.3a and b are uniform and of mass m and length L.

Solution:

(a) From Eq. (1.6) the velocity of the mass center is expressed as

$$\bar{v} = \frac{L}{2}\dot{\theta}$$

The kinetic energy is calculated from Eq. (1.14) as

$$T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\dot{\theta}^2\right) = \frac{1}{6}mL^2\dot{\theta}^2$$

(*b*) The kinetic energy of the two-degree-of-freedom system is expressed as

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}\left(\tfrac{1}{12}mL^2\dot{\theta}^2\right)$$

(*c*) The kinetic energy of the system is the sum of the kinetic energies of the three bodies:

$$T = \tfrac{1}{2}I_p\dot{\theta}^2 + \tfrac{1}{2}m_1(r\dot{\theta})^2 + \tfrac{1}{2}m_2(2r\dot{\theta})^2 = \tfrac{1}{2}(I_p + m_1r^2 + 4m_2r^2)\dot{\theta}^2$$

(*d*) The kinetic energy of the three-degree-of-freedom system is

$$T = \tfrac{1}{2}I_p\dot{\theta}^2 + \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2$$

1.4.4 PRINCIPLE OF IMPULSE AND MOMENTUM

The impulse of the force $\mathbf{F}$ between t_1 and t_2 is defined as

$$I_{1\to2} = \int_{t_1}^{t_2} \mathbf{F}\,dt \qquad [1.21]$$

The total angular impulse of a system of forces and moments about a point O is

$$J_{O_{1\to2}} = \int_{t_1}^{t_2} \sum \mathbf{M}_O\,dt \qquad [1.22]$$

The system momenta at a given time are defined by the system's linear momentum

$$\mathbf{L} = m\bar{\mathbf{v}} \qquad [1.23]$$

and its angular momentum about its mass center

$$\mathbf{H}_G = \bar{I}\boldsymbol{\omega} \qquad [1.24]$$

Integrating Eqs. (1.10) and (1.11) between arbitrary times t_1 and t_2 leads to

$$\mathbf{L}_1 + \mathbf{I}_{1\to2} = \mathbf{L}_2 \qquad [1.25]$$

and

$$\mathbf{H}_{G_1} + \mathbf{J}_{G_{1\to2}} = \mathbf{H}_{G_2} \qquad [1.26]$$

Using an equivalent force system argument similar to that used to obtain Eqs. (1.12) and (1.13), it is deduced from Eqs. (1.25) and (1.26) that the system of applied impulses is equivalent to the difference between the system momenta at t_1 and the system momenta at t_2. This form of the principle of impulse and momentum, convenient for problem solution, is illustrated in Fig. 1.10.

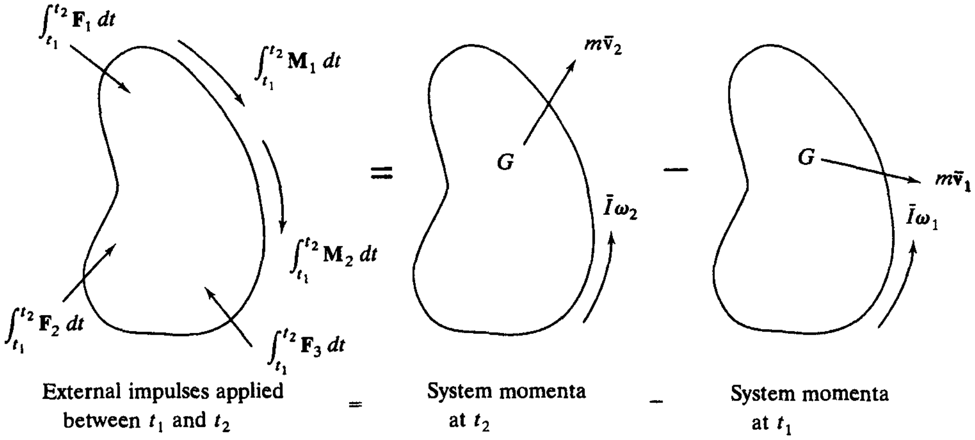


Figure 1.10 Illustration of the principle of impulse and momentum.

.5 The slender rod of mass m of Fig. 1.11 is swinging through a vertical position with an angular velocity ω_1 when it is struck at A by a particle of mass $m/4$ moving with a velocity v_p. Upon impact the particle sticks to the bar. Determine (a) the angular velocity of the bar and particle immediately after impact, (b) the maximum angle through which the bar and particle will swing after impact, and (c) the angular acceleration of the bar and particle when they reach the maximum angle.

Solution:

(a) Let t_1 occur immediately before impact and t_2 occur immediately after impact. Consider the bar and the particle as a system. During the time of impact, the only external impulses are due to gravity and the reactions at the pin support. The principle of impulse and momentum is used in the following form:

$$\begin{pmatrix}\text{External angular}\\ \text{impulses about } O\\ \text{between } t_1 \text{ and } t_2\end{pmatrix} = \begin{pmatrix}\text{Angular momentum}\\ \text{about } O\\ \text{at } t_2\end{pmatrix} - \begin{pmatrix}\text{Angular momentum}\\ \text{about } O\\ \text{at } t_1\end{pmatrix}$$

Using the momentum diagrams of Fig. 1.11b, this becomes

$$0 = \left(m\frac{L}{2}\omega_2\right)\left(\frac{L}{2}\right) + \left(\frac{m}{4}a\omega_2\right)(a) + \frac{1}{12}mL^2\omega_2$$
$$-\left[\left(m\frac{L}{2}\omega_1\right)\left(\frac{L}{2}\right) - \left(\frac{m}{4}v_p\right)(a) + \frac{1}{12}mL^2\omega_1\right]$$

which is solved to yield

$$\omega_2 = \frac{4L^2\omega_1 - 3v_p a}{4L^2 + 3a^2}$$

(b) Let t_3 be the time when the bar and particle assembly attains its maximum angle. Gravity forces are the only external forces that do work; hence conservation of energy

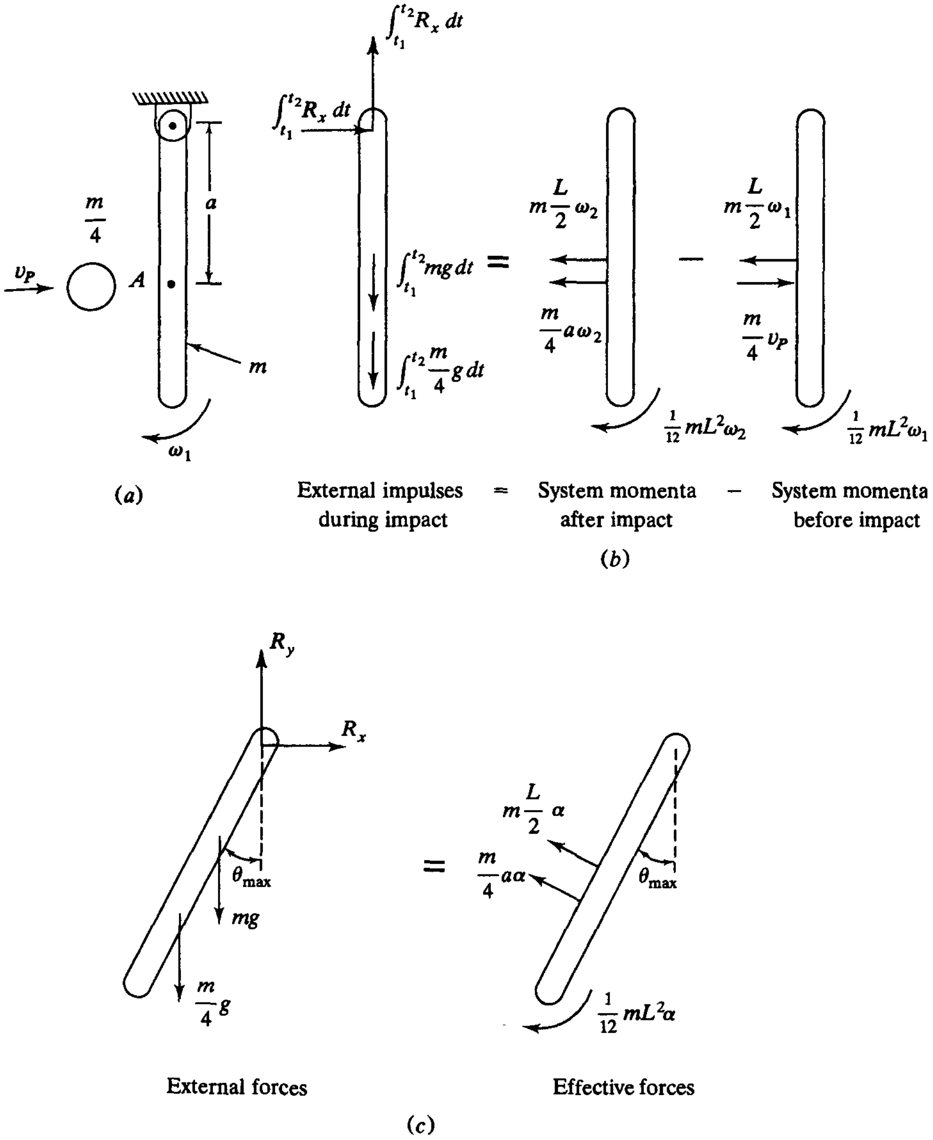


Figure 1.11 (a) Slender rod of Example 1.5 is swinging through vertical with angular velocity ω_1 when struck at A by particle moving with horizontal velocity v_P; (b) impulse and momentum diagrams between the time immediately before impact and the time immediately after impact; (c) free-body diagrams of bar as it swings through maximum angle.

applies between t_2 and t_3. Thus from Eq. (1.20)

$$T_2 + V_2 = T_3 + V_3$$

The potential energy of a gravity force is the magnitude of the force times the distance its point of application is above a horizontal datum plane. Choosing the datum as the horizontal plane through the support, using Eq. (1.14) for the kinetic energy of a rigid body, and noting $T_3 = 0$ yields

$$\frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\omega_2^2 + \frac{1}{2}\frac{m}{4}(a\omega_2)^2 - mg\frac{L}{2} - \frac{mg}{4}a$$
$$= -mg\frac{L}{2}\cos\theta_{max} - \frac{m}{4}ga\cos\theta_{max}$$

which is solved to yield

$$\theta_{max} = \cos^{-1}\left[1 - \frac{(4L^2 + 3a^2)\omega_2^2}{g(12L + 6a)}\right]$$

(*c*) The bar attains its maximum angle at t_3, $\omega_3 = 0$. Summing moments about O using the free-body diagrams of Fig. 1.11*c* gives

$$\sum \overset{\curvearrowright}{\underset{+}{}}M_{O_{ext}} = \sum \overset{\curvearrowright}{\underset{+}{}}M_{O_{eff}}$$

$$-(mg)\left(\frac{L}{2}\sin\theta_{max}\right) - \left(\frac{mg}{4}\right)(a\sin\theta_{max})$$
$$= \left(m\frac{L}{2}\alpha\right)\left(\frac{L}{2}\right) + \left(\frac{m}{4}a\alpha\right)(a) + \frac{1}{12}mL^2\alpha$$

which is solved to yield

$$\alpha = -\frac{(6L + 3a)g\sin\theta_{max}}{4L^2 + 3a^2}$$

1.5 CLASSIFICATION OF VIBRATION

The method of analysis used to solve the mathematical problem resulting from a mathematical model of a vibrating system depends on a number of factors. A system with a finite number of degrees of freedom is a *discrete system*. The vibrations of a *one-degree-of-freedom system* are governed by an ordinary differential equation in which time is the independent variable and the chosen generalized coordinate is the dependent variable. The vibrations of a *multi-degree-of-freedom system* (MDOF system) are governed by a system of n differential equations, where n is the number of degrees of freedom. The dependent variables are the chosen generalized coordinates while time is the independent variable. The differential equations for an MDOF system are, in general, coupled.

A system with an infinite number of degrees of freedom is called a *continuous system* or *distributed parameter system*. The vibrations of a continuous system are governed by partial differential equations. The displacement of a particle is a continuous function of time and the particle's location when the system is in equilibrium. Spatial coordinates are used to describe the distribution of inertia when the system is in equilibrium. All systems are, in reality, continuous systems. Particle and rigid body assumptions are often made to approximate a continuous system by a discrete system. A vibrations problem may be formulated for a continuous system but a discrete approximation method like the finite-element method is used to solve the problem.

A system is undergoing *free vibrations* when the vibrations occur in the absence of an external excitation. The vibrations are initiated by developing an initial kinetic energy or potential energy in the system. In the absence of nonconservative forces, free vibrations sustain themselves and are periodic. Vibrations which occur in the presence of an external excitation are called *forced vibrations*. If the excitation force is periodic the excitation is said to be *harmonic*. Forced nonperiodic vibrations are called *transient vibrations*.

A system is *linear* if its motion is governed by linear differential equations. A system is *nonlinear* if its motion is governed by nonlinear differential equations. Under certain conditions the vibrations of a nonlinear system subject to a periodic excitation may not be periodic. Such systems are said to be *chaotic*.

If the excitation force is known at all instants of time, the excitation is said to be *deterministic*. If the excitation force is unknown, but averages and standard deviations are known, the excitation is said to be *random*. In this case the resulting vibrations are also random, and cannot be determined exactly at any instant of time.

1.6 SPRINGS

1.6.1 INTRODUCTION

A *spring* is a flexible mechanical link between two particles in a mechanical system. In reality a spring itself is a continuous system. However, the inertia of the spring is usually small compared to other elements in the mechanical system and is neglected. Under this assumption the force applied to each end of the spring is the same.

The length of a spring when it is not subject to external forces is called its *unstretched length*. Since the spring is made of a flexible material, the applied force F that must be applied to the spring to change its length by x is some continuous function of x,

$$F = f(x) \tag{1.27}$$

The appropriate form of $f(x)$ is determined by using the constitutive equation for the spring's material. Since $f(x)$ is infinitely differentiable at $x = 0$, it can be expanded by a Taylor series about $x = 0$ (a MacLaurin expansion):

$$F = k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \cdots \tag{1.28}$$

Since x is the spring's change in length from its unstretched length, when $x = 0$, $F = 0$. Thus $k_0 = 0$. When x is positive, the spring is in tension. When x is negative, the spring is in compression. Many materials have the same properties in tension and compression. That is, if a tensile force F is required to lengthen the spring by δ, then a compressive force of the same magnitude F is required to shorten the spring by δ. For these materials, $f(x)$ must be an odd function and cannot contain even powers. Thus Eq. (1.28) becomes

$$F = k_1 x + k_3 x^3 + k_5 x^5 + \cdots \tag{1.29}$$

All springs are inherently nonlinear. However in many situations x is small enough that the nonlinear terms of Eq. (1.29) are small compared with $k_1 x$. A *linear spring* obeys a force-displacement law of

$$F = kx \tag{1.30}$$

where k is called the *spring stiffness* or *spring constant* and has dimensions of force per length.

The force in a spring whose force-displacement law is given by Eq. (1.29) is conservative; the work done by the spring force between two arbitrary displacements is independent of the path taken between the two displacements. Thus a potential energy function exists and for a linear spring is determined as

$$V(x) = \tfrac{1}{2}kx^2 \tag{1.31}$$

A *torsional spring* is a link in a mechanical system where application of a torque leads to an angular displacement between the ends of the torsional spring. A linear torsional spring has a relationship between an applied moment M and the angular displacement θ of

$$M = k_t\theta \tag{1.32}$$

where the *torsional stiffness* k_t has dimensions of force times length. The potential energy function for a torsional spring is

$$V = \tfrac{1}{2}k_t\theta^2 \tag{1.33}$$

1.6 The numerical values for the parameters in the system of Fig. 1.3c are $m_1 = 20$ kg, $m_2 = 10$ kg, $I_p = 0.4$ kg · m^2, $r = 10$ cm, $k = 1300$ N/m. (*a*) Determine the potential energy in the spring when the system is in equilibrium. (*b*) Develop an expression for the potential energy in the spring when the pulley is rotated θ clockwise from the equilibrium position.

Solution:

(*a*) Let Δ be the *static deflection* of the spring, its change in length from its unstretched length when the system is in equilibrium. The static deflection is determined by applying the equations of equilibrium to the free-body diagram of Fig. 1.12.

$$\sum \overset{\curvearrowleft +}{M_O} = 0$$

$$m_1 g(r) + k\Delta(2r) - m_2 g(2r) = 0$$

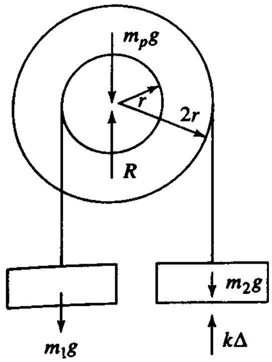


Figure 1.12 Free-body diagram of static equilibrium position of system of Fig. 1.3c and Example 1.6.

$$\Delta = \frac{(2m_2 - m_1)g}{2k} = \frac{[2(10\text{ kg}) - 25\text{ kg}]\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}{2\left(1300\ \frac{\text{N}}{\text{m}}\right)}$$
$$= -0.0188\text{ m} = -18.8\text{ mm}$$

The free-body diagram shows that it was assumed that the spring is compressed when the system is in equilibrium. The negative sign indicates that the spring is actually stretched. Its potential energy when the system is in equilibrium is

$$V = \frac{1}{2}k\Delta^2 = \frac{1}{2}\left(1300\ \frac{\text{N}}{\text{m}}\right)(-0.0188\text{ m})^2 = 0.230\text{ N}\cdot\text{m}$$

(*b*) When the pulley is rotated through a clockwise angle θ from the system's equilibrium position, the spring is shortened in length $2r\theta$ from its length when the system is in equilibrium. Thus the total change in length of the spring is

$$x = \Delta + 2r\theta$$

and the potential energy in the spring is

$$V = \tfrac{1}{2}k(\Delta + 2r\theta)^2$$
$$= \frac{1}{2}\left(1300\ \frac{\text{N}}{\text{m}}\right)[-0.0188\text{ m} + 2(0.1\text{ m})\theta]^2$$
$$= 0.230 - 4.89\theta + 26\theta^2\text{ N}\cdot\text{m}$$

1.6.2 Helical Coil Springs

The helical coil spring is used in applications such as industrial machines and vehicle suspension systems. Consider a spring manufactured from a rod of circular cross section of diameter D. The shear modulus of the rod is G. The rod is formed into a coil of N turns of radius r. It is assumed that the coil radius is much larger than the

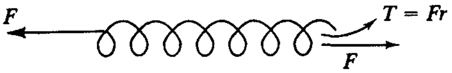


Figure 1.13 Free-body diagram of cut coil spring exposes resultant shear force and resultant torque.

radius of the rod and that the normal to the plane of one coil nearly coincides with the axis of the spring.

Consider a helical coil spring when subject to an axial load F. Imagine cutting the rod with a knife at an arbitrary location in a coil, slicing the spring in two sections. The cut exposes an internal shear force F and an internal resisting torque Fr, as illustrated in Fig. 1.13. Assuming elastic behavior, the shear stress due to the resisting torque varies linearly with distance from the center of the rod to a maximum of

$$\tau_{\max} = \frac{FrD}{2J} = \frac{16Fr}{\pi D^3} \qquad [1.34]$$

where $J = (\pi D^4)/32$ is the polar moment of inertia of the rod. The shear stress due to the shear force varies nonlinearly with distance from the neutral axis. For $r/D \gg 1$ the maximum shear stress due to the internal shear force is much less than the maximum shear stress due to the resisting torque, and its effect is neglected.

Principles of mechanics of materials can be used to show that the total change in length of the spring due to an applied force F is

$$x = \frac{64Fr^3N}{GD^4} \qquad [1.35]$$

Comparing Eq. (1.35) with Eq. (1.30) leads to the conclusion that under the assumptions stated a helical coil spring can be modeled as a linear spring of stiffness

$$k = \frac{GD^4}{64Nr^3} \qquad [1.36]$$

1.7 A tightly wound spring is made from a 20-mm-diameter bar of 0.2% C-hardened steel ($G = 80 \times 10^9$ N/m^2). The coil diameter is 20 cm. The spring has 30 coils. What is the largest force that can be applied such that the elastic strength in shear of 220×10^6 N/m^2 is not exceeded? What is the change in length of the spring when this force is applied?

Solution:

Assuming the shear stress due to the shear force is negligible, the maximum shear stress in the spring when a force F is applied is

$$\tau = \frac{FrD}{2J} = F\frac{(0.1\text{ m})(0.02\text{ m})}{\frac{2\pi}{32}(0.02\text{m})^4} = 6.37 \times 10^4 F$$

Thus the maximum allowable force is

$$F_{\max} = \frac{\tau_{\max}}{6.37 \times 10^4} = 3.45 \times 10^3\text{ N}$$

The stiffness of this spring is calculated by using Eq. 1.36:

$$k = \frac{(80 \times 10^9\ \text{N/m}^2)(0.02\ \text{m})^4}{(64)(30)(0.1\ \text{m})^3} = 6.67 \times 10^3\ \frac{\text{N}}{\text{m}}$$

The total change in length of the spring due to application of the maximum allowable force is

$$\Delta = \frac{F}{k} = 0.517\ \text{m}$$

1.6.3 ELASTIC ELEMENTS AS SPRINGS

Application of a force F to the block of mass m of Fig. 1.14 results in a displacement x. The block is attached to a uniform thin rod of elastic modulus E, unstretched length L, and cross-sectional area A. Application of the force results in a uniform normal strain in the rod of

$$\varepsilon = \frac{F}{AE} = \frac{x}{L} \qquad [1.37]$$

The total strain energy developed due to the work of the force in stretching the rod is

$$S = \tfrac{1}{2}EAL\varepsilon^2 = \tfrac{1}{2}Fx \qquad [1.38]$$

If the force is suddenly removed, the block will oscillate about its equilibrium position. The initial strain energy is converted to kinetic energy and vice versa, a process which continues indefinitely. If the mass of the rod is small compared to the mass of the block, then inertia of the rod is negligible and the rod behaves as a discrete spring. From strength of materials, the force F required to change the length of the rod by x is

$$F = \frac{AE}{L}x \qquad [1.39]$$

A comparison of Eq. (1.39) with Eq. (1.30) implies that the stiffness of the rod is

$$k = \frac{AE}{L} \qquad [1.40]$$

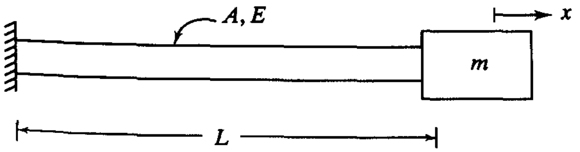


Figure 1.14 Longitudinal vibrations of mass attached to end of uniform thin rod can be modeled as a linear mass-spring system with $k = AE/L$.

The motion of a particle attached to an elastic element can be modeled as a particle attached to a linear spring, provided the mass of the element is small compared to the mass of the particle and a linear relationship between force and displacement exists for the element. In Fig. 1.15 a particle of mass m is attached to the midspan of a simply supported beam of length L, elastic modulus E, and cross-sectional moment of inertia I. The transverse displacement of the midspan of the beam due to an applied static load F is

$$x = \frac{L^3}{48EI}F \tag{1.41}$$

Thus a linear relationship exists between transverse displacement and static load. Hence if the mass of the beam is small, the vibrations of the particle can be modeled as the vertical motion of a particle attached to a spring of stiffness

$$k = \frac{48EI}{L^3} \tag{1.42}$$

In general the transverse vibrations of a particle attached to a beam can be modeled as those of a particle attached to a linear spring. Let $w(z)$ represent the displacement function of the beam due to a concentrated unit load applied at $z = a$. Then the displacement at $z = a$ due to a load F applied at $z = a$ is

$$x = w(a)F \tag{1.43}$$

Then the spring stiffness for a particle placed at $z = a$ is

$$k = \frac{1}{w(a)} \tag{1.44}$$

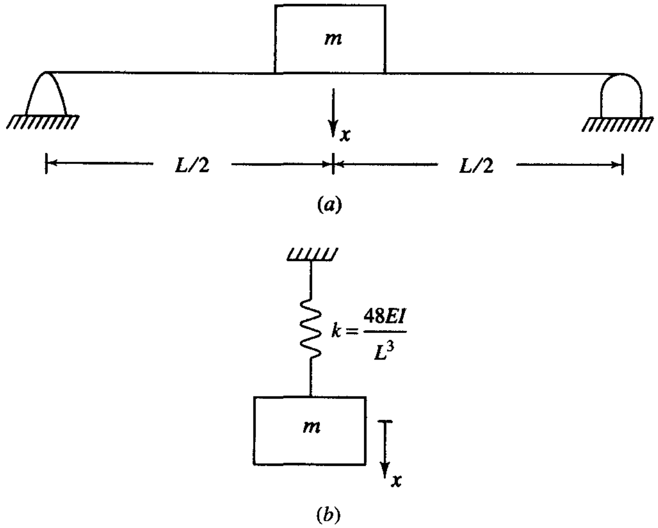


Figure 1.15 The transverse vibrations of a block attached to a simply supported beam (a) are modeled by the mass-spring system of (b), provided the mass of the beam is small compared to the mass of the block.

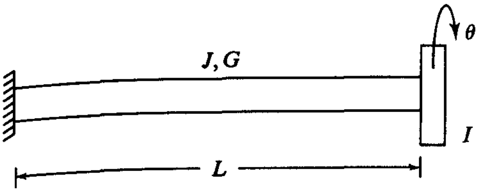


Figure 1.16 The torsional stiffness of the shaft is JG/L.

Torsional oscillations occur in the system of Fig. 1.16. A thin disk of mass moment of inertia I is attached to a circular shaft of length L, shear modulus G, and polar moment of inertia J. When the disk is rotated through an angle θ from its equilibrium position, a moment

$$M = \frac{JG}{L}\theta \qquad [1.45]$$

develops between the disk and the shaft. Thus, if the polar mass moment of inertia of the shaft is small compared with I, then the shaft acts as a torsional spring of stiffness

$$k_t = \frac{JG}{L} \qquad [1.46]$$

Exampl

A 200-kg machine is attached to the end of a cantilever beam of length $L = 2.5$ m, elastic modulus $E = 200 \times 10^9$ N/m^2, and cross-sectional moment of inertia 1.8×10^{-6} m^4. Assuming the mass of the beam is small compared to the mass of the machine, what is the stiffness of the beam?

Solution:

From Table D.2 the deflection equation for a cantilever beam with a concentrated unit load at $z = L$ is

$$w(z) = \frac{1}{EI}\left(-\frac{1}{6}z^3 + \frac{L}{2}z^2\right)$$

The deflection at the end of the beam is

$$w(L) = \frac{1}{EI}\left(-\frac{L^3}{6} + \frac{L}{2}L^2\right) = \frac{L^3}{3EI}$$

The stiffness of the cantilever beam at its end is

$$k = \frac{3EI}{L^3} = \frac{3\left(200 \times 10^9\ \text{N/m}^2\right)(1.8 \times 10^{-6}\ \text{m}^4)}{(2.5\ \text{m})^3} = 6.91 \times 10^4\ \frac{\text{N}}{\text{m}}$$

1.6.4 Springs in Combination

Often, in applications, springs are placed in combination. It is convenient, for purposes of modeling and analysis, to replace the combination of springs by a single spring of an equivalent stiffness, k_{eq}. The equivalent stiffness is determined such that the system with a combination of springs has the same displacement, x, as the equivalent system when both systems are subject to the same force, F. A model one-degree-of-freedom system consisting of a block attached to a spring of an equivalent stiffness is illustrated in Fig. 1.17. The resultant force acting on the block is

$$F = k_{eq}x \tag{1.47}$$

The springs in the system of Fig. 1.18 are in *parallel*. The displacement of each spring in the system is the same, but the resultant force acting on the block is the sum of the forces developed in the parallel springs. If x is the displacement of the block, then the force developed in the ith spring is $k_i x$ and the resultant is

$$F = k_1 x + k_2 x + \cdots + k_n x = \left(\sum_{i=1}^{n} k_i\right) x \tag{1.48}$$

Equating the forces from Eqs. (1.47) and (1.48) leads to

$$k_{eq} = \sum_{i=1}^{n} k_i \tag{1.49}$$

The springs in Fig. 1.19 are in *series*. The force developed in each spring is the same and equal to the force acting on the block. The displacement of the block is the sum of the changes in length of the springs in the series combination. If x_i is the change in length of the ith spring, then

$$x = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i \tag{1.50}$$

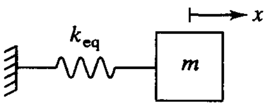


Figure 1.17 Combination of springs is replaced by a single spring such that the behavior of the system with an equivalent spring is identical to the behavior of the original system.

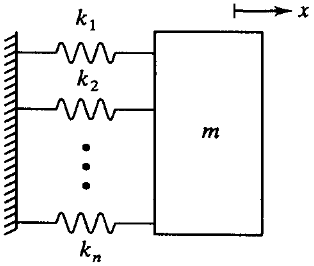


Figure 1.18 Each of the n springs in parallel has the same displacement, but the resultant force acting on the block is the sum of the spring forces.

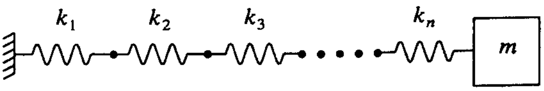


Figure 1.19 The n springs in the series combination each have the same force, but the total displacement is the sum of the changes in length of the series springs.

Since the force is the same in each spring, $x_i = F/k_i$ and Eq. (1.50) becomes

$$x = \sum_{i=1}^{n} \frac{F}{k_i} \tag{1.51}$$

Since the series combination is to be replaced by a spring of an equivalent stiffness, Eq. (1.47) is used in Eq. (1.51), leading to

$$k_{eq} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{k_i}} \tag{1.52}$$

Electrical circuit components can also be placed in series and parallel and the effect of the combination replaced by a single component with an equivalent value. The equivalent capacitance of capacitors in parallel or series is calculated like that of springs in parallel or series. The equivalent resistance of resistors in series is the sum of the resistances, whereas the equivalent resistance of resistors in parallel is calculated by using an equation similar to Eq. (1.52).

Model each of the systems of Fig. 1.20 by a mass attached to a single spring of an equivalent stiffness. The system of Fig. 1.20*c* is to be modeled by a disk attacked to a torsional spring of an equivalent stiffness.

Solution:

(*a*) The steps involved in modeling the system of Fig. 1.20*a* by the system of Fig. 1.17 are shown in Fig. 1.21. Equation (1.49) is used to replace the two parallel springs by an equivalent spring of stiffness $3k$. The three springs on the left of the mass are then in series and Eq. (1.52) is used to obtain an equivalent stiffness.

If the mass in Fig. 1.21*b* is given a displacement x to the right, then the spring on the left of the mass will increase in length by x, while the spring on the right of the mass will decrease in length by x. Thus each spring will exert a force to the left on the mass. The spring forces add; the springs behave as if they are in parallel. Hence Eq. (1.49) is used to replace these springs by the equivalent spring shown in Fig. 1.21*c*.

(*b*) The deflection of the simply supported beam due to a unit load at $x = 2m$ is calculated using Table D.2

$$w(z = 2m) = w\left(\frac{2L}{3}\right) = \frac{4L^3}{243EI}$$

from which the equivalent stiffness is obtained

$$k_1 = \frac{243EI}{4L^3} = \frac{243(210 \times 10^9 \text{ N/m}^2)(5 \times 10^{-4} \text{ m}^4)}{4(3 \text{ m})^3} = 2.36 \times 10^8 \frac{\text{N}}{\text{m}}$$

The displacement of the block of mass m equals the displacement of the beam at the location where the spring is attached plus the change in length of the spring. Hence the

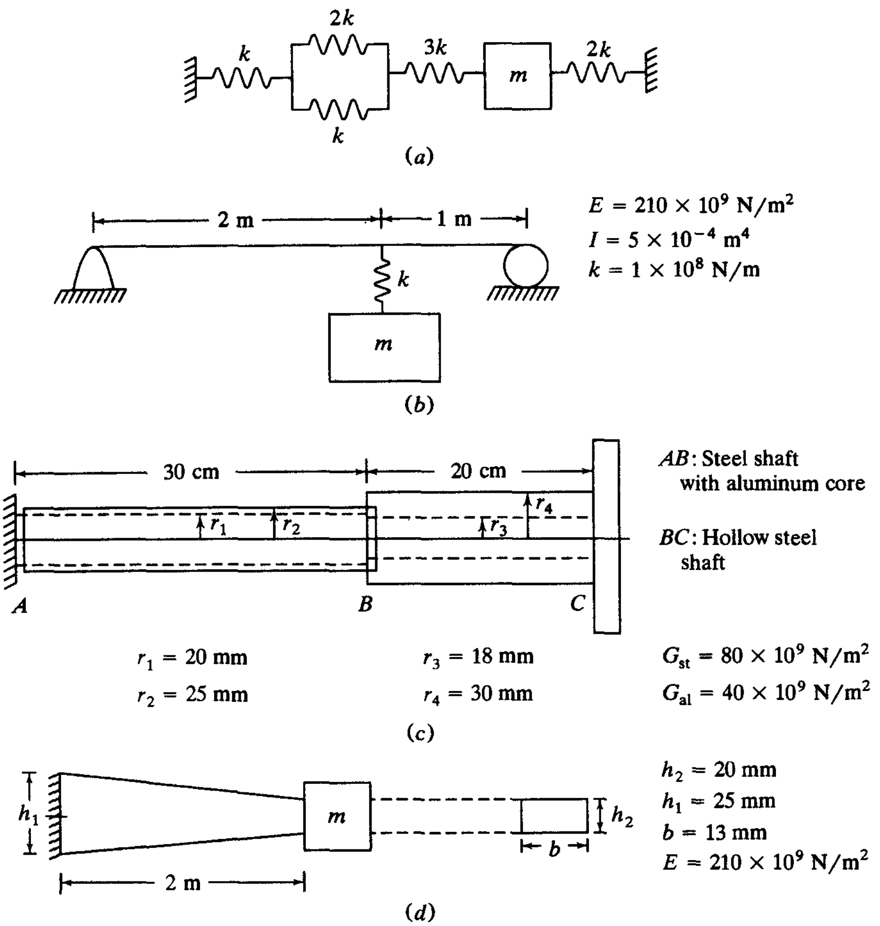


Figure 1.20 Systems for Example 1.9.

beam and spring act as a series combination. Equation (1.52) is used to calculate the equivalent stiffness

$$k_{eq} = \frac{1}{\dfrac{1}{2.36 \times 10^8 \text{ N/m}} + \dfrac{1}{1 \times 10^8 \text{ N/m}}} = 7.02 \times 10^7 \, \frac{\text{N}}{\text{m}}$$

(*c*) The aluminum core of shaft *AB* is rigidly bonded to the steel shell. Thus the angular rotation at *B* is the same for both materials. The total resisting torque transmitted to section *BC* is the sum of the torque developed in the aluminum core and the torque developed in the steel shell. Thus the aluminum core and steel shell of shaft *AB* behave as two torsional springs in parallel. The resisting torque in shaft *AB* is the same as the resisting torque in shaft *BC*. The angular displacement at *C* is the angular displacement of *B* plus the angular displacement of *C* relative to *B*. Thus shafts *AB* and *BC* behave

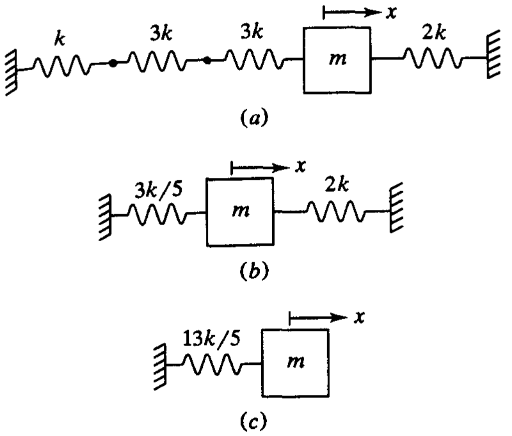


Figure 1.21 Steps in replacing the combination of springs of Fig. 1.20a by a single spring of an equivalent stiffness.

as two torsional springs in series. In view of the preceding discussion and using Eqs. (1.49) and (1.52), the equivalent stiffness of shaft AC is

$$k_{t_{\mathrm{eq}}} = \cfrac{1}{\cfrac{1}{k_{t_{AB_{\mathrm{al}}}} + k_{t_{AB_{\mathrm{st}}}}} + \cfrac{1}{k_{t_{BC}}}}$$

where the torsional stiffness of a shaft is $k_t = JG/L$ and

$$k_{t_{AB_{\mathrm{al}}}} = \frac{\frac{\pi}{32}(0.04\ \mathrm{m})^4\left(40 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.3\ \mathrm{m}} = 3.35 \times 10^4\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

$$k_{t_{AB_{\mathrm{st}}}} = \frac{\frac{\pi}{32}\left[(0.05\ \mathrm{m})^4 - (0.04\ \mathrm{m})^4\right]\left(80 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.3\ \mathrm{m}} = 9.66 \times 10^4\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

$$k_{t_{BC}} = \frac{\frac{\pi}{32}\left[(0.06\ \mathrm{m})^4 - (0.036\ \mathrm{m})^4\right]\left(80 \times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)}{0.2\ \mathrm{m}} = 4.43 \times 10^5\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

Substitution of these values into the equation for k_{eq} gives

$$k_{t_{\mathrm{eq}}} = 1.01 \times 10^5\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}}$$

(*d*) Under the assumption that the rate of taper of the bar is small the following mechanics of materials equation is used to calculate the change in length of the bar due to a unit load applied at its end:

$$\Delta = \int_0^L \frac{dz}{AE}$$

where the cross-sectional area A is given by

$$A(z) = \left(\frac{h_2 - h_1}{L}z + h_1\right)b$$

Substituting and integrating yields

$$\Delta = \frac{L}{Eb(h_2 - h_1)}\ln\left(\frac{h_2}{h_1}\right) = 3.27 \times 10^{-8}\ \text{m}$$

Since Δ is the displacement of the end of the bar due to a unit axial load, the displacement due to an axial load of magnitude F is $x = F\Delta$. Thus

$$k = \frac{1}{\Delta} = 3.06 \times 10^{7}\ \frac{\text{N}}{\text{m}}$$

1.6.5 INERTIA EFFECTS OF SPRINGS

When a force is applied to displace the block of Fig. 1.17 from its equilibrium position, the work done by the force is converted into strain energy stored in the spring. If the block is held in this position and then released, the strain energy is converted to kinetic energy of both the block and the spring. If the mass of the spring is much smaller than the mass of the block, its kinetic energy is negligible. In this case the inertia of the spring has negligible effect on the motion of the block, and the system is modeled using one degree of freedom. The generalized coordinate is usually chosen as the displacement of the block.

If the mass of the spring is comparable to the mass of the block, the one-degree-of-freedom assumption is not valid. The particles along the axis of the spring are kinematically independent from each other and from the block. The spring should be modeled as a continuous system.

If the mass of the spring is much smaller than the mass of the block, but not negligible, a reasonable one-degree-of-freedom approximation can be made by approximating the spring's inertia effects. The actual system is modeled by the ideal system of Fig. 1.22. The spring in Fig. 1.22 is massless. The mass of the block in Fig. 1.22 is greater than the mass of the actual block to account for inertia effects of the spring. The value of m_{eq} is calculated such that the kinetic energy of the system of Fig. 1.22 is the same as the kinetic energy of the system of Fig. 1.17, including the kinetic energy of the spring, when the velocities of both blocks are equal. Unfortunately, calculation of the exact kinetic energy of the spring requires

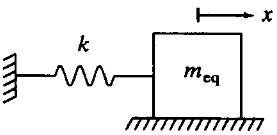


Figure 1.22 An equivalent mass of $m + m_s/3$ is used to approximate inertia effects of the spring.

a continuous system analysis. Thus an approximation to the spring's kinetic energy is used.

Let $x(t)$ be the generalized coordinate describing the motion of both the block of Fig. 1.17 and the block of Fig. 1.22. The kinetic energy of the system of Fig. 1.17 is

$$T = T_s + \tfrac{1}{2}m\dot{x}^2 \qquad [1.53]$$

where T_s is the kinetic energy of the spring. The kinetic energy of the system of Fig. 1.22 is

$$T = \tfrac{1}{2}m_{\text{eq}}\dot{x}^2 \qquad [1.54]$$

The spring in Fig. 1.17 is uniform, has an unstretched length l and a total mass m_s. Define the coordinate z along the axis of the spring, measured from its fixed end, as defined in Fig. 1.23. The coordinate z measures the distance of a particle from the fixed end in the spring's unstretched state. The displacement of a particle on the spring, $u(z)$, is assumed explicitly independent of time and a linear function of z such that $u(0) = 0$ and $u(l) = x$,

$$u(z) = \frac{x}{l}z \qquad [1.55]$$

Equation (1.55) represents the displacement function of a uniform spring when it is statically stretched. Consider a differential element of length dz, located a distance z from the spring's fixed end. The kinetic energy of the differential element is

$$dT_s = \frac{1}{2}\dot{u}^2(z)\,dm = \frac{1}{2}\dot{u}^2(z)\frac{m_s}{l}\,dz \qquad [1.56]$$

The total kinetic energy of the spring is

$$T_s = \int dT_s = \int_0^l \frac{1}{2}\frac{m_s}{l}\left(\frac{\dot{x}z}{l}\right)^2 dz = \frac{1}{2}\frac{m_s}{l^3}\dot{x}^2\frac{z^3}{3}\bigg|_0^l = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2 \qquad [1.57]$$

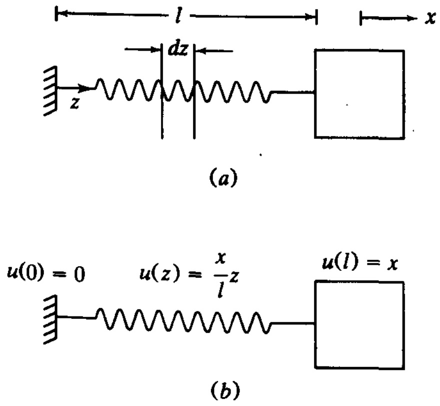


Figure 1.23 (a) The coordinate z is measured along axis of spring from its fixed end; (b) the displacement in the spring is assumed as a linear function of z.

Equating T from Eqs. (1.53) and (1.54) and using T_s from Eq. (1.57) gives

$$m_{\text{eq}} = m + \frac{m_s}{3} \tag{1.58}$$

Equation (1.58) can be interpreted as follows: The inertia effects of a linear spring with one end fixed and the other end connected to a moving body can be approximated by placing a particle whose mass is one-third of the mass of the spring at the point where the spring is connected to the body.

The preceding statement is true for all springs where use of a linear displacement function of the form of Eq. (1.55) is justified. This is valid for helical coil springs, bars that are modeled as springs for longitudinal vibrations, and shafts acting as torsional springs.

1.10 The springs in the system of Fig. 1.24*a* are all identical, with stiffness k and mass m_s. Calculate the kinetic energy of the system in terms of $\theta(t)$, including the inertia effects of the springs.

Solution:

Each spring is replaced by a massless spring and a particle of mass $m_s/3$ at the point on the bar where the spring is attached as shown in Fig. 1.24*b*. The total kinetic energy of

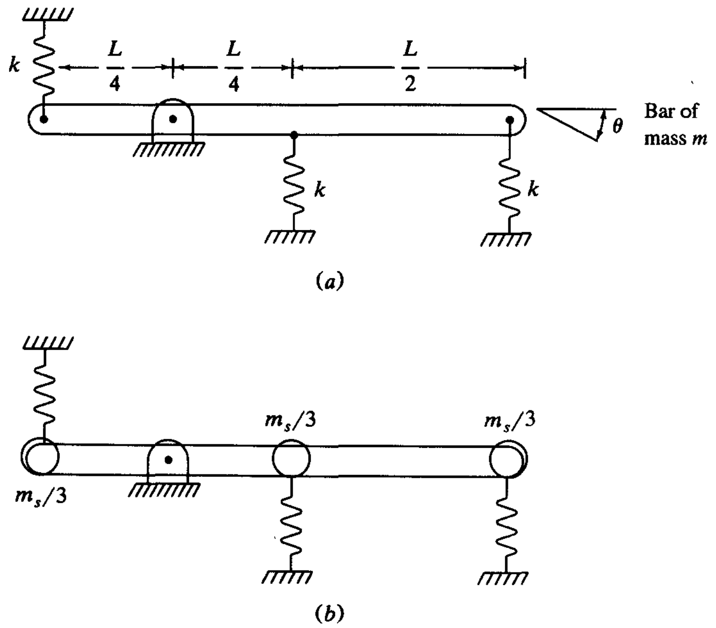


Figure 1.24 (a) System of Example 1.10; (b) inertia effects of springs are approximated by placing particles of mass $m_s/3$ at locations on bar where springs are attached.

the system of Fig. 1.24*b* is the kinetic energy of the bar plus the kinetic energy of each of the particles

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2 + T_1 + T_2 + T_3$$

$$= \frac{1}{2}m\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m_s}{3}\left(\frac{3L}{4}\dot{\theta}\right)^2$$

$$= \frac{1}{2}\frac{7m + 11m_s}{48}L^2\dot{\theta}^2$$

Exampl

The simply supported beam of Fig. 1.25 is uniform and has a total mass of 100 kg. A machine of mass 350 kg is attached at B, as shown. What is the mass of a particle that should be placed at B to approximate the beam's inertia effects?

Solution:

Since the exact expression for the dynamic beam deflection is not known, an approximate displacement function must be used in the calculation of the beam's kinetic energy. Let z be a coordinate along the beam's neutral axis. Assume that the time-dependent displacement of any particle can be expressed as

$$y(z, t) = x(t)w(z)$$

where $x(t)$ is the deflection of B. An appropriate approximation for $w(z)$ is the static deflection of the beam due to a concentrated load, P, applied at B, such that B has a unit deflection.

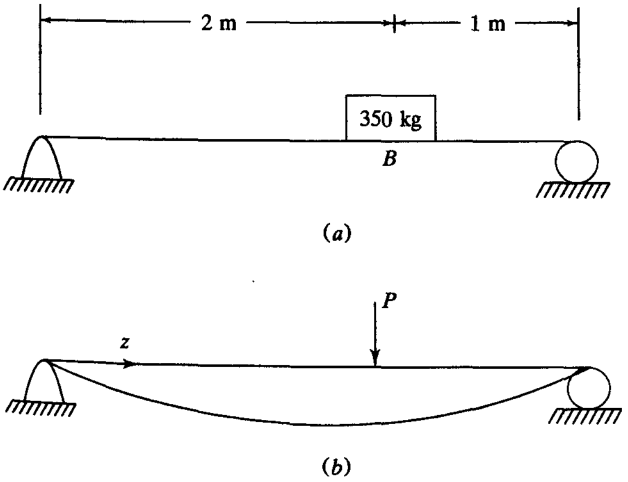


Figure 1.25 (*a*) System of Example 1.11; (*b*) static deflection of beam due to concentrated load at *B*.

By using the methods of App. D, the static deflection due to a concentrated load at B is found to be

$$w(z) = \begin{cases} \dfrac{P}{18EI} z\left(\dfrac{8L^2}{9} - z^2\right) & 0 \le z \le \dfrac{2L}{3} \\ \dfrac{P}{18EI}\left(2z^3 - 6z^2L + \dfrac{44}{9}zL^2 - \dfrac{8}{9}L^3\right) & \dfrac{2L}{3} \le z \le L \end{cases}$$

The load required to cause a unit deflection at $z = 2L/3$

$$P = \frac{243EI}{4L^3}$$

Consider a differential element of length dz, located a distance z from the left support The kinetic energy of the element is

$$dT = \tfrac{1}{2}\dot{y}^2(z,t)\rho A\, dz$$

where ρ is the mass density of the beam and A is its cross-sectional area. The beam's total kinetic energy is calculated by integrating dT over the entire beam. Substituting the previous results for $w(x, t)$ in this integral leads to

$$T = \frac{1}{2}\rho A\left(\frac{27}{8L^3}\right)^2 \dot{x}^2\left[\int_0^{2L/3} z^2\left(\frac{8L^2}{9} - z^2\right)^2 dz + \int_{2L/3}^{L}\left(2z^3 - 6z^2L + \frac{44}{9}zL^2 - \frac{8}{9}L^3\right)^2 dz\right]$$

which after considerable algebra gives

$$T = \tfrac{1}{2}0.586\rho AL\dot{x}^2$$

Noting that the total mass of the beam is ρAL, a particle of mass 58.6 kg should be added at B to approximate the inertia effects of the beam. The system of Fig. 1.25a is modeled as a one-degree-of-freedom system with a particle of 408.6 kg located at B.

1.7 VISCOUS DAMPERS

Viscous damping occurs in a mechanical system because of viscous friction that results from the contact of a system component and a viscous liquid. The damping force produced when a rigid body is in contact with a viscous liquid is usually proportional to the velocity of the body

$$F = cv \qquad [1.59]$$

where c is called the damping coefficient and has dimensions of mass per time.

Viscous damping can occur naturally, as when a buoyant body oscillates on the surface of a lake or a column of liquid oscillates in a U-tube manometer. Viscous damping is often added to mechanical systems as a means of vibration control. Viscous damping leads to an exponential decay in amplitude of free vibrations and a reduction in amplitude in forced vibrations caused by a harmonic excitation. In addition, the presence of viscous damping gives rise to a linear term in the governing differential equation, and thus does not significantly complicate the mathematical modeling of the system. A mechanical device called a *dashpot* is added to mechanical systems to provide viscous damping. A schematic of a dashpot in a one-degree-of-freedom system is shown in Fig. 1.26a. The free-body diagram of the rigid body, Fig. 1.26b, shows the viscous force in the opposite direction of the positive velocity.

A simple dashpot configuration is shown in Fig. 1.27a. The upper plate of the dashpot is connected to a rigid body. As the body moves, the plate slides over a reservoir of viscous liquid of dynamic viscosity μ. The area of the plate in contact with the liquid is A. The shear stress developed between the fluid and the plate creates a resultant friction force acting on the plate. Assume the reservoir is stationary and the upper plate slides over the liquid with a velocity v. The reservoir depth h is small enough that the velocity profile in the liquid can be approximated as linear, as illustrated in Fig. 1.27b. If y is a coordinate measured upward from the bottom of the reservoir,

$$u(y) = v\frac{y}{h} \tag{1.60}$$

The shear stress developed on the plate is determined from Newton's viscosity law

$$\tau = \mu\frac{du}{dy} = \mu\frac{v}{h} \tag{1.61}$$

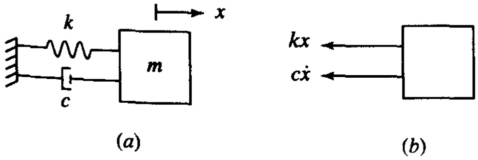


Figure 1.26 (a) Schematic of one-degree-of-freedom mass-spring-dashpot system. (b) Dashpot force is $c\dot{x}$ and opposes the direction of positive $\dot{x}$.

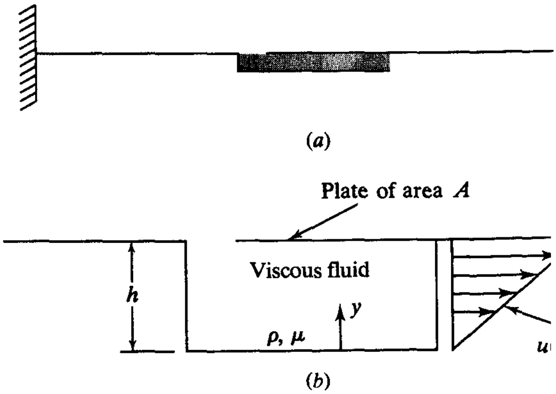


Figure 1.27 (a) Simple dashpot model where plate s a fixed reservoir of viscous liquid. (b) Since h is small, a linec profile is assumed in the liquid.

The viscous force acting on the plate is

$$F = \tau A = \frac{\mu A}{h} v \tag{1.62}$$

Comparison of Eq. (1.62) with Eq. (1.59) shows that the damping coefficient for this dashpot is

$$c = \frac{\mu A}{h} \tag{1.63}$$

Equation (1.63) shows that a large damping force is achieved with a very viscous fluid, a small h, and a large A. A dashpot design with these parameters is often impractical and thus the device of Fig. 1.27*a* is rarely actually used as a dashpot.

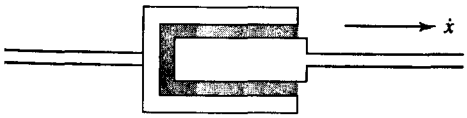
The above analysis assumes the plate moves with a constant velocity. During the motion of a mechanical system the dashpot is connected to a particle which has a time-dependent velocity. The changing velocity of the plate leads to unsteady effects in the liquid. If the reservoir depth h is small, the unsteady effects are small and can be neglected.

A piston-cylinder design for a dashpot is illustrated in Fig. 1.28. As the piston moves with a velocity v into the cylinder containing viscous liquid, shear stresses develop on the side of the piston and a pressure force develops on the surface of the piston. Both effects lead to a force proportional to the velocity of the piston.

A torsional viscous damper is illustrated in Fig. 1.29. The shaft is rigidly connected to a point on a body undergoing torsional oscillations. As the disk rotates in a dish of viscous liquid, a net moment due to the shear stresses developed on the face of the disk acts about the axis of rotation. The moment is proportional to the angular velocity of the shaft

$$M = c_t \dot{\theta} \tag{1.64}$$

where c_t is the torsional viscous damping coefficient and has dimensions of force-length-time.



A piston and cylinder device that serves as a dashpot.

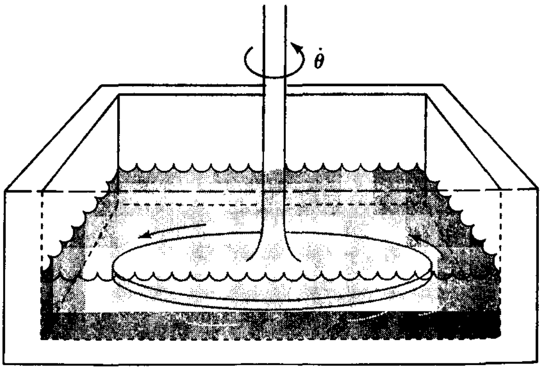


Figure 1.29 Disk rotates in a dish of viscous liquid, producing a moment about the axis of the shaft and acting as a torsional viscous domper.

1.8 FLOATING AND IMMERSED BODIES

Vibrations of a body immersed in a liquid or floating on the interface of a liquid and a gas can be modeled by using the methods of this chapter with special considerations.

1.8.1 BUOYANCY

When a solid body is submerged in a liquid or floating on the interface of a liquid and air, a force acts vertically upward on the body because of the variation of hydrostatic pressure. This force is called the buoyant force. Archimedes' principle states that the buoyant force acting on a floating or submerged body is equal to the weight of the liquid displaced by the body.

Exampl

A sphere of mass 2.5 kg and radius 10 cm is hanging from a spring of stiffness 1000 N/m in a fluid of mass density 1200 kg/m^3. What is the static deflection of the spring?

Solution:

The spring force must balance with the gravity force and the buoyancy force as shown on the free-body diagram in Fig. 1.30.

$$k\Delta_{\text{st}} + F_B - mg = 0$$

Archimedes' principle is used to calculate the buoyant force as

$$F_B = \tfrac{4}{3}\rho g \pi r^3 = \tfrac{4}{3}(1200\ \text{kg/m}^3)\pi(9.81\ \text{m/s}^2)(0.1\ \text{m})^3 = 49.3\ \text{N}$$

The static deflection is calculated as

$$\Delta_{\text{st}} = \frac{mg - F_B}{k} = \frac{(2.5\ \text{kg})(9.81\ \text{m/s}^2) - 49.3\ \text{N}}{1000\ \text{N/m}} = -24.8\ \text{mm}$$

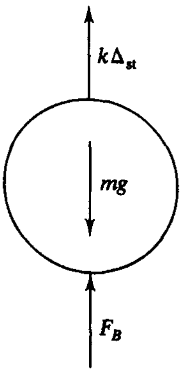


Figure 1.30 Free-body diagram of sphere, attached to spring, and submerged in a liquid.

Consider a body floating stably on a liquid-air interface. The buoyant force balances with the gravity force. If the body is pushed farther into the liquid, the buoyant force increases. If the body is then released, it seeks to return to its equilibrium configuration. The buoyant force does work, which is converted into kinetic energy and oscillations about the equilibrium position ensue.

The circular cylinder of Fig. 1.31 has a cross-sectional area A and floats stably on the surface of a fluid of density ρ. When the cylinder is in equilibrium, it is subject to a buoyant force mg and its center of gravity is a distance Δ from the surface. Let x be the vertical displacement of the center of gravity of the cylinder from this position. The additional volume displaced by the cylinder is xA. According to Archimedes' principle, the buoyant force is

$$F_B = mg + \rho g A x$$

Calculations show that the work done by the buoyant force as the cylinder's center of gravity moves between positions x_1 and x_2 is

$$U_{1\rightarrow 2} = \tfrac{1}{2}\rho g A x_2^2 - \tfrac{1}{2}\rho g A x_1^2$$

and is independent of path. Hence the buoyant force is conservative. Its effect on the cylinder is the same as that of a linear spring of stiffness $\rho g A$. The oscillations of the cylinder on the liquid-gas interface can be modeled by a one-degree-of-freedom mass-spring system.

1.8.2 ADDED MASS

Consider a mass-spring system immersed in an inviscid fluid, as shown in Fig. 1.32. The spring is stretched from its equilibrium configuration and the mass released. The ensuing motion of the mass causes motion in the surrounding fluid. The strain energy initially stored in the spring is converted to kinetic energy for both the mass

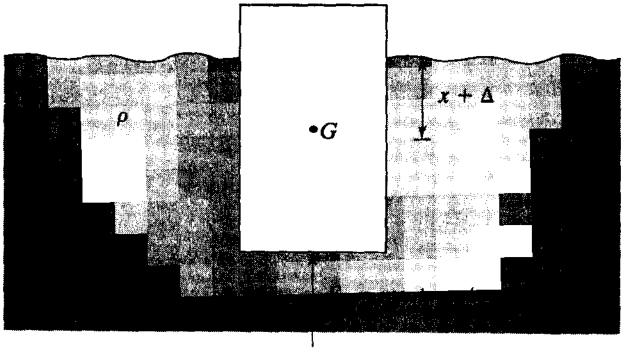


Figure 1.31 Oscillations of circular cylinder on free surface can be modeled by a one-degree-of-freedom mass-spring system.

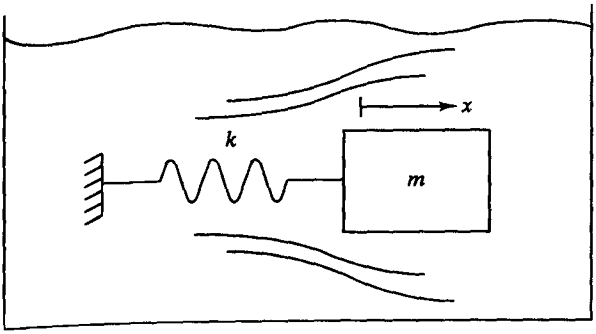


Figure 1.32 Oscillations of a submerged body create kinetic energy in a fluid. The inertia of the fluid can be approximated with added mass to the body.

and the fluid. Since the fluid is inviscid, energy is conserved

$$T_m + T_f + V = C \tag{1.65}$$

The inertia effects of the fluid can be included in an analysis by using a method similar to that used in Sec. 1.6 to account for the inertia effects of springs. An imagined particle is attached to the mass such that the kinetic energy of the particle is equal to the total kinetic energy of the fluid. If x is the displacement of the mass, the total kinetic energy of the system is $\frac{1}{2}m_{\text{eq}}\dot{x}^2$, where

$$m_{\text{eq}} = m + m_a \tag{1.66}$$

The mass of the particle is called the *added mass.*

The kinetic energy of the fluid is difficult to quantify. The motion of the body theoretically entrains fluid infinitely far away in all directions. The total kinetic energy of the fluid is calculated from

$$T_f = \frac{1}{2}\int\int\int \rho v^2\, dV \tag{1.67}$$

where v is the velocity of the fluid set in motion by the motion of the body. The integration is carried out from the body surface to infinity in all directions. If the integration of Eq. (1.67) is carried out, the added mass is calculated from

$$m_a = \frac{T_f}{\frac{1}{2}\dot{x}^2} \tag{1.68}$$

Potential flow theory can be used to develop the velocity distribution in a fluid for a body moving through the fluid at a constant velocity. This velocity distribution is used in Eqs. (1.67) and (1.68) to calculate the added mass. Table 1.1 is adapted from Wendel (1956) and Patton (1965) and presents the added mass for common body shapes.

able 1.1 Added mass for common two- and three-dimensional bodies (ρ is the mass density of the fluid)

ody	Added Mass
phere of diameter D	$\frac{1}{12}\pi\rho D^3$
hin Circular disk of diameter D	$\frac{1}{3}\rho D^3$
hin square plate of side h	$0.1195\pi\rho h^3$
ircular cylinder of length L, diameter D	$\frac{1}{4}\pi\rho D^2 L$
hin flat plate of length L, width w	$\frac{1}{4}\pi\rho w^2 L$
quare cylinder of side h, length L	$0.3775\rho\pi h^2 L$
ube of side h	$2.33\rho h^3$

Table 1.2 Added moments of inertia for common bodies (ρ is the mass density of the fluid)

Body	Added moment of inertia
Sphere	0
Circular cylinder	0
Any body rotating about axis of symmetry	0
Thin plate of length L, rotating about axis in the plane of the surface area of plate, perpendicular to direction for which L is defined	$0.0078125\pi\rho L^4$
Disk of diameter D rotating about a diameter	$\frac{1}{90}\rho D^5$

Rotational motion of a body in a fluid also imparts motion to the fluid resulting in rotational kinetic energy of the fluid. The inertia effects of the fluid are taken into account by adding a disk of an appropriate moment of inertia to the rotating body. If ω is the angular velocity of the body, the added mass moment of inertia is calculated from

$$I_a = \frac{T_f}{\frac{1}{2}\omega^2} \tag{1.69}$$

Note that the added mass moment of inertia is zero if the body is rotating about an axis of symmetry. Both the added mass and added moment of inertia terms are negligible for bodies moving in gases. Table 1.2 presents added moments of inertia for a few common bodies. It is adapted from Wendel (1956).

1.9 SUMMARY

The solution of vibrations problems requires mathematical modeling of the vibrating system. It is necessary to make certain assumptions regarding the elements composing the system. In addition to the assumptions listed in Sec. 1.2, unless otherwise specified the following assumptions will be used throughout the text:

1. All springs are linear with a force-displacement relation $F = kx$
2. Inertia effects of discrete springs are negligible.

3. Inertia elements of discrete systems are particles or rigid bodies.
4. All dashpots are linear with a force velocity relation $F = cv$
5. Mechanical systems are undergoing planar motion.
6. All forms of friction besides viscous damping are neglected.

PROBLEMS

1.1. A vibrating body is undergoing simple harmonic motion with an amplitude of 3 mm ar frequency of 30 cycles/s.

(*a*) What is the maximum velocity of the particle?

(*b*) What is the maximum acceleration of the body?

1.2. The maximum velocity of a vibrating body undergoing simple harmonic motion is 3.2 r The period of motion is measured as 0.15 s.

(*a*) What is the amplitude of motion?

(*b*) What is the maximum acceleration of the body?

1.3. A particle is traveling in a circular path of radius 30 cm at a constant angular speed of rad/s, as illustrated in Fig. P1.3. The particle starts at $\theta = 30°$ at $t = 0$. Determine $x(t$

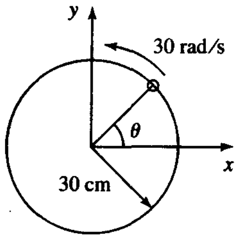


FIGURE P1.3

1.4. A particle starts at the origin of a cartesian coordinate system and moves with a veloc vector

$$\mathbf{v} = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + 0.4t\mathbf{k} \text{ m/s}$$

where t is in seconds.

(*a*) What is the magnitude of the particle's acceleration at $t = \pi$ s?

(*b*) What is the particle's position vector at $t = \pi$ s?

1.5. The displacement of a particle undergoing free underdamped vibrations is

$$x(t) = 0.5e^{-1.2t}\sin(15t + 0.24) \text{ m}$$

What is the maximum acceleration magnitude of the particle?

1.6. At $t = 0$ a particle of mass 1.2 kg has a velocity of zero but its speed is increasing a constant rate of 0.5 m/s^2. After the particle travels 3 m, the local radius of curvature of t particle's path is 25 m.

(*a*) What is the speed of the particle after it travels 3 m?

(*b*) What is the magnitude of the particle's acceleration after it has traveled 3 m?

(*c*) How long does it take the particle to travel 3 m?

1.7. At the instant shown, the slender rod of Fig. P1.7 has a clockwise angular velocity of 5 rad/s and a counterclockwise angular acceleration of 14 rad/s^2. What is the acceleration of the particle at P?

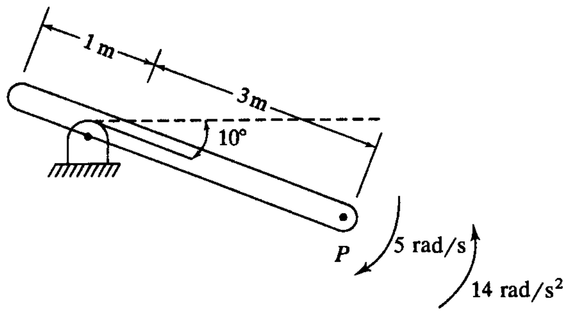


FIGURE P1.7

1.8. An automobile is traveling with a horizontal velocity of 40 m/s when it encounters a pothole whose depth is approximated by

$$y(x) = 0.02(x^2 - 6x)\ \mathrm{m}$$

where x is the distance from the leading edge of the pothole. When the driver encounters the pothole, he begins a constant deceleration of 10 m/s^2. What is the maximum vertical velocity and acceleration attained by the automobile as it traverses the pothole?

1.9. The contour of a bumpy road is approximated as

$$y(x) = 0.03 \sin(0.125x)\ \mathrm{m}$$

What is the amplitude of the vertical acceleration of the wheels of an automobile as it travels over this road at a constant horizontal speed of 40 m/s?

1.10. The machine of Fig. P1.10 has a vertical displacement $x(t)$. The machine has a component that rotates with a constant angular speed ω relative to the machine. The center of mass of the rotating component is a distance e from its axis of rotation. If the center of mass is shown at $t = 0$, determine its vertical component of acceleration as a function of time.

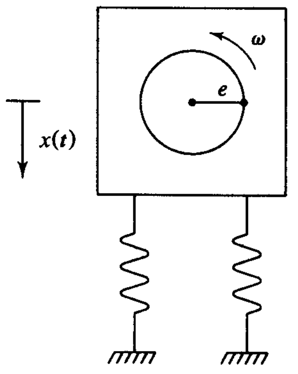


FIGURE P1.10

1.11. A 2-ton truck is traveling down a 10° icy hill at 50 mi/h when the driver sees a car stalled at the bottom of the hill, 250 ft away. Because of icy conditions, a braking force of only 2000 lb is generated when the driver applies the brakes. Does the truck stop before hitting the car?

1.12. Figure P1.12 shows the schematic of a one-cylinder reciprocating engine. If the piston has a velocity v and an acceleration a, determine the angular acceleration of the crank in terms of v, a, the crank radius r, the connecting rod length l, and the crank angle θ.

1.13. The helicopter of Fig. P1.13 has a horizontal speed of 110 ft/s and a horizontal acceleration of 3.1 ft/s². The main blades rotate at a constant speed of 135 rpm. At the instant shown determine the velocity and acceleration of particle A.

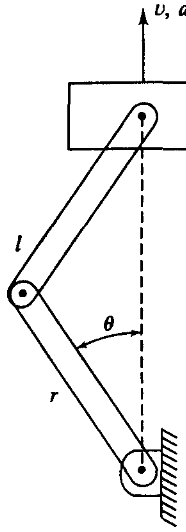


FIGURE P1.12

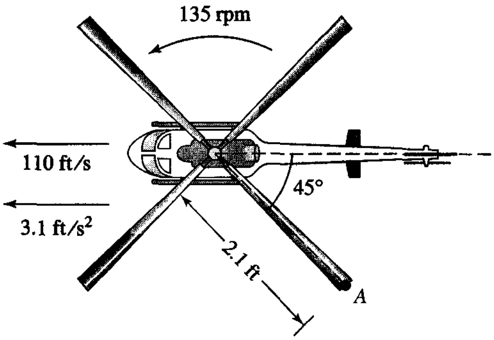


FIGURE P1.13

1.14. The mechanism of Fig. P1.14 is in equilibrium in the position shown. The horizontal displacement of the collar from this position is

$$x(t) = 0.05 \sin 20t \text{ m}$$

Determine the angular velocity and angular acceleration of bar AB as functions of time.

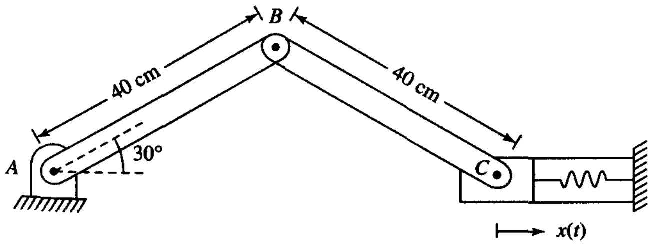


FIGURE P1.14

1.15. A 60-lb block of Fig. P1.15 is connected by an inextensible cable through the pulley to the fixed surface. A 40-lb weight is attached to the pulley which is free to move vertically. A force of magnitude $P = (70 + 30e^{-t})$ lb tows the block. The system is released from rest at $t = 0$.

(*a*) What is the acceleration of the 60-lb block as a function of time?

(*b*) How far will the block travel up the incline before it attains a velocity of 4 ft/s?

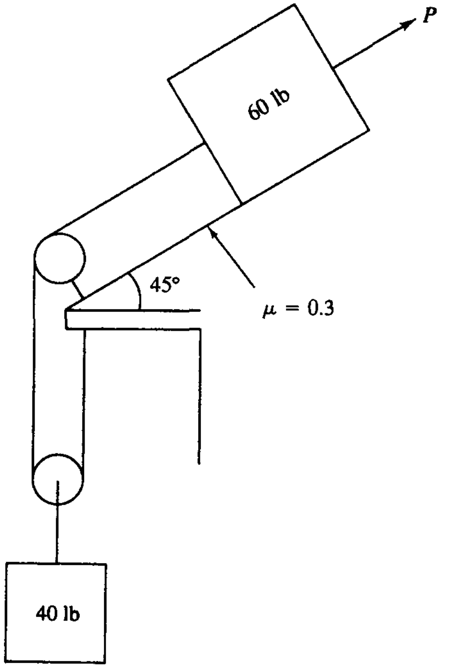


FIGURE P1.15

1.16. Repeat Prob. 1.15 if $P(t) = 200t$ lb.

1.17. Determine the angular acceleration of each of the disks of Fig. P1.17.

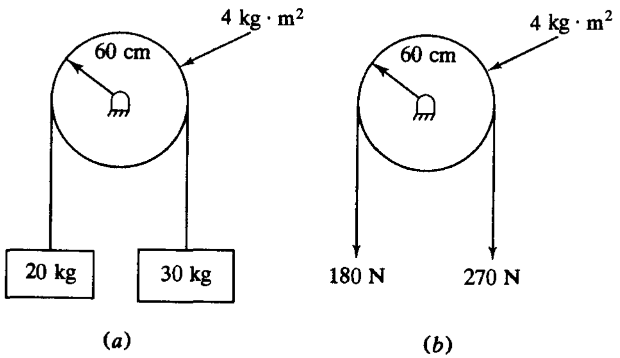


FIGURE P1.17

1.18. Determine the reactions at A for the two-link mechanism of Fig. P1.18. The roller at C rolls on a frictionless surface.

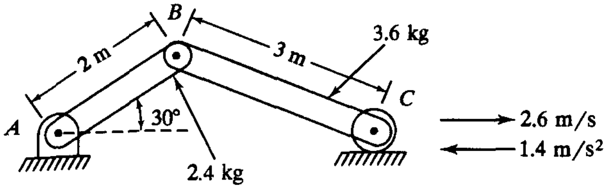


FIGURE P1.18

1.19. The slender bar of Fig. P1.19 has a mass of 50 kg. Determine the applied moment M and the reactions at the pin support at the instant shown.

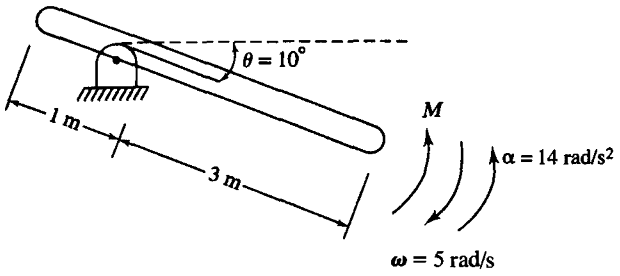


FIGURE P1.19

1.20. The disk of Fig. P1.20 rolls without slip. Determine the acceleration of the mass center of the disk if $P = 18$ N.

1.21. The coefficient of friction between the surface and the disk of Fig. P1.20 is 0.12. What is the largest value of P such that the disk rolls without slip?

1.22. The coefficient of friction between the surface and the disk of Fig. P1.20 is 0.12. Determine the angular acceleration of the disk if $P = 15$ N.

1.23. What is the maximum angular velocity attained by the disk of Fig. P1.23 if the 3-kg block is displaced 10 mm and released?

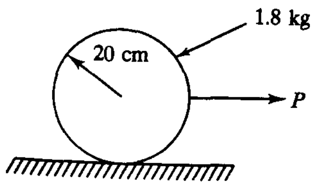


FIGURE P1.20

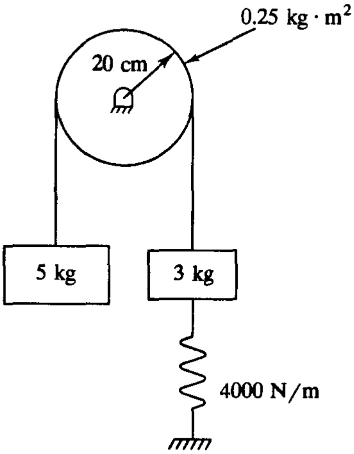


FIGURE P1.23

1.24. The five-blade ceiling fan of Fig. P1.24 operates at 60 rpm. What is its total kinetic energy?

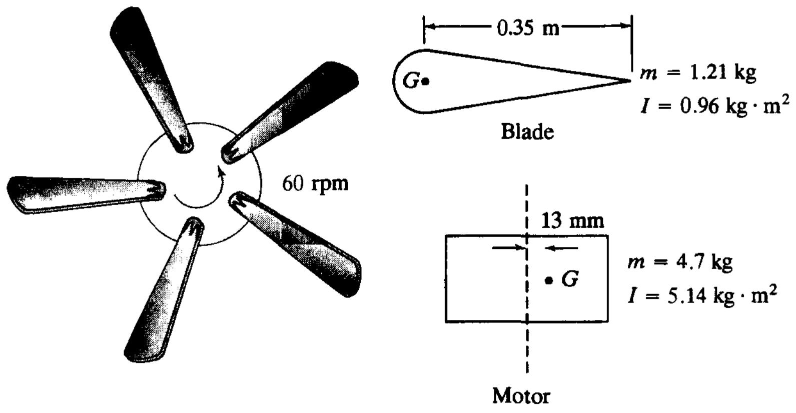


FIGURE P1.24

1.25. The U-tube manometer of Fig. P1.25 rotates about axis A-A at a speed of 40 rad/s. At the instant shown, the column of liquid moves in the manometer with a velocity of 20 m/s relative to the manometer. Calculate the total kinetic energy of the column of liquid at this instant.

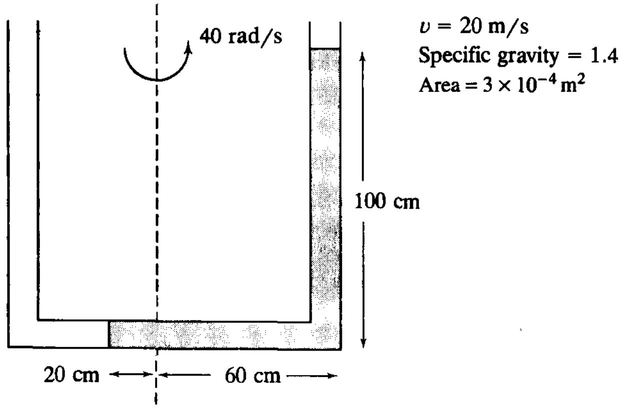


FIGURE P1.25

1.26. The displacement function for the simply supported beam of Fig. P1.26 is

$$y(x,t) = c \sin\left(\pi \frac{x}{L}\right) \cos\left(\pi^2 \sqrt{\frac{EI}{\rho A L^4}}\, t\right)$$

Where $c = 0.003$ m and t is in seconds. Determine the total kinetic energy of the beam.

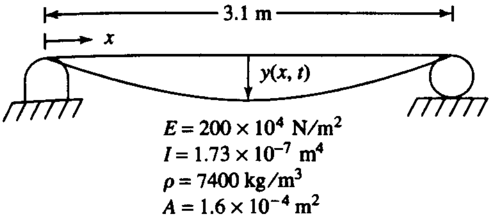


FIGURE P1.26

1.27. The block of Fig. P1.27 is given a displacement δ and then released.
(a) What is the minimum value of δ such that motion ensues?
(b) What is the minimum value of δ such that the block returns to its original equilibrium position before stopping?

1.28. The center of the thin disk of Fig. P1.28 is displaced 15 mm and released. What is the maximum velocity attained by the center of the disk, assuming no slip between the disk and the surface?

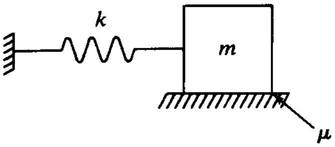


FIGURE P1.27

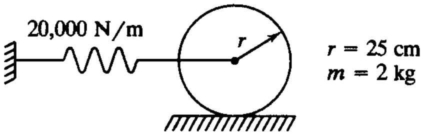


FIGURE P1.28

1.29. The slender bar of Fig. P1.29 is moved to the horizontal position and then released. When the bar is horizontal the spring is compressed 22 mm. What is the maximum angle through which the bar will swing?

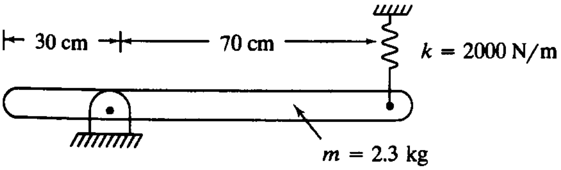


FIGURE P1.29

1.30. The block of Fig. P1.30 is displaced 1.5 cm from equilibrium and released.
(a) What is the maximum velocity attained by the block?
(b) What is the acceleration of the block immediately after it is released?

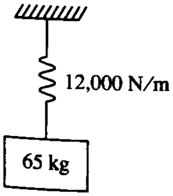


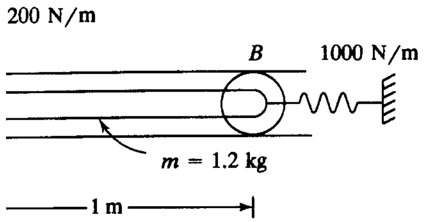
FIGURE P1.30

1.31. The slender rod of Fig. P1.31 is released from the horizontal position where the spring attached at A is stretched 10 mm and the spring attached at B is unstretched.

(*a*) What is the angular acceleration of the bar immediately after it is released?

(*b*) What is the maximum angular velocity attained by the bar?

1.32. Let x be the displacement of the left end of the bar of Fig. P1.32. Let θ be the clockwise angular rotation of the bar. Express the kinetic energy of the system in terms of $\dot{x}$ and $\dot{\theta}$.



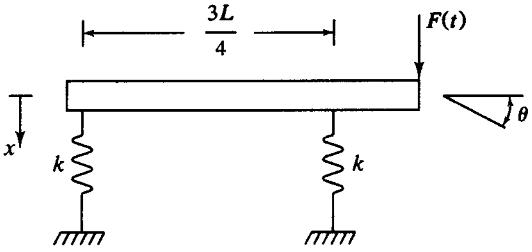


FIGURE P1.32

1.33. Express the potential energy of the system of Fig. P1.32 in terms of x and θ.

1.34. Let θ be the clockwise angular displacement of the pulley of the system of Fig. P1.34 from the system's equilibrium position.

(*a*) Express the potential energy of the system at an arbitrary instant in terms of θ.

(*b*) Express the kinetic energy of the system at an arbitrary instant in terms of $\dot{\theta}$.

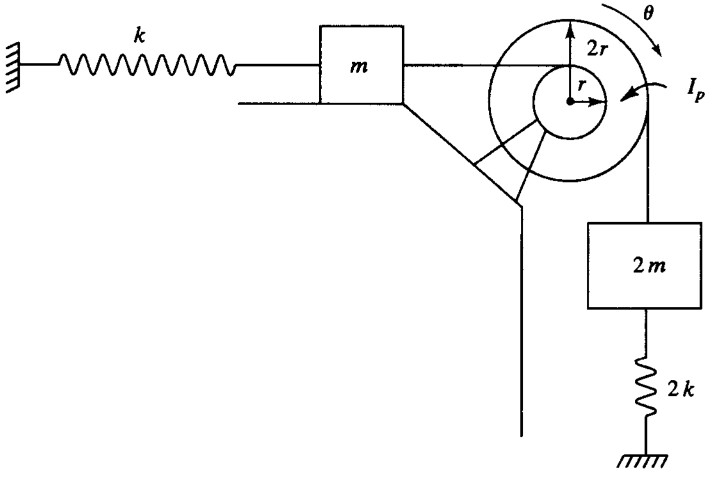


FIGURE P1.34

1.35. A 20-ton railroad car is coupled to a 15-ton car by moving the 20-ton car at 5 mph toward the stationary 15-ton car. What is the speed of the two-car coupling?

1.36. The 15-kg block of Fig. P1.36 is moving with a velocity of 3 m/s at $t = 0$ when the force $F(t)$ is applied to the block.
(*a*) Determine the velocity of the block at $t = 2$ s.
(*b*) Determine the velocity of the block at $t = 4$ s.

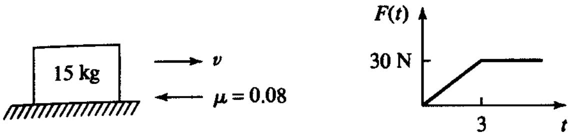


FIGURE P1.36

1.37. A 400-kg forge hammer is mounted on four identical springs, each of stiffness $k = 4200$ N/m. During the forging process, a 110-kg component is dropped from a height of 1.4 m onto an anvil. What is the maximum displacement of the machine after the impact?

1.38. The motion of a baseball bat in a ballplayer's hands is approximated as a rigid-body rotation about an axis through the player's hands (Fig. P1.38). The bat has a centroidal moment of inertia I. The player's "bat speed" is ω and the velocity of the pitched ball is v. Determine the distance from the player's hands along the bat where the batter should strike the ball to minimize the impulse felt by the player's hands.

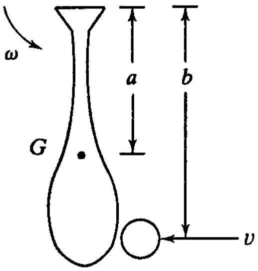


FIGURE P1.38

1.39. A playground ride has a centroidal moment of inertia of 17 slug · ft^2. Three children of weights 40 lb, 50 lb, and 55 lb are on the ride, which is rotating at 110 rpm. The children are 20 in from the center of the ride. A father stops the ride by grabbing it with his hands. What angular impulse is felt by the father?

1.40–1.46. How many degrees of freedom are required to model the systems? Identify a set of generalized coordinates which can be used to analyze the systems' vibrations.

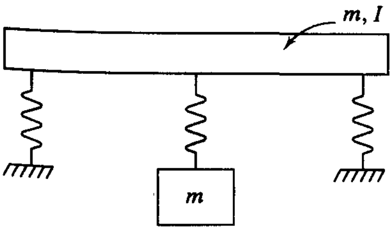
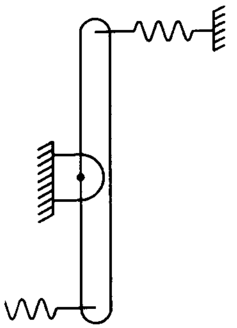


FIGURE P1.40



RE P1.41

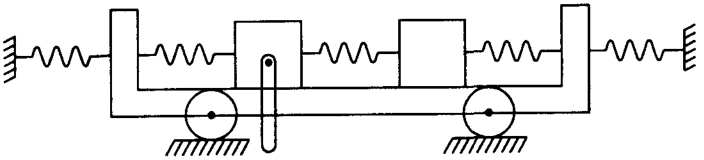
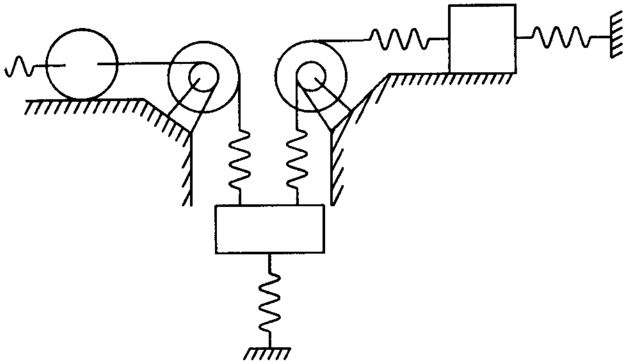


FIGURE P1.42



RE P1.43

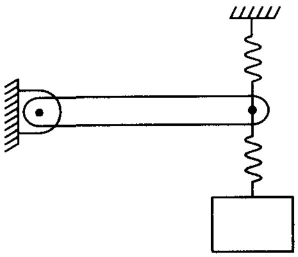
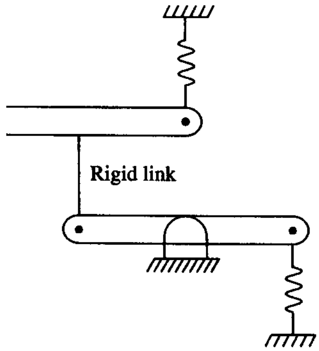


FIGURE P1.44



RE P1.45

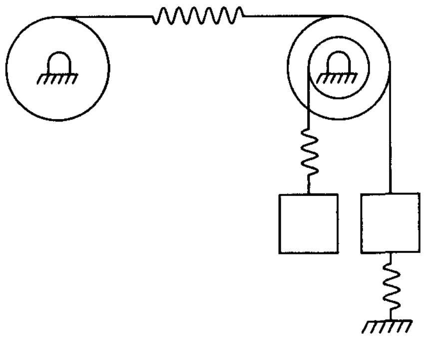


FIGURE P1.46

1.47. A 5-kg block is attached to the end of a spring with a cubic nonlinearity with $F = k_1 x + k_3 x^3$, where $k_1 = 3 \times 10^5$ N/m and $k_3 = 1 \times 10^8$ N/m^3. The other end of the spring is fixed. The block, which slides on a frictionless surface, is displaced 22 mm and released from rest. Calculate the maximum velocity attained by the block during its subsequent motion.

1.48. If the coefficient of friction between the block of Prob. 1.47 and the surface is 0.15, calcul the maximum compression of the spring if it is initially stretched 22 mm.

1.49. The block of Prob. 1.47 is hit by a 20-g particle traveling at 100 m/s After impact the parti becomes embedded in the block. What is the maximum compression of the spring?

1.50. A 10-kg mass is hung from a spring whose force-displacement relationship is $F = (2$ $10^4)x - (4 \times 10^7)x^3$ N, where x is in meters. What is the static deflection of the sprin

1.51. The ends of a 20-kg bar are connected to collars which slide along tracks as shown Fig. P1.51. The coefficient of friction between the collars and the tracks is 0.15. T collar sliding on the vertical track is to be attached to a spring such that the system is equilibrium when $\theta = 30°$ and the spring is compressed 10 cm when $\theta = 0°$. Desi a steel spring ($G = 80 \times 10^9$ N/m^2) to meet these specifications. The coil radius of t spring should be 5 cm. Specify the radius of the bar from which the spring is made and t number of active turns. If the maximum displacement from equilibrium of the collar is cm, what is the maximum shear stress developed in the spring?

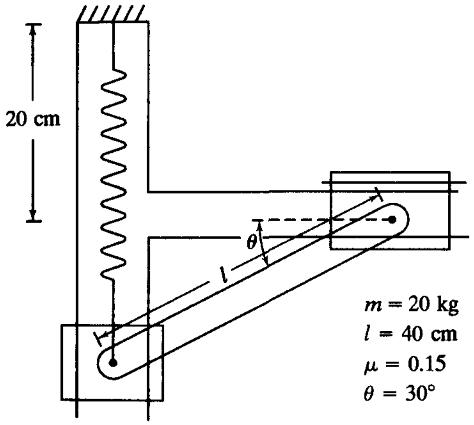


FIGURE P1.51

1.52. A 150-kg fan is to be placed in an industrial plant. The fan must be mounted on isolator in order to protect the plant floor from large forces generated during its operati Calculations show that the fan is to be placed on four springs in parallel, each with a stiffn of $k = 4 \times 10^5$ N/m. During operation the fan will have a displacement from its equilibri position given by $x(t) = 0.003 \sin(50t)$ m. Design a steel spring ($G = 80 \times 10^9$ N/m by specifying N, r, and D such that it has a stiffness of k and the maximum shear str ($\tau_{max} = 200 \times 10^6$ N/m^2) is not exceeded during operation of the fan.

1.53. A helical coil spring is made of a steel with a shear modulus of 80×10^9 N/m^2 and elastic shear strength of 200×10^6 N/m^2. The spring is to have a coil radius of 20 cm. 10-kg block is to be suspended from the spring. The block is subject to a displacement r to exceed 20 mm. Design a spring (specify the diameter of each coil and the number active turns) such that the spring's elastic strength is not exceeded.

1.54. To achieve effective isolation, a 100-kg loudspeaker system is to be mounted on springs three locations. From the vibration control theory of Chap. 8 it has been determined tl the minimum static deflection of each spring should be 2.5 mm. What is the maximu allowable stiffness of each element?

1.55–1.59. Determine the deflection of each spring from its unstretched length when the system shown is in equilibrium.

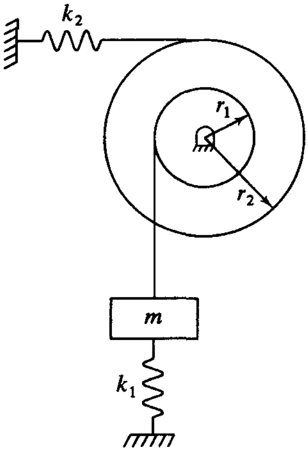


FIGURE P1.55

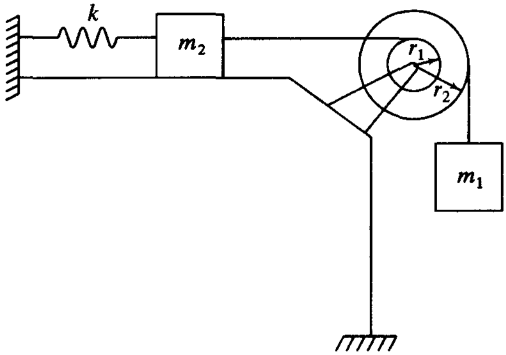


FIGURE P1.56

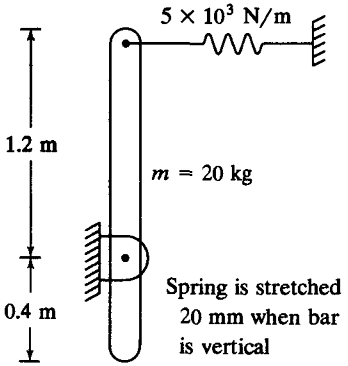


FIGURE P1.57

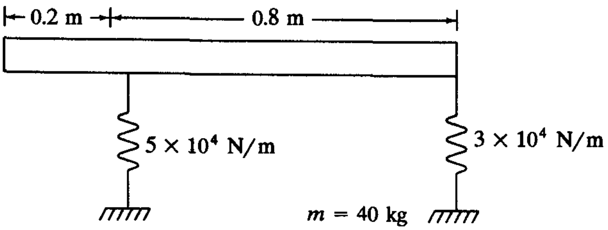


FIGURE P1.58

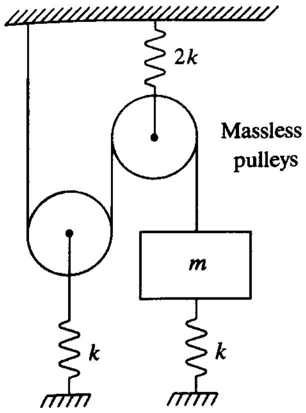


FIGURE P1.59

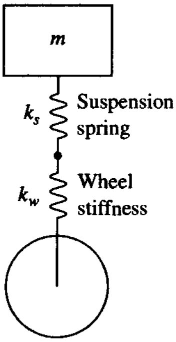


FIGURE P1.60

1.60. A simplified one-degree-of-freedom model of a vehicle suspension system is shown in Fig. P1.60. The vehicle mass is 1000 kg, which is much greater than the axle mass. The suspension spring has a stiffness of 80,000 N/m. The wheel is modeled as a spring placed in series with the suspension spring. When the vehicle is empty, the wheel deflection is measured as 4.1 cm.

(a) Determine the stiffness of the wheel, k_w.

(b) Determine the equivalent stiffness of the spring combination.

1.61. Determine the static deflection of each of the springs in the system of Fig. P1.61.

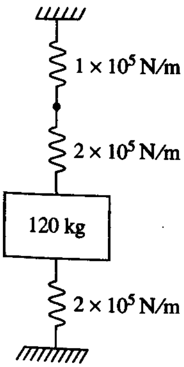


FIGURE P1.61

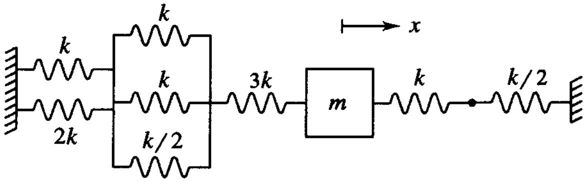


FIGURE P1.62

1.62–1.66. Calculate the equivalent stiffness of a linear spring when a linear one-degree-of-freedom mass-spring model is used to model the system shown and x is the chosen generalized coordinate.

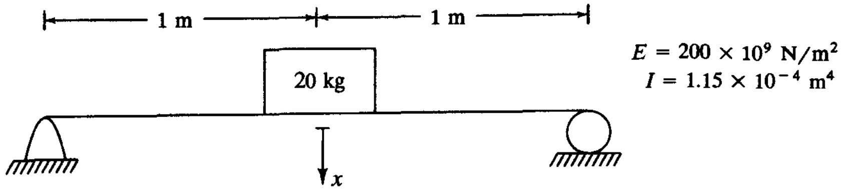


FIGURE P1.63

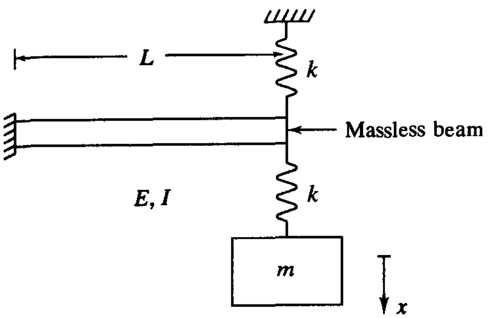


FIGURE P1.64

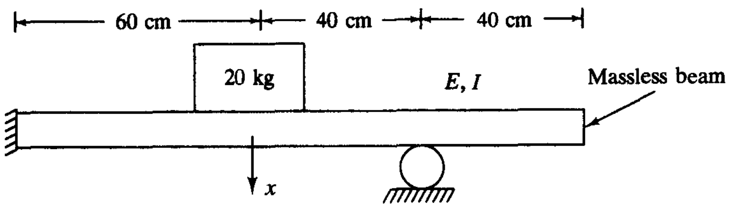


FIGURE P1.65

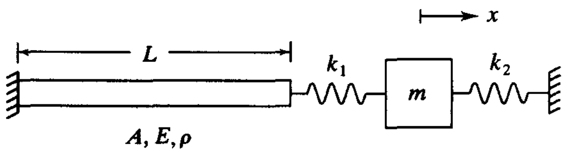


FIGURE P1.66

1.67. Calculate the torsional stiffness of a solid aluminum shaft ($G = 40 \times 10^9$ N/m^2) of length 1.8 m and radius 25 cm.

1.68. Calculate the torsional stiffness of an annular steel shaft ($G = 80 \times 10^9$ N/m^2) of length 2.5 m, inner radius 10 cm, and outer radius 15 cm.

1.69. The disk attached to the end of the circular beam of Fig. P1.69 has three degrees of freedom. The longitudinal displacement, transverse deflection, and angular rotation are kinematically independent. In fact, the degrees of freedom are also kinetically independent. For example, application of a torque does not induce longitudinal or transverse displacement of the disk. Calculate the longitudinal stiffness, torsional stiffness, and transverse stiffness for this beam.

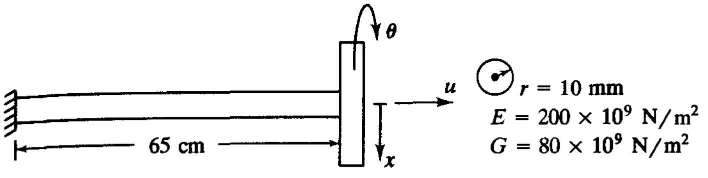


FIGURE P1.69

1.70. The propeller shaft of a ship is a tapered circular cylinder, as shown in Fig. P1.70. When installed in the ship, one end is constrained against longitudinal motion relative to the ship while a 500-kg propeller is attached to the other end.

(*a*) Determine the equivalent stiffness of the shaft for a one-degree-of-freedom model.

(*b*) Assuming a linear displacement function along the length of the shaft, determine the equivalent mass of the shaft to use in a one-degree-of-freedom model including inertia effects of the shaft.

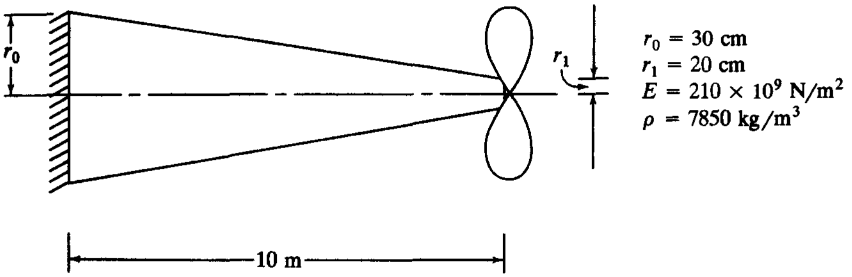


FIGURE P1.70

1.71. A tightly wound helical coil spring is made from an 1.8-mm-diameter bar of 0.2 percent hardened steel ($G = 80 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3). The spring has 80 active coils with a coil diameter of 1.6 cm. A 100-g particle is hung from the spring. Determine:

(*a*) The static deflection of the spring.

(*b*) The equivalent mass of the system, including an approximation for inertia effects of the spring.

1.72. One end of a spring of stiffness k_1 and mass m_{s1} is attached to a wall while its other end is connected to a spring of stiffness k_2 and mass m_{s2}. The end of the second spring is connected to a block of mass m which has a displacement $x(t)$. Determine the equivalent mass of these two springs in series.

1.73. A block of mass m is attached to two identical springs in series. Each spring has a mass m_s and a stiffness k. Determine the mass of a particle that should be added to the block to approximate the inertia effects of the springs.

1.74. Show that the inertia effects of a torsional shaft of mass moment of inertia J_m can be approximated by adding a thin disk of mass moment of inertia $J_m/3$ to the free end of the shaft.

1.75. Use the static deflection function of a simply supported beam to determine the mass of a particle that should be added to the mass of a machine at the midspan of the beam to approximate inertia effects of the beam.

1.76. Determine the kinetic energy of the system of Fig. P1.76 at an arbitrary instant in terms of $\dot{x}$ including inertia effects of the springs.

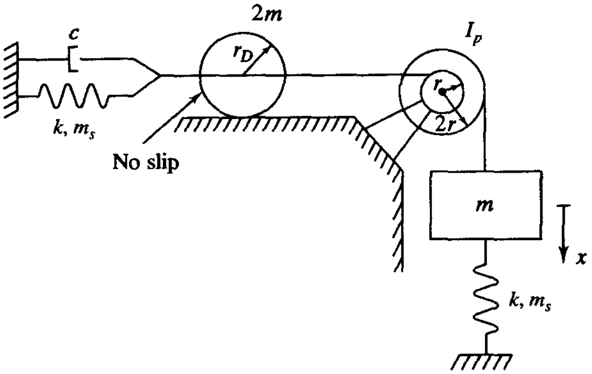


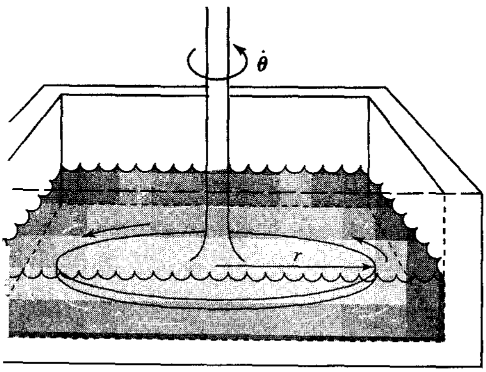
FIGURE P1.76

1.77. The time-dependent displacement of the block of mass m of Fig. P1.76 is $x(t) = 0.03e^{-1.35t}$ $\sin(4t)$ m. Determine the time-dependent force in the viscous damper if $c = 125$ N · s/m.

1.78. Calculate the work done by the viscous damper of Prob. 1.77 between $t = 0$ and $t = 1$ s.

1.79. Determine the torsional viscous damping coefficient for the torsional viscous damper of Fig. P1.79. Assume a linear velocity profile between the bottom of the dish and the disk.

1.80. Determine the torsional viscous damping coefficient for the torsional viscous damper of Fig. P1.80. Assume a linear velocity profile in the liquid between the fixed surface and the rotating cone.



Disk of radius r
Oil of density ρ, viscosity μ
Depth of oil = h

ɌE P1.79

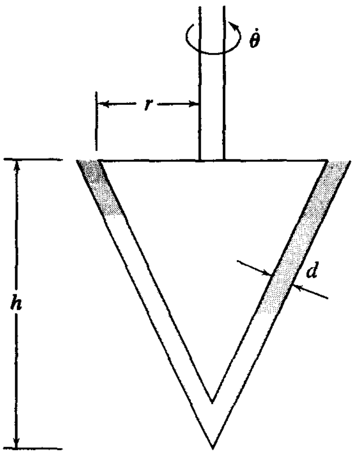


FIGURE P1.80

1.81. Shock absorbers and many other forms of viscous dampers use a piston moving in a cylinde of viscous liquid as illustrated in Fig. P1.81. For this configuration the force developed o the piston is the sum of the viscous forces acting on the side of the piston and the force du to the pressure difference between the top and bottom surfaces of the piston.

(*a*) Assume the piston moves with a constant velocity v_p. Draw a free-body diagram of th piston and mathematically relate the damping force, the viscous force, and the pressur force.

(*b*) Assume steady flow between the side of the piston and the side of the cylinder. Sho that the equation governing the velocity profile between the piston and the cylinder i

$$\frac{dp}{dx} = \mu \frac{\partial v^2}{\partial r^2}$$

(*c*) Assume the vertical pressure gradient is constant. Use the preceding results to dete mine the velocity profile in terms of the damping force and the shear stress on the sid of the piston.

(*d*) Use the results of part (*c*) to determine the wall shear stress in terms of the dampin force.

(*e*) Note that the flow rate between the piston and the cylinder is equal to the rate at whic liquid is displaced by the piston. Use this information to determine the damping forc in terms of the velocity and thus the damping coefficient.

(*f*) Use the results of part (*e*) to design a shock absorber for a motorcycle that uses SA 1040 oil and requires a damping coefficient of 1000 N · m/s.

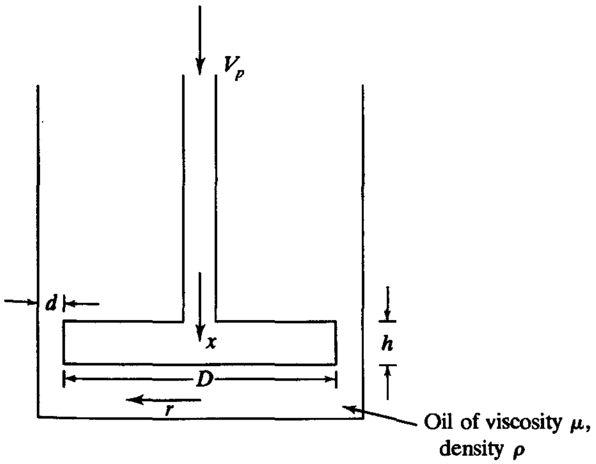


FIGURE P1.81

1.82. The spring of Fig. P1.82 (see page 58) is unstretched in the position shown. What is tl deflection of the spring when the system is in equilibrium?

1.83. A 20 mm × 20 mm × 80 mm block is attached to a spring of stiffness 5 × 10^4 N/m. Tl assembly is immersed in a liquid of specific gravity 1.05. What is the added mass requir to approximate the inertia of the liquid?

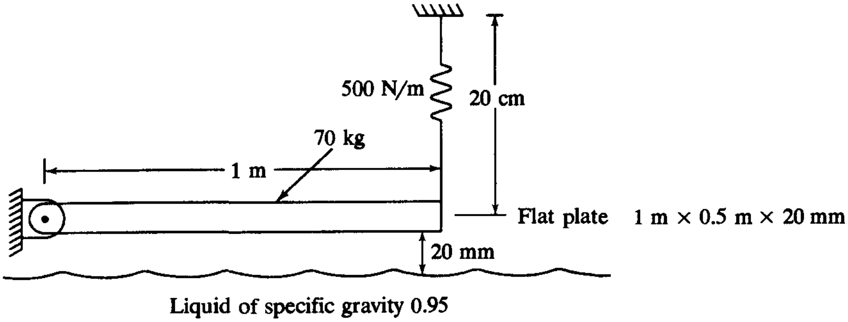


FIGURE P1.82

1.84. A wedge is floating stably on the interface between a liquid of mass density ρ, as shown in Fig. P1.84. When the wedge is disturbed from equilibrium, let x be the displacement of the wedge's mass center.

(*a*) What is the buoyant force acting on the wedge?

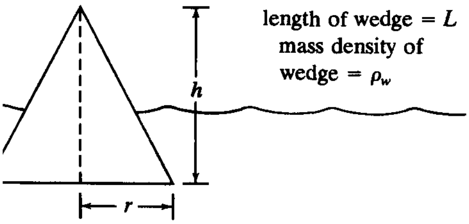
(*b*) What is the work done by the buoyant force as the mass center of the wedge moves from x_1 to x_2?

(*c*) Can the oscillations of the wedge on the surface be modeled as a mass attached to a linear spring?

1.85. A bar of length L and cross-sectional area A is made of a material whose stress-strain diagram is shown in Fig. P1.85. If the internal force developed in the bar is such that $\sigma < \sigma_p$, then the bar's stiffness for a one-degree-of-freedom model is given by Eq. (1.40). Consider the case where $\sigma > \sigma_p$. Let $P = \sigma_p A + \delta P$ be the applied load which results in a deflection $\Delta = \sigma_p L/E + \delta\Delta$.

(*a*) The work done ($W = P\Delta/2$) by the applied force is equal to the strain energy developed in the bar. The strain energy per unit volume is the area under the stress-strain curve. Use this information to relate δP to $\delta\Delta$.

(*b*) What is an approximation to the linear stiffness for small $\delta\Delta$?



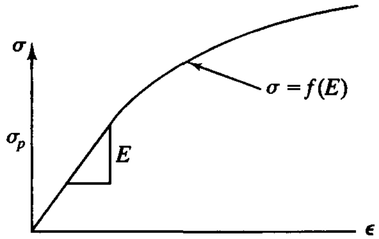


FIGURE P1.85

1.86. Consider a solid circular shaft of length L and radius c made of an elastoplastic material whose shear stress—shear strain diagram is given in Fig. P1.86. If the applied torque is such that the shear stress at the outer radius of the shaft is less than τ_p, a linear relationship exists between the torque and the angular displacement resulting in Eq. (1.46). When the applied torque is large enough to cause plastic behavior, a plastic shell develops around an elastic core of radius $r < c$. Let $T = \pi\tau_p c^3/2 + \delta T$ be the applied torque which results in an angular displacement of $\theta = \tau_p L/(cG) + \delta\theta$.

(*a*) The shear strain at the outer radius of the shaft is related to the angular displacement by $\theta = \gamma_c L/c$. The shear strain distribution is linear over a given cross section. Show that this implies

$$\theta = \frac{\tau_p L}{rG}$$

(*b*) The torque is the resultant moment of the shear stress distribution over the cross section of the shaft,

$$T = \int_0^c 2\pi\tau\rho^2\, d\rho$$

Use this to relate the torque to the radius of the elastic core.

(*c*) Determine the relationship between δT and $\delta\theta$.

(*d*) Determine a linear approximation to the stiffness for small $\delta\theta$.

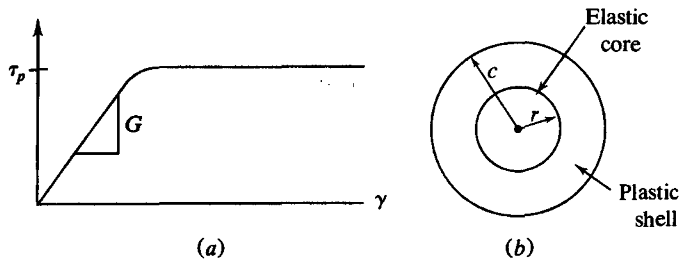


FIGURE P1.86

1.87. A gas spring consists of a piston of area A moving in a cylinder of gas. As the cylinder moves, the gas expands and contracts changing the pressure exerted on the piston. If the process occurs adiabatically (without heat transfer), then

$$p = C\rho^{\gamma}$$

where p is the gas pressure, ρ is the gas density, γ is the constant ratio of specific heats, and C is a constant dependent on the initial state. Consider a spring where the initial pressure is p_0 at a temperature T_0. At this pressure the height of the gas column in the cylinder is h. Let $F = p_0 A + \delta F$ be the pressure force on the piston when the piston has displaced a distance x into the gas from its initial height.

(*a*) Determine the relation between δF and x.

(*b*) Linearize the relationship of part (*a*) to approximate the air spring by a linear spring. What is the equivalent stiffness of the spring?

(*c*) What is the required piston area for an air spring ($\gamma = 1.4$) to have a stiffness of 300 N/m for a pressure of 150 kPa (absolute) with $h = 30$ cm?

MATLAB PROBLEMS

M1.1. File VIBES_1A.m provides the solution of Example 1.5 in terms of the parameters of the problem. Use the program to

(*a*) Investigate the dependence of $\theta_{\max}$ and α on a/L.

(*b*) Investigate the dependence of $\theta_{\max}$ and α on m_p/m where m_p is the mass of the particle.

M1.2. A 500-kg machine is to be mounted on four identical helical coil springs in parallel. The static deflection of each spring is required to be 3.6 mm. As the machine vibrates, its maximum displacement from its equilibrium position will be 10 mm. Design the springs such that the ratio of the coil diameter to the rod diameter is at least 10 and the maximum yield shear stress of the spring is not exceeded during operation of the machine. Use file VIBES_1B.m to design the spring by specifying the material from which the spring should be made, the coil diameter, the rod diameter, and the number of active turns.

M1.3. File VIBES_1C.m provides the stiffness of a fixed-free beam at a location a distance a from the fixed support. Use VIBES_1C.m to

(*a*) Design the cross section of a fixed-free steel beam of length 4.5 m such that its stiffness is between 1×10^5 and 4.5×10^5 N/m at $a = 1.8$ m.

(*b*) Plot the stiffness of the beam designed in (*a*) as a function of a for $0.5 \text{ m} < a < 4 \text{ m}$.

M1.4. File VIBES_1D.m provides the equivalent mass of a fixed-free beam at a location a distance a from the fixed support. Use VIBES_1D.m to design the cross section of a fixed-free steel beam of length 4.1 m such that the equivalent mass of the beam is less than 3 kg at $a = 1.5$ m.

M1.5. Write a MATLAB program to determine the maximum acceleration of a particle whose displacement is

$$x(t) = 0.003e^{-1.5t} \sin(14t + 0.14) \text{ m}$$

Plot the displacement of the particle, the velocity of the particle, and the acceleration of the particle as functions of time.

M1.6. Write a MATLAB program to determine the stiffness of a simply supported beam of length L, elastic modulus E, and cross-sectional moment of inertia I, at a distance a from the left support. Use the program to design a simply supported aluminum beam of length 5 m that has a stiffness between 4×10^4 and 1×10^5 N/m when a 500-kg machine is mounted on the beam a distance 2 m from the left support.

M1.7. Write a MATLAB program to determine the equivalent mass of a simply supported beam of length L, mass density ρ, cross-sectional area A, and elastic modulus E, at a distance a from the left support. Use the program to design an aluminum beam of length 5 m that has an equivalent mass of less than 5 kg when a 150-kg machine is mounted on the beam a distance 2 m from the left support.

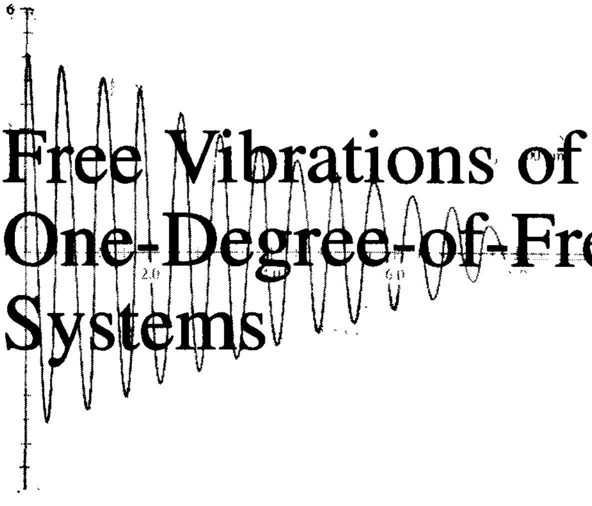
M1.8. The damping coefficient of a piston-cylinder type damper is

$$c = \mu\left[\frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)\right]$$

where μ is the dynamic viscosity of the fluid, D is the piston diameter, d is the clearance between the piston and the cylinder, and l is the length of the piston head. Write a MATLAB program that determines the damping coefficient of the piston-cylinder damper, given values of the parameters. Use the program to design a shock absorber using SAE 1040 oil to provide a damping coefficient of 1000 N · m/s. The maximum piston diameter is 8 cm.

chapter

2 Free Vibrations of One-Degree-of-Freedom Systems



2.1 INTRODUCTION

Free vibrations are oscillations about a system's equilibrium position that occur in the absence of an external excitation. Free vibrations are a result of a kinetic energy imparted to the system or of a displacement from the equilibrium position that leads to a difference in potential energy from the system's equilibrium position.

Consider the model one-degree-of-freedom system of Fig. 2.1. When the block is displaced a distance x_0 from its equilibrium position, a potential energy $kx_0^2/2$ is developed in the spring. When the system is released from equilibrium, the spring force draws the block toward the system's equilibrium position, with the potential energy being converted to kinetic energy. When the block reaches its equilibrium position, the kinetic energy reaches a maximum and motion continues. The kinetic energy is converted to potential energy until the spring is compressed a distance x_0. This process of transfer of potential energy to kinetic energy and vice versa is continual in the absence of nonconservative forces. In a physical system, such perpetual motion is impossible. Dry friction, internal friction in the spring, aerodynamic drag, and other nonconservative mechanisms eventually dissipate the energy.

Examples of free vibrations of systems that can be modeled using one degree of freedom include the oscillations of a pendulum about a vertical equilibrium position, the motion of a recoil mechanism of a firearm once it has been fired, and the motion of a vehicle suspension system after the vehicle encounters a pothole.

Free vibrations of a one-degree-of-freedom system are described by a homogeneous second-order ordinary differential equation. The independent variable is time, while the dependent variable is the chosen generalized coordinate. The chosen generalized coordinate represents the displacement of a particle in the system or an angular displacement and is measured from the system's equilibrium position.

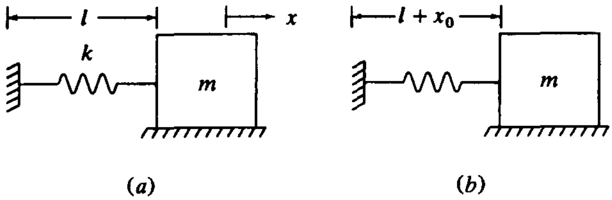


Figure 2.1 (a) When the mass-spring system is at rest in equilibrium the spring has an unstretched length l; (b) when the mass is displaced a distance x_0, a force kx_0 and potential energy $\frac{1}{2}kx_0^2$ develops in the spring.

Two methods are introduced to derive the differential equation governing the motion of a one-degree-of-freedom system: the free-body diagram method introduced in Chap. 1 and the equivalent systems method, which is based on the principle of work and energy. If a system is nonlinear, a linearizing assumption will be made.

The general solution of the second-order differential equation is a linear combination of two linearly independent solutions. The arbitrary constants, called *constants of integration*, are uniquely determined on application of two initial conditions. The necessary initial conditions are values of the generalized coordinate and its first time derivative at a specified time, usually $t = 0$.

The form of the solution of the differential equation depends on system parameters. For example, the mathematical form of the solution for an undamped system is different from the solution for a system with viscous damping. Solutions are examined for all possible values of the parameters. Examples and applications are presented.

2.2 FREE-BODY DIAGRAM METHOD

Newton's laws, as formulated in Chap. 1, are applied to free-body diagrams of vibrating systems to derive the governing differential equation. The following steps are used in application to a one-degree-of-freedom system.

1. A generalized coordinate is chosen. This variable should represent the displacement of a particle in the system. If rotational motion is involved, the generalized coordinate could represent an angular displacement.
2. Free-body diagrams are drawn showing the system at an arbitrary instant of time. In line with the methods of Sec. 1.4, two free-body diagrams are drawn. One free-body diagram shows all external forces acting on the system. The second free-body diagram shows all effective forces acting on the system. Recall that the effective forces are a force equal to $m\bar{a}$, applied at the mass center and a couple equal to $\bar{I}\alpha$.

The forces drawn on each free-body diagram are annotated for an arbitrary instant. The direction of each force and moment are drawn consistent with the positive direction of the generalized coordinate. Geometry, kinematics, constitutive equations, and other laws valid for specific systems can be used to specify the external and effective forces.

3. The appropriate form of Newton's law is applied to the free-body diagrams.
4. Applicable assumptions are used along with algebraic manipulation. The result is the governing differential equation.

Examp

Derive the differential equation governing the motion of the block of Fig. 2.2*a*.

Solution:

Let $x(t)$ represent the displacement of the block, measured positive downward, from its static-equilibrium position. Free-body diagrams showing the external and effective forces acting on the block at an arbitrary instant of time are given in Fig. 2.2*b*. Thus the force developed in the spring is given by Eq. (1.30) where x, in that equation, represents the change in length of the spring from its unstretched length. Since x is measured from the static-equilibrium position of the system, the spring force developed for the system of Fig. 2.2 is

$$F_s = k(x + \Delta_{\text{st}})$$

where Δ_{st} is the static deflection of the spring.

Note that since x is measured positive downward, when x is positive the spring is stretched further from its equilibrium position and pulls on the mass, as illustrated on the free-body diagram of external forces. The effective force is drawn downward to be consistent with the choice of positive x.

The appropriate form of Newton's law for this problem is

$$\sum F_{\text{ext}} = \sum F_{\text{eff}}$$

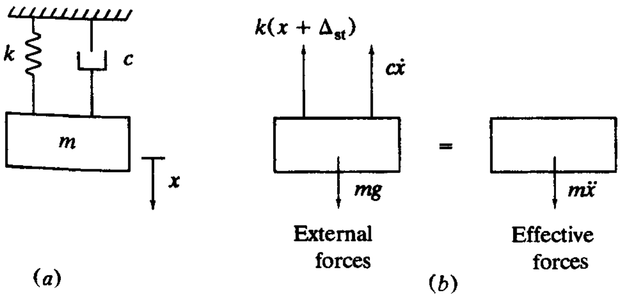


Figure 2.2 (*a*) Mass-spring-dashpot system of Example 2.1; (*b*) free-body diagrams at an arbitrary instant. Directions of external and effective forces are consistent with positive direction of generalized coordinate x.

which when applied to the free-body diagrams of Fig. 2.2*b* gives

$$mg - k(x + \Delta_{st}) - c\dot{x} = m\ddot{x}$$

Analysis of the static-equilibrium position reveals

$$\Delta_{st} = \frac{mg}{k}$$

When this result is substituted into the previous equation, the static-deflection term cancels with the gravity term leaving

$$m\ddot{x} + c\dot{x} + kx = 0$$

The time history of motion of the system in Fig. 2.2 is obtained by solving the preceding second-order linear homogeneous ordinary differential equation subject to appropriate initial conditions.

[Exam]ple 2.2 Derive the differential equation governing the angular oscillations of the compound pendulum of Fig. 2.3*a*.

Let $\theta(t)$ be the counterclockwise angular displacement of the rod measured from its vertical equilibrium position. Summing moments about O using the free-body diagrams of Fig. 2.3*b*,

$$\left(\sum \overset{\curvearrowleft}{\underset{+}{M}}_O\right)_{\text{ext}} = \left(\sum \overset{\curvearrowleft}{\underset{+}{M}}_O\right)_{\text{eff}}$$

yields

$$-mg\frac{L}{2}\sin\theta = m\frac{L^2}{12}\ddot{\theta} + m\frac{L}{2}\ddot{\theta}\frac{L}{2}$$

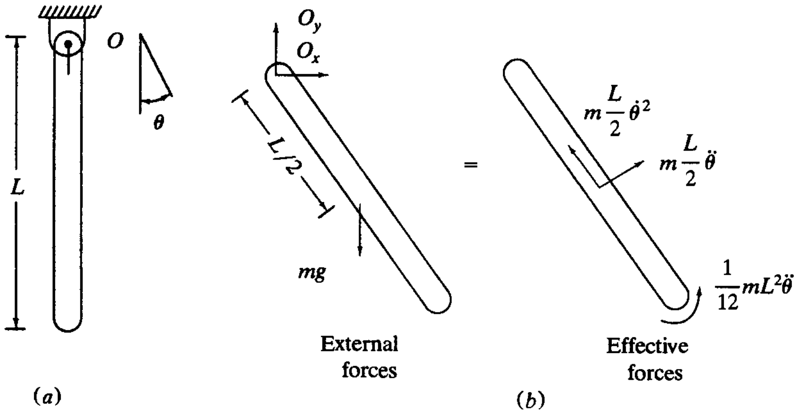


Figure 2.3 (*a*) Compound pendulum of Example 2.2 is a slender rod pinned at one end. The generalized coordinate θ is the counterclockwise angular displacement from equilibrium; (*b*) free-body diagrams at an arbitrary instant of time.

which becomes

$$m\frac{L^2}{3}\ddot{\theta} + mg\frac{L}{2}\sin\theta = 0$$

The differential equation obtained in Example 2.2 is a second-order nonlinear ordinary differential equation. While an exact solution of this equation exists in terms of elliptic integrals, exact solutions for most nonlinear equations have not been found.

Approximations for solutions of problems governed by nonlinear differential equations are obtained by one of two approaches. An approximate solution of the exact equation can be obtained by a numerical method, or, if conditions are right, the differential equation can be approximated by a linear equation whose exact solution is easily obtained. The latter approach is used here.

Consider the Taylor series expansion for $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots$$

For small θ,

$$\sin\theta \approx \theta$$

Similar truncations of the Taylor series expansions for $\cos\theta$ and $\tan\theta$ for small θ lead to

$$\cos\theta \approx 1 \qquad \tan\theta \approx \theta$$

The small-angle approximations of the Taylor series expansions are used to linearize nonlinear differential equations. When the small-angle assumption is made for Example 2.2, the resulting linearized differential equation is

$$\ddot{\theta} + \frac{3g}{2L}\theta = 0$$

Exam

A flywheel of mass moment of inertia I is attached to the end of a solid circular shaft of radius r, length L, shear modulus G, and mass m, as shown in Fig. 2.4a. A moment is applied to the disk, rotating it from its static-equilibrium configuration. The disk is released and torsional oscillations about the equilibrium position ensue. Derive the differential equation governing the torsional oscillations. Include an approximation for inertia effects of the shaft.

Solution:

Let $\theta(t)$ be the angular displacement of the disk from its equilibrium position. The shaft acts as a torsional spring. As the disk oscillates, a moment

$$M = \frac{JG}{L}\theta$$

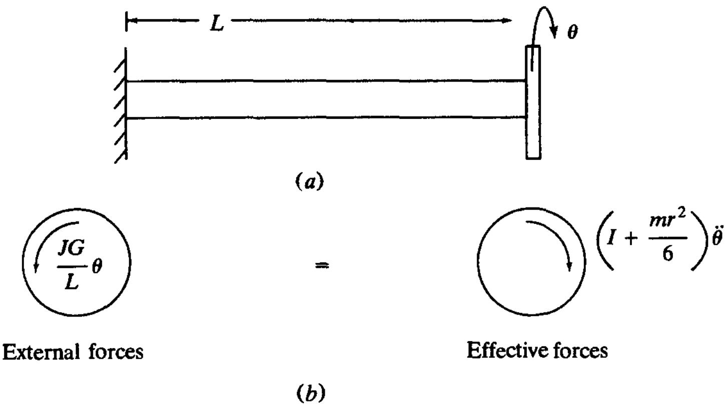


Figure 2.4 (a) The system of Example 2.3 is a flywheel of mass moment of inertia I attached to the end of a shaft of length L, radius r, and mass m; (b) free-body diagrams at an arbitrary instant.

is developed between the shaft and the disk. The moment acting on the shaft is in the direction of the rotation. From Newton's third law, the moment from the shaft on the disk resists the rotation.

Using the results of Sec. 1.6.5, the inertia effects of the shaft are approximated by adding a disk at the end of the shaft whose mass moment of inertia is one-third of the mass moment of inertia of the shaft.

$$I_{D_1} = \tfrac{1}{6}mr^2$$

Summing moments about the center of the disk

$$\left(\sum \overset{\curvearrowright +}{M_C}\right)_{\text{ext}} = \left(\sum \overset{\curvearrowright +}{M_C}\right)_{\text{eff}}$$

using the free-body diagrams of Fig. 2.4b leads to

$$-\frac{JG}{L}\theta = \left(I + \frac{mr^2}{6}\right)\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{JG}{\left(I + \dfrac{mr^2}{6}\right)L}\theta = 0$$

4 A slender rod of length L and mass m is pinned at O, as shown in Fig. 2.5. A spring of stiffness k is connected to the rod at point P while a dashpot of damping coefficient c is connected at point Q. Assuming small displacements, derive a linear differential equation governing the free vibrations of this system. Use x, the displacement of particle P, measured from the system's equilibrium position, as the generalized coordinate.

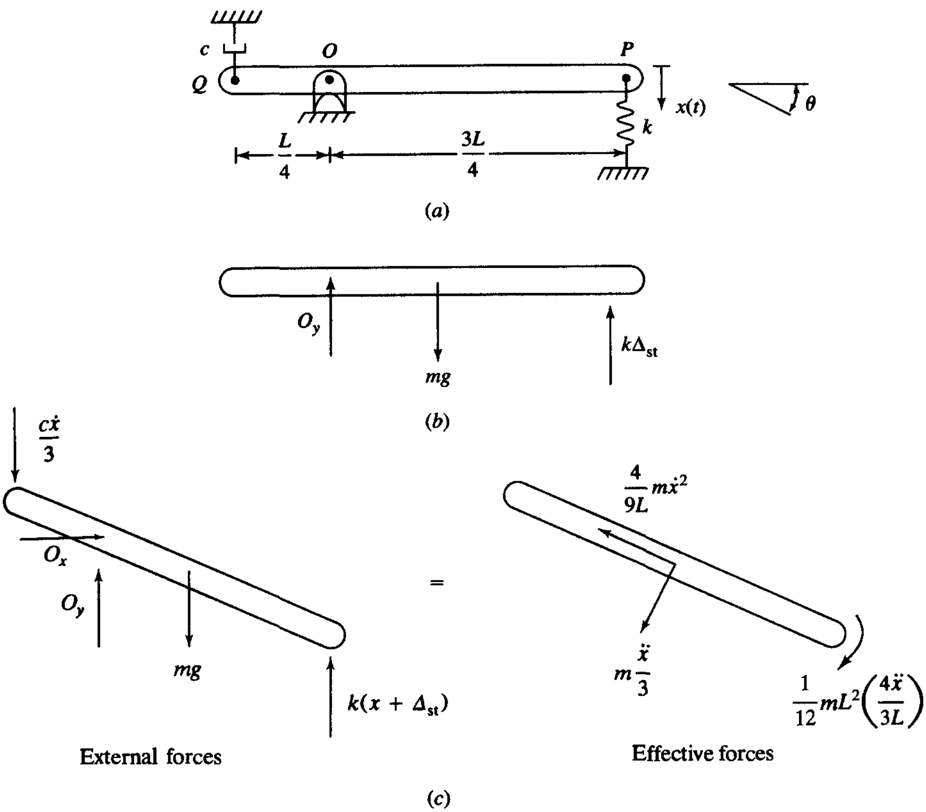


Figure 2.5 (*a*) System of Example 2.4; (*b*) free-body diagram of static equilibrium position; (*c*) free-body diagrams at an arbitrary instant.

Solution:

Summing moments about O on the free-body diagram of Fig. 2.5*b* leads to the following equation defining the equilibrium position of the system.

$$-3k\frac{L}{4}\Delta_{st} + mg\frac{L}{4} = 0$$

Consistent with the assumption that x is small, the lines of action of the damping force and the spring force are assumed to be vertical. From the geometry of Fig. 2.5,

$$x = \frac{3L}{4}\sin\theta$$

Using the small-angle approximation, for $\sin\theta$ gives

$$x \approx \frac{3L}{4}\theta$$

The appropriate equation for summation of moments about O is

$$\left(\sum \overset{\curvearrowright}{\underset{+}{}} M_O\right)_{\text{ext}} = \left(\sum \overset{\curvearrowright}{\underset{+}{}} M_O\right)_{\text{eff}}$$

which, when applied to the free-body diagrams of Fig. 2.5c, gives

$$mg\frac{L}{4}\cos\left(\frac{4x}{3L}\right) - k(x + \Delta_{\text{st}})\frac{3L}{4}\cos\left(\frac{4x}{3L}\right) - c\frac{\dot{x}}{3}\frac{L}{4}\cos\left(\frac{4x}{3L}\right)$$
$$= \frac{mL^2}{12}\frac{4\ddot{x}}{3L} + m\frac{\ddot{x}}{3}\frac{L}{4}$$

The small-angle approximation is used to approximate the cosine terms by one. The static-equilibrium condition is used to cancel the static-deflection terms with the gravity terms. The resulting differential equation becomes

$$\frac{7m}{36}\ddot{x} + \frac{c}{12}\dot{x} + \frac{3k}{4}x = 0$$

2.5 Using x, the displacement of the block measured positive downward from its equilibrium position as the generalized coordinate, derive the differential equation governing the free vibrations of the system of Fig. 2.6a.

Solution:

The static-equilibrium position of the system is analyzed and yields the following relation between gravity and static deflection of the spring:

$$2k\Delta_{\text{st}} - mg = 0$$

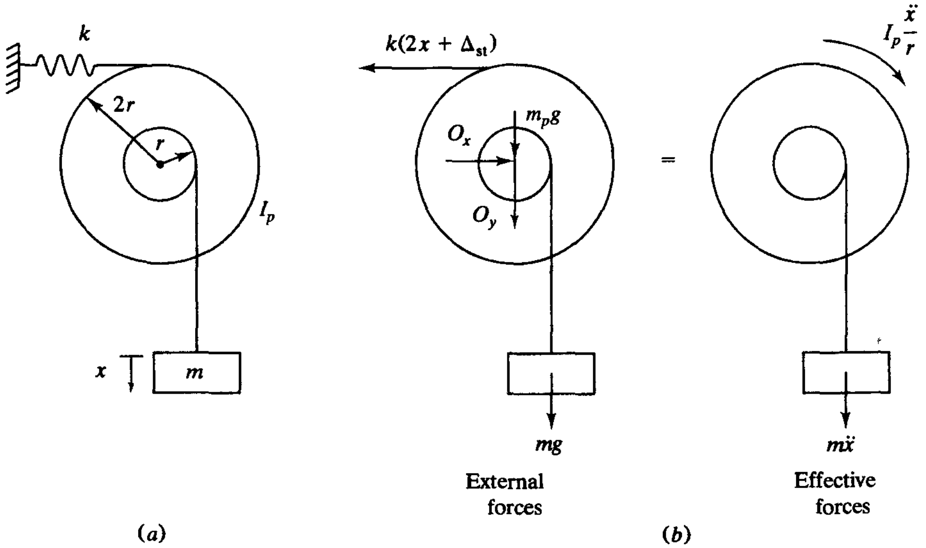


Figure 2.6 (a) System of Example 2.5; (b) free-body diagrams at an arbitrary instant.

Free-body diagrams showing the external and effective forces acting on the mass-pulley system at an arbitrary instant of time are shown in Fig. 2.6*b*. No slip is assumed between the pulley and the cables, and friction is neglected. Kinematics is used to express the relation between x and the change in length of the spring.

The appropriate form of Newton's law is

$$\left(\sum \overset{\curvearrowright}{\overset{+}{M}}_O\right)_{\text{ext}} = \left(\sum \overset{\curvearrowright}{\overset{+}{M}}_O\right)_{\text{eff}}$$

which, applied to the free-body diagrams, gives

$$-k(2x + \Delta_{\text{st}})2r + mgr = \frac{I_p}{r}\ddot{x} + m\ddot{x}r$$

The static-equilibrium condition is used to eliminate gravity and static deflection from the equation resulting in

$$\left(\frac{I_p}{r} + mr\right)\ddot{x} + 4krx = 0$$

Example

A sphere of radius r and mass m is attached to a spring of stiffness k (Fig. 2.7). The assembly is placed in a highly viscous fluid of dynamic viscosity μ and mass density ρ. The sphere is displaced from its equilibrium configuration and released from rest. Derive the differential equation governing the resulting oscillations about the equilibrium position. Note that the drag force on the sphere from the fluid is $D = 6\pi r\mu v$, where v is the velocity of the sphere.

Solution:

When the system is in equilibrium, a balance exists between the gravity force, the buoyant force, and the spring force

$$mg - F_B - k\Delta_{\text{st}} = 0$$

As the sphere oscillates, fluid surrounding the sphere is set in motion. Using the results of Sec. 1.7 and Table 1.1, the kinetic energy of the fluid can be taken into account by adding a particle of mass $m_a = \frac{3}{4}\pi\rho r^3$ to the sphere. Let $x(t)$ be the displacement of the sphere from its equilibrium position. Application of Newton's law

$$\sum F_{\text{ext}} = \sum F_{\text{eff}}$$

to the free-body diagrams of Fig. 2.7*b* gives

$$-(kx + \Delta_{\text{st}}) - 6\pi\mu r\dot{x} + mg - F_B = (m + \tfrac{3}{4}\rho\pi r^3)\ddot{x}$$

Use of the static-equilibrium condition eliminates gravity, buoyancy, and static spring force terms and leads to

$$(m + \tfrac{3}{4}\rho\pi r^3)\ddot{x} + 6\pi\mu r\dot{x} + kx = 0$$

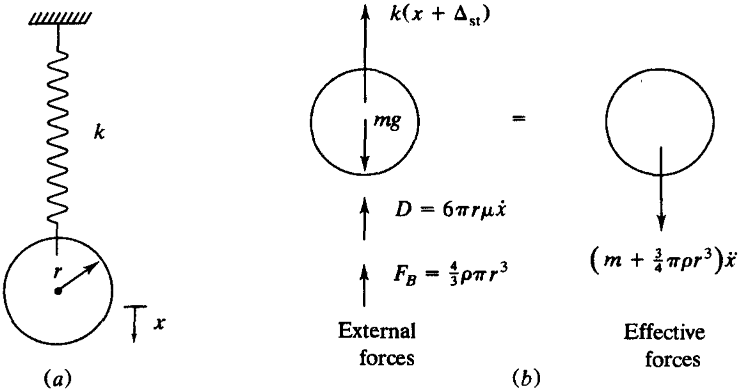


Figure 2.7 (a) Solid sphere of radius r is suspended from spring of stiffness k in a fluid of mass density ρ and viscosity μ; (b) free-body diagrams at an arbitrary instant. Effective force includes added mass to account for inertia effects of entrained fluid.

A static deflection is developed in a spring whenever a spring force is required to balance an external force such as gravity or buoyancy for the system to exist in equilibrium. A static analysis of the equilibrium condition leads to a static equilibrium condition relating the static deflection and the external force that is the cause of the static deflection. In the previous examples, and for all linear systems, the static equilibrium condition is replicated in the differential equation derived from applying conservation laws to the free-body diagrams drawn for an arbitrary instant. The static equilibrium condition is subtracted from the differential equation. Thus the static deflection developed in a spring will always cancel with the external force causing the static deflection when the differential equation governing the motion of a linear system is formulated. Since this is now recognized, it can be assumed a priori, and thus the static deflection and the external force causing the static deflection will no longer be included in the analysis used to derive the differential equation governing the motion of a linear system.

In many systems the force system composed of the static spring force and the external force causing the static deflection is equivalent to the zero force. Consider, though, the system of Example 2.4. The moment of the gravity force balances with the moment of the static spring force. However the summation of the two forces is not the zero force. Even though these forces cancel with one another when moments are summed to derive the governing differential equation, they do not cancel when forces are summed to determine the reactions at the pin supports.

2.7 Derive the differential equation governing the motion of the system of Fig. 2.8, using x, the downward displacement of pulley A from the system's equilibrium position as the generalized coordinate.

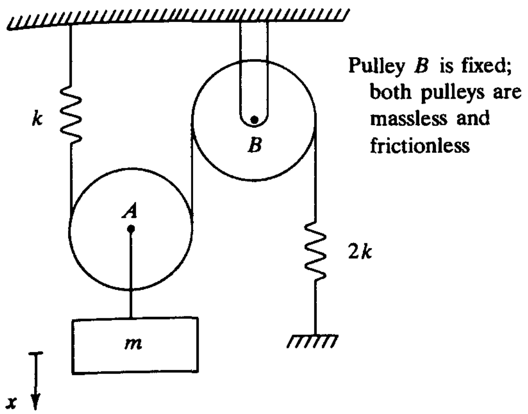


Figure 2.8 System of Example 2.7.

Solution:

The total change in length of the cable from its equilibrium length is $2x$. Let x_1 be the change in length of the spring connected to pulley A. Let x_2 be the change in length of the spring connected to pulley B. Then

$$2x = x_1 + x_2$$

Free-body diagrams of the pulleys at an arbitrary instant are shown in Fig. 2.9. Note that for this linear system, gravity will cancel with the static spring force from the governing differential equation and thus neither is included on the free-body diagrams.

Assuming no friction the tension is the same throughout the cable. This implies that the force developed in each spring is the same

$$kx_1 = 2kx_2$$

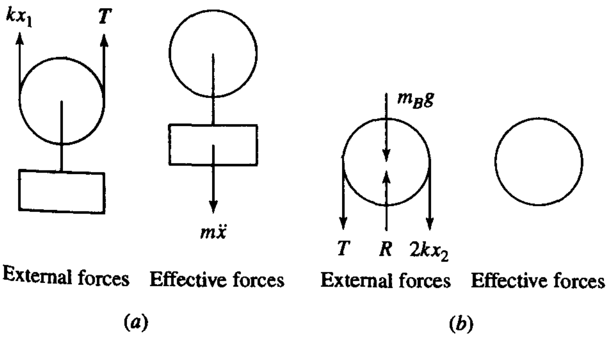


Figure 2.9 Free-body diagrams of (a) pulley A and (b) pulley B at an arbitrary instant. Gravity of block and static spring forces are not included, as they will cancel with one another in differential equation.

Combining the two equations leads to

$$x_1 = \tfrac{4}{3}x \qquad x_2 = \tfrac{2}{3}x$$

Summing forces on pulley A

$$\sum F_{\text{ext}} = \sum F_{\text{eff}}$$

leads to

$$-kx_1 - 2kx_2 = m\ddot{x} \qquad -k\left(\tfrac{4}{3}x\right) - 2k\left(\tfrac{2}{3}x\right) = m\ddot{x} \qquad m\ddot{x} + \tfrac{8}{3}kx = 0$$

2.3 EQUIVALENT SYSTEMS METHOD

Let x be the generalized coordinate chosen to model a linear one-degree-of-freedom system. The system is composed of n rigid bodies and particles. The total kinetic energy of the system is

$$T = \sum_{i=1}^{n} \frac{1}{2}(m_i \bar{v}_i^2 + \bar{I}_i \omega_i^2) \qquad \textbf{[2.1]}$$

Since the system has only one degree of freedom, the velocity of the mass center of each rigid body and the angular velocity of each rigid body must be kinematically related to the generalized coordinate, x

$$\bar{v}_i = \alpha_i \dot{x}_i \qquad \omega_i = \beta_i \dot{x}_i \qquad \textbf{[2.2]}$$

Substitution of Eq. (2.2) into Eq. (2.1) leads to

$$T = \frac{1}{2}\sum_{i=1}^{n}(m_i \alpha_i^2 + \bar{I}_i \beta_i^2)\dot{x}^2 \qquad \textbf{[2.3]}$$

or

$$T = \frac{1}{2} m_{\text{eq}} \dot{x}^2 \qquad \textbf{[2.4]}$$

where

$$m_{\text{eq}} = \sum_{i=1}^{n}(m_i \alpha_i^2 + \bar{I}_i \beta_i^2) \qquad \textbf{[2.5]}$$

m_{eq} is a constant called the *equivalent mass.*

Assume the linear one-degree-of-freedom system has m discrete elastic elements. The ith spring is connected to a particle in the system whose displacement is x_i. Let Δ_i represent the static deflection in the ith spring. The total potential energy of the system at an arbitrary instant includes the potential energy developed in each of the springs as well as the potential energy due to gravity

$$V = \sum_{i=1}^{m} k_i (x_i + \Delta_i)^2 + \sum_{i=1}^{n} m_i g \bar{y}_i \qquad \textbf{[2.6]}$$

where $\bar{y}_i$ is the vertical displacement of the mass center of rigid body i above its position when the system is in equilibrium. Since the system has only one degree of freedom, kinematic relations must exist between x and x_i for each spring and x and $\bar{y}_i$ for each rigid body. To this end, assume

$$x_i = \gamma_i x \qquad \bar{y}_i = \varepsilon_i x \tag{2.7}$$

Substitution of Eq. (2.7) into Eq. (2.6) leads to

$$V = \frac{1}{2}\left(\sum_{i=1}^{m} k_i \gamma_i^2\right)x^2 + \left[\sum_{i=1}^{m}(k_i \gamma_i \Delta_i) + \sum_{i=1}^{n}(\varepsilon_i m_i g)\right]x + \frac{1}{2}\sum_{i=1}^{m} k_i \Delta_i^2 \tag{2.8}$$

It can be shown, by an analysis of the system's static equilibrium position, that the term multiplying x in Eq. (2.8) is identically zero. The last term in the potential energy expression is the potential energy of the system in its equilibrium position, V_0. Equation (2.8) can be rewritten as

$$V = \tfrac{1}{2}k_{\text{eq}}x^2 + V_0 \tag{2.9}$$

where k_{eq} is a constant called the *equivalent stiffness.*

In a similar fashion it can be shown that the work done by all viscous damping forces between two arbitrary positions defined by x_1 and x_2 is

$$U_{1\to2} = -\int_{x_1}^{x_2} c_{\text{eq}}\dot{x}\,dx \tag{2.10}$$

where c_{eq} is a constant called the *equivalent viscous damping coefficient.*

The system has an initial position in which its kinetic energy is T_1 and its potential energy is V_1. The principle of work and energy is applied between the initial time and an arbitrary instant

$$\begin{gathered} T_1 + V_1 + U_{1\to2} = T + V \\ T_1 + V_1 - \int_{x_1}^{x_2} c_{\text{eq}}\dot{x}\,dx = \tfrac{1}{2}m_{\text{eq}}\dot{x}^2 + \tfrac{1}{2}k_{\text{eq}}x^2 + V_0 \end{gathered} \tag{2.11}$$

Noting that T_1, V_1, and V_0 are constants and

$$\int_{x_1}^{x_2} c_{\text{eq}}\dot{x}\,dx = \int_{t_1}^{t_2} c_{\text{eq}}\dot{x}^2\,dt \tag{2.12}$$

differentiation of Eq. (2.11) with respect to time leads to

$$-c_{\text{eq}}\dot{x}^2 = m_{\text{eq}}\ddot{x}\dot{x} + k_{\text{eq}}x\dot{x}$$

$$m_{\text{eq}}\ddot{x} + c_{\text{eq}}\dot{x} + k_{\text{eq}}x = 0 \tag{2.13}$$

Equation (2.13) is the differential equation governing the motion of any linear one-degree-of-freedom system undergoing free vibrations.

A similar analysis can be performed if the chosen generalized coordinate is an angular coordinate θ. In this case the kinetic energy of a linear one-degree-of-freedom system can be written as

$$T = \tfrac{1}{2}I_{\text{eq}}\dot{\theta}^2 \tag{2.14}$$

where I_{eq} is the *equivalent mass moment of inertia*. The potential energy for the system is

$$V = \frac{1}{2}k_{t_{eq}}\theta^2 + V_0 \tag{2.15}$$

where $k_{t_{eq}}$ is the *equivalent torsional stiffness*. The work done by viscous damping forces between two arbitrary positions defined by θ_1 and θ_2 is

$$U_{1\to 2} = -\int_{\theta_1}^{\theta_2} c_{t_{eq}}\dot{\theta}\,d\theta \tag{2.16}$$

where $c_{t_{eq}}$ is the *equivalent torsional damping coefficient*. The differential equation governing the free vibrations of a linear one-degree-of-freedom system when an angular coordinate is chosen as the generalized coordinate is

$$I_{eq}\ddot{\theta} + c_{t_{eq}}\dot{\theta} + k_{t_{eq}}\theta = 0 \tag{2.17}$$

The above analysis shows that the system of Fig. 2.10 serves as a model system for any linear one-degree-of-freedom system when a linear displacement is chosen as the generalized coordinate. The torsional system of Fig. 2.11 serves as a model system for any linear one-degree-of-freedom system when an angular displacement is chosen as the generalized coordinate.

The equivalent systems method, or the energy method can be used to derive the differential equation governing the motion of any linear one-degree-of-freedom system. The kinetic energy and the potential energy of the system are determined at an arbitrary instant in terms of the chosen generalized coordinate. The expression for the kinetic energy is compared with Eq. (2.4) from which the equivalent mass is determined. The potential energy expression is compared with Eq. (2.9) to determine the equivalent stiffness. If the system has viscous damping, the work done

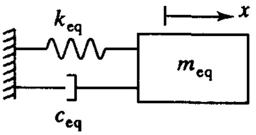


Figure 2.10 The equivalent mass-spring-dashpot system is used as a model for a linear one-degree-of-freedom linear system with viscous damping.

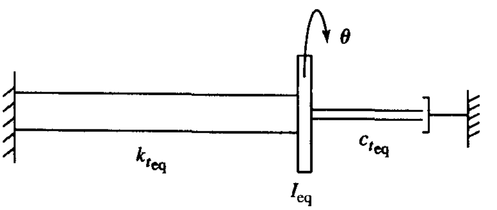


Figure 2.11 Equivalent system model of a linear one-degree-of-freedom system with angular coordinate as generalized coordinate.

by the system between two arbitrary positions is determined. This expression is compared with Eq. (2.10) to determine the equivalent viscous damping coefficient. The governing differential equation is written as Eq. (2.13). If an angular coordinate is chosen as the generalized coordinate, then Eqs. (2.14) to (2.17) are used.

The quadratic form of the potential energy for a linear system, Eq. (2.9), was derived assuming that linear springs were the only source of potential energy. Equation (2.9) is applicable for any linear system and can be used to determine the equivalent stiffness when any source of potential energy is present. Examples of other sources of potential energy include buoyancy forces and gravity forces when a small displacement assumption is applied. It is not necessary to determine V_0, the potential energy of the system when it is in equilibrium, as it does not appear in the governing differential equation, Eq. (2.13).

Use the equivalent systems method to derive the differential equation governing the motion of the system of Fig. 2.5 and Example 2.4, using x as the generalized coordinate and assuming small displacements.

Solution:

The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}m\left(\frac{\dot{x}}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{4}{3L}\dot{x}\right)^2 = \frac{1}{2}\left(\frac{7}{27}m\right)\dot{x}^2$$

Comparison with Eq. (2.4) leads to $m_{\mathrm{eq}} = \frac{7}{27}$ m.

The potential energy of the system at an arbitrary instant is

$$V = \tfrac{1}{2}kx^2 + V_0$$

Comparison with Eq. (2.9) leads to $k_{\mathrm{eq}} = k$.

The work done by the viscous damping force between two arbitrary positions is

$$U_{1\to2} = -\int_{x_1}^{x_2} c\frac{\dot{x}}{3}d\left(\frac{x}{3}\right) = -\int_{x_1}^{x_2} \frac{c}{9}\dot{x}\,dx$$

Comparison with Eq. (2.10) leads to $c_{\mathrm{eq}} = c/9$.

Equation (2.13) is used to write the differential equation governing the free vibrations of the system as

$$\tfrac{7}{27}m\ddot{x} + \tfrac{1}{9}c\dot{x} + kx = 0$$

Note that the differential equation derived in Example 2.4 is obtained when the above equation is multiplied by $\frac{3}{4}$.

The slender rod of Fig. 2.12 will be subject only to small displacements from equilibrium. Use the equivalent systems method to derive the differential equation governing the motion of the rod using θ, the counterclockwise angular displacement of the rod from its equilibrium position, as the generalized coordinate.

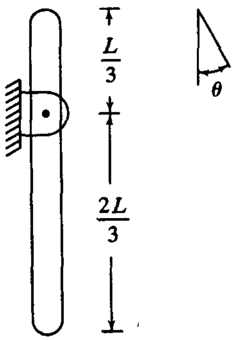


Figure 2.12 For small θ, the nonlinear system can be approximated using the linear system of Fig. 2.11 with $c_t = 0$.

Solution:

The kinetic energy of the bar at an arbitrary instant, from Eq. (1.14) is

$$T = \frac{1}{2}m\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}mL^2\right)\dot{\theta}^2$$

Comparison with Eq. (2.14) leads to $I_{\text{eq}} = mL^2/9$.

The potential energy in the system is due to gravity. Choosing the plane of the pin support as the datum, the potential energy of the system at an arbitrary instant is

$$V = -mg\frac{L}{6}\cos\theta$$

For small θ, the Taylor series expansion for $\cos\theta$ truncated after the second term leads to an approximation for the potential energy as

$$V = -mg\frac{L}{6}\left(1 - \frac{1}{2}\theta^2\right) = \frac{1}{2}mg\frac{L}{6}\theta^2 - mg\frac{L}{6}$$

Comparison with Eq. (2.15) leads to $k_{t_{\text{eq}}} = mgL/6$. Since the datum was chosen as the plane of the pin support, the system has a potential energy of $V_0 = -mgL/6$ when it is in equilibrium.

Equation (2.17) is used to write the differential equation governing the motion of the system as

$$\frac{1}{9}mL^2\ddot{\theta} + \frac{1}{6}mgL\theta = 0 \qquad \ddot{\theta} + \frac{3g}{2L}\theta = 0$$

2.10 Use the equivalent system method to derive the differential equation governing the free vibrations of the system of Fig. 2.13. Use x, the displacement of the mass center of the disk from the system's equilibrium position, as the generalized coordinate. The disk rolls without slip, no slip occurs at the pulley, and the pulley is frictionless. Include an approximation for the inertia effects of the springs. Each spring has a mass m_s.

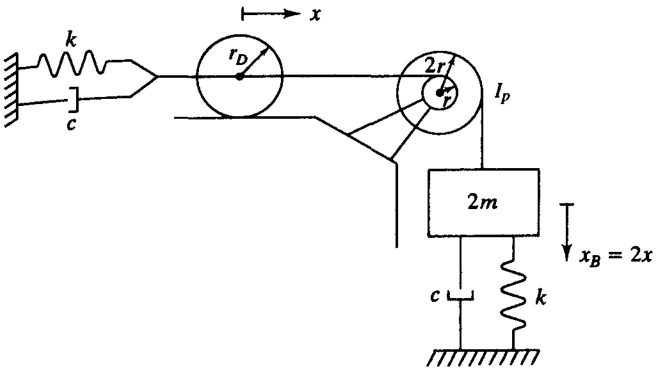


Figure 2.13 The system of Example 2.10 is modeled by the equivalent system of Fig. 2.10.

Solution:

Let θ be the clockwise angular rotation of the pulley from the system's equilibrium position and x_B be the downward displacement of the block, also measured from equilibrium. Then

$$x = r\theta \qquad x_B = 2r\theta$$

Eliminating θ between these equations leads to $x_B = 2x$. Since the disk rolls without slip, its angular velocity is $\omega_D = \dot{x}/r_D$. As shown in Sec. 1.6, the inertia effect of each spring is approximated by placing a particle of mass $m_s/3$ at the location where the spring is attached to the system. To this end it is imagined that a particle of mass $m_s/3$ is attached to the center of the disk and a particle of mass $m_s/3$ is attached to the block. The total kinetic energy of the system, including the kinetic energies of the imagined attached particles is

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_D\omega_D^2 + \frac{1}{2}I_P\dot{\theta}^2 + \frac{1}{2}(2m)\dot{x}_B^2 + T_{s_1} + T_{s_2} \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mr_D^2\right)\left(\frac{\dot{x}}{r_D}\right)^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}(2m)(2\dot{x})^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}(2\dot{x})^2 \\ &= \frac{1}{2}\left(\frac{19}{2}m + \frac{I_P}{r^2} + \frac{5}{3}m_s\right)\dot{x}^2 \end{aligned}$$

Comparison with Eq. (2.4) leads to

$$m_{\text{eq}} = \frac{19}{2}m + \frac{I_P}{r^2} + \frac{5}{3}m_s$$

The potential energy of the system at an arbitrary instant is

$$V = \tfrac{1}{2}kx^2 + \tfrac{1}{2}k(2x)^2 = \tfrac{1}{2}(5k)x^2$$

Comparison with Eq. (2.9) leads to $k_{\text{eq}} = 5k$.

The work done by the viscous dampers between two arbitrary instants is

$$U_{1\to 2} = -\int_{x_1}^{x_2} c\dot{x}\,dx - \int_{x_1}^{x_2} c(2\dot{x})\,d(2x) = -\int_{x_1}^{x_2} 5c\dot{x}\,dx$$

Comparison with Eq. (2.10) leads to $c_{eq} = 5c$.

The differential equation governing free vibration of the system is obtained by using Eq. (2.13)

$$\left(\frac{19}{2}m + \frac{I_P}{r^2} + \frac{5}{3}m_s\right)\ddot{x} + 5c\dot{x} + 5kx = 0$$

2.11 A simplified model of a rack-and-pinion steering system is shown in Fig. 2.14. A gear of radius r and polar moment of inertia J is attached to a shaft of torsional stiffness k_t. The gear rolls without slip on the rack of mass m. The rack is attached to a spring of stiffness k. Derive the differential equation governing the motion of the system using x, the horizontal displacement of the rack from the system's equilibrium position, as the generalized coordinate.

Solution:

Since there is no slip between the rack and the gear, $\theta = x/r$, where θ is the angular displacement of the gear from equilibrium. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\dot{x}^2$$

from which the equivalent mass is determined as $m_{eq} = m + J/r^2$. The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k_t\left(\frac{x}{r^2}\right) = \frac{1}{2}\left(k + \frac{k_t}{r^2}\right)x^2$$

from which the equivalent stiffness is determined as $k_{eq} = k + k_t/r^2$. The differential

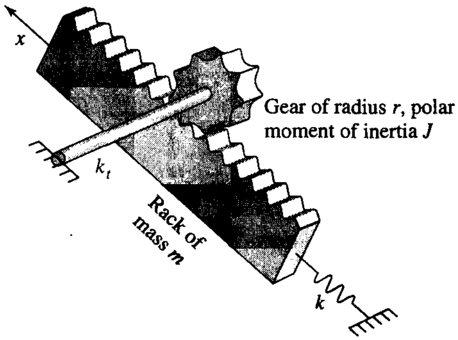


Figure 2.14 Model of rack-and-pinion system of Example 2.11.

equation governing the motion of the system is

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + \left(k + \frac{k_t}{r^2}\right)x = 0$$

2.4 FREE VIBRATIONS OF UNDAMPED ONE-DEGREE-OF-FREEDOM SYSTEMS

The general form of the differential equation for undamped free vibrations of a one-degree-of-freedom system is

$$\tilde{m}\ddot{x} + \tilde{k}x = 0 \qquad [2.18]$$

where $\tilde{m}$ and $\tilde{k}$ are coefficients specific to the system determined during the derivation of the differential equation. If the equivalent systems method is used to derive the differential equation, then $\tilde{m} = m_{eq}$ and $\tilde{k} = k_{eq}$. If the free-body diagram approach is used to derive the differential equation then, as illustrated in Example 2.8 $\tilde{m}/\tilde{k} = m_{eq}/k_{eq}$. Equation (2.18) is subject to initial conditions of the form

$$x(0) = x_0 \qquad [2.19a]$$

and

$$\dot{x}(0) = \dot{x}_0 \qquad [2.19b]$$

The solution of Eq. (2.18) subject to Eq. (2.19) is

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \qquad [2.20]$$

where

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} \qquad [2.21]$$

An alternate and more instructive form of Eq. (2.20) is

$$x(t) = A\sin(\omega_n t + \phi) \qquad [2.22]$$

Expanding Eq. (2.22) using the trigonometric identity for the sine of the sum of angles

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \qquad [2.23]$$

gives

$$x(t) = A\cos\phi \sin\omega_n t + A\sin\phi\cos\omega_n t \qquad [2.24]$$

Equating coefficients of like trigonometric terms of Eqs. (2.20) and (2.24) leads to

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \qquad [2.25]$$

and

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{\dot{x}_0}\right) \qquad [2.26]$$

The free-vibration response of a one-degree-of-freedom system, described mathematically by Eq. (2.22), is plotted in Fig. 2.15. The initial conditions determine the energy initially present in the system. Potential energy is continually converted to kinetic energy and vice versa. Since energy is conserved, the system eventually returns to its initial state with its original kinetic and potential energies, completing the first cycle of motion. The subsequent motion duplicates the previous motion. The system takes the same amount of time to execute its second cycle as it does its first. Since no energy is dissipated from the system, the system executes cycles of motion indefinitely.

A motion which exactly repeats after some time is said to be periodic. A *period* is the amount of time it takes the system to execute one cycle. The *frequency* is the number of cycles the system executes in a period of time and is the reciprocal of the period.

Figure 2.15 shows that the free undamped vibrations of a one-degree-of-freedom system are periodic of period

$$T = \frac{2\pi}{\omega_n} \qquad \textbf{[2.27]}$$

and frequency

$$f = \frac{\omega_n}{2\pi} \qquad \textbf{[2.28]}$$

The frequency of Eq. (2.28) is usually calculated in units of cycles per second or hertz (Hz). The frequency is converted into radians per second by noting

$$1 \text{ cycle/s} = 2\pi \text{ rad/s}$$

Thus ω_n is the frequency in radians per second. The parameter ω_n is called the *natural frequency* of free vibration. It is a property of the system and is determined directly

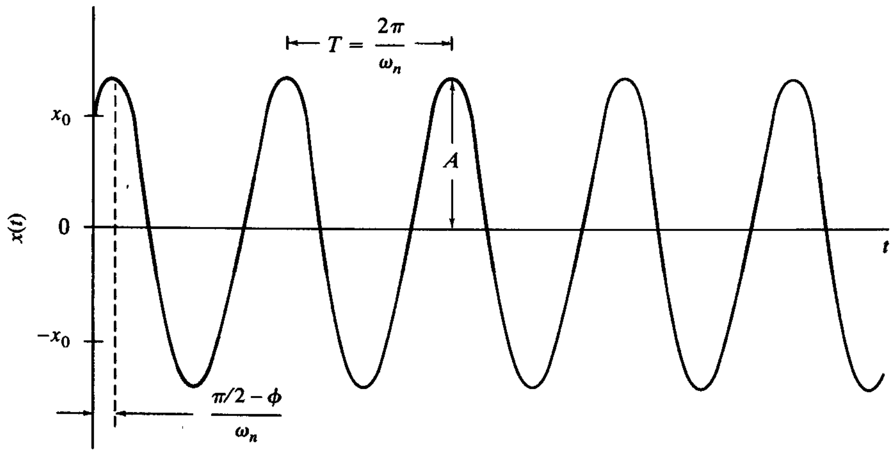


Figure 2.15 Free-vibration response of an undamped one-degree-of-freedom system is periodic of period $2\pi/\omega_n$, amplitude A, and phase angle ϕ.

from the governing differential equation and Eq. (2.21). The natural frequency is a function of system parameters and independent of initial conditions.

Equation (2.18) is divided by $\tilde{m}$, yielding

$$\ddot{x} + \omega_n^2 x = 0 \tag{2.29}$$

Equation (2.29) is the standard form of the differential equation for free vibrations of an undamped one-degree-of-freedom system. The derived differential equation for any system can be put in this standard form. The natural frequency is determined directly from the equation.

The *amplitude* A, defined by Eq. (2.25), is the maximum displacement from equilibrium. The amplitude is a function of the system parameters and the initial conditions. The amplitude is a measure of the energy imparted to the system through the initial conditions. For a linear system

$$A = \sqrt{\frac{2E}{\tilde{k}}} \tag{2.30}$$

where E is the sum of kinetic and potential energies.

The phase angle ϕ, calculated from Eq. (2.26), is an indication of the lead or lag between the response and a pure sinusoidal response. The response is purely sinusoidal with $\phi = 0$ if $x_0 = 0$. The response leads a pure sinusoidal response by $\pi/2$ rad if $\dot{x}_0 = 0$. The system takes a time of

$$t = \begin{cases} \dfrac{\pi - \phi}{\omega_n} & \phi > 0 \\[2ex] -\dfrac{\phi}{\omega_n} & \phi \le 0 \end{cases} \tag{2.31}$$

to reach its equilibrium position from its initial position.

Example 2.'

An engine of mass 500 kg is mounted on an elastic foundation of equivalent stiffness 7×10^5 N/m. Determine the natural frequency of the system.

Solution:

The system is modeled as a hanging mass-spring system. Equation (2.18) governs the displacement of the engine from its static-equilibrium position. The natural frequency is determined by using Eq. (2.21)

$$\omega_n = \sqrt{\frac{7 \times 10^5\ \text{N/m}}{500\ \text{kg}}} = 37.4\ \frac{\text{rad}}{\text{s}} = 5.96\ \text{Hz}$$

Example 2.1

A wheel is mounted on a steel shaft ($G = 83 \times 10^9$ N/m^2) of length 1.5 m and radius 0.80 cm. The wheel is rotated 5° and released. The period of oscillation is observed as 2.3 s. Determine the mass moment of inertia of the wheel.

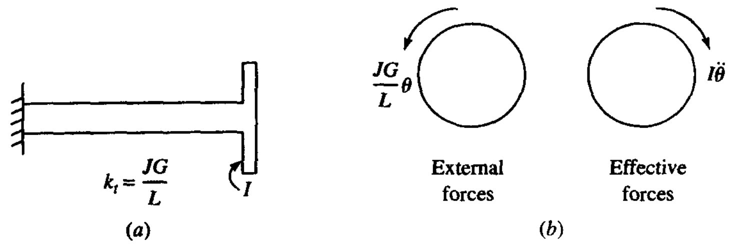


Figure 2.16 (a) Wheel is modeled as disk attached to torsional spring; (b) free-body diagrams at an arbitrary instant.

Solution:

The oscillations of the wheel about its equilibrium position are modeled as the torsional oscillations of a disk on a massless shaft, as illustrated in Fig. 2.16*a*. The differential equation is derived by using the free-body diagrams of Fig. 2.16*b*.

$$\left(\sum \overset{\curvearrowleft +}{M_O}\right)_{\text{ext}} = \left(\sum \overset{\curvearrowleft +}{M_O}\right)_{\text{eff}}$$

$$-\frac{JG}{L}\theta = I\ddot{\theta}$$

$$I\ddot{\theta} + \frac{JG}{L}\theta = \theta$$

The last equation is put in the form of Eq. (2.18) by dividing by I

$$\ddot{\theta} + \frac{JG}{IL}\theta = 0$$

Comparison of the preceding equation with Eq. (2.29) gives

$$\omega_n = \sqrt{\frac{JG}{IL}}$$

The observed natural frequency is

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi \text{ rad/cycle}}{2.3 \text{ s/cycle}} = 2.73 \frac{\text{rad}}{\text{s}}$$

Thus the moment of inertia of the wheel is calculated from

$$I = \frac{JG}{L\omega_n^2} = \frac{\frac{\pi}{2}(0.008 \text{ m})^4 \left(83 \times 10^9 \frac{\text{N}}{\text{m}^2}\right)}{(1.5 \text{ m})\left(2.73 \frac{\text{rad}}{\text{s}}\right)^2} = 47.8 \text{ kg} \cdot \text{m}^2$$

2.14 A mass of 45 kg is dropped onto the end of a cantilever beam with a velocity of 0.5 m/s, as shown in Fig. 2.17. The mass sticks to the beam and vibrates with the beam. The I beam is made of steel ($E = 210 \times 10^9$ N/m^2), is 3.5 m long, has a depth of 20 cm,

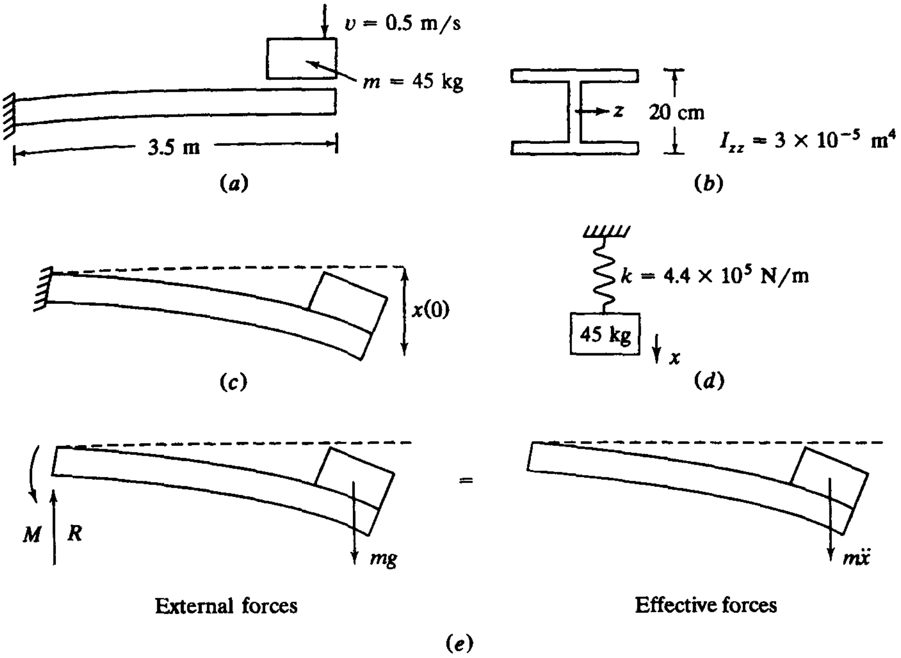


Figure 2.17 (a) System of Example 2.14. (b) Cross section of I beam. (c) $x(t)$ is measured from static-equilibrium position when mass is attached to beam. Thus $x(0) = -\Delta_{st}$. (d) Equivalent mass-spring model. (e) Free-body diagrams of an arbitrary instant.

and a cross-sectional moment of inertia of 3×10^{-5} m^4. Determine the maximum stress developed in the beam as it vibrates. Neglect the inertia of the beam.

Solution:

Let $x(t)$ be the displacement of the mass, measured positive downward from the static-equilibrium position of the mass after it is attached to the beam. The system is modeled as a mass of 45 kg hanging from a spring of stiffness

$$k_{eq} = \frac{3EI}{L^3} = \frac{3(210 \times 10^9\ \text{N/m}^2)(3 \times 10^{-5}\ \text{m}^4)}{(3.5\ \text{m})^3} = 4.41 \times 10^5\ \frac{\text{N}}{\text{m}}$$

The natural frequency of free vibrations is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4.41 \times 10^5\ \text{N/m}}{45\ \text{kg}}} = 99.0\ \frac{\text{rad}}{\text{s}}$$

The beam is undeflected at $t = 0$, when the mass strikes. If the system were in static equilibrium, the end of the beam would have a static deflection. Thus

$$x(0) = -\Delta_{st} = -\frac{mg}{k_{eq}} = -\frac{g}{\omega_n^2} = -\frac{9.81\ \text{m/s}^2}{(99.0\ \text{rad/s})^2} = -1.00 \times 10^{-3}\ \text{m}$$

The initial velocity is the velocity at which the mass strikes the beam

$$\dot{x}(0) = 0.5 \text{ m/s}$$

The time history of the mass's displacement is calculated using Eq. (2.22)

$$x(t) = A\sin(99.0t + \phi)$$

where the amplitude A and the phase angle ϕ are calculated from Eqs. (2.25) and (2.26) as

$$A = \sqrt{(-1.00 \times 10^{-3}\text{ m})^2 + \left(\frac{0.5\text{ m/s}}{99.0\text{ rad/s}}\right)^2} = 5.15 \times 10^{-3}\text{ m}$$

$$\phi = \tan^{-1}\left(\frac{(99.0\text{ rad/s})(-1 \times 10^{-3}\text{ m})}{0.5\text{ m/s}}\right) = -0.195\text{ rad} = -11.2^\circ$$

The maximum normal stress due to bending occurs in the outer fibers of the beam as its fixed end. The reaction moment at the support is obtained by summing moments on the free-body diagrams of Fig. 2.17*e*.

$$\left(\sum \overset{\frown}{\overset{+}{M}}_A\right)_{\text{ext}} = \left(\sum \overset{\frown}{\overset{+}{M}}_A\right)_{\text{eff}}$$

$$-M + mgL = m\ddot{x}L$$

The maximum acceleration is calculated by

$$\ddot{x}_{\max} = \omega_n^2 A = (99.0\text{ rad/s})^2(5.15 \times 10^{-3}\text{ m}) = 50.5\text{ m/s}^2$$

The maximum absolute value of the reaction moment is

$$\begin{aligned}|M| &= mgL + m\ddot{x}_{\max}L = 45\text{ kg}(9.81\text{ m/s}^2 + 50.5\text{ m/s}^2)(3.5\text{ m})\\ &= 9.5 \times 10^3\text{ N}\cdot\text{m}\end{aligned}$$

The maximum normal stress is calculated by using the elastic flexure formula

$$\alpha_{\max} = \frac{Mc}{I} = \frac{(9.5 \times 10^3\text{ N}\cdot\text{m})(0.1\text{ m})}{3 \times 10^{-5}\text{ m}^4} = 3.2 \times 10^7\ \frac{\text{N}}{\text{m}^2}$$

2.15 An assembly plant uses a hoist to raise and maneuver large objects. The hoist is a winch attached to a beam that can move along a track. The model of the hoist is shown in Fig. 2.18. Determine the natural frequency of the system when the hoist is used to raise a 800-kg machine part at a cable length of 9 m.

Solution:

The beam is modeled as a pinned-pinned beam. If the hoist is at its midspan, its stiffness is

$$k_b = \frac{48EI}{L^3} = \frac{48\,(200 \times 10^9\text{ N/m}^2)\,(3.5 \times 10^{-4}\text{ m}^4)}{(3.1\text{ m})^3} = 1.13 \times 10^8\ \frac{\text{N}}{\text{m}}$$

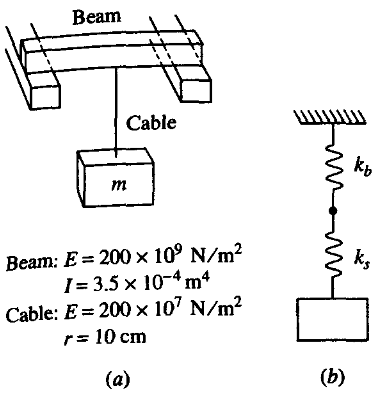


Figure 2.18 (*a*) Hoist is used to lift large objects; (*b*) beam acts in series with cable.

The stiffness of the cable is

$$k_c = \frac{AE}{L} = \frac{\pi(0.1\ \text{m})^2\,(200 \times 10^9\ \text{N/m}^2)}{9\ \text{m}} = 6.98 \times 10^8\ \frac{\text{N}}{\text{m}}$$

The beam and the cable act as springs in series with an equivalent stiffness of

$$k_{eq} = \frac{1}{\dfrac{1}{k_b} + \dfrac{1}{k_c}} = k\frac{1}{\dfrac{1}{1.13 \times 10^8\ \text{N/m}} + \dfrac{1}{6.98 \times 10^8\ \text{N/m}}} = 9.73 \times 10^7\ \frac{\text{N}}{\text{m}}$$

The system's natural frequency is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{9.73 \times 10^7\ \text{N/m}}{800\ \text{kg}}} = 3.49 \times 10^2\ \frac{\text{rad}}{\text{s}}$$

Example

The pendulum of a cuckoo clock consists of a slender rod on which an aesthetically designed mass slides. If the clock gains time, should the mass be moved closer to or farther away from the support to correct the tuning?

Solution:

The pendulum is modeled as a particle of mass m on a rigid, massless rod. The particle is assumed to be a distance l from its axis of rotation. Summing moments about the point of support on the free-body diagrams of Fig. 2.19 leads to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Application of the small-angle assumption yields the linearized equation of motion

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

The differential equation is compared with the standard form, Eq. (2.29), yielding

$$\omega_n = \sqrt{\frac{g}{l}}$$

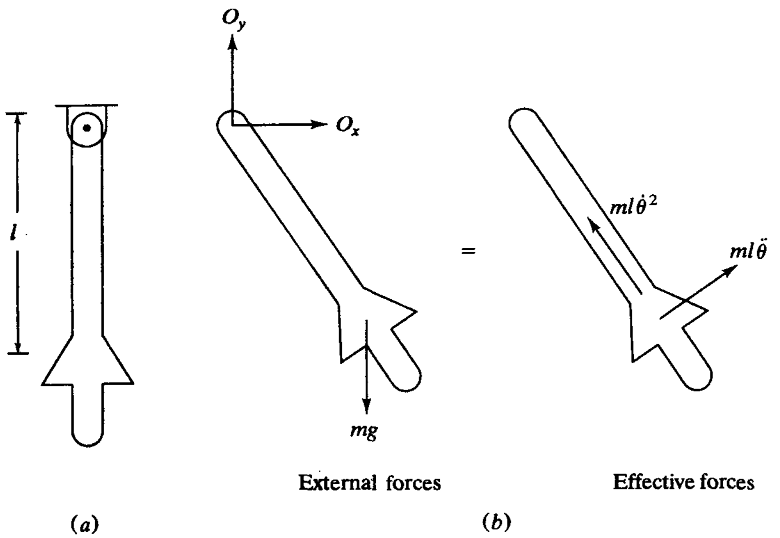


Figure 2.19 (a) Pendulum of cuckoo clock; (b) free-body diagrams at an arbitrary instant.

The period of oscillation is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Since the clock is running fast, the period of the pendulum needs to be increased. Thus l should be increased and the mass moved farther away from the axis of rotation.

The nonlinear differential equation derived in Example 2.16 is linearized by assuming small θ and replacing $\sin\theta$ by θ. The exact nonlinear pendulum equation

$$\ddot{\theta} + \omega_n^2 \sin\theta = 0 \qquad \textbf{[2.32]}$$

is one of the few nonlinear equations for which an exact solution is known. The solution of Eq. (2.32) subject to $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ is developed in terms of elliptic integrals, which are well-known tabulated functions.

The period of motion of a nonlinear system is dependent upon the initial conditions, while the period of a linear system is independent of initial conditions. One method of assessing the validity of the small-angle approximation for a given amplitude is to compare the period calculated using the exact solution to the period calculated using the linearized differential equations for different initial displacements. This comparison is given in Table 2.1, which shows that the small-angle approximation leads to accurate prediction of the period for amplitudes as large as 40°. For an initial angular displacement of 40°, the error in the period from using the small-angle approximation is only 3.1%.

Table 2.1 Ratio of period of simple pendulum, T, calculated from exact nonlinear solution to period calculated from linearized equation as a function of initial angle, θ_0, $\frac{1}{2}T\sqrt{\frac{g}{l}}$. Nonlinear period is $4K$ where K is the complete elliptic integral of the first kind with a parameter of $\sin(\theta_0/2)$

$\theta_0(°)$	$\frac{T}{2\pi}\sqrt{g/l}$	$\theta_0(°)$	$\frac{T}{2\pi}\sqrt{g/l}$
2	1.00007	48	1.04571
4	1.00032	50	1.04978
6	1.00070	52	1.05405
8	1.00120	54	1.05851
10	1.00191	56	1.06328
12	1.00274	58	1.06806
14	1.00376	60	1.07321
16	1.00490	62	1.07850
18	1.00618	64	1.08404
20	1.00764	66	1.08982
22	1.00930	68	1.09588
24	1.01108	70	1.10211
26	1.01305	72	1.10867
28	1.01515	74	1.11548
30	1.01738	76	1.12255
32	1.01987	78	1.12987
34	1.02248	80	1.13751
36	1.02528	82	1.14540
38	1.02821	84	1.15368
40	1.03132	86	1.16221
42	1.03463	88	1.17112
44	1.03814	90	1.18035
46	1.04183		

The success of the use of the small-angle approximation in the pendulum example should give confidence to its use in other problems, where an exact solution is not available. An alternative to its use, for most problems, is to find a numerical solution to the exact nonlinear equation.

2.5 Free Vibrations of One-Degree-of-Freedom Systems with Viscous Damping

The general form of the differential equation for the displacement of a particle in a one-degree-of-freedom linear system where viscous damping is present is

$$\tilde{m}\ddot{x} + \tilde{c}\dot{x} + \tilde{k}x = 0 \qquad [2.33]$$

where the coefficients are determined during the derivation of the differential equation. Dividing Eq. (2.33) by $\tilde{m}$ gives

$$\ddot{x} + \frac{\tilde{c}}{\tilde{m}}\dot{x} + \frac{\tilde{k}}{\tilde{m}}x = 0 \qquad [2.34]$$

The general solution of Eq. (2.34) is obtained by assuming

$$x(t) = Be^{\alpha t} \qquad [2.35]$$

Substitution of Eq. (2.35) into Eq. (2.34) leads to the following quadratic equation for α:

$$\alpha^2 + \frac{\tilde{c}}{\tilde{m}}\alpha + \frac{\tilde{k}}{\tilde{m}} = 0 \qquad [2.36]$$

The quadratic formula is used to obtain the roots of Eq. (2.36) as

$$\alpha_{1,2} = -\frac{\tilde{c}}{2\tilde{m}} \pm \sqrt{\left(\frac{\tilde{c}}{2\tilde{m}}\right)^2 - \frac{\tilde{k}}{\tilde{m}}} \qquad [2.37]$$

The mathematical form of the solution of Eq. (2.34) and the physical behavior of the system depends on the sign of the discriminant of Eq. (2.37). If the discriminant is positive, Eq. (2.36) has two real roots. If the discriminant is negative, Eq. (2.36) has two complex conjugate roots. If the discriminant is zero, Eq. (2.36) has only a double real root.

The physical nature of the vibrations is dependent on the sign of the discriminant. The case when the discriminant is zero is a special case and occurs only for a certain combination of parameters. When this occurs the system is said to be *critically damped*. For fixed values of $\tilde{k}$ and $\tilde{m}$, the value of $\tilde{c}$ which leads to critical damping is called the critical damping coefficient, $\tilde{c}_c$. From Eq. (2.37)

$$\tilde{c}_c = 2\sqrt{\tilde{k}\tilde{m}} \qquad [2.38]$$

The nondimensional damping ratio, ζ, is defined as the ratio of the actual value of $\tilde{c}$ to the critical damping coefficient,

$$\zeta = \frac{\tilde{c}}{\tilde{c}_c} = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} \qquad [2.39]$$

The damping ratio is a property of the system parameters.

Using Eqs. (2.39) and (2.21), Eq. (2.37) is rewritten in terms of ζ and ω_n as

$$\alpha_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad [2.40]$$

For $\zeta \neq 1$, the general solution of Eq. (2.34) is

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} + C_2 e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right) \qquad [2.41]$$

where C_1 and C_2 are arbitrary constants of integration. From Eq. (2.41) it is evident that the nature of the motion depends on the value of ζ. Using Eqs. (2.21) and (2.39), Eq. (2.34) becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \textbf{[2.42]}$$

Equation (2.42) is the standard form of the differential equation governing the free vibrations of a one-degree-of-freedom system with viscous damping.

Three cases must be examined to explore completely the behavior of a one-degree-of-freedom system with viscous damping.

Case 1: $\zeta < 1$ (Underdamped Free Vibrations) For $\zeta < 1$ the roots of Eq. (2.36) exist as a complex conjugate pair

$$\alpha_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right) \qquad \textbf{[2.43]}$$

Euler's identity is used to replace the complex exponentials that occur in Eq. (2.41) by a linear combination of trigonometric terms of real argument,

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 \cos\omega_n\sqrt{1-\zeta^2}\,t + C_2 \sin\omega_n\sqrt{1-\zeta^2}\,t\right) \qquad \textbf{[2.44]}$$

The constants of integration are determined by applying the initial conditions, Eq. (2.19), resulting in

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_n\sqrt{1-\zeta^2}} \sin\omega_n\sqrt{1-\zeta^2}\,t\right) \qquad \textbf{[2.45]}$$

An alternative form of the solution is developed by using the trigonometric identity, Eq. (2.23)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d) \qquad \textbf{[2.46]}$$

where

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \qquad \textbf{[2.47]}$$

$$\phi_d = \tan^{-1}\left(\frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right) \qquad \textbf{[2.48]}$$

and

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \textbf{[2.49]}$$

Equation (2.46) is plotted in Fig. 2.20. Once free oscillations of a viscously damped system commence, the nonconservative viscous damping force continually dissipates energy. Since no work is being done on the system, this leads to a continual decrease in the sum of the potential and kinetic energies. For underdamped free vibrations, the system oscillates about an equilibrium position. However, each time it reaches equilibrium, the system's total energy level is less than at the previous time. The maximum displacement on each cycle of motion is continually decreasing. Equation (2.46) and Fig. 2.20 show that the amplitude decreases exponentially with time.

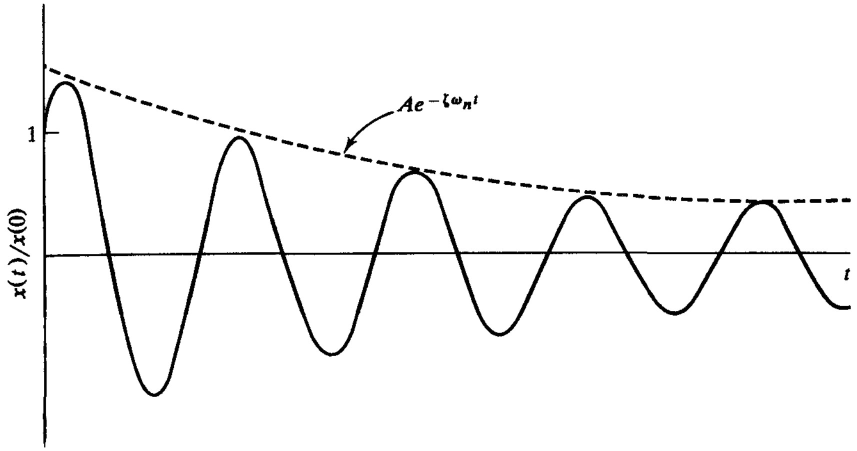


Figure 2.20 Free vibrations of an underdamped one-degree-of-freedom system decay exponentially.

The free vibrations of an underdamped system are oscillatory, but not periodic. The vibrations would be periodic if it were not for the decay in amplitude. Even though the amplitude decreases between cycles, the system takes the same amount of time to execute each cycle. This time is called the *period of free underdamped vibrations* or the *damped period* and is given by

$$T_d = \frac{2\pi}{\omega_d} \tag{2.50}$$

Thus ω_d is called the *damped natural frequency*. Note that $\omega_d < \omega_n$.

Consider a mass-spring-dashpot system with $x(0) = 0$, $x(0) = x_0$. Then

$$\phi_d = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \sin\phi_d = \sqrt{1-\zeta^2} \qquad \cos\phi_d = \zeta$$

and

$$A = \frac{x_0}{\sqrt{1-\zeta^2}}$$

Using Eq. (2.46) the total energy present in an underdamped system is

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}\frac{kx_0^2e^{-2\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\Big[(1+\zeta^2)\sin^2(\omega_d t + \phi_d) \tag{2.51}$$

$$- 2\zeta\sqrt{1-\zeta^2}\sin(\omega_d t + \phi_d)\cos(\omega_d t + \phi_d)$$

$$+ (1-\zeta^2)\cos^2(\omega_d t + \phi_d)\Big]$$

The total energy in the system at the end of the nth cycle is

$$E_n = E\left(t = \frac{2n\pi}{\omega_d}\right) = \frac{1}{2}kx_0^2\sqrt{1-\zeta^2}e^{-4n\pi/\sqrt{1-\zeta^2}} \qquad [2.52]$$

The energy dissipated as the system executes one cycle of motion is

$$\Delta E_n = E_n - E_{n+1} \qquad [2.53]$$

$$= \tfrac{1}{2}kx_0^2\sqrt{1-\zeta^2}e^{-4n\pi\zeta/\sqrt{1-\zeta^2}}\left(1 - e^{-4\pi\zeta/\sqrt{1-\zeta^2}}\right)$$

The ratio of the energy dissipated over a cycle to the total energy at the beginning of the cycle is

$$\frac{\Delta E_n}{E_n} = 1 - e^{-4\pi\zeta/\sqrt{1-\zeta^2}} \qquad [2.54]$$

Equations (2.53) and (2.54) show that the energy dissipated over one cycle of motion is a fraction of the total energy at the beginning of the cycle. The fraction of energy dissipated over a cycle is constant and depends only on the damping ratio. The larger the damping ratio, the larger the fraction of energy dissipated over a single cycle. Equation (2.54) shows that as the damping ratio approaches 1, the fraction of energy dissipated over a single cycle approaches 1. Thus, for underdamped free vibrations, the energy dissipated over a single cycle is a constant fraction of the energy in the system at the beginning of the cycle; the total energy is never completely dissipated. This indicates that free vibrations for an underdamped system continue indefinitely with exponentially decaying amplitude.

The *logarithmic decrement*, δ, is defined for underdamped free vibrations as the natural logarithm of the ratio of the amplitudes of vibration on successive cycles.

$$\delta = \ln\left(\frac{x(t)}{x(t+T_d)}\right) = \ln\left(\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_d)}{Ae^{-\zeta\omega_n(t+T_d)}\sin[\omega_d(t+T_d)+\phi_d]}\right) \qquad [2.55]$$

$$= \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

For small ζ,

$$\delta = 2\pi\zeta \qquad [2.56]$$

The logarithmic decrement is often measured by experiment and damping ratio determined from

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \qquad [2.57]$$

It can be shown that the following equations can also be used to calculate the logarithmic decrement:

$$\delta = \frac{1}{n}\ln\left(\frac{x(t)}{x(t+nT_d)}\right) \qquad [2.58]$$

for any integer n and

$$\delta = \ln\left(\frac{\dot{x}(t)}{\dot{x}(t+T_d)}\right) \quad [2.59]$$

$$\delta = \ln\left(\frac{\ddot{x}(t)}{\ddot{x}(t+T_d)}\right) \quad [2.60]$$

Equation (2.58) implies that the logarithmic decrement can be determined from amplitudes measured on nonsuccessive cycles, while Eqs. (2.59) and (2.60) imply that velocity and acceleration data can also be used to determine the logarithmic decrement.

2.17 The The slender rod of Example 2.4 has a mass of 31 kg and a length of 2.6 m. A 50-N force is statically applied to the bar at P and then removed. The ensuing oscillations of P are monitored and an oscilloscope provides the acceleration data shown in Fig. 2.21b where the time scale is calibrated but the acceleration scale is not calibrated. Use the data to find the spring constant k and the damping coefficient c. Also calibrate the acceleration scale.

Solution:

The differential equation obtained in Example 2.4 is divided by the coefficient of its highest derivative

$$\ddot{x} + \frac{3c}{7m}\dot{x} + \frac{27k}{7m}x = 0$$

The natural frequency and damping ratio are determined by comparing the preceding equation with the standard form of the differential equation for damped free vibrations, Eq. (2.42)

$$\omega_n = \sqrt{\frac{27k}{7m}} \qquad \zeta = \frac{3c}{14m\omega_n}$$

The period of damped free vibrations is determined from the oscilloscope data as 0.1 s. The value of the logarithmic decrement is determined from the oscilloscope data and Eq. (2.60)

$$\delta = \ln\left(\frac{\ddot{x}(0)}{\ddot{x}(0.1\text{ s})}\right) = \ln\frac{3}{2} = 0.405$$

The damping ratio is calculated using Eq. (2.57).

$$\zeta = \frac{0.405}{\sqrt{4\pi^2 + (0.405)^2}} = 0.0643$$

Equation (2.50) is used to calculate the damped natural frequency as 62.83 rad/s. Equation (2.49) is used to calculate the natural frequency as 62.96 rad/s.

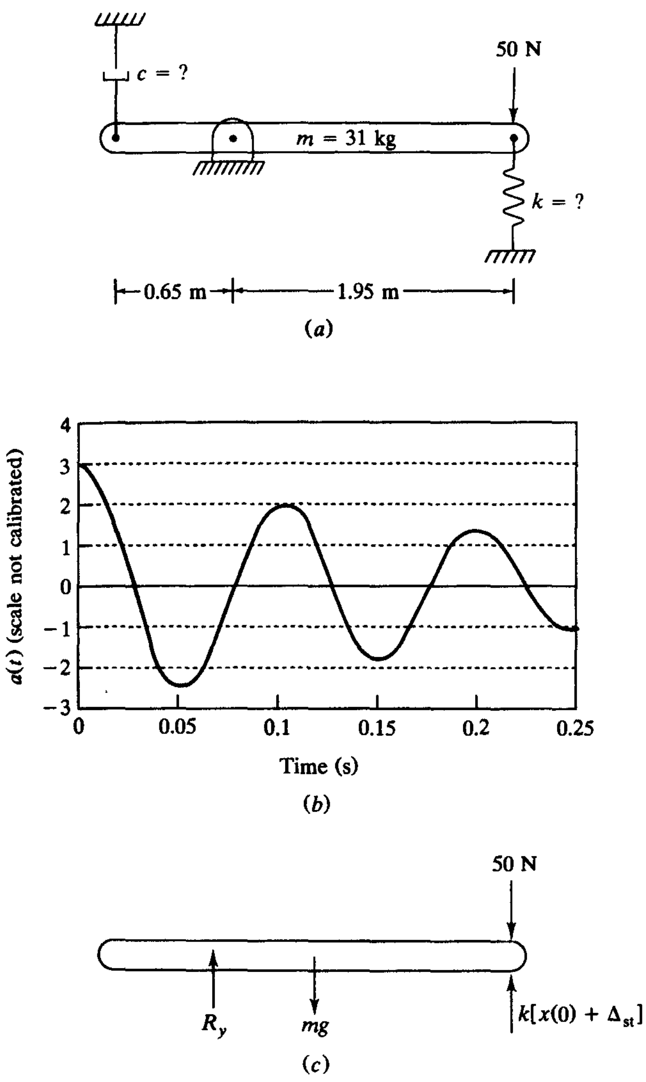


Figure 2.21 (*a*) Initial position of bar of Example 2.17; (*b*) oscilloscope data for Example 2.17; (*c*) free-body diagram of initial position.

The spring stiffness and damping coefficient are determined as

$$k = \frac{7m\omega_n^2}{27} = \frac{7(31\text{ kg})(62.96\text{ rad/s})^2}{27} = 3.19 \times 10^4\ \frac{\text{N}}{\text{m}}$$

$$c = \frac{14m\omega_n\zeta}{3} = \frac{14(31\text{ kg})(62.96\text{ rad/s})(0.0643)}{3} = 585.7\ \frac{\text{N}\cdot\text{s}}{\text{m}}$$

A static analysis of the equilibrium position provides the initial displacement of P from equilibrium as

$$x(0) = \frac{F}{k} = \frac{50\ \text{N}}{3.19 \times 10^4\ \text{N/m}} = 1.6\ \text{mm}$$

The initial acceleration is calculated using the governing differential equation

$$\ddot{x}(0) = -\frac{3c}{7m}\dot{x}(0) - \frac{27k}{7m}x(0) = -\frac{27(3.19 \times 10^4\ \text{N/m})}{7(31\ \text{kg})}(0.0016\ \text{m}) = -6.35\ \frac{\text{m}}{\text{s}^2}$$

The acceleration scale is then calibrated:

$$1\ \text{unit} = \frac{6.35\ \text{m/s}^2}{3} = 2.12\ \frac{\text{m}}{\text{s}^2}$$

2.18 A simplified model of a vehicle suspension system is illustrated in Fig. 2.22. The vehicle of mass m is connected to its axles through a spring of stiffness k in parallel with a viscous damper of damping coefficient c. When the system is designed to be underdamped, the overshoot when the vehicle encounters a pothole of depth h is defined as the displacement at the end of the first half cycle, $-x(T_d/2)$. The suspension system for a 1200-kg vehicle is to be designed such that it has a static deflection of 5 cm when empty and has no more than 7 percent overshoot when the vehicle is either empty or full. It is estimated that when fully loaded the vehicle will carry 800 kg of passengers and cargo.

(a) Determine the required stiffness and damping coefficient of the suspension system.

(b) Plot the overshoot as a function of vehicle mass.

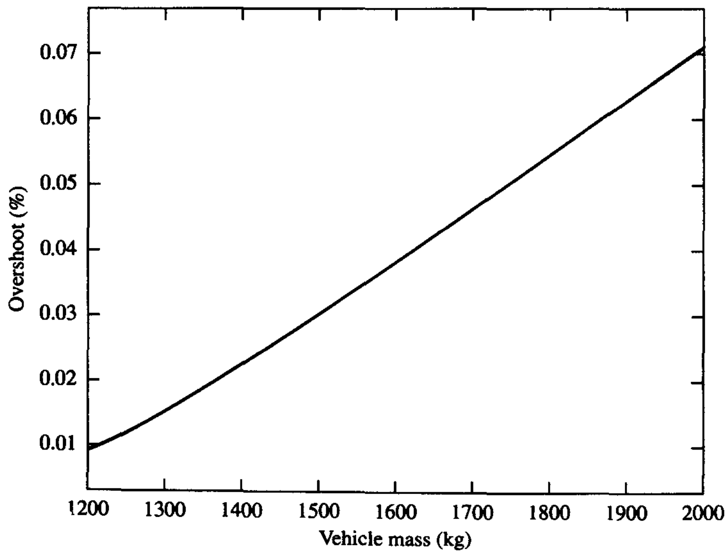


Figure 2.23 Overshoot increases as mass increases.

Solution:

The required stiffness is calculated as

$$k = \frac{mg}{\Delta_{st}} = \frac{(1200\text{ kg})(9.81\text{ m/s}^2)}{(0.05\text{ m})} = 2.35 \times 10^5\text{ N/m}$$

For an underdamped system the overshoot is

$$\eta = -x(T_d/2) = -Ae^{-\zeta\omega_n T_d/2}\sin(\omega_d T_d/2 + \phi_d)$$

Noting that $T_d = 2\pi/\omega_d$ leads to

$$\eta = -Ae^{-\zeta\pi/\sqrt{1-\zeta^2}}\sin(\pi + \phi_d) = Ae^{-\zeta\pi/\sqrt{1-\zeta^2}}\sin\phi_d$$

The initial conditions when the vehicle encounters a pothole of depth h are $x(0) = h$ and $\dot{x}(0) = 0$. Thus $h = A\sin\phi_d$. And

$$\eta = he^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

The damping ratio in terms of overshoot is

$$\zeta = \frac{-\frac{1}{\pi}\ln\left(\frac{\eta}{h}\right)}{\sqrt{1 + \left[-\frac{1}{\pi}\ln\left(\frac{\eta}{h}\right)\right]^2}}$$

For 7 percent overshoot, $\eta/h = 0.07$, leading to a minimum damping ratio of $\zeta = 0.645$.

Note that for the suspension system

$$\zeta = \frac{c}{2\sqrt{mk}}$$

Thus, as the passengers and cargo are added to the vehicle the damping ratio decreases. Hence the suspension system should be designed so that the damping ratio is 0.645 when it is fully loaded. To this end

$$c = 2\zeta\sqrt{mk} = 2(0.645)\sqrt{(2000\text{ kg})(2.35 \times 10^5)\text{ N/m}} = 2.79 \times 10^4\text{ N}\cdot\text{s/m}$$

Substitution for ζ in terms of c, m, and k into the expression for overshoot leads to

$$\frac{\eta}{h} = e^{-c\pi/\left(2\sqrt{mk}\sqrt{1-(c^2/4mk)}\right)}$$

The overshoot as a function of mass is plotted in Fig. 2.23.

Case 2: $\zeta = 1$ (Critically Damped Vibrations) For $\zeta = 1$, Eq. (2.36) has only one real root

$$\alpha_1 = \alpha_2 = -\omega_n \qquad [2.61]$$

The general solution of Eq. (2.42) is

$$x(t) = e^{-\omega_n t}(C_1 + C_2 t)$$

Application of the initial conditions leads to

$$x(t) = e^{-\omega_n t}[x_0 + (\dot{x}_0 + \omega_n x_0)t] \qquad \textbf{[2.62]}$$

The response of a one-degree-of-freedom system subject to critical viscous damping is plotted in Fig. 2.24 for different initial conditions. If the initial conditions are of opposite sign or if $\dot{x}_0 = 0$, the motion decays immediately. If both initial conditions have the same sign or if $x_0 = 0$, the absolute value of x initially increases and reaches a maximum value of

$$x_{\max} = e^{-\dot{x}_0/(\dot{x}_0+\omega_n x_0)}\left(x_0 + \frac{\dot{x}_0}{\omega_n}\right)$$

at

$$t = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}$$

If the signs of the initial conditions are opposite and

$$\frac{x_0}{\dot{x}_0 + \omega_n x_0} < 0$$

then the response overshoots the equilibrium position before eventually decaying.

Equation (2.52) with $n = 1$ shows that a damping force that leads to critical damping is sufficient to dissipate all of a system's initial energy before one cycle of motion is complete. A critically damped system can thus pass through equilibrium at most once before the motion decays. The total energy decays exponentially,

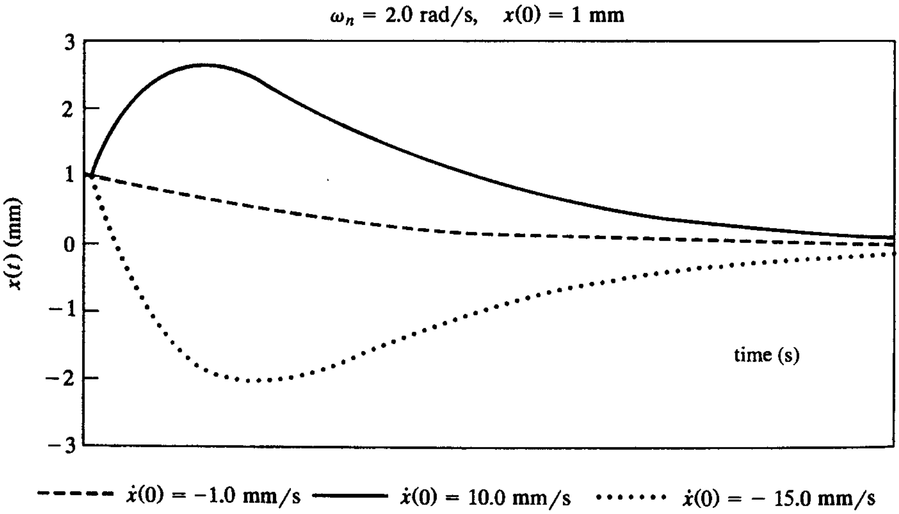


Figure 2.24 The free-vibration response of a critically damped one-degree-of-freedom system is aperiodic and rapidly decays. The system may pass through its equilibrium position if initial conditions are of opposite sign.

but never reaches zero. Thus critically damped motion is predicted to continue indefinitely.

Case 3: $\zeta > 1$ (Overdamped Free Vibrations) For $\zeta > 1$, Eq. (2.37) has two real, distinct roots

$$\alpha_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The general solution of Eq. (2.34) is given by Eq. (2.41). Application of Eq. (2.19) leads to

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2-1}}\left\{\left[\frac{\dot{x}_0}{\omega_n} + x_0\left(\zeta + \sqrt{\zeta^2-1}\right)\right]e^{\omega_n\sqrt{\zeta^2-1}\,t} + \left[-\frac{\dot{x}_0}{\omega_n} + x_0\left(-\zeta + \sqrt{\zeta^2-1}\right)\right]e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right\} \quad \textbf{[2.63]}$$

Equation (2.63) is plotted in Fig. 2.25. The response of an overdamped one-degree-of-freedom system is not periodic. It attains its maximum either at $t = 0$ or at

$$t = -\frac{1}{2\omega_n\sqrt{\zeta^2-1}}\ln\left[\frac{\zeta - \sqrt{\zeta^2-1}}{\zeta + \sqrt{\zeta^2-1}}\,\frac{\dfrac{\dot{x}_0}{\omega_n} + x_0\left(\zeta + \sqrt{\zeta^2-1}\right)}{\dfrac{\dot{x}_0}{\omega_n} + x_0\left(\zeta - \sqrt{\zeta^2-1}\right)}\right] \quad \textbf{[2.64]}$$

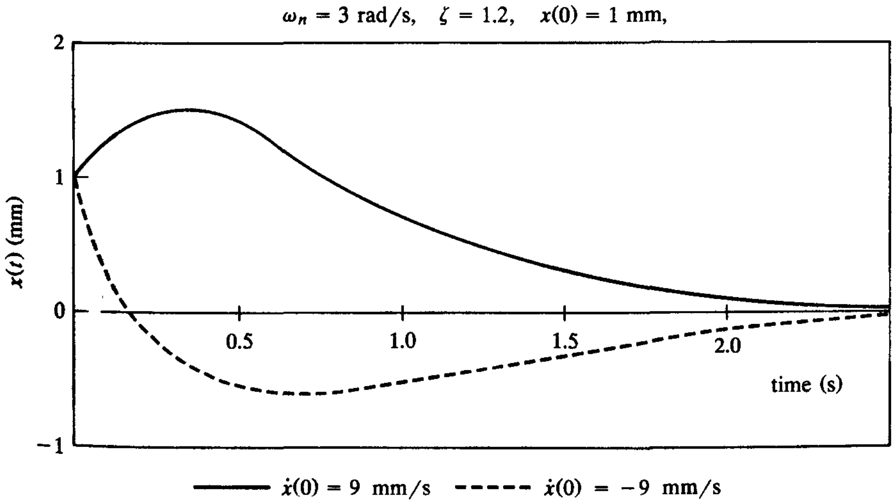


Figure 2.25 The free-vibration response of an overdamped one-degree-of-freedom system is aperiodic. The response quickly decays after reaching a maximum.

Viscous damping leads to the decay of free oscillations. When added to a linear one-degree-of-freedom system, it adds a force linearly proportional to the velocity, and the differential equation remains linear. Viscous damping also leads to positive effects when added to systems undergoing forced excitation. For these reasons viscous damping is often artificially added to many mechanical systems.

Many systems are designed with viscous damping such that the free vibrations are underdamped because of its favorable effect in reducing amplitudes of forced vibration. When a system is subject to free vibrations only, the system may be designed with critical damping or overdamping to alleviate the oscillatory motion. A critically damped system returns to its equilibrium quicker than an overdamped system. Thus recoil mechanisms of guns are designed with critical damping to allow rapid firing.

Automobile suspension systems are often subject to both free and forced vibrations. Free vibrations occur when the vehicle is subject to a sudden change in road contour. In this case the shock absorbers should have a damping ratio near one. However, if the vehicle is traveling on a bumpy road, the vehicle is subject to a possible random excitation. In this case the system should be underdamped. For these reasons vehicle shock absorbers are often self-adaptive.

2.19 The restroom door of Fig. 2.26 is equipped with a torsional spring and a torsional viscous damper so that it automatically returns to its closed position after being opened. The door has a mass of 60 kg and a centroidal moment of inertia about an axis parallel to the axis of the door's rotation of 7.2 kg · m². The torsional spring has a stiffness of 25 N · m/rad.

(*a*) What is the damping coefficient such that the system is critically damped?

(*b*) A man with an armload of packages, but in a hurry, kicks the door to cause it to open. What angular velocity must his kick impart to cause the door to open 70°?

(*c*) How long after his kick will the door return to within 5° of completely closing?

(*d*) Repeat parts *a–c* if the door is designed with a damping ratio, $\zeta = 1.3$.

The differential equation is derived from the free-body diagrams of Fig. 2.26*b*,

$$(\bar{I} + md^2)\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$$

The differential equation is put in the standard form of Eq. (2.42) by dividing by $\bar{I} + md^2$. Then it is evident that

$$\omega_n = \sqrt{\frac{k_t}{\bar{I} + md^2}} = \sqrt{\frac{25\ \mathrm{N\cdot m/rad}}{7.2\ \mathrm{kg\cdot m^2} + (60\ \mathrm{kg})(0.45\ \mathrm{m})^2}} = 1.14\ \frac{\mathrm{rad}}{\mathrm{s}}$$

and

$$\zeta = \frac{c_t}{2\omega_n(\bar{I} + md^2)}$$

(*a*) For critical damping the damping ratio is 1. Thus

$$c_t = 2\omega_n(\bar{I} + md^2) = 44.1\ \mathrm{N\cdot m\cdot s}$$

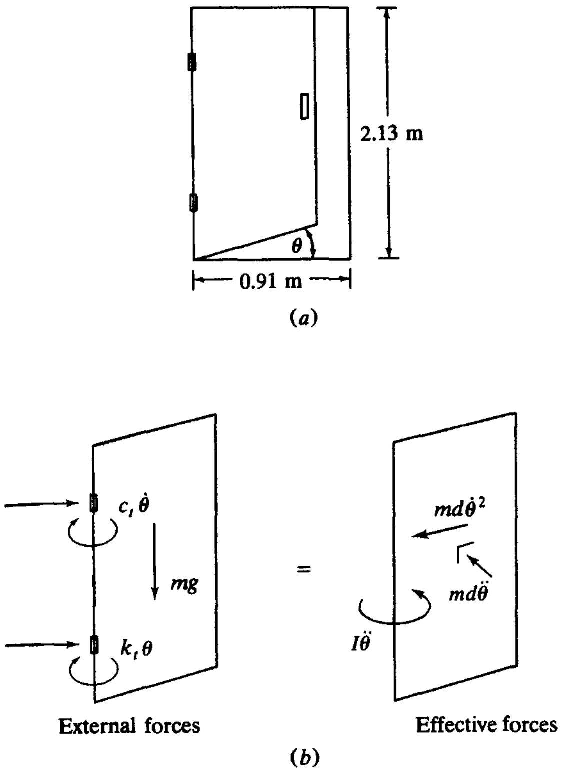


Figure 2.26 (a) The restroom door of Example 2.19 is modeled as a one-degree-of-freedom system with a torsional spring and a torsional viscous damper; (b) free-body diagrams of restroom door at an arbitrary instant.

(*b*) If the kick is given when the door is closed, $\theta(0) = 0$, the maximum displacement occurs at

$$t = \frac{1}{\omega_n} = 0.88 \text{ s}$$

and is

$$\theta_{\max} = \frac{\dot{\theta}_0}{e\omega_n}$$

Requiring $\theta_{\max} = 70°$ yields

$$\dot{\theta}_0 = 70° \left(\frac{2\pi \text{ rad}}{360°}\right) \left(1.14 \frac{\text{rad}}{\text{s}}\right) e = 3.79 \frac{\text{rad}}{\text{s}}$$

```
% Part d of Example 2.19
x=sym('x');
z=sym('z');
k=25;          %Torsional stiffness (N-m/rad)
I=7.2;         %Moment of inertia (kg-m^2)
m=60;          %Mass (kg)
d=0.45;        %Distance to center of gravity (m)
zeta=1.3;      %Damping ratio
theta_m=70;    %Maximum angle for door to open (degrees)
theta_f=5;     %Closing angle (degrees)
% Natural frequency calculation (rad/s)
omega_n=sqrt(k/(I+m*d^2));
% Torsional damping coefficient (N-m-s)
disp('Torsional damping coefficient in N-m-s/rad')
c=2*zeta*omega_n*(I+m*d^2);
disp (c)
% Time to maximum displacement from Eq.(2.64)
C_1=sqrt(zeta^2-1);
t_m=-1/(2*omega_n*C_1)*log((zeta-C_1)/(zeta+C_1));
% Equation(2.63) is used to solve for angular velocity
% required for theta=70 degrees at maximum opening
theta_m=theta_m*(2*pi)/360;   %Converting degrees to radians
C_2=omega_n*t_m*C_1;
disp(' Required initial angular velocity in rad/s')
thetadot_0=2*C_1*theta_m*omega_n*exp(zeta*omega_n*t_m);
thetadot_0=thetadot_0/(exp(C_2)-exp(-C_2));
disp (thetadot_0)
%Applying Eq.(2.63) leads to a transcendental
%equation to solve for closing time, t
theta_f=theta_f*(2*pi)/360;
theta=thetadot_0/(2*omega_n*C_1)*exp(-omega_n*zeta*x);
theta=theta*(exp(omega_n*C_1*x)-exp(-omega_n*C_1*x));
thet=vpa(theta);
ezplot(thet,[0 6])
xlabel('time (s)')
ylabel('theta (rad)')
% The second time when theta=5 degrees is difficult to obtain
% using MATLAB
% An approximation is obtained by neglecting the term with the
% negative exponent
tz=thetadot_0/(2*omega_n*C_1)*exp(-omega_n*zeta*x)*exp(omega_n*C_1*x);
dz=tz-theta_f;
fz=solve(dz);
qz=double(fz);
sz=thetadot_0/(2*omega_n*C_1)*exp(-omega_n*zeta*x);
sz=sz*(exp(omega_n*C_1*x)-exp(-omega_n*C_1*x));
disp('Time to close to within 5 degrees in sec')
```

Figure 2.26 (Con't) (c) MATLAB script.

```
disp(qz)
w=subs(sz,qz,x)-theta_f;
error=double(w);
disp('error=    ')
disp(error)
```

(c)

```
EDU» Ex2_19
Torsional damping coefficient in N-m-s/rac
   57.1852

 Required initial angular velocity in rad/
    4.5367

Time to close to within 5 degrees in sec
    6.2145

error=
 -6.9878e-007
```

(d)

Figure 2.26 (Con't) (c) Con't; (d) MATLAB output.

(c) Applying Eq. (2.62) with $\theta = 5°$ gives

$$5° \left(\frac{2\pi \text{ rad}}{360°}\right) = e^{-(1.14 \text{ rad/s})t} \left(3.79 \frac{\text{rad}}{\text{s}}\right) t$$

which is solved by trial and error to yield $t = 4.658$ s.

(d) The solution for $\zeta = 1.3$ can be obtained by using MATLAB. The MATLAB script and output are given in Fig. 2.26c and d. The time for which the maximum occurs is calculated from Eq. (2.64). The initial angular velocity required to open the door to 70° is obtained from Eq. (2.63). Subsequent application of Eq. (2.63) to obtain the time required for the door to close to within 5° leads to a transcendental equation for t, which is plotted by MATLAB in Fig 2.26e. The first root of this equation corresponds to when the door is open to 5° after being kicked. The second root is difficult to obtain with MATLAB, but can be approximated by neglecting the term $e^{-\omega_n\sqrt{\zeta^2-1}\,t}$ compared to its reciprocal. The MATLAB output shows that this leads to an excellent approximation for the second root.

The solution can also be obtained manually. Setting $\zeta = 1.3$ yields

$$c_t = 2\zeta(\bar{I} + md^2)\omega_n = 57.4 \text{ N} \cdot \text{m} \cdot \text{s}$$

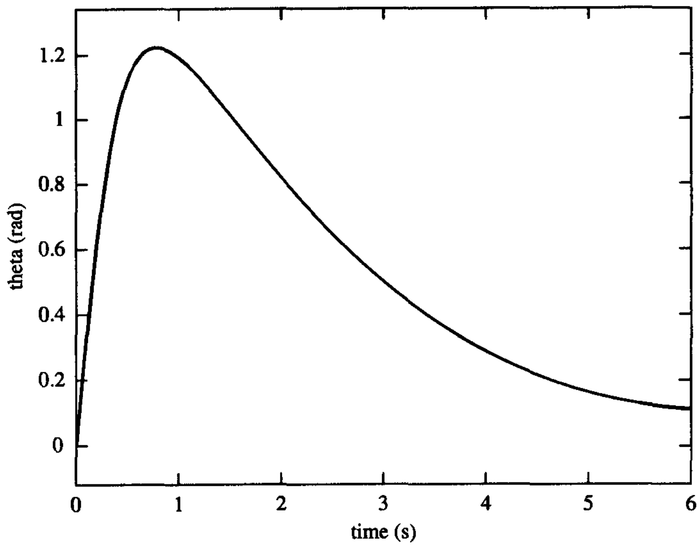


Figure 2.26 (Con't) (e) MATLAB plot for $\zeta = 1.3$.

From Eq. (2.64) the maximum displacement occurs at

$$t = -\frac{1}{2(1.14\text{ rad/s})\sqrt{(1.3)^2-1}}\ln\left(\frac{1.3-\sqrt{(1.3)^2-1}}{1.3+\sqrt{(1.3)^2-1}}\right) = 0.80\text{ s}$$

Substituting the preceding result in Eq. (2.63) and setting $\theta = 70°$ yields

$$70°\left(\frac{2\pi\text{ rad}}{360°}\right) = \left(\frac{\dot{\theta}_0}{1.14\text{ rad/s}}\right)\frac{1}{2\sqrt{(1.3)^2-1}}e^{-1.3(1.14\text{ rad/s})(0.8\text{ s})}$$
$$\times\left(e^{1.14\text{ rad/s}\sqrt{(1.3)^2-1}\,(0.8\text{ s})} - e^{-1.14\text{ rad/s}\sqrt{(1.3)^2-1}\,(0.8\text{ s})}\right)$$

which gives

$$\dot{\theta}_0 = 4.55\text{ rad/s}$$

Applying Eq. (2.63) with $\theta = 5°$ yields

$$5°\left(\frac{2\pi\text{ rad}}{360°}\right) = \left(\frac{e^{-1.14(1.3)t}}{2\sqrt{(1.3)^2-1}}\right)\left(\frac{4.55\text{ rad/s}}{1.14\text{ rad/s}}\right)$$
$$\times\left(e^{1.14\sqrt{(1.3)^2-1}\,t} - e^{-1.14\sqrt{(1.3)^2-1}\,t}\right)$$

This equation could be solved by trial and error. However, a good approximation is obtained by neglecting the smaller exponential to give $t = 6.2$ s. The neglected term at this time is 0.00081 rad which is only 0.9% of the total angular displacement.

Note that a harder kick is required to open the door when the system is overdamped than when the system is critically damped even though the time required to open the door is approximately the same. This reflects the increase in the viscous resistance moment.

2.6 COULOMB DAMPING

Coulomb damping is the damping that occurs due to dry friction when two surfaces slide against one another. Coulomb damping can be the result of a mass sliding on a dry surface, axle friction in a journal bearing, belt friction, or rolling resistance. The case of a mass sliding on a dry surface is analyzed here, but the qualitative results apply to all forms of Coulomb damping.

As the mass of Fig. 2.27 slides on a dry surface, a friction force that resists the motion develops between the mass and the surface. Coulomb's law states that the friction force is proportional to the normal force developed between the mass and the surface. The constant of proportionality μ, is called the *kinetic coefficient of friction*. Since the friction force always resists the motion, its direction depends on the sign of the velocity.

Application of Newton's law to the free-body diagrams of Fig. 2.27*b* yields the following differential equations:

$$m\ddot{x} + kx = -\mu mg \qquad \dot{x} > 0 \tag{2.65a}$$

$$m\ddot{x} + kx = \mu mg \qquad \dot{x} < 0 \tag{2.65b}$$

Equation (2.65*a* and *b*) are generalized by using a single equation

$$m\ddot{x} + kx = -\mu mg\frac{|\dot{x}|}{\dot{x}} \tag{2.65c}$$

The right-hand side of Eq. (2.65*c*) is a nonlinear function of the generalized coordinate. Thus the free vibrations of a one-degree-of-freedom system with Coulomb damping are governed by a nonlinear differential equation. However, an analytical solution exists and is obtained by solving Eq. (2.65*a*, *b*).

Without loss of generality, assume that free vibrations of the system of Fig. 2.27 are initiated by displacing the mass a distance δ to the right, from equilibrium, and releasing it from rest. The spring force draws the mass toward equilibrium; thus the velocity is initially negative. Equation (2.65*b*) applies over the first half-cycle of motion, until the velocity again becomes zero.

The solution of Eq. (2.65*b*) subject to

$$x(0) = \delta$$

and

$$\dot{x}(0) = 0$$

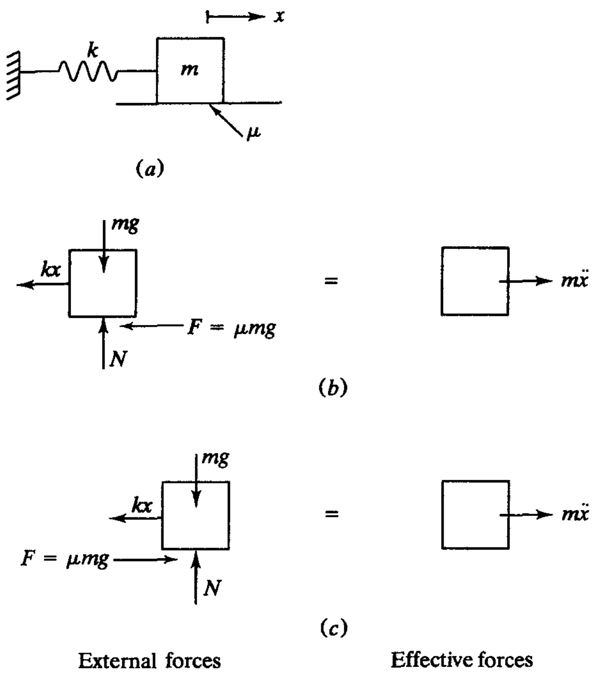


Figure 2.27 (a) Mass slides on a surface with a kinetic coefficient of friction μ; (b) free-body diagrams at an arbitrary instant of time when $\dot{x} > 0$; (c) free-body diagrams at an arbitrary instant of time with $\dot{x} < 0$.

is

$$x(t) = \left(\delta - \frac{\mu mg}{k}\right)\cos\omega_n t + \frac{\mu mg}{k} \tag{2.66}$$

Equation (2.66) describes the motion until the velocity changes sign at $t = \pi/\omega_n$ when

$$x\left(\frac{\pi}{\omega_n}\right) = -\delta + \frac{2\mu mg}{k} \tag{2.67}$$

Equation (2.65*a*) governs the motion until the velocity next changes sign. The solution of Eq. (2.65*a*) using Eq. (2.67) and

$$\dot{x}\left(\frac{\pi}{\omega_n}\right) = 0 \tag{2.68}$$

as initial conditions is

$$x(t) = \left(\delta - \frac{3\mu mg}{k}\right)\cos\omega_n t - \frac{\mu mg}{k} \qquad \frac{\pi}{\omega_n} \le t \le \frac{2\pi}{\omega_n} \tag{2.69}$$

The velocity again changes sign at $t = 2\pi/\omega_n$ when

$$x\left(\frac{2\pi}{\omega_n}\right) = \delta - \frac{4\mu mg}{k} \tag{2.70}$$

The motion during the first complete cycle is described by Eqs. (2.66) and (2.69). The amplitude change between the beginning and the end of the cycle is

$$x(0) - x\left(\frac{2\pi}{\omega_n}\right) = \frac{4\mu mg}{k} \tag{2.71}$$

The analysis of the subsequent and each successive cycle continues in the same fashion. Equation (2.65b) governs during the first half of the cycle, while Eq. (2.65a) governs during the second half of the cycle. The initial conditions used to solve for the displacement during a half-cycle are that the velocity is zero and the displacement is the displacement calculated at the end of the previous half-cycle.

The period of each cycle is

$$T = \frac{2\pi}{\omega_n} \tag{2.72}$$

Thus Coulomb damping has no effect on the natural frequency.

Mathematical induction is used to develop the following expressions for the displacement of the mass during each half-cycle:

$$x(t) = \left[\delta - (4n-3)\frac{\mu mg}{k}\right]\cos\omega_n t + \frac{\mu mg}{k}$$

$$2(n-1)\frac{\pi}{\omega_n} \le t \le 2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n} \tag{2.73}$$

$$x(t) = \left[\delta - (4n-1)\frac{\mu mg}{k}\right]\cos\omega_n t - \frac{\mu mg}{k}$$

$$2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n} \le t \le 2n\frac{\pi}{\omega_n} \tag{2.74}$$

$$x\left(2n\frac{\pi}{\omega_n}\right) = \delta - 4n\frac{\mu mg}{k} \tag{2.75}$$

Equation (2.75) shows that the displacement at the end of each cycle is $4\mu mg/k$ less than the displacement at the end of the previous cycle. Thus the amplitude of free vibration decays linearly as shown, when Eqs. (2.73) and (2.74) are plotted in Fig. 2.28.

The motion continues with this constant decrease in amplitude as long as the restoring force is sufficient to overcome the resisting friction force. However, since the friction causes a decrease in amplitude, the restoring force eventually becomes less than the friction force. This occurs when

$$k\left|x\left(2n\frac{\pi}{\omega_n}\right)\right| \le \mu mg \tag{2.76}$$

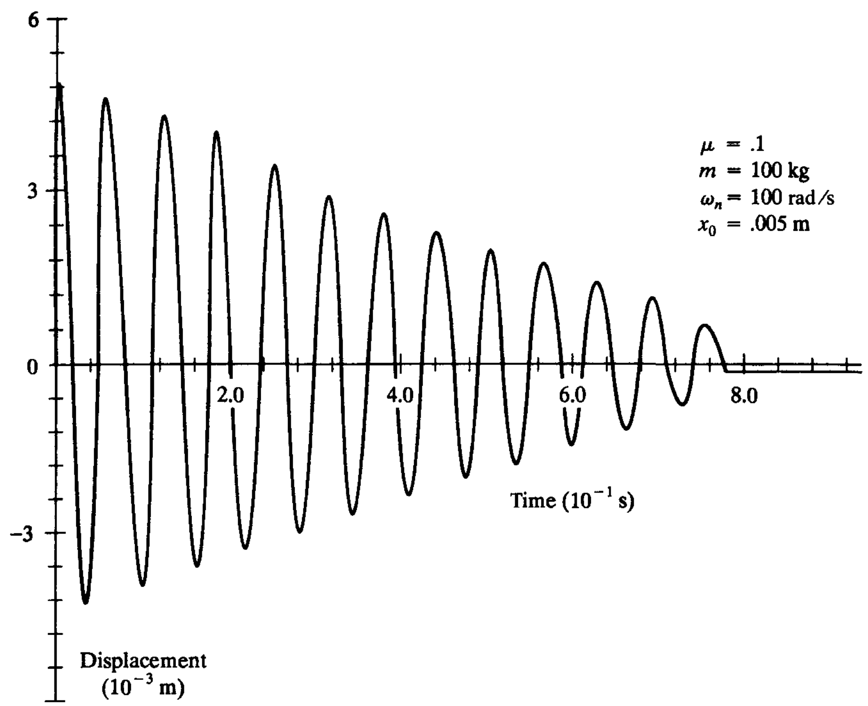


Figure 2.28 Coulomb damping does not alter the natural period of free vibration of a one-degree-of-freedom system, but causes a linear decay in amplitude. The motion of the system ceases when the amplitude is too small for the spring force to overcome the friction force.

Motion ceases during the nth cycle, where n is the smallest integer such that

$$n > \frac{k\delta}{4\mu mg} - \frac{1}{4} \tag{2.77}$$

When motion ceases a constant displacement from equilibrium of $\mu mg/k$ is maintained.

The effect of Coulomb damping differs from the effect of viscous damping in these respects:

1. Viscous damping causes a linear term proportional to the velocity in the governing differential equation, while Coulomb damping gives rise to a nonlinear term.
2. The natural frequency of an undamped system is unchanged when Coulomb damping is added, but is decreased when viscous damping is added.
3. Motion is not periodic if the viscous damping coefficient is large enough, whereas the motion is always periodic when Coulomb damping is the only source of damping.

4. The amplitude decreases linearly because of Coulomb damping and exponentially because of viscous damping.
5. Coulomb damping leads to a cessation of motion with a resulting permanent displacement from equilibrium, while motion of a system with only viscous damping continues indefinitely with a decaying amplitude.

Since the motion of all physical systems ceases in the absence of continuing external excitation, Coulomb damping is always present. Coulomb damping appears in many forms, such as axle friction in journal bearings and friction due to belts in contact with pulleys or flywheels. The response of systems to these and other forms of Coulomb damping can be obtained in the same manner as the response for dry sliding friction.

The general form of the differential equation governing the free vibrations of a linear system where Coulomb damping is the only source of damping is

$$\ddot{x} + \omega_n^2 x = \begin{cases} \dfrac{F_f}{\tilde{m}} & \dot{x} < 0 \\ -\dfrac{F_f}{\tilde{m}} & \dot{x} > 0 \end{cases} \tag{2.78}$$

where F_f is the magnitude of the Coulomb damping force. The decrease in amplitude per cycle of motion is

$$\Delta A = \frac{4F_f}{\tilde{m}\omega_n^2} \tag{2.79}$$

Example 2.2

An experiment is run to determine the kinetic coefficient of friction between a block and a surface. The block is attached to a spring and displaced 150 mm from equilibrium. It is observed that the period of motion is 0.5 s and that the amplitude decreases by 10 mm on successive cycles. Determine the coefficient of friction and how many cycles of motion the block executes before motion ceases.

Solution:

The natural frequency is calculated as

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.5\ \text{s}} = 12.57\ \frac{\text{rad}}{\text{s}}$$

The decrease in amplitude is expressed as

$$\Delta A = \frac{4\mu m g}{k} = \frac{4\mu g}{\omega_n^2}$$

which is rearranged to yield

$$\mu = \frac{\Delta A}{4g}\omega_n^2 = \frac{(0.01\ \text{m})(12.57\ \text{rad/s})^2}{4(9.81\ \text{m/s}^2)} = 0.04$$

From Eq. (2.77) the motion ceases during the 15th cycle. The mass has a permanent displacement of 2.5 mm from its original equilibrium position.

2.21 A father builds a swing for his children. The swing consists of a board attached to two ropes, as shown in Fig. 2.29. The swing is mounted on a tree branch, with the board 3.5 m below the branch. The diameter of the branch is 8.2 cm and the kinetic coefficient of friction between the ropes and the branch is 0.1. After the swing is installed and his child is seated, the father pulls the swing back 10° and releases. What is the decrease in angle of each swing and how many swings will the child receive before Dad needs to give another push?

Solution:

Because of the friction between the tree branch and the ropes, the tension on opposite sides of a rope will be different. These tensions can be related using the principles of belt friction. When the swing is swinging clockwise, $T_2 > T_1$, and

$$T_2 = T_1 e^{\mu\beta}$$

where β is the angle of contact between the tree branch and the rope. As the child swings the angle of contact may vary. However, this complication is too much to handle with a simplified analysis. A good approximation is to assume β is constant and $\beta = \pi$ rad. When the swing is swinging counterclockwise $T_1 > T_2$ and

$$T_1 = T_2 e^{\mu\beta}$$

Let θ be the clockwise angular displacement of the swing from equilibrium. Summing forces in the direction of the tensions gives

$$\sum F_{\text{ext}} = \sum F_{\text{eff}}$$

$$2T_1 + 2T_2 - mg\cos\theta = ml\dot{\theta}^2$$

The swing is pulled back only 10°. Thus the usual small-angle approximation is valid, with $\cos\theta \approx 1$ and the nonlinear inertia term ignored in comparison to the tensions and gravity. The belt friction relations and the normal force equation are solved simultaneously to yield

$$\dot{\theta} > 0, \qquad T_1 = \frac{mg}{2(1 + e^{\mu\pi})}$$

$$T_2 = \frac{mge^{\mu\pi}}{2(1 + e^{\mu\pi})}$$

$$\dot{\theta} < 0, \qquad T_1 = \frac{mge^{\mu\pi}}{2(1 + e^{\mu\pi})}$$

$$T_2 = \frac{mg}{2(1 + e^{\mu\pi})}$$

Summing moments about the center of the tree branch, using the free-body diagrams of

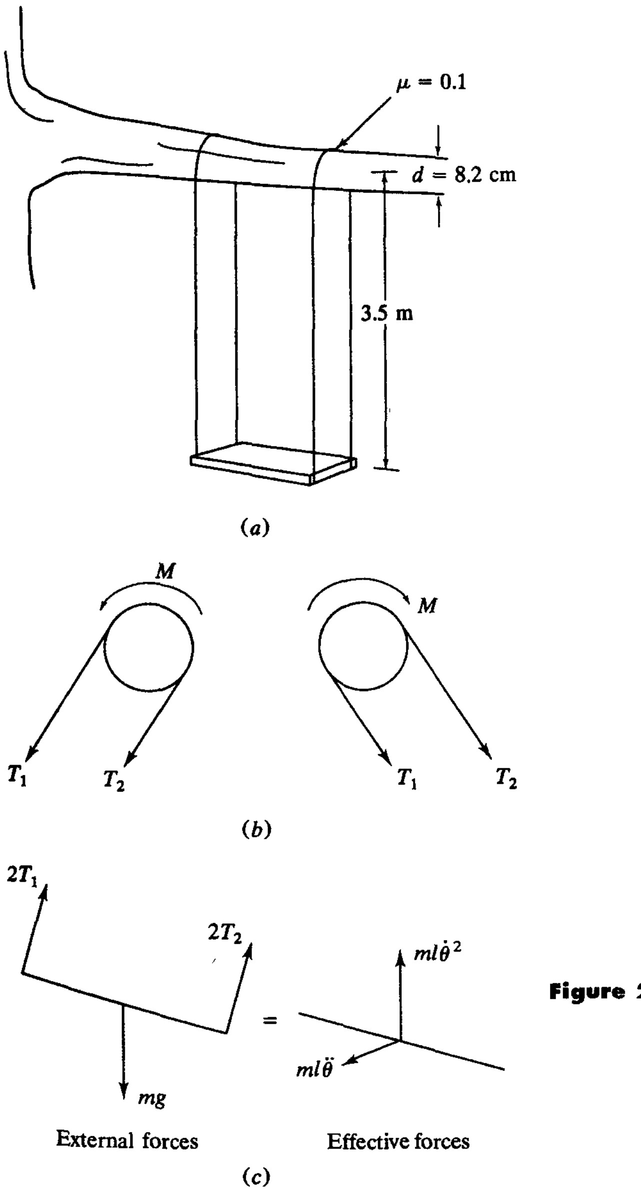


Figure 2.29 (a) Tree swing of Example 2.21; (b) because of friction the tension developed in opposite sides of a rope are unequal; (c) free-body diagrams at an arbitrary instant.

Fig. 2.29c, and the small-angle assumption yields

$$\left(\sum \overset{\frown}{\overset{+}{M}}_O\right)_{\text{ext}} = \left(\sum \overset{\frown}{\overset{+}{M}}_O\right)_{\text{eff}}$$

$$(2T_1 - 2T_2)\frac{d}{2} - mgl\theta = ml^2\ddot{\theta}$$

Substituting for the tensions into the preceding equation and rearranging leads to

$$\ddot{\theta} + \frac{g}{l}\theta = \begin{cases} \dfrac{gd}{2l^2}\dfrac{1 - e^{\mu\pi}}{1 + e^{\mu\pi}} & \dot{\theta} > 0 \\ -\dfrac{gd}{2l^2}\dfrac{1 - e^{\mu\pi}}{1 + e^{\mu\pi}} & \dot{\theta} < 0 \end{cases}$$

The frequency of the swinging is

$$\omega_n = \sqrt{\frac{g}{l}} = 1.67\ \frac{\text{rad}}{\text{s}}$$

which is the same as it would be in the absence of friction.

The governing differential equation is of the form of Eq. (2.78). Thus, from Eq. (2.79), the decrease in amplitude per swing is

$$\frac{2d}{l}\frac{e^{\mu\pi} - 1}{e^{\mu\pi} + 1} = 2\left(\frac{0.082\ \text{m}}{3.5\ \text{m}}\right)\frac{e^{0.1\pi} - 1}{e^{0.1\pi} + 1} = 0.0073\ \text{rad} = 0.42^\circ$$

Motion ceases when, at the end of a cycle, the moment of the gravity force about the center of the branch is insufficient to overcome the frictional moment. This occurs when

$$mgl\theta < |T_2 - T_1|d$$

or

$$\theta < \frac{d}{2l}\frac{e^{\mu\pi} - 1}{e^{\mu\pi} + 1} = 0.10^\circ$$

Thus if Dad does not give the swing another push after 23 swings, the swing will come to rest with an angle of response of 0.1°.

2.7 HYSTERETIC DAMPING

The stress-strain diagram for a typical linearly elastic material is shown in Fig. 2.30. Ideally, if the material is stressed below its yield point and then unloaded, the stress-strain curve for the unloading follows the same curve for the loading. However, in a real engineering material, internal planes slide relative to one another and molecular bonds are broken, causing conversion of strain energy into thermal energy and causing the process to be irreversible. A more realistic stress-strain curve for the loading-unloading process is shown in Fig. 2.31.

The curve in Fig. 2.31 is a hysteresis loop. The area enclosed by the curve is the dissipated strain energy per unit volume. The area enclosed by the hysteresis loop from a force-displacement curve is the total strain energy dissipated during a loading-unloading cycle. In general, the area under a hysteresis curve is independent of the rate of the loading-unloading cycle.

In a vibrating mechanical system an elastic member undergoes a cyclic load-displacement relationship as shown in Fig. 2.32. The loading is repeated over each cycle. The existence of the hysteresis loop leads to energy dissipation from the system during each cycle, which causes natural damping, called *hysteretic damping*.

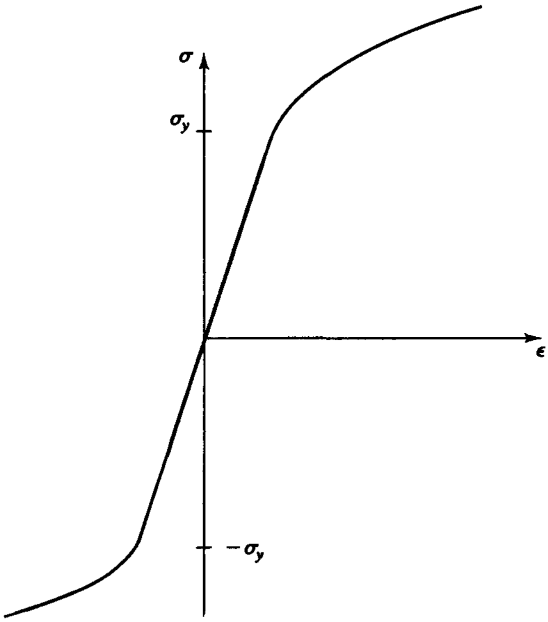


Figure 2.30 Stress-strain diagram for a linearly elastic isotropic material with the same behavior in compression and tension. Material behavior is linear when $|\sigma| < \sigma_y$.

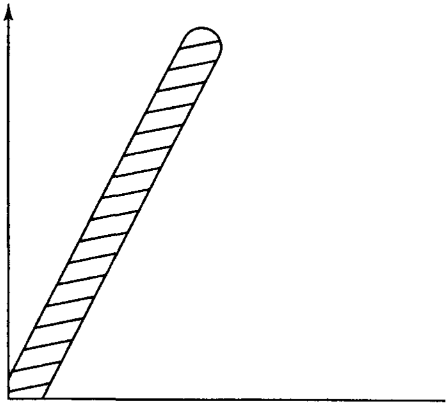


Figure 2.31 Imperfections such as dislocations cause the loading process in most rea materials to be irreversible resulting in a hysteresi loop. The energy dissipated is the area within the hysteresis loop.

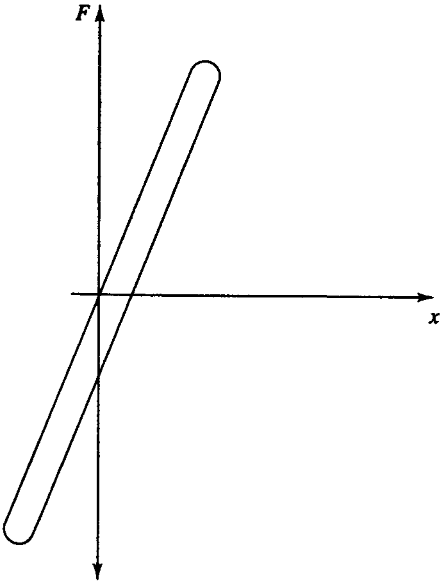


Figure 2.32 A system undergoing periodic vibrations has a recurring load-displacement diagram. The amplitude decreases over each cycle of motion due to hysteretic damping.

It has been shown experimentally that the energy dissipated per cycle of motion is independent of the frequency and proportional to the square of the amplitude. An empirical relationship is

$$\Delta E = \pi k h X^2 \tag{2.80}$$

where X is the amplitude of motion during the cycle and h is a constant, called the *hysteretic damping coefficient*.

The hysteretic damping coefficient cannot be simply specified for a given material. It is dependent upon other considerations such as how the material is prepared and the geometry of the structure under consideration. Existing data cannot be extended to apply to every situation. Thus it is usually necessary to empirically determine the hysteretic damping coefficient.

Mathematical modeling of hysteretic damping is developed from a work-energy analysis. Consider a simple mass-spring system with hysteretic damping. Let X_1 be the amplitude at a time when the velocity is zero and all energy is potential energy stored in the spring. Hysteretic damping dissipates some of that energy over the next cycle of motion. Let X_2 be the displacement of the mass at the next time when the velocity is zero, after the system executes one half-cycle of motion. Let X_3 be the displacement at the subsequent time when the velocity is zero, one full cycle later. Application of the work-energy principle over the first half-cycle of motion gives

$$T_1 + V_1 + U_{1\to 2} = T_2 + V_2$$

$$T_1 + V_1 = T_2 + V_2 + \frac{\Delta E}{2}$$

The energy dissipated by hysteretic damping is approximated by Eq. (2.80) with X as the amplitude at the beginning of the half-cycle.

$$\tfrac{1}{2}kX_1^2 = \tfrac{1}{2}kX_2^2 + \tfrac{1}{2}\pi k h X_1^2$$

This yields

$$X_2 = \sqrt{1 - \pi h}\, X_1$$

A work-energy analysis over the second half-cycle leads to

$$X_3 = \sqrt{1 - \pi h}\, X_2 = (1 - \pi h) X_1 \tag{2.81}$$

Thus the rate of decrease of amplitude on successive cycles is constant, as it is for viscous damping. By analogy a logarithmic decrement is defined for hysteretic damping as

$$\delta = \ln \frac{X_1}{X_3} = -\ln(1 - \pi h) \tag{2.82}$$

which for small h is approximated as

$$\delta = \pi h \tag{2.83}$$

By analogy with viscous damping an equivalent damping ratio for hysteretic damping is defined as

$$\zeta = \frac{\delta}{2\pi} = \frac{h}{2} \qquad [2.84]$$

and an equivalent viscous damping coefficient is defined as

$$c_{eq} = 2\zeta\sqrt{\tilde{m}\tilde{k}} = \frac{h\tilde{k}}{\omega_n} \qquad [2.85]$$

The free vibrations response of a system subject to hysteretic damping is the same as the response of the system when subject to viscous damping with an equivalent viscous damping coefficient given by Eq. (2.85). This is true only for small hysteretic damping, as subsequent plastic behavior leads to a highly nonlinear system. The analogy between viscous damping and hysteretic damping is also only true for linearly elastic materials and for materials where the energy dissipated per unit cycle is proportional to the square of the amplitude. In addition, the hysteretic damping coefficient is a function of geometry as well as the material.

The response of a system subject to hysteretic or viscous damping continues indefinitely with exponentially decaying amplitude. However, hysteretic damping is significantly different from viscous damping in that the energy dissipated per cycle for hysteretic damping is independent of frequency, whereas the energy dissipated per cycle increases with frequency for viscous damping. Thus while the mathematical treatments of viscous damping and hysteretic damping are the same, they have significant physical differences.

Example 2.22

The force-displacement curve for a structure is shown in Fig. 2.33. The structure is modeled as a one-degree-of-freedom system with an equivalent mass 500 kg located at the position where the measurements are made. Describe the response of this structure when a shock imparts a velocity of 20 m/s to this point on the structure.

Solution:

The area under the hysteresis curve is approximated by counting the squares inside the hysteresis loop. Each square represents $(1 \times 10^4 \text{ N})(0.002 \text{ m}) = 20 \text{ N} \cdot \text{m}$ of dissipated energy. There are approximately 38.5 squares inside the hysteresis loop resulting in 770 $\text{N} \cdot \text{m}$ dissipated over one cycle of motion with an amplitude of 20 mm.

The equivalent stiffness is the slope of the force deflection curve and is determined as 5×10^6 N/m. Application of Eq. (2.80) leads to

$$h = \frac{\Delta E}{\pi k X^2} = \frac{770 \text{ N} \cdot \text{m}}{\pi(5 \times 10^6 \text{ N/m})(0.02 \text{ m})^2} = 0.123$$

The logarithmic decrement, damping ratio, and natural frequency are calculated by using Eqs. (2.83) and (2.84)

$$\delta = \pi h = 0.386$$

$$\zeta = \frac{h}{2} = 0.0615$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{5 \times 10^6 \text{ N/m}}{500 \text{ kg}}} = 100 \frac{\text{rad}}{\text{s}}$$

The response of this structure with hysteretic damping is approximately the same as the response of a simple mass-spring-dashpot system with a damping ratio of 0.0615 and a

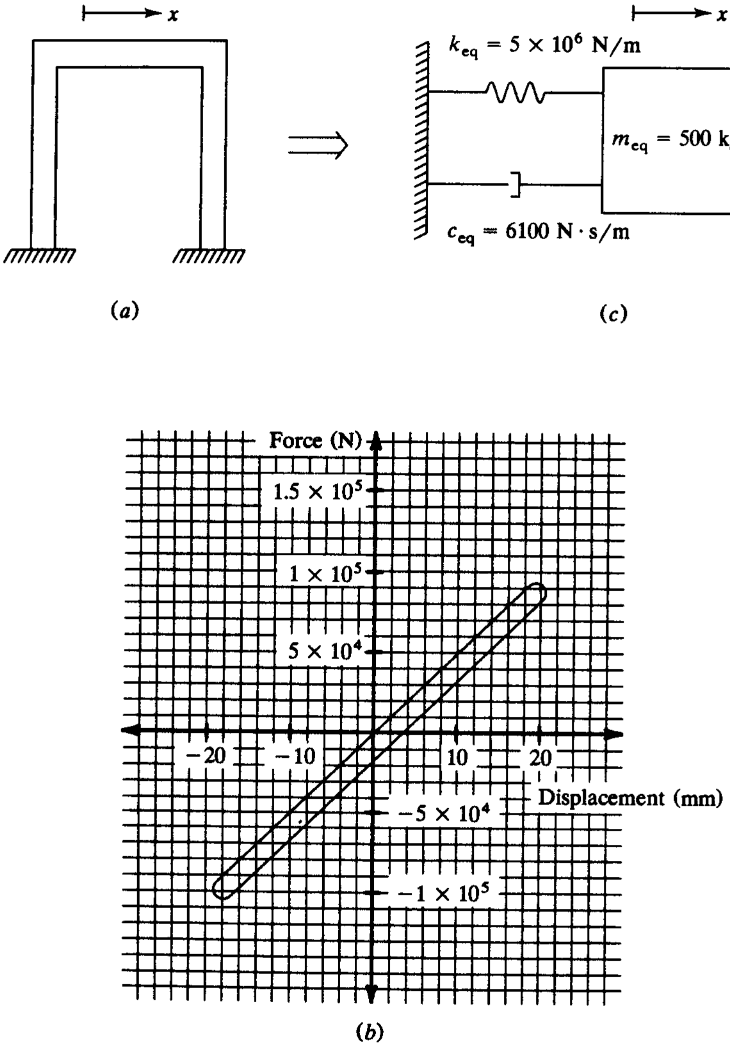


Figure 2.33 (*a*) One-story frame structure modeled as a one-degree-of-freedom system; (*b*) force-displacement curve for the structure of Example 2.20; (*c*) hysteretic damping leads to an equivalent viscous damping coefficient of 6100 N · s/m for this example.

natural frequency of 100 rad/s. Then from Eq. (2.45) with $\dot{x}_0 = 20$ m/s, and $x_0 = 0$, the response is

$$x(t) = 0.20e^{-6.15t} \sin(99.81t) \text{ m}$$

2.8 OTHER FORMS OF DAMPING

A mechanical or structural system may be subject to other forms of damping such as aerodynamic drag, radiation damping, or anelastic damping. However, these give rise to nonlinear terms in the governing differential equations. Exact solutions do not exist for these forms of damping. The periodic motion of systems subject to these forms of damping can be approximated by developing an equivalent viscous damping coefficient. The equivalent viscous damping coefficient is obtained by equating the energy dissipated over one cycle of motion, assuming harmonic motion at a specific amplitude and frequency, for the particular form of damping with the energy dissipated over one cycle of motion because of the force in a dashpot of the equivalent viscous damping coefficient.

For a harmonic motion of the form

$$x(t) = X \sin \omega t$$

the energy dissipated over one cycle of motion due to a damping force F_D is

$$\Delta E = \int_0^{2\pi/\omega} F_D \dot{x}\, dt = \int_0^{2\pi/\omega} F_D X \omega \cos \omega t\, dt \tag{2.86}$$

For viscous damping Eq. (2.86) yields

$$\Delta E = \int_0^{2\pi/\omega} c\dot{x}^2\, dt = \int_0^{2\pi/\omega} c\omega^2 X^2 \cos^2 \omega t\, dt = c\omega\pi X^2 \tag{2.87}$$

Thus, by analogy, the equivalent viscous damping coefficient for another form of damping is

$$c_{eq} = \frac{\Delta E}{\pi \omega X^2} \tag{2.88}$$

Aerodynamic drag is present in all real problems. However, its effect is often ignored. The determination of the correct form of the drag force is a problem in fluid mechanics. At high Reynolds numbers the drag is very nearly proportional to the square of the velocity and can be written as

$$F_D = C_D \dot{x} |\dot{x}| \tag{2.89}$$

where C_D is a coefficient that is a function of body geometry and air properties. For moderate Reynolds numbers appropriate forms of the drag force have been proposed as

$$F_D = C_D |\dot{x}|^\alpha \dot{x} \tag{2.90}$$

where $0 < \alpha \leq 1$. In either case the resulting differential equation is nonlinear.

Some materials (e.g., rubber) are viscoelastic and obey a constitutive equation in which stress is related to strain and strain rate. It is shown in Chap. 3 that for an undamped system the forced response is in phase with a harmonic excitation, whereas a phase lag occurs for a damped system. This phase lag also occurs for many viscoelastic materials. Indeed, many viscoelastic materials have constitutive equations that are derived by modeling the material as a spring in parallel with a dashpot. This is called a Kelvin model. The phase lag results in energy dissipation and the resulting damping is called anelastic damping.

Damping occurs when energy is dissipated from a vibrating body by any means. Another example is radiation damping that occurs for a body vibrating on the free surface between two fluids. The vibrating body causes pressure waves to be radiated outward, causing energy transfer from the body to the surrounding fluids.

Most physical systems are subject to a combination of forms of damping. Indeed, a simple mass-spring-dashpot system is subject to viscous damping from the dashpot, Coulomb damping from the dry sliding friction, hysteretic damping from the spring, and aerodynamic drag. The presence of Coulomb damping leads to cessation of free vibrations after a finite time. The aerodynamic drag is usually neglected in an analysis as its effect is negligible and it leads to a nonlinear differential equation. The hysteretic damping acts in parallel with the viscous damping. The equivalent damping coefficient is the sum of the viscous damping coefficient for the dashpot and the equivalent viscous damping coefficient for the hysteretic damping. For small amplitudes the effect of viscous damping is much greater than the effect of hysteretic damping. For large amplitudes the hysteretic damping can be dominant.

2.23 A block of mass 1 kg is attached to a spring of stiffness 3×10^5 N/m. The block is displaced 20 mm from equilibrium and released from rest. The block is in a fluid where the drag force is given by Eq. (2.89) with $C_D = 0.86\ \mathrm{N \cdot s^2/m}$. Approximate the number of cycles before the amplitude is reduced to 15 mm.

Solution:

The energy lost per cycle of motion due to aerodynamic drag is calculated from Eq. (2.86)

$$\Delta E = \int_0^{2\pi/\omega} C_D X^3 \omega^3 \cos^2 \omega t |\cos \omega t|\, dt$$

$$= 4 \int_0^{\pi/2\omega} C_D X^3 \omega^3 \cos^3 \omega t\, dt = \frac{8}{3} C_D \omega^2 X^3$$

From Eq. (2.88) the equivalent viscous damping coefficient is calculated as

$$c_{\text{eq}} = 0.730 \omega X$$

If the equivalent viscous damping is small, the frequency is approximately equal to the natural frequency of free undamped vibrations

$$\omega = \sqrt{\frac{k}{m}} = 547.7\ \frac{\text{rad}}{\text{s}}$$

Table 2.2 Viscous approximation used to predict decay in amplitude for Example 2.21

Cycle	Amplitude at beginning of cycle $X_n = X_{n-1}e^{-2.32X_{n-1}}$
1	20.0
2	19.09
3	18.26
4	17.50
5	16.81
6	16.16
7	15.56
8	15.00

The damping ratio on a given cycle is

$$\zeta = \frac{c_{eq}}{2\sqrt{km}} = \frac{0.73(547.7\ \text{rad/s})X}{2\sqrt{(1\ \text{kg})(3\times 10^5\ \text{N/m})}} = 0.37X$$

From Eq. (2.56) the logarithmic decrement is

$$\delta = 2\pi\zeta = 2.32X$$

Since the equivalent viscous damping coefficient, and hence the damping ratio and the logarithmic decrement, depend on the amplitude, the decrease in amplitude is not constant on each cycle. Using an amplitude of 20 mm for the first cycle, the amplitude at the beginning of the second cycle is obtained using the logarithmic decrement, which in turn is used to predict the amplitude at the beginning of the third cycle. Table 2.2 is developed in this fashion. The amplitude of vibration is reduced to 15 mm in seven cycles.

PROBLEMS

2.1–2.10. Derive the differential equation governing the motion of the one-degree-of-freedom system by applying the appropriate form(s) of Newton's laws to the appropriate free-body diagrams. Use the generalized coordinate shown. Linearize nonlinear differential equations by assuming small displacements. Determine the undamped natural frequency for each system.

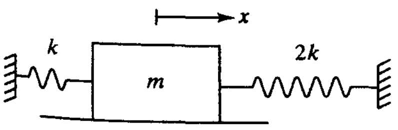


FIGURE P2.1, P2.11

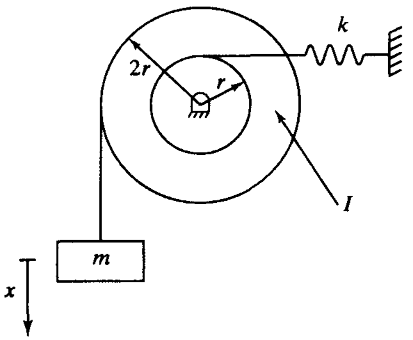


FIGURE P2.2, P2.12

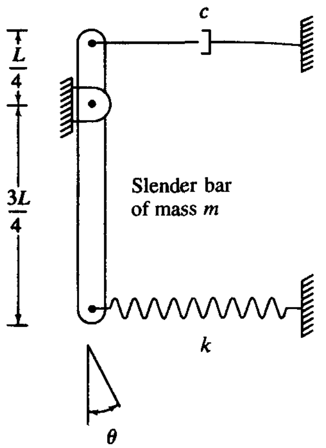


FIGURE P2.3, P2.13

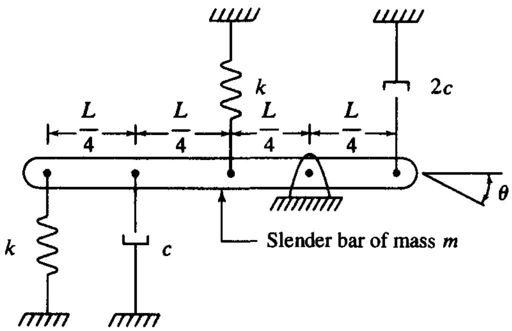
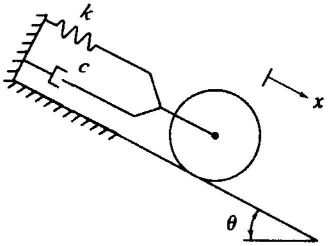
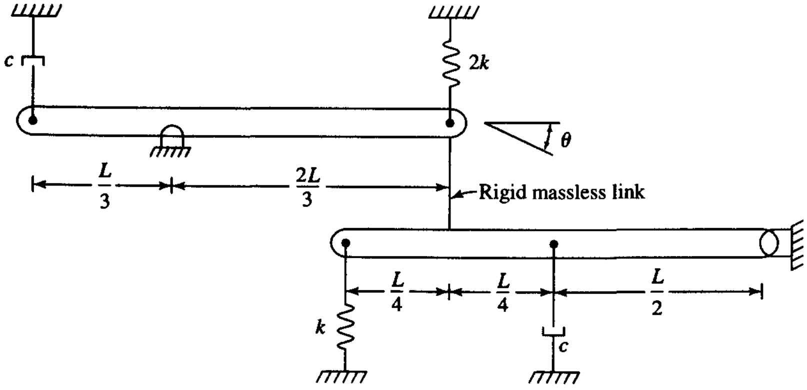


FIGURE P2.4, P2.14



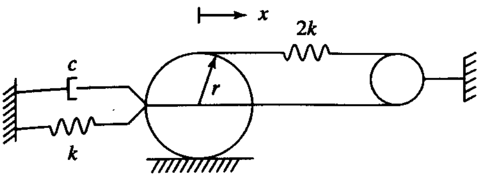
Thin disk of mass m radius r rolls without slip

FIGURE P2.5, P2.15



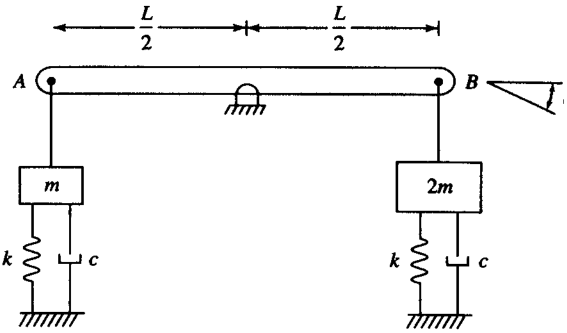
Identical slender bars of mass m, length L

FIGURE P2.6, P2.16



Thin disk of mass m, radius r

FIGURE P2.7, P2.17



Slender bar of mass m connected to blocks through rigid links at A and B

FIGURE P2.8, P2.18

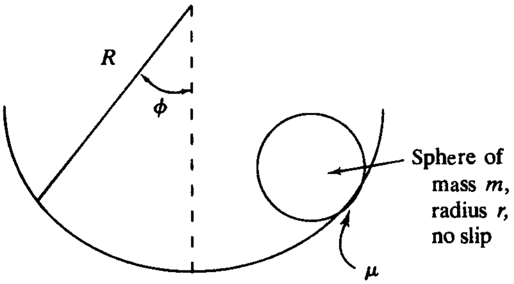


FIGURE P2.9, P2.19

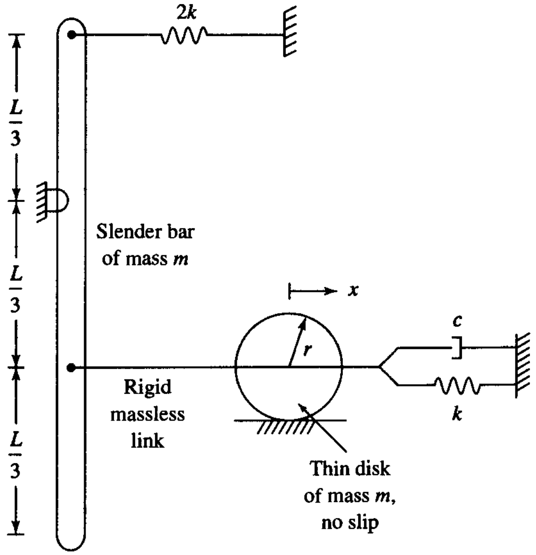


FIGURE P2.10, P2.20

2.11–2.20. Derive the differential equation governing the motion of the one-degree-of-freedom system using the equivalent systems method. Use the generalized coordinate shown. Linearize nonlinear differential equations by assuming small displacements. Determine the undamped natural frequency for each system.

2.21–2.25. Assuming the inertia of the elastic elements is small, use a one-degree-of-freedom model to approximate the natural frequency of the system.

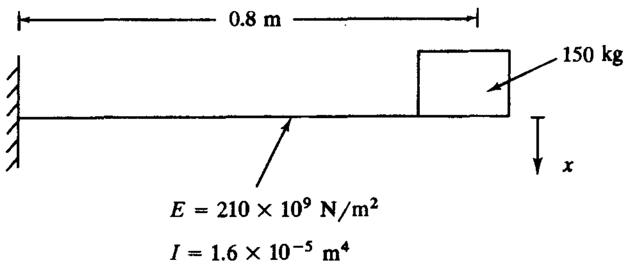


FIGURE P2.21

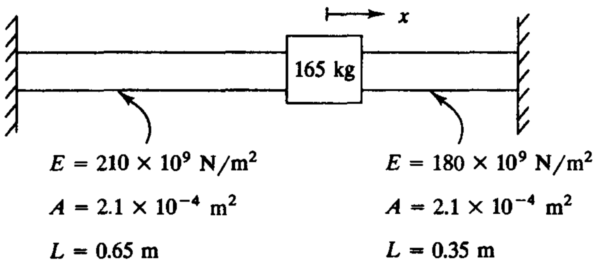


FIGURE P2.22

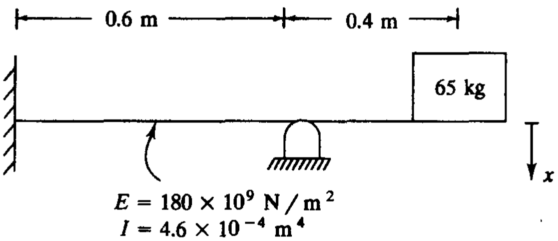


FIGURE P2.23

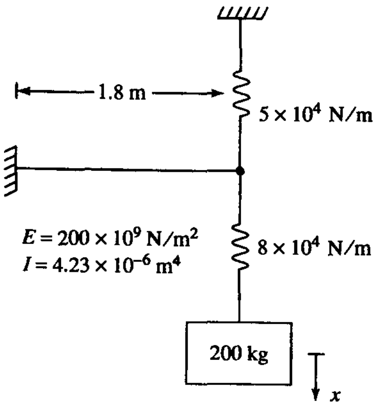


FIGURE P2.24

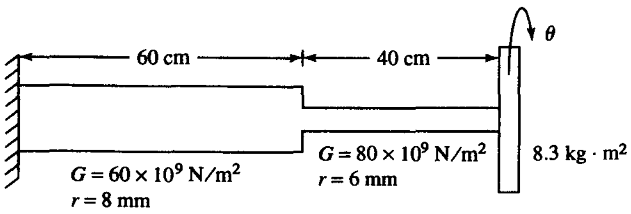


FIGURE P2.25

2.26. When an engine of mass 400 kg is mounted on an elastic foundation, the foundation deflects 5 mm. What is the natural frequency of the system in rpm?

2.27. The cylindrical container of Fig. P2.27 has a mass of 25 and floats stably on the surface of an unknown fluid. When disturbed, the period of free oscillations is measured as 0.2 s. What is the specific gravity of the liquid?

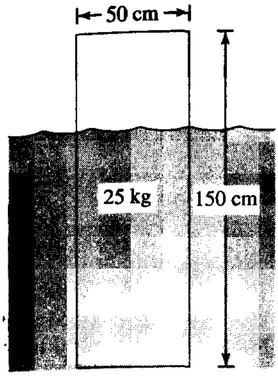


FIGURE P2.27

2.28. A ceiling fan is an assembly of five blades driven by a motor. The assembly is attached to the end of a thin shaft whose other end is fixed to the ceiling. What is the natural frequency of torsional oscillations of the fan of Fig. P2.28?

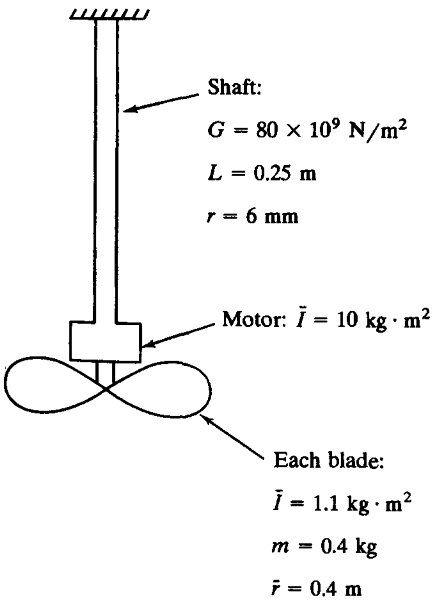


FIGURE P2.28

2.29. The mass of the pendulum bob of a cuckoo clock is 0.49 kg. How far from the pin should the bob be placed such that the pendulum's period of oscillation is 1.0 s?

2.30. When the 5.1 kg connecting rod of Fig. P2.30 is placed in the position shown, the spring deflects 0.5 mm. When the end of the rod is displaced and released, the resulting period of oscillation is observed as 0.15 s. Determine the location of the center of mass of the connecting rod and the centroidal mass moment of inertia of the rod.

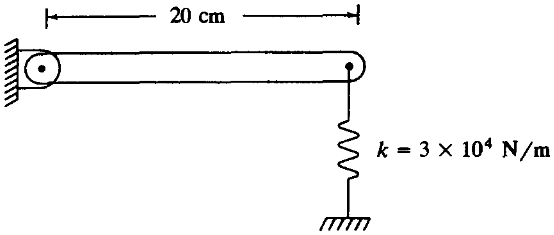


FIGURE P2.30

2.31. When a 2000-lb vehicle is empty, the static deflection of its suspension system is measured as 0.8 in. What is the natural frequency of the vehicle when it is carrying 700 lb of passengers and cargo?

2.32. A 400-kg machine is placed at the midspan of a 3.2-m simply supported steel ($E = 200 \times 10^9$ N/m^2) beam. The machine is observed to vibrate with a natural frequency of 9.3 Hz. What is the moment of inertia of the beam's cross section about its neutral axis?

2.33. A one-degree-of-freedom model of a 9-m steel flagpole ($\rho = 7400$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $G = 80 \times 10^9$ N/m^2) is that of a beam fixed at one end and free at one end. The flagpole has an inner diameter of 4 cm and an outer diameter of 5 cm.
(*a*) Approximate the natural frequency of transverse vibration.
(*b*) Approximate the natural frequency of torsional oscillation.

2.34. A 250-kg compressor is to be placed at the end of a 2.5-m fixed-free steel ($E = 200 \times 10^9$ N/m^2) beam. Specify the allowable moment of inertia of the beam's cross section about its neutral axis such that the natural frequency of the machine is outside the range of 100 to 130 Hz.

2.35. A 50-kg pump is to be placed at the midspan of a 2.8-m simply supported steel ($E = 200 \times 10^9$ N/m^2) beam. The beam is of rectangular cross section of width 25 cm. What are the allowable values of the cross-sectional height such that the natural frequency is outside the range of 50 to 75 Hz?

2.36. A diving board is modeled as a simply supported beam with an overhang. What is the natural frequency of a 140-lb diver at the end of the diving board of Fig. P2.36?

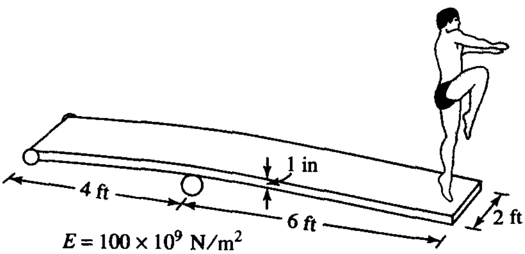


FIGURE P2.36

2.37. A diver is able to slightly adjust the location of the intermediate support on the diving board in Fig. P2.36. What is the range of natural frequencies a 140-lb diver can attain if the distance between the supports can be adjusted between 4 and 6.5 ft?

2.38. A 60-kg drum of waste material is being hoisted by an overhead crane and winch system as illustrated in Fig. P2.38. The system is modeled as a simply supported beam to which the cable is attached. The drum of waste material is attached to the end of the cable. When the length of the cable is 6 m, the natural period of the system is measured as 0.3 s. What is the mass of the waste material?

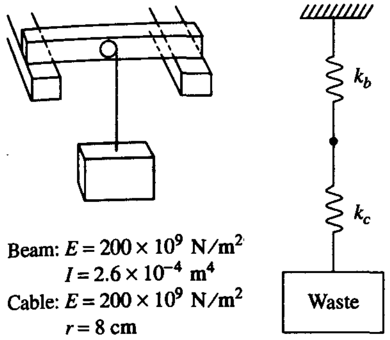


FIGURE P2.38

2.39. One end of the mercury-filled U-tube manometer of Fig. P2.39 is open to the atmosphere while the other end is capped and under a pressure of 20 psig. The cap is suddenly removed. Determine $x(t)$, the displacement of the mercury-air interface from the column's equilibrium position.

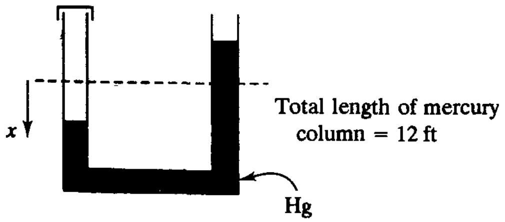


FIGURE P2.39

2.40. A 200-kg package is being hoisted by a 120-mm-diameter steel cable ($E = 200 \times 10^9$ N/m^2) at a constant velocity v. What is the largest value of v such that the cable's elastic strength of 560×10^6 N/m^2 is not exceeded if the hoisting mechanism suddenly fails when the cable has a length of 10 m.

2.41. A 3-kg block is hanging in equilibrium from a coil spring of diameter 5 cm made from a steel rod of diameter 5 mm. The shear modulus of the spring's material is $G = 80 \times 10^9$ N/m^2 and the spring has 25 active coils. A weight W is dropped from 20 cm onto the block. On impact the weight attaches itself to the block. What is the largest weight that can be dropped such that the spring's elastic shear strength of 100×10^6 N/m^2 is not exceeded?

2.42. The center of the disk Fig. P2.42 is displaced a distance δ and the system released. Determine $x(t)$ if the disk rolls without slip.

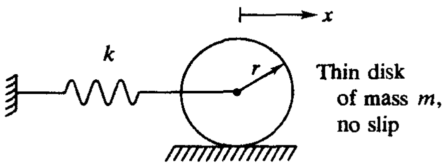


FIGURE P2.42

2.43. The coefficient of friction between the disk and the surface in Fig. P2.42 is μ. What is the largest initial velocity of the mass center that can be imparted such that the disk rolls without slip for its entire motion?

2.44–2.47. For the systems shown

(a) Determine the damping ratio

(b) State whether the system is underdamped, critically damped, or overdamped

(c) Determine $x(t)$ or $\theta(t)$ for the given initial conditions

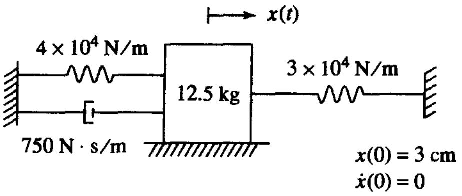


FIGURE P2.44

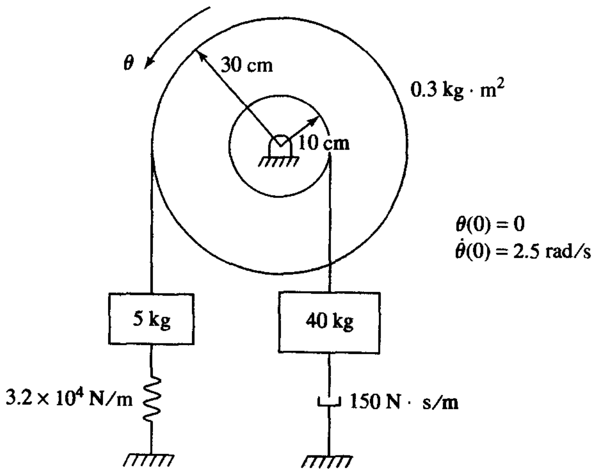


FIGURE P2.45

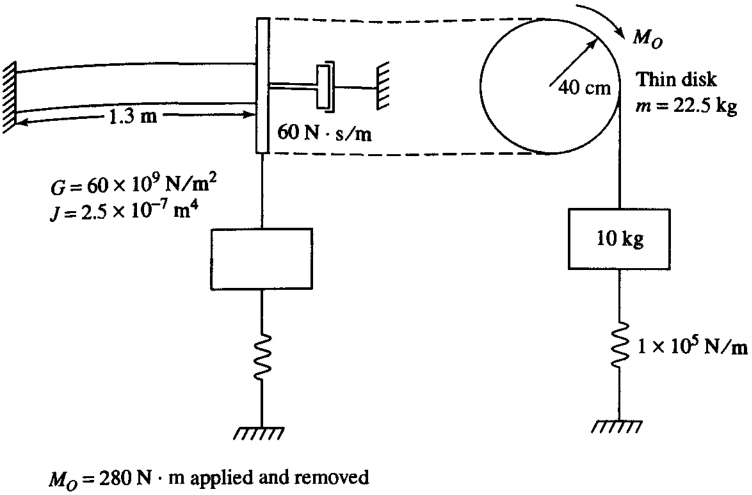


FIGURE P2.46

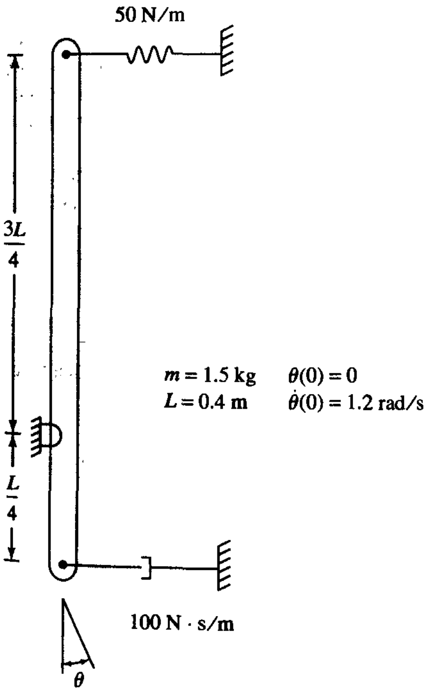


FIGURE P2.47

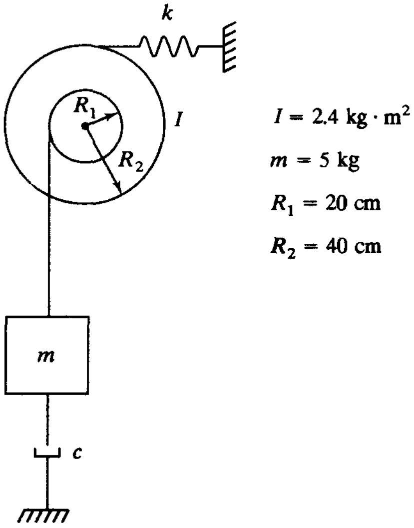


FIGURE P2.48

2.48. The amplitude of vibration of the system of Fig. P2.48 decays to half of its initial value in 11 cycles with a period of 0.3 s. Determine the spring stiffness and the viscous damping coefficient.

2.49. The damping ratio of the system of Fig. P2.49 is 0.3. How long will it take for the amplitude of free oscillation to be reduced to 2 percent of its initial value?

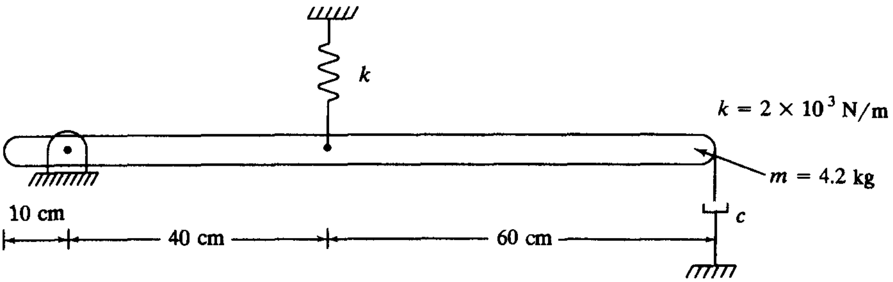


FIGURE P2.49

2.50. When a 40-kg machine is placed on an elastic foundation, its free vibrations appear to decay exponentially with a frequency of 91.7 rad/s. When a 60-kg machine is placed on the same foundation, the frequency of the exponentially decaying oscillations is 75.5 rad/s. Determine the equivalent stiffness and equivalent viscous damping coefficient for the foundation.

2.51. A suspension system is being designed for a 1300-kg vehicle. When the vehicle is empty, its static deflection is measured as 2.5 mm. It is estimated that the largest cargo carried by the vehicle will be 1000 kg. What is the minimum value of the damping coefficient such that the vehicle will be subject to no more than 5 percent overshoot, whether it is empty or fully loaded.

2.52. During operation a 500-kg press machine is subject to an impulse of magnitude 5000 N · s. The machine is mounted on an elastic foundation that can be modeled as a spring of stiffness 8×10^5 N/m in parallel with a viscous damper of damping coefficient 6000 N · s/m. What is the maximum displacement of the press after the impulse is applied. Assume the press is at rest when the impulse is applied.

2.53. The disk of Fig. P2.53 rolls without slip.

(*a*) What is the critical damping coefficient, c_c, for the system?

(*b*) If $c = c_c/2$, plot the response of the system when the center of the disk is displaced 5 mm from equilibrium and released from rest.

(*c*) Repeat part (*b*) if $c = 3c_c/2$.

(*d*) If the coefficient of friction between the disk and surface is 0.15, is the no-slip assumption still valid for the systems of parts (*b*) and (*c*)?

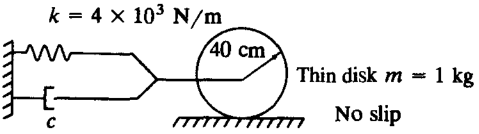


FIGURE P2.53

2.54. A recoil mechanism of a gun is designed as a spring and viscous damper in parallel such that the system has critical damping. A 52-kg cannon has a maximum recoil of 50 cm after firing. Specify the stiffness and damping coefficient of the recoil mechanism such that the mechanism returns to within 5 mm of firing position within 0.5 s after firing.

2.55. The initial recoil velocity of a 1.4-kg gun is 2.5 m/s. Design a recoil mechanism that is critically damped such that the mechanism returns to within 0.5 mm of firing within 0.5 s after firing.

2.56. A railroad bumper is modeled as a linear spring in parallel with a viscous damper. What is the damping coefficient of a bumper of stiffness 2×10^5 N/m such that the system has a damping ratio of 1.15 when it is engaged by a 22,000-kg railroad car.

2.57. What is the maximum deflection of the bumper of Prob. 2.56 when it is engaged by a 22,000-kg railroad car traveling at 5 mph?

2.58. A block of mass m is attached to a spring of stiffness k and slides on a horizontal surface with a coefficient of friction μ. At some time t, the velocity is zero and the block is displaced a distance δ from equilibrium. Use the principle of work-energy to calculate the spring deflection at the next instant when the velocity is zero. Can this result be generalized to determine the decrease in amplitude between successive cycles?

2.59. Reconsider Example 2.21 using a work-energy analysis. That is, assume the amplitude of the swing is θ at the end of an arbitrary cycle. Use the principle of work-energy to determine the amplitude at the end of the next half-cycle.

2.60. The center of the thin disk of Fig. P2.60 is displaced a distance δ and the disk released. The coefficient of friction between the disk and the surface is μ. The initial displacement is sufficient to cause the disk to roll and slip.

(*a*) Derive the differential equation governing the motion when the disk rolls and slips.

(*b*) When the displacement of the mass center from equilibrium becomes small enough, the disk rolls without slip. At what displacement does this occur?

(*c*) Derive the differential equation governing the motion when the disk rolls without slip.

(*d*) What is the change in amplitude per cycle of motion?

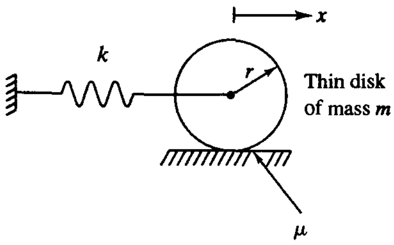


FIGURE P2.60

2.61. A 10-kg block is attached to a spring of stiffness 3×10^4 N/m. The block slides on a horizontal surface with a coefficient of friction of 0.2. The block is displaced 30 mm and released. How long will it take before the block returns to rest?

2.62. The block of Prob. 2.61 is displaced 30 mm and released. What is the range of values of the coefficient of friction such that the block comes to rest during the 14th cycle?

2.63. A 2.2-kg block is attached to a spring of stiffness 1000 N/m and slides on a surface that makes an angle of 7° with the horizontal. When displaced from equilibrium and released, the decrease in amplitude per cycle of motion is observed to be 2 mm. Determine the coefficient of friction.

2.64. A block of mass m is attached to a spring of stiffness k and viscous damper of damping coefficient c and slides on a horizontal surface with a coefficient of friction μ. Let $x(t)$ represent the displacement of the block from equilibrium.

(*a*) Derive the differential equation governing $x(t)$.

(*b*) Solve the equation and sketch the response over two periods of motion.

2.65. A connecting rod is fitted around a cylinder with a connecting rod between the cylinder and bearing. The coefficient of friction between the cylinder and bearing is 0.08. If the rod is rotated 12° counterclockwise and then released, how many cycles of motion will it execute before it comes to rest? The ratio of the diameter of the cylinder to the distance to the center of mass of the connecting rod from the center of the cylinder is 0.01.

2.66. A one-degree-of-freedom structure has a mass of 65 kg and a stiffness of 238 N/m. After 10 cycles of motion the amplitude of free vibrations is decreased by 75 percent. Calculate the hysteretic damping coefficient and the total energy lost during the first 10 cycles if the initial amplitude is 20 mm.

2.67. The end of a steel cantilever beam ($E = 210 \times 10^9$ N/m^2) of $I = 1.5 \times 10^{-4}$ m^4 is given an initial amplitude of 4.5 mm. After 20 cycles of motion the amplitude is observed as 3.7 mm. Determine the hysteretic damping coefficient and the equivalent viscous damping ratio for the beam.

2.68. A 500-kg press is placed at the midspan of a simply supported beam of length 3 m, elastic modulus 200×10^9 N/m^2, and cross-sectional moment of inertia 1.83×10^{-5} m^4. It is observed that free vibrations of the beam decay to half of the initial amplitude in 35 cycles. Determine the response of the press, $x(t)$, if it is subject to an impulse of magnitude 10,000 N · s.

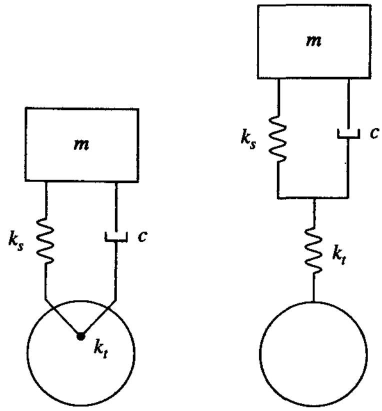
2.69. Use the theory of Sec. 2.7 to derive the equivalent viscous damping coefficient for Coulomb damping. Compare the response of a one-degree-of-freedom system of natural frequency 35 rad/s and friction coefficient 0.12 using the exact theory to that obtained using the approximate theory with an equivalent viscous damping coefficient.

2.70. A 0.5-kg sphere is attached to a spring of stiffness 6000 N. The sphere is given an initial displacement of 8 mm from its equilibrium position and released. If aerodynamic drag is the only source of friction, how many cycles will the system execute before the amplitude is reduced to 1 mm?

2.71. A one-degree-of-freedom model of a suspension system is shown in Fig. P2.71a. For this model the mass of the vehicle is much greater than the axle mass, but the tire has characteristics which should be included in the analysis. In the model of Fig. P2.71b, the tire is assumed to be elastic with a stiffness k_t. The tire stiffness acts in series with the spring and viscous damper of the suspension system.

(*a*) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.

(*b*) Solve the differential equation to determine the response of the system when the wheel encounters a pothole of depth h.

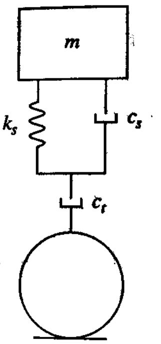


(*a*) (*b*) **FIGURE P2.71**

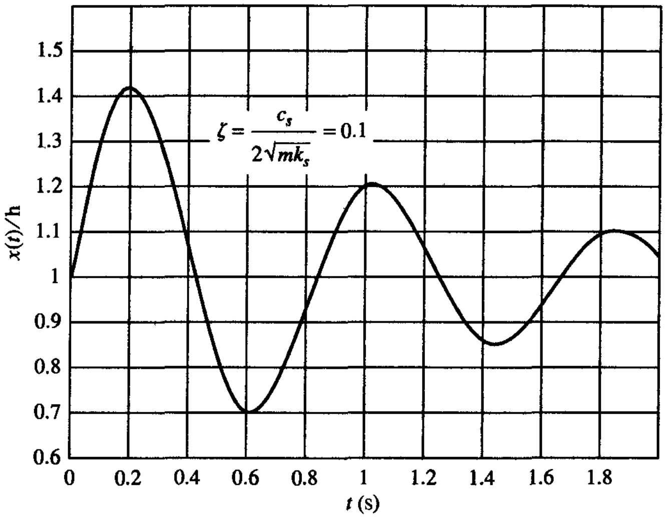
2.72. A one-degree-of-freedom model of a suspension system is shown in Fig. P2.72*a*. Consider a model in which the tire is modeled by a viscous damper of damping coefficient c_t and is placed in series with the spring and viscous damper modeling the suspension system, as illustrated in Fig. P2.72*a*.

(*a*) Derive a third-order differential equation governing the displacement of the vehicle from the system's equilibrium position.

(*b*) A plot of the suspension system when the wheel encounters a pothole is given in Fig. P2.72*b*. The plot is made for a suspension system that is designed to have a damping ratio of 0.1. Use this information to find c_t.



(*a*)



(*b*)

FIGURE P2.72

MATLAB PROBLEMS

M2.1. File VIBES_2A.m provides the natural frequency calculations for a machine mounted on a fixed-pinned beam, including an approximation for inertia effects of the beam. Use VIBES_2A to help decide upon an appropriate I beam of W shape for the following cases

(*a*) A 400-kg turbine is to be mounted on a 5.4-m steel beam a distance 2.3 m from the fixed end. The natural frequency of the system must be outside the range of 40 to 200 Hz.

(*b*) A 215-kg compressor is to be mounted on a 6.5 m steel beam a distance 2.8 m from the fixed end. The natural frequency of the system must be outside the range of 20 to 600 Hz.

M2.2. Use VIBES_2A.m to select a material to use for the beam in the following situation. A 400-kg pump is to be placed at the midspan of a 4.4 m standard I beam, W27 $\times$ 114. The natural frequency of the system must be greater than 600 Hz.

M2.3. File VIBES_2B.m provides the natural frequency calculations for the diver of Probs. 2.36 and 2.37. Use the file to

(*a*) Plot the natural frequency of the diver assuming the intermediate support is 4 ft from the pinned end

(*b*) Plot the distance of the intermediate support from the pinned end as a function of the diver's weight in order for the diver's natural frequency to be 3 Hz.

M2.4. File VIBES_2C.m provides the free-vibration response of a one-degree-of-freedom system with viscous damping. Use the file to determine the maximum displacement, maximum velocity, and maximum acceleration for a one-degree-of-freedom mass–spring–viscous damper system under the following conditions

(*a*) $m = 150$ kg, $k = 10,000$ N/m, $c = 120$ N · s/m, $x(0) = 0.01$ m, $\dot{x}(0) = 0$

(*b*) $m = 150$ kg, $k = 50,000$ N/m, $c = 7000$ N · s/m, $x(0) = 0$, $\dot{x}(0) = 0.5$ m/s

(*c*) $m = 100$ kg, $k = 10,000$ N/m, $c = 2000$ N · s/m, $x(0) = 0.05$ m, $\dot{x}(0) = 0.05$ m/s

(*d*) $m = 100$ kg, $k = 10,000$ N/m, $c = 2000$ N · s/m, $x(0) = 0.05$ m, $\dot{x}(0) = 0.2$ m/s

M2.5. File VIBES_2D.m provides the free-vibration response of a one-degree-of-freedom system with Coulomb damping. Use the file to plot the free-vibration response for a one-degree-of-freedom system under the following conditions

(*a*) $m = 100$ kg, $k = 1 \times 10^6$ N/m, $\mu = 0.08$, $x(0) = 0.08$ m, $\dot{x}(0) = 0$

(*b*) $m = 100$ kg, $k = 1 \times 10^6$ N/m, $\mu = 0.12$, $x(0) = 0$, $\dot{x}(0) = 0.85$ m/s

(*c*) $m = 20$ kg, $k = 1 \times 10^5$ N/m, $\mu = 0.04$, $x(0) = 0.06$ m, $\dot{x}(0) = 2.4$ m/s

M2.6. File VIBES_2E.m provides the free vibration response for the suspension system model of Prob. 2.71. Use VIBES_2E to help design an appropriate suspension system for the following situation. A 1500-kg vehicle is to be mounted on tires of stiffness 280,000 N/m. When empty the static deflection of the suspension should be 2.6 cm. It is estimated that the vehicle will carry a maximum of 1000 kg in passengers and cargo. The maximum overshoot when the vehicle encounters a pothole is to be 5 percent, whether the vehicle is empty or fully loaded.

M2.7. For the suspension system designed in M2.6, plot the overshoot as a function of tire stiffness. Use VIBES_2E to help with the calculations.

M2.8. Use MATLAB to plot the response of an undamped one-degree-of-freedom system given initial conditions. Use the program to plot the response of the system of Example 2.15 if the machine part is being raised at a rate of 0.8 m/s when it suddenly stops. Repeat the problem if the hoist is located 1 m from the pinned end.

M2.9. Write a MATLAB program that simulates the motion of the disk of Prob. 2.53, part (*b*), assuming

(*a*) The disk rolls without slip

(*b*) The disk rolls and slips

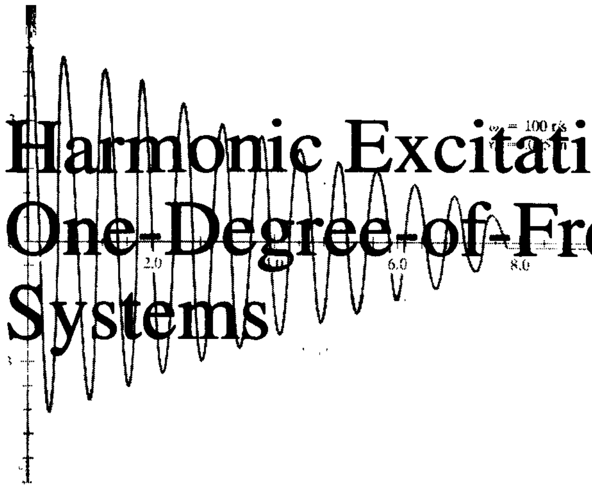
M2.10. Write a MATLAB program that simulates the free vibration response of a one-degree-of-freedom system with hysteretic damping.

M2.11. Write a MATLAB program that provides the natural frequency calculations for a machine located along the span of a pinned-pinned beam. Use the file to solve problems M2.1–M2.3 assuming the beam is pinned-pinned.

M2.12. Write a MATLAB program that plots the free vibration response of a vehicle using the suspension system model of Prob. 2.72. Use the file to determine the effect of c_t on the vehicle of Example 2.18.

chapter

3 Harmonic Excitation of One-Degree-of-Freedom Systems



3.1 INTRODUCTION

Forced vibrations of a one-degree-of-freedom system occur when work is being done on the system while the vibrations occur. Examples of forced excitation include the ground motion during an earthquake or the motion caused by an unbalanced reciprocating component. Figure 3.1 illustrates the equivalent systems model for the forced vibrations of a linear one-degree-of-freedom system when a linear displacement is chosen as the generalized coordinate. The equivalent systems model when an angular coordinate is chosen as the generalized coordinate is illustrated in Fig. 3.2.

Application of Newton's law to the free-body diagrams of Fig. 3.3 lead to the differential equation governing the motion of the model system of Fig. 3.1

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F_{eq}(t) \qquad [3.1]$$

The differential equation governing the motion of the model system of Fig. 3.2 is

$$I_{eq}\ddot{\theta} + c_{t_{eq}}\dot{\theta} + k_{t_{eq}}\theta = M_{eq}(t) \qquad [3.2]$$

The excitation is periodic (or harmonic) if there exists a T such that

$$F_{eq}(t + T) = F_{eq}(t) \qquad [3.3]$$

for all t. Examples of periodic excitations are given in Fig. 3.4. The frequency of a periodic excitation is

$$\omega = \frac{2\pi}{T} \qquad [3.4]$$

A single-frequency excitation has the form

$$F_{eq}(t) = F_0 \sin(\omega t + \psi) \qquad [3.5]$$

where F_0 is the amplitude of the excitation, ω is its frequency, and ψ is its phase. The frequency of the excitation is independent of the system's natural frequency.

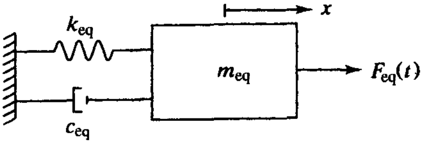


Figure 3.1 Equivalent systems model of a linear one-degree-of-freedom system with linear displacement as generalized coordinate.

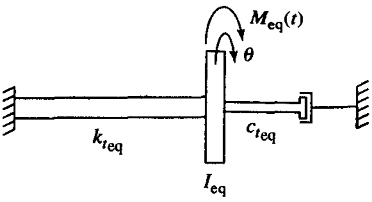


Figure 3.2 Equivalent systems model of a linear one-degree-of-freedom system with angular coordinate as generalized coordinate.

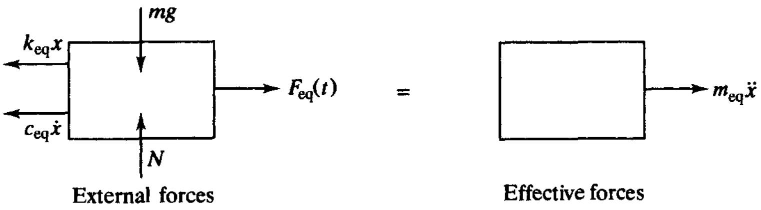
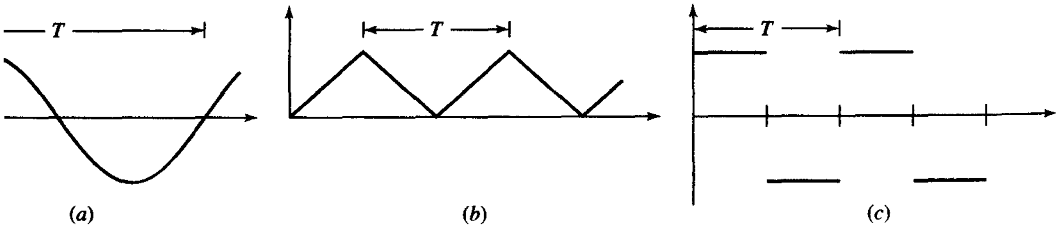


Figure 3.3 Free-body diagrams of model system of Fig. 3.1 at arbitrary instant.



3.4 Examples of periodic excitation include (a) sinusoidal excitation, (b) repeated triangular pulses, (c) repeated loading reversal.

3.2 DIFFERENTIAL EQUATIONS GOVERNING FORCED VIBRATION

The differential equation governing forced vibrations of a one-degree-of-freedom system can be derived by either the free-body diagram method or the equivalent systems method. Application of the free-body diagram method is as for free vibrations except with the excitation forces illustrated at an arbitrary instant included on the free-body diagram showing the external forces.

The excitation forces are nonconservative. The work done by the excitation force in the model system of Fig. 3.1 as the block moves between two arbitrary positions x_1 and x_2 is

$$U_{1\to 2_{\text{ext}}} = \int_{x_1}^{x_2} F_{\text{eq}}(t)\,dx \tag{3.6}$$

The work done by the nonconservative external force can be included in the work-energy analysis of Sec. 2.3 and Eq. (2.11). Application of the method leads to Eq. (3.1), the differential equation governing the motion of the model one-degree-of-freedom system. Thus the equivalent systems method can be used to derive the differential equation governing the forced vibrations of a linear one-degree-of-freedom system. The equivalent external force is obtained by calculating the work done by the external forces between two arbitrary times. The equivalent force is determined when the work is written in the form of Eq. (3.6).

Examp[le 3.1]

Assuming small displacements and using θ as the generalized coordinate, derive the differential equation governing the forced vibrations of the system of Fig. 3.5 using

(a) The free-body diagram method

(b) The equivalent-systems method

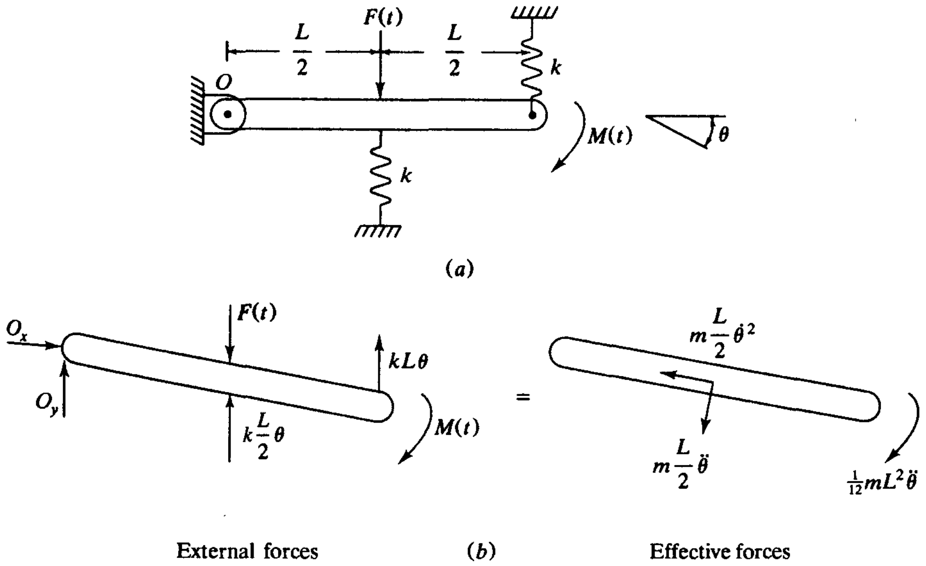


Figure 3.5 (a) System of Example 3.1; (b) free-body diagrams at an arbitrary instant.

Solution:

(a) The free-body diagrams of Fig. 3.5b are drawn with the knowledge that application of the static equilibrium condition leads to the gravity force and static deflections canceling from the governing differential equation. Summing moments about the point of support using the free-body diagrams of Fig. 3.5b

$$\left(\sum \overset{\curvearrowright}{\overset{+}{M}}_O\right)_{\text{ext}} = \left(\sum \overset{\curvearrowright}{\overset{+}{M}}_O\right)_{\text{eff}}$$

and using the small-angle assumption leads to

$$F(t)\frac{L}{2} - k\left(\frac{L}{2}\theta\right)\frac{L}{2} - k(L\theta)L + M(t) = \left(m\frac{L}{2}\ddot{\theta}\right)\frac{L}{2} + \frac{1}{12}mL^2\ddot{\theta}$$

Rearrangement leads to

$$m\frac{L^2}{3}\ddot{\theta} + \frac{5}{4}kL^2\theta = M(t) + \frac{L}{2}F(t)$$

(b) Since θ is an angular displacement, the appropriate model is the torsional system of Fig. 3.2. The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2$$

from which the equivalent moment of inertia is obtained as

$$I_{\text{eq}} = \tfrac{1}{3}mL^2$$

The potential energy of the system at an arbitrary instant is

$$V = \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k\left(\frac{L}{2}\theta\right)^2 = \frac{1}{2}\left(\frac{5}{4}kL^2\right)\theta^2$$

From which the equivalent torsional stiffness is obtained as

$$k_{t_{\text{eq}}} = \tfrac{5}{4}kL^2$$

The work done by the external forces between two arbitrary positions is

$$U_{1\to 2_{\text{ext}}} = \int_{\theta_1}^{\theta_2} F(t)\,d\left(\frac{L}{2}\theta\right) + \int_{\theta_1}^{\theta_2} M(t)\,d\theta = \int_{\theta_1}^{\theta_2}\left(M(t) + \frac{L}{2}F(t)\right)d\theta$$

from which the equivalent external moment is obtained as

$$M_{\text{eq}}(t) = M(t) + \frac{L}{2}F(t)$$

Thus the governing differential equation is

$$m\frac{L^2}{3}\ddot{\theta} + \frac{5}{4}kL^2\theta = M(t) + \frac{L}{2}F(t)$$

which is the same result derived by the free-body diagram method.

Dividing Eq. (3.1) by m_{eq} leads to the standard form of the differential equation governing the forced vibrations of a linear one-degree-of-freedom system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m_{\text{eq}}} F_{\text{eq}}(t) \tag{3.7}$$

The general solution of Eq. (3.7) is the sum of the homogeneous solution, the solution obtained if the right-hand side is set to zero, and the particular solution. The homogeneous solution of Eq. (3.7) for $\zeta < 1$ is

$$x_h(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + C_2 \sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \tag{3.8}$$

The form of the particular solution depends on the form of $F_{\text{eq}}(t)$. If $\zeta > 0$, $x_h(t)$ approaches zero as t gets large. The presence of the homogeneous solution in the general solution leads to an initial transient motion which quickly decays. The *steady-state response* is the system response after the transient motion has decayed sufficiently. Only the particular solution contributes to the steady-state response for a linear system.

3.3 FORCED RESPONSE OF AN UNDAMPED SYSTEM DUE TO A SINGLE-FREQUENCY EXCITATION

The differential equation for undamped forced vibrations of a one-degree-of-freedom system subject to a single-frequency harmonic excitation of the form of Eq. (3.5) is

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m_{\text{eq}}} \sin(\omega t + \psi) \tag{3.9}$$

If $\omega \neq \omega_n$ the method of undetermined coefficients is used to obtain the particular solution of Eq. (3.9)

$$x_p(t) = \frac{F_0}{m_{\text{eq}}\left(\omega_n^2 - \omega^2\right)} \sin(\omega t + \psi) \tag{3.10}$$

The homogeneous solution, Eq. (3.8), is added to the particular solution and the initial conditions are applied, yielding

$$x(t) = \left[x_0 - \frac{F_0 \sin\psi}{m_{\text{eq}}(\omega_n^2 - \omega^2)}\right]\cos(\omega_n t) + \frac{1}{\omega_n}\left[\dot{x}_0 - \frac{F_0\omega\cos\psi}{m_{\text{eq}}(\omega_n^2 - \omega^2)}\right]\sin(\omega_n t) + \frac{F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)}\sin(\omega t + \psi) \tag{3.11}$$

The response, plotted in Fig. 3.6, is the sum of two trigonometric terms of different frequencies.

The case when $\omega = \omega_n$ is special. The nonhomogeneous term in Eq. (3.9) and the homogeneous solution are not linearly independent. Thus when the method of

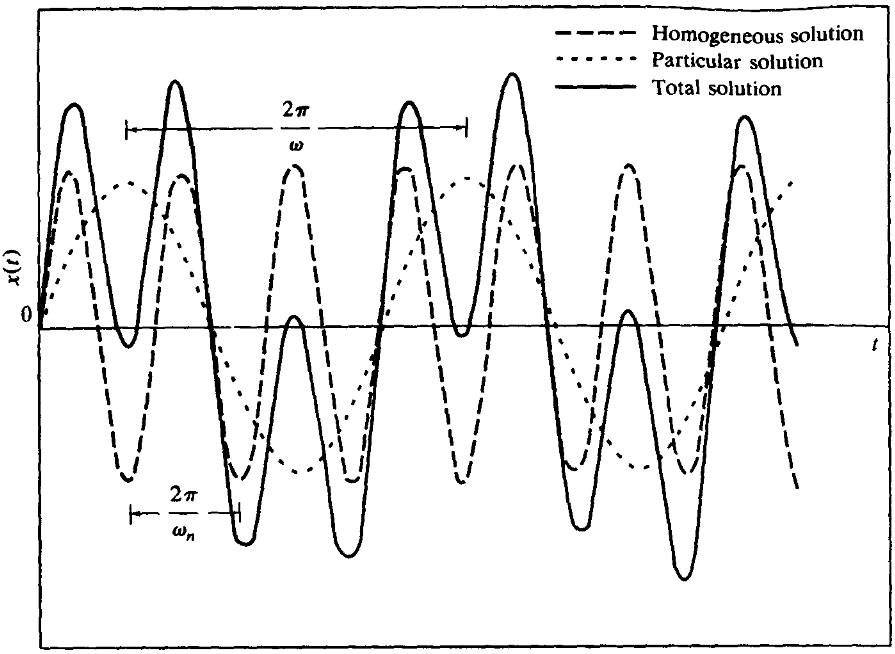


Figure 3.6 Response of an undamped system to a single-frequency harmonic excitation when $\omega < \omega_n$.

undetermined coefficients is used to determine the particular solution the appropriate trial solution is

$$x_p(t) = At \sin(\omega_n t + \psi) + Bt \cos(\omega_n t + \psi) \tag{3.12}$$

Substitution of Eq. (3.12) in Eq. (3.9) leads to

$$x_p(t) = -\frac{F_0}{2m_{\text{eq}}\omega_n} t \cos(\omega_n t + \psi) \tag{3.13}$$

Application of initial conditions to the sum of the homogenous and particular solution yields

$$x(t) = x_0 \cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} + \frac{F_0 \cos\psi}{2m_{\text{eq}}\omega_n^2}\right) \sin(\omega_n t) - \frac{F_0}{2m_{\text{eq}}\omega_n} t \cos(\omega_n t + \psi) \tag{3.14}$$

The response of a system in which the excitation frequency equals the natu frequency grows without bounds, as shown in Fig. 3.7. This condition, where amplitude increases without bound, is called *resonance*. In a real physical syst the amplitude is bounded. In many cases as the amplitude grows, assumptions us in modeling the physical system become invalid. In a system with an elastic elem such as a coil spring, the proportional limit of the spring's material is eventua reached. After this time the motion of the system is governed by a nonlinear diff ential equation reflecting the nonlinear force-displacement relation in the spring. I a system such as a pendulum in which a small displacement assumption is used linearize the governing differential equation, the assumption becomes invalid wh resonance occurs. The original nonlinear differential equation must be used.

Resonance is a dangerous condition in a mechanical or structural system a will produce unwanted large displacements or lead to failure. Resonant torsio oscillations were partially the cause of the famous Tacoma Narrows Bridge disast It is suspected that the frequency at which vortices were shed from the bridge c incided with a torsional natural frequency, leading to oscillations that grew withc bound. Recent investigations have shown that nonlinear phenomena resulting frc improper installation may have also played a role in the disaster. Resonance c also lead to unwanted large-amplitude vibrations in machinery.

When vibrations of a conservative system are initiated, the motion is sustain at the system's natural frequency without additional energy input. Thus, when t frequency of excitation is the same as the natural frequency, the work done by t external force is not needed to sustain motion. The total energy increases because

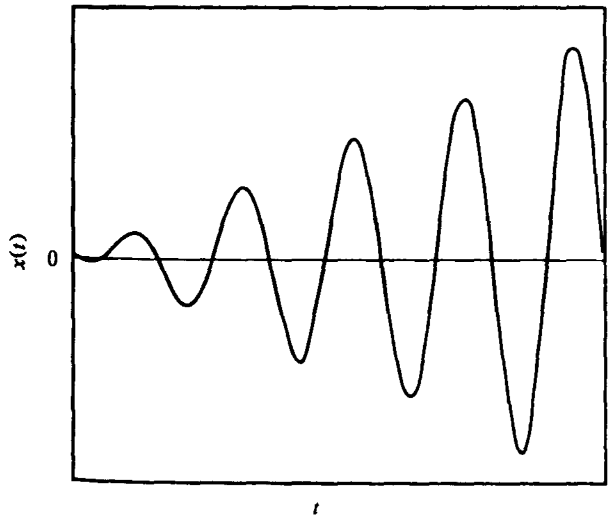


Figure 3.7 Undamped response when excitation frequency equals natural frequency. The response grows without bound producing resonance.

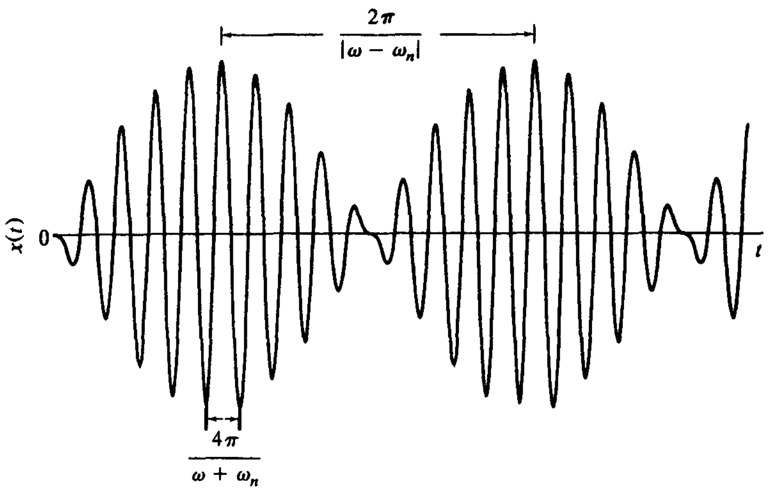


Figure 3.8 Beating phenomenon occurs when $\omega \approx \omega_n$.

the work input and leads to a continual increase in amplitude. When the frequency of excitation is different from the natural frequency, the work done by the external force is necessary to sustain motion at the excitation frequency.

When the excitation frequency is close, but not exactly equal, to the natural frequency, an interesting phenomenon called *beating* occurs. Beating is a continuous buildup and decrease of amplitude as shown in Fig. 3.8. When ω is very close to ω_n and $x_0 = \dot{x}_0 = 0$ and $\psi = 0$, Eq. (3.11) can be written as

$$x(t) = \frac{2F_0}{m_{\text{eq}}(\omega_n^2 - \omega^2)} \sin\left(\frac{\omega - \omega_n}{2}\right) t \cos\left(\frac{\omega + \omega_n}{2}\right) t \qquad [3.15]$$

From Eq. (3.15), since $|\omega - \omega_n|$ is small, the solution can be viewed as a cosine wave with a slowly varying amplitude. The period of the amplitude is called the period of the beating and equals $2\pi/|\omega - \omega_n|$. The period of vibration is $4\pi/(\omega + \omega_n)$.

3.4 FORCED RESPONSE OF A VISCOUSLY DAMPED SYSTEM SUBJECT TO A SINGLE-FREQUENCY HARMONIC EXCITATION

The standard form of the differential equation governing the motion of a viscously damped one-degree-of-freedom system with the single-frequency harmonic excitation of Eq. (3.5) is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m_{\text{eq}}}\sin(\omega t + \psi) \qquad [3.16]$$

The particular solution of Eq. (3.16) is

$$x_p(t) = \frac{F_0}{m_{\text{eq}}\left[\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega\omega_n)^2\right]}\left[-2\zeta\omega\omega_n\cos(\omega t + \psi) + \left(\omega_n^2 - \omega^2\right)\sin(\omega t + \psi)\right] \qquad [3.17]$$

Use of the trigonometric identity for the sine of the difference of angles and algebraic manipulation leads to the following alternate form of Eq. (3.17):

$$x_p(t) = X\sin(\omega t + \psi - \phi) \qquad [3.18]$$

where

$$X = \frac{F_0}{m_{\text{eq}}\left[\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega\omega_n)^2\right]^{1/2}} \qquad [3.19]$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right) \qquad [3.20]$$

X is the amplitude of the forced response and ϕ is the phase angle between the response and the excitation.

Only the long-term behavior is of interest for most systems subject to harmonic excitation. As $t \to \infty$, the homogeneous solution of Eq. (3.7) goes to zero and only the forced response remains. Thus, for harmonic excitation, the free-vibration response is neglected and only the forced response or the steady-state response considered.

The amplitude and phase angle provide important information about the forced response. Formulation of Eqs. (3.19) and (3.20) in nondimensional form allows better qualitative interpretation of the response. It is noted from these equations that

$$X = f(F_0, m_{\text{eq}}, \omega, \omega_n, \zeta) \qquad [3.21]$$

and

$$\phi = g(\omega, \omega_n, \zeta) \qquad [3.22]$$

The parameters use three basic dimensions: mass, length, and time. Multiplying Eq. (3.19) by $m_{\text{eq}}\omega_n^2/F_0$ gives

$$\frac{m_{\text{eq}}\omega_n^2 X}{F_0} = \frac{1}{\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]^{1/2}} \qquad [3.23]$$

where

$$r = \frac{\omega}{\omega_n} \qquad [3.24]$$

is the frequency ratio. The ratio

$$M = \frac{m_{\text{eq}}\omega_n^2 X}{F_0} \tag{3.25}$$

is dimensionless and is often called the *amplitude ratio* or *magnification factor*. While it can be construed as the ratio of the amplitude of forced response to the amplitude of excitation, a more direct physical interpretation is that of the maximum force developed in the spring of a mass-spring-dashpot system to the maximum value of the exciting force. Thus the nondimensional form of Eq. (3.19) is

$$M(r, \zeta) = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}} \tag{3.26}$$

The magnification factor as a function of frequency ratio for different values of the damping ratio is shown in Fig. 3.9. These curves are called frequency response curves. The following are noted about Eq. (3.26) and Fig. 3.9:

1. $M = 1$ when $r = 0$. In this case the excitation force is a constant and the maximum force developed in the spring of a mass-spring-dashpot system is equal to the value of the exciting force.
2. $M \to 0$ as $r \to \infty$. The amplitude of the forced response is very small for high-frequency excitations.
3. For a given value of r, M decreases with increasing ζ.
4. The magnification factor grows without bound only for $\zeta = 0$. For $0 < \zeta \leq 1/\sqrt{2}$, the magnification factor has a maximum for some value of ζ.
5. For $0 < \zeta \leq 1/\sqrt{2}$, the maximum value of the magnification factor occurs for a frequency ratio of

$$r_m = \sqrt{1 - 2\zeta^2} \tag{3.27}$$

   Equation (3.27) is obtained from Eq. (3.26) by determining the value of r such that $dM/dr = 0$.
6. The corresponding maximum value of M is

$$M_{\max} = \frac{1}{2\zeta\left(1 - \zeta^2\right)^{1/2}} \tag{3.28}$$

7. For $\zeta = 1/\sqrt{2}$, $dM/dr = 0$ for $r = 0$. For $\zeta > 1/\sqrt{2}$, M monotonically decreases with increasing r.

   The nondimensional form of Eq. (3.20) is

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.29}$$

The phase angle from Eq. (3.29) is plotted as a function of frequency ratio for different values of the damping ratio in Fig. 3.10. The following are noted from Eq. (3.29) and Fig. 3.10:

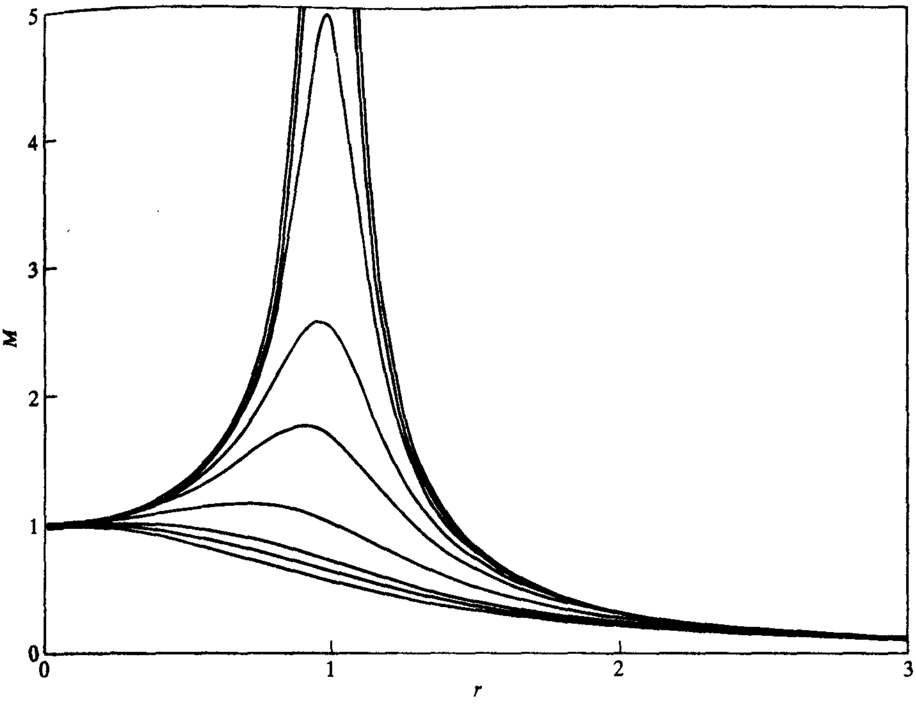


Figure 3.9 Magnification factor versus frequency ratio for different damping ratios.

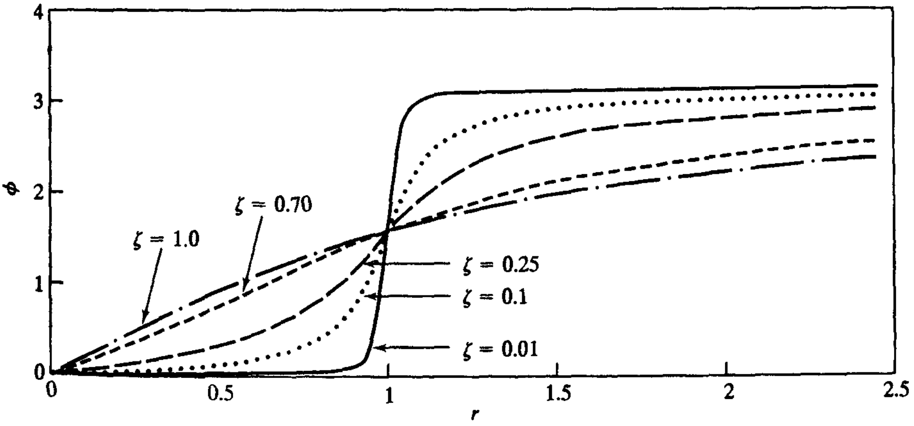


Figure 3.10 Phase angle versus frequency ratio for different damping ratios.

1. The forced response and the excitation force are in phase for $\zeta = 0$. For $\zeta > 0$, the response and excitation are in phase only for $r = 0$.
2. If $\zeta > 0$ and $0 < r < 1$, then $0 < \phi < \pi/2$. The response lags the excitation as shown in Fig. 3.11.

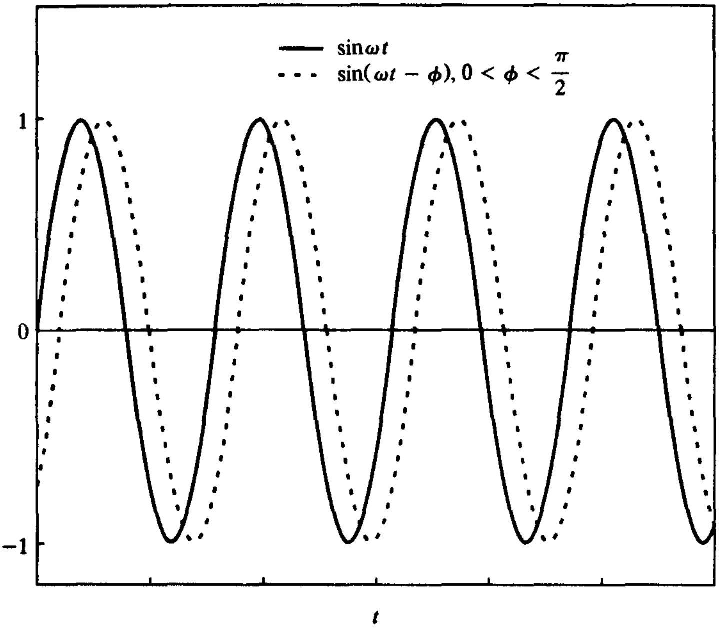


Figure 3.11 Forced steady-state response lags excitation, for $r < 1$.

3. If $\zeta > 0$ and $r = 1$, then $\phi = \pi/2$. If $\psi = 0$, then the excitation is a pure sine wave while the steady-state response is a pure cosine wave. The excitation is in phase with the velocity. The direction of the excitation is always the same as the direction of motion.
4. If $\zeta > 0$ and $r > 1$, then $\pi/2 < \phi < \pi$. The response leads the excitation as shown in Fig. 3.12.
5. If $\zeta > 0$ and $r \gg 1$, then $\phi \approx \pi$. The sign of the steady-state response is opposite that of the excitation.
6. For $\zeta = 0$, the response is in phase with the excitation for $r < 1$ and π radians (180°) out of phase for $r > 1$.

   For an undamped system

$$M(r, 0) = \frac{1}{\sqrt{(1 - r^2)^2}} = \frac{1}{|1 - r^2|} \tag{3.30}$$

Equations (3.10) and (3.30) are reconciled by noting that for $r > 1$ ($\omega > \omega_n$) the negative sign that would occur in Eq. (3.10) is incorporated into the phase ($\phi = \pi$ for $r > 1$); $\sin(\omega t - \pi) = -\sin(\omega t)$.

3.2 ▲ A moment, $M_0 \sin \omega t$, is applied to the end of the bar of Fig. 3.13. Determine the maximum value of M_0 such that the steady-state amplitude of angular oscillation does

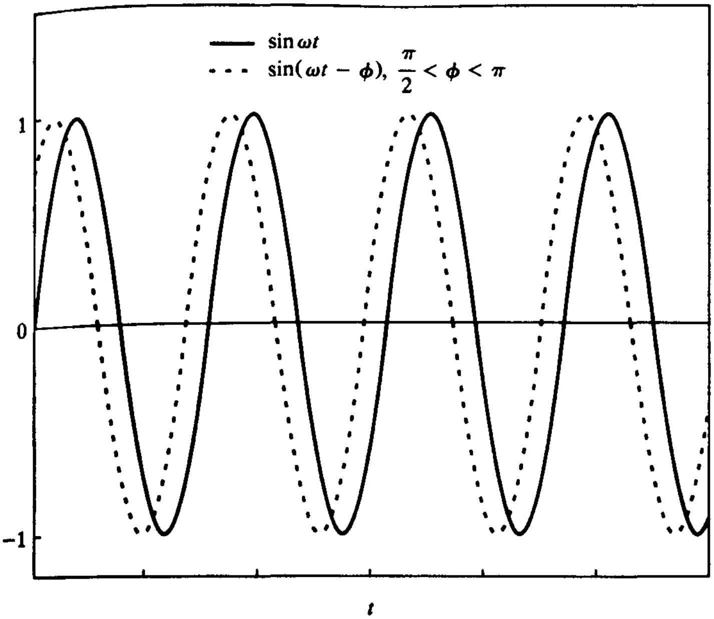


Figure 3.12 Response leads excitatian when $r > 1$.

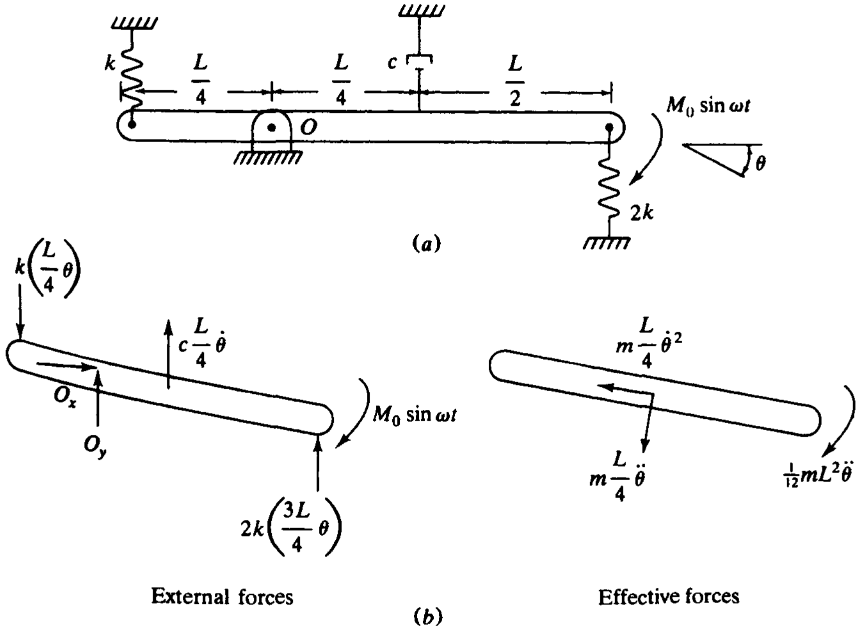


Figure 3.13 (a) System of Example 3.2; (b) free-body diagrams at an arbitrary instant.

not exceed 10° if $\omega = 500$ rpm, $k = 7000$ N/m, $c = 650\ \mathrm{N \cdot s/m}$, $L = 1.2$ m, and the mass of the bar is 15 kg.

Solution:

The differential equation obtained by summing moments about 0 using the free-body diagrams of Fig. 3.13*b* is

$$\tfrac{7}{48} mL^2 \ddot{\theta} + \tfrac{1}{16} cL^2 \dot{\theta} + \tfrac{19}{16} kL^2 \theta = M_0 \sin \omega t$$

Using the notation of Eq. (3.2),

$$I_{eq} = \tfrac{7}{48} mL^2 = \tfrac{7}{48}(15\ \mathrm{kg})(1.2\ \mathrm{m})^2 = 3.15\ \mathrm{kg \cdot m^2}$$

The differential equation is rewritten in the form of Eq. (3.7) by dividing by I_{eq}:

$$\ddot{\theta} + \frac{3}{7}\frac{c}{m}\dot{\theta} + \frac{57}{7}\frac{k}{m}\theta = \frac{M_0}{I_{eq}} \sin \omega t$$

The preceding equation has a steady-state solution of the form

$$\theta(t) = \Theta \sin(\omega t - \phi)$$

The natural frequency and damping ratio are obtained by comparison to Eq. (3.7)

$$\omega_n = \sqrt{\frac{57}{7}\frac{k}{m}} = \sqrt{\frac{(57)(7000\ \mathrm{N/m})}{(7)(15\ \mathrm{kg})}} = 61.6\ \frac{\mathrm{rad}}{\mathrm{s}}$$

$$\zeta = \frac{3}{14}\frac{c}{m\omega_n} = \frac{(3)(650\ \mathrm{N \cdot s/m})}{(14)(15\ \mathrm{kg})(61.6\ \mathrm{rad/s})} = 0.15$$

The frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{(500\ \mathrm{rev/min})(2\pi\ \mathrm{rad/rev})(1\ \mathrm{min}/60\ \mathrm{s})}{61.6\ \mathrm{rad/s}} = 0.85$$

The magnification factor is calculated from Eq. (3.26)

$$M(0.85, 0.15) = \frac{1}{\sqrt{\left[1 - (0.85)^2\right]^2 + [2(0.15)(0.85)]^2}} = 2.65$$

The maximum allowable magnitude of the applied moment is calculated from Eq. (3.25),

$$\frac{I_{eq}\omega_n^2 \Theta}{M_0} = M(0.85, 0.15) = 2.65$$

Requiring $\Theta < 10°$ leads to

$$M_0 < \frac{\left(3.15\ \mathrm{kg \cdot m^2}\right)(61.6\ \mathrm{rad/s})^2(10°)(2\pi\ \mathrm{rad}/360°)}{2.65} = 787.2\ \mathrm{N \cdot m}$$

3.3 A machine of mass 25.0 kg is placed on an elastic foundation. A sinusoidal force of magnitude 25 N is applied to the machine. A frequency sweep reveals that the maximum steady-state amplitude of 1.3 mm occurs when the period of response is 0.22 s. Determine the equivalent stiffness and damping ratio of the foundation.

Solution:

The system is modeled as a mass attached to a spring and dashpot in parallel with an applied sinusoidal force of magnitude 25 N. For a linear system the frequency of response is the same as the frequency of excitation. Thus the maximum response occurs for a period of $(2\pi)/\omega = 0.22$ s which leads to $\omega = 28.6$ rad/s. From Eq. (3.27) the maximum response occurs when

$$\frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2}$$

or

$$\omega_n = \frac{\omega}{\sqrt{1 - 2\zeta^2}} = \frac{28.6 \text{ rad/s}}{\sqrt{1 - 2\zeta^2}}$$

Equation (3.28) implies

$$\frac{(25.0 \text{ kg})(0.0013 \text{ m})\omega_n^2}{25 \text{ N}} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

which upon substitution for ω_n becomes

$$\frac{1.063}{1 - 2\zeta^2} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Squaring the preceding equation and rearranging leads to

$$\zeta^4 - \zeta^2 + 0.117 = 0$$

The quadratic formula is used to solve for ζ^2, yielding

$$\zeta = 0.368, 0.930$$

The larger value is discarded because a frequency sweep would yield a maximum only for $\zeta < 1/\sqrt{2}$. Thus $\zeta = 0.368$. The natural frequency is calculated as

$$\omega_n = \frac{28.6 \text{ rad/s}}{\sqrt{1 - 2(0.368)^2}} = 33.5 \frac{\text{rad}}{\text{s}}$$

The stiffness of the elastic foundation is

$$k = m\omega_n^2 = (25 \text{ kg})(33.5 \text{ rad/s})^2 = 2.80 \times 10^4 \text{ N/m}$$

3.5 FREQUENCY-SQUARED EXCITATIONS

3.5.1 GENERAL THEORY

Many one-degree-of-freedom systems are subject to a single-frequency harmonic excitation whose amplitude is proportional to the square of its frequency.

$$F_{eq}(t) = A\omega^2 \sin(\omega t + \psi) \qquad [3.31]$$

When $F_{eq}(t)$ represents a force, A has dimensions of $[F][T^2]$ or $[M][L]$. When $F_{eq}(t)$ represents a moment, A has dimensions of $[F][L][T^2]$ or $[M][L^2]$. The steady-state response due to this type of excitation is developed by applying equations developed in Sec. 3.4 with

$$F_0 = A\omega^2 \tag{3.32}$$

Substitution of Eq. (3.32) into Eq. (3.23) yields

$$\left(\frac{m_{eq}X}{A}\right)\left(\frac{\omega_n}{\omega}\right)^2 = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

or

$$m_{eq}\frac{X}{A} = \Lambda(r, \zeta) \tag{3.33}$$

where

$$\Lambda(r, \zeta) = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}} \tag{3.34}$$

Λ is, like M, a nondimensional function of the frequency ratio and the damping ratio. Λ is related to M by

$$\Lambda(r, \zeta) = r^2 M(r, \zeta) \tag{3.35}$$

The steady-state response is given by Eq. (3.18) where X is determined from Eqs. (3.33) and (3.34) and ϕ is determined using Eq. (3.29).

Λ is plotted as a function of r for various values of ζ in Fig. 3.14. The following are noted from Eq. (3.34) and Fig. 3.14:

1. $\Lambda = 0$ if and only if $r = 0$ for all values of ζ.
2. $\Lambda \approx 1$ for large r for all values of ζ.
3. Λ grows without bound near $r = 1$ for $\zeta = 0$.
4. For $0 < \zeta < 1/\sqrt{2}$, Λ has a maximum for a frequency ratio of

$$r_m = \frac{1}{\sqrt{1-2\zeta^2}} \tag{3.36}$$

   Equation (3.36) is derived by finding the value of r such that $d\Lambda/dr = 0$.

5. For a given $0 < \zeta < 1/\sqrt{2}$, the maximum value of Λ corresponds to the frequency ratio of Eq. (3.36) and is given by

$$\Lambda_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{3.37}$$

6. For $\zeta > 1/\sqrt{2}$, Λ does not reach a maximum. Λ grows slowly from zero near $r = 0$ and approaches one for large r.

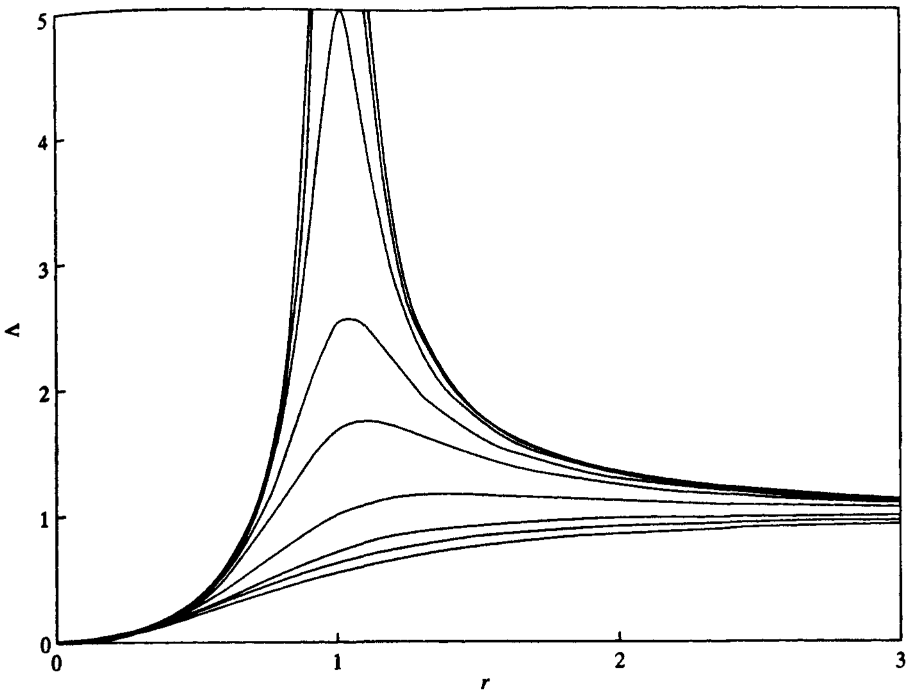


Figure 3.14 Λ versus r for different values of ζ.

Example 3

A one-degree-of-freedom system is subject to a harmonic excitation whose magnitude is proportional to the square of its frequency. The frequency of excitation is varied and the steady-state amplitude noted. A maximum amplitude of 8.5 mm occurs at a frequency of 200 Hz. When the frequency is much higher than 200 Hz, the steady-state amplitude is 1.5 mm. Determine the damping ratio for the system.

Solution:

From Fig. 3.14, $\Lambda \to 1$ as $r \to \infty$. Thus, from Eq. (3.33) and the given information,

$$\frac{m_{eq}}{A} = \frac{1}{1.5\text{ mm}}$$

Substituting the preceding equation into Eq. (3.37) yields

$$\Lambda_{max} = \frac{m}{A} X_{max} = \frac{8.5\text{ mm}}{1.5\text{ mm}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Inverting, squaring, and rearranging leads to

$$\zeta^4 - \zeta^2 + 0.00778 = 0$$

The appropriate root of the preceding equation is $\zeta = 0.089$.

3.5.2 ROTATING UNBALANCE

The machine of Fig. 3.15 has a component which rotates at a constant speed, ω. Its center of mass is located a distance e, called the eccentricity, from the axis of rotation. The mass of the rotating component is m_0, while the total mass of the machine, including the rotating component, is m. The machine is constrained to move vertically. The acceleration of the rotating component relative to the machine is $m_0 e\omega^2$ directed from the center of mass of the rotating component to its axis of rotation. Since the location of the center of mass of the rotating component moves as the component rotates, the direction of this component of acceleration also changes.

Summation of forces applied to the free-body diagrams of Fig. 3.16 yields

$$\sum \overset{\downarrow +}{F}_{\text{ext}} = \sum \overset{\downarrow +}{F}_{\text{eff}}$$

$$-kx - c\dot{x} = m\ddot{x} + m_0 e\omega^2 \sin\theta \tag{3.38}$$

For constant ω,

$$\theta = \omega t + \theta_0 \tag{3.39}$$

where θ_0 is an angle between the initial position of the center of mass of the rotating component and the horizontal. Using Eq. (3.39) in Eq. (3.38), and rearranging yields

$$m\ddot{x} + c\dot{x} + kx = -m_0 e\omega^2 \sin(\omega t + \theta_0) \tag{3.40}$$

The negative sign is incorporated into the sine function by defining

$$\psi = \theta_0 + \pi$$

Then Eq. (3.40) becomes

$$m\ddot{x} + c\dot{x} + kx = m_0 e\omega^2 \sin(\omega t + \psi) \tag{3.41}$$

It is apparent from Eq. (3.41) that the unbalanced rotating component leads to a harmonic excitation whose amplitude is proportional to the square of its frequency. The constant of proportionality is

$$A = m_0 e \tag{3.42}$$

Using Eq. (3.33) gives

$$\frac{mX}{m_0 e} = \Lambda(r, \zeta) \tag{3.43}$$

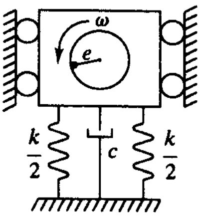


Figure 3.15 Rotating unbalance produces harmonic excitation whose amplitude is proportional to the square of the frequency.

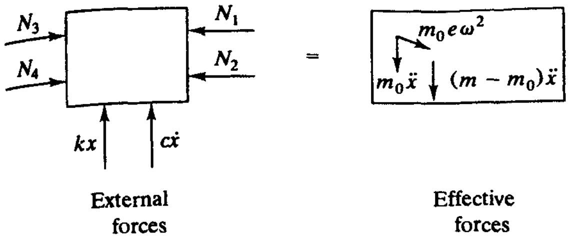


Figure 3.16 Free-body diagrams of machine with rotating unbalance at an arbitrary instant.

A 150-kg electric motor has a rotating unbalance of 0.5 kg, 0.2 m from the center of rotation. The motor is to be mounted at the end of a steel ($E = 210 \times 10^9$ N/m^2) cantilever beam of length 1 m. The operating range of the motor is from 500 to 1200 rpm. For what values of I, the beam's cross-sectional moment of inertia, will the steady-state amplitude of vibration be less than 1 mm? Assume the damping ratio is 0.1.

Solution:

The maximum allowable value of Λ is

$$\Lambda_{\text{allow}} = \frac{(150\text{ kg})(0.001\text{ m})}{(0.5\text{ kg})(0.2\text{ m})} = 1.5$$

Since $\Lambda_{\text{allow}} > 1$ and $\zeta < 1/\sqrt{2}$, Fig. 3.14 shows that two values of r correspond to $\Lambda = \Lambda_{\text{allow}}$. These are determined using Eq. (3.34)

$$1.5 = \frac{r^2}{\sqrt{(1-r^2)^2 + (0.2r)^2}}$$

Rearrangement leads to the following equation:

$$0.556r^4 - 1.96r^2 + 1 = 0$$

whose positive roots are

$$r = 0.787, 1.71$$

Thus $\Lambda < 1.5$ if $r < 0.787$ or $r > 1.71$. Requiring $r < 0.792$ over the entire operating range yields,

$$\frac{(1200\text{ rev/min})(2\pi\text{ rad/rev})(1\text{ min/60 s})}{\omega_n} < 0.787$$

or $\omega_n > 159.7$ rad/s. The one-degree-of-freedom approximation for the natural frequency of the motor attached to the end of a cantilever beam of negligible mass is

$$\omega_n = \sqrt{\frac{3EI}{mL^3}}$$

Thus

$$I > \frac{(159.7 \text{ rad/s})^2 L^3 m}{3E} = \frac{(159.7 \text{ rad/s})^2 (1 \text{ m})^3 (150 \text{ kg})}{3(210 \times 10^9 \text{ N/m}^2)} = 6.07 \times 10^{-6} \text{ m}^4$$

Requiring $r > 1.71$ over the entire operating range

$$\frac{(500 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{\omega_n} > 1.71$$

or $\omega_n < 30.6$ rad/s. This requirement leads to $I < 2.23 \times 10^{-7}$ m^4.

Thus the amplitude of vibration will be limited to 1 mm if $I > 6.08 \times 10^{-6}$ m^4 or $I < 2.23 \times 10^{-7}$ m^4. However, other considerations limit the design of the beam. The smaller the moment of inertia, the larger the bending stress in the outer fibers of the beam at the support.

3.5.3 VORTEX SHEDDING FROM CIRCULAR CYLINDERS

When a circular cylinder is placed in a steady uniform stream at sufficient velocity, flow separation occurs on the cylinder's surface, as illustrated in Fig. 3.17. The separation leads to vortex shedding from the cylinder and the formation of a wake behind the cylinder. Vortices are shed alternately from the upper and lower surfaces of the cylinder at a constant frequency. The alternate shedding of vortices causes oscillating streamlines in the wake which, in turn, lead to an oscillating pressure distribution. The oscillating pressure distribution, in turn, gives rise to an oscillating force acting normal to the cylinder,

$$F(t) = F_0 \sin(\omega t) \qquad [3.44]$$

where F_0 is the magnitude of the force and ω is the frequency of vortex shedding.

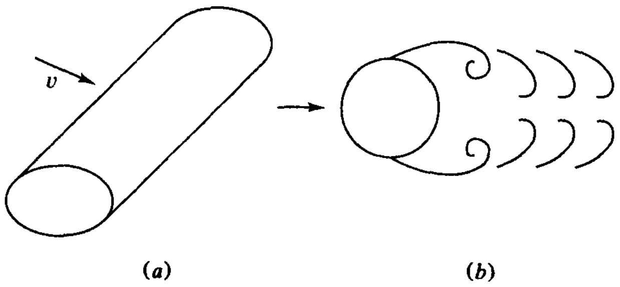


Figure 3.17 (a) Circular cylinder in steady flow; (b) cross section of cylinder, showing vortices shed alternately from each surface of the cylinder resulting in a wake behind the cylinder and a harmonic force acting on the cylinder.

These parameters are dependent upon the fluid properties and the geometry of the cylinder. That is,

$$F_0 = F_0(v, \rho, \mu, D, L) \tag{3.45}$$

and

$$\omega = \omega(v, \rho, \mu, D, L) \tag{3.46}$$

where v = the magnitude of fluid velocity, $[L]/[T]$
ρ = the fluid density, $[M]/[L]^3$
μ = the dynamic viscosity of fluid, $[M]/([L][T])$
D = the diameter of cylinder, $[L]$
L = the length of cylinder, $[L]$

The dependent parameters F_0 and ω are both functions of five independent parameters. Dimensional analysis theory implies that Eqs. (3.45) and (3.46) can be rewritten as relationships between three dimensionless parameters. Indeed, nondimensional forms of Eqs. (3.45) and (3.46) are

$$C_D = f\left(\mathrm{Re}, \frac{D}{L}\right) \tag{3.47}$$

$$S = f\left(\mathrm{Re}, \frac{D}{L}\right) \tag{3.48}$$

The dependent dimensionless parameters are (1) the drag coefficient

$$C_D = \frac{F_0}{\frac{1}{2}\rho v^2 DL} \tag{3.49}$$

which is the ratio of the drag force to the inertia force, and (2) the Strouhal number

$$S = \frac{\omega D}{2\pi v} \tag{3.50}$$

which is the ratio of the inertia force due to the local acceleration to the inertia force due to the convective acceleration.

The independent dimensionless parameters are (1) the Reynolds number

$$\mathrm{Re} = \frac{\rho v D}{\mu} \tag{3.51}$$

which is the ratio of the inertia force to the viscous force and (2) the diameter-to-length ratio D/L.

For long cylinders ($D/L \ll 1$), a two-dimensional approximation is used. Then the effect of D/L on the drag coefficient and Strouhal number is negligible. Empirical data are used to determine the forms of Eqs. (3.47) and (3.48), assuming that both the drag coefficient and Strouhal number are independent of D/L. Empirical curves are shown in Figs. 3.18 and 3.19.

The density and dynamic viscosity of air at 20°C are 1.204 kg/m^3 and 1.82×10^{-5} N · s/m, respectively. Thus for air at 20°C the Reynolds number for flow over

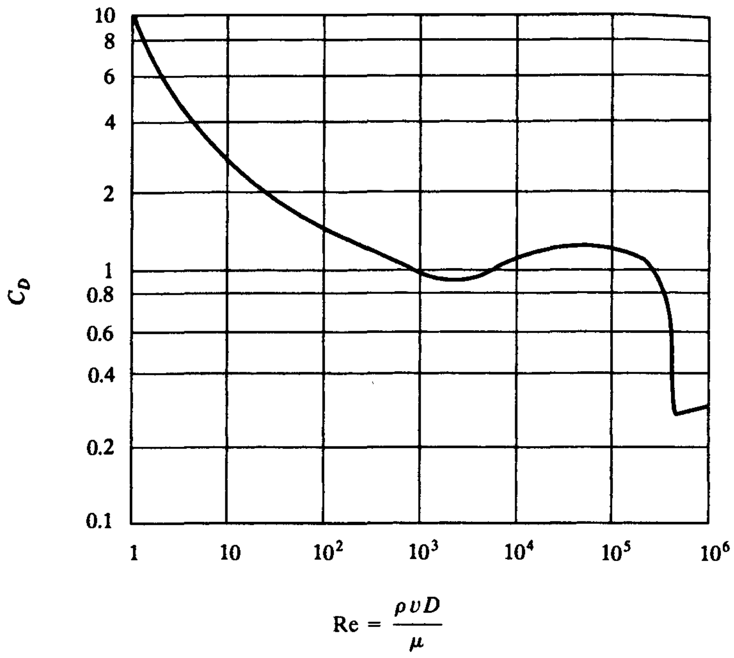


Figure 3.18 Drag coefficient versus Reynolds number for a circular cylinder $C_D \approx 1$ for $1 \times 10^3 < \text{Re} < 2 \times 10^5$. (*From Shames.*)

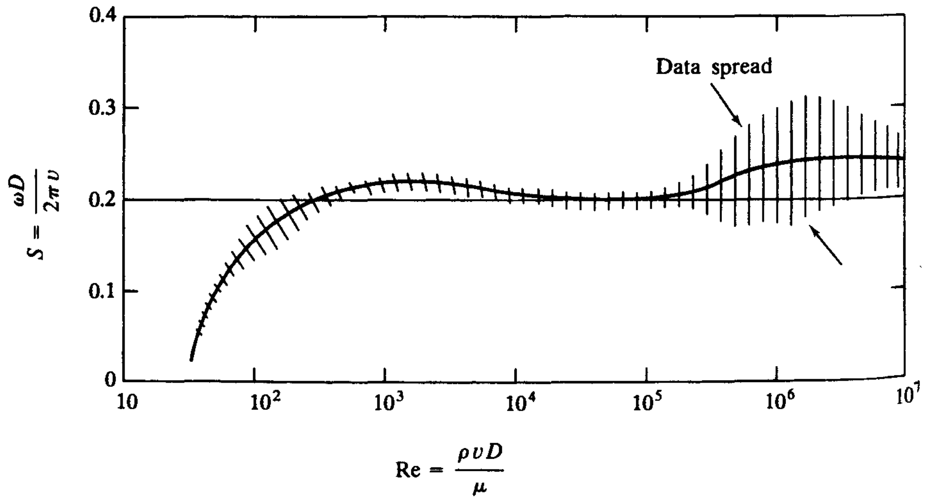


Figure 3.19 Strouhal number as a function of Reynolds number. The Strouhal number is approximately 0.2 for $1 \times 10^3 < \text{Re} < 2 \times 10^5$. [*From White, based on experimental data in Roshko* (1954) *and Jones* (1968).]

a 10-cm-diameter circular cylinder at 20 m/s is

$$\mathrm{Re} = \frac{(1.204\ \mathrm{kg/m^3})(20\ \mathrm{m/s})(0.1\ \mathrm{m})}{1.82 \times 10^{-5}\ \mathrm{N \cdot s/m}} = 1.3 \times 10^5$$

The Reynolds number for many situations involving wind-induced oscillations is between 1×10^3 and 2×10^5. Over this Reynolds number regime, both the drag coefficient and the Strouhal number are approximately constant. From Figs. 3.18 and 3.19,

$$C_D \approx 1 \qquad 1 \times 10^3 < \mathrm{Re} < 2 \times 10^5 \tag{3.52}$$

$$S \approx 0.2 \qquad 1 \times 10^3 < \mathrm{Re} < 2 \times 10^5 \tag{3.53}$$

From Eq. (3.53) and the definition of the Strouhal number, Eq. (3.50),

$$v = \frac{\omega D}{0.4\pi} \tag{3.54}$$

Then from Eqs. (3.49), (3.52), and (3.54),

$$F_0 = 0.317\rho D^3 L\omega^2 \tag{3.55}$$

Hence the harmonic excitation to a circular cylinder provided by vortex shedding when the Reynolds number is between 1×10^3 and 2×10^5 has a magnitude that is proportional to the square of its frequency. Using the notation of Eqs. (3.32) and (3.33) gives

$$A = 0.317\rho D^3 L \tag{3.56}$$

and

$$\frac{3.16mX}{\rho D^3 L} = \Lambda(r, \zeta) \tag{3.57}$$

The theory is presented for vortex shedding from circular cylinders. If the frequency at which the vortices are shed is near the natural frequency of the structure, then large-amplitude vibrations exist. The effects of vortex shedding must be taken into account when designing structures such as street lamp posts, transmission towers, chimneys, and tall buildings. Vortex shedding also occurs from noncircular structures such as buildings and bridges. Vortex shedding at a frequency near a torsional natural frequency is thought to be partially responsible for the famous Tacoma Narrows Bridge disaster. Amplitudes of a torsional mode were observed to be as large as 45°.

Example

A street lamp consists of a 60-kg light fixture attached at the end of a 3-m-tall solid steel ($E = 210 \times 10^9\ \mathrm{N/m^2}$) cylinder with a diameter of 20 cm. Use a one-degree-of-freedom model consisting of a cantilever beam with a concentrated mass at its end to analyze the response of the light fixture to wind excitation. Assume the beam has an equivalent viscous damping ratio of 0.2.

(a) At what wind speed will the maximum steady-state amplitude of vibration due to vortex shedding occur?

(*b*) What is the corresponding maximum amplitude?

(*c*) Redesign the light by changing its diameter such that the maximum amplitude of vibration does not exceed 0.10 mm for any wind speed.

Solution:

Before proceeding with the analysis, there are several questions associated with the modeling that must be addressed. Vortices are shed along the entire length of the cylinder. The two-dimensional assumption implies that the force per unit length is constant along the entire length of the light post. Thus the force given by Eq. (3.44) is really the resultant of this force per unit length distribution. Its point of application should be the midpoint of the light post. However, the problem is not really two dimensional because of among other things, the boundary layer of the earth. The presence of a boundary layer causes a varying wind velocity over the length of the light post, which, in turn, causes a nonuniform force per unit length distribution, as shown in Fig. 3.20*a*. Thus the actual point of application of the resultant force will be somewhat higher than the midpoint of the light post. In addition, the mass is assumed to be lumped at the end of the beam, while the point of application of the applied force is elsewhere. The resultant force can be replaced by a force of the same magnitude located at the end of the beam and a moment. However, the moment causes rotational effects which are not adequately taken into account in a one-degree-of-freedom model. At least a two-degree-of-freedom model should be used. In order to attain an approximate result, these effects are neglected. A one-degree-of-freedom model is used where the excitation is provided by a concentrated harmonic load located at the light of fixture, as shown in Fig. 3.20*b*.

Assume air at 20°C. The Reynolds number for a velocity of 20 m/s is

$$\mathrm{Re} = \frac{(1.204\ \mathrm{kg/m^3})(20\ \mathrm{m/s})(0.20\ \mathrm{m})}{(1.82 \times 10^{-5}\ \mathrm{N \cdot s/m})} = 2.6 \times 10^5$$

This Reynolds number is higher than the 2×10^5 upper limit on the range of strict

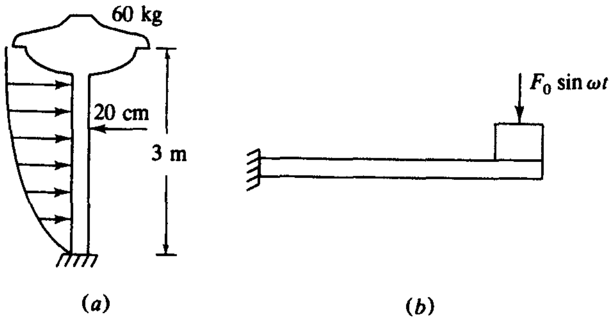


Figure 3.20 (*a*) Street light post in steady wind is subject to harmonic excitation whose amplitude is proportional to the square of the frequency because of vortex shedding; (*b*) the problem is modeled as a mass concentrated at the end of a cantilever beam.

applicability of the theory presented previously. However, from Fig. 3.19 the Strouhal number is only slightly higher than 0.2. Using 0.2 as an approximation for the Strouhal number is in line with other approximations made in the modeling.

(*a*) Using a one-degree-of-freedom model, the natural frequency of the cantilever beam is

$$\omega_n = \sqrt{\frac{3EI}{mL^3}} = \sqrt{\frac{3(210 \times 10^9\ \mathrm{N/m^2})(\pi/64)(0.2\ \mathrm{m})^4}{(60\ \mathrm{kg})(3\ \mathrm{m})^3}} = 174.8\ \frac{\mathrm{rad}}{\mathrm{s}}$$

The magnitude of the excitation force is proportional to the square of its frequency. Thus, from Eq. (3.36), the maximum steady-state amplitude occurs for a frequency ratio of

$$r_{\max} = \frac{1}{\sqrt{1 - 2\zeta^2}} = 1.043$$

Thus the frequency at which the maximum amplitude occurs is

$$\omega = 1.043(174.8\ \mathrm{rad/s}) = 182.2\ \mathrm{rad/s}$$

The wind velocity that gives rise to this frequency is calculated using the definition of the Strouhal number

$$v = \frac{\omega D}{2\pi S} = \frac{(182.2\ \mathrm{rad/s})(0.2\ \mathrm{m})}{2\pi(0.2)} = 29.0\ \frac{\mathrm{m}}{\mathrm{s}}$$

(*b*) The value of Λ corresponding to this frequency ratio is calculated from Eq. (3.37)

$$\Lambda_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} = 2.55$$

The corresponding maximum amplitude is calculated by using Eq. (3.57)

$$X = \frac{\rho D^3 L\Lambda}{3.16m} = \frac{(1.204\ \mathrm{kg/m^3})(0.2\ \mathrm{m})^3(3\ \mathrm{m})(2.55)}{3.16(60\ \mathrm{kg})} = 3.9 \times 10^{-4}\ \mathrm{m}$$

(*c*) The maximum value of Λ is a function of ζ only and does not change with ω_n. The steady-state amplitude can be limited to 0.1 mm for all wind speeds by requiring that $\Lambda = 2.55$ for $X = 0.1$ mm. This leads to

$$D = \left(\frac{3.16mX}{\rho L\Lambda}\right)^{1/3} = 12.7\ \mathrm{cm}$$

Thus the maximum diameter of the light pole should be 12.7 cm.

3.6 RESPONSE DUE TO HARMONIC EXCITATION OF SUPPORT

Consider the mass-spring-dashpot system of Fig. 3.21. The spring and dashpot are in parallel with one end of each connected to the mass and the other end of each connected to a moveable support. Let $y(t)$ denote the known displacement of the

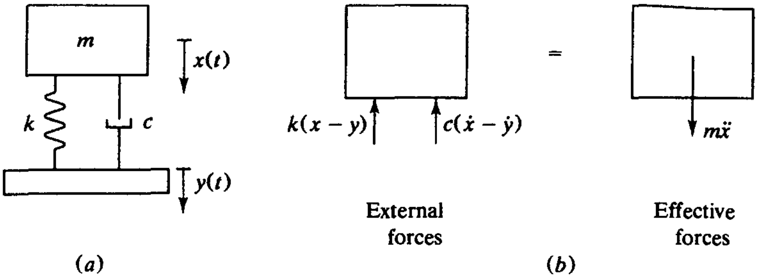


Figure 3.21 (a) Block is connected through spring and viscous damper in parallel to moveable support; (b) free-body diagrams at an arbitrary instant including effects of support displacement, $y(t)$.

support and let $x(t)$ denote the absolute displacement of the mass. Application of Newton's law to the free-body diagrams of Fig. 3.21b yields

$$-k(x-y)-c(\dot{x}-\dot{y})=m\ddot{x} \tag{3.58}$$

or

$$m\ddot{x}+c\dot{x}+kx=c\dot{y}+ky \tag{3.59}$$

Define

$$z(t)=x(t)-y(t) \tag{3.60}$$

as the displacement of the mass relative to the displacement of its support. Equation (3.59) is rewritten using z as the dependent variable

$$m\ddot{z}+c\dot{z}+kz=-m\ddot{y} \tag{3.61}$$

Dividing Eqs. (3.59) and (3.61) by m yields

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=2\zeta\omega_n\dot{y}+\omega_n^2y \tag{3.62}$$

and

$$\ddot{z}+2\zeta\omega_n\dot{z}+\omega_n^2z=-\ddot{y} \tag{3.63}$$

If the base displacement is given by a single-frequency harmonic of the form

$$y(t)=Y\sin\omega t \tag{3.64}$$

then Eqs. (3.62) and (3.63) become

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=2\zeta\omega_n\omega Y\cos\omega t+\omega_n^2Y\sin\omega t \tag{3.65}$$

and

$$\ddot{z}+2\zeta\omega_n\dot{z}+\omega_n^2z=\omega^2Y\sin\omega t \tag{3.66}$$

Equation (3.66) shows that a mass-spring-dashpot system subject to harmonic base motion is yet another example in which the magnitude of a harmonic excitation is proportional to the square of its frequency. Using the theory of Sec. 3.5,

$$z(t)=Z\sin(\omega t-\phi) \tag{3.67}$$

where
$$Z = Y\Lambda(r, \zeta) \qquad [3.68]$$

where Λ is defined in Eq. (3.34) and ϕ defined by Eq. (3.29).

When Eqs. (3.67) and (3.68) are substituted into Eq. (3.60), the absolute displacement becomes

$$x(t) = Y[\Lambda \sin(\omega t - \phi) + \sin \omega t] \qquad [3.69]$$

Using the trigonometric relationship for the sine of the difference of two angles, it is possible to express Eq. (3.69) in the form

$$x(t) = X \sin(\omega t - \lambda) \qquad [3.70]$$

where
$$\frac{X}{Y} = T(r, \zeta) \qquad [3.71]$$

and
$$\lambda = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1) r^2}\right] \qquad [3.72]$$

where $T(r, \zeta)$ is yet another nondimensional function of the frequency ratio and the damping ratio defined by

$$T(r, \zeta) = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \qquad [3.73]$$

X/Y is the amplitude of the absolute displacement of the mass to the amplitude of displacement of the base. Multiplying the numerator and denominator of this ratio by ω^2 shows that $T(r, \zeta)$ also represents the ratio of the amplitude of absolute acceleration of the mass to the amplitude of acceleration of the base. Equation (3.73) is plotted in Fig. 3.22. The following are noted about $T(r, \zeta)$:

1. $T(r, \zeta)$ is near one for small r.
2. For all ζ, $0 < \zeta < 1$, $T(r, \zeta)$ grows until it reaches a maximum for a frequency ratio of

$$r_{\max} = \frac{1}{2\zeta}\left(\sqrt{1 + 8\zeta^2} - 1\right)^{1/2} \qquad [3.74]$$

3. The maximum of $T(r, \zeta)$ corresponding to the frequency ratio of Eq. (3.74)

$$T_{\max} = 4\zeta^2\left[\frac{\sqrt{1 + 8\zeta^2}}{2 + 16\zeta^2 + (16\zeta^4 - 8\zeta^2 - 2)\sqrt{1 + 8\zeta^2}}\right]^{1/2} \qquad [3.75]$$

4. $T(\sqrt{2}, \zeta) = 1$, independent of the value of ζ.
5. For $r < \sqrt{2}$, $T(r, \zeta)$ is larger for smaller values of ζ. However, for $r > \sqrt{2}$, $T(r, \zeta)$ is smaller for smaller values of ζ.
6. For all values of ζ, $T(r, \zeta)$ is less than one when and only when $r > \sqrt{2}$.

The phase angle between the absolute displacement and the displacement of the base is given by Eq. (3.72).

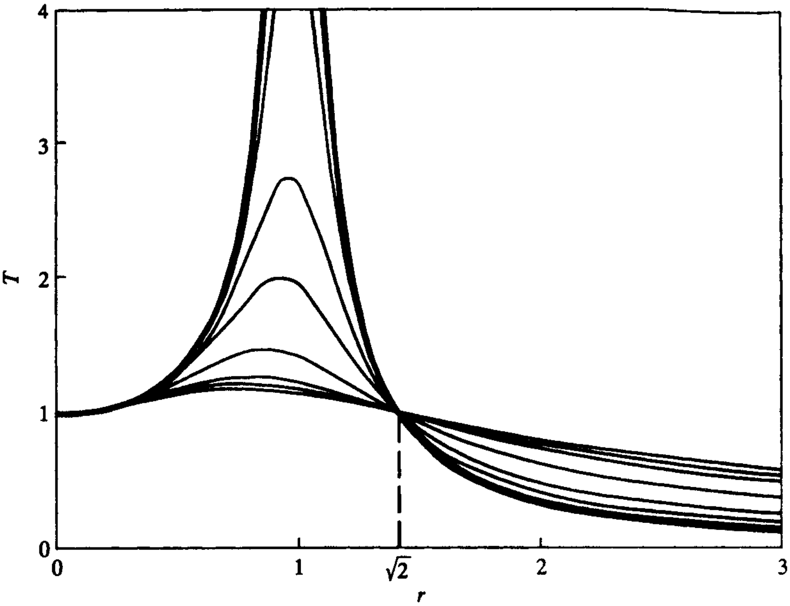


Figure 3.22 T versus r for different values of ζ.

3.7 ▲ A simple one-degree-of-freedom model of a vehicle suspension system is shown in Fig. 3.23. The vehicle is traveling over a rough road at a constant horizontal speed v. The road contour is approximated by a sinusoid as shown in Fig. 3.23b. The mass of the vehicle is 900 kg. The stiffness of the suspension system is 8×10^4 N/m and the system has a damping ratio of 0.20.

(a) Determine the acceleration and displacement amplitudes of the body of the vehicle when $v = 40$ m/s.

(b) Plot the displacement amplitude and the acceleration amplitude as functions of vehicle speed.

Solution:

Let ξ be a coordinate that measures, in meters, the horizontal distance from a reference location. The equation for the road contour is

$$y(\xi) = d \sin\left(\frac{2\pi}{l}\xi\right) = 0.010 \sin(1.25\pi\xi) \text{ m}$$

If the vehicle is at the reference location at $t = 0$, the instantaneous horizontal location of the vehicle is

$$\xi = vt$$

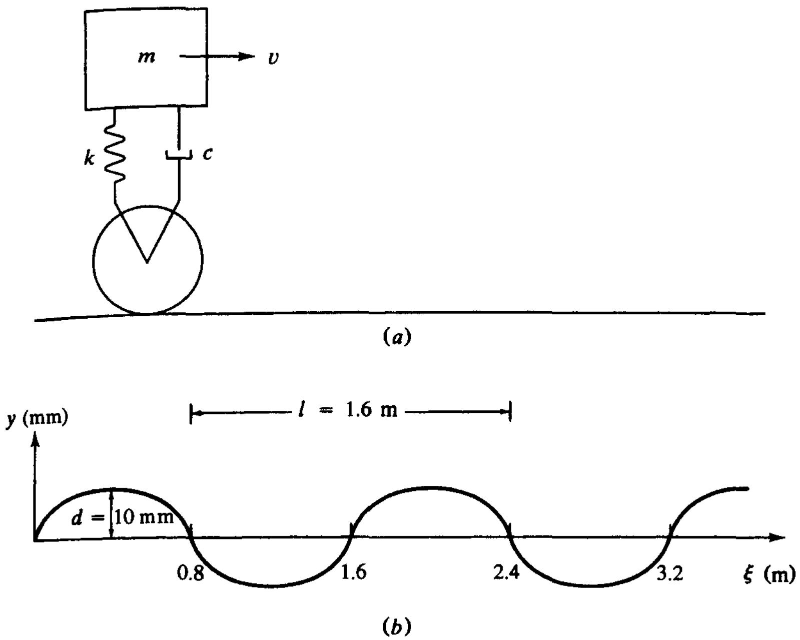


Figure 3.23 (a) Simplified one-degree-of-freedom model of vehicle suspension system; (b) sinusoidal road contour of Example 3.7.

Thus the vertical displacement of the wheel as a function of time is

$$y(t) = 0.010 \sin(1.25\pi vt) \text{ m}$$

The motion of the wheel provides a vertical input which provides a harmonic base displacement of amplitude 0.010 m and frequency $1.25\pi v$ to the body of the vehicle. If the suspension system were not present, the body would have an amplitude of displacement of 0.010 m and an acceleration amplitude of $(0.010 \text{ m})(1.25\pi v)^2 = 246.7 \text{ m/s}^2$ when $v = 40$ m/s. The system's natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8 \times 10^4 \text{ N/m}}{900 \text{ kg}}} = 9.43 \frac{\text{rad}}{\text{s}}$$

(a) For a vehicle speed of 40 m/s, the excitation frequency is

$$\omega = 1.25\pi(40 \text{ m/s}) = 157.1 \text{ rad/s}$$

leading to a frequency ratio of

$$r = \frac{\omega}{\omega_n} = \frac{157.1 \text{ rad/s}}{9.43 \text{ rad/s}} = 16.67$$

The amplitude of absolute displacement of the body is obtained by using Eq. (3.71)

$$X = dT(16.67, 0.20) = (0.010\ \text{m})\sqrt{\frac{1 + [2(0.20)(16.67)^2]}{[1 - (16.67)^2]^2 + [2(0.20)(16.67)]^2}}$$
$$= 2.43 \times 10^{-4}\ \text{m}$$

The acceleration amplitude is

$$A = \omega^2 X = \left(157.1\ \frac{\text{rad}}{\text{s}}\right)^2 (2.43 \times 10^{-4}\ \text{m}) = 6.01\ \frac{\text{m}}{\text{s}^2}$$

(*b*) The displacement amplitude as a function of vehicle speed is

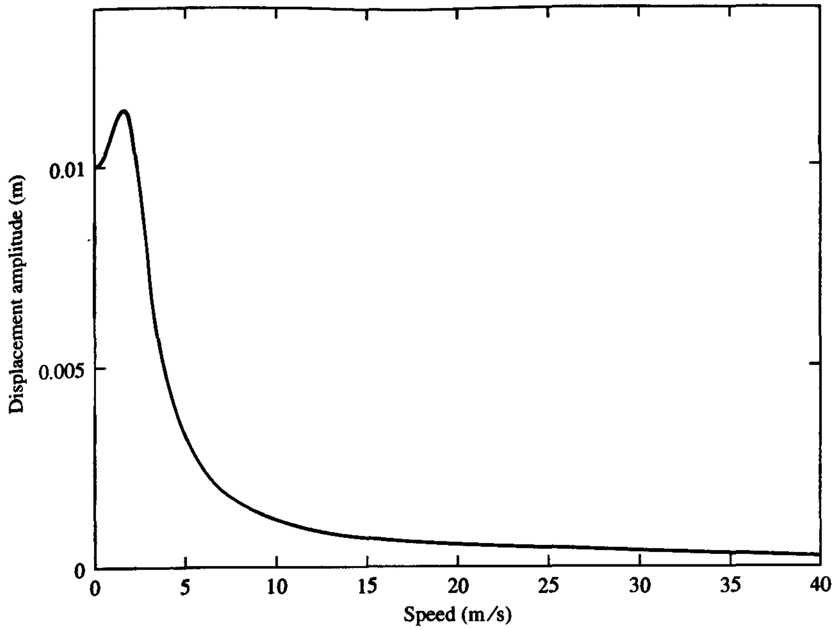
$$X(v) = (0.01)\sqrt{\frac{1 + \left[2(0.20)\dfrac{1.25\pi v}{9.43}\right]^2}{\left[1 - \left(\dfrac{1.25\pi}{9.43}\right)^2\right]^2 + \left[2(0.20)\dfrac{1.25\pi v}{9.43}\right]^2}}$$
$$= 0.01\sqrt{\frac{1 + 0.028v^2}{1 - 0.145v^2 + 0.030v^4}}$$

The acceleration amplitude is

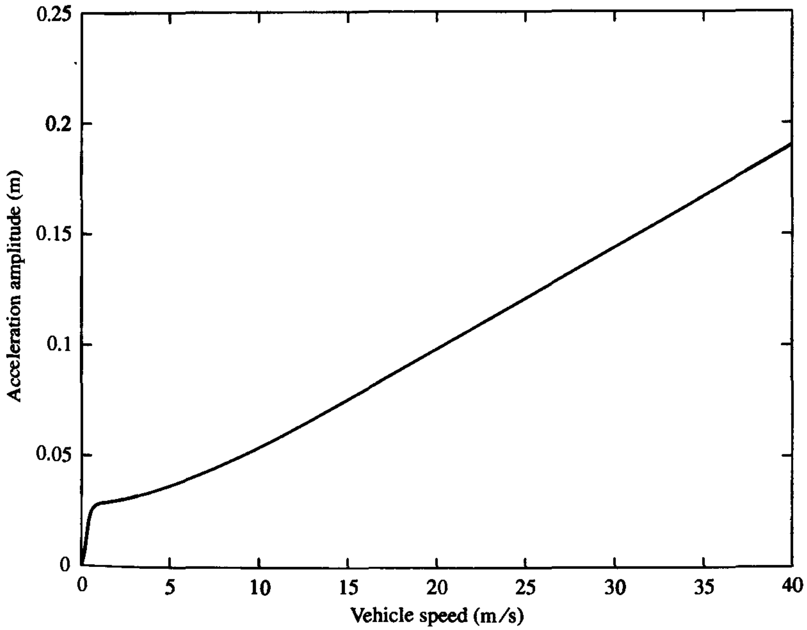
$$A = (1.25\pi v)^2(0.01)\sqrt{\frac{1 + \left[2(0.20)\dfrac{1.25\pi v}{9.43}\right]^2}{\left[1 - \left(\dfrac{1.25\pi v}{9.43}\right)^2\right]^2 + \left[2(0.20)\dfrac{1.25\pi v}{9.43}\right]^2}}$$
$$= 0.154v^2\sqrt{\frac{1 + 0.028v^2}{1 - 0.145v^2 + 0.030v^4}}$$

Amplitude plots generated by MATLAB are shown in Fig. 3.24. The displacement amplitude reaches a peak for v less than 5 m/s and then decreases with increasing speed. The acceleration amplitude increases with increasing speed.

Mechanisms can be used to produce harmonic base excitations. One simple example is the eccentric circular cam of Fig. 3.25. When rotating at a constant speed, the cam produces a displacement of $e \sin \omega t$ to its follower, which, in turn, produces a harmonic base excitation in the arrangement shown. The Scotch yoke of Fig. 3.26 is another mechanism that produces simple harmonic motion. When the crank is rotating at a constant speed the base is given a displacement of $l \sin \omega t$.



(a)



(b)

Figure 3.24 MATLAB plot of (a) displacement amplitude versus vehicle speed; (b) occeleration amplitude versus speed.

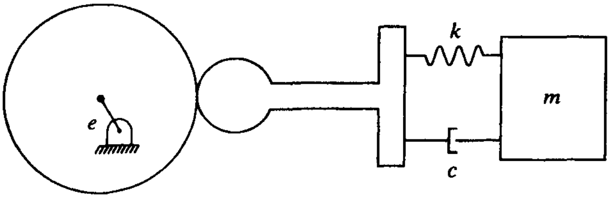


Figure 3.25 Eccentric circular cam produces harmonic motion of follower which provides support excitation to mass-spring-dashpot system.

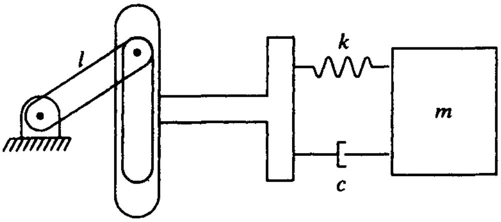


Figure 3.26 Scotch yoke mechanism produces simple harmonic motion and provides support excitation to mass-spring-dashpot system.

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8 A Scotch yoke mechanism provides a harmonic base excitation for the mass-spring-dashpot system of Fig. 3.26. The crank arm is 80 mm long. The speed of rotation of the crank arm is varied and the resulting steady-state amplitude is recorded at each speed. The maximum recorded amplitude of the 14.73-kg block is 13 cm at 1000 rpm. Determine the spring stiffness and damping coefficient.

Solution:

The amplitude of the base displacement is 0.08 m. The maximum displacement of the mass is 0.13 m. Thus

$$T_{\max} = \frac{X_{\max}}{Y} = \frac{0.13 \text{ m}}{0.08 \text{ m}} = 1.625$$

The value of ζ which corresponds to this $T_{\max}$ is determined by solving Eq. (3.75). However, algebraic manipulation of Eq. (3.75) yields a fifth-order polynomial equation for ζ^2. A numerical method must be used to find ζ. An easier trial-and-error approach is outlined in the following discussion, and then used to find the value of ζ for this example.

Equation (3.74) is rearranged as

$$\zeta = \sqrt{\frac{1 - r_{\max}^2}{2r_{\max}^4}}$$

A value of $r_{\max} < 1$ is guessed and its corresponding value of ζ calculated from the preceding equation. Equation (3.73) or (3.75) is then used to calculate the value of T_{max} corresponding to the guessed value of r_{max}. However, small changes in the accuracy of an intermediate calculation using Eq. (3.75) lead to large changes in the result. Thus Eq. (3.73) is usually used. The calculated value of T_{max} is compared against the desired value of 1.625. If $T_{max} > 1.625$ another guess for r_{max}, smaller than the previous one, should be made. Other iteration schemes are possible, but the method presented is the most direct using the equations as presented. The trial-and-error scheme is illustrated in the following table:

r_{max}(guess)	ζ	T_{max} [from Eq. (3.73)]
0.98	0.147	3.180
0.90	0.381	1.702
0.89	0.407	1.640
0.88	0.437	1.573

Then, for $r_{max} = 0.89$,

$$\omega_n = \frac{\omega}{r_{max}} = \left(1000\ \frac{\text{rev}}{\text{min}}\right)\left(2\pi\ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)\frac{1}{0.89} = 117.7\ \frac{\text{rad}}{\text{s}}$$

and

$$k = m\omega_n^2 = 2.04 \times 10^5\ \text{N/m}$$

3.7 SYSTEMS WITH COULOMB DAMPING

The differential equations derived using the free-body diagram of Fig. 3.27 governing the response of a one-degree-of-freedom system with Coulomb damping due to a harmonic excitation are

$$m\ddot{x} + kx = F_0 \sin(\omega t + \psi) - F_f \qquad \dot{x} > 0 \tag{3.76a}$$

$$m\ddot{x} + kx = F_0 \sin(\omega t + \psi) + F_f \qquad \dot{x} < 0 \tag{3.76b}$$

where $F_f = \mu m g$ is the magnitude of the friction force.

If the initial displacement and velocity are both zero, motion commences only when the excitation force is as large as the friction force. Motion will continue until the resultant of the spring force and the excitation force is less than the friction force,

$$|kx - F_0 \sin \omega t| < F_f \Rightarrow \dot{x} = 0 \tag{3.77}$$

The resultant eventually grows large enough such that the inequality in Eq. (3.77) is no longer satisfied, when motion again commences. This process is known as stick-slip and can occur several times during one cycle of motion.

Equation (3.76) is nonlinear. Thus the principles guiding the solution of linear differential equations are not applicable. Specifically, the general solution cannot be written as a homogeneous solution independent of the excitation plus a particular

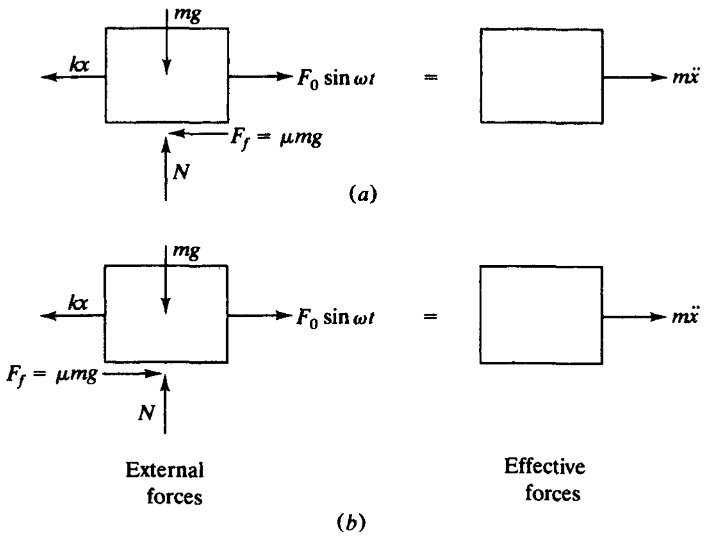


Figure 3.27 Free-body diagrams, at an arbitrary instant of time, for block subject to harmonic excitation and Coulomb damping: (a) $\dot{x} > 0$; (b) $\dot{x} < 0$.

solution. Thus, even though free vibrations of a system with Coulomb damping decay linearly and eventually cease, it is not possible to predict the particular solution as a steady-state solution. Indeed, from the preceding discussion, the stick-slip process should occur for large time and cannot be predicted by a particular solution.

The analytical solution to Eq. (3.76) can be attained using a procedure similar to that of Sec. 2.6 used to obtain the free-vibration response of a system subject to Coulomb damping. The solution of Eqs. (3.76*a*) and (3.76*b*) are readily available over the time that the equation governs. The constants of integration are determined by noting that the velocity is zero and the displacement is continuous at the time when the equation first begins to govern. Equation (3.77) must be checked over each half-cycle to determine if and when the mass sticks.

The analytical solution is very involved and difficult to use to predict long-term behavior. In many applications only the maximum displacement is of interest. It is a function of five parameters

$$X = f(m, \omega, \omega_n, F_0, F_f) \tag{3.78}$$

The nondimensional form of Eq. (3.78) is

$$\frac{m\omega_n^2 X}{F_0} = f(r, \iota) \tag{3.79}$$

where

$$\iota = \frac{F_f}{F_0} \tag{3.80}$$

For small ι, the friction force is much less than the magnitude of the excitation force, and it is expected that the transient solution will decrease as t increases and a harmonic steady state of the form

$$x(t) = X_c \sin(\omega t - \phi_c) \tag{3.81}$$

exists for large t. In this case the effects of Coulomb damping can be reasonably approximated by an equivalent viscous damping model as discussed in Sec. 2.8. The equivalent viscous damping coefficient for Coulomb damping is

$$c_{eq} = \frac{4F_f}{\pi \omega X_c} \tag{3.82}$$

An equivalent damping ratio is defined by

$$\zeta_{eq} = \frac{c_{eq}}{2m\omega_n} = \frac{2F_f}{\pi m \omega \omega_n X_c} \tag{3.83a}$$

Rearrangement of Eq. (3.82) leads to

$$\zeta_{eq} = \frac{2\iota F_0}{\pi r m \omega_n^2 X} = \frac{2\iota}{\pi r M_c} \tag{3.83b}$$

where M_c, the magnification factor for Coulomb damping, is

$$M_c = \frac{m\omega_n^2 X}{F_0}$$

Using ζ_{eq} in place of ζ in Eq. (3.26) leads to

$$M_c(r, \iota) = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + \left(\dfrac{4\iota}{\pi M_c}\right)^2}}$$

which is solved for M_c, yielding

$$M_c(r, \iota) = \sqrt{\frac{1 - \left(\dfrac{4\iota}{\pi}\right)^2}{\left(1 - r^2\right)^2}} \tag{3.84}$$

The magnification factor for Coulomb damping is plotted in Fig. 3.28 as a function of r for several values of ι. The following are noted from Eq. (3.84) and Fig. 3.28:

1. The small ι theory predicts that $M_c(r, \iota)$ exists only for $\iota < \pi/4$. The equivalent viscous damping theory cannot be used to predict the maximum displacement for $\iota > \pi/4$.
2. Resonance occurs for systems with Coulomb damping with small ι when $r = 1$. Resonance occurs because, for small ι, the excitation provides more energy per cycle of motion than is dissipated by the friction. Since free vibrations sustain themselves, the extra energy leads to an amplitude buildup.
3. For all values of r, M_c is smaller for larger ι.

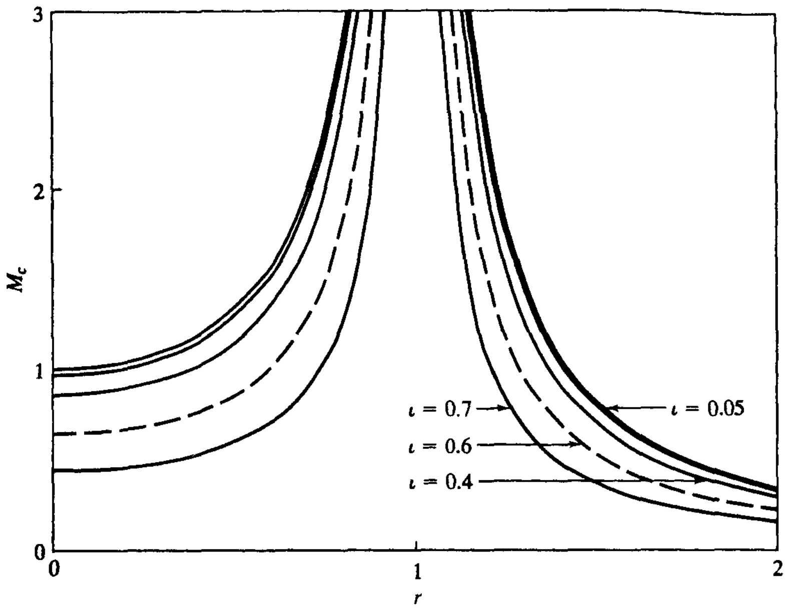


Figure 3.28 M_c versus r for different values of ι, using equivalent viscous damping coefficient.

When Eq. (3.83*b*) is substituted into Eq. (3.29) and the resulting equation manipulated, the following result for the phase angle occurs:

$$\phi_c = \tan^{-1}\left[\frac{\dfrac{4\iota}{\pi}}{\sqrt{1-\left(\dfrac{4\iota}{\pi}\right)^2}}\right] \qquad r < 1 \tag{3.85a}$$

$$\phi_c = -\tan^{-1}\left[\frac{\dfrac{4\iota}{\pi}}{\sqrt{1-\left(\dfrac{4\iota}{\pi}\right)^2}}\right] \qquad r > 1 \tag{3.85b}$$

The phase angle is constant with r, except that it is positive for $r < 1$ and negative for $r > 1$.

The preceding theory is sufficient for small ι. For larger ι, the equation is truly nonlinear and the results more complex. However, it is expected that larger ι leads to smaller-amplitude vibrations, and less serious problems. In the absence of initial energy, vibrations will not be initiated for $\iota > 1$.

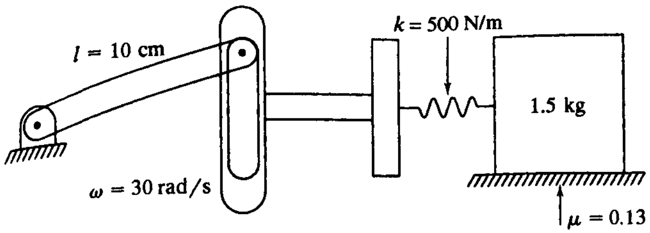


Figure 3.29 Scotch yoke mechanism provides base displacement to system with Coulomb damping.

Example

A Scotch yoke mechanism operating at 30 rad/s is used to provide base excitation to a block as shown in Fig. 3.29. The block has a mass of 1.5 kg and is connected to the Scotch yoke through a spring of stiffness 500 N/m. The coefficient of friction between the block and the surface is 0.13. Approximate the steady-state amplitude of the block.

Solution:

The differential equation governing the motion of the block is

$$m\ddot{x} + kx = kl \sin \omega t \mp \mu mg$$

The amplitude of the excitation is kl. Thus

$$\iota = \frac{\mu mg}{kl} = \frac{(0.13)(1.5 \text{ kg})(9.81 \text{ m/s}^2)}{(500 \text{ N/m})(0.1 \text{ m})} = 0.038$$

The system's natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = 18.26 \text{ rad/s} \qquad r = \frac{\omega}{\omega_n} = 1.64$$

The Coulomb damping magnification factor is

$$M_c(1.64, 0.038) = \sqrt{\frac{1 - \left[\frac{4(0.038)}{\pi}\right]^2}{[1 - (1.64)^2]^2}} = 0.591$$

The steady-state amplitude is calculated from

$$\frac{m\omega_n^2 X}{kl} = \frac{X}{l} = M_c(1.64, 0.038) \qquad X = (0.1 \text{ m})(0.591) = 0.0591 \text{ m}$$

3.8 Systems with Hysteretic Damping

Recall from Sec. 2.7 that the energy dissipated per cycle of motion for a system with hysteretic damping is independent of frequency but proportional to the square of the amplitude. This leads to the direct analogy between viscous damping and hysteretic damping and the development of an equivalent viscous damping coefficient

$$c_{\mathrm{eq}} = \frac{hk}{\omega} \quad [3.86]$$

The true forced response of a mass-spring system with hysteretic damping is nonlinear. The equivalent viscous damping coefficient of Eq. (3.86) is valid only when the excitation consists of a single-frequency harmonic. During the initial part of the response, the transient solution and the particular solution have harmonic terms with different frequencies. On the basis of the viscous damping analogy, it is suspected that the transient solution decays leaving only the steady-state solution after a long time. The differential equation governing the steady-state response of a mass-spring system with hysteretic damping due to a single-frequency harmonic excitation is assumed to be

$$m\ddot{x} + \frac{kh}{\omega}\dot{x} + kx = F_0 \sin(\omega t + \psi) \quad [3.87]$$

It is noted that the generalization of Eq. (3.87) to a more general excitation is not permissible because the damping approximation is valid only for a single-frequency harmonic excitation. The equation is also nonlinear so that the method of superposition is not applicable to determine particular solutions for multifrequency excitations.

The steady-state solution of Eq. (3.87) is obtained by comparison with Eq. (3.7). The equivalent damping ratio is

$$\zeta_{\mathrm{eq}} = \frac{h}{2r} \quad [3.88]$$

The steady-state response is

$$x(t) = X_h \sin(\omega t - \phi_h) \quad [3.89]$$

where X_h and ϕ_h are obtained by analogy with Eqs. (3.23), (3.26), and (3.29)

$$\frac{m\omega_n^2 X_h}{F_D} = M_h(r, h) \quad [3.90]$$

$$M_h(r, h) = \frac{1}{\sqrt{(1 - r^2)^2 + h^2}} \quad [3.91]$$

$$\phi_h = \tan^{-1}\left(\frac{h}{1 - r^2}\right) \quad [3.92]$$

Equations (3.91) and (3.92) are plotted in Figs. 3.30 and 3.31. The following are noted from these equations and figures:

1. For a given h, the $M_h(r, h)$ attains a maximum value of $1/h$ for $r = 1$.
2. The phase angle is nonzero for $r = 0$. The response is never in phase with the excitation.

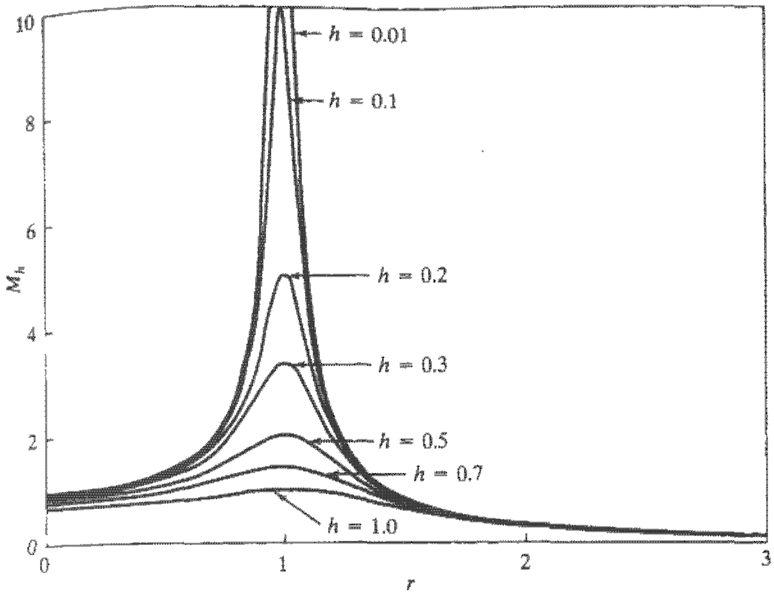


Figure 3.30 Magnification factor for hysteretic damping for different values of h. $M_{h,\,max} = 1/h$ for $r = 1$.

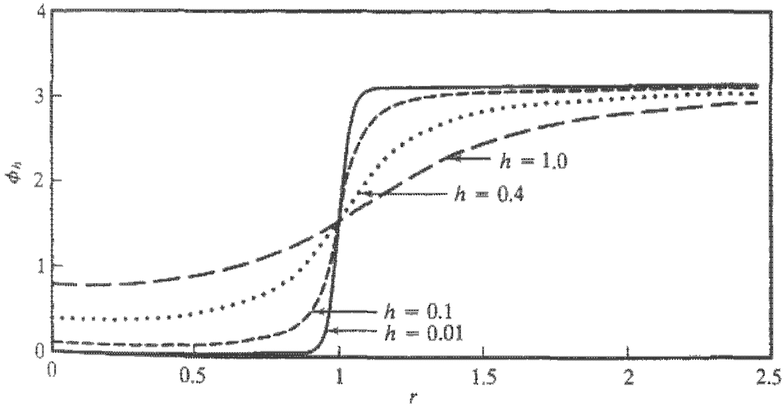


Figure 3.31 ϕ_h versus r for different values of h. The response of a system with hysteretic damping is never in phase with the excitation.

Most damping is not viscous, but hysteretic. The differences are slight, but noticeable. Viscous damping is often assumed, even when hysteretic damping is present. The viscous damping assumption is easier to use because the damping ratio is independent of frequency. For hysteretic damping the damping ratio is higher for lower frequencies.

3.10 A 100-kg lathe is mounted at the midspan of a 1.8-m simply supported beam ($E = 200 \times 10^9$ N/m, $I = 4.3 \times 10^{-6}$ m^4). The lathe has a rotating unbalance of 0.43 kg · m and operates at 2000 rpm. When a free vibrations test is performed on the system it is found that the ratio of amplitudes on successive cycles is 1.8 to 1. Determine the steady-state amplitude of vibration induced by the rotating unbalance. Assume the damping is hysteretic.

Solution:

The beam's stiffness is

$$k = \frac{48EI}{L^3} = \frac{48\left(200 \times 10^9 \text{ N/m}^2\right)\left(4.3 \times 10^{-6} \text{ m}^4\right)}{(1.8 \text{ m})^3} = 7.08 \times 10^6 \frac{\text{N}}{\text{m}}$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7.08 \times 10^6 \text{ N/m}}{100 \text{ kg}}} = 266.1 \frac{\text{rad}}{\text{s}}$$

$$r = \frac{\omega}{\omega_n} = \frac{(2000 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min/60 s})}{266.1 \text{ rad/s}} = 0.787$$

The logarithmic decrement and hysteretic damping coefficient are calculated as

$$\delta = \ln 1.8 = 0.588 \qquad h = \frac{\delta}{\pi} = 0.187$$

The appropriate form of Λ for hysteretic damping is

$$\Lambda_h(r, h) = \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + h^2}}$$

$$\Lambda_h(0.787, 0.187) = \frac{(0.787)^2}{\sqrt{\left[1 - (0.787)^2\right]^2 + (0.187)^2}} = 1.46$$

The lathe's steady-state amplitude is

$$X = \frac{m_0 e}{m}\Lambda_h(0.787, 0.187) = \frac{0.43 \text{ kg} \cdot \text{m}}{100 \text{ kg}}(1.46) = 6.3 \text{ mm}$$

3.9 MULTIFREQUENCY EXCITATIONS

A multifrequency excitation has the form

$$F(t) = \sum_{i=1}^{n} F_i \sin\left(\omega_i t + \psi_i\right) \qquad [3.93]$$

Without loss of generality, it is assumed that $F_i > 0$ for each i. The steady-state response due to a multifrequency excitation is obtained using the response for a single-frequency excitation and the principle of linear superposition. The total response is the sum of the responses due to each of the individual frequency terms. Thus the solution of Eq. (3.7) with the excitation of Eq. (3.93) is

$$x(t) = \sum_{i=1}^{n} X_i \sin(\omega_i t + \psi_i - \phi_i) \tag{3.94}$$

where

$$X_i = \frac{M_i F_i}{m_{eq}\omega_n^2} \tag{3.95}$$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1 - r_i^2}\right) \tag{3.96}$$

$$r_i = \frac{\omega_i}{\omega_n} \tag{3.97}$$

and

$$M_i = M(r_i, \zeta) = \frac{1}{\sqrt{(1 - r_i^2)^2 + (2\zeta r_i)^2}} \tag{3.98}$$

The maximum displacement from equilibrium is difficult to obtain. The maxima of the trigonometric terms in Eq. (3.94) do not occur simultaneously. An upper bound on the maximum is

$$X_{max} \le \sum_{i=1}^{n} X_i \tag{3.99}$$

Example 3.1

A slider-crank mechanism is used to provide a base motion for the block shown in Fig. 3.32. Plot the maximum absolute displacement of the block as a function of frequency ratio for a damping ratio of 0.05. The crank rotates with a constant speed, ω.

Solution:

The instantaneous position of the block relative to point O is

$$y(t) = \hat{r}\cos\omega t + l\cos\alpha$$

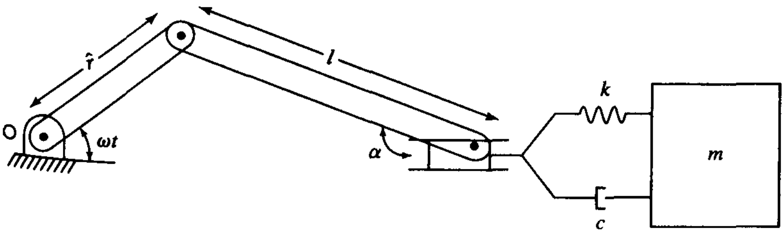


Figure 3.32 Slider-crank mechanism produces multifrequency excitation to system of Example 3.11.

Application of the law of sines gives

$$\sin\alpha = \frac{\hat{r}}{l}\sin\omega t$$

Thus

$$y(t) = \hat{r}\cos\omega t + l\sqrt{1 - \left(\frac{\hat{r}}{l}\sin\omega t\right)^2}$$

Assuming $\hat{r}/l$ is small, the binomial expansion is used to expand the square root

$$y(t) = l - \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2 + \hat{r}\cos\omega t + \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2\cos 2\omega t + \cdots$$

where the expansion has been terminated after the term proportional to $\sin^2\omega t$ and the double-angle formula is used to replace $\sin^2\omega t$. The principle of linear superposition and the theory of Sec. 3.6 are used to solve for the absolute displacement of the mass

$$x(t) = l\left[1 - \frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2\right] + \hat{r}T_1\sin\left(\omega t - \lambda_1 + \frac{\pi}{2}\right) + \frac{l}{4}\left(\frac{\hat{r}}{l}\right)^2 T_2\sin\left(2\omega t - \lambda_2 + \frac{\pi}{2}\right)$$

where
$$T_i = T(r_i, \zeta) = \sqrt{\frac{1 + (2\zeta r_i)^2}{\left(1 - r_i^2\right)^2 + (2\zeta r_i)^2}}$$

and
$$\lambda_i = \tan^{-1}\left(\frac{2\zeta r_i^3}{1 + \left(4\zeta^2 - 1\right)r_i^2}\right)$$

with
$$r_1 = \frac{\omega}{\omega_n}$$

and
$$r_2 = \frac{2\omega}{\omega_n}$$

The response is the sum of the responses due to each frequency term plus the response due to the constant term. The maximum displacement is difficult to attain. Instead an upper bound is calculated

$$x_{\max} < l\left[1 - \frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2\right] + \hat{r}T_1 + \frac{1}{4}\left(\frac{\hat{r}}{l}\right)^2 T_2$$

$x_{\max}/l$ versus ω/ω_n is plotted in Fig. 3.33 for $\hat{r}/l = \frac{1}{2}$ and $\zeta = 0.05$. The graph has two peaks. The first peak near $\omega/\omega_n = \frac{1}{2}$ is smaller than the second peak near $\omega/\omega_n = 1$. If additional terms from the binomial expansion were used, higher harmonics would appear in the solution. Small peaks on the frequency response curve will appear near values of $\omega/\omega_n = 1/i$ where i is an even integer. The magnitude of the peaks grows smaller with increasing i.

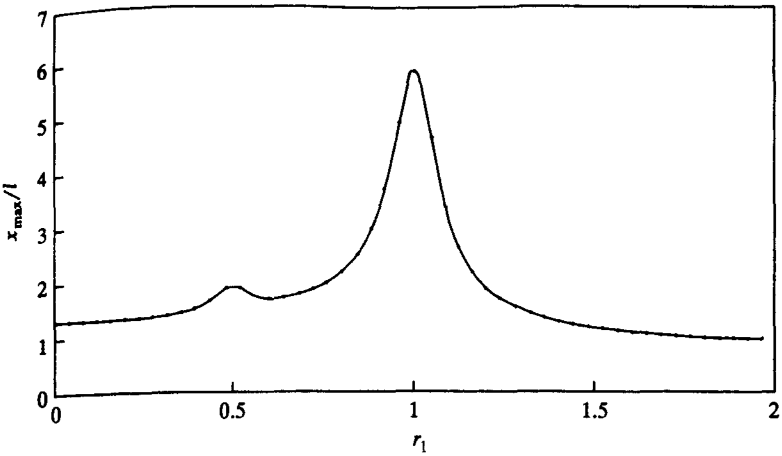


Figure 3.33 Upper bound on absolute displacement as a function of frequency ratio for Example 3.11.

3.10 FOURIER SERIES REPRESENTATION OF PERIODIC FUNCTIONS

If $F(t)$ is a piecewise continuous function of period T, then the Fourier series representation for $F(t)$ is

$$F(t) = \frac{a_0}{2} + \sum_{l=1}^{\infty} (a_l \cos \omega_l t + b_l \sin \omega_l t) \tag{3.100}$$

where

$$a_l = \frac{2}{T} \int_0^T F(t) \cos \omega_l t \, dt \qquad l = 0, 1, 2, \ldots \tag{3.101}$$

$$b_l = \frac{2}{T} \int_0^T F(t) \sin \omega_l t \, dt \qquad l = 1, 2, \ldots \tag{3.102}$$

and

$$\omega_l = \frac{2l\pi}{T} \tag{3.103}$$

The Fourier series representation has the following properties:

1. The Fourier series converges to $F(t)$ for all t where $F(t)$ is continuous for $0 \le t \le T$.
2. If $F(t)$ has a finite jump discontinuity at t, then the Fourier series converges to $\frac{1}{2}[F(t^+) + F(t^-)]$.
3. The Fourier series converges to the periodic extension of $F(t)$ for $t > T$.
4. If $F(t)$ is an odd function, that is, $F(-t) = -F(t)$ for all t, $0 \le t \le T$, then $a_l = 0, l = 0, 1, 2, 3, \ldots$.

5. If $F(t)$ is an even function, that is, $F(-t) = F(t)$ for all t, $0 \le t \le T$, then $b_l = 0, l = 1, 2, 3, \ldots$.

12 Decide whether each of the functions in Fig. 3.34 is an even function, an odd function, or neither and draw the Fourier series representation of each function on the interval [0, 6]. Each function is shown over one period.

Solution:

The function in Fig. 3.34*a* is an even function, the function in Fig. 3.34*b* is neither even nor odd, the function in Fig. 3.34*c* is an odd function. The functions to which the Fourier series representation converges are given in Fig. 3.35.

Use of the trigonometric identity for the sine of the sum of two angles and algebraic manipulation leads to an alternative form for the Fourier series representation

$$F(t) = \frac{a_0}{2} + \sum_{l=1}^{\infty} c_l \sin(\omega_l t + \kappa_l) \tag{3.104}$$

where

$$c_l = \sqrt{a_l^2 + b_l^2} \tag{3.105}$$

and

$$\kappa_l = \tan^{-1} \frac{a_l}{b_l} \tag{3.106}$$

If $F(t)$ is a periodic excitation for a one-degree-of-freedom system with viscous damping, the differential equation governing the response of the system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m_{\text{eq}}}\left[\frac{a_0}{2} + \sum_{l=1}^{\infty} c_l \sin(\omega_l t + \kappa_l)\right] \tag{3.107}$$

The principle of linear superposition is used to determine the response as

$$x(t) = \frac{1}{m_{\text{eq}}\omega_n^2}\left[\frac{a_0}{2} + \sum_{l=1}^{\infty} c_l M_l \sin(\omega_l t + \kappa_l - \phi_l)\right] \tag{3.108}$$

where M_l and ϕ_l are defined in Eqs. (3.98) and (3.96), respectively.

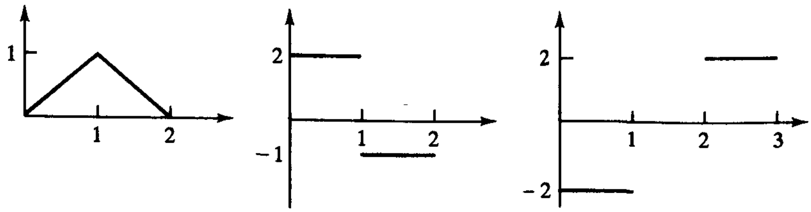


Figure 3.34 One period of periodic excitations. Function in (*a*) is even; function in (*c*) is odd; function in (*b*) is neither even nor odd.

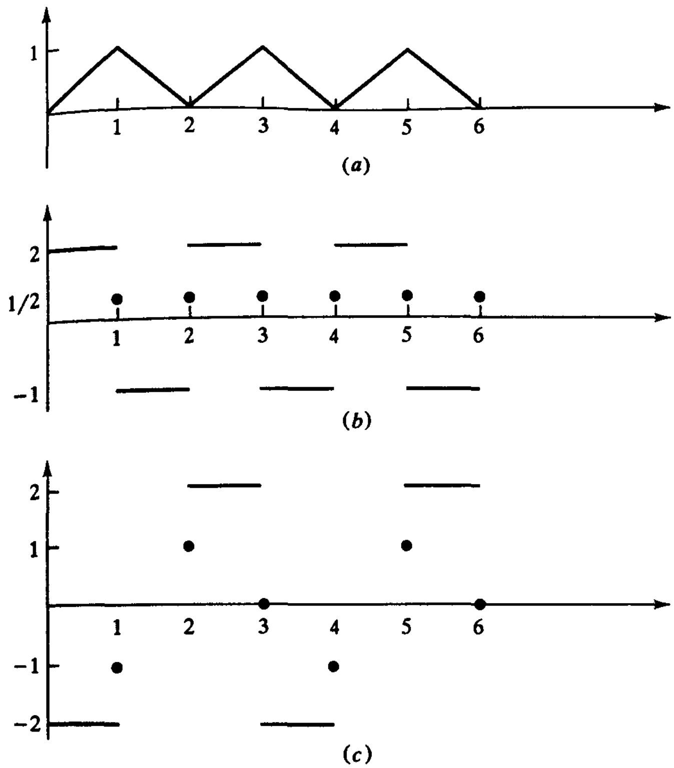


Figure 3.35 Convergence of Fourier representations on [0, 6] of functions of Fig. 3.34.

Example 3.13

A punch press of mass 500 kg sits on an elastic foundation of stiffness $k = 1.25 \times 10^6$ N/m and damping ratio $\zeta = 0.1$. The press operates at a speed of 120 rpm. The punching operation occurs over 40 percent of each cycle and provides a force of 5000 N to the machine. The excitation force is approximated as the periodic function of Fig. 3.36. Estimate the maximum displacement of the elastic foundation.

Solution:

From the given information, the period of one cycle is 0.5 s and the natural frequency of the system is 50 rad/s.

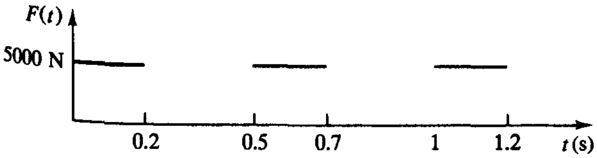


Figure 3.36 Force developed during punching operation is periodic.

The excitation force is periodic, but it is neither an even function nor an odd function. Its mathematical representation is

$$F(t) = \begin{cases} 5000\text{ N} & 0 < t < 0.2\text{ s} \\ 0 & 0.2\text{ s} < t < 0.5\text{ s} \end{cases}$$

The Fourier coefficients for the Fourier series representation for $F(t)$ are

$$a_0 = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N}\,dt + \int_{0.2\text{ s}}^{0.5\text{ s}} (0)\,dt\right) = 4000\text{ N}$$

$$a_l = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N}\cos 4\pi l t\,dt\right) = \frac{5000}{\pi l}\sin 4\pi l t\,\Big|_0^{0.2\text{ s}}\text{ N} = \frac{5000}{\pi l}\sin 0.8\pi l\text{ N}$$

and
$$b_l = \frac{2}{0.5\text{ s}}\left(\int_0^{0.2\text{ s}} 5000\text{ N}\sin 4\pi l t\,dt\right)$$

$$= -\frac{5000}{\pi l}\cos 4\pi l t\,\Big|_0^{0.2\text{ s}}\text{ N} = \frac{5000}{\pi l}(1 - \cos 0.8\pi l)\text{ N}$$

The Fourier series representation of the excitation force is

$$F(t) = \frac{a_0}{2} + \sum_{l=1}^{\infty} c_l \sin(4\pi l t + \kappa_l)$$

where
$$c_l = \frac{5000}{\pi l}\sqrt{2(1 - \cos 0.8\pi l)}\text{ N}$$

and
$$\kappa_l = \tan^{-1}\left(\frac{\sin 0.8\pi l}{1 - \cos 0.8\pi l}\right)$$

An upper bound on the displacement is

$$x_{\max} < \frac{1}{m\omega_n^2}\left(\frac{a_0}{2} + \sum_{l=1}^{\infty} c_l M_l\right)$$

The MATLAB script for evaluation of the Fourier series and system response, and the corresponding plots, are shown in Fig. 3.37. Fifty terms are used in the evaluation of the Fourier series and system response.

3.11 SEISMIC VIBRATION-MEASURING INSTRUMENTS

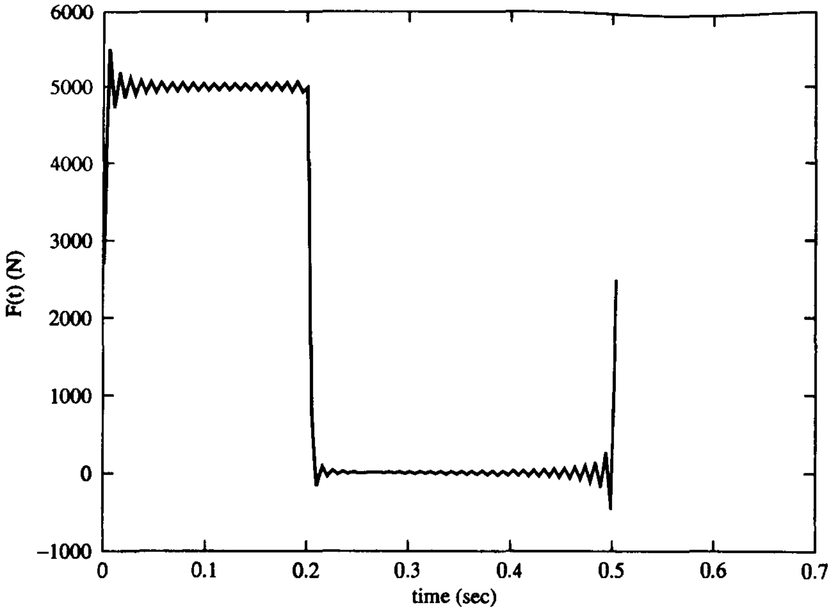
Time histories of vibrations are sensed by using seismic transducers. A *transducer* is a device that converts mechanical motion into voltage. A schematic of a piezoelectric transducer is shown in Fig. 3.38. The transducer is mounted on a body whose vibrations are to be measured. As the vibrations occur, the seismic mass moves relative to the transducer housing, causing deformation in the piezoelectric crystal. A charge is produced in the piezoelectric crystal that is proportional to its deformation. The charge is amplified and displayed on an output device. The measured signal is the motion of the seismic mass relative to the transducer housing.

```
% Program to evaluate Fourier Series and System Response
% for Example 3.13; F(t) is given in Fig 3.36
t=linspace(0,0.5,100);              % Defining t over one period
m=500;                              % Mass of punch press
k=1.25*10^6;                        % Stiffness of foundation
zeta=0.1;                           % Damping ratio
omega_n=sqrt(k/m);                  % Natural frequency
omega=120*(2*pi)/60;                % Excitation frequency
% Defining Fourier coefficients and Magnification factors and phases
a0=4000;
for i=1:50
   c(i)=5000/(pi*i)*sqrt(2*(1-cos(0.8*pi*i)));  %Equation (3.105)
   if c(i)==0                        % If c=0, kap is indeterminate
      kap(i)=0;
   else
      kap(i)=atan(sin(0.8*pi*i)/(1-cos(0.8*pi*i))); %Eq.(3.106)
   end
   r(i)=i*omega/omega_n;                          % Equation (3.97)
   M(i)=1/sqrt((1-r(i)^2)^2+(2*zeta*r(i))^2);     % Equation (3.98)
   phi(i)=atan((2*zeta*r(i))/(1-r(i)^2));         % Equation (3.96)
end
% Evaluating Fourier series and system response over one period
% Equation (3.104) is used for Fourier series
% Equation (3.108) is used to evaluate system response
for j=1:100
   F(j)=a0/2;
   x(j)=a0/(2*m*omega_n^2);
   for i=1:50
      F(j)=F(j)+c(i)*sin(4*pi*i*t(j)+kap(i));
      x(j)=x(j)+c(i)*M(i)*sin(4*pi*i*t(j)+kap(i)-phi(i))/(m*omega_n^
   end
end
% Plotting Fourier series and system response over one period
figure(1)
plot(t,F)
xlabel('time (sec)')
ylabel('F(t) (N)')
figure(2)
plot(t,x)
xlabel('time (sec)')
ylabel('x(t) (m)')
```

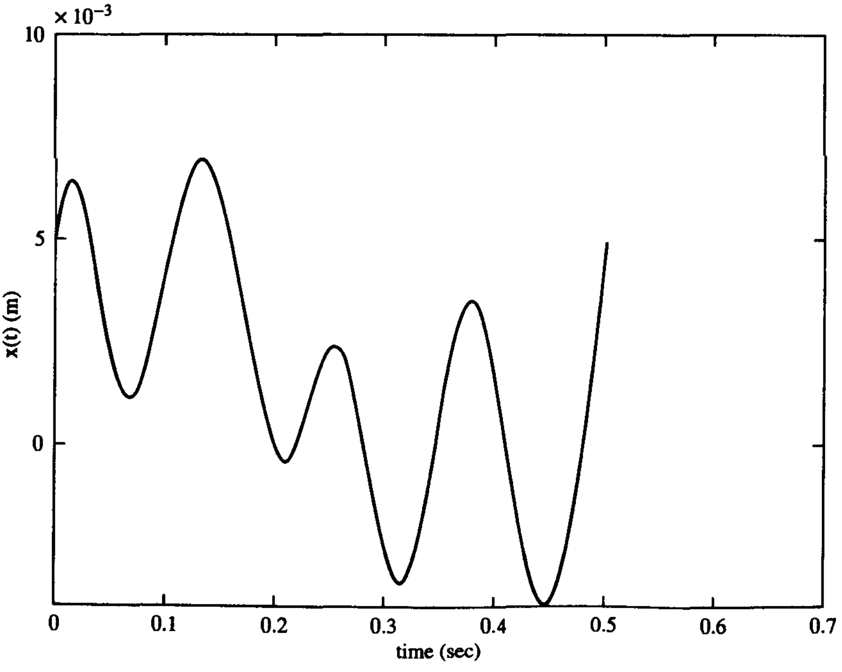
(*a*)

Figure 3.37 (*a*) MATLAB script for Example 3.13.

A model of the transducer is shown in Fig. 3.39. The piezoelectric crystal is assumed to provide viscous damping. The purpose of the transducer is to measure the motion of the body, $y(t)$. However it actually measures $z(t)$, the displacement of the seismic mass relative to the body. Assume the vibrations of the body are a



(*b*)



(*c*)

Figure 3.37 (Con't) (*b*) Fourier series representation for $F(t)$ with 50 terms; (*c*) approximation to $x(t)$ with 50 terms.

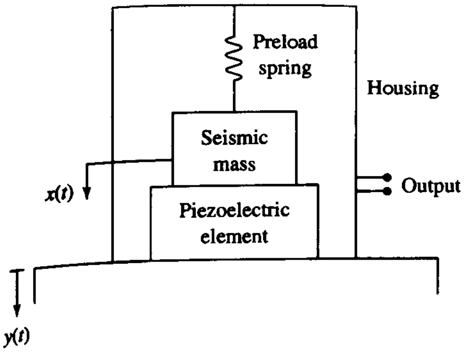


Figure 3.38 Schematic of piezoelectric crystal transducer. As seismic mass moves a charge is produced in the piezoelectric element that is proportional to its deflection. The transducer measures $z(t) = x(t) - y(t)$.

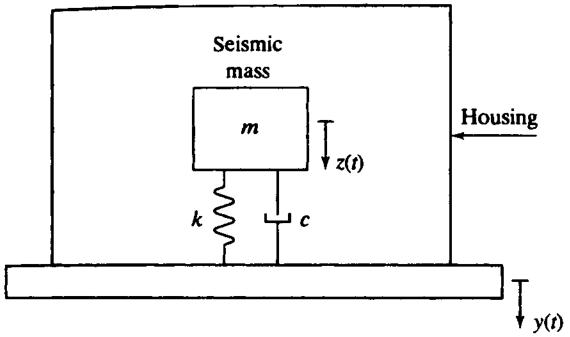


Figure 3.39 Schematic diagram of seismic instrument.

single-frequency harmonic of the form

$$y(t) = Y \sin \omega t \qquad [3.109]$$

The displacement of the seismic mass relative to the vibrating body is

$$z(t) = Z \sin(\omega t - \phi) \qquad [3.110]$$

where $$Z = Y\Lambda(r, \zeta) \qquad \phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

where $\Lambda(r, \zeta)$ is defined by Eq. (3.34) and $r = \omega/\omega_n$, where ω_n and ζ are the natural frequency and damping ratio of the transducer.

Figure 3.14 shows that Λ is approximately 1 for large $r (r > 3)$. In this case the amplitude of the relative displacement which is monitored by the transducer is approximately the same as the vibration amplitude of the body. From Fig. 3.10, it is noted that for large r, ϕ is approximately π. Thus for large r, the transducer response is approximately that of the response to be measured, but out of phase by π radians.

A seismic transducer that requires a large frequency ratio for accurate measurement is called a *seismometer*. A large frequency ratio requires a small natural frequency for the transducer. This, in turn, requires a large seismic mass and a very flexible spring. Because of the required size for accurate measurement, seismometers are not practical for many applications.

The percentage error in using a seismic transducer is

$$E = 100\left|\frac{Y_{\text{actual}} - Y_{\text{measured}}}{Y_{\text{actual}}}\right| \qquad [3.111]$$

When using a seismometer the percentage error is

$$E = 100\left|\frac{Y - Z}{Y}\right| = 100|1 - \Lambda| \qquad [3.112]$$

The acceleration of the body is

$$\ddot{y}(t) = -\omega^2 Y \sin \omega t \qquad [3.113]$$

Noting that $Z/Y = \Lambda(r, \zeta)$ and $\Lambda = r^2 M(r, \zeta)$ leads to

$$\ddot{y}(t) = -\omega^2 \frac{Z}{\Lambda(r, \zeta)} \sin \omega t = -\omega^2 \frac{Z}{r^2 M(r, \zeta)} \sin \omega t = -\omega_n^2 \frac{Z}{M} \sin \omega t \qquad [3.114]$$

Comparing Eq. (3.110) to Eq. (3.114) makes it apparent that

$$\ddot{y}(t) = \frac{\omega_n^2}{M(r, \zeta)} z\left(t - \frac{\phi}{\omega} - \frac{\pi}{\omega}\right) \qquad [3.115]$$

The negative sign in Eq. (3.114) is taken into account in Eq. (3.115) by subtracting π from the phase. For small r, $M(r, \zeta)$ is approximately 1, and

$$\ddot{y}(t) \approx \omega_n^2 z\left(t - \frac{\phi}{\omega} - \frac{\pi}{\omega}\right) \qquad [3.116]$$

Thus, for small r, the acceleration of the particle to which the seismic instrument is attached is approximately proportional to the relative displacement between the particle and the seismic mass, but on a shifted time scale. A vibration measuring instrument that works on this principle is called an *accelerometer*. The transducer in an accelerometer records the relative displacement which is electronically multiplied by ω_n^2. The acceleration is integrated twice to yield the displacement.

The natural frequency of an accelerometer must be high to measure vibrations accurately over a wide range of frequencies. The seismic mass must be small and the spring stiffness must be large. The error in using an accelerometer is

$$E = 100\left|\frac{\omega^2 Y - \omega_n^2 Z}{\omega^2 Y}\right| = 100\left|\frac{\omega^2 Y - \omega_n^2 \Lambda(r, \zeta) Y}{\omega^2 Y}\right| = 100|1 - M(r, \zeta)| \qquad [3.117]$$

Consider the measurement of the vibration of a multifrequency vibration,

$$y(t) = \sum_{i=1}^{n} Y_i \sin(\omega_i t + \psi_i) \qquad [3.118]$$

According to the theory of Sec. 3.10, the displacement of a seismic mass relative to the housing of a seismic instrument is

$$z(t) = \sum_{i=1}^{n} \Lambda(r_i, \zeta) Y_i \sin(\omega_i t + \psi_i - \phi_i) \qquad [3.119]$$

$$= \frac{1}{\omega_n^2} \sum_{i=1}^{n} \omega_i^2 M(r_i, \zeta) Y_i \sin(\omega_i t + \psi_i - \phi_i)$$

The accelerometer measures $-\omega_n^2 z(t)$. Note that each term in the summation of Eq. (3.119) has a different phase shift. When summed with different phase shifts, the accelerometer output will be distorted from the true measurement. This phase distortion is illustrated in Fig. 3.40a, which compares the accelerometer output to the signal to be measured for a 10-frequency vibration. The damping ratio of the accelerometer is 0.25, and the largest frequency ratio in the measurement is 0.66.

Accelerometers are used only when $r < 1$. In this frequency range, the phase shift is approximately linear with r for $\zeta = 0.7$ (See Fig. 3.10). Then

$$\phi_i = \alpha \frac{\omega_i}{\omega_n} \qquad [3.120]$$

where α is the constant of proportionality. Using Eq. (3.120) in Eq. (3.119) leads to

$$z(t) = -\frac{1}{\omega_n^2} \sum_{i=1}^{n} M(r_i, \zeta) Y_i \sin\left[\omega_i \left(t - \frac{\alpha}{\omega_n}\right) + \psi_i\right]$$

If $r_i \ll 1$, then $M(r_i, \zeta) \approx 1$ for $i = 1, 2, \ldots, n$ and

$$z(t) \approx -\frac{1}{\omega_n^2} \ddot{y}\left(t - \frac{\alpha}{\omega_n}\right) \qquad [3.121]$$

Thus, when an accelerometer with $\zeta = 0.7$ is used, its output device duplicates the actual acceleration, but on a shifted time scale. This is illustrated in Fig. 3.40b, which compares the use of Eq. (3.119) with $\zeta = 0.7$ to the actual acceleration for the example of Fig. 3.40a.

3.12 COMPLEX REPRESENTATIONS

The use of complex algebra provides an alternative method to the solution of the differential equations governing the forced response of systems subject to harmonic excitation. It can prove to be less tedious than the use of trigonometric solutions. Recall that if Q is a complex number, it has the representation

$$Q = Q_r + i Q_i \qquad [3.122]$$

where $Q_r = \text{Re}(Q)$ is the real part of Q and $Q_i = \text{Im}(Q)$ is the imaginary part of Q. The complex number also has the polar form

$$Q = A e^{i\phi} \qquad [3.123]$$

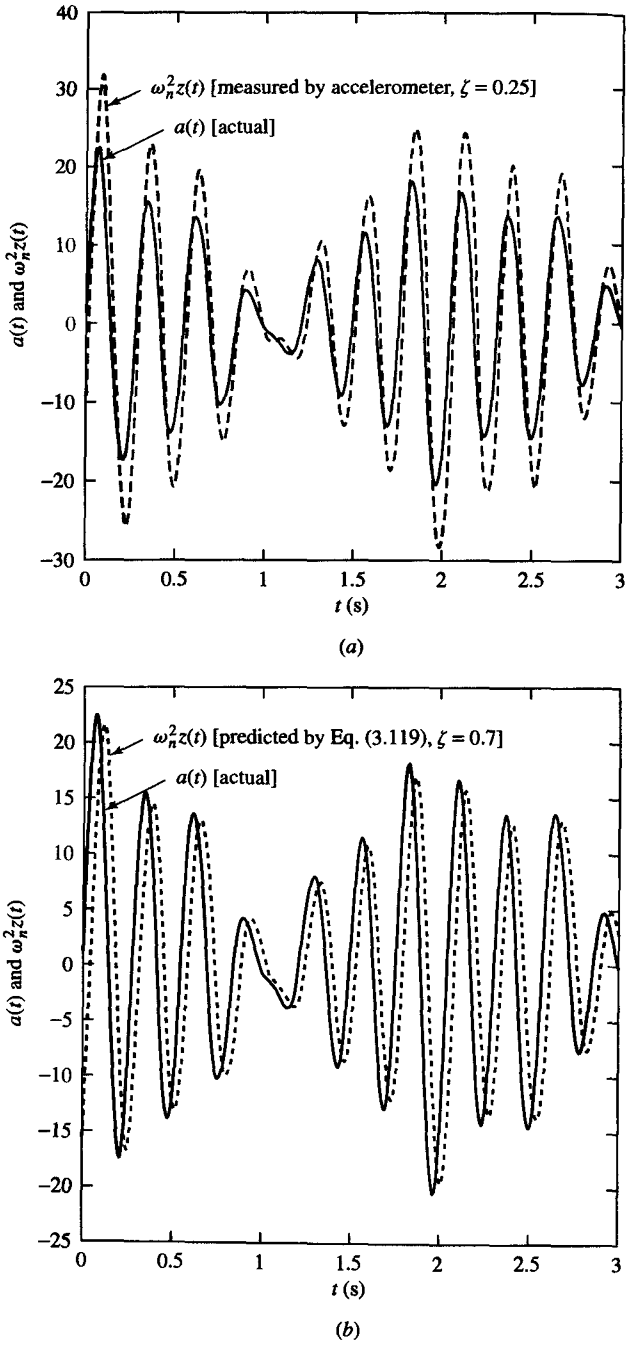


Figure 3.40 Comparison of $a(t)$, the acceleration to be measured (solid line), and $\omega_n^2 z(t)$, the acceleration actually measured or predicted (dashed line), for a 10-frequency vibration. (a) The phase distortion, which is converted into a time scale shift in (b); $\zeta = 0.25$ in (a) $\zeta = 0.7$ in (b).

where A is the magnitude of Q and ϕ is the phase of Q. Euler's identity

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{3.124}$$

leads to

$$A = \sqrt{Q_r^2 + Q_i^2} \tag{3.125}$$

$$\phi = \tan^{-1}\left(\frac{Q_i}{Q_r}\right) \tag{3.126}$$

In view of Euler's identity it is noted that

$$\cos(\omega t) = \text{Re}\,(e^{i\omega t}) \qquad \sin(\omega t) = \text{Im}\,(e^{i\omega t}) \tag{3.127}$$

Thus the standard form of the differential equation governing the motion of a linear one-degree-of-freedom system subject to a single-frequency sinusoidal excitation can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\,\text{Im}\,(e^{i\omega t}) \tag{3.128}$$

Then the solution of Eq. (3.128) is the imaginary part of the solution of

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}e^{i\omega t} \tag{3.129}$$

A solution of Eq. (3.129) is assumed as

$$x(t) = He^{i\omega t} \tag{3.130}$$

where H is complex. Substitution of Eq. (3.130) into Eq. (3.129) leads to

$$H = \frac{F_0}{m(\omega_n^2 - \omega^2 + 2i\zeta\omega\omega_n)} \tag{3.131}$$

Equation (3.131) can be rewritten by using the definition of the frequency ratio $r = \omega/\omega_n$:

$$H = \frac{F_0}{m\omega_n^2(1 - r^2 + 2i\zeta r)} \tag{3.132}$$

Multiplying the denominator by its complex conjugate puts H in its proper form as

$$H = \frac{F_0}{m\omega_n^2[(1-r^2)^2 + (2\zeta r)^2]}(1 - r^2 - 2i\zeta r) \tag{3.133}$$

Then, from Eqs. (3.125) and (3.126), H can be written as

$$H = Xe^{-i\phi} \tag{3.134}$$

where

$$X = \frac{F_0}{m\omega_n^2}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3.135}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.136}$$

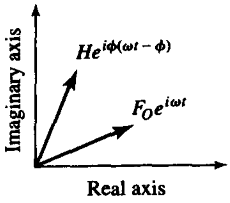


Figure 3.41 Graphical representation of excitation and response.

Eqs. (3.135) and (3.136) are the same as those derived by using a trigonometric solution. The system response is

$$x(t) = \mathrm{Im}\,(Xe^{-i\phi}e^{\omega t}) = X\sin(\omega t - \phi) \tag{3.137}$$

A graphical interpretation of the complex representation of the excitation and response is shown in Fig. 3.41.

The equation governing the motion of a system with hysteretic damping subject to a single frequency harmonic excitation can be written as

$$m\ddot{x} + \frac{hk}{\omega}\dot{x} + kx = F_0e^{i\omega t} \tag{3.138}$$

Assumption of a solution of the form of Eq. (3.130) leads to

$$H = \frac{F_0}{-m\omega^2 + k(1+ih)} \tag{3.139}$$

The same response is obtained using the differential equation

$$m\ddot{x} + k(1+ih)x = F_0e^{i\omega t} \tag{3.140}$$

Thus the forced response of a system with hysteretic damping can be modeled as a system with a *complex stiffness* of $k(1+ih)$.

PROBLEMS

3.1–3.3. Use the free-body diagram method to derive the differential equation governing the forced vibrations of the linear one-degree-of-freedom systems shown using the indicated generalized coordinate.

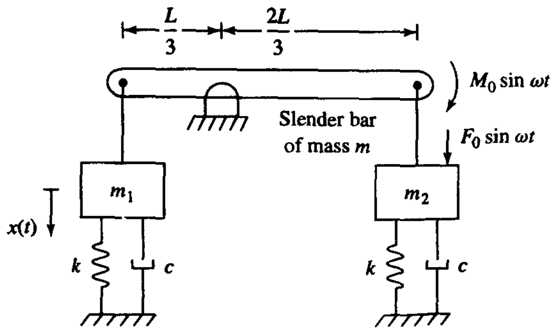


FIGURE P3.1, P3.4

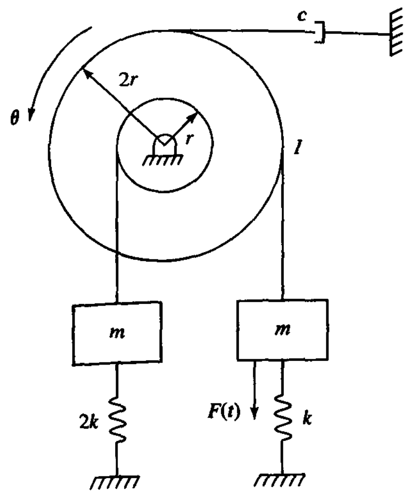


FIGURE P3.2, P3.5

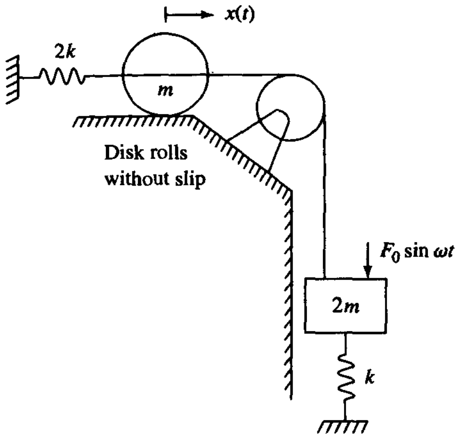


FIGURE P3.3, P3.6

3.4–3.6. Use the equivalent systems method to derive the differential equation governing the forced vibrations of the linear one-degree-of-freedom systems shown using the indicated generalized coordinate.

3.7. A 40-kg mass is hanging from a spring of stiffness 4×10^4 N/m. A harmonic force of magnitude 100 N and frequency 120 rad/s is applied. Determine the amplitude of the forced response.

3.8. Determine the amplitude of forced oscillations of the 30-kg block of Fig. P3.8.

3.9. For what values of M_0 will the forced amplitude of angular displacement of the bar of Fig. P3.9 be less than 3° if $\omega = 25$ rad/s?

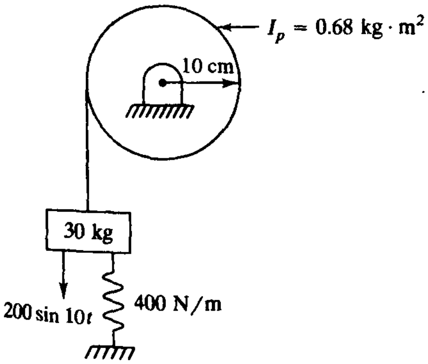


FIGURE P3.8

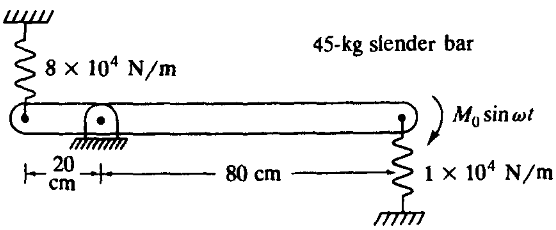


FIGURE P3.9

3.10. For what values of ω will the forced amplitude of the angular displacement of the bar of Fig. P3.9 be less than 3° if $M_0 = 300$ N · m?

3.11. For what values of M_0 will the forced amplitude of angular displacement of the bar of Fig. P3.9 be less than 3° for all values of ω between 10 rad/s and 30 rad/s?

3.12. A 2-kg gear of radius 20 cm is attached to the end of a 1-m-long steel shaft ($G = 80 \times 10^9$ N/m^2). A moment $M = 100 \sin 150t$ N is applied to the gear. For what shaft radii will the forced amplitude of torsional oscillations be less than 4°?

3.13. A 1-kg block is to be suspended from a coil spring and subject to a harmonic excitation. The magnitude of the excitation is 360 N, and its frequency ranges from 30 rad/s to 120 rad/s. The coil spring is to be made from a 1-cm-diameter steel bar ($G = 80 \times 10^9$ N/m^2). The coil radius is to be 20 cm. Specify the number of active coils required if the maximum shear stress developed in the spring is less than 50×10^7 N/m^2 at all frequencies.

3.14. A helical coil spring is made from a steel bar ($G = 80 \times 10^9$ N/m^2) of radius 5 mm. The spring has 40 active coils and has a coil radius of 10 cm. A 2-kg block is to be suspended from the spring.

(*a*) What is the resonant frequency for the system?

(*b*) If the block is subject to a harmonic force of amplitude 25 N at the resonant frequency, how long will it take for the shear stress in the spring to reach its yield strength of 50×10^7 N/m^2?

3.15. A 40-kg pump is to be placed at the midspan of a 2.5-m-long steel beam ($E = 200 \times 10^9$ N/m^2). The pump is to operate at 3000 rpm. For what values of the cross-sectional moment of inertia will the pump be operating within 3 Hz of resonance?

3.16. A mass-spring system of natural frequency 22 Hz is subject to a harmonic excitation at a frequency of 24 Hz. Does beating occur? If so, what is the period of beating?

3.17. A 5-kg block is mounted on a helical coil spring such that the system's natural frequency is 50 rad/s. The block is subject to a harmonic excitation of amplitude 45 N at a frequency of 50.8 rad/s. What is the maximum displacement of the block from its equilibrium position?

3.18. A 50-kg turbine is mounted on four parallel springs, each of stiffness 3×10^5 N/m. When the machine operates at 40 Hz, its steady-state amplitude is observed as 1.8 mm. What is the magnitude of the excitation?

3.19. A system of equivalent mass 30 kg has a natural frequency of 120 rad/s and a damping ratio of 0.12 and is subject to a harmonic excitation of amplitude 2000 N and frequency 150 rad/s. What are the steady-state amplitude and phase angle for the response?

3.20. A 30-kg block is suspended from a spring of stiffness 300 N/m and attached to a dashpot of damping coefficient 120 N · s/m. The block is subject to a harmonic excitation of amplitude 1150 N at a frequency of 20 Hz. What is the block's steady-state amplitude?

3.21. What is the amplitude of steady-state oscillation of the 30-kg block of the system of Fig. P3.21?

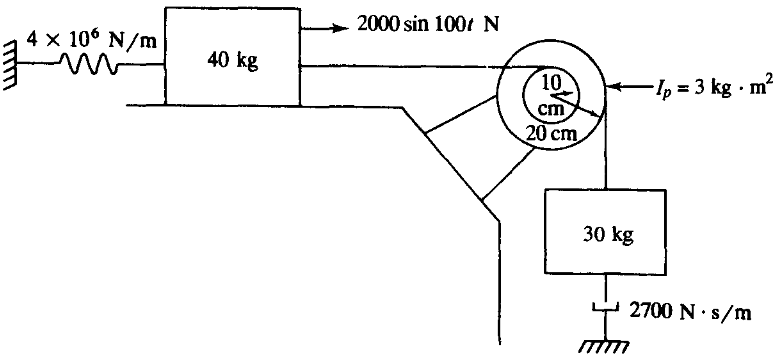


FIGURE P3.21

3.22. If $\omega = 16.5$ rad/s, what is the maximum value of M_0 such that the disk of Fig. P3.22 rolls without slip?

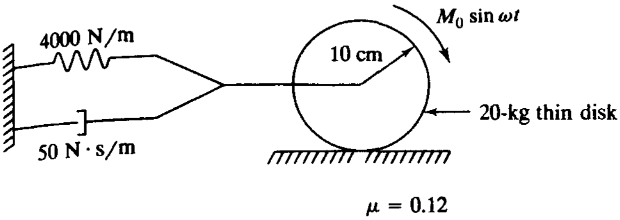


FIGURE P3.22

3.23. If $M_0 = 2$ N · m, for what values of ω will the disk of Fig. P3.22 roll without slip?

3.24. For what values of d will the steady-state amplitude of angular oscillations be less than 1° for the rod of Fig. P3.24?

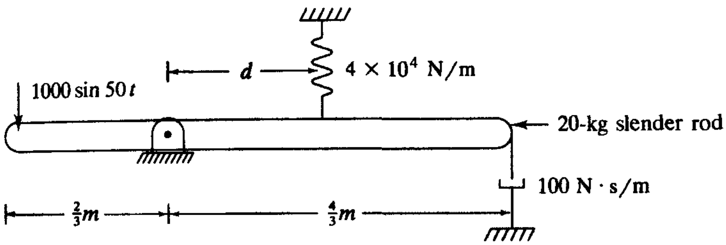


FIGURE P3.24

3.25. A 30-kg compressor is mounted on an isolator pad of stiffness 6×10^5 N/m. When subject to a harmonic excitation of magnitude 350 N and frequency 100 rad/s, the phase difference between the excitation and steady-state response is 24.3°. What is the damping ratio of the isolator and its maximum deflection due to this excitation?

3.26. A thin disk of mass 5 kg and radius 10 cm is connected to a torsional damper of coefficient 4.1 N · s · m/rad and a solid circular shaft of radius 10 mm, length 40 cm, and shear modulus 80×10^9 N/m^2. The disk is subject to a harmonic moment of magnitude 250 N · m and frequency 600 Hz. What is the amplitude of steady-state torsional oscillations?

3.27. A 50-kg machine tool is mounted on an elastic foundation. An experiment is run to determine the stiffness and damping properties of the foundation. When the tool is excited with a harmonic force of magnitude 8000 N at a variety of frequencies, the maximum steady-state amplitude obtained is 2.5 mm, occurring at a frequency of 32 Hz. Use this information to determine the stiffness and damping ratio of the foundation.

3.28. A 100-kg machine tool has a 2-kg rotating component. When the machine is mounted on an isolator and its operating speed is very large, the steady-state vibration amplitude is 0.7 mm. How far is the center of mass of the rotating component from its axis of rotation?

3.29. A 1000-kg turbine with a rotating unbalance is placed on springs and viscous dampers in parallel. When the operating speed is 20 Hz, the observed steady-state amplitude is 0.08 mm. As the operating speed is increased, the steady-state amplitude increases with an amplitude of 0.25 mm at 40 Hz and an amplitude of 0.5 mm for much larger speeds. Determine the equivalent stiffness and damping coefficient of this system.

3.30. A 120-kg fan with a rotating unbalance of 0.35 kg · m is to be placed at the midspan of a 2.6-m simply supported beam. The beam is made of steel ($E = 210 \times 10^9$ N/m^2) with a uniform rectangular cross section of height of 5 cm. For what values of the cross-sectional depth will the steady-state amplitude of the machine be limited to 5 mm for all operating speeds between 50 and 125 rad/s?

3.31. Solve Prob. 3.30 assuming the damping ratio of the beam is 0.04.

3.32. A 620-kg fan has a rotating unbalance of 0.25 kg · m. What is the maximum stiffness of the fan's mounting such that the steady-state amplitude is 0.5 mm or less at all operating speeds greater than 100 Hz? Assume a damping ratio of 0.08.

Problems 3.33 and 3.34 refer to the following situation: The tail rotor section of the helicopter of Fig. P3.33 consists of four blades, each of mass 2.1 kg, and an engine box of mass 25 kg. The center of gravity of each blade is 170 mm from the rotational axis. The tail section is connected to the main body of the helicopter by an elastic structure. The natural frequency of the tail section has been observed as 150 rad/s. During flight the rotor operates at 900 rpm. Assume the system has a damping ratio of 0.05.

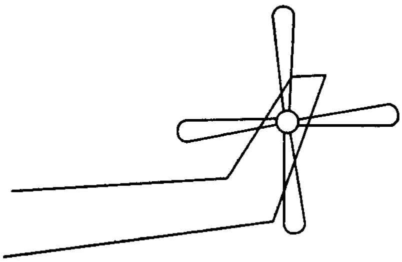


FIGURE P3.33

3.33. During flight a 75-g particle becomes stuck to one of the blades, 25 cm from the axis of rotation. What is the steady-state amplitude of vibration caused by the resulting rotating unbalance?

3.34. Determine the steady-state amplitude of vibration if one of the blades snaps off during flight.

3.35. Whirling is a phenomenon that occurs in a rotating shaft when an attached rotor is unbalanced. The motion of the shaft and the eccentricity of the rotor cause an unbalanced inertia force, pulling the shaft away from its centerline, causing it to bow. Use Fig. P3.35 and the theory of Sec. 3.5 to show that the amplitude of whirling is

$$X = e\Lambda(r, \zeta)$$

where e is the distance from the center of mass of the rotor to the axis of the shaft.

3.36. A 30-kg rotor has an eccentricity of 1.2 cm. It is mounted on a shaft and bearing system whose stiffness is 2.8×10^4 N/m and damping ratio is 0.07. What is the amplitude of whirling when the rotor operates at 850 rpm? Refer to Prob. 3.35 for an explanation of whirling.

3.37. An engine flywheel has an eccentricity of 0.8 cm and mass 38 kg. Assuming a damping ratio of 0.05, what is the necessary stiffness of the bearings to limit its whirl amplitude to

0.8 mm at all speeds between 1000 and 2000 rpm? Refer to Prob. 3.35 for an explanation of whirling.

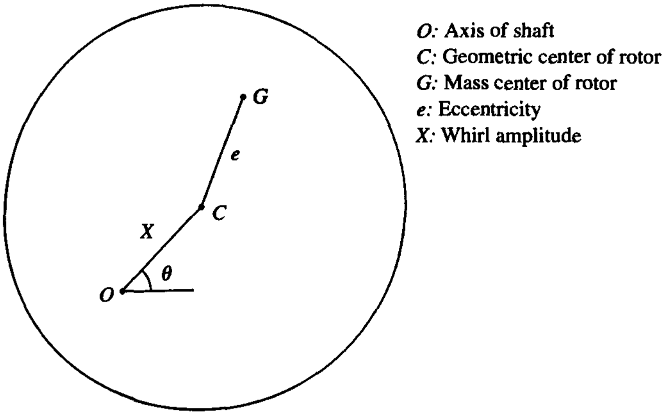


FIGURE P3.35

3.38. It is proposed to build a 6-m smokestack on the top of a 60-m factory. The smokestack will be made of steel ($\rho = 7850\ \mathrm{kg/m^3}$) and will have an inner radius of 40 cm and an outer radius of 45 cm. What is the maximum amplitude of vibration due to vortex shedding and at what wind speed will it occur? Use a one-degree-of-freedom model for the smokestack with a concentrated mass at its end to account for inertia effects. Use $\zeta = 0.05$.

3.39. What is the steady-state amplitude of oscillation due to vortex shedding of the smokestack of Prob. 3.38 if the wind speed is 22 mph?

3.40. Repeat parts (*a*) and (*b*) of Example 3.6 if the light pole is hollow with an inner diameter of 15 cm and outer diameter of 20 cm.

3.41. Repeat Prob. 3.40 including the inertia effects of the light pole.

3.42. A factory is using the piping system of Fig. P3.42 to discharge environmentally safe wastewater into a small river. The velocity of the river is estimated as 5.5 m/s. Determine the allowable values of l such that the amplitude of torsional oscillations of the vertical pipe due to vortex shedding is less than 1°. Assume the vertical pipe is rigid and rotates about an axis perpendicular to the page through the elbow. The horizontal pipe is restrained from rotation at the river bank. Assume a damping ratio of 0.05.

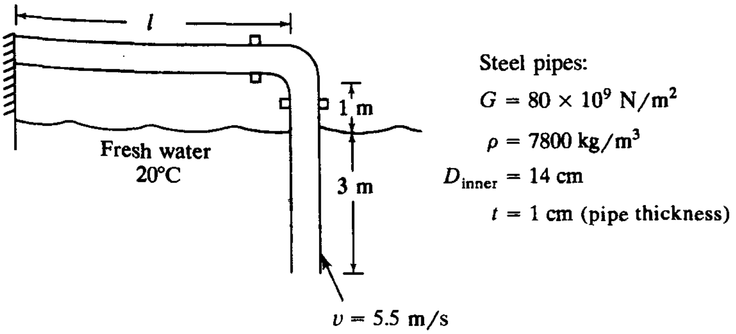
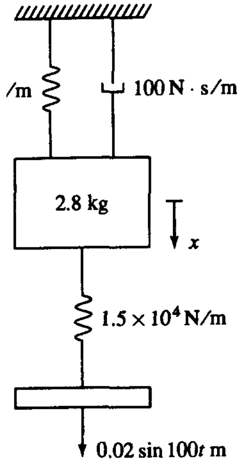


FIGURE P3.42

3.43–3.47. Determine the amplitude of steady-state vibration for the systems shown. Use the indicated generalized coordinate.



P3.43

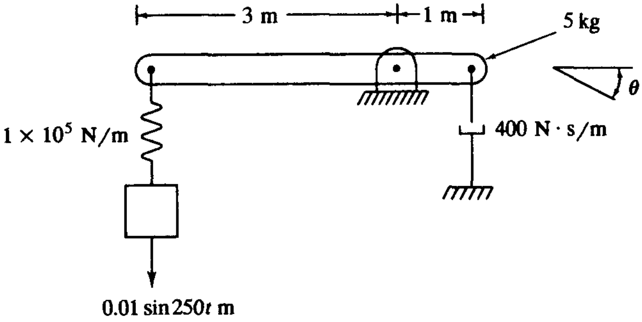


FIGURE P3.44

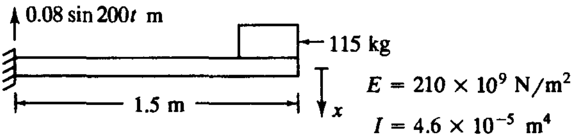


FIGURE P3.45

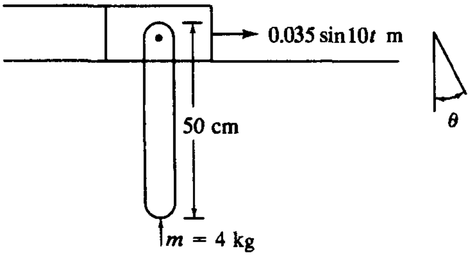


FIGURE P3.46

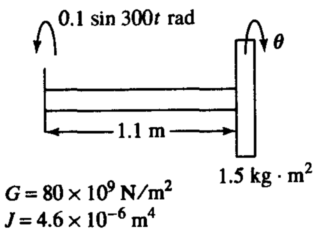


FIGURE P3.47

3.48. A 40-kg machine is attached to a base through a spring of stiffness 2×10^4 N/m in parallel with a dashpot of damping coefficient 150 N · s · m. The base is given a time-dependent displacement $0.15 \sin 30.1t$ m. Determine the amplitude of the absolute displacement of the machine and the amplitude of displacement of the machine relative to the base.

3.49. A 5-kg rotor-balancing machine is mounted on a table through an elastic foundation of stiffness 3.1×10^4 N/m and damping ratio 0.04. Transducers indicate that the table on which the machine is placed vibrates at a frequency of 110 rad/s with an amplitude of 0.62 mm. What is the steady-state amplitude of acceleration of the balancing machine?

3.50. During a long earthquake the one-story frame structure of Fig. P3.50 is subject to a ground acceleration of amplitude 50 mm/s^2 at a frequency of 88 rad/s. Determine the acceleration amplitude of the structure. Assume the girder is rigid and the structure has a damping ratio of 0.03.

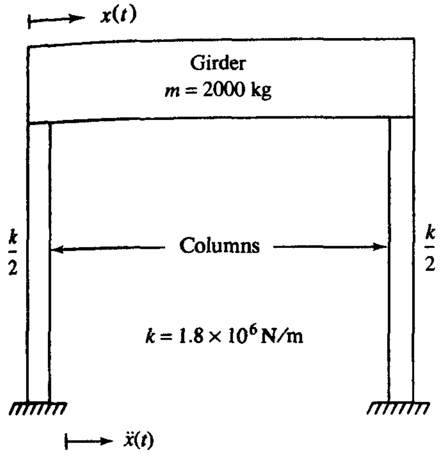


FIGURE P3.50

3.51. What is the required column stiffness of a one-story structure to limit its acceleration amplitude to 2.1 m/s^2 during an earthquake whose acceleration amplitude is 150 mm/s^2 at a frequency of 50 rad/s. The mass of structure is 1800 kg. Assume a damping ratio of 0.05.

3.52. In a rough sea the heave of a ship is approximated as harmonic of amplitude 20 cm at a frequency of 1.5 Hz. What is the acceleration amplitude of a 20-kg computer workstation mounted on an elastic foundation in the ship of stiffness 700 N/m and damping ratio 0.04?

3.53. In the rough sea of Prob. 3.52 what is the required stiffness of an elastic foundation of damping ratio 0.05 to limit the acceleration amplitude of a 5-kg radio set to 1.5 m/s^2?

3.54. Consider the one-degree-of-freedom model of a vehicle suspension system of Example 3.7 and Fig. 3.23. Consider a motorcycle of mass 250 kg. The suspension stiffness is 70,000 N/m and the damping ratio is 0.15. The motorcycle travels over a terrain that is approximately sinusoidal with a distance between peaks of 10 m and the distance from peak to valley is 10 cm. What is the acceleration amplitude felt by the motorcycle rider when she is traveling at

(*a*) 30 m/s

(*b*) 60 m/s

(*c*) 120 m/s

3.55. A 20-kg block is connected to a spring of stiffness 1×10^5 N/m and placed on a surface which makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is 0.15. A force $300 \sin 80t$ N is applied to the block. What is the steady-state amplitude of the resulting oscillations?

3.56. A 20-kg block is connected to a spring of stiffness 1×10^5 N/m and placed on a surface which makes an angle of 30° with the horizontal. A force $300 \sin 80t$ N is applied to the block. The steady-state amplitude is measured as 10.6 mm. What is the coefficient of friction between the block and the surface?

3.57. A 40-kg block is connected to a spring of stiffness 1×10^5 N/m and slides on a surface with a coefficient of friction 0.2. When a harmonic force of frequency 60 rad/s is applied to the block, the resulting amplitude of steady-state vibrations is 3 mm. What is the amplitude of the excitation?

3.58. The 2-kg swing of Example 2.21 is subject to a harmonic moment of $1.5 \sin 5t$ N · m. What is the amplitude of steady-state oscillations?

3.59–3.60. Determine the steady-state amplitude of motion of the 5-kg block. The coefficient of friction between the block and surface is 0.11.

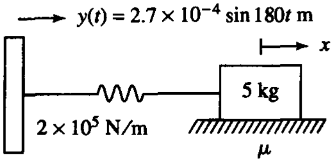


FIGURE P3.59

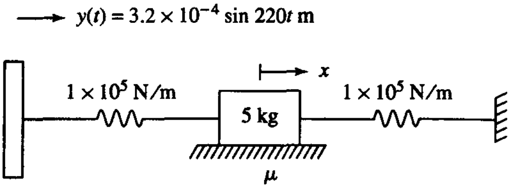


FIGURE P3.60

3.61. Use the equivalent viscous damping approach to determine the steady-state response of a system subject to both viscous damping and Coulomb damping.

3.62. The area under the hysteresis curve for a particular helical coil spring is 0.2 N · m when subject to a 350-N load. The spring has a stiffness of 4×10^5 N/m. If a 44-kg block is hung from the spring and subject to an excitation force of $350 \sin 35t$ N, what is the amplitude of the resulting steady-state oscillations?

3.63. When a free-vibration test is run on the system of Fig. P3.63, the ratio of amplitudes on successive cycles is 2.8 to 1. Determine the response of the engine when it has an excitation force of magnitude 3000 N at a frequency of 2000 rpm. Assume the damping is hysteretic.

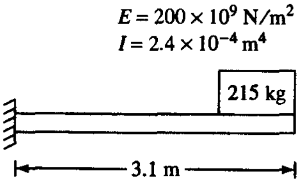


FIGURE P3.63

3.64. When a free-vibration test is run on the system of Fig. P3.63, the ratio of amplitudes on successive cycles is 2.8 to 1. When operating, the pump has a rotating unbalance of magnitude 0.25 kg · m. The pump operates at speeds between 500 and 2500 rpm. For what value of ω within the operating range will the pump's steady-state amplitude be largest? What is the maximum amplitude? Assume the damping is hysteretic.

3.65. When the pump at the end of the beam of Fig. P3.63 operates at 1860 rpm, it is noted that the phase angle between the excitation and response is 18°. What is the steady-state amplitude of the pump if it has a rotating unbalance of 0.8 kg · m and operates at 1860 rpm? Assume hysteretic damping.

3.66. A schematic of a single-cylinder engine mounted on springs and a viscous damper is shown in Fig. P3.66. The crank rotates about O with a constant speed ω. The connecting rod of mass m_r connects the crank and the piston of mass m_p such that the piston moves in a vertical plane. The center of gravity of the crank is at its axis of rotation.

(*a*) Derive the differential equation governing the absolute vertical displacement of the engine including the inertia forces of the crank and piston, but ignoring forces due to combustion. Use an exact expression for the inertia forces in terms of m_r, m_p, ω, the crank length r, and the connecting rod length l. Write the differential equation in the form of Eq. (3.1).

(*b*) Since $F(t)$ is periodic, a Fourier series representation can be used. Set up, but do not evaluate, the integrals required for a Fourier series expansion for $F(t)$.

(*c*) Assume $r/l \ll 1$. Rearrange $F(t)$ and use a binomial expansion such that

$$F(t) = \sum_{i=1}^{\infty} a_i \left(\frac{r}{l}\right)^i$$

(*d*) Truncate the preceding series after $i = 3$. Use trigonometric identities to approximate

$$F(t) \approx b_1 \cos \omega t + b_2 \cos 2\omega t + b_3 \cos 3\omega t$$

(*e*) Find an approximation to the steady-state form of $x(t)$.

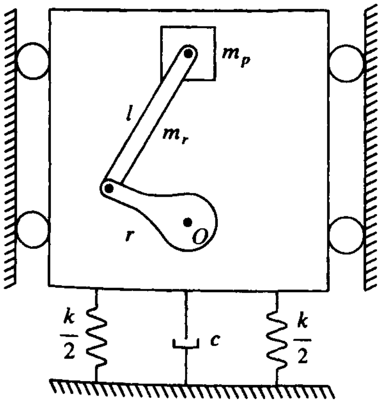


FIGURE P3.66

3.67. Using the results of Prob. 3.66, determine the maximum steady-state response of a single-cylinder engine with $m_r = 1.5$ kg, $m_p = 1.7$ kg, $r = 5.0$ cm, $l = 15.0$ cm, $\omega = 800$ rpm, $k = 1 \times 10^5$ N/m, $c = 500$ N · s/m, and total mass 7.2 kg.

3.68. A 5-kg rotor-balancing machine is mounted to a table through an elastic foundation of stiffness 10,000 N/m and damping ratio 0.04. Use of a transducer reveals that the table's vibration has two main components: an amplitude of 0.8 mm at a frequency of 140 rad/s and an amplitude of 1.2 mm at a frequency of 200 rad/s. Determine the steady-state response of the rotor balancing machine.

3.69–3.73. During operation a 100-kg press is subject to the periodic excitations shown. The press is mounted on an elastic foundation of stiffness 1.6×10^5 N/m and damping ratio 0.2. Determine the steady-state response of the press and approximate its maximum displacement from equilibrium. Each excitation is shown over one period.

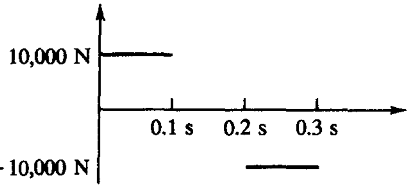


FIGURE P3.69

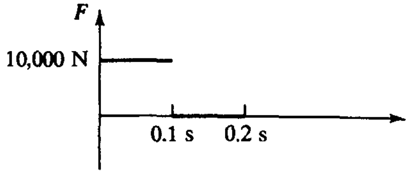


FIGURE P3.70

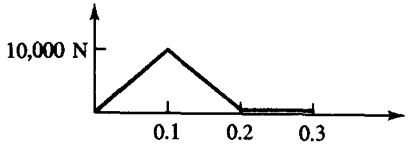


FIGURE P3.71

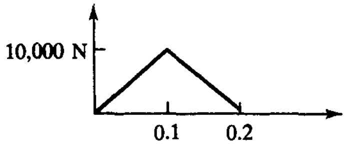


FIGURE P3.72

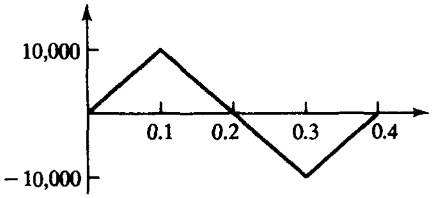


FIGURE P3.73

3.74. Use of an accelerometer of natural frequency 100 Hz and damping ratio 0.15 reveals that an engine vibrates at a frequency of 20 Hz and has an acceleration amplitude of 14.3 m/s^2. Determine

(a) The percent error in the measurement

(b) The actual acceleration amplitude

(c) The displacement amplitude

3.75. An accelerometer of natural frequency 200 Hz and damping ratio 0.7 is used to measure the vibrations of a system whose actual displacement is $x(t) = 1.6 \sin 45.1t$ mm. What is the accelerometer output?

3.76. An accelerometer of natural frequency 200 Hz and damping ratio 0.2 is used to measure the vibrations of an engine operating at 1000 rpm. What is the percent error in the measurement?

3.77. A 550-kg industrial sewing machine has a rotating unbalance of 0.24 kg · m. The machine operates at speeds between 2000 and 3000 rpm. The machine is placed on an isolator pad of stiffness 5×10^6 N/m and damping ratio 0.12. What is the maximum natural frequency of an undamped seismometer that can be used to measure the steady-state vibrations at all operating speeds with an error less than 4 percent. If this seismometer is used, what is its output when the machine is operating at 2500 rpm?

3.78. The system of Fig. P3.78 is subject to the excitation

$$F(t) = 1000 \sin 25.4t + 800 \sin(48t + 0.35) - 300 \sin(100t + 0.21) \text{ N}$$

What is the output in mm/s^2 of an accelerometer of natural frequency 100 Hz and damping ratio 0.7 placed at A?

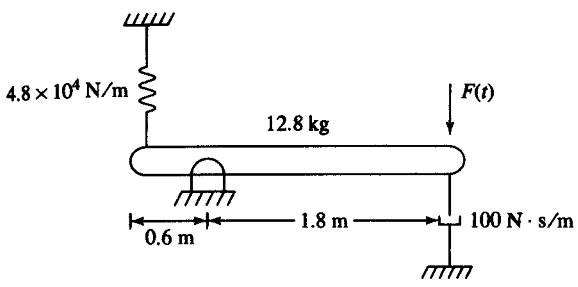


FIGURE P3.78

3.79. What is the output, in mm, of a seismometer with a natural frequency of 2.5 Hz and a damping ratio of 0.05 placed at point A for the system of Fig. P3.78?

3.80. A 20-kg block is connected to a moveable support through a spring of stiffness 1×10^5 N/m in parallel with a viscous damper of damping coefficient 600 N · s/m. The support is given a harmonic displacement of amplitude 25 mm and frequency 40 rad/s. An accelerometer of natural frequency 25 Hz and damping ratio 0.2 is attached to the block. What is the output of the accelerometer in mm/s^2?

3.81. An accelerometer has a natural frequency of 80 Hz and a damping coefficient of 8.0 N · s/m. When attached to a vibrating structure, it measures an amplitude of 8.0 m/s^2 and a frequency of 50 Hz. The true acceleration of the structure is 7.5 m/s^2. Determine the mass and stiffness of the accelerometer.

3.82. An accelerometer of natural frequency 200 rad/s and damping ratio 0.7 is attached to the engine of Prob. 3.67. What is the output of the accelerometer in mm/s^2?

3.83. Use complex notation to derive Eqs. (3.72) and (3.73) from Eqs. (3.69) and (3.70).

3.84. Use complex notation to derive the solution of Eq. (3.62) when $y(t) = Y \sin \omega t$.

MATLAB PROBLEMS

File VIBES_3A.m provides analysis of problems using the function $M(r, \zeta)$. Use VIBES_3A.m to solve Probs. M3.1 to M3.5.

M3.1. Plot $M(r, 0.1)$, $M(r, 0.5)$, and $M(r, 0.8)$ on the same set of axes.

M3.2. Use VIBES_3A to solve Prob. 3.10.

M3.3. Use VIBES_3A to solve Prob. 3.17.

M3.4. Use VIBES_3A to solve Prob. 3.24.

M3.5. Use VIBES_3A to solve Prob. 3.26.

File VIBES_3B.m provides analysis of problems using the function $\Lambda(r, \zeta)$. Use VIBES_3B to solve Probs. M3.6 to M3.11.

M3.6. Use VIBES_3B to plot $\Lambda(r, 0.1)$, $\Lambda(r, 0.3)$, and $\Lambda(r, 0.8)$ on the same set of axes.

M3.7. Use VIBES_3B to solve Prob. 3.31.

M3.8. Use VIBES_3B to solve Prob. 3.32.

M3.9. Use VIBES_3B to solve Prob. 3.33.

M3.10. Use VIBES_3B to solve Prob. 3.36.

M3.11. Use VIBES_3B to solve Prob. 3.40.

File VIBES_3C.m provides analysis of problems using the function $T(r, \zeta)$. Use VIBES_3C to solve Probs. M3.12 to M3.15.

M3.12. Use VIBES_3C to plot $T(r, 0.1)$, $T(r, 0.3)$ and $T(r, 0.8)$ on the same set of axes.

M3.13. Use VIBES_3C to solve Prob. 3.50.

M3.14. Use VIBES_3C to solve Prob. 3.52.

M3.15. Use VIBES_3C to solve Prob. 3.53.

M3.16. File VIBES_3D.m provides the solution to Example 3.7. Use VIBES_3D to investigate the effect of ζ on the maximum displacement and maximum acceleration. Plot the acceleration amplitude versus vehicle speed for several damping ratios on the same set of axes. Draw conclusions regarding the best damping ratio to use for steady-state operation.

M3.17. File VIBES_3E.m provides the analysis of problems using $M_c(r, \iota)$. Use VIBES_3E to solve Prob. 3.56.

M3.18. File VIBES_3F.m provides the solution of Example 3.13. Use VIBES_3F to determine the maximum steady-state displacement of the press assuming the period T remains constant, but the fraction of time during punching ($f = t_0/T$) changes. Note in Example 3.13 that $t_0 = 0.2$ s and $T = 0.5$ s, thus $f = 0.4$. Plot $x_{\max}$ versus f for $0.1 \leq f \leq 0.8$.

M3.19–M3.22. File VIBES_3G.m uses symbolic algebra to determine the Fourier series representation of an arbitrary periodic function of period T. The user must provide the function in a script file V3.m. VIBES_3G also determines symbolically the steady-state response of a one-degree-of-freedom system. The instructions for writing V3.m are given in the comment statements of VIBES_3G. Use VIBES_3G to determine and plot the Fourier series representation and steady-state response of a one-degree-of-freedom system of mass 20 kg, natural frequency 10 rad/s, and damping ratio 0.2 to the excitation shown. Each excitation is shown over one period.

M3.23. File VIBES_3H.m provides the response of a vehicle mounted on a suspension system modeled as in Prob. 2.71 as the vehicle traverses a round countour that is sinusoidal.

(*a*) Use VIBES_3H to solve Example 3.7 using the suspension stiffness and damping of Example 3.7, but with $k_t = 1.2 \times 10^5$ N/m.

(*b*) Use VIBES_3H to examine the effect of k_t on the steady-state amplitude of the vehicle of Example 3.7. Plot the maximum displacement and maximum acceleration as a function of k_t for $v = 3$ m/s and $v = 40$ m/s.

M3.24. Write a MATLAB script file that will plot the phase angle $\phi(r, \zeta)$ as a function of r for several values of ζ.

M3.25. Write a MATLAB script file that provides analysis for problems using the function $M_h(r, h)$. Use the program to solve Probs. 3.63 and 3.64.

M3.26. Write a MATLAB script file that uses symbolic algebra to determine the Fourier series representation of an arbitrary odd function of period T. Use the file to determine the Fourier series representation for the excitation of Prob. 3.73.

M3.27. Write a MATLAB script file that uses numerical integration to determine the Fourier coefficients for an arbitrary periodic function. Use the MATLAB program trap.m for the integration. Plot the approximation after every five terms to see how the approximation improves. Define a criterion for stopping the evaluation of Fourier coefficients. Use something to the effect that if $F_i(t)$ is the Fourier series approximation to $F(t)$ after i terms, then Fourier coefficients will cease being calculated when

$$\max\left|\frac{F_i(t) - F(t)}{F(t)}\right| < \varepsilon$$

for a predefined ε. Use the program on each of the excitations of Probs. M3.19 to M3.22.

M3.28. Write a MATLAB script, using the script of Prob. M3.26, to determine the response of a one-degree-of-freedom excitation to a periodic excitation. Use the program to solve Probs. M3.19 to M3.22.

M3.29. Write a MATLAB script to solve Probs. 3.66 and 3.67. Use symbolic algebra to continue the binomial expansion of $F(t)$ to include terms up through $\cos 5\omega t$.

chapter

4 Transient Vibrations of One-Degree-of-Freedom Systems

4.1 INTRODUCTION

When vibrations of a mechanical or structural system are initiated by a periodic excitation, an initial transient period occurs where the free-vibration response is as large as the forced response. The free vibration response quickly decays, resulting in a steady-state motion. Systems subject to a nonperiodic excitation do not usually achieve a non-zero steady state. In many cases, when a system is subject to a nonperiodic excitation, the free vibration response interacts with the forced response and is important throughout the duration of the motion of the system. Such is the case when a system is subject to a pulse of finite duration where the period of free vibration is greater than the pulse duration.

One example of a nonperiodic excitation is the ground motion of an earthquake. The response of structures due to ground motion is obtained by using the methods of this chapter. An earthquake is usually of short duration, but maximum displacements and stresses occur while the earthquake takes place. The terrain traveled by a vehicle is usually nonperiodic. Suspension systems must be designed to protect passengers from sudden changes in road contour. Forces produced in operation of machines in manufacturing processes are often nonperiodic. Sudden changes in forces occur in presses and milling machines.

Forced vibrations of one-degree-of-freedom systems are described by the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_{\text{eq}}(t)}{m_{\text{eq}}} \qquad [4.1]$$

Initial conditions, values of $x(0)$ and $\dot{x}(0)$, complete the problem formulation. Solution of Eq. (4.1) for periodic forms of $\tilde{F}(t)$ is discussed in Chap. 3.

The purpose of this chapter is to analyze the motion of systems undergoing transient vibrations. Equation (4.1) is a second-order linear nonhomogeneous ordinary differential equation. For certain forms of $F_{eq}(t)$, the method of undetermined coefficients, as applied in Chap. 3, can be used to determine the particular solution. The homogeneous solution is added to the particular solution, resulting in a general solution involving two constants of integration. Initial conditions are applied to evaluate the constants of integration. If damping is present the homogeneous solution dies out, leaving the particular solution as a steady-state solution. The method of undetermined coefficients is best suited for harmonic, polynomial, or exponential excitations and not useful for most excitations studied in this chapter.

The initial conditions and the homogeneous solution have an important effect on the short-term transient motion of vibrating systems. For these problems, it is convenient to use a solution method in which the homogeneous solution and particular solution are obtained simultaneously and the initial conditions are incorporated in the solution.

Many excitations are of short duration. For short-duration responses, the maximum response may occur after the excitation has ceased. Thus it is necessary to develop a solution method which determines the response of a system for all time, even after the excitation is removed. In addition, many excitations change form at discrete times. For these excitations a solution method in which a unified mathematical form of the response is determined is a great convenience.

The primary method of solution presented in this chapter is use of the convolution integral. The convolution integral is derived using the principle of impulse and momentum and linear superposition. It can also be derived by application of the method of variation of parameters. The convolution integral provides the most general closed-form solution of Eq. (4.1). The initial conditions are applied in the derivation of the integral, and need not be applied during every application. The convolution integral can be used to generate a unified mathematical response for excitations whose form changes at discrete times. Since it only requires evaluation of an integral, it is easy to apply.

A second method presented in this chapter is the Laplace transform method. Initial conditions are applied during the transform procedure and the Laplace transform can be used to develop a unified mathematical response for excitations whose form changes at discrete times. Use of tables of transforms makes application of the method convenient. The algebraic effort can be less than that using the convolution integral for damped systems, if appropriate transforms are available in a table. However, if the appropriate transforms are not available in a table, determination of the response is difficult.

There are some excitations in which a closed-form solution of Eq. (4.1) does not exist. In these cases, the convolution integral does not have a closed-form evaluation and application of Laplace transform method leads only to the convolution integral. In addition, situations exist when the excitation is not known explicitly at all values of time. The excitation may be obtained empirically. In these situations, numerical methods must be used to develop approximations to the response at discrete times. These numerical methods include numerical evaluation of the convolution integral and direct numerical solution of Eq. (4.1).

4.2 DERIVATION OF CONVOLUTION INTEGRAL

4.2.1 RESPONSE DUE TO A UNIT IMPULSE

Consider a one-degree-of-freedom system initially at rest in equilibrium. Let $x(t)$ be a generalized coordinate, representing the displacement of a particle. A linear one-degree-of-freedom system has the equivalent systems model of Fig. 4.1. An impulse of magnitude I is applied to the particle at $t = 0^-$. The principle of impulse and momentum is used to determine the velocity of the particle immediately after application of the impulse

$$v = \frac{I}{m_{\text{eq}}} \tag{4.2}$$

Application of the impulse initiates free vibration of the system. The motion is governed by Eq. (4.1) with $F_{\text{eq}}(t) = 0$ and $x(0) = 0$, $\dot{x}(0) = v$. If $\zeta < 1$, the resulting motion of the system is determined using Eq. (2.45) as

$$x(t) = \frac{I}{m_{\text{eq}}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{4.3}$$

or

$$x(t) = Ih(t) \tag{4.4}$$

where

$$h(t) = \frac{1}{m_{\text{eq}}\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{4.5}$$

is the response of the system due to a unit impulse applied at $t = 0$.

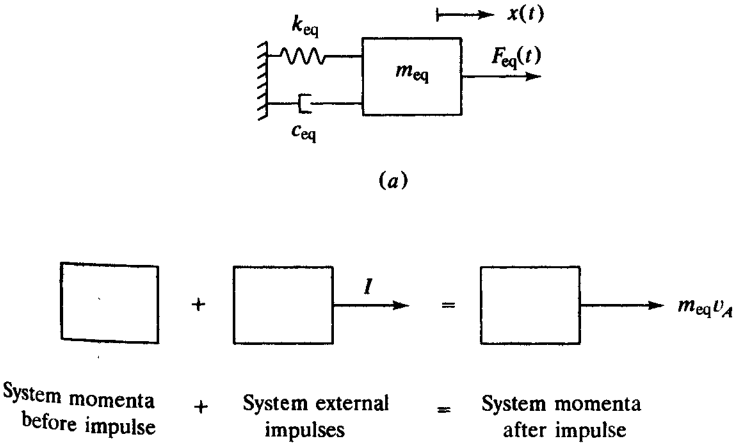


Figure 4.1 (a) Equivalent system used to model linear one-degree-of-freedom systems undergoing forced vibrations; (b) impulse and momentum diagrams used to obtain velocity immediately after impulse is applied if system is initially at rest in its equilibrium position.

4.2.2 RESPONSE DUE TO A GENERAL EXCITATION

Consider a one-degree-of-freedom system subject to an arbitrary external excitation as illustrated in Fig. 4.2. The interval from 0 to an arbitrary time t is broken into n equal intervals, each of duration $\Delta\tau = t/n$. The total impulse applied to the system during the interval between $k\Delta\tau$ and $(k+1)\Delta\tau$ is

$$I_k^n = \int_{k\Delta\tau}^{(k+1)\Delta\tau} F(t)\,dt$$

The mean value theorem of integral calculus implies that there exists a τ_k^*, $k\Delta\tau \leq \tau_k^* \leq (k+1)\Delta\tau$ such that

$$I_k^n = F\left(\tau_k^*\right)\Delta\tau \qquad [4.6]$$

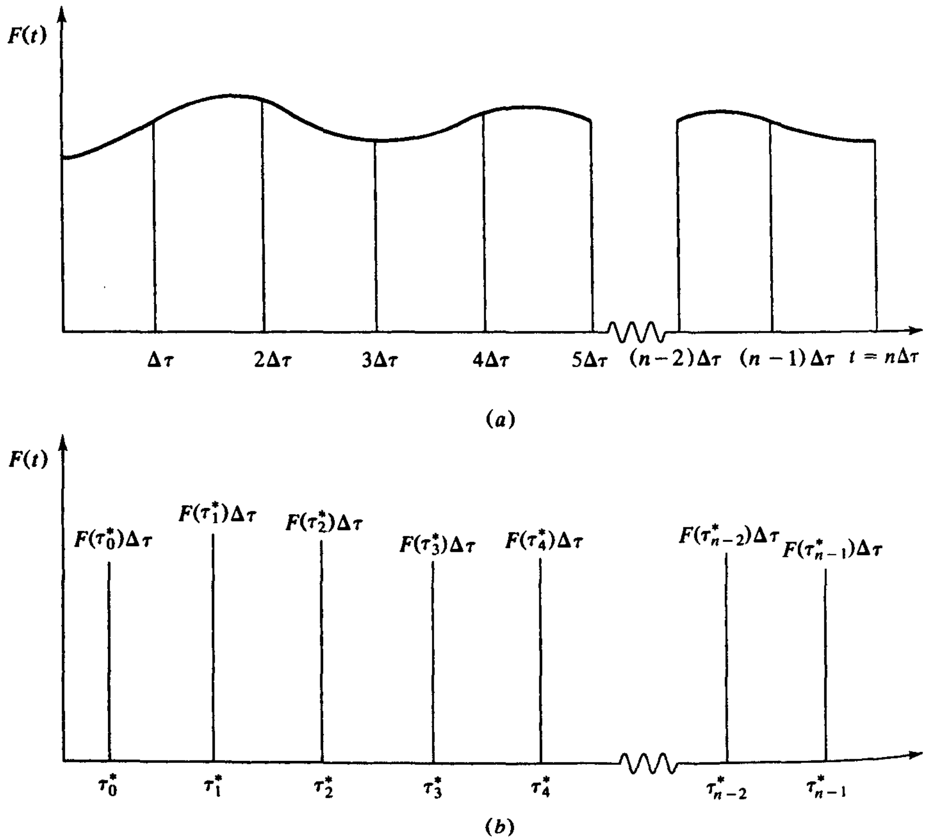


Figure 4.2 (a) Discretization of interval form 0 to t by n intervals of duration $\Delta\tau = t/n$; (b) approximation of $F(t)$ by impulses applied at τ_k, $k = 0, 1, \ldots, n-1$. As $n \to \infty$ the series of impulses approaches $F(t)$.

If $\Delta\tau$ is small, the effect of the force applied between $k\Delta\tau$ and $(k+1)\Delta\tau$ can be approximated by an impulse of magnitude I_k^n applied at $\tau_k = \left(k+\frac{1}{2}\right)\Delta\tau$. Thus, as illustrated in Fig. 4.2*b*, the excitation $F(t)$ applied between 0 and t is approximated by the sequence of impulses I_l^n, $l = 0, 1, 2, \ldots, n-1$.

The response of the system at time t due to an impulse of magnitude I_k^n applied at time τ_k is obtained from Eq. (4.4)

$$x_{k,n}(t) = I_k^n h\,(t-\tau_k) \tag{4.7}$$

Since Eq. (4.1) is linear, the principle of linear superposition is applied to determine the response at time t as

$$x_n(t) = \sum_{k=1}^{n-1} x_{k,n}(t) = \sum_{k=1}^{n-1} F\left(\tau_k^*\right)\Delta\tau h\,(t-\tau_k) \tag{4.8}$$

The approximation provided by Eq. (4.8) to the response of the system at time t becomes exact in the limit as $n \to \infty$ (or $\Delta\tau \to 0$). To this end

$$x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n-1} F\left(\tau_k^*\right)\Delta\tau h\,(t-\tau_k) \tag{4.9}$$

In the limit as $n \to \infty$, τ_k and τ_k^* approach $k\Delta\tau$ and τ becomes a continuous variable. Also in the limit the sum becomes an integral and Eq. (4.9) becomes

$$x(t) = \int_0^t F(\tau)h(t-\tau)\,d\tau \tag{4.10}$$

For a system whose free vibrations are underdamped, Eq. (4.5) is used in Eq. (4.10), leading to

$$x(t) = \frac{1}{m_{\text{eq}}\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d\,(t-\tau)\,d\tau \tag{4.11}$$

The integral representation of Eq. (4.10) is called the *convolution integral.* It can be used to determine the response of a one-degree-of-freedom system initially at rest in equilibrium subject to any form of excitation. The convolution integral solution is value for all linear systems when $h(t)$ is viewed as the response of the system due to a unit impulse at $t = 0$. The appropriate forms of $h(t)$ for systems whose free vibrations are critically damped and overdamped are, respectively,

$$h(t) = \frac{te^{-\omega_n t}}{m_{\text{eq}}} \qquad \zeta = 1 \tag{4.12}$$

$$h(t) = \frac{e^{-\zeta\omega_n t}}{m_{\text{eq}}\omega_n\sqrt{\zeta^2-1}}\sinh\left(\omega_n\sqrt{\zeta^2-1}\,t\right) \qquad \zeta > 1 \tag{4.13}$$

The response of a system with a nonzero initial velocity is obtained by adding to the convolution integral of Eq. (4.10) the response of the system due to a unit impulse at $t = 0$, necessary to cause the initial velocity. The response of a system that is not in its equilibrium position at $t = 0$ is obtained by defining a new independent variable,

$$y = x - x(0)$$

The differential equation governing $y(t)$ is

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\frac{k_{\text{eq}}}{m_{\text{eq}}}x(0) + \frac{F_{\text{eq}}(t)}{m_{\text{eq}}}$$

The convolution integral is used to obtain

$$y(t) = \int_0^t \left[-k_{\text{eq}}x(0) + F_{\text{eq}}(t)\right] h(t-\tau)\,d\tau$$

The resulting general solution for a system whose free vibrations are underdamped is

$$x(t) = x(0)e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$
$$+ \frac{1}{m_{\text{eq}}\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \qquad \textbf{[4.14]}$$

Find the response of a one-degree-of-freedom mass-spring-dashpot system initially at rest in equilibrium when the force

$$F(t) = F_0 e^{-\alpha t}$$

is applied.

Solution:

Application of Eq. (4.10) for this particular form of $F(t)$ gives

$$x(t) = \int_0^t \frac{F_0 e^{-\alpha\tau}}{m_{\text{eq}}\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$
$$= \frac{F_0}{m_{\text{eq}}\omega_d\left(\omega_n^2 - 2\zeta\omega_n\alpha + \alpha^2\right)}$$
$$\times \left\{e^{-\zeta\omega_n t}\left[(\alpha - \zeta\omega_n)\sin\omega_d t - \omega_d\cos\omega_d t\right] - \omega_d e^{-\alpha t}\right\}$$

A press of mass m is mounted on an elastic foundation of stiffness k. During operation the force applied to the press builds up to its final value F_0 in a time t_0, as illustrated in Fig. 4.3. Determine the response of the press for (a) $t < t_0$, and (b) $t > t_0$.

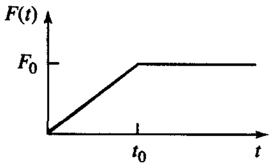


Figure 4.3 Excitation for Example 4.2.

Solution:

The force applied to the press can be expressed as

$$F(t) = \begin{cases} F_0 \dfrac{t}{t_0} & t < t_0 \\ F_0 & t \geq t_0 \end{cases}$$

For an undamped system the convolution integral, Eq. (4.11) becomes

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t - \tau)\, d\tau$$

(*a*) For $t < t_0$, the convolution integral yields

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n} \int_0^t F_0 \frac{\tau}{t_0} \sin \omega_n (t - \tau)\, d\tau \\ &= \frac{F_0}{m\omega_n t_0} \left[\frac{\tau}{\omega_n} \cos \omega_n (t - \tau) + \frac{1}{\omega_n^2} \sin \omega_n (t - \tau) \right]_{\tau=0}^{\tau=t} \\ &= \frac{F_0}{m\omega_n^2 t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right) \end{aligned}$$

(*b*) For $t > t_0$, application of the convolution integral leads to

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n} \left[\int_0^{t_0} F_0 \frac{\tau}{t_0} \sin \omega_n (t - \tau)\, d\tau + \int_{t_0}^t F_0 \sin \omega_n (t - \tau)\, d\tau \right] \\ &= \frac{F_0}{m\omega_n} \left\{ \left[\frac{\tau}{\omega_n} \cos \omega_n (t - \tau) + \frac{1}{\omega_n^2} \sin \omega_n (t - \tau) \right]_{\tau=0}^{\tau=t_0} + \left[\frac{1}{\omega_n} \cos \omega_n (t - \tau) \right]_{\tau=t_0}^{\tau=t} \right\} \\ &= \frac{F_0}{m\omega_n^2 t_0} \left[t_0 \cos \omega_n (t - t_0) + \frac{1}{\omega_n} \sin \omega_n (t - t_0) - \frac{1}{\omega_n} \sin \omega_n t + \frac{1}{\omega_n} \right. \\ &\qquad \left. - \frac{1}{\omega_n} \cos \omega_n (t - t_0) \right] \end{aligned}$$

4.3 EXCITATIONS WHOSE FORMS CHANGE AT DISCRETE TIMES

Many engineering systems are subject to a force whose mathematical form changes at discrete values of time. Such is the case with the force applied to the press in Example 4.2. The force linearly increases to its maximum value in a time t_0. The mathematical form of the response of the press is different for $t < t_0$ than it is for $t > t_0$. It is more convenient to have unified mathematical forms for the excitation and response. To this end, the unit step function, introduced in App. A, is used

$$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases} \qquad \text{[4.15]}$$

If a constant force F_0 is not applied until time t_0, it can be represented using a delayed unit step function

$$F(t) = \begin{cases} 0 & t \le t_0 \\ F_0 & t > t_0 \end{cases} = F_0 u\,(t - t_0)$$

4.3 Use the unit step function to write a unified mathematical expression for each of the forces of Fig. 4.4.

Solution:

Each of the forces of Fig. 4.4 can be written as the sum and/or difference of functions that are nonzero only after a discrete time. The graphical breakdown for each function is shown in Fig. 4.5. The unit step function is used to write a mathematical expression for each term in the forcing functions, leading to

$$(a)\quad F(t) = F_0\,[u(t) - u\,(t - t_0)]$$

$$(b)\quad F(t) = \frac{F_0 t}{t_0}\,[u(t) - u\,(t - t_0)] + F_0\,[u\,(t - t_0) - u\,(t - 3t_0)] + F_0\left(4 - \frac{t}{t_0}\right)[u\,(t - 3t_0) - u\,(t - 4t_0)]$$

$$(c)\quad F(t) = \frac{F_0 t}{t_0}\,[u(t) - u\,(t - t_0)] + F_0 e^{-\alpha(t - t_0)} u\,(t - t_0)$$

The unit impulse function, defined in App. A, is used to represent the excitation when subject to a unit impulse. The response of a system at time t due to a unit impulse applied at time t_0 is $h\,(t - t_0)\,u\,(t - t_0)$.

Many functions found in practice can be written as combinations of impulses, step functions, ramp functions, exponentially decaying functions, and sinusoidal pulses. Many functions which cannot be mathematically defined in terms of these functions are often approximated by these functions for estimation purposes.

Table 4.1 provides the response of an undamped one-degree-of-freedom system to common excitation terms delayed by a time t_0. The responses are derived from the convolution integral making use of the following formula:

$$\int_0^t F(\tau) u\,(\tau - t_0)\, d\tau = u\,(t - t_0) \int_{t_0}^t F(\tau)\, d\tau \qquad [4.16]$$

4.4 Use the convolution integral to derive the responses of an undamped linear one-degree-of-freedom system of mass m and natural frequency ω_n when subject to the delayed exponential excitation illustrated in Table 4.1.

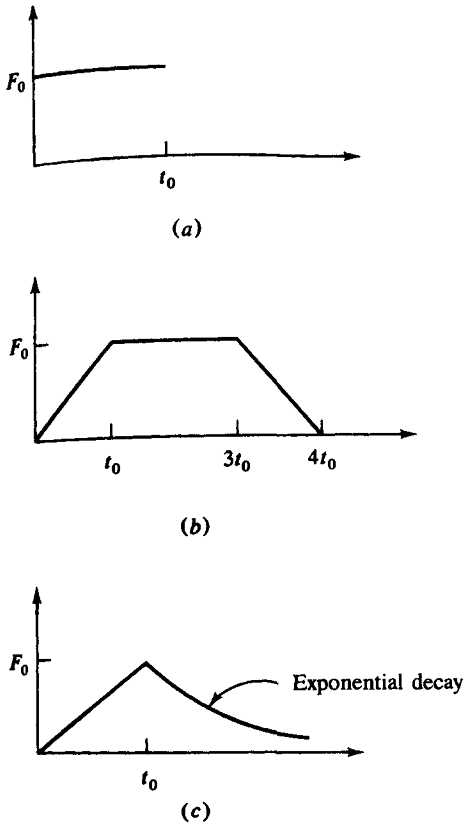


Figure 4.4 Excitations of Example 4.3.

Solution:

The mathematical representation of the forcing function is

$$F(t) = F_0 e^{-\alpha(t-t_0)} u\,(t - t_0)$$

The convolution integral, Eq. (4.11), is used to write the solution as

$$x(t) = \frac{F_0}{m_{\text{eq}}\omega_n} \int_0^t e^{-\alpha(\tau-t_0)} u\,(\tau - t_0) \sin \omega_n (t - \tau)\, d\tau$$

which using Eq. (4.16) is rearranged as

$$\begin{aligned} x(t) &= u\,(t - t_0) \frac{F_0}{m_{\text{eq}}\omega_n} \int_{t_0}^t e^{-\alpha(\tau-t_0)} \sin \omega_n (t - \tau)\, d\tau \\ &= u\,(t - t_0) \frac{F_0}{m_{\text{eq}}\omega_n \left(\alpha^2 + \omega_n^2\right)} \left[\omega_n e^{-\alpha(t-t_0)} + \alpha \sin \omega_n\,(t - t_0) \right. \\ &\qquad \left. - \ \omega_n \cos \omega_n\,(t - t_0)\right] \end{aligned}$$

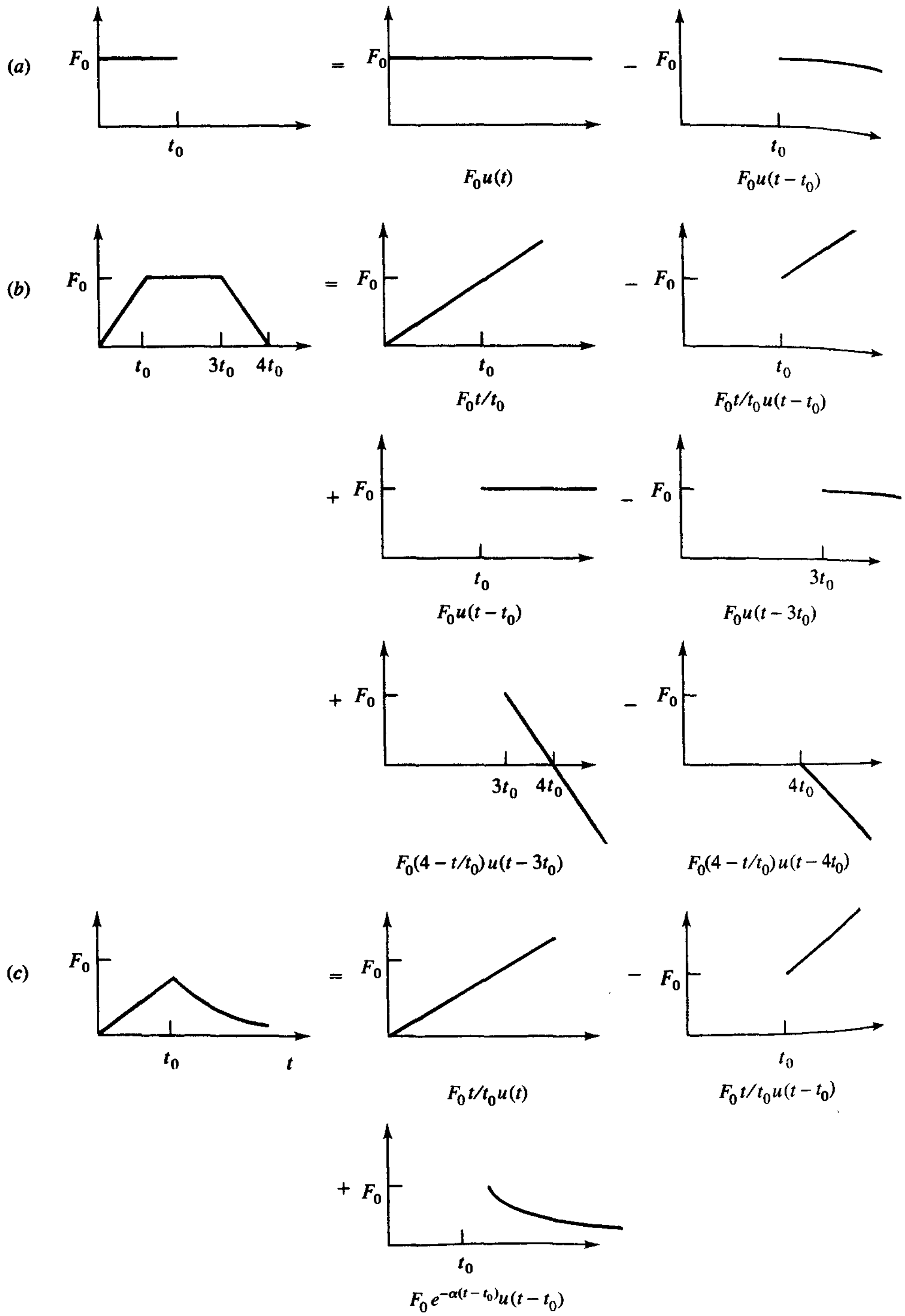
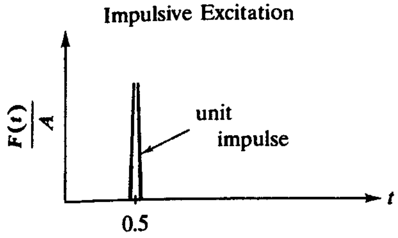
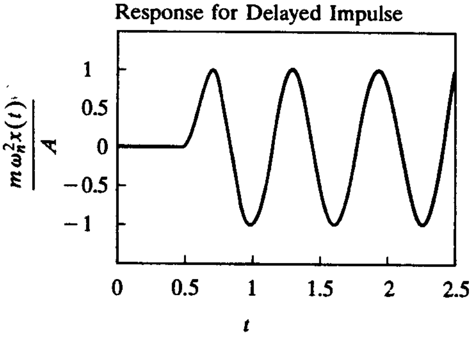


Figure 4.5 Graphical breakdown of excitations of Fig. 4.4 into functions that can be written by using unit step functions.

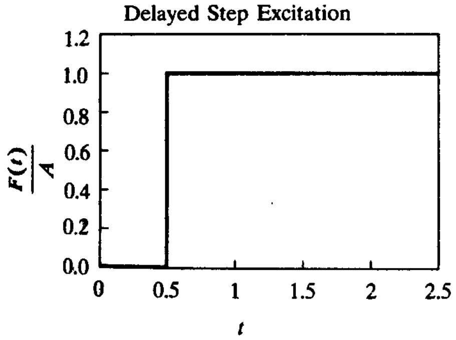
Table 4.1 Response of an undamped one-degree-of-freedom to common forms of excitation

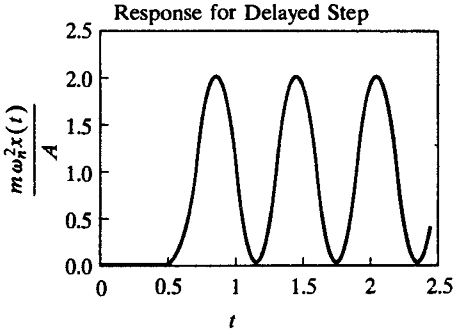
Delayed impulse
Excitation: $F(t) = A\delta\,(t - t_0)$
Response: $m_{eq}\omega_n^2 x(t)/A = \omega_n \sin\omega_n\,(t - t_0)\,u\,(t - t_0)$



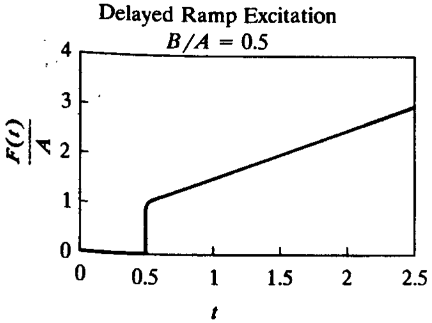


Delayed step function
Excitation: $F(t) = Au\,(t - t_0)$
Response: $m_{eq}\omega_n^2 x(t)/A = [1 - \cos\omega_n\,(t - t_0)]\,u\,(t - t_0)$





Delayed ramp function
Excitation: $F(t) = (At + B)u\,(t - t_0)$
Response: $m_{eq}\omega_n^2 x(t)/A = [t + B/A - (t_0 + B/A)\cos\omega_n\,(t - t_0) - \sin\omega_n\,(t - t_0)\,/\omega_n]\,u\,(t - t_0)$



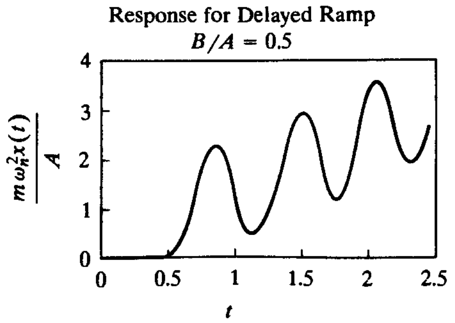
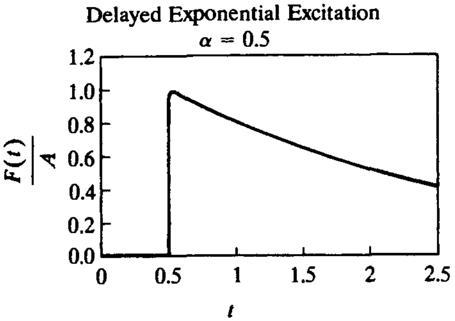


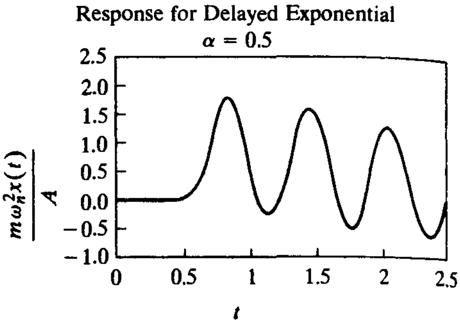
Table 4.1 (Con't)

Delayed exponential function

Excitation: $F(t) = Ae^{-\alpha(t-t_0)}u(t-t_0)$

Response: $m_{eq}\omega_n^2 x(t)/A = \left[e^{-\alpha(t-t_0)} + \alpha/\omega_n \sin\omega_n(t-t_0) - \cos\omega_n(t-t_0)\right] / \left(1+\alpha^2/\omega_n^2\right) u(t-t_0)$

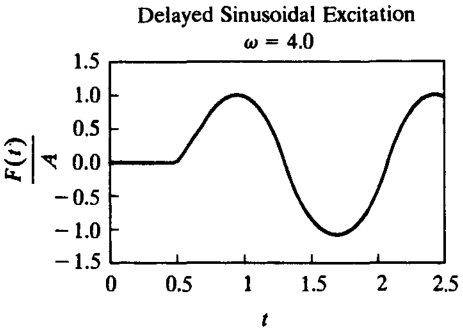


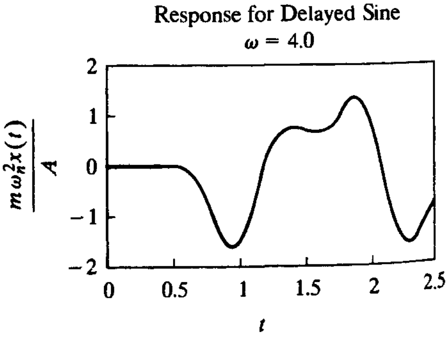


Delayed sine function:

Excitation: $F(t) = A\sin\omega(t-t_0)u(t-t_0)$

Response: $$\frac{m_{eq}\omega_n^2 x(t)}{A} = \frac{1}{2}\left\{\left(\frac{1}{\omega/\omega_n - 1}\right)[\sin\omega(t-t_0) - \sin\omega_n] - \left(\frac{1}{\omega/\omega_n + 1}\right)[\sin\omega(t-t_0) + \sin\omega_n]\right\} u(t-t_0)$$





NOTE: This table provides the response of an undamped one-degree-of-freedom system to common forms of excitation. Many forms of excitation can be written as combinations of the excitations whose system responses are provided in the table. Superposition can be used to determine the response due to these excitations. In other cases excitations can be approximated by combinations of excitations in this table. Then this table and superposition is used to approximate the response of an undamped one-degree-of-freedom system.

The table provides the mathematical form of the excitation and response as well as graphical representations. In all cases values of $\omega_n = 10$ rad/s and $t_0 = 0.5$ s were used to generate the graphs. The values of specific parameters used for specific excitations are given.

Often, excitations are linear combinations of the function whose responses are presented in Table 4.1. The general form of an excitation that changes form at discrete times $t_1, t_2, \ldots, t_n$ is

$$F(t) = \sum_{i=1}^{n} f_i(t)u(t - t_i) \qquad \textbf{[4.17]}$$

Application of the convolution integral to the excitation of Eq. (4.17), using Eq. (4.16), yields

$$x(t) = \sum_{i=1}^{n} u(t - t_i) \int_{t_i}^{t} f_i(\tau)h(t - \tau)\, d\tau \qquad \textbf{[4.18]}$$

Equation (4.18) shows that the total response is the sum of the responses due to the individual terms of the excitation. This result is due to the linearity of Eq. (4.1). The effects of any nonzero initial conditions are included with the response due to $f_1(t)$.

Example 4.5

Use Table 4.1 to develop the response of a linear one-degree-of-freedom system of mass m and natural frequency ω_n when subject to the triangular pulse excitation of Fig. 4.6.

Solution:

The triangular pulse can be written as the sum and difference of ramp functions as shown. The response due to the triangular pulse is obtained by adding and subtracting

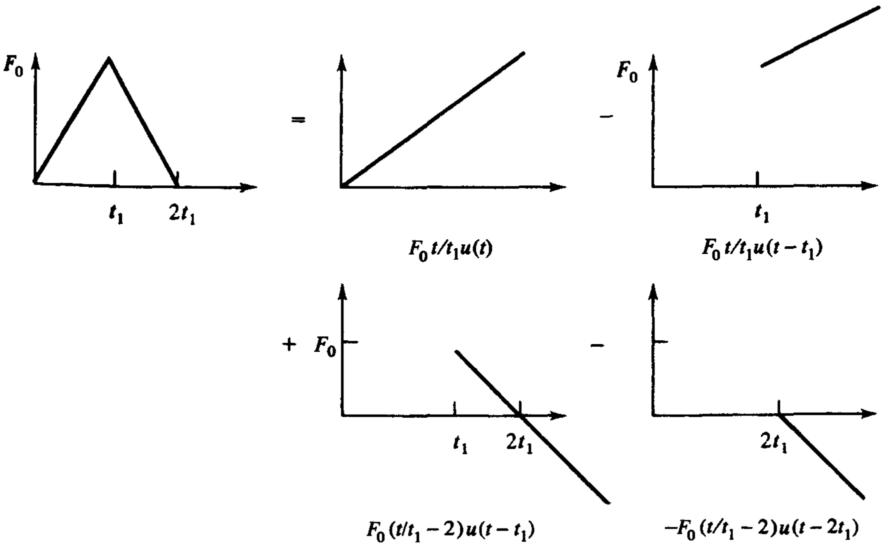


Figure 4.6 Triangular pulse of Example 4.5 and its graphical breakdown.

the responses due to each ramp function according to

$$x(t) = x_a(t) - x_b(t) + x_c(t) - x_d(t)$$

where the individual responses are determined from Table 4.1.

For $x_a(t)$, the ramp function entry of Table 4.1 is used with $A = F_0/t_1$, $B = 0$, and $t_0 = 0$ leading to

$$x_a(t) = \frac{F_0}{m\omega_n^2}\left[\frac{t}{t_1} - \frac{1}{\omega_n t_1}\sin\omega_n t\right]$$

$x_b(t)$ is determined from the ramp function entry of Table 4.1 with $A = F_0/t_1$, $B = 0$, $t_0 = t_1$. This gives

$$x_b(t) = \frac{F_0}{m\omega_n^2}\left[\frac{t}{t_1} - \cos\omega_n(t-t_1) - \frac{1}{\omega_n t_1}\sin\omega_n(t-t_1)\right]u(t-t_1)$$

For $x_c(t)$, the ramp function entry of Table 4.1 is used with $A = -F_0/t_1$, $B = 2F_0$, and $t_0 = t_1$. This leads to

$$x_c(t) = \frac{F_0}{m\omega_n^2}\left[\left(2 - \frac{t}{t_1}\right) - \cos\omega_n(t-t_1) + \frac{1}{\omega_n t_1}\sin\omega_n(t-t_1)\right]u(t-t_1)$$

$x_d(t)$ is determined using the ramp function entry of Table 4.1 with $A = -F_0/t_1$, $B = 2F_0$, and $t_0 = 2t_1$. This gives

$$x_d(t) = \frac{F_0}{m\omega_n^2}\left[\left(2 - \frac{t}{t_1}\right) + \frac{1}{\omega_n t_1}\sin\omega_n(t-2t_1)\right]u(t-2t_1)$$

Simplifying the resulting expression in each interval of time yields

$$x(t) = \frac{F_0}{m\omega_n^2}\begin{cases} \dfrac{t}{t_1} - \dfrac{1}{\omega_n t_1}\sin\omega_n t & 0 \le t \le t_1 \\[2ex] 2 - \dfrac{t}{t_1} + \dfrac{1}{\omega_n t_1}[2\sin\omega_n(t-t_1) - \sin\omega_n t] & t_1 \le t \le 2t_1 \\[2ex] \dfrac{1}{\omega_n t_1}[2\sin\omega_n(t-t_1) - \sin\omega_n t - \sin\omega_n(t-2t_1)] & t > 2t_1 \end{cases}$$

The response of each component part and the total response is shown in Fig. 4.7.

4.4 TRANSIENT MOTION DUE TO BASE EXCITATION

Many mechanical systems and structures are subject to nonperiodic base excitation. A rigid wheel traveling along a road contour excites motion of a vehicle through the suspension system. Earthquakes excite structures through base motion.

Recall the governing equation for the relative displacement between a mass and its base when the mass is connected to the base through a spring and viscous damper

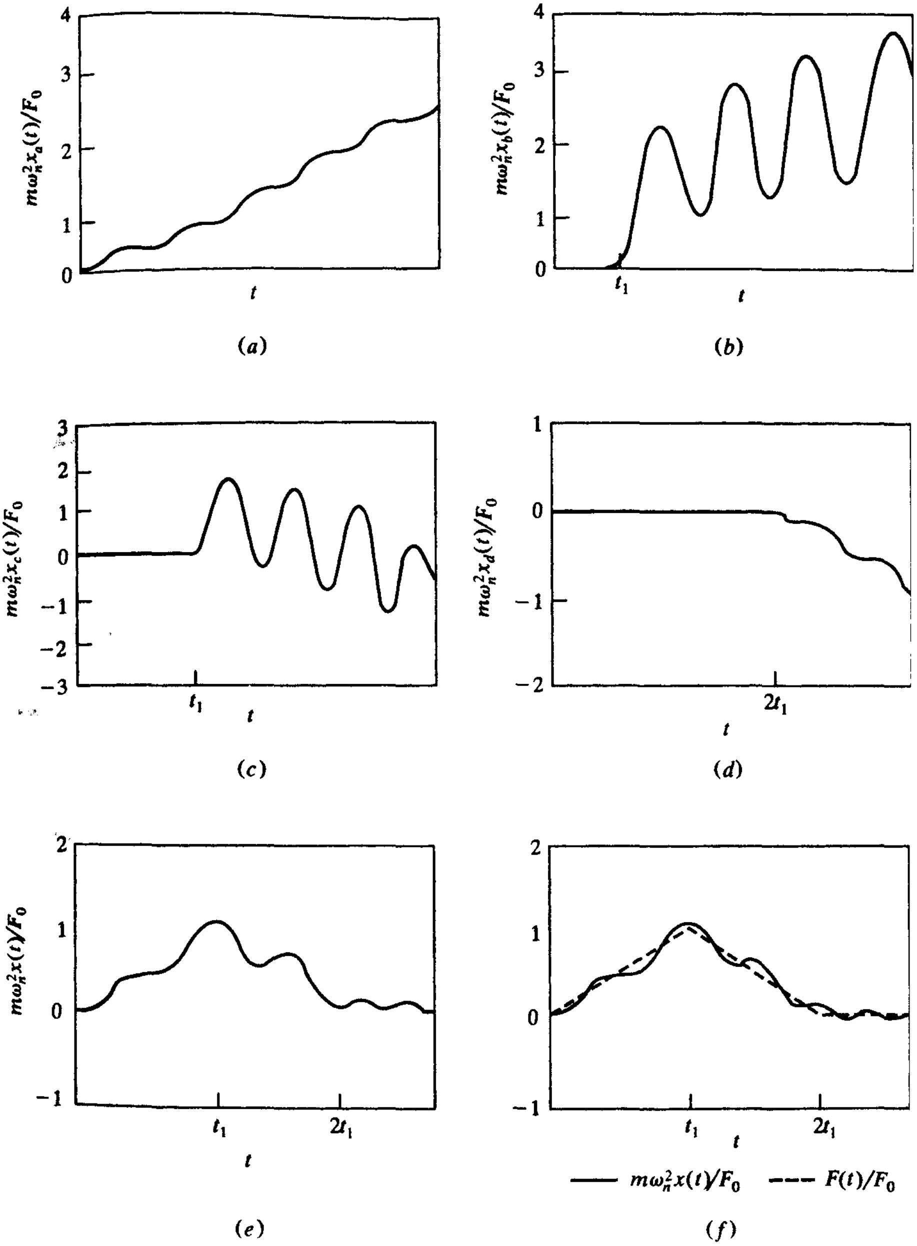


Figure 4.7 (a)–(d) Response of a one-degree-of-freedom system due to the component parts of a triangular pulse excitation obtained by using Table 4.1; (e) response of a one-degree-of-freedom system due to a triangular pulse excitation obtained using superposition; (f) comparison of triangular pulse and the resulting excitation.

in parallel

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \qquad \text{[4.19]}$$

where y is the prescribed base motion. If $z(0) = 0$ and $\dot{z}(0) = 0$, the convolution integral is used to solve Eq. (4.19), yielding

$$z(t) = -m_{\text{eq}} \int_0^t \ddot{y}(\tau) h(t-\tau)\,d\tau \qquad \text{[4.20]}$$

Equation (4.20) is integrated by parts to write the solution in terms of the base velocity

$$z(t) = m_{\text{eq}} \left[\dot{y}(0)h(t) - \int_0^t \dot{y}(\tau)\dot{h}(t-\tau)\,d\tau \right] \qquad \text{[4.21]}$$

where

$$\dot{h}(t) = -\frac{e^{-\zeta\omega_n t}}{m_{\text{eq}}\sqrt{1-\zeta^2}} \sin(\omega_d t - \chi) \qquad \text{[4.22]}$$

$$\chi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

If the base displacement is known, it can be differentiated to calculate the velocity and Eq. (4.21) can be used to determine the relative displacement. Alternatively, the absolute displacement of the base can be attained by solving

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -2\zeta\omega_n\dot{y} - \omega_n^2 y \qquad \text{[4.23]}$$

When applied to Eq. (4.23) the convolution integral yields

$$x(t) = -m_{\text{eq}} \int_0^t \left[2\zeta\omega_n\dot{y}(\tau) + \omega_n^2 y(\tau)\right] h(t-\tau)\,d\tau \qquad \text{[4.24]}$$

Determine the response of a block of mass m connected through a spring of stiffness k to a base when the base is subject to the rectangular velocity pulse of Fig. 4.8. Use (*a*) Eq. (4.21) and (*b*) Eq. (4.20).

Solution:

The mathematical expression for the velocity pulse is

$$\dot{y}(t) = v\,[u(t) - u(t - t_0)]$$

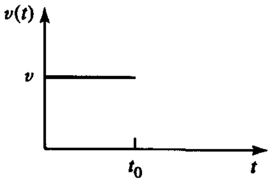


Figure 4.8 Velocity pulse for Example 4.6.

(*a*) By definition $u(0) = 0$, thus $\dot{y}(0) = 0$. In using Eq. (4.21) for an undamped system, note that $\chi = 0$ and $\sin(\omega_n t - \pi/2) = \cos\omega_n t$. Application of Eq. (4.21) then yields

$$z(t) = -v\int_0^t [u(\tau) - u(\tau - t_0)]\cos\omega_n(t-\tau)\,d\tau$$

Using Eq. (4.16) to evaluate the integral leads to

$$z(t) = -v\left[u(t)\int_0^t \cos\omega_n(t-\tau)\,d\tau - u(t-t_0)\int_{t_0}^t \cos\omega_n(t-\tau)\,d\tau\right]$$
$$= \frac{v}{\omega_n}[\sin\omega_n(t-t_0)\,u(t-t_0) - \sin(\omega_n t)\,u(t)]$$

(*b*) The base acceleration is obtained by differentiating the base velocity with respect to time. Noting that the derivative of the unit step function is the unit impulse function, differentiation gives

$$\ddot{y}(t) = v[\delta(t) - \delta(t-t_0)]$$

The base velocity changes instantaneously at $t = t$ and $t = t_0$. Instantaneous velocity changes result only from applied impulses.

Substituting the base acceleration into Eq. (4.20) gives

$$z(t) = -\frac{v}{\omega_n}\int_0^t [\delta(\tau) - \delta(\tau - t_0)]\sin\omega_n(t-\tau)\,d\tau$$

The integrals are evaluated after noting

$$\int_0^t \delta(\tau - t_0)\,f(\tau)\,d\tau = f(t_0)\,u(t-t_0)$$

The relative displacement is determined as

$$z(t) = \frac{v}{\omega_n}[\sin\omega_n(t-t_0)\,u(t-t_0) - \sin(\omega_n t)\,u(t)]$$

Example 4.7

The vehicle of Example 3.7 encounters a bump in the road of the shape shown in Fig. 4.9. Determine the response of the vehicle if it encounters the bump when it is traveling at a speed of 20 m/s. Recall from Example 3.7 that $m = 900$ kg, $k = 80{,}000$ N/m, and $\zeta = 0.2$.

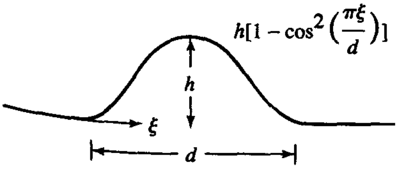


Figure 4.9 Versed sine pulse model for bump of Example 4.7.

Solution:

The mathematical form of the bump, in terms of the horizontal road coordinate ξ is

$$y(\xi) = h\left[1 - \cos^2\left(\frac{\pi\xi}{d}\right)\right][1 - u(\xi - d)]$$

where $h = 0.01$ m and $d = 0.8$ m. If the vehicle encounters the bump at $t = 0$ and maintains a constant horizontal speed v, then $\xi = vt$ and the vertical displacement of the wheels is

$$y(t) = h\left[1 - \cos^2\left(\frac{\pi v}{d}t\right)\right]\left[1 - u\left(t - \frac{d}{v}\right)\right]$$

The convolution integral of the form of Eq. (4.24) is used to determine the system response. The wheel velocity is determined as

$$\dot{y}(t) = 2\left(\frac{\pi v}{d}\right)\sin\left(\frac{2\pi v}{d}t\right)\left[1 - u\left(t - \frac{d}{v}\right)\right]$$

Note that, since $y(0)$ and $y(d/v) = 0$, it is not necessary to include derivatives of the unit step functions.

The problem is solved by using the symbolic capabilities of MATLAB. The MATLAB script and the resulting displacement plot are given in Fig. 4.10.

4.5 LAPLACE TRANSFORM SOLUTIONS

The Laplace transform method is a convenient method for finding the response of a system due to any excitation. The basic method is to use known properties of the transform to transform an ordinary differential equation into an algebraic equation, using the initial conditions. The algebraic equation is solved to find the transform of the solution. This transform is inverted by using properties of the transform and a table of known transform pairs.

The Laplace transform can be used to solve linear ordinary differential equations with constant or polynomial coefficients. The method easily handles excitations whose form changes with time. Such excitations are written in a unified mathematical expression by using the unit step functions. The shifting theorems help perform the transform and evaluate the inversions.

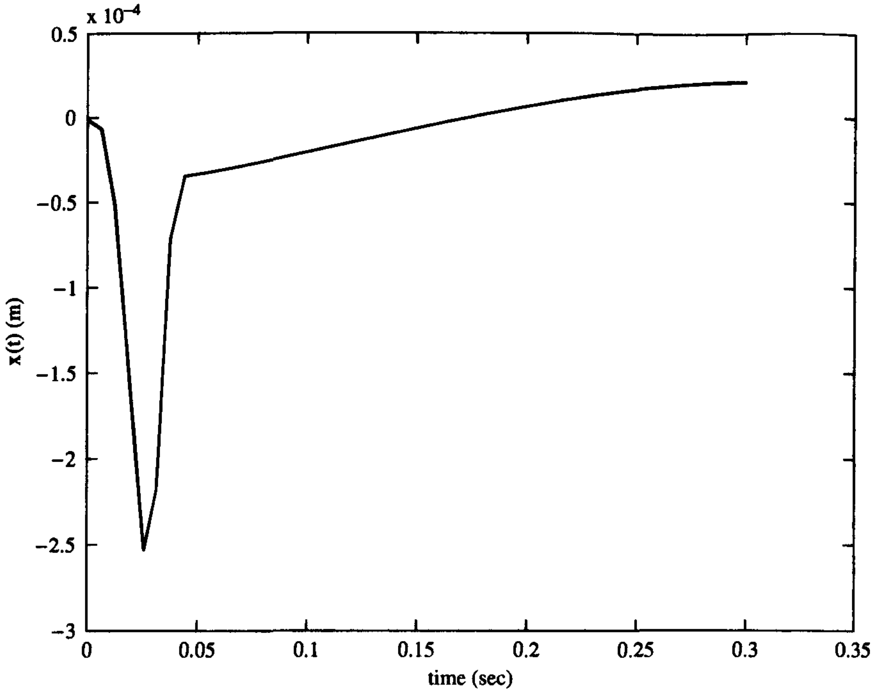
The Laplace transform is not as easy to apply as the convolution integral unless one has extensive experience in its use. The main drawback of the method is the difficulty in inverting the transform. A formal inversion theorem, involving contour integration in the complex plane, is available, but is beyond the scope of this text.

The transform pairs and properties used in the following discussion are summarized and explained in App. B.

```
% Example 4.7
% Simplified one-degree-of-freedom model of vehicle suspension system
% Vehicle encounters bump in road modeled as a versed sinusoidal pulse
% y(t)=h(1-(cos(pi*v*t/t0))^2)*(u(t)-u(t-d/v))
% Convolution integral is used to evaluate system response
syms t tau                    % Defining symbolic variables
% Input parameters
digits(10)
format short e
m=900;                        % Vehicle mass (kg)
k=80000;                      % Suspension stiffness (N/m)
zeta=0.2;                     % Damping ratio
hb=0.01;                      % Maximum height of bump (m)
d=0.8;                        % Length of bump (m)
v=20;                         % Velocity of vehicle (m/s)
% Calculation of system parameters and constants
omega_n=sqrt(k/m);            % Natural frequency (rad/s)
omega_d=omega_n*sqrt(1-zeta^2);%Damped natural frequency
C1=pi/d;
% Definition of time dependent wheel displacement and velocity
% MATLAB uses 'Heaviside' for the unit step function
y=hb*(1-cos(C1*v*t)^2)*(1-sym('Heaviside(t-0.04)'));
ydot=hb*C1*sin(2*C1*v*tau)*(1-sym('Heaviside(tau-.04)'))
% Convolution integral evaluation of response
h=exp(-zeta*omega_n*(t-tau))*sin(omega_d*(t-tau))/(m*omega_d);
g1=-2*zeta*m*omega_n*ydot*h;
g2=-omega_n^2*m*y*h;
g1a=vpa(g1,5);
g2a=vpa(g2,5);
I1=int(g1a,tau,0,t);
I1a=vpa(I1,5)
I2=int(g2a,tau,0,t);
I2a=vpa(I2,5);
x1=I1a+I2a;
x=vpa(x1,5)
vel=diff(x)
acc=diff(vel)
time=linspace(0,0.3,50);
for i=1:50
   x1=subs(x,t,time(i));
   xa(i)=vpa(x1);
end
xp=double(xa);
plot(time,xp,'-')
xlabel('time (sec)')
ylabel('x(t) (m)')
```

(*a*)

Figure 4.10 (*a*) MATLAB script for solution of Example 4.7 using symbolic integration of convolution integral.



(*b*)

Figure 4.10 (Con't) (*b*) plot of displacement versus time.

Let $\bar{x}(s)$ be the Laplace transform of the generalized coordinate for a one-degree-of-freedom system. That is,

$$\bar{x}(s) = \int_0^\infty x(t)e^{-st}\,dt \tag{4.25}$$

Let $\bar{F}(s)$ be the Laplace transform of the known forcing function which, for a specific form of $F_{eq}(t)$, is calculated from the transform definition, referring to a table of transform pairs, or using basic properties in conjunction with a table.

Taking the Laplace transform of Eq. (4.1) and using linearity of the transform,

$$\mathcal{L}\{\ddot{x}\} + 2\zeta\omega_n\mathcal{L}\{\dot{x}\} + \omega_n^2\bar{x}(s) = \frac{\bar{F}(s)}{m_{eq}} \tag{4.26}$$

The property for transform of derivatives allows the transform of the differential equation for $x(t)$ into an algebraic equation for $\bar{x}(s)$. Its application to Eq. (4.26) gives

$$s^2\bar{x}(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n[s\bar{x}(s) - x(0)] + \omega_n^2\bar{x}(s) = \frac{\bar{F}(s)}{m_{eq}}$$

which rearranges to

$$\bar{x}(s) = \frac{\frac{\bar{F}(s)}{m_{eq}} + (s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad [4.27]$$

The definition and linearity of the inverse transform is used to find $x(t)$,

$$x(t) = \frac{1}{m_{eq}}\mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \quad [4.28]$$

The inverse transform of each term of Eq. (4.28) depends on the types of roots in the denominator, which, in turn, depend on the value of ζ. For a given ζ, the inverse transform of the last term of Eq. (4.28) is directly determined. The inverse transform of the first term is determined only after specifying $F_{eq}(t)$ and taking its Laplace transform.

If the free vibrations are underdamped, then the denominator has two complex roots. In this case it is convenient to complete the square of the denominator and write it as

$$(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)$$

The inverse transform of the last term of Eq. (4.28) is found by applying the first shifting theorem and known transform pairs $B4$ and $B5$,

$$\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \quad [4.29]$$

$$= e^{-\zeta\omega_n t}\left[x(0)\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t\right]$$

The inverse transform of the first term of Eq. (4.28) is found by finding $\bar{F}(s)$ for the particular form of $F(t)$, forming $\bar{F}(s)/\left(s_2 + s\zeta\omega_n s + \omega_n^2\right)$, and inverting using algebraic manipulations, transform properties, and a table of known transform pairs.

Example 4

A 200-kg machine is to be mounted on an elastic surface of equivalent stiffness 2×10^5 N/m with no damping. During operation, the machine is subject to a constant force of 2000 N for 3 s. Can steady-state vibrations be eliminated without adding damping? If so, what is the maximum deflection of the machine?

Solution:

The differential equation governing motion of the machine is

$$\ddot{x} + \omega_n^2 x = F_0[u(t) - u(t-3)]$$

where $F_0 = 2000$ N and $\omega_n = 31.63$ rad/s. The Laplace transform of $F(t)$ is obtained by using the second shifting theorem

$$\mathcal{L}\{F_0[u(t) - u(t-3)]\} = \frac{F_0}{s}\left(1 - e^{-3s}\right)$$

Then from Eq. (4.28) with $x(0) = 0$ and $\dot{x}(0) = 0$,

$$\bar{x}(s) = \frac{F_0}{m}\mathcal{L}^{-1}\left\{\frac{1 - e^{-3s}}{s\left(s^2 + \omega_n^2\right)}\right\}$$

Partial fraction decomposition yields

$$\bar{x}(s) = \frac{F_0}{m\omega_n^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right)\left(1 - e^{-3s}\right)$$

The second shifting theorem is used to help invert the transform

$$x(t) = \frac{F_0}{m\omega_n^2}\left[1 - \cos\omega_n t - u(t-3)\left(1 - \cos\omega_n(t-3)\right)\right]$$

The solution for $t > 3$ s is

$$x(t) = \frac{F_0}{m\omega_n^2}\left[\cos\omega_n(t-3) - \cos\omega_n t\right] \qquad t > 3 \text{ s}$$

For no steady-state motion,

$$\cos\omega_n t = \cos\omega_n(t-3)$$

which is satisfied if $3\omega_n = 2n\pi$ for any positive integer n. Thus steady-state vibrations are eliminated by requiring

$$\omega_n = \frac{2n\pi}{3} = 2.09n\ \frac{\text{rad}}{\text{s}}$$

For $n = 15$, $\omega_n = 31.35$ rad/s, which is attained if $m = 203.5$ kg. Thus steady-state vibrations are eliminated if 3.5 kg is rigidly added to the machine.

The machine undergoes 15 cycles while the force is applied, and motion ceases when the force is removed. The maximum displacement during operation is

$$x_{\max} = \frac{2F_0}{m\omega_n^2} = \frac{2F_0}{k} = 0.02 \text{ m}$$

Use MATLAB to determine the response of a one-degree-of-freedom system with viscous damping to an exponential excitation $F(t) = e^{-at}$.

Solution:

The MATLAB script and output when the script is run are given in Fig. 4.11. MATLAB can determine the Laplace transform of many excitations symbolically and then invert the transform of the displacement to symbolically determine the system response. Since the output for a damped system is lengthy, the output is presented for an undamped system.

```
% MATLAB script for Example 4.9
% Laplace transform solution for response of one-degree-of-freedom
% system due to exponential excitation.
syms s t a z w
% Excitation
f=exp(-a*t);
% Laplace transform of excitation
fs=laplace(f);
hs=1/(s^2+2*z*w*s+w^2);
% Laplace transform of x(t)
gs=fs*hs;
disp('L(x(t))=')
disp(gs)
% Inverse transform to find x(t)
x1=ilaplace(gs,s,t);
disp('system response')
x=simplify(x1);
x2=subs(x,z,0);
x3=simplify(x2);
disp(x3)
```

(*a*)

```
L(x(t))=
1/(s+a)/(s^2+2*z*w*s+w^2)

system response
1/w*(exp(-a*t)*w+csgn(w)*a*sin(csgn(w)*w*t)
-cos(csgn(w)*w*t)*w)/(a^2+w^2)
```

(*b*)

Figure 4.11 (*a*) MATLAB script for Example 4.9; symbolic solution using Laplace transform method; (*b*) MATLAB output.

Example 4.10

Use the Laplace transform method to determine the response of an underdamped one-degree-of-freedom system to the rectangular velocity pulse of Fig. 4.8.

Solution:

From the analysis in Example 4.6, the differential equation governing displacement of the mass relative to its base when the base is subject to a rectangular velocity pulse is

$$\ddot{z} + 2\zeta\omega_n z + \omega_n^2 z = -v\left[\delta(t) - \delta(t - t_0)\right]$$

Using transform pair $B1$, and assuming $z(0) = 0$ and $\dot{z}(0) = 0$, Eq. (4.27) becomes

$$\bar{z}(s) = \frac{-v\left(1 - e^{-st_0}\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The transform is inverted by completing the square in the denominator and using both the first shifting theorem and the second shifting theorem to obtain

$$z(t) = \frac{-v}{\omega_n}\left[e^{-\zeta\omega_n t}\sin\omega_d t - e^{-\zeta\omega_n(t-t_0)}\sin\omega_d\left(t - t_0\right)u\left(t - t_0\right)\right]$$

4.6 SHOCK SPECTRUM

Design problems often require the determination of system parameters such that constraints are satisfied. In many problems the design criteria involve limiting maximum displacements and/or maximum stresses for a given type of excitation. For example, if it is determined that all earthquakes in a given area have similar forms of excitations, only with different levels of severity, then knowledge of the maximum displacement as a function of system parameters is useful in the design of a structure to withstand a certain level of earthquake. The structure's ability to withstand the earthquake depends on the maximum displacement developed in the structure during the earthquake and the maximum stresses developed. A structure in California along the San Andreas fault will usually be designed to withstand a more severe earthquake than a structure in Ohio. This, of course, depends on the use of the structure.

Thus it is useful for the designer to know the maximum response of a structure as a function of system parameters. The transmissibility curves presented in Chap. 3 and discussed in more detail in Chap. 8 actually do this for the steady-state response due to harmonic excitations. For a given value of the damping ratio, the transmissibility curve plots the nondimensional ratio of the amplitude of the transmitted force to the maximum amplitude of the excitation force against the nondimensional frequency ratio.

Similar curves are useful for analysis and design of systems that are subject to shock excitations. A shock is a large force applied over a short interval resulting in transient vibration. The maximum response is a function of the type of shock and system parameters.

A *shock spectrum* (*response spectrum*) is a nondimensional plot of the maximum response of a one-degree-of-freedom system for a specified excitation as a function of a nondimensional time ratio. The vertical axis of the plot is the maximum value of the force developed in the spring divided by the maximum of the excitation force. The horizontal axis is the ratio of a characteristic time for the excitation divided by the natural period. For a shock excitation the characteristic time is usually taken as the duration of the shock.

Shock spectra are often plotted only for undamped systems as damping tends to act favorably to reduce the maximum response. Also, a shock spectrum is very tedious to calculate and plot. Inclusion of damping in the development of a shock spectrum greatly increases the amount of algebra performed. The resulting complexity may obscure the usefulness of the results.

Example 4.

A one-story frame structure is to be built to house a chemical laboratory. The experiments performed in the laboratory involve highly volatile chemicals and the possibility of explosion is great. It is estimated that the worst explosion will generate a force of 5×10^6 N lasting 0.5 s. The structure is to be designed such that the maximum displacement due to such an explosion is 10 mm. The equivalent mass of the structure is 500,000 kg. Draw the shock spectrum for the structure subject to a rectangular pulse and determine the minimum allowable stiffness for the structure.

Solution:

The laboratory frame structure of Fig. 4.12 is modeled as an undamped one-degree-of-freedom system with $x(t)$ representing the displacement at the top of the columns. The excitation is modeled as a rectangular pulse of magnitude $F_0 = 5 \times 10^6$ N and duration $t_0 = 0.5$ s. The response of an undamped one-degree-of-freedom system to a rectangular pulse with zero initial conditions is calculated using superposition and Table 4.1 as

$$x(t) = \frac{F_0}{k}\{1 - \cos\omega_n t - u(t - t_0)[1 - \cos\omega_n(t - t_0)]\}$$

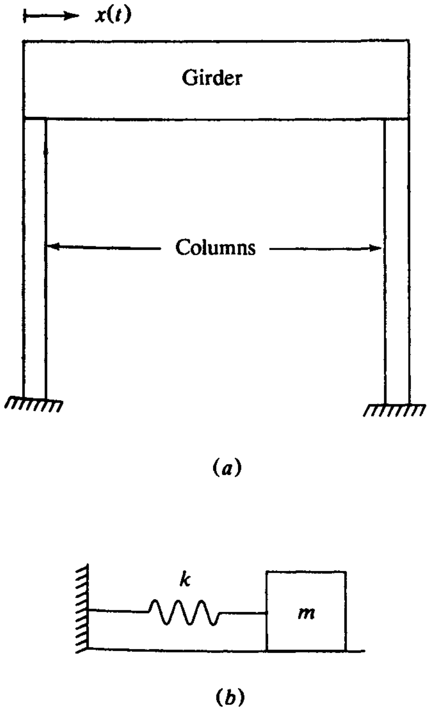


Figure 4.12 (a) The one-story chemical laboratory of Example 4.11 is modeled as a frame structure; (b) the frame structure is modeled as a one-degree-of-freedom mass-spring system, assuming the girder is very rigid compared to the columns.

For $t < t_0$ the nondimensional force ratio is

$$\frac{kx}{F_0} = 1 - \cos \omega_n t$$

The preceding function increases until $t = \pi/\omega_n$ when it reaches a maximum value of 2. If $t_0 < \pi/\omega_n$, the maximum nondimensional force ratio in this interval is

$$\frac{kx_{\max}}{F_0} = 1 - \cos \omega_n t_0$$

However, since the response is continuous, the maximum response for $t > \pi/\omega_n$ must be at least this large. For $t > t_0$ the nondimensional force ratio is

$$\frac{kx}{F_0} = \cos \omega_n (t - t_0) - \cos \omega_n t$$

Trigonometric identities are used on the above equation to obtain

$$\frac{kx}{F_0} = 2 \sin \frac{\omega_n t_0}{2} \sin (\omega_n t - \alpha)$$

where
$$\tan \alpha = \frac{\cos \omega_n t_0 - 1}{\sin \omega_n t_0}$$

Thus

$$\frac{kx_{\max}}{F_0} = 2 \sin \frac{\omega_n t_0}{2} \qquad t_0 < \frac{\pi}{\omega_n}$$

In summary,

$$\frac{kx_{\max}}{F_0} = \begin{cases} 2 \sin \dfrac{\omega_n t_0}{2} & t_0 < \dfrac{\pi}{\omega_n} \quad \left(\dfrac{t_0}{\tau} \le \dfrac{1}{2}\right) \\[2ex] 2 & t_0 > \dfrac{\pi}{\omega_n} \quad \left(\dfrac{t_0}{\tau} > \dfrac{1}{2}\right) \end{cases}$$

The shock spectrum is plotted in Fig. 4.13.

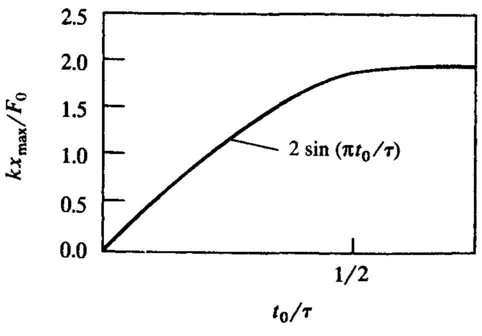


Figure 4.13 The shock spectrum for a rectangular pulse.

Returning to the specific problem, assume first that $\pi/\omega_n > 0.5$ s. The smallest stiffness is obtained by setting $x_{max} = 0.01$ m, leading to

$$1 \times 10^{-8}k = \sin\left(3.53 \times 10^{-4}\sqrt{k}\right)$$

The smallest solution of the preceding equation is $k = 5.33 \times 10^7$ N/m. Then

$$\frac{\pi}{\omega_n} = \pi\sqrt{\frac{m}{k}} = 0.304 \text{ s} < 0.5 \text{ s}$$

Hence there is no solution for $\pi/\omega_n > 0.5$ s.

Now let $\pi/\omega_n < 0.5$ s. Setting $x_{max} = 0.01$ m leads to

$$k = \frac{2F_0}{x_{max}} = \frac{2\,(5 \times 10^6 \text{ N})}{0.01 \text{ m}} = 1 \times 10^9 \frac{\text{N}}{\text{m}}$$

Then

$$\frac{\pi}{\omega_n} = 0.070 \text{ s} < 0.5 \text{ s}$$

Hence the maximum deflection will be less than 10 mm if the structure is designed such that $k > 1 \times 10^9$ N/m.

4.7 NUMERICAL METHODS

The convolution integral and Laplace transform methods are easy methods of solving Eq. (4.1) for any excitation. However, closed-form solutions using these methods are limited to cases where the forcing function has an explicit mathematical formulation and closed-form evaluation of the convolution integral is possible. In addition, there are explicitly defined forcing functions such as those proportional to nonintegral powers of time where a closed-form evaluation of the convolution integral or evaluation of the inverse Laplace transform is very difficult. When these situations occur, numerical methods must be used to obtain an approximate solution to the differential equation at discrete values of time.

Numerical solutions of forced one-degree-of-freedom vibrations problems are of two classes: numerical evaluation of the convolution integral and direct numerical evaluation of Eq. (4.1).

4.7.1 NUMERICAL EVALUATION OF CONVOLUTION INTEGRAL

Many numerical integration techniques are available for evaluation of integrals. Most numerical integration techniques use piecewise defined functions to interpolate the integrand. A closed-form integration of the interpolated integrand is performed. The method described here uses an interpolation for $F_{eq}(t)$ from which an approximation to the convolution integral is obtained. The discretization of a time interval and possible interpolations to $F_{eq}(t)$ are shown in Fig. 4.14.

Let $t_1, t_2, \ldots$ be values of time at which an approximate solution is to be obtained. Let $F_1(t), F_2(t), \ldots$ be the interpolating functions such that $F_k(t)$ interpolates $F_{eq}(t)$ on the interval $t_{k-1} < t < t_k$. Let x_k be the numerical approximation for $x\,(t_k)$. Also define

$$\Delta_j = t_j - t_{j-1}$$

The convolution integral is used to obtain the response of an underamped one-degree-of-freedom system as

$$x(t) = x(0)e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t + \int_0^t \frac{F_{eq}(\tau)}{m_{eq}\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \qquad [4.30]$$

The trigonometric identity for the sine of the difference of angles is used to rewrite Eq. (4.30) as

$$\begin{aligned} x(t) = {} & e^{-\zeta\omega_n t}\left[x(0)\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t\right] \\ & + \frac{1}{m_{eq}\omega_d}e^{-\zeta\omega_n t}\left[\sin\omega_d t\int_0^t F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau \right. \\ & \left. - \cos\omega_d t\int_0^t F_{eq}(\tau)e^{\zeta\omega_n\tau}\sin\omega_d\tau\,d\tau\right] \end{aligned} \qquad [4.31]$$

Define

$$G_{1j} = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)e^{\zeta\omega_n\tau}\cos\omega_d\tau\,d\tau \qquad [4.32]$$

and

$$G_{2j} = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)e^{\zeta\omega_n\tau}\sin\omega_d\tau\,d\tau \qquad [4.33]$$

Using the definitions in Eqs. (4.32) and (4.33) in Eq. (4.30) leads to

$$\begin{aligned} x_k = {} & e^{-\zeta\omega_n t_k}\left[x(0)\cos\omega_d t_k + \frac{\zeta\omega_n x(0) + \dot{x}(0)}{\omega_d}\sin\omega_d t_k\right] \\ & + \frac{1}{m_{eq}\omega_d}e^{-\zeta\omega_n t_k}\left[\sin\omega_d t_k\sum_{j=1}^{k}G_{1j} - \cos\omega_d t_k\sum_{j=1}^{k}G_{2j}\right] \end{aligned} \qquad [4.34]$$

Interpolating functions are chosen for $F_{eq}(t)$ such that Eqs. (4.32) and (4.33) have closed-form evaluations when the interpolating function is used in place of $F_{eq}(t)$. Then Eq. (4.34) is used to calculate approximations to the solution at discrete times.

First, consider the case where $F_{eq}(t)$ is interpolated by a series of impulses, as illustrated in Fig. 4.14*b*. During the interval between t_{j-1} and t_j, application of

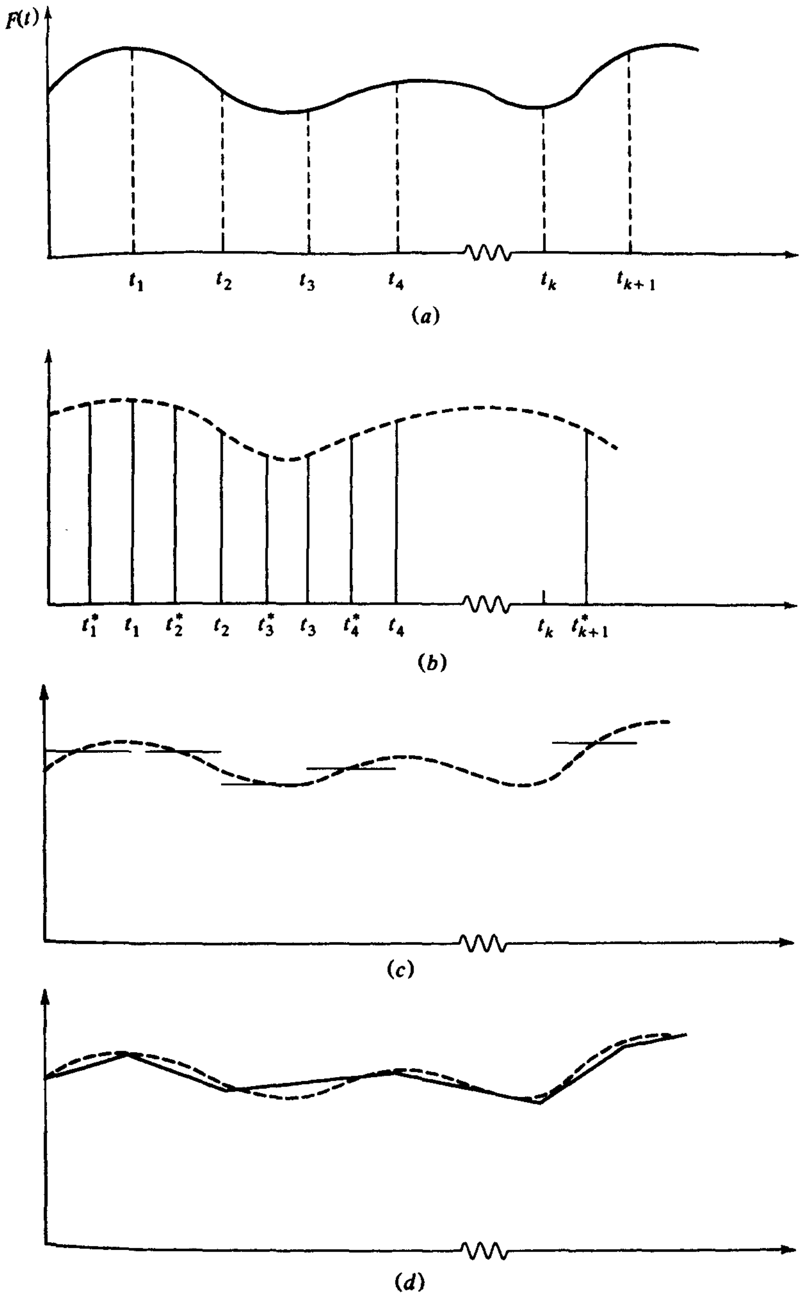


Figure 4.14 (*a*) Discretization of time for numerical integration of convolution integral; (*b*) $F(t)$ can be interpolated by impulses of magnitude $F(t_j^*)\Delta\, t_j$ applied at t_j^*, $t_{j-1} < t_j^* < t_j$, (*c*) Interpolation of $F(t)$ by piecewise constants. For $t_{j-1} < t < t_j$, $F(t) = F(t_j^*)$ for some t_j^*, $t_{j-1} < t_j^* < t_j$. (*d*) Interpolation of $F(t)$ by piecewise linear functions.

$F_{eq}(t)$ results in an impulse of magnitude

$$I_j = \int_{t_{j-1}}^{t_j} F_{eq}(\tau)\,d\tau$$

The mean value theorem of integral calculus implies that there exists a t_j^*, $t_{j-1} < t_j^* < t_j$, such that

$$I_j = F_{eq}\left(t_j^*\right)\Delta_j$$

For the sake of interpolation, approximate t_j^* by

$$t_j^* \approx \frac{t_j + t_{j-1}}{2}$$

Thus, on the interval $t_{j-1} < t < t_j$, $F(t)$ is interpolated by an impulse of magnitude I_j applied at the midpoint of the interval. With this choice of interpolation, Eqs. (4.32) and (4.33) are evaluated as

$$G_{1j} = F_{eq}\left(t_j^*\right)\Delta_j e^{\zeta\omega_n t_j^*}\cos\omega_d t_j^* \tag{4.35}$$

$$G_{2j} = F_{eq}\left(t_j^*\right)\Delta_j e^{\zeta\omega_n t_j^*}\sin\omega_d t_j^* \tag{4.36}$$

It is also possible to interpolate $F_{eq}(t)$ with piecewise constants. Over the interval from t_{j-1} to t_j, the interpolate for $F_{eq}(t)$ assumes the value of $F_{eq}(t)$ at the interval's midpoint, as illustrated in Fig. 4.14*c*. Call the value of the interpolate f_j. Then

$$G_{1j} = f_j C_j \tag{4.37}$$

$$G_{2j} = f_j D_j \tag{4.38}$$

where
$$C_j = \frac{1-\zeta^2}{\omega_d}\left[e^{\zeta\omega_n t_j}\left(\sin\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_j\right) - e^{\zeta\omega_n t_{j-1}}\left(\sin\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_{j-1}\right)\right] \tag{4.39}$$

$$D_j = \frac{1-\zeta^2}{\omega_d}\left[e^{\zeta\omega_n t_j}\left(-\cos\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_j\right) - e^{\zeta\omega_n t_{j-1}}\left(-\cos\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_{j-1}\right)\right] \tag{4.40}$$

Finally, consider the case where $F_{eq}(t)$ is interpolated linearly between t_{j-1} and t_j, as illustrated in Fig. 4.14*d*. Then if $g_j = f\left(t_j\right)$,

$$G_{1j} = \frac{1}{\Delta_j}\left[\left(g_j - g_{j-1}\right)A_j + \left(g_{j-1}t_j - g_j t_{j-1}\right)C_j\right] \tag{4.41}$$

$$G_{2j} = \frac{1}{\Delta_j}\left[\left(g_j - g_{j-1}\right)B_j + \left(g_{j-1}t_j - g_j t_{j-1}\right)D_j\right] \tag{4.42}$$

where C_j and D_j are given by Eqs. (4.39) and (4.40), respectively, and

$$A_j = \frac{1-\zeta^2}{\omega_d}\left[t_j e^{\zeta\omega_n t_j}\left(\sin\omega_d t_j + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_j\right)\right.$$
$$\left. - t_{j-1}e^{\zeta\omega_n t_{j-1}}\left(\sin\omega_d t_{j-1} + \frac{\zeta\omega_n}{\omega_d}\cos\omega_d t_{j-1}\right) - \left(D_j + \frac{\zeta\omega_n}{\omega_d}C_j\right)\right] \tag{4.43}$$

$$B_j = \frac{1-\zeta^2}{\omega_d}\left[t_j e^{\zeta\omega_n t_j}\left(\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_j - \cos\omega_d t_j\right)\right.$$
$$\left. - t_{j-1}e^{\zeta\omega_n t_{j-1}}\left(\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_{j-1} - \cos\omega_d t_{j-1}\right) + \left(C_j - \frac{\zeta\omega_n}{\omega_d}D_j\right)\right] \tag{4.44}$$

Other choices for interpolating functions for $F_{eq}(t)$ are possible. Higher-order piecewise polynomials may be used, as well as interpolates which require more smoothness at each t_j, such as splines. Any form of interpolating function can be chosen as long as Eqs. (4.32) and (4.33) have closed-form evaluations. However, the more complicated the interpolating function, the more algebra is involved in the evaluation of G_{1j} and G_{2j}. The numerical evaluation of the convolution integral also requires more computations for more complicated interpolating functions.

If $F_{eq}(t)$ is known empirically, any of the methods presented may be used to evaluate the convolution integral. If piecewise impulses or piecewise constants are used, the times where $F_{eq}(t)$ is known are taken as midpoints of the intervals. If piecewise linear interpolates are used, the times where $F_{eq}(t)$ is known are taken as the t_j's.

Error analysis of the preceding methods is beyond the scope of this text. Better accuracy for the response is, of course, obtained with better accuracy of the interpolate. Error analysis usually involves comparing the interpolation with a Taylor series expansion to estimate the error in the interpolation. The error is usually expressed as being the order of some power of Δ_j. Bounds on the error in using the convolution integral are obtained. Integration tends to smooth errors.

Determination of the response using these methods requires evaluation of the convolution integral at discrete values of time. Since errors are introduced in the evaluation of G_{1j} and G_{2j}, the more of these terms used in the evaluation, the larger is the error. Hence the error in approximation grows with increasing t. Reduction of error can be achieved by using smaller time intervals, if possible, or by using more accurate interpolates.

4.7.2 NUMERICAL SOLUTION OF EQ. (4.1)

An alternative to numerical evaluation of the convolution integral is to approximate the solution of Eq. (4.1) by direct numerical integration. Many methods are available for numerical solution of ordinary differential equations.

Since vibrations of discrete systems are governed by initial value problems, it is best to use a numerical method that is self-starting. That is, previous knowledge of the solution at only one time is required to start the procedure.

Best application of self-starting methods required the rewriting of an nth-order differential equation as n first-order differential equations. This is done for Eq. (4.1) by defining

$$y_1(t) = x(t) \tag{4.45a}$$

$$y_2(t) = \dot{x}(t) \tag{4.45b}$$

Thus

$$\dot{y}_1(t) = y_2(t) \tag{4.46a}$$

and from Eq. (4.1)

$$\dot{y}_2(t) = \frac{F_{\text{eq}}}{m_{\text{eq}}} - 2\zeta\omega_n y_2(t) - \omega_n^2 y_1(t) \tag{4.46b}$$

Equations (4.46a and 4.46b) are two simultaneous linear first-order ordinary differential equations whose numerical solution yields the values of displacement and velocity at discrete times.

In the following let t_i, $i = 1, 2, \ldots$, be the discrete times at which the solution is obtained and let $y_{1,i}$ and $y_{2,i}$ be the displacements and velocities at these times and define

$$\Delta_j = t_{j+1} - t_j$$

The recurrence relations for the simplest self-starting method, called the Euler method, are obtained from truncating Taylor series expansions for $y_{k,i+1}$ about $y_{k,i}$ after the linear terms. These recurrence relations are

$$y_{1,i+1} = y_{1,i} + (t_{i+1} - t_i)\, y_{2,i} \tag{4.47a}$$

$$y_{2,i+1} = y_{2,i} + (t_{i+1} - t_i)\left[\frac{F_{\text{eq}}(t_i)}{m_{\text{eq}}} - 2\zeta\omega_n y_{2,i} - \omega_n^2 y_{1,i}\right] \tag{4.47b}$$

Given initial values of y_1 and y_2, Eqs. (4.47a and 4.47b) are used to calculate recursively the displacement and velocity at increasing times. The Euler method is first-order accurate meaning that the error is of the order of Δ_j.

Runge-Kutta methods are more popular than the Euler method because of their better accuracy, while still being easy to use. A Runge-Kutta formula for the solution of the first-order differential equation

$$\dot{y} = f(y, t)$$

is of the form

$$y_{i+1} = y_i + \sum_{j=1}^{n} a_j k_j \tag{4.48}$$

where
$$k_1 = (t_{i+1} - t_i)\, f(y_i, t_i) \qquad [4.49a]$$
$$k_2 = (t_{i+1} - t_i)\, f\left(y_1 + q_{1,1}k_1, t_i + p_1\right) \qquad [4.49b]$$
$$k_3 = (t_{i+1} - t_i)\, f\left(y_i + q_{2,1}k_1 + q_{2,2}k_2, t_i + p_2\right) \qquad [4.49c]$$
$$\vdots$$
$$k_n = (t_{i+1} - t_i)\, f\left(y_i + q_{n-1,1}k_1 + q_{n-2,2}k_2 + \cdots + q_{n-1,n-1}k_{n-1}, t_i + p_{n-1}\right)$$

and the a, q, and p coefficients are chosen by using Taylor series expansions to approximate the differential equation to the desired accuracy.

The error for a fourth-order Runge-Kutta formula is proportional to Δ_j^4. A fourth-order Runge-Kutta formula is

$$y_{i+1} = y_i + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad [4.50]$$

where
$$\begin{aligned} k_1 &= (t_{i+1} - t_i)\, f(y_i, t_i) \\ k_2 &= (t_{i+1} - t_i)\, f\left(y_i + \tfrac{1}{2}k_1, \tfrac{1}{2}(t_i + t_{i+1})\right) \\ k_3 &= (t_{i+1} - t_i)\, f\left(y_i + \tfrac{1}{2}k_2, \tfrac{1}{2}(t_i + t_{i+1})\right) \\ k_4 &= (t_{i+1} - t_i)\, f(y_i + k_3, t_{i+1}) \end{aligned} \qquad [4.51]$$

Equation (4.50) can be used for higher-order differential equations by rewriting it as a system of first-order equations as has been done in Eq. (4.46) for a one-degree-of-freedom system. The result is

$$y_{1,i+1} = y_{1,i} + \tfrac{1}{6}\left(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}\right) \qquad [4.52a]$$
$$y_{2,i+1} = y_{2,i} + \tfrac{1}{6}\left(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}\right) \qquad [4.52b]$$

where
$$k_{1,1} = (t_{i+1} - t_i)\, y_{2,i} \qquad [4.53a]$$
$$k_{1,2} = (t_{i+1} - t_i)\left(y_{2,i} + \tfrac{1}{2}k_{2,1}\right) \qquad [4.53b]$$
$$k_{1,3} = (t_{i+1} - t_i)\left(y_{2,i} + \tfrac{1}{2}k_{2,2}\right) \qquad [4.53c]$$
$$k_{1,4} = (t_{i+1} - t_i)\left(y_{2,i} + k_{2,3}\right) \qquad [4.53d]$$
$$k_{2,1} = (t_{i+1} - t_i)\left[\frac{F_{eq}(t_i)}{m_{eq}} - 2\zeta\omega_n y_{2,i} - \omega_n^2 y_{1,i}\right] \qquad [4.53e]$$
$$k_{2,2} = (t_{i+1} - t_i)\left[\frac{F_{eq}\left(\frac{1}{2}(t_i + t_{i+1})\right)}{m_{eq}} - 2\zeta\omega_n\left(y_{2,i} + \tfrac{1}{2}k_{2,1}\right) - \omega_n^2\left(y_{1,i} + \tfrac{1}{2}k_{1,1}\right)\right] \qquad [4.53f]$$
$$k_{2,3} = (t_{i+1} - t_i)\left[\frac{F_{eq}\left(\frac{1}{2}(t_i + t_{i+1})\right)}{m_{eq}} - 2\zeta\omega_n\left(y_{2,i} + \tfrac{1}{2}k_{2,2}\right) - \omega_n^2\left(y_{1,i} + \tfrac{1}{2}k_{1,2}\right)\right] \qquad [4.53g]$$

$$k_{2,4} = (t_{i+1} - t_i)\left[\frac{F_{eq}(t_{i+1})}{m_{eq}} - 2\zeta\omega_n\left(y_{2,i} + k_{2,3}\right) - \omega_n^2\left(y_{1,i} + k_{1,3}\right)\right] \tag{4.53h}$$

The Runge-Kutta method is often used because it is easy to program for a digital computer. Its most restrictive limitation is that extension of the approximation between two discrete times requires evaluation of the excitation at an intermediate time. If the forcing function is known only at discrete times, evaluation at the appropriate intermediate times is often impossible. In addition, a large number of function evaluations can lead to large computer times.

Adams' formulas provide more accurate approximations of ordinary differential equations. An open Adams formula requires knowledge of the functions at the two previous time steps to calculate the approximation at the desired time. A closed Adams formula requires knowledge of the function at only the previous time step, but the formula involves the evaluation of the function at the time step of interest. Thus a closed Adams formula requires an iterative solution at each time step. The closed Adams formula is much more accurate than an open formula of the same order. The closed formula is self-starting, whereas the open formula is not self-starting.

A predictor-corrector method is a compromise that uses the closed formula for increased accuracy but uses the open formula to reduce computation time. The open formula is used to "predict" the solution at the desired time, then the closed formula is used to "correct" by using the predicted value as an initial guess. Iterations are not necessary as the first correction is very accurate. Since the open Adams formulas are not self-starting, a self-starting method such as the Runge-Kutta method of the same order is used to calculate the solution at the first time. The predictor-corrector method is used for the remainder of the calculations.

ple 4.12 A 200-kg milling machine is subject to the versed sine pulse of Fig. 4.15 during operation. The machine is mounted on an elastic foundation of stiffness 1×10^6 N/m and damping ratio 0.2. Write a MATLAB script that uses piecewise constants as interpolating functions to numerically integrate the convolution integral to obtain the response of the machine up to $t = 0.5$ s.

Solution:

The MATLAB script and the resulting plot of displacement are illustrated in Fig. 4.16. The MATLAB script is written in a general form. When the script is run by MATLAB, the user will be prompted for input. The form of the excitation is provided in a separate MATLAB m file. This file is included on the accompanying CD.

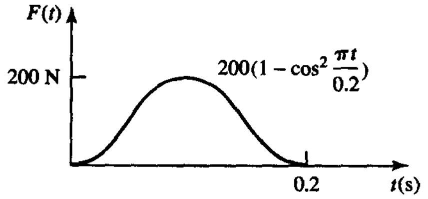
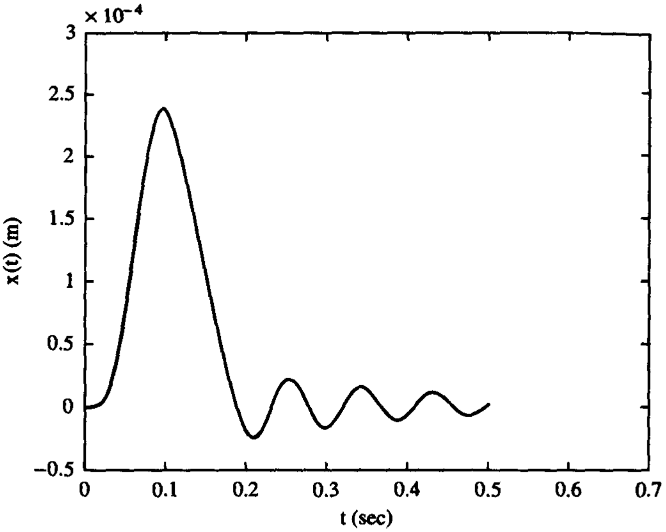


Figure 4.15 Versed sine pulse excitation of Examples 4.12 and 4.13.

```
% Example 4.12
% Numerical integration of convolution integral using
% piecewise constants to interpolate excitation
m=200;                              % Mass of system
k=1.*10^6;                          % Stiffness
zeta=0.06;                          % Damping ratio
omega_n=sqrt(k/m);                  % Natural frequency
omega_d=omega_n*sqrt(1-zeta^2);     % Damped natural frequency
F0=200;                             % Magnitude of pulse
t0=0.2;                             % Duration of pulse
x0=0;                               % Initial displacement
xdot0=0;                            % Initial velocity
t=linspace(0,.5,1001);              % Discretization of time scale
sum1=0;                             % Initialization of sum for G1
sum2=0;                             % Initialization of sum for G2
x(1)=x0;                            % Initialization of x
C1=(1-zeta^2)/omega_d;
C2=zeta*omega_n;
C3=C2/omega_d;
for k=2:1001
   % Calculating F(t)
   if t(k)<=t0
      F=F0*(1-(cos(pi*t(k)/t0)^2)); % F(t)
   else
      F=0;
   end
   % Numerical integration formula Eqs.(4.37)-(4.40)
   EK=exp(C2*t(k));
   EK1=exp(C2*t(k-1));
   SK=sin(omega_d*t(k));
   SK1=sin(omega_d*t(k-1));
   CK=cos(omega_d*t(k));
   CK1=cos(omega_d*t(k-1));
   G1=F*C1*(EK*(SK+C3*CK)-EK1*(SK1+C3*CK1));
   G2=F*C1*(EK*(-CK+C3*SK)-EK1*(-CK1+C3*SK1));
   sum1=sum1+G1;
   sum2=sum2+G2;
   % Eq.(4.34)
   xK=(x0*CK+(C2*x0+xdot0)/omega_d*SK)/EK;
   x(k)=xK+(SK*sum1-CK*sum2)/(EK*m*omega_d);
end
plot(t,x)
xlabel('t (sec)')
ylabel('x(t) (m)')
```

(a)

Figure 4.16 (a) MATLAB script for Example 4.12, numerical integration of convolution integral using piecewise constants for interpolation of excitation force.



(*b*)

Figure 4.16 (Con't) (*b*) plot of displacement versus time obtained by running the script.

ıple 4.13 Write a MATLAB script using the program ODE45 to determine the response of the system of Example 4.12.

Solution:

The MATLAB script for the development of the response is given in Fig. 4.17*a*. The script uses the MATLAB function ODE45, which uses a Runge-Kutta-Fehlberg method to numerically approximate the response.

The resulting response generated from MATLAB is shown in Fig. 4.17*b*. The response is very close to that generated in Example 4.12 by numerical integration of the convolution integral.

```
ıge-Kutta solution to Example 4.13 using MATLAB program ODE45
tial conditions

=0;
)=x;    y(2)=xdot
=x0;
=xdot0;
0(1);y0(2)];
=[0 0.5];
```

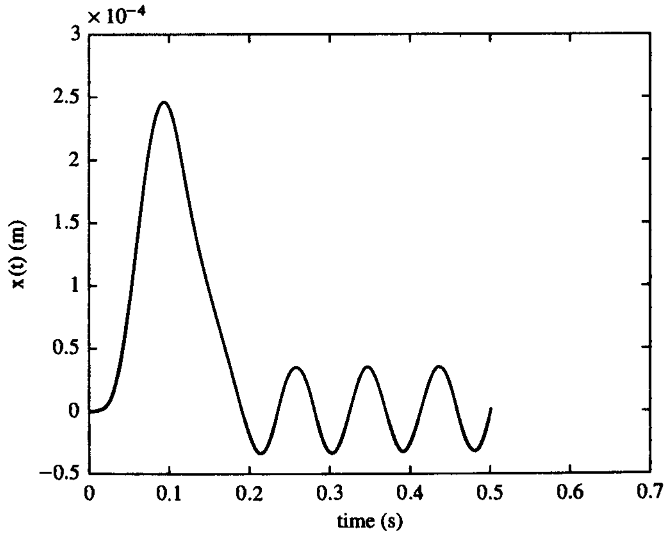
4.17 (a) MATLAB script for solving differential equation for Example 4.13 by using ODE45, a Runge–Kutta solution.

```
[T,Y]=ode45('fun412',TSPAN,y0);
plot(T,Y(:,1))
xlabel('time (s)')
ylabel('x(t) (m)')
```

(*a*)

```
% Defining file for function of Example 4.13
function F=fun412(T,Y)
m=200;                              % Mass of system
k=1.*10^6;                          % Stiffness
zeta=0.06;                          % Damping ratio
omega_n=sqrt(k/m);                  % Natural frequency
F0=200;                             % Magnitude of pulse
t0=0.2;                             % Duration of pulse
F(1)=Y(2);
% Calculating F(T)
if T<=t0
   f1=F0/m*(1-(cos(pi*T/t0))^2);
else
   f1=0;
end
% xdot=F(1), xddot=F(2)
F(2)=-2*zeta*Y(2)-omega_n^2*Y(1)+f1;
F=[F(1); F(2)];
```

(*b*)



(*c*)

Figure 4.17 (Con't) (a) Con't. (*b*) user-provided function for Example 4.13; (c) plot of displacement versus time obtained by running script.

PROBLEMS

4.1. Use the method of variation of parameters to obtain the general solution of Eq. (4.1) and show that it can be written in the form of the convolution integral, Eq. (4.11).

4.2. Use the convolution integral to determine the response of an underdamped one-degree-of-freedom system of mass m and natural frequency ω_n when the excitation is the unit step function, $u(t)$.

4.3. Let $g(t)$ be the response of an underdamped system to a unit step function and $h(t)$ the response of an underdamped system to a unit impulse function. Show

$$h(t) = \frac{dg}{dt}$$

4.4. Use the convolution integral and the notation and results of Prob. 4.3 to derive the following alternative expression for the response of a system subject to an excitation, $F(t)$:

$$x(t) = F(0)g(t) + \int_0^t \frac{dF(\tau)}{d\tau} g(t-\tau)\,d\tau$$

4.5. A one-degree-of-freedom undamped system is initially at rest in equilibrium and subject to a force $F(t) = F_0 t e^{-t/2}$. Use the convolution integral to determine the response of the system.

4.6. The mass of Fig. P4.6 has a velocity v when it engages the spring-dashpot mechanism. Let $x(t)$ be the displacement of the mass from the position where the mechanism is engaged. Use the convolution integral to determine $x(t)$. Assume the system is underdamped.

4.7. Use the convolution integral to determine the response of the system of Fig. P4.7.

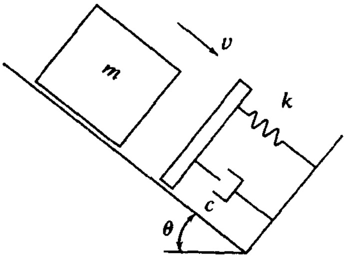


FIGURE P4.6

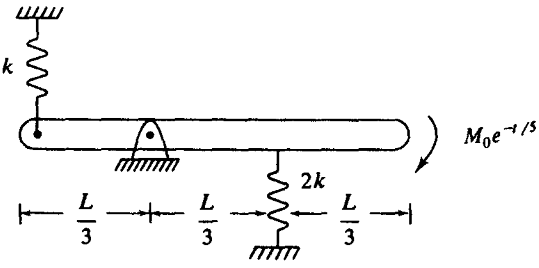


FIGURE P4.7

4.8. Use the convolution integral to determine the response of an underdamped one-degree-of-freedom system of natural frequency ω_n and damping ratio ζ when subject to a harmonic excitation $F(t) = F_0 \sin \omega t$.

4.9. A one-degree-of-freedom system of natural frequency ω_n and damping ratio $\zeta < 1$ is subject to a rectangular pulse of magnitude F_0 and duration t_0. Under what conditions will the maximum response occur after the pulse is removed?

4.10–4.16. A machine tool of mass 30 kg is mounted on an undamped foundation of stiffness 1500 N/m. During operation it is subject to one of the machining forces shown. Use the principle of superposition and the convolution integral to determine the response of the system to each force.

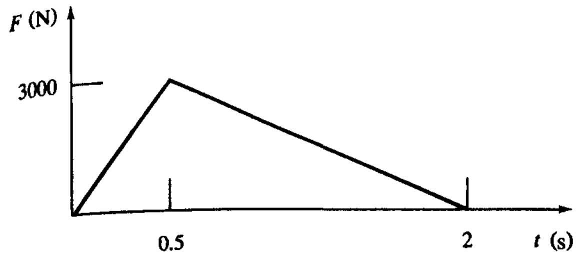


FIGURE P4.10

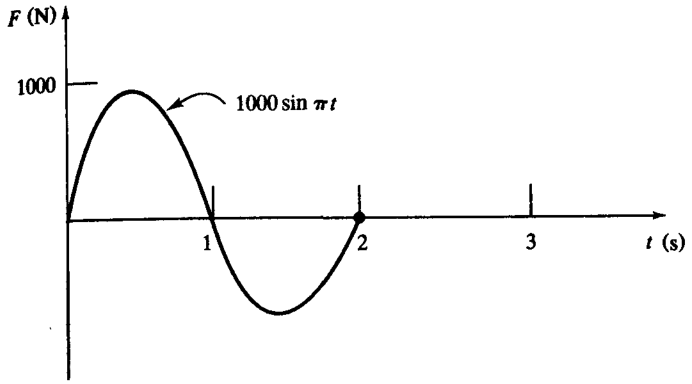


FIGURE P4.11

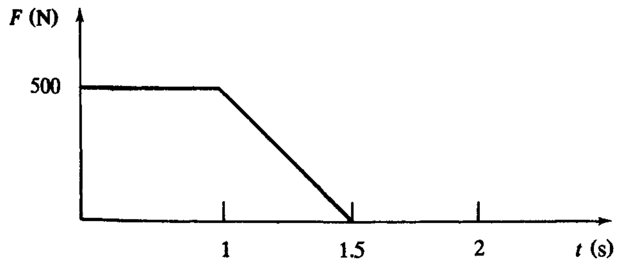


FIGURE P4.12

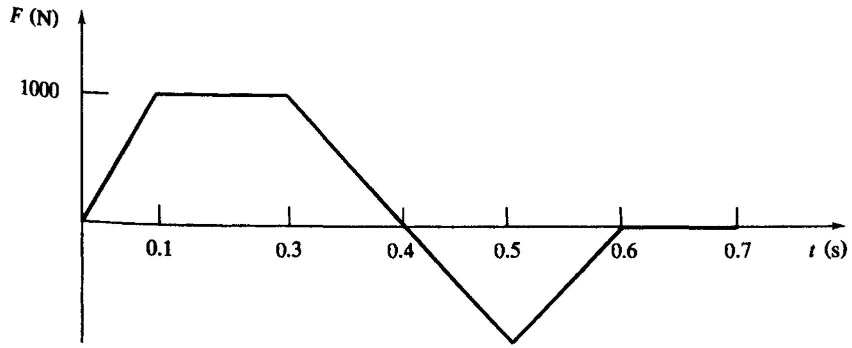


FIGURE P4.1

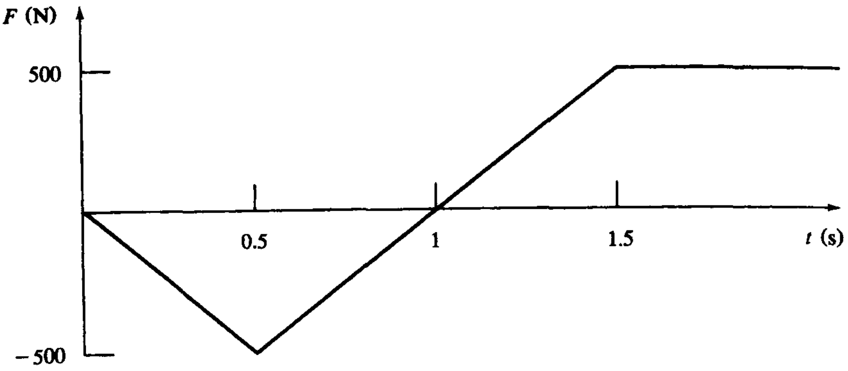


FIGURE P4.14

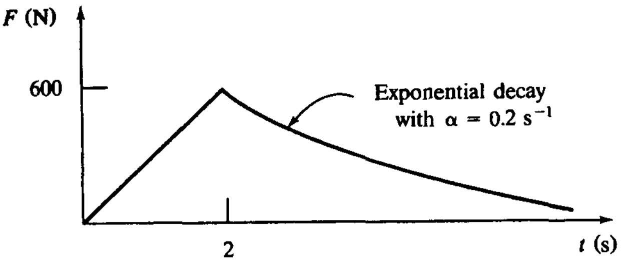


FIGURE P4.15

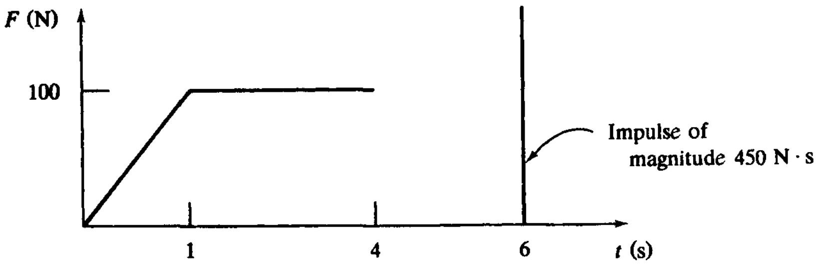
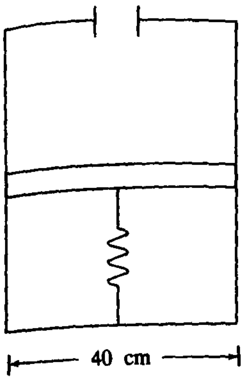


FIGURE P4.16

4.17. The force applied to the 120-kg anvil of a forge hammer during operation is approximated as a rectangular pulse of magnitude 2000 N for a duration of 0.3 s. The anvil is mounted on a foundation of stiffness 2000 N/m and damping ratio 0.4. What is the maximum displacement of the anvil?

4.18. The system of Fig. P4.18 is initially in equilibrium in a cylinder of compressed air at a gauge pressure of 300 kPa. A small hole is punctured in the casing, causing the gauge pressure to decrease exponentially. After 5 s the pressure drops to 200 kPa. Determine $x(t)$, the displacement of the piston from its initial position.



$m_p = 5$ kg

$k = 5 \times 10^5$ N/m

$P(t = 0) = 300$ kPa

$P(t = 5\text{ s}) = 200$ kPa

FIGURE P4.18

4.19. A one-story frame structure houses a chemical laboratory. Figure P4.19 shows the results of a model test to predict the transient force to which the structure would be subject if an explosion would occur. The equivalent mass of the structure is 2000 kg and its equivalent stiffness is 5×10^6 N/m. Approximate the maximum displacement of the structure due to this blast.

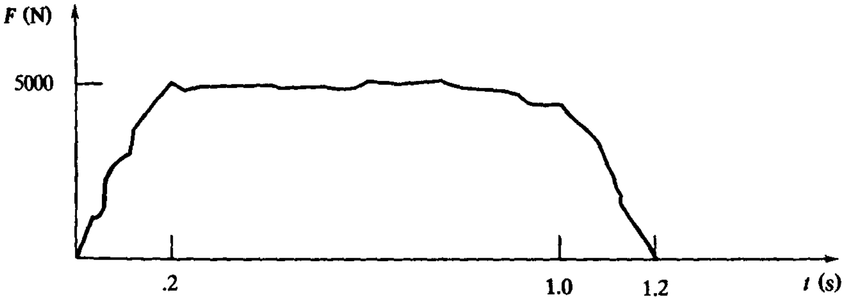


FIGURE P4.19

4.20–4.23. A 500-kg vehicle is traveling at 45 m/s when it encounters a pothole or bump in the road, as illustrated. The deflection of the suspension spring under the weight of the vehicle is 7.2 mm. The shock absorber has a damping ratio of 0.3. Using a one-degree-of-freedom system to model the vehicle and its suspension system, approximate the maximum vertical displacement of the vehicle, assuming it maintains its horizontal speed as it encounters the pothole or bump.

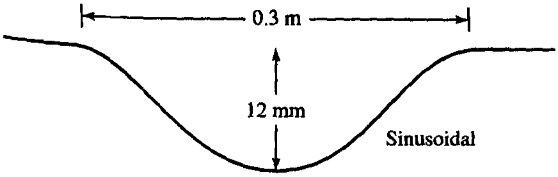


FIGURE P4.20

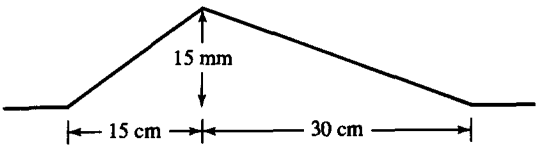


FIGURE P4.21

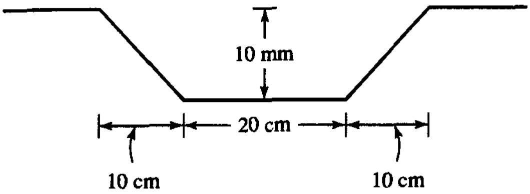


FIGURE P4.22

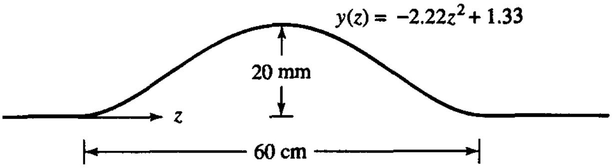


FIGURE P4.23

4.24. A 20-kg radio set is mounted in a ship on an undamped foundation of stiffness 1000 N/m. The ship is loosely tied to a dock. During a storm the ship experiences the displacement of Fig. P4.24. Determine the maximum acceleration of the radio.

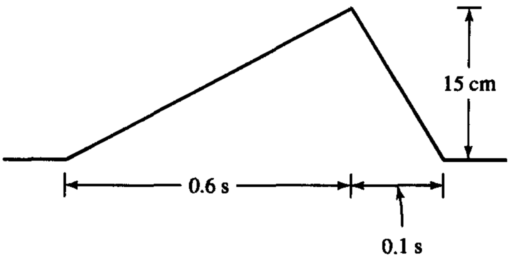


FIGURE P4.24

4.25. A personal computer of mass m is packed inside a box such that the stiffness and damping coefficient of the packing material are k and c respectively. The package is accidentally dropped from a height h and lands on a hard surface without rebound. Set up the convolution integral whose evaluation leads to the displacement of the computer relative to the package.

4.26. Use the Laplace transform method to determine the response of a system at rest in equilibrium when subject to

$$F(t) = F_0 \cos \omega t \,[1 - u\,(t - t_0)]$$

for (*a*) $\zeta = 0$, (*b*) $0 < \zeta < 1$, (*c*) $\zeta = 1$, (*d*) $\zeta > 1$.

4.27. Use the Laplace transform method to determine the response of an undamped one-degree-of-freedom system initially at rest in equilibrium when subject to a symmetric triangular pulse of magnitude F_0 and total duration t_0.

4.28. Use the Laplace transform method to determine the response of an underdamped one-degree-of-freedom system to a rectangular pulse of magnitude F_0 and time t_0.

4.29. Use the Laplace transform method to determine the steady-state response of an underdamped mass-spring system due to the periodic force of Fig. P4.29.

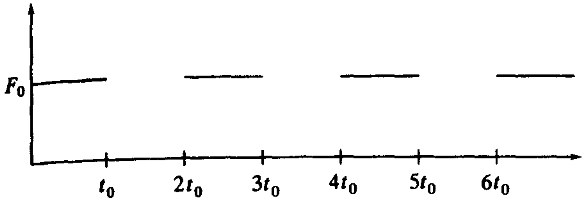


FIGURE P4.29

4.30. Use the Laplace transform method to derive the response of a one-degree-of-freedom system initially at rest in equilibrium when subject to a harmonic force $F_0 \sin \omega t$, when (*a*) $\omega \neq \omega_n$, (*b*) $\omega = \omega_n$.

4.31. Repeat Example 4.10 if the force is modeled as a triangular pulse of magnitude 5×10^6 N and total duration 0.5 s.

4.32. Determine the response spectrum for a rectangular velocity pulse.

4.33. Determine the response spectrum for a rectangular acceleration pulse.

4.34. Derive Eq. (4.35).

4.35. Derive Eq. (4.42).

MATLAB PROBLEMS

M4.1. File VIBES_4A.m provides the convolution integral response of an undamped one-degree-of-freedom system due to an arbitrary excitation. The convolution integral is evaluated symbolically. The excitation is provided by the user in the script file v4A.m. Use VIBES_4A.m to determine the response of an undamped system initially at rest in equilibrium due to

(*a*) $F(t) = e^{-t/5}$

(*b*) $F(t) = te^{-t/5}$

(*c*) $F(t) = 0.01 \sin(2t + 0.1)$

(*d*) $F(t) = 0.01e^{-t/10} \sin(2t + 0.01)$

M4.2. File VIBES_4B.m provides the convolution integral solution for an undamped one-degree-of-freedom system to the excitation of Fig. PM4.2. Use VIBES_4B.m to determine the response spectrum for this excitation.

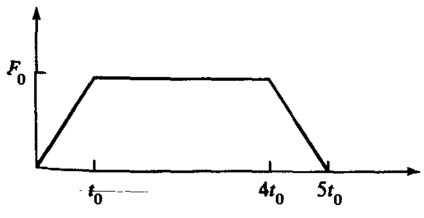


FIGURE PM4.2

M4.3. File VIBES_4C.m provides the Laplace transform solution for a one-degree-of-freedom system due to an arbitrary excitation. The excitation is provided in the script file v4A.m. Use VIBES_4C.m to determine the response of a system when subject to the excitations of Prob. M4.1.

File VIBES_4D.m is the file which is used in Example 4.12, providing the numerical integration of the convolution integral using piecewise constants. The excitation is provided to the program through a user-defined script file, v4D.m. Use VIBES_4D.m to solve the Prob. M4.4 to M4.9.

M4.4. A 100-kg punch press is mounted on an elastic foundation of stiffness 10,000 N/m and damping ratio 0.3. During operation the press is subject to the excitation of Fig. PM4.4. Determine the maximum response of the press during operation.

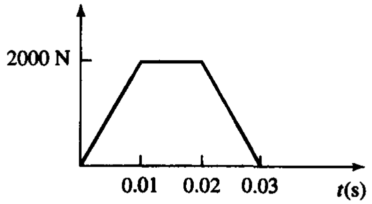


FIGURE PM4.4

M4.5. Determine the time-dependent response of the system of Fig. PM4.5.

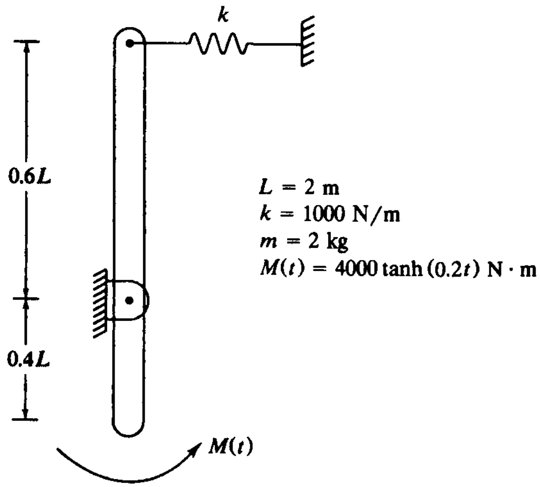


FIGURE PM4.5

M4.6. Determine the time dependent response of a 100-kg milling machine mounted on an elastic foundation of stiffness 30,000 N · m and damping ratio 0.3 subject to the force $F(t) = 250e^{t^2}$ N during operation. Determine the time-dependent response of the machine.

M4.7. A 300-kg reciprocating machine has a rotating unbalance of 1.5 kg · m. The machine is mounted on an isolator of stiffness 3×10^6 N/m and damping ratio 0.07. The machine has a steady-state operating speed of 2000 rpm. Once the machine begins to operate, it takes 1.8 s to reach this operating speed. Assuming the operating speed increases linearly to its final value, use VIBES_4D to determine the response of the machine until a steady-state is reached.

M4.8. A 2000-kg vehicle is traveling along a smooth road at 60 km/h when it encounters a bump whose contour is given in Fig. PM4.8. The driver begins to decelerate linearly when the vehicle first encounters the bump, reaching a speed of 40 km/h in 1 s. Model the vehicle and its suspension as a one-degree-of-freedom system with a stiffness of 4×10^6 N/m and a damping ratio of 0.16. Use VIBES_4D to determine the response of the system.

M4.9–M4.12. File VIBES_4E.m is used to develop the response spectrum of an undamped one-degree-of-freedom system due to an excitation of characteristic time t_0. VIBES_4E uses MATLAB program ODE45 for a Runge-Kutta solution of Eq. (4.1) and requires a user supplied script file V4E.m. Use VIBES_4E to solve Example 4.11 assuming the blast is modeled by the pulse shown with $F_0 = 5 \times 10^6$ N and $t_0 = 0.5$ s.

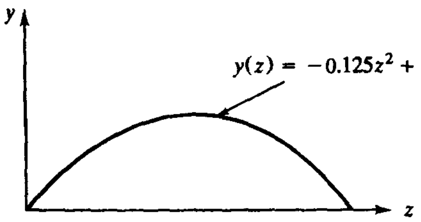


FIGURE PM4.8

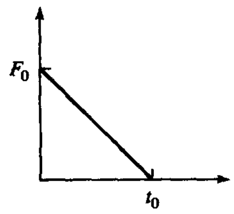


FIGURE PM4.9

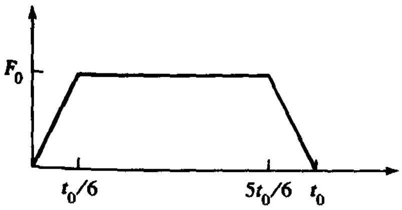


FIGURE PM4.10

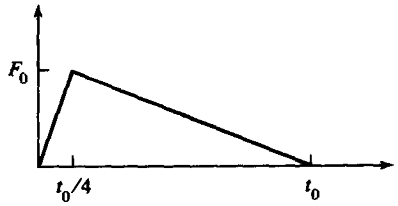


FIGURE PM4.11

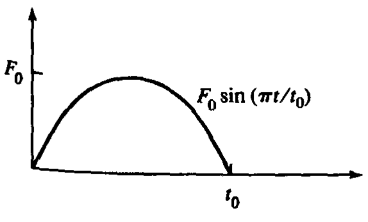


FIGURE PM4.12

M4.13. Write a MATLAB script file that uses symbolic computation to determine the response of an underdamped one-degree-of-freedom system to an excitation of the form $F(t) = F_0 t e^{-\alpha t}$.

M4.14. Write a MATLAB script file that uses symbolic computation to determine the response of an undamped one-degree-of-freedom system to an excitation of the form

$$F(t) = F_0 \left(1 - \frac{t}{t_0}\right)[1 - u(t - t_0)]$$

Note that MATLAB uses sym('Heaviside(t-a)') to symbolically represent $u(t - a)$.

M4.15. Write a MATLAB script file that uses the Laplace transform method to determine the response of an undamped one-degree-of-freedom system due to the excitation of Prob. M4.14.

M4.16. Write a MATLAB script file using the symbolic differential equation solver dsolve to determine the response of an undamped one-degree-of-freedom system with the excitation $F(t) = F_0 e^{-\alpha t}$.

M4.17. Write a MATLAB script file that uses VIBES_4D that will determine the absolute acceleration of a body connected through an elastic foundation to a movable support. Use the program to determine and plot the absolute acceleration as a function of time for Prob. M4.8.

M4.18. If $x(t)$ is the displacement of a machine on an elastic foundation, the force transmitted from the system to the fixed foundation is $F_T = kx + c\dot{x}$. Write a MATLAB script file that uses VIBES_4D that will determine F_T once $x(t)$ has been calculated. Use the program to determine the maximum value of the transmitted forces for the vibrations in Probs. M4.4, M4.6, and M4.7.

M4.19. In Chap. 11 it is suggested that the best isolator design for a system with a short-duration pulse ($t_0 \ll 2\pi/\omega_n$) is for $\zeta = 0.4$. This minimizes the maximum displacement for a fixed value of the maximum value of the transmitted force. If an isolator of damping ratio 0.4 is used, the stiffness is obtained from $\omega_n = 0.88 I F_{T_{\max}}$, where I is the total impulse imparted to the system by the pulse. Use the MATLAB program in Prob. M4.18 to determine the maximum transmitted force and maximum displacement for the pulse of Fig. PM4.9 with $F_0 = 2000$ N and $t_0 = 0.5$ s for $\zeta = 0.1, 0.2, 0.3$, and 0.4 and $t_0\omega_n/2\pi = 0.05, 0.1, 0.15, 0.3$, and 0.5. Use the table of results to comment on the proposed hypothesis.

M4.20. Write a MATLAB script to numerically integrate the convolution integral using piecewise linear interpolates for the excitation. Use the program to compare the approximation obtained by piecewise linear interpolates and piecewise constant interpolates for the excitation of Example 4.12. Compare the accuracy as the time interval between approximations decreases.

M4.21. Write a MATLAB script program to numerically integrate the differential equation governing the motion of a damped one-degree-of-freedom system due to an arbitrary excitation, using the MATLAB function ODE45.

M4.22. Use the MATLAB program of Prob. M4.21 to solve Prob. M4.6.

M4.23. Use the MATLAB program of Prob. M4.21 to solve Prob. M4.7.

M4.24. Use the MATLAB program of Prob. M4.21 to solve Prob. M4.8.

M4.25. A performance test is run on a model of a rocket. The model is connected to an elastic medium of stiffness k and damping coefficient c. The rocket is fired and thrust is developed as exhaust gases leave the nozzle. The thrust is $\dot{m}_e v_e$ where $\dot{m}_e = \rho_e v_e A_e$

is the mass flow rate of the exhaust gases, v_e is the exit velocity, ρ_e is the density of the exit gas, and A_e is the exit area of the nozzle. The velocity of the exhaust gases are predicted in the following table:

t (s)	v_e (m/s)	t (s)	v_e (m/s)	t (s)	v_e (m/s)
0	0	0.35	125.0	0.70	150
0.05	10.0	0.40	144.0	0.75	120
0.10	22.0	0.45	160.0	0.80	95
0.15	40.0	0.50	169.0	0.85	70.0
0.20	61.0	0.55	174.0	0.90	50.0
0.25	89.0	0.60	180.0	0.85	20.0
0.30	100.0	0.65	180.0	1.00	0.0

Burnout is achieved at 1.0 s. The total mass of the rocket and fuel is 3000 kg, the spring stiffness is 40,000 N/m, the damping coefficient is 1000 N · s/m, and $\rho_e A_e = 0.41$ kg/m. The governing differential equation is

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = \dot{m}_e v_e$$

Write a MATLAB script using ODE45 to predict the response of the system. Assume the mass remains constant at 3000 kg and use a linear interpolation to predict the velocity of the exhaust gases between given values.

M4.26. The mass of the rocket of Prob. M4.25 is actually a function of time given by

$$M(t) = M_0 - \int_0^t \dot{m}_e\, dt$$

where $M_0 = 3000$ kg. Modify the Runge-Kutta method to account for variable mass. Use a numerical integration scheme to calculate the mass as a function of time. Modify the program of Prob. M4.25 to account for the change in mass.

chapter

5

Multi-Degree-of-Freedom Systems: Derivation of Governing Equations

5.1 Introduction

The analysis of the vibrations of a multi-degree-of-freedom system is significantly more difficult and time-consuming than the analysis of the vibrations of a one-degree-of-freedom system. Recall from Chap. 1 that the number of degrees of freedom necessary for the analysis of vibrations of a system is the number of kinematically independent coordinates necessary to specify the motion of every particle contained in the system. The number of degrees of freedom of many mechanical systems is easy to identify. The motion of the rigid bar in Fig. 5.1*a* is analyzed using one degree of freedom. However, when the pin support is removed, as shown in Fig. 5.1*b*, the motion must be analyzed using two degrees of freedom. This is because the kinematic relationship, $x_a = L\theta$, obtained using the small-angle assumption is no longer valid and x_a and θ are kinematically independent. If the cable in the mass-pulley system of Fig. 5.2 is flexible, then the motion of the system is analyzed using two degrees of freedom. The no-slip assumption between the cable and the pulley is still used but kinetics, not kinematics, provides the relationship between the displacement of the mass and the angular displacement of the pulley.

Modeling a structural system with a finite number of degrees of freedom provides approximations to the behavior of the system. All structural systems are continuous systems with an infinite number of degrees of freedom. The continuous change in internal forces and moments across the length of a beam prevents a kinematic relationship between the displacement of any two particles on the beam's neutral axis. However, the analysis of continuous systems requires the solution of partial differential equations. The analysis of a multi-degree-of-freedom system is easier. Modeling of a structural element using a one-degree-of-freedom system was introduced in Chap. 1. The one-degree-of-freedom model provides only an approximation to the lowest natural frequency, whereas a continuous system possesses an

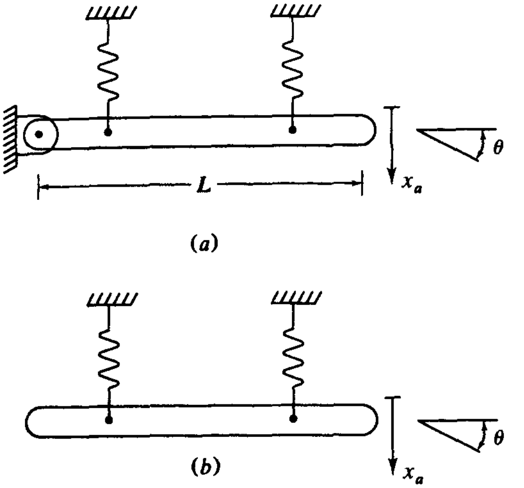


Figure 5.1 (a) One-degree-of-freedom system, $x_a \approx L\theta$; (b) when pin support is removed, system has two degrees of freedom. No kinemotic relotion exists between x_a ond θ.

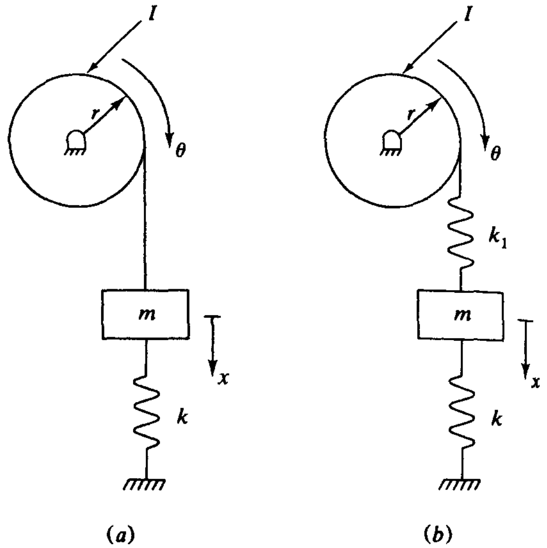


Figure 5.2 (a) If there is no slip between cable ond pulley, $x = r\theta$ ond system is modeled using one degree of freedom; (b) if coble is elastic, no kinematic relotion exists between x and θ.

infinite but countable number of free vibration modes. The use of a multi-degree-of-freedom model yields a better approximation for the lowest natural frequency and also provides approximations for higher frequencies.

This chapter deals with the derivation of the differential equations governing the vibrations of multi-degree-of-freedom systems. Chapter 6 is concerned with the free-vibration response of a multi-degree-of-freedom system, while Chap. 7 is concerned with the forced-vibration response of a multi-degree-of-freedom system.

Two methods of deriving the differential equations are presented: application of Newton's laws to free-body diagrams and energy methods. The application of Newton's laws to free-body diagrams is straightforward, similar to their application for one-degree-of-freedom systems. However, their application can be tedious and the resulting equations may have to be manipulated to be put into a usable form. Energy methods are based on the use of Lagrange's equation, a general equation derived from energy methods that is used to formulate differential equations for possibly nonlinear systems. A complete understanding of Lagrange's equation and its full use requires knowledge of the calculus of variations and is beyond the scope of this book.

Differential equations for linear systems are summarized in a matrix form. Lagrange's equations are used to show how knowledge of the quadratic forms of potential and kinetic energies for a linear system are used to define the elements of the stiffness and mass matrices. Influence coefficients are developed as an alternative to formally calculating the potential energy.

The formulation of the differential equations for discrete models of structural systems is handled differently. The inverse of the stiffness matrix, called the flexibility matrix, is calculated and approximations are used for the mass matrix.

5.2 DERIVATION OF DIFFERENTIAL EQUATIONS BY USING BASIC PRINCIPLES OF DYNAMICS

Governing differential equations for multi-degree-of-freedom mechanical systems can be derived by applying the basic principles of rigid-body dynamics to the appropriate free-body diagrams The method of application is very similar to that presented in Chap. 2 for one-degree-of-freedom systems. The method used in this textbook requires two free-body diagrams drawn for each rigid body or system of rigid bodies. One free-body diagram shows the external forces acting on the rigid body and a separate free-body diagram shows the effective forces acting on the body. The effective forces are equivalent to a force equal to the mass times the acceleration of the mass center of the rigid body placed at its mass center, and a couple equal to the centroidal mass moment of inertia of the body times the angular acceleration of the body. Conservation laws are then applied to the free body-diagrams. The following examples illustrate this procedure.

The blocks in Fig. 5.3 slide on a frictionless surface. Derive the differential equations governing free vibrations using x_1, x_2, and x_3 as generalized coordinates.

Solution:

The following form of Newton's law:

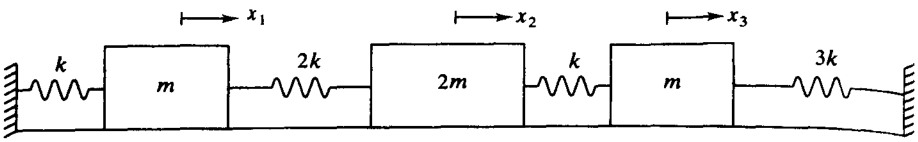
$$\sum \overset{+}{\vec{F}}_{\text{eff}} = \sum \overset{+}{\vec{F}}_{\text{ext}}$$

is applied to the free-body diagrams of Fig. 5.3*b*. The resulting differential equations are

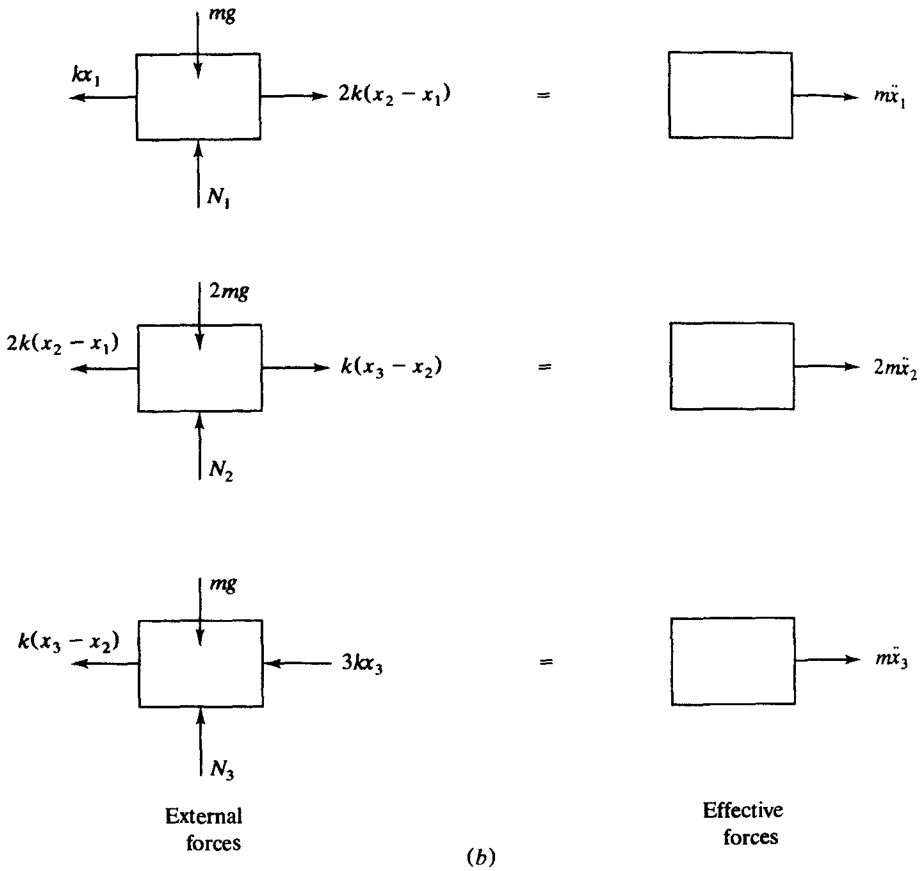
$$-kx_1 + 2k(x_2 - x_1) = m\ddot{x}_1$$
$$-2k(x_2 - x_1) + k(x_3 - x_2) = 2m\ddot{x}_2$$
$$-k(x_3 - x_2) - 3kx_3 = m\ddot{x}_3$$

The preceding equations are rearranged and written in matrix form as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



(a)



(b)

Figure 5.3 (a) System of Example 5.1; (b) free-body diagrams of each block at an arbitrary instant.

5.2 Consider the uniform slender rod of Fig. 5.4. Assume the springs remain vertical, the bar is in equilibrium when it is horizontal, and the small-angle assumption applies. Using x, the displacement of the mass center of the bar, and θ, the clockwise angular displacement of the bar form equilibrium, as the generalized coordinates, derive the differential equations governing the forced vibrations of this two-degree-of-freedom system.

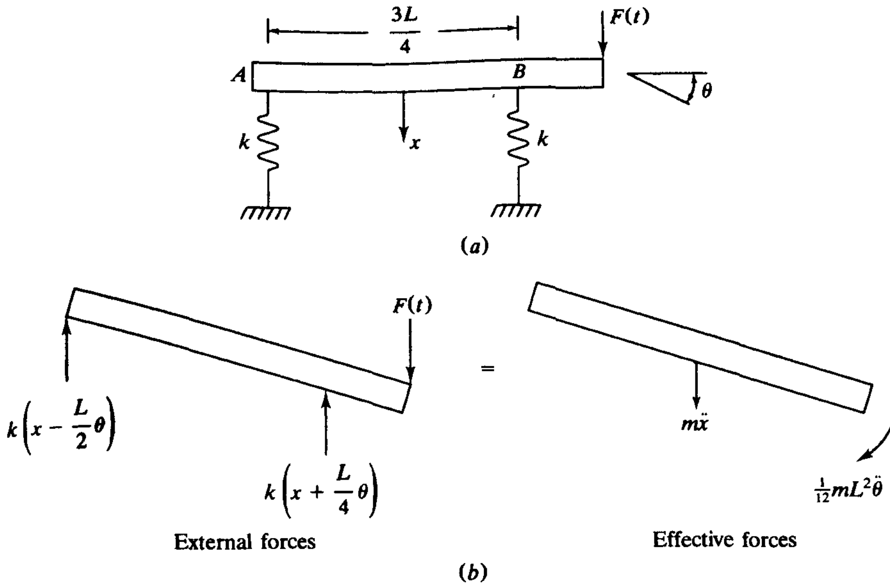


Figure 5.4 (*a*) System of Example 5.2; x and θ are chosen as generalized coordinates; (*b*) free-body diagrams at an arbitrary instant, assuming small θ.

Solution:

Free-body diagrams of the bar are shown in Fig. 5.4*b* where θ is exaggerated for illustration. The differential equations are derived by using summation of moments using the small-angle assumption

$$\sum \overset{+}{\overleftarrow{M}}_{A_{\text{ext}}} = \sum \overset{+}{\overleftarrow{M}}_{A_{\text{eff}}}$$

$$F(t)L - k\left(x + \frac{L}{4}\theta\right)\frac{3L}{4} = m\ddot{x}\frac{L}{2} + \frac{1}{12}mL^2\ddot{\theta}$$

$$\sum \overset{+}{\overleftarrow{M}}_{B_{\text{ext}}} = \sum \overset{+}{\overleftarrow{M}}_{B_{\text{eff}}}$$

$$F(t)\frac{L}{4} + k\left(x - \frac{L}{2}\theta\right)\frac{3L}{4} = -m\ddot{x}\frac{L}{4} + \frac{1}{12}mL^2\ddot{\theta}$$

Rearranging and writing the equations in matrix form gives

$$\begin{bmatrix} m\frac{L}{2} & m\frac{L^2}{12} \\ -m\frac{L}{4} & m\frac{L^2}{12} \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 3k\frac{L}{4} & 3k\frac{L^2}{16} \\ -3k\frac{L}{4} & 3k\frac{L^2}{8} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} LF(t) \\ \frac{L}{4}F(t) \end{bmatrix}$$

5.3 The block of Fig. 5.5 slides on a frictionless surface. The slender rod is pinned at the center of the block. Using x, the displacement of the block from equilibrium, and θ, the angular displacement of the bar from the vertical, as generalized coordinates, derive the nonlinear differential equations governing the motion of this system. Linearize the differential equations using the small-angle assumption.

Solution:

Free-body diagrams of the block-rod assembly and the rod alone are shown in Fig. 5.5*b*. A free-body diagram of the block alone is not analyzed as it includes the unknown pin reactions. These reactions must be known in terms of the generalized coordinates to apply Newton's law to the block. Application of

$$\overset{+}{\sum} \vec{F}_{\text{ext}} = \overset{+}{\sum} \vec{F}_{\text{eff}}$$

to the block-rod assembly yields

$$F(t) - kx = 2m\ddot{x} + m\ddot{x} + m\frac{L}{2}\ddot{\theta}\cos\theta - m\frac{L}{2}\dot{\theta}^2\sin\theta$$

Application of

$$\overset{+}{\sum} \vec{M}_{O_{\text{ext}}} = \overset{+}{\sum} \vec{M}_{O_{\text{eff}}}$$

to the free-body diagram of the rod alone yields

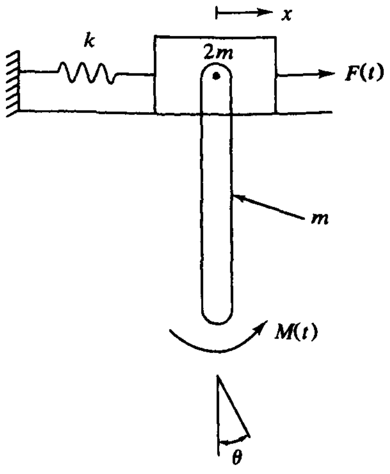
$$M(t) - mg\ \sin\theta\left(\frac{L}{2}\right) = m\frac{L}{2}\ddot{\theta}\left(\frac{L}{2}\right) + m\frac{L^2}{12}\ddot{\theta} + m\ddot{x}\cos\theta\left(\frac{L}{2}\right)$$

These differential equations are nonlinear because of the presence of the $\sin\theta$, $\cos\theta$, and $\dot{\theta}^2$ terms. The small-angle assumption is used to linearize the differential equations with $\cos\theta \approx 1$, $\sin\theta \approx \theta$, and neglecting $\dot{\theta}^2$ in comparison to linear terms. The resulting equations are written in matrix form as

$$\begin{bmatrix} 3m & m\frac{L}{2} \\ m\frac{L}{4} & m\frac{L^2}{3} \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mg\frac{L}{2} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F(t) \\ M(t) \end{bmatrix}$$

5.3 LAGRANGE'S EQUATIONS

Energy methods are often more useful than application of Newton's laws for deriving differential equations governing vibrations of multi-degree-of-freedom systems. A full understanding of the most convenient energy method first requires knowledge of the calculus of variations. Thus the method and examples of its application are presented without theory.



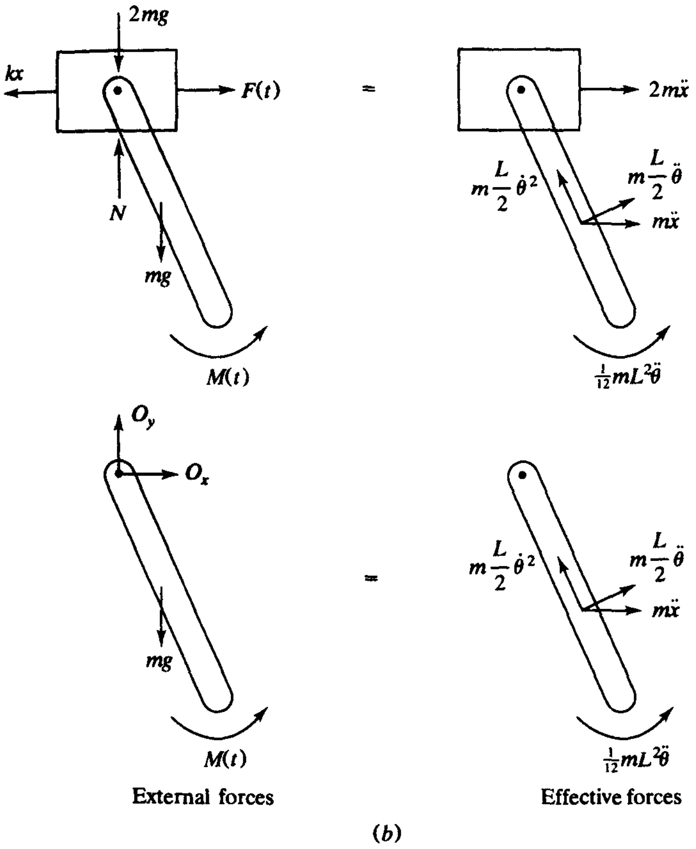


Figure 5.5 (a) System of Example 5.3; (b) free-body diagrams at an arbitrary instant.

The lagrangian of a dynamic system is defined as the difference between its kinetic and potential energy at an arbitrary instant

$$L = T - V \tag{5.1}$$

The lagrangian is a function of the generalized coordinates and their time derivatives

$$L = L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \tag{5.2}$$

The lagrangian is treated as a function of $2n$ independent variables; the time derivatives of the generalized coordinates are viewed as being independent of the generalized coordinates.

For a conservative system, the dot product of Newton's laws is taken with a virtual displacement vector defined by one of the generalized coordinates. The resulting energy equation can be manipulated to yield Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n \tag{5.3}$$

Equation (5.3) is applied to derive n differential equations for an n-degree-of-freedom conservative system. Lagrange's equations can be used to derive differential equations for linear and nonlinear systems.

.4 **Use** Lagrange's equations to derive the differential equations governing the motion of the system of Example 5.1 using x_1, x_2, and x_3 as generalized coordinates.

Solution:

The kinetic energy of the system at an arbitrary instant is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}2m\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2$$

The potential energy of the system at an arbitrary instant is

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}2k(x_2 - x_1)^2 + \tfrac{1}{2}k(x_3 - x_2)^2 + \tfrac{1}{2}3kx_3^2$$

The lagrangian is

$$L = \tfrac{1}{2}\left[m\dot{x}_1^2 + 2m\dot{x}_2^2 + m\dot{x}_3^2 - kx_1^2 - 2k(x_2 - x_1)^2 - k(x_3 - x_2)^2 - 3kx_3^2\right]$$

Application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}(m\dot{x}_1) - [-kx_1 - 2k(x_2 - x_1)(-1)] = 0$$

$$m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}(2m\dot{x}_2) - [-2k(x_2 - x_1) - k(x_3 - x_2)(-1)] = 0$$

$$2m\ddot{x}_2 - 2kx_1 + 3kx_2 - kx_3 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0$$

$$\frac{d}{dt}(m\dot{x}_3) - [-k(x_3 - x_2) - 3kx_3] = 0$$

$$m\ddot{x}_3 - kx_2 + 4kx_3 = 0$$

The differential equations derived from Lagrange's equations are identical to those obtained in Example 5.1 by the free-body diagram method.

Example

Use Lagrange's equations to derive the differential equations for the system of Example 5.3 if $M(t) = 0$ and $F(t) = 0$.

Solution:

The only external forces acting on the system are the spring force, gravity, and the normal force between the block and the surface on which it slides. All external forces are conservative. Thus Eq. (5.3) is used with x and θ as generalized coordinates. If the pin support is used as the datum for potential energy calculations, then

$$V = \frac{1}{2}kx^2 - mg\frac{L}{2}\cos\theta$$

Recall that the kinetic energy of a rigid body is

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$

where $\bar{v}$ is the velocity of the mass center of the body and ω is the angular velocity of the body. The velocity of the mass center of the bar has both a vertical component and a horizontal component. Thus

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta\right)^2 + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)^2\right] + \frac{1}{2}\left(\frac{1}{12}mL^2\dot{\theta}^2\right)$$

$$= \frac{1}{2}m\left(3\dot{x}^2 + \dot{x}\dot{\theta}L\cos\theta + \frac{L^2}{3}\dot{\theta}^2\right)$$

and

$$L = \frac{1}{2}m\left(3\dot{x}^2 + \dot{x}\dot{\theta}L\cos\theta + \frac{L^2}{3}\dot{\theta}^2\right) - \frac{1}{2}kx^2 + mg\frac{L}{2}\cos\theta$$

With $x_1 = x$, Eq. (5.3) becomes

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

which gives

$$0 = \frac{d}{dt}\left(3m\dot{x} + m\frac{L}{2}\dot{\theta}\cos\theta\right) + kx = 3m\ddot{x} + m\frac{L}{2}\ddot{\theta}\cos\theta - m\frac{L}{2}\dot{\theta}^2\sin\theta + kx$$

Application of Eq. (5.3) with $x_2 = \theta$ gives

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \\ &= \frac{d}{dt}\left(m\frac{L}{2}\dot{x}\cos\theta + \frac{L^2}{3}\dot{\theta}\right) + m\frac{L}{2}\dot{x}\dot{\theta}\sin\theta + mg\frac{L}{2}\sin\theta \\ &= m\frac{L}{2}\ddot{x}\cos\theta + m\frac{L^2}{3}\ddot{\theta} + mg\frac{L}{2}\sin\theta \end{aligned}$$

The differential equations derived using Lagrange's equations are identical to the nonlinear equations derived using Newton's laws in Example 5.3.

Consider an n-degree-of-freedom system with generalized coordinates x_1, $x_2, \ldots, x_n$ acted on by external nonconservative forces. The system is moved through small displacements to a new arbitrary state specified by $x_1 + \delta x_1$, $x_2 + \delta x_2, \ldots, x_n + \delta x_n$. The changes in displacements are called *virtual displacements.* The work done by the nonconservative forces as the system moves through the virtual displacements is called the *virtual work* and is calculated by using the usual definition of work done by a force. The virtual work can be written in the form

$$\delta W = \sum_{i=1}^{n} Q_i \delta x_i \tag{5.4}$$

The Q_i terms are called generalized forces. It can be shown that Lagrange's equations for a nonconservative system take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i \qquad i = 1, \ldots, n \tag{5.5}$$

Use Lagrange's equations to derive the differential equations governing the motion of the nonconservative system of Fig. 5.6, using x and θ as generalized coordinates.

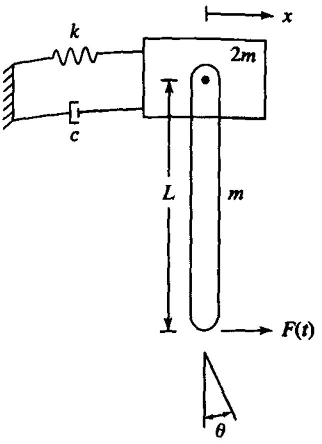


Figure 5.6 Two-degree-of-freedom nonconservative system of Example 5.6.

Solution:

The lagrangian for the system of Fig. 5.6 was developed in Example 5.5. Imagine a virtual displacement for the block δx and a virtual counterclockwise rotation of the bar $\delta\theta$. The work done by the nonconservative forces as the system moves through the virtual displacement is

$$\delta W = -c\dot{x}\delta x + F(t)(\delta x + L\delta\theta) = (-c\dot{x} + F(t))\delta x + LF(t)\delta\theta$$

from which the generalized forces are determined as

$$Q_1 = -c\dot{x} + F(t) \qquad Q_2 = LF(t)$$

Then, from the results of Example 5.5, the differential equations governing the motion of the system of Fig. 5.6 are

$$3m\ddot{x} + m\frac{L}{2}\ddot{\theta}\cos\theta - m\frac{L}{2}\dot{\theta}^2\sin\theta + c\dot{x} + kx = F(t)$$

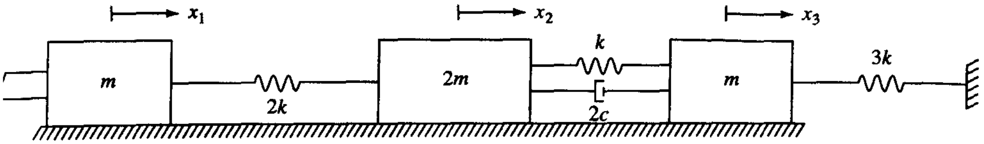
$$m\frac{L}{2}\ddot{x}\cos\theta + m\frac{L^2}{3}\ddot{\theta} + mg\frac{L}{2}\sin\theta = LF(t)$$

Example 5.

Use Lagrange's equations to derive the differential equations governing the motion of the system of Fig. 5.7 using x_1, x_2, and x_3 as generalized coordinates.

Solution:

Lagrange's equations were used to derive the differential equations governing the motion of the system but without the viscous dampers in Example 5.4. The method of virtual work is used to derive the generalized forces. To this end imagine virtual displacements δx_1, δx_2, and δx_3. The force developed in the leftmost viscous damper is $c\dot{x}_1$ and its virtual work is $-c\dot{x}_1\delta x_1$. The force developed in the rightmost viscous damper is $2c(\dot{x}_3 - \dot{x}_2)$. The work done by the damping force due to the virtual displacement δx_2 is



5.7 System of Example 5.7.

$2c(\dot{x}_3 - \dot{x}_2)\delta x_2$. Since the force opposes the direction of δx_3, the work done due to δx_3 is $-2c(\dot{x}_3 - \dot{x}_2)\delta x_3$. Thus the total virtual work is

$$\delta W = -c\dot{x}_1\delta x_1 + 2c(\dot{x}_3 - \dot{x}_2)\delta x_2 - 2c(\dot{x}_3 - \dot{x}_2)\delta x_3$$

Thus the generalized forces are

$$Q_1 = -c\dot{x}_1 \qquad Q_2 = 2c(\dot{x}_3 - \dot{x}_2) \qquad Q_3 = -2c(\dot{x}_3 - \dot{x}_2)$$

Then, from the results of Example 5.4, the differential equations are

$$m\ddot{x}_1 + 3kx_1 - 2kx_2 = -c\dot{x}_1$$

$$2m\ddot{x}_2 - 2kx_1 + 3kx_2 - kx_3 = 2c(\dot{x}_3 - \dot{x}_2)$$

$$m\ddot{x}_3 - kx_2 + 4kx_3 = -2c(\dot{x}_3 - \dot{x}_2)$$

which can be rewritten in a matrix form as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c & 0 & 0 \\ 0 & 2c & -2c \\ 0 & -2c & 2c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5.4 MATRIX FORMULATION OF DIFFERENTIAL EQUATIONS FOR LINEAR SYSTEMS

It can be shown that for an n-degree-of-freedom linear system the potential and kinetic energies must have the quadratic forms

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j \tag{5.6}$$

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{x}_i \dot{x}_j \tag{5.7}$$

...grangian for a linear system becomes

$$L = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(m_{ij}\dot{x}_i\dot{x}_j - k_{ij}x_i x_j\right)\right]$$

Application of Eq. (5.5) yields

$$Q_l = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_l}\right) - \frac{\partial L}{\partial x_l} \qquad l = 1, 2, \dots, n$$

$$Q_l = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m_{ij}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}_l}(\dot{x}_i\dot{x}_j)\right] + k_{ij}\frac{\partial}{\partial x_l}(x_i x_j)\right\} \tag{5.8}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m_{ij}\frac{d}{dt}\left[\dot{x}_i\frac{\partial \dot{x}_j}{\partial \dot{x}_l} + \dot{x}_j\frac{\partial \dot{x}_i}{\partial \dot{x}_l}\right] + k_{ij}\left(x_i\frac{\partial x_j}{\partial x_l} + x_j\frac{\partial x_i}{\partial x_l}\right)\right\}$$

Since

$$\frac{\partial x_i}{\partial x_l} = \delta_{il} = \begin{cases} 0 & i \neq l \\ 1 & i = l \end{cases}$$

Eq. (5.8) becomes

$$Q_l = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[m_{ij}\frac{d}{dt}(\dot{x}_i\delta_{jl} + \dot{x}_j\delta_{il}) + k_{ij}(x_i\delta_{jl} + x_j\delta_{il})\right]$$

The right-hand side of the preceding equation is broken into four terms and the order of summation interchanged on the second and fourth terms. Then, because of the presence of the δ terms, the value of the term on the inner summation is nonzero only for one value of the summation index. Thus the preceding equation can be rewritten using single summations as

$$Q_l = \frac{1}{2}\left(\sum_{i=1}^{n} m_{il}\ddot{x}_i + \sum_{j=1}^{n} m_{lj}\ddot{x}_j + \sum_{i=1}^{n} k_{il}x_i + \sum_{j=1}^{n} k_{lj}x_j\right)$$

The name of a summation index is arbitrary. Thus these summations are combined, yielding

$$Q_l = \frac{1}{2}\left[\sum_{i=1}^{n}(m_{il} + m_{li})\ddot{x}_i + \sum_{i=1}^{n}(k_{il} + k_{li})x_i\right] \tag{5.9}$$

Note that in Eq. (5.6) k_{il} and k_{li} both multiply the product $x_i x_l$ and without loss of generality can be set equal. The same reasoning for Eq. (5.7) leads to $m_{il} = m_{li}$. Thus

$$\sum_{i=1}^{n} m_{li}\ddot{x}_i + \sum_{i=1}^{n} k_{li}x_i = Q_l \qquad l = 1, \dots, n \tag{5.10}$$

Equation (5.10) represents a system of n simultaneous linear differential equations. The matrix formulation of Eq. (5.10) is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{5.11}$$

where $\mathbf{M}$ is the $n \times n$ *mass matrix*, $\mathbf{K}$ is the $n \times n$ *stiffness matrix*, $\mathbf{F}$ is the $n \times 1$ *force vector*, $\mathbf{x}$ is the $n \times 1$ displacement vector, and $\ddot{\mathbf{x}}$ is the $n \times 1$ *acceleration vector*. Note from Eq. (5.9) that for the lth equation the coefficient multiplying $\ddot{x}_i$ is $(m_{il} + m_{li})/2$ which is m_{li}, the element in the lth row and ith column of $\mathbf{M}$. Similarly m_{il}, the element in the ith row and lth column is determined as $(m_{li} + m_{il})/2$. Hence $m_{il} = m_{li}$ for each $i, l = 1, 2, \ldots, n$. Thus the mass matrix is *symmetric*. The element in the ith row and jth column of the mass matrix is m_{ij}, the same coefficient that multiplies $\dot{x}_i \dot{x}_j$ in the quadratic form of the kinetic energy, Eq. (5.7). A similar argument can be used to show that the stiffness matrix is symmetric and that the element in the ith row and jth column of $\mathbf{K}$ is the coefficient that multiplies $x_i x_j$ in the quadratic form of the potential energy, Eq. (5.6). The ith element of the force vector is the generalized force Q_i, as determined by the method of virtual work.

The matrix formulation of the differential equations governing the motion of a linear n-degree-of-freedom system is used in deriving the free and forced responses of the system. If the mass and stiffness matrices and the force vector are known for a chosen set of generalized coordinates, differential equations of the form of Eq. (5.11) can be directly written. Thus if the quadratic forms of the kinetic and potential energies can be determined, the elements of the mass and stiffness matrices are the coefficients in these quadratic forms. Formal application of Lagrange's equations to derive the differential equations governing the motion of a linear system is not necessary.

The coupling of a system relative to the choice of generalized coordinate is specified according to how the mass and stiffness matrices are populated. A *diagonal matrix* is a matrix in which the only nonzero elements are along the main diagonal of the matrix. If the stiffness matrix is not a diagonal matrix, the system is said to be *statically coupled* relative to the choice of generalized coordinates. If the system is statically coupled with respect to a set of generalized coordinates $x_i, i = 1, 2, \ldots, n$, then there is at least one i such that application of a static force to the particle whose displacement is x_i results in a static displacement of the particle whose displacement is x_j, for some $j \neq i$.

If the mass matrix is not a diagonal matrix, the system is said to be *dynamically coupled*. If the system is dynamically coupled, then there exists at least one i such that application of an impulse to the particle whose displacement is x_i instantaneously induces a velocity $\dot{x}_j$ for some $j \neq i$.

Use the quadratic forms of kinetic and potential energy to derive the differential equations governing free vibration of the system of Fig. 5.8 and discuss the coupling using (a) x and θ as generalized coordinates and (b) x_A, the vertical displacement of particle A, and x_B, the vertical displacement of particle B, as generalized coordinates.

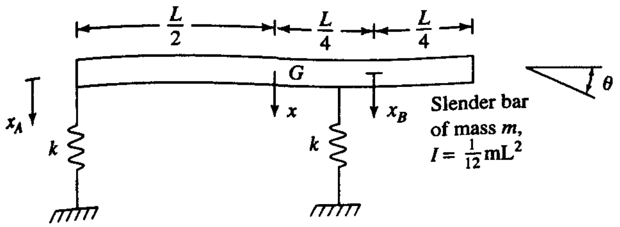


Figure 5.8 Two-degree-of-freedom system of Example 5.8. The nature of the coupling depends on the choice of generolized coordinates.

Solution:

(a) With x and θ as generalized coordinates, the kinetic and potential energies of the system at an arbitrary instant are

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}k\left(x + \frac{L}{4}\theta\right)^2 = \frac{1}{2}\left(2kx^2 - k\frac{L}{2}x\theta + \frac{5}{16}kL^2\theta^2\right)$$

Comparing the above equations with the quadratic forms of kinetic and potential energies, Eqs. (5.7) and (5.6) respectively, using x for x_1 and θ for x_2 leads to

$$m_{11} = m \qquad m_{12} = m_{21} = 0 \qquad m_{22} = \frac{1}{12}mL^2$$

$$k_{11} = 2k \qquad k_{12} = k_{21} = -k\frac{L}{4} \qquad k_{22} = \frac{5}{16}kL^2$$

Note that the term multiplying $x\theta$ in the quadratic form of potential energy is $2k_{12} = 2k_{21}$. Thus the governing differential equations are

$$\begin{bmatrix} m & 0 \\ 0 & \frac{1}{12}mL^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & \frac{5}{16}kL^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the stiffness matrix is not a diagonal matrix and the mass matrix is a diagonal matrix the system is statically coupled, but not dynamically coupled.

(*b*) With x_A and x_B as generalized coordinates, the quadratic forms of kinetic and potential energies at an arbitrary instant are

$$T = \frac{1}{2}m\left(\frac{\ddot{x}_A}{3} + \frac{2\ddot{x}_B}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{\ddot{x}_B - \ddot{x}_A}{\frac{3L}{4}}\right)^2$$

$$= \frac{1}{2}\left(\frac{7}{27}\dot{x}_A^2 + \frac{4}{27}\dot{x}_A\dot{x}_B + \frac{16}{27}\dot{x}_B^2\right)$$

$$V = \frac{1}{2}kx_A^2 + \frac{1}{2}kx_B^2$$

The elements of the mass and stiffness matrices are obtained by comparing the above equations to Eqs. (5.7) and (5.6) respectively, leading to the following differential equations

$$\begin{bmatrix} \frac{7}{27}m & \frac{2}{27}m \\ \frac{2}{27}m & \frac{16}{27}m \end{bmatrix} \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus the system is dynamically coupled, but not statically coupled, when x_A and x_E are used as generalized coordinates.

The method presented in this section to determine the mass and stiffness matrices for linear systems is the multi-degree-of-freedom analogy to the equivalent systems method presented in Sec. 2.3 to derive the differential equations governing the motion of a linear one-degree-of-freedom system. The equivalent systems method uses the kinetic energy to determine an equivalent mass and the potential energy to determine an equivalent stiffness. The mass and stiffness matrices are analogous to the equivalent mass and the equivalent stiffness.

Viscous damping adds linear terms to the differential equations. The application of Lagrange's equation to a system with viscous damping is illustrated in Examples 5.6 and 5.7. In these examples the method of virtual work is used to determine the generalized forces resulting from the nonconservative viscous damping forces. The viscous damping terms are incorporated in a matrix formulation and the equations are written in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \qquad [5.12]$$

where **M** and **K** are the previously defined mass and stiffness matrices and **C** is the $n \times n$ *damping matrix.*

As illustrated earlier in this section, formal application of Lagrange's equations is not necessary to derive the governing differential equations for a linear system. This is true for systems with viscous damping, as the virtual work for viscous damping forces can be written as

$$\delta W = -\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{x}_i\delta x_j \qquad [5.13]$$

where c_{ij} is the element in the ith row and jth column of the damping matrix.

The equivalent damping coefficient for a linear one-degree-of-freedom system is determined by using the energy dissipated by the viscous damping forces between two arbitrary times. Extending this analogy to a multi-degree-of-freedom system it is possible to write the work done by the viscous damping forces as

$$U_{1\to 2} = -\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{x_{j1}}^{x_{j2}} c_{ij}\dot{x}_i\,dx_j \qquad [5.14]$$

Using $dx_j = (dx_j/dt)\,dt$ in Eq. (5.14) leads to

$$U_{1\to 2} = -\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t_1}^{t_2} c_{ij}\dot{x}_i\dot{x}_j\,dt = -\int_{t_1}^{t_2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{x}_i\dot{x}_j\,dt \qquad [5.15]$$

Equation (5.15) can be used to argue that $c_{ij} = c_{ji}$ and hence the damping matrix is symmetric.

5.5 STIFFNESS INFLUENCE COEFFICIENTS

It is shown in Sec. 5.4 that the elements of the stiffness matrix for a linear system can be determined as the coefficients in the quadratic form of the potential energy. The work done by a conservative force is independent of path and can be expressed as the difference in potential energy between the initial position and the final position of the system. The potential energy function is a function only of the position of the system. Thus when evaluating the potential energy for a specific system configuration, one can look at any means of arriving at that configuration.

Stiffness influence coefficients provide an alternate means of determining the elements of the stiffness matrix. It is based on determining the potential energy for a system configuration that is obtained through static application of concentrated forces. To illustrate the development of the method consider three particles along the span of a fixed-free beam as illustrated in Fig. 5.9*a*. The beam is initially in its static equilibrium configuration. Let x_1, x_2, and x_3 be the chosen generalized coordinates which represent the displacements of the particles.

Consider the static application of a set of concentrated loads f_{11} applied to particle 1, f_{21} applied to particle 2, and f_{31} applied to particle 3 such that after their application, $x_1 = x_1$, $x_2 = 0$, and $x_3 = 0$ as illustrated in Fig. 5.9*b*. Since particles 2 and 3 do not change position during application of these loads, the forces applied to these particles do no work. The total work done by the external loads during this application is

$$U_{0\to 1} = \tfrac{1}{2} f_{11} x_1 \qquad [5.16]$$

Now add a second set of forces f_{12} applied to particle 1, f_{22} applied to particle 2, and f_{32} applied to particle 3 such that after static application of these loads, $x_1 = x_1$, $x_2 = x_2$, and $x_3 = 0$ as illustrated in Fig. 5.9*c*. Since particles 1 and 3 do not change position during application of these loads, only the forces applied to particle 2 do work. Note that the force f_{21} was already fully applied when the displacement

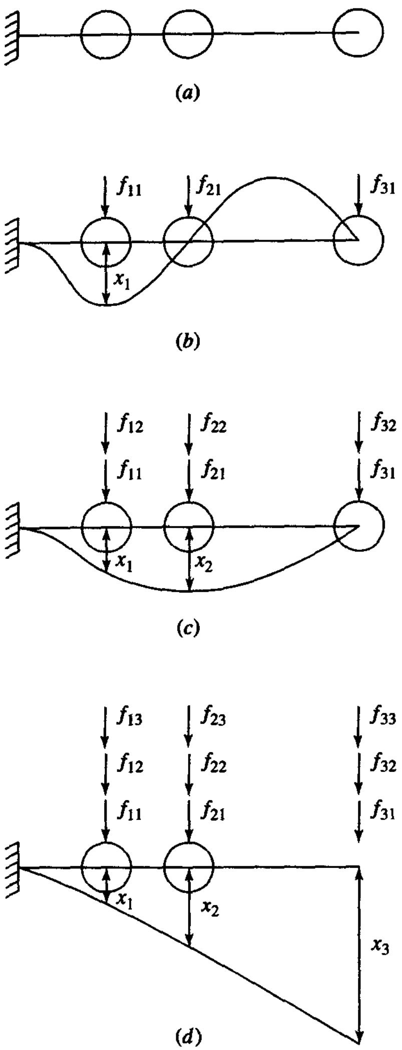


Figure 5.9 (*a*) Fixed-free beam with three particles along span; (*b*) configuration after first set of loads; (*c*) configuration after second set of loads; (*d*) final configuration after application of third set of loads.

occurred and the displacement occurred as f_{22} was being applied. Hence the work done during application of these forces is

$$U_{1\to 2} = f_{21}x_2 + \tfrac{1}{2}f_{22}x_2 \tag{5.17}$$

Next add a third set of forces f_{13} applied to particle 1, f_{23} applied to particle 2, and f_{33} applied to particle 3 such that after static application of these loads $x_1 = x_1$, $x_2 = x_2$, and $x_3 = x_3$, as illustrated in Fig. 5.9*d*. The work done during application of these forces is

$$U_{2\to 3} = f_{31}x_3 + f_{32}x_3 + \tfrac{1}{2}f_{33}x_3 \tag{5.18}$$

Thus, after application of the three sets of forces, the particles have arbitrary displacements. According to the principle of work and energy, the potential energy in the system is equal to the work done by the external forces between configuration 0 and configuration 3,

$$V = \frac{1}{2} f_{11} x_1 + f_{21} x_2 + \frac{1}{2} f_{22} x_2 + f_{31} x_3 + f_{32} x_3 + \frac{1}{2} f_{33} x_3 \qquad [5.19]$$

The system is linear, thus a proportional change in the system of forces applied on any step leads to a proportional change in displacements. Define k_{11}, k_{21}, and k_{31} as the set of forces required to cause a unit displacement for the first particle. Then, due to the linearity of the system

$$f_{11} = k_{11} x_1 \qquad f_{21} = k_{21} x_1 \qquad f_{31} = k_{31} x_1 \qquad [5.20]$$

Similarly define k_{12}, k_{22}, and k_{32} as the set of forces required to cause a unit displacement for particle 2 and k_{13}, k_{23}, and k_{33} as the set of forces required to cause a unit displacement for particle 3. Then, in general

$$f_{ij} = k_{ij} x_j \qquad [5.21]$$

Using Eq. (5.21) in Eq. (5.19) leads to

$$V = \frac{1}{2} k_{11} x_1 x_1 + k_{21} x_1 x_2 + \frac{1}{2} k_{22} x_2 x_2 + k_{31} x_1 x_3 + k_{32} x_2 x_3 + \frac{1}{2} k_{33} x_3 x_3 \qquad [5.22]$$

The potential energy is a function only of the beam's configuration, not of how the configuration is attained. Thus the potential energy would be the same if the order of the loading were reversed. Suppose the forces f_{12}, f_{22}, and f_{32} are applied first, resulting in $x_1 = 0$, $x_2 = x_2$, and $x_3 = 0$. Then the forces f_{21}, f_{22}, and f_{32} are applied such that after their static application the beam's configuration is defined by $x_1 = x_1$, $x_2 = x_2$, and $x_3 = 0$. Then, from Eq. (5.21), the potential energy is calculated as

$$V = \frac{1}{2} k_{22} x_2 x_2 + k_{12} x_2 x_1 + \frac{1}{2} k_{11} x_1 x_1 + k_{31} x_3 x_1 + k_{32} x_3 x_2 + \frac{1}{2} k_{33} x_3 x_3 \qquad [5.23]$$

Since the potential energy calculated by Eq. (5.22) must be the same as that calculated by Eq. (5.23) for arbitrary values of x_1, x_2, and x_3, $k_{12} = k_{21}$. Other combinations of the order of loading can be studied to show that in general

$$k_{ij} = k_{ji} \qquad [5.24]$$

Then, using Eq. (5.24) in Eq. (5.22) leads to

$$V = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} k_{ij} x_i x_j \qquad [5.25]$$

Equation (5.25) is identical to the quadratic form of the potential energy for this three-degree-of-freedom system. Thus the coefficients k_{ij}, i, $j = 1, 2, 3$ are the elements of the stiffness matrix. The k_{ij} calculated in this fashion are called *stiffness influence coefficients*. Equation (5.24) shows that the stiffness matrix is symmetric when stiffness influence coefficients are used in its determination.

The concept of stiffness influence coefficients can be generalized to any linear system. Each column of the stiffness matrix has a physical interpretation. The jth column of the stiffness matrix is the set of forces acting on the particles whose displacements are described by the chosen generalized coordinates such that after static application of these forces, $x_j = 1$ and $x_i = 0$ for $i \neq j$.

In summary the influence coefficient method for determining the elements of an n-degree-of-freedom system is as follows:

1. Assign a unit displacement for x_1, maintaining $x_2, x_3, \ldots, x_n$ in their static-equilibrium position. Calculate the system of forces required to maintain this as an equilibrium position. The forces, k_{i1}, are applied at the locations whose displacements define the generalized coordinates. This set of forces yields the first column of the stiffness matrix.
2. Continue this procedure to find all columns of the stiffness matrix. The jth column is found by prescribing $x_j = 1$ and $x_i = 0$, $i \neq j$, and calculating the system of forces necessary to maintain this as an equilibrium position
3. If x_j is an angular coordinate, then k_{ji} is an applied moment. When calculating the jth column of the stiffness matrix, a unit rotation in radians must be applied to the angle defined by x_j. If the small-angle assumption is necessary to achieve a linear system, it is also used to calculate the stiffness influence coefficients.
4. Reciprocity implies the stiffness matrix must be symmetric: $k_{ij} = k_{ji}$. The symmetry can be used as a check.
5. When deriving differential equations for linear systems, note that static deflections in springs cancel with the gravity forces or other conservative forces that cause the static deflections. Thus static deflections and their sources do not need to be considered in determining stiffness influence coefficients.

Use the stiffness influence coefficient method to calculate the stiffness matrix for the system of Fig. 5.3 in Example 5.1.

Solution:

The first column of the stiffness matrix is obtained by setting $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and calculating the system of applied forces necessary to maintain this position in equilibrium. Free-body diagrams of the blocks are shown in Fig. 5.10. Setting $\Sigma F = 0$ yields

$$\begin{aligned}
&\text{Block } a: && -k - 2k + k_{11} = 0 \Rightarrow k_{11} = 3k \\
&\text{Block } b: && 2k + k_{21} = 0 \Rightarrow k_{21} = -2k \\
&\text{Block } c: && \Rightarrow k_{31} = 0
\end{aligned}$$

The second column is obtained by setting $x_2 = 0$, $x_1 = 1$, $x_3 = 0$. Summing forces on the free-body diagrams yields

$$\begin{aligned}
&\text{Block } a: && 2k + k_{12} = 0 \Rightarrow k_{12} = -2k \\
&\text{Block } b: && -2k - k + k_{22} = 0 \Rightarrow k_{22} = 3k \\
&\text{Block } c: && k + k_{32} = 0 \Rightarrow k_{32} = -k
\end{aligned}$$

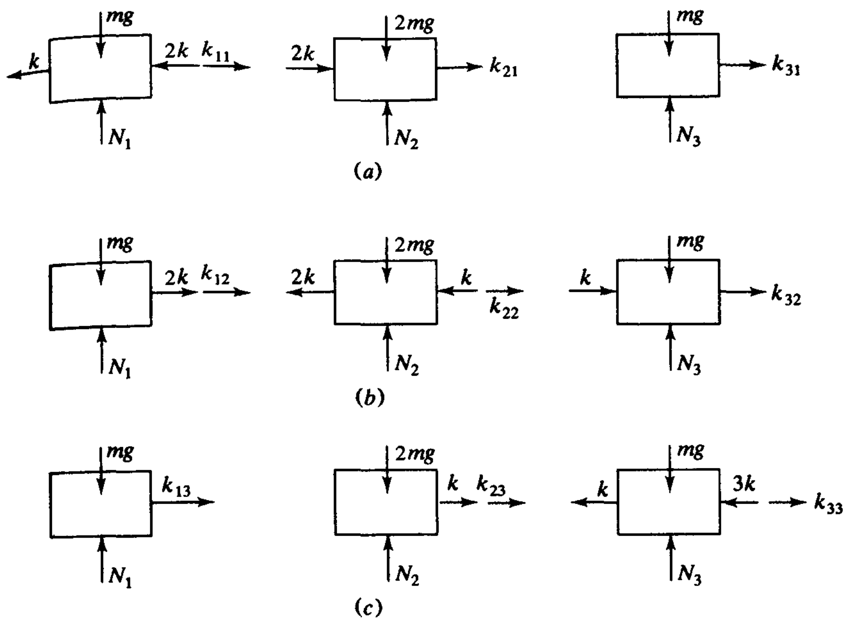


Figure 5.10 (a) First column of stiffness matrix is calculated by setting $x_1 = 1, x_2 = 0, x_3 = 0$; (b) second column of stiffness matrix is calculated by setting $x_1 = 0, x_2 = 1, x_3 = 0$; (c) third column of stiffness matrix is calculated by setting $x_1 = 0, x_2 = 0, x_3 = 1$.

The third column is obtained by setting $x_1 = 0, x_2 = 0, x_3 = 1$. Summing forces on the free-body diagrams yields

$$\begin{aligned}
&\text{Block } a: && \Rightarrow k_{13} = 0 \\
&\text{Block } b: && k + k_{23} = 0 \Rightarrow k_{23} = -k \\
&\text{Block } c: && -k - 3k + k_{33} = 0 \Rightarrow k_{33} = 4k
\end{aligned}$$

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{bmatrix}$$

Example 5

Use the stiffness influence coefficient method to find the stiffness matrix for the system in Fig. 5.11. Use x_A, the downward displacement of block A, x_B, the upward displacement of block B, and θ, the counterclockwise angular rotation of the pulley, as generalized coordinates.

Solution:

The first column of the stiffness matrix is obtained by setting $x_A = 1$, $x_B = 0$, and $\theta = 0$ and finding the resulting system of forces and moments to maintain this as an equilibrium position. Note that since θ is an angular coordinate k_{31} is a moment.

Block A: $\sum F = 0 \Rightarrow -k + k_{11} = 0 \Rightarrow k_{11} = k$

Block B: $\sum F = 0 \Rightarrow k_{21} = 0$

Pulley: $\sum M_O = 0 \Rightarrow k(r) + k_{31} = 0 \Rightarrow k_{31} = -kr$

The second column is obtained by setting $x_A = 0$, $x_B = 1$, $\theta = 0$. The equations of equilibrium yield

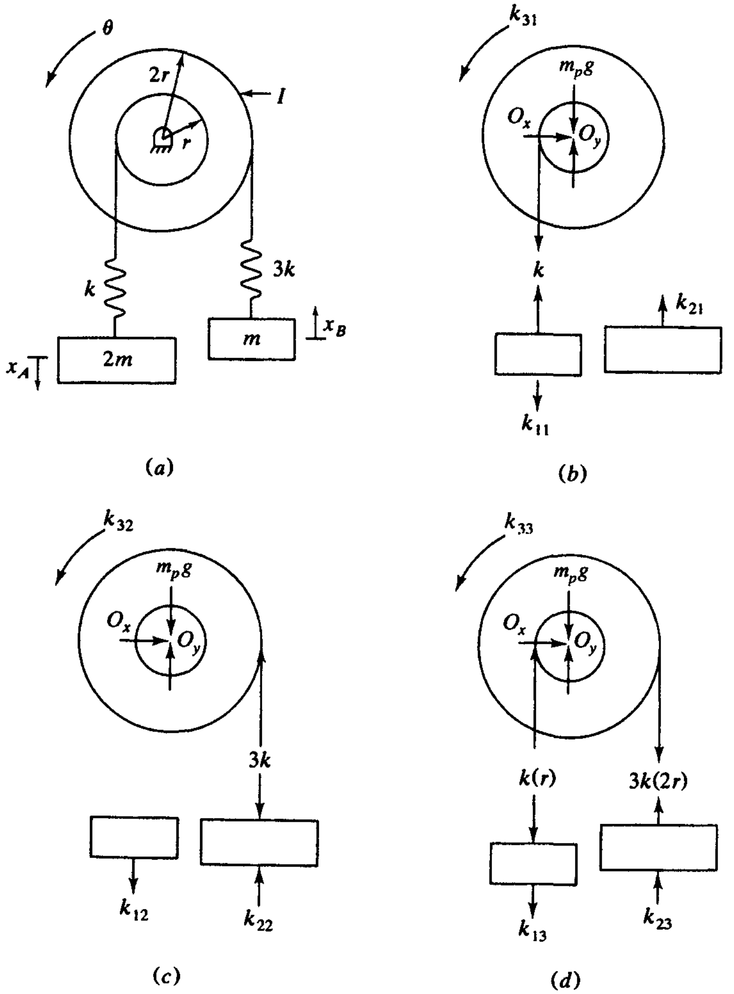


Figure 5.11 (*a*) System of Example 5.10; (*b*) first column of stiffness matrix is calculated by setting $x_A = 1$, $x_B = 1$, $\theta = 0$; (*c*) second column of stiffness matrix is calculated by setting $x_A = 0$, $x_B = 1$, $\theta = 0$; (*d*) third column of stiffness matrix is calculated by setting $x_A = 0$, $x_B = 0$, $\theta = 1$.

Block A: $\sum F = 0 \Rightarrow k_{12} = 0$

Block B: $\sum F = 0 \Rightarrow 3k - k_{22} = 0 \Rightarrow k_{22} = 3k$

Pulley: $\sum M_O = 0 \Rightarrow 3k(2r) + k_{32} = 0 \Rightarrow k_{32} = -6kr$

The third column is obtained by setting $x_A = 0$, $x_B = 0$, $\theta = 1$. The equations of equilibrium yield

Block A: $\sum F = 0 \Rightarrow kr + k_{13} = 0 \Rightarrow k_{13} = -kr$

Block B: $\sum F = 0 \Rightarrow 3k(2r) + k_{23} = 0 \Rightarrow k_{23} = -6kr$

Pulley: $\sum M_O = 0 \Rightarrow -k(r)(r) - 3k(2r)(2r) + k_{33} = 0 \Rightarrow k_{33} = 13kr^2$

Thus the stiffness matrix for this choice of generalized coordinates is

$$\mathbf{K} = \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix}$$

Example 5.1

Use the influence coefficient method to find the stiffness matrix for the system of Fig. 5.12 using θ_1, the clockwise angular displacement of bar AB, and θ_2, the counterclockwise angular displacement of bar CD, as generalized coordinates.

Solution:

The first column of the stiffness matrix is obtained by setting $\theta_1 = 1$ and $\theta_2 = 0$ and finding the moments that must be applied to the bars to maintain this as an equilibrium position. The small-angle assumption is used. Equilibrium equations are applied to the free-body diagrams of Fig. 5.12b.

$$\sum \overset{+}{\overleftarrow{M}}_A = 0 = -2k\frac{L}{2}\left(\frac{L}{2}\right) - 5k\frac{L}{6}\left(5\frac{L}{6}\right) - kL(L) + k_{11} \Rightarrow k_{11} = \frac{79}{36}kL^2$$

$$\sum \overset{+}{\overrightarrow{M}}_D = 0 = 5k\frac{L}{6}(L) + kL\left(5\frac{L}{6}\right) + k_{21} \Rightarrow k_{21} = -5k\frac{L^2}{3}$$

The second column is obtained by setting $\theta_1 = 0$ and $\theta_2 = 1$. The equilibrium equations are applied to the free-body diagrams to yield

$$\sum \overset{+}{\overleftarrow{M}}_A = 0 = kL\left(5\frac{L}{6}\right) + 5k\frac{L}{6}(L) + k_{12} \Rightarrow k_{12} = -5k\frac{L^2}{3}$$

$$\sum \overset{+}{\overrightarrow{M}}_D = 0 = -kL(L) - 5k\frac{L}{6}\left(5\frac{L}{6}\right) - 3k\frac{L}{2}\left(\frac{L}{2}\right) + k_{22} \Rightarrow k_{22} = 22k\frac{L^2}{9}$$

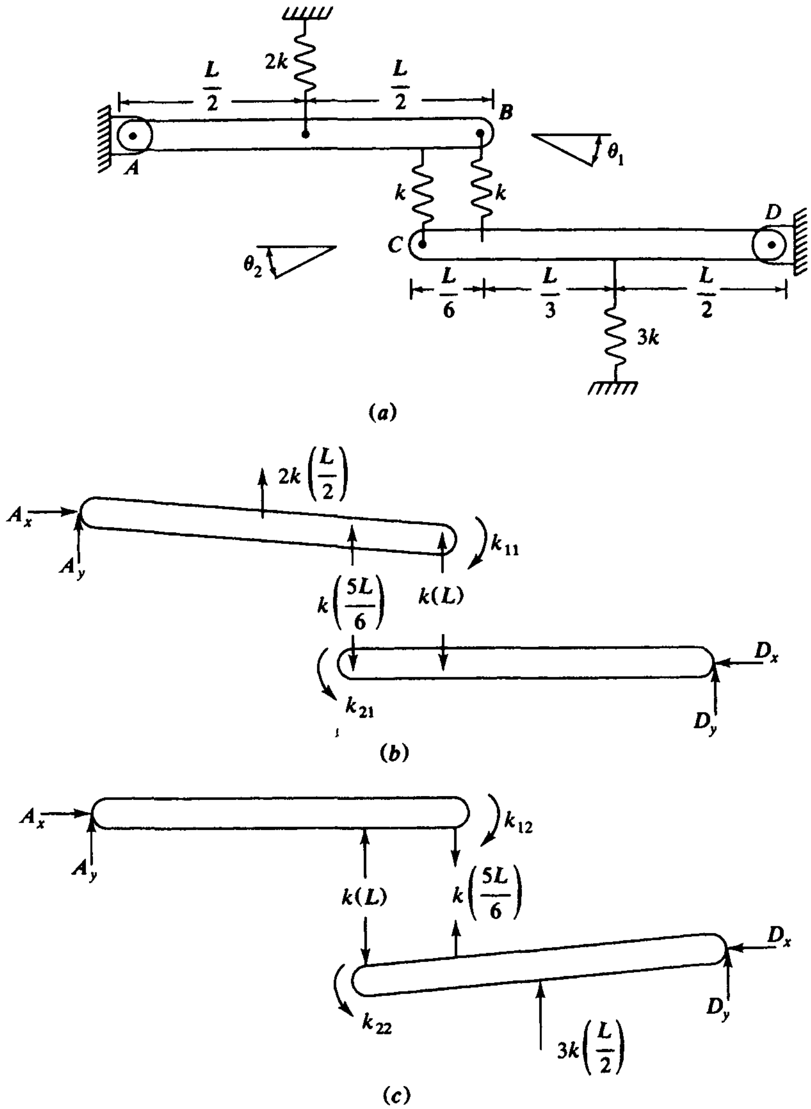


Figure 5.12 (*a*) System of Example 5.12; (*b*) first column of stiffness matrix is calculated by setting $\theta_1 = 1, \theta_2 = 0$; (c) second column of stiffness matrix is calculated by setting $\theta_1 = 0, \theta_2 = 1$.

The stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} \frac{79}{36}kL^2 & -5k\frac{L^2}{3} \\ -5k\frac{L^2}{3} & \frac{22}{9}kL^2 \end{bmatrix}$$

Example

The transverse vibrations of the cantilever beam of Fig. 5.13 are to be approximated by modeling the beam as a two-degree-of-freedom system. The inertia of the beam is modeled by placing discrete masses at the beam's midspan and end. Calculate the stiffness matrix for this two-degree-of-freedom model using the displacements of the midspan and end of the beam as generalized coordinates.

Solution:

Calculation of the stiffness matrix requires the evaluation of the deflection of the beam due to a concentrated load at the midspan and a concentrated load at the end of the beam. Perhaps the best way of handling the beam deflection problem is to use the method of superposition as shown in Fig. 5.13*b*. The elements of the *i*th column of the stiffness matrix are calculated from

$$y\left(\frac{L}{2}\right) = k_{1i} y_1\left(\frac{L}{2}\right) + k_{2i} y_2\left(\frac{L}{2}\right)$$

$$y(L) = k_{1i} y_1(L) + k_{2i} y_2(L)$$

where $y(z)$ is the total deflected shape of the beam, $y_1(z)$ is the deflected shape of the beam due to a concentrated unit load at the midspan, and $y_2(z)$ is the deflected shape of the beam due to a concentrated unit load at the end of the beam. From Table D.2 these are evaluated as

$$y_1\left(\frac{L}{2}\right) = \frac{L^3}{24EI} \qquad y_2\left(\frac{L}{2}\right) = \frac{5L^3}{48EI}$$

$$y_1(L) = \frac{5L^3}{48EI} \qquad y_2(L) = \frac{L^3}{3EI}$$

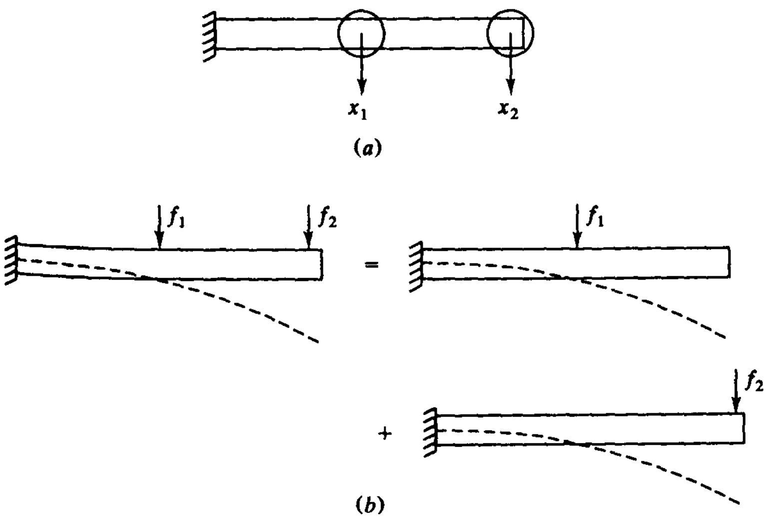


Figure 5.13 (a) Two-degree-of-freedom model of cantilever beam of Example 5.12; (b) illustration of superposition method used to calculate stiffness matrix.

To determine the first column, set $y(L/2) = 1$ and $y(L) = 0$. The equations are solved simultaneously, yielding

$$k_{11} = \frac{768EI}{7L^3} \qquad k_{21} = -\frac{240EI}{7L^3}$$

To determine the second column, set $y(L/2) = 0$ and $y(L) = 1$. The equations are solved simultaneously, yielding

$$k_{12} = -\frac{240EI}{7L^3} \qquad k_{22} = \frac{96EI}{7L^3}$$

5.6 FLEXIBILITY INFLUENCE COEFFICIENTS

Development of the stiffness matrix using stiffness influence coefficients is straightforward. For mechanical systems the calculation of stiffness influence coefficients requires the application of the principles of statics and little algebra. However, as shown in Example 5.12, the calculation of a column of stiffness influence coefficients for a structural system modeled with n degrees of freedom requires the solution of n simultaneous equations. This leads to significant computation time for systems with many degrees of freedom. Flexibility influence coefficients provide a convenient alternative. They are easier to calculate than stiffness influence coefficients for structural systems and their knowledge is sufficient for solution of the free-vibration problem.

If the stiffness matrix, $\mathbf{K}$, is nonsingular, then its inverse exists. The flexibility matrix, $\mathbf{A}$, is defined by

$$\mathbf{A} = \mathbf{K}^{-1} \qquad [5.26]$$

Premultiplying Eq. (5.1) by $\mathbf{A}$ gives

$$\mathbf{A}\mathbf{M}\ddot{\mathbf{x}} + \mathbf{A}\mathbf{C}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{A}\mathbf{F} \qquad [5.27]$$

Equation (5.27) shows that knowledge of $\mathbf{A}$ instead of $\mathbf{K}$ is sufficient for solution of the vibration problem.

The elements of $\mathbf{K}$ are determined by using stiffness influence coefficients. Analogously, flexibility influence coefficients can be used to determine $\mathbf{A}$. The flexibility influence coefficient a_{ij} is defined as the displacement of the particle whose displacement is represented by x_i when a unit load is applied to the particle whose displacement is represented by x_j and no other loading is applied to the system. If x_j represents an angular coordinate, then a unit moment is applied.

Suppose an arbitrary set of concentrated loads $\{f_1, f_2, \ldots, f_n\}$ is applied statically to an n-degree-of-freedom system. The load f_i is applied to the particle whose displacement is represented by x_i. Using the definition of flexibility influence coefficients, x_j is calculated from

$$x_j = \sum_{i=1}^{n} a_{ji} f_i \tag{5.28}$$

Equation (5.28) is summarized in matrix form as

$$\mathbf{x} = \mathbf{A}\mathbf{f} \tag{5.29}$$

Multiplying Eq. (5.29) by $\mathbf{A}^{-1}$ yields

$$\mathbf{f} = \mathbf{A}^{-1}\mathbf{x} = \mathbf{K}\mathbf{x} \tag{5.30}$$

which defines the static relationship between force and displacement. Equation (5.30) shows that the flexibility influence coefficients as defined are the elements of the inverse of the stiffness matrix, defined as the flexibility matrix.

The procedure for determining the flexibility matrix using influence coefficients is as follows:

1. Apply a unit load at the location whose displacement is defined by x_1. The flexibility influence coefficient in the first column, a_{i1}, is the resulting displacement of the particle whose displacement is x_i.
2. Successively apply concentrated unit loads to particles whose displacements define the remaining generalized coordinates. Calculate columns of flexibility influence coefficients using the principles of statics.
3. If x_l is an angular displacement, then a unit moment is applied to calculate a_{jl}, $j = 1, \ldots, n$. The displacements calculated for a_{li}, $i = 1, \ldots, n$, are angular displacements.
4. Since the stiffness matrix is symmetric, the flexibility matrix must also be symmetric. This condition serves as a check on the analysis.

Example

Determine the flexibility matrix for the system in Fig. 5.12 of Example 5.11 using flexibility influence coefficients.

Solution:

The free-body diagrams of Fig. 5.14 show the external forces, in terms of angular displacements, acting on each bar when an arbitrary set of moments is applied. The equations of equilibrium are used to derive equations relating the displacements to the applied forces

$$\text{Bar } AB: \quad \sum \overset{+}{\circlearrowleft} M_A = 0 \Rightarrow m_1 = \frac{79kL^2}{36}\theta_1 - \frac{5kL^2}{3}\theta_2$$

$$\text{Bar } BC: \quad \sum \overset{+}{\circlearrowleft} M_D = 0 \Rightarrow m_2 = -\frac{5kL^2}{3}\theta_1 + \frac{22kL^2}{9}\theta_2$$

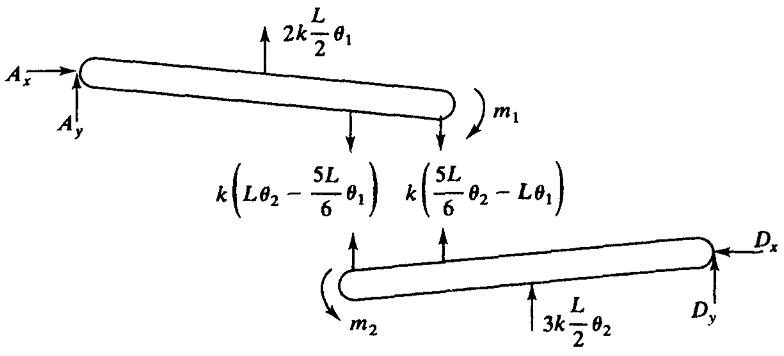


Figure 5.14 Free-body diagram of a static position used to calculate flexibility influence coefficients for system of Examples 5.11 and 5.13. For the first column θ_1 and θ_2 are determined by setting $m_1 = 1$, $m_2 = 0$. For the second column θ_1 and θ_2 are determined by setting $m_1 = 0$, $m_2 = 1$.

The first column of the flexibility matrix is obtained by setting $m_1 = 1$, $m_2 = 0$, $\theta_1 = a_{11}$, $\theta_2 = a_{21}$, and solving the resulting equations simultaneously. The second column is obtained by setting $m_1 = 0$, $m_2 = 1$, $\theta_1 = a_{12}$, $\theta_2 = a_{22}$, and solving the resulting simultaneous equations. The flexibility matrix is

$$\mathbf{A} = \begin{bmatrix} \frac{396}{419kL^2} & \frac{270}{419kL^2} \\ \frac{270}{419kL^2} & \frac{711}{838kL^2} \end{bmatrix}$$

5.14 Two small machines are to be bolted to an overhanging beam as shown in Fig. 5.15. The beam is nonuniform; thus prediction of influence coefficients from strength-of-materials concepts is difficult. Instead, the project engineer performs static measurements. After the first machine is installed, the engineer notes that the deflection directly below the machine is 10mm and the deflection of the end of the beam is 2 mm. After the second machine is also installed, the deflection of the end of the beam is 0.8 mm.

(*a*) What is the deflection at the location where the first machine is installed after the second machine is installed?

(*b*) What is the flexibility matrix for this system?

Solution:

(*a*) Assuming a linear system, the principle of superposition yields the following relationships between the static loads, the influence coefficients, and the deflections:

$$x_1 = a_{11}f_1 + a_{12}f_2 \qquad x_2 = a_{21}f_1 + a_{22}f_2$$

When only the first machine is installed, $f_1 = (60\ \text{kg})(9.81\ \text{m/s}^2) = 588.6\ \text{N}$, $f_2 = 0$, $x_1 = 0.01$ m, $x_2 = -0.002$ m. Substitution into the preceding equations yields $a_{11} = 1.7 \times 10^{-5}$ m/N, $a_{21} = -3.4 \times 10^{-6}$ m/N. When the second machine is also installed,

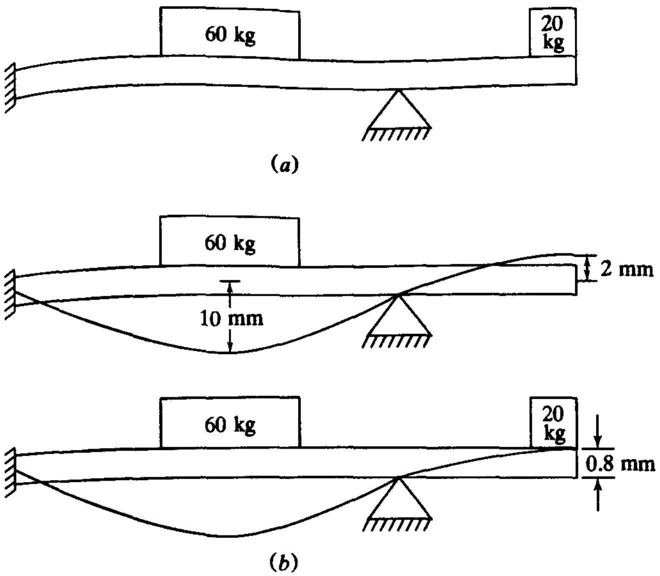


Figure 5.15 (a) System of Example 5.14; (b) as each machine is bolted to the beam, static deflection measurements are made.

$f_1 = 588.6$ N, $f_2 = (20\text{ kg})(9.81\text{ m/s}^2) = 196.2$ N, and $x_2 = -0.0008$ m. Then, since $a_{12} = a_{21}$, the displacement at the location of the first machine when both machines are installed is

$$x_1 = (1.7 \times 10^{-5}\text{ m/N})(588.6\text{ N}) + (-3.4 \times 10^{-6}\text{ m/N})(196.2\text{ N}) = 9.3\text{ mm}$$

(b) The second of the preceding equations yields

$$a_{22} = \frac{x_2 - a_{21}f_1}{f_2} = \frac{[-0.0008\text{ m} - (-3.4 \times 10^{-6}\text{ m/N})(588.6\text{ N})]}{196.2\text{ N}} = 6.1 \times 10^{-6}\ \frac{\text{m}}{\text{N}}$$

The flexibility matrix is

$$\mathbf{A} = \begin{bmatrix} 1.7 & -0.34 \\ -0.34 & 0.61 \end{bmatrix} 10^{-6}\ \frac{\text{m}}{\text{N}}$$

Example 5

Four machines are equally spaced along the length of an 8-m fixed-free beam of elastic modulus 210×10^9 N/m^2 and cross-section moment of inertia 1.6×10^{-5} m^4. Write a MATLAB script to determine the flexibility matrix for a four-degree-of-freedom model of the system with the generalized coordinates representing the displacements of the machines.

Solution:

The MATLAB script and the output when the script is run are shown in Fig. 5.16. The MATLAB script is written to accommodate an n-degree-of-freedom model for a fixed-free beam with the nodes equally spaced along the length of the beam. The beam

```
% Example 5.15
% Development of the flexibility matrix for a n-degree-of-freedom model
% of a fixed-free beam. The program uses flexiblity influence coefficients
% calculated using the deflection equations of Appendix D.2. The program
% assumes the nodes are equally spaced along the length of the beam.

% Input beam parameters

n=input('Enter number of degrees of freedom (must be an integer)');
E=input('Enter elastic modulus of beam (N/m^2)');
I=input('Enter area moment of inertia of cross section (m^4)');
L=input('Enter length of beam (m)');
% Specify location of nodes
for i=1:n
   z(i)=i*L/n;
end
% Determine columns of flexibility influence coefficients beginning with
% last column. Symmetry of flexibility matrix is used.
for i=1:n
   k=n-i+1;
   a=z(k);
   for j=1:k
      x=z(j);
      A(j,k)=(-x^3/6+a*x^2/2)/(E*I);
      A(k,j)=A(j,k);
   end
end
disp('Flexibility matrix for fixed-free beam')
disp('The system parameters are')
disp('Number of degrees of freedom='),disp(n)
disp('Elastic modulus (N/m^2)'),disp(E)
disp('Moment of inertia (m^4)'), disp(I)
disp('Length (m)'), disp(L)
disp('The locations of the nodes are (m)')
disp(z)
disp('The flexibility martrix is (m/N)')
disp(A)
```

(*a*)

Figure 5.16 (*a*) MATLAB script for solution of Example 5.15.

deflection equation is taken from Table D.2. The algorithm calculates the columns in reverse order sequentially, beginning with the last column. The guaranteed symmetry of the flexibility matrix is used to generate a_{ij} once a_{ji} is calculated. This algorithm minimizes the computations required in that the location where the deflection is calculated is always at or to the left of the location where the concentrated load is applied. In this case the $(z-a)^3/6u(z-a)$ term is always zero, as $z \le a$.

```
Flexibility matrix for fixed-free beam
The system parameters are
Number of degrees of freedom=
     4

Elastic modulus (N/m^2)
  2.1000e+011

Moment of inertia (m^4)
  1.6000e-005

Length (m)
     8

The locations of the nodes are (m)
     2      4      6      8

The flexibility matrix is (m/N)
  1.0e-004 *

    0.0079    0.0198    0.0317    0.0437
    0.0198    0.0635    0.1111    0.1587
    0.0317    0.1111    0.2143    0.3214
    0.0437    0.1587    0.3214    0.5079
```

(b)

Figure 5.16 (Con't) (b) Output when program is run.

Systems exist in which the stiffness matrix is singular and hence the flexil matrix does not exist. These systems are called *semidefinite* or *unconstraine* is shown in Chap. 6 that these systems have a natural frequency of zero a corresponding mode where the system moves as a rigid body.

The system of Fig. 5.17 has two degrees of freedom and is unconstrained. stiffness matrix for this system is calculated as

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

The second row of the stiffness matrix is a multiple of the first row, which implie the matrix is singular and a flexibility matrix for this system does not exist. In when the definition of flexibility influence coefficients is applied in an atten calculate the flexibility matrix, as shown in Fig. 5.18, no solution is found. S the system is unconstrained, when a unit force is applied to either mass, the sy

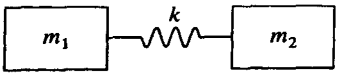


Figure 5.17 Unconstrained two-degree-of-freedom system.

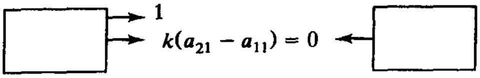


Figure 5.18 Free-body diagrams of system of Fig. 5.17 are used to show that the flexibility matrix does not exist.

cannot remain in equilibrium. Instead, the system will behave as a rigid body with uniform acceleration.

Another example of an unconstrained system is the system of Fig. 5.10 in Example 5.9. The stiffness matrix for this example is repeated here

$$\mathbf{K} = \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix}$$

Inspection of this matrix reveals that the first row plus two times the second row is proportional to the third row. Thus the three rows of the stiffness matrix are dependent, which implies that the stiffness matrix is singular, which, in turn, implies that the flexibility matrix does not exist. If, for example, a unit moment were applied to the pulley, then there are no other external forces which develop a moment about the center of the pulley. Hence equilibrium cannot be maintained.

A beam pinned at one end with no other support is an example of an unconstrained structural system. Application of a force or moment will lead to rigid body rotation about the pin support. A free-free beam is doubly unconstrained, in that it has two independent rigid-body motions. A free-free beam is unconstrained from transverse motion as well as rigid-body rotation.

Flexibility influence coefficients can be used to calculate the flexibility matrix. Equation (5.27) shows that knowledge of the flexibility matrix instead of knowledge of the stiffness matrix is sufficient to proceed with solution of the system of differential equations governing the vibrations of a multi-degree-of-freedom system. The choice of whether to determine the stiffness matrix or the flexibility matrix is usually easy.

For structural systems, calculation of the flexibility matrix is easier than calculation of the stiffness matrix. For these systems deflection equations from mechanics of solids are used to determine the deflection of a particle due to an applied concentrated load. The deflection equation for the structure is often available in a textbook or handbook (e.g., App. D). Thus calculation of the flexibility matrix is direct, whereas the solution of a system of simultaneous equations is necessary to determine each column of the stiffness matrix. However, calculation of the stiffness matrix is easier than calculation of the flexibility matrix for mechanical systems that comprise rigid bodies connected by flexible elements. For these systems application of the equations of static equilibrium to appropriate free-body diagrams is sufficient to calculate the stiffness matrix, while calculation of a column of the flexibility matrix also requires the solution of a system of simultaneous equations.

The stiffness matrix must be calculated for unconstrained systems.

5.7 LUMPED-MASS MODELING OF CONTINUOUS SYSTEMS

Vibrations of continuous systems are governed by partial differential equations. Analytical solutions to partial differential equations are often difficult to obtain. Thus approximate and numerical methods are often used to approximate the vibration properties and systems response of continuous systems. Some of these, such as Rayleigh-Ritz method and the finite-element method, are discussed in Chaps. 9 and 10. A simpler method of approximation is to replace the distributed inertia of the continuous system by a finite number of lumped inertia elements. A point where a lumped mass is placed is called a *node*. All inertia effects are concentrated at the nodes. The nodes are assumed to be connected by elastic, but massless elements. Generalized coordinates are chosen as the displacements of the nodes.

A lumped-mass model of a continuous system is a discrete model of a continuous system. A system with n nodes is modeled as an n-degree-of-freedom system. Differential equations of the form of Eq. (5.1) or Eq. (5.27) are derived to approximate the vibrations of the continuous system. It is necessary to determine the mass matrix, either the stiffness matrix or the flexibility matrix, and the force vector for the discrete approximation.

Unless the system is unconstrained the flexibility matrix is used in lumped mass modeling of a continuous system. The flexibility matrix is obtained by using flexibility influence coefficients, as described in Sec. 5.6. If the system is unconstrained, the stiffness matrix must be determined.

Lumped-mass approximations for modeling a continuous system using one-degree-of-freedom were considered in Chap. 1. Recall that the inertia effects of a linear spring are approximated by placing a particle of mass equal to one-third of the mass of the spring at its end. The one-third approximation determined by calculating the particle mass such that the kinetic energy of the model system is equal to the kinetic energy of the spring, assuming a linear displacement function along the axis of the spring. This model illustrates that it is incorrect to model the inertia effects of the spring by using the full mass of the spring. The kinetic energy of particles near its fixed support is much less than the kinetic energy of the particles near the point of attachment to the system. Kinetic energy considerations could be used to determine the mass matrix for a discrete approximation. However, such a mass matrix, called the *consistent mass matrix*, is difficult to obtain and is not a diagonal matrix. The amount of effort used in determining a consistent mass matrix would be better used in developing a finite-element model for the system.

For simplicity it is desirable to specify a diagonal mass matrix for a lumped-mass approximation of a continuous system. If a discretization is used where the mass of the system is lumped at nodes, then an obvious approximation to the mass matrix is a diagonal matrix with the nodal masses along the diagonal. In such a situation, the values of the nodal masses affects the accuracy of the system response. Using the one-degree-of-freedom approximation of the inertia effects of a linear spring as a guide, it is clear that using the entire mass of the system in the approximation will lead to errors in the approximation.

When a diagonal matrix is used to model the inertia effects of a continuous system, the mass lumped at each node should represent the mass of an identifiable region of the structure. A good scheme, whose accuracy will be questioned later, is to define the nodal mass as the mass of a region whose boundaries are halfway between the node and neighboring nodes on its right and left. If the node has no neighbor on one side, but is adjacent to a free end, then all of the mass between the node and the free end is used in calculating the nodal mass. If the particle is adjacent to a support that prevents motion, then only half of the mass between the node and the support is used. The accuracy of this method of approximation is considered in Chap. 6.

Calculation of the force vector may also require additional approximations. As shown in Sec. 6.3, the force vector is obtained by calculating the generalized forces, which occur when the method of virtual work is used. If a concentrated load is applied at a node, then the generalized force for the node's generalized coordinate is the value of the concentrated load and the generalized forces for all other coordinates are zero. However, if a concentrated load is applied at a location other than a node or the loading is distributed, calculation of the generalized forces requires additional approximations. The dynamic displacement is not available to apply the method of virtual work. In these cases it is suggested that the loading be replaced by a series of concentrated loads, calculated as follows, such that the resulting system is approximately statically equivalent to the applied loading. Static equivalence does not imply dynamic equivalence.

If the applied loading is replaced by a system of concentrated loads, the following method is used. The loading between any two nodes is replaced by a concentrated load at each of the nodes. The two concentrated loads are statically equivalent to the loading between the nodes. The sum of the concentrated loads is the resultant of the loading between the nodes. The moment of the distributed loading about either node is the same as the moment of the two concentrated loads about that point. Thus the total generalized force applied at a node is approximated by the sum of the contribution from the loading between the node and its neighbor to the left and the contribution from the loading between the node and its neighbor to the right. If the node is adjacent to a free end, the contribution to the loading between the node and the free end is the resultant of the loading. If the particle is adjacent to a support that prevents displacement, only the resultant of the loading between the node and the point halfway between the node and the support is used. In this case the work done by particles near supports is ignored in modeling the system, just as these particles' kinetic energy is ignored. The concentrated load is not statically equivalent to the actual loading if the particle is adjacent to a free end or a support.

Examp

Derive the differential equations whose solution approximates the forced response of the cantilever beam of Fig. 5.19. Use four degrees of freedom to discretize the system. The beam is made of a material of elastic modulus E and mass density ρ. It has a cross-sectional area A and moment of inertia I. Neglect damping.

Solution:

The beam is discretized by lumping its mass in four particles as shown in Fig. 5.19*b*. The nodes are chosen to be equally spaced. The generalized coordinates are the displacements of the nodes. The mass of each particle models the inertia effects of the regions shown in the figure. The loading is replaced by time-dependent concentrated loads at the nodes, as shown in Fig. 5.19*c*.

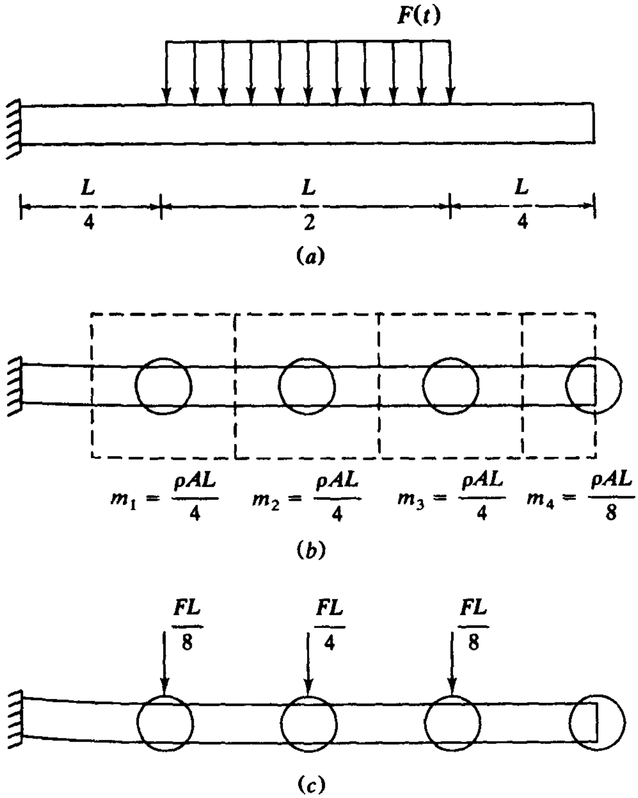


Figure 5.19 (*a*) System of Example 5.16; (*b*) nodal masses; (*c*) nodal forces.

The flexibility matrix for this discretized system is determined from flexibility influence coefficients, as described in Sec. 5.6. The first column is obtained by placing a unit load at the first node and calculating the resulting deflections at each of the nodes. The result is

$$\mathbf{A} = \frac{L^3}{384EI}\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}$$

The mass matrix is a diagonal matrix with the nodal masses along the diagonal. The force vector is simply the vector of concentrated loads from Fig. 5.19*c*. Then Eq. (5.27) becomes

$$\left(\frac{\rho AL}{4}\right)\left(\frac{L^3}{384EI}\right)\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix}$$

$$+\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left(\frac{L^3}{384EI}\right)\left(\frac{FL}{8}\right)\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 28 & 40 \\ 8 & 28 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

which simplifies to

$$\frac{\rho AL^3}{1536EI}\begin{bmatrix} 4 & 10 & 16 & 11 \\ 10 & 32 & 56 & 40 \\ 16 & 56 & 108 & 81 \\ 22 & 80 & 162 & 128 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{\rho AL^4 F(t)}{3072EI}\begin{bmatrix} 18 \\ 65 \\ 118 \\ 172 \end{bmatrix}$$

PROBLEMS

5.1–5.15. Use the free-body diagram method to derive the differential equations governing the motion of the systems shown in Figs. P5.1 to P5.15 using the indicated generalized coordinates. Make linearizing assumptions and write the resulting equations in matrix form.

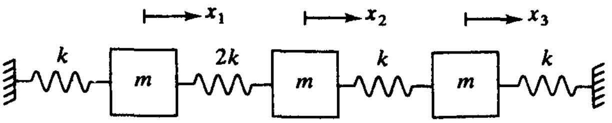


FIGURE P5.1
(Problems 5.1, 5.16, 5.31, 5.37, 5.45, 5.60)

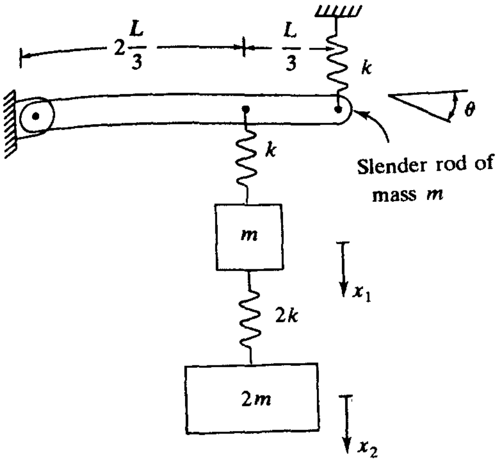


FIGURE P5.2
(Problems 5.2, 5.17, 5.38, 5.46)

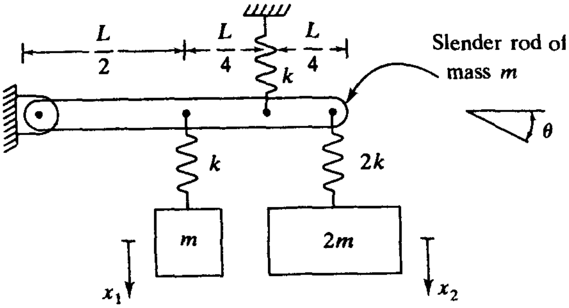


FIGURE P5.3
(Problems 5.3, 5.18, 5.47, 5.61)

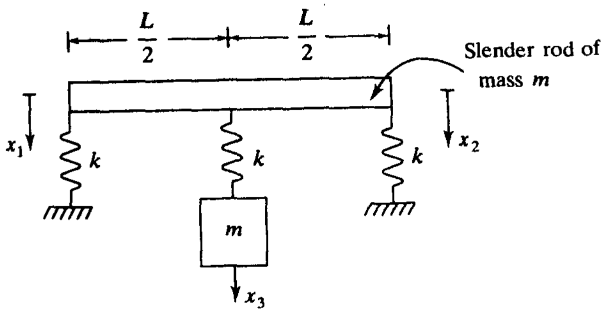


FIGURE P5.4
(Problems 5.4, 5.19, 5.32, 5.48)

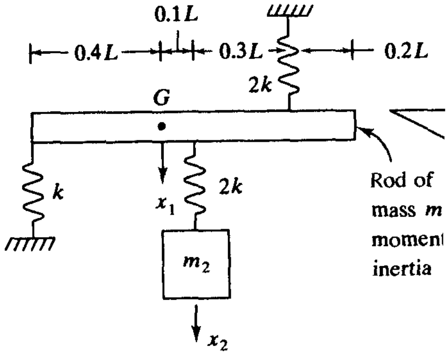
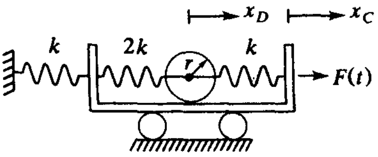


FIGURE P5.5
(Problems 5.5, 5.20, 5.33, 5.39, 5.49, 5.62)



Thin disk of mass m and radius r rolls without slip relative to center of mass $2m$. x_D is absolute displacement of mass center of disk

FIGURE P5.6
(Problems 5.6, 5.21, 5.34, 5.40, 5.50, 5.65)

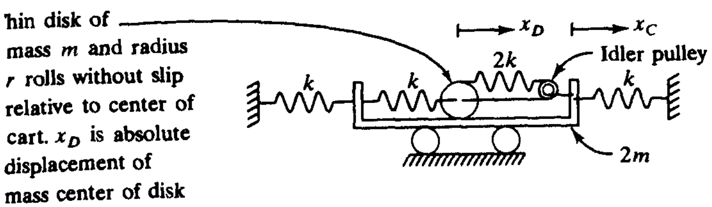


FIGURE P5.7
(Problems 5.7, 5.22, 5.41, 5.51)

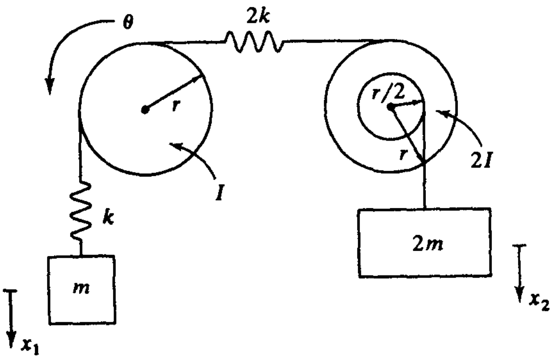


FIGURE P5.8
(Problems 5.8, 5.23, 5.42, 5.52)

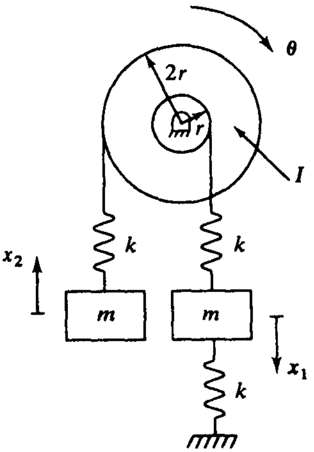


FIGURE P5.9
(Problems 5.9, 5.24, 5.53, 5.63)

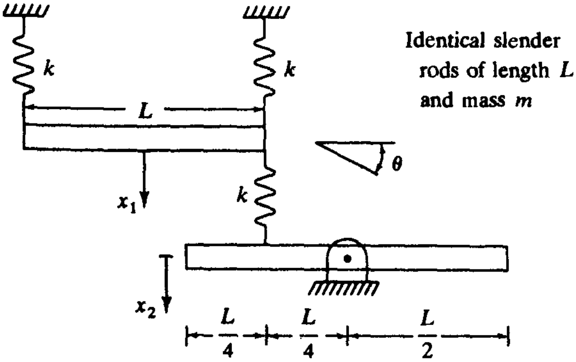


FIGURE P5.10
(Problems 5.10, 5.25, 5.35, 5.43, 5.54, 5.64)

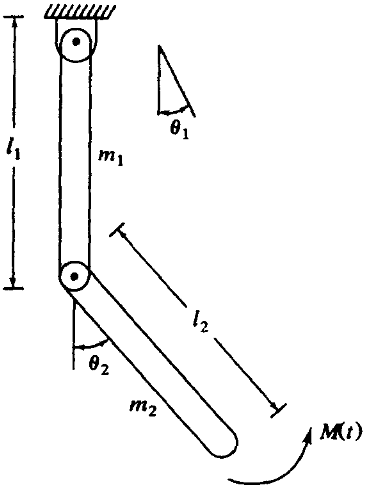


FIGURE P5.11
(Problems 5.11, 5.26, 5.36, 5.44, 5.55)

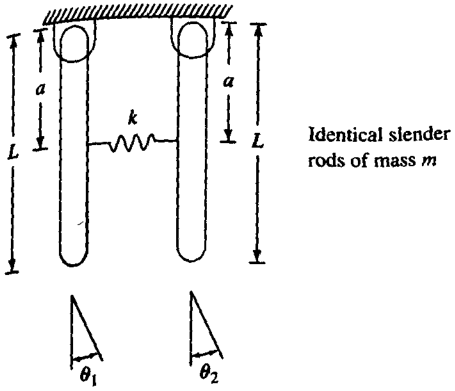


FIGURE P5.12
(Problems 5.12, 5.27, 5.56)

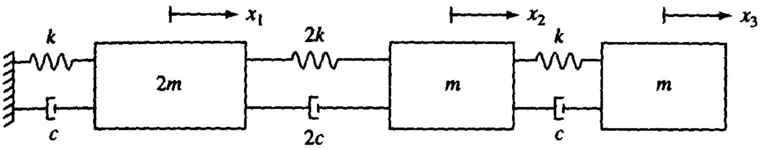


FIGURE P5.13
(Problems 5.13, 5.28, 5.57)

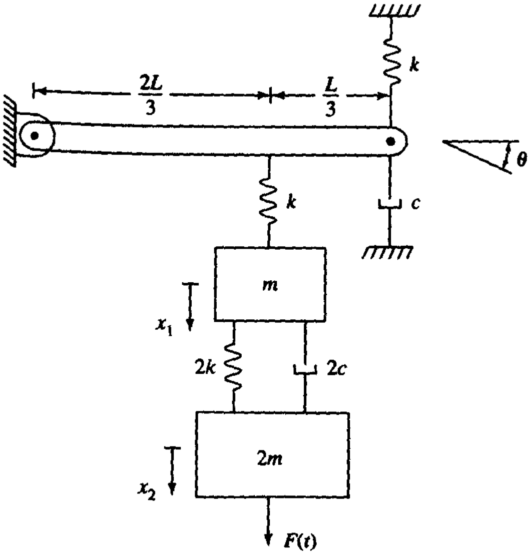


FIGURE P5.14
(Problems 5.14, 5.29, 5.58)

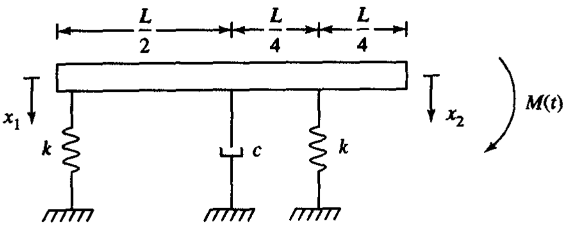


FIGURE P5.15
(Problems 5.15, 5.30, 5.59)

5.16–5.30. Use Lagrange's equations to derive the differential equations governing the motion of the systems shown in Figs. P5.1 to P5.15 using the indicated generalized coordinates. Make linearizing assumptions and write the differential equations in matrix form. Indicate whether the system is statically coupled, dynamically coupled, neither, or both.

5.31–5.36. Determine the kinetic energy of the systems in Figs. P5.1, P5.4, P5.5, P5.6, P5.10, and P5.11 at an arbitrary instant in terms of the indicated generalized coordinates. Make necessary linearizing assumptions and put the kinetic energy in the quadratic form of Eq. (5.7). Use the quadratic form of the kinetic energy to determine the mass matrix for the system.

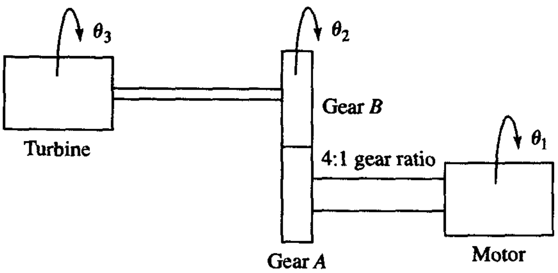
5.37–5.44. Determine the potential energy of the systems in Figs. P5.1, P5.2, P5.5, P5.6, P5.7, P5.8, P5.10, and P5.11 at an arbitrary instant in terms of the indicated generalized coordinates. Make necessary linearizing assumptions and put the potential energy in the quadratic form of Eq. (5.6). Use the quadratic form of the potential energy to determine the stiffness matrix for the system.

5.45–5.59. Derive the stiffness matrix for the systems in Figs. P5.1 to P5.15 using the indicated generalized coordinates and stiffness influence coefficients.

5.60–5.64. Derive the flexibility matrix for the systems in Figs. P5.1, P5.3, P5.5, P5.9, and P5.10 using the indicated generalized coordinates and flexibility influence coefficients.

5.65. Derive the differential equations governing the motion of the system of Fig. P5.6 using x_c, the displacement of the cart, and x_r, the displacement of the disk relative to the cart, as the generalized coordinates.

5.66. Derive the differential equations governing the torsional oscillations of the turbomotor of Fig. P5.66. The motor operates at 800 rpm and the turbine shaft turns at 3200 rpm.



Moments of inertia:	Turbine shaft	Motor shaft
Motor 1800 kg · m^2	$G = 80 \times 10^9$ N/m^2	$G = 80 \times 10^9$ N/m^2
Turbine 600 kg · m^2	$L = 2.1$ m	$L = 1.4$ m
Gear A 400 kg · m^2	$d = 180$ mm	$d = 305$ mm
Gear B 80 kg · m^2		

FIGURE P5.66

5.67. Derive the differential equations governing the torsional oscillations of the system of Fig. P5.67.

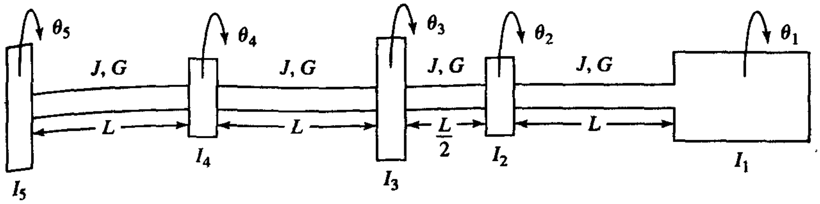


FIGURE P5.67

5.68. A rotor of mass m is mounted on an elastic shaft with journal bearings at both ends. A three-degree-of-freedom model of the system is shown in Fig. P5.68. Each journal bearing is modeled as a spring in parallel with a viscous damper. Derive the differential equations governing the transverse motion of the system.

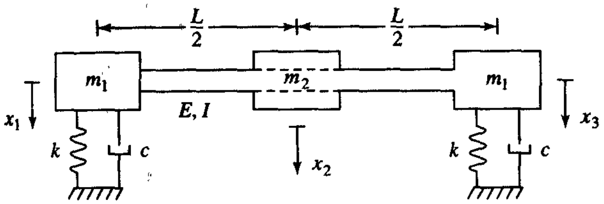


FIGURE P5.68

5.69. A two-degree-of-freedom model of an airfoil, shown in Fig. P5.69, is used for flutter analysis. Derive the differential equations governing the motion using the indicated generalized coordinates.

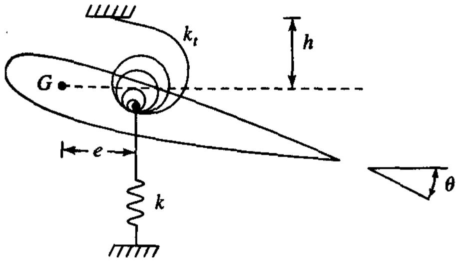


FIGURE P5.69

5.70. A three-degree-of-freedom model of a railroad bridge is shown in Fig. P5.70. The bridge is composed of three rigid spans. Each span is pinned at its base. Using the angular displacements of the spans as generalized coordinates, derive the differential equations governing the motion of bridge.

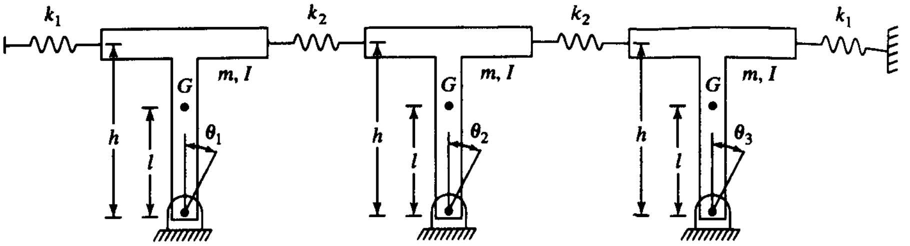


FIGURE P5.70

5.71. A five-degree-of-freedom model of a railroad bridge is shown in Fig. P5.71. The bridge is composed of five rigid spans. The connection between each span and its base is modeled as a torsional spring. Using the angular displacements of the spans as the generalized coordinates, derive the differential equations governing the motion of the bridge.

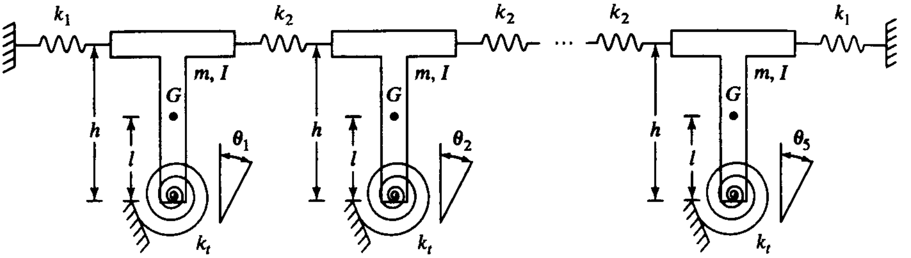


FIGURE P5.71

5.72. A two-degree-of-freedom model of a vehicle suspension system is shown in Fig. P5.72. Derive the differential equations governing the motion of the system.

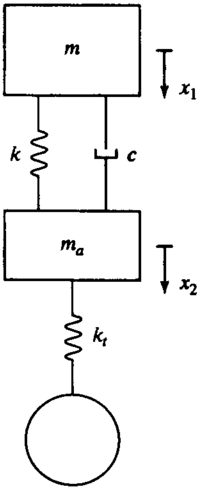


FIGURE P5.72

5.73. A four-degree-of-freedom model of a vehicle suspension system is shown in Fig. P5.73. Derive the differential equations governing the motion of the system using the indicated generalized coordinates.

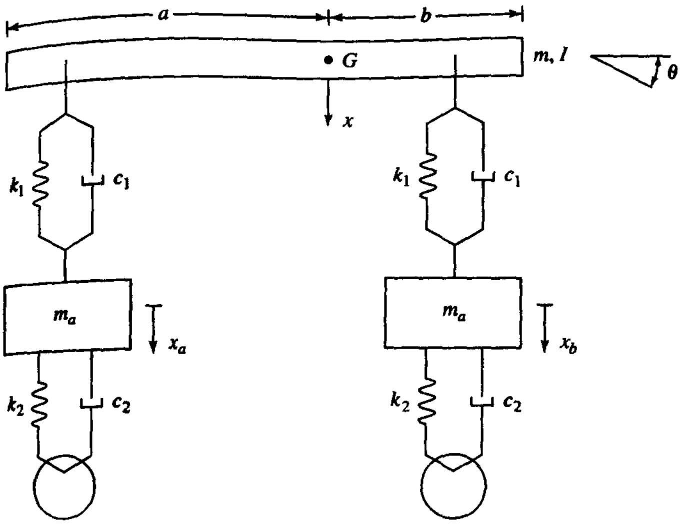


FIGURE P5.73

5.74. A four-degree-of-freedom model of an aircraft wing is shown in Fig. P5.74. Derive the flexibility matrix for the model.

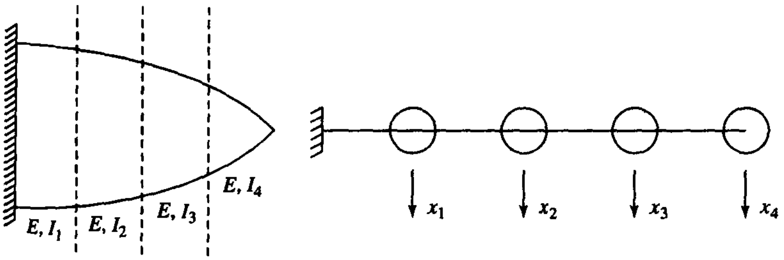


FIGURE P5.74

5.75. Figure P5.75 illustrates a three-degree-of-freedom model of an aircraft. A rigid fuselage is attached to two thin flexible wings. An engine is attached to each wing, but the wings themselves are of negligible mass. Derive the differential equations governing the motion of the system.

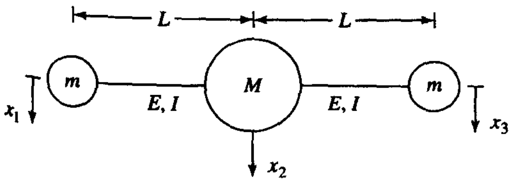


FIGURE P5.75

5.76. An airplane is modeled as two flexible wings attached to a rigid fuselage (Fig. P5.76). Use two degrees of freedom to model each wing and derive the differential equations governing the motion of the five-degree-of-freedom system.

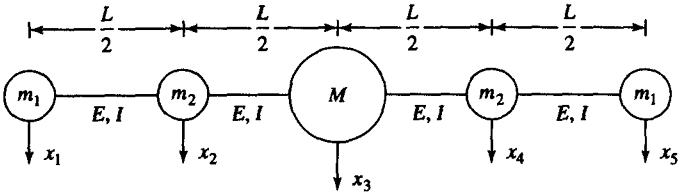


FIGURE P5.76

5.77. A drum of mass m is being hoisted by an overhead crane as illustrated in Fig. P5.77. The crane is modeled as a simply supported beam with a winch at its midspan. The cable connecting the crane to the drum is of stiffness k. Derive the differential equations governing the motion of the system using three degrees of freedom to model the beam.

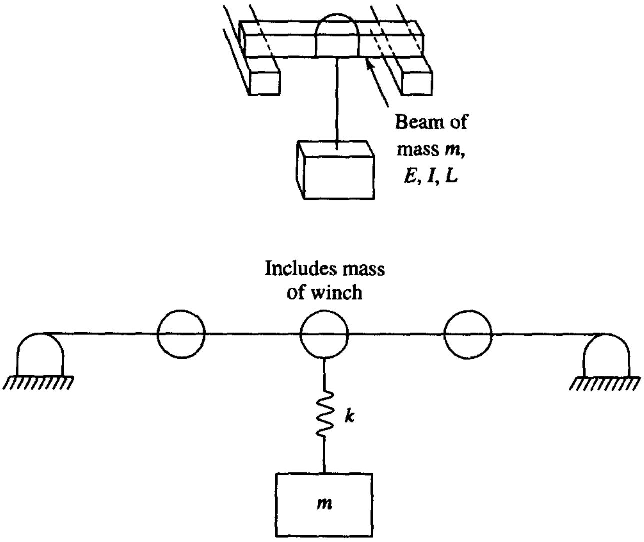


FIGURE P5.77

5.78–5.81. The beams shown are made of an elastic material of elastic modulus 210×10^9 N/m^2 and have a cross-sectional moment of inertia 1.3×10^{-5} m^4. Determine the flexibility matrix when a three-degree-of-freedom model is used to analyze the beam's vibrations. Use the displacements of the particles shown as generalized coordinates. Use Table D.2 for deflection calculations.

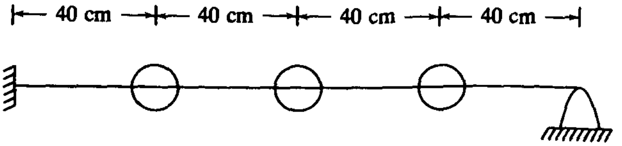


FIGURE P5.78

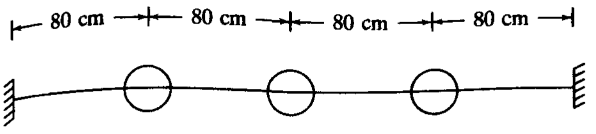


FIGURE P5.79

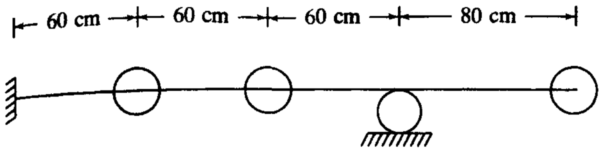


FIGURE P5.80

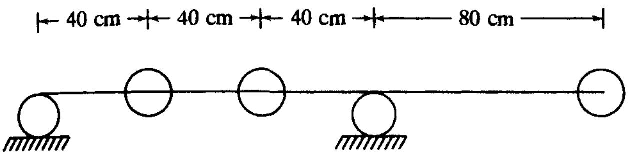


FIGURE P5.81

5.82. Determine the stiffness matrix for the three-degree-of-freedom model of the free-free beam of Fig. P5.82.

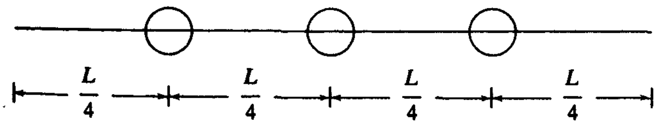


FIGURE P5.82

5.83. Using a two-degree-of-freedom model, derive the differential equations governing the forced vibration of the system of Fig. P5.83.

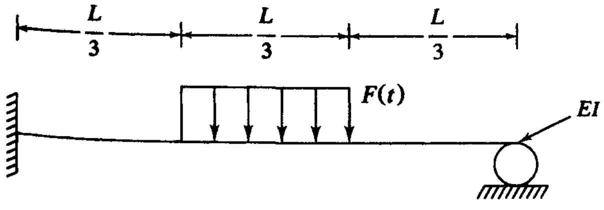


FIGURE P5.83

5.84. Use a two-degree-of-freedom model to derive the differential equations governing the motion of the system of Fig. P5.84. A thin disk of mass moment of inertia I_D is attached to the end of the fixed-free beam. Use x, the vertical displacement of the disk, and θ, the slope of the end of the beam, as generalized coordinates.

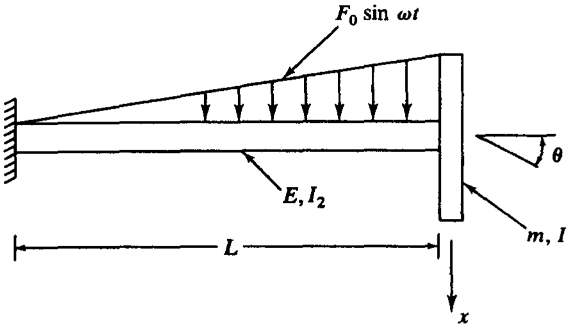


FIGURE P5.84

MATLAB PROBLEMS

M5.1. The file VIBES_5A.m contains the MATLAB script used in Example 5.15. Use the file to determine the flexibility matrix for a six-degree-of-freedom model of a fixed-free beam where the nodes are equally spaced along the span of the beam. If you use unit values of E, I, and L, the resulting matrix should be multiplied by L^3/EI.

M5.2. The file VIBES_5B.m contains the MATLAB script for calculation of the flexibility matrix of an n-degree-of-freedom model of a fixed-fixed beam. The program allows for nodes at any location along the span of the beam. Use VIBES_5B.m to calculate the flexibility matrix for the four-degree-of-freedom system of Fig. PM5.2.

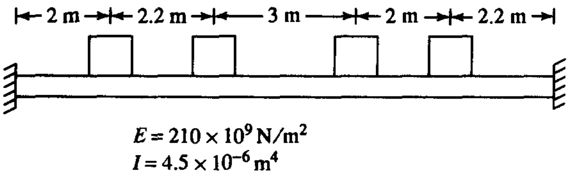


FIGURE PM5.2

M5.3. The file VIBES_5C.m contains the MATLAB script for determination of the stiffness matrix of a free-free beam. A free-free beam is unrestrained, and thus its flexibility matrix does not exist. Use VIBES_5C.m to determine the stiffness matrix for a four-degree-of-freedom model of the missile of Fig. PM5.3.

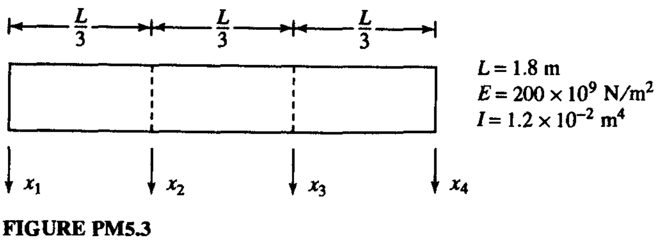


FIGURE PM5.3

M5.4. Write a MATLAB script file to determine the flexibility matrix for an n-degree-of-freedom model of a fixed-pinned beam. Use the program to determine the flexibility matrix for a three-degree-of-freedom model of a fixed-pinned beam.

M5.5. Write a MATLAB script file to determine the flexibility matrix for an n-degree-of-freedom model of a beam pinned at one end, free at one end, but with an intermediate simple support. Use the program to determine the flexibility matrix for a three-degree-of-freedom model of the diving board of Prob. 2.36. Place one node midway between the two supports, one node midway between the intermediate support and the end, and one node at the end of the board.

M5.6. Write a MATLAB script file to determine the stiffness matrix for an n-degree-of-freedom model of a pinned-free beam. A pinned-free beam is unrestrained and thus its flexibility matrix does not exist. Use the program to determine the flexibility matrix for a five-degree-of-freedom model of a pinned-free beam.

chapter 6

Free Vibrations of Multi-Degree-of-Freedom Systems

6.1 Introduction

The vibrations of an n-degree-of-freedom system are governed by a system of n differential equations. The general solution of these differential equations is the sum of a homogeneous solution and a particular solution. The homogeneous solution is the free-vibration response, and its determination is often necessary before the forced response can be determined.

The free-vibration analysis of a multi-degree-of-freedom system is significantly more complicated than the free-vibration analysis of a one-degree-of-freedom system. Determination of the free-vibration response requires matrix algebra. A reader unfamiliar with matrix algebra and terminology is encouraged to read App. C before proceeding with Chaps. 6 and 7.

In the following discussion it is assumed that the mass matrix, the stiffness matrix, and, for systems with viscous damping, the damping matrix are all symmetric. The symmetry of these matrices is guaranteed if energy methods are used to derive the differential equations. If energy methods are not used and **M**, **C**, and **K** are not symmetric, then the differential equations can be algebraically manipulated until a reformulation involves symmetric matrices.

6.2 Normal-Mode Solution

The general formulation of the differential equations governing free vibrations of a linear undamped n-degree-of-freedom system is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \qquad [6.1]$$

where **M** and **K** are the symmetric $n \times n$ mass and stiffness matrices, respectively, and $\mathbf{x}$ is the n-dimensional column vector of generalized coordinates.

Free vibrations of a multi-degree-of-freedom system are initiated by the presence of an initial potential or kinetic energy. If the system is undamped, there are no dissipative mechanisms and it is expected that the free vibrations described by the solution of Eq. (6.1) are periodic. It is assumed that the vibrations are synchronous in that all dependent variables execute motion with the same time-dependent behavior. Thus, when free vibrations at a single frequency are initiated for a particular system, the ratio of any two dependent variables is independent of time. These assumptions lead to hypothesizing the normal-mode solution of Eq. (6.1) in the form

$$\mathbf{x}(t) = \mathbf{X}e^{i\omega t} \tag{6.2}$$

where ω is the frequency of vibration and $\mathbf{X}$ is an n-dimensional vector of constants, called a *mode shape*. This hypothesis implies that certain initial conditions lead to a solution of the form of Eq. (6.2) for specific values of ω. The values of ω such that Eq. (6.2) is a solution of Eq. (6.1) are called the *natural frequencies*. Each natural frequency has at least one corresponding mode shape. Since the differential equations represented by Eq. (6.1) are linear and homogeneous, their general solution is a linear superposition over all possible modes.

Substitution of Eq. (6.2) into Eq. (6.1) leads to

$$(-\omega^2\mathbf{MX} + \mathbf{KX})e^{i\omega t} = \mathbf{0} \tag{6.3}$$

Since $e^{i\omega t} \neq 0$, for any real value of t,

$$-\omega^2\mathbf{MX} + \mathbf{KX} = \mathbf{0} \tag{6.4}$$

The mass matrix is nonsingular, and thus $\mathbf{M}^{-1}$ exists. Premultiplying Eq. (6.4) by $\mathbf{M}^{-1}$ and rearranging gives

$$(\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I})\mathbf{X} = 0 \tag{6.5}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Equation (6.5) is the matrix representation of a system of n simultaneous linear algebraic equations for the n components of the mode shape vector. The system is homogeneous. Application of Cramer's rule gives the solution for the jth component of $\mathbf{X}$, X_j, as

$$X_j = \frac{0}{\det|\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}|} \tag{6.6}$$

Thus the trivial solution ($\mathbf{X} = \mathbf{0}$) is obtained unless

$$\det|\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}| = 0 \tag{6.7}$$

Hence, applying the definitions of App. C, ω^2 must be an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$. The square root of a real positive eigenvalue has two possible values, one positive and one negative. While both are used to develop the general solution, the positive square root is identified as a natural frequency. The mode shape is the corresponding eigenvector.

It is shown in Sec. 5.6 that when the stiffness matrix, $\mathbf{K}$, is nonsingular, its inverse is the flexibility matrix, $\mathbf{A}$. Premultiplying Eq. (6.4) by $\mathbf{A}$ leads to

$$(-\omega^2\mathbf{AM} + \mathbf{I})\mathbf{X} = \mathbf{0} \tag{6.8}$$

Dividing by ω^2 gives

$$\left(\mathbf{AM} - \frac{1}{\omega^2}\mathbf{I}\right)\mathbf{X} = \mathbf{0} \tag{6.9}$$

Thus the natural frequencies are the reciprocals of the positive square roots of the eigenvalues of **AM** and the mode shapes are its eigenvectors. The matrix, **AM**, is often called the *dynamical matrix.*

Natural frequencies of multi-degree-of-freedom systems are calculated as either the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ or as the reciprocals of the square roots of the eigenvalues of **AM**. The mode shapes are the corresponding eigenvectors of either matrix.

6.3 NATURAL FREQUENCIES AND MODE SHAPES

In the previous section it is shown that the natural frequencies of an n-degree-of-freedom system are the positive square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ or the reciprocals of the positive square roots of the eigenvalues of **AM**. The mode shape vectors are the corresponding eigenvectors. As shown in App. C, the evaluation of Eq. (6.7) leads to an nth-order polynomial equation, called the *characteristic equation,* whose roots are the eigenvalues. Since all elements of the mass and stiffness matrices are real, all coefficients in the characteristic equation are real and thus if complex roots occur, they must occur in complex conjugate pairs. However, it can be shown (see Sec. 6.7) that, because of the symmetry of **M** and **K**, the characteristic equation has only real roots. Negative roots are possible, but lead to imaginary values of the natural frequency. When the negative square root of a negative eigenvalue is multiplied by i to form the exponent in the normal-mode solution, Eq. (6.2), a real positive exponent is developed. This term grows without bound as time increases. Such a system is unstable.

Assume that all eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ corresponding to symmetric mass and stiffness matrices are nonnegative. Then there exist n real natural frequencies that can be ordered by $\omega_1 \le \omega_2 \le \cdots \le \omega_n$. Each distinct eigenvalue ω_i^2, $i = 1, 2, \ldots, n$, has a corresponding nontrivial eigenvector, $\mathbf{X}_i$, which satisfies

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{X}_i = \omega_i^2\mathbf{X}_i \tag{6.10}$$

This mode shape, $\mathbf{X}_i$, is an n-dimensional column vector of the form

$$\mathbf{X}_i = \begin{bmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{in} \end{bmatrix} \tag{6.11}$$

Since the system of equations represented by Eq. (6.10) is homogeneous, the mode shape is not unique. However, if ω_1^2 is not a repeated root of the characteristic equation, then there is only one linearly independent nontrivial solution of Eq. (6.10).

The eigenvector is unique only to an arbitrary multiplicative constant. Normalization schemes exist such that the constant is chosen so the eigenvector satisfies an externally imposed condition.

If ω_i^2 is a repeated root of the characteristic equation of multiplicity r, there are r linearly independent nontrivial solutions of Eq. (6.10). Each of these mode shapes is also unique to a multiplicative constant.

Solution of the eigenvalue-eigenvector problem is an important part of the vibration analysis of multi-degree-of-freedom systems. The quadratic formula is used to find the roots of the characteristic equation for a two-degree-of-freedom system. The natural frequencies of a three-degree-of-freedom system are obtained by finding the roots of a cubic polynomial, which can be done by trial and error or an iterative method. The algebraic complexity of the solution grows exponentially with the number of degrees of freedom. The development of a characteristic equation for an n-degree-of-freedom system requires the evaluation of an $n \times n$ determinant and the natural frequencies are the n roots of the characteristic equation. The determination of each eigenvector requires the solution of n homogeneous simultaneous algebraic equations. Thus numerical methods which do not require the evaluation of the characteristic equation are used for systems with a large number of degrees of freedom.

Find the natural frequencies and mode shapes for the slender rod of Fig. 5.4. Use x and θ as generalized coordinates.

Solution:

The differential equations of motion using x and θ as generalized coordinates are obtained in Example 5.8.

$$\begin{bmatrix} m & 0 \\ 0 & m\frac{L^2}{12} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & 5k\frac{L^2}{16} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix $\mathbf{M}^{-1}\mathbf{K}$ is calculated as

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{12}{mL^2} \end{bmatrix} \begin{bmatrix} 2k & -k\frac{L}{4} \\ -k\frac{L}{4} & 5k\frac{L^2}{16} \end{bmatrix} = \begin{bmatrix} 8\phi & -\phi L \\ -\phi\frac{12}{L} & 15\phi \end{bmatrix}$$

where
$$\phi = \frac{k}{4m}$$

The determinant whose evaluation yields the characteristic equation is

$$\det \begin{bmatrix} 8\phi - \lambda & -\phi L \\ -\phi\frac{12}{L} & 15\phi - \lambda \end{bmatrix} = 0$$

The resulting characteristic equation is

$$\beta^2 - 23\beta + 108 = 0$$

where

$$\beta = \frac{\lambda}{\phi}$$

Application of the quadratic formula yields

$$\beta = \frac{23 \pm \sqrt{(23)^2 - 4(108)}}{2} = 6.58,\ 16.42$$

Since the natural frequencies are the positive square roots of the eigenvalues,

$$\omega_1 = 1.28\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.03\sqrt{\frac{k}{m}}$$

The equations from which the mode shapes are obtained are

$$\begin{bmatrix} 8\phi - \lambda_i & -\phi L \\ -\phi\dfrac{12}{L} & 15\phi - \lambda_i \end{bmatrix} \begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the determinant of the matrix is zero, these equations are not independent and only one is used. The first equation gives the following relation between the components of each mode shape:

$$X_{i2} = \frac{8\phi - \lambda_i}{\phi L} X_{i1}$$

Arbitrarily choosing $X_{i1} = 1$ leads to $X_{12} = 1.42/L$ for $\lambda_1 = 6.58\phi$ and $X_{22} = -8.42/L$ for $\lambda_2 = 16.42\phi$. The mode shape vectors are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ \dfrac{1.43}{L} \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -\dfrac{8.42}{L} \end{bmatrix}$$

Graphical representations of the mode shapes are shown in Fig. 6.1. A node, or point of zero velocity, occurs at a distance $0.119L$ to the right of the mass center for the second mode. There are no nodes on the bar for the first mode.

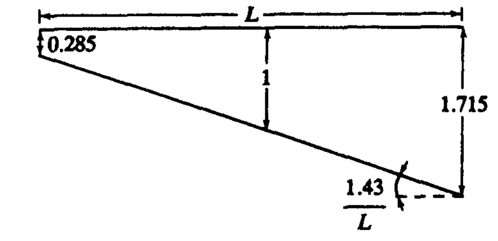
Example

Resolve Example 6.1 using x_A and x_B as generalized coordinates.

Solution:

The differential equations using x_A and x_B as generalized coordinates are

$$\begin{bmatrix} \dfrac{7}{27}m & \dfrac{2}{27}m \\ \dfrac{2}{27}m & \dfrac{16}{27}m \end{bmatrix} \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



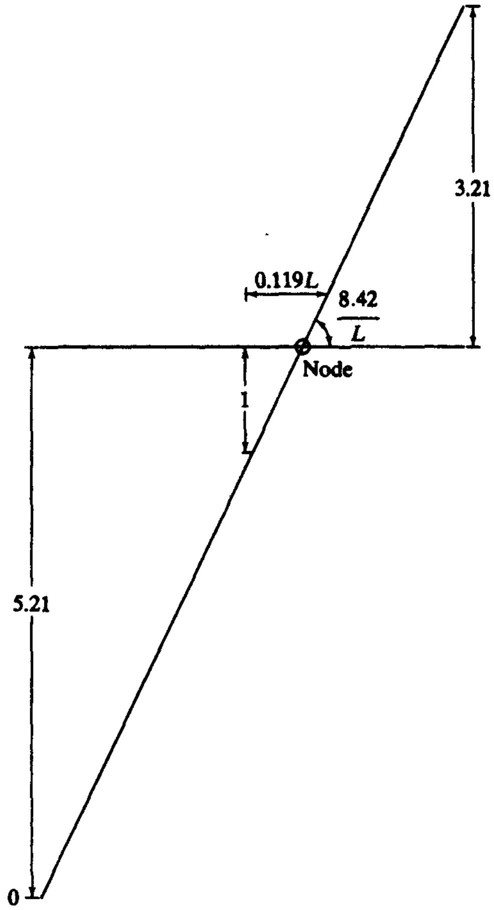


Figure 6.1 Mode shapes for Example 6.1. Displacement of mass center is taken as one. (a) $\omega = 1.28\sqrt{k/m}$; (b) $\omega = 2.03\sqrt{k/m}$, a node occurs a distance $0.119L$ to right of mass center. Mode shape represents rigid body rotation about node.

Since the system is dynamically coupled, but not statically coupled, it is easiest to invert the stiffness matrix to determine the flexibility matrix and calculate **AM**

$$\mathbf{AM} = \frac{m}{27k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 7\phi & 2\phi \\ 2\phi & 16\phi \end{bmatrix}$$

where $\phi = m/27k$. The eigenvalues of **AM** are determined from

$$\begin{vmatrix} 7\phi - \lambda & 2\phi \\ 2\phi & 16\phi - \lambda \end{vmatrix} = 0$$

$$\beta^2 - 23\beta + 108 = 0$$

where $\beta = \lambda/\phi$. Note that the characteristic equation is the same as obtained in Example 6.1 except for a different definition of ϕ. The eigenvalues are

$$\lambda_1 = 6.58\phi = 6.58\left(\frac{m}{27k}\right) = 0.244\frac{m}{k}$$

$$\lambda_2 = 16.42\phi = 16.42\left(\frac{m}{27k}\right) = 0.608\frac{m}{k}$$

The system's natural frequencies are

$$\omega_1 = \frac{1}{\sqrt{\lambda_2}} = 1.28\sqrt{\frac{k}{m}} \qquad \omega_2 = \frac{1}{\sqrt{\lambda_1}} = 2.03\sqrt{\frac{k}{m}}$$

which are the same natural frequencies calculated in Example 6.1. The mode shape vectors are obtained by solving

$$\begin{bmatrix} 7\phi - \omega_i^2 & 2\phi \\ 2\phi & 16\phi - \omega_i^2 \end{bmatrix}\begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first of these equations leads to

$$X_{i2} = \frac{7\phi - \omega_i^2}{-2\phi} X_{i1}$$

which leads to the mode shape vectors

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 4.71 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.21 \end{bmatrix}$$

The graphical representation of these mode shape vectors is given in Fig. 6.2.

Example 6

A simplified model of an automobile suspension system is shown in Fig. 6.3. The body of the vehicle weighs 3000 lb, has a centroidal moment of inertia of 300 slug · ft^2, and is connected to four massless springs. The springs are connected to the axle-wheel assembly. If the inertia of the axle-wheel assembly is ignored, then the elasticity of the tires is included by adding a spring in series with each of the existing springs. The result is the two-degree-of-freedom model with equivalent spring stiffness

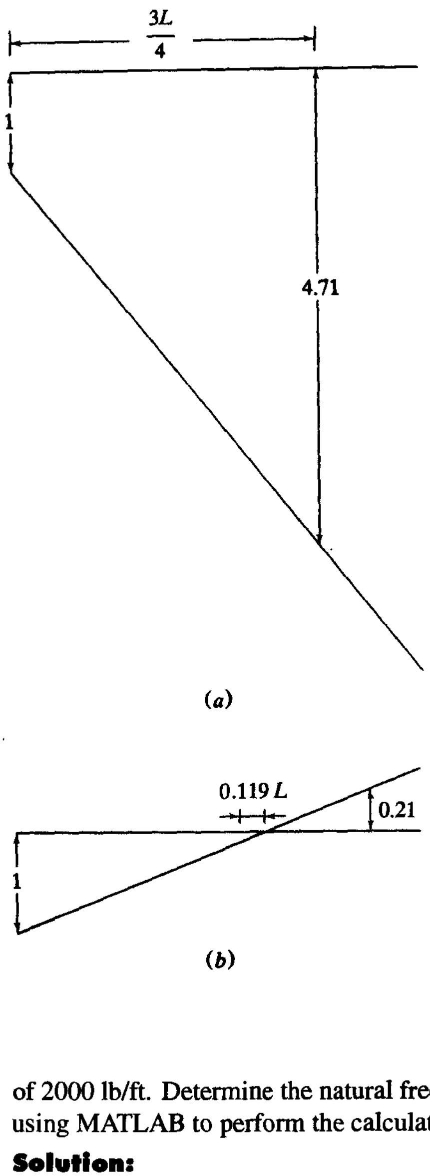


Figure 6.2 Mode shapes for Example 6.2. These are identical to mode shapes of Fig. 6.1 except that displacement of left end is taken to be one. (a) $\omega = 1.28\sqrt{k/m}$; (b) $\omega = 2.03\sqrt{k/m}$.

of 2000 lb/ft. Determine the natural frequencies for this two-degree-of-freedom model, using MATLAB to perform the calculations.

Solution:

Let x be the displacement of the mass center and θ be the angular rotation of the bar from its horizontal position. The differential equations derived by either summing forces in the vertical direction and summing moments about the mass center or by using energy methods are

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & (b-a)k \\ (b-a)k & (b^2+a^2)k \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

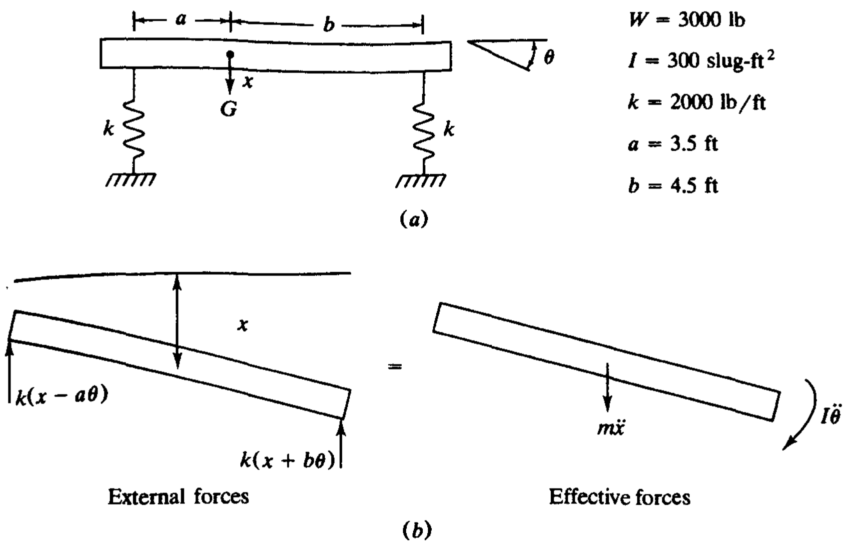


Figure 6.3 (a) Two-degree-of-freedom model of automobile suspension system; (b) free-body diagrams used to derive differential equations.

The MATLAB script for solution of this problem and the output when the script is run are given in Fig. 6.4.

Example 6.

Calculate the natural frequencies and the mode shapes for the three-degree-of-freedom system of Fig. 6.5.

Solution:

The differential equations for free vibrations using the displacements of the masses from equilibrium as the generalized coordinates are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Application of Eq. (6.7) gives

$$\det \begin{bmatrix} 3\phi - \lambda & -2\phi & 0 \\ -2\phi & 3\phi - \lambda & -\phi \\ 0 & -2\phi & 6\phi - \lambda \end{bmatrix} = 0$$

where

$$\phi = \frac{k}{m}$$

```
Solution of Example 6.3
Two-degree-of-freedom model of suspension system
Natural frequency and mode shape calculations
Input parameters
=input('Input vehicle weight in lb   ');
-input('Input mass moment of inertia in slugs-ft^2   ');
input('Input stiffness in lb/ft   ');
input('Distance from rear springs to center of gravity in ft   ');
input('Distance from front springs to center of gravity in ft   ');
Set up mass matrix
32.2;                          % Acceleration due to gravity in ft/s^2
W/g;                           % Vehicle mass
[m,0;0,I];
Set up stiffness matrix
[2*k,(b-a)*k;(b-a)*k,(b^2+a^2)*k];
Calculate eigenvalues and eigenvectors of M^(-1)*K
inv(M)*K;
,D]=eig(C);
_1=sqrt(D(1,1));
_2=sqrt(D(2,2));
=[V(1,1);V(2,1)];
=[V(1,2);V(2,2)];
Output
sp('Weight of vehicle in lb=   ');disp(W)
sp('Mass moment of inertia in slugs-ft^2=   ');disp(I)
sp('Stiffness in lb/ft=   ');disp(k)
sp('Distance from rear springs to center of gravity in ft=   ');disp(a)
sp('Distance from front springs to center of gravity in ft=   ');disp(b)
sp('Mass matrix   ');disp(M)
sp('Stiffness matrix   ');disp(K)
sp('Natural frequencies in rad/s=   ');disp(om_1);disp(om_2)
sp('Mode shape vectors   ');disp(X1);disp(X2)
```

(*a*)

gure 6.4 (*a*) MATLAB script for solution of Example 6.3. Eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are calculated by using MATLAB program eig. (*b*) Output produced by the script.

```
Input vehicle weight in lb   3000
Input mass moment of inertia in slugs-ft^2   300
Input stiffness in lb/ft   2000
Distance from rear springs to center of gravity in ft
Distance from front springs to center of gravity in ft
Weight of vehicle in lb=
        3000

Mass moment of inertia in slugs-ft^2=
   300

Stiffness in lb/ft=
        2000

Distance from rear springs to center of gravity in ft=
    3.5000

Distance from front springs to center of gravity in ft=
    4.5000

Mass matrix
   93.1677          0
         0   300.0000

Stiffness matrix
        4000        2000
        2000       65000

Natural frequencies in rad/s=
    6.4895

   14.7474
```

Figure 6.4 (Con't) (*b*) Output produced by the script.

```
Mode shape vectors
   -0.9993
    0.0382

   -0.1221
   -0.9925
```

(b)

Figure 6.4B (Con't)

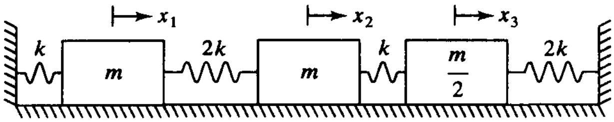


Figure 6.5 Three-degree-of-freedom system of Example 6.4.

Expansion of the determinant yields the characteristic equation

$$-\beta^3 + 12\beta^2 - 39\beta + 24 = 0$$

where $\beta = \lambda/\phi$. A plot of the preceding cubic polynomial is given in Fig. 6.6. The roots of this equation are

$$\beta = 0.798, 4.455, 6.747$$

which leads to the natural frequencies

$$\omega_1 = 0.893\sqrt{\frac{k}{m}} \qquad \omega_2 = 2.110\sqrt{\frac{k}{m}} \qquad \omega_3 = 2.597\sqrt{\frac{k}{m}}$$

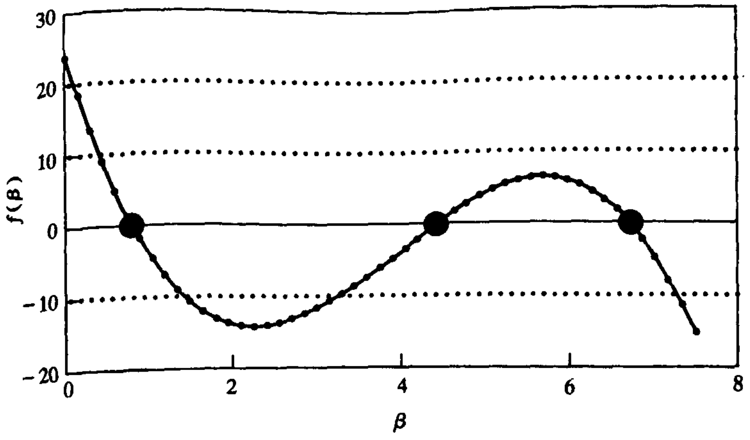


Figure 6.6 Plot of characteristic equation of Example 6.4. Roots occur for values of β where curve intersects β axis.

The mode shapes are obtained by finding the nontrivial solutions of

$$\begin{bmatrix} 3\phi - \lambda_i & -2\phi & 0 \\ -2\phi & 3\phi - \lambda_i & -\phi \\ 0 & -2\phi & 6\phi - \lambda_i \end{bmatrix} \begin{bmatrix} X_{i1} \\ X_{i2} \\ X_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation leads to

$$X_{i1} = \frac{2\phi}{3\phi - \lambda_i} X_{i2}$$

while the third equation leads to

$$X_{i3} = \frac{2\phi}{6\phi - \lambda_i} X_{i2}$$

Arbitrarily choosing $X_{i2} = 1$ leads to the following mode shape vectors:

$$\mathbf{X}_1 = \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix}$$

The graphical representations of the mode shapes in Fig. 6.7 are based on the assumption that the displacement in each spring is a linear function of position along the length of the spring. There are no nodes for the first mode. The second mode has a node in the spring between the first and second mass. The third mode has one node in the spring between the first and second mass and one node in the spring between the second and third masses.

le 6.5 An engineer is designing an 18-ft-long steel fixed-pinned beam ($E = 30 \times 10^6$ lb/in^2, $\gamma = 394$ lb/ft^3) for use in an industrial plant. The beam is to support a machine at its midspan. The machine may weigh up to 5 tons and will operate at speeds between 1000 rad/s and 2000 rad/s. The engineer is considering using either a W-shape W16 $\times$ 100 ($I = 712$ in^4, $A = 29.4$ in^2) beam or a W-shape W27 $\times$ 114 beam ($I = 4090$ in^4, $A =$ 33.5 in^2) in the design. Use a three-degree-of-freedom model of the beam to help decide which shape is the better choice in this design.

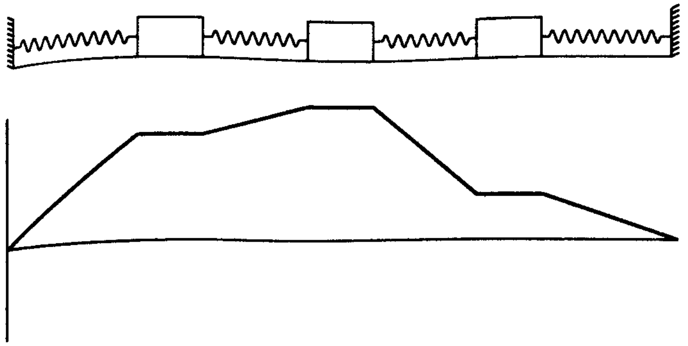
Solution:

Using a three-degree-of-freedom model the mass of the beam is lumped at three equally spaced locations along the span of the beam. The mass of each particle is $m_b/4$, where m_b is the total mass of the beam. If β is the mass of the machine, the mass matrix for a three-degree-of-freedom model is

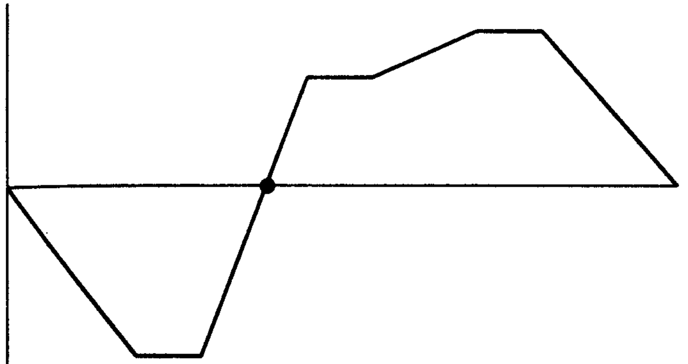
$$\mathbf{M} = \begin{bmatrix} \frac{m_b}{4} & 0 & 0 \\ 0 & \frac{m_b}{4} + \beta & 0 \\ 0 & 0 & \frac{m_b}{4} \end{bmatrix}$$

The flexibility matrix **A** for the model is determined from Table D.2.

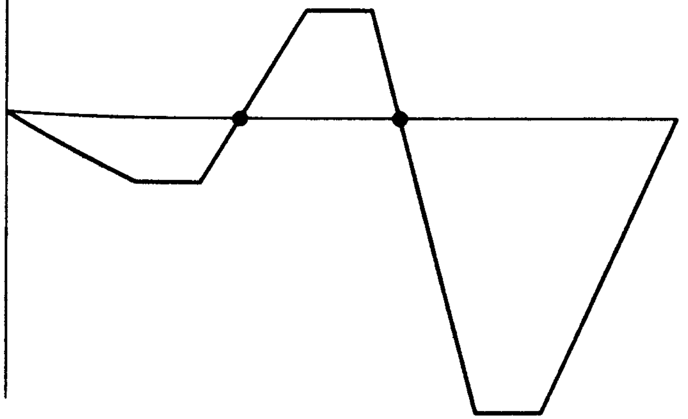
The MATLAB script of Fig. 6.8*a* is a program to symbolically determine the eigenvalues of **AM** as a function of the machine mass. The natural frequencies are the reciprocal of the square roots of the eigenvalues. The MATLAB generated plots of the natural frequency approximations as a function of the machine mass for each of the beams under consideration are given in Figs. 6.8*b* and *c*. These plots show that using the W16 $\times$ 100 shape is not a good choice, as the system's second natural frequency is in this range. The W27 $\times$ 100 shape is a better choice, as the specified operating range of 1000 rpm to 2000 rpm is between the system's two lowest natural frequencies for all machines up to 5 tons.



(a)



(b)



(c)

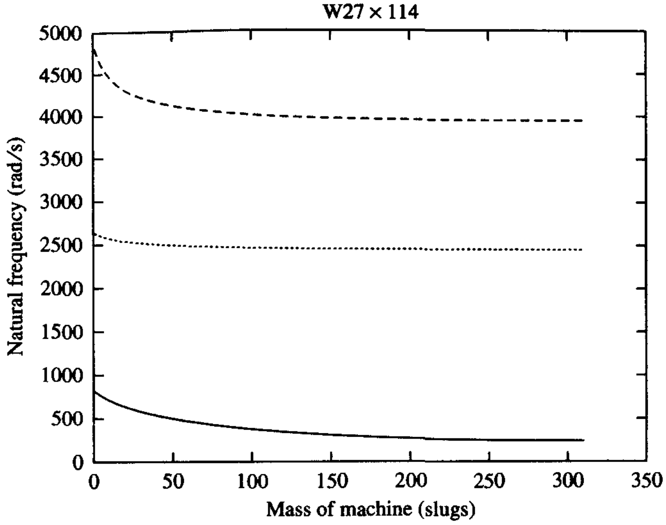
Figure 6.7 Mode shapes for Example 6.4. Nodes are shown for second and third modes. (a) $\omega = 0.893\sqrt{k/m}$; (b) $\omega = 2.110\sqrt{k/m}$; (c) $\omega = 2.597\sqrt{k/m}$. Mode shapes are chosen such that $\mathbf{X}_1 = 1$.

```
% MATLAB solution of Example 6.5
digits(5)
syms beta
% Beta is the mass of the machine attached to the beam at its midspan
% Input paramters
E=input(' Input elastic modulus in lb/in^2 ');
I=input(' Input moment of inertia in in^4 ');
Area=input(' Input area in in^2 ');
L=input(' Input length of beam in ft ');
gamma=input(' Input specific weight of beam in lb/ft^3 ');
m2=input(' Maximum weight of machine in lb ');
%Unit conversions
rho=gamma/32.2;                          % Mass density
E=E*144;                                 % E in lb/ft^2
I=I/12^4;                                % I in ft^4
Area=Area/144;                           % Area in ft^2
mb=rho*Area*L;                           % Mass of beam in slugs
m2=m2/32.2;                              % Max. machine mass in slugs
% Develop flexibility matrix for three-degree-of-freedom model of
% fixed-pinned beam
for i=1:3
   k=4-i;
   aL=k/4;
   C1=(1-aL)*(aL^2-2*aL-2)/2;
   C2=aL*L*(1-aL)*(2-aL)/2;
   for j=1:k
      x=j/4*L;
      A(j,k)=C1*x^3/6+C2*x^2/2;
      A(k,j)=A(j,k);                     % Using symmetry
   end
end
% Develop mass matrix for model
M=[mb/4,0,0;0,mb/4+beta,0;0,0,mb/4];     % mb/4 is lumped at each node
C=A*M;
V=eig(C);
Q=vpa(V);
s1=real(Q);
b=linspace(0,m2,51);
for i=1:51
   sa=subs(s1,beta,b(i));
   sb=double(sa);
   s2=sort(sb);
   om1(i)=sqrt(E*I/s2(1));               % Natural frequencies are
   om2(i)=sqrt(E*I/s2(2));               % reiprocals of square roots of
   om3(i)=sqrt(E*I/s2(3));               % eigenvalues of AM
end
om11=double(om1);
om22=double(om2);
```

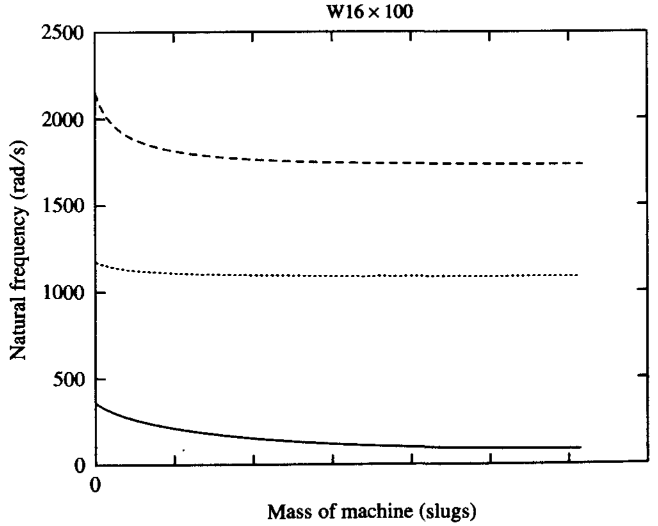
Figure 6.8 (a) MATLAB script for solution of Example 6.5.

```
om33=double(om3);
plot(b,om11,'-',b,om22,'--',b,om33,':')
xlabel(' Mass of machine (slugs)')
ylabel(' Natural frequencies (rad/s)')
```

(*a*)



(*b*)



(*c*)

Figure 6.8 (Con't) (*a*) Con't. (*b*) Natural frequencies versus machine mass for W27 × 114 bean frequencies versus machine mass for W16 × 100 beam section.

6.4 GENERAL SOLUTION

Equation (6.1) is a homogeneous system of n second-order linear differential equations. The normal-mode assumption, Eq. (6.2), leads to the determination of n natural frequencies. If λ is an eigenvalue of $\mathbf{M}^{-1}\mathbf{K}$, then both $\omega = +\sqrt{\lambda}$ and $\omega = -\sqrt{\lambda}$ satisfy Eq. (6.7) and given rise to the same solution, $\mathbf{X}$, of Eq. (6.5). The functions $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independent with each other and linearly independent with other functions of the same form with different values of ω. Thus the normal-mode solution generates $2n$ linearly independent solutions of Eq. (6.1). The most general solution of a linear homogeneous problem is a linear combination of all possible solutions. To this end,

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i \left(\tilde{C}_{i1} e^{i\omega t} + \tilde{C}_{i2} e^{-i\omega t}\right) \tag{6.12}$$

Using Euler's identity to replace the complex exponential by trigonometric functions and redefining the arbitrary constants gives

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i \left(C_{i1} \cos \omega_i t + C_{i2} \sin \omega_i t\right) \tag{6.13}$$

Trigonometric identities are used to write Eq. (6.13) in the alternate form

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{X}_i A_i \sin\left(\omega_i t - \phi_i\right) \tag{6.14}$$

Initial conditions must be specified for each dependent variable

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \qquad \dot{\mathbf{x}}(0) = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{bmatrix}$$

Application of the $2n$ initial conditions to Eq. (6.14) yields $2n$ equations to be solved for the $2n$ integration constants.

$$\mathbf{x}(0) = -\sum_{i=1}^{n} \mathbf{X}_i A_i \sin \phi_i \tag{6.15}$$

and

$$\dot{\mathbf{x}}(0) = \sum_{i=1}^{n} \mathbf{X}_i \omega_i A_i \cos \phi_i \tag{6.16}$$

6.6 Consider again the system of Examples 6.1 and 6.2. The springs are designed such that the static-equilibrium position coincides with the horizontal orientation of the bar. Both ends of the bar are displaced a distance δ from the equilibrium position. The bar is held in this position and released. Solve for the time-dependent history of the resulting free vibrations.

Solution:

With x and θ as generalized coordinates, the initial conditions are

$$\begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}(0) \\ \dot{\theta}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying the initial conditions and using the results of Example 6.1 in Eqs. (6.15) and (6.16) lead to

$$\begin{bmatrix} \delta \\ 0 \end{bmatrix} = -A_1 \begin{bmatrix} 1 \\ \dfrac{1.43}{L} \end{bmatrix} \sin \phi_1 - A_2 \begin{bmatrix} 1 \\ -\dfrac{8.42}{L} \end{bmatrix} \sin \phi_2$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left(1.28\sqrt{\frac{k}{m}}\right) A_1 \begin{bmatrix} 1 \\ \dfrac{1.43}{L} \end{bmatrix} \cos \phi_1 + \left(2.03\sqrt{\frac{k}{m}}\right) \begin{bmatrix} 1 \\ -\dfrac{8.42}{L} \end{bmatrix} \cos \phi_2$$

The last two equations are satisfied by taking $\phi_1 = \phi_2 = -\pi/2$. The solution of the first two equations is $A_1 = 0.855\delta$ and $A_2 = 0.145\delta$. The free vibration response of the system is

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = 0.855\delta \begin{bmatrix} 1 \\ \dfrac{1.43}{L} \end{bmatrix} \sin \left(1.28\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)$$

$$+ 0.145\delta \begin{bmatrix} 1 \\ -\dfrac{8.42}{L} \end{bmatrix} \sin \left(2.03\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)$$

or

$$x(t) = \delta \left[0.855 \sin \left(1.28\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)\right] + 0.145 \sin \left(2.03\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)$$

$$\theta(t) = 1.22\frac{\delta}{L}\left[\sin \left(1.28\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right) - \sin \left(2.03\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right)\right]$$

6.5 SPECIAL CASES

Repeated eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{AM}$ occur when the natural frequencies of two distinct modes coincide. It is usually possible to identify the separate modes of vibration. For example, consider the circular cantilever beam of Fig. 6.9. The beam has a thin disk attacked at its end. If the disk is vertically displaced and released,

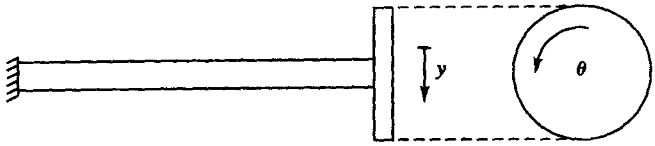


Figure 6.9 For certain combination of parameters the natural frequency for transverse vibration coincides with the natural frequency for torsional oscillation.

the disk undergoes free transverse vibrations. For a one-degree-of-freedom model, with inertia effects of the beam ignored, the natural frequency of free transverse vibrations of the disk is

$$\omega_1 = \sqrt{\frac{3EI}{mL^3}}$$

where E is the elastic modulus of the beam, I is the cross-sectional moment of inertia of the beam, L is the length of the beam, and m is the mass of the disk. If the disk is twisted and released, it undergoes free torsional oscillations. For a one-degree-of-freedom model, with inertia effects of the beam ignored, the natural frequency of free torsional oscillations is

$$\omega_2 = \sqrt{\frac{JG}{I_D L}}$$

where J is the polar moment of inertia of the cross section of the beam, G is the beam's shear modulus, and I_D is the mass moment of inertia of the disk. These two natural frequencies are equal for a steel shaft when the ratio of the length of the beam to the radius of the disk is 1.40. The two modes of vibration are independent but happen to have the same natural frequency.

If ω_i is a natural frequency calculated from an eigenvalue of multiplicity m, then only $n - m$ of the linear algebraic equations from which the mode shape is calculated are independent. Thus m elements of the mode shape can be arbitrarily chosen. The most general mode shape involves m arbitrary constants. Then m linearly independent mode shapes, $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+m}$, are specified. The general solution of Eq. (6.1) is still given by Eq. (6.14), but $\omega_i = \omega_{i+1} = \cdots = \omega_{i+m-1}$.

The two-degree-of-freedom system of Fig. 6.10 has a natural frequency of $\sqrt{6k/m}$ corresponding to a rotational mode and a natural frequency of $\sqrt{2k/m}$ corresponding to a translational mode. The system is neither statically nor dynamically coupled. A block of mass m is attached to the mass center of the bar through a spring as shown in Fig. 6.11, adding a degree of freedom and leading to static coupling. The differential equations

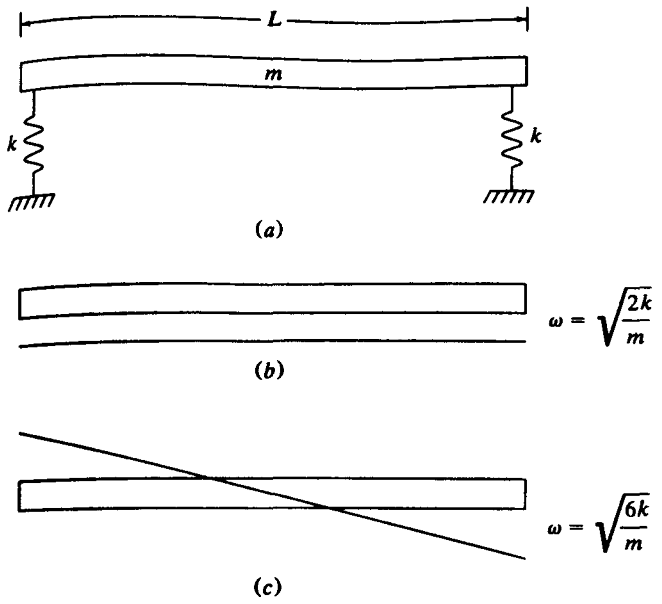


Figure 6.10 (a) Original system of Example 6.7; (b) mode shape for translational mode, $\omega = \sqrt{2k/m}$; (c) mode shape for rotational mode $\omega = \sqrt{6k/m}$.

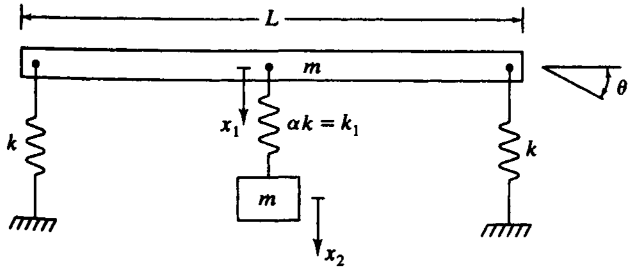


Figure 6.11 System of Example 6.7.

governing free vibration of this vibration of this three-degree-of-freedom system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\frac{L^2}{12} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k+k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & k\frac{L^2}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rotational mode is still uncoupled from the other modes. Find a value of k_1 such that another natural frequency of the system coincides with the natural frequency of the rotational mode. Find the mode shapes for all modes.

Solution:

The determinant leading to the characteristic equation is

$$\det\begin{bmatrix}(2+\alpha)\phi-\lambda & -\alpha\phi & 0\\ -\alpha\phi & \alpha\phi-\lambda & 0\\ 0 & 0 & 6\phi-\lambda\end{bmatrix}=0$$

where
$$\phi=\frac{k}{m}$$

and
$$\alpha=\frac{k_1}{k}$$

The characteristic equation obtained by row expansion of the determinant, using the third row, is

$$(6-\beta)\left[\beta^2-2(1+\alpha)\beta+2\alpha\right]=0$$

where
$$\beta=\frac{\lambda}{\phi}$$

The roots of the characteristic equation are

$$\beta=6,\ 1+\alpha\pm\sqrt{1+\alpha^2}$$

The $\beta=6$ root corresponds to the natural frequency of the rotational mode. Requiring one of the other natural frequencies to be equal to the natural frequency of the rotational mode leads to

$$1+\alpha\pm\sqrt{1+\alpha^2}=6\Rightarrow\alpha=\tfrac{12}{5}$$

Then the natural frequencies become

$$\omega_1=\sqrt{\frac{4k}{5m}}\qquad\omega_2=\omega_3=\sqrt{6\frac{k}{m}}$$

The mode shape corresponding to the lowest natural frequency is

$$\mathbf{X}_1=\begin{bmatrix}1\\1.5\\0\end{bmatrix}$$

For $\beta=6$ the mode shapes are determined from

$$\begin{bmatrix}-1.6\phi & -2.4\phi & 0\\ -2.4\phi & -3.6\phi & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}X_{21}\\X_{22}\\X_{23}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The general solution of this system contains two arbitrary constants and can be written as

$$\begin{bmatrix} a \\ -\frac{2}{3}a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the two linearly independent mode shapes corresponding to $\omega = \sqrt{6k/m}$ are

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that the mode corresponding to the lowest natural frequency is a translational mode with extension of the spring. One mode corresponding to $\omega = \sqrt{6k/m}$ is a translational mode with extension in the spring, but with a node in the spring. The second independent mode for $\omega = \sqrt{6k/m}$ is a rigid-body rotation of the bar about its mass center, with no extension in the spring.

A second special case occurs when one of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ is zero. The general solution for a system with a zero eigenvalue is

$$x(t) = (C_1 + C_2 t)\mathbf{X}_1 + \sum_{i=2}^{n} A_i \mathbf{X}_i \sin(\omega_i t - \phi_i) \qquad \textbf{[6.17]}$$

where C_1, C_2, and A_i are constants determined from application of the initial conditions. The first part of the solution corresponds to a rigid-body motion. The summation term corresponds to oscillatory motion.

A system has a natural frequency of zero only when it is unrestrained. For example, if both masses of the two-degree-of-freedom system of Fig. 6.12 are given the same initial displacement with no initial velocity, they will remain in their displaced positions indefinitely. If the shaft connecting the two flywheels of Fig. 6.13 is rotating at a constant speed, both flywheels will continue to rotate at this speed.

When motion of an unrestrained system occurs, either linear or angular momentum is conserved for the entire system. Application of the principle of conservation of linear momentum or the principle of conservation of angular momentum provides a relationship between the generalized coordinates of the form

$$\sum_{l=1}^{n} \alpha_l \dot{x}_l = C_1 \qquad \textbf{[6.18]}$$

where C_1 is a constant determined from the initial state. Equation (6.18) can be

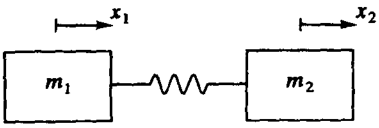


Figure 6.12 A two-degree-of-freedom unrestrained system. If both blocks are given the same displacement, they will move as a rigid body. If blocks are given different displacements, free oscillations occur.

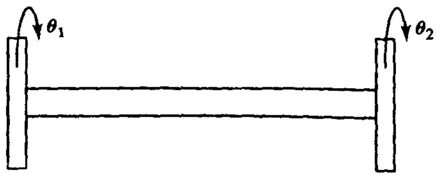


Figure 6.13 An unrestrained torsional system.

integrated to provide a constraint between the generalized coordinates of the form

$$\sum_{l=1}^{n} \alpha_l x_l = C_1 t + C_2 \tag{6.19}$$

Equation (6.19) could be used to reduce the number of degrees of freedom by one.

A railroad car of mass 1500 kg is to be coupled to an assembly of two precoupled identical railroad cars. The couplers are elastic connections of stiffness 4.2×10^7 N/m. The single car is rolled toward the other cars with a velocity of 7 m/s. Describe the motion of the three railroad cars after coupling is achieved.

Solution:

After coupling, the motion of the three railroad cars is modeled by using three degrees of freedom, as shown in Fig. 6.14*b*. The differential equations of motion are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{bmatrix} = \begin{bmatrix} 7 \text{ m/s} \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are determined from

$$\det \begin{bmatrix} \phi - \lambda & -\phi & 0 \\ -\phi & 2\phi - \lambda & -\phi \\ 0 & -\phi & \phi - \lambda \end{bmatrix} = 0$$

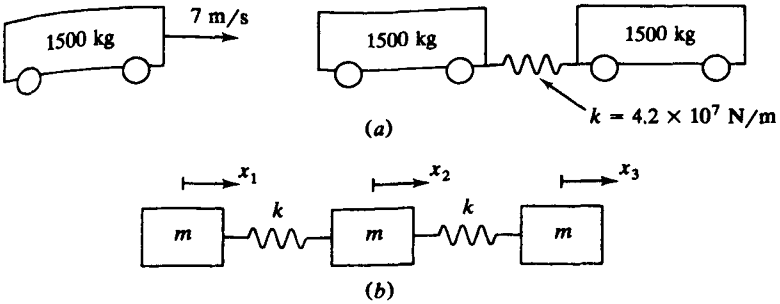


Figure 6.14 (a) Shunting of railroad cars; (b) three-degree-of-freedom model of shunting.

where $\phi = \sqrt{k/m}$. The resulting characteristic equation is solved to yield

$$\omega_1 = 0 \qquad \omega_2 = \sqrt{\frac{k}{m}} = 167.3\ \frac{\text{rad}}{\text{s}} \qquad \omega_3 = \sqrt{\frac{3k}{m}} = 289.8\ \frac{\text{rad}}{\text{s}}$$

The corresponding mode shapes are

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The general solution of the differential equations is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (C_1 + C_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \sin(167.3t + \phi_1)$$

$$+ C_4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \sin(289.8t + \phi_2)$$

Application of the initial conditions leads to

$$C_1 = \phi_1 = \phi_2 = 0 \qquad C_2 = 2.32\ \text{m/s} \qquad C_3 = 0.021\ \text{m} \qquad C_4 = 0.004\ \text{m}$$

The equation expressing conservation of linear momentum of the railroad cars after coupling is achieved is

$$m\dot{x}_1(t) + m\dot{x}_2(t) + m\dot{x}_3(t) = C$$

Example 6

Consider the unrestrained three-degree-of-freedom system of Example 5.10 and Fig. 5.11. Let $mr^2/I = 2$. Calculate the natural frequencies and illustrate the development of the constraint from momentum considerations.

Solution:

The differential equations are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 & -kr \\ 0 & 3k & -6kr \\ -kr & -6kr & 13kr^2 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The characteristic equation is developed from

$$\det \begin{bmatrix} \frac{1}{2}\phi - \lambda & 0 & -\frac{r}{2}\phi \\ 0 & 3\phi - \lambda & -6r\phi \\ -\frac{mr}{I}\phi & -\frac{6mr}{I}\phi & \frac{13mr^2}{I}\phi - \lambda \end{bmatrix} = 0$$

where $\phi = \sqrt{k/m}$. The characteristic equation is

$$-\beta^3 + \frac{59}{2}\beta^2 - \frac{39}{2}\beta = 0$$

where $\beta = \lambda/\phi$. The roots of this equation are

$$\beta = 0, 0.677, 28.82$$

Which lead to natural frequencies of

$$\omega_1 = 0 \qquad \omega_2 = 0.823\sqrt{\frac{k}{m}} \qquad \omega_3 = 5.369\sqrt{\frac{k}{m}}$$

Application of the principle of conservation of angular momentum about the center of the pulley leads to

$$2mr\dot{x}_A(t) + 2mr\dot{x}_B(t) + I\dot{\theta} = 2mr\dot{x}_A(0) + 2mr\dot{x}_B(0) + I\dot{\theta}(0)$$

6.6 ENERGY SCALAR PRODUCTS

A scalar product is an operation performed on two vectors such that the result is a scalar. In order for the operation to be termed a scalar product, it must satisfy certain rules as outlined in App. C. When the differential equations governing the motion of a linear n-degree-of-freedom system are formulated by using energy methods, the mass and stiffness matrices are symmetric. Then for a stable restrained system the following two operations satisfy all requirements to be called scalar products. Let $\mathbf{y}$ and $\mathbf{z}$ be any two n-dimensional vectors; define

$$(\mathbf{y}, \mathbf{z})_K = \mathbf{y}^T\mathbf{K}\mathbf{z} \tag{6.20}$$

and

$$(\mathbf{y}, \mathbf{z})_M = \mathbf{y}^T\mathbf{M}\mathbf{z} \tag{6.21}$$

The scalar product defined by Eq. (6.20) is called the *potential energy scalar product*. Let $\mathbf{X}_i$ be the mode shape corresponding to a natural frequency ω_i. If the system response includes only this mode, then from Eq. (6.14)

$$\mathbf{x}(t) = A_i\mathbf{X}_i \sin(\omega_i t - \phi_i) \tag{6.22}$$

From Eq. (5.6) the potential energy is calculated as

$$V = \frac{A_i^2}{2}\sin^2(\omega_i t - \phi_i)\sum_{r=1}^{n}\sum_{s=1}^{n} k_{rs}X_{ir}X_{is} = \frac{A_i^2}{2}\sin^2(\omega_i t - \phi_i)(\mathbf{X}_i, \mathbf{X}_i)_K \tag{6.23}$$

Thus, at a given instant of time, the potential energy scalar product of a mode shape with itself is proportional to the potential energy associated with that mode.

The scalar product defined by Eq. (6.21) is called the *kinetic energy scalar product*. It can be shown by using Eqs. (5.7) and (6.21) that

$$T = \frac{A_i^2}{2}\omega_i^2\cos^2(\omega_i t - \phi_i)(\mathbf{X}_i, \mathbf{X}_i)_M \tag{6.24}$$

or that for a linear system the kinetic energy scalar product of a mode shape with itself is proportional to the kinetic energy associated with that mode.

The mass and stiffness matrices for a linear system are guaranteed to be symmetric. In addition, the mass matrix is positive definite. The stiffness matrix for a stable system is positive definite unless it is unrestrained. The stiffness matrix for an unstable system is not positive definite. Thus, from Example C.5 of App. C, Eq. (6.21) defines a valid scalar product for all n-degree-of-freedom systems and Eq. (6.20) defines a valid scalar product for all stable constrained n-degree-of-freedom systems.

The fact that Eqs. (6.20) and (6.21) are valid scalar products leads to the following useful properties:

1. *Communtivity of scalar products*. For all real n-dimensional vectors $\mathbf{y}$ and $\mathbf{z}$:

$$(\mathbf{y}, \mathbf{z})_K = (\mathbf{z}, \mathbf{y})_K \tag{6.25a}$$

and

$$(\mathbf{y}, \mathbf{z})_M = (\mathbf{z}, \mathbf{y})_M \tag{6.25b}$$

2. *Linearity*. For all real n-dimensional vectors $\mathbf{w}$, $\mathbf{y}$, and $\mathbf{z}$ and all real scalars α and β:

$$(\alpha\mathbf{w} + \beta\mathbf{y}, \mathbf{z})_K = \alpha(\mathbf{w}, \mathbf{z})_K + \beta(\mathbf{y}, \mathbf{z})_K \tag{6.26a}$$

and

$$(\alpha\mathbf{w} + \beta\mathbf{y}, \mathbf{z})_M = \alpha(\mathbf{w}, \mathbf{z})_M + \beta(\mathbf{y}, \mathbf{z})_M \tag{6.26b}$$

Two vectors are said to be orthogonal with respect to a scalar product if their scalar product is zero. The n-dimensional vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal with respect to the potential energy scalar product if

$$(\mathbf{y}, \mathbf{z})_K = 0 \tag{6.27a}$$

The vectors are orthogonal with respect to the kinetic energy scalar product if

$$(\mathbf{y}, \mathbf{z})_M = 0 \tag{6.27b}$$

The use of scalar product notation is not essential to analyze and understand free and forced vibrations of multi-degree-of-freedom systems. However, writing equations in scalar product notation is usually less confusing than using matrix and vector notation. In addition, since the scalar products have identifiable physical meaning, it may be easier to identify the physical significance of an equation when it is written in scalar product notation. At the very least the energy scalar products can be thought of as shorthand notation for the products defined by Eqs. (6.20) and (6.21). For these reasons, the remainder of the discussion in Chap. 6 and the entire discussion in Chap. 7 use scalar product notation. In addition, a scalar product is developed for use with continuous systems in Chap. 9. Many equations are also written using matrix notation for those not comfortable with scalar product notation.

6.7 PROPERTIES OF NATURAL FREQUENCIES AND MODE SHAPES

Let ω_i and ω_j be distinct natural frequencies of an n-degree-of-freedom system. Let $\mathbf{X}_i$ and $\mathbf{X}_j$ be their respective mode shapes. From Eq. (6.4) the equations satisfied by these natural frequencies and mode shapes are

$$\omega_i^2\mathbf{M}\mathbf{X}_i = \mathbf{K}\mathbf{X}_i \tag{6.28}$$

and

$$\omega_j^2\mathbf{M}\mathbf{X}_j = \mathbf{K}\mathbf{X}_j \tag{6.29}$$

Premultiplying Eq. (6.28) by $\mathbf{X}_j^T$ gives

$$\omega_i^2\mathbf{X}_j^T\mathbf{M}\mathbf{X}_i = \mathbf{X}_j^T\mathbf{K}\mathbf{X}_i$$

or in scalar product notation

$$\omega_i^2(\mathbf{X}_j, \mathbf{X}_i)_M = (\mathbf{X}_j, \mathbf{X}_i)_K \tag{6.30}$$

Premultiplying Eq. (6.29) by $\mathbf{X}_i^T$ gives

$$\omega_j^2(\mathbf{X}_i, \mathbf{X}_j)_M = (\mathbf{X}_i, \mathbf{X}_j)_K \tag{6.31}$$

Subtracting Eq. (6.31) from Eq. (6.30) gives

$$\omega_i^2(\mathbf{X}_j, \mathbf{X}_i)_M - \omega_j^2(\mathbf{X}_i, \mathbf{X}_j)_M = (\mathbf{X}_j, \mathbf{X}_i)_K - (\mathbf{X}_i, \mathbf{X}_j)_K \tag{6.32}$$

On the basis of the commutivity of the scalar products, Eq. (6.32) reduces to

$$(\omega_i^2 - \omega_j^2)(\mathbf{X}_i, \mathbf{X}_j)_M = 0 \tag{6.33}$$

Since $\omega_i \neq \omega_j$,

$$(\mathbf{X}_i, \mathbf{X}_j)_M = 0 \tag{6.34}$$

or mode shapes corresponding to distinct natural frequencies are orthogonal with respect to the kinetic energy scalar product. Then from Eq. (6.30), these mode shapes are also orthogonal with respect to the potential energy scalar product, or

$$(\mathbf{X}_i, \mathbf{X}_j)_K = 0 \qquad \textbf{[6.35]}$$

If a system has a zero natural frequency, then it is strictly improper to define a potential energy scalar product. Property 3 required of scalar products is violated. However, it can be shown that the mode shape for the rigid-body mode for an unrestrained system is orthogonal to all other mode shapes for the system.

If an eigenvalue is not distinct, but has a multiplicity $m > 1$, then there are m linearly independent mode shapes corresponding to that eigenvalue. The preceding analysis shows that each of these mode shapes is orthogonal to mode shapes corresponding to different natural frequencies. Independent mode shapes obtained by solving Eq. (6.5) for the same eigenvalue may or may not be mutually orthogonal with respect to the energy scalar products. However, a procedure known as the Gram-Schmidt orthogonalization process can be used to replace these mode shapes with a set of m mutually orthogonal mode shapes. These orthogonalized mode shapes are linearly dependent with the original mode shapes.

Example

Demonstrate orthogonality of the mode shapes with respect to the kinetic energy scalar product for the system of Example 6.4.

Solution:

The mass matrix, stiffness matrix, and mode shapes are as given in Example 6.4. Orthogonality with respect to the kinetic energy inner product is as follows:

$$
\begin{aligned}
(\mathbf{X}_1, \mathbf{X}_2)_M &= \mathbf{X}_1^T \mathbf{M} \mathbf{X}_2 \\
&= [0.908 \quad 1 \quad 0.384] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m}{2} \end{bmatrix} \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \\
&= [0.908 \quad 1 \quad 0.384] \begin{bmatrix} -1.375m \\ m \\ 0.647m \end{bmatrix} \\
&= (0.908)(-1.375m) + (1)(m) + (0.384)(0.647m) \\
&= -0.000052m \approx 0
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{X}_1, \mathbf{X}_3)_M &= \mathbf{X}_1^T \mathbf{M} \mathbf{X}_3 \\
&= [0.908 \quad 1 \quad 0.384] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m}{2} \end{bmatrix} \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix}
\end{aligned}
$$

$$= [0.908 \quad 1 \quad 0.384] \begin{bmatrix} -0.534m \\ m \\ -1.339m \end{bmatrix}$$

$$= (0.908)(-0.534m) + (1)(m) + (0.384)(-1.339m)$$

$$= 0.00095m \approx 0$$

$$(\mathbf{X}_2, \mathbf{X}_3)_M = \mathbf{X}_2^T \mathbf{M} \mathbf{X}_3$$

$$= [-1.375 \quad 1 \quad 1.294] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} -0.534 \\ 1 \\ -2.677 \end{bmatrix}$$

$$= [1.375 \quad 1 \quad 1.294] \begin{bmatrix} -0.535m \\ m \\ -1.339m \end{bmatrix}$$

$$= (-1.375)(-0.534m) + (1)(m) + (1.294)(-1.339m)$$

$$= -0.00159m \approx 0$$

A variation of the preceding argument is used to prove that the eigenvalues are all real. The formal proof of this statement involves the introduction of a complex energy scalar product and is beyond the scope of this text.

The argument can also be used to show that if **M** and **K** are positive definite then the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are all positive. Let $\mathbf{X}_i = \mathbf{X}_j$ in Eq. (6.30)

$$\omega_i^2 = \frac{(\mathbf{X}_i, \mathbf{X}_i)_K}{(\mathbf{X}_i, \mathbf{X}_i)_M} \tag{6.36}$$

If **M** and **K** are positive definite, then both scalar products in the quotient of Eq. (6.36) are positive. Hence

$$\omega_i^2 > 0 \tag{6.37}$$

This, in turn, shows that a system in which both the mass and stiffness matrices are positive definite is stable.

The ratio of Eq. (6.36) is called *Rayleigh's quotient.* For a given mode it is the ratio of the potential energy to the kinetic energy.

It is possible to construct n orthogonal, and hence linearly independent, mode shapes for an n-degree-of-freedom system. Thus any n-dimensional vector can be written as a linear combination of these n mode shapes. To this end, if $\mathbf{y}$ is any

n-dimensional vector there exist constants $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{X}_i \qquad [6.38]$$

Equation (6.38) is a representation of the *expansion theorem*. Premultiplying Eq. (6.38) by $\mathbf{X}_j^T\mathbf{M}$ for some j, $1 \le j \le n$ gives, in scalar product notation

$$(\mathbf{X}_j, \mathbf{y})_M = \left(\mathbf{X}_j, \sum_{i=1}^{n} c_i \mathbf{X}_i\right)_M \qquad [6.39]$$

Interchanging the scalar product operation with the summation and using the property of Eq. (6.26) gives

$$(\mathbf{X}_j, \mathbf{y})_M = \sum_{i=1}^{n} c_i (\mathbf{X}_j, \mathbf{X}_i)_M \qquad [6.40]$$

The orthogonality of the mode shapes implies that the only nonzero term in the summation occurs when $i = j$. Then Eq. (6.40) reduces to

$$c_j = \frac{(\mathbf{X}_j, \mathbf{y})_M}{(\mathbf{X}_j, \mathbf{X}_j)_M} \qquad [6.41]$$

6.8 NORMALIZED MODE SHAPES

A mode shape corresponding to a specific natural frequencies of an n-degree-of-freedom system is unique only to a multiplicative constant. The arbitrariness can be alleviated by requiring the mode shape to satisfy the normalization constraint. A mode shape chosen to satisfy the normalization constraint is called a *normalized mode shape*. The normalization constraint, itself, is arbitrary. However, all mode shapes are required to satisfy the same normalization constraint. The constraint should be chosen such that subsequent use of the normalized mode shape is convenient.

It is convenient to normalize mode shapes by requiring that the kinetic energy scalar product of a mode shape with itself is equal to one. That is,

$$(\mathbf{X}_i, \mathbf{X}_i)_M = \mathbf{X}_i^T\mathbf{M}\mathbf{X}_i = 1 \qquad [6.42]$$

If the mode shape, $\mathbf{X}_i$, is normalized according to Eq. (6.42), then from Rayleigh's quotient, Eq. (6.36)

$$\mathbf{X}_i^T\mathbf{K}\mathbf{X}_i = (\mathbf{X}_i, \mathbf{X}_i)_K = \omega_i^2 \qquad [6.43]$$

The orthogonality relations, Eqs. (6.34) and (6.35), the normalization constraint, Eq. (6.42), and the subsequent result of the choice of normalization, Eq. (6.43), are summarized by

$$(\mathbf{X}_i, \mathbf{X}_j)_M = \delta_{ij} \tag{6.44}$$

and
$$(\mathbf{X}_i, \mathbf{X}_j)_K = \omega_i^2 \delta_{ij} \tag{6.45}$$

where δ_{ij} is the Kronecker delta. From this point, mode shapes will be assumed to be normalized by Eq. (6.42).

With the normalization scheme of Eq. (6.42), the expansion theorem, Eqs. (6.38) and (6.41), becomes

$$\mathbf{y} = \sum_{i=1}^{n} (\mathbf{X}_i, \mathbf{y})_M \mathbf{X}_i \tag{6.46}$$

.11 Expand the vector

$$\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

using the normalized mode shapes of Example 6.4.

Solution:

The general mode shapes of Example 6.4 are

$$\mathbf{X}_1 = B_1 \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix} \qquad \mathbf{X}_2 = B_2 \begin{bmatrix} -1.375 \\ 1 \\ 1.294 \end{bmatrix} \qquad \mathbf{X}_3 = B_3 \begin{bmatrix} -0.534 \\ 1 \\ 2.677 \end{bmatrix}$$

where B_1, B_2, and B_3 are arbitrary constants. The normalization of the first mode shape proceeds as follows

$$1 = (\mathbf{X}_1, \mathbf{X}_1)_M = B_1^2 [0.908 \quad 1 \quad 0.384] \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} 0.908 \\ 1 \\ 0.384 \end{bmatrix}$$

which yields $B_1 = 0.726/\sqrt{m}$ and

$$\mathbf{X}_1 = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.659 \\ 0.726 \\ 0.279 \end{bmatrix}$$

The other mode shapes are normalized in the same manner yielding

$$\mathbf{X}_2 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.712 \\ 0.518 \\ 0.670 \end{bmatrix} \qquad \mathbf{X}_3 = \frac{1}{\sqrt{m}}\begin{bmatrix} -0.242 \\ 0.453 \\ -1.213 \end{bmatrix}$$

The first coefficient in the expansion is calculated by

$$c_1 = (\mathbf{X}_1, \mathbf{y})_M = \frac{1}{\sqrt{m}}[0.659 \quad 0.726 \quad 0.279]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = 3.284\sqrt{m}$$

The other coefficients are calculated in a similar manner, yielding $c_2 = 0.690\sqrt{m}$, $c_3 = 2.777\sqrt{m}$. Thus

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = 3.284\begin{bmatrix} 0.659 \\ 0.726 \\ 0.279 \end{bmatrix} + 0.690\begin{bmatrix} -0.712 \\ 0.518 \\ 0.670 \end{bmatrix} + 2.777\begin{bmatrix} -0.242 \\ 0.455 \\ -1.213 \end{bmatrix}$$

6.9 RAYLEIGH'S QUOTIENT

Consider a situation where the free vibrations of a one-degree-of-freedom system are generated such that only one mode is present. The frequency of the mode is ω and its mode shape is $\mathbf{X}$. The maximum potential energy associated with this mode of vibration is determined from Eq. (6.23) as

$$V_{\max} = \tfrac{1}{2}(\mathbf{X}, \mathbf{X})_K \tag{6.47}$$

The maximum kinetic energy associated with this mode is determined from Eq. (6.24) as

$$T_{\max} = \tfrac{1}{2}\omega^2(\mathbf{X}, \mathbf{X})_M \tag{6.48}$$

For a conservative system, where a continual process of transfer of kinetic and potential energy occurs without dissipation, the maximum potential energy equals the maximum kinetic energy. Thus from Eqs. (6.47) and (6.48)

$$\omega^2(\mathbf{X}, \mathbf{X})_M = (\mathbf{X}, \mathbf{X})_K$$

or

$$\omega_2 = \frac{(\mathbf{X}, \mathbf{X})_K}{(\mathbf{X}, \mathbf{X})_M} \tag{6.49}$$

For a general n-dimensional vector $\mathbf{X}$, not necessarily a mode shape, Eq. (6.49) is generalized to

$$R(\mathbf{X}) = \frac{(\mathbf{X}, \mathbf{X})_K}{(\mathbf{X}, \mathbf{X})_M} \tag{6.50}$$

The scalar function defined in Eq. (6.50) is called *Rayleigh's quotient.* If $\mathbf{X}$ is a mode shape of the linear n degree of freedom whose stiffness and mass matrices are $\mathbf{K}$ and $\mathbf{M}$, respectively, then $R(\mathbf{X})$ takes on the value of the natural frequency associated with that mode. If $\mathbf{X}$ is not a mode shape, then $R(\mathbf{X})$ takes on some other value.

Rayleigh's quotient can be useful in determining an upper bound on the lowest natural frequency. In some cases it can be used to attain a good approximation to the lowest natural frequency.

From the expansion theorem, an arbitrary vector $\mathbf{X}$ can be written as a linear combination of the normalized mode shapes

$$\mathbf{X} = \sum_{i=1}^{n} c_i \mathbf{X}_i \tag{6.51}$$

Substituting Eq. (6.51) in Rayleigh's quotient, using properties of the scalar products and orthonormality of the mode shapes, leads to

$$R(\mathbf{X}) = \frac{\sum_{i=1}^{n} c_i^2 \omega_i^2}{\sum_{i=1}^{n} c_i^2} \tag{6.52}$$

Stationary values of $R(\mathbf{X})$ occur when

$$\frac{\partial R}{\partial c_1} = \frac{\partial R}{\partial c_2} = \cdots = \frac{\partial R}{\partial c_n} = 0 \tag{6.53}$$

The n solutions of Eq. (6.53) are summarized by $c_i = \delta_{ij}$ for $j = 1, \ldots, n$. That is, Rayleigh's quotient is stationary only when $\mathbf{X}$ is an eigenvector. It is also possible to show that these stationary values are minimums. Hence ω_1^2 is the minimum value of Rayleigh's quotient.

The preceding result implies that an upper bound and perhaps an approximation for the lowest natural frequency can be obtained by using Rayleigh's quotient. Rayleigh's quotient can be calculated for several trial vectors. The lowest natural frequency can be no greater than the square root of the smallest value obtained. The closer a trial vector is to the actual mode shape, the closer the value of Rayleigh's quotient is to the square of the lowest natural frequency.

.12 Use Rayleigh's quotient to obtain an approximation to the lowest natural frequency of the system of Example 6.4. Use the trial vectors

$$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Solution:

Calculate Rayleigh's quotient:

$$R(\mathbf{X}) = \frac{[1 \quad 1 \quad 0.5]\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}}{[1 \quad 1 \quad 0.5]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}} = 0.823\frac{k}{m}$$

Similar calculations yield

$$R(\mathbf{Y}) = 6.0\frac{k}{m} \qquad R(\mathbf{Z}) = 2.57\frac{k}{m}$$

From the preceding equations an upper bound on the lowest natural frequency is

$$\omega_1 < 0.907\sqrt{\frac{k}{m}}$$

From Example 6.4 the lowest natural frequency for this system is $0.893\sqrt{k/m}$.

6.10 PRINCIPAL COORDINATES

Let $\omega_1, \omega_2, \ldots, \omega_n$, be the natural frequencies of a linear n-degree-of-freedom system with corresponding normalized mode shapes $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$. The expansion theorem implies that a general solution to Eq. (6.1) can be written as

$$\mathbf{x}(t) = \sum_{i=1}^{n} p_i(t)\mathbf{X}_i \tag{6.54}$$

Substitution of Eq. (6.54) into Eq. (6.1) leads to

$$\mathbf{M}\left(\sum_{i=1}^{n} \ddot{p}_i\mathbf{X}_i\right) + \mathbf{K}\left(\sum_{i=1}^{n} p_i\mathbf{X}_i\right) = 0 \tag{6.55}$$

Taking the standard scalar product of Eq. (6.55) with $\mathbf{X}_j$ for an arbitrary j leads to

$$\left(\mathbf{X}_j, \sum_{i=1}^{n} \ddot{p}_i\mathbf{M}\mathbf{X}_i\right) + \left(\mathbf{X}_j, \sum_{i=1}^{n} p_i\mathbf{K}\mathbf{X}_i\right) = 0$$

which, after the properties of scalar products are invoked, becomes

$$\sum_{i=1}^{n} \ddot{p}_i(\mathbf{X}_j, \mathbf{MX}_i) + \sum_{i=1}^{n} p_i(\mathbf{X}_j, \mathbf{KX}_i) = 0 \tag{6.56}$$

Using the definitions of the energy scalar products, Eqs. (6.21) and (6.22), in Eq. (6.56) leads to

$$\sum_{i=1}^{n} \ddot{p}_i(\mathbf{X}_j, \mathbf{X}_i)_M + \sum_{i=1}^{n} p_i(\mathbf{X}_j, \mathbf{X}_i)_K = 0 \tag{6.57}$$

Orthogonality and normalization of mode shapes, Eqs. (6.44) and (6.45), are used in Eq. (6.57), leading to

$$\ddot{p}_j + \omega_i^2 p_j = 0 \tag{6.58}$$

Since j was arbitrarily chosen an equation of the form of Eq. (6.58) can be written for each $j = 1, 2, \ldots, n$.

Equation (6.54) can be viewed as a linear transformation between the chosen generalized coordinates, $\mathbf{x}$, and the coordinates $\mathbf{p} = [p_1 \quad p_2 \quad \cdots \quad p_n]^T$, called the *principal coordinates.* The transformation matrix is the matrix whose columns are the normalized mode shapes. This matrix, $\mathbf{P} = [\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_n]$ is called the *modal matrix.* Since the columns of the modal matrix are linearly independent the modal matrix is nonsingular and the transformations

$$\mathbf{x} = \mathbf{Pp} \qquad \mathbf{p} = \mathbf{P}^{-1}\mathbf{x} \tag{6.59}$$

have a one-to-one correspondence.

The differential equations governing the vibrations of a linear n-degree-of-freedom system are uncoupled when the principal coordinates are used as dependent variables.

6.13 Calculate the principal coordinates for the system of Example 6.1.

Solution:

Using x and θ as generalized coordinates, the mode shapes are calculated in Example 6.1 as

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ \dfrac{1.43}{L} \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -\dfrac{8.42}{L} \end{bmatrix}$$

Each mode shape is normalized by dividing by the kinetic energy scalar product of the mode shape with itself. This leads to

$$\mathbf{X}_1 = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.925 \\ \dfrac{1.33}{L} \end{bmatrix} \qquad \mathbf{X}_2 = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.411 \\ -\dfrac{3.461}{L} \end{bmatrix}$$

The modal matrix is the matrix whose columns are the normalized mode shapes

$$\mathbf{P} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.925 & 0.411 \\ \dfrac{1.33}{L} & -\dfrac{3.46}{L} \end{bmatrix}$$

The inverse of **P** is calculated, using the methods of App. C, as

$$\mathbf{P}^{-1} = \sqrt{m}\begin{bmatrix} 0.924 & 0.110L \\ 0.353 & -0.247L \end{bmatrix}$$

Then from Eq. (6.59)

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \sqrt{m}\begin{bmatrix} 0.924 & 0.110L \\ 0.353 & -0.247L \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix}$$

The preceding equations can be rewritten as

$$p_1(t) = 0.924\sqrt{m}(x + 0.119L\theta)$$

and

$$p_2(t) = 0.353\sqrt{m}(x - 0.700L\theta)$$

The preceding equations show that p_1 is proportional to the displacement of a particle a distance $0.119L$ to the right of the mass center. This particle is identified in Example 6.1 as a node for the second mode. The principal coordinate p_2 is proportional to the displacement of a particle $0.700L$ to the left of the mass center. However, this particle does not exist on the rigid bar.

The preceding discussion shows that vibrations at the lowest natural frequency of this system are rigid-body oscillations about the node for the second mode. Vibrations at the higher natural frequency are rigid-body oscillations about a point lying off the bar.

Equation (6.59) shows that the generalized coordinates are linear combinations of the principal coordinates. The generalized coordinates for a linear system are chosen such that the displacement of any particle in the system is a linear combination of the generalized coordinates. Thus the displacement of any particle in the system is a linear combination of the principal coordinates. This implies that if a particle is a node for the higher mode of a two-degree-of-freedom system, then p_1 is proportional to the displacement of that particle. If a particle is a node for the second mode of a three-degree-of-freedom system, then a linear combination of the first and third principal coordinates represents the displacement of that point. Nothing can be inferred about the physical interpretation of either principal coordinate.

6.11 DETERMINATION OF NATURAL FREQUENCIES AND MODE SHAPES

The determination of the natural frequencies and mode shapes for a multi-degree-of-freedom system requires the solution of a matrix eigenvalue-eigenvector problem. If the system has more than three degrees of freedom, the algebraic and computational burden usually leads one to seek approximate, numerical, or computer solutions. Rayleigh's quotient, presented in Sec. 6.9, may be used to provide an upper bound to the lowest natural frequency. In the Rayleigh-Ritz method for discrete systems, a linear combination of linearly independent vectors is used in Rayleigh's quotient. The coefficients in the linear combination are chosen to render Rayleigh's quotient stationary.

Most applications require more accurate determination of the natural frequencies and mode shapes than can be provided by Rayleigh's quotient or the Rayleigh-Ritz method. A number of numerical methods lead to accurate numerical determination of natural frequencies and mode shapes. One such is the matrix iteration method. Beginning with a trial mode shape vector $\mathbf{x}_0$ a sequence of vectors $\mathbf{x}_i$ is generated by

$$\mathbf{x}_i = \mathbf{AMx}_{i-1} \tag{6.60}$$

It can be shown that the ratio of two corresponding elements of $\mathbf{x}_i$ and $\mathbf{x}_{i-1}$ approaches ω_1^2 as i gets large and that $\mathbf{x}_i$ approaches the corresponding mode shape vector. Higher natural frequencies and mode shape vectors can be obtained by requiring trial vectors to be orthogonal with respect to the kinetic energy scalar product to all previously obtained mode shape vectors. Matrix iteration has the advantage that natural frequencies and mode shape vectors are determined sequentially and that only the number desired need to be determined.

Jacobi's method is a powerful iterative method that determines all eigenvalues and eigenvectors of a matrix. Jacobi's method uses a series of transformations to convert a symmetric matrix into a diagonal matrix with the eigenvalues along the diagonal. The product of the matrices used in the transformation produces a matrix whose columns are the eigenvectors. The mass and stiffness matrices for a multi-degree-of-freedom system are guaranteed to be symmetric, but the matrix $\mathbf{M}^{-1}\mathbf{K}$, whose eigenvalues are the squares of the natural frequencies, is not necessarily symmetric. In this case, it can be shown that there exists a symmetric matrix $\mathbf{D}$, that can be obtained by a method called *Choleski decomposition*, such that the eigenvalues and eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$ are the same as the eigenvalues and eigenvectors of $\mathbf{D}$.

The above methods are described in other texts on vibrations (Meirovitch, Rao) or numerical analysis texts. (Hoffman). These methods are tools that can be used to solve eigenvalue-eigenvector problems and thus lead to natural frequencies and mode shapes for multi-degree-of-freedom systems. However, understanding the mechanics of these methods does not enhance the understanding of vibrations. The interested reader is referred to the cited texts. These methods have been incorporated into the eigenvalue routines used in MATLAB. These MATLAB routines are easy to use and are sufficient for the purposes of this text.

Example 6.1

Determine the natural frequencies and mode shapes of the automobile suspension system shown in Fig. 6.15. A six-degree-of-freedom model is used, including the inertia of the axle and the wheels, and the stiffness of the seats and the passengers in the seats (assuming they are wearing seat belts). The springs of stiffness k_t model the tires.

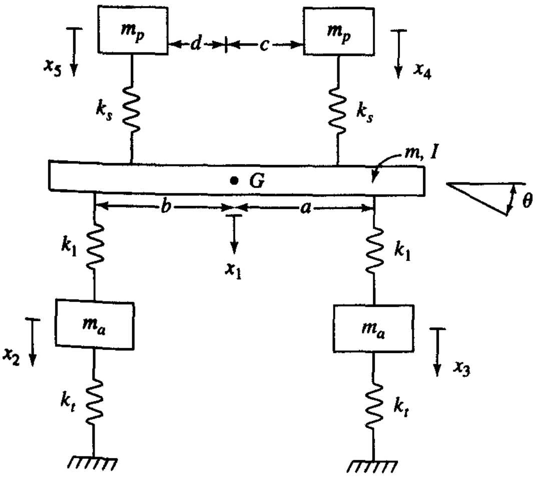
Solution:

The generalized coordinates are as illustrated in Fig. 6.15. The displacement vector is defined as $\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad \theta]$. Kinetic energy is used to determine the mass matrix

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m_a & 0 & 0 & 0 & 0 \\ 0 & 0 & m_a & 0 & 0 & 0 \\ 0 & 0 & 0 & m_p & 0 & 0 \\ 0 & 0 & 0 & 0 & m_p & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

Stiffness influence coefficients are used to determine the stiffness matrix

$$\mathbf{K} = \begin{bmatrix} 2k_1 + 2k_s & -k_1 & -k_1 & -k_s & -k_s & k_1(a-b) + k_s(c-d) \\ -k_1 & k_1 + k_t & 0 & 0 & 0 & -k_1 a \\ -k_1 & 0 & k_1 + k_t & 0 & 0 & k_1 b \\ -k_s & 0 & 0 & k_s & 0 & k_s c \\ -k_s & 0 & 0 & 0 & k_s & -k_s d \\ k_1(b-a) + k_s(d-c) & -k_1 a & k_1 b & k_s c & -k_s d & k_1(a^2 + b^2) + k_s(c^2 + d^2) \end{bmatrix}$$



$m = 1000$ kg
$m_a = 150$ kg
$m_p = 60$ kg

$I = 360\ \mathrm{kg \cdot m^2}$
$k_1 = 1.2 \times 10^5$ N/m
$k_s = 2 \times 10^5$ N/m

$k_t = 8 \times 10^4$ N/m
$a = 1.8$ m
$b = 2.0$ m

$c = 60$ cm
$d = 40$ cm

Figure 6.15 Six-degree-of-freedom model of suspension system and passengers.

The MATLAB script for determining the natural frequencies and modal matrix and the corresponding output when the program is run are given in Fig. 6.16.

```
% Example 6.14. Natural frequencies and modal matrix for six-degree-of-
% freedom model of suspension system and passengers
% Parameters
m=1000;          % Mass of vehicle in kg
ma=150;          % Mass of axle in kg
mp=60;           % Mass of each passenger in kg
I=360;           % Mass moment of inertia of vehicle in kg-m2
k1=120000;       % Stiffness of suspension spring in N/m
ks=200000;       % Stiffness of seat in N/m
kt=80000;        % Stiffness of tire in N/m
a=1.8;           % Distance from front springs to center of gravity in m
b=2.0;           % Distance from rear springs to center of gravity in m
c=0.6;           % Distance from front passenger to center of graity in m
d=0.4;           % Distance from rear passenger to center of gravity in m
%Mass matrix
disp('Mass matrix')
M=[m,0,0,0,0,0;
   0,ma,0,0,0,0;
   0,0,ma,0,0,0;
   0,0,0,mp,0,0;
   0,0,0,0,mp,0;
   0,0,0,0,0,I];
disp(M)
% Stiffness matrix
disp('Stiffness matrix')
K=[2*k1+2*ks,-k1,-k1,-ks,-ks,k1*(a-b)+ks*(c-d);
   -k1,k1+kt,0,0,0,-k1*a;
   -k1,0,k1+kt,0,0,k1*b;
   -ks,0,0,ks,0,ks*c;
   -ks,0,0,0,ks,-ks*d;
   k1*(a-b)+ks*(c-d),-k1*a,k1*b,ks*c,-ks*d,k1*(a^2+b^2)+ks*(c^2+d^2)];
lisp(K)
; Eigenvalues and eigenvectors of M^(-1)*K
:=inv(M)*K;
V,D]=eig(C);
 Sorting to put eigenvalues in ascending order and to develop modal
 matrix corresponding to ascending order of eigenvalues
=[D(1,1),D(2,2),D(3,3),D(4,4),D(5,5),D(6,6)];
h=max(E)+0.01;
```

gure 6.16 (a) MATLAB script for solution of Exomple 6.14, six-degree-of-freedom model of suspension system.

```
Ql=0;
for i=1:6
   for j=1:6
      if E(j)>Ql & E(j)<Qh
         k=j;
         Qh=E(j);
      else
      end
   end
      Ql=Qh;
      Qh=max(E)+0.01;
      Oml(i)=E(k);
      omega(i)=sqrt(E(k));
      for m=1:6
         P1(m,i)=V(m,k);
      end
end
% Normalizing mode shapes, L is a diagonal matrix with kinet.
% Scalar products of mode shapes along diagonal
L=P1'*M*P1;
% Developing modal matrix, a matrix whose columns are the no
% mode shapes
for i=1:6
   for j=1:6
      P(i,j)=P1(i,j)/L(j,j);
   end
end
disp('Natural frequencies in rad/s')
disp(omega)
disp('Modal matrix')
disp(P)
```

(*a*)

```
Mass matrix
        1000           0           0           0           0
           0         150           0           0           0
           0           0         150           0           0
           0           0           0          60           0
           0           0           0           0          60
           0           0           0           0           0
```

Figure 6.16 (Con't) (*a*) Con't. (*b*) natural frequencies and modal matrix determined by running

```
:ness matrix
)e+005 *

 .4000   -1.2000   -1.2000   -2.0000   -2.0000    0.1600
 .2000    2.0000         0         0         0   -2.1600
 .2000         0    2.0000         0         0    2.4000
 .0000         0         0    2.0000         0    1.2000
 .0000         0         0         0    2.0000   -0.8000
 .1600   -2.1600    2.4000    1.2000   -0.8000    9.7280

al frequencies in rad/s
 .9736   19.6281   38.1937   51.7955   61.2744   65.8075

 matrix
 .0016    0.0004    0.0007    0.0001    0.0013   -0.0002
 .0006    0.0036   -0.0045   -0.0029   -0.0008   -0.0016
 .0015   -0.0033   -0.0043    0.0031   -0.0001    0.0019
 .0019   -0.0010    0.0012   -0.0075   -0.0078    0.0075
 .0015    0.0015    0.0012    0.0062   -0.0122   -0.0040
 .0004    0.0022    0.0000    0.0027    0.0006    0.0034
```

(*b*)

6.16B (Con't)

ple 6.15 Study the accuracy of lumped-mass models to approximate the natural frequencies of a simply supported beam. Model the beam using 2, 3, 4, 5, 6, and 7 lumped masses. Compare the natural frequency approximations obtained when each lumped mass is m_b/n, where m_b is the total mass of the beam and n is the number of nodes, to the natural frequencies obtained when the method of Sec. 5.7 is used to obtain the nodal masses. Write a MATLAB script that performs the natural frequency calculations.

Solution:

A simply supported beam modeled with n lumped masses is illustrated in Fig. 6.17. The nodal masses are of equal value

$$m = \frac{m_b}{\beta}$$

where β is a parameter dependent on the method of discretization. If the sum of the nodal masses equals the total mass of the beam, then $\beta = n$. If each nodal mass represents the mass of a region surrounding the particle, as described and illustrated in Sec. 5.7, then $\beta = n + 1$.

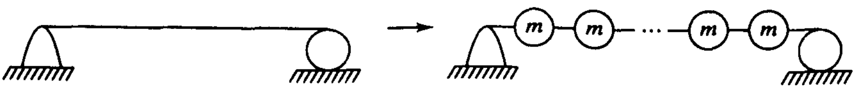


Figure 6.17 Simply supported beam modeled with n equal discrete masses.

Flexibility influence coefficients are used to determine the elements of the flexibility matrix. These elements are in the form

$$a_{ij} = \frac{EI}{L^3}\alpha_{ij}$$

where α_{ij} is determined from Table D.2. The coefficients are determined for $j \geq 1$, using $z = i/(n+1)$ and $a = j/(n+1)$. Flexibility matrix symmetry is used to determine the coefficients for $j < 1$.

The differential equations governing the free vibrations of the approximate system are written as

$$\ddot{\mathbf{x}} + \phi^2 \mathbf{A}\mathbf{x} = \mathbf{0}$$

where

$$\phi = \sqrt{\frac{\beta EI}{m_b L^3}}$$

The natural frequencies are obtained as $1/\omega_i^2 = \phi\lambda_i$ where λ_i, $i = 1, 2, \ldots, n$, are the eigenvalues of $\mathbf{A}$.

The MATLAB script for setting up the flexibility matrix and calculating the natural frequencies is given in Fig. 6.18. The results, the eigenvalues of **A**, are summarized

```
% Example 6.15
% Natural frequency calculations for a discrete lumped
% mass approximation to a simply supported beam.
% Input number of degrees of freedom to be used in model.
n=input('Input number of degrees of freedom')
% Modal masses are of equal value. Using the
% non-dimensional formulation of Example 6.15, the mass
% matrix is the nxn identity matrix. Thus it is not
% necessary to specify. The natural frequencies are the
% reciprocals of the square roots of the eigenvalues of
% the flexibility matrix A
% Setting up the flexibility matrix
for i=1:n
   k=n-i+1;
   for j=1:k
      A(j,k)=(k/(n+1)-1)*j^3/(6*(n+1)^3);
      A(j,k)=A(j,k)+k/(n+1)*(1-k/(n+1))*(2-k/(n+1))*j/(6*(n+1
      A(k,j)=A(j,k);
   end
end
% Calcualting eigenvalues
V=eigs(A);
disp('Nondimensional flexibility matrix'),disp(A)
% Nondimensional natural frequencies are reciprocals
% of square roots of eigenvalues
for i=1:n
   omega(i)=1/sqrt(V(i));
end
disp('Nondimensional natural frequencies'), disp(omega)
```

Figure 6.18 MATLAB script for solution of Example 6.15.

in Table 6.1. The natural frequency approximations using $\beta = n + 1$ are summarized in Table 6.2, while the natural frequency approximations for $\beta = n$ are summarized in Table 6.3. When the results are compared to the exact natural frequencies, obtained by the method of Chap. 9, it is clear that using $\beta = n + 1$ leads to a better approximation.

Table 6.1 Nondimensional frequencies for simply supported beam

	Mode number						
ω	1	2	3	4	5	6	7
$n = 2$	5.6922	22.046	—	—	—	—	—
$n = 3$	4.9333	19.596	41.607	—	—	—	—
$n = 4$	4.4133	17.637	39.988	64.202	—	—	—
$n = 5$	4.0290	16.100	36.000	62.356	89.194	—	—
$n = 6$	3.7302	14.913	33.456	58.826	88.776	116.19	—
$n = 7$	3.4894	13.954	31.348	55.427	85.221	117.68	145.52

Table 6.2 Dimensional frequencies assuming $\beta = n + 1$

	Mode number						
$\hat{\omega}$	1	2	3	4	5	6	7
Exact	9.8696	39.478	88.826	157.91	246.74	355.31	483.61
$n = 2$	9.8591	38.184	—	—	—	—	—
$n = 3$	9.8666	39.192	83.214	—	—	—	—
$n = 4$	9.8685	39.381	87.179	143.56	—	—	—
$n = 5$	9.8691	39.437	88.182	152.74	218.48	—	—
$n = 6$	9.8693	39.457	88.523	155.64	234.88	307.40	—
$n = 7$	9.8694	39.467	88.664	156.77	241.04	332.85	411.60

$\omega = \hat{\omega}\sqrt{\dfrac{EI}{\rho A L^4}}$ where ω is the dimensional natural frequency.

Table 6.3 Dimensional frequencies assuming $\beta = n$

	Mode number						
$\hat{\omega}$	1	2	3	4	5	6	7
Exact	9.8696	39.478	88.826	157.91	246.74	355.31	483.61
$n = 2$	8.0499	31.177	—	—	—	—	—
$n = 3$	8.5447	33.941	72.065	—	—	—	—
$n = 4$	8.8267	35.223	77.973	128.40	—	—	—
$n = 5$	9.0092	36.000	80.499	139.43	199.44	—	—
$n = 6$	9.1372	33.820	81.956	144.09	217.46	284.60	—
$n = 7$	9.2320	36.918	82.938	146.64	225.47	311.35	295.93

$\omega = \hat{\omega}\sqrt{\dfrac{EI}{\rho A L^4}}$ where ω is the natural frequency of a simply supported beam.

6.12 PROPORTIONAL DAMPING

A multi-degree-of-freedom system is said to have *proportional damping* if the viscous damping matrix is a linear combination of the mass matrix and the stiffness matrix,

$$\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M} \tag{6.61}$$

where α and β are constants. The differential equations governing the free vibrations of a linear system with proportional damping are

$$\mathbf{M}\ddot{\mathbf{x}} + (\alpha\mathbf{K} + \beta\mathbf{M})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{6.62}$$

Let $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ be the natural frequencies of an undamped system whose mass matrix is $\mathbf{M}$ and whose stiffness matrix is $\mathbf{K}$. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be the corresponding normalized mode shapes. The expansion theorem implies that $\mathbf{x}(t)$ can be written as a linear combination of the mode shape vectors, as in Eq. (6.54). Substituting Eq. (6.54) in Eq. (6.62) leads to

$$\mathbf{M}\left(\sum_{i=1}^{n} \ddot{p}_i \mathbf{X}_i\right) + (\alpha\mathbf{K} + \beta\mathbf{M})\left(\sum_{i=1}^{n} \dot{p}_i \mathbf{X}_i\right) + \mathbf{K}\left(\sum_{i=1}^{n} p_i \mathbf{X}_i\right) \tag{6.63}$$

Taking the standard scalar product of Eq. (6.63) with $\mathbf{X}_j$ for an arbitrary j, and using properties of scalar products and the definitions of energy scalar products, leads to

$$\sum_{i=1}^{n} \ddot{p}_i \left(\mathbf{X}_j, \mathbf{X}_i\right)_M + \sum_{i=1}^{n} \dot{p}_i \left[\alpha\left(\mathbf{X}_j, \mathbf{X}_i\right)_K + \beta\left(\mathbf{X}_j, \mathbf{X}_i\right)_M\right] + \sum_{i=1}^{n} p_i \left(\mathbf{X}_j, \mathbf{X}_i\right)_K = 0 \tag{6.64}$$

Use of the orthonormality relations, Eqs. (6.44) and (6.45), in Eq. (6.64) leads to

$$\ddot{p}_j + (\alpha\omega_j^2 + \beta)\dot{p}_j + \omega_j^2 p_j = 0 \qquad j = 1, 2, \ldots, n \tag{6.65}$$

The principal coordinates are related to the original generalized coordinates through the linear transformation, Eq. (6.59). Thus the same principal coordinates that uncouple the undamped system uncouple the system when proportional damping is added.

Equation (6.65) is analogous to the differential equation governing free vibrations of a one-degree-of-freedom system and by analogy is rewritten as

$$\ddot{p}_j + 2\zeta_j\omega_j\dot{p}_j + \omega_j^2 p_j = 0 \tag{6.66}$$

where

$$\zeta_j = \frac{1}{2}\left(\alpha\omega_j + \frac{\beta}{\omega_j}\right) \tag{6.67}$$

is called the *modal damping ratio*.

The general solution of Eq. (6.66) for $\zeta_j < 1$ is

$$p_j(t) = A_j e^{-\zeta_j\omega_j t} \sin\left(\omega_j\sqrt{1-\zeta_j^2}\,t - \phi_j\right) \tag{6.68}$$

where A_j and ϕ_j are determined from initial conditions. The generalized coordinates are obtained by using Eq. (6.59).

Damping in structural systems is mostly hysteretic and hard to quantify. Lacking a better model, proportional damping is often assumed. The modal damping ratios are usually determined experimentally. The equivalent damping ratio for a harmonically excited one-degree-of-freedom system with hysteretic damping is proportional to the natural frequency, and inversely proportional to the excitation frequency. This model fits proportional damping where the damping matrix is proportional to the stiffness matrix. In these cases the modes with higher frequencies are damped more than modes with lower frequencies. The natural frequencies in stiff structural systems are usually greatly separated. The effect of the higher modes in the free vibration response is often negligible.

16 A six-degree-of-freedom model of passengers in an automobile is shown in Fig. 6.19, which is the model of Fig. 6.15 with viscous damping added. The damping of the system of Fig. 6.19 is proportional. Determine the modal damping ratios and the time-dependent form of the principal coordinates in terms of arbitrary constants of integration.

Solution:

The damping matrix for the six-degree-of-freedom model is

$$\mathbf{C} = \begin{bmatrix} 2c_1 + 2c_s & -c_1 & -c_1 & -c_s & -c_s & c_1(b-a) + c_s(d-c) \\ -c_1 & c_1 + c_t & 0 & 0 & 0 & c_1 a \\ -c_1 & 0 & c_1 + c_t & 0 & 0 & -c_1 b \\ -c_s & 0 & 0 & c_s & 0 & c_s c \\ -c_s & 0 & 0 & 0 & c_s & -c_s d \\ c_1(b-a) + c_s(d-c) & -c_1 a & c_1 b & -c_s c & c_s d & c_1(a^2+b^2) + c_s(c^2+d^2) \end{bmatrix}$$

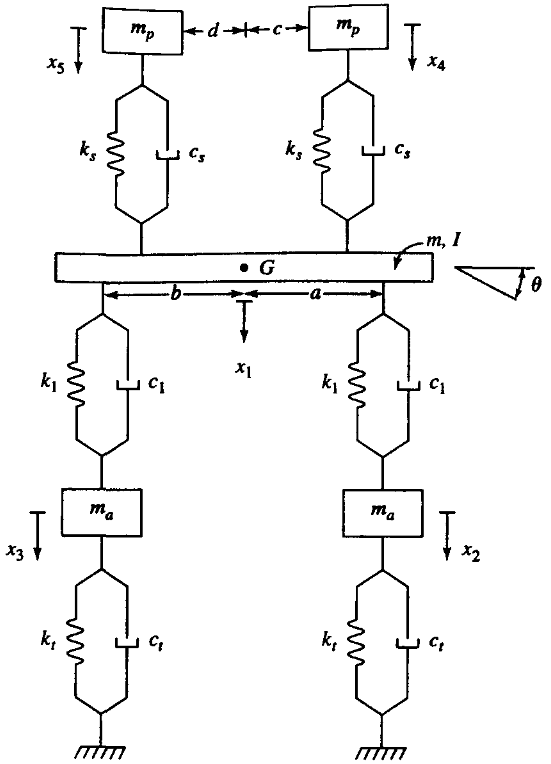
Figure 6.19 illustrates a model in which viscous dampers are in parallel with springs. Each parallel combination has the same ratio of its viscous damping coefficient to its spring stiffness, $\alpha = 0.015$. Thus the damping matrix is proportional to the stiffness matrix with $\alpha = 0.015$.

Using the results of Example 6.14 and Eq. (6.67), the modal damping ratios are calculated as $\zeta_i = \alpha\omega_i/2$, leading to

$$\zeta_1 = 0.060 \qquad \zeta_2 = 0.147 \qquad \zeta_3 = 0.286$$
$$\zeta_4 = 0.388 \qquad \zeta_5 = 0.460 \qquad \zeta_6 = 0.494$$

Thus all modes are underdamped. The free vibration response for the principal coordinates is

$$\mathbf{p} = \begin{bmatrix} C_1 e^{-0.478t} \sin(7.96t + \phi_1) \\ C_2 e^{-2.86t} \sin(19.41t + \phi_2) \\ C_3 e^{-10.92t} \sin(36.60t + \phi_3) \\ C_4 e^{-20.10t} \sin(47.74t + \phi_4) \\ C_5 e^{-28.19t} \sin(54.41t + \phi_5) \\ C_6 e^{-32.51t} \sin(57.22t + \phi_6) \end{bmatrix}$$



$m = 1000$ kg	$I = 360$ kg · m^2	$k_t = 8 \times 10^4$ N/m	$c_s = 3.0 \times 10^3$ N · s/m	$c = 60$ cm
$m_a = 150$ kg	$k_1 = 1.2 \times 10^5$ N/m	$c_1 = 1.8 \times 10^3$ N · s/m	$a = 1.8$ m	$d = 40$ cm
$m_p = 60$ kg	$k_s = 2 \times 10^5$ N/m	$c_t = 1.2 \times 10^3$ N · s/m	$b = 2.0$ m	

Figure 6.19 Six-degree-of-freedom model of suspension system with proportional damping.

6.13 GENERAL VISCOUS DAMPING

The differential equations governing the free vibrations of a multi-degree-of-freedom system with viscous damping is

$$\mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{0} \tag{6.69}$$

If the damping matrix is a linear combination of the mass matrix and the stiffness matrix, the system is proportionally damped. In this case the principal coordinates of the undamped system are used to uncouple the differential equations, Eq. (6.64). The differential equation defining each principal coordinate is analogous to the differential equation governing the motion of a linear one-degree-of-freedom system with viscous damping.

If the damping matrix is arbitrary, the principal coordinates of the undamped system do not uncouple Eq. (6.69). A more general procedure must be used. Equation

(6.69) can be reformulated as $2n$ first-order differential equations by writing

$$\tilde{\mathbf{M}}\dot{\mathbf{y}} + \tilde{\mathbf{K}}\mathbf{y} = \mathbf{0} \tag{6.70}$$

where $$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \quad \tilde{\mathbf{K}} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} \tag{6.71}$$

A solution to Eq. (6.70) is assumed as

$$\mathbf{y} = \mathbf{\Phi} e^{-\gamma t} \tag{6.72}$$

Substitution of Eq. (6.72) into Eq. (6.70) leads to

$$\gamma \tilde{\mathbf{M}} \mathbf{\Phi} = \tilde{\mathbf{K}} \mathbf{\Phi} \tag{6.73}$$

or $$\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}\mathbf{\Phi} = \gamma \mathbf{\Phi} \tag{6.74}$$

Thus the values of γ are the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ and the vectors are the corresponding eigenvectors $\mathbf{\Phi}$.

The values of γ occur in complex conjugate pairs. The system is stable only if all eigenvalues have nonnegative real parts. Eigenvectors corresponding to complex conjugate eigenvalues are also complex conjugates of one another. Eigenvectors corresponding to eigenvalues which are not complex conjugates satisfy the orthogonality relation

$$\bar{\mathbf{\Phi}}_i^T \tilde{\mathbf{M}} \mathbf{\Phi}_j = 0 \tag{6.75}$$

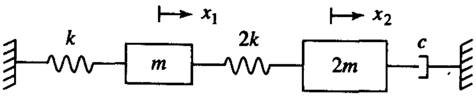
6.17 Plot the free-vibration response to the system of Fig. 6.20 under the initial conditions $x_1(0) = 0$, $x_2(0) = 0.01$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$.

Solution:

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The damping matrix for this system is not a linear combination of the mass matrix and the stiffness matrix. Hence the principal coordinates of the undamped system cannot be



$k = 10{,}000$ N/m $\quad m = 20$ kg $\quad c = 80$ N · s/m

Figure 6.20 System of Example 6.17 has a general viscous damping matrix.

used to uncouple the differential equations. These equations are written in the form of Eq. (6.70) where

$$\mathbf{y} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{bmatrix} \qquad \tilde{\mathbf{M}} = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \\ m & 0 & 0 & 0 \\ 0 & 2m & 0 & c \end{bmatrix} \qquad \tilde{\mathbf{K}} = \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 3k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}$$

A solution of Eq. (6.70) is assumed in the form of Eq. (6.72). The resulting values of γ are the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$. The eigenvalues are obtained by using MATLAB and are given in Fig. 6.21*b*. The eigenvectors, $\mathbf{\Phi}$, are the columns of the matrix $\mathbf{V}$ of Fig. 6.21*b*. The initial condition vector is $\mathbf{y}_0 = [0 \quad 0 \quad 0.01 \quad 0]$. The general solution is a linear combination over all solutions

$$\mathbf{y} = \sum_{j=1}^{4} C_j \mathbf{\Phi}_j e^{-\gamma_j t}$$

```
% Example 6.17
% Parameters
digits(5)
m=20;
k=10000;
c=80;
% Construction of 4x4 matrices
disp('4x4 Mass matrix')
MT=[0,0,m,0;0,0,0,2*m;m,0,0,0;0,2*m,0,c]
disp('4x4 Stiffness matrix')
KT=[-m,0,0,0;0,-2*m,0,0;0,0,3*k,-2*k;0,0,-2*k,2*k]
Z=inv(MT)*KT;
[V,D]=eig(Z);
disp('Eigenvalues')
DS=[D(1,1),D(2,2),D(3,3),D(4,4)]
disp('Eigenvectors')
V
disp('Initial conditions')
x0=[0;0;0.01;0]
disp('Constants of integration')
S=inv(V)*x0
tk=linspace(0,2,101);
% Evaluation of time dependent response
% Recall that x1=y3 and x2=y4
for k=1:101
   t=tk(k);
```

Figure 6.21 (a) MATLAB script for solution of Example 6.17 free-vibration response of system with general damping.

```
for i=3:4
   x(k,i-2)=0;
   for j=1:4
      x(k,i-2)=x(k,i-2)+(real(S(j))*real(V(i,j))-
imag(S(j))*imag(V(i,j)))*cos(imag(D(j,j))*t);
      x(k,i-2)=x(k,i-2)+(imag(S(j))*real(V(i,j))-
real(S(j))*imag(V(i,j)))*sin(imag(V(i,j))*t);
      x(k,i-2)=x(k,i-2)*exp(-real(D(j,j))*t);
   end
end
end
plot(tk,x(:,1),'-',tk,x(:,2),':')
title('Solution of Example 6.16')
xlabel('t (sec)')
ylabel('x (m)')
legend('x1(t)','x2(t)')
```

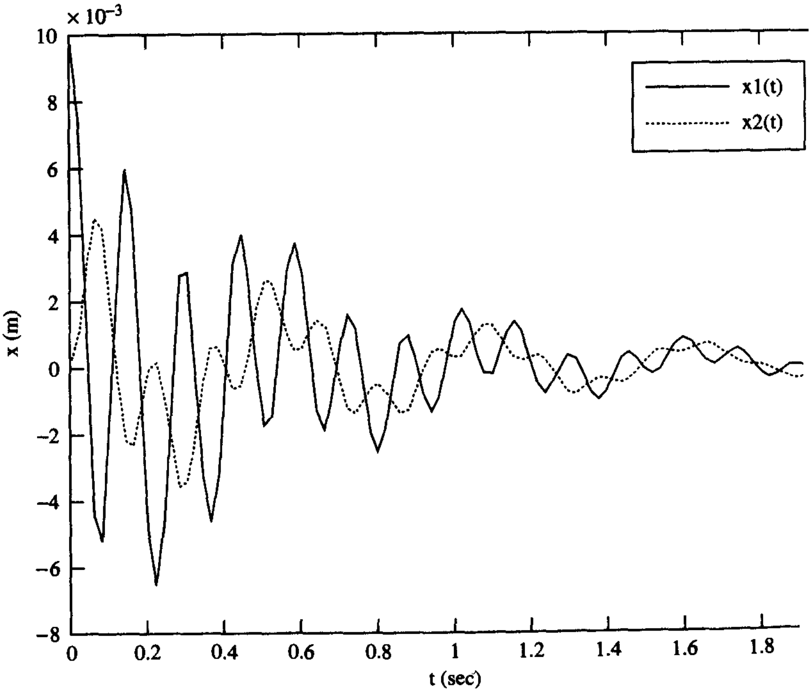
(a)

```
EDU» Ex6_16
4x4 Mass matrix
MT =

     0     0    20     0
     0     0     0    40
    20     0     0     0
     0    40     0    80

4x4 Stiffness matrix

KT =
         -20             0             0             0
           0           -40             0             0
           0             0         30000        -20000
           0             0        -20000         20000

Eigenvalues

DS =

   0.2110+43.1887i   0.2110-43.1887i   0.7890+11.5500i   0.7890-11.550

Eigenvectors

V =

  -0.9242- 0.1659i  -0.9242+ 0.1659i   0.4984- 0.3123i   0.4984+ 0.312
   0.3405+ 0.0437i   0.3405- 0.0437i   0.6871- 0.4179i   0.6871+ 0.417
   0.0039- 0.0214i   0.0039+ 0.0214i   0.0240+ 0.0448i   0.0240- 0.044
  -0.0011+ 0.0079i  -0.0011- 0.0079i   0.0320+ 0.0617i   0.0320- 0.061
```

Figure 6.21 (Con't) (a) Con't; (b) Output including eigenvalues and eigenvectors obtained by running script.

```
Initial conditions

x0 =

         0
         0
    0.0100
         0

Constants of integration

S =

   0.0301+ 0.1790i
   0.0301- 0.1790i
   0.0082- 0.0192i
   0.0082+ 0.0192i

EDU»
```

(*b*)



(*c*)

Figure 6.21 (Con't) (*b*) Con't; (*c*) Free-vibration response of system of Fig. 6.20.

where C_j are constants of integration. Application of initial conditions leads to

$$\mathbf{y}_0 = \sum_{j=1}^{4} C_j \mathbf{\Phi}_j = \mathbf{VC} \qquad \mathbf{C} = \mathbf{V}^{-1}\mathbf{y}_0$$

Since the eigenvalues and eigenvectors are complex conjugate pairs, evaluation of the solution leads to a real response. Its construction is illustrated in the MATLAB script. Evaluation and plotting the response over a period of time leads to Fig. 6.21c.

PROBLEMS

6.1–6.5. Determine the natural frequencies and mode shapes for the two-degree-of-freedom systems shown. Graphically illustrate the mode shapes and identify any nodes.

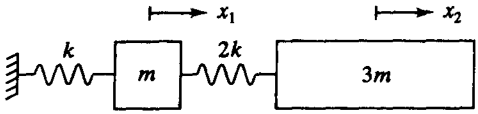


FIGURE P6.1
(Problems 6.1, 6.20, 6.21.)

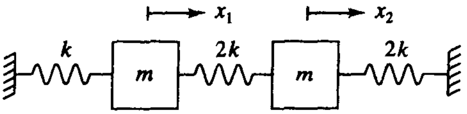


FIGURE P6.2

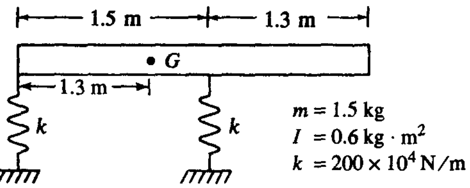


FIGURE P6.3
(Problems 6.3, 6.22.)

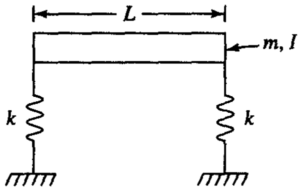


FIGURE P6.4

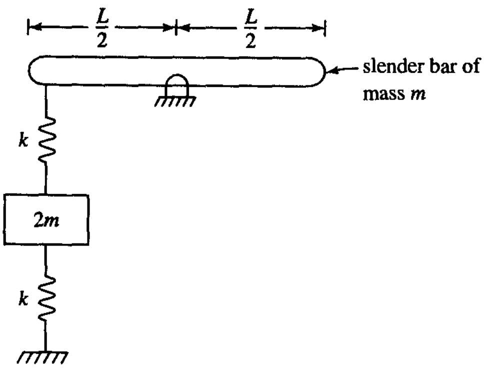


FIGURE P6.5

6.6–6.8. Determine the natural frequencies and mode shapes for the three-degree-of-freedom system shown. Graphically illustrate the mode shapes and identify any nodes.

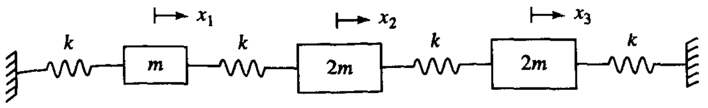


FIGURE P6.6

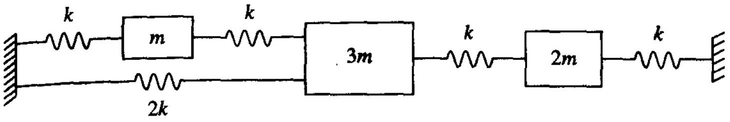


FIGURE P6.7

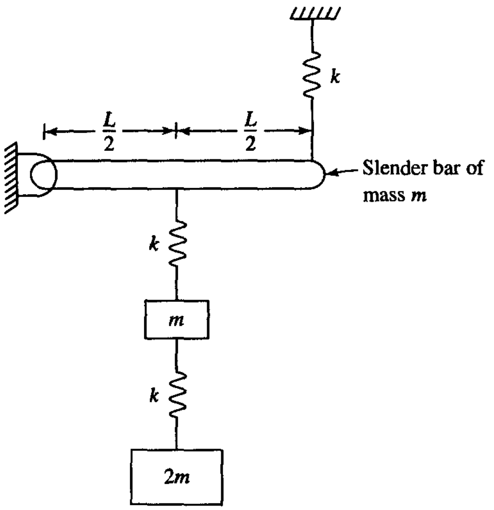


FIGURE P6.8

6.9. Two machines are placed on the massless fixed-pinned beam of Fig. P6.9. Determine the natural frequencies for the system.

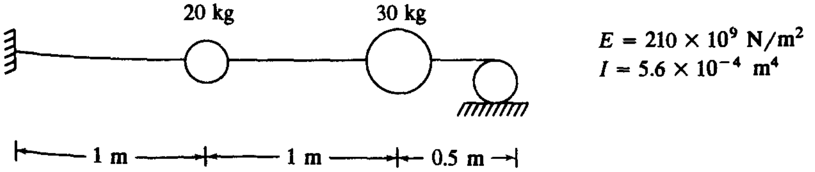


FIGURE P6.9

6.10. Determine the natural frequencies and mode shapes for the system of Fig. P5.2 if $k = 3.4 \times 10^5$ N/m, $L = 1.5$ m and $m = 4.6$ kg.

6.11. Determine the natural frequencies of the system of Fig. P5.5 if $k = 2500$ N/m, $m_1 = 2.4$ kg, $m_2 = 1.6$ kg, $I = 0.65$ kg · m², and $L = 1$ m.

6.12. Determine the natural frequencies and mode shapes for the system of Fig. P5.8 If $k = 10{,}000$ N/m, $m = 3$ kg, $I = 0.6$ kg · m², and $r = 80$ cm.

6.13. Determine the natural frequencies and mode shapes of the system of Fig. P5.10 if $k = 12{,}000$ N/m and each bar is of mass 12 kg and length 4 m.

6.14. A 400-kg machine is placed at the midspan of a 3-m-long, 200-kg simply supported beam. The beam is made of a material of elastic modulus 200×10^9 N/m² and has a cross-sectional moment of inertia of 1.4×10^{-5} m⁴. Use a three-degree-of-freedom model to approximate the system's lowest natural frequency.

6.15. A 500-kg machine is placed at the end of a 3.8-m-long, 190-kg fixed-free beam. The beam is made of a material of elastic modulus 200×10^9 N/m² and has a cross-sectional moment of inertia of 1.4×10^{-5} m⁴. Use a three-degree-of-freedom model to approximate the two lowest natural frequencies of the system.

6.16. A 1500-kg compressor is mounted on springs of stiffness 4×10^5 N/m at the middle of a floor in an industrial plant, which can be modeled as a fixed-fixed beam of length 10 m, elastic modulus 200×10^9 N/m², and has a cross-sectional moment of inertia of 2.3×10^{-4} m⁴ and mass of 7500 kg. The beam rests on soil whose equivalent stiffness is 6×10^6 N/m. Use the two-degree-of-freedom model of Fig. P6.16 to approximate the natural frequencies of the system.

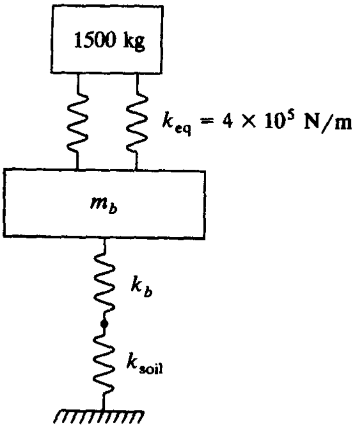


FIGURE P6.16

6.17. Determine the two lowest natural frequencies of the railroad bridge of Prob. 5.70 if $k_1 = 5.5 \times 10^7$ N/m, $k_2 = 1.2 \times 10^7$ N/m, $m = 15{,}000$ kg, $I = 1.6 \times 10^6$ kg · m², $l = 6.7$ m, and $h = 8.8$ m.

6.18. Determine the natural frequencies of the system of Prob. 5.77. The beam is of length 5 m, made of a material of elastic modulus 200×10^9 N/m², and has a cross-sectional moment of inertia of 1.4×10^{-5} m⁴. The total mass of the beam is 320 kg. The mass of the winch is 115 kg. The winch cable is made of a material of elastic modulus 200×10^9 N/m² and has a cross-sectional area of 3.4×10^{-2} m². The length of the cable is 5.5 m and the mass being lifted is 715 kg.

6.19. The natural frequencies of the system of Fig. P6.19 are measured as 5.0 rad/s and 7.8 rad/s. When free vibrations occur at the lower natural frequency only, the two blocks always move in the same direction and the displacement of the block of mass m_2 is 0.5 times the displacement of the block of mass m_1. When free vibrations occur at the larger natural frequency only, the blocks move in opposite directions and the displacement of the block of mass m_2 is 1.3 times the displacement of the block of mass m_1. If $m_1 = 4$ kg, determine the stiffness of each spring.

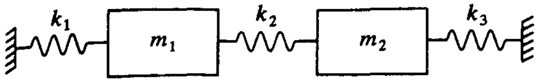


FIGURE P6.19

6.20. Free vibrations of the system of Fig. P6.1 are initiated by moving the block of mass $3m$ a distance δ to the right while holding the block of mass m in its equilibrium position and releasing the system from rest. Mathematically describe the resulting motion.

6.21. Determine a set of initial conditions for the system of Prob. 6.20 such that free vibrations occur only at the system's lowest natural frequency.

6.22. Free vibrations of the system of Fig. P6.3 are initiated by applying a vertical impulse of 1.5 N · s to the left end of the bar when the bar is in equilibrium. Mathematically describe the resulting motion of the bar.

6.23. Determine the free vibration response of the railroad bridge of Prob. 6.17 if a ground disturbance initially leads to $\theta_1 = 0.8°$ with $\theta_2 = \theta_3 = 0$.

6.24. A robot arm is 60 cm long, made of a material of elastic modulus 200×10^9 N/m^2, and has the cross section of Fig. P6.24. The total mass of the arm is 850 g. A tool of mass 1 kg is attached to the end of the arm. Assume one end of the arm is pinned and the other end is free. Use a three-degree-of-freedom model to determine the arm's natural frequencies.

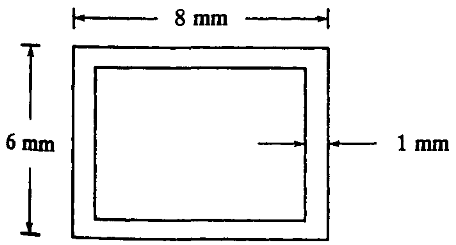


FIGURE P6.24

6.25. A 30,000-kg locomotive is coupled to a fully loaded 20,000-kg boxcar and moving at 6.5 m/s. The assembly is coupled to a stationary and empty 5,000-kg cattle car. The stiffness of each coupling is 5.7×10^5 N/m.

(*a*) What are the natural frequencies of the three-car assembly?

(*b*) Mathematically describe the motion of the cattle car after coupling.

6.26. Determine the natural frequencies and mode shapes for the three-degree-of-freedom model of an airplane of Prob. 5.75. Assume $M = 3.5$ m.

6.27. Determine the natural frequencies and mode shapes of the torsional system of Prob. 5.66.

6.28. Use a four-degree-of-freedom model to approximate the two lowest nonzero natural frequencies of a free-free beam.

6.29. A pipe extends from a wall as shown in Fig. P7.29. The pipe is supported at A to prevent transverse displacement, but not to prevent rotation. Under what conditions will the pipe's lowest natural frequency of transverse vibrations coincide with its frequency of free torsional vibrations?

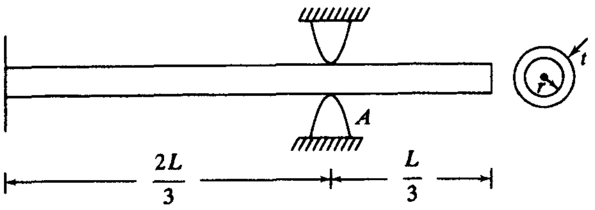


FIGURE P6.29

6.30. The mode shape for the lowest mode of a two-degree-of-freedom system is

$$\mathbf{X}_1 = \begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix}$$

The mass matrix for the system is

$$\mathbf{M} = \begin{bmatrix} 5.4 & 1.2 \\ 1.2 & 3.8 \end{bmatrix}$$

Determine the mode shape corresponding to the system's higher natural frequency.

6.31. The normalized mode shape vector for the lowest mode of a three-degree-of-freedom system is

$$\mathbf{X}_1 = \begin{bmatrix} 1.2 \\ -0.8 \\ 1.1 \end{bmatrix}$$

The stiffness matrix for the system is

$$\mathbf{K} = 10^5 \begin{bmatrix} 3.1 & -1.4 & 0 \\ -1.4 & 3.5 & -1.9 \\ 0 & -1.9 & 2.6 \end{bmatrix}$$

Determine the natural frequency corresponding to this mode.

6.32. The mass matrix for a three-degree-of-freedom system is

$$\mathbf{M} = \begin{bmatrix} 1.5 & 0.6 & 0 \\ 0.6 & 2.4 & 0 \\ 0 & 0 & 3.1 \end{bmatrix}$$

Mode shape vectors corresponding to two-modes of this system are

$$\mathbf{X}_1 = \begin{bmatrix} 1.0 \\ 1.5 \\ 0.6 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} 1.3 \\ -2.1 \\ 2.39 \end{bmatrix}$$

Determine (*a*) the mode shape vector for the third mode and (*b*) a set of normalized mode shape vectors.

6.33. Show that Rayleigh's quotient $R(\mathbf{X})$ is stationary if and only if $\mathbf{X}$ is a mode shape vector.

6.34. Use Rayleigh's quotient to determine an upper bound on the lowest natural frequency of the system of Fig. P6.7. Use at least four trial vectors.

6.35. An alternative method to derive the uncoupled equations governing the motion of the free vibrations of a n-degree-of-freedom system in terms of principal coordinates is to introduce a linear transformation between the generalized coordinates $\mathbf{x}$ and the principal coordinates $\mathbf{p}$ as $\mathbf{x} = \mathbf{P}\mathbf{p}$, where $\mathbf{P}$ is the modal matrix, the matrix whose columns are the normalized mode shapes. Follow these steps to derive the equations governing the principal coordinates:

(*a*) Rewrite Eq. (6.1) using the principal coordinates as dependent variables by introducing the linear transformation in Eq. (6.1).

(*b*) Premultiply the resulting equation by $\mathbf{P}^T$.

(*c*) Use the orthonormality of mode shapes to show that $\mathbf{P}^T\mathbf{M}\mathbf{P}$ and $\mathbf{P}^T\mathbf{K}\mathbf{P}$ are diagonal matrices.

(*d*) write the uncoupled equations for the principal coordinates.

6.36. Use the method of Prob. 6.35 to derive the uncoupled equations governing the principal coordinates for a system with proportional damping.

6.37. Determine the free vibration response of the system of Fig. P6.37 if the system is released from rest after the 3-kg block is displaced 5 mm.

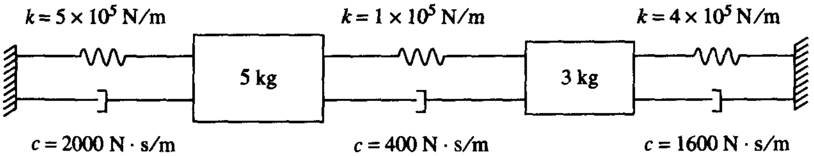


FIGURE P6.37

6.38. If the modal damping ratio for the lowest mode of Prob. 6.14 is 0.03, determine the modal damping ratio for the higher modes and determine the response of the system if the machine is displaced 2 mm and released.

6.39. Determine the free-vibration response of the bar of Fig. P6.39 is the mass center is displaced 1 cm from equilibrium while the bar is held horizontal and the system released from this position.

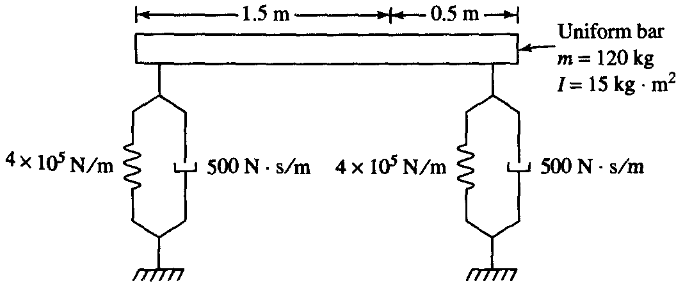


FIGURE P6.39

6.40. Determine the free-vibration response of the system of Fig. P6.40.

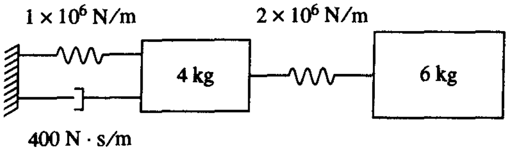


FIGURE P6.40

6.41. Determine the free-vibration response of the system of Fig. P6.41.

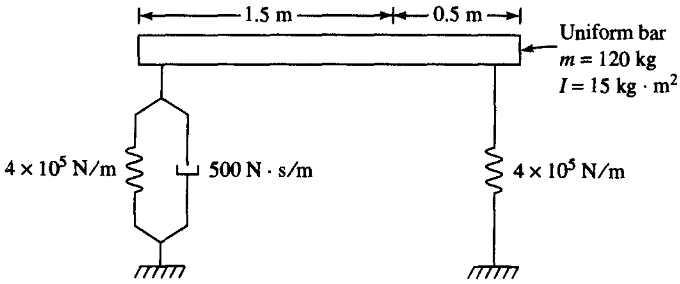


FIGURE P6.41

6.42. Determine the free-vibration response of the system of Prob. 5.68 when $E = 200 \times 10^9$ N/m^2, $I = 1.5 \times 10^{-6}$ m^4, $L = 0.8$ m, $k = 1.5 \times 10^5$ N/m, $c = 250$ N · s/m, $m_1 = 4$ kg, $m_2 = 6.1$ kg.

MATLAB PROBLEMS

M6.1. File VIBES_6A.m contains the MATLAB script used in the solution of Example 6.3. Use VIBES_6A to determine an appropriate value of k such that both natural frequencies are between 20 rad/s and 50 rad/s.

M6.2. File VIBES_6B.m contains a MATLAB script used in the solution of Example 6.5. Use the file to determine an appropriate W section such that the beam's natural frequencies are

(*a*) Outside the range of 300 to 500 rad/s

(*b*) All greater than 200 rad/s

(*c*) Outside the range 1000 to 1200 rad/s.

M6.3. File VIBES_6C.m contains a MATLAB script file that is used to determine the natural frequencies and normalized mode shapes for an n-degree-of-freedom system, given the system's mass matrix and stiffness matrix. Use VIBES_6C to determine the natural frequencies and mode shapes for

(*a*) The system of Fig. P6.3

(*b*) The system of Fig. P6.7 given $m = 2.8$ kg and $k = 10{,}000$ N/m

(*c*) The system of Fig. P6.8 given $m = 4.6$ kg, $k = 18{,}000$ N/m, and $L = 80$ cm

M6.4. File VIBES_6D.m contains a MATLAB script file that is used to determine the natural frequencies and normalized mode shapes for an n-degree-of-freedom model of the system of Fig. PM6.4. Use the file to:

(*a*) Study the effect of the number of degrees of freedom used in the model on the natural frequency approximations for a beam with $\beta = m/m_b = 0.8$ and $\alpha = kL^3/3EI = 0.6$. Develop a table similar to Table 6.1 for the study.

(*b*) Study the effect of β on the natural frequencies for $\alpha = 0.6$. Use a five-degree-of-freedom model (four for the beam and one for the discrete mass) with $\alpha = 0.6$. Plot the natural frequency approximations as a function of β for $0.1 < \beta < 4$.

M6.5. File VIBES_6E.m contains a MATLAB script file that is used to determine the free vibration response of a four-degree-of-freedom system with proportional damping. Use VIBES_6E to determine the response of the four-degree-of-freedom model of the suspension system of Fig. PM6.5 when the vehicle encounters a pothole of depth h. Assume both wheels encounter the pothole at the same time. The front seat is 1.2 m from the front of the vehicle. Determine the effect of the location of the center of gravity on the maximum displacement of the front seat.

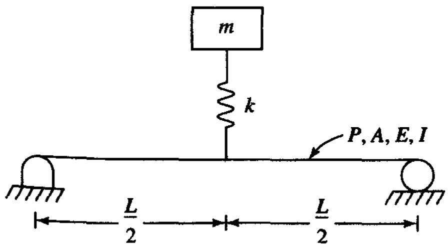
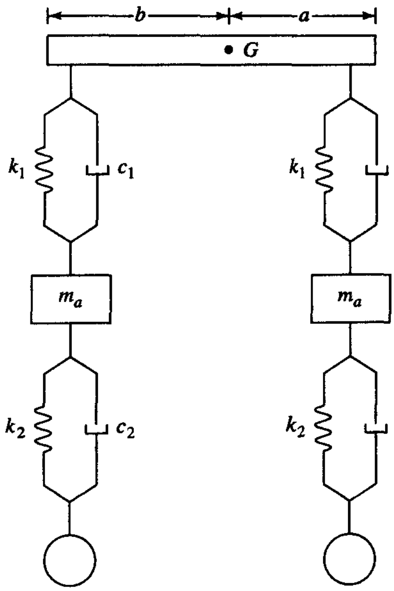


FIGURE PM6.4



$m = 1000$ kg $\quad m_a = 165$ kg

$I = 200$ kg · m^2 $\quad a + b = 2.4$ m

$k_1 = 1.2 \times 10^5$ N/m

$k_2 = 8 \times 10^4$ N/m

Problem M6.5:	$a = 1.0$ m	$b = 1.4$ m
	$c_1 = 1320$ N · s/m	$c_2 = 880$ N · s/m
Problem M6.6:	$c_1 = 1500$ N · s/m	$c_2 = 1300$ N · s/m

FIGURE PM6.5

M6.6. File VIBES_6F.m contains a MATLAB script file that is used to determine the free vibration response of the four-degree-of-freedom model of the suspension system of Fig. PM6.5. Use VIBES6_F to repeat the study of Prob. M6.5 for the parameters given in the figure.

M6.7. Write a MATLAB script file that is used to determine the natural frequencies and normalized mode shapes of an n-degree-of-freedom model of a system given the mass matrix and the flexibility matrix. Use the file to solve (*a*) Prob. 6.14, (*b*) Prob. 6.15.

M6.8. Write a MATLAB script file to determine the natural frequencies and normalized mode shapes for an n-degree-of-freedom model of a fixed-free beam with a machine at its free end. Use the file to approximate the lowest natural frequencies for the system of Fig. PM6.8.

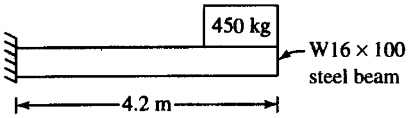


FIGURE PM6.8

M6.9. Write a MATLAB script file that determines the free vibration response of a n-degree-of-freedom system subject to appropriate initial conditions. Use the file to solve Prob. 6.23.

M6.10. Write a MATLAB script file that solves Prob. 6.42.

chapter

7 Forced Vibrations of Multi-Degree-of-Freedom Systems

7.1 INTRODUCTION

The forced response of a linear multi-degree-of-freedom system, as for a linear one-degree-of-freedom system, is the sum of a homogeneous solution and a particular solution. The homogeneous solution depends on system properties, while the particular solution is the response due to the particular form of the excitation. The free-vibration response is often ignored for a system whose long-term behavior is important, such as a system subject to a periodic excitation. The free-vibration solution is important for systems in which the short-term behavior is important, such as a system subject to a shock excitation.

Several methods are available to determine the forced response of a multi-degree-of-freedom system. The method of undetermined coefficients can be applied to any system subject to a periodic excitation. However, because of algebraic complexity, its usefulness is restricted to systems with only a few degrees of freedom. The Laplace transform method can be applied to determine system properties, but its usefulness is limited because its application requires the solution of a system of simultaneous equations whose coefficients are functions of the transform variable. Both the method of undetermined coefficients and the Laplace transform method can be used to determine the forced response of a system with a general damping matrix.

The most useful method for determining the forced-vibrations response of a linear multi-degree-of-freedom system is modal analysis, which is based on using the principal coordinates to uncouple the differential equations governing the motion of an undamped or proportionally damped system. The uncoupled differential equations are solved by the standard techniques for solution of ordinary differential equations. A more general form of modal analysis involving complex algebra is developed for systems with a general damping matrix.

Often the differential equations cannot be solved in closed form. Modal analysis can still be used to uncouple the differential equations. The differential equations for the principal coordinates can be solved by numerical integration of the convolution integral or direct numerical simulation of the differential equation by a method such as a Runge-Kutta method.

7.2 HARMONIC EXCITATIONS

The response of a multi-degree-of-freedom system due to a harmonic excitation is the sum of the homogeneous solution and the particular solution. Even if damping is not included, the homogeneous solution is often ignored. In a real situation, damping is present, causing the homogeneous solution to decay with time. The long-time or steady-state solution is only the particular solution.

The method of undetermined coefficients can be adapted to find the particular solution for a multi-degree-of-freedom system subject to a harmonic excitation. The method of undetermined coefficients can be used for damped or undamped systems. Its application for an n-degree-of-freedom system requires the solution of at least one set of n simultaneous equations. Thus the method of undetermined coefficients is efficient for systems with only a few degrees of freedom.

The differential equations governing the motion of an n-degree-of-freedom undamped system subject to a single-frequency excitation with all excitation terms at the same phase are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}\sin\omega t \quad [7.1]$$

where $\mathbf{F}$ is an n-dimensional vector of constants. The method of undetermined coefficients is used and assumes a particular solution of the form

$$\mathbf{x}(t) = \mathbf{U}\sin\omega t \quad [7.2]$$

where $\mathbf{U}$ is an n-dimensional vector of undetermined coefficients. Substituting Eq. (7.2) in Eq. (7.1) leads to

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{U} = \mathbf{F} \quad [7.3]$$

Equation (7.3) represents a set of n simultaneous algebraic equation to solve for the components of the vector $\mathbf{U}$. A unique solution of Eq. (7.3) exists unless

$$|-\omega^2\mathbf{M} - \mathbf{K}| = 0 \quad [7.4]$$

Equation (7.4) is satisfied only when the excitation frequency coincides with one of the system's natural frequencies. When this occurs the use of Eq. (7.2) is inappropriate. The response grows linearly with time, producing a resonance condition.

When a solution of Eq. (7.3) exists it can be written as

$$\mathbf{U} = (-\omega^2\mathbf{M} + \mathbf{K})^{-1}\mathbf{F} \quad [7.5]$$

7.1 Determine the response of the two-degree-of-freedom system shown in Fig. 7.1. Formulate the equations for the steady-state amplitudes in nondimensional form.

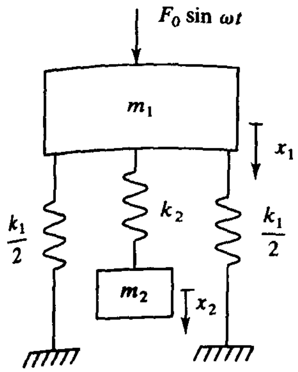


Figure 7.1 System of Example 7.1. Response for harmonic excitation is determined using undetermined coefficients.

Solution:

The differential equations governing the behavior of the system of Fig. 7.1 are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_0 \sin \omega t$$
$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

The particular solution is of the format of Eq. (7.2)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin \omega t$$

Equation (7.3) becomes

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

The equations are solved simultaneously, yielding

$$U_1 = \frac{(-\omega^2 m_2 + k_2)F_0}{D(\omega)} \qquad U_2 = \frac{k_2 F_0}{D(\omega)}$$

where $$D(\omega) = m_1 m_2 \omega^4 - (k_2 m_1 + k_1 m_2 + k_2 m_2)\omega^2 + k_1 k_2$$

is the determinant of the coefficient matrix. The roots of the equation $D(\omega) = 0$ are the natural frequencies.

The steady-state amplitudes are each functions of six parameters:

$$U_1 = f(m_1, m_2, k_1, k_2, F_0, \omega) \qquad U_2 = g(m_1, m_2, k_1, k_2, F_0, \omega)$$

The dimensions of the six parameters involve mass, length, and time. Nondimensional relationships between the steady-state amplitudes and the independent parameters involve four nondimensional parameters. Nondimensional forms of the dimensional equations are obtained by multiplying each equation by k_1/F_0. The resulting nondimensional relationships are

$$M_1 = \frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1+\mu) r_1^2 + 1}$$

and

$$M_2 = \frac{1}{r_1^2 r_2^2 - r_2^2 - (1+\mu) r_1^2 + 1}$$

where

$$M_1 = \frac{k_1 U_1}{F_0} \qquad M_2 = \frac{k_1 U_2}{F_0}$$

$$r_1 = \omega\sqrt{\frac{m_1}{k_1}} \qquad r_2 = \omega\sqrt{\frac{m_2}{k_2}}$$

$$\mu = \frac{m_2}{m_1}$$

It is noted from these equations that the steady-state amplitude of the mass m_1 is zero when $r_2 = 1$. Thus when correctly tuned the second mass-spring system absorbs the steady-state motion of the first mass. This is the concept of the dynamic vibration absorber, which is considered in more detail in Chap. 8.

The differential equations governing the motion of an n-degree-of-freedom system with viscous damping subject to a single-frequency harmonic excitation are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathrm{Im}\,(\mathbf{F}e^{i\omega t}) \tag{7.6}$$

where $\mathbf{F}$ is an n-dimensional vector of constants. The constants could be complex if each generalized force is not of the same phase and are of the form

$$F_i = f_i e^{i\phi} \tag{7.7}$$

The solution of Eq. (7.6) is assumed as

$$\mathbf{x}(t) = \mathrm{Im}\,(\mathbf{U}e^{i\omega t}) \tag{7.8}$$

where $\mathbf{U}$ is an n-dimensional vector of complex constants. Substitution of Eq. (7.8) in Eq. (7.6) leads to

$$(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})\mathbf{U} = \mathbf{F} \tag{7.9}$$

The solution of Eq. (7.9) is obtained as

$$\mathbf{U} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\mathbf{F} \tag{7.10}$$

---

ple 7.2 A two-degree-of-freedom model of a vehicle suspension system is illustrated in Fig. 7.2a. The vehicle travels over a road whose contour is approximated by the sinusoidal contour of Fig. 7.2b. The horizontal speed of the vehicle is v.

(a) Determine the response of the vehicle in terms of the system parameters.

(b) Plot the amplitude of steady-state acceleration for the values of the parameters given.

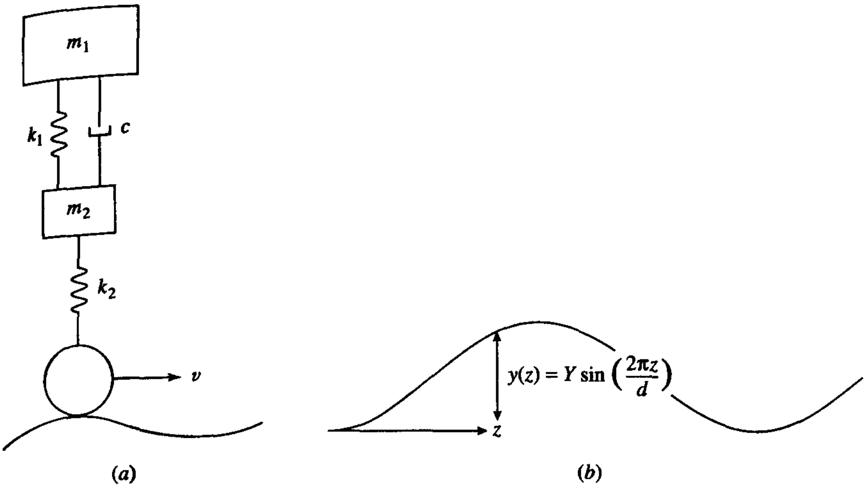


Figure 7.2 (a) Two-degree-of-freedom model of suspension system; (b) sinusoidal road contour.

Solution:

The differential equations governing the motion of the system are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 Y \end{bmatrix} \sin \omega t$$

A solution to the above equations in the form of Eq. (7.8) is assumed leading to Eqs. (7.9) and (7.10). Figure 7.3 shows the MATLAB script that symbolically determines **U**. Running the script leads to

$$U_1 = \frac{k_2 Y(k_1 + i\omega c)}{(\omega^4 m_1 m_2 - \omega^2 m_1 k_1 - \omega^2 m_1 k_2 - \omega^2 k_1 m_2 + k_1 k_2) + i(-\omega^3 m_1 c - \omega^3 m_2 c + \omega c k_2)}$$

U_1 is put into proper form by multiplying and dividing by the complex conjugate of the denominator

$$U_1 = \mathrm{Re}\,(U_1) + i\,\mathrm{Im}\,(U_1)$$

The polar form of U_1 is

$$U_1 = |U_1| e^{-i\phi_1}$$

where

$$|U_1| = \sqrt{\mathrm{Re}\,(U_1)^2 + \mathrm{Im}\,(U_1)^2}$$

$$\phi_1 = -\tan^{-1}\left(\frac{\mathrm{Im}\,(U_1)}{\mathrm{Re}\,(U_1)}\right)$$

Algebraic manipulations lead to

$$|U_1| = \sqrt{\frac{k_1^2 + (\omega c)^2}{(\mathrm{Re}\,(D)^2 + \mathrm{Im}\,(D)^2}}$$

$$\phi_1 = -\tan^{-1}\left(\frac{-k_1\,\mathrm{Im}\,(D) + \omega c\,\mathrm{Re}\,(D)}{\omega c\,\mathrm{Im}\,(D) + k_1\,\mathrm{Re}\,(D)}\right)$$

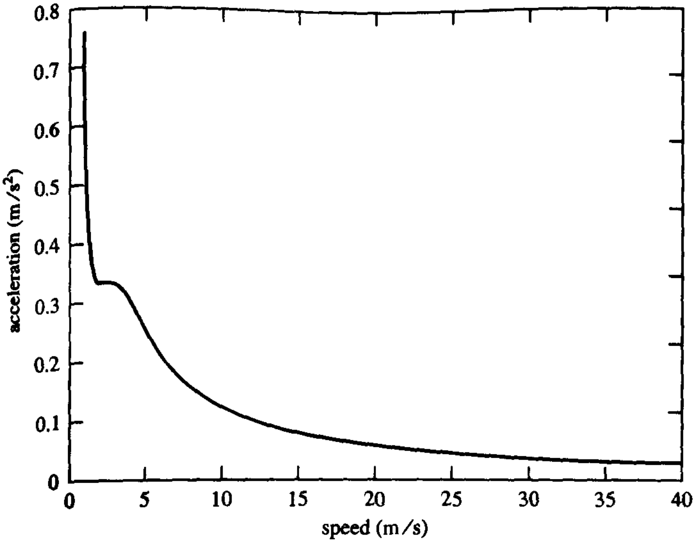
```
% Symbolic determination of steady-state amplitude for two-degree-of-
% freedom suspension model of Example 7.2.
syms m1 m2 k1 k2 c Y w
% Mass matrix
M=[m1,0;0,m2];
% Damping matrix
C=[c,-c;-c,c];
% Stiffness matrix
K=[k1,-k1;-k1,k1+k2];
% Generalized force vector
F=[0;k2*Y];
% Impedance matrix
S=-w^2*M+i*w*C+K;
% Steady-state solution
U=inv(S)*F
```

(*a*)

```
% MATLAB script to plot steady-state amplitude vs. vehicle speed
% for Example 7.2
% Input parameters
Y=0.01;                % Road contour amplitude (m)
k1=80000;              % Suspension stiffness (N/m)
k2=40000;              % Tire stiffness (n/m)
m1=1200;               % Vehicle mass (kg)
m2=400;                % Axle mass (kg)
c=7000;                % Suspension damping coefficient (N-s/m)
d=1.2;                 % Road period (m)
Set up v
=linspace(1,40,501);
or i=1:501
   % Frequency
   w=2*pi*v(i)/d;
   % Re(D)
   DR=w^4*m1*m2-w^2*m1*k1-w^2*m1*k2-w^2*k1*m2+k1*k2;
   % Im(D)
   DI=-w^3*m1*c-w^3*m2*c+w*c*k2;
   % Acceleration amplitude
   Acc(i)=abs(w^2*k2*Y*sqrt((k1^2+(w*c)^2)/(DR^2+DI^2)));
nd
lot(v,Acc,'-')
label('speed (m/s)')
label('acceleration (m/s^2)')
```

(*b*)

ure 7.3 (*a*) MATLAB script for symbolic determination of steady-state amplitude of Example 7.2. Complex representation is used; (*b*) MATLAB script to plot steady-state amplitude of acceleration of vehicle as function of speed.



(c)

Figure 7.3 (Con't) (c) Acceleration amplitude versus speed.

where $$\mathrm{Re}\,(D) = \omega^4 m_1 m_2 - \omega^2 m_1 k_1 - \omega^2 m_1 k_2 - \omega^2 k_1 m_2 + k_1 k_2$$
$$\mathrm{Im}\,(D) = -\omega^3 m_1 c - \omega^3 m_2 c + \omega c k_2$$

The solution for $x_1(t)$ is

$$x_1(t) = \mathrm{Im}\left(U_1 e^{i\omega t}\right) = \mathrm{Im}\left(|U_1| e^{i(\omega t - \phi_1)}\right) = |U_1| \sin(\omega t - \phi_1)$$

If the vehicle speed is v, the frequency at which the vehicle traverses the road contour is $\omega = (2\pi v)/d$. The acceleration is $A = \omega^2 U_1$. The MATLAB script to plot the steady-state amplitude for the given parameters is shown in Fig. 7.3*b* and the resulting plot is given in Fig. 7.3*c*.

7.3 LAPLACE TRANSFORM SOLUTIONS

Let $\bar{\mathbf{x}}(s)$ be the vector of Laplace transforms of generalized coordinates for an n-degree-of-freedom system. Taking the Laplace transform of Eq. (5.12), using linearity of the transform and the property of transforms of derivatives, gives

$$(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})\bar{\mathbf{x}}(s) = \bar{\mathbf{F}}(s) + (s\mathbf{M} + \mathbf{C})\mathbf{x}(0) + \mathbf{M}\dot{\mathbf{x}}(0) \qquad [7.11]$$

where $\bar{\mathbf{F}}(s)$ is the vector of Laplace transforms of $\mathbf{F}(t)$. If $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$, Eq. (7.11) becomes

$$\mathbf{z}(s)\bar{\mathbf{x}}(s) = \bar{\mathbf{F}}(s) \qquad [7.12]$$

where
$$\mathbf{Z}(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{7.13}$$
is called the *impedance matrix*. Premultiplying Eq. (7.13) by $\mathbf{Z}^{-1}$ yields
$$\bar{\mathbf{x}}(s) = \mathbf{Z}^{-1}(s)\bar{\mathbf{F}}(s) \tag{7.14}$$

Using the methods of App. C, the inverse of the impedance matrix is written as
$$\mathbf{Z}^{-1}(s) = \frac{1}{\det(\mathbf{Z}(s))}\mathbf{H}(s) \tag{7.15}$$
where the components of $\mathbf{H}(s)$ are polynomials in s of order $n-1$ or less. The determinant of the impedance matrix is a polynomial in s of order $2n$, called the *characteristic polynomial*, and is used to determine the free-vibration characteristics of the system. The roots of this polynomial occur in complex conjugate pairs,
$$s_j = s_{r_j} + i s_{i_j} \qquad j = 1, 2, \ldots, n \tag{7.16}$$
leading to the following factorization:
$$\det(\mathbf{Z}(s)) = \prod_{j=1}^{n}\left(s^2 - 2s_{r_j}s + s_{r_j}^2 + s_{i_j}^2\right) \tag{7.17}$$

Partial fraction decomposition is used to develop
$$\bar{x}_k(s) = A_k(s) + \sum_{j=1}^{n}\frac{\alpha_{kj}s + \beta_{kj}}{s^2 - 2s_{r_j}s + s_{r_j}^2 + s_{i_j}^2} \qquad k = 1, \ldots, n \tag{7.18}$$
Inversion of Eq. (7.18) leads to
$$x_k(t) = x_{k_p}(t) + \sum_{j=1}^{n} e^{s_{r_j}t}\left[C_{kj}\sin\left(\sqrt{s_{r_j}^2 + s_{i_j}^2}\,t\right) + D_{kj}\cos\left(\sqrt{s_{r_j}^2 + s_{i_j}^2}\,t\right)\right] \tag{7.19}$$
where
$$x_{k_p}(t) = \mathcal{L}^{-1}\{A_k(s)\}$$

If the real part of all roots of the characteristic polynomial are negative, all terms in the summation in Eq. (7.19) decay with time. The steady state has only a contribution from the particular solution. If any root has a positive real part, then its corresponding term in the summation in Eq. (7.19) grows exponentially and the system is unstable.

ample 7.3 Determine the steady-state amplitudes for the two-degree-of-freedom system of Fig. 7.4.

Solution:

The matrix form of the governing differential equations for the system of Fig. 7.4 is
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \sin\omega t \\ 0 \end{bmatrix}$$

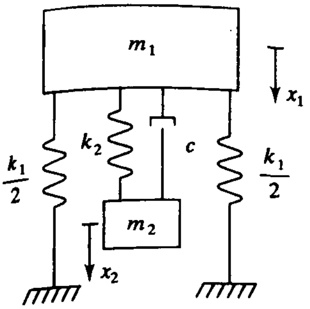


Figure 7.4 System of Example 7.3.

The system impedance matrix is

$$\mathbf{Z}(s) = \begin{bmatrix} m_1 s^2 + cs + k_1 + k_2 & -cs - k_2 \\ -cs - k_2 & m_2 s^2 + cs + k_2 \end{bmatrix}$$

and its determinant is

$$\begin{aligned} D(s) &= (m_1 s^2 + cs + k_1 + k_2)(m_2 s^2 + cs + k_2) - (cs + k_2)^2 \\ &= m_1 m_2 s^4 + (m_1 + m_2)cs^3 + (m_1 k_2 + k_1 m_2 + m_2 k_2)s^2 + k_1 cs + k_1 k_2 \end{aligned}$$

Cramer's rule is used to solve for $\bar{\mathbf{x}}(s)$, leading to

$$\bar{x}_1(s) = \frac{F_0 \omega (m_2 s^2 + cs + k_2)}{(s^2 + \omega^2)D(s)}$$

and

$$\bar{x}_2(s) = \frac{F_0 \omega (cs + k)}{(s^2 + \omega^2)D(s)}$$

A partial fraction decomposition of $\bar{x}_1(s)$ allows it to be written as

$$\bar{x}_1(s) = \frac{U_1 s + V_1}{s^2 + \omega^2} + \frac{Q(s)}{D(s)}$$

where the constants U_1 and V_1 and the function $Q(s)$ are to be determined. Inversion of the term with $D(s)$ in the denominator leads to the transient part of the solution represented by the summation in Eq. (7.19). The steady-state solution for $x_1(t)$ is obtained by solving for U_1 and V_1 and inverting only the first term in the preceding equation. Multiplication by the common denominator leads to

$$(U_1 s + V_1)D(s) + Q(s)(s^2 + \omega^2) = F_0 \omega (m_2 s^2 + cs + k_2)$$

Since the preceding equation is valid for all complex s, set $s = i\omega$. Then

$$(i\omega U_1 + V_1)D(i\omega) = F_0 \omega (-m_2 \omega^2 + ic\omega + k_2)$$

or
$$i\omega U_1 + V_1 = \frac{F_0\omega(-m_2\omega^2 + ic\omega + k_2)\bar{D}(i\omega)}{D(i\omega)\bar{D}(i\omega)}$$

The constant V_1 is equal to the real part of the right-hand side of the preceding equation, while ωU_1 is equal to the imaginary part. Performing this algebra leads to

$$U_1 = \frac{F_0\left[c\omega M(\omega) - (k_2 - m_2\omega^2)N(\omega)\right]}{M^2(\omega) + N^2(\omega)}$$

and
$$V_1 = \frac{F_0\omega\left[(k_2 - m_2\omega^2)M(\omega) + c\omega N(\omega)\right]}{M^2(\omega) + N^2(\omega)}$$

where
$$M(\omega) = m_1 m_2\omega^2 - (m_1 k_2 + k_1 m_2 + m_2 k_2)\omega^2 + k_1 k_2$$

and
$$N(\omega) = -(m_1 + m_2)c\omega^3 + k_1 c\omega$$

Inversion of the transform yields the steady-state solution for $x_1(t)$ as

$$x_1(t) = U_1\cos\omega t + \frac{V_1}{\omega}\sin\omega t$$

An alternative form for the preceding equation is

$$x_1(t) = X_1\sin(\omega t - \phi_1)$$

where
$$X_1 = \sqrt{U_1^2 + \left(\frac{V_1}{\omega}\right)^2} = \sqrt{\frac{(c\omega)^2 + (k_2 - m_2\omega^2)^2}{M^2 + N^2}}$$

and
$$\phi_1 = \frac{\omega U_1}{V_1}$$

A similar procedure is applied to solve for the steady-state solution for $x_2(t)$, yielding

$$x_2(t) = U_2\cos\omega t + \frac{V_2}{\omega}\sin\omega t = X_2\sin(\omega t - \phi_2)$$

where
$$U_2 = \frac{F_0[k_2 M(\omega) - c\omega N(\omega)]}{M^2(\omega) + N^2(\omega)}$$

$$V_2 = \frac{F_0\omega[c\omega M(\omega) + k_2 N(\omega)]}{M^2(\omega) + N^2(\omega)}$$

and
$$X_2 = \sqrt{\frac{k_2^2 + (c\omega^2)^2}{M^2 + N^2}}$$

The system in this example is an undamped one-degree-of-freedom system with a damped vibration absorber attached. This system is discussed in more detail in Sec. 8.7.

7.4 MODAL ANALYSIS FOR UNDAMPED SYSTEMS AND SYSTEMS WITH PROPORTIONAL DAMPING

The differential equations governing the forced vibrations motion of an undamped linear n-degree-of-freedom system are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.20}$$

The method of *modal analysis* uses the principal coordinates of the system to uncouple the differential equations of Eq. (7.20).

Let $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ be the natural frequencies of the system whose equations are given by Eq. (7.20). Let $\mathbf{P}$ be the system's modal matrix, the matrix whose columns are the normalized mode shapes, $\mathbf{P} = [\mathbf{X}_1 \;\; \mathbf{X}_2 \;\; \cdots \;\; \mathbf{X}_n]$. Using the expansion theorem, as in Sec. 6.8, the response at any instant of time can be expanded as

$$\mathbf{x}(t) = \sum_{i=1}^{n} p_i(t)\mathbf{X}_i \tag{7.21}$$

where $p_i(t)$ are the system's principal coordinates. Equation (7.21) is equivalent to a linear transformation between the original generalized coordinates and the principal coordinates

$$\mathbf{x} = \mathbf{P}\mathbf{p} \tag{7.22}$$

Substitution of Eq. (7.21) in Eq. (7.20) leads to

$$\sum_{i=1}^{n} \ddot{p}_i \mathbf{M}\mathbf{X}_i + \sum_{i=1}^{n} p_i \mathbf{K}\mathbf{X}_i = \mathbf{F} \tag{7.23}$$

Taking the standard scalar product of Eq. (7.23) with $\mathbf{X}_j$ for an arbitrary j leads to

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{M}\mathbf{X}_i) + \sum_{i=1}^{n} p_i (\mathbf{X}_j, \mathbf{K}\mathbf{X}_i) = (\mathbf{X}_j, \mathbf{F}) \tag{7.24}$$

On the basis of the definitions of energy scalar products, Eq. (7.24) becomes

$$\sum_{i=1}^{n} \ddot{p}_i (\mathbf{X}_j, \mathbf{X}_i)_M + \sum_{i=1}^{n} p_i (\mathbf{X}_j, \mathbf{X}_i)_K = (\mathbf{X}_j, \mathbf{F}) \tag{7.25}$$

Application of mode shape orthogonality leads to only one nonzero term in each summation, the term corresponding to $i = j$. Since the mode shapes are normalized, Eq. (7.25) leads to

$$\ddot{p}_j + \omega_j^2 p_j = g_j(t) \tag{7.26}$$

where

$$g_j(t) = (\mathbf{X}_j, \mathbf{F}) \tag{7.27}$$

An equation of the form of Eq. (7.26) can be written for each $j = 1, 2, \ldots, n$. This shows that the principal coordinates that are used to uncouple the differential equations governing free vibrations can also be used to uncouple the differential equations governing forced vibrations. The differential equations of Eq. (7.26) can be solved by any useful means. If the initial conditions for p_i are both zero then the convolution integral solution of Eq. (7.26) is

$$p_i(t) = \frac{1}{\omega_i}\int_0^t g_i(\tau)\sin[\omega_i(t-\tau)]\,d\tau \tag{7.28}$$

Once the solutions for each p_i have been obtained, Eq. (7.21) is used to determine the original generalized coordinates.

The modal analysis procedure to determine the forced response of an undamped linear n-degree-of-freedom system is summarized below.

1. A set of generalized coordinates is chosen. The differential equations governing the motion of the system are derived by any appropriate method. The differential equations are written in the matrix form of Eq. (7.20).
2. The natural frequencies and normalized mode shapes are obtained. The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ and the mode shapes are the corresponding eigenvectors. The mode shapes are normalized by requiring that the kinetic energy scalar product of a mode shape with itself be equal to one.
3. The elements of the column vector $\mathbf{G}$ are obtained by using Eq. (7.27). An alternative method to obtain $\mathbf{G}$ is

$$\mathbf{G} = \mathbf{P}^T\mathbf{F} \tag{7.29}$$

4. Equations of the form of Eq. (7.26) are solved to obtain the time-dependent form of the principal coordinates. Equation (7.28) gives the convolution integral solution of Eq. (7.26).
5. The time-dependent form of the original generalized coordinates is obtained by using Eq. (7.21) or Eq. (7.22).

7.4 Use modal analysis to determine the time-dependent response of the system of Fig. 7.5*a* subject to the excitation of Fig. 7.5*b*.

Solution:

The differential equations governing the motion of the system of Fig. 7.5*a* are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m}{2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$$

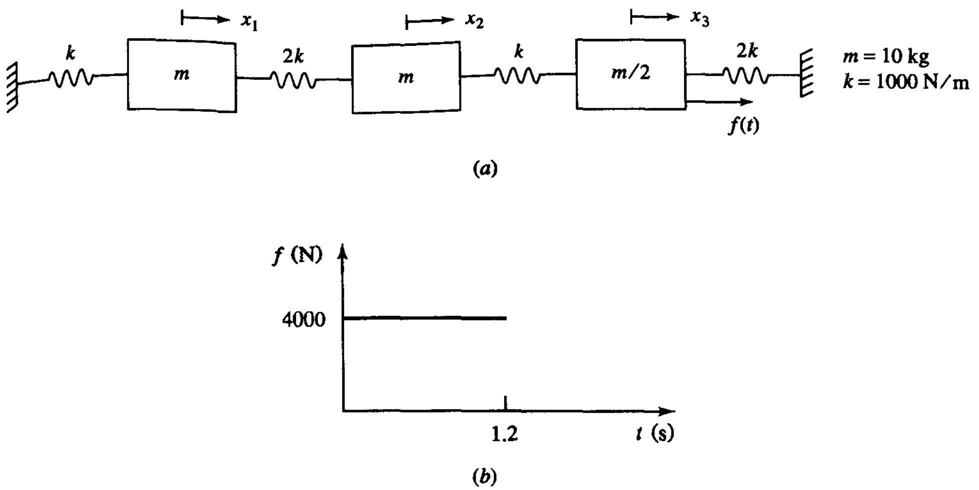


Figure 7.5 (a) Three-degree-of-freedom system of Example 7.4; (b) excitation for system of Example 7.4

where from Fig. 7.5b

$$f(t) = 4000\,[1 - u(t - 1.2)]\ \text{N}$$

where t is in seconds.

The natural frequencies for this system are determined in Example 6.4 and the normalized mode shapes are determined in Example 6.11. Substituting $m = 10$ kg and $k = 1000$ N/m in these results leads to natural frequencies of

$$\omega_1 = 8.936 \text{ rad/s} \qquad \omega_2 = 21.107 \text{ rad/s} \qquad \omega_3 = 25.974 \text{ rad/s}$$

and a modal matrix of

$$\mathbf{P} = \begin{bmatrix} 0.2085 & 0.2252 & 0.0765 \\ 0.2295 & -0.1638 & -0.1432 \\ 0.0882 & -0.2120 & 0.3838 \end{bmatrix} (\text{kg})^{-1/2}$$

The vector $\mathbf{G}(t)$ is then calculated by using Eq. (7.29)

$$\mathbf{G}(t) = \mathbf{P}^T\mathbf{F} = \begin{bmatrix} 0.0882 \\ -0.2120 \\ 0.3838 \end{bmatrix} f(t)$$

The differential equations satisfied by the principal coordinates are written by using Eq. (7.26)

$$\begin{aligned} \ddot{p}_1 + 79.852p_1 &= 352.8\,[1 - u(t - 1.2)] \\ \ddot{p}_2 + 445.5p_2 &= -848.0\,[1 - u(t - 1.2)] \\ \ddot{p}_3 + 674.6p_3 &= 1535.2\,[1 - u(t - 1.2)] \end{aligned}$$

The convolution integral is used to solve for p_1 as

$$\begin{aligned} p_1(t) &= \frac{1}{8.936}\int_0^t 352.8\,[1-u(\tau-1.2)]\sin 8.936(t-\tau)\,d\tau \\ &= 4.418\,\{\cos 8.936t - 1 + u(t-1.2)\,[1-\cos 8.936(t-1.2)]\} \end{aligned}$$

The convolution integral is also used to solve for p_2 and p_3, yielding

$$\begin{aligned} p_2(t) &= -1.903\,\{\cos 21.107t - 1 + u(t-1.2)\,[1-\cos 21.107(t-1.2)]\} \\ p_3(t) &= 2.276\{\cos 25.974t - 1 + u(t-1.2)[1-\cos 25.974(t-1.2)]\} \end{aligned}$$

The solution in terms of the original generalized coordinates is obtained by using Eq. (7.22)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2085 & 0.2252 & 0.0765 \\ 0.2295 & -0.1638 & -0.1432 \\ 0.0882 & -0.2120 & 0.3838 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

which leads to

$$\begin{aligned} x_1(t) &= 0.921h_1(t) - 0.429h_2(t) + 0.174h_3(t) \\ x_2(t) &= 1.014h_1(t) + 0.312h_2(t) - 0.326h_3(t) \\ x_3(t) &= 0.390h_1(t) + 0.403h_2(t) + 0.874h_3(t) \end{aligned}$$

where

$$\begin{aligned} h_1(t) &= \cos 8.936t - 1 + u(t-1.2)[1-\cos 8.936(t-1.2)] \\ h_2(t) &= \cos 21.107t - 1 + u(t-1.2)[1-\cos 21.107(t-1.2)] \\ h_3(t) &= \cos 25.974t - 1 + u(t-1.2)[1-\cos 25.974(t-1.2)] \end{aligned}$$

.5 A machine of mass 150 kg is placed as shown on the simply supported beam of Fig. 7.6. The machine has a rotating unbalance of 0.965 kg · m and operates at 1250 rpm. The beam has a total mass of 280 kg, a cross-sectional moment of inertia of 1.2×10^{-4} m^4, a length of 3 m, and an elastic modulus of 210×10^9 N/m^2. Model the beam with three degrees of freedom and use modal analysis to predict the steady-state amplitude of displacement for the point where the machine is attached.

Solution:

The beam is modeled three particles of mass 70 kg, as shown in Fig. 7.6*b*. The mass matrix for this model is

$$\mathbf{M} = \begin{bmatrix} 70 & 0 & 0 \\ 0 & 70 & 0 \\ 0 & 0 & 220 \end{bmatrix} \text{kg}$$

Flexibility influence coefficients are used to determine the flexibility matrix as

$$\mathbf{A} = 10^{-9} \begin{bmatrix} 12.53 & 15.33 & 9.75 \\ 15.33 & 22.29 & 15.33 \\ 9.75 & 15.33 & 12.53 \end{bmatrix} \frac{\text{m}}{\text{N}}$$

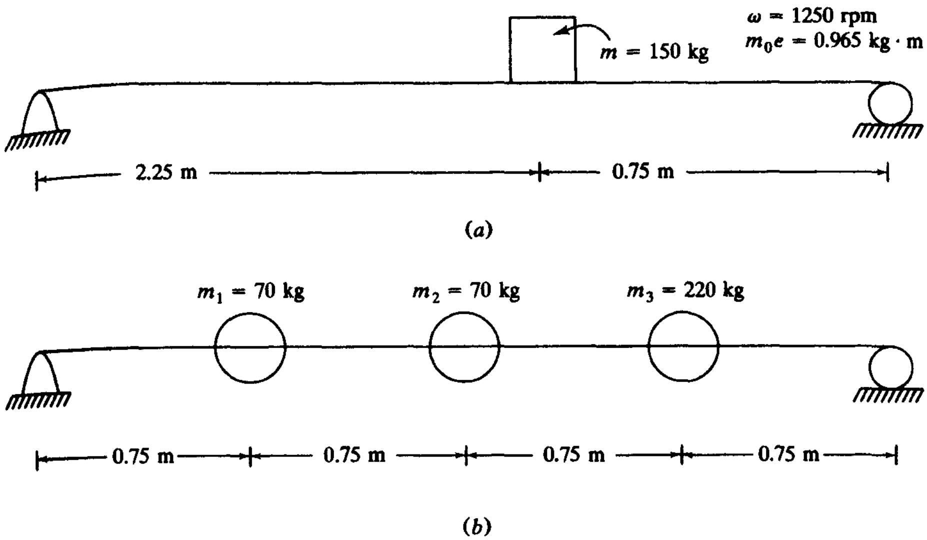


Figure 7.6 (a) Machine with rotating unbalance is attached to pinned-pinned beam; (b) three-degree-of-freedom model of beam.

The governing differential equations are

$$\mathbf{AM\ddot{x}} + \mathbf{x} = \mathbf{AF}$$

where

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 0 \\ 16500 \sin 130.9t \end{bmatrix} \text{N}$$

The natural frequencies and normalized mode shapes are determined by using the MATLAB script of Fig. 7.7*a*. The MATLAB results are given in Fig. 7.7*b*. The vector $\mathbf{G}(t)$ is calculated as

$$\mathbf{G} = \mathbf{P}^T\mathbf{F} = \begin{bmatrix} 0.0453 & 0.0666 & 0.0498 \\ -0.0851 & -0.4000 & 0.0416 \\ -0.0707 & 0.0908 & -0.0182 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 16500 \sin 130.9t \end{bmatrix}$$

$$= \begin{bmatrix} 821.8 \\ 687.0 \\ -300.3 \end{bmatrix} \sin 130.9t \text{ N (kg)}^{-1/2}$$

The differential equations for the principal coordinates are written by using Eq. (7.26)

$$\ddot{p}_1 + (455.8)^2 p_1 = 821.8 \sin 130.9t$$

$$\ddot{p}_2 + (1736.5)^2 p_2 = 687.0 \sin 130.9t$$

$$\ddot{p}_3 + (4474)^2 p_3 = -300.3 \sin 130.9t$$

The steady-state solution of

$$\ddot{P}_i + \omega_i^2 P_i = F_i \sin \omega t$$

is

$$p_i(t) = \frac{F_i}{\omega_i^2 - \omega^2} \sin \omega t$$

The steady-state solution for the principal coordinates is

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 10^{-5} \begin{bmatrix} 432.0 \\ 22.93 \\ -1.501 \end{bmatrix} \sin 130.9t \ (\text{kg})^{1/2}$$

Equation (7.22) is used to determine $x_3(t)$ as

$$x_3(t) = 0.0498 p_1(t) + 0.0416 p_2(t) - 0.0182 p_3(t) = 2.25 \times 10^{-4} \sin 130.9t \text{ m}$$

Thus the maximum steady-state displacement of the point on the beam where the machine is placed is 0.225 mm.

```
.ample 7.5
.ss and flexibility matrices
 0,0,0;0,70,0;0,0,220];
1^-9*[12.53,15.33,9.75;15.33,22.29,15.33;9.75,15.33,12.53];
 M;
 envalues and eigenvectors
 ]=eig(C);
 tural frequencies are reciprocals of square roots of eigenvalues
 1/sqrt(D(1,1));
 1/sqrt(D(2,2));
 1/sqrt(D(3,3));
 rmalizing mode shapes, E is a diagonal matrix with energy scalar products
  mode shape vectors with themselves along diagonal
 *M*V;
 j=1:3
 =sqrt(E(j,j));
 or i=1:3
   P(i,j)=V(i,j)/c;
 nd

 tput results
 (' Natural frequencies in rad/sec'), disp(om1), disp(om2), disp(om3)
 (' Modal matrix in kg^(-1/2)'), disp(P)
```

(a)

7.7 (a) MATLAB script used to determine natural frequencies and modal matrix for system of Example 7.5.

```
Natural frequencies in rad/sec
 455.8455

 1.7365e+003

 4.4740e+003

Modal matrix in kg^(-1/2)
   0.0453   -0.0851   -0.0707
   0.0666   -0.0400    0.0908
   0.0498    0.0416   -0.0182
```

(*b*)

Figure 7.7 (Con't) (*b*) output obtained by running script.

The differential equations governing the forced vibrations of a linear system with viscous damping are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.30}$$

If the system is proportionally damped, the damping matrix is a linear combination of the mass matrix and the stiffness matrix as in Eq. (6.61).

Modal analysis using the principal coordinates of the undamped system can be used to uncouple the differential equations of a system with proportional damping. Substitution of Eq. (7.21) into Eq. (7.30) and following a procedure similar to that used for the undamped system leads to the differential equations for the principal coordinates as

$$\ddot{p}_i + 2\zeta_i\omega_i\dot{p}_i + \omega_i^2 p_i = g_i(t) \tag{7.31}$$

where the modal damping ratio ζ_i is defined in Eq. (6.67).

The convolution integral solution of Eq. (7.31) is

$$p_i(t) = \frac{1}{\omega_i\sqrt{1-\zeta_i^2}}\int_0^t g_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\left[\omega_i\sqrt{1-\zeta_i^2}(t-\tau)\right]d\tau \qquad [7.32]$$

The procedure for application of modal analysis to a system with proportional damping is the same as that for an undamped system with the addition of the determination of the modal damping ratios to step 2 and the use of Eq. (7.32) as the convolution integral solution.

Damping in structural systems is mostly hysteretic and hard to quantify. Lacking a better model, proportional damping is often assumed. The modal damping ratios are usually determined experimentally. The equivalent damping ratio for a harmonically excited one-degree-of-freedom system with hysteretic damping is proportional to the natural frequency, and inversely proportional to the excitation frequency. This model fits proportional damping where the damping matrix is proportional to the stiffness matrix. In these cases the higher modes are damped more than the lower modes. The natural frequencies in stiff structural systems are usually greatly separated. The effect of the higher modes in the total response is less than the modes with lower natural frequencies. For these reasons, damping ratios are often specified only for the lower modes.

If proportional damping is assumed, the higher modes are damped more than the lower modes and have a lesser effect on the overall solution. Modes with higher damping ratios die out more quickly when the system is subject to any short-term or shock excitation. If the system is subject to a harmonic excitation, the modes with higher frequencies have lesser effect because their amplitudes are inversely proportional to the square of their frequencies. Thus fewer modes can be calculated without losing significant accuracy. Hence, in practice, Eq. (7.21) is often replaced by

$$\mathbf{x}(t) = \sum_{i=1}^{m} p_i\mathbf{X}_i \qquad [7.33]$$

for some $m < n$. Equation (7.33) is often used in situations where the mode shapes are determined experimentally and an experimental modal analysis method is used to determine the response of a system.

---

nple 7.6 The three-degree-of-freedom system of Example 7.4 is modified by the addition of dashpots as shown in Fig. 7.8. Determine the forced response of the damped system.

Solution:

The damping matrix is

$$\mathbf{C} = \begin{bmatrix} 3c & -2c & 0 \\ -2c & 3c & -c \\ 0 & -c & 3c \end{bmatrix}$$

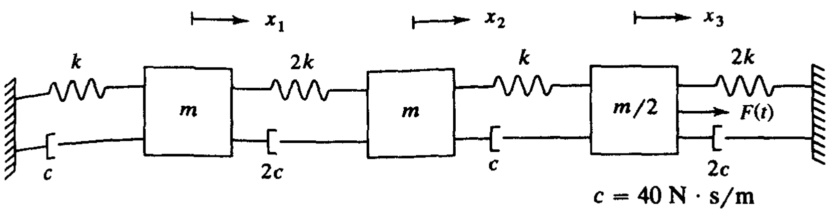


Figure 7.8 System of Example 7.6 has viscous damping with the damping matrix proportional to the stiffness matrix.

and is proportional to the stiffness matrix with

$$\alpha = \frac{c}{k} = \frac{40 \text{ N} \cdot \text{s/m}}{1000 \text{ N/m}} = 0.04 \text{ s}$$

Thus the modal damping ratios are given by

$$\zeta_1 = \frac{\alpha}{2}\omega_1 = 0.178 \qquad \zeta_2 = \frac{\alpha}{2}\omega_2 = 0.422 \qquad \zeta_3 = \frac{\alpha}{2}\omega_3 = 0.520$$

All modes are underdamped. The differential equations governing the principal coordinates are

$$\ddot{p}_1 + 1.60\dot{p}_1 + 79.85 p_1 = 0.0882 f(t)$$
$$\ddot{p}_2 + 8.91\dot{p}_2 + 445.5 p_2 = -0.2120 f(t)$$
$$\ddot{p}_3 + 13.49\dot{p}_3 + 674.6 p_3 = 0.3838 f(t)$$

The solution for the principal coordinates is obtained from the convolution integral. It is noted that

$$\int_0^t [1 - u(\tau - 1.2)]\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$
$$= -\frac{1-\zeta^2}{\omega_d}\left[1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]\right.$$
$$- u(t-1.2)\left\{1 - e^{-\zeta\omega_n(t-1.2)}\left[\cos\omega_d(t-1.2)\right.\right.$$
$$\left.\left.\left. + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d(t-1.2)\right]\right\}\right]$$

Application of the convolution integral to the first equation leads to

$$p_1(t) = 4.43\left[1 - e^{-1.60t}(\cos\,8.79t + 0.181\sin\,(8.79t)\right]$$
$$- 4.43u(t-1.2)\left\{1 - 6.77e^{-1.60t}\left[\cos\,(8.79t - 10.55)\right.\right.$$
$$\left.\left. + 0.181\sin\,(8.79t - 10.55)\right]\right\}$$

A MATLAB script used to apply modal analysis to Example 7.6 is given in Fig. 7.9. Given the mass matrix, the stiffness matrix, and α, the program determines the natural frequencies and modal matrix for the system, develops the vector $\mathbf{G}(t)$, and uses symbolic algebra to apply the convolution integral to determine the responses of the principal coordinates and uses modal superposition to determine the time-dependent response of the original generalized coordinates. The response for an undamped system is plotted as well as the response for a system with $\alpha = 0.04$.

```
% Example 7.6
digits(5)
syms t tau
% Input parameters
m=10;                                     % Mass in kg
k=1000;                                   % Stiffness in N/m
c=4;                                      % Damping coefficient in N-s/m
F0=4000;                                  % Excitation magnitude in N
alpha=c/k;                                % Proportional damping coefficient
M=[m,0,0;0,m,0;0,0,m/2];                  % Mass matrix
K=[3*k,-2*k,0;-2*k,3*k,-k;0,-k,3*k];      % Stiffness matrix
C=[3*c,-2*c,0;-2*c,3*c,-c;0,-c,3*c];      % Damping matrix
W=inv(M)*K;                               % M^(-1)*K
[V,D]=eig(W);                             % Eigenvalues and eigenvectors
for i=1:3
   w(i)=sqrt(D(i,i));                     % Natural frequencies
   z(i)=alpha/2*w(i);                     % Modal damping ratios
end
% Normalization of mode shapes
E=V'*M*V;
for j=1:3
   cc=1/sqrt(E(j,j));
   for i=1:3
      P(i,j)=V(i,j)*cc;                   % Modal matrix
   end
end
disp('Natural frequencies (r/s)');disp(w)
disp('Modal matrix');disp(P)
f=F0*(1-sym('Heaviside(tau-1.2)'));  % Excitation vector using symbolics
```

Figure 7.9 (a) MATLAB script for system of Example 7.6.

```
F=[0;0;f];
G=P'*F;                                         % G(t)
% Symbolic integration of convolution integral
for i=1:3
   k=G(i)*exp(-z(i)*w(i)*(t-tau))*sin(w(i)*sqrt(1-z(i)^
   p(i)=int(k,tau,0,t);
   p(i)=p(i)/(w(i)*sqrt(1-z(i)^2));
end
p=[p(1);p(2);p(3)];
% Original generalized coordinates
x=P*p;
% Plotting
time=linspace(0,1.5,101);
for i=1:101
   xp1(i)=subs(x(1),t,time(i));
   xp2(i)=subs(x(2),t,time(i));
   xp3(i)=subs(x(3),t,time(i));
   xp1(i)=vpa(xp1(i));
   xp2(i)=vpa(xp2(i));
   xp3(i)=vpa(xp3(i));
end
xp1=double(xp1);
xp2=double(xp2);
xp3=double(xp3);
plot(time,xp1,'-',time,xp2,':',time,xp3,'-.')
xlabel('t (s)')
ylabel('response (m)')
legend('x1(t)','x2(t)','x3(t)')
```

(*a*)

```
Natural frequencies (r/s)
    8.9360   21.1066   25.9742

Modal matrix
   -0.2085    0.2252   -0.0765
   -0.2295   -0.1638    0.1432
   -0.0882   -0.2120   -0.3838
```

(*b*)

Figure 7.9 (Con't) (*a*) Con't. (*b*) output from running script.

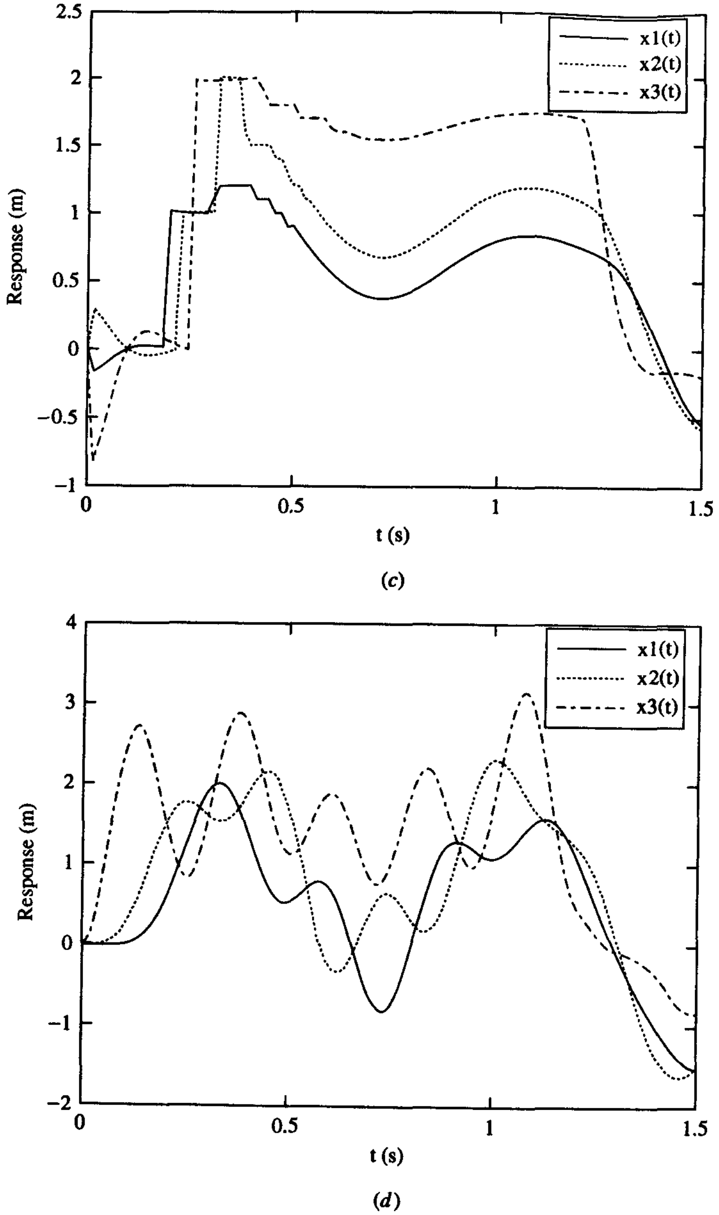


Figure 7.9 (Con't) (*c*) system response for $\alpha = 0.04$; (*d*) system response for $\alpha = 0$.

7.5 MODAL ANALYSIS FOR SYSTEMS WITH GENERAL DAMPING

The differential equations governing the forced vibrations of a linear n-degree-of-freedom system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{7.34}$$

can be rewritten as a system of $2n$ linear first-order equations

$$\tilde{\mathbf{M}}\dot{\mathbf{y}} + \tilde{\mathbf{K}}\mathbf{y} = \tilde{\mathbf{F}} \tag{7.35}$$

where $\mathbf{y}$, $\tilde{\mathbf{M}}$, and $\tilde{\mathbf{K}}$ are defined in Eq. (6.71) and

$$\tilde{\mathbf{F}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix} \tag{7.36}$$

The homogeneous solution of Eq. (7.35) is obtained in Sec. 6.13. The solution uses eigenvalues and eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$. Eigenvalues occur in complex conjugate pairs. Eigenvectors satisfy the orthogonality relation of Eq. (6.75). The eigenvectors can be normalized by requiring

$$\bar{\mathbf{\Phi}}_i^T \tilde{\mathbf{M}} \mathbf{\Phi}_i = 1 \tag{7.37}$$

The modal matrix $\tilde{\mathbf{P}}$ is the matrix whose columns are the normalized eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$. The principal coordinates are defined by

$$\mathbf{y} = \tilde{\mathbf{P}}\tilde{\mathbf{p}} \tag{7.38}$$

Substituting Eq. (7.38) in Eq. (7.35) leads to

$$\tilde{\mathbf{M}}\tilde{\mathbf{P}}\dot{\tilde{\mathbf{p}}} + \tilde{\mathbf{K}}\tilde{\mathbf{P}}\tilde{\mathbf{p}} = \tilde{\mathbf{F}} \tag{7.39}$$

Premultiplying Eq. (7.39) by $\tilde{\mathbf{P}}^T$ leads to

$$\tilde{\mathbf{P}}^T\tilde{\mathbf{M}}\tilde{\mathbf{P}}\dot{\tilde{\mathbf{p}}} + \tilde{\mathbf{P}}^T\tilde{\mathbf{K}}\tilde{\mathbf{P}}\tilde{\mathbf{p}} = \tilde{\mathbf{P}}^T\tilde{\mathbf{F}} = \tilde{\mathbf{G}} \tag{7.40}$$

Use of mode shape orthonormality in Eq. (7.40) results in

$$\dot{\tilde{\mathbf{p}}} - \mathbf{\Lambda}\mathbf{p} = \mathbf{G} \tag{7.41}$$

where $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ along the diagonal. Thus the differential equations represented by Eq. (7.41) are uncoupled and written as

$$\dot{\tilde{p}}_i - \gamma_i \tilde{p}_i = \tilde{g}_i(t) \qquad i = 1, 2, \ldots, 2n \tag{7.42}$$

The convolution integral solution of Eq. (7.42) is

$$\tilde{p}_i = \int_0^t \tilde{g}_i(\tau) e^{-\gamma_i(t-\tau)}\, d\tau \tag{7.43}$$

Application of modal analysis to systems with general damping is very similar to its application to systems with proportional damping. The procedure is summarized below.

1. The differential equations governing the forced vibrations of the system are derived in terms of a chosen set of generalized coordinates and written in the form of Eq. (7.34).
2. The differential equations are reformulated in the form of Eq. (7.35), using Eqs. (6.71) and (7.36).
3. The eigenvalues and eigenvectors of $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}$ are obtained. The eigenvectors are normalized by using Eq. (6.75). The modal matrix $\tilde{\mathbf{P}}$ is formed as the matrix whose columns are the normalized mode shapes.
4. The vector $\tilde{\mathbf{G}} = \tilde{\mathbf{P}}^T\mathbf{F}$ is determined.
5. Differential equations of the form of Eq. (7.42) are written for each principal coordinate.
6. The differential equations are solved by any convenient method. The convolution integral solution is given by Eq. (7.43).
7. The time-dependent behavior of the chosen generalized coordinates is obtained by using Eq. (7.38).

le 7.7 Determine the response of the system of Fig. 7.10 when $F(t) = 50e^{-1.5t}$ N.

Solution:

The differential equations governing the motion of the system are

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

The differential equations are written in the form of Eq. (7.35) as

$$\begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \\ m & 0 & 0 & 0 \\ 0 & 2m & 0 & c \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} + \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 3k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix}$$

where $\mathbf{y} = [\dot{x}_1 \quad \dot{x}_2 \quad x_1 \quad x_2]^T$.

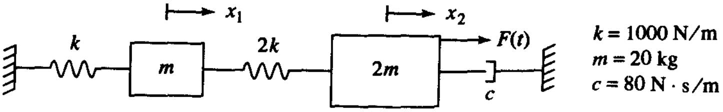


Figure 7.10 Two-degree-of-freedom system with external excitation and general damping.

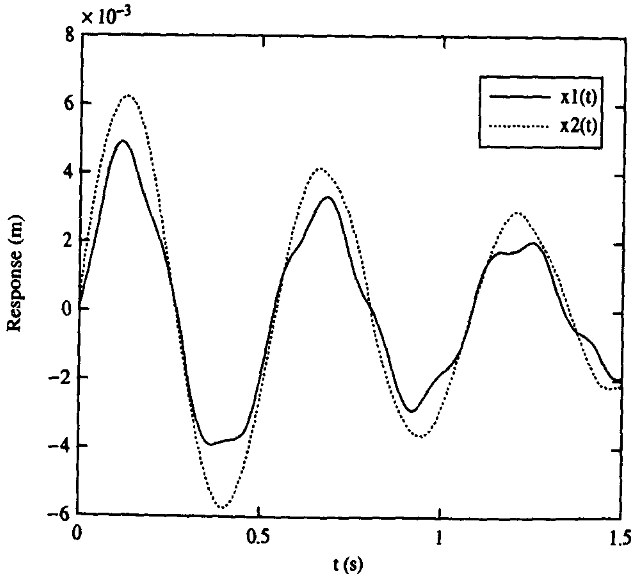
The free vibration response for this example is given in Example 6.16. The MATLAB script using the modal analysis procedure to determine the forced response is given in Fig. 7.11*a*. The program symbolically evaluates the convolution integral for each principal coordinate, then uses Eq. (7.38) to determine the time-dependent response of the original generalized coordinates. The MATLAB output is too lengthy to be printed here, but can be obtained by using VIBES_7C.m on the accompanying software package. The output plot is given in Fig. 7.11*b*.

```
% Example 7.7- Forced response of system with general damping
% Parameters
syms t tau
digits(5)
m=20;
k=10000;
c=80;
F0=50;
alpha=1.5;
% Construction of 4x4 matrices
disp('4x4 Mass matrix')
MT=[0,0,m,0;0,0,0,2*m;m,0,0,0;0,2*m,0,c];disp(MT)
disp('4x4 Stiffness matrix')
KT=[-m,0,0,0;0,-2*m,0,0;0,0,3*k,-2*k;0,0,-2*k,2*k];disp(KT)
Z=inv(MT)*KT;
[V,D]=eig(Z);
disp('Eigenvalues')
DS=[D(1,1),D(2,2),D(3,3),D(4,4)];disp(DS)
% Determination of modal matrix
L=conj(V)'*MT*V;for j=1:4
   ss=1/sqrt(L(j,j));
   for i=1:4
      P(i,j)=V(i,j)*ss;
   end
end
disp(' Modal matrix');disp(P)
% Definition of force vector
F=[0;0;0;F0*exp(-alpha*tau)]
% Calculation of G(t)
G=P'*F;
G=vpa(G);
% Convolution integral solution
for i=1:4
   f(i)=G(i)*exp(-D(i,i)*(t-tau));
   p(i)=int(f(i),tau,0,t);
end
```

Figure 7.11 (*a*) MATLAB script for Example 7.7 uses modal analysis with a general damping matrix to dete forced response of a two-degree-of-freedom systems.

```
disp('Time dependent solution for generalized coordinates')
p=[p(1);p(2);p(3);p(4)];disp(p)
% Back to original coordinates
y=P*p;
disp('Time dependent response of original generalized coordinates')
disp(' x1=y3,     x2=y4     ')
y=vpa(y);
% Plotting system response
time=linspace(0,1.5,151);
for k=1:151
   x1a=subs(y(3),t,time(k));
   x2a=subs(y(4),t,time(k));
   x1b(k)=vpa(real(x1a));
   x2b(k)=vpa(real(x2a));
end
x1=double(x1b);
x2=double(x2b);
plot(time,x1,'-',time,x2,':')
xlabel('t (s)')
ylabel('response (m)')
legend('x1(t)','x2(t)')
```

(*a*)



(*b*)

Figure 7.11 (Con't) (*a*) Con't. (*b*) time-dependent response.

7.6 NUMERICAL SOLUTIONS

An exact solution for the forced response of an n-degree-of-freedom linear system is not always possible. The excitation may be such that the convolution integral cannot be evaluated in closed form or the excitation may be known exactly only at discrete values of time. While a closed-form solution is always preferable to a numerical solution, it may be easier to obtain a numerical solution. Even when a closed-form solution is available, it must be evaluated numerically to plot the response.

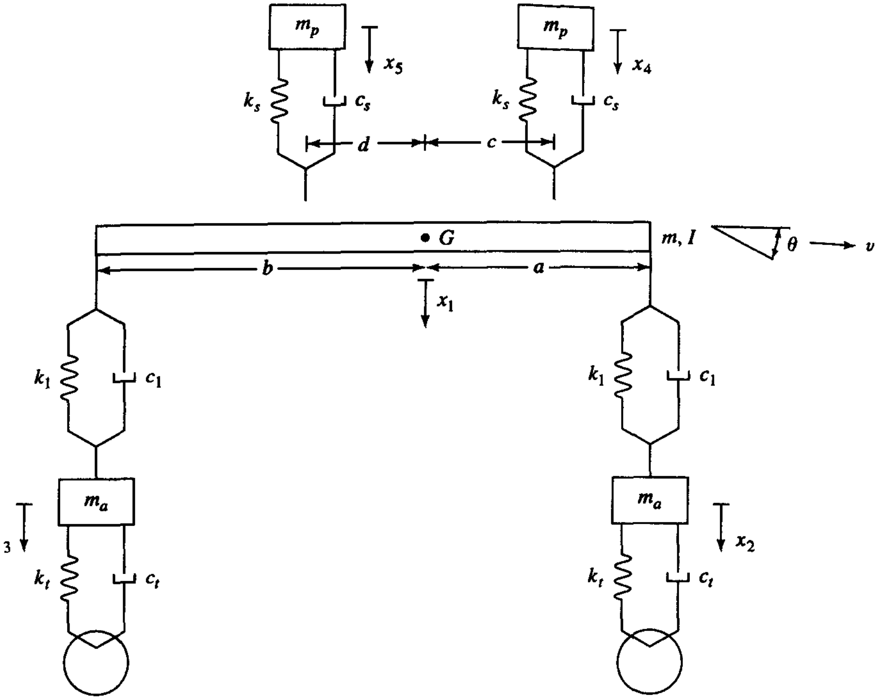
Numerical difficulties may arise if a direct numerical simulation of Eq. (7.20) is used. An n-degree-of-freedom system has n natural frequencies and n natural periods. Hence there are n time scales implicit in the response. The time step in a numerical simulation must be chosen such that a sufficient number of time steps are taken over each natural period. Thus the natural periods should be determined before any numerical simulation is attempted.

Since the natural frequencies should be determined before a numerical simulation is attempted, it is suggested that modal analysis be applied before a numerical simulation is attempted. Numerical solutions for the modal equations can be obtained, and Eq. (7.22) can be used to obtain the response in terms of the chosen generalized coordinates. This approach has several advantages over direct numerical simulation of Eq. (7.20):

1. The natural frequencies and mode shapes are known before the numerical solution begins. This makes it easier to determine an appropriate time step in a numerical approximation.
2. The use of modal analysis provides a choice of numerical solutions. Numerical integration of the convolution integral may be employed or numerical integration of the modal equations based on a method like Runge-Kutta may be used.
3. The numerical solution of n uncoupled equations is simpler and quicker than the numerical solution of n coupled equations.
4. It is not necessary to include all modes in the forced response. If the system is proportionally damped, the higher modes are more highly damped and will contribute less to the overall response. If a large number of degrees of freedom are used in modeling a structural system in order to assure high accuracy for the lowest modes, it is not desirable to include the higher modes in the response, since they provide inaccurate approximations.

Example

Consider again the six-degree-of-freedom model of the suspension system of Example 6.13 and Fig. 7.12. The vehicle is traveling on a smooth road when it encounters a bump whose contour is approximated by the sinusoidal bump of Fig. 7.13. Use modal analysis with numerical integration of the convolution integral to determine the response of the system assuming (a) the system is undamped and (b) the system is proportionally damped with $\alpha = 0.02$.



$m = 1000$ kg	$k_1 = 120{,}000$ N/m	$a = 1.8$ m
$m_a = 150$ kg	$k_t = 80{,}000$ N/m	$b = 2.0$ m
$m_p = 60$ kg	$k_s = 200{,}000$ N/m	$c = 0.6$ m
$I = 360\ \text{kg}\cdot\text{m}^2$	$v = 40$ m/s	$d = 0.4$ m

'igure 7.12 Six-degree-of-freedom model of suspension system of Example 7.8.

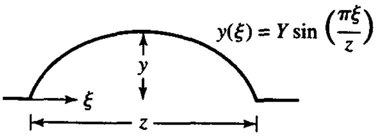


Figure 7.13 Model of bump encountered by vehicle of Example 7.8.

olution:

'he excitation is provided by the displacement encountered by the wheels. The front 'heel encounters the bump at $t = 0$ and takes a time of z/v to traverse the bump. The ·ar wheel encounters the bump at $t = (a + b)/v$ and takes a time of z/v to traverse ıe bump. If the system is proportionally damped, the damping coefficient of the tire is $_t = \alpha k_t$. Thus the generalized force vector is

$$\mathbf{F} = \begin{bmatrix} 0 \\ k_t y_2(t) + c_t \dot{y}_2(t) \\ k_t y_3(t) + c_t \dot{y}_3(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where

$$y_2(t) = Y \sin(\pi v t/z)\,[1 - u(t - z/v)]$$
$$y_3(t) = Y \sin(\pi v(t - (a+b)/v)/z)\,[u(t - (a+b)/v) - u(t - (a+b+z)/v)]$$

The natural frequencies, modal matrix, and modal damping ratios are obtained as described in Examples 6.13 and 6.15. Modal analysis is used to uncouple the differential equations. The convolution integral is used to determine the response of each mode. The convolution integrals are numerically integrated by the method described in Sec. 4.7 with piecewise constants as interpolating functions. Equation (7.22) is used to obtain the chosen generalized coordinates at each time step.

The MATLAB script to determine the response of the system by modal analysis and numerical integration of the convolution integral is given in Fig. 7.14*a*. The script requires the use of an additional MATLAB script, shown in Fig. 7.14*b*. The second script develops the vector $G(t)$ at each time step. The MATLAB-generated plots of the system response for the undamped system are given in Fig. 7.15. The MATLAB generated plot for the proportionally damped system is given in Fig. 7.16.

```
% Example 7.8 Natural frequencies and modal matrix for six-degree-of-
% freedom model of suspension system and passengers
% Parameters
m=1000;          % Mass of vehicle in kg
ma=150;          % Mass of axle in kg
mp=60;           % Mass of each passenger in kg
I=360;           % Mass moment of inertia of vehicle in kg-m^2
k1=60000;        % Stiffness of suspension spring in N/m
ks=200000;       % Stiffness of seat in N/m
kt=80000;        % Stiffness of tire in N/m
a=1.8;           % Distance from front springs to center of gravity in m
b=2.0;           % Distance from rear springs to center of gravity in m
c=0.6;           % Distance from front passenger to center of graity in
d=0.4;           % Distance from rear passenger to center of gravity in
Y=0.06;          % Height of bump in m
z=1.7;           % Width of bump in m
v=40;            % Vehicle speed in m/s
```

Figure 7.14 (*a*) MATLAB script for solution of Example 7.8 by modal analysis and numerical integration of the convolution integral.

```
alpha=0.02;          % Proportional damping ratio
%Mass matrix
disp('Mass matrix')
M=[m,0,0,0,0,0;
   0,ma,0,0,0,0;
   0,0,ma,0,0,0;
   0,0,0,mp,0,0;
   0,0,0,0,mp,0;
   0,0,0,0,0,I];disp(M)
% Stiffness matrix
disp('Stiffness matrix')
K=[2*k1+2*ks,-k1,-k1,-ks,-ks,k1*(a-b)+ks*(c-d);
   -k1,k1+kt,0,0,0,-k1*a;
   -k1,0,k1+kt,0,0,k1*b;
   -ks,0,0,ks,0,ks*c;
   -ks,0,0,0,ks,-ks*d;
   k1*(a-b)+ks*(c-d),-k1*a,k1*b,ks*c,-ks*d,k1*(a^2+b^2)+ks*(c^2+d^2)];disp(K)
% Eigenvalues and eigenvectors of M^(-1)*K
C=inv(M)*K;
[V,D]=eig(C);
% Sorting to put eigenvalues in ascending order and to develop modal
% matrix corresponding to ascending order of eigenvalues
E=[D(1,1),D(2,2),D(3,3),D(4,4),D(5,5),D(6,6)];
Qh=max(E)+0.01;
Q1=0;
for i=1:6
   for j=1:6
      if E(j)>Ql & E(j)<Qh
         k=j;
         Qh=E(j);
      else
      end
   end
      Ql=Qh;
      Qh=max(E)+0.01;
      Om1(i)=E(k);
      omega(i)=sqrt(E(k));
      zeta(i)=alpha*omega(i)/2
      for m=1:6
         P1(m,i)=V(m,k);
      end
end
% Normalizing mode shapes, L is a diagonal matrix with kinetic energy
% Scalar products of mode shapes along diagonal
L=P1'*M*P1;
% Developing modal matrix, a matrix whose columns are the normalized
% mode shapes
for i=1:6
   for j=1:6
```

Figure 7.14A (Con't)

```
        P(i,j)=P1(i,j)/L(j,j);
    end
end
disp('Natural frequencies in rad/s');disp(omega)
disp('Modal damping ratios');disp(zeta)
disp('Modal matrix');disp(P)
time=linspace(0,0.4,401);
% Initial conditions
for i=1:6
   x(i,1)=0;
   G1(i)=0;
   G2(i)=0;
end
% Defining constants for each mode
for j=1:6
   omd(j)=omega(j)*sqrt(1-zeta(j)^2);
   C1(j)=(1-zeta(j)^2)/omega(j);
   C2(j)=zeta(j)*omega(j)/omd(j);
end
% Numerical integration of convolution integral
   for k=2:401
   t=time(k);
   t1=time(k-1);
   Fun7_8;
   for j=1:6
      a1=omd(j)*t;
      b1=omd(j)*t1;
      C=exp(zeta(j)*omega(j)*t)*(sin(a1)+C2(j)*cos(a1));
      C=(C-exp(zeta(j)*omega(j)*t1)*(sin(b1)+C2(j)*cos(b1
      D=exp(zeta(j)*omega(j)*t)*(-cos(a1)+C2(j)*sin(a1));
      D=(D-exp(zeta(j)*omega(j)*t1)*(-cos(b1)+C2(j)*sin(b
      G1(j)=G1(j)+h(j)*C;
      G2(j)=G2(j)+h(j)*D;
      p(j)=exp(-zeta(j)*omega(j)*t)*(sin(a1)*G1(j)-cos(a1
   end
   % Calculating original generalized coordinates
   p=[p(1);p(2);p(3);p(4);p(5);p(6)];
   y=P*p;
   for i=1:6
      x(i,k)=y(i);
   end
end
plot(time,x(1,:),'-',time,x(2,:),':',time,x(3,:),'-.')
xlabel('t (s)')
ylabel('x (m)')
legend('x1(t)','x2(t)','x3(t)')
figure
plot(time,x(4,:),'-',time,x(5,:),':',time,x(6,:),'-.')
xlabel('t (s)')
ylabel('x (m) or theta (rad)')
legend('x4(t)','x5(t)','theta(t)')
```

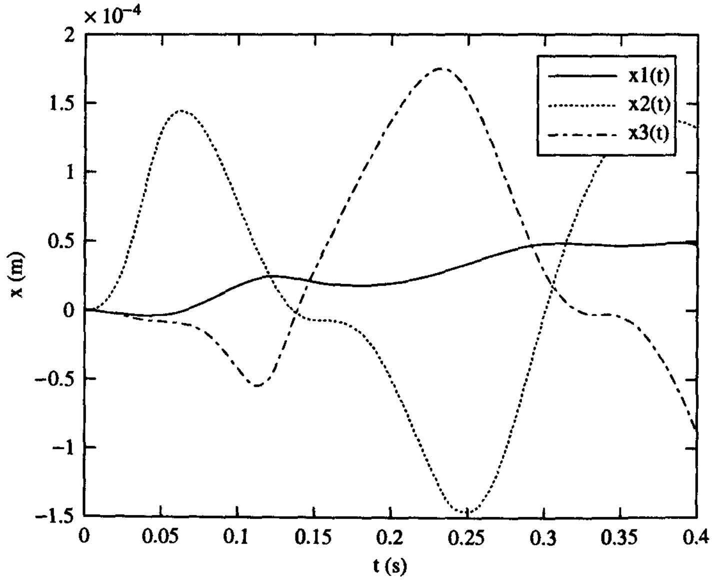
(*a*)

Figure 7.14A (Con't)

```
% MATLAB script file to provide excitation for Example 7.8
% f is the generalized force vector
% h is the modal force vector
f=[0;0;0;0;0;0];
if t<z/v
   f(2)=kt*Y*sin(pi*v*t/z)+alpha*kt*Y*(pi*v/z)*cos(pi*v*t/d);
else
   f(2)=0;
   end
if t>(a+b)/v & t<(a+b+z)/v
   f(3)=kt*Y*sin(pi*v/z*(t-(a+b)/v));
   f(3)=f(3)+alpha*kt*Y*(pi*v/z)*cos(pi*v/z*(t-(a+b)/v));
else
   f(3)=0;
   end
h=P'*f;
```

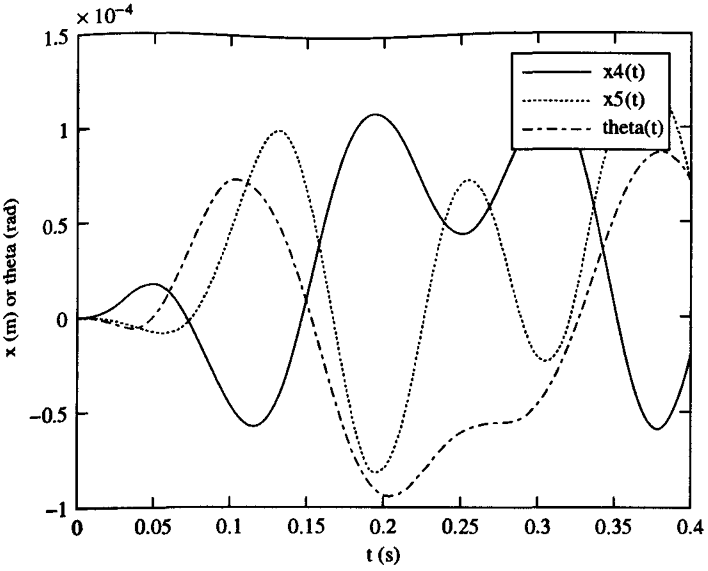
(*b*)

Figure 7.14 (Con't) (*b*) Additional MATLAB script necessary for solution of Example 7.8. This script provides $G(t)$.



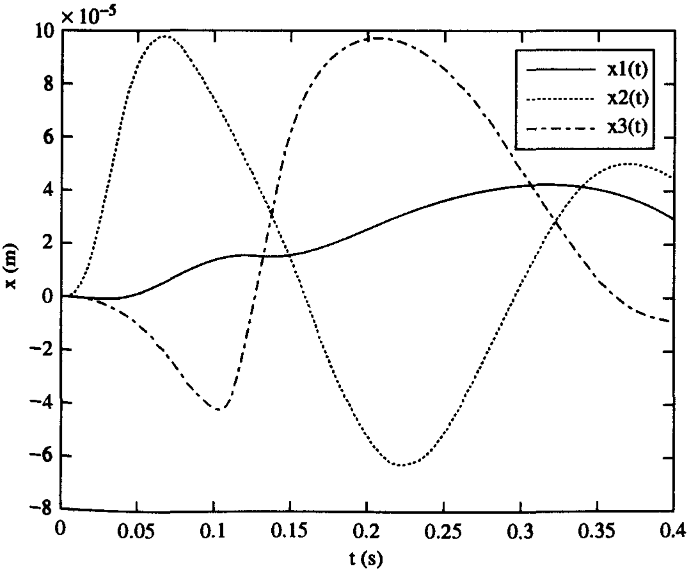
(*a*)

Figure 7.15 System response for $\alpha = 0$ (undamped system).



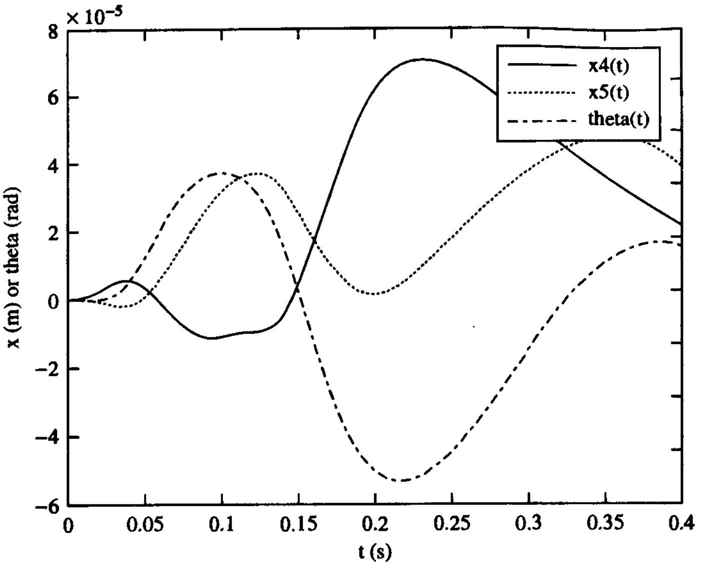
(*b*)

Figure 7.15B (Con't)



(*a*)

Figure 7.16 System response for $\alpha = 0.020$ (proportionally damped system). $x_4(t)$ is displacement of front seat passenger; $x_5(t)$ is displacement of rear seat passenger.



(*b*)

Figure 7.16B (Con't)

PROBLEMS

7.1. Determine the steady-state amplitudes of each of the blocks of Fig. P7.1.

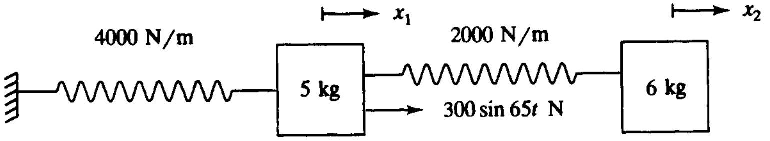


FIGURE P7.1

7.2. Determine the steady-state amplitudes of each of the blocks of Fig. P7.2. Write the frequency response in terms of the nondimensional parameters introduced in Example 7.1.

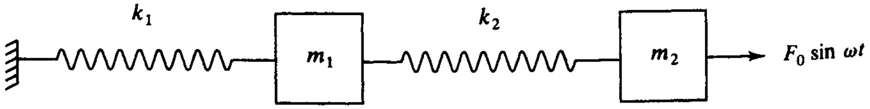


FIGURE P7.2

7.3. Determine the steady-state amplitudes of each of the blocks of Fig. P7.3. Write the frequency response equations in terms of the nondimensional parameters introduced in Example 7.1 and $\zeta = c/(2\sqrt{m_1 k_1})$.

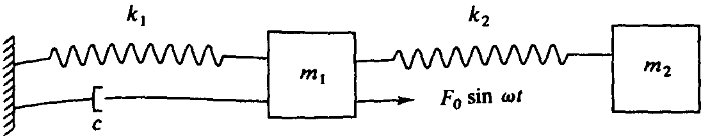


FIGURE P7.3

7.4. For what values of ω will the steady-state amplitude of the block of mass m_2 of Fig. P7.3 be a maximum if $m_1 = 1$ kg, $m_2 = 1.5$ kg, $k_1 = 200$ N/m, $k_2 = 300$ N/m, and $c = 3$ N · s/m?

7.5. Determine the steady-state amplitude of the mass center of the bar of Fig. P6.3 when its right end is subject to a moment $M = 4000 \sin 105t$ N · m.

7.6. A machine of mass m is attached at the end of a cantilever beam of elastic modulus E, moment of inertia I, length L, and mass density ρ. The machine has a rotating unbalance of magnitude $m_0 e$ and operates at a speed ω. Use two degrees of freedom to model the beam and determine the steady-state amplitude of the machine.

7.7. Nondimensionalize the steady-state amplitude of the machine of Prob. 7.6. Plot the nondimensional amplitude as a function of a nondimensional parameter involving ω. The excitation due to the rotating unbalance is a frequency-squared excitation. For a one-degree-of-freedom system the steady-state amplitude approaches a constant nonzero value as the operating speed is increased. Is this the case for the two-degree-of-freedom model? If so, what is the limiting amplitude?

7.8. The system of Fig. P7.8 represents a simplified model of a vehicle suspension system and a passenger in the vehicle. The seat is modeled as a spring and viscous damper in parallel. For the suspension system shown, plot the acceleration amplitude of the passenger as a function of vehicle speed.

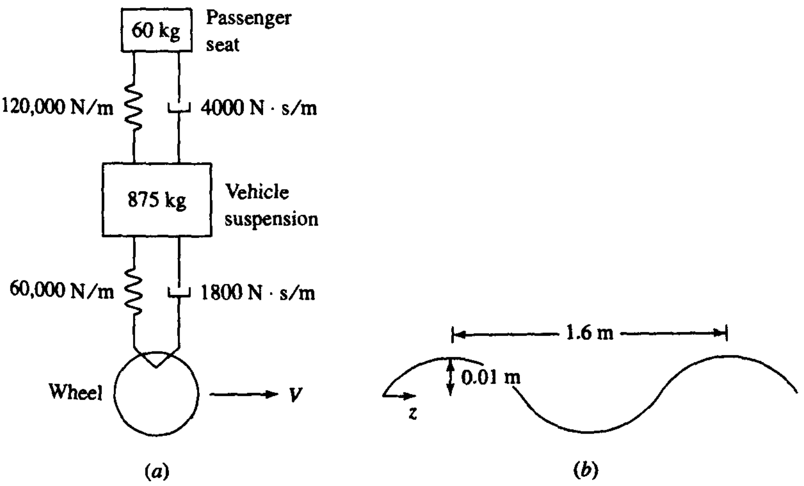


FIGURE P7.8

(*a*) Model of vehicle suspension system and passenger; (*b*) road contour.

7.9. A seismometer is mounted on an engine to monitor its vibrations. The 110-kg engine is mounted on an elastic foundation of stiffness 5×10^5 N/m. The engine operates at 100 rad/s. The seismometer has a damping ratio of 0.2 and the maximum error of measurement is 3 percent. What is the maximum mass of the seismometer such that the vibrations of the seismic mass do not affect the vibrations of the engine?

7.10. Determine the steady-state amplitude of the machine of Prob. 6.16 if during operation it is subject to a harmonic force $2000 \sin(120t)$ N.

7.11. Use the Laplace transform method to solve Prob. 7.1.

7.12. Use the Laplace transform method to determine the response of the system of Fig. P7.12. Both blocks are at rest in equilibrium at $t = 0$.

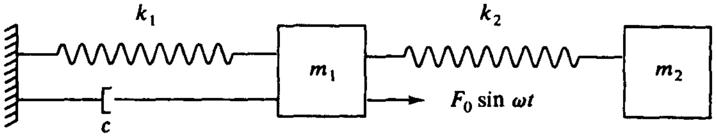


FIGURE P7.12

7.13. Use the Laplace transform method to determine the response of the system of Fig. P7.13.

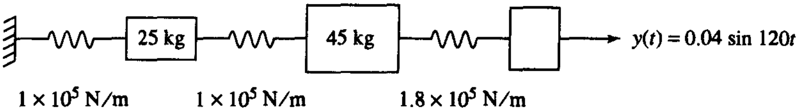


FIGURE P7.13

7.14. Use modal analysis to determine the steady-state response of the system of Prob. 7.1.

7.15. Use modal analysis to solve Prob. 7.6.

7.16. Three machines are on the floor of an industrial plant, which is modeled as a simply supported beam, as shown in Fig. P7.16*a*. During operation the machine at the beam's midspan is subject to the impulsive force of Fig. P7.16*b*. Use modal analysis to determine the maximum displacement of each machine. Assume the beam is undamped and massless.

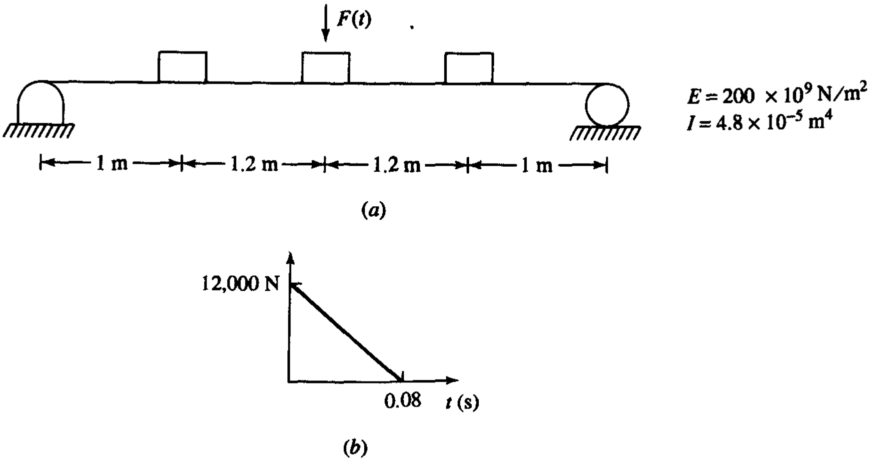


FIGURE P7.16

7.17. Repeat Prob. 7.16 assuming proportional damping with $\alpha = 0.011$.

7.18. Determine the response of the system of Fig. P7.18

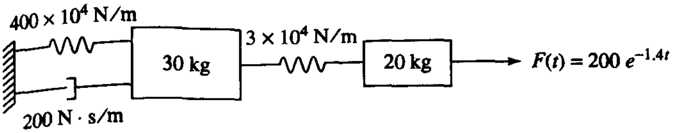
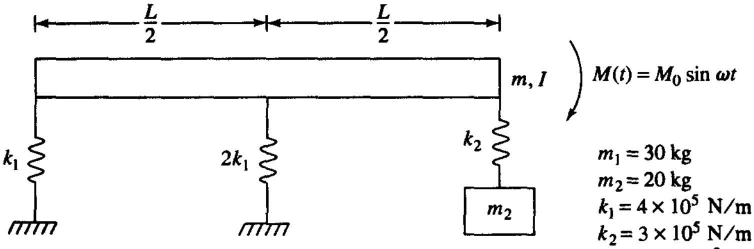


FIGURE P7.18

MATLAB PROBLEMS

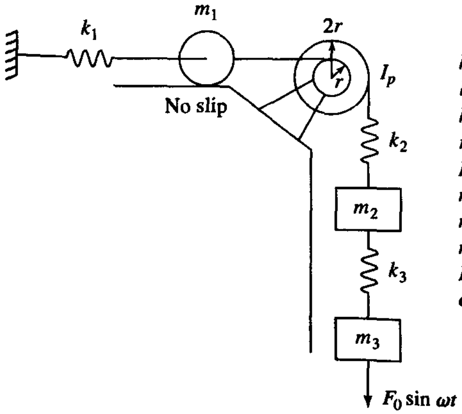
M7.1. The file VIBES_7A.m is used to symbolically develop the steady-state response of a two-degree-of-freedom system, given the mass matrix, stiffness matrix, damping matrix, and harmonic force vector. Use VIBES_7A to determine the steady-state amplitude of Prob. 7.3.

M7.2–M7.5. The file VIBES_7B.m is used to numerically determine the steady-state response of an n-degree-of-freedom system subject to a single-frequency harmonic excitation. Use VIBES_7B to determine the steady-state amplitude of each of the systems of Figs. PM7.2 to PM7.5.



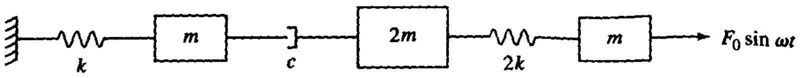
$m_1 = 30$ kg
$m_2 = 20$ kg
$k_1 = 4 \times 10^5$ N/m
$k_2 = 3 \times 10^5$ N/m
$I = 1.8$ kg · m^2
$L = 80$ cm
$M_0 = 20$ N · m
$\omega = 45$ rad/s

FIGURE PM7.2



$k_1 = 3 \times 10^4$ N/m
$k_2 = 4.5 \times 10^4$ N/m
$k_3 = 1 \times 10^4$ N/m
$r = 25$ cm
$I_p = 1.4$ kg · m^2
$m_1 = 18$ kg
$m_2 = 20$ kg
$m_3 = 45$ kg
$F_0 = 25$ N
$\omega = 35$ rad/s

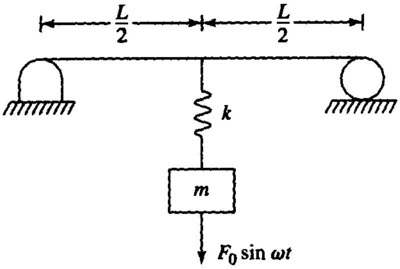
FIGURE PM7.3



$k = 1 \times 10^4$ N/m
$c = 100$ N · s/m
$m = 10$ kg

$F_0 = 20$ N
$\omega = 15$ rad/s

FIGURE PM7.4



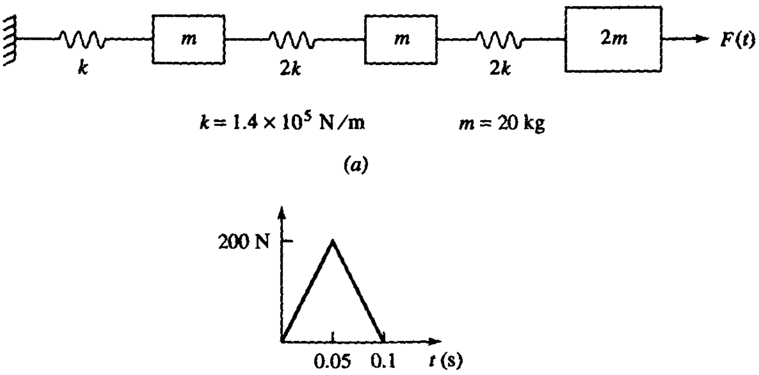
$A = 1.5 \times 10^{-3}$ m^2
$\rho = 6000$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$I = 3.0 \times 10^{-6}$ m^4
$L = 2.4$ m
$m = 125$ kg
$k = 3 \times 10^5$ N/m
$F_0 = 100$ N
$\omega = 80$ rad/s

Use one degree of freedom for beam

FIGURE PM7.5

M7.6. It is shown in Chap. 11 that the steady-state amplitude of the right end of the bar of the system of Fig. PM7.2 is zero when $\sqrt{k_2/m_2} = \omega$. Use VIBES_7B to determine the minimum values of k_2 and m_2 such that the steady-state amplitude of the right end of the bar is zero and the steady-state amplitude of the mass m_2 is less than 1.5 cm.

M7.7–M7.9. The file VIBES_7C.m is used to perform modal analysis on the three-degree-of-freedom system of Fig. PM7.7*a*. Use VIBES_7C to determine the time-dependent response of each of the system due to the excitations of Figs. PM7.7 to PM7.9.



$k = 1.4 \times 10^5$ N/m $\quad$ $m = 20$ kg

(*a*)

(*b*)

FIGURE PM7.7

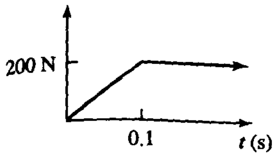
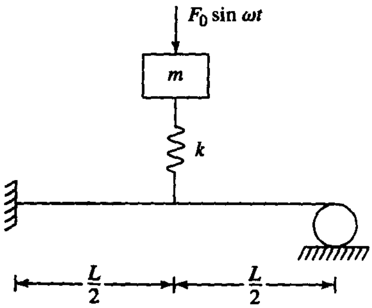


FIGURE PM7.8

FIGURE PM7.9

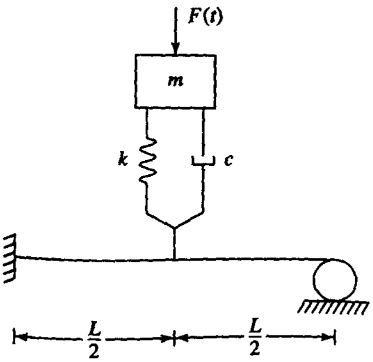
M7.10. The file VIBES_7D.m is used to determine the steady-state response of the system of Fig. PM7.10 when the discrete mass is subject to a harmonic excitation $F_0 \sin \omega t$. The system models a machine subject to a harmonic excitation attached to a beam. An undamped isolator modeled as a discrete spring is placed between the machine and the beam to reduce the magnitude of the force transmitted to the beam. Use VIBES_7D to determine an appropriate value of k such that the magnitude of the force transmitted to the beam is reduced by 80 percent when (*a*) $\omega = 200$ rad/s; (*b*) $\omega = 1000$ rad/s.



$m = 120$ kg
$L = 4$ m
$E = 150 \times 10^9$ N/m^2
$A = 1.5 \times 10^{-3}$ m^2
$\rho = 7000$ kg/m^3
$I = 4.6 \times 10^{-6}$ m^4

FIGURE PM7.10

M7.11. The file VIBES_7E.m is used to determine the steady-state response of the system in Fig. PM7.11 when the discrete mass is subject to a harmonic excitation. Use VIBES_7E to repeat Prob. M7.10 if the isolator is to be designed with a damping ratio of 0.06.



$m = 120$ kg
$L = 4$ m
$E = 150 \times 10^9$ N/m^2
$A = 1.5 \times 10^{-3}$ m^2
$\rho = 7000$ kg/m^3
$I = 4.6 \times 10^{-6}$ m^4

FIGURE PM7.11

M7.12. The file VIBES_7F.m uses modal analysis to determine the response of the system of Fig. PM7.11 when the machine is subject to a force $F(t)$. If, according to the theory of Chap. 8, the stiffness and inertia of the beam are ignored, then, when the machine is subject to the excitation of Fig. PM7.12, the isolator should have a damping ratio of 0.4. If the beam is designed with a stiffness of 2.34×10^4 N/m the maximum force transmitted to the beam will then be 500 N. Use VIBES_7F to determine how much effect including the stiffness and inertia of the beam has on the transmitted force. Model the beam using three degrees of freedom.

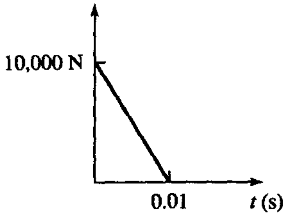


FIGURE PM7.12

The file VIBES_7G.m is the MATLAB script for the solution of Example 7.8. The script uses modal analysis and numerical integration of the convolution integral to determine the response of a six-degree-of-freedom model of a suspension system. Use the script to solve problems M7.13 to M7.15. You will need a user-written script file g.m, which determines the vector $G(t)$. A sample g.m is given in Fig. 7.14*b*.

M7.13. Use VIBES_7G to determine the response of the suspension system to the bump of Fig. PM7.13.

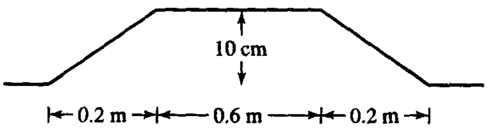


FIGURE PM7.13

M7.14. Use VIBES_7G to study the dependence of the passenger displacement as a function of the location of the center of gravity. Run the program using all of the same parameters as in Example 7.8 except vary the ratio of a/b. Is there a ratio that minimizes the displacement of the passenger in the front seat? Is there a different ratio that minimizes the displacement of the passenger in the rear seat?

M7.15. Use VIBES_7G to study the response of the system as the vehicle traverses a road with pavement joints. The joints are spaced 15 m apart and each is of length 3 cm. Assume that as the wheel traverses a joint it displaces 15 mm. The joints provide a periodic excitation to the suspension system. Determine the maximum steady-state displacement of the passengers in the front seat and the rear seat using the parameters of Example 7.8.

M7.16. Write a MATLAB script file to determine the steady-state amplitude of each of the machines of the system of Fig. PM7.16.

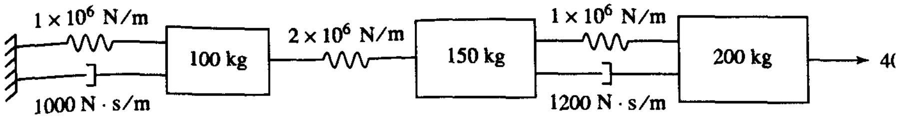


FIGURE PM7.16

M7.17. Write a MATLAB script file to solve Prob. 7.8

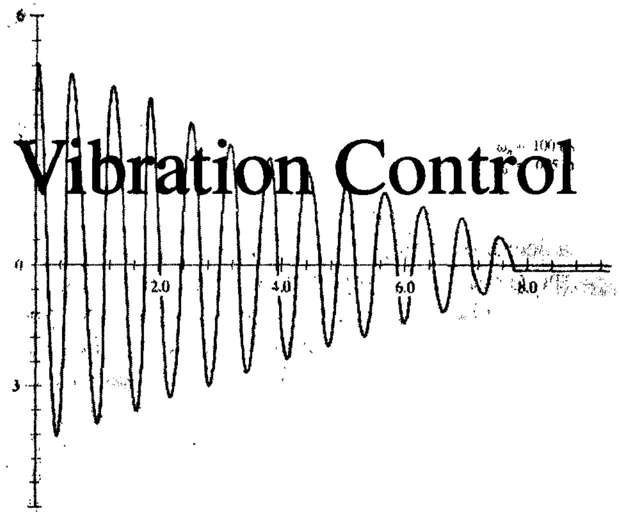
M7.18. Write a MATLAB script file to solve Prob. 7.10.

M7.19. Write a MATLAB script file to solve Prob. 7.18.

chapter

8

Vibration Control



8.1 INTRODUCTION

As illustrated in previous chapters, unwanted vibration in mechanical and structural systems may lead to catastrophic results. Unwanted vibrations may be induced by large impulsive forces in machines such as hammers and presses; unbalanced reciprocating components such as engines, motors, compressors, and pumps; or effects such as wind loading on structures or fluid flow in pipes. One of the major objectives of vibration analysis is to apply its results to understand how unwanted vibrations can be eliminated or reduced. It is possible to introduce vibrations to a system to protect system components from unwanted force or motion transmission. Such is the function of vehicle suspension systems. Vibration analysis is used to develop design principles for these isolation systems. *Vibration control* is the use of vibration analysis to develop methods to eliminate or reduce unwanted vibrations or to use vibrations to protect against unwanted force or motion transmission. There are several ways in which vibration control can be achieved.

1. *Control of the excitation.* Often the external excitation causing the vibration can be eliminated, reduced in magnitude, or altered in some other way. Consider, for example, a machine with an unbalanced reciprocating component. It is shown in Sec. 3.5 that the rotating unbalance produces a harmonic excitation force whose amplitude is proportional to the magnitude of the unbalance and to the square of its angular speed. Since the steady-state amplitude of response is proportional to the amplitude of excitation, a reduction in the magnitude of the unbalance or the rotational speed leads to a reduction in the steady-state amplitude. Thus improved balancing of the machine leads to a reduction in vibrations. Methods of balancing reciprocating components are described in books on machine dynamics. A single-cylinder engine can never be balanced, while multicylinder engines can be balanced by proper positioning of the crankshafts. The cost of

achieving perfect balancing is often too high, and some vibrations will remain. In addition, as a machine ages, wear and degradation lead to unbalance.

A change in excitation frequency of a harmonic excitation will also lead to a change in steady-state amplitude. For a one-degree-of-freedom system, an increase in excitation frequency beyond the system's natural frequency leads to a decrease in amplitude. If the excitation amplitude is independent of frequency, the steady-state amplitude approaches zero as the excitation frequency grows large, whereas the steady-state amplitude for a system subject to a frequency-squared excitation asymptotically approaches a constant value as the excitation frequency grows large. Thus, if the frequency of excitation can be adjusted, the steady-state amplitude can be significantly decreased. If the excitation frequency is much greater than all natural frequencies of a multi-degree-of-freedom system, the steady-state amplitudes will be small. It is also possible to examine the frequency response curves to determine excitation frequencies between the natural frequencies for which the steady-state amplitude is acceptable.

There are many systems where it is not possible to control the excitation. Reciprocating machines are designed to operate at certain speeds for reasons beyond vibration control. It is not possible to control excitations due to earthquakes or to control the size and frequency of waves in an ocean which provide excitation to a boat or ship.

2. *Control of system parameters.* The steady-state amplitude of response of the one degree of freedom system of Fig. 8.1 is

$$X = \frac{F_0}{k}M(r,\zeta) = \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad \textbf{[8.1]}$$

where $$r = \frac{\omega}{\omega_n} = \omega\sqrt{\frac{m}{k}} \qquad \zeta = \frac{c}{2\sqrt{mk}}$$

Thus the steady-state amplitude is a function of the system stiffness k, its mass m, and damping coefficient c. In general the system response is a function of the system's inertia, stiffness, and damping parameters. A change in any parameter leads to a change in steady-state amplitude. If the excitation provided to a system is known, then it may be possible to design the system such that the steady-state amplitude is acceptable. An increase in damping ratio leads to a decrease in steady-state amplitude. In many structural systems, such as the tail section of a helicopter, it is not possible to add additional viscous damping. When damping

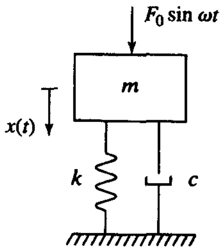


Figure 8.1 Steady-state amplitude of machine is $F_0/m\omega_n^2\, M(r,\zeta)$, and depends on m, k, and c.

is low, large amplitude oscillations are avoided when the frequency ratio is large. This requires a large natural frequency and thus a low stiffness or large mass. It is usually more feasible to control stiffness properties than inertia properties. For example, increasing the mass of an aircraft wing to reduce flutter amplitude is not a feasible solution, whereas an optimized structural design may be employed to increase the stiffness.

3. *Change of system configuration.* An alternative to altering the stiffness, inertia, or damping parameters of a system to achieve vibration control is to change the configuration of the system. The one-degree-of-freedom system of Fig. 8.1 becomes the two-degree-of-freedom system of Fig. 8.2 when an auxiliary mass-spring system, called a *vibration absorber*, is attached. If the absorber is designed correctly, the natural frequencies of the two-degree-of-freedom system are away from the excitation frequency. The addition of the absorber alters the frequency response of the original mass. Indeed, steady-state vibrations of the original machine can be eliminated when a correctly tuned vibration absorber is added.

   A *torsional vibration damper* is illustrated in Fig. 8.3. The damper is an inertia element connected to a shaft through a torsional damper. Such devices are used on crankshafts.

4. *Reduction of force or motion transmission.* Figure 8.4 illustrates the use of a *vibration isolator* between two components of a mechanical system. If designed correctly, an isolator can reduce the transmission of force between a machine and its foundation when the machine is subject to a harmonic excitation. Isolators can also be designed to protect the foundation from impulsive forces experienced during operation of the machine. The same theory is used to design isolators to protect against motion transmission.

Examples of how vibration analysis is used to determine appropriate amplitudes and frequencies of excitation to limit steady-state response are presented in Chap. 3. Examples of how vibration analysis is used to determine system parameters to limit the response for a given excitation are presented in Chaps. 3 and 4. Analysis and design of vibration isolators and vibration absorbers are the subject of this chapter.

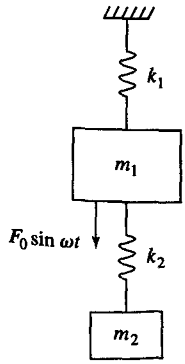


Figure 8.2 An auxiliary mass-spring system added to a machine with harmonic excitation serves as a vibration absorber.

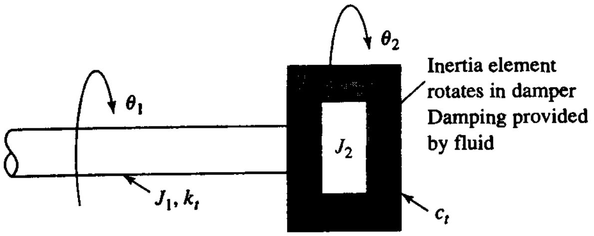


Figure 8.3 Torsional viscous damper is used on crankshafts to reduce steady-state torsional oscillations.

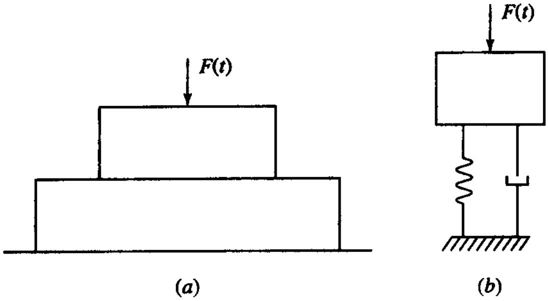


Figure 8.4 (*a*) Elastic mounting used as a vibration isolator to protect a foundation from large transmitted forces; (*b*) one-degree-of-freedom model of a machine on isolator.

The focus of this chapter is on *passive vibration control*, where the control devices have constant properties. *Active vibration control* occurs when the control devices are operated through external power. These devices have sensors and feedback loops that allow properties of the control system to change when system conditions change. Active control systems are discussed in books on control system design.

8.2 VIBRATION ISOLATION THEORY

Problems in vibration isolation are of two classes.

1. Foundations and mountings are protected against large forces produced in equipment. Examples of this class of problems include protecting mountings from large inertia forces generated by reciprocating machines and protecting a foundation from a large impulsive force produced by a drop hammer.

A model one-degree-of-freedom system is shown in Fig. 8.4b. The differential equation governing the displacement of the mass is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \qquad \textbf{[8.2]}$$

where $\omega_n = \sqrt{k/m}$ is the system's natural frequency and $\zeta = c/(2m\omega_n)$ is the system's damping ratio. The force transmitted to the foundation through the spring and viscous damper is

$$F_T(t) = kx + c\dot{x} \qquad \textbf{[8.3]}$$

2. Equipment is protected against motion of its foundation. Examples of this class of problems include protecting sensitive computer equipment from harmonic structural vibrations caused by other equipment or protecting a computer on a ship from sudden motions due to waves and rough seas.

   A model one-degree-of-freedom system for this class of problems is shown in Fig. 8.5. The differential equation governing the relative displacement between the mass and its foundation is

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \qquad \textbf{[8.4]}$$

   where $z = x - y$. The acceleration transmitted to the mass due to the motion of the base is given by

$$\ddot{x} = \ddot{z} + \ddot{y} = -(2\zeta\omega_n z + \omega_n^2 z) = -\frac{kz + c\dot{z}}{m}$$

   or

$$F_T = -m\ddot{x} \qquad \textbf{[8.5]}$$

   where F_T is the total force developed in the spring and viscous damper.

For force transmission problems, given the excitation force $F(t)$, the two quantities of most concern are the maximum displacement of the machine, $x_{\max}$, and the maximum force transmitted to the foundation, $F_{T_{\max}}$. For motion transmission problems, given the base acceleration $\ddot{y}(t)$, the two quantities of most concern are the displacement of the equipment relative to the foundation, $z_{\max}$, and the maximum absolute acceleration of the equipment, $\ddot{x}_{\max}$. Comparison of Eqs. (8.2) and (8.4) shows that if the excitation force for a force transmission problem is equivalent to $-m\ddot{x}$ for a motion transmission problem, then $x_{\max} = z_{\max}$. Comparison of Eqs.

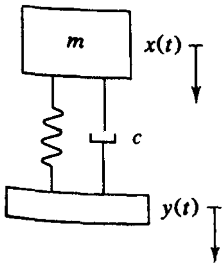


Figure 8.5 One-degree-of-freedom model of system protected from motion of base by isolator.

(8.3) and (8.5) further reveals that under these conditions $F_{T_{max}} = m\ddot{x}_{max}$. Thus the same theory can be used to analyze both force and motion transmission problems. Then without loss of generality the remainder of the discussion will focus on force transmission problems.

For a given excitation Eq. (8.2) is solved and Eq. (8.3) is used to determine the time-dependent transmitted force. If $F(t)$ is harmonic, the steady-state amplitudes are constant and the important considerations. If $F(t)$ is impulsive, the short-time behavior is important. In either case, vibration control involves minimizing the maximum transmitted force $F_{T_{max}}$, the maximum displacement x_{max}, or both.

Let F_0 be a characteristic magnitude and let t_0 be a characteristic time of the excitation. The maximum transmitted force and displacement are dependent on these characteristic values and system properties

$$F_{T_{max}} = f_1(m, \zeta, \omega_n, F_0, t_0) \tag{8.6}$$

$$x_{max} = f_2(m, \zeta, \omega_n, F_0, t_0) \tag{8.7}$$

Equations (8.6) and (8.7) can be rearranged in nondimensional form

$$\frac{F_{T_{max}}}{F_0} = \bar{f}_1(\zeta, \omega_n t_0) \tag{8.8}$$

$$\frac{m\omega_n^2 x_{max}}{F_0} = \bar{f}_2(\zeta, \omega_n t_0) \tag{8.9}$$

If the excitation is harmonic the characteristic time is taken as the inverse of the excitation frequency and the frequency ratio $r = \omega/\omega_n$ is used in place of $\omega_n t_0$. General principles of vibration isolation are obtained by studying the equations in their nondimensional form.

8.3 VIBRATION ISOLATION THEORY FOR HARMONIC EXCITATION

8.3.1 GENERAL THEORY

If the machine of Fig. 8.4 were bolted to the floor and subject to a harmonic excitation of the form

$$F(t) = F_0 \sin \omega t \tag{8.10}$$

then the floor is subject to a load with a static component load equal to the weight of the machine and a harmonic component of magnitude F_0 and frequency ω. The harmonic component can lead to fatigue damage of the bolts and the structure and cause unwanted noise. The harmonic force applied to the floor can also induce vibrations of the structure that may affect operation of neighboring machines.

The magnitude of the harmonic component of the transmitted force can be reduced by isolating the floor from the machine by placing the machine on an elastic foundation modeled as springs in parallel with a viscous damper as shown in Fig. 8.4*b*. The steady-state response of the machine is

$$x(t) = X \sin(\omega t - \phi) \tag{8.11}$$

where X is given by Eq. (3.25) and ϕ is given in Eq. (3.29). Use of Eq. (8.11) in Eq. (8.3) leads to

$$F_T(t) = F_{T_{\max}} \sin(\omega t - \lambda) \tag{8.12}$$

where λ is given by Eq. (3.72) and

$$F_{T_{\max}} = F_0 T(r, \zeta) \tag{8.13}$$

The function $T(r, \zeta)$ is defined in Eq. (3.74), which is duplicated below

$$T(r, \zeta) = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{8.14}$$

The nondimensional function of Eq. (8.14) is called the *transmissibility*. If the transmissibility is less than one, the magnitude of the repeating force transmitted to the floor is less than the magnitude of the excitation force. When this occurs, the vibrations are said to be isolated. Figure 8.6 shows the transmissibility as a function of frequency ratio for different values of the damping ratio. Isolation occurs only when $r > \sqrt{2}$. When isolation occurs, increased damping increases

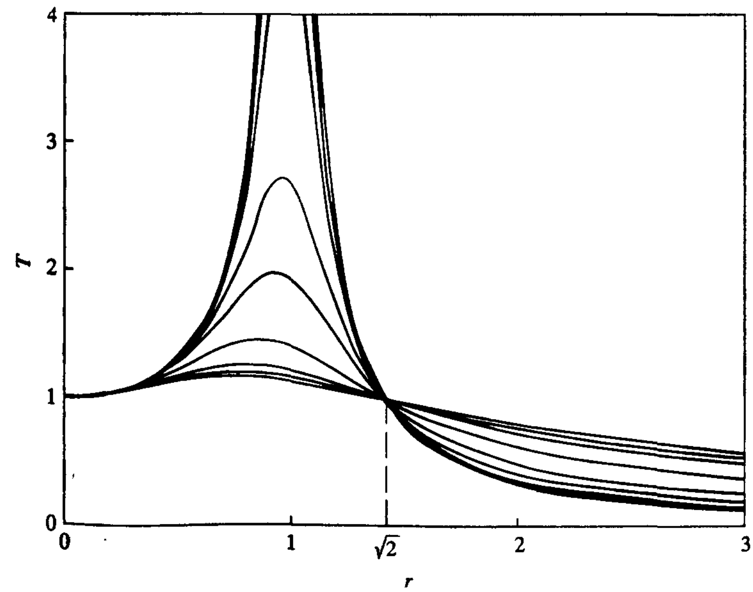


Figure 8.6 T versus r for different values of ζ.

the transmissibility ratio. However, viscous damping is still necessary to limit the amplitude of vibration as the system passes through resonance.

The theory of motion transmission for harmonic base excitation is presented in Sec. 3.6. The ratio of the equipment acceleration to the base acceleration is

$$\frac{\omega^2 X}{\omega^2 Y} = \frac{X}{Y} = T(r, \zeta) \tag{8.15}$$

8.1 **UNDAMPED ISOLATOR DESIGN** An air conditioner weighs 250 lb and is driven by a motor at 500 rpm. What is the required static deflection of an undamped isolator to achieve 80 percent isolation?

Solution:

Eighty percent isolation requires a transmissibility ratio of 0.2. For an undamped isolator

$$0.2 = \sqrt{\frac{1}{(1-r^2)^2}}$$

Since $r > \sqrt{2}$ to achieve isolation, and a positive result is required from the square root, the appropriate form of the preceding equation after the square root is taken is

$$0.2 = \frac{1}{r^2 - 1}$$

which yields $r = 2.45$. The maximum natural frequency for the air conditioner–isolator system to achieve 80 percent isolation is calculated as

$$\omega_n = \frac{\omega}{r} = \frac{(500\ \text{rev/min})(2\pi\ \text{rad/rev})(1\ \text{min}/60\ \text{s})}{2.45} = 21.4\ \frac{\text{rad}}{\text{s}}$$

The required static deflection is obtained from

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta_{st}}}$$

or

$$\Delta_{st} = \frac{g}{\omega_n^2} = \frac{32.2\ \text{ft/s}^2}{(21.4\ \text{rad/s})^2} = 0.07\ \text{ft}$$

8.2 **DAMPED ISOLATOR DESIGN** An industrial sewing machine has a mass of 430 kg and operates at 1500 rpm (157 rad/s). It appears to have a rotating unbalance of magnitude $m_0 e = 0.8\ \text{kg}\cdot\text{m}$. Structural engineers suggest that the maximum repeated force transmitted to the floor is 10,000 N. The only isolator available has a stiffness of 7×10^6 N/m and a damping ratio of 0.1. If the isolator is placed between the machine and the floor, will the transmitted force be reduced to an acceptable level? If not, what can be done?

Solution:

The maximum allowable transmissibility ratio is

$$T_{\max} = \frac{F_{T_{\max}}}{m_0 e \omega^2} = \frac{10{,}000\ \text{N}}{(0.8\ \text{kg}\cdot\text{m})(157\ \text{rad/s})^2} = 0.507$$

The natural frequency with the isolator in place is

$$\omega_n = \sqrt{\frac{7 \times 10^6 \text{ N/m}}{430 \text{ kg}}} = 127.6 \frac{\text{rad}}{\text{s}}$$

which leads to a frequency ratio of $1.24 < \sqrt{2}$. Use of this isolator actually amplifies the force transmitted to the floor.

Adequate isolation is achieved only by increasing the frequency ratio, thus decreasing the natural frequency. The maximum allowable natural frequency is obtained by solving for r from

$$0.507 = \sqrt{\frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2}}$$

The preceding equation is squared and rearranged to yield the following quadratic equation for r^2:

$$r^4 - 2.12r^2 - 2.89 = 0$$

The appropriate solution is $r = 1.75$. Thus the maximum natural frequency is

$$\omega_n = \frac{157 \text{ rad/s}}{1.75} = 89.7 \frac{\text{rad}}{\text{s}}$$

If more than one of the described isolator were available, the natural frequency of the system can be decreased by placing isolators in series. The equivalent stiffness for n isolators in series is k/n. Further calculations show that at least two isolator pads in series are necessary to reduce the natural frequency below 89.7 rad/s.

If only one isolator pad is available, the natural frequency is decreased by adding mass to the machine. A mass of at least 440 kg must be rigidly attached to the machine and the assembly placed on the existing isolator.

Example

A flow-monitoring device of mass 10 kg is to be installed to monitor the flow of a gas in a manufacturing process. Because of the operation of pumps and compressors, the floor of the plant vibrates with an amplitude of 4 mm at a frequency of 2500 rpm. Effective operation of the flow-monitoring device requires that its acceleration amplitude be limited to 5g. What is the equivalent stiffness of an isolator with a damping ratio of 0.05 to limit the transmitted acceleration to an acceptable level? What is the maximum displacement of the flow-monitoring device and what is the maximum deformation of the isolator?

Solution:

The acceleration amplitude of the floor is

$$\omega^2 Y = \left(2500 \frac{\text{rev}}{\text{min}} 2\pi \frac{\text{rad}}{\text{rev}} 1 \frac{\text{min}}{60 \text{ s}}\right)^2 (0.004 \text{ m}) = 274.1 \frac{\text{m}}{\text{s}^2} = 27.95g$$

The maximum allowable transmissibility ratio is

$$T_{\max} = \frac{\omega^2 X}{\omega^2 Y} = \frac{5g}{27.95g} = 0.179$$

From Eq. (8.14) with $\zeta = 0.05$,

$$0.179 = \sqrt{\frac{1 + 0.01r^2}{1 - 1.99r^2 + r^4}}$$

Solution of the preceding equation gives the minimum frequency ratio for which vibrations are sufficiently isolated. It yields $r > 2.60$. Thus

$$\omega_n < \frac{\omega}{2.60} = 100.7 \ \frac{\text{rad}}{\text{s}}$$

The maximum stiffness of the isolator is

$$k = m\omega_n^2 = 1.01 \times 10^5 \text{ N/m}$$

When $T = 0.179$, Eq. (3.71) is used to calculate the steady-state amplitude of the flow-monitoring device as

$$X = YT = (0.004 \text{ m})(0.179) = 0.72 \text{ mm}$$

Since the isolator is placed between the floor and the flow-monitoring device, its deformation is equal to the relative displacement between the floor and the device.

The steady-state amplitude of the relative displacement is calculated by using Eq. (3.68)

$$Z = \Lambda Y = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = 4.69 \text{ mm}$$

When the magnitude of the excitation force is independent of the excitation frequency, if sufficient isolation is achieved at a certain speed, then greater isolation is achieved at higher speeds. The transmitted force can theoretically be made as small as desired by making the natural frequency of the machine-isolator system small enough.

8.3.2 FREQUENCY-SQUARED EXCITATION

A special case occurs when the amplitude of the excitation force is proportional to the square of the excitation frequency, as for the harmonic excitation due to a rotating unbalance. Since the maximum allowable force transmitted to the foundation is independent of the frequency of excitation, the percentage of isolation required varies with the frequency. When the excitation is caused by a rotating unbalance, Eq. (8.13) becomes

$$\frac{F_T}{m_0 e \omega^2} = T(r, \zeta)$$

or

$$\frac{F_T}{m_0 e \omega_n^2} = r^2 T(r, \zeta) = R(r, \zeta) \tag{8.16}$$

The nondimensional function $R(r, \zeta)$ is defined as

$$R(r, \zeta) = r^2\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \quad \textbf{[8.17]}$$

$R(r, \zeta)$ is plotted in Fig. 8.7. The following is noted about its behavior

1. $R(r, \zeta)$ is asymptotic to the line $f(r) = 2\zeta r$ for large r. That is,

$$\lim_{r \to \infty} R(r, \zeta) = 2\zeta r \quad \textbf{[8.18]}$$

2. For $\zeta < \sqrt{2}/4 = 0.354$, $R(r, \zeta)$ increases with increasing r, from 0 at $r = 0$ and reaches a maximum value. R then decreases and reaches a relative minimum. As r increases from the value where the minimum occurs, R grows without bound and approaches the asymptotic limit given by Eq. (8.18). The values of r where the maximum and relative minimum occur are obtained by setting $dR/dr = 0$, yielding

$$1 + (8\zeta^2 - 1)r^2 + 8\zeta^2(2\zeta^2 - 1)r^4 + 2\zeta^2 r^6 = 0 \quad \textbf{[8.19]}$$

Equation (8.19) is a cubic polynomial in r^2. It has three roots. One root is the value of r where the maximum occurs, another is the value of r where the relative minimum occurs, and one root is negative and irrelevant. Figure 8.8 shows the value of r for which the minimum occurs as a function of ζ. Figure 8.9 shows the corresponding value of R at its relative minimum.

3. $R = 2$ for $r = \sqrt{2}$ for all values of ζ.
4. Equation (8.19) has a double root of $r = \sqrt{2}$ for $\zeta = \sqrt{2}/4 = 0.354$. The maximum and minimum coalesce for this value of ζ. For $\zeta = 0.354$, $R = 2$ at $r = \sqrt{2}$ is an inflection point.

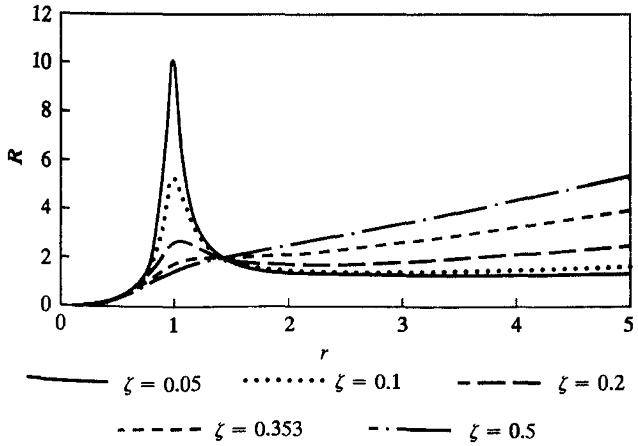


Figure 8.7 *R* versus *r* for several values of ζ.

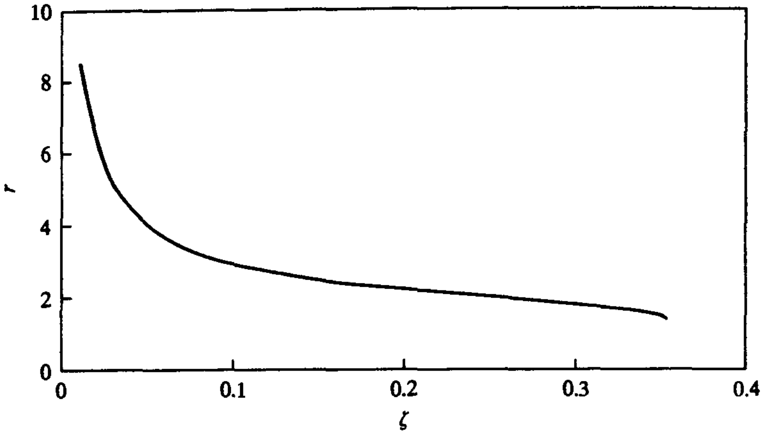


Figure 8.8 Value of r for which minimum R occurs as a function of ζ.

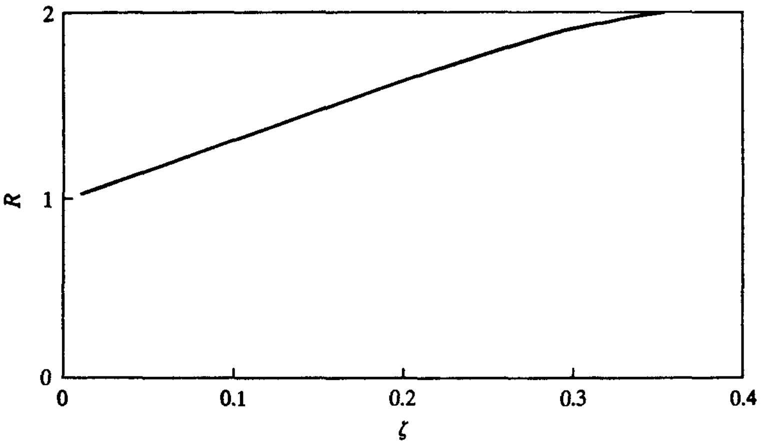


Figure 8.9 $R_{\min}(\zeta)$.

5. For $\zeta > \sqrt{2}/4$, Eq. (8.19) has no positive roots. Thus R does not reach a maximum, but grows without bound from $R = 0$ at $r = 0$.

If the natural frequency of a system whose vibrations are due to a rotating unbalance is fixed, Fig. 8.7 shows that the transmitted force has a minimum for some value of r. If r exceeds this value, the force increases without bound as r increases. If ζ is small, the curve in the vicinity of the relative minimum is flat. The transmitted force varies little over a range of r. This suggests that for situations where vibrations must be isolated over a range of excitation frequencies, it is best to choose ω_n such that the value of r at the center of the operating range is near the value of r for which the relative minimum occurs.

The limit process used to develop Eq. (8.18) is performed for a fixed value of ω_n as ω is increased. Thus for a fixed ω_n the transmitted force approaches $m_0 e\omega\omega_n$. The limit of F_T as ω_n goes to zero for a fixed ω is zero. Thus decreasing the natural frequency decreases the magnitude of the transmitted force for a specific excitation frequency. Decreasing the natural frequency such that the minimum is to the left of the operating range reduces the magnitude of the repeating component of the transmitted force over a portion of the operating range. However, the transmitted force may vary greatly over the operating range.

Exam

A 250-kg pump operates at speeds between 1000 and 2400 rpm and has a rotating unbalance of 2.5 kg·m. The pump is placed at a location in an industrial plant where it has been determined that the maximum repeated force that should be applied to the floor is $F_{\max}$. Specify the stiffness of an isolator of damping ratio 0.1 that can be used to reduce the repeating component of the transmitted force to an acceptable level. Solve for (*a*) $F_{\max} = 15,000$ N; (*b*) $F_{\max} = 10,000$ N.

Solution:

If the pump is placed directly on the floor, the repeating component of the transmitted force is 27,400 N at 1000 rpm and 157,800 N at 2400 rpm. Thus isolation is necessary.

(*a*) From Fig. 8.9, for $\zeta = 0.1$ the minimum value of R occurs for $r = 2.94$. If ω_n is chosen such that $r = 2.94$ is at the center of the operating range, then

$$\omega_n = \frac{1700 \text{ rpm}}{2.94} = 578.2 \text{ rpm} = 60.55 \frac{\text{rad}}{\text{s}}$$

At the lower end of the operating range, the frequency ratio is 1.73 and the transmitted force is

$$\begin{aligned} F_T &= m_0 e\omega_n^2 R(1.73, 0.1) \\ &= 2.5 \text{ kg}\cdot\text{m}\left(60.55 \frac{\text{rad}}{\text{s}}\right)^2 (1.73)^2 \sqrt{\frac{1+(0.346)^2}{[1-(1.73)^2]^2+(0.346)^2}} \\ &= 14,350 \text{ N} \end{aligned}$$

At the upper end of the operating range, the frequency ratio is 4.15 and the transmitted force is

$$\begin{aligned} F_T &= m_0 e\omega_n^2 R(4.15, 0.1) \\ &= (2.5 \text{ kg}\cdot\text{m})\left(60.55 \frac{\text{rad}}{\text{s}}\right)^2 (4.15)^2 \sqrt{\frac{1+(0.830)^2}{[1-(4.15)^2]^2+(0.830)^2}} \\ &= 12,630 \text{ N} \end{aligned}$$

(*b*) The above analysis works for $F_{T_{\max}} = 15,000$ N, but will not work for $F_{T_{\max}} = 10,000$ N, as the transmitted forces at both ends of the operating range are larger than

10,000 N when the center of the operating range corresponds to the minimum of $R(r, \zeta)$. Setting $F_{T_{\max}} = 15{,}000$ N for $\omega = 1000$ rpm leads to

$$T(r, 0.1) = \frac{F_{T_{\max}}}{m_0 e \omega^2} = \frac{10{,}000 \text{ N}}{(2.5 \text{ kg} \cdot \text{m})(104.7 \text{ rad/s})^2} = 0.365$$

which leads to $r = 2.012$. Then $\omega_n = (104.7 \text{ rad/s})/2.102 = 52.02$ rad/s. Then for $\omega = 2400$ rpm, $r = 4.83$, and from Eq. (8.16)

$$F_T = m_0 e \omega_n^2 R(4.83, 0.1) = 9810 \text{ N}$$

8.3.3 MULTIFREQUENCY AND GENERAL PERIODIC EXCITATIONS

Vibration isolation of a system subject to a multifrequency excitation can be difficult, especially if the lowest frequency is very low. Consider a system subject to an excitation composed of n harmonics

$$F(t) = \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \qquad \textbf{[8.20]}$$

The principle of linear superposition is used to calculate the total response of the system due to this excitation. Equation (8.3) is used to calculate transmitted force, leading to

$$F_T(t) = \sum_{i=1}^{n} T(r_i, \zeta) F_i \sin(\omega_i t + \psi_i - \lambda_i) \qquad \textbf{[8.21]}$$

where

$$r_i = \frac{\omega_i}{\omega_n}$$

Since the harmonic terms of Eq. (8.21) are out of phase, their maxima occur at different times. A closed-form expression for the absolute maximum is difficult to attain. The following is used as an upper bound:

$$F_{T_{\max}} < \sum_{i=1}^{n} F_i T(r_i, \zeta) \qquad \textbf{[8.22]}$$

Equation (8.22) can be viewed as an equation which provides an approximation to the upper bound of the natural frequency. If the natural frequency is less than the calculated upper bound, the repeating force transmitted to the floor is always less than the allowable force. An initial guess for the upper bound is obtained by determining the natural frequency such that the transmitted force due to the lowest-frequency harmonic only is reduced to F_T. Since additional forces at higher frequencies are present, greater isolation is required. The natural frequency can be systematically reduced from this initial guess, checking Eq. (8.22), until an upper bound is obtained.

Example 8

The 500-kg punch press of Example 3.13 is to be mounted on an isolator such that the maximum of the repeating force transmitted to the floor is 1000 N. Determine the required static deflection of an isolator assuming a damping ratio of 0.1. What is the resulting maximum deflection of the isolator during the punching operation?

Solution:

From Example 3.13 the excitation force is periodic and is expressed by a Fourier series as

$$F(t) = 2000 + \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1-\cos 0.8\pi i}\,\sin(4\pi i t + \psi_i)\text{ N}$$

The 2000-N term is the average force applied to the punch during one cycle. It contributes to the total static load applied to the floor and is not part of the repeating load. Application of Eq. (8.22) to the repeated components of loading gives

$$1000 > \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1-\cos 0.8\pi i}\;T(r_i, \zeta)$$

where

$$r_i = \frac{4\pi i}{\omega_n} = i r_1$$

An initial guess for an upper bound for the natural frequency is obtained by calculating r_1 such that the transmitted force due to the lowest-frequency harmonic is less than 1000 N. This leads to

$$1000 = \frac{5000}{\pi}\sqrt{2(1-\cos 0.8\pi)}\sqrt{\frac{1+(0.2r_1)^2}{(1-r_1^2)^2+(0.2r_1)^2}}$$

which gives $r_1 = 1.54$. Defining

$$f(r_1) = \frac{5000\sqrt{2}}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sqrt{1-\cos 0.8\pi i}\;T(i r_1, \zeta)$$

it is desired to solve

$$f(r_1) = 1000$$

A lower bound on the value of r_1 that solves the preceding equation is 1.54. A trial-and-error solution is used to determine r_1, leading to $r_1 = 1.95$. For $r_1 = 1.95$, an upper bound for the natural frequency is calculated as

$$\omega_n = \frac{\omega_1}{1.95} = \frac{4\pi}{1.95} = 6.44\ \frac{\text{rad}}{\text{s}}$$

The required static deflection of the isolator is $\Delta_{st} = g/\omega_n^2 = 236$ mm. The static deflection is excessive, and a flexible foundation is required. The total static load on the isolator is the weight of the machine plus the average value of the excitation force, $a_0/2 = 2000$ N. Thus the total static load to be supported is

$$F_{\text{static}} = (500\text{ kg})(9.81\text{ m/s}^2) + 2000\text{ N} = 6905\text{ N}$$

8.4 PRACTICAL ASPECTS OF VIBRATION ISOLATION

Vibration isolation is required in a variety of military and industrial applications. Isolation is required to reduce the force transmitted between a machine and its foundation during ordinary operation or to isolate a machine from vibrations of its surroundings. Motors are often isolated to protect mountings from forces arising from harmonic variation of torque and unbalanced rotors. Electrical components such as transformers and circuit breakers are isolated to protect surroundings from electromagnetic forces generated in solenoids or as a result of alternating current. Large harmonic inertia forces are developed by rotating components of single-cylinder reciprocating engines. Isolation is required to protect the engine mounting from these forces. Other machines with rotating components such as fans, pumps, and presses are often isolated to protect against inherent rotating unbalance.

The maximum stiffness of an isolator required for a particular application is calculated by using the theory of Sec. 8.3. A one-degree-of-freedom system using an isolator is modeled as the simple mass-spring-dashpot system of Fig. 8.4.

Specifications provided in catalogs of commercially available isolators include allowable static deflections. If the isolated system of Fig. 8.4 has a minimum required natural frequency ω_n, the required minimum static deflection of the isolator is

$$\Delta_{st} = \frac{g}{\omega_n^2} \tag{8.23}$$

Isolation of low-frequency vibrations requires a small natural frequency, which leads to a large isolator static deflection.

The vibration amplitude of a machine during operation is calculated from Eq. (3.25)

$$\frac{m\omega_n^2 X}{F_0} = M(r, \zeta)$$

Multiplying both sides of the preceding equation by r^2 leads to

$$\frac{m\omega^2 X}{F_0} = r^2 M(r, \zeta) = \Lambda(r, \zeta) \tag{8.24}$$

where $\Lambda(r, \zeta)$ is defined in Eq. (3.34). Since vibration isolation requires $r > \sqrt{2}$ and $\Lambda(r, \zeta)$ decreases and approaches 1 as r increases, the steady-state amplitude decreases as isolation is improved. However, for fixed m, F_0, and ω the steady-state amplitude has a lower bound given by

$$X > \frac{F_0}{m\omega^2} \tag{8.25}$$

Equations (8.24) and (8.25) show that if an isolator is being designed to provide isolation over a range of frequencies, the steady-state amplitude is greatest at the lowest operating speed.

Since vibration isolation requires $r > \sqrt{2}$, the speed at which the maximum vibration amplitude occurs must be passed during start-up and stopping. The maximum vibration amplitude for a fixed ω_n is obtained using Eq. (3.28) as

$$X_{\max} = \frac{F_0}{m\omega_n^2} \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad [8.26]$$

The smaller the natural frequency, the larger the resonant amplitude. In addition, the larger the damping ratio, the smaller the resonant amplitude.

A large vibration amplitude can lead to ineffective operation of machinery. Large-amplitude vibrations of machines which must be properly aligned with devices that feed materials to the machine can lead to improper alignment and improper operation. Many machine tools require a rigid foundation for effective operation. Equation (8.24) shows that one way to reduce the amplitude of vibration during operation and the resonant amplitude is to increase the mass of the isolated system. Equation (8.23) shows that the only way to reduce the amplitude below a calculated value at a given operating speed is to increase the system mass. Increasing the mass allows a proportional increase in the stiffness required to achieve sufficient isolation.

The mass of a system can be increased by rigidly mounting the machine on a block of concrete. A small machine can be mounted above ground, while a large machine is usually mounted in a specially designed pit. The static load applied to the isolator and the mounting is increased when the mass of the system is increased.

Exan

A milling machine of mass 450 kg operates at 1800 rpm and has an unbalance which causes a harmonic repeated force of magnitude 20,000 N. Design an isolation system to limit the transmitted force to 4000 N, the amplitude of vibration during operation to 1 mm, and the amplitude of vibration during start-up to 10 mm. Specify the required stiffness of the isolator and the minimum mass that should be added to the machine. Assume a damping ratio of 0.05.

Solution:

The maximum allowable transmissibility is

$$T = \frac{4000\ \text{N}}{20{,}000\ \text{N}} = 0.2$$

The minimum frequency ratio is determined by solving

$$0.2 = \sqrt{\frac{1 + 0.01r^2}{1 - 1.99r^2 + r^4}}$$

which yields $r = 2.48$ and a maximum natural frequency of

$$\omega_n = \frac{\omega}{2.48} = 76.0\ \frac{\text{rad}}{\text{s}}$$

The maximum amplitude during start-up for the 450-kg machine mounted on an isolator such that the system natural frequency is 76.0 rad/s is

$$X_{\max} = \frac{20{,}000\ \text{N}}{(450\ \text{kg})(76.0\ \text{rad/s})^2} \frac{1}{2(0.05)\sqrt{1-(0.05)^2}} = 76.9\ \text{mm}$$

The resonant amplitude can be decreased to 10 mm only by increasing the mass to

$$m = \frac{20{,}000 \text{ N}}{(0.01 \text{ m})(76.0 \text{ rad/s})^2} \frac{1}{2(0.05)\sqrt{1-(0.05)^2}} = 3460 \text{ kg}$$

When the mass is increased to 3460 kg, the amplitude of vibration of the milling machine, when operating at 1800 rpm is

$$X = \frac{20{,}000 \text{ N}}{(3460 \text{ kg})(76.0 \text{ rad/s})^2} \frac{1}{\sqrt{[1-(2.48)^2]^2 + [2(0.05)(2.48)]^2}} = 0.19 \text{ mm}$$

The isolator stiffness is calculated by

$$k = m\omega_n^2 = (3460 \text{ kg})(76.0 \text{ rad/s})^2 = 2.0 \times 10^7 \text{ N/m}$$

The milling machine should be mounted on a concrete block of mass 3010 kg and the system isolated by springs with an equivalent stiffness of 2×10^7 N/m.

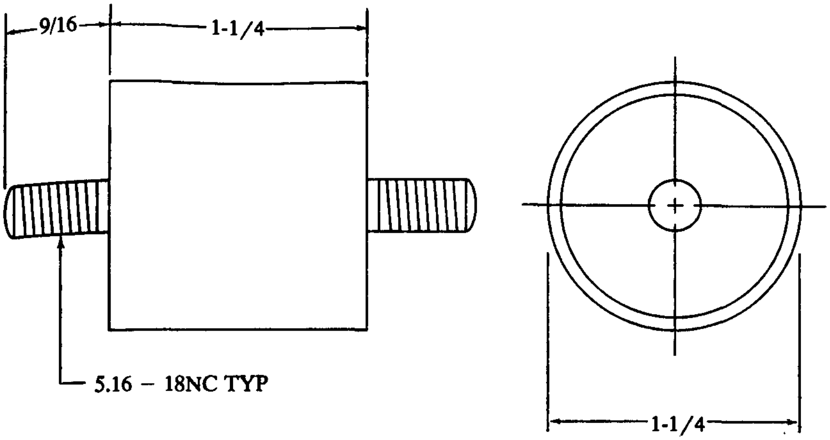
There are three classes of isolators in general use. The choice of an isolator for a particular application depends on the constraints noted previously (static deflection, vibration amplitude, and resonant amplitude), as well as other factors such as cost, weight, and space limitations, the amount of damping required, and environmental conditions.

Helical coil steel springs are used as isolators when large static deflections (> 1 in. or 3 cm) are required and a flexible foundation is acceptable. This occurs when good isolation is required at low operating speeds. Hysteresis in steel springs is low, so discrete viscous dampers are used in parallel with the springs to provide adequate damping. Steel springs may be used in combination with other isolation methods when a machine must be mounted on a concrete block. These isolators can be designed for specific use or can be obtained commercially.

Isolators made of elastomers are used in applications where small static deflections are required. If used for larger static loads, the elastomers are subject to creep, reducing their effectiveness after a period of time. Caution should be taken in using these isolators in extreme temperatures. Hysteretic damping inherent in the isolators is usually sufficient. However, discrete dampers can be employed in conjunction with these isolators. The damping ratio of an isolator depends on the elastomeric material from which it is made, the steady-state frequency, and the amplitude. The damping ratio for isolators made of natural rubber varies little with amplitude but is highly dependent on frequency. The damping ratio of a natural rubber isolator at 200 Hz is $\zeta = 0.03$ while $\zeta = 0.09$ at 1200 Hz.

Figure 8.10 is a copy of a page from the catalog of Stock Drive Products, a commercial manufacturer of vibration isolators. These isolators are usually used for

FOR COMPRESSION LOADS TO 120 POUNDS
SHEAR LOADS TO 63 POUNDS

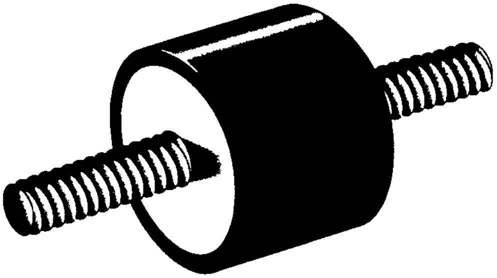


Compression		Minimum Load for 81% Isolation (lbs)									
		Forcing Frequency in Cycles per Minute									
Load Rating	Maximum Load (lbs)	600	850	950	1100	1250	1500	1750	2000	2500	3000
A	41				34.5	27.5	19	14	10	7	
B	64					48	32	24	17.5	12	8.5
C	90					80	55	41.5	30	20	14
D	120						89	70.5	53	38.5	26.5

Shear		Minimum Load for 81% Isolation (lbs)									
		Forcing Frequency in Cycles per Minute									
Load Rating	Maximum Load (lbs)	600	850	950	1100	1250	1500	1750	2000	2500	3000
A	21	20	11.0	8.5	6.7	5.5	*	*	*	*	*
B	31		18	14	10.5	8	5.5	*	*	*	*
C	48		31.5	25	19.5	15.5	11	8.5	*	*	*
D	63		50	41	32.6	27.5	20.5	16	14	8	*

*At these forcing frequencies lesser loads will yield 81% isolation.

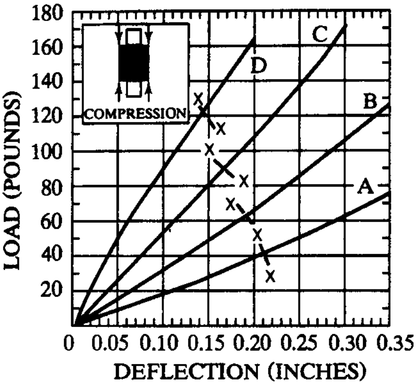
Figure 8.10 Example page from *Stock Drive Products Catalog* showing vibration isolator and its properties. (*Courtesy of Stock Drive Products.*)

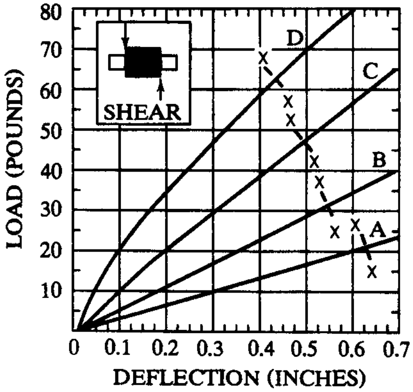


MATERIAL: Fastener—Steel, Brass Plated
Isolator—Natural Rubber

LOAD DEFLECTION GRAPHS

Deflections below the line X---X are considered safe practice for static loads; data above that line are useful for calculating deflections under dynamic loads.





CATALOG NUMBER DESIGNATION:

10Z2-310 ☐

Load Rating A
B
C
D

Figure 8.10 (Con't) Example page from *Stock Drive Products Catalog* showing vibration isolotor and its properties. (*Courtesy of Stock Drive Products.*)

small machines where the required static deflection does not exceed half an inch. The load rating given refers to the static load that the isolator can tolerate. An isolator that can tolerate a larger static load can supply a smaller static deflection. Isolators with different load ratings are made from different structural rubber compounds. The isolators with a higher load rating are harder and have a higher damping ratio. The damping ratios generally range from 0.02 for those with load rating A to 0.1 for those with load rating D.

Pads made of materials such as cork, felt, or elastomeric resin are often used to isolate large machines. Pads used to isolate a specific machine can be cut from larger pads. Pads of prescribed thicknesses can be placed on top of one another, acting as springs in series, to provide increased flexibility.

A small rotor balancing machine of weight 80 lb is to be placed on a table in a laboratory. The table vibrates because of operation of other equipment in the laboratory. An accelerometer was used to monitor the vibrations of the table. Analysis of its output showed that the vibrations of the table are made up of a number of components at different frequencies. However, the largest contribution is from the lowest frequency, an acceleration amplitude of 10 in./s^2 at a frequency of 150 rad/s. Accurate use of the rotor balancing machine requires that its base acceleration be limited to 1.5 in./s^2. If balanced four-point mounting is used, can the isolator of Fig. 8.10, mounted in compression, be used to sufficiently isolate the rotor balancing machine from the vibrations of the table?

Solution:

The maximum transmissibility ratio is

$$T = \frac{1.5 \text{ in.}/\text{s}^2}{10.0 \text{ in.}/\text{s}^2} = 0.15$$

Assuming a damping ratio of 0.1, in order for $T(r, 0.1)$ to be less than 0.15, a frequency ratio greater than 2.95 is required. The maximum natural frequency is (150 rad/s)/2.95 = 50.85 rad/s and the required isolator static deflection is $g/\omega_n^2 = 0.15$ in.

If four-point mounting is used, each isolator is subject to a static load of 20 lb. From the force-deflection curve for compression mounting of Fig. 8.10, a 20-lb load produces an isolator deflection of only 0.1 in. However, this isolator can tolerate a static deflection of 0.15 in., which will occur if the static load is 30 lb. This suggests that the rotor balancing machine can be mounted on a 40-lb block. Then the isolator of Fig. 8.10 can be used to sufficiently isolate the rotor balancing machine from the table vibrations.

Since an isolator of load rating A is used, the preceding calculations can be modified by assuming a damping ratio of 0.02. Then a frequency ratio of 2.78 sufficiently isolates vibrations. This leads to an isolator static deflection of 0.133 in. A static load of 26.6 lb on each isolator is required to provide the required deflection. Thus the machine needs to be mounted on a block of only 26.4 lb.

8.5 SHOCK ISOLATION

8.5.1 SHORT-DURATION PULSES

If the forge hammer of Fig. 8.11 is rigidly mounted to the foundation, the foundation is subject to a large impulsive force when the hammer impacts the anvil. An isolation system modeled as a spring and viscous damper in parallel can be designed to reduce the magnitude of the force to which the foundation is subject. The principles used in the design of a shock isolation system are similar to the principles used to design an isolation system to protect against harmonic excitation, but the equations are different.

If the duration t_0 of a transient excitation $F(t)$ is small, say $t_0 < T/5$ where T is the natural period of the system, then the system response can be adequately approximated by the response due to an impulse of magnitude

$$I = \int_0^{t_0} F(t)\,dt \qquad [8.27]$$

If the system is at rest in equilibrium when a pulse of short duration is applied, the principle of impulse-momentum is used to calculate the velocity imparted to the mass as

$$v = \frac{I}{m} \qquad [8.28]$$

The impulse provides external energy to initiate vibrations. Time is measured beginning immediately after the excitation is removed. The ensuing response is the free-vibration response due to an impulse providing the mass with an initial velocity v.

$$x(t) = \frac{v}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \qquad [8.29]$$

The maximum displacement occurs at a time

$$t_m = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad [8.30]$$

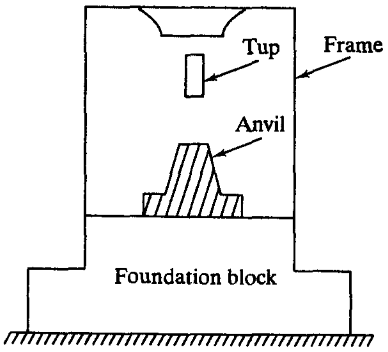


Figure 8.11 Schematic of a forge hammer. When tup impacts anvil, a large impulsive force is produced.

and is equal to

$$x_{\max} = \frac{v}{\omega_n} \exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \tag{8.31}$$

Equations (8.3) and (8.29) and trigonometric identities are used to calculate the force transmitted to the foundation through the isolator as

$$F_T(t) = \tilde{F} e^{-\zeta\omega_n t} \sin(\omega_d t - \beta) \tag{8.32}$$

where

$$\tilde{F} = \frac{m\omega_n v}{\sqrt{1-\zeta^2}} \tag{8.33}$$

and

$$\beta = -\tan^{-1}\left(\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}\right) \tag{8.34}$$

The maximum value of the transmitted force is obtained by differentiating Eq. (8.32) with respect to time, solving for the smallest time for which the derivative is zero, and finding the transmitted force at this time. The time for which the maximum transmitted force occurs is

$$t_{m_F} = \frac{1}{\omega_d} \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}\,(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right] \tag{8.35}$$

The corresponding maximum transmitted force is

$$F_{T_{\max}} = mv\omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}\,(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right]\right) \tag{8.36}$$

Equation (8.35) shows that the maximum transmitted force occurs at $t = 0$ for $\zeta = 0.5$. For $\zeta > 0.5$, the first time where $dF/dt = 0$ corresponds to a minimum. Thus, for $\zeta \geq 0.5$, the maximum transmitted force occurs at $t = 0$ and is given by

$$F_T(0) = cv = 2\zeta m\omega_n v \tag{8.37}$$

Equations (8.36) and (8.37) are combined to develop a nondimensional function $Q(\zeta)$ that is a measure of the maximum transmitted force, defined by

$$Q(\zeta) = \frac{F_{T_{\max}}}{mv\omega_n} = \begin{cases} \exp\left(-\dfrac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left[\dfrac{\sqrt{1-\zeta^2}\,(1-4\zeta^2)}{\zeta(3-4\zeta^2)}\right]\right) & \zeta < 0.5 \\ 2\zeta & 0.5 \leq \zeta < 1 \end{cases} \tag{8.38}$$

Figure 8.12 shows that $Q(\zeta)$ is flat and approximately equal to 0.81 for $0.23 < \zeta < 0.30$. If minimization of the transmitted force is the sole criterion for the isolator design, the isolator should have a damping ratio near 0.25.

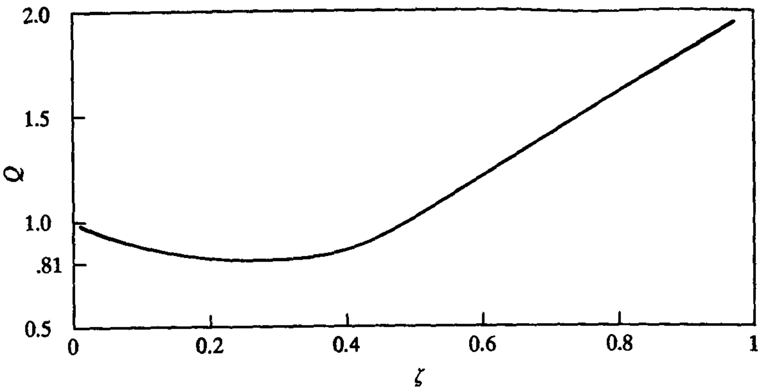


Figure 8.12 Shock isolation function for short-duration pulse.

Equation (8.38) shows that, for a given ζ, the transmitted force is proportional to the natural frequency. Thus a low natural frequency and large natural period is necessary and the short-duration assumption is often valid.

Equation (8.31) shows that the maximum displacement varies inversely with the natural frequency. Thus requiring a small transmitted force leads to a large displacement. The natural frequency is eliminated between Eqs. (8.31) and (8.38), yielding

$$\frac{F_{T_{\max}} x_{\max}}{\frac{1}{2} m v^2} = S(\zeta) \qquad [8.39]$$

where

$$S(\zeta) = \begin{cases} 2\exp\left(-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left[\dfrac{\zeta\sqrt{1-\zeta^2}\,(4-8\zeta^2)}{8\zeta^2-8\zeta^4-1}\right]\right) & \zeta \le 0.5 \\ 4\zeta\exp\left[-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] & 0.5 < \zeta < 1 \end{cases} \qquad [8.40]$$

The denominator of the nondimensional ratio of Eq. (8.39) is the initial kinetic energy of the system. The numerator is a measure of the work done by the transmitted force. The inverse of this ratio is the fraction of energy absorbed by the isolator, the isolator efficiency. Figure 8.13 shows that the maximum isolator efficiency occurs for $\zeta = 0.40$ where $S = 1.04$.

The 200-kg hammer of a 1000-kg forge hammer is dropped from a height of 1 m. Design an isolator to minimize the maximum displacement when the maximum force transmitted to the foundation is 20,000 N. What is the maximum displacement of the hammer when placed on this isolator?

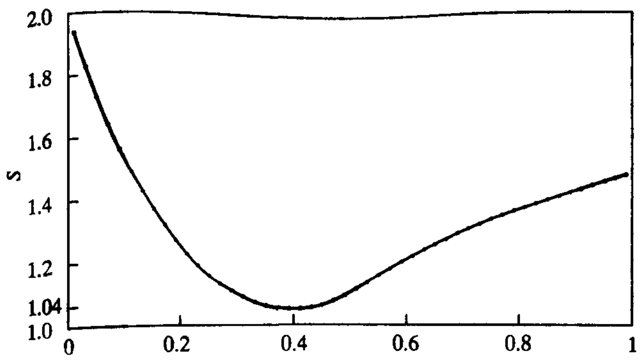


Figure 8.13 Energy absorption for short-duration pulse.

Solution:

The excitation is a result of the impact of the hammer with the anvil, and thus of short duration. The velocity of the anvil at the time of impact is

$$v = \sqrt{2(9.81\ \mathrm{m/s^2})(1\ \mathrm{m})} = 4.43\ \mathrm{m/s}$$

The velocity of the machine after impact is determined by using the principle of impulse and momentum

$$v = \frac{(200\ \mathrm{kg})(4.43\ \mathrm{m/s})}{1000\ \mathrm{kg}} = 0.886\ \frac{\mathrm{m}}{\mathrm{s}}$$

The product of the maximum transmitted force and the maximum displacement is minimized by selecting $\zeta = 0.4$. Then if the transmitted force is limited to 20,000 N, the maximum displacement is obtained by using Eq. (8.39)

$$x_{\max} = \frac{\frac{1}{2}mv^2}{F_{T_{\max}}}S(0.4) = \frac{\frac{1}{2}(1000\ \mathrm{kg})(0.886\ \mathrm{m/s})^2}{20{,}000\ \mathrm{N}}1.04 = 0.02\ \mathrm{m}$$

The natural frequency of the isolator is calculated by using Eq. (8.38)

$$\omega_n = \frac{F_{T_{\max}}}{mvQ(0.4)} = \frac{20{,}000\ \mathrm{N}}{(1000\ \mathrm{kg})(0.886\ \mathrm{m/s})(0.88)} = 25.65\ \frac{\mathrm{rad}}{\mathrm{s}}$$

and the maximum isolator stiffness is calculated as

$$k = m\omega_n^2 = (1000\ \mathrm{kg})(25.65\ \mathrm{rad/s})^2 = 6.58 \times 10^5\ \mathrm{N/m}$$

8.5.2 Long-Duration Pulses

For pulses of longer duration, it is possible that the maximum displacement or transmitted force could occur while the excitation is being applied. In addition, the shape of the pulse has an effect on these extreme values. Shock isolation calculations, for pulses of duration longer than $T/5$ are made by using the shock spectrum for the pulse. The shock spectrum, first introduced in Sec. 4.6, is a nondimensional plot of the maximum displacement as a function of t_0/T

Shock spectra are often calculated only for undamped systems. Algebraic complexity usually prevents analytical determination of shock spectra for damped systems. The maximum response is obtained either by numerical evaluation of the exact expression for the displacement or by numerical solution of the differential equation. Damping does not have as much effect on the transient response due to a pulse of longer duration as it does on the steady-state response due to a harmonic excitation or on the response due to a short-duration pulse.

Since shock isolation often involves minimizing the force transmitted between a system and its support, a plot similar to the shock spectrum, but involving the maximum value of the transmitted force, is useful. The vertical coordinate of the force spectrum is the ratio of the maximum value of the transmitted force to the maximum value of the excitation force. When the system is undamped, the force spectrum is the same as the shock spectrum.

Figures 8.14 through 8.19 present displacement spectra and force (acceleration) spectra for common pulse shapes. These spectra were obtained by using a Runge-Kutta solution of the governing differential equation. A system with $\omega_n = 1$ and $m = 1$ was arbitrarily used. A time increment of the smaller of $t_0/50$ and $T/50$ was used. The Runge-Kutta solution was carried out until the larger of $4t_0$ or $4T$. The displacement and transmitted force were calculated at each time step and compared to maxima from the previous times. The spectra were developed for several values of ζ.

The force spectra for the rectangular pulse, the triangular pulse, the sinusoidal pulse, the versed sine pulse, the negative-slope ramp pulse, and the reversed loading pulse show that shock isolation is achieved only for small natural frequencies. The shock spectra for these excitations show that the nondimensional displacement is small for small natural frequencies. However, the dimensional displacement is calculated by using the nondimensional displacement from

$$x_{\max} = \frac{F_0}{m\omega_n^2}\left(\frac{m\omega_n^2 x_{\max}}{F_0}\right) \qquad \textbf{[8.41]}$$

Thus a small natural frequency leads to a large displacement.

8.9 A 1000-kg machine is subject to a triangular pulse of duration 0.05 s and peak of 20,000 N. What is the range of isolator stiffness for an undamped isolator such that the maximum transmitted force is less than 8000 N and the maximum displacement is less than 2.8 cm?

Solution:

The force spectrum for the triangular pulse shows that for $F_T/F_0 < 0.4$, $\omega_n t_0/(2\pi) < 0.16$, which gives

$$\omega_n < \frac{2\pi(0.16)}{0.05 \text{ s}} = 20.1 \frac{\text{rad}}{\text{s}}$$

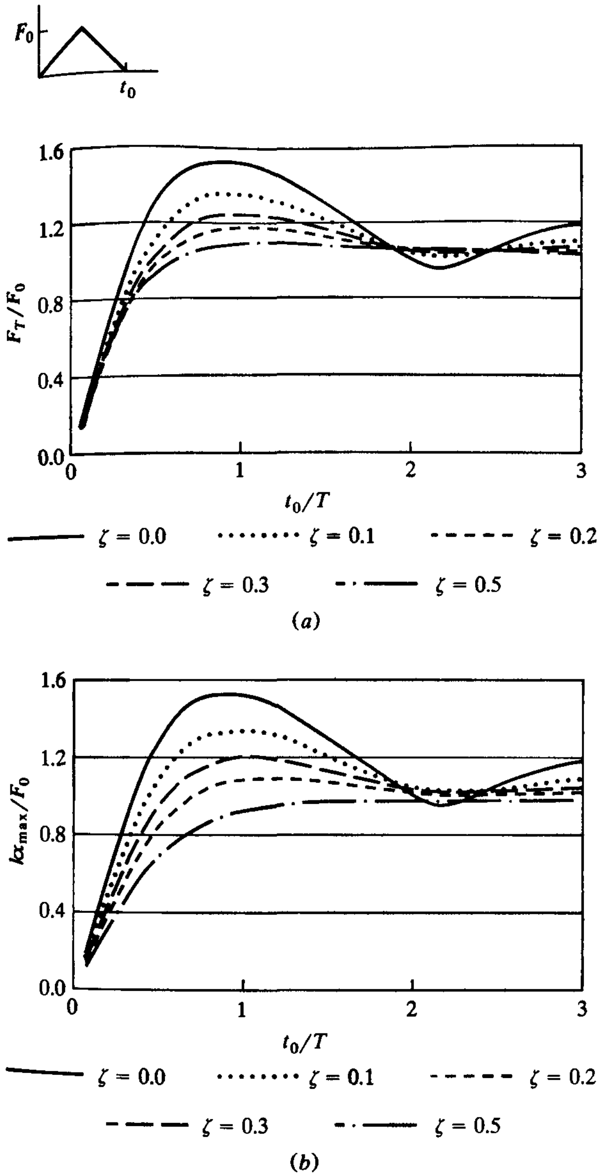
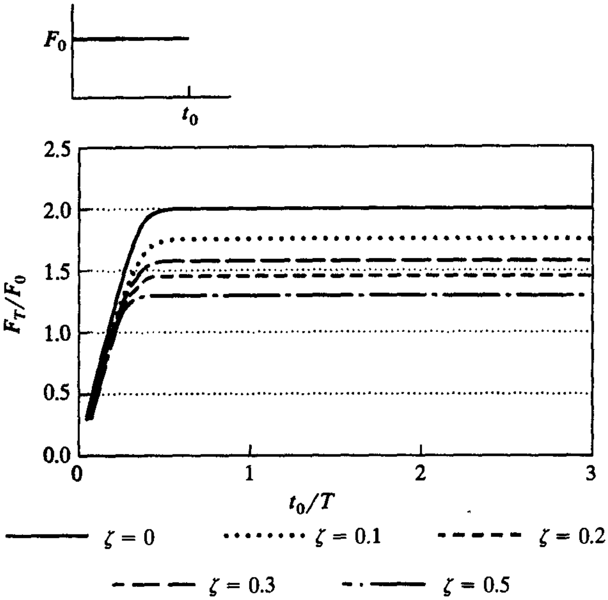
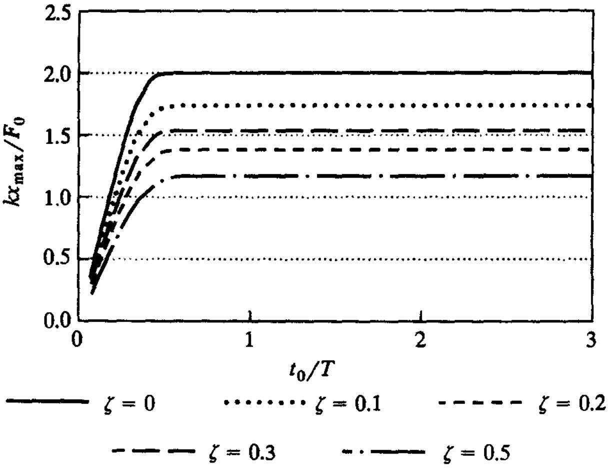


Figure 8.14 (*a*) Force spectrum triangular pulse; (*b*) displacement spectrum triangular pulse.



(*a*)



(*b*)

Figure 8.15 (*a*) Force spectrum rectangular pulse; (*b*) displacement spectrum rectangular pulse.

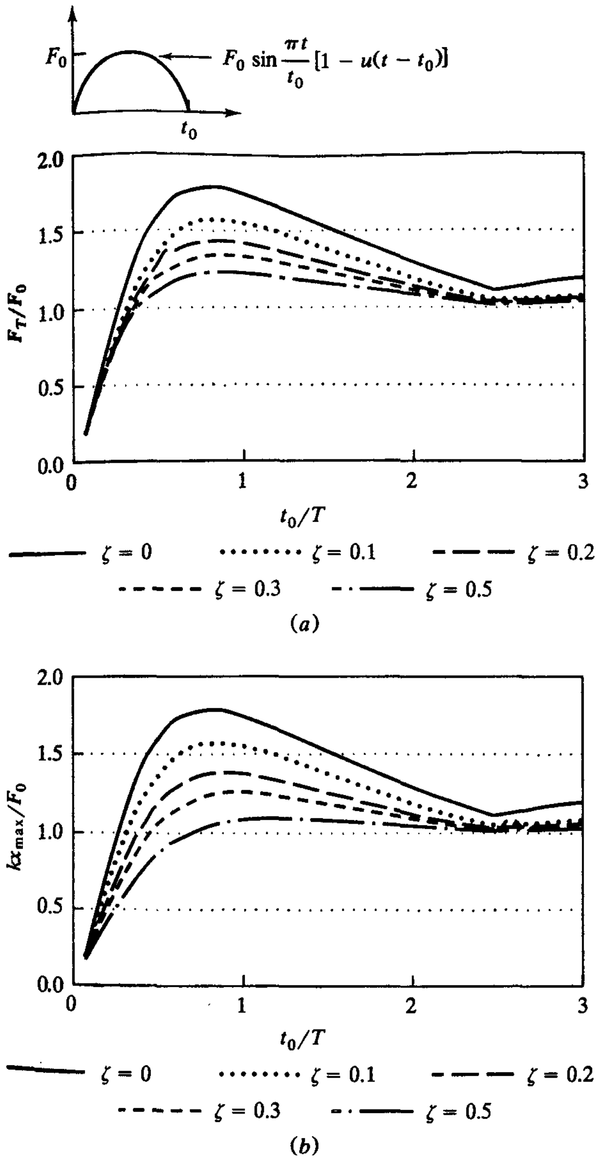


Figure 8.16 (*a*) Force spectrum sinusoidal pulse; (*b*) displacement spectrum sinusoidal pulse.

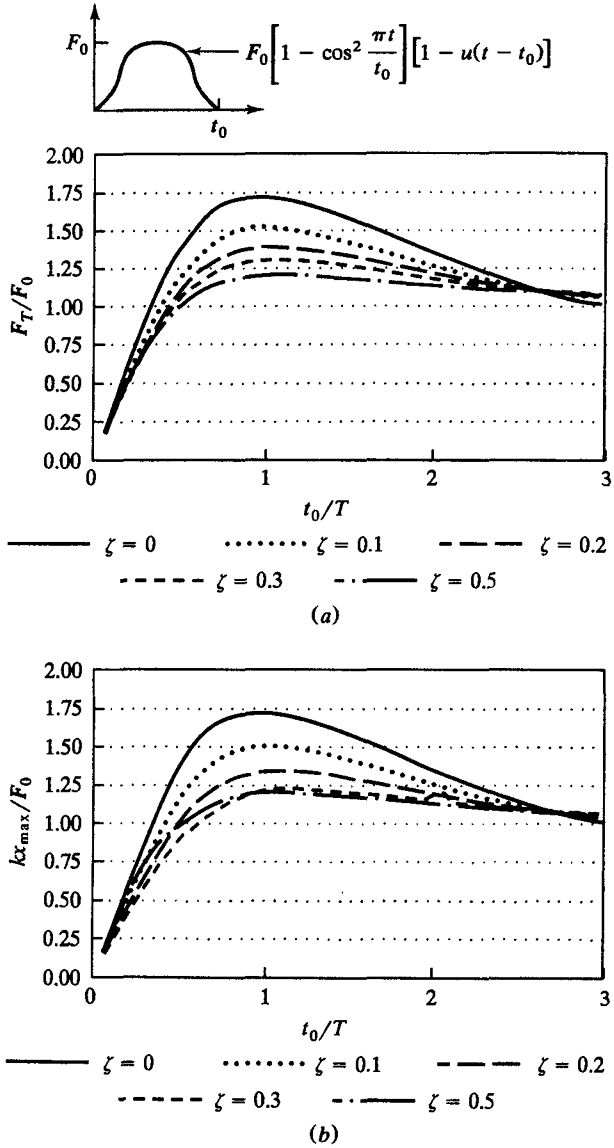
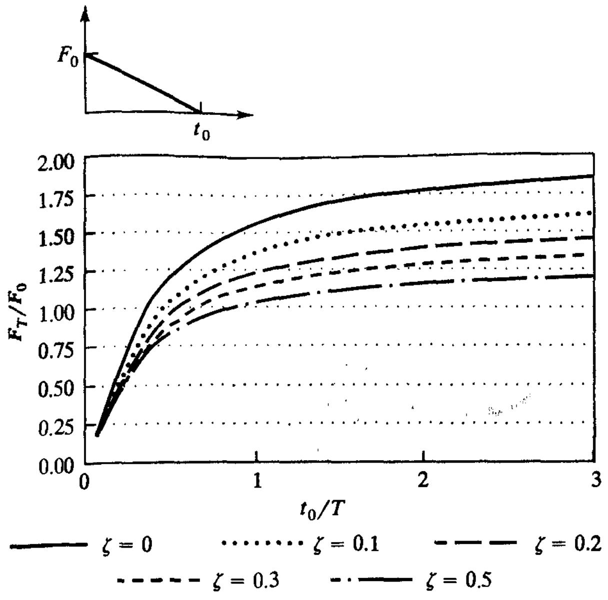
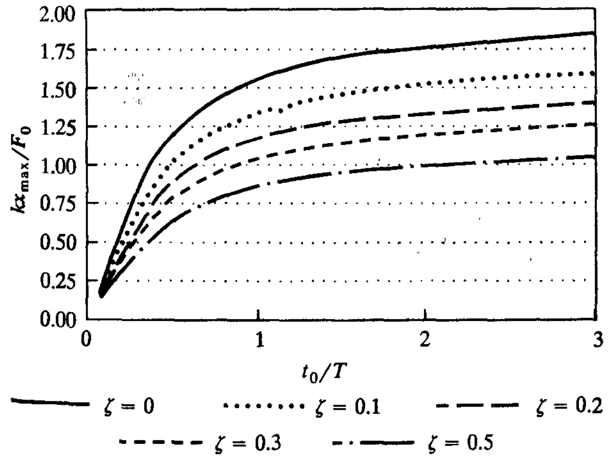


Figure 8.17 (*a*) Force spectrum versed sine pulse; (*b*) displacement spectrum versed sine pulse.



(*a*)



(*b*)

Figure 8.18 (*a*) Force spectrum negative slope ramp; (*b*) displacement spectrum negative slope ramp.

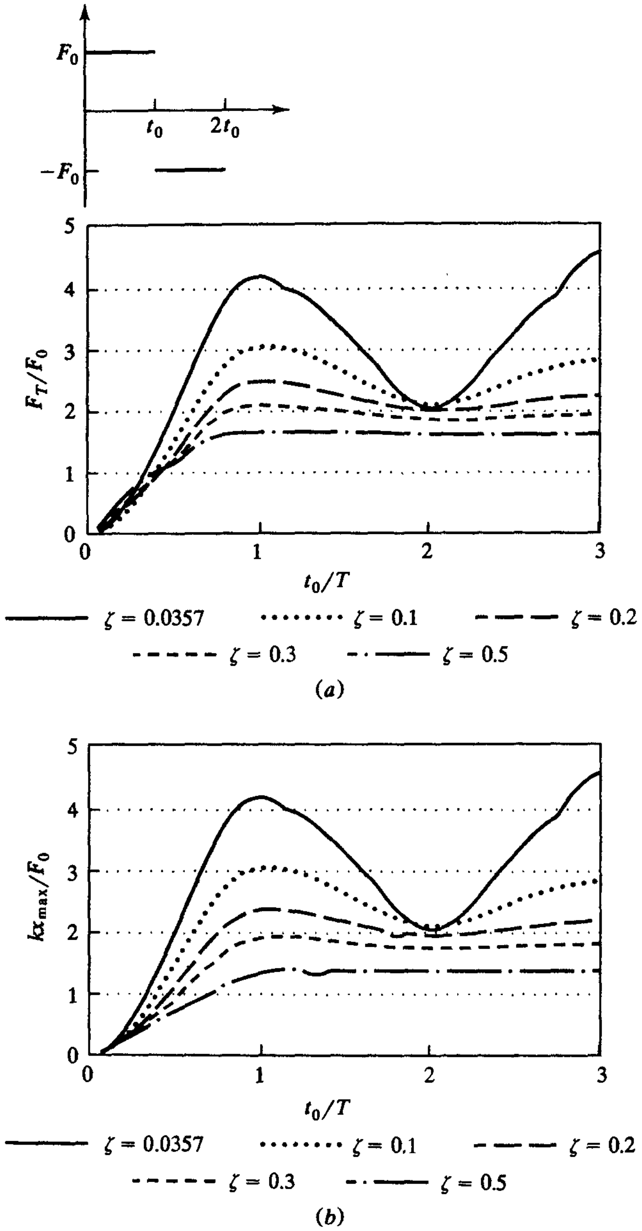


Figure 8.19 (*a*) Reversed step pulse force spectrum; (*b*) reversed step pulse displacement spectrum.

The lower bound on the natural frequency is obtained by trial and error, using the displacement spectrum for the triangular pulse. For a guessed value of ω_n, $\omega_n t_0/(2\pi)$ is calculated and the corresponding value of the maximum nondimensional displacement is found from the displacement spectrum. The maximum dimensional displacement is calculated from Eq. (8.41). If the displacement is greater than the allowable displacement, the guess for the lower bound must be increased. The calculations for this example are given in Table 8.1. The lower bound is calculated as 17 rad/s. Thus the allowable stiffness range is

$$2.89 \times 10^5 \text{ N/m} < k < 4.04 \times 10^5 \text{ N/m}$$

8.6 DYNAMIC VIBRATION ABSORBERS

8.6.1 UNDAMPED ABSORBERS

Large-amplitude vibrations of a one-degree-of-freedom system subject to a harmonic excitation occur when an excitation frequency is near a natural frequency. The amplitude of response is reduced when the system properties are changed such that the natural frequency is away from the excitation frequency. Alternatively, an additional degree of freedom can be added such that the natural frequencies of the resulting two-degree-of-freedom system are away from the excitation frequency.

A vibration absorber is an auxiliary mass-spring system that, when correctly tuned and attached to a vibrating body subject to a harmonic excitation, causes steady-state motion of the point to which it is attached to cease. Figure 8.20 shows a schematic of a one-degree-of-freedom system with an absorber added. A mass m_1, called the primary mass, attached to a rigid foundation through a spring of stiffness k_1 is excited by a harmonic excitation, whose frequency is near $\sqrt{k_1/m_1}$. An absorber of mass m_2 is connected to the original mass through an elastic element of stiffness k_2. The resulting system has two degrees of freedom. If correctly designed, the frequency response curve for the primary mass is altered such that the steady-state amplitude of the primary mass is small at the excitation frequency.

The solution for the steady-state vibrations of the resulting two-degree-of-freedom system is presented in Example 7.1. The steady-state amplitude of the

Table 8.1

ω_n, rad/s	$\frac{\omega_n t_0}{2\pi}$	$\frac{m\omega_n^2 x_{max}}{F_0}$	x_{max}, cm
10	0.08	0.25	5.0
15	0.12	0.38	3.4
18	0.14	0.42	2.6
17	0.135	0.40	2.8

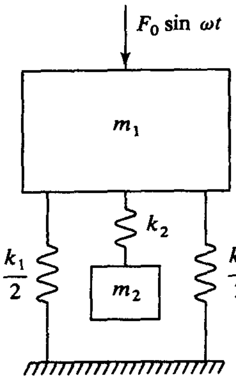


Figure 8.20 When $\omega \approx \sqrt{k_1/m_1}$, large-amplitude vibrations of the primary system occur. The addition of an auxiliary mass-spring system (the vibration absorber) changes the system to two degrees of freedom with natural frequencies away from $\sqrt{k_1/m_1}$.

primary mass is given by

$$\frac{k_1 X_1}{F_0} = \left| \frac{1 - r_2^2}{r_1^2 r_2^2 - r_2^2 - (1+\mu) r_1^2 + 1} \right| \qquad [8.42]$$

and the steady-state amplitude of the absorber mass is

$$\frac{k_1 X_2}{F_0} = \left| \frac{1}{r_1^2 r_2^2 - r_2^2 - (1+\mu) r_1^2 + 1} \right| \qquad [8.43]$$

where

$$r_1 = \omega \sqrt{\frac{m_1}{k_1}} = \frac{\omega}{\omega_{11}} \qquad [8.44]$$

is the ratio of the excitation frequency to the natural frequency of the primary system,

$$r_2 = \omega \sqrt{\frac{m_2}{k_2}} = \frac{\omega}{\omega_{22}} \qquad [8.45]$$

is the ratio of the excitation frequency to the natural frequency of the absorber system, if taken by itself, and

$$\mu = \frac{m_2}{m_1} \qquad [8.46]$$

The natural frequencies of the two-degree-of-freedom system are the values of ω such that the denominator of Eqs. (8.42) and (8.43) is zero. They are obtained as

$$\omega_{1,2} = \frac{\omega_{11}}{\sqrt{2}} \sqrt{1 + q^2(1+\mu) \pm \sqrt{q^4(1+\mu)^2 + 2(\mu - 1)q^2 + 1}} \qquad [8.47]$$

where

$$q = \frac{\omega_{22}}{\omega_{11}} \qquad [8.48]$$

From Eq. (8.42) the steady-state amplitude of the original mass is zero when

$$\sqrt{\frac{k_2}{m_2}} = \omega \qquad [8.49]$$

Under this condition the steady-state amplitude of the absorber mass is

$$X_2 = \frac{F_0}{k_2} \tag{8.50}$$

A vibration absorber can be used to eliminate unwanted steady-state vibrations from a one-degree-of-freedom freedom system if the natural frequency of the absorber is tuned to the excitation frequency. The vibration absorber has many practical applications. However, the following must be kept in mind in designing an undamped vibration absorber.

1. When the absorber is tuned to the excitation frequency, Eq. (8.47) shows that one of the two degree of freedom's natural frequencies is less than the absorber's natural frequency while the other is greater. Hence the lower natural frequency must be passed during start-up and stopping, leading to large-amplitude vibrations during these transient periods.
2. The steady-state vibrations of the original mass are eliminated only at a single operating speed. If the system operates over a range of frequencies, the steady-state amplitudes at frequencies away from the absorber frequency may be large. Figure 8.21 shows $k_1 X_1/F_0$ as a function of r_1 for $q = 1$ and $\mu = 0.15$ and $\mu = 0.25$. If r_1 is much less than or greater than q, the steady-state amplitude of the primary mass is large. An effective operating range should be defined for each application by limiting the amplitude of the vibrations or the transmitted force to an acceptable maximum.
3. Figure 8.22 shows the natural frequencies as a function of the mass ratio, μ, for fixed q. The separation of the two natural frequencies is small for small μ, resulting in a narrow operating range. The separation of natural frequencies and the effective operating range increases for larger μ. However, a small μ is usually desired for practical reasons.
4. If the absorber is tuned to the excitation frequency and a given mass ratio μ is not to be exceeded, then the maximum value of the absorber spring stiffness is

   $$k_{2_{\max}} = \mu m_1 \omega^2 \tag{8.51}$$

   and the minimum steady-state amplitude of the absorber mass is

   $$X_{2_{\min}} = \frac{F_0}{\mu m_1 \omega^2} \tag{8.52}$$

5. The preceding analysis is valid only for an undamped system. If damping is present in the absorber it is not possible to eliminate steady-state vibrations of the original mass. The amplitude of vibrations can only be reduced.

Example

A machine of mass 150 kg with a rotating unbalance of 0.5 kg · m is placed at the midspan of a 2-m-long simply supported beam. The machine operates at a speed of 1200 rpm. The beam has an elastic modulus of 210×10^9 N/m^2 and a cross-sectional moment of

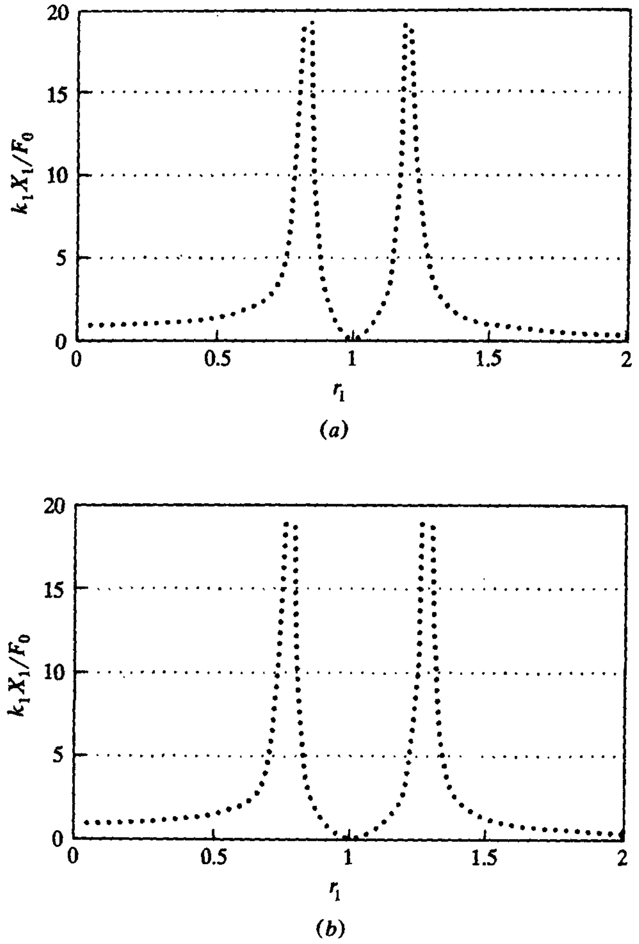


Figure 8.21 Frequency response of the primary mass with the addition of a vibration absorber; (a) $\mu = 0.15$ and $q = 1.0$; (b) $\mu = 0.25$, and $q = 1.0$.

inertia of 2.1×10^{-6} m^4. Design a dynamic vibration absorber such that when attached to the midspan of the beam, the vibrations of the beam will cease and the steady-state absorber amplitude will be less than 20 mm. What are the system's natural frequencies with the absorber in place? What is the effective operating range such that the midspan deflection does not exceed 5 mm when the absorber is in place?

Solution:

Modeling the beam vibrations using one degree of freedom and ignoring the mass of the beam, the stiffness and natural frequency of the primary system are calculated as

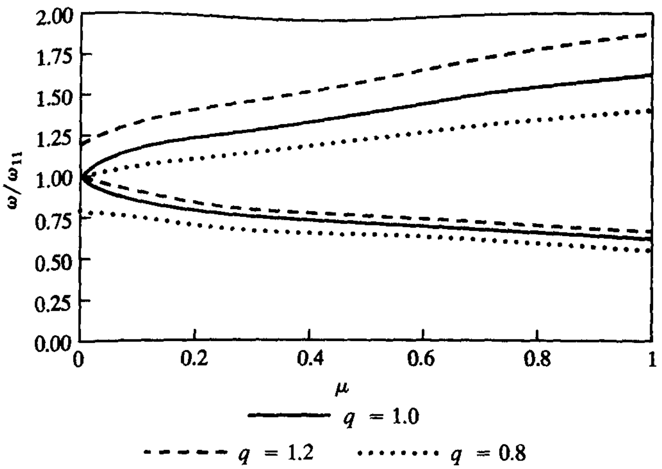


Figure 8.22 The natural frequencies of the system with absorber added are farther apart for larger mass ratios.

$$k_1 = \frac{48EI}{L^3} = \frac{48(210 \times 10^9\ \mathrm{N/m^2})(2.1 \times 10^{-6}\ \mathrm{m^4})}{(2\ \mathrm{m})^3} = 2.65 \times 10^6\ \frac{\mathrm{N}}{\mathrm{m}}$$

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2.65 \times 10^6\ \mathrm{N/m}}{150\ \mathrm{kg}}} = 132.9\ \frac{\mathrm{rad}}{\mathrm{s}}$$

The operating speed is

$$\omega = (1200\ \mathrm{rpm})\left(2\pi\ \frac{\mathrm{rad}}{\mathrm{rev}}\right)\left(1\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = 125.7\ \frac{\mathrm{rad}}{\mathrm{s}}$$

Thus large-amplitude vibrations are expected to occur in the absence of an absorber. The natural frequency of the absorber is chosen to coincide with the operating speed

$$\omega_{22}\sqrt{\frac{k-2}{m_2}} = 125.7\ \frac{\mathrm{rad}}{\mathrm{s}}$$

Under this condition the steady-state amplitude of the absorber mass is given by Eq. (8.50). Requiring this amplitude to be less than 20 mm leads to

$$k_2 > \frac{F_0}{X_{2_{\max}}} = \frac{(0.5\ \mathrm{kg}\cdot\mathrm{m})(125.7\ \mathrm{rad/s})^2}{0.02\ \mathrm{m}} = 3.95 \times 10^5\ \frac{\mathrm{N}}{\mathrm{m}}$$

The required absorber mass is

$$m_2 = \frac{k_2}{\omega_{22}^2} = 25 \text{ kg}$$

The natural frequencies of the two-degree-of-freedom system are calculated from Eq. (8.47) with $\mu = 1/6$ and $q = 0.946$. They are

$$\omega_1 = 105.8 \text{ rad/s} \qquad \omega_2 = 157.6 \text{ rad/s}$$

The effective operating range is obtained from Eq. (8.42), setting $F_0 = 0.5\omega^2$, and using Eqs. (8.44) and (8.45). The denominator of Eq. (8.42) is negative for all ω between the two natural frequencies. When $r_2 < 1$, the numerator is positive and X_1 is set equal to -0.005 m. When $r_2 > 1$ the numerator is negative and X_1 is set equal to 0.005 m. The following quadratic equations in ω_2 are obtained to determine the bounds of the operating range:

$$\omega^4 - 2.79 \times 10^4\omega^2 + 1.67 \times 10^8 = 0 \qquad r_2 > 1$$
$$\omega^4 - 7.63 \times 10^4\omega^2 + 8.28 \times 10^8 = 0 \qquad r_2 < 1$$

The resulting operating range is

$$114.8 \text{ rad/s} < \omega < 138.5 \text{ rad/s}$$

The lower end of the operating range is closer to the lower natural frequency than the upper end is to the higher natural frequency because the excitation force is less for lower excitation frequencies.

8.6.2 DAMPED VIBRATION ABSORBERS

Viscous damping may be added in parallel to the elastic element of a vibration absorber to limit the amplitude as the lower natural frequency is passed during system start-up and stopping or to increase the effective operating range of the resulting two-degree-of-freedom system. A steady-state analysis of the system of Fig. 8.23 when excited by a harmonic force is presented in Example 7.2. The equations for calculating the steady-state amplitudes are nondimensionalized, yielding

$$\frac{k_1 X_1}{F_0} = G(r_1, q, \zeta, \mu) \qquad [8.53]$$

where

$$G = \sqrt{\frac{(2\zeta r_1 q)^2 + (r_1^2 - q^2)^2}{\{r_1^4 - [1 + (1+\mu)q^2]r_1^2 + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}} \qquad [8.54]$$

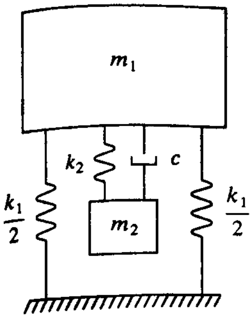


Figure 8.23 The damped vibration absorber consists of an auxiliary mass connected to the primary mass by an elastic element in parallel with a viscous damper.

and

$$\frac{k_1 X_2}{F_0} = \sqrt{\frac{q^4 + (2\zeta q)^2}{\{r_1^4 - [1 + (1+\mu)q^2]r_1^2 + q^2\}^2 + (2\zeta r_1 q)^2[1 - r_1^2(1+\mu)]^2}} \qquad \text{[8.55]}$$

where r_1 and q are defined in Eqs. (8.44) and (8.48) respectively, and

$$\zeta = \frac{c}{2\sqrt{k_2 m_2}} \qquad \text{[8.56]}$$

The nondimensional function G defined by Eq. (8.54) is shown in Fig. 8.24 for $\mu = 0.25$ and $q = 1$ for several values of ζ. The steady-state motion of the original mass is not zero for any r_1. A minimum, near $r_1 = 1$, is reached between two peaks. An absorber using this choice of parameters is not effective because the peak at the lower resonant frequency is still large. It is noted that each curve, for different ζ, passes through the same two points.

G is plotted in Fig. 8.25 for $\mu = 0.25$ and $q = 0.8$. The peak at the lower resonant frequency is smaller than the peak at the higher resonant frequency. However, the higher peak occurs near $r_1 = 1$, which is the region where an absorber is usually needed. Also, the effective operating range is still small. It is noted again that there are two fixed points through which each curve passes. These fixed points are different than those in Fig. 8.24.

Since it is not possible to eliminate steady-state motion of the original system when damping is present, a damped vibration absorber must be designed to reduce the peak at the lower resonant frequency and to widen the effective operating range. Absorbers using the parameters used to generate Fig. 8.24 and 8.25 are not suitable for these purposes.

Widening the operating range requires that the two peaks have approximately the same magnitude. Since the locations of the fixed points are dependent on q, it should be possible to tune the absorber such that the values of G at the fixed points are the same. Since curves for all values of ζ pass through the fixed points, it should be possible to find a value of ζ such that the fixed points are near the peaks.

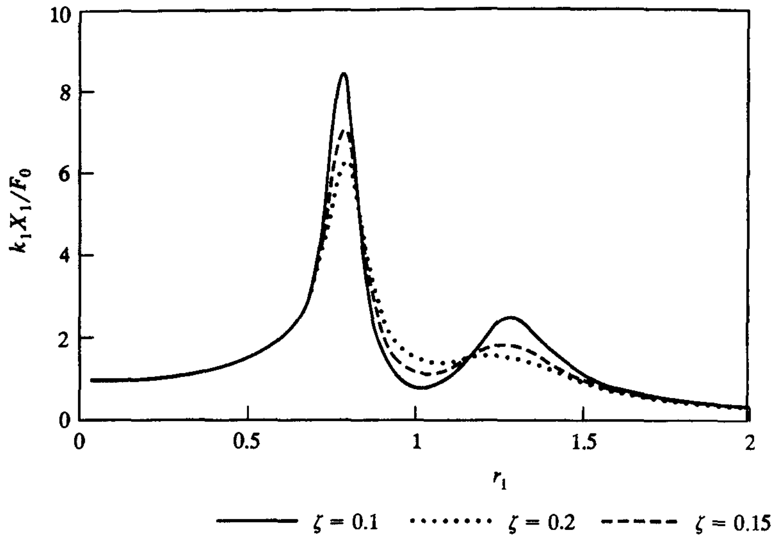


Figure 8.24 Frequency response of primary mass with additions of damped vibration absorber with $\mu = 0.25$, $q = 1.0$.

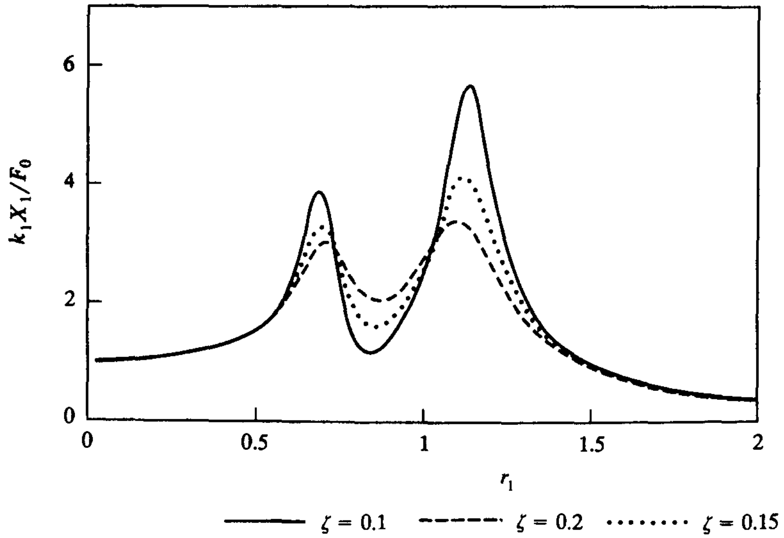


Figure 8.25 Frequency response of primary mass with addition of damped vibration absorber with $\mu = 0.25$, $q = 0.80$.

For fixed values of μ and q, there are two values of r_1 which yield a value of G, independent of ζ. The value of G at these points is written as

$$G = \sqrt{\frac{A(\mu, q)\zeta^2 + B(\mu, q)}{C(\mu, q)\zeta^2 + D(\mu, q)}} \tag{8.57}$$

Since Eq. (8.57) holds for all ζ and powers of ζ are linearly independent,

$$\frac{A}{C} = \frac{B}{D} \tag{8.58}$$

Using Eq. (8.54) to determine the forms of A, B, C, and D, substituting into Eq. (8.58), and rearranging leads to

$$r_1^4\left(1 + \frac{\mu}{2}\right) - [1 + q^2(1 + \mu)]r_1^2 + q^2 = 0 \tag{8.59}$$

The solution of Eq. (8.59) places the fixed points at

$$r_1 = \sqrt{\frac{1 + (1 + \mu)q^2 \pm \sqrt{1 - 2q^2 + (1 + \mu)^2 q^4}}{2 + \mu}} \tag{8.60}$$

Since Eq. (8.57) yields the same value of G, independent of ζ for r_1 given by Eq. (8.60), letting $\zeta \rightarrow \infty$ gives

$$G = \sqrt{\frac{1}{[1 - r_1^2(1 + \mu)]^2}} \tag{8.61}$$

Requiring G to be the same at both fixed points leads to

$$q = \frac{1}{1 + \mu} \tag{8.62}$$

An optimum absorber could be designed with an appropriate value of ζ such that the smaller r_1 given by Eq. (8.60) corresponds to both a fixed point and a peak on the frequency response curve. The appropriate value of ζ is obtained by setting $dG/d\zeta = 0$, using q from Eq. (8.62). The same procedure can be followed to yield the value of ζ such that the larger value of r_1 given by Eq. (8.60) corresponds to both a fixed point and a peak. Since the values of ζ are not equal, their average is usually used to define the optimum damping ratio

$$\zeta_{\text{opt}} = \sqrt{\frac{3\mu}{8(1 + \mu)}} \tag{8.63}$$

In summary, the optimum design of a damped vibration absorber requires that the absorber be tuned to the frequency calculated from Eq. (8.62) with the damping ratio of Eq. (8.63). For $\mu = 0.25$, Eq. (8.63) gives an optimum damping ratio of $\zeta = 0.2379$ and an optimum $q = 0.80$. Figure 8.26 shows G for these values as a

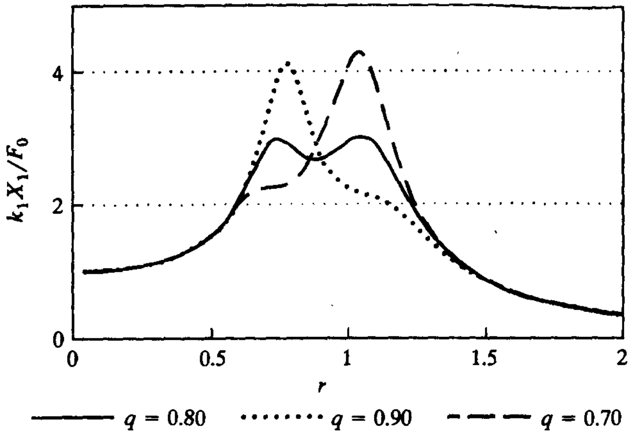


Figure 8.26 For $\mu = 0.25$, $\zeta_{opt} = 0.2739$ and $q_{opt} = 0.80$. This comparison shows that values of q other than q_{opt} lead to larger peaks and a narrower operating range.

function of r_1. This figure also shows G for the same μ and ζ but with values of q, one on each side of the optimum. The curve corresponding to the optimum value of q has smaller resonant peaks and the value of G does not vary much between the peaks.

8.11 A milling machine has a mass of 250 kg and a natural frequency of 120 rad/s and is subject to a harmonic excitation of magnitude 10,000 N at speeds between 95 rad/s and 120 rad/s. Design a damped vibration absorber of mass 50 kg such that the steady-state amplitude is no greater than 15 mm at all operating speeds.

Solution:

The mass ratio is

$$\mu = \frac{50 \text{ kg}}{250 \text{ kg}} = 0.2$$

Since a wide operating range is required, the optimum absorber design is tried. From Eqs. (8.62) and (8.63),

$$q = \frac{1}{1.2} = 0.833 \qquad \zeta = \sqrt{\frac{3(0.2)}{8(1.2)}} = 0.25$$

The steady-state amplitude at any operating speed for this absorber design is calculated by Eqs. (8.53) and (8.54). The results are used to generate the frequency response curve of Fig. 8.27. Since the extremes of the operating range lie between the peaks and the

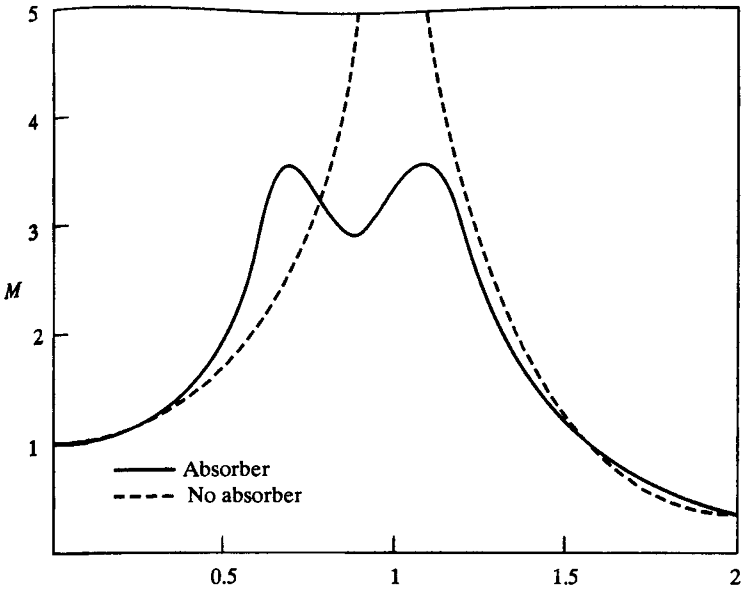


Figure 8.27 The addition of the optimum damped vibration absorber to the system of Example 8.11 reduces the steady-state amplitude to an acceptable level over the entire operating range.

steady-state amplitudes at the extremes are

$$X(\omega = 95 \text{ rad/s}) = 9.13 \text{ mm} \qquad X(\omega = 120 \text{ rad/s}) = 8.86 \text{ mm}$$

and both are less than 15 mm, the optimum design is acceptable. The absorber stiffness and damping ratio are calculated from Eqs. (8.48) and (8.56), respectively, as

$$k_2 = m_2\omega_{22}^2 = \mu q^2 k_1 = (0.2)(0.833)^2(3.6 \times 10^6 \text{ N/m}) = 5.08 \times 10^5 \text{ N/m}$$
$$c = 2\zeta\sqrt{k_2 m_2} = 2500 \text{ N} \cdot \text{s/m}$$

8.6.3 Multi-Degree-of-Freedom Systems

Large-amplitude responses occur for an n-degree-of-freedom system when a harmonic excitation frequency is near one of the system's n natural frequencies. Vibration absorbers can be designed to reduce the steady-state amplitudes near these resonances.

Consider an n-degree-of-freedom system subject to a harmonic excitation near one of the system's natural frequencies. Let $x_1, x_2, \ldots, x_n$ be the chosen generalized coordinates. Let k_{ij} and m_{ij} be the elements of the stiffness and mass matrices, respectively, for this choice of generalized coordinates. A vibration absorber of mass $\tilde{m}$ and stiffness $\tilde{k}$ is attached to the system at the particle whose displacement is given by x_1. The generalized coordinate x_{n+1} is defined as the displacement of the absorber mass. The mass and stiffness matrices for the resulting $n+1$-degree-of-freedom system are

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} & 0 \\ m_{21} & m_{22} & \cdots & m_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} & 0 \\ 0 & 0 & \cdots & 0 & \tilde{m} \end{bmatrix} \qquad [8.64]$$

$$\mathbf{K} = \begin{bmatrix} k_{11}+\tilde{k} & k_{12} & \cdots & k_{1n} & -\tilde{k} \\ k_{21} & k_{22} & \cdots & k_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} & 0 \\ -\tilde{k} & 0 & \cdots & 0 & \tilde{k} \end{bmatrix} \qquad [8.65]$$

For a single-frequency harmonic excitation, the force vector takes the form

$$\mathbf{F} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \\ 0 \end{bmatrix} \sin \omega t \qquad [8.66]$$

By the methods of Sec. 7.2, the steady-state amplitude of the particle on the original system to which the absorber is attached is

$$X_1 = \frac{1}{D(\omega)} \det \begin{bmatrix} A_1 & k_{12}-m_{12}\omega^2 & k_{13}-m_{13}\omega^2 & \cdots & k_{1n}-m_{1n}\omega^2 & -\tilde{k} \\ A_2 & k_{22}-m_{22}\omega^2 & k_{23}-m_{23}\omega^2 & \cdots & k_{2n}-m_{2n}\omega^2 & 0 \\ A_3 & k_{32}-m_{32}\omega^2 & k_{33}-m_{33}\omega^2 & \cdots & k_{3n}-m_{3n}\omega^2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_1 & k_{n2}-m_{n2}\omega^2 & k_{n3}-m_{n3}\omega^2 & \cdots & k_{nn}-m_{nn}\omega^2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \tilde{k}-\tilde{m}\omega^2 \end{bmatrix} \qquad [8.67]$$

where $D(\omega) = \det\{-\omega^2\mathbf{M} - \mathbf{K}\}$. The determinant of the numerator of Eq. (8.67) is evaluated by expanding by the last row. The result, in the notation of App. C, is

$$X_1 = \frac{(\tilde{k} - \tilde{m}\omega^2)C_{n+1,\,n+1}}{D(\omega)} \qquad [8.68]$$

If the absorber is tuned such that its frequency

$$\tilde{\omega} = \sqrt{\frac{\tilde{k}}{\tilde{m}}} \tag{8.69}$$

is equal to the excitation frequency, then $X_1 = 0$, from Eq. (8.68).

The preceding result is summarized as follows: Addition of a vibration absorber tuned to the excitation frequency of a multi-degree-of-freedom system leads to no steady-state motion of the system particle to which it is attached. That particle becomes a node. The steady-state amplitude is nonzero for all other generalized coordinates.

The vibration absorber attached to a multi-degree-of-freedom system works by shifting the natural frequencies of the system away from the excitation frequency. The natural frequency near the excitation frequency is decreased and a natural frequency is added between the excitation frequency and the next higher natural frequency. The higher and lower natural frequencies are only slightly altered by the addition of a vibration absorber.

Steady-state motion of the particle to which the absorber is attached ceases when the excitation frequency coincides with the absorber frequency, but steady-state motion exists for all particles which are not rigidly connected to this particle. However, since the excitation frequency is away from all natural frequencies when the absorber is added, the steady-state amplitudes corresponding to all other generalized coordinates are significantly reduced from when the absorber is not added.

The addition of damping to the absorber leads to a much more complicated analysis. The steady-state amplitude cannot be eliminated for any particle when damping is present. An optimum tuning frequency and optimum damping ratio can be determined for a specific system.

Example

The slender rod of Fig. 8.28 is subject to a harmonic excitation force at 75 rad/s. Design a dynamic vibration absorber of mass 3 kg such that steady-state motion of the mass center of the bar ceases when the absorber is added. What is the corresponding steady-state amplitude of the point where the force is applied? Draw frequency response curves for the steady-state amplitude of the mass center and the steady-state amplitude of angular oscillations with and without the absorber.

Solution:

Let x_1, the displacement of the mass center, and θ, the clockwise angular rotation of the rod, be chosen as generalized coordinates to describe the motion of the original system. The governing differential equations are

$$\begin{bmatrix} m & 0 \\ 0 & m\dfrac{L^2}{12} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -k\dfrac{L}{4} \\ -k\dfrac{L}{4} & \dfrac{5kL^2}{16} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} F_0 \\ F_0\dfrac{L}{2} \end{bmatrix} \sin \omega t$$

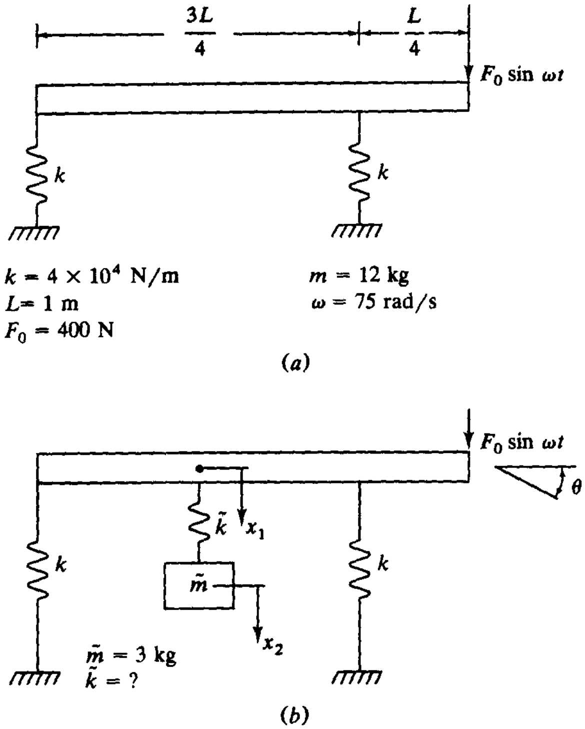


Figure 8.28 System of Example 8.12: (*a*) without absorber; (*b*) with absorber attached to mass center.

For the values given in Fig. 8.28, the natural frequencies are calculated as

$$\omega_1 = 74.02 \text{ rad/s} \qquad \omega_2 = 117.0 \text{ rad/s}$$

An absorber of mass 3 kg is attached to the center of mass of the bar. If the absorber is tuned to 75 rad/s, steady-state vibrations of the center of mass are eliminated, and the bar rotates about its mass center. This requires

$$\tilde{k} = (3 \text{ kg})(75 \text{ rad/s})^2 = 1.6 \times 10^4 \text{ N/m}$$

Let x_2 be the absorber displacement. When the absorber is added to the system, the differential equations become

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \tilde{m} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k+\tilde{k} & -k\frac{L}{4} & -\tilde{k} \\ -k\frac{L}{4} & 5k\frac{L^2}{16} & 0 \\ -\tilde{k} & 0 & \tilde{k} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ F_0\frac{L}{2} \\ 0 \end{bmatrix} \sin\omega t$$

The natural frequencies are obtained by solving $D(\omega) = 0$ or

$$(2k + \tilde{k} - m\omega^2)\left(5k\frac{L^2}{16} - m\frac{L^2}{12}\omega^2\right)(\tilde{k} - \tilde{m}\omega^2)$$
$$- \tilde{k}^2\left(5k\frac{L^2}{16} - m\frac{L^2}{12}\omega^2\right) - \left(\frac{kL}{4}\right)^2(\tilde{k} - \tilde{m}\omega^2) = 0$$

Substitution of the given and previously calculated values leads to

$$\omega_1 = 58.98 \text{ rad/s} \qquad \omega_2 = 92.78 \text{ rad/s} \qquad \omega_3 = 118.86 \text{ rad/s}$$

and
$$D(\omega) = -m\left(m\frac{L^2}{12}\right)\tilde{m}(\omega^2 - 58.98^2)(\omega^2 - 92.78^2)(\omega^2 - 118.86^2)$$

The steady-state amplitudes for an arbitrary value of ω are obtained by the method of Sec. 7.2 and Cramer's rule as

$$X_1 = \frac{F_0}{D(\omega)}(\tilde{k} - \tilde{m}\omega^2)\left(7k\frac{L^2}{16} - m\frac{L^2}{12}\omega^2\right)$$

and
$$\Theta = \frac{F_0 L}{2D(\omega)}\left[\tilde{k}^2 + \left(\tilde{k} - \tilde{m}\omega^2\right)\left(\frac{5}{2}k + \tilde{k} - m\omega^2\right)\right]$$

The steady-state amplitude of the particle where the force is applied is $\Theta L/2$ when the excitation frequency equals with return frequency. Substituting the given values for the absorber frequency and the excitation frequency equal to 75 rad/s leads to the steady-state amplitude of A as 0.01 m.

Dimensional plots of the steady-state amplitudes as functions of the excitation frequency are shown in Figs. 8.29 and 8.30.

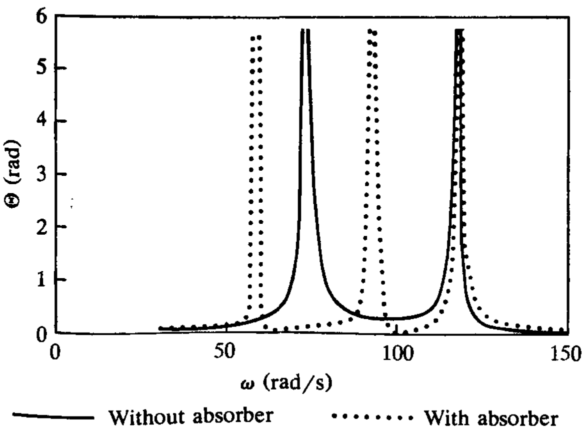


Figure 8.29 Steady-state Θ for Example 8.12.

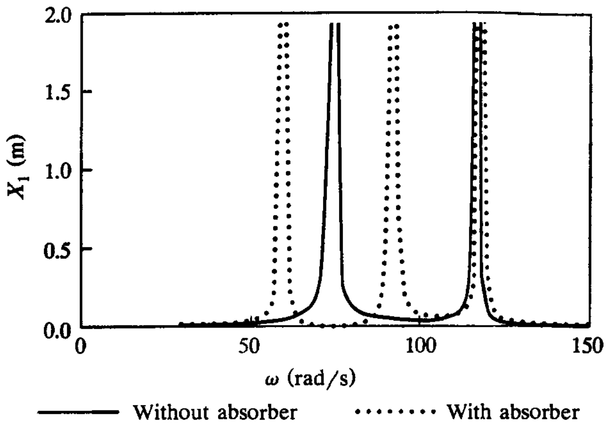


Figure 8.30 Steady-state X_1 for Example 8.12.

8.7 VIBRATION DAMPERS

A *vibration damper* is an auxiliary system composed of an inertia element and a viscous damper that is connected to a primary system as a means of vibration control. Vibration dampers are used in situations where vibration control is required over a range of frequencies.

The Houdaille damper of Fig. 8.3 is an example of a vibration damper that is used for vibration control of rotating devices such as engine crankshafts. The damper is inside a casing that is attached to the end of the shaft. The casing contains a viscous fluid and a mass that is free to rotate in the casing. The differential equations governing the motion of the two-degree-of-freedom torsional system are

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_0 \sin \omega t \\ 0 \end{bmatrix} \tag{8.70}$$

The steady-state amplitude of the primary system is obtained by the methods of Chap. 7 as

$$\Theta_1 = \frac{M_0}{k}\sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} \tag{8.71}$$

where

$$r = \frac{\omega}{\sqrt{\dfrac{k}{J_1}}} \qquad \zeta = \frac{c}{2J_2\sqrt{\dfrac{k}{J_1}}} \qquad \mu = \frac{J_2}{J_1} \tag{8.72}$$

The optimum damping ratio is defined as the damping ratio for which the maximum value of Θ_1 is smallest. The peak amplitude, $\Theta_{1p}(\zeta)$, is the value of $\Theta_1(r_m)$ where r_m is the value of r that yields $d\Theta_1/dr = 0$. The optimum damping ratio is the value of ζ such that $d\Theta_{1p}/d\zeta = 0$. Extensive algebra leads to

$$\zeta_{\text{opt}} = \frac{1}{\sqrt{2(\mu+1)(\mu+2)}} \tag{8.73}$$

If the optimum damping ratio is used in the design of a Houdaille damper then

$$r_m = \sqrt{\frac{2}{2+\mu}} \tag{8.74}$$

and

$$\Theta_{1p} = \frac{M_0}{k}\frac{2+\mu}{\mu} \tag{8.75}$$

PROBLEMS

8.1. What is the minimum static deflection of an undamped isolator that provides 75 percen isolation to a 200-kg washing machine operating at 500 rpm?

8.2. What is the maximum allowable stiffness of an isolator of damping ratio 0.05 that provide 81 percent isolation to a 40-kg printing press operating at 850 rpm?

8.3. When set on a rigid foundation and operating at 800 rpm, a machine tool provides harmonic force of magnitude 18,000 N to the foundation. An engineer has determined th the maximum magnitude of a harmonic force to which the foundation should be subjecte at this frequency is 2600 N. What is the maximum stiffness of an undamped isolator th provides sufficient isolation between the tool and the foundation?

8.4. Repeat Prob. 8.3 if the isolator is to be designed with a damping ratio of 0.11.

8.5. A 150-kg engine operates at 1500 rpm. What percent isolation is achieved if the engine mounted on four identical springs each of stiffness 1.2×10^5 N/m?

8.6. Repeat Prob. 8.5 if the springs are in parallel with a viscous damper of damping coeffici 1000 N · s/m.

8.7. A 150-kg engine operates at speeds between 1000 and 2000 rpm. It is desired to achieve least 85 percent isolation at all speeds. The only readily available isolator has a stiffness 5×10^5 N/m. How much mass must be added to the engine to achieve the desired isolati

8.8. Cork pads of stiffness 6×10^5 N/m and damping ratio 0.2 are to be used to isolate a 40 machine tool from its foundation. The machine tool operates at 1400 rpm and produ a harmonic force of magnitude 80,000 N. If the pads are placed in series, how many required such that the magnitude of the transmitted force is less than 10,000 N?

8.9. A 100-kg machine operates at 1400 rpm and produces a harmonic force of magnit 80,000 N. The magnitude of the force transmitted to the foundation is to be reduce 20,000 N by mounting the machine on four undamped isolators in parallel. What is maximum stiffness of each isolator?

8.10. A 10-kg laser flow-measuring device is used on a table in a laboratory. Because of oper of other equipment, the table is subject to external vibrations. Accelerometer measuren show that the dominant component of the table vibration is at 300 Hz and has an ampl of 4.3 m/s^2. For effective operation the laser can be subject to an acceleration amplitude of only 0.7 m/s^2. Design an undamped isolator to reduce the transmitted acceleration.

8.11. Repeat Prob. 8.10 if the isolator has a damping ratio of 0.04.

8.12. Rough seas cause a ship to heave with an amplitude of 0.4 m at a frequency of 20 rad/s. Design an isolation system with a damping ratio of 0.13 such that a 45 kg navigational computer is subject to an acceleration of only 20 m/s^2.

8.13. A sensitive computer is being transported by rail in a boxcar. Accelerometer measurements indicate that when the train is traveling at its normal speed of 85 m/s the dominant component of the boxcar's vertical acceleration is 8.5 m/s^2 at a frequency of 36 rad/s. The crate in which the computer is being transported is tied to the floor of the boxcar. What is the required stiffness of an isolator with a damping ratio of 0.05 such that the acceleration amplitude of the 60-kg computer is less than 0.5 m/s^2? With this isolator, what is the displacement of the computer relative to the crate?

8.14. A 200-kg engine operates at 1200 rpm. Design an isolator such that the transmissibility ratio during start-up is less than 4.6 and the system achieves 80 percent isolation.

8.15. A 150-kg machine tool operates at speeds between 500 and 1500 rpm. At each speed a harmonic force of magnitude 15,000 N is produced. Design an isolator such that the maximum transmitted force during start-up is 60,000 N and the maximum transmitted steady-state force is 2000 N.

8.16. A 200-kg testing machine operates at 500 rpm and produces a harmonic force of magnitude 40,000 N. An isolation system for the machine consists of a damped isolator and a concrete block for mounting the machine. Design the isolation system such that all of the following are met:

(*a*) The maximum transmitted force during start-up is 100,000 N.

(*b*) The maximum transmitted force in the steady-state is 5000 N.

(*c*) The maximum steady-state amplitude of the machine is 2 cm.

8.17. A 150-kg washing machine has a rotating unbalance of 0.45 kg · m. The machine is placed on isolators of equivalent stiffness 4×10^5 N/m and damping ratio 0.08. Over what range of operating speeds will the transmitted force between the washing machine and the floor be less than 3000 N?

8.18. A 54-kg air compressor operates at speeds between 800 and 2000 rpm and has a rotating unbalance of 0.23 kg · m. Design an isolator with a damping ratio of 0.15 such that the transmitted force is less than 1000 N at all operating speeds.

8.19. A 1000-kg turbomachine has a rotating unbalance of 0.1 kg · m. The machine operates at speeds between 500 and 750 rpm. What is the maximum isolator stiffness of an undamped isolator that can be used to reduce the transmitted force to 300 N at all operating speeds?

8.20. A motorcycle travels over a road whose contour is approximately sinusoidal, $y(z) = 0.2\sin(0.4z)$ m where z is measured in meters. Using a simple one-degree-of-freedom model, design a suspension system with a damping ratio of 0.1 such that the acceleration felt by the rider is less than 15 m/s^2 at all horizontal speeds between 30 and 80 m/s. The mass of the motorcycle and the rider is 225 kg.

8.21. A suspension system is being designed for a 1000-kg vehicle. A first model of the system used in the design process is a spring of stiffness k in parallel with a viscous damper of damping coefficient c. The model is being analyzed as the vehicle traverses a road with a sinusoidal contour, $y(z) = Y\sin(2\pi z/d)$ when the vehicle has a constant horizontal speed v. The suspension system is to be designed such that the maximum acceleration of the passengers is 2.5 m/s^2 for all vehicle speeds less than 60 m/s for all reasonable road contours. It is estimated that for such contours, $Y < 0.01$ m and $0.2 \text{ m} < d < 1$ m. Specify k and c for such a design.

8.22. Rework Example 8.5 when the total period of the punching operation is 1.2 s and a force of 4000 N is applied over 0.3 s.

8.23. When a machine tool is placed directly on a rigid floor, it provides an excitation of the form

$$F(t) = (4000 \sin 100t + 5100 \sin 150t) \text{ N}$$

to the floor. Determine the natural frequency of the system with an undamped isolator with the minimum possible static deflection such that when the machine is mounted on the isolator the amplitude of the force transmitted to the floor is less than 3500 N.

8.24. Use the force shown in Fig. P8.24 as an approximation to the force provided by the punch press during its operation and rework Example 8.5.

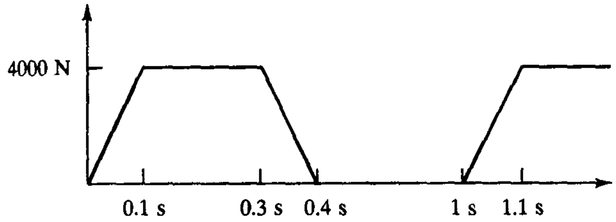


FIGURE P8.24

8.25. Solve Prob. 8.10 if the accelerometer measurements indicate that the three most dominant components of the table vibration are at 300 Hz, 500 Hz, and 750 Hz with amplitudes of 4.3, 3.8, and 6.9 m/s^2 respectively.

8.26. A 120-kg tumbler is to be mounted using four point balanced mounting using the isolator of Fig. 8.10. It has been decided to use isolators of load rating B mounted in compression. Assume $\zeta = 0.04$. If the tumbler has a rotating unbalance of 0.25 kg · m during operation, what is the force transmitted to the foundation when the tumbler operates at (*a*) 1000 rpm, (*b*) 1200 rpm, (*c*) 1500 rpm?

8.27. An isolator is being selected for use as an engine mount. The 40-kg engine operates at speeds from 1000 to 2000 rpm. Can the isolator of Fig. 8.10 be used with balanced four-point mounting to achieve 81 percent isolation at all operating speeds?

8.28. Rework Example 8.8 if the table's acceleration amplitude is 8 m/s^2 at a frequency of 100 rad/s.

8.29. Rework Example 8.8 if the table's acceleration has two dominant frequencies of 100 rad/s and 150 rad/s. The acceleration amplitudes at these frequencies are 10 and 12 m/s^2 respectively.

8.30. A 110-kg machine tool is placed on the floor of an industrial plant. It has been determined that beyond the static load the maximum transmitted force should be 500 N. During operation the machine is subject to a variety of excitations. For each of the excitations shown in Fig. P8.30, determine the maximum allowable stiffness of an isolator of damping ratio 0.1 such that the transmitted force is limited to 500 N.

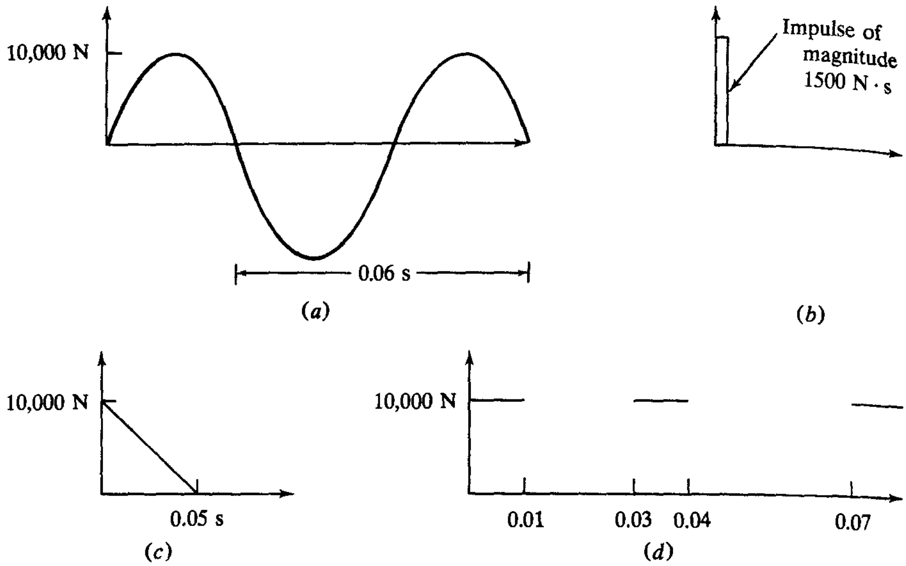


FIGURE P8.30

8.31. A 200-kg machine tool is mounted on an isolator of stiffness 5×10^5 N/m and damping ratio 0.1. What is the maximum transmitted force and maximum displacement when the machine tool is subject to each of the excitations of Fig. P8.30?

8.32. During its normal operation, a 144-kg machine tool is subject to a 15,000 N · s impulse. Design an efficient isolator such that the maximum force transmitted through the isolator is 2500 N and the maximum displacement is minimized.

8.33. A 110-kg pump is mounted on an isolator of stiffness 4×10^5 N/m and a damping ratio 0.15. The pump is given a sudden velocity of 30 m/s. What is the maximum force transmitted through the isolator and what is the maximum displacement of the pump?

8.34. During operation, a 50-kg machine tool is subject to the short-duration pulse of Fig. P8.34. Design an isolator that minimizes the maximum displacement and reduces the maximum transmitted force to 5000 N. What is the maximum displacement of the machine tool when this isolator is used?

8.35. Repeat Prob. 8.34 for the short-duration pulse of Fig. P8.35.

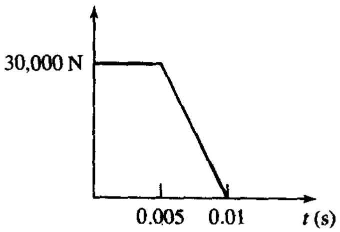


FIGURE P8.34

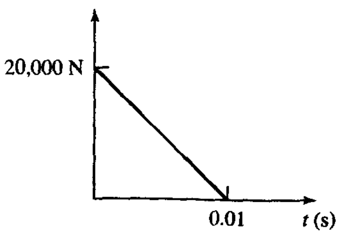


FIGURE P8.35

8.36. A ship is moored at a dock in rough seas and frequently impacts the dock. The maximum velocity change caused by the impact is 15 m/s. Design an isolator to protect a sensitive 80-kg navigational control system such that its maximum acceleration is 30 m/s^2.

8.37. A one-story frame structure of equivalent mass 12,000 kg and stiffness 1.8×10^6 N/m is subject to a blast whose force is given in Fig. P8.37. What is the maximum deflection of the structure?

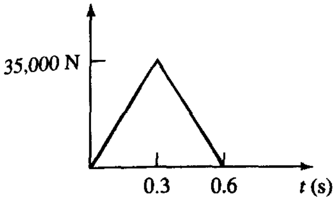


FIGURE P8.37

8.38. A 20-kg machine tool is on a foundation that is subject to an acceleration that is modeled as a versed sine pulse of magnitude 20 m/s^2 and duration 0.4 s. Design an undamped isolator such that the maximum acceleration felt by the machine is 15 m/s^2. What is the maximum displacement of the machine tool relative to its foundation when this isolator is used?

8.39. During operation, a 100-kg machine tool is exposed to a force that is modeled as a sinusoidal pulse of magnitude 3100 N and duration 0.05 s. Design an isolator with a damping ratio 0.1 such that the maximum force transmitted through the isolator is 2000 N and the maximum displacement of the machine tool is 3 cm.

8.40. During operation a 80-kg machine tool is subject to a triangular pulse of magnitude 30,000 N and duration 0.15 s. What is the range of undamped isolator stiffness such that the maximum transmitted force is 15,000 N and the maximum displacement is 5 cm?

8.41. A 50-kg lathe machine mounted on an elastic foundation of stiffness 4×10^5 N/m has a vibration amplitude of 35 cm when the motor speed is 95 rad/s. Design an undamped dynamic vibration absorber such that steady-state vibrations are completely eliminated at 95 rad/s and the maximum displacement of the absorber mass at this speed is 5 cm.

8.42. What is the lowest natural frequency of the lathe of Prob. 8.41 with the absorber in place? What is the operating range around 95 rad/s such that the steady-state amplitude of the lathe is less than 1 cm?

8.43. What is the required stiffness of an undamped dynamic vibration absorber whose mass is 5 kg to eliminate vibrations of a 25-kg machine of natural frequency 125 rad/s when the machine operates at 110 rad/s?

8.44. A 35-kg machine is attached to the end of a cantilever beam of length 2 m, elastic modulus 210×10^9 N/m^2, and moment of inertia 1.3×10^{-7} m^4. The machine operates at 180 rpm and has a rotating unbalance of 0.3 kg · m.

(*a*) What is the required stiffness of an undamped absorber of mass 5 kg such that steady-state vibrations are eliminated at 180 rpm?

(*b*) With the absorber in place what are the natural frequencies of the system?

(*c*) For what range of operating speeds will the steady-state amplitude of the machine be less than 8 mm?

8.45. A 150-kg pump experiences large-amplitude vibrations when operating at 1500 rpm. Assuming this is the natural frequency of a one-degree-of-freedom system, design a dynamic vibration absorber such that the lower natural frequency of the two-degree-of-freedom system is less than 1300 rpm and the higher natural frequency is greater than 1700 rpm.

8.46. A solid disk of diameter 30 cm and mass 10 kg is attached to the end of a solid 3-cm-diameter, 1-m-long steel shaft ($G = 80 \times 10^9$ N/m^2). A torsional vibration absorber consists of a disk attached to a shaft that is then attached to the primary system. If the absorber disk has a mass of 3 kg and a diameter of 10 cm, what is the required diameter of a 50-cm-long absorber shaft to eliminate steady-state vibrations of the original system when excited at 500 rad/s?

8.47. A 200-kg machine is placed on a massless simply supported beam as shown in Fig. P8.47. The machine has a rotating unbalance of 1.41 kg · m and operates at 3000 rpm. The steady-state vibrations of the machine are to be absorbed by hanging a mass attached to a 40-cm steel cable from the location on the beam where the machine is attached. What is the required diameter of the cable such that machine vibrations are eliminated at 3000 rpm and the amplitude of the absorber mass is less than 50 mm?

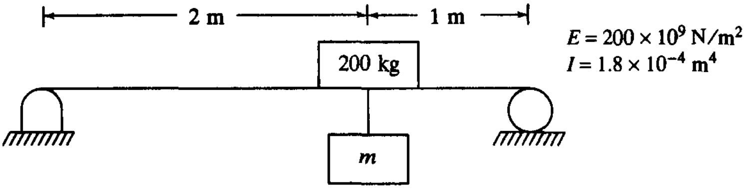


FIGURE P8.47

8.48. The disk in Fig. P8.48 rolls without slip and the pulley is massless. What is the mass of the block that should be hung from the cable such that steady-state vibrations of the cylinder are eliminated when $\omega = 120$ rad/s?

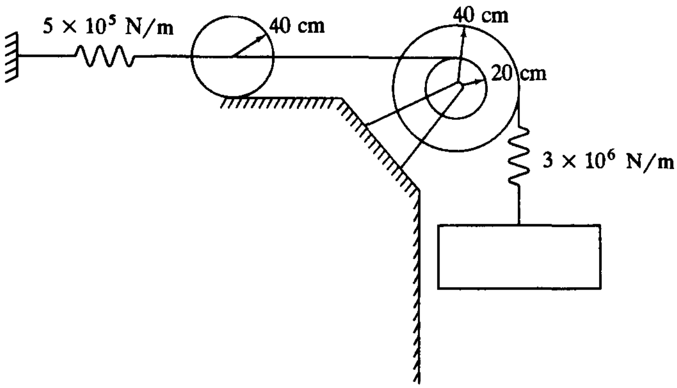


FIGURE P8.48

8.49. Vibration absorbers are used in boxcars to protect sensitive cargo from large accelerations due to periodic excitations provided by rail joints. For a particular railway, joints are spaced 5 m apart. The boxcar, when empty, has a mass of 25,000 kg. Two absorbers, each of mass 12,000 kg, are used. Absorbers for a particular boxcar are designed to eliminate vibrations of the main mass when the boxcar is loaded with a 12,000-kg cargo and travels at 100 m/s. The natural frequency of the unloaded boxcar is 165 rad/s.

(a) At what speeds will resonance occur for the boxcar with a 12,000-kg cargo?

(b) What is the best speed for the boxcar when it is loaded with a 25,000-kg cargo?

.50. A 500-kg reciprocating machine is mounted on a foundation of equivalent stiffness 5×10^6 N/m. When operating at 800 rpm, the machine produces an unbalanced harmonic force of magnitude 50,000 N. Two cantilever beams with end masses are added to the machine to act as absorbers. The beams are made of steel ($E = 210 \times 10^9$ N/m^2) and have a moment of inertia of 4×10^{-6} m^4. A 10-kg mass is attached to each beam. The absorbers are adjustable in that the location of the mass on the absorber can be varied.

(a) How far away from the support should the masses be located when the machine is operating at 800 rpm? What is the amplitude of the absorber mass?

(b) If the machine operates at 1000 rpm and produces a harmonic force of amplitude 100,000 N, where should the absorber masses be placed and what is their vibration amplitude?

8.51. A 100-kg machine is placed at the midspan of a 2-m-long cantilever beam ($E = 210 \times 10^9$ N/m^2, $I = 2.3 \times 10^{-6}$ m^4). The machine produces a harmonic force of amplitude 60,000 N. Design a damped vibration absorber of mass 30 kg such that when hung from the beam at midspan, the steady-state amplitude of the machine is less than 8 mm at all speeds between 1300 and 2000 rpm.

8.52. Repeat Prob. 8.51 if the excitation is due to a rotating unbalance of magnitude 0.33 kg · m.

8.53. For the absorber designed in Prob. 8.51 what is the minimum steady-state amplitude of the machine and at what speed does it occur?

8.54. Determine values of k and c such that the steady-state amplitude of the center of the cylinder in Fig. P8.54 is less than 4 mm for 60 rad/s $< \omega <$ 110 rad/s?

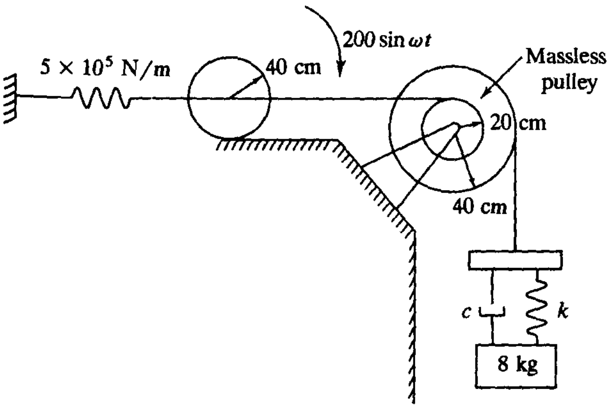


FIGURE P8.54

8.55. Use the Laplace transform method to analyze the situation of an undamped absorber attached to a viscously damped system, as shown in Fig. P8.55.

(a) Determine the steady-state amplitude of the mass m_1.

(b) Use the results of part (a) to design an absorber for a 123-kg machine of natural frequency 87 rad/s and damping ratio 0.13. Use an absorber mass of 35 kg.

8.56. What is the steady-state amplitude of the mass center of the bar of Example 8.12 if an adsorber is added to eliminate steady-state vibrations of the right end of the bar?

8.57. Design an undamped absorber such that steady-state the motion of the 25-kg machine component in Fig. P8.57 ceases when the absorber is added. What is the steady-state amplitude of the 31 kg component?

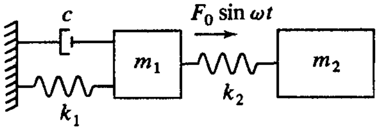


FIGURE P8.55

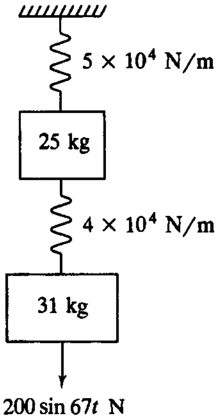


FIGURE P8.57

8.58. A 300-kg compressor is placed at the end of a cantilever beam of length 1.8 m, elastic modulus 200×10^9 N/m^2, and moment of inertia 1.8×10^{-5} m^4. When the compressor operates at 1000 rpm, it has a steady-state amplitude of 1.2 mm. What is the compressor's steady-state amplitude when a 30-kg absorber of damping coefficient 500 N · s/m and stiffness 1.3×10^5 N/m is added to the end of the beam?

8.59. An engine has a moment of inertia of 7.5 kg · m^2 and a natural frequency of 125 Hz. Design a Houdaille damper such that the engine's maximum magnification factor is 4.8. During operation the engine is subject to a harmonic torque of magnitude 150 N · m at a frequency of 120 Hz. What is the engine's steady-state amplitude when the absorber is used?

8.60. A 200-kg machine is subjected to an excitation of magnitude 1500 N. The machine is mounted on a foundation of stiffness 2.8×10^6 N/m. What are the mass and damping coefficient of an optimally designed vibration damper such that the maximum amplitude is 3 mm?

MATLAB PROBLEMS

M8.1. The MATLAB script file VIBES_8A.m is used to aid in the design of a vibration isolator for isolation from harmonic excitation. Given the required isolation, operating speed, mass, and damping ratio, VIBES_8A determines the maximum allowable isolator stiffness. Use VIBES_8A to solve (*a*) Prob. 8.2, (*b*) Prob. 8.4, (*c*) Prob. 8.9, (*d*) Prob. 8.10.

M8.2. The MATLAB file VIBES_8B.m is used to aid in the design of a vibration isolator for isolation from frequency-squared excitation over a range of frequencies. Given the required isolation, damping ratio, mass, and range of operating speed, VIBES_8B determines the stiffness of an isolator that provides adequate isolation and plots the frequency response curve using the designed isolator. Use VIBES_8B to solve (*a*) Prob. 8.17, (*b*) Prob. 8.19, (*c*) Prob. 8.20.

M8.3–M8.5. The MATLAB file VIBES_8C.m is used to aid in the design of a vibration isolator for isolation from a periodic excitation that is represented as a Fourier series. The user provides the excitation in a function file. VIBES_8C symbolically evaluates the integrals for the Fourier coefficients. Given the required isolation, the operating speed, mass, and damping ratio, VIBES_8C then determines the required isolator stiffness. Use VIBES_8C to design an isolator for a 200-kg press that is subject to the periodic excitations of Figs. PM8.3 to PM8.5. It is desired to design an isolator of damping ratio 0.2 that limits the transmitted force to 1000 N.

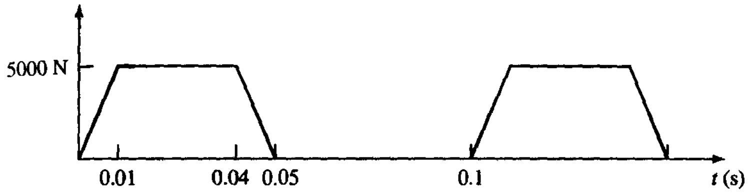


FIGURE PM8.3

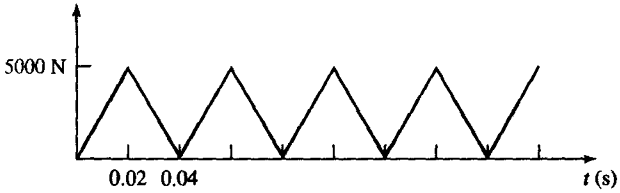


FIGURE PM8.4

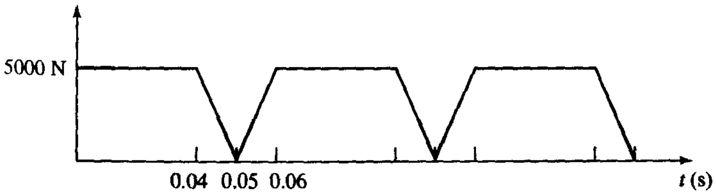


FIGURE PM8.5

M8.6. The MATLAB file VIBES_8D.m provides the values of $Q(\zeta)$ and $S(\zeta)$ given the value of ζ. Use VIBES_8D to determine the maximum transmitted force and maximum displacement for a 150-kg hammer mounted on an isolator of stiffness 1.3×10^6 N/m when subject to an impulsive load of total impulse 250 N·s.

M8.7–M8.9. The MATLAB file VIBES_8E.m determines the response and force spectra of a system of damping ratio ζ due to an arbitrary form of excitation. Use VIBES_8E to determine the response and force spectra for the excitation shown. Use these spectra to solve Prob. 8.39, assuming the excitation shown in Figs. PM8.7 to PM8.9 instead of the sinusoidal pulse. (Use $F_0 = 3100$ N and $t_0 = 0.05$ s.)

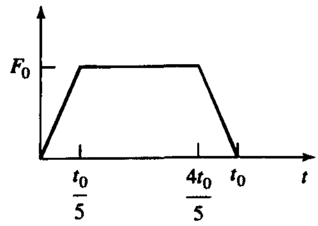


FIGURE PM8.7

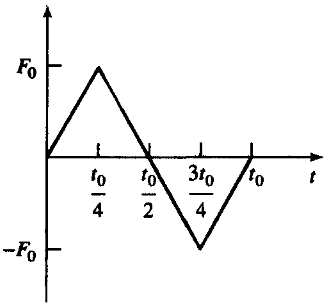


FIGURE PM8.8

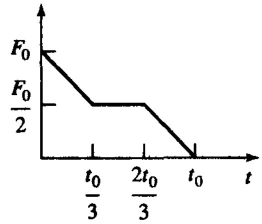


FIGURE PM8.9

M8.10. The MATLAB file VIBES_8F.m is used in the design of a vibration absorber to be added to a one-degree-of-freedom system. The program also plots the frequency response for an absorber design. Use VIBES_8F to solve (*a*) Prob. 8.41 and 8.42, (*b*) Prob. 8.43, (*c*) Prob. 8.44.

M8.11. Write a MATLAB script file that designs a vibration isolator for isolation from a harmonic excitation that meets the following criteria:

(*a*) The maximum transmitted force during start-up is FS_{max}

(*b*) The maximum transmitted force at steady state is F_{max}

Additionally, given the machine mass, excitation amplitude, and excitation frequency, the program should specify the isolator's stiffness and damping ratio. Use the file to design an isolator for the following situation. During operation at 1200 rpm, a 250-kg compressor is subjected to a harmonic force of magnitude 10,000 N. It is desired to limit the transmitted force during start-up to 25,000 N and during steady-state operation to 2500 N.

M8.12. Write a MATLAB script file similar to that of Prob. M8.11 except that the harmonic excitation is produced by a rotating unbalance. Use the file to design an isolator for the system of Prob. M8.11 if the compressor has a rotating unbalance of 0.85 kg · m and operates at speeds between 1200 and 1800 rpm.

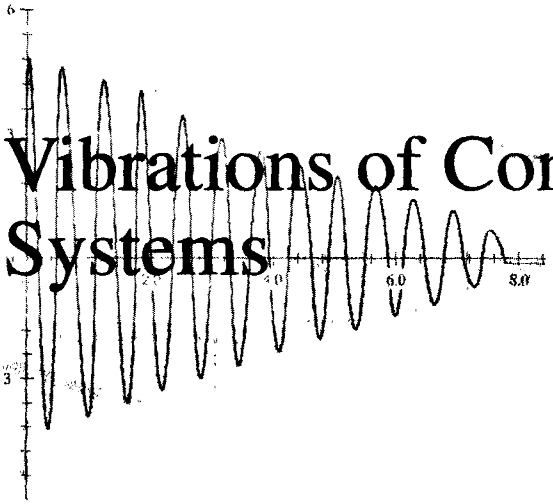
M8.13. Write a MATLAB script file for the design of an optimal damped vibration absorber. The program should calculate the parameters for the design of the absorber, given all relevant information. The program should also provide the frequency response of both the primary system and the auxiliary system when the absorber is in place. Use the file to solve Prob. 8.51–8.53.

M8.14. Write a MATLAB script file that is used in the design of an optimal Houdaille damper. Use the file to solve Prob. 8.59.

chapter

9

Vibrations of Continu
Systems



9.1 Introduction

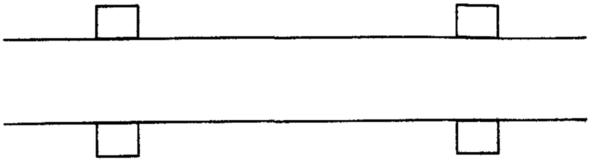
All solid objects are made of deformable materials. Often a solid is assumed to be rigid. This allows for simpler modeling and leads to information about essential vibrational characteristics. The validity of a rigid-body assumption in modeling the vibrations of a system depends on many factors such as geometry and frequency range. For example, consider a machine mounted on springs and operating in an industrial plant. The floor of the industrial plant is often assumed to be rigid and the vibrations of the machine considered by analyzing a one-degree-of-freedom system. However, if the forces developed in the springs are large, then since the floor is really deformable, vibrations are excited in the floor and perhaps the entire structure. In this case the vibrations of the machine are coupled to the structural vibrations.

Examples of continuous systems are shown in Fig. 9.1. All structural elements such as beams, columns, and plates are continuous systems. This includes the suspended piping system of Fig. 9.1*a*, simply supported at locations along its length. Vibrations of the pipeline are excited by the fluid flowing through the pipe, the operation of pumps, or structural vibrations. The vibrations are analyzed by considering a continuous beam with simple supports.

All elements of the frame structure of Fig. 9.1*b* are continuous structural elements. Often the columns of a frame structure are much more flexible than the girders, and the girders are considered rigid, resulting in the model shown.

The spring of Fig. 9.1*c* is a simple continuous system. As one end of the spring is moved relative to the other, a compression wave is generated and travels throughout the spring. If the excitation frequency is near the frequency of the compression waves, a phenomenon called *surge* develops. Surge can be a problem in mechanical systems where one end of a spring is given a harmonic displacement.

The free and forced vibrations of a rigid body attached to a continuous system are approximated by using one degree of freedom in Chaps. 1 through 4. The inertia



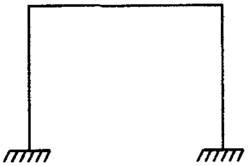


Figure 9.1 Examples of continuous systems: (*a*) simply supported piping system; (*b*) one-story frame structure; (*c*) helical coil spring.

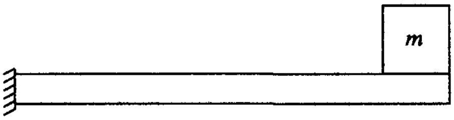


Figure 9.2 Discrete approximation works well when m is large compared to the mass of the beam.

effects of a continuous element are approximated by adding a particle of a calculated equivalent mass at the location of the rigid body. Multi-degree-of-freedom approximations are considered in Chaps. 5 through 7.

The ordinary differential equations obtained by using a discrete model of the continuous system are easier to solve than the governing partial differential equation. Thus discrete approximations are often used, but have limitations. A continuous system has an infinite, but countable, number of natural frequencies and corresponding mode shapes. A discrete approximation predicts only a finite number of modes. Often a large number of degrees of freedom are needed to attain accurate approximations for higher natural frequencies. Consider, for example, the cantilever beam of Fig. 9.2 with a concentrated mass at its end. Figure 9.3 shows the nondimensional lowest natural frequency as a function of β, the ratio of the concentrated mass to the mass of the beam. Figure 9.3 shows natural frequencies calculated using up to six degrees of freedom, including a one-degree-of-freedom approximation.

The methods used in this chapter are analogous to those used for multi-degree-of-freedom systems. The separation-of-variables method used to determine the natural

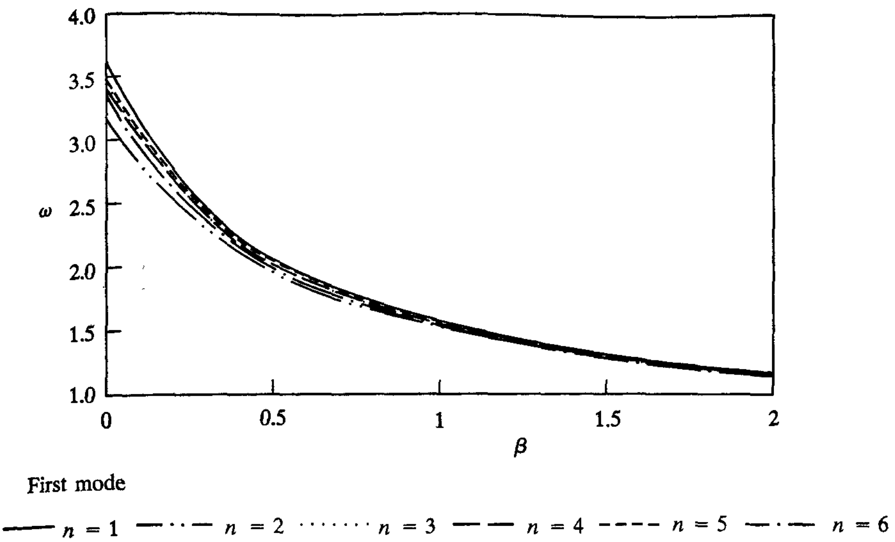


Figure 9.3 As the ratio of the concentrated mass to the mass of the beam grows larger, the approximation for the lowest natural frequency using a discrete model with n degrees of freedom improves.

frequencies is analogous to the normal-mode solution of Eq. (6.2). The method used for the analysis of forced vibrations is a direct result of an expansion theorem and is directly analogous to modal analysis. The approximate methods presented are based on energy methods. Indeed, similar notation using energy scalar products can be used. The continuous functions used in the analysis of continuous systems are analogous to the column vectors of generalized coordinates used for discrete systems. Energy scalar products are defined for continuous systems using definite integrals.

9.2 General Method

This section presents an outline of an exact closed-form method for analyzing vibrations of continuous systems. The method is applied to analyze the torsional oscillations of a circular shaft and the transverse vibrations of a beam in Secs. 9.3 and 9.4, respectively. This chapter is intended only as an introduction to vibrations of continuous systems. Thus it is assumed that the dependent displacement is a function of only one spatial variable and time, all material properties are constant, and all geometries are uniform.

The analysis procedure is broken into three parts: problem formulation, free-vibration analysis, and forced-vibration analysis. The mathematical theory underlying the analysis of vibrations of continuous systems is developed by using an infinite-dimensional vector space, while the mathematical foundation for a multi-degree-of-freedom system is developed by using a finite-dimensional vector space. Many of the concepts developed for finite-dimensional spaces have direct extension to infinite-dimensional spaces.

Part I: Problem Formulation

1. An independent spatial variable is chosen, call it x. This independent spatial variable represents the displacement of a particle from a reference position when the system is in its equilibrium position. A continuous system has an infinite number of degrees of freedom and hence an infinite number of generalized coordinates are required. These are chosen as the displacement of the particles in the system. They can be summarized by a single dependent variable, $w(x, t)$.
2. Free-body diagrams of a representative differential element are drawn at an arbitrary instant. One free-body diagram shows the external forces acting on the element. Another free-body diagram shows the effective forces for that element. The external forces include forces on the surface of the element that are resultants of stress distributions on these surfaces.
3. The appropriate form of Newton's law is applied to the free-body diagrams. Appropriate kinematic conditions and constitutive equations are applied to derive a partial differential equation governing $w(x, t)$.
4. Appropriate boundary conditions, dependent on the end supports of the structural member, are formulated.
5. Appropriate initial conditions are formulated.
6. An optional step is to nondimensionalize the governing equation and boundary conditions by introducing nondimensional forms of the independent and dependent variables. This leads to the formulation of dimensionless parameters which are important in the physical understanding of the results. Assume for the remainder of this discussion that nondimensional variables are introduced and all variables referred to are nondimensional. Also assume that the nondimensional spatial variable x ranges from 0 to 1.

   The governing equations and boundary conditions can also be derived by energy methods. Kinetic and potential energy scalar products directly analogous to those formed for multi-degree-of-freedom systems can be defined.

Part II: Free-Vibration Solution A free-vibration problem is one where $w(x, 0)$ or $\partial w/\partial t(x, 0)$ are nonzero and the partial differential equation and all boundary conditions are homogeneous. The initial potential or kinetic energy drives the vibrations, during which no external forces are applied.

As for multi-degree-of-freedom systems, the free-vibration problem is considered to determine the system's natural frequencies and mode shapes. The method presented to solve free vibrations problems for continuous systems is called *separation of variables*. Application of this method requires that the partial differential equation be of an appropriate form, called *separable*. The governing partial differential equations for torsional vibrations of a uniform shaft, longitudinal vibrations of a uniform elastic bar, and transverse vibrations of a uniform beam are all separable.

1. The dependent variable is assumed to be a product of functions of the independent variables,

$$w(x, t) = X(x)T(t) \tag{9.1}$$

   Equation (9.1) is substituted into the governing partial differential equation. If the governing partial differential equation is separable then the resulting equation can be written in the form of $[L_x X(x)]/X(x) = [L_t T(t)]/T(t)$ where L_x and L_t are linear ordinary differential operators. Note that the left-hand side of this equation is a function of x only and the right-hand side is a function of t only. Since x and t are independent, this can only be true if both sides are equal to the same constant, call it $-\lambda$. The above argument is called the *separation argument.* Its application leads to ordinary differential equations for $X(x)$ and $T(t)$, both in terms of λ, called the *separation constant.*
2. Equation (9.1) is applied to the boundary conditions to obtain homogeneous boundary conditions for $X(x)$.
3. If the system is undamped, the solution for $T(t)$ is harmonic. It becomes obvious that the natural frequencies are related to the separation constant and the mode shapes are related to $X(x)$.
4. The problem for $X(x)$ is a homogeneous ordinary differential equation with homogeneous boundary conditions. This is called a *differential eigenvalue problem.* A nontrivial solution is available only for certain values of the separation constant. Standard solution techniques for ordinary differential equations are applied to determine $X(x)$ in terms of arbitrary constants of integration.
5. Application of the boundary conditions leads to a solvability condition of the form $f(\lambda) = 0$. Nontrivial solutions of the eigenvalue problem exist only for values of λ such that $f(\lambda) = 0$. This results in an infinite, but countable, number of solutions, $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$. Corresponding to each λ_k, there is an $X_k(x)$, which is unique only to a multiplicative constant.

   If only the natural frequencies and mode shapes are necessary, the solution ends here.
6. An energy scalar product, (X_i, X_j), is defined such that (X_i, X_i) is proportional to the kinetic energy of the ith mode at any instant. It can be shown that for systems governed by the wave equation (torsional vibrations of shafts, longitudinal vibrations of bars) and for uniform beam vibrations, mode shapes for distinct modes are mutually orthogonal with respect to this energy scalar product. For a uniform continuous system, in the absence of discrete masses, the appropriate kinetic energy scalar product is

$$(X_i, X_j)_T = \int_0^1 X_i(x)X_j(x)\,dx \tag{9.2}$$

   If the system has discrete masses, additional terms are added to the integral of

Eq. (9.2) to account for the kinetic energy of the discrete masses. The mode shapes are normalized by requiring

$$(X_i, X_i)_T = 1 \tag{9.3}$$

7. If the mode shapes are normalized with respect to a scalar product for which they are also mutually orthogonal, then an expansion theorem exists which states that any continuous function, $f(x)$, can be expanded in a series of the mode shapes as

$$f(x) = \sum_{k=1}^{\infty} (f, X_k)_T X_k \tag{9.4}$$

The expansion converges to $f(x)$ at all x except perhaps at $x = 0$ and $x = 1$.

If a forced-vibration solution is required, the expansion theorem of Eq. (9.4) is noted and the solution proceeds to step 1 of the forced response. If a free-vibration solution is required, the solution continues as follows.

8. The general solution is formed by taking a linear combination over all modes.

$$w(x, t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) \tag{9.5}$$

Two arbitrary constants for each mode are present from the solution for $T_k(t)$. These constants are determined from application of initial conditions. Often the functions involved in the initial conditions must be expanded by the expansion theorem, Eq. (9.4). For example, if $w(x, 0)$ is nonzero and is equal to $f(x)$, then $f(x)$ is expanded by Eq. (9.4) and compared to $w(x, 0)$ obtained from Eq. (9.5), in terms of arbitrary constants. The linear independence of each $X_k(x)$ is used to determine the constants.

Part III: Forced-Vibration Solution As for discrete systems, there are several methods available to determine the forced response of continuous systems. As for discrete systems, these include application of the method of undetermined coefficients for harmonic excitations, the Laplace transform method, and modal analysis. Again, as for discrete systems, modal analysis is the most powerful and most often used.

A modal analysis procedure can be developed for forced vibrations of continuous systems. Let $f(x, t)$ represent the nondimensional nonhomogeneous term arising in the partial differential equation as a result of the external forcing. Nonhomogeneous terms can also occur in the boundary conditions.

1. The expansion theorem, Eq. (9.4) is used to expand $f(x, t)$ as

$$f(x, t) = \sum_{k=1}^{\infty} C_k(t) X_k(x) \tag{9.6}$$

where
$$C_k(t) = (f(x, t), X_k(x))_T \tag{9.7}$$

2. The expansion theorem is also used to expand

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) X_k(x) \tag{9.8}$$

where the $p_k(t)$ are called the *principal coordinates* for the continuous system. Equations (9.6) and (9.8) are substituted into the governing partial differential equation.

3. The scalar product of the resulting partial differential equation is taken with $X_j(x)$ for an arbitrary j. For a problem whose appropriate scalar product is given by Eq. (9.2), this is equivalent to multiplying the equation by $X_j(x)$ and integrating from 0 to 1. Application of the orthogonality condition leads to uncoupled differential equations for the principal coordinates.

4. The uncoupled differential equations are solved to determine each $p_k(t)$.

9.3 TORSIONAL OSCILLATIONS OF A CIRCULAR SHAFT

9.3.1 PROBLEM FORMULATION

The circular shaft of Fig. 9.4 is made of a material of mass density ρ and shear modulus G and has a length L, cross-sectional area A, and polar moment of inertia J. Let x be the coordinate along the axis of the shaft, measured from its left end. The shaft is subject to a time-dependent torque per unit length, $T(x,t)$. Let $\theta(x,t)$ measure the resulting torsional oscillations where θ is chosen positive clockwise.

Figure 9.5 shows free-body diagrams of a differential element of the shaft at an arbitrary instant of time. The element is of infinitesimal thickness dx and its left face is a distance x from the left end of the shaft.

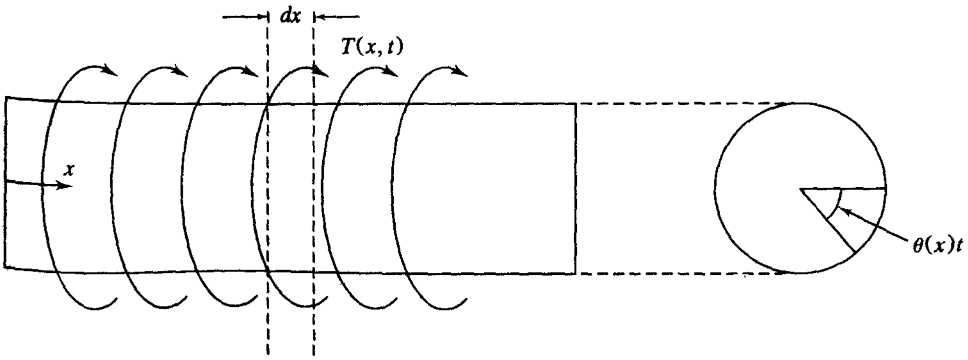


Figure 9.4 Circular shaft is subject to torsional loading $T(x, t)$; $\theta(x, t)$ measures torsional oscillations.

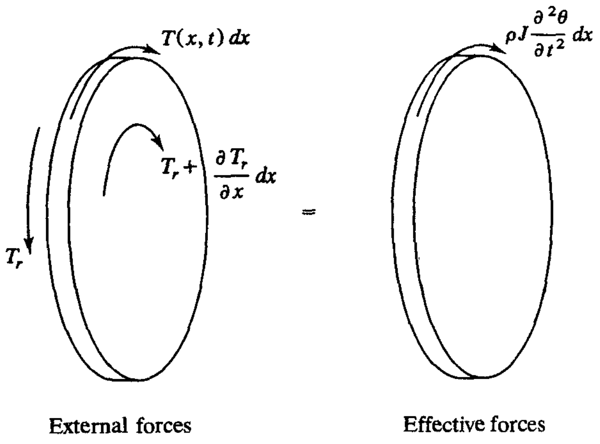


Figure 9.5 Free-body diagram of differential element of shaft at arbitrary instant.

The free-body diagram of the external forces shows the time-dependent torque loading as well as the internal resisting torques developed in the cross sections. The internal resisting torques are the resultant moments of the shear stress distributions. If $T_r(x, t)$ is the resisting torque acting on the left face of the element, then a Taylor series expansion truncated after the linear terms gives

$$T_r(x + dx, t) = T_r(x, t) + \frac{\partial T_r(x, t)}{\partial t}\,dx \qquad [9.9]$$

The directions of the torques shown on the free-body diagram are consistent with the choice of θ positive clockwise.

Since the disk is infinitesimal, the angular acceleration is assumed constant across the thickness. Thus the free-body diagram of the effective forces simply shows a moment equal to the mass moment of inertia of the disk times its angular acceleration.

Summation of moments about the mass center of the disk

$$\left(\sum M\right)_{\text{ext}} = \left(\sum M\right)_{\text{eff}}$$

gives

$$T(x, t)\,dx - T_r(x, t) + T_r(x, t) + \frac{\partial T_r(x, t)}{\partial x}\,dx = \rho J\,dx\frac{\partial^2 \theta(x, t)}{\partial t^2}$$

or

$$T(x, t) + \frac{\partial T_r(x, t)}{\partial x} = \rho J\frac{\partial^2 \theta}{\partial t^2} \qquad [9.10]$$

From mechanics of materials,

$$T_r(x, t) = JG\frac{\partial \theta(x, t)}{\partial x} \qquad [9.11]$$

which, when substituted in Eq. (9.10) for a uniform shaft, leads to

$$T(x,t) + JG\frac{\partial^2\theta}{\partial x^2} = \rho J\frac{\partial^2\theta}{\partial t^2} \qquad \textbf{[9.12]}$$

The following nondimensional variables are introduced:

$$x^* = \frac{x}{L} \qquad t^* = \sqrt{\frac{G}{\rho}}\frac{t}{L} \qquad \textbf{[9.13]}$$

and

$$T^*(x^*, t^*) = \frac{T(x,t)}{T_m} \qquad \textbf{[9.14]}$$

where T_m is the maximum value of T. Introduction of Eqs. (9.13) and (9.14) in Eq. (9.12) leads to

$$\left(\frac{L^2T_m}{JG}\right)T(x,t) + \frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \qquad \textbf{[9.15]}$$

where the $*$ has been dropped from the nondimensional variables.

Table 9.1 provides nondimensional boundary conditions for different types of shaft ends. The problem formulation is completed by specifying appropriate initial conditions of the form

$$\theta(x,0) = g_1(x) \qquad \textbf{[9.16]}$$

and

$$\frac{\partial\theta(x,0)}{\partial t} = g_2(x) \qquad \textbf{[9.17]}$$

Consider the homogeneous form of Eq. (9.15),

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \qquad \textbf{[9.18]}$$

Equation (9.18) is a hyperbolic partial differential equation, called the *wave equation*. The wave equation also governs such variables as the axial displacement during the longitudinal motion of a bar, the axial displacement of a particle on a coil spring during a compression wave, and the free vibrations of a taut string. Applications in areas other than vibrations include propagation of surface waves on the interface of two fluids and the velocity potential for supersonic flow in an ideal fluid.

Solutions of the wave equation are rich in physical phenomena. It can be shown that the solutions of the wave equation represent waves propagating through the medium. The speed of propagation is determined from the governing partial differential equation in dimensional form or in the definition of t^*. In general, to arrive at a partial differential equation of the form of Eq. (9.18), in which no parameters appear,

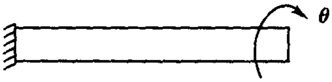
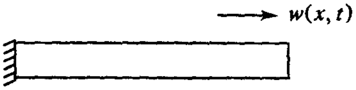
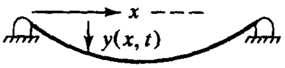
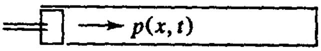
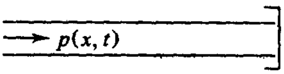
$$t^* = \frac{c}{L}t \qquad \textbf{[9.19]}$$

where c is the wave speed. Thus, for torsional oscillations, the wave speed is $\sqrt{G/\rho}$. Table 9.2 gives the wave speed for other situations governed by the wave equation.

Table 9.1 Boundary conditions for torsional oscillations of a circular shaft

End condition	Boundary condition	Remarks
Fixed, $x = 0$ or $x = 1$	$\theta = 0$	
Free, $x = 0$ or $x = 1$	$\frac{\partial \theta}{\partial x} = 0$	
Torsional spring, $x = 0$	$\frac{\partial \theta}{\partial x} = \beta\theta$	$\beta = \frac{k_t L}{JG}$
Torsional spring, $x = 1$	$\frac{\partial \theta}{\partial x} = -\beta\theta$	$\beta = \frac{k_t L}{JG}$
Torsional damper, $x = 0$	$\frac{\partial \theta}{\partial x} = \beta\frac{\partial \theta}{\partial t}$	$\beta = c_t\sqrt{\frac{J}{\rho G}}$
Torsional damper, $x = 1$	$\frac{\partial \theta}{\partial x} = -\beta\frac{\partial \theta}{\partial t}$	$\beta = c_t\sqrt{\frac{J}{\rho G}}$
Attached disk, $x = 0$	$\frac{\partial \theta}{\partial x} = \beta\frac{\partial^2 \theta}{\partial t^2}$	$\beta = \frac{I_D}{\rho JL}$
Attached disk, $x = 1$	$\frac{\partial \theta}{\partial x} = -\beta\frac{\partial^2 \theta}{\partial t^2}$	$\beta = \frac{I_D}{\rho JL}$

Table 9.2 Physical problems governed by the wave equation

Problem	Schematic	Nondimensional wave equation		Wave speed
Torsional oscillations of circular cylinder		$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$	$c = \sqrt{\frac{G}{\rho}}$	G = shear modulus ρ = mass density
Longitudinal oscillations of bar		$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}$	$c = \sqrt{\frac{E}{\rho}}$	E = elastic modulus ρ = mass density
Transverse vibrations of taut string		$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$	$c = \sqrt{\frac{T}{\mu}}$	T = tension μ = linear density
Pressure waves in an ideal gas		$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}$	$c = \sqrt{kRT}$	k = ratio of specific heats R = gas constant T = temperature
Waterhammer waves in rigid pipe		$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}$	$c = \sqrt{\frac{k}{\rho}}$	k = bulk modulus of fluid ρ = mass density

9.3.2 FREE-VIBRATION SOLUTIONS

A moment M is statically applied to the end of a circular shaft, fixed at $x = 0$ and free at $x = 1$, causing the angle of twist to vary linearly over the length of the shaft. Determine the resulting free torsional response when the moment is suddenly removed.

Solution:

The free torsional oscillations are governed by Eq. (9.18). The boundary condition corresponding to a fixed end at $x = 0$ is

$$\theta(0, t) = 0 \qquad \textbf{[9.20]}$$

and corresponding to a free end at $x = 1$ is

$$\frac{\partial\theta(1, t)}{\partial x} = 0 \qquad \textbf{[9.21}$$

Static application of the moment M leads to the initial condition

$$\theta(x, 0) = \frac{M}{JG}x = \gamma x \qquad \textbf{[9.22}$$

Since the shaft is released from rest a second initial condition is

$$\frac{\partial\theta(x, 0)}{\partial t} = 0 \qquad \textbf{[9.23}$$

A separation-of-variables solution is assumed as

$$\theta(x, t) = X(x)T(t) \qquad \textbf{[9.24}$$

Substituting Eq. (9.24) into Eq. (9.18) and rearranging leads to

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{T(t)}\frac{d^2T}{dt^2} \qquad \textbf{[9.2}$$

The left-hand side of Eq. (9.25) is a function of x only, while the right-hand side is function of t only. However, x and t are independent. Thus Eq. (9.25) is true only both sides are equal to the same constant, call it $-\lambda$, where λ is called the *separati* *constant*. Then Eq. (9.25) leads to

$$\frac{d^2T}{dt^2} + \lambda T = 0 \qquad \textbf{[9.2}$$

and

$$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad \textbf{[9.2}$$

The solution of Eq. (9.26) is

$$T(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \qquad \textbf{[9.2}$$

where A and B are arbitrary constants of integration. From Eq. (9.28) it is obvious t the natural frequencies are the square roots of the separation constant.

The solution of Eq. (9.27) is

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x \tag{9.29}$$

Application of Eq. (9.20) to Eq. (9.24) yields

$$X(0) = 0 \tag{9.30}$$

and its subsequent application to Eq. (9.29) gives $C = 0$.

Application of Eq. (9.21) to Eq. (9.24) yields

$$\frac{dX(1)}{dx} = 0 \tag{9.31}$$

Application of Eq. (9.31) to Eq. (9.29) with $C = 0$ leads to

$$D\sqrt{\lambda}\,\cos\sqrt{\lambda} = 0 \tag{9.32}$$

Choosing either $D = 0$ or $\lambda = 0$ leads to the trivial solution. Thus a nontrivial solution is obtained only when

$$\cos\sqrt{\lambda} = 0 \tag{9.33}$$

or

$$\lambda_k = \left[(2k-1)\frac{\pi}{2}\right]^2 \qquad k = 1, 2, \ldots \tag{9.34}$$

There are an infinity of solutions of Eq. (9.33), but as evidenced by Eq. (9.34), they are countable. The mode shape corresponding to λ_k is

$$X_k(x) = D_k \sin(2k-1)\frac{\pi}{2}x \tag{9.35}$$

for any D_k. The mode shapes are orthogonal with respect to the scalar product of Eq. (9.2) as follows:

$$\begin{aligned}(X_k(x), X_j(x)) &= \int_0^1 D_j D_k \sin(2k-1)\frac{\pi}{2}x \sin(2j-1)\frac{\pi}{2}x\,dx \\ &= \frac{D_j D_k}{\pi}\left[\frac{1}{j-k}\sin(j-k)\pi - \frac{1}{j+k+1}\sin(j+k+1)\pi\right] \\ &= 0\end{aligned}$$

The mode shapes are normalized by requiring

$$1 = (X_k, X_k) = \int_0^1 D_k^2 \sin^2(2k-1)\frac{\pi}{2}x\,dx = \frac{D_k^2}{2}$$

which leads to

$$X_k(x) = \sqrt{2}\sin(2k-1)\frac{\pi}{2}x \tag{9.36}$$

The first three normalized mode shapes are shown in Fig. 9.6.

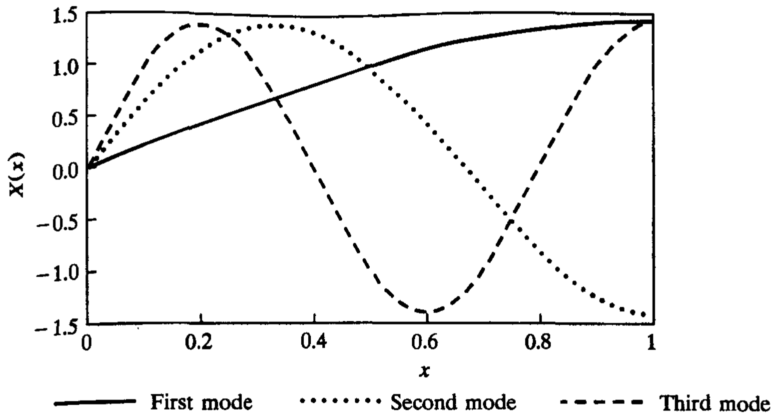


Figure 9.6 First three normalized mode shapes of fixed-free shaft.

The general solution to the free-vibration problem is

$$\theta(x, t) = \sum_{k=1}^{\infty} \sqrt{2} \sin (2k-1)\frac{\pi}{2}x \left[A_k \cos (2k-1)\frac{\pi}{2}t + B_k \sin (2k-1)\frac{\pi}{2}t\right] \tag{9.37}$$

Application of the initial condition, Eq. (9.23), yields $B_k = 0$. Application of Eq. (9.22) then gives

$$\gamma x = \sum_{k=1}^{\infty} A_k\sqrt{2} \sin (2k-1)\frac{\pi}{2}x \tag{9.38}$$

The expansion theorem, Eq. (9.4), is used to expand

$$\gamma x = \sum_{k=1}^{\infty}(\gamma x, X_k)X_k \tag{9.39}$$

where
$$\begin{aligned}(\gamma x, X_k) &= \int_0^1 \gamma x\sqrt{2} \sin (2k-1)\frac{\pi}{2}x\, dx \\ &= \frac{4\gamma\sqrt{2}}{\pi^2(2k-1)^2} \sin (2k-1)\frac{\pi}{2} \\ &= \frac{4\gamma\sqrt{2}}{\pi^2(2k-1)^2}(-1)^{k+1}\end{aligned}$$

Comparison of Eqs. (9.38) and (9.39) yields

$$A_k = (\gamma x, X_k) = \frac{4\gamma\sqrt{2}(-1)^{k+1}}{\pi^2(2k-1)^2} \tag{9.40}$$

Equation (9.37) becomes

$$\theta(x,t) = \frac{8\gamma}{\pi^2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{(2k-1)^2}\sin(2k-1)\frac{\pi}{2}x\cos(2k-1)\frac{\pi}{2}t \qquad [9.41]$$

The time-dependent angles of twist at four locations along the axis of the shaft, obtained by numerical evaluation of Eq. (9.41), are plotted in Fig. 9.7.

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mple 9.2 The circular shaft of Fig. 9.8 is fixed at $x = 0$ and has a thin disk of mass moment of inertia I attached at $x = 1$. Determine the natural frequencies for this system, identify the orthogonality condition satisfied by the mode shapes, and determine the normalized mode shapes.

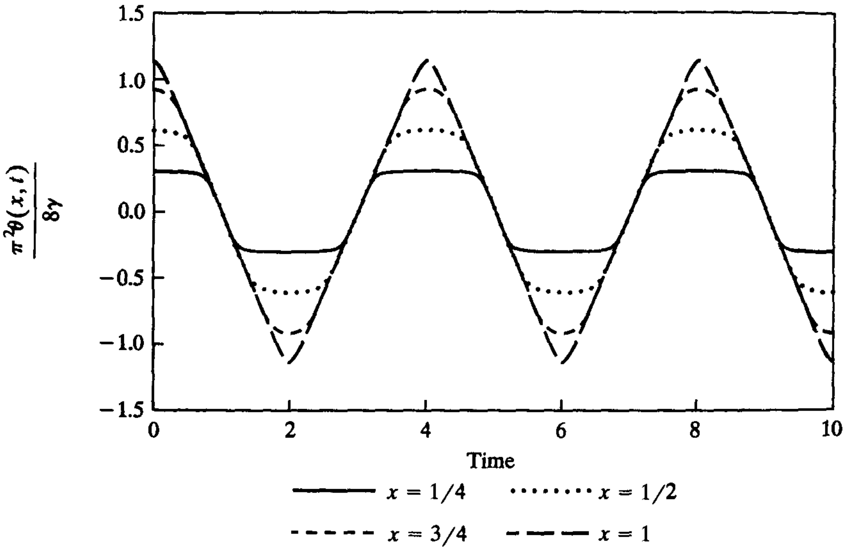


Figure 9.7 Time-dependent torsional oscillations of circular shaft fixed at $x = 0$, free at $x = 1$.

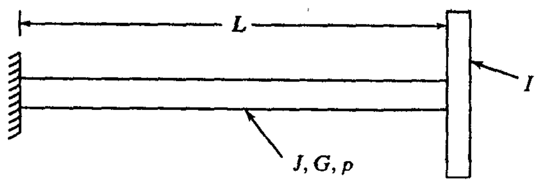


Figure 9.8 Shaft has disk of moment of inertia I attached at its free end.

Solution:

The partial differential equation governing this system is Eq. (9.18). It is subject to Eq. (9.20), and from Table 9.1

$$\frac{\partial\theta(1,t)}{\partial x} = -\beta\frac{\partial^2\theta(1,t)}{\partial t^2} \tag{9.42}$$

where

$$\beta = \frac{I}{\rho JL}$$

The separation-of-variables assumption of Eq. (9.24) leads to Eq. (9.27) subject to Eq. (9.30) and

$$\frac{dX(1)}{dx} = \beta\lambda X(1) \tag{9.43}$$

The solution satisfying Eqs. (9.24) and (9.27) is

$$X(x) = D\sin\sqrt{\lambda}x \tag{9.44}$$

Application of Eq. (9.43) to Eq. (9.44) yields

$$\sqrt{\lambda}\cos\sqrt{\lambda} = \beta\lambda\sin\sqrt{\lambda} \tag{9.45}$$

or

$$\tan\sqrt{\lambda} = \frac{1}{\beta\sqrt{\lambda}} \tag{9.46}$$

A graphical solution of the transcendental equation, Eq. (9.46), is shown in Fig. 9.9. The values of λ where the curves $\tan\sqrt{\lambda}$ and $1/\beta\sqrt{\lambda}$ intersect are the solutions of Eq. (9.46), and are the values of the separation constant for which nontrivial solutions for $X(x)$ occur. Figure 9.9 shows that there are infinite, but countable, values of λ where these curves intersect. Figure 9.9 also shows that for large k, λ_k approaches $[(k+1)\pi]^2$.

The natural frequencies are the square roots of the separation constants. Figure 9.10 shows the first four natural frequencies as a function of β. The first four mode shapes are plotted in Figure 9.11 for $\beta = 0.4$.

Let λ_i and λ_j be distinct solutions of Eq. (9.45) with corresponding mode shapes $X_i(x)$ and $X_j(x)$, respectively. The mode shapes satisfy the following problems

$$\begin{aligned} &\frac{d^2X_i}{dx^2} + \lambda_i X_i = 0 \qquad X_i(0) = 0 \qquad \frac{dX_i}{dx}(1) = \beta\lambda_i X_i(1) \\ &\frac{d^2X_j}{dx^2} + \lambda_j X_j = 0 \qquad X_j(0) = 0 \qquad \frac{dX_j}{dx}(1) = \beta\lambda_j X_j(1) \end{aligned} \tag{9.47}$$

Multiplying the first of Eqs. (9.47) by $X_j(x)$ and integrating from 0 to 1 leads to

$$\int_0^1 \frac{d^2X_i}{dx^2}X_j\,dx + \lambda_i\int_0^1 X_iX_j\,dx = 0 \tag{9.48}$$

Using integration by parts on the first integral leads to

$$X_j(1)\frac{dX_i}{dx}(1) - X_j(0)\frac{dX_i}{dx}(0) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}\,dx + \lambda_i\int_0^1 X_iX_j\,dx = 0 \tag{9.49}$$

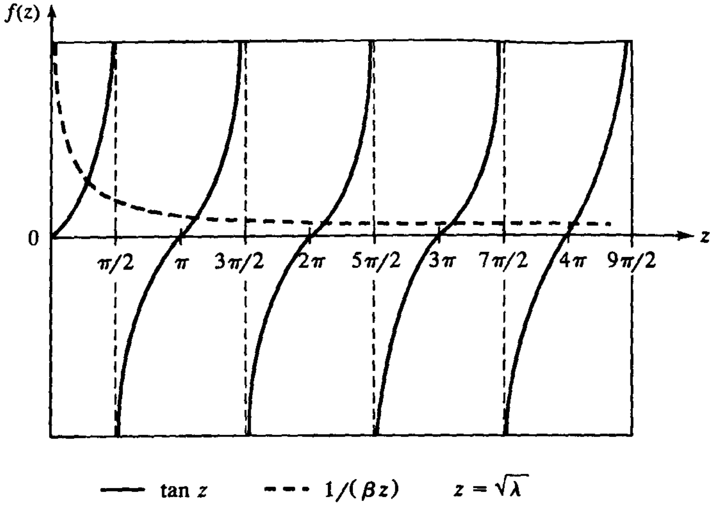


Figure 9.9 Graphical solution of transcendental equation $\tan\sqrt{\lambda} = 1/\beta\sqrt{\lambda}$ to determine natural frequencies of system of Example 9.2. Values of separation constants correspond to points of intersection of two curves.

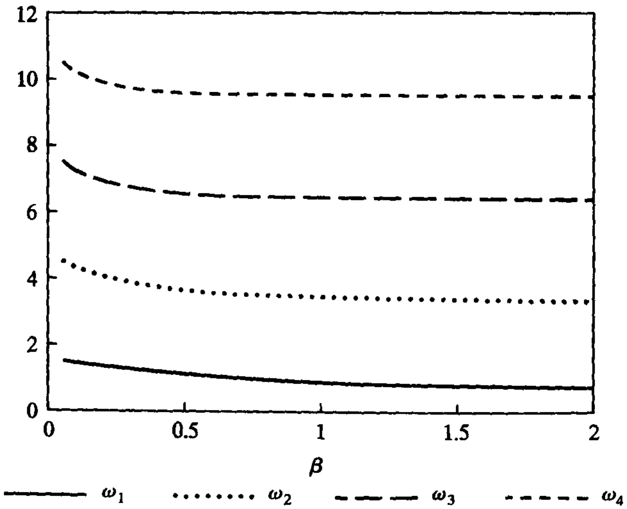


Figure 9.10 Nondimensional natural frequencies of Example 9.2 as function of nondimensional parameter β obtained by solving Eq. (9.46).

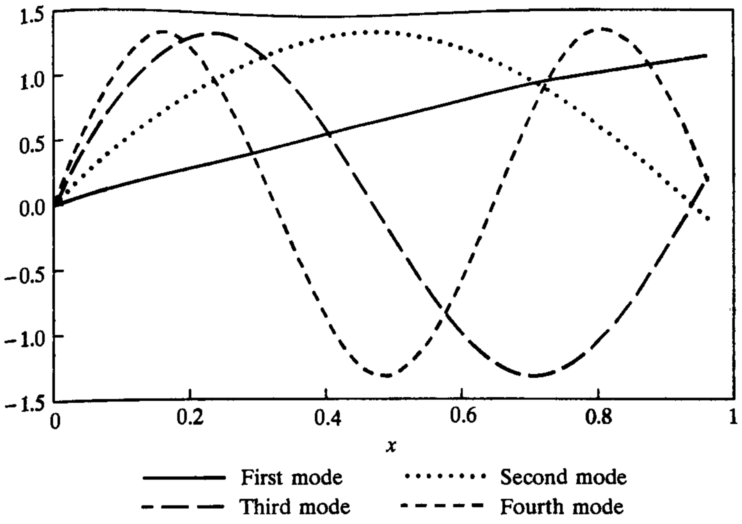


Figure 9.11 Mode shapes of Example 9.2 with $\beta = 0.4$.

Application of the boundary conditions in Eq. (9.49) leads to

$$\beta\lambda_i X_i(1)X_j(1) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}\,dx + \lambda_i \int_0^1 X_i X_j\,dx = 0 \qquad [9.50]$$

Multiplying the second of Eqs. (9.47) by $X_i(x)$, integrating from 0 to 1, and performing algebra similar to that above leads to

$$\beta\lambda_j X_j(1)X_i(1) - \int_0^1 \frac{dX_i}{dx}\frac{dX_j}{dx}\,dx + \lambda_j \int_0^1 X_i X_j\,dx = 0 \qquad [9.51]$$

Subtracting Eq. (9.51) from Eq. (9.50) leads to

$$(\lambda_i - \lambda_j)\left(\beta X_i(1)X_j(1) + \int_0^1 X_i X_j\,dx\right) = 0 \qquad [9.52]$$

Since $\lambda_i \neq \lambda_j$, Eq. (9.52) implies

$$\beta X_i(1)X_j(1) + \int_0^1 X_i X_j\,dx = 0 \qquad [9.53]$$

If the scalar product of f and g is defined by

$$(f, g) = \int_0^1 f(x)g(x)\,dx + \beta f(1)g(1) \qquad [9.54]$$

then

$$(X_j, X_k) = 0 \qquad [9.55]$$

Equation (9.54) defines the energy scalar product with which the mode shapes are mutually orthogonal.

Normalization of the mode shape requires

$$1 = (X_k, X_k) = \int_0^1 D_k^2 \sin^2 \sqrt{\lambda_k} x \, dx + D_k^2 \beta \sin^2 \sqrt{\lambda_k}$$
$$= D_k^2 \left[\int_0^1 \frac{1}{2}(1 - \cos 2\sqrt{\lambda_k} x) \, dx + \beta \sin^2 \sqrt{\lambda_k} \right]$$
$$= D_k^2 \left[\frac{1}{2} \left(1 - \frac{1}{2\sqrt{\lambda_k}} \sin 2\sqrt{\lambda_k} \right) + \beta \sin^2 \sqrt{\lambda_k} \right]$$

Using the trigonometric identity

$$\sin 2\sqrt{\lambda_k} = 2 \sin \sqrt{\lambda_k} \cos \sqrt{\lambda_k}$$

and replacing $\cos \sqrt{\lambda_k}$ from Eq. (9.45) leads to

$$D_k = \sqrt{2} \left(1 + \beta \sin^2 \sqrt{\lambda_k} \right)^{-1/2}$$

where λ_k is the kth real solution of Eq. (9.46).

9.3.3 FORCED VIBRATIONS

The application of undetermined coefficients for harmonic excitations is illustrated in the following example. Modal analysis is illustrated with examples in Sec. 9.4. Application of the Laplace transform method is beyond the scope of this book.

9.3 The thin disk of Example 9.2 is subject to a harmonic torque,

$$T(t) = T_0 \sin \omega t$$

Determine the steady-state response of the system.

Solution:

The torsional oscillations, in terms of nondimensional variables, are governed by Eq. (9.18) with

$$\theta(0, t) = 0 \qquad \textbf{[9.56a]}$$

and

$$\frac{\partial \theta}{\partial x}(1, t) = -\beta \frac{\partial^2 \theta}{\partial t^2}(1, t) + \frac{T_0 L}{JG} \sin \tilde{\omega} t \qquad \textbf{[9.56b]}$$

where

$$\tilde{\omega} = L \sqrt{\frac{\rho}{G}} \omega \qquad \textbf{[9.57]}$$

Since the external excitation is harmonic, the steady-state response is assumed as

$$\theta(x,t) = u(x)\sin\tilde{\omega}t \tag{9.58}$$

Substituting Eq. (9.58) into Eq. (9.18) leads to

$$\frac{d^2u}{dx^2}\sin\tilde{\omega}t = -\tilde{\omega}^2 u \sin\tilde{\omega}t$$

or

$$\frac{d^2u}{dx^2} + \tilde{\omega}^2 u = 0 \tag{9.59}$$

Substituting Eq. (9.58) into the boundary conditions, Eqs. (9.56*a*) and (9.56*b*), leads to

$$u(0) = 0 \tag{9.60}$$

and

$$\frac{du}{dx}(1) - \beta\tilde{\omega}^2 u(1) = \frac{T_0 L}{JG} \tag{9.61}$$

The solution of Eq. (9.59) subject to Eqs. (9.60) and (9.61) is

$$u(x) = \frac{T_0 L}{(\tilde{\omega}\cos\tilde{\omega} - \beta\tilde{\omega}^2\sin\tilde{\omega})JG}\sin\tilde{\omega}x \tag{9.62}$$

Note that if $\tilde{\omega}$ is equal to any of the system's natural frequencies, the denominator vanishes. The assumed form of the solution, Eq. (9.58), must be modified to account for this resonance condition.

The steady-state solution is given by Eq. (9.58), where $u(x)$ is given in Eq. (9.62). The total solution is the steady-state solution plus the homogeneous solution, which is a summation over all free-vibration modes. Initial conditions can then be applied to determine the constants in the linear combination.

9.4 TRANSVERSE BEAM VIBRATIONS

9.4.1 PROBLEM FORMULATION

The uniform beam of Fig. 9.12 is made of a material of mass density ρ and elastic modulus E, and has a length L, cross-sectional area A, and centroidal moment of inertia I. Let x be a coordinate along the neutral axis of the beam, measured from its left end. The beam has an external load per unit length, $f(x,t)$. Let $w(x,t)$ be the transverse deflection of the beam, measured from its equilibrium position.

Free-body diagrams of an arbitrary differential element of the beam at an arbitrary instant of time are shown in Fig. 9.13. The element is a slice of the beam of thickness dx and its left face is a distance x from the beam's left end. The external forces shown are the external loading, the internal bending moment which is the resultant moment of the normal stress distribution, and the internal shear force, which

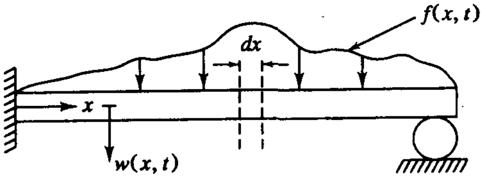


Figure 9.12 $w(x, t)$ is the transverse deflection of the beam measured from its static equilibrium position.

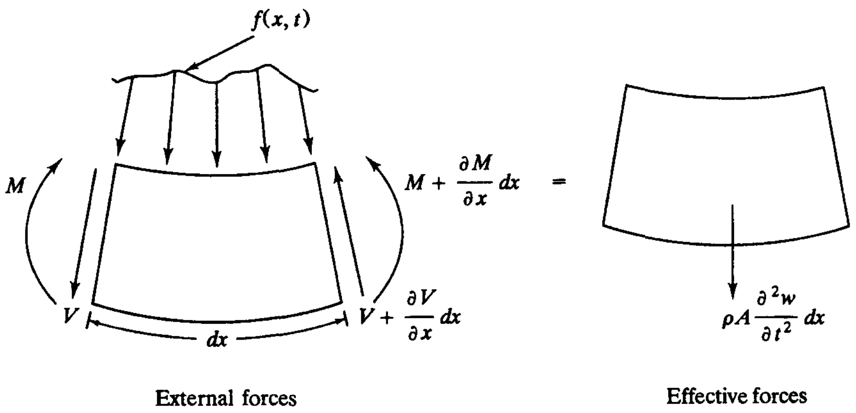


Figure 9.13 Free-body diagrams of differential beam element at an arbitrary time.

is the resultant of the shear stress distribution. It is assumed that the resultant of the normal stress distribution is zero. The effective force is the element mass times its acceleration. The element's longitudinal acceleration and angular acceleration are small in comparison to other effects and are thus ignored.

Sum forces in the vertical direction:

$$\left(\sum \overset{\downarrow}{F}\right)_{\text{ext}} = \left(\sum \overset{\downarrow}{F}\right)_{\text{eff}} \qquad [9.63]$$

$$V - \left(V + \frac{\partial V}{\partial x} dx\right) + \int_x^{x+dx} f(\zeta, t)\, d\zeta = \rho A \frac{\partial^2 w}{dt^2}\, dx$$

The mean value theorem implies that there is an $\tilde{x}$, $x < \tilde{x} < x + dx$, such that

$$\int_x^{x+dx} f(\zeta, t)\, d\zeta = f(\tilde{x}, t)\, dx$$

Since dx is infinitesimal, $\tilde{x} \approx x$. Equation (9.63) becomes

$$f(x, t) - \frac{\partial V}{\partial x} = \rho A \frac{\partial^2 w}{\partial x^2} \qquad [9.64]$$

Sum moments about the neutral axis of the left face of the element:

$$\left(\sum \overset{\curvearrowright}{\overset{+}{M}}_0\right)_{\text{ext}} = \left(\sum \overset{\curvearrowright}{\overset{+}{M}}_0\right)_{\text{eff}}$$

$$M - \left(M + \frac{\partial M}{\partial x}dx\right) - \left(V + \frac{\partial V}{\partial x}dx\right)dx + \int_x^{x+dx} (\zeta - x) f(\zeta, t)\,d\zeta = \rho A \frac{\partial^2 w}{\partial x^2}dx\left(\frac{dx}{2}\right) \qquad [9.65]$$

Since dx is infinitesimal, terms of order dx^2 are negligible compared to terms of order dx. When the mean value theorem is used on the integral, since $\zeta - x$ is less than dx over the entire range of integration, it becomes apparent that the term is of order dx^2. Then Eq. (9.65) simplifies to

$$V = -\frac{\partial M}{\partial x} \qquad [9.66]$$

From mechanics of materials, with the chosen sign conventions,

$$M = -EI\frac{\partial^2 w}{\partial x^2} \qquad [9.67]$$

Substitution of Eqs. (9.66) and (9.67) into Eq. (9.64) leads to

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f(x, t) \qquad [9.68]$$

Equation (9.68) is nondimensionalized by introducing

$$x^* = \frac{x}{L} \qquad t^* = t\sqrt{\frac{EI}{\rho A L^4}} \qquad w^* = \frac{w}{L} \qquad f^* = \frac{f}{f_m} \qquad [9.69]$$

where f_m is the maximum value of f. The resulting nondimensional form of Eq. (9.68) is

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = \frac{f_m L^3}{EI} f(x, t) \qquad [9.70]$$

Four boundary conditions, two at $x = 0$ and two at $x = 1$, must be specified. The forms of the boundary conditions depend on the type of end supports. Nondimensional boundary conditions associated with different support conditions are given in Table 9.3.

The formulation of the mathematical problem is completed by specifying two initial conditions.

Equation (9.70) is the governing nondimensional partial differential equation for forced vibrations of a beam assuming no axial load, longitudinal effects are negligible, rotary inertia and transverse shear are negligible, and other standard assumptions of beam theory from mechanics of materials apply.

Table 9.3 Boundary conditions for transverse vibrations of a beam

End condition	Boundary condition A	Boundary condition B	Remarks
Free, $x = 0$ or $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = 0$	
Pinned, $x = 0$ or $x = 1$	$w = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	
Fixed, $x = 0$ or $x = 1$	$w = 0$	$\frac{\partial w}{\partial x} = 0$	
Linear spring, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta w$	$\beta = \frac{kL^3}{EI}$
Linear spring, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta w$	$\beta = \frac{kL^3}{EI}$
Viscous damper, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta \frac{\partial w}{\partial t}$	$\beta = \frac{cL}{\sqrt{\rho EIA}}$
Viscous damper, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta \frac{\partial w}{\partial t}$	$\beta = \frac{cL}{\sqrt{\rho EIA}}$
Attached mass, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = -\beta \frac{\partial^2 w}{\partial t^2}$	$\beta = \frac{m}{\rho AL}$
Attached mass, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = 0$	$\frac{\partial^3 w}{\partial x^3} = \beta \frac{\partial^2 w}{\partial t^2}$	$\beta = \frac{m}{\rho AL}$
Attached inertia element, $x = 0$	$\frac{\partial^2 w}{\partial x^2} = -\beta \frac{\partial^3 w}{\partial x \partial t^2}$	$\frac{\partial^3 w}{\partial x^3} = 0$	$\beta = \frac{J}{\rho AL^3}$
Attached inertia element, $x = 1$	$\frac{\partial^2 w}{\partial x^2} = \beta \frac{\partial^3 w}{\partial x \partial t^2}$	$\frac{\partial^3 w}{\partial x^3} = 0$	$\beta = \frac{J}{\rho AL^3}$

9.4.2 FREE VIBRATIONS

When the product solution

$$w(x, t) = X(x)T(t) \tag{9.71}$$

is substituted into Eq. (9.70) with $f = 0$, the result is

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = -\frac{1}{X(x)}\frac{d^4X}{dx^4} \tag{9.72}$$

The usual separation argument is used to set both sides of Eq. (9.72) equal to the same constant, say $-\lambda$. This leads to

$$\frac{d^2T}{dt^2} + \lambda T = 0 \tag{9.73}$$

and

$$\frac{d^4X}{dx^4} - \lambda X = 0 \tag{9.74}$$

The solution of Eq. (9.73)

$$T(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \tag{9.75}$$

from which it is obvious that the natural frequencies are the square roots of the separation constants. The general solution of Eq. (9.74)

$$X(x) = C_1\cos\lambda^{1/4}x + C_2\sin\lambda^{1/4}x + C_3\cosh\lambda^{1/4}x + C_4\sinh\lambda^{1/4}x \tag{9.76}$$

The solvability condition is determined by applying the homogeneous boundary conditions to Eq. (9.76). Table 9.4 summarizes the solvability conditions for different types of end supports, provides the first five nondimensional natural frequencies for each entry, their corresponding mode shapes, and specifies the scalar product for which the mode shapes are orthogonal.

Free-free and pinned-free beams are unrestrained and thus their lowest natural frequency is zero, corresponding to a rigid-body mode. The fixed-pinned beam has the same characteristic equations as the pinned-free beam, and $\lambda = 0$ is a solution of this equation. However, $\lambda = 0$ leads to a trivial mode shape for the fixed-pinned beam.

Determine the natural frequencies and normalized mode shapes for a simply supported beam.

Solution:

The boundary conditions for a simply supported beam are

$$w(0,t) = 0 \qquad \frac{\partial^2 w(0,t)}{\partial x^2} = 0$$

and

$$w(1,t) = 0 \qquad \frac{\partial^2 w(1,t)}{\partial x^2} = 0$$

which when applied to Eq. (9.76) gives

$$0 = C_1 + C_3$$

$$0 = -\sqrt{\lambda}C_1 + \sqrt{\lambda}C_3$$

$$0 = C_1\cos\lambda^{1/4} + C_2\sin\lambda^{1/4} + C_3\cosh\lambda^{1/4} + C_4\sinh\lambda^{1/4}$$

$$0 = -\sqrt{\lambda}C_1\cos\lambda^{1/4} - \sqrt{\lambda}C_2\sin\lambda^{1/4} + \sqrt{\lambda}C_3\cosh\lambda^{1/4} + \sqrt{\lambda}C_4\sinh\lambda^{1/4}$$

The first two of these equations imply $C_1 = C_3 = 0$. Then the last two equations become

$$C_2\sin\lambda^{1/4} + C_4\sinh\lambda^{1/4} = 0$$

$$-C_2\sin\lambda^{1/4} + C_4\sinh\lambda^{1/4} = 0$$

These equations have a nontrivial solution if and only if

$$\sin\lambda^{1/4} = 0$$

Table 9.4 Natural frequencies and mode shapes for beams.

End conditions $X = 0$ $X = 1$	Characteristic equation	Five lowest natural frequencies $\omega_k = \sqrt{\lambda_k}$	Mode shape	Kinetic energy scalar product $(X_j(x), X_k(x))$
Fixed-fixed	$\cos\lambda^{1/4}\cosh\lambda^{1/4} = 1$	$\omega_1 = 22.37$ $\omega_2 = 61.66$ $\omega_3 = 120.9$ $\omega_4 = 199.9$ $\omega_5 = 298.6$	$C_k\left[\cosh\lambda_k^{1/4}x - \cos\lambda_k^{1/4}x - \alpha_k(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cosh\lambda_k^{1/4} - \cos\lambda_k^{1/4}}{\sinh\lambda_k^{1/4} - \sin\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$
Pinned-pinned	$\sin\lambda^{1/4} = 0$	$\omega_1 = 9.870$ $\omega_2 = 39.48$ $\omega_3 = 88.83$ $\omega_4 = 157.9$ $\omega_5 = 246.7$	$C_k\sin\lambda_k^{1/4}x$	$\int_0^1 X_j(x)X_k(x)\,dx$
Fixed-free	$\cos\lambda^{1/4}\cosh^{1/4} = -1$	$\omega_1 = 3.51$ $\omega_2 = 22.03$ $\omega_3 = 61.70$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$C_k\left[\cosh\lambda_k^{1/4}x - \cos\lambda_k^{1/4}x - \alpha_k(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$
Free-free	$\cosh\lambda^{1/4}\cos\lambda^{1/4} = 1$	$\omega_1 = 0$ $\omega_2 = 22.37$ $\omega_3 = 61.66$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$1, \sqrt{3}x\,(k = 1)$ $C_k\left[\cosh\lambda_k^{1/4}x + \cos\lambda_k^{1/4}x + \alpha_k(\sinh\lambda_k^{1/4}x + \sin\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cosh\lambda_k^{1/4} - \cos\lambda_k^{1/4}}{\sin\lambda_k^{1/4} - \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$
Fixed-linear spring	$\lambda^{3/4}(\cosh\lambda^{1/4}\cos\lambda^{1/4} + 1)$ $- \beta(\cos\lambda^{1/4}\sinh\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4})$ $= 0$	For $\beta = 0.25$ $\omega_1 = 3.65$ $\omega_2 = 22.08$ $\omega_3 = 61.70$ $\omega_4 = 120.9$ $\omega_5 = 199.9$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x - \alpha_k(\sin\lambda^{1/4}x - \sinh\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$

Pinned-linear spring	$\cot\lambda^{1/4} - \coth\lambda^{1/4} = -\dfrac{2\beta}{\lambda^{3/4}}$	For $\beta = 0.25$ $\omega_1 = 0.8636$ $\omega_2 = 15.41$ $\omega_3 = 49.47$ $\omega_4 = 104.25$ $\omega_5 = 178.27$	$C_k\left[\sin\lambda_k^{1/4}x + \dfrac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right]$	$\int_0^1 X_j(x)X_k(x)\,dx$
Fixed-attached mass	$\lambda^{1/4}(\cos\lambda^{1/4}\cosh\lambda^{1/4}+1)$ $+\beta(\cos\lambda^{1/4}\sinh\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4})$ $=0$	For $\beta = 0.25$ $\omega_1 = 3.047$ $\omega_2 = 21.54$ $\omega_3 = 61.21$ $\omega_4 = 120.4$ $\omega_5 = 199.4$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x + \alpha_k(\sinh\lambda_k^{1/4}x - \sin\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} + \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$ $+\beta X_j(1)X_k(1)$
Pinned-free	$\tan\lambda^{1/4} = \tanh\lambda^{1/4}$	$\omega_1 = 0$ $\omega_2 = 15.42$ $\omega_3 = 49.96$ $\omega_4 = 104.2$ $\omega_5 = 178.3$	$\sqrt{3}x,\ (k=1)$ $C_k\left[\sin\lambda_k^{1/4}x + \dfrac{\sin\lambda_k^{1/4}}{\sinh\lambda_k^{1/4}}\sinh\lambda_k^{1/4}x\right]$ $(k>1)$	$\int_0^1 X_j(x)X_k(x)\,dx$
Fixed-pinned	$\tan\lambda^{1/4} = \tanh\lambda^{1/4}$	$\omega_1 = 15.42$ $\omega_2 = 49.96$ $\omega_3 = 104.2$ $\omega_4 = 178.3$ $\omega_5 = 272.0$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x - \alpha_k(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\cos\lambda_k^{1/4} - \cosh\lambda_k^{1/4}}{\sin\lambda_k^{1/4} - \sinh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$
Fixed-attached inertia element	$\cos\lambda^{1/4}\cosh\lambda^{1/4}$ $+\beta(\sin\lambda^{1/4}\cosh\lambda^{1/4} + \cos\lambda^{1/4}\sinh\lambda^{1/4})$ $=-1$	For $\beta = 0.25$ $\omega_1 = 4.425$ $\omega_2 = 27.28$ $\omega_3 = 71.41$ $\omega_4 = 135.4$ $\omega_5 = 219.2$	$C_k\left[\cos\lambda_k^{1/4}x - \cosh\lambda_k^{1/4}x + \alpha_k(\sin\lambda_k^{1/4}x - \sinh\lambda_k^{1/4}x)\right]$ $\alpha_k = \dfrac{\sin\lambda_k^{1/4} - \sinh\lambda_k^{1/4}}{\cos\lambda_k^{1/4} + \cosh\lambda_k^{1/4}}$	$\int_0^1 X_j(x)X_k(x)\,dx$ $+\beta X_j(1)X_k(1)$

| The dimensional natural frequencies are obtained by multiplying the given nondimensional natural frequencies by $\sqrt{EI/\rho AL^4}$; for a given beam β is as defined in Table 9.3.

which is satisfied by

$$\lambda_k = (k\pi)^4 \qquad k = 1, 2, \ldots$$

For these values of λ, $C_4 = 0$ and C_2 remains arbitrary, leading to the mode shape

$$X_k(x) = C_k \sin k\pi x$$

The mode shapes are orthogonal with respect to the scalar product of Eq. (9.2), as evidenced by

$$\int_0^1 C_k C_j \sin k\pi x \sin j\pi x \, dx = 0 \qquad k \neq j$$

Normalization with respect to this scalar product yields $C_k = \sqrt{2}$.

9.5 Determine the first four natural frequencies for the beam of Fig. 9.14.

Solution:

From Table 9.3, the appropriate boundary conditions are

$$w(0,t) = 0 \qquad \frac{\partial w(0,t)}{\partial x} = 0$$

and
$$\frac{\partial^2 w(1,t)}{\partial^2 x} = 0 \qquad \frac{\partial^3 w(1,t)}{\partial x^3} = \beta w(1,t)$$

where
$$\beta = \frac{kL^3}{EI} = \frac{(2 \times 10^6 \text{ N/m})(1 \text{ m})^3}{(210 \times 10^9 \text{ N/m}^2)(5 \times 10^{-5} \text{ m}^4)} = 0.190$$

Application of the boundary conditions to Eq. (9.76) gives

$$0 = C_1 + C_3$$
$$0 = C_2 + C_4$$
$$0 = -C_1 \cos \lambda^{1/4} - C_2 \sin \lambda^{1/4} + C_3 \cosh \lambda^{1/4} + C_4 \sinh \lambda^{1/4}$$
$$0 = (\lambda^{3/4} \sin \lambda^{1/4} - \beta \cos \lambda^{1/4})C_1 + (-\lambda^{3/4} \cos \lambda^{1/4} - \beta \sin \lambda^{1/4})C_2 + (\lambda^{3/4} \sinh \lambda^{1/4} - \beta \cosh \lambda^{1/4})C_3 + (\lambda^{3/4} \cosh \lambda^{1/4} - \beta \sinh \lambda^{1/4})C_4$$

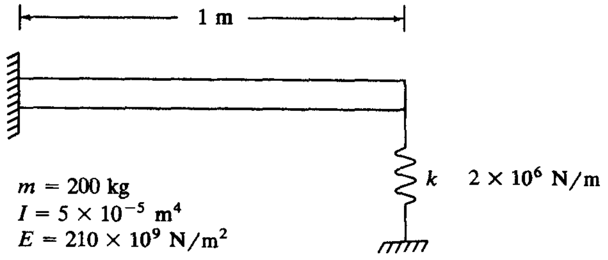


Figure 9.14 Beam of Example 9.5.

which leads to the solvability condition

$$\lambda^{3/4}(1+\cos\lambda^{1/4}\cosh\lambda^{1/4}) = -\beta(\cosh\lambda^{1/4}\sin\lambda^{1/4} - \cos\lambda^{1/4}\sinh\lambda^{1/4})$$

For $\beta = 0.190$ the first four roots of this equation are

$$\lambda = 13.10, 486.2, 3807.0, 14161.6, \ldots$$

The nondimensional natural frequencies are the square roots of the values of λ that solve the characteristic equation. The dimensional natural frequencies are obtained by noting the relationship between the dimensional time and the nondimensional time and its application to Eq. (9.69),

$$\omega = \sqrt{\lambda\frac{EI}{\rho AL^4}} = 229.1\sqrt{\lambda}$$

The first four natural frequencies for this beam are

$$\omega_1 = 829.2 \text{ rad/s} \qquad \omega_2 = 5051 \text{ rad/s}$$
$$\omega_3 = 14140 \text{ rad/s} \qquad \omega_4 = 27260 \text{ rad/s}$$

9.4.3 FORCED VIBRATIONS

The modal analysis method, described in Sec. 9.2, for analyzing the forced vibrations of a continuous system is applied to the following examples.

Examp

The simply supported beam of Fig. 9.15 is subject to a harmonic excitation over half of its span. Determine the beam's steady-state response.

Solution:

The nondimensional force per unit length is

$$f(x,t) = \sin\tilde{\omega}t\left[u\left(x-\tfrac{1}{4}\right) - u\left(x-\tfrac{3}{4}\right)\right]$$

where

$$\tilde{\omega} = \omega\sqrt{\frac{\rho AL^4}{EI}}$$

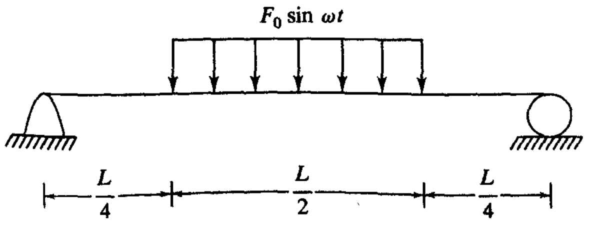


Figure 9.15 Beam of Example 9.6.

The expansion theorem is used to expand $f(x, t)$ in terms of the normalized mode shapes of the corresponding free-vibration problem, which are determined in Example 9.4. The expansion coefficients are determined using Eq. (9.6), with the scalar product defined by Eq. (9.2),

$$
\begin{aligned}
C_k &= \int_0^1 f(x,t)\sqrt{2}\sin k\pi x\,dx \\
&= \sqrt{2}\sin\tilde{\omega}t\int_{1/4}^{3/4}\sin k\pi x\,dx \\
&= \frac{\sqrt{2}}{k\pi}\sin\tilde{\omega}t\left(\cos k\frac{\pi}{4} - \cos k\frac{3\pi}{4}\right) \\
&= \frac{2}{k\pi}\sin\tilde{\omega}t\begin{cases} 0 & k = 2, 4, 6, \ldots \\ 1 & k = 1, 7, 9, 15, 17, 23, \ldots \\ -1 & k = 3, 5, 11, 13, 19, 21, \ldots \end{cases} \\
&= \frac{2}{k\pi}a_k\sin\tilde{\omega}t
\end{aligned}
$$

The displacement is expanded as

$$
w(x,t) = \sum_{k=1}^{\infty}\sqrt{2}\sin k\pi x p_k(t)
$$

Substituting for w and f in Eq. (9.70) leads to

$$
\sum_{k=1}^{\infty}[\ddot{p}_k + (k\pi)^4 p_k]\sqrt{2}\,\sin k\pi x = \Lambda\sum_{k=1}^{\infty}C_k(t)\sqrt{2}\,\sin k\pi x
$$

where

$$
\Lambda = \frac{F_0L^3}{EI}
$$

The preceding equation is multiplied by $\sqrt{2}\sin j\pi x$ for an arbitrary j and integrated from 0 to 1. This is equivalent to taking the scalar product of both sides of the equation with $X_j(x)$. The orthogonality condition, Eq. (9.6), is used such that each sum collapses to a single term, yielding

$$
\ddot{p}_j + (j\pi)^4 p_j = \Lambda C_j \qquad j = 1, 2, \ldots
$$

whose steady-state solution is

$$
p_j(t) = \frac{\Lambda}{(j\pi)^4 - \tilde{\omega}^2}\frac{2}{j\pi}a_j\sin\tilde{\omega}t
$$

The steady-state response of the beam is

$$
\begin{aligned}
w(x,t) = \frac{2\sqrt{2}\Lambda}{\pi}\sin\tilde{\omega}t\Bigg[&\frac{1}{\pi^4 - \tilde{\omega}^2}\sin\pi x \\
&- \frac{1}{3(81\pi^4 - \tilde{\omega}^2)}\sin 3\pi x - \frac{1}{5(625\pi^4 - \tilde{\omega}^2)}\sin 5\pi x
\end{aligned}
$$

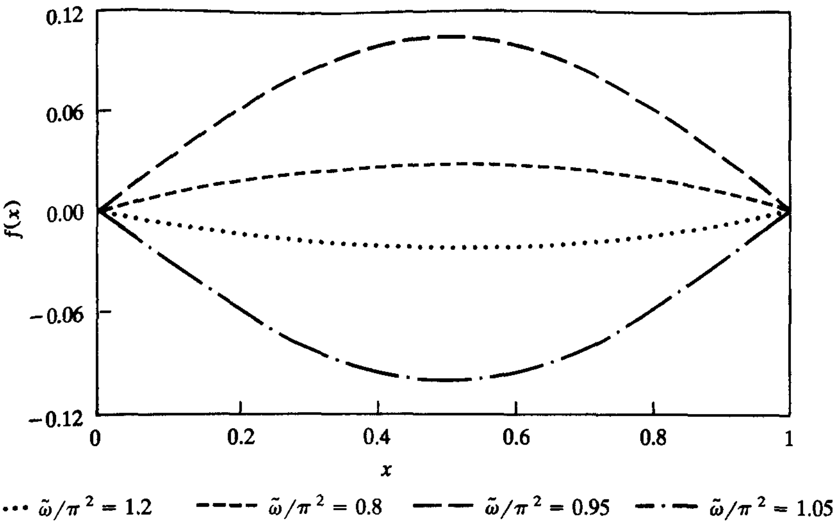


Figure 9.16 Steady-state response for Example 9.6.

$$+ \frac{1}{7(1501\pi^4 - \tilde{\omega}^2)} \sin 7\pi x + \cdots \Bigg]$$

$$= \frac{2\sqrt{2}\Lambda}{\pi} f(x) \sin \tilde{\omega} t$$

The function $f(x)$ is shown in Fig. 9.16 for several values of $\tilde{\omega}$. Note that when $\tilde{\omega}$ is close to π^2 the steady-state amplitude is large at the midspan.

A machine of mass 150 kg is attached to the end of the cantilever beam of Fig. 9.17. The machine operates at 2000 rpm and has a rotating unbalance of 0.965 kg · m. What is the steady-state amplitude of vibration of the end of the beam?

Solution:

The nondimensional formulation of the governing mathematical problem is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0$$

subject to

$$w(0, t) = 0 \qquad \frac{\partial w(0, t)}{\partial x} = 0 \qquad \frac{\partial^2 w(1, t)}{\partial x^2} = 0$$

$$\frac{\partial^3 w(1, t)}{\partial x^3} = \beta \frac{\partial^2 w(1, t)}{\partial t^2} + \alpha \sin \tilde{\omega} t$$

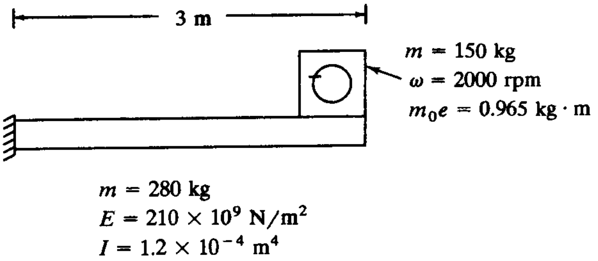


Figure 9.17 Cantilever beam of Example 9.7 has machine with rotating unbalance at its end.

where
$$\tilde{\omega} = \omega\sqrt{\frac{\rho A L^4}{EI}}$$
$$= 209.4\ \frac{\text{rad}}{\text{s}}\sqrt{\frac{(280\text{ kg})(3\text{ m})^3}{(210\times 10^9\text{ N/m}^2)(1.2\times 10^{-4}\text{ m}^4)}}$$
$$= 3.63$$
$$\beta = \frac{m}{\rho A L} = \frac{150\text{ kg}}{280\text{ kg}} = 0.536$$

and
$$\alpha = \frac{m_0 e\omega^2 L^2}{EI}$$
$$= \frac{(0.965\text{ kg}\cdot\text{m})(209.4\text{ rad/s})^2(3\text{ m})^2}{(210\times 10^9\text{ N/m}^2)(1.2\times 10^{-4}\text{ m}^4)}$$
$$= 0.015$$

The last boundary condition is developed by applying Newton's law to the machine as shown in Fig. 9.18. The problem is nonhomogeneous due to this boundary condition. From Table 9.4 the characteristic equation for the homogeneous problem of a beam with a concentrated mass at its end is

$$\lambda^{1/4}(1+\cos\lambda^{1/4}\cosh\lambda^{1/4}) + \beta(\cos\lambda^{1/4}\sinh\lambda^{1/4} - \cosh\lambda^{1/4}\sin\lambda^{1/4}) = 0$$

The corresponding mode shapes for the homogeneous problem are

$$X_k(x) = C_k\left[\cos\lambda^{1/4}x - \cosh\lambda^{1/4}x + \frac{\cos\lambda^{1/4}+\cosh\lambda^{1/4}}{\sin\lambda^{1/4}+\sinh\lambda^{1/4}}(\sinh\lambda^{1/4}x - \sin\lambda^{1/4}x)\right]$$

where C_k is chosen to normalize the mode shape with respect to the scalar product defined by

$$(X_j(x), X_k(x)) = \int_0^1 X_j(x)X_k(x)\,dx + \beta X_j(1)X_k(1)$$

The first six nondimensional natural frequencies and normalization constants are given in Table 9.5.

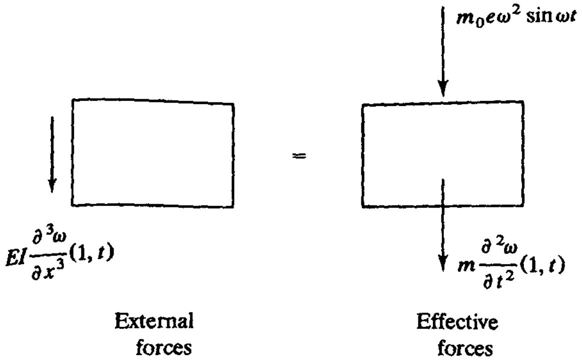


Figure 9.18 Free-body diagrams used to derive boundary condition for Example 9.7.

Table 9.5 Free vibration properties for Example 9.7

λ_k	Nondimensional natural frequency	Natural frequency ω_k(rad/s)	C_k
6.71	2.59	149.55	0.715
443.5	21.06	1216.0	0.617
3682.1	60.68	3483.0	0.593
14,371.2	119.88	6922.0	0.584
39,533.3	198.83	11,480.0	0.582
88,513.2	297.51	17,178.0	0.434

The expansion theorem implies that the solution of the nonhomogeneous problem can be expanded in a series of normalized mode shapes. To this end,

$$w(x, t) = \sum_{k=1}^{\infty} p_k(t)X_k(x)$$

Substituting for $w(x, t)$ into the governing partial differential equation, multiplying by $X_j(x)$ for an arbitrary j, and integrating from 0 to 1 leads to

$$\sum_{k=1}^{\infty}(\ddot{p}_k + \lambda p_k) \int_0^1 X_j(x)X_k(X)\,dx = 0$$

The mutual orthonormality of the mode shapes implies

$$\int_0^1 X_j(x)X_k(x)\,dx = \delta_{jk} - \beta X_j(1)X_k(1)$$

Use of this orthogonality condition leads to

$$\ddot{p}_j + \lambda_j p_j = \sum_{k=1}^{\infty} (\ddot{p}_k + \lambda_k p_k)\beta X_j(1) X_k(1)$$

Substituting for $w(x, t)$ from the expansion theorem in the nonhomogeneous boundary condition leads to

$$\sum_{k=1}^{\infty} \frac{d^3 X_k(1)}{dx^3} p_k(t) = \alpha \sin \tilde{\omega} t + \beta \sum_{k=1}^{\infty} X_k(1) \ddot{p}_k(t)$$

The mode shapes satisfy the boundary conditions for the nonhomogeneous problem. Thus

$$\frac{d^3 X(1)}{dx^3} = -\lambda_k \beta X_k(1)$$

which when used in the preceding equation gives

$$\sum_{k=1}^{\infty} (\ddot{p}_k + \lambda_k p_k) X_k(1) = \alpha \sin \tilde{\omega} t$$

and which when substituted into the previously derived differential equations for the principal coordinates uncouples these equations and gives

$$\ddot{p}_j + \lambda_j p_j = \alpha X_j(1) \sin \tilde{\omega} t \qquad j = 1, 2, \ldots$$

The steady-state solution for each of the principal coordinates is now easily obtained and the expansion theorem is used to write the steady-state solution as

$$w(x, t) = \left[\sum_{k=1}^{\infty} \frac{\alpha X_k(1)}{\lambda_k - \tilde{\omega}^2} X_k(x) \right] \sin \tilde{\omega} t$$

The nondimensional steady-state amplitude of the end of the beam is

$$\alpha \sum_{k=1}^{\infty} \frac{X_k^2(1)}{\lambda_k - \tilde{\omega}^2} = 1.67 \times 10^{-4}$$

The dimensional amplitude is obtained using Eq. (9.69) as $1.67 \times 10^{-4}(3 \text{ m}) = 4.0$ mm.

9.5 ENERGY METHODS

Consider a differential element of the shaft of Fig. 9.4. Assuming elastic behavior throughout, a shear stress distribution is developed across the cross section of the shaft according to

$$\tau = \frac{Tr}{J} \qquad [9.77]$$

where $T(x, t)$ is the resisting torque in the cross section and r is the distance from the center of the shaft to a point in its cross section. The total strain energy in the element is

$$dV = \frac{1}{2G}\left(\int \tau^2\, dA\right) dx \qquad [9.78]$$

Substitution of Eqs. (9.77) and (9.11) into Eq. (9.78) leads to

$$dV = \frac{G}{2}\left(\frac{\partial \theta}{\partial x}\right)^2 \left(\int r^2\, dA\right) dx \qquad [9.79]$$

Noting that $J = \int r^2\, dA$ and integrating over the entire length of the shaft, the total strain energy becomes

$$V = \frac{1}{2}\int_0^L JG\left(\frac{\partial \theta}{\partial x}\right)^2 dx \qquad [9.80]$$

The kinetic energy of the differential element is

$$dT = \frac{1}{2}\rho J\left(\frac{\partial \theta}{\partial t}\right)^2 dx \qquad [9.81]$$

where ρ is the mass density of the shaft's material. The total kinetic energy of the shaft is

$$T = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial \theta}{\partial t}\right)^2 dx \qquad [9.82]$$

For a conservative system the maximum potential energy is equal to the maximum kinetic energy. Thus if the free oscillations of the shaft are described by

$$\theta(x, t) = u(x)\sin \omega t \qquad [9.83]$$

then

$$\omega^2 = \frac{\displaystyle\int_0^L JG\left(\frac{du}{dx}\right)^2 dx}{\displaystyle\int_0^L \rho J u^2\, dx} \qquad [9.84]$$

Introducing the nondimensional variables of Eq. (9.69) into Eq. (9.84) and assuming the shaft is uniform leads to

$$\tilde{\omega}^2 = \frac{\displaystyle\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\displaystyle\int_0^1 u^2\, dx} \qquad [9.85]$$

where

$$\tilde{\omega} = L\sqrt{\frac{\rho}{G}}\,\omega \qquad [9.86]$$

For any function $w(x)$ which satisfies the boundary conditions specified for the shaft, define

$$R(w) = \frac{\displaystyle\int_0^L JG\left(\frac{dw}{dx}\right)^2 dx}{\displaystyle\int_0^L \rho J w^2\, dx} \tag{9.87}$$

$R(w)$ is Rayleigh's quotient for this continuous system. If $w(x)$ is a mode shape, then $R(w)$ is equal to the square of the natural frequency of that mode. If w is not a mode shape, then $R(w)$ is a scalar function of w. As for discrete systems, $R(w)$ is a minimum when w is a mode shape. Hence Rayleigh's quotient can be used to approximate the lowest natural frequency for the continuous system.

9.8 Use Rayleigh's quotient to approximate the lowest natural frequency of the tapered circular shaft of Fig. 9.19.

Solution:

The polar moment of inertia varies over the length of the shaft as

$$J(x) = \frac{\pi}{2}(0.2 - 0.05x)^4$$

A trial function which satisfies the boundary conditions $w(0) = 0$ and $dw/dx(3\text{ m}) = 0$ is

$$w(x) = \sin\frac{\pi}{6}x$$

An upper bound and approximation on the lowest natural frequency is

$$R(w) = \frac{80 \times 10^9 \dfrac{\text{N}}{\text{m}^2}\dfrac{\pi}{2}\displaystyle\int_0^3 (0.2 - 0.05x)^4\left(\frac{\pi}{6}\right)^2 \cos^2\frac{\pi}{6}x\, dx}{7850 \dfrac{\text{kg}}{\text{m}^3}\dfrac{\pi}{2}\displaystyle\int_0^3 (0.2 - 0.05x)^4 \sin^2\frac{\pi}{6}x\, dx}$$

$$\omega_1 \le [R(w)]^{1/2} = 3767 \text{ rad/s}$$

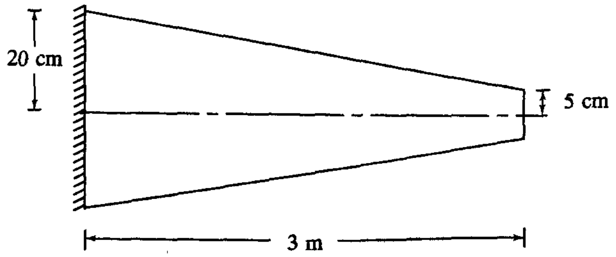


Figure 9.19 System of Example 9.8.

Rayleigh's quotient can be generalized as the ratio of a potential energy scalar to a kinetic energy scalar product, where the energy products are defined by integrals, perhaps with additional terms to account for discrete masses or springs.

$$R(w) = \frac{(w, w)_V}{(w, w)_T} \tag{9.88}$$

Rayleigh's quotient can be applied to any continuous system. Table 9.6 gives the appropriate form of the scalar products for several continuous systems.

A method based on Rayleigh's quotient, called the *Rayleigh-Ritz method*, can be used to approximate a finite number of the lowest natural frequencies of a continuous system. Let $u_1(x)$, $u_2(x)$, $\ldots$, $u_n(x)$ be n linearly independent functions, each of which satisfies the boundary conditions for a specific continuous system. An approximation to the free-vibration response of the continuous system is assumed as

$$w(x) = \sum_{i=1}^{n} c_i u_i(x) \tag{9.89}$$

Table 9.6 Scalar products for Rayleigh-Ritz method

Structural element	Case	$(u, v)_T$	$(u, v)_V$
Torsional shaft	No added disks or springs	$\int_0^L \rho J u(x) v(x)\, dx$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx} dx$
	Added disk at $x = \tilde{x}$	$\int_0^L \rho J u(x) v(x)\, dx + I_D u(\tilde{x}) v(\tilde{x})$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx} dx$
	Torsional spring at $x = \tilde{x}$	$\int_0^L \rho J u(x) v(x)\, dx$	$\int_0^L GJ \frac{du}{dx}\frac{dv}{dx} dx + k_t u(\tilde{x})v$
Longitudinal bar	No added masses or springs	$\int_0^L \rho A u(x) v(x)\, dx$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx} dx$
	Added mass at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\, dx + m u(\tilde{x}) v(\tilde{x})$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx} dx$
	Spring at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\, dx$	$\int_0^L EA \frac{du}{dx}\frac{dv}{dx} dx + k u(\tilde{x}) v$
Beam	No added masses, disks, or springs	$\int_0^L \rho A u(x) v(x)\, dx$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx$
	Added mass at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\, dx + m u(\tilde{x}) v(\tilde{x})$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx$
	Added spring at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\, dx$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx + k u(\tilde{x}$
	Added disk (I_D) at $x = \tilde{x}$	$\int_0^L \rho A u(x) v(x)\, dx + I_D \frac{du(\tilde{x})}{dx}\frac{dv(\tilde{x})}{dx}$	$\int_0^L EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx$

Equation (9.89) is substituted into Rayleigh's quotient which is rewritten as

$$R(w)(w, w)_T = (w, w)_V \qquad [9.90]$$

Since $R(w)$ is stationary near a mode shape, a good approximation to the natural frequencies and mode shapes is obtained by setting

$$\frac{\partial R}{\partial c_1} = \frac{\partial R}{\partial c_2} = \cdots = \frac{\partial R}{\partial c_n} = 0 \qquad [9.91]$$

Differentiating Eq. (9.90) with respect to c_k for any $k = 1, 2, \ldots, n$ and using Eq. (9.91) gives

$$R(w)\frac{\partial (w, w)_T}{\partial c_k} = \frac{\partial (w, w)_V}{\partial c_k} \qquad [9.92]$$

Developing Eq. (9.92) for each $k = 1, 2, \ldots, n$ leads to n linear homogeneous equations to solve for $c_1, c_2, \ldots, c_n$ in terms of the parameter $R(w)$. Since the equations are homogeneous, a nontrivial solution is available if and only if the determinant is set equal to zero, yielding an nth-order polynomial equation for $R(w)$. The roots of the polynomial are the squares of the approximations to the lowest natural frequencies. Approximations for the mode shapes can be obtained by returning to the homogeneous equations. The method is illustrated in the following example.

---

9 **Use** the Rayleigh-Ritz method to approximate the two lowest natural frequencies of Example 9.1.

Solution:

Two polynomials which satisfy the boundary conditions of Example 9.1 are

$$u_1(x) = 2x - x^2 \qquad u_2(x) = 3x - x^3$$

An approximation to the mode shape is developed as

$$w(x) = c_1(2x - x^2) + c_2(3x - x^3)$$

Calculation of the energy scalar products gives

$$(w, w)_T = \int_0^1 \left[c_1(2x - x^2) + c_2(3x - x^3)\right]^2 dx = \frac{8}{15}c_1^2 + \frac{61}{30}c_1c_2 + \frac{204}{105}c_2^2$$

$$(w, w)_V = \int_0^1 \left[c_1(2 - 2x) + c_2(3 - 3x^2)\right]^2 dx = \frac{4}{3}c_1^2 + 5c_1c_2 + \frac{24}{5}c_2^2$$

Application of Eq. (9.92) leads to

$$\left(\frac{8}{3} - \frac{16}{15}R\right)c_1 + \left(5 - \frac{61}{30}R\right)c_2 = 0$$

$$\left(5 - \frac{61}{30}R\right)c_1 + \left(\frac{48}{5} - \frac{136}{35}R\right)c_2 = 0$$

A nontrivial solution of the preceding equations is obtained if and only if

$$\det \begin{bmatrix} \frac{8}{3} - \frac{16}{15}R & 5 - \frac{61}{30}R \\ 5 - \frac{61}{30}R & \frac{48}{5} - \frac{136}{35}R \end{bmatrix} = 0$$

Evaluation of the determinant leads to

$$9.24R^2 - 241.0R + 538.0 = 0$$

whose roots are

$$R = 2.467, 23.610$$

The natural frequency approximations are

$$\omega_1 \approx 1.571 \qquad \omega_2 \approx 4.859$$

The approximation to the lowest natural frequency is excellent. The approximation to the second natural frequency is also very good, being only 3.3 percent higher than the exact value.

The mode shape approximations are obtained by solving for c_2 in terms of c_1 for each R and then substituting into the expression for $w(x)$ with c_1 remaining arbitrary. This leads to

$$w_1(x) = 7.58x - x^2 - 1.86x^3$$
$$w_2(x) = 0.4295x - x^2 + 0.5235x^3$$

The approximate mode shapes plotted in Fig. 9.20 have been normalized such that $w_i(1) = 1$. These compare favorably to the first two mode shapes for a fixed-free torsional shaft plotted in Fig. 9.6.

PROBLEMS

9.1. Calculate the speed of longitudinal waves in a 3-m-long steel bar ($E = 210 \times 10^9$ N/m^2, $\rho = 7850$ kg/m^3) of a circular cross section of 20 mm radius.

9.2. Calculate the three lowest torsional natural frequencies of a solid 20-cm-radius steel shaft ($\rho = 7500$ kg/m^3, $G = 80 \times 10^9$ N/m^2) of length 1.5 m that is fixed at one end and free at its other end.

9.3. A 5000-N · m torque is statically applied to the free end of the shaft of Prob. 9.2 and suddenly removed. Plot the time-dependent angular displacement of the free end.

9.4. A 5000-N · m torque is statically applied to the midspan of the shaft of Prob. 9.2 and suddenly removed. Determine an expression for the time-dependent angular displacement of the free end of the shaft.

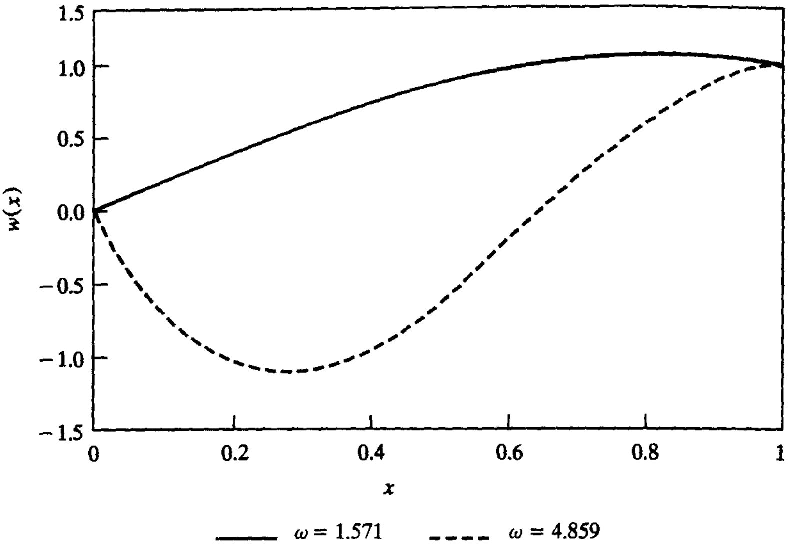


Figure 9.20 Rayleigh-Ritz approximations to the two lowest mode shapes of a fixed-free torsional shaft.

9.5. A steel shaft ($\rho = 7850\ \text{kg/m}^3$, $G = 85 \times 10^9\ \text{N/m}^2$) of inner radius 30 mm and outer radius 50 mm and length 1.0 m is fixed at both ends. Determine the three lowest natural frequencies of the shaft.

9.6. A 10,000-N · m torque is applied to the midspan of the shaft of Prob. 9.5 and suddenly removed. Determine the time-dependent angular displacement of the midspan of the shaft.

9.7. A motor of mass moment of inertia 85 kg · m^2 is attached to the end of the shaft of Prob. 9.2. Determine the three lowest natural frequencies of the shaft and motor assembly. Compare the lowest natural frequency to that obtained by making a one-degree-of-freedom model and approximating the inertia effects of the shaft.

9.8. Show the orthogonality of the two lowest mode shapes of the system in Prob. 9.7.

9.9. Operation of the motor attached to the shaft of Prob. 9.7 produces a harmonic torque of amplitude 2000 N · m at a frequency of 110 Hz. Determine the steady-state angular displacement of the end of the shaft.

9.10. A 20-cm-diameter, 2-m-long steel shaft ($\rho = 7600\ \text{kg/m}^3$, $G = 80 \times 10^9\ \text{N/m}^2$) has rotors of mass moment of inertia 110 kg · m^2 and 65 kg · m^2 attached to its ends. Determine the three lowest natural frequencies of the shaft. Compare the lowest nonzero natural frequency to that obtained by using a two-degree-of-freedom model, ignoring the inertia of the shaft.

9.11. Determine an expression for the natural frequencies of the shaft of Fig. P9.11.

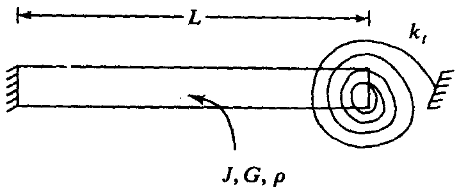


FIGURE P9.11

9.12. An oil well drilling tool is modeled as a bit attached to the end of a long shaft, unrestrained from rotation at its fixed end.

(a) Determine the equation defining the natural frequencies of the drilling tool.

(b) For a particular operation, the shaft ($\rho = 7500$ kg/m^3, $G = 80 \times 10^9$ N/m^2) is 20 m long with a 20-cm diameter. The tool operates at a speed of 400 rad/s. What are the limits on the moment of inertia of the drill bit such that the two lowest nonzero natural frequencies of the tool are not within 20 percent of the operating speed?

9.13. The shaft of Prob. 9.2 is at rest in equilibrium when the time-dependent moment of Fig. P9.13 is applied to the end of the shaft. Determine the time-dependent form of the resulting torsional oscillations.

9.14. The shaft of Prob. 9.2 is at rest in equilibrium when it is subject to the uniform time-dependent torque loading per unit length of Fig. P9.14. Determine the time-dependent form of the resulting torsional oscillations.

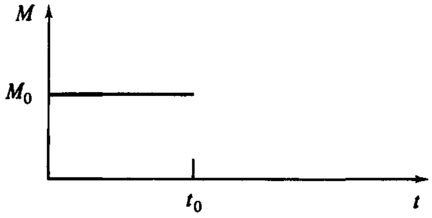


FIGURE P9.13

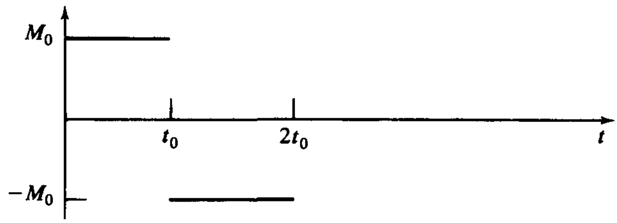


FIGURE P9.14

9.15. The elastic bar of Fig. P9.15 is undergoing longitudinal vibrations. Let $u(x, t)$ be the time-dependent displacement of a particle along the centroidal axis of the bar, initially a distance x from the left support.

(a) Draw free-body diagrams showing the external and effective forces acting on a differential element of thickness dx, a distance x from the left end of the bar at an arbitrary instant of time.

(b) Show that the governing partial differential equation is

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

(c) Introduce nondimensional variables to derive a nondimensional partial differential equation.

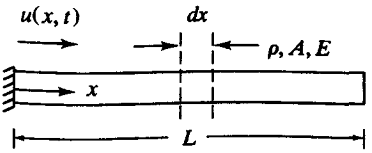


FIGURE P9.15

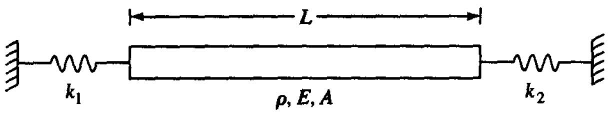
9.16. Using the results of Prob. 9.15, determine the natural frequencies of longitudinal vibrations of a bar fixed at one end and free at the other.

9.17. Show the orthogonality of mode shapes of longitudinal vibration of a bar fixed at one end and free at its other end.

9.18. A large industrial piston operates at 1000 Hz. The piston head has a mass of 20 kg. The shaft is made from steel ($\rho = 7500$ kg/m^3, $E = 210 \times 10^9$ N/m^2). For what shaft diameters will all natural frequencies be out of the range of 900 to 1100 Hz?

9.19. The free end of the piston of Prob. 9.18 is subject to a force $1000 \sin \omega t$ N, where $\omega =$ 100 Hz. If the diameter of the shaft is 8 cm, determine the steady-state response of the piston.

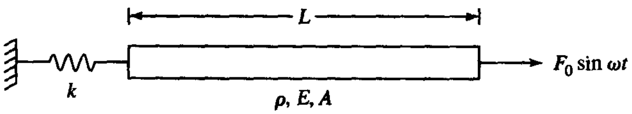
9.20. Determine the five lowest natural frequencies of the system of Fig. P9.20.



$\rho = 7500$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$A = 1.5 \times 10^{-5}$ m^2
$L = 3$ m
$k_1 = 1 \times 10^6$ N/m
$k_2 = 1.5 \times 10^6$ N/m

FIGURE P9.20

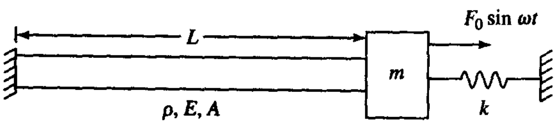
9.21. Determine the steady-state response of the system of Fig. P9.21.



$\rho = 7500$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$A = 4.5 \times 10^{-5}$ m^2
$k = 9 \times 10^5$ N/m
$F_0 = 800$ N
$\omega = 100$ rad/s
$L = 3.3$ m

FIGURE P9.21

9.22. Determine the steady-state response of the system of Fig. P9.22.



$\rho = 7500$ kg/m^3
$E = 200 \times 10^9$ N/m^2
$A = 4.5 \times 10^{-5}$ m^2
$k = 9 \times 10^5$ N/m
$m = 2.5$ kg
$L = 3.5$ m
$F_0 = 600$ N
$\omega = 450$ rad/s

FIGURE P9.22

9.23. Draw frequency response curves for the response of the disk at the end of the shaft in Example 9.3. Plot the curves for $\beta = 0.5$, $\beta = 2$, and $\beta = 20.0$.

9.24. Determine the steady-state response of a circular shaft subject to a uniform torque per unit length $T_0 \sin \omega t$ applied over its entire length.

9.25. Determine the steady-state response of the system of Fig. P9.25.

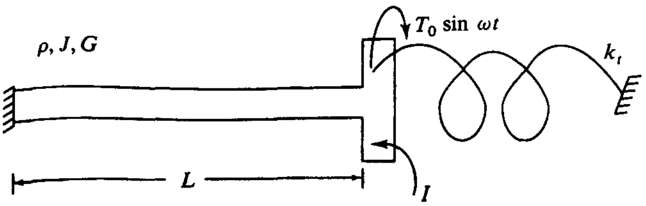


FIGURE P9.25

9.26. Propeller blades totaling 1200 kg with a total mass moment of inertia of 155 kg · m^2 are attached to a solid circular shaft ($\rho = 5000$ kg/m^3, $G = 60 \times 10^9$ N/m^2, $E = 140 \times 10^9$ N/m^2) of radius 40 cm and length 20 m. The other end of the shaft is fixed in an ocean liner. Determine

(*a*) The lowest natural frequency for torsional oscillations of the propeller.

(*b*) The lowest natural frequency for longitudinal motion of the propeller.

9.27. A pipe used to convey fluid is cantilevered from a wall. The steel pipe ($\rho = 7500$ kg/m^3, $G = 80 \times 10^9$ N/m^2, $E = 200 \times 10^9$ N/m^2) has an inner radius of 20 cm, a thickness of 1 cm, and a length of 4.6 m. For an empty pipe determine

(*a*) The five lowest natural frequencies for torsional oscillation.

(*b*) The five lowest natural frequencies for longitudinal vibration.

(*c*) The five lowest natural frequencies for transverse motion.

9.28–9.31. Each of the beams has $\rho = 8000$ kg/m^3, $E = 200 \times 10^9$ N/m^2, $I = 4 \times 10^{-5}$ m^4, $A = 1.2 \times 10^{-2}$ m^2, $L = 1.4$ m. Use Table 9.4 to calculate the beam's three lowest natural frequencies of transverse vibration.

FIGURE P9.28

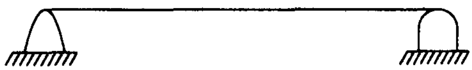


FIGURE P9.29

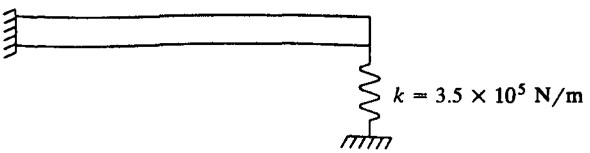


FIGURE P9.30

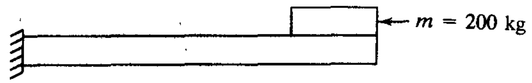


FIGURE P9.31

9.32. Verify the characteristic equation given in Table 9.4 for a pinned-free beam.

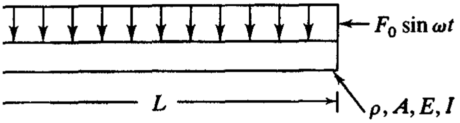
9.33. Verify the characteristic equation given in Table 9.4 for a fixed-fixed beam.

9.34. The characteristic equations given in Table 9.4 for a free-free beam and a fixed-fixed beam are identical. Explain both mathematically and physically why the lowest natural frequency for a free-free beam is zero, but not for a fixed-fixed beam.

9.35. Verify the orthogonality of the eigenfunctions given in Table 9.4 for a pinned-free beam.

9.36. Verify the orthogonality of the eigenfunctions given in Table 9.4 for a fixed-attached mass beam.

9.37–9.41. Determine the time-dependent displacement for the beam shown.



;URE P9.37

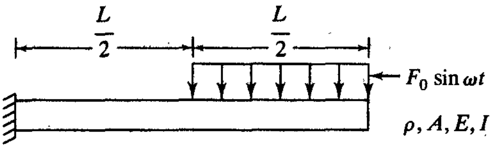
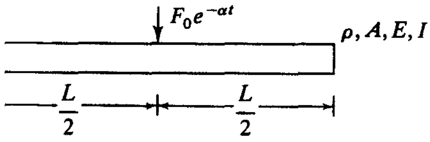


FIGURE P9.38



URE P9.39

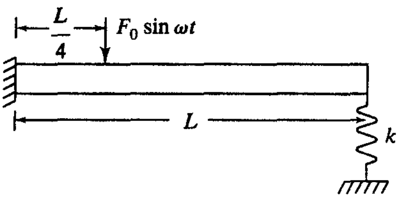


FIGURE P9.40

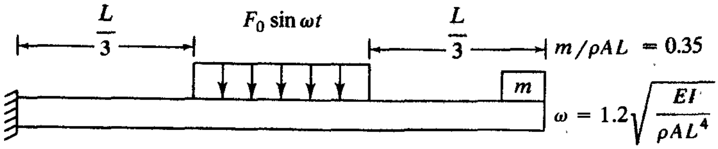


FIGURE P9.41

9.42. A root manipulator is 60 cm long, made of steel ($E = 210 \times 10^9$ N/m^2, $\rho = 7500$ kg/m^3), and has the cross section of Fig. P9.42. One end of the manipulator is fixed and a 1-kg mass is attached to its opposite end. Determine the three lowest natural frequencies for transverse vibration of the manipulator.

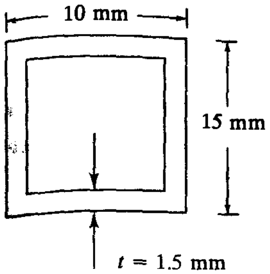


FIGURE P9.42

Problems 9.43 to 9.45 refer to the problem of vortex shedding from a street light fixture considere in Example 3.6.

9.43. Model the light structure as a cantilever beam with an attached mass. Determine the fi\ lowest natural frequencies of the structure.

9.44. Assume the vortex shedding produces a uniform force per unit length of $F_0 \sin \omega t$ applie over the upper three-fourths of the structure. For a wind speed of 60 mph, calculate tl maximum displacement of the light fixture.

9.45. For a wind speed of 20 mph, calculate the maximum displacement of the light fixture.

9.46. The steam pipe of Fig. P9.46 is suspended from the ceiling in an industrial plant. heavy machine with a rotating unbalance is placed on the floor above the machine causi vibrations of the ceiling. If the frequency of the oscillations is 150 Hz and the amplitu of displacement of the pipe's left support is 0.5 mm and the amplitude of displacement the pipe's right support is 0.8 mm, determine the maximum deflection of the center of t pipe.

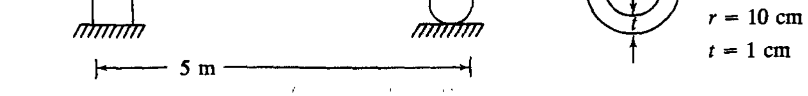


FIGURE P9.46

9.47. A simplified model of the rocket of Fig. P9.47 is a free-free beam.
(*a*) Calculate the five lowest natural frequencies for longitudinal vibration.
(*b*) Calculate the five lowest natural frequencies for transverse vibration.

9.48. Longitudinal vibrations are initiated in the rocket of Fig. P9.47 when thrust is de oped. Determine the Laplace transform of the transient response $\bar{u}(x, s)$ when the th of Fig. P9.48 is developed. Do not invert the transform.

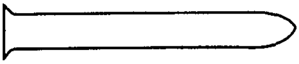


FIGURE P9.47

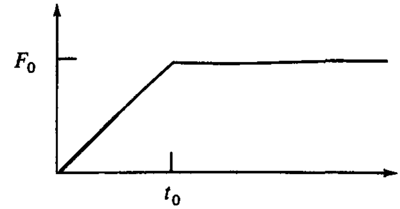


FIGURE P9.48

9.49. Determine the response of a cantilever beam when the fixed support is subject to a displacement $w(t) = A \sin \omega t$. Use the Laplace transform method and determine $\bar{y}(x, s)$. Do not invert.

9.50. The tail rotor blades of a helicopter have a rotating unbalance of magnitude 0.5 kg · m and operate at a speed of 1200 rpm. Modeling the tail section as a cantilever beam of length 3.5 m with $EI = 3.1 \times 10^6$ N · m^2, determine the steady-state response of the tail section.

9.51. Repeat Prob. 9.50 when the helicopter has a constant vertical acceleration of 12.1 m/s^2. Use the Laplace transform method to determine $\bar{y}(x, s)$. Do not invert.

9.52. Solve Prob. 3.45 assuming the beam is a continuous system and is made of a material of $\rho = 7500$ kg/m^3.

9.53. Determine the steady-state amplitude of the engine of Fig. P9.53.

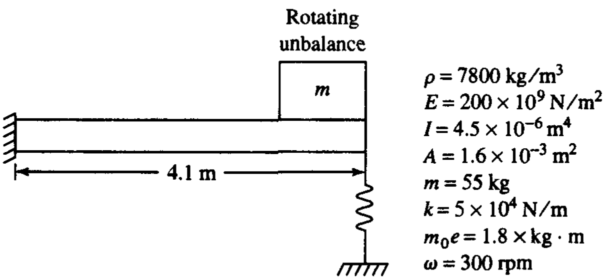


FIGURE P9.53

9.54. Show that the differential equation governing free vibration of a uniform beam subject to a constant axial load, P, is

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial w}{\partial t^2} = 0$$

9.55. Determine the frequency equation for a simply supported beam subject to an axial load.

9.56. Determine the frequency equation for a fixed-free beam subject to an axial load.

9.57. A fixed-fixed beam is made of a material with a coefficient of thermal expansion α. After installed, the temperature is decreased by ΔT. Determine the beam's frequency equation.

9.58. Show orthogonality of the mode shapes for a simply supported beam subject to an axial load.

9.59. Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft fixed at both ends.

9.60. Use Rayleigh's quotient to approximate the lowest natural frequency of a torsional shaft with a disk of mass moment of inertia I placed at its midspan. The shaft is fixed at both ends.

9.61. Use Rayleigh's quotient to approximate the lowest natural frequency of a fixed-free beam.

9.62. Use Rayleigh's quotient to approximate the lowest natural frequency of a simply supported beam with a mass m at its midspan. Use $w(x) = \sin(\pi x/L)$ as the trial function.

9.63. Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of a fixed-free beam.

9.64. Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies of the system of Fig. P9.64.

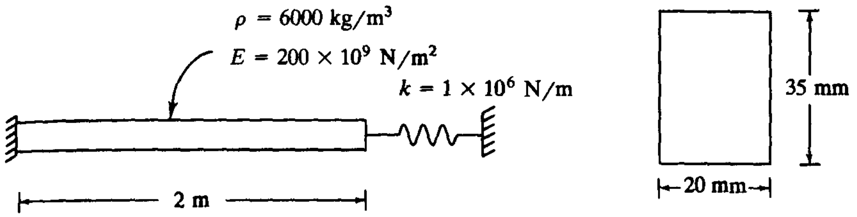


FIGURE P9.64

9.65. Use the Rayleigh-Ritz method to approximate the two lowest natural frequencies for the system of Fig. P9.65.

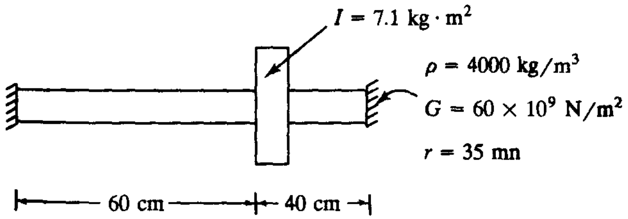


FIGURE P9.65

MATLAB PROBLEMS

The files VIBES_9A.m, VIBES_9B.m, VIBES_9C.m, and VIBES_9D.m are used to provide the natural frequencies, mode shapes, and normalization constants for four of the beams listed in Table 9.4. The beams and the corresponding files are: fixed-free-VIBES_9A, fixed-pinned-VIBES_9B, fixed-linear spring-VIBES_9C, pinned-linear spring-VIBES_9D. These files may be used in solving many of the problems of Chap. 9.

M9.1. The file VIBES_9E.m is used to plot the mode shapes for the system of Example 9.2. Use VIBES_9E to plot the first three mode shapes for (*a*) $\beta = 0.1$, (*b*) $\beta = 1.4$, (*c*) $\beta = 5.0$.

M9.2. The file VIBES_9F.m is used to determine the frequency response for the system of Example 9.7. Use the file to choose a standard beam cross section such that the maximum steady-state displacement of the machine is less than 5 mm for all operating speeds between 1500 and 2500 rpm. Assume all parameters are as given in Example 9.7 except the mass of the beam and I. Assume the beam will be made of steel ($\rho = 7600 \text{ kg/m}^3$).

M9.3. The file VIBES_9G.m provides a Rayleigh-Ritz approximation for the solution of Example 6.5. Use VIBES_9G to

(*a*) Determine whether the W16 × 100 section is better than the W27 × 114 section to meet the requirements of Example 6.5.

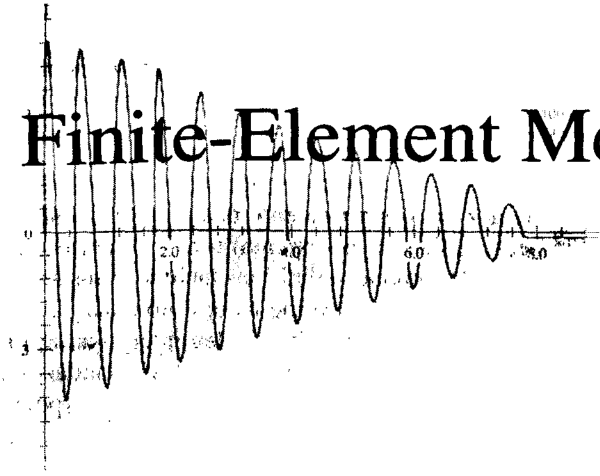
(*b*) Using the W27 × 100 section, plot the lowest natural frequency as a function of mass of the machine (up to 5 tons).

M9.4. Write a MATLAB file that uses numerical integration to illustrate mode shape orthogonality for the system of Example 9.2.

M9.5. Write a MATLAB file that uses numerical integration to illustrate mode shape orthogonality for the system of Example 9.7.

chapter

10 Finite-Element Method



10.1 Introduction

The finite-element method is a powerful numerical method that is used to provide approximations to solutions of static and dynamic problems for continuous systems. The disciplines in which the finite-element method can be applied include stress analysis, heat transfer, electromagnetics, fluid flow, and vibrations. Application of the finite-element method to a continuous system requires the system be divided into a finite number of discrete elements. Interpolations for the dependent variables are assumed across each element and are chosen to assure appropriate interelement continuity. The interpolating functions are developed in terms of the unknown values of the dependent variables at discrete points, called *nodes*. The nodes for a one-dimensional system are located at element boundaries. A variational principle is applied to derive equations whose solution leads to approximations to the dependent variables at the nodes. The defined interpolations are used to provide approximations to the dependent variables across the system. Lagrange's equations, derived by the calculus of variations, is applied for vibrations problems, resulting in a set of differential equations for the dependent variables at the nodes.

The finite-element method for vibration problems could be derived by applying the Rayleigh-Ritz method of Sec. 9.5 with the interpolating functions, $u_1(x)$, $u_2(x), \ldots, u_n(x)$, chosen to be defined piecewise over each element. Consider application of the Rayleigh-Ritz method to approximate the natural frequencies and mode shapes of a bar. The governing equation, the wave equation, has second-order spatial derivatives. Thus the exact solution is at least twice differentiable. However, the energy scalar products used in evaluation of Rayleigh's quotient, given in Table 9.6, only require that approximate solutions be first-order differentiable. Thus functions that are only first-order differentiable are permissible interpolating functions for Rayleigh-Ritz approximations.

Boundary conditions for continuous systems are classified as being of two types. *Geometric boundary conditions* are those that must be satisfied according to geometric constraints. For example, $u(0) = 0$, if $x = 0$ is a fixed support for a bar problem, is a geometric boundary condition. *Natural boundary conditions* are those that must be satisfied as a result of force and moment balances. For example, $du/dx = 0$ at $x = 1$, if $x = 1$ is a free end, is a natural boundary condition. This condition occurs because there is no external force at the free end. Note from Table 9.6 that the energy scalar product definitions include terms arising because of natural boundary conditions. Thus, since the natural boundary conditions are incorporated into the energy scalar products, the chosen interpolating functions for a Rayleigh-Ritz approximation must satisfy only geometric boundary conditions.

The set of *admissible functions* for use as interpolating functions in a Rayleigh-Ritz approximation to solutions of the wave equation consists of those that are first-order differentiable and satisfy all geometric boundary conditions (displacement conditions). By similar arguments, it is determined that the set of admissible functions for use as basis functions in a Rayleigh-Ritz approximation to solutions of the beam equation consists of those that are second-order differentiable and satisfy all geometric boundary conditions (displacement and slope conditions).

The Rayleigh-Ritz can be difficult to apply for vibrations problems. The *assumed modes method*, introduced in the next section, is based on application of Lagrange's equations and leads to the same approximation for the same set of interpolating functions as the Rayleigh-Ritz method. The finite-element method will be developed from the assumed modes method.

10.2 ASSUMED MODES METHOD

Consider the forced vibrations of a longitudinal bar of Fig. 10.1. The displacement u is a function of the spatial coordinate x and time t, $u(x, t)$. Let $u_1(x)$, $u_2(x), \ldots, u_n(x)$ be a set of n linearly independent functions that are at least first-order differentiable and satisfy all of the system's geometric boundary conditions. An approximate solution is assumed as

$$u(x,t) = \sum_{i=1}^{n} w_i(t) u_i(x) \tag{10.1}$$

The kinetic energy of the bar, according to the approximation of Eq. (10.1), is calculated as

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2} m \left(\frac{\partial u}{\partial t}(x = L)\right)^2 \\ &= \frac{1}{2}\int_0^L \rho A \left(\sum_{i=1}^{n} \dot{w}_i u_i(x)\right)^2 dx + \frac{1}{2} m \left(\sum_{i=1}^{n} \dot{w}_i u_i(L)\right)^2 \end{aligned}$$

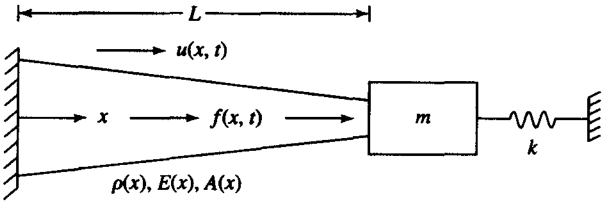
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{w}_i\dot{w}_j\left[\int_0^L \rho A u_i(x)u_j(x)\,dx + m u_i(L)u_j(L)\right]$$

Thus the kinetic energy has the quadratic form

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{w}_i\dot{w}_j \qquad [10.2]$$

where

$$m_{ij} = \int_0^L \rho A u_i(x)u_j(x)\,dx + m u_i(L)u_j(L) \qquad [10.3]$$



(a)

```
% Example 10.1
% Assumed modes method to determine natural frequencies, modes
%    and forced response of tapered bar with attached mass and
digits(5)
x=sym('x');
% Parameters
E=200*10^9;                    % Elastic modulus in N/m^2
rho=7600;                      % Mass density in kg/m^3
L=3.6;                         % Length in m
m=12;                          % Attached mass in kg
k=4*10^7;                      % Stiffness in N/m
% Functions
A=0.001*(1-0.002*x);           % Area in m^2
u(1)=sin(pi*x/(2*L));
u(2)=sin(3*pi*x/(2*L));        % Assumed modes
u(3)=sin(5*pi*x/(2*L));
% Mass and stiffness matrices
for i=1:3
   for j=1:i
      c1=m*subs(u(i),x,L)*subs(u(j),x,L);
      Mint=rho*A*u(i)*u(j);
      Kint=E*A*diff(u(i),x)*diff(u(j),x);
      M(i,j)=int(Mint,x,0,L)+m*c1;
      K(i,j)=int(Kint,x,0,L)+k*c1;
      M(j,i)=M(i,j);
```

Figure 10.1 (a) Forced longitudinal vibrations of a bar are described by a displacement function, $u(x, t)$; (b) MATLAB script for Example 10.1.

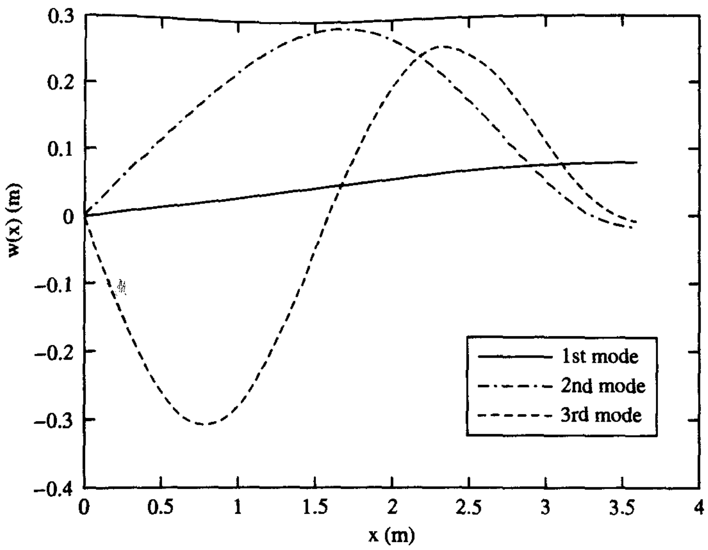
```
      K(j,i)=K(i,j);
   end
end
K1=double(K);
M1=double(M);
C=inv(M1)*K1;
[V,D]=eig(C);
for i=1:3
   w(i)=sqrt(D(i,i));
end
% Normalize mode shape vectors
E=V'*M*V;
for j=1:3
   for i=1:3
      P(i,j)=V(i,j)/sqrt(E(j,j));
   end
end
disp('Natural frequencies in rad/s')
w=vpa(w);disp(w)
disp('Modal matrix')
P=vpa(P);disp(P)
% Mode shapes
xx=linspace(0,L,37);
for k=1:37
   for i=1:3
      v(i,k)=0;
      for j=1:3
         v(i,k)=v(i,k)+P(j,i)*subs(u(j),x,xx(k));
      end
   end
end
plot(xx,v(1,:),'-',xx,v(2,:),'-.',xx,v(3,:),'--')
xlabel('x (m)')
ylabel('w(x) (m)')
legend('1st mode','2nd mode','3rd mode')
```

(*b*)

```
Natural frequencies in rad/s
[ 1863.6, 4898.0, 9720.4]

Modal matrix
[  .76137e-1,      .19529,  .75794e-1]
[ -.28668e-2,      .17555,    -.13695]
[  .96798e-3, -.36689e-1,    -.21976]
```

(*c*)

Figure 10.1 (Con't) (*b*) Con't; (*c*) Output from running script.



(*d*)

Figure 10.1 (Con't) (*d*) Assumed mode approximations to mode shape.

The potential energy of the system, according to the approximation of Eq. (10
is

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}k(u(L))^2$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \left[\int_0^L EA\frac{du_i}{dx}\frac{du_j}{dx}\,dx + ku_i(L)u_j(L)\right]$$

The potential energy has the quadratic form

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} w_i w_j \qquad [10$$

where
$$k_{ij} = \int_0^L EA \frac{du_i}{dx}\frac{du_j}{dx}\,dx + ku_i(L)u_j(L) \tag{10.5}$$

The virtual work done by the external force $f(x,t)$ due to a virtual displacement $\delta u(x,t)$ is

$$\delta W = \int_0^L f(x,t)\delta u(x,t)\,dx = \sum_{i=1}^{n} \delta w_i \int_0^L f(x,t)u_i(x)\,dx$$

The virtual work can be written as

$$\delta W = \sum_{i=1}^{n} Q_i \delta w_i \tag{10.6}$$

where
$$Q_i = \int_0^L f(x,t)u_i(x)\,dx \tag{10.7}$$

The assumed modes method approximates the solution to the forced vibrations of a continuous system with n degrees of freedom. The generalized coordinates for the n-degree-of-freedom model are the coefficient functions $w_1(t), w_2(t), \ldots, w_n(t)$. Quadratic forms of the kinetic and potential energies in terms of the generalized coordinates have been obtained. Use of Lagrange's equations, as applied in Sec. 5.4 to linear systems with quadratic energy forms, leads to differential equations of the form

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{K}\mathbf{w} = \mathbf{F} \tag{10.8}$$

where the elements of the mass matrix $\mathbf{M}$ are the coefficients of Eq. (10.3), the elements of the stiffness matrix $\mathbf{K}$ are the coefficients of Eq. (10.5), and the elements of the force vector $\mathbf{F}$ are the generalized forces of Eq. (10.7). If scalar product notation is used

$$m_{ij} = (u_i, u_j)_T \qquad k_{ij} = (u_i, u_j)_V \qquad Q_i = (f, u_i)$$

Approximations to the n lowest natural frequencies are obtained as the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The corresponding mode shape vectors are used in Eq. (10.1) to approximate the mode shapes for these frequencies. An approximation to the forced response is obtained by solving Eq. (10.8) using the methods of Chap. 7.

10.1 Use the assumed modes method to approximate the three lowest natural frequencies and mode shapes for the bar of Fig. 10.1*a* with $A(x) = 0.001(1 - 0.002x)$ m^2, $E = 200 \times 10^9$ N/m^2, $\rho = 7600$ kg/m^3, $L = 3.6$ m, $m = 12$ kg, and $k = 4 \times 10^7$ N/m. Use the interpolating functions

$$u_1(x) = \sin\left(\frac{\pi x}{2L}\right) \qquad u_2(x) = \sin\left(\frac{3\pi x}{2L}\right) \qquad u_3(x) = \sin\left(\frac{5\pi x}{2L}\right)$$

which are the first three mode shapes of a uniform fixed-free bar.

Solution:

Equations (10.3) and (10.5) are used to determine the elements of the mass and stiffness matrices, respectively, for the assumed modes approximation. For example,

$$m_{12} = \int_0^L \rho[0.001(1-0.002x)] \sin\left(\frac{\pi x}{2L}\right) \sin\left(\frac{3\pi x}{2L}\right) dx - m$$

$$k_{12} = \int_0^L E[0.001(1-0.002x)] \left(\frac{\pi}{2L}\right)\left(\frac{3\pi}{2L}\right) \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{3\pi x}{2L}\right) dx - k$$

The MATLAB script of Fig. 10.1*b* illustrates the use of symbolic algebra to determine the mass and stiffness matrices for this assumed modes approximation. The natural frequency approximations are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The eigenvectors are used to develop approximations to the mode shapes. If $\mathbf{X}_1 = [X_{11} \quad X_{12} \quad X_{13}]^T$ is the eigenvector corresponding to the eigenvalue that gives an approximation to the lowest natural frequency, then the approximation to the corresponding mode shape is

$$w_1(x) = X_{11}u_1(x) + X_{12}u_2(x) + X_{13}u_3(x)$$

The natural frequency approximations obtained by running the MATLAB script are given in Fig. 10.1*c*. The corresponding mode shape approximations are plotted in Fig. 10.1*d*.

10.3 GENERAL METHOD

Consider again the bar of Fig. 10.1. The bar is divided into n discrete segments, or elements. For purposes of discussion assume the elements are of equal length $l = L/n$. The discretization of a uniform bar into n elements of equal length l is shown in Fig. 10.2*a*. The piecewise defined interpolating functions of Fig. 10.2*b* are mathematically described as

$$\begin{aligned}
u_0(x) &= \left(-\frac{x}{l} + 1\right) u(x-l) \\
u_j(x) &= \left[\frac{x}{l} + (1-j)\right][u(x-(j-1)l) - u(x-jl)] \\
&\quad + \left[-\frac{x}{l} + (1+j)\right][u(x-jl) - u(x-(j+1)l] \qquad 1 \le j < n \\
u_n(x) &= \left[\frac{x}{l} + (1-n)\right] u(x-(n-1)l)
\end{aligned} \tag{10.9}$$

When the functions of Eq. (10.9) are used in an assumed modes approximation of the form

$$u(x,t) = \sum_{i=0}^{n} W_i(t)u_i(x) \tag{10.10}$$

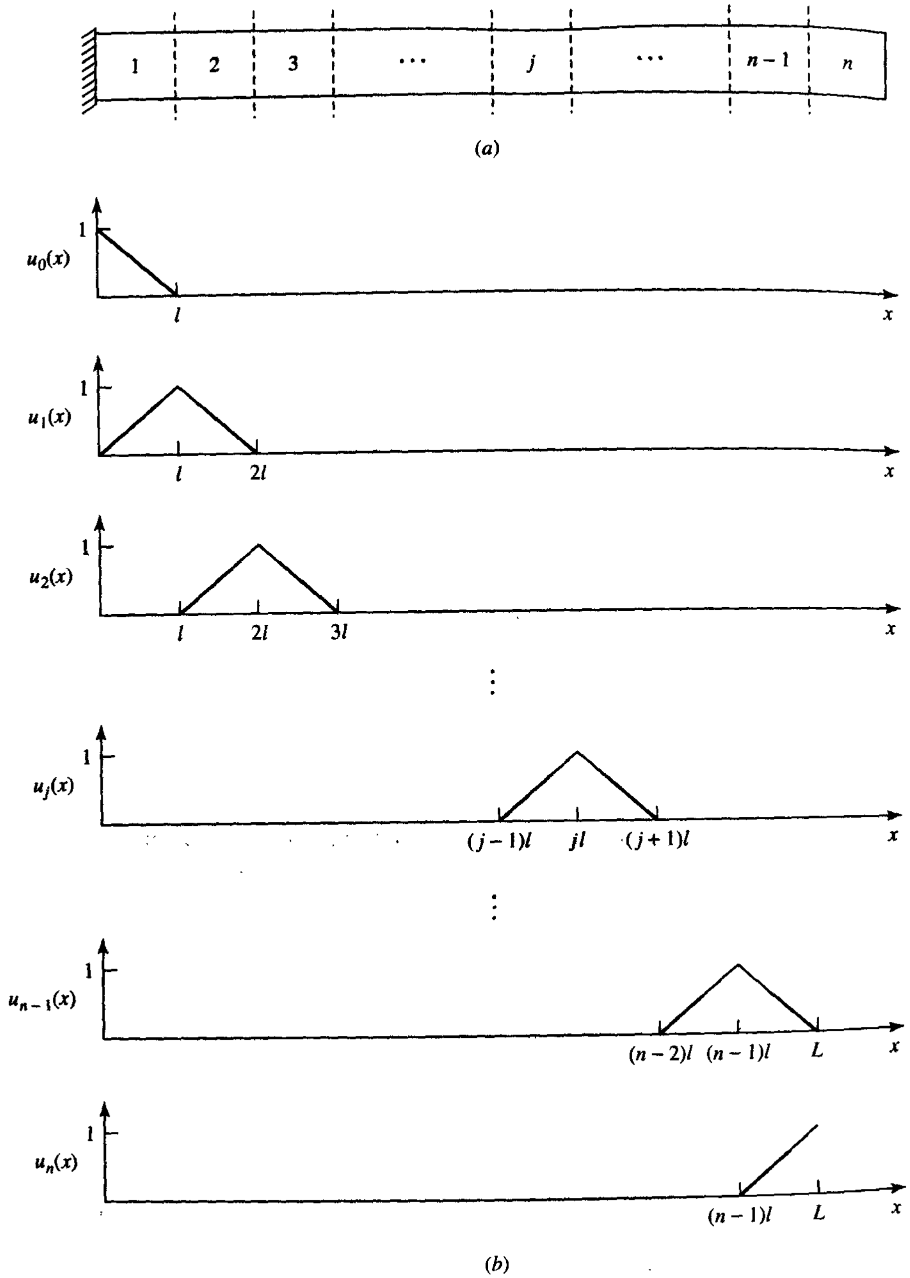


Figure 10.2 *(a)* Discretization of uniform bar into n elements of equal length l. *(b)* Interpolating functions that can be used in an assumed modes approximation.

then

$$u(jl, t) = W_j \tag{10.11}$$

Thus the generalized coordinates are the displacements at the element boundaries. The geometric boundary condition $w(0, t) = 0$ can be imposed simply by setting $W_0 = 0$.

The finite-element method is an application of the assumed modes method using piecewise-defined basis functions. The basis function $u_j(x)$ is nonzero only over the jth and $(j + 1)$st elements. The assumed modes method as described in Sec. 10.2 is applied. The mass matrix is developed from the kinetic energy, the stiffness matrix is developed from the potential energy, and the force vector is developed from the virtual work of the external forces. As a result of the piecewise definition of the interpolating functions, it is noted that $m_{ij} = (u_i, u_j)_T = 0$ unless $i = j - 1, j$, or $j + 1$.

When the interpolating functions of Eq. (10.9) are used in the assumed modes method, Eq. (10.10) can be rearranged to

$$u(x, t) = \sum_{i=1}^{n} \phi_i(x, t) \tag{10.12}$$

where

$$\begin{aligned}
\phi_1(x, t) &= W_1(t)\frac{x}{l}[1 - u(x - l)] \\
\phi_j(x, t) &= \frac{1}{l}\left[\left(W_{j+1}(t) - W_j(t)\right)x + (j + 1)lW_j(t) - jlW_{j+1}(t)\right] \\
&\qquad [u(x - jl) - u(x - (j + 1)l] \qquad 2 \le j < n \\
\phi_n(x, t) &= W_n(t)u(x - (n - 1)l)
\end{aligned} \tag{10.13}$$

Equations (10.12) and (10.13) are illustrated in Fig. 10.3. Equation (10.12) rewrites the assumed modes approximation as a linear combination of functions that are each nonzero only over one element. The functions are in terms of the displacements at the element boundaries. The finite-element method obtains approximations to the displacements of the nodes (the element boundaries). Fig. 10.3 illustrates that the finite-element method, as applied to this problem, assumes a linear interpolation between the nodal displacements.

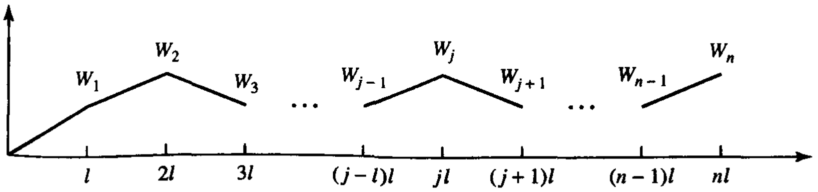


Figure 10.3 Linear interpolation between nodes for finite-element model of a bar.

Often a large number of elements are required to obtain accurate results for complex structures. Application of the finite-element method is more convenient when formulated as in Eq. (10.12). This allows an approximation function to be defined for each element in terms of the displacements at the element boundaries. The kinetic energy, potential energy, and work done by external forces are calculated for the element in terms of the generalized coordinates representing displacements at element boundaries. For example, the kinetic energy of element j can be written in the quadratic form

$$T_j = \frac{1}{2}\dot{\mathbf{w}}_j^T \mathbf{m}_j \dot{\mathbf{w}}_j \tag{10.14}$$

where $\mathbf{w} = [W_{j-1} \quad W_j]^T$ is the element displacement vector, the vector of boundary displacements, written in terms of global generalized coordinates and $\mathbf{m}_j$ is the element mass matrix written in local coordinates. The total kinetic energy of the system is the sum of the element kinetic energies

$$T = \sum_{j=1}^{n} T_j$$

and has the quadratic form

$$T = \frac{1}{2}\dot{\mathbf{W}}^T \mathbf{M} \dot{\mathbf{W}} \tag{10.15}$$

where $\mathbf{W} = [W_1 \quad W_2 \quad \cdots \quad W_n]^T$ is the global displacement vector, the vector of generalized coordinates, and $\mathbf{M}$ is the global mass matrix.

The above development suggests a computational procedure where the energy methods are used to develop the finite-element model. The system is divided into a finite number of discrete elements. The *global generalized coordinates* are the coordinates representing the degrees of freedom at the nodes. Each element has a specific number of degrees of freedom. The bar element, for example, has two degrees of freedom, the displacements of the ends of the element. A *local coordinate system* is defined for each element in the finite-element model. The kinetic energy, potential energy, and virtual work are determined for each element. The potential energy, for example, is written in quadratic form in terms of an element displacement vector and element stiffness matrix. Model elements for a bar, a torsional system, and a beam are developed in this fashion. The element mass and stiffness matrices are assembled into global mass and stiffness matrices. The differential equations are written in terms of the global generalized coordinates by using the global matrices. The homogeneous solution of the differential equations provides approximations to the natural frequencies and mode shapes. Nonhomogeneous equations are solved to provide approximation to the forced response.

The following sections provide the details of the method. The standard bar element and standard beam element, written in terms of local coordinates, are developed. Methods of assembling the element matrices into global matrices are discussed. Examples of application of the finite-element method to bar, beam, and truss problems are presented.

This chapter provides only an overview of the finite-element method. There is much more to the method that is beyond the scope of the discussion. This includes error analysis, element selection, substructuring, and algorithm development. Many excellent finite-element software packages exist.

10.4 THE BAR ELEMENT

A bar element of length l is illustrated in Fig. 10.4. The element has two degrees of freedom represented by w_1, the displacement of the left end of the element, and w_2, the displacement of the right end of the element. Define a local coordinate ξ, $0 \le \xi \le l$, along the axis of element. The linear displacement function for the element is

$$u(\xi, t) = \frac{1}{l}(w_2 - w_1)\xi + w_1 \qquad [10.16]$$

The kinetic energy of the element, assuming uniform properties, is

$$\begin{aligned} T &= \frac{1}{2}\int_0^l \rho A \left(\frac{\partial u}{\partial t}\right)^2 d\xi \\ &= \frac{1}{2}\int_0^l \rho A \left[\frac{1}{l}(\dot{w}_2 - \dot{w}_1)\xi + \dot{w}_1\right]^2 d\xi \\ &= \frac{1}{2}\frac{\rho A l}{3}(\dot{w}_1^2 + \dot{w}_1\dot{w}_2 + \dot{w}_2^2) \end{aligned} \qquad [10.17]$$

Equation (10.17) can be rewritten in the quadratic form

$$T = \frac{1}{2}\dot{\mathbf{w}}^T\mathbf{m}\dot{\mathbf{w}} = \frac{1}{2}\frac{\rho A l}{6}[\dot{w}_1 \quad \dot{w}_2]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} \qquad [10.18]$$

Thus the element mass matrix is

$$\mathbf{m} = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad [10.19]$$

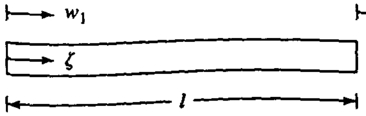


Figure 10.4 A bar element of length l has two degrees of freedom. A linear function in terms of local coordinate ξ interpolates displacement.

The potential energy of the element, assuming uniform properties, is

$$V = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial \xi}\right)^2 d\xi$$

$$= \frac{1}{2}\int_0^l EA\left[\frac{1}{l}(w_2 - w_1)\right]^2 d\xi \qquad [10.20]$$

$$= \frac{1}{2}\frac{EA}{l}\left(w_2^2 - 2w_1w_2 + w_1^2\right)$$

The potential energy can be written in the quadratic form

$$V = \frac{1}{2}\frac{EA}{l}[w_1 \quad w_2]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad [10.21]$$

from which the element stiffness matrix is determined as

$$\mathbf{k} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [10.22]$$

If the element has an external axial load $f(\xi, t)$, then the virtual work done by the load is

$$\delta W = \int_0^l f(\xi, t)\delta u(\xi, t)\, d\xi$$

$$= \int_0^l f(\xi, t)\left[\frac{1}{l}(\delta w_2 - \delta w_1)\xi + \delta w_1\right] d\xi$$

$$= \delta w_1 \int_0^l f(\xi, t)\left(1 - \frac{\xi}{l}\right) d\xi + \delta w_2 \int_0^l f(\xi, t)\frac{1}{l}\, d\xi$$

and the element generalized forces are

$$q_1 = \int_0^l f(\xi, t)\left(1 - \frac{\xi}{l}\right) d\xi \qquad q_2 = \int_0^l f(\xi, t)\frac{1}{l} d\xi \qquad [10.23]$$

The torsion element of Fig. 10.5 is developed in the same manner as the bar element. If w_1 is the angular displacement at the left end of the element and w_2 the angular displacement at the right end of the element, then the finite-element approximation to the angular displacement over the element is given by

$$\theta(\xi, t) = \left(1 - \frac{\xi}{l}\right) w_1 + \frac{1}{l} w_2 \qquad [10.24]$$

Application of Eq. (10.24) to the kinetic energy

$$T = \frac{1}{2}\int_0^l \rho J\left(\frac{\partial \theta}{\partial t}\right)^2 d\xi$$

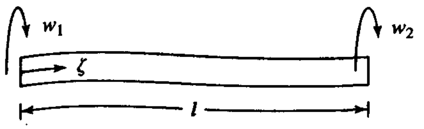


Figure 10.5 Uniform torsion element of length l has two degrees of freedom represented by w_1 and w_2, angular displacements at the ends of the element.

leads to the element mass matrix

$$\mathbf{m} = \frac{\rho J l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{10.25}$$

Application of Eq. (10.24) to the potential energy

$$V = \frac{1}{2}\int_0^l JG\left(\frac{\partial\theta}{\partial\xi}\right)^2 d\xi$$

leads to the element stiffness matrix

$$\mathbf{k} = \frac{JG}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.26}$$

Example 1

Use a one-element finite-element model to approximate the lowest natural frequency of a fixed-free bar.

Solution:

From Eqs. (10.19) and (10.22), the differential equations for a one-degree-of-freedom model of a bar are

$$\frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the bar is fixed at $x = 0$, $w_1 = 0$ and the above equations reduce to a single equation

$$\frac{\rho A l}{6}(2\ddot{w}_2) + \frac{EA}{l}w_2 = 0 \qquad \ddot{w}_2 + \frac{3E}{\rho l^2}w_2 = 0$$

from which the natural frequency is obtained as

$$\omega_1 = \sqrt{\frac{3E}{\rho l^2}}$$

Note that the one-element finite-element model of the fixed-free bar leads to the same natural frequency approximation that is obtained by using a one-degree-of-freedom model with an equivalent mass of $\rho A l/3$ lumped at the end of the bar.

10.5 BEAM ELEMENT

The potential energy scalar product for a beam involves the second spatial derivative of the displacement. Thus a Rayleigh-Ritz or assumed modes approximation must be twice differentiable. When a finite-element model of the beam is developed by the assumed modes method, the requirement that the interpolation be twice differentiable leads to requiring that displacements and slopes (first spatial derivatives) be continuous at element boundaries. In order to enforce this requirement over the entire beam, each beam element has four degrees of freedom represented by the displacements and slopes at the ends of the element. Let w_1 represent the transverse displacement of the left end of the element, w_2 the slope at the left end of the element, w_3 the transverse displacement of the right end of the element, and w_4 the slope at the right end of the element, as illustrated in Fig. 10.6. If ξ is the local coordinate over the beam element the finite element approximation for the displacement across the beam element must satisfy

$$u(0,t) = w_1 \qquad \frac{\partial u}{\partial \xi}(0,t) = w_2 \qquad u(l,t) = w_3 \qquad \frac{\partial u}{\partial \xi}(l,t) = w_4 \tag{10.27}$$

The deflection of a beam element without transverse loading across its span, but with prescribed displacements and slopes at its ends, is

$$u(\xi) = C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4 \tag{10.28}$$

Using Eq. (10.27) in Eq. (10.28) to determine the constants leads to

$$\begin{aligned} C_1 &= \frac{1}{l^3}(2w_1 + lw_2 - 2w_3 + lw_4) \\ C_2 &= \frac{1}{l^2}(-3w_1 - 2lw_2 + 3w_3 - lw_4) \\ C_3 &= w_2/l \\ C_4 &= w_1 \end{aligned} \tag{10.29}$$

Use of Eq. (10.29) in Eq. (10.28) and rearranging leads to

$$\begin{aligned} u(\xi,t) = {} & \left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}\right)w_1 + \left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)w_2 \\ & + \left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right)w_3 + \left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right)w_4 \end{aligned} \tag{10.30}$$

The kinetic energy of the beam element is

$$T = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial u}{\partial t}\right)^2 d\xi \tag{10.31}$$

Use of Eq. (10.30) in Eq. (10.31) leads to a quadratic form of kinetic energy

$$T = \tfrac{1}{2}\dot{\mathbf{w}}^T\mathbf{m}\dot{\mathbf{w}}$$

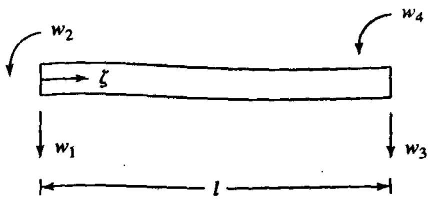


Figure 10.6 Beam element has four degrees of freedom, represented by displacements and slopes at the ends of the element.

where $\dot{\mathbf{w}}^T = [\dot{w}_1 \quad \dot{w}_2 \quad \dot{w}_3 \quad \dot{w}_4]$ and the element (local) mass matrix for a uniform beam element is

$$\mathbf{m} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \qquad \textbf{[10.32]}$$

The potential energy of the beam element is

$$V = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 d\xi \qquad \textbf{[10.33]}$$

Use of Eq. (10.30) in Eq. (10.33) leads to the quadratic form of potential energy

$$V = \tfrac{1}{2}\mathbf{w}^T\mathbf{k}\mathbf{w}$$

where the element (local) stiffness matrix for a uniform beam element is

$$\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \qquad \textbf{[10.34]}$$

The method of virtual work is used to obtain the generalized forces as

$$q_1 = \int_0^l f(\xi, t)\left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3}\right) d\xi \qquad \textbf{[10.35]}$$

$$q_2 = \int_0^l f(\xi, t)\left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right) l\, d\xi$$

$$q_3 = \int_0^l f(\xi, t)\left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3}\right) d\xi$$

$$q_4 = \int_0^l f(\xi, t)\left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3}\right) l\, d\xi$$

10.6 GLOBAL MATRICES

Local mass and stiffness matrices are derived for bar, torsion, and beam elements in Secs. 10.4 and 10.5. The accuracy of the finite-element method improves as the number of elements used increases. The use of many elements is necessary in the approximation of complicated systems. Local mass and stiffness matrices are calculated for each element and assembled into global matrices. When many elements are used, an efficient assembly algorithm is necessary.

A bar element has two degrees of freedom. The local generalized coordinates are the displacements of the ends of the elements. An n-element finite-element model of a bar, as illustrated in Fig. 10.7, has at most $n+1$ degrees of freedom. The global generalized coordinates are the displacements of the boundaries between elements and the ends of the bar. Each geometric boundary condition reduces by one the number of global degrees of freedom. For example if the left end of the bar is fixed, then its displacement is zero and the model has n degrees of freedom.

Let $W_1, W_2, \ldots, W_n$ represent the global generalized coordinates. Each local generalized coordinate is one of the global generalized coordinates, unless that element is subject to a geometric boundary condition. The local mass and stiffness matrices can be expanded to include all global generalized coordinates. The total kinetic energy of the system is the sum of the kinetic energies of the elements. Let

$$T_i = \tfrac{1}{2}\dot{\mathbf{w}}_i^T \mathbf{m}_i \dot{\mathbf{w}}_i \qquad [10.36]$$

be the kinetic energy of the ith element. The local mass matrix can be enlarged and the kinetic energy written in terms of the global generalized coordinates as

$$T_i = \tfrac{1}{2}\dot{\mathbf{W}}^T \tilde{\mathbf{M}}_i \dot{\mathbf{W}} \qquad [10.37]$$

The total kinetic energy of the system is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2}\sum_{i=1}^{n} \dot{\mathbf{W}}^T \tilde{\mathbf{M}}_i \dot{\mathbf{W}} = \frac{1}{2}\dot{\mathbf{W}}^T \left(\sum_{i=1}^{n} \tilde{\mathbf{M}}_i\right) \dot{\mathbf{W}} \qquad [10.38]$$

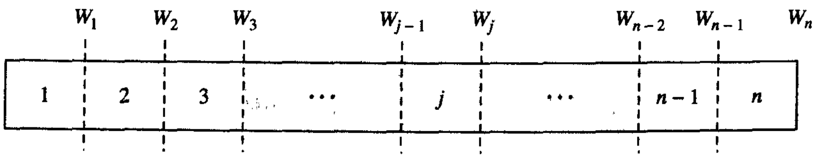


Figure 10.7 An n-element model of a fixed-free bar. Global generalized coordinates are displacements of nodes, located at element boundaries, or ends of the bar. Since the bar is fixed at $x = 0$, $W_0 = 0$.

Thus the global mass matrix is

$$\mathbf{M} = \sum_{i=1}^{n} \tilde{\mathbf{M}}_i \tag{10.39}$$

The global stiffness matrix and the global force vector can be obtained in an analogous manner.

Derive the global mass matrix for a three-element model of a fixed-free bar.

Solution:

The three-element model of a fixed-free bar is shown in Fig. 10.8. The three-element model has three degrees of freedom, noting that $u(0) = 0$. The global generalized coordinates are the displacements of the ends of the elements. The assembly of the global mass matrix from the local mass matrices is shown. The global displacement vector is $\mathbf{W} = [W_1 \quad W_2 \quad W_3]^T$. The quadratic form of the kinetic energy is $T = \frac{1}{2}\dot{\mathbf{W}}^T\mathbf{M}\dot{\mathbf{W}}$.

Element 1: Local generalized coordinates:

$$w_1 = 0 \qquad w_2 = W_1$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_1 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

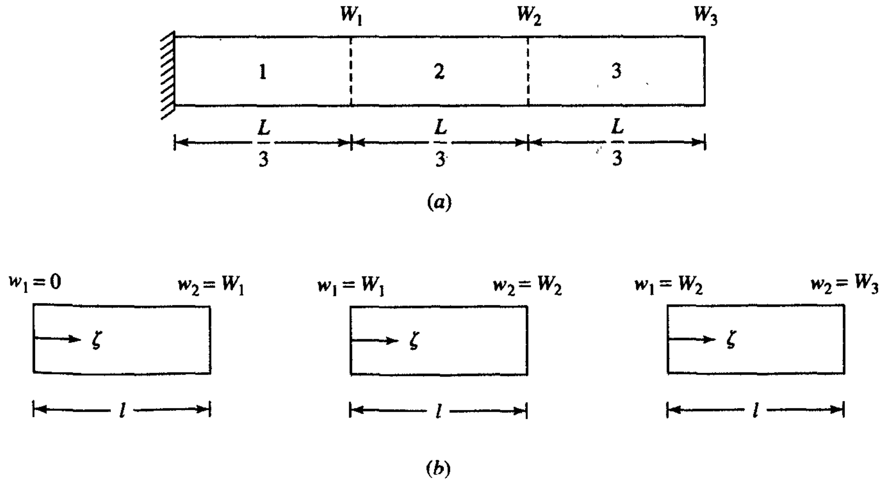


Figure 10.8 (a) Three-element model of fixed-free bar has three degrees of freedom. The elements are of equal length $l = L/3$. (b) Local coordinates for each element.

Kinetic energy of element:

$$T = \frac{1}{2}\frac{\rho Al}{6}(2\dot{w}_2^2) = \frac{1}{2}\frac{\rho Al}{6}(2\dot{W}_1^2) = \frac{1}{2}[\dot{W}_1 \quad \dot{W}_2 \quad \dot{W}_3]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \\ \dot{W}_3 \end{bmatrix}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_1 = \frac{\rho Al}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates:

$$w_1 = W_1 \qquad w_2 = W_2$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_2 = \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_2 = \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 3: Local generalized coordinates:

$$w_1 = W_2 \qquad w_2 = W_3$$

Element mass matrix in terms of local generalized coordinates:

$$\mathbf{m}_3 = \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element mass matrix in terms of global generalized coordinates:

$$\tilde{\mathbf{M}}_3 = \frac{\rho Al}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Thus the global mass matrix is

$$\begin{aligned}\mathbf{M} &= \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 \\ &= \frac{\rho Al}{6}\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) \\ &= \frac{\rho Al}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\end{aligned}$$

The model of Example 10.3 has only three degrees of freedom, and it is easy to construct $\tilde{\mathbf{M}}$. It is more difficult for systems with a large number of degrees of freedom. For such systems computer analysis will be used to formulate the model and solve the resulting differential equations. Thus it is important to have an efficient algorithm for assembly of the global mass matrices.

Let $\mathbf{S}_i$ be a transformation matrix between the local generalized coordinates for element i and the global generalized coordinates,

$$\mathbf{w}_i = \mathbf{S}_i \mathbf{W} \tag{10.40}$$

The total kinetic energy of the system is

$$T = \frac{1}{2} \sum_{i=1}^{n} \dot{\mathbf{w}}_i^T \mathbf{m}_i \dot{\mathbf{w}}_i \tag{10.41}$$

Using Eq. (10.40) in Eq. (10.41) leads to

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^{n} (\mathbf{S}_i \dot{\mathbf{W}})^T \mathbf{m}_i (\mathbf{S}_i \dot{\mathbf{W}}) \\ &= \frac{1}{2} \sum_{i=1}^{n} \dot{\mathbf{W}}^T \mathbf{S}_i^T \mathbf{m}_i \mathbf{S}_i \dot{\mathbf{W}} \\ &= \frac{1}{2} \dot{\mathbf{W}}^T \left(\sum_{i=1}^{n} \mathbf{S}_i^T \mathbf{m}_i \mathbf{S}_i \right) \dot{\mathbf{W}} \end{aligned} \tag{10.42}$$

Thus the global mass matrix is

$$\mathbf{M} = \sum_{i=1}^{n} \mathbf{S}_i^T \mathbf{m}_i \mathbf{S}_i \tag{10.43}$$

Exam

Illustrate the development of $\tilde{\mathbf{M}}_2$ for the system of Example 10.3 using the transformation matrix.

Solution:

The transformation between the local generalized coordinates and the global generalized coordinates for element 2 of Example 10.3 is

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

Thus

$$\tilde{\mathbf{M}}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{\rho A l}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10.7 EXAMPLES

10.5 Use a three-element finite-element model to approximate the lowest natural frequency and its corresponding mode shape for a uniform fixed-free bar.

Solution:

The three-element model of a fixed-free bar is illustrated in Fig. 10.8 on page 517. The global mass matrix was derived in Example 10.3. Using the same method, the global stiffness matrix is determined as

$$\mathbf{K} = \frac{EA}{l} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The differential equations for the bar in the finite-element model are

$$\frac{\rho A l}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \end{bmatrix} + \frac{EA}{l} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The mode shape vectors are the corresponding eigenvectors. The lowest natural frequency and mode shape vector are calculated as

$$\omega_1 = 1.584\sqrt{\frac{E}{\rho L^2}} \qquad \mathbf{W} = \begin{bmatrix} 0.577 \\ 1 \\ 1.155 \end{bmatrix}$$

The mode shape vector provides the displacements at the element boundaries. The finite-element approximation to the mode shape is a piecewise linear approximation between the element boundaries.

Example 1

Use a two-element finite-element model to approximate the four lowest natural frequencies for the system of Fig. 10.9*a*. Note that the exact solution for this system was obtained in Example 9.5.

Solution:

The two-element finite-element model for the fixed-free beam illustrating the global generalized coordinates is shown in Fig. 10.9*b*. The beam element of Sec. 10.5 is used.

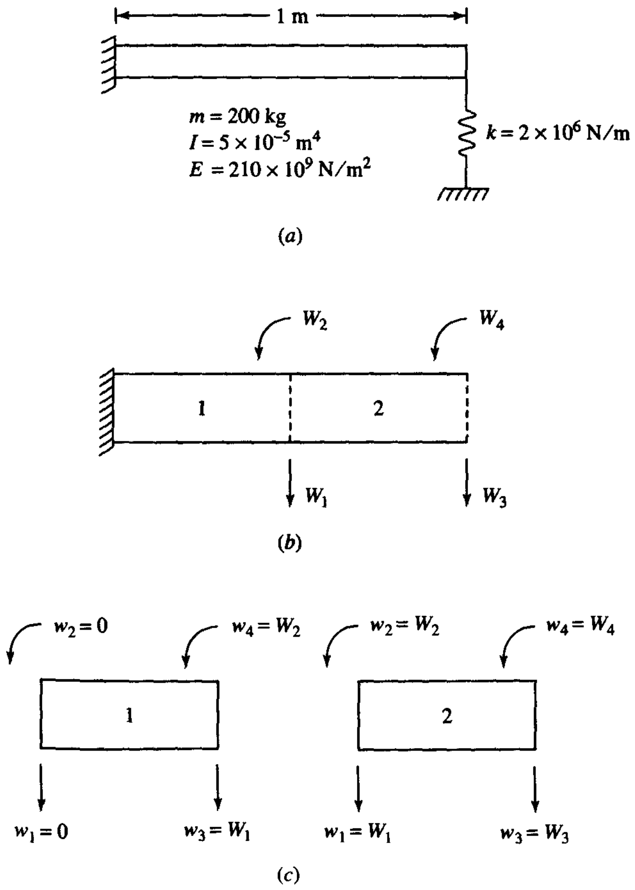


Figure 10.9 (*a*) System of Example 10.5; (*b*) two-element finite-element model of beam illustrating global generalized coordinates; (*c*) local generalized coordinates for each element.

Note that since the left end of the beam is fixed, the geometric boundary conditions of zero slope and zero displacement must be imposed. The generic element mass and stiffness matrices for a beam element are

$$\mathbf{m} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

$$\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

The potential energy for the discrete spring is incorporated into the local stiffness matrix for element 2. For this model $l = L/2$.

Element 1: Local generalized coordinates:

$$w_1 = 0 \qquad w_2 = 0 \qquad w_3 = W_1 \qquad w_4 = W_2$$

Element global matrices are

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l}{420}\begin{bmatrix} 156 & -22l & 0 & 0 \\ -22l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_1 = \frac{EI}{l^3}\begin{bmatrix} 12 & -6l & 0 & 0 \\ -6l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: Local generalized coordinates:

$$w_1 = W_1 \qquad w_2 = W_2 \qquad w_3 = W_3 \qquad w_4 = W_4$$

The element stiffness matrix for element 2 must be modified to account for the potential energy of the spring, $V = \frac{1}{2}kw_3^2$. The stiffness matrix term k_{33} is the only term affected by the discrete spring. The global mass and stiffness matrices for element 2 are

$$\tilde{\mathbf{M}}_2 = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22 & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

$$\tilde{\mathbf{K}}_2 = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 + \dfrac{kl^3}{EI} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

The global mass and stiffness matrices are

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 = \frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

$$\mathbf{K} = \tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2 = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 + \dfrac{kl^3}{EI} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Substitution of given values leads to

$$\mathbf{M} = \begin{bmatrix} 74.29 & 0 & 12.86 & -1.55 \\ 0 & 0.476 & 1.55 & -0.179 \\ 12.86 & 1.55 & 37.14 & -0.262 \\ -1.55 & -0.179 & -2.62 & 0.238 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2.016 & 0 & -1.008 & 0.252 \\ 0 & 0.108 & -0.252 & 0.042 \\ -1.008 & -0.252 & 1.008 & -0.252 \\ 0.252 & 0.042 & -0.252 & 0.084 \end{bmatrix} 10^9$$

The natural frequency approximations, the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, are obtained as

$$\omega_1 = 806.0 \text{ rad/s} \qquad \omega_2 = 5.09 \times 10^3 \text{ rad/s}$$
$$\omega_3 = 1.72 \times 10^4 \text{ rad/s} \qquad \omega_4 = 5.00 \times 10^4 \text{ rad/s}$$

The exact natural frequencies for this system, obtained in Example 9.5, are

$$\omega_1 = 829 \text{ rad/s} \qquad \omega_2 = 5.05 \times 10^3 \text{ rad/s}$$
$$\omega_3 = 1.41 \times 10^4 \text{ rad/s} \qquad \omega_4 = 2.73 \times 10^4 \text{ rad/s}$$

Examp

Use a two-element finite-element model for the beam to determine the steady-state response of the system of Fig. 10.10*a*. Use MATLAB to perform the computations and to plot the steady-state response.

Solution:

For a two-element finite-element model of the beam, the system has five degrees of freedom. The global generalized coordinates are illustrated in Fig. 10.10*b*. The local mass and stiffness matrices for each element are given by Eqs. (10.32) and (10.34) respectively.

Element 1:

$$w_1 = 0 \qquad w_2 = W_1 \qquad w_3 = W_2 \qquad w_4 = W_3$$

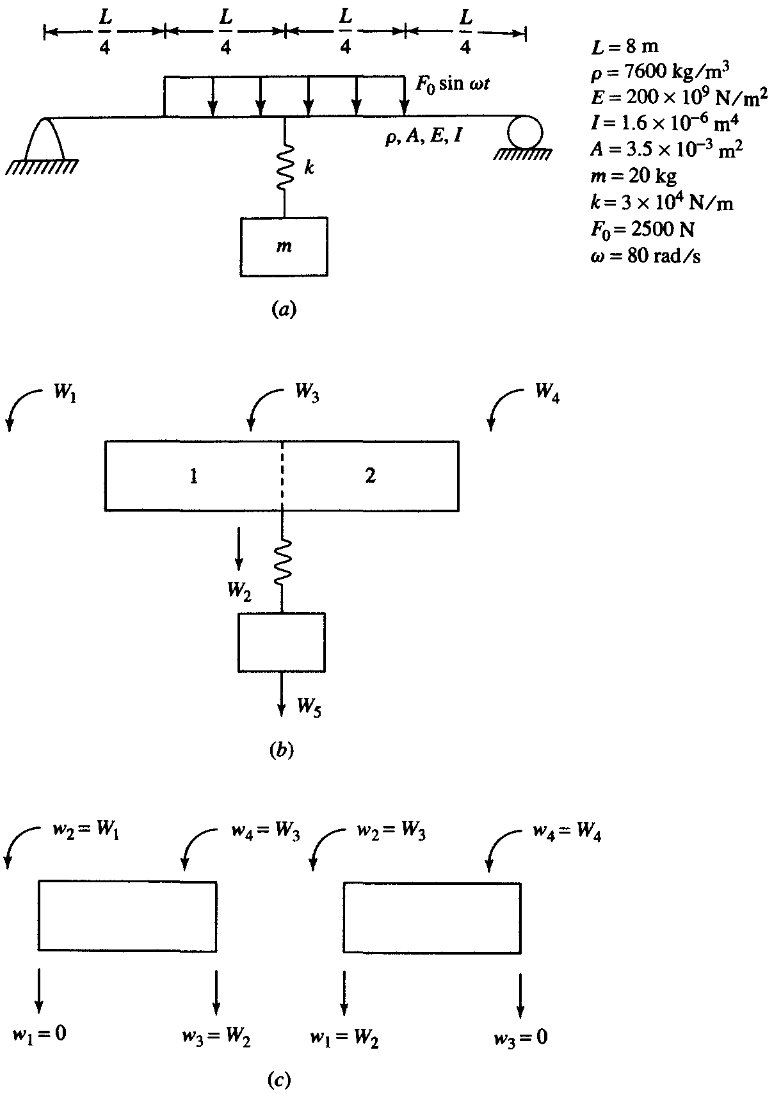


Figure 10.10 (*a*) System of Example 10.7; (*b*) two-element model for beam illustrating global generalized coordinates; (*c*) relations between local coordinates and global coordinates for each element.

Global element matrices are

$$\tilde{\mathbf{M}}_1 = \frac{\rho A l}{420}\begin{bmatrix} 4l^2 & 13l & -3l^2 & 0 & 0 \\ 13l & 156 & -22l & 0 & 0 \\ -3l^2 & -22l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_1 = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 \\ -6l & 12 & -6l & 0 & 0 \\ 2l^2 & -6l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The generalized force vector for element 1 is calculated by using Eqs. (10.35). Since $w_1 = 0$, q_1 is not calculated.

$$q_2(t) = \int_{l/2}^{l} F_0 \sin \omega t \left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3} \right) d\xi = -\frac{1}{48} l F_0 \sin \omega t$$

$$q_3(t) = \int_{l/2}^{l} F_0 \sin \omega t \left(3\frac{\xi^2}{l^2} - 2\frac{\xi^3}{l^3} \right) d\xi = \frac{13}{32} l F_0 \sin \omega t$$

$$q_4 = \int_{l/2}^{l} F_0 \sin \omega t \left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3} \right) d\xi = -\frac{11}{192} l F_0 \sin \omega t$$

The global generalized force vector for element 1 is

$$\mathbf{F}_1 = \begin{bmatrix} -\frac{1}{48} \\ \frac{13}{32} \\ -\frac{11}{192} \\ 0 \\ 0 \end{bmatrix} l F_0 \sin \omega t$$

Element 2: Local generalized coordinates:

$$w_1 = W_2 \qquad w_2 = W_3 \qquad w_3 = 0 \qquad w_4 = W_4$$

Global mass and stiffness matrices are

$$\tilde{\mathbf{M}}_2 = \frac{\rho A l}{420} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22l & -13l & 0 \\ 0 & 22l & 4l^2 & -3l^2 & 0 \\ 0 & -13l & -3l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_2 = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l & 6l & 0 \\ 0 & 6l & 4l^2 & 2l^2 & 0 \\ 0 & 6l & 2l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The generalized force vector for element 2 is calculated by using Eqs. (10.35). Since $w_3 = 0$, q_3 is not calculated.

$$q_1(t) = \int_{0}^{l/2} F_0 \sin \omega t \left(1 - 3\frac{\xi^2}{l^2} + 2\frac{\xi^3}{l^3} \right) d\xi = \frac{13}{32} l F_0 \sin \omega t$$

$$q_2(t) = \int_{0}^{l/2} F_0 \sin \omega t \left(\frac{\xi}{l} - 2\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3} \right) d\xi = \frac{11}{192} l F_0 \sin \omega t$$

$$q_4(t) = \int_{0}^{l/2} F_0 \sin \omega t \left(-\frac{\xi^2}{l^2} + \frac{\xi^3}{l^3} \right) d\xi = -\frac{5}{192} l F_0 \sin \omega t$$

The global generalized force vector for element 2 is,

$$\mathbf{F}_2 = \begin{bmatrix} 0 \\ \frac{13}{32} \\ \frac{35}{192} \\ -\frac{5}{192} \\ 0 \end{bmatrix} lF_0 \sin \omega t$$

For the discrete spring-mass system.

Potential energy: $V = \frac{1}{2}k(W_2^2 - W_5^2)$

Kinetic energy: $T = \frac{1}{2}m\dot{W}_5^2$

The contributions to the global matrices due to the discrete mass-spring system are

$$\mathbf{M}_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \qquad \mathbf{K}_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & k \end{bmatrix}$$

Assembling the global mass matrix, global stiffness matrix, and global generalized force vector leads to the following differential equations

$$\frac{\rho A l}{420}\begin{bmatrix} 4l^2 & 13l & -3l^2 & 0 & 0 \\ 13l & 312 & 0 & -13l & 0 \\ -3l^2 & 0 & 8l^2 & -3l^2 & 0 \\ 0 & -13l & -3l^2 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & \frac{420m}{\rho A l} \end{bmatrix}\begin{bmatrix} \ddot{W}_1 \\ \ddot{W}_2 \\ \ddot{W}_3 \\ \ddot{W}_4 \\ \ddot{W}_5 \end{bmatrix}$$

$$+\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 \\ -6l & 24 + \frac{kl^3}{EI} & 0 & 6l & -\frac{kl^3}{EI} \\ 2l^2 & 0 & 8l^2 & 2l^2 & 0 \\ 0 & 6l & 2l^2 & 4l^2 & 0 \\ 0 & -\frac{kl^3}{EI} & 0 & 0 & \frac{kl^3}{EI} \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{48} \\ \frac{13}{16} \\ -\frac{1}{8} \\ -\frac{5}{192} \\ 0 \end{bmatrix} lF_0 \sin \omega t$$

The method of undetermined coefficients is used to approximate the steady-state response of the system. The steady-state response is assumed as $\mathbf{W}(t) = \mathbf{S}\sin \omega t$ where $\mathbf{S}$ is the vector of undetermined coefficients. The relations between the global coordinates and the local coordinates for each element and Eq. (10.30) are used to determine the steady-state mode shape. The MATLAB script written to determine the natural frequencies and steady-state response is given in Fig. 10.11*a*. The output from running the script is given in Fig. 10.11*b*, while the MATLAB-generated plot of the steady-state mode shape is given in Fig. 10.11*c*. The steady-state amplitude of the discrete mass is $W_5 = 3.3$ mm.

```
% Example 10.7
% Two-element finite-element model for forced response of
%    a simply-supported beam with discrete mass-spring
%    system attached at midspan. Part of span of beam is
%    subject to a harmonic excitation
% Parameters
digits(5)
L=8;                        % Length in m
rho=7600;                   % Density in kg/m^3
E=200*10^9;                 % Elastic modulus in N/m^2
I=1.6*10^-6;                % Moment of inertia in m^4
A=3.6*10^-3;                % Area in m^2
m=20;                       % Mass of hanging block in kg
k=3*10^4;                   % Stiffness of discrete spring in N/m
s=L/2;                      % Element length
F0=2500;                    % Excitation amplitude in N
om=80;                      % Excitation frequency in rad/s
% Global mass matrix
disp(' Global mass matrix')
M=rho*A*s/420*[4*s^2,13*s,-3*s^2,0,0;13*s,312,0,-13*s,0;
   -3*s^2,0,8*s^2,-3*s^2,0;0,-13*s,-3*s^2,4*s^2,0;
   0,0,0,0,420*m/(rho*A*s)];
M1=vpa(M);disp(M1)
% Global stiffness matrix
disp(' Global stiffness matrix')
K=E*I/s^3*[4*s^2,-6*s,2*s^2,0,0;-6*s,24+k*s^3/(E*I),0,6*s,-k*s^3/(E*I);
   2*s^2,0,8*s^2,2*s^2,0;0,6*s,2*s^2,4*s^2,0;
   0,-k*s^3/(E*I),0,0,k*s^3/(E*I)];
K1=vpa(K);disp(K1)
% Natural frequencies
W2=eigs(inv(M)*K);
for i=1:5
   w(i)=sqrt(W2(i));
end
% Force vector
```

Figure 10.11 (*a*) MATLAB script for two-element finite-element model of system of Fig. 10.10.

```
disp(' Force vector')
F=F0*s*[-1/48;13/16;-1/8;-5/192;0];disp(F)
% Use of undetermined coefficients to determine steady-state response
Z=-om^2*M+K;
W=inv(Z)*F;
x=linspace(0,L,21);
for k=1:21
   if x(k)<s
      xi=x(k)/s;
      y(k)=(xi-2*xi^2+xi^3)*W(1)+(3*xi^2-2*xi^3)*W(2);
      y(k)=y(k)+(-xi^2+xi^3)*W(3);
   else
      xi=(x(k)-s)/s;
      y(k)=(1-3*xi^2+2*xi^3)*W(2)+(xi-2*xi^2+xi^3)*W(3);
      y(k)=y(k)+(-xi^2+xi^3)*W(4);
   end
end
plot(x,y,'-')
xlabel('x (m)')
ylabel('W(x) (m)')
w=vpa(w);
W=vpa(W)
disp(' Natural frequencies in rad/s ');disp(w)
disp(' Steady-state amplitudes in m ');disp(W)
```

(a)

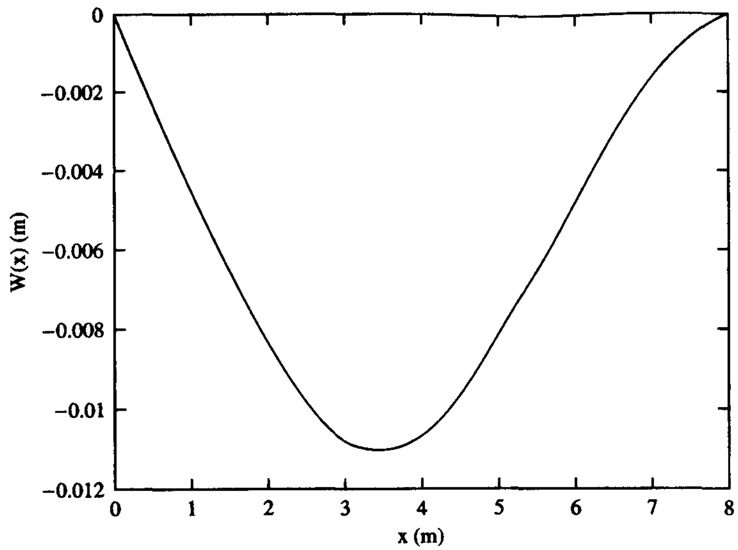
```
 Global mass matrix
[  16.677,  13.550, -12.507,        0,        0]
[  13.550,  81.298,        0, -13.550,        0]
[ -12.507,       0,  33.353, -12.507,        0]
[       0, -13.550, -12.507,  16.677,        0]
[       0,       0,        0,       0,      20.]

 Global stiffness matrix
[  .32000e6, -.12000e6,  .16000e6,          0,          0]
[ -.12000e6,  .15000e6,         0,   .12000e6,    -30000.]
[  .16000e6,         0,  .64000e6,   .16000e6,          0]
[         0,  .12000e6,  .16000e6,   .32000e6,          0]
[         0,   -30000.,         0,          0,     30000.]

iter =

     2
```

Figure 10.11 (Con't) (*a*) Con't; (*b*) Output from running script.

```
eigs =

    1.151315789473683e+005
    3.487849420597156e+004
    5.482456140350870e+003
    1.816727650409610e+003
    2.298793581937252e+002


stopcrit =

    1.245169782336117e-015

 Force vector
   -2.083333333333333e+002
    8.125000000000000e+003
   -1.250000000000000e+003
   -2.604166666666667e+002
                          0


W =

[ -.18521e-1]
[ -.10731e-1]
[  .63460e-2]
[  .20374e-2]
[  .32850e-2]


 Natural frequencies in rad/s
[ 339.31, 186.76, 74.044, 42.623, 15.162]


 Steady-state amplitudes in m
[ -.18521e-1]
[ -.10731e-1]
[  .63460e-2]
[  .20374e-2]
[  .32850e-2]
```

(*b*)

Figure 10.11B (Con't)



(c)

Figure 10.11 (Con't) (c) Steady-state mode shape.

10.8 Use the finite-element method to approximate the lowest natural frequency for the truss of Fig. 10.12*a*. Use one bar element for each truss member.

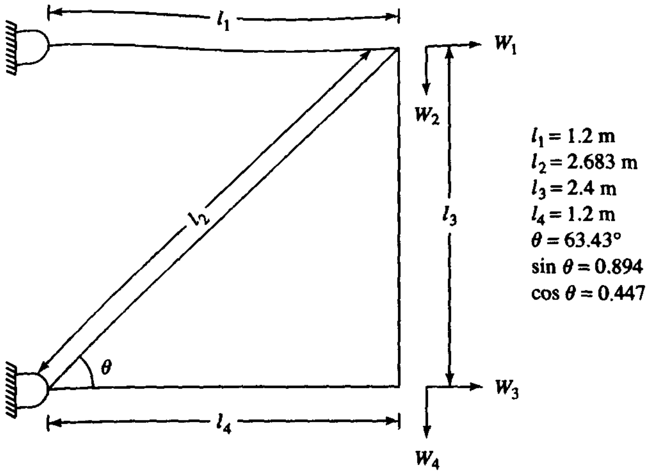
Solution:

The finite-element model of the four-bar truss using one bar element for each member has four degrees of freedom. The global generalized coordinates are illustrated in Fig. 10.12*b*.

Member 1: The relations between the local generalized coordinates and the global generalized coordinates are $w_1 = 0$, $w_2 = W_1$. The contributions to the global mass and stiffness matrices for element 1 are

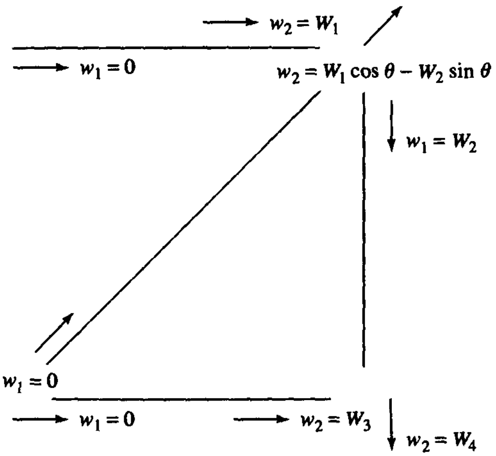
$$\tilde{\mathbf{M}}_1 = \frac{\rho A l_1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_1 = \frac{EA}{l_1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Member 2: The relations between the local generalized coordinates and the global generalized coordinates for member 2 are $w_1 = 0$, $w_2 = W_1 \cos\theta - W_2 \sin\theta$. The transformation between the nonzero local generalized coordinate and the global generalized



All members are made of material of elastic modulus E and have cross-sectional area A.

(*a*)



(*b*)

Figure 10.12 (a) Four-bar truss of Example 10.8 illustrating global coordinates; (b) relationships between local coordinates and global coordinates for each truss member.

coordinates written in matrix form is

$$[w_2] = [\cos\theta \; -\sin\theta \quad 0 \quad 0] \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

The contributions to the global mass and stiffness matrices from element 2 are obtained by using Eq. (10.43) with $\mathbf{S}_2 = [\cos\theta \; -\sin\theta \;\; 0 \;\; 0]$. Note that since $w_1 = 0$, the element mass and stiffness matrices in terms of the local generalized coordinate are

$$\mathbf{m}_2 = \frac{\rho A l_2}{6}[2] \qquad \mathbf{k}_2 = \frac{EA}{l_2}[1]$$

Thus the contribution to the global mass matrix for element 2 is

$$\tilde{\mathbf{M}}_2 = \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \\ 0 \end{bmatrix} \left(\frac{\rho A l_2}{6}\right)[2][\cos\theta \; -\sin\theta \;\; 0 \;\; 0]$$

$$= \frac{\rho A l_2}{3}\begin{bmatrix} \cos^2\theta & -\cos\theta\sin\theta & 0 & 0 \\ -\cos\theta\sin\theta & \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The contribution to the global stiffness matrix for element 2 is calculated as

$$\tilde{\mathbf{K}}_2 = \frac{EA}{l_2}\begin{bmatrix} \cos^2\theta & -\cos\theta\sin\theta & 0 & 0 \\ -\cos\theta\sin\theta & \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3: The relations between the local generalized coordinates and the global generalized coordinates for element 3 are $w_1 = W_2$, $w_2 = W_4$. The contributions to the global mass and stiffness matrices from element 3 are

$$\tilde{\mathbf{M}}_3 = \frac{\rho A l_3}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \tilde{\mathbf{K}}_3 = \frac{EA}{l_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 4: The relations between the local generalized coordinates and the global generalized coordinates for element 4 are $w_1 = 0$, $w_2 = W_4$. The contributions to the global mass and stiffness matrices from element 4 are

$$\tilde{\mathbf{M}}_4 = \frac{\rho A l_4}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{K}}_4 = \frac{EA}{l_4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The global mass matrix is

$$\mathbf{M} = \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \tilde{\mathbf{M}}_3 + \tilde{\mathbf{M}}_4 = \frac{\rho A}{6}\begin{bmatrix} 2l_1 + 2l_2\cos^2\theta & -2l_2\cos\theta\sin\theta & 0 & 0 \\ -2l_2\cos\theta\sin\theta & 2l_3 + 2l_2\sin^2\theta & 0 & l_3 \\ 0 & 0 & 2l_4 & 0 \\ 0 & l_3 & 0 & 2l_3 \end{bmatrix}$$

Similar calculations lead to the global stiffness matrix

$$\mathbf{K} = EA\begin{bmatrix} \frac{1}{l_1}+\frac{(\cos^2\theta)}{l_2} & -\frac{(\cos\theta\sin\theta)}{l_2} & 0 & 0 \\ -\frac{(\cos\theta\sin\theta)}{l_2} & \frac{(\sin^2\theta)}{l_2}+\frac{1}{l_3} & 0 & -\frac{1}{l_3} \\ 0 & 0 & \frac{1}{l_4} & 0 \\ 0 & -\frac{1}{l_3} & 0 & \frac{1}{l_3} \end{bmatrix}$$

The natural frequencies are the square roots of the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. Output from a MATLAB script to determine the natural frequencies and mode shapes is given in Fig. 10.13. Note that the results show only three distinct natural frequencies.

```
Global mass matrix
      176.01      -108.78            0            0
     -108.78       460.67            0        121.6
           0            0        121.6            0
           0        121.6            0        243.2

Global stiffness matrix
  7.2634e+009  -1.193e+009            0            0
  -1.193e+009  5.7184e+009            0 -3.3333e+009
            0            0  6.6667e+009            0
            0 -3.3333e+009            0  3.3333e+009
```

Figure 10.13 Output from MATLAB script run for Example 10.8.

```
ter =

    2


igs =

   5.482456140350878e+007
   5.482456140350876e+007
   3.162555149229576e+007
   2.146804007227498e+006


:opcrit =

   4.239315146409638e-016

Jatural frequencies in rad/s
      7404.4        7404.4        5623.7        1465.2
```

jure 10.13 (Con't)

PROBLEMS

It may be convenient to use MATLAB to perform natural frequency calculations as well as to solve for forced responses.

10.1. The potential energy scalar product for a uniform bar is defined as

$$(f, g)_V = \int_0^L EAf(x)\frac{d^2g}{dx^2}\,dx$$

Consider the cases where (*a*) the bar is fixed at $x = 0$ and free at $x = L$ and (*b*) the bar is fixed at $x = 0$ and attached to a linear spring of stiffness k at $x = L$. Discuss, in each case, the implication of requiring $f(x)$ and $g(x)$ to satisfy only the geometric boundary conditions.

10.2. Use the assumed modes method with trial functions

$$w_1(x) = \sin\left(\pi\frac{x}{L}\right) \qquad w_2(x) = \sin\left(2\pi\frac{x}{L}\right) \qquad w_3(x) = \sin\left(3\pi\frac{x}{L}\right)$$

to approximate the lowest natural frequency and its corresponding mode shape for a uniform fixed-fixed bar of length L.

10.3. Let $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$ be linearly independent polynomials of degree four or less that satisfy the geometric boundary conditions for a bar fixed at $x = 0$ and attached to a spring of stiffness k at $x = L$.

(*a*) Determine a set of $w_1(x)$, $w_2(x)$, $w_3(x)$, $w_4(x)$.

(*b*) Use the assumed modes method with the functions obtained in part (*a*) as trial functions and $kL^3/EI = 0.5$ to approximate the system's lowest natural frequencies and mode shapes.

10.4. Use the assumed modes method with trial functions

$$w_1(x) = x(x - L) \qquad w_2(x) = x(x - L)^2 \qquad w_3(x) = x(x - L)^3$$

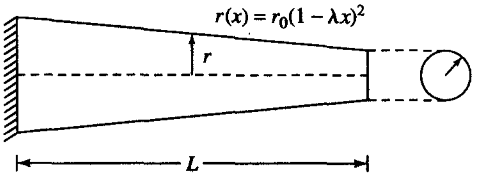
to approximate the two lowest natural frequencies and mode shapes for a simply supported beam.

10.5. Repeat Prob. 10.4 if the beam has a machine of mass $m = 2.0\rho AL$ where ρAL is the total mass of the beam. The machine is placed at the midspan of the beam.

10.6. The mode shapes of a uniform fixed-free bar are of the form

$$\phi_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad n = 1, 2, 3, \ldots$$

Use the assumed modes method with $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ as trial functions to approximate the lowest natural frequency and mode shapes for the tapered bar of Fig. P10.6



$r_0 = 5$ cm
$\lambda = 1$ cm/m
$L = 3$ m
$E = 200 \times 10^9$ N/m^2
$\rho = 7500$ kg/m^3

FIGURE P10.6

10.7. Use a one-element finite-element model to approximate the lowest natural frequency of a uniform bar of mass density ρ, cross-sectional area A, elastic modulus E, and length L that is fixed at one end and has a block of mass m attached at one end.

10.8. Use a one-element finite-element model to approximate the lowest natural frequency of a uniform bar of mass density ρ, cross-sectional area A, elastic modulus E, and length L that is fixed at one end and is attached to a linear spring of stiffness k at one end.

10.9. Use a one-element finite-element model to approximate the lowest nonzero torsional natural frequency of a uniform shaft of mass density ρ, polar moment of inertia J, shear modulus G, and length L that has a thin disk of mass moment of inertia I_1 attached at one end and a thin disk of mass moment of inertia I_2 attached at the other end.

10.10. Use a one-element finite-element model to approximate the lowest natural frequencies of a uniform beam of mass density ρ, cross-sectional area A, cross-sectional moment of inertia I, elastic modulus E, and length L that is free at both ends.

10.11. Derive the element m_{34} of the element mass matrix for a beam element.

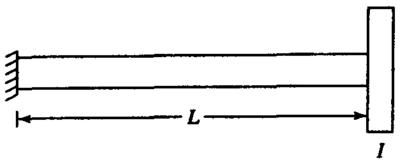
10.12. Derive the element k_{23} of the element stiffness matrix for a beam element.

10.13. Use a two-element finite-element model to approximate the two lowest natural frequencies and their corresponding mode shapes for the system of Fig. P10.13.

$E = 200 \times 10^9$ N/m^2
$A = 1.6 \times 10^{-4}$ m^2
$L = 2.5$ m
$k = 1 \times 10^7$ N/m

FIGURE P10.13

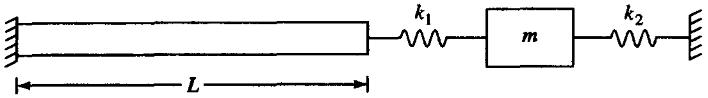
10.14. Use a two-element finite-element model to approximate the two lowest torsional natural frequencies for the system of Fig. P10.14.



$J = 3.2 \times 10^{-5}\ \text{m}^4$
$G = 80 \times 10^9\ \text{N/m}^2$
$\rho = 7000\ \text{kg/m}^3$
$L = 4.2\ \text{m}$
$I = 1.5\ \text{kg} \cdot \text{m}^2$

FIGURE P10.14

10.15. Use a three-element finite-element model to approximate the lowest natural frequency and its corresponding mode shape for the system of Fig. P10.15.

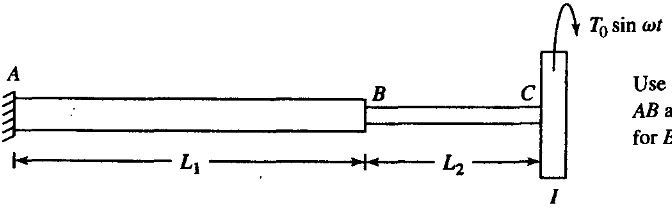


$E = 200 \times 10^9\ \text{N/m}^2$
$A = 3.5 \times 10^{-5}\ \text{m}^2$
$L = 2.5\ \text{m}$
$\rho = 7000\ \text{kg/m}^3$

$m = 1.2\ \text{kg}$
$k_1 = 2 \times 10^6\ \text{N/m}$
$k_2 = 1.4 \times 10^6\ \text{N/m}$

FIGURE P10.15

10.16. Use a three-element finite-element model to approximate the steady-state response of the system of Fig. P10.16.



Use two elements for *AB* and one element for *BC*

$L_1 = 2.1\ \text{m}$
$G_1 = 40 \times 10^9\ \text{N/m}^2$
$J_1 = 1.8 \times 10^{-5}\ \text{m}^4$
$\rho_1 = 5000\ \text{kg/m}^3$

$L_2 = 1.0\ \text{m}$
$G_2 = 80 \times 10^9\ \text{N/m}^2$
$J_2 = 4.3 \times 10^{-6}\ \text{m}^4$
$\rho_2 = 7000\ \text{kg/m}^3$

$I = 0.25\ \text{kg} \cdot \text{m}^2$
$T_0 = 100\ \text{N} \cdot \text{m}$
$\omega = 500\ \text{rad/s}$

FIGURE P10.16

10.17. Use a three-element finite-element model to approximate the forced response of the system of Fig. P10.15 when the end of the bar is subject to the excitation of Fig. P10.17.

10.18. Use a two-element finite-element model to approximate the two lowest natural frequencies of transverse vibration of the beam of Fig. P10.18.

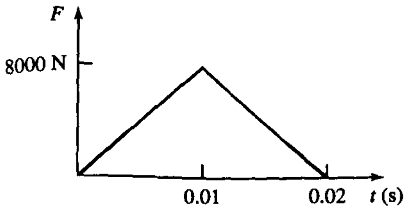


FIGURE P10.17

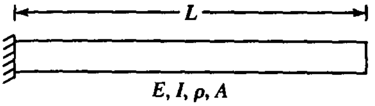


FIGURE P10.18

10.19. Use a two-element finite-element model to approximate the lowest natural frequencies of the beam of Fig. P10.19.

10.20. Use a two-element finite-element model to approximate the two lowest natural frequencies of the system of Fig. P10.20. Use elements of equal length.

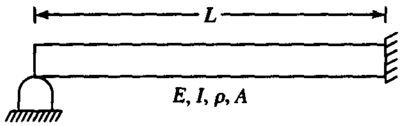


FIGURE P10.19

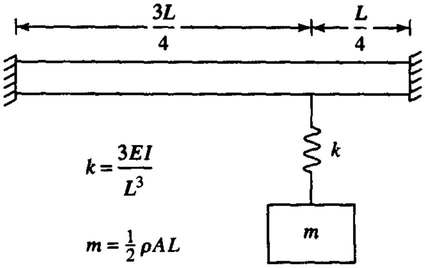
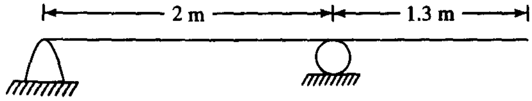


FIGURE P10.20

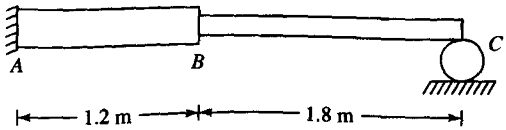
10.21. Use a three-element finite-element model to approximate the three lowest natural frequencies of the system of Fig. P10.21.



$E = 200 \times 10^9$ N/m
$I = 4.6 \times 10^{-6}$ m^4
$\rho = 7500$ kg/m^3
$A = 1.5 \times 10^{-2}$ m^2

FIGURE P10.21

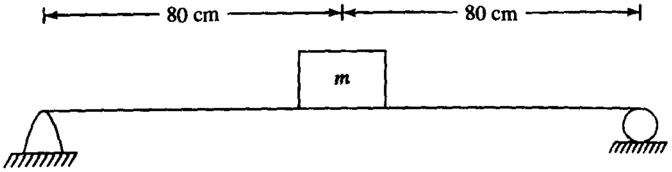
10.22. Use a two-element finite-element model to approximate the lowest natural frequency of the system of Fig. P10.22.



$I_{AB} = 4.1 \times 10^{-6}$ m^4
$E_{AB} = 200 \times 10^9$ N/m^2
$A_{AB} = 6.3 \times 10^{-4}$ m^2
$\rho_{AB} = 7500$ kg/m^2
$E_{BC} = 140 \times 10^9$ N/m^2
$A_{BC} = 5.4 \times 10^{-5}$ m^2
$\rho_{BC} = 5600$ kg/m^3
$I_{BC} = 5.3 \times 10^{-7}$ m^4

FIGURE P10.22

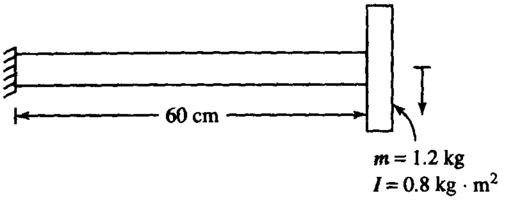
10.23. Use a two-element finite-element model for the beam to approximate the two lowest natural frequencies of the system of Fig. P10.23.



$\rho = 7600$ kg/m^3
$A = 4.5 \times 10^{-3}$ m^4
$m = 100$ kg
$E = 200 \times 10^9$ N/m^2
$I = 1.8 \times 10^{-6}$ m^4

FIGURE P10.23

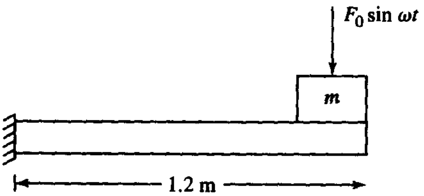
10.24. Use a two-element finite-element model to approximate the two lowest natural frequencies of the system of Fig. P10.24.



$\rho = 8000$ kg/m^3
$A = 1.1 \times 10^{-4}$ m^2
$E = 200 \times 10^9$ N/m^2
$I_b = 1.9 \times 10^{-6}$ m^4

FIGURE P10.24

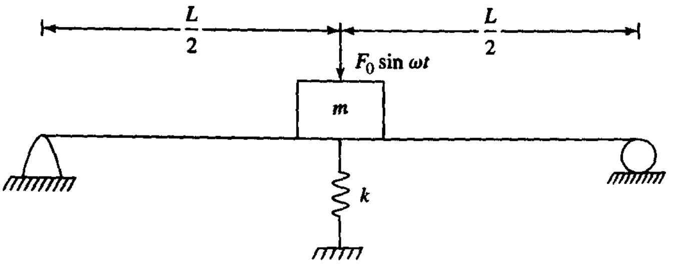
10.25. Use a three-element finite-element model to approximate the steady-state amplitude of the machine of the system of Fig. P10.25.



$I = 1.9 \times 10^{-5}$ m^4
$m = 210$ kg
$E = 200 \times 10^9$ N/m^2
$\rho = 7500$ kg/m^3
$A = 1.4 \times 10^{-2}$ m^2
$F_0 = 1500$ N
$\omega = 200$ rad/s

FIGURE P10.25

10.26. Use a three-element finite-element model to approximate the steady-state amplitude of the machine of the system of Fig. P10.26.



$L = 2.6$ m
$E = 200 \times 10^9$ N/m^2
$I = 4.5 \times 10^{-6}$ m^4
$A = 3.1 \times 10^{-2}$ m^2
$\rho = 7000$ kg/m^3
$m = 20$ kg
$k = 2 \times 10^5$ N/m

FIGURE P10.26

10.27. Reconsider the street light problem of Example 3.6. The street light has a mass of 25 kg. The wind velocity is 60 m/s, but the force distribution is as shown in Fig. P10.27. Use a three-element finite-element model of the structure to approximate the steady-state amplitude of the light.

10.28. Use a three-element finite-element model to approximate the steady-state response of the system of Fig. P10.28.

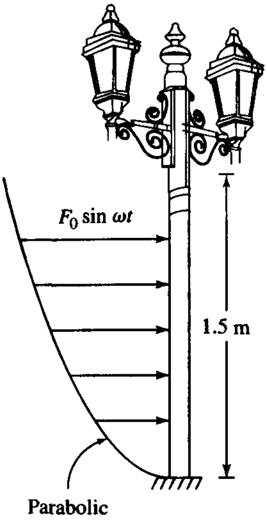
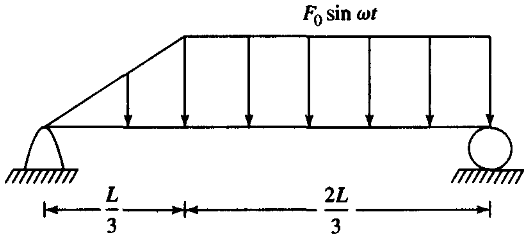


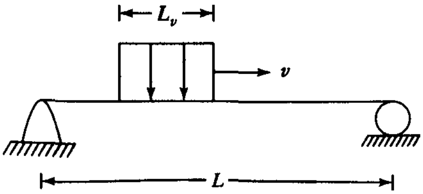
FIGURE P10.27



$F_0 = 800$ N
$\omega = 120$ rad/s
$E = 200 \times 10^9$ N/m^2
$A = 4.1 \times 10^{-3}$ m^2
$I = 8.6 \times 10^{-5}$ m^4
$\rho = 7500$ kg/m^3
$L = 3$ m

FIGURE P10.28

10.29. A plate and girder bridge is modeled as a simply supported beam, as illustrated in Fig. P10.29. A vehicle is traveling across the bridge with the velocity v. Use a three-element finite-element model of the bridge to determine the time-dependent response of the structure as the vehicle is crossing the bridge.



Vehicle of mass 1100 kg

$L_v = 2.5$ m
$v = 60$ m/s
$L = 8$ m
$I = 1.5 \times 10^{-6}$ m^4
$\rho = 7500$ kg/m^3
$A = 3.2 \times 10^{-2}$ m^2
$E = 200 \times 10^9$ N/m^2

FIGURE P10.29

10.30. A simple model of a one-story frame structure is shown in Fig. P10.30a. Use one beam element to model each of the columns and two bar elements to model the girder. Determine the response of the structure if it is subject to the blast force of Fig. P10.30b.

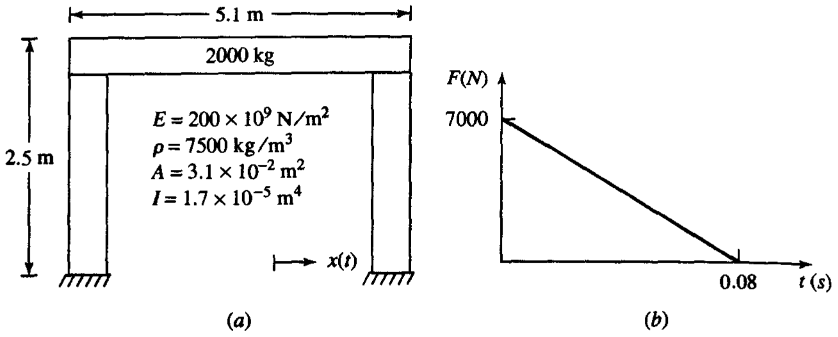


FIGURE P10.30

10.31. Use the finite-element model of Prob. 10.30 to determine the response of the structure if it is subject to the earthquake of Fig. P10.31.

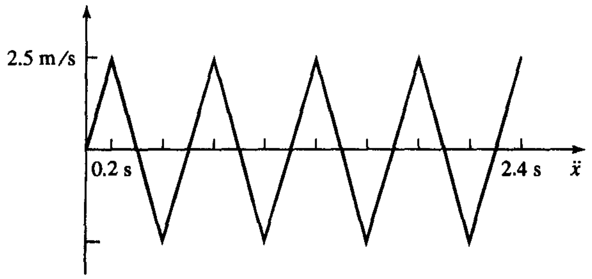


FIGURE P10.31

10.32. Use the finite-element model of Prob. 10.30 to determine the response of the structure if HVAC equipment on the girder produces a lateral harmonic force of magnitude 3000 N at a frequency of 500 rpm.

10.33. Use two bar elements to model each member of the truss of Example 10.8 and approximate the three lowest natural frequencies of the truss.

10.34. Use one bar element to model each member of the truss of Fig. P10.34 and approximate its two lowest natural frequencies.

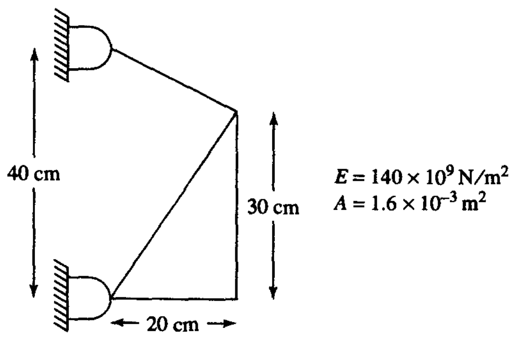
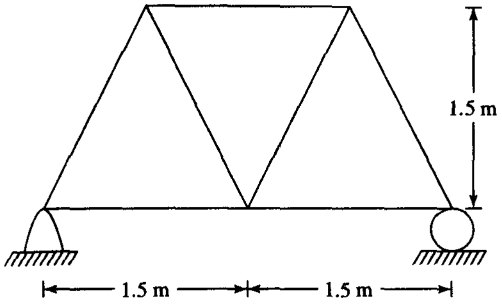


FIGURE P10.34

10.35. Use one bar element to model each member of the truss of Fig. P10.35 and approximate its two lowest natural frequencies.



$E = 140 \times 10^9\ \mathrm{N/m^2}$
$A = 1.6 \times 10^{-3}\ \mathrm{m^2}$

FIGURE P10.35

10.36. A beam is placed on an elastic foundation whose stiffness per unit length is k. Derive the element k_{23} of the local stiffness matrix for a beam element of length l including the stiffness of the elastic foundation. Use the local displacement functions of Eq. (10.30).

10.37. A beam is subject to a constant axial load of magnitude P, applied along the beam's neutral axis. Derive the element k_{31} of the local stiffness matrix for a beam element of length l, including the effect of transverse displacement due to the axial load.

10.38. A beam is rotating about an axis with an angular velocity ω. Determine the element m_{13} of the local mass matrix for a beam element of length l, including the kinetic energy due to the rotation of the beam. The left end of the element is a distance r from the axis of rotation.

MATLAB PROBLEMS

M10.1. The file VIBES_10A.m contains the MATLAB script for the solution of Example 10.1, modified such that the area is supplied in a user-defined function Area10A. Use VIBES10_A.m to approximate the lowest natural frequencies and mode shapes of the system of Example 10.1 with

$$A(x) = \pi(0.05 - 0.001x)^2$$

M10.2. The file VIBES_10B.m contains the file VIBES10_A.m extended to use the assumed modes method to develop the forced response of the system of Example 10.1 when a force $F(t)$ is applied at the end of the bar. The force is supplied in a user-defined function Force10B. Use VIBES10_B.m to approximate the forced response of Example 10.1 when

(*a*) $F(t) = 5000 \sin(1000t)$.

(*b*) $F(t) = 5000e^{-1.2t}$.

(*c*) $F(t)$ is as given in Fig. PM10.2.

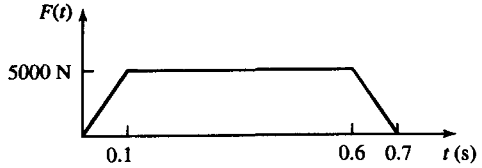


FIGURE PM10.2

M10.3. The file VIBES_10C.m contains the MATLAB script for the development of the local stiffness and mass matrices for a beam element whose cross-sectional properties vary across the span of the beam. The area and moment of inertia are supplied in user-defined functions Area10C and Inertia10C. Use VIBES_10C.m to develop the local mass and stiffness matrices for each element of a four-element finite-element model for the beam of Fig. PM10.3.

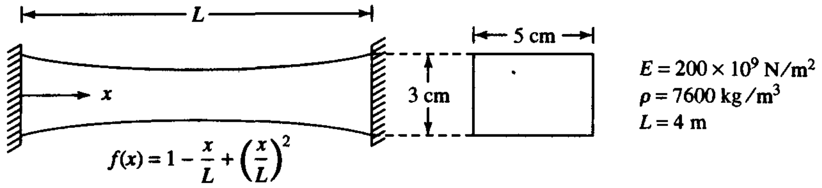


FIGURE PM10.3

M10.4. The file VIBES_10D.m contains the MATLAB script for the development of an n-element finite-element model for the natural frequencies and mode shapes of a uniform shaft with 2 rotors, as illustrated in Fig. PM10.4. Use VIBES_10D.m to study the dependence of the lowest natural frequency of the number of elements in the approximation for $n = 2$ to $n = 8$.

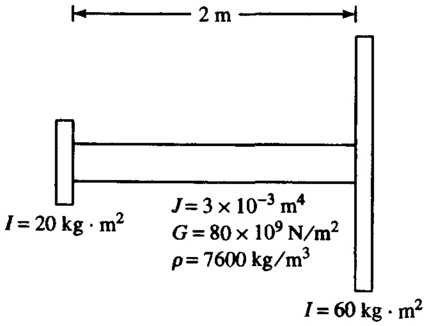


FIGURE PM10.4

M10.5. The file VIBES_10E.m contains the MATLAB script for the development of an n-element finite-element model of a fixed-pinned beam with a machine at its midspan. Use VIBES_10E.m to study the dependence of the three lowest natural frequencies on β, the ratio of the mass of the machine to the mass of the beam. Plot the natural frequencies as a function of β for $0.1 < \beta < 5$. Use VIBES_10E.m to solve Example 6.5.

M10.6. The file VIBES_10F.m contains the MATLAB script for the development of an n-element finite element model for the forced response of a fixed-pinned beam with a machine at its midspan which is subject to a time dependent force. The force is provided in a user-supplied function Force10F. Use the program to determine the time-dependent response of the beam of Fig. PM10.6a when the machine is subject to

(a) $F(t) = 5000 \sin(1000t)$.

(b) The force of Fig. PM10.6b.

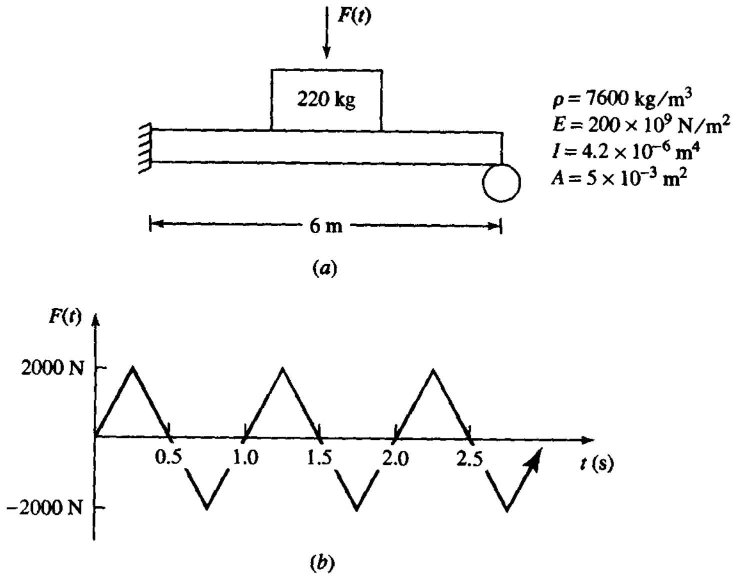


FIGURE PM10.6

M10.7. The file VIBES_10G.m contains the MATLAB script for the solution of Example 10.7. Run the program for a range of ω, 20 rad/s $< \omega <$ 2000 rad/s to determine a frequency response curve for the system of Example 10.7 with

(a) $F_0 = 2500$ N at all speeds.

(b) $F_0 = 0.8\omega^2$.

M10.8. The file VIBES_10H.m contains the MATLAB script for a four-element finite-element model of the system of Fig. PM10.8. An undamped isolator is used to protect the beam from large harmonic forces transmitted during operation of the machine. Recall that when the system is modeled using a one-degree-of-freedom approximation isolation occurs when $\omega/\omega_n > \sqrt{2}$. The stiffness of the spring is chosen to achieve isolation. Use VIBES_10H.m to study the effect of the elasticity of the beam on the transmitted force. Specifically study the problem where a 250-lb reciprocating engine is to be mounted at the midspan of a 12 ft W21 × 62 steel beam ($A = 18.3$ in^2, $I = 1330$ in^4). Assuming a one-degree-of-freedom system an undamped isolator has been designed to provide 81 percent isolation when the engine operates at 3000 rpm. Use VIBES_10H.m to determine the percent isolation when a four-element finite-element model is used for the beam and to determine the maximum steady-state displacement of the midspan if the excitation amplitude is 10000 N.

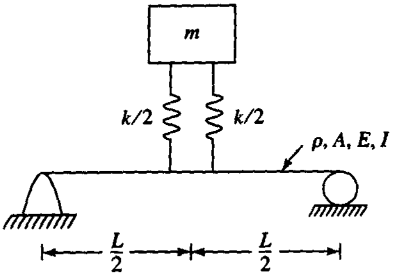
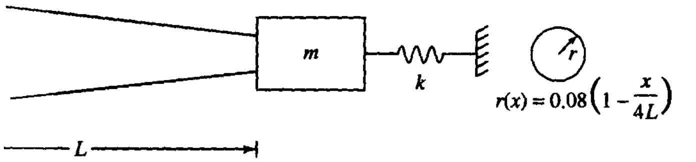


FIGURE PM10.8

M10.9. Use file VIBES10_H.m of Prob. M10.8 to determine the maximum transmitted force and midspan deflections as functions of ω if the excitation is caused by a rotating unbalance of magnitude 0.4 kg-m.

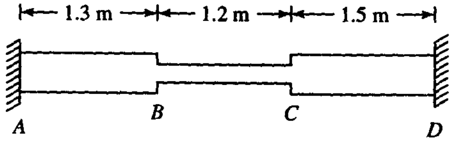
M10.10. Write a MATLAB script to determine the local mass and stiffness matrices for each element of the four-element finite-element model of the system of Fig. PM10.10.



$L = 5.4$ m
$E = 200 \times 10^9$ N/m^2
$\rho = 7600$ kg/m^3
$m = 300$ kg
$k = 3 \times 10^6$ N/m

I10.10

M10.11. Write a MATLAB script for the determination of the natural frequencies and mode shapes of the system of Fig. PM10.11. Use two elements to model each section of the bar.



$E_{AB} = E_{BC} = E_{CD} = 200 \times 10^9$ N/m^2
$\rho_{AB} = \rho_{BC} = \rho_{CD} = 7600$ kg/m^3
$A_{AB} = 3 \times 10^{-4}$ m^2
$A_{BC} = 2 \times 10^{-4}$ m^2
$A_{CD} = 3.5 \times 10^{-4}$ m^2

FIGURE PM10.11

M10.12. Write a MATLAB script for the determination of the natural frequencies and mode shapes of a fixed-free beam using an n-element finite-element model. Use the script to determine the dependence of the lowest natural frequency on the number of elements used in the approximation.

M10.13. Write a MATLAB script for an n-element finite-element model to determine the forced response of the system of Fig. PM10.13a. Write the script such that $f(x, t)$ is provided in a user supplied function. When the script is run, it should provide plots of the beam displacement at selected values of time. Run the script for $f(x, t)$ as illustrated in Fig. PM10.13b and c.

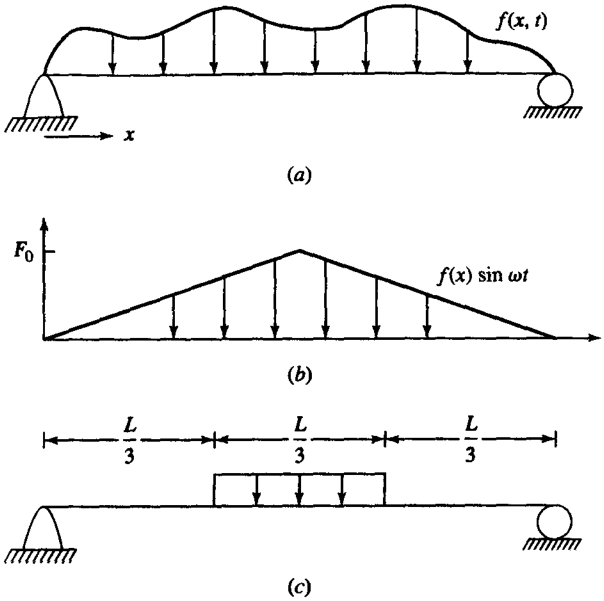
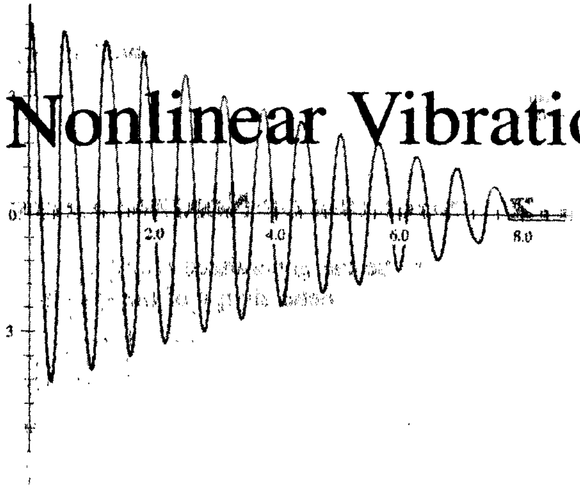


FIGURE PM10.13

chapter

11 Nonlinear Vibrations



11.1 INTRODUCTION

All physical systems are inherently nonlinear. Often assumptions and approximations are made such that the mathematical problem governing the behavior of the system is linear. This is done for an obvious reason; the solution of a linear problem is much easier than the solution of a nonlinear problem. Often, the results obtained using the linear approximation are sufficient for engineering work. Except for the discussions of free and forced oscillations when Coulomb damping is present, this text has thus far considered only linear systems.

Nonlinear systems are much more difficult to analyze than linear systems because the principle of linear superposition is not valid for nonlinear systems. Among the ramifications of the absence of a superposition principle are

1. The homogeneous solution of a second-order nonlinear differential equation is not a linear combination of two linearly independent solutions.
2. The general solution of a nonlinear differential equation cannot be written as the sum of a homogeneous solution and a particular solution, which is independent of initial conditions. The forced response of a nonlinear system cannot be separated from its free-vibration response.
3. The method of superposition cannot be used to add the forced responses due to a combination of excitations. The nonlinearity causes the responses to interact.
4. Since the convolution integral is derived by using linear superposition, it does not apply to nonlinear systems. There is no equivalent of the convolution integral for nonlinear systems.
5. The Laplace transform cannot be used to derive the solution of nonlinear differential equations.

The focus of this chapter is on the qualitative analysis of nonlinear systems. Quantitative results are presented to show how the nonlinearities act to produce nonlinear phenomena.

11.2 SOURCES OF NONLINEARITY

Let $x_1, x_2, \ldots, x_n$ be the generalized coordinates for a conservative n-degree-of-freedom system. The kinetic energy of the system is a function of the generalized coordinates and their derivatives

$$T = T(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \qquad [11.1]$$

The potential energy of the system is a function of the generalized coordinates

$$V = V(x_1, x_2, \ldots, x_n) \qquad [11.2]$$

If the system is linear, then its kinetic energy is independent of the generalized coordinates and a quadratic function of their derivatives. A conservative system is nonlinear if either the kinetic or potential energy cannot be written in a quadratic form.

The kinetic energy function contains terms other than quadratic terms when the inertia properties of the system are dependent on the generalized coordinates or other kinematic relationships between the generalized coordinates are nonlinear. Nonlinear terms due to the latter are called *geometric nonlinearities.*

Terms other than quadratic terms appear in the potential energy function because of geometric nonlinearities or nonlinear force-displacement relations in flexible elements. Nonlinear terms due to the latter are called *material nonlinearities.*

11.1 Derive the governing differential equation for the simple pendulum of Fig. 11.1.

Solution:

The kinetic energy function for the pendulum is

$$T = \tfrac{1}{2}m(l\dot{\theta})^2$$

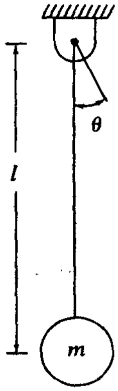


Figure 11.1 The differential equation governing oscillations of the simple pendulum of Example 11.1 is nonlinear.

With the plane of the support as the datum,

$$V = -mgl\cos\theta$$

The kinetic energy function is quadratic, but the potential energy function is not. The nonquadratic term in the potential energy function is a result of the geometric relationship between the instantaneous position of the particle and the datum.

Lagrange's equation, Eq. (5.3), is applied

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

giving

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

The nonlinear term in the differential equation of Example 11.1 is a transcendental function of the dependent variable. Approximate solutions to such equations are made by replacing the transcendental function by its Taylor series expansion. For the equation of Example 11.1, this leads to

$$\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)$$

Approximations can be made by assuming θ is small. A linear approximation is obtained by ignoring all but the linear terms. The simplest nonlinear approximation is obtained by keeping only the largest nonlinear term. Since this term is proportional to the cube of the dependent variable, the nonlinearity is called a *cubic nonlinearity*.

Exampl

Derive the differential equations governing the motion of the system of Fig. 11.2.

Solution:

Let x, the change in length of the spring from its length when the system is in equilibrium with a length l, and θ be the generalized coordinates. The system's kinetic energy function is

$$T = \tfrac{1}{2}m[\dot{x}^2 + (l+x)^2\dot{\theta}^2]$$

Assuming the spring is linear and using the plane of the support as the datum, the system's potential energy function is

$$V = \frac{1}{2}k\left(x + \frac{mg}{k}\right)^2 - mg(l+x)\cos\theta$$

Application of Lagrange's equations leads to

$$m\ddot{x} + kx - m(l+x)\dot{\theta}^2 + mg(1-\cos\theta) = 0$$

and

$$m(l+x)^2\ddot{\theta} + m(l+x)g\sin\theta + 2m(l+x)\dot{x}\dot{\theta} = 0$$

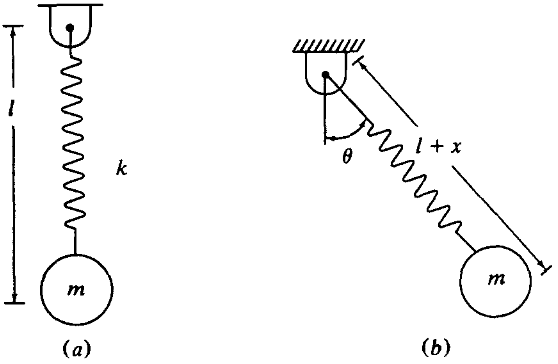


Figure 11.2 (a) The "swinging spring" in equlibrium; (b) the oscillations of the swinging spring are described by coupled nonlinear differential equations. The coupling occurs only in the nonlinear terms. A linear approximation predicts the extensional mode is uncoupled from the swinging mode.

If x and θ are assumed small, Taylor series expansions used for the transcendental functions, and only linear terms retained, the differential equations of Example 11.2 becomes

$$m\ddot{x} + kx = 0$$
$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Thus a linear approximation predicts two uncoupled modes, a spring mode with a natural frequency of $\sqrt{k/m}$ and a pendulum mode with a natural frequency of $\sqrt{g/l}$. Coupling occurs only in the nonlinear terms. If only the largest nonlinear terms are retained, the governing differential equations become

$$m\ddot{x} + kx - ml\dot{\theta}^2 + \frac{mg}{2}\theta^2 = 0$$
$$l\ddot{\theta} + g\theta + \frac{g}{l}\theta x + 2\dot{x}\dot{\theta} = 0$$

Since the largest nonlinear terms involve quadratic products of the generalized coordinates and their derivatives, the nonlinearities are termed *quadratic*.

Note that l is not the unstretched length of the spring, but its length when the system is in static equilibrium, $l = l_0 + mg/k$. Hence the effect of gravity causing a static spring force does not cancel with the static spring force in a nonlinear differential equation. Both must be included in the potential energy formulation.

A material nonlinearity occurs when a flexible component has a nonlinear constitutive equation. The system of Fig. 11.3 is used to model most one-degree-of-freedom systems with viscous damping and harmonic excitation. If the spring has a force-displacement relation of the form

$$F = f(x) \tag{11.3}$$

where f is a nonlinear function of x, then the governing differential equation is

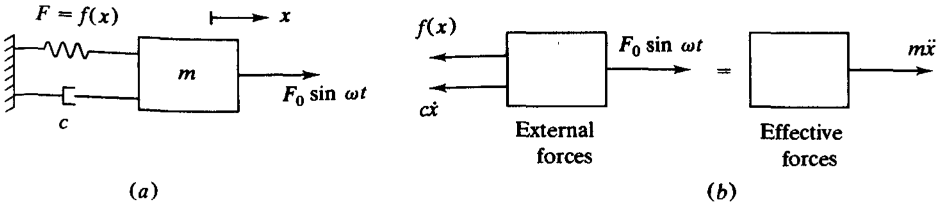


Figure 11.3 (a) Model system for one-degree-of-freedom system with a nonlinear elastic element, viscous damping and harmonic excitation; (b) free-body diagrams used to derive Eq. (11.4). Nonlinear terms are due to a material nonlinearity.

nonlinear,

$$m\ddot{x} + c\dot{x} + f(x) = F_0 \sin \omega t \qquad [11.4]$$

If the spring is unstretched when it is unloaded, then a Taylor series expansion is used to expand $f(x)$ about $x = 0$. If the spring has the same properties in compression as in tension, only odd powers of x appear in the expansion,

$$m\ddot{x} + c\dot{x} + k_1 x + k_3 x^3 + \cdots = F_0 \sin \omega t \qquad [11.5]$$

The values of the coefficients in the Taylor series expansion should decrease as the power increases. The expansion is usually truncated after the cubic term, leading to

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x + \alpha\omega_n^2 x^3 = \frac{F_0}{m} \sin \omega t \qquad [11.6]$$

where ω_n is the natural frequency of the corresponding linear system, ζ is the damping ratio for the linear system, and

$$\alpha = \frac{k_3}{k_1} \qquad [11.7]$$

A spring for which α is positive is called a *hardening spring*. A spring for which α is negative is called a *softening spring*.

Equation (11.6) is called *Duffing's equation*. Duffing's equation is nondimensionalized by introducing

$$x^* = \frac{x}{\Delta} \qquad t^* = \omega_n t \qquad [11.8]$$

where

$$\Delta = \frac{mg}{k_1}$$

is the static deflection of a linear spring of stiffness k_1. Substituting Eq. (11.8) into Eq. (11.6), rearranging, and dropping the * from the nondimensional variables leads to

$$\ddot{x} + 2\zeta\dot{x} + x + \epsilon x^3 = \Lambda \sin rt \qquad [11.9]$$

where

$$r = \frac{\omega}{\omega_n}$$

$$\Lambda = \frac{F_0}{m\omega_n^2 \Delta}$$

and

$$\epsilon = \alpha\Delta^2$$

It is shown in Chap. 2 that the presence of some forms of damping causes nonlinear terms in the differential equation. If the damping force is a function of the velocity,

$$F_d = g(\dot{x})$$

then for Coulomb damping

$$g(\dot{x}) = \mu mg \frac{\dot{x}}{|\dot{x}|}$$

and for aerodynamic drag

$$g(\dot{x}) = c\dot{x}^2$$

The general form of the differential equation for a system subject to a harmonic excitation with nonlinear damping and a nonlinear flexible element is

$$m\ddot{x} + g(\dot{x}) + f(x) = F_0 \sin \omega t \qquad [11.10]$$

Nonlinear terms can arise in differential equations because of an external excitation, as in the following example.

11.3 The U-tube manometer of Fig. 11.4 rotates about an axis other than its centroidal axis with an angular velocity $\omega(t)$. The liquid is incompressible with a mass density ρ, the column has a total length l, and the tube has a cross-sectional area A. If the rotational speed is greater than a critical speed, then all of the fluid is drained from the left leg. Assume the column of liquid moves in the manometer as a rigid body and let $h(t)$ represent the instantaneous height of the column in the right leg. The potential energy function for this system is

$$V = \tfrac{1}{2}\rho g A h^2$$

The system's kinetic energy function is

$$T = \tfrac{1}{2}\rho A l \dot{h}^2 + \tfrac{1}{2}\rho A b^2 h \omega^2 + \int_0^b \rho A r^2 \omega^2\, dr + \int_0^{l-b-h} \rho A r^2 \omega^2\, dr$$

Neglecting viscous friction, Lagrange's equation is applied to derive

$$l\ddot{h} + gh + \frac{\omega^2}{2}(l - b - h)^2 = \frac{\omega^2 b^2}{2}$$

The preceding equation has a quadratic nonlinearity which is the result of the externally imposed rotation. If the speed of rotation is time-dependent, the differential equation has variable coefficients and the system is said to *parametrically excited.*

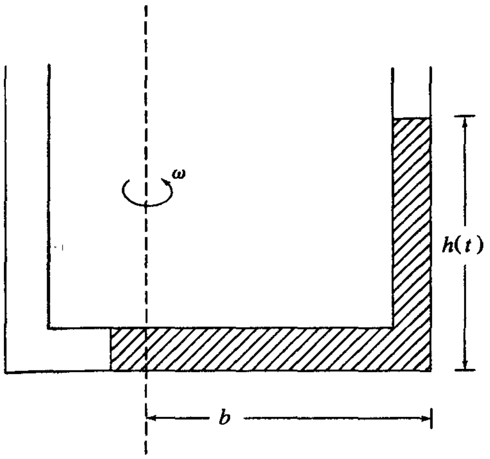


Figure 11.4 The oscillations of a column of liquid in a U-tube manometer rotating about a noncentroidal axis, when the angular velocity is large enough to drain fluid from the left leg, are governed by a nonlinear differential equation.

11.3 QUALITATIVE ANALYSIS OF NONLINEAR SYSTEMS

Qualitative analysis of nonlinear systems is of importance since exact analytical solutions are often not available. Qualitative analysis is used to predict general features of the motion including stability and long-time behavior.

The most useful tool for qualitative analysis of a nonlinear system is the state plane, a graphical time history of the relationship between two variables. The state plane for a one-degree-of-freedom system is a family of curves showing the history of the relation between velocity and displacement. The curves in the state plane are called *trajectories*. Attractors are points or curves to which the trajectories eventually approach.

Exam

Draw the state plane for the unforced Duffing's equation with no damping for a hardening spring.

Solution:

Let $v = x$. Then

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Duffing's equation, Eq. (11.6), becomes

$$v\frac{dv}{dx} = -x - \epsilon x^3$$

Integrating both sides with respect to x gives

$$\tfrac{1}{2}v^2 = C - \tfrac{1}{2}x^2 - \tfrac{1}{4}\epsilon x^4$$

where C is the constant of integration, dependent on initial conditions. The state plane for $\epsilon = \frac{1}{2}$ is shown in Fig. 11.5. Different trajectories correspond to different values of C.

The system of Fig. 11.3 is in equilibrium when its velocity is zero and the sum of the spring force and damping force is zero. For a linear system, this occurs only when $v = 0$ and $x = 0$. A nonlinear system may have more than one equilibrium point. An equilibrium point is *stable* if trajectories approach the equilibrium point for large time. An equilibrium point is *unstable* if trajectories diverge from the equilibrium point for large time.

The equilibrium points for a system governed by Eq. (11.10) are $v = 0$ and the values of x such that $f = 0$. The stability of an equilibrium point is determined by analyzing the trajectories in the vicinity of the equilibrium point. Let

$$x = x_0 + \Delta x \tag{11.11}$$

be a point in the phase plane in the vicinity of the equilibrium point, x_0. Substituting Eq. (11.11) into Eq. (11.10) with $F_0 = 0$ leads to

$$\Delta\ddot{x} + g(\Delta\dot{x}) + f(x_0 + \Delta x) = 0$$

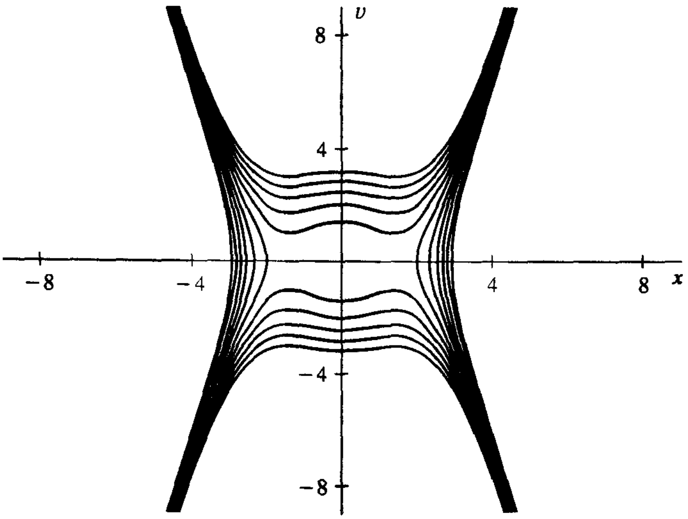


Figure 11.5 State plane for unforced undamped Duffing's equation.

Expanding f and g about $x = x_0$ and $\dot{x} = 0$, respectively, and keeping only linear terms gives

$$\Delta\ddot{x} + \frac{dg(0)}{d\dot{x}}\Delta\dot{x} + \frac{df(x_0)}{dx}\Delta x = 0 \tag{11.12}$$

The general solution of Eq. (11.12) is

$$\Delta x = Ae^{\beta_1 t} + Be^{\beta_2 t} \tag{11.13}$$

If either β_1 or β_2 have a positive real part, then the equilibrium point is unstable.

If β_1 and β_2 are real and have the same sign, the equilibrium point is called a *node*. If β_1 and β_2 are real and have different signs, the equilibrium point is called a *saddle point*, and is, by definition, unstable. If β_1 and β_2 are complex conjugates, the equilibrium point is called a *focus*. A special case of a focus occurs when β_1 and β_2 are purely imaginary, in which case the equilibrium point is called a *center*. Sketches of state planes in the vicinity of a node, saddle point, focus, and center are given in Fig. 11.6.

Examp

Determine the equilibrium points and their nature for the damped unforced Duffing's equation.

Solution:

The equilibrium points are the values of x such that

$$x + \epsilon x^3 = 0$$

For a hardening spring, the only equilibrium point for Duffing's equation is $x = 0$. For a softening spring, the system has the additional equilibrium points

$$x_0 = \pm\sqrt{\frac{1}{-\epsilon}}$$

The nature of the equilibrium point corresponding to $x_0 = 0$ is investigated by assuming $x = \Delta x$, which leads to

$$\beta_{1,2} = -\zeta \pm \sqrt{\zeta^2 - 1}$$

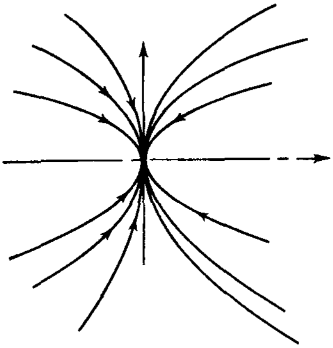
Hence the equilibrium point $x = 0$ is a stable node if $\zeta \geq 1$, and is a stable focus if $\zeta < 1$.

For a softening spring, the natures of the additional equilibrium points are determined using

$$x = \pm\sqrt{-\frac{1}{\epsilon}} + \Delta x$$

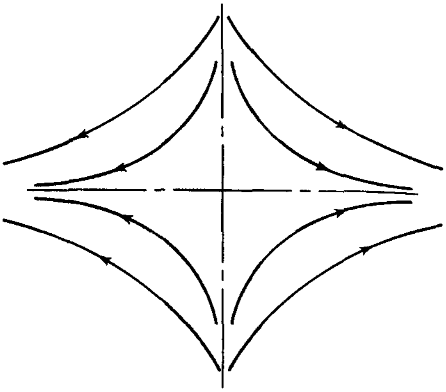
Substituting into Duffing's equation and linearizing leads to

$$\Delta\ddot{x} + 2\zeta\Delta\dot{x} - 2\Delta x = 0$$



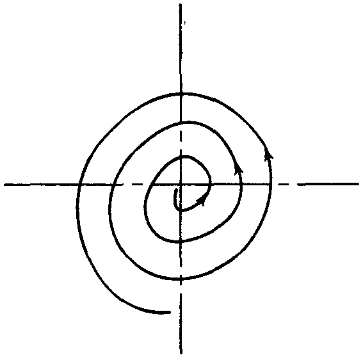
Stable node

(*a*)



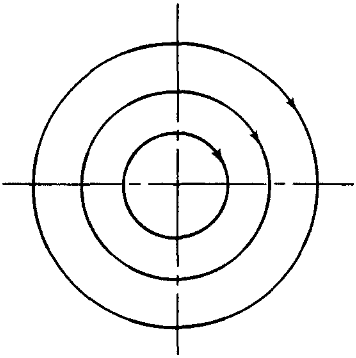
Saddle point

(*b*)



Unstable focus

(*c*)



Center

(*d*)

Figure 11.6 State planes in the vicinity of equilibrium points.

and
$$\beta = -\zeta \pm \sqrt{\zeta^2 + 2}$$

Since the two values of β are real with opposite signs, these equilibrium points are saddle points and thus, by their very nature, unstable.

The phase plane for a system subject to a forced excitation is usually difficult to determine solely by analytical methods. Often, these phase planes must be drawn by graphical methods or numerical results. Figure 11.7 shows several phase planes corresponding to the forced Duffing's equation.

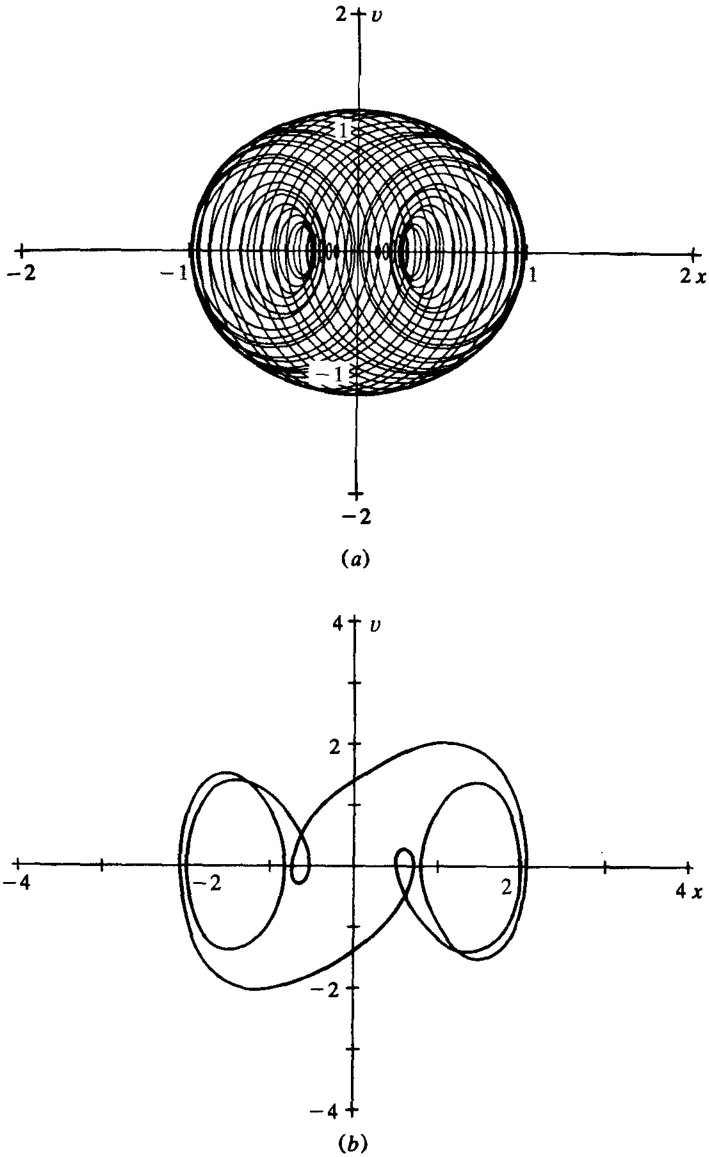


Figure 11.7 Examples of state planes for (*a*) forced, undamped Duffing's equation; (*b*) forced, damped Duffing's equation.

11.4 QUANTITATIVE METHODS OF ANALYSIS

Exact solutions to nonlinear vibration problems exist only for a few special free-vibration problems. Exact solutions for nonlinear forced-vibration problems are almost nonexistent. Consider Eq. (11.10) with $F_0 = 0$. Let $v = x$. Then, using the chain rule for differentiation, as in Example 11.4, Eq. (11.10) can be written as

$$v\frac{dv}{dx} + g(v) + f(x) = 0 \tag{11.14}$$

For certain forms of $g(v)$ and $f(x)$, Eq. (11.14) can be integrated, yielding $v(x)$, which, in turn, can be integrated, yielding $t(x)$.

Consider an undamped system, $g(v) = 0$. Integrating Eq. (11.14) with respect to x and using $x = x_0$ and $v = 0$ when $t = 0$ yields

$$v(x) = \left[2\int_x^{x_0} f(\eta)\,d\eta\right]^{1/2} \tag{11.15}$$

Rearranging and integrating with respect to x gives

$$t = \int_{x_0}^{x} \frac{d\lambda}{\left[2\int_\lambda^{x_0} f(\eta)\,d\eta\right]^{1/2}} \tag{11.16}$$

Since Eq. (11.16) gives t as a function of x, it is not useful for computing the time history of motion, but can be used for frequency calculations. For many forms of $f(x)$, closed-form evaluation of the integral does not exist, and numerical integration is used. Care must be taken when evaluating Eq. (11.16) numerically because the integrand is singular for $\lambda = 0$.

Since exact solutions are not often available, numerical solutions are used. Self-starting methods such as Runge-Kutta are convenient for numerical solution of nonlinear equations.

The general form of the equations for a nonlinear n-degree-of-freedom system is

$$\begin{aligned}
\ddot{x}_1 &= h_1(\mathbf{x}, \dot{\mathbf{x}}, t) \\
\ddot{x}_2 &= h_2(\mathbf{x}, \dot{\mathbf{x}}, t) \\
&\vdots \\
\ddot{x}_n &= h_n(\mathbf{x}, \dot{\mathbf{x}}, t)
\end{aligned} \tag{11.17}$$

Let $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{x}$ be independent n-dimensional vectors. Equation (11.17) can be rewritten as two systems of first-order equations

$$\begin{aligned}
\frac{dx_1}{dt} &= v_1 & \frac{dv_1}{dt} &= h_1(\mathbf{x}, \mathbf{v}, t) \\
\frac{dx_2}{dt} &= v_2 & \frac{dv_2}{dt} &= h_2(\mathbf{x}, \mathbf{v}, t) \\
&\vdots & &\vdots \\
\frac{dx_n}{dt} &= v_n & \frac{dv_n}{dt} &= h_n(\mathbf{x}, \mathbf{v}, t)
\end{aligned}$$

Analytical solutions are preferable to numerical solutions because they can be used to predict trends, analyze the effects of parameters, and develop qualitative results. Thus approximate analytical methods are often used to approximate the solution of nonlinear problems.

If the magnitude of the nonlinear term is small or the amplitude of motion is small, then a perturbation method can be used to develop an approximate solution. Let ϵ be a small nondimensional parameter, $\epsilon \ll 1$. The small parameter may be a measure of the amplitude or a measure of the nonlinearity. For a one-degree-of-freedom system, the generalized coordinate is expanded in a series of powers of ϵ,

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{11.18}$$

Equation (11.18) is substituted into the governing differential equation. Coefficients of like powers of ϵ are collected and set to zero independently. The result is a set of linear differential equations that are successively solved for $x_i(t)$, $i = 1, 2, \ldots$.

The series of Eq. (11.18) is convergent. However, it converges slowly and thus a finite number of terms are inadequate to represent the solution for all t. When only a few terms are included, nonperiodic terms appear which cause the solution to be unbounded for large t. The terms which produce these nonuniformities are called *secular terms*. Since it is impossible to include an infinite number of terms in the evaluation, the secular terms must be removed. A variety of perturbation methods have been developed to remove secular terms. These include the method of strained parameters, the method of renormalization, the method of multiple scales, and the method of averaging. The application of these methods to nonlinear oscillation problems is beyond the scope of this book, but an exhaustive treatment is found in Nayfeh and Mook. The method of renormalization is illustrated in Sec. 11.5.

11.5 FREE VIBRATIONS OF ONE-DEGREE-OF-FREEDOM SYSTEMS

The free vibrations of a conservative system are periodic. If the spring in the system of Fig. 11.3 has the same properties in compression as in tension, then each period of motion can be broken into four parts, each of which takes the same amount of time. If the mass is displaced a distance x_0 from equilibrium and released from rest, the period of the resulting motion can be calculated by using Eq. (11.16) as four times the time it takes the mass to go from its initial position to $x = 0$,

$$T = \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{\left[\int_{\lambda}^{x_0} f(\eta)\, d\eta\right]^{1/2}} \tag{11.19}$$

Equation (11.19) shows that, in contrast to a linear system, the period and the corresponding natural frequency for a nonlinear system depend on the initial conditions.

11.6 A mass, attached to a softening spring with a cubic nonlinearity, is displaced a nondimensional distance x_0 from equilibrium and released from rest. Determine the period of the resulting oscillations as a function of ϵ and x_0.

Solution:

In the notation of Sec. 11.2 and Eqs. (11.5) through (11.9), the nondimensional force developed in a softening spring is

$$f(x) = x - \epsilon x^3 \qquad \epsilon = \alpha \Delta^2$$

Thus the nondimensional period is determined from Eq. (11.19),

$$T = \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{\left[\int_{\lambda}^{x_0} (\eta - \epsilon \eta^3)\, d\eta\right]^{1/2}}$$

where x_0 is the nondimensional initial displacement. The dimensional period is the nondimensional period divided by the linear natural frequency. Proceeding with the algebra gives

$$T = \frac{4}{\sqrt{2}} \int_{x_0}^{0} \frac{d\lambda}{\left[\frac{x_0^2}{2} - \epsilon \frac{x_0^4}{4} - \frac{\lambda^2}{2} + \epsilon \frac{\lambda^4}{4}\right]^{1/2}} = \frac{4\sqrt{2}}{x_0\sqrt{\epsilon}} \int_0^1 \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} - 1 - \frac{2}{\epsilon x_0^2}\phi^2 + \phi^4}}$$

$$= \frac{4\sqrt{2}}{\sqrt{2 - \epsilon x_0^2}} \int_0^1 \frac{d\phi}{(1-\phi^2)(1-k^2\phi^2)} = \frac{4\sqrt{2}}{\sqrt{2 - \epsilon x_0^2}} F\left(k, \frac{\pi}{2}\right)$$

where $F(k, \pi/2)$ is the complete elliptic integral of the first kind of argument k, where

$$k = \sqrt{\frac{2 - \epsilon x_0^2}{\epsilon x_0^2}}$$

A table of elliptic integrals, such as in Abramowitz and Stegun, is used to generate Fig. 11.8.

When the integral of Eq. (11.19) cannot be evaluated in closed form, numerical integration must be used. However, the integrand is singular at $\lambda = x_0$. Let δ be a small nondimensional value. Then for the system of Ex. 11.6,

$$T = \frac{4\sqrt{2}}{x_0\sqrt{\epsilon}} \left[\int_0^{1-\delta} \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} - 1 - \frac{2}{\epsilon x_0^2}\phi^2 + \phi^4}} + \int_{1-\delta}^{1} \frac{d\phi}{\sqrt{\frac{2}{\epsilon x_0^2} - 1 - \frac{2}{\epsilon x_0^2}\phi^2 + \phi^4}} \right]$$

The first integral is evaluated by numerical integration. The integrand of the second integral is expanded by the binomial theorem, and the resulting expansion is

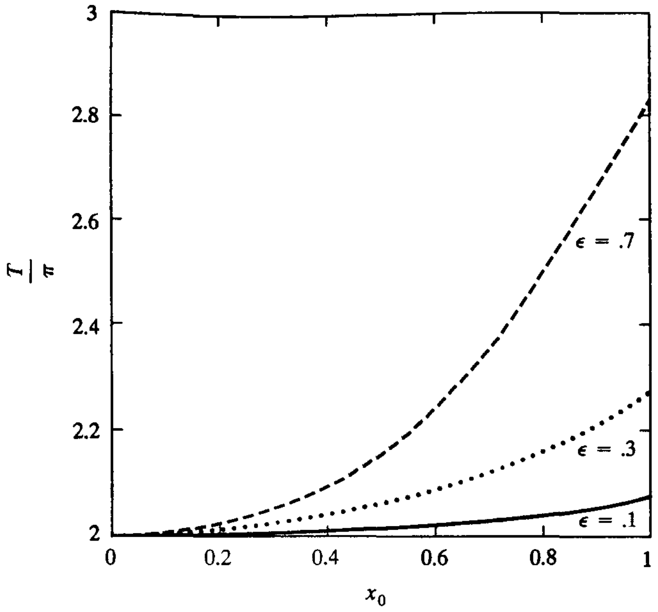


Figure 11.8 Period of Duffing's equation as a function of initial displacement, x_0, for several values of ϵ.

integrated term by term. The expansion is truncated such that desired accuracy is achieved.

Perturbation methods can be applied to approximate the period of a nonlinear system. When the straightforward expansion, Eq. (11.18), is substituted into the unforced, undamped Duffing's equation, the following results:

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_1 + x_0^3) + \epsilon^2(\ddot{x}_2 + x_2 + 3x_0^2 x_1) + \cdots = 0 \qquad [11.20]$$

Coefficients of powers of ϵ are set to zero independently, leading to a set of hierarchical equations

$$\begin{aligned} \ddot{x}_0 + x_0 &= 0 \\ \ddot{x}_1 + x_1 &= -x_0^3 \\ \ddot{x}_2 + x_2 &= -3x_0^2 x_1 \\ &\vdots \end{aligned} \qquad [11.21]$$

The solution for x_0 is

$$x_0 = A \sin(t + \phi) \qquad [11.22]$$

where A and ϕ are determined using initial conditions. Substitution of Eq. (11.22) into the second of Eqs. (11.21) and use of trigonometric identities lead to

$$\ddot{x}_1 + x_1 = -\frac{A^3}{4}[3\sin(t+\phi) - \sin 3(t+\phi)] \qquad [11.23]$$

The particular solution of Eq. (11.23) is

$$x_1(t) = \frac{A^3}{8} t \cos(t+\phi) - \frac{A^3}{32}\sin 3(t+\phi) \qquad [11.24]$$

and the resulting two-term approximation for $x(t)$ is

$$x(t) = A\sin(t+\phi) + \epsilon\left[\frac{3}{8}A^3 t\cos(t+\phi) - \frac{A^3}{32}\sin 3(t+\phi)\right] + \cdots \qquad [11.25]$$

Unfortunately, the expansion of Eq. (11.25) is not periodic and grows without bound. Indeed, when t is as large as $1/\epsilon$, the second term in the expansion is as large as the first term, rendering it invalid.

The problem with the straightforward expansion is that it cannot account for the variation of the period with initial conditions, as mandated by the exact solution. The method of renormalization is used to take this variation into account and render the two-term straightforward expansion uniform. A new time scale is introduced according to

$$t = w(1 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots) \qquad [11.26]$$

Equation (11.25) is rewritten with w as the independent variable

$$\begin{aligned} x &= A\sin(w + \epsilon\lambda_1 w + \cdots + \phi) \qquad [11.27] \\ &\quad + \epsilon\left[\frac{3}{8}A^3(w + \epsilon\lambda_1 w + \cdots)\cos(w + \epsilon\lambda_1 w + \cdots + \phi)\right. \\ &\quad \left. - \frac{A^3}{32}\sin 3(w + \epsilon\lambda_1 w + \cdots + \phi)\right] + \cdots \end{aligned}$$

Taylor series expansions are used to expand the trigonometric functions and coefficients of powers of ϵ are recollected, leading to

$$\begin{aligned} x = A\sin(w+\phi) + \epsilon\Big[& A\lambda_1 w\cos(w+\phi) \\ & + \frac{3}{8}A^3 w\cos(w+\phi) - \frac{A^3}{32}\sin 3(w+\phi)\Big] + \cdots \end{aligned} \qquad [11.28]$$

The secular term is removed from Eq. (11.28) by choosing

$$\lambda_1 = -\tfrac{3}{8}A^2 \qquad [11.29]$$

leading to

$$x = A\sin(w+\phi) - \epsilon\frac{A^3}{32}\sin 3(w+\phi) + \cdots \qquad [11.30]$$

where
$$t = w(1 - \epsilon \tfrac{3}{8} A^2 + \cdots) \tag{11.31}$$

The binomial expansion is used to invert Eq. (11.31)
$$w = t(1 + \epsilon \tfrac{3}{8} A^2 + \cdots) \tag{11.32}$$

The amplitude is determined by application of the initial conditions. If $x(0) = \delta$ and $\dot{x}(0) = 0$, then
$$\phi = \frac{\pi}{2}$$
$$\delta = A - \epsilon \frac{A^3}{32}$$

A natural frequency approximation can be obtained to greater accuracy by calculating higher-order terms in the expansion for x, and choosing the λ_i from Eq. (11.31) to eliminate secular terms.

For damped systems, the damping term is often small enough to be ordered with the nonlinearity. To this end, define
$$\zeta = 2\epsilon\mu \tag{11.33}$$

where μ is of order 1. When the straightforward expansion is used in the damped, unforced version of Duffing's equation, the following equations result, defining x_0 and x_1:
$$\ddot{x}_0 + x_0 = 0 \tag{11.34}$$
$$\ddot{x}_1 + x_1 = -x_0^3 - 2\mu\dot{x}_0$$

In order to use the method of renormalization for damped systems, the solutions of Eqs. (11.34) are written using complex exponentials
$$x_0 = A\cos(t + \phi) = \tfrac{1}{2} A[e^{i(t+\phi)} + e^{-i(t+\phi}] \tag{11.35}$$

When Eq. (11.31) is used to remove secular terms from the two-term expansion,
$$\lambda_1 = -\frac{3}{8}A^2 - i\frac{\mu}{2} \tag{11.36}$$

and the resulting two-term uniformly valid expansion is
$$x = Ae^{-\zeta t}\sin\left[(1 + \epsilon \tfrac{3}{8} A^2)t + \phi\right] \tag{11.37}$$

Thus, when secular terms are removed through x_1, damping has no effect on the natural period. The exponential decay, comparable to that of a linear system, is apparent.

In summary, the natural frequency of a nonlinear system depends on its initial conditions. The straightforward perturbation expansion and the method of renormalization can be used to determine an approximation to the natural frequency when the nonlinearity is small or when the amplitude is small. Small viscous damping has an effect on free vibrations of nonlinear systems similar to that on free vibrations of linear systems, causing an exponential decay of amplitude.

11.6 FORCED VIBRATIONS OF ONE-DEGREE-OF-FREEDOM SYSTEMS WITH CUBIC NONLINEARITIES

Consider the damped Duffing's equation subject to a two-frequency excitation,

$$\ddot{x} + 2\mu\epsilon\dot{x} + x + \epsilon x^3 = F_1 \sin r_1 t + F_2 \sin r_2 t \qquad r_1 \neq r_2 \tag{11.38}$$

Use of the straightforward expansion, Eq. (11.18), produces the following two-term approximation to the solution of Eq. (11.38):

$$\begin{aligned}
x = {} & A\sin(t+\phi) + F_1 M_1 \sin r_1 t + F_2 M_2 \sin r_2 t \\
& + \epsilon \Bigg\{ -\mu A t \sin(t+\phi) - \left(\tfrac{3}{8}A^3 + \tfrac{3}{4}AF_1^2M_1^2 + \tfrac{3}{4}AF_2^2M_2^2\right) t\cos(t+\phi) \\
& - \frac{2\mu F_1 M_1 r_1}{1-r_1^2}\cos r_1 t - \frac{2\mu F_2 M_2 r_2}{1-r_2^2}\cos r_2 t \\
& + \frac{3(2A^2F_1M_1 + F_1^3M_1^3 + 2F_1F_2^2M_1M_2^2)}{4(1-r_1^2)}\sin r_1 t \\
& + \frac{3(2A^2F_2M_2 + F_2^3M_2^3 + 2F_1^2F_2M_1^2M_2)}{4(1-r_2^2)}\sin r_2 t \\
& + \frac{A^3}{32}\sin 3(t+\phi) - \frac{3A^2F_1M_1}{4\left[1-(2+r_1)^2\right]}\sin\left[(2+r_1)t + 2\phi\right] \\
& + \frac{3A^2F_1M_1}{4\left[1-(2-r_1)^2\right]}\sin\left[(2-r_1)t + 2\phi\right] \\
& - \frac{3AF_1^2M_1^2}{4\left[1-(1+2r_1)^2\right]}\sin\left[(1+2r_1)t + \phi\right] \\
& + \frac{3AF_1^2M_1^2}{4\left[1-(1-2r_1)^2\right]}\sin\left[(1-2r_1)t + \phi\right] \\
& - \frac{3AF_2M_2}{4\left[1-(2+r_2)^2\right]}\sin\left[(2+r_2) + 2\phi\right] \\
& + \frac{3A^2F_2M_2}{4\left[1-(2-r_2)^2\right]}\sin\left[(2-r_2) + 2\phi\right] \\
& - \frac{3AF_2^2M_2^2}{4\left[1-(1+2r_2)^2\right]}\sin\left[(1+2r_2)t + \phi\right] \\
& + \frac{3AF_2^2M_2^2}{4\left[1-(1-2r_2)^2\right]}\sin\left[(1-2r_2)t + \phi\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3F_1^2 F_2 M_1^2 M_2}{4\left[1-(2r_1+r_2)^2\right]} \sin(2r_1+r_2)t \\
&+ \frac{3F_1^2 F_2 M_1^2 M_2}{4\left[1-(2r_1-r_2)^2\right]} \sin(2r_1-r_2)t \\
&- \frac{3F_1 F_2^2 M_1 M_2^2}{4\left[1-(2r_2+r_1)^2\right]} \sin(2r_2+r_1)t \\
&+ \frac{3F_1 F_2^2 M_1 M_2^2}{4\left[1-(2r_2-r_1)^2\right]} \sin(2r_2-r_1)t \\
&- \frac{F_1^3 M_1^3}{4(1-9r_1^2)} \sin 3r_1 t - \frac{F_2^3 M_2^3}{4(1-9r_2^3)} \sin 3r_2 t \\
&+ \frac{3AF_1F_2M_1M_2}{2\left[1-(r_1-r_2+1)^2\right]} \sin\left[(r_1-r_2+1)t+\phi\right] \\
&- \frac{3AF_1F_2M_1M_2}{2\left[1-(r_1-r_2-1)^2\right]} \sin\left[(r_2-r_1-1)t-\phi\right] \\
&- \frac{3AF-1F-2M_1M_2}{2\left[1-(r_1+r_2+1)^2\right]} \sin\left[(r_1+r_2+1)t+\phi\right] \\
&\left.+ \frac{3AF_1F_2M_1M_2}{2\left[1-(r_1+r_2-1)^2\right]} \sin\left[(r_1+r_2-1)t-\phi\right]\right\} \cdots
\end{aligned}
\qquad [11.39]
$$

where

$$M_i = \frac{1}{1-r_i^2}$$

The expansion of Eq. (11.39) is nonuniform because of the secular terms arising from the free-vibration solution. Additional nonuniformities occur when the values of r_1 and r_2 are such that the denominators of other terms are very small. Examination of Eq. (11.39) suggests that an exhaustive study of the frequency response of a one-degree-of-freedom system with a cubic nonlinearity requires the following cases be considered:

1. No resonances.
2. $r_1 = 1$ or $r_2 = 1$, primary resonance.
3. $r_1 = \frac{1}{3}$ or $r_2 = \frac{1}{3}$, superharmonic resonance.
4. $r_1 = 3$ or $r_2 = 3$, subharmonic resonance.
5. $2r_2 + r_1 = 1, 2r_1 - r_2 = \pm 1, 2r_2 - r_1 = \pm 1, r_1 - r_2 + 1 = -1, r_1 - r_2 - 1 = \pm 1$, or $r_1 + r_2 - 1 = 1$, combination resonances.
6. Conditions when two resonances occur simultaneously. For example, when $r_1 = \frac{1}{3}$ and $r_2 = \frac{5}{3}$, both superharmonic and combination resonances occur.

A resonance condition occurs when the free-vibration contribution to the solution does not decay with time. The steady-state solution has a contribution from the free vibrations as well as the forced steady-state response. For a linear system the free-vibration response is periodic with a frequency equal to the natural frequency and the forced response due to a harmonic excitation is periodic with a frequency equal to the excitation frequency. For a linear system only the primary resonance can occur, when the excitation frequency is near the natural frequency.

For a system with a cubic nonlinearity, Eq. (11.28) shows that the free-vibration response includes a periodic term whose frequency is three times the linear natural frequency. Thus oscillations at this frequency are sustained in the absence of an external excitation. Any additional energy input may lead to growth of the free oscillations and thus produce the subharmonic resonance.

The forced response of a system with a cubic nonlinearity to a harmonic excitation includes a periodic term whose frequency is three times the excitation frequency. Thus, when the excitation frequency is one-third of the natural frequency, this term tends to excite the free vibrations and causes the free-vibration term to be sustained, even in the presence of small damping. This produces the superharmonic resonance.

When a system with a cubic nonlinearity is subject to a multifrequency excitation, the forced response includes periodic terms at frequencies that are combinations of the excitation frequencies. When this combination of frequencies is close to the natural frequency, free oscillations are sustained and a combination resonance exists.

The straightforward expansion is nonuniform for all r_1 and r_2, even when no resonance conditions exist. The method of renormalization can be used to render the two-term expansion uniform, but it can only be used to predict periodic responses, and cannot provide information about the stability of equilibrium points. Possibly the best method for obtaining uniform expansions to approximate the solution of nonlinear forced-vibration problems is the method of multiple scales. The results provided in the following discussion can be obtained using the method of multiple scales. Since its application is beyond the scope of this text, the discussion focuses on qualitative behavior. More detail is available in Nayfeh and Mook.

1. *No resonances.* For most values of r_1 and r_2, no resonance conditions exist. However, the expansion of Eq. (11.39) is still nonuniform. When secular terms are removed, the solution is the sum of the free-vibration response and the forced response. The free vibrations decay exponentially, but the frequency of free vibrations depends on the initial conditions and the amplitudes and frequencies of the excitation.

2. *Primary resonance.* A primary resonance occurs when an excitation frequency is near the system's linear natural frequency, corresponding to the nondimensional frequency being near 1. When the amplitude of the excitation is of order 1, the straightforward perturbation expansion predicts an infinite amplitude response, even in the presence of small damping. When the amplitude of the excitation is the same order as the nonlinearity and the damping, secular terms occur in x_1.

The frequency response in the vicinity of the primary resonance is studied by introducing a *detuning parameter*, defined by

$$r_1 = 1 + \epsilon\sigma \tag{11.40}$$

The amplitude and phase of the resulting motion vary with time, but possible steady states can be identified. The following approximate equations can be derived for the steady-state amplitude and the steady-state phase angle:

$$4A^2\left[\mu^2 + \left(\sigma - \tfrac{3}{8}A^2\right)^2\right] = \hat{F}_1^2 \tag{11.41}$$

$$\phi = -\tan^{-1}\left(\frac{\mu}{\sigma - \frac{3}{8}A^2}\right) \tag{11.42}$$

where

$$\hat{F}_1 = \frac{F_1}{\epsilon}$$

Equations (11.41) and (11.42) are plotted in Figs. 11.9 and 11.10. Note from these figures that there is a frequency range where three possible steady-state amplitudes and phases exist for a single frequency. This leads to an interesting phenomenon, peculiar to nonlinear systems, called the jump phenomenon. Imagine that the amplitude of excitation is fixed, but its frequency is slowly increased, starting slightly below the linear natural frequency. As the frequency is increased the steady-state amplitude follows the upper branch of the frequency response curve, until the point of vertical tangency is reached. When the frequency is increased beyond this critical value, the only possible steady-state amplitude is finitely lower than the amplitude at the critical frequency, and the amplitude will "jump" to this lower value. Now if the frequency is decreased from this value, the steady-state amplitude will follow the lower branch of the frequency response curve, until the point of vertical tangency is reached, when it will "jump" to the upper branch.

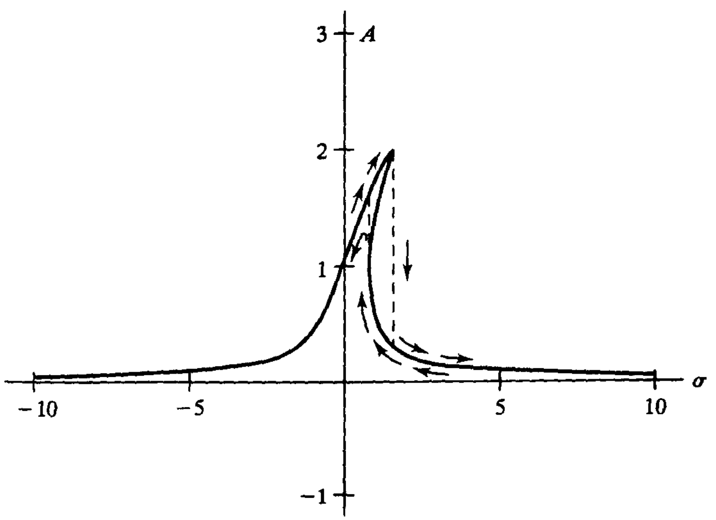


Figure 11.9 Frequency response curve for primary resonance of Duffing's equation illustrates the jump phenomenon ($\mu = 0.25$, $\hat{F}_1 = 1$).

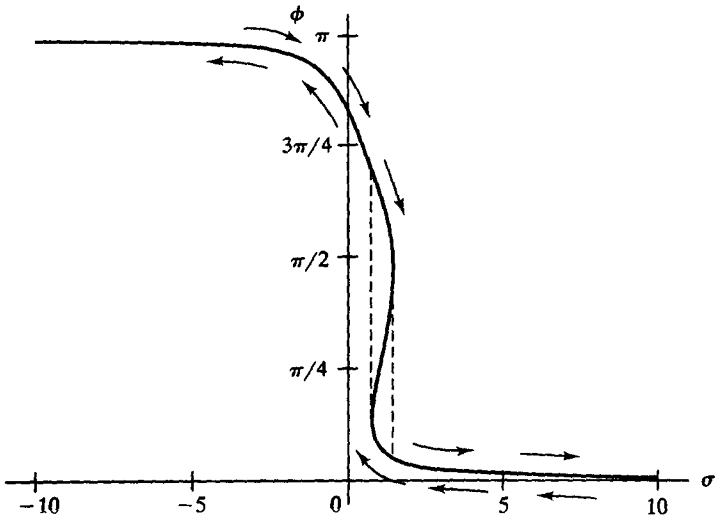


Figure 11.10 Phase versus frequency curve for primary resonance of Duffing's equation also shows a jump phenomenon ($\mu = 0.25$, $\hat{F}_1 = 1$).

A state plane showing the relation between the amplitude and phase can be plotted for Duffing's equation with a primary resonance for parameters where the triple valuedness exists. Two equilibrium points are stable foci corresponding to the points on the upper and lower branches of the frequency response curve. A third equilibrium point is a saddle point corresponding to the intermediate amplitude between the points of vertical tangency. Since this equilibrium point is unstable, it can never be physically attained. The initial conditions dictate which of the two stable foci corresponds to the steady-state solution.

3. *Superharmonic resonance.* When either r_1 or r_2 is near $\frac{1}{3}$, the free-oscillation term does not decay exponentially. The steady-state response then consists of the forced response whose period is three times that of the linear natural period plus the free response, whose frequency is adjusted to three times that of the excitation. Thus the total response is periodic with the period equal to that of the excitation.

Introduction of a detuning parameter when r_1 is near $\frac{1}{3}$, $3r_i = 1 + \epsilon\sigma$, leads to the frequency response equation

$$\left[\mu^2 + (\sigma - 3F_i^2 - \tfrac{3}{8}A^2)^2\right]A^2 = F_i^6 \qquad \textbf{[11.43]}$$

which is cubic in A^2 and hence has three solutions. For a certain frequency range, three real solutions exist. The triple valuedness of the amplitude leads to a jump phenomenon similar to that for the primary resonance, as shown in Fig. 11.11.

4. *Subharmonic resonance.* When an excitation frequency is near three times the linear natural frequency, a subharmonic resonance may occur. The frequency

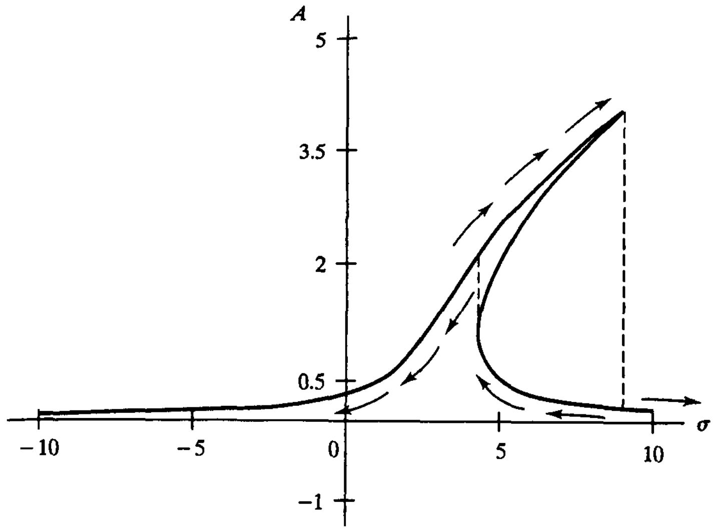


Figure 11.11 Frequency response curve for superharmonic resonance ($\mu = 0.25$, $F_1 = 1$).

response curve when r_i is near 3, $r_i = 3 + \epsilon\sigma$ is given by

$$\left[9\mu^2 + \left(\sigma - 9F_i^2 - \tfrac{9}{8}A^2\right)^2\right]A^2 = \tfrac{81}{16}F_i^2 A^4 \qquad [11.44]$$

Equation (11.44) has the trivial solution, $A = 0$, and two solutions obtained as roots of a quadratic equation in A^2. The quadratic equation yields real solutions for A if and only if the parameters satisfy the following inequality:

$$\frac{\sigma}{\mu} - \sqrt{\left(\frac{\sigma}{\mu}\right)^2 - 63} \le \frac{63F_1^2M_1^2}{4\mu} \le \frac{\sigma}{\mu} + \sqrt{\left(\frac{\sigma}{\mu}\right)^2 - 63} \qquad [11.45]$$

When nontrivial solutions exist, one corresponds to a stable focus and one corresponds to a saddle point. The initial conditions determine whether the steady-state contribution from the free-oscillation term is trivial or approaches the stable focus.

Thus, if Eq. (11.45) is satisfied and the initial conditions are appropriate, the free-vibration term will not decay, but will exist with an adjusted frequency of one-third of that of the excitation. The total response is periodic with the period equal to that of the excitation. The frequency response curve is illustrated in Fig. 11.12.

5. *Combination resonances.* Combination resonances are unique to nonlinear systems and occur because of the nonlinear interaction of the particular solutions from x_0 when x_1 is calculated. When a combination resonance is present, a nontrivial free-vibration solution exists. The nonlinearity tunes the free-vibration response to the appropriate combination of frequencies.

The jump phenomenon does occur when a combination resonance is present.

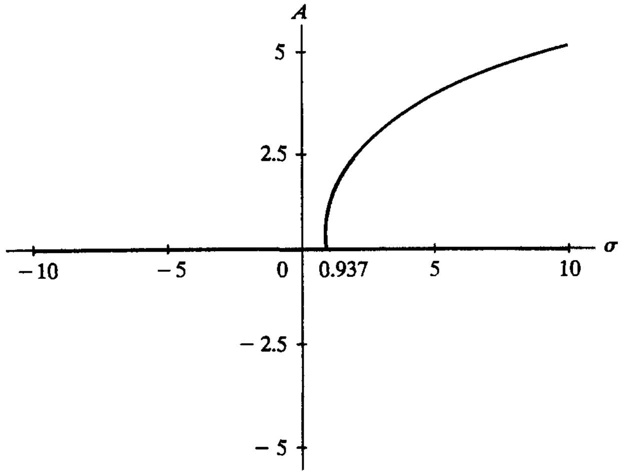


Figure 11.12 Frequency response curve for superharmonic resonance with $\mu = 0.25$ and $F_1 = 1$. For $\sigma < 0.937$, only the trivial steady-state response exists.

6. *Simultaneous resonances.* Simultaneous resonances occur when two resonance conditions occur simultaneously. A detuning parameter is introduced for each resonance condition. Analysis of the steady state is much more complicated. For some simultaneous resonances, as many as seven equilibrium points exist in the state plane for the same frequency.

11.7 MULTI-DEGREE-OF-FREEDOM SYSTEMS

Nonlinear multi-degree-of-freedom systems exhibit behaviors which are not present for linear systems. It is instructive to consider free and forced vibrations of systems with quadratic nonlinearities and systems with cubic nonlinearities. Let p_1, $p_2, \ldots, p_n$ be the principal coordinates for a linearized system with natural frequencies $\omega_1 < \omega_2 < \cdots < \omega_n$, respectively. Principal coordinates that uncouple a linear system do not uncouple the system when nonlinearities are considered. The differential equations for the principal coordinates are coupled through nonlinear terms. For example, the free vibrations of an undamped two-degree-of-freedom system with quadratic nonlinearities are governed by

$$\ddot{p}_1 + \omega_1^2 p_1 + \alpha_1 p_1^2 + \alpha_2 p_1 p_2 + \alpha_3 p_2^2 = 0 \qquad [11.46]$$

$$\ddot{p}_2 + \omega_2^2 p_2 + \beta_1 p_1^2 + \beta_2 p_1 p_2 + \beta_3 p_2^2 = 0$$

11.7.1 FREE VIBRATIONS

The free-vibration response of a system with quadratic nonlinearities includes periodic terms with frequencies of $\omega_1+\omega_2$, $\omega_1-\omega_2$, $2\omega_1$, and $2\omega_2$. If $\omega_2 \approx 2\omega_1$, then the nonlinearity acts as if it excites the system with a harmonic excitation of frequencies ω_1 and ω_2, producing a self-sustaining free oscillation, called an *internal resonance.*

In the absence of the internal resonance, and in the presence of damping, the free oscillations of both modes decay exponentially, and are to first approximation independent. When an internal resonance is present, free oscillations are sustained, even when damping is present and causes coupling between the two modes. Even if only one mode is initially excited, the internal resonance excites the other mode as well. Energy is continually exchanged between the two modes.

An internal resonance occurs in a two-degree-of-freedom system with cubic nonlinearities when $\omega_2 \approx 3\omega_1$.

Exampl

Reconsider the spring pendulum of Example 11.2. The spring has a stiffness 1×10^3 N/m and an unstretched length of 0.5 m. For what values of m will an internal resonance occur?

Solution:

Since l is the length of the spring when the system is in equilibrium,

$$l = \left(0.5 + \frac{mg}{k}\right) m$$

Since the approximate linear system is uncoupled when x and θ are used as generalized coordinates, these are also the principal coordinates and the linear natural frequencies are

$$\omega_1 = \sqrt{\frac{g}{0.5 + \dfrac{mg}{k}}} \qquad \omega_2 = \sqrt{\frac{k}{m}}$$

Setting $\omega_2 = 2\omega_1$ gives $m = 12.74$ kg.

11.7.2 FORCED VIBRATIONS

The free oscillations are self-sustaining in multi-degree-of-freedom systems subject to harmonic excitations when the frequency of excitation is near certain values. A primary resonance occurs if the excitation frequency is near any of the system's natural frequencies. Subharmonic and superharmonic resonances occur as for one-degree-of-freedom systems. Other secondary resonances occur when the excitation frequency is near a certain combination of natural frequencies.

For a system with quadratic nonlinearities, these resonances occur when the excitation frequency is near the sum or difference of two natural frequencies.

Combination resonances occur for multifrequency excitations. Simultaneous resonance conditions can also exist.

A complete summary of the phenomena present in nonlinear multi-degree-of-freedom systems is too extensive. The jump phenomenon occurs for certain types of resonances. Quenching can also occur in certain systems with simultaneous resonances where introduction of the second resonance causes the total response to decrease.

A saturation phenomenon can also occur for systems with quadratic nonlinearities. The amplitude of a specific mode may build up as the amplitude of excitation is increased. When the excitation amplitude reaches a certain value, the mode may become saturated; its amplitude of response remains constant as the excitation amplitude is further increased. The amplitudes of the other modes will continue to grow with the excitation amplitude.

In addition to primary resonances, subharmonic resonances, and superharmonic resonances, combination resonances occur in a two-degree-of-freedom system with cubic nonlinearities when one of the following conditions is met:

$$\begin{aligned} \Omega &\approx 2\omega_1 \pm \omega_2 \\ \Omega &\approx 2\omega_2 \pm \omega_1 \\ \Omega &\approx \tfrac{1}{2}(\omega_2 \pm \omega_1) \end{aligned} \qquad [11.47]$$

where Ω is the excitation frequency.

11.8 CONTINUOUS SYSTEMS

The nonlinear dimensionless partial differential equation governing transverse vibrations of a uniform beam of length L and radius of gyration r, subject to a transverse load per unit length $F(x, t)$, is

$$\left(\frac{r}{L}\right)^2 \left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4}\right) = \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx \frac{\partial^2 w}{\partial x^2} + F(x, t) \qquad [11.48]$$

The nonlinear term is a result of the midplane stretching and is often ignored.

Let $\omega_1, \omega_2, \ldots$ be the natural frequencies of the linearized system and $\phi_1, \phi_2, \ldots$ be their corresponding normalized mode shapes such that

$$(\phi_i(x), \phi_j(x)) = \delta_{ij}$$

for an appropriate scalar product.

The expansion theorem is used to develop an approximation to the solution of Eq. (11.48) as

$$w(x, t) = \epsilon \sum_{i=1}^{\infty} p_i(t)\phi_i(x) \qquad [11.49]$$

where $\epsilon \ll 1$ is a small dimensionless amplitude. Substituting Eq. (11.49) into Eq. (11.48), taking the scalar product with respect to $\phi_j(x)$ for an arbitrary j, and using algebra and mode shape orthonormality lead to

$$\ddot{p}_j + \omega_j^2 p_j = \epsilon \left(\frac{L}{r}\right)^2 \left[\frac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\phi_j, \frac{\partial^2 \phi_k}{\partial x^2}\right)\int_0^1 \frac{\partial \phi_l}{\partial x}\frac{\partial \phi_m}{\partial x}\,dx\, p_k p_l p_m\right] + (F(x,t), \phi_j(x)) \tag{11.50}$$

The preceding procedure is similar to the modal analysis method of Chap. 9, except that the members of the resulting set of ordinary differential equations are still coupled through the nonlinear terms. The nonlinear terms, due to midplane stretching are cubic nonlinearities. If the excitation is harmonic with a frequency Ω, then from the results of Sec. 11.7, the following resonances can occur:

1. Internal resonances occur if $\omega_i \approx 3\omega_j$, or $\omega \approx 2\omega_j + \omega_k$ for any i, j, and k. From Table 9.4, for a fixed-pinned beam, $\omega_2 = 3\omega_1 + 2.30$, and for a fixed-fixed beam $\omega_5 = 2\omega_3 + \omega_2 - 4.86$. Internal resonances occur in each of these beams. It is noted that for a pinned-pinned beam $\omega_3 = 2\omega_2 + \omega_1$. However, the coefficient multiplying $p_2^2 p_1$ in Eq. (11.50) is zero for a pinned-pinned beam.
2. Primary resonance occurs if $\Omega \approx \omega_i$ for any i.
3. Superharmonic resonance occurs if $\Omega = \omega_i/3$ for any i.
4. Subharmonic resonance occurs if $\Omega = 3\omega_i$ for any i.
5. Combination resonances occur if $\Omega \approx 2\omega_i \pm \omega_j$, $\Omega \approx \omega_i \pm \omega_j \pm \omega_k$, or $\Omega \approx (\omega_i \pm \omega_j)/2$ for any i, j, and k.

11.9 CHAOS

Recent research in nonlinear phenomena has led to the development of a new branch of physics called chaos. The term *chaos* is not well defined but refers to the seemingly random response of a nonlinear system due to deterministic excitation. Chaos occurs when a periodic excitation leads to a nonperiodic response. Chaos occurs when slightly different initial conditions lead to divergent responses.

Chaos has been observed and predicted in nonlinear systems in such diverse fields as physics, medicine, economics, and meteorology. Chaos occurs in mechanical systems, electrical systems, and chemical systems. Researchers observed that chicken pox epidemics are periodic while measles epidemics are chaotic. Others have used chaos to model stock market fluctuations. Many researchers hope that chaos may unlock some of the mysteries of fluid turbulence.

Chaotic motion has been observed in many mechanical systems. Chaotic vibrations for systems modeled by Duffing's equation are well documented, as are chaotic motions of a forced pendulum.

Analytical tools have been developed to identify and classify chaotic behavior. These tools can be applied to analytical solutions for vibrating systems as well as experimental observations. Some are described in the following discussion.

1. *State space.* Observation of the state space can indicate whether a system is chaotic. A chaotic motion will have trajectories that do not repeat, when viewed in the phase plane. The trajectories will fill a region of the phase plane without ever repeating. However, viewing of the state plane is by itself insufficient to speculate that a motion is chaotic. An example of a chaotic response from Duffing's equation, Eq. (11.9) as viewed in a state plane is shown in Fig. 11.13.

2. *Poincaré sections.* A Poincaré section is a graph of the phase plane response taken or sampled only at fixed intervals of time. If the response is periodic and the time interval is equal to the period, then the Poincaré section is only a point, as the same response is obtained on each sampling. If the response is periodic and the time interval is less than the period, but commensurate with the period, the Poincaré section is a finite number of points.

The Poincaré section of a nonlinear system with a quadratic subharmonic resonance, sampled at the period of excitation should have two points. The presence of the subharmonic resonance doubles the period of response. If a system subject to a periodic excitation is sampled at intervals equal to the period of excitation, and the Poincaré section is a seemingly random collection of points, the response can be guessed to be chaotic. Poincaré sections for responses of Duffing's equation, Eq. (11.9) are given in Fig. 11.14. These Poincaré sections illustrate that values of parameters determine whether a response is chaotic.

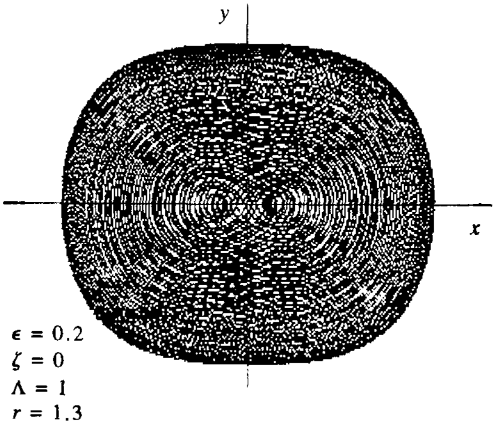


Figure 11.13 State plane for an apparently chaotic motion.

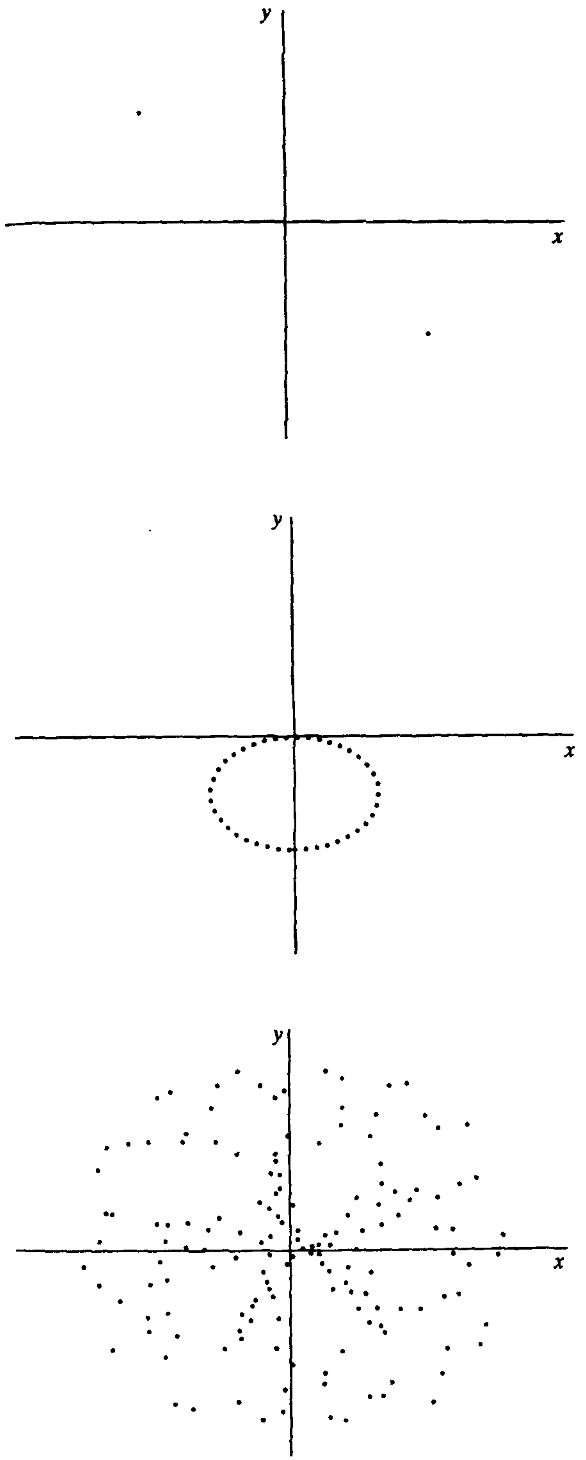


Figure 11.14 (a) Poincaré section for peri motion when sampling inter equal to half the period; (b) section for periodic motion sampling interval is incomm with period; (c) Poincaré se a chaotic motion.

3. *Fourier transforms.* The Fourier transform of a nonperiodic continuous function is an extension of the Fourier series defined for periodic functions. The Fourier transform is obtained from the Fourier series by allowing the period to become infinite. The resulting Fourier transform of $f(t)$ is defined as

$$\bar{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \qquad [11.51]$$

The transform function, $\bar{f}(\omega)$, is a function of the transform variable, ω. If the Fourier transform of a periodic function is taken, then $\bar{f}(\omega) = 0$ unless ω is a multiple of the function's fundamental frequency.

The Fourier transform decomposes a function into its harmonic components. The strength of a component is given by the magnitude of $\bar{f}(\omega)$. The values of ω which have significant nonzero values of $\bar{f}(\omega)$ are called the *spectrum* of the function. If the Fourier transform of the response of a nonlinear system due to a periodic response is a continuous spectrum, then the response is chaotic.

For computational purposes the Fourier transform is replaced by the fast Fourier transform. If $f(t)$ is known at k times, $t_1, t_2, \ldots, t_k$, then the discrete fast Fourier transform is given by

$$\bar{f}(j) = \sum_{l=1}^{k} f(t_i)e^{-2\pi i(l-1)(j-1)/k} \qquad [11.52]$$

Examples of Fourier transforms are given in Fig. 11.15.

4. *Bifurcation diagrams.* Bifurcation diagrams can be used to identify one route to chaos. The steady-state amplitude (or phase) of a nonlinear system as a function of a system parameter is plotted as the parameter is slowly changed. For a nonlinear system the steady-state solution may split at a certain value of the parameter and two possible steady states exist for greater values of the parameter. A bifurcation is said to occur for the value of the parameter where the split occurs. The bifurcation is often the result of the sudden presence of a subharmonic resonance. When this occurs the period of motion doubles. As the parameter is increased, additional bifurcations may occur, where the period again doubles. If the system is chaotic, as the parameter increases, bifurcations and period doubling occur more rapidly. The chaotic response bounces between amplitudes and has no discernible period. The plot of steady-state amplitudes (or phases) as the parameter increases becomes a blur. It is often the case that as the parameter is increased much further, the motion again becomes periodic.

While chaotic motion is characterized by its unpredictable nature, it has some universal features. Feigenbaum showed that, as the number of bifurcations increases, the values of the parameter, call it A, for which the bifurcations occur are given by

$$A_n - A_{n-1} = (4.669\ldots)(A_{n+1} - A_n) \qquad [11.53]$$

There are many routes to chaos. The one described here applies to systems undergoing nonlinear oscillations subject to a harmonic excitation and is illustrated

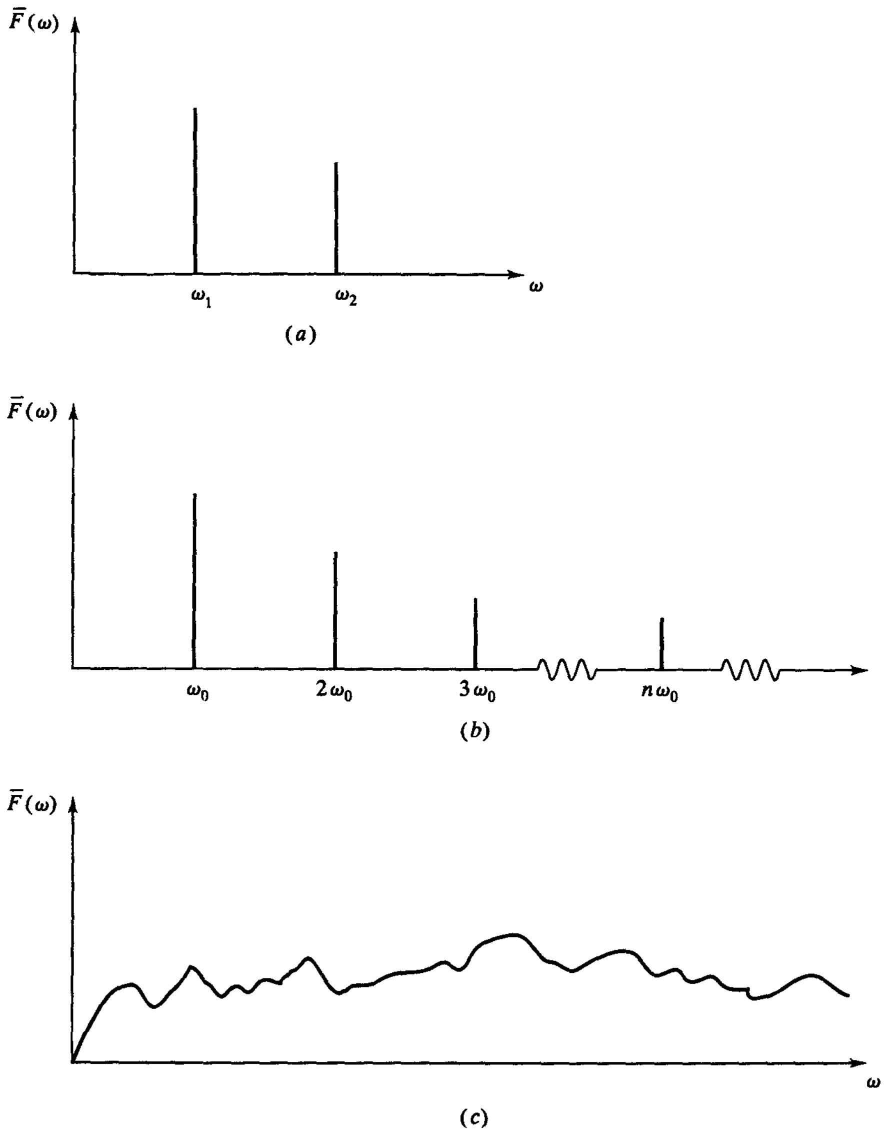


Figure 11.15 (a) Fourier transform of a periodic function with two distinct frequencies; (b) Fourier transform of a periodic function of fundamental frequency ω_0; (c) Fourier transform of a chaotic response.

by the rotating U-tube manometer of Example 11.3 and Fig. 11.16. The manometer is rotated about a vertical axis other than its centroidal axis. The rotational speed of the manometer varies as

$$\omega(t) = A \sin \lambda t \qquad [11.54]$$

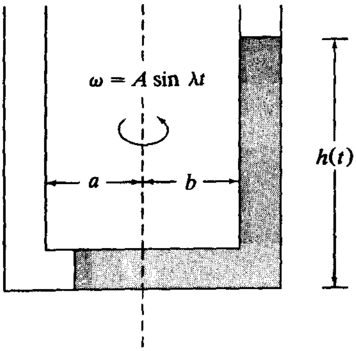


Figure 11.16 For certain values of λ and ε the motion of the column of liquid in the U tube manometer can be chaotic when the manometer rotates about a non-centroidal axis.

where A is large enough to cause the fluid to be completely drained from the left leg during an initial transient period. The system is subject to viscous damping from the interaction of the fluid with the wall of the manometer.

The behavior of a nonlinear system is heavily influenced by the system parameters. This is evidenced by the state planes of Figs. 11.17 and 11.18. Figure 10.17 shows the state planes for two slightly different values of the frequency for the same amplitude. A steady state is evident for the motion of Fig. 10.17*a*, while the motion of Fig. 10.17*b* appears chaotic. Chaos is also induced by small amplitude changes for the same frequency as shown in Fig. 10.18*a*.

A bifurcation diagram for the parameter A is shown in Fig. 11.19. The frequency of excitation is fixed as its amplitude varies. For $A < 3.33$ the steady-state motion is periodic. The stationary response is periodic of frequency 2λ and a certain amplitude.

For $A \approx 3.33$ the parameters change such that a subharmonic resonance becomes present. A bifurcation is said to occur. The presence of the subharmonic resonance means that the steady-state response is the sum of a free-vibration term and a forced-vibration term and that the period of motion is doubled. Two amplitudes are evident in the stationary oscillations.

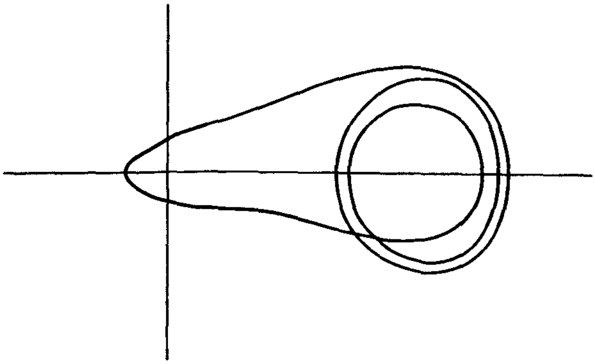
For $A \approx 3.35$ another bifurcation occurs. A higher-order subharmonic resonance is induced. The response has a period of four times the original period and is made up of four distinct amplitudes.

As A increases bifurcations occur more rapidly with the period doubling with each bifurcation. Eventually, the response is chaotic. The chaotic response shown in Fig. 11.20 bounces between amplitudes and has no discernible period.

For $A \approx 3.36$ the motion ceases to be chaotic and returns to the doubled period. However, bifurcations begin to occur again at $A \approx 3.37$.

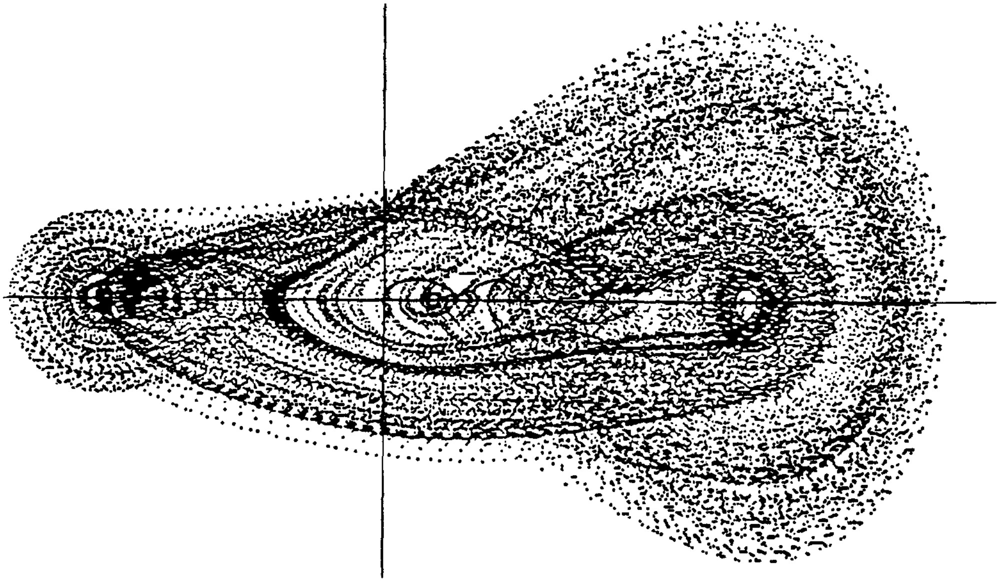
The process described previously is called *period doubling through a subharmonic cascade.*

Chaos is the subject of much current research. It is hoped that studying chaos can lead to the better understanding of nonlinear systems like turbulent fluid flows, the flow and pumping of blood through a human heart, weather patterns, and nonlinear vibrations.



(*a*)

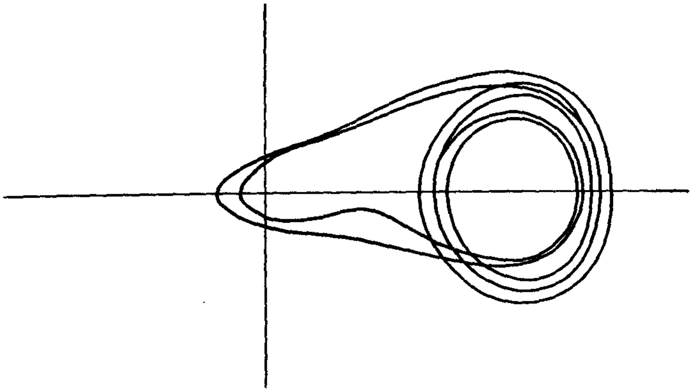
$\lambda = 0.27 \qquad A = 4.5$



(*b*)

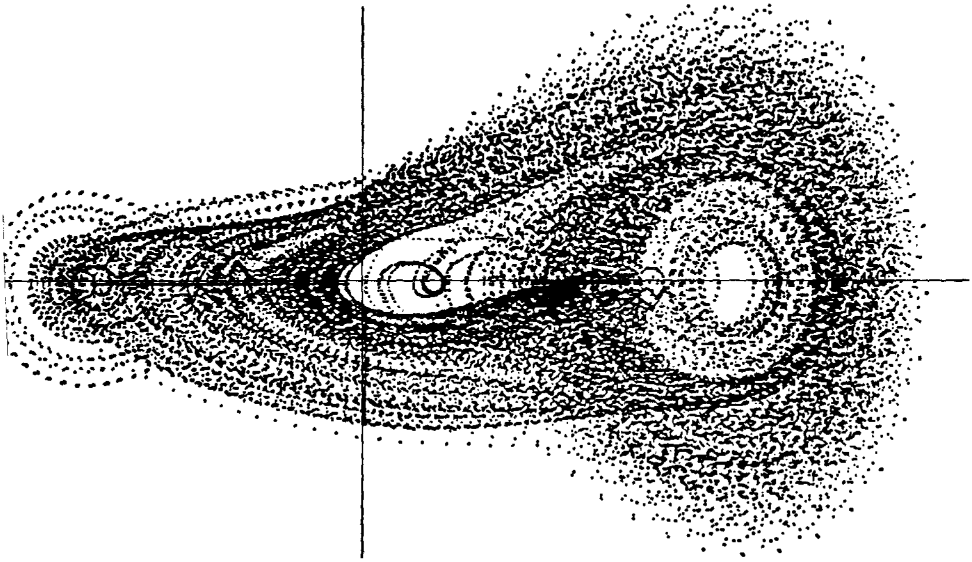
$\lambda = 0.28 \qquad A = 4.5$

Figure 11.17 These state planes show that a small frequency change can cause a change from a periodic response to a chaotic response.



$\lambda = 0.27 \quad A = 4.6$

(*a*)



(*b*)

$\lambda = 0.27 \quad A = 4.7$

ure 11.18 These state planes show that a small amplitude change can cause a change from a periodic response to a chaotic response.

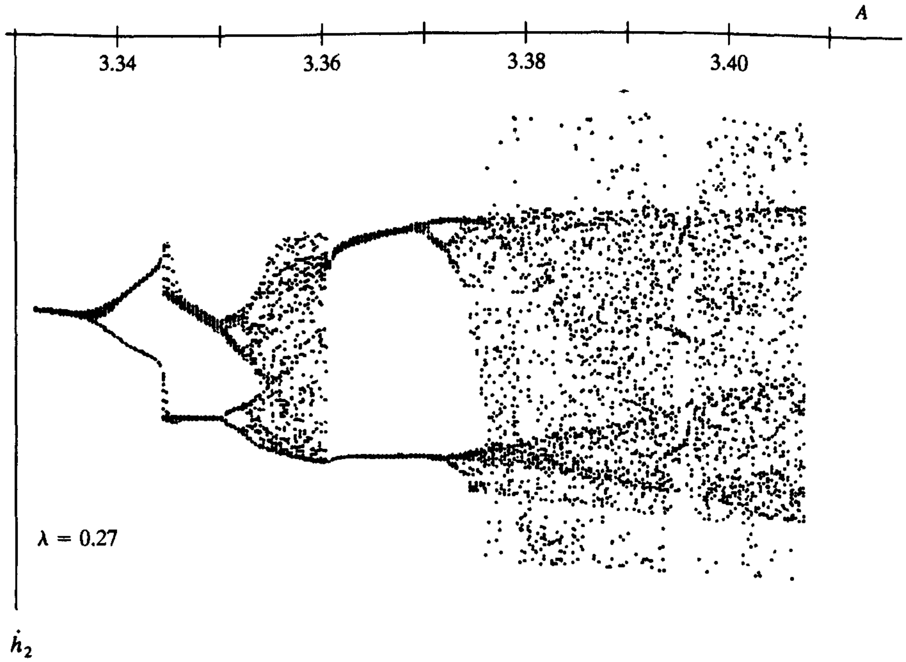


Figure 11.19 Bifurcation diagram for rotating manometer. First bifurcation occurs near A = 3.34. As A increases chaos develops. Motion is not chaotic for a range of A, the process to chaos begins again.

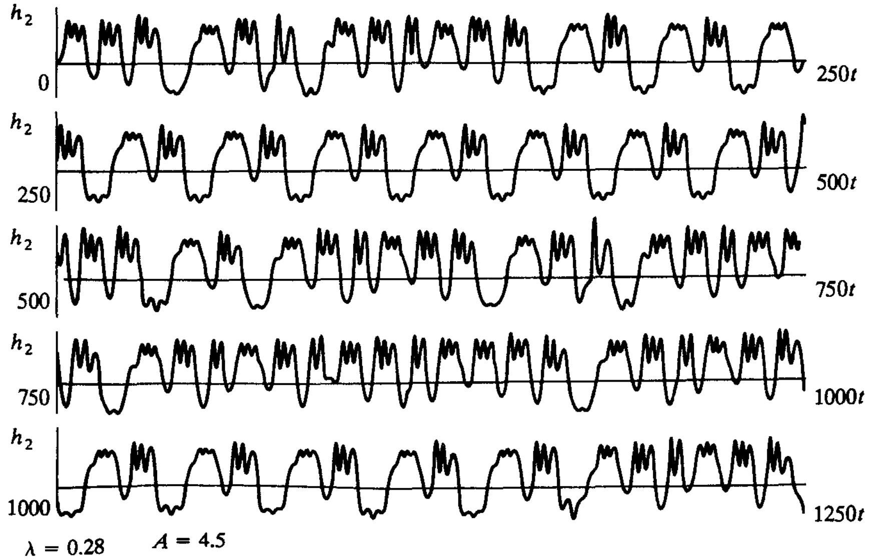


Figure 11.20 The time history of motion for these parameters has no discernable period.

PROBLEMS

11.1. The free-vibration response of a block hanging from a linear spring is the same as that of the block attached to the same spring, but sliding on a frictionless surface. Is the response the same if the spring has a force-displacement relation given by

(*a*) $F = k_1x + k_3x^3$.

(*b*) $F = k_1x + k_2x^2$.

(*c*) $F = k_1x, x < x_0$.
$F = k_2x, x > x_0$.

11.2. The system of Fig. P11.2 is one of the few for which an exact solution is available. Its solution is obtained in a manner analogous to that of free vibrations with Coulomb damping described in Sec. 16. The block is displaced a distance $x_0 > \delta$ to the right from equilibrium and released. Determine the period of the resulting oscillations.

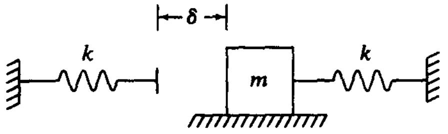


FIGURE P11.2

11.3. The block in Fig. P11.3 is not attached to the springs. Determine the period of the resulting oscillations if the block is displaced a distance x_0 to the right from equilibrium and released.

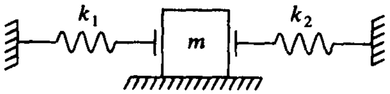


FIGURE P11.3

11.4–11.7. Without making linearizing assumptions, use Lagrange's equations to derive the nonlinear differential equation(s) governing the motion of the systems shown. Use the generalized coordinates indicated.

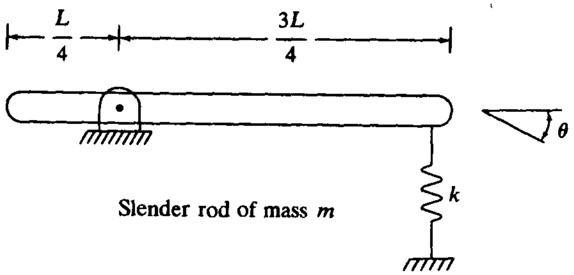


FIGURE P11.4

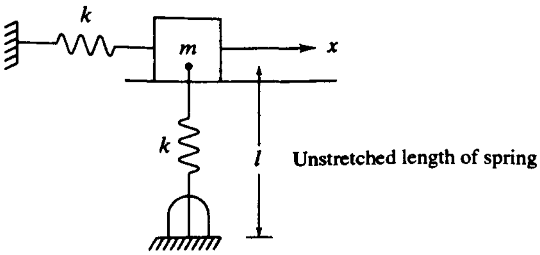


FIGURE P11.5

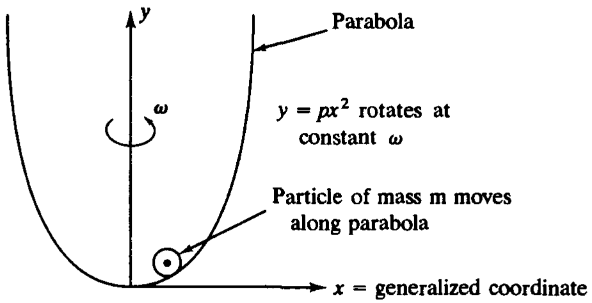


FIGURE P11.6

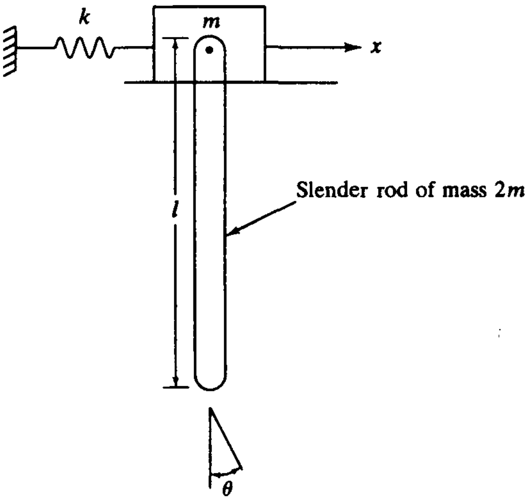


FIGURE P11.7

11.8. A wedge of specific weight γ floats stably on the free surface of a fluid of specific weight γ_w (Fig. P11.8). The wedge is given a vertical displacement δ from this equilibrium position.

(*a*) Derive the differential equation governing the resulting free oscillations of the wedge. Neglect viscous effects and the added mass of the fluid.

(*b*) What is the equation of the trajectory in the phase plane which describes the resulting motion. Sketch the trajectory.

(*c*) Assume δ is small and use the method of renormalization to determine a two-term approximation for the frequency-amplitude relationship.

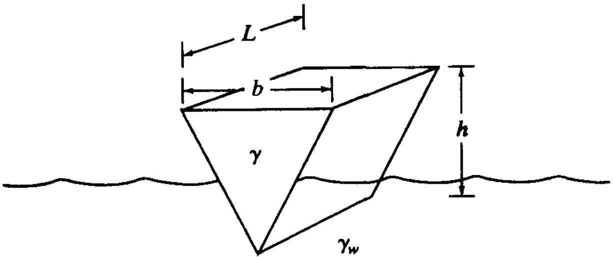


FIGURE P11.8

11.9. Repeat Prob. 11.8 for the inverted cone of Fig. P11.9.

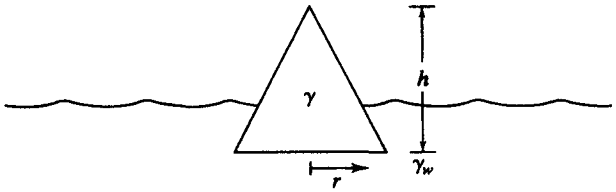


FIGURE P11.9

11.10. Determine the equation defining the state plane for the system of Fig. P11.6. Sketch trajectories in the phase plane when

(*a*) $p = 1.5\ \text{m}^{-1}$, $\omega = 5$ rad/s.

(*b*) $p = 1.0\ \text{m}^{-1}$, $\omega = 5$ rad/s.

(*c*) $p = 5.097\ \text{m}^{-1}$, $\omega = 10$ rad/s.

11.11. Plot the trajectory in the state plane corresponding to the motion of a mass attached to a linear spring free to slide on a surface with Coulomb damping when the mass is displaced from equilibrium and released from rest.

11.12. Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x + \epsilon x^3 = 0$$

11.13. Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} - x - \epsilon x^3 = 0$$

11.14. Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} + x + \epsilon x^2 = 0$$

11.15. Determine the equilibrium points and their type for the differential equation

$$\ddot{x} + 2\zeta\dot{x} + x - \epsilon x^2 = 0$$

11.16. The equation of motion for the free oscillations of a pendulum subject to quadratic damping is

$$\ddot{\theta} + 2\zeta\dot{\theta}^2 + \sin\theta = 0$$

(*a*) Determine an exact equation defining the state plane.

(*b*) Determine the equilibrium points and their type.

11.17. Determine the period of oscillation of a mass attached to a hardening spring with a cubic nonlinearity.

11.18. Determine an integral expression for the period of oscillation of the system of Fig. P11.6.

11.19. Use the method of renormalization to determine a two-term approximation for the frequency-amplitude relation for the system of Fig. P11.4. If the bar is rotated 4° from equilibrium and released, what is the period for $L = 4$ m, $k = 1000$ N/m, and $m = 10$ kg?

11.20. A 25-kg mass is attached to a hardening spring with $k_1 = 1000$ N/m and $k_3 = 4{,}000$ N/m^3. The mass is displaced 15 mm from equilibrium and released from rest. What is the period of the ensuing oscillations?

11.21. Suppose the mass of Prob. 11.20 is subject to an impulse which imparts a velocity of 3.1 m/s to the mass when the mass is in equilibrium. What is the period of the ensuing oscillations?

11.22. Suppose the mass of Prob. 11.20 is attached to the same spring when a 50-N force is statically applied and suddenly removed. What is the period of the ensuing oscillations?

11.23. Use the method of renormalization to determine a two-term frequency-amplitude relationship for the particle on the rotating parabola of Fig. P11.6, assuming the amplitude is small.

11.24. Use the method of renormalization to determine a two-term frequency-amplitude relationship for a block of mass m attached to a spring with a quadratic nonlinearity. When nondimensionalized the differential equation governing free vibrations of the system is

$$\ddot{x} + \omega^2 x + \epsilon x^2 = 0 \qquad \epsilon \ll 1$$

Problems 11.25 to 11.31 refer to the system of Fig. P11.25.

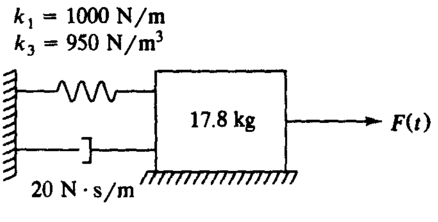


FIGURE P11.25

11.25. If $F(t) = F_0 \sin \omega t$, what values of ω will lead to the presence of

(*a*) A primary resonance.

(*b*) A superharmonic resonance.

(*c*) A subharmonic resonance.

11.26. When $F(t) = 5 \sin 8t$ N a primary resonance condition occurs. Determine the amplitude of the forced response.

11.27. When $F(t) = 150 \sin 2.5t$ N a superharmonic resonance condition occurs. Determine the amplitude of the forced response.

11.28. If $F(t) = F_0 \sin \omega t$ N, for what value of ω will a jump in amplitude occur when ω is increased slightly beyond this value when

(*a*) $F_0 = 5$ N and a primary resonance occurs.

(*b*) $F_0 = 150$ N and a superharmonic resonance occurs.

11.29. If $F(t) = 25 \sin 22t$ N, will a nontrivial subharmonic response exist?

11.30. If $F(t) = 30 \sin 15t + 25 \sin \omega t$ N, what values of ω lead to a combination resonance?

11.31. If $F(t) = 30 \sin 2.5t + 25 \sin \omega t$ N, what values of ω lead to simultaneous resonances?

Problems 11.32 to 11.35 refer to the systems of Fig. P11.32. The spring of stiffness k_2 is a linear spring.

11.32. If $m_2 = 10$ kg, for what values of k_2 will internal resonances exist?

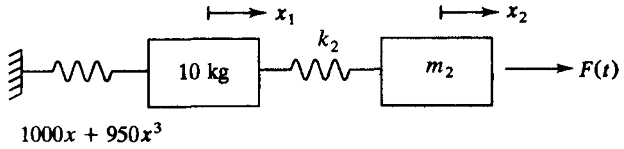


FIGURE P11.32

11.33. For what values of m_2 are internal resonances possible? If an internal resonance is possible, in terms of m_2, for what values of k_2 will they exist?

11.34. Consider the system with $m_2 = 10$ kg and $k_2 = 2000$ N/m. The right mass is displaced 10 mm from equilibrium while the left mass is held in place. The system is released from rest from this configuration.

(*a*) Determine the natural frequencies, mode shapes, and principal coordinates for the linearized system.

(*b*) Write the nonlinear differential equations governing the system using the principal coordinates of the linearized system as dependent variables.

11.35. If $m_2 = 10$ kg, $k_2 = 1000$ N/m, and $F(t) = 150 \sin \omega t$ N, for what values of ω will the following resonances exist?

(*a*) Primary resonance.

(*b*) Superharmonic resonance.

(*c*) Subharmonic resonance.

(*d*) Combination resonance.

11.36. Consider the system of Fig. P11.36.

(*a*) Derive the nonlinear differential equations governing the motion of the system using the generalized coordinates shown.

(*b*) Expand trigonometric functions of the generalized coordinates using Taylor series expansions. Rewrite the differential equations keeping only quadratic and cubic nonlinearities.

(*c*) For what values of l in terms of the other parameters will an internal resonance exist?

(*d*) In the absence of an internal resonance, for what values of ω will resonance conditions exist?

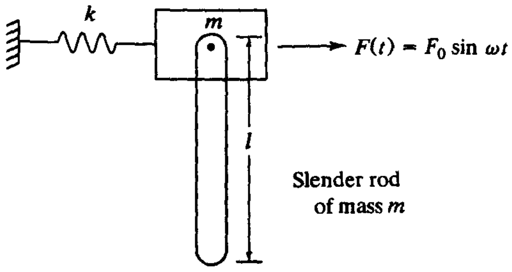


FIGURE P11.36

11.37. Show that the coefficient multiplying $p_2^2 p_1$ for a pinned-pinned beam is zero in Eq. (11.50).

11.38. A fixed-free rectangular steel beam ($\rho = 7850$ kg/m^3, $E = 210 \times 10^9$ N/m^2) of length 1 m, base 2 cm, and height 5 cm is subject to a single-frequency harmonic excitation. List all excitation frequencies that should be avoided to avoid all primary, secondary, and combination resonances involving the three lowest modes.

11.39. If the beam of Prob. 11.38 is fixed-fixed, which of the following excitation frequencies should be avoided and why?

(*a*) 180 rad/s.

(*b*) 1530 rad/s.

(*c*) 2200 rad/s.

(*d*) 7940 rad/s.

MATLAB PROBLEMS

M11.1. The file VIBES_11A.m is used to numerically integrate Duffing's equation. The file uses the MATLAB ordinary differential equation solver ODE45. Use VIBES_11A to investigate the parameters that lead to chaos. Use VIBES_11A to help develop a bifurcation diagram for the parameter Λ.

M11.2. Write a MATLAB file that uses the ordinary differential equation solver to solve the nonlinear equation

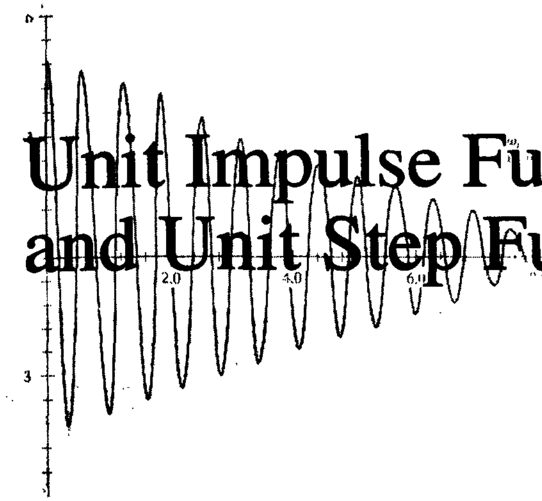
$$\ddot{x} + 2\mu\dot{x} + x + \alpha x^2 + \beta x^3 = F_0 \sin \omega t$$

The program should also plot the state plane for the solution. Use the program to investigate how the addition of the quadratic nonlinearity affects chaotic behavior.

appendix

A

Unit Impulse Function and Unit Step Function



Consider the function, $f_\Delta(x; a)$, where $f_\Delta(x; a)$ as shown in Fig. A.1 is defined by

$$f_\Delta(x; a) = \begin{cases} 0 & -\infty < x < a - \dfrac{\Delta}{2} \\ \dfrac{1}{\Delta} & a - \dfrac{\Delta}{2} \leq x \leq a + \dfrac{\Delta}{2} \\ 0 & a + \dfrac{\Delta}{2} < x < \infty \end{cases} \tag{A.1}$$

The function has the property

$$\int_{-\infty}^{\infty} f_\Delta(x; a)\, dx = 1 \tag{A.2}$$

Taking the limit of $f_\Delta(x; a)$ as $\Delta \to 0$ yields

$$\lim_{\Delta \to 0} f_\Delta(x; a) = \delta(x - a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases} \tag{A.3}$$

From Eq. (A.2)

$$\int_{-\infty}^{\infty} \delta(x - a)\, dx = 1 \tag{A.4}$$

The function defined in Eq. (A.3) and whose valuable property is given in Eq. (A.4) is called the *unit impulse function*. It has many applications in physics and engineering. It is used to mathematically represent the force that is applied to cause a unit impulse applied at a time $t = a$ in a mechanical system. It is used to represent a unit concentrated load applied at a location $x = a$ to a structure. The unit impulse function is also used to represent a unit heat source in a heat transfer problem.

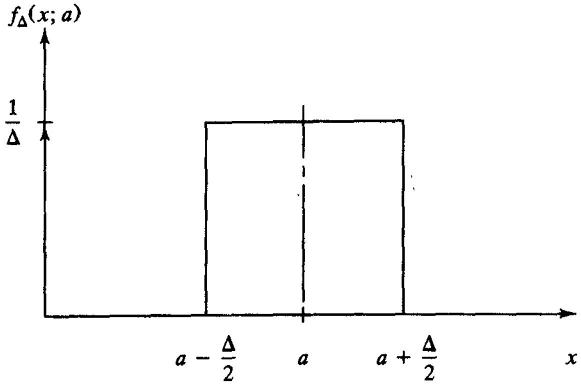


Figure A.1 $\delta(x - a) = \lim_{\Delta \to 0} f_\Delta(x; a)$.

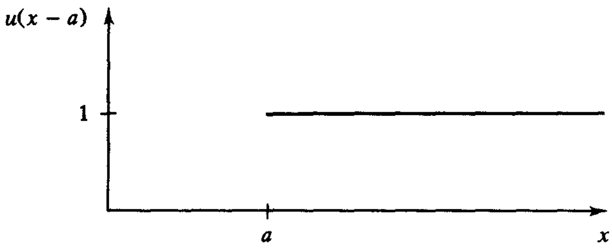


Figure A.2 The unit step function $u(x - a)$.

Now define

$$u(x-a) = \int_0^x \delta(x-a)\,dx = \int_0^x \lim_{\Delta \to 0} f_\Delta(x; a)\,dx = \begin{cases} 0 & x \le a \\ 1 & x > a \end{cases} \quad \textbf{[A.5]}$$

The function defined in Eq. (A.5) is called the *unit step function* and is illustrated in Fig. A.2. Differentiating Eq. (A.5) gives

$$\frac{du(x-a)}{dx} = \delta(x-a) \quad \textbf{[A.6]}$$

The definitions of the unit impulse function and unit step function can also be used to derive the following integral formulas. For any function $g(t)$,

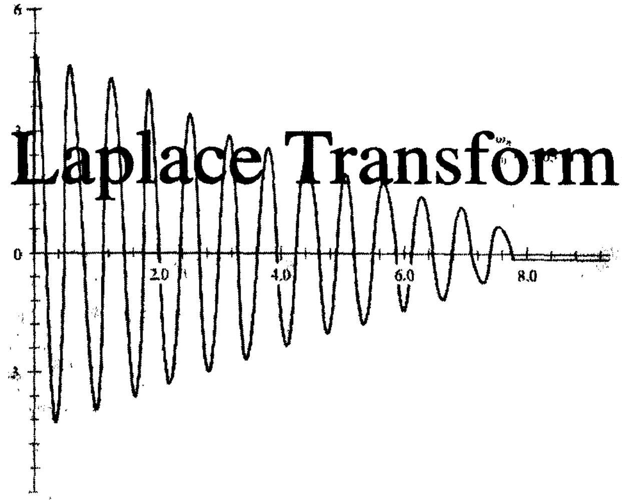
$$\int_0^t \delta(\tau - a) g(\tau)\,d\tau = u(t-a) g(a) \quad \textbf{[A.7]}$$

and

$$\int_0^t u(\tau - a) g(\tau)\,d\tau = u(t-a) \int_a^t g(\tau)\,d\tau \quad \textbf{[A.8]}$$

appendix

B Laplace Transforms



B.1 DEFINITION

The Laplace transform of a function $f(t)$ is defined as

$$\bar{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{B.1}$$

If there exist values of α, M, and T such that

$$e^{-\alpha t}|f(t)| < M \qquad \text{for all } t > T \tag{B.2}$$

then $\bar{f}(s)$ exists for $s > \alpha$. Equation (B.2) is satisfied for all excitations and responses in this text.

The Laplace transform transforms a real-valued function into a function of a complex variable, s. For many functions the Laplace transform can be obtained by direct integration.

Determine the Laplace transform of $f(t) = e^{\alpha t}$.

Solution:

$$\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{\alpha t}e^{-st}\,dt = \frac{1}{\alpha - s}e^{(\alpha - s)t}\Big|_0^\infty = \frac{1}{s-\alpha} \qquad s > \alpha$$

Table B.1

Number	$f(t)$	$\bar{f}(s)$
1	1	$\frac{1}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	$e^{\alpha t}$	$\frac{1}{s-\alpha}$
4	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
5	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
6	$\delta(t-a)$	e^{-as}
7	$u(t-a)$	$\frac{e^{-as}}{s}$

B.2 TABLE OF TRANSFORMS

Equation (B.1) is used to develop a table of Laplace transforms of common functions. Laplace transforms of other functions can be developed using Table B.1 in conjunction with properties of the transform.

B.3 LINEARITY

The Laplace transform operator is a linear operator. Let $\bar{f}(s) = \mathcal{L}\{f(t)\}$ and $\bar{g}(s) = \mathcal{L}\{g(t)\}$ and let α and β be any real numbers. Then

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \bar{f}(s) + \beta \bar{g}(s) \tag{B.3}$$

B.4 TRANSFORM OF DERIVATIVES

The property of the Laplace transforms of derivatives allows easy application of the Laplace transform to the solution of differential equations

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2}\dot{f}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \tag{B.4}$$

Exa

Use transform pair 5 from Table B.1 and Eq. (B.4) to determine $\mathcal{L}\{\sin 2t\}$.

Solution:

Noting that

$$\sin 2t = -\frac{1}{2}\frac{d(\cos 2t)}{dt}$$

and applying properties (B.3) and (B.4) with $n = 1$ gives

$$\mathcal{L}\{\sin 2t\} = -\tfrac{1}{2}(s\mathcal{L}\{\cos 2t\} - 1)$$

Using transform pair 5 from Table B.1,

$$\mathcal{L}\{\sin 2t\} = -\frac{1}{2}\left(\frac{s^2}{s^2+4} - 1\right) = \frac{2}{s^2+4}$$

B.5 FIRST SHIFTING THEOREM

If $\bar{f}(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{e^{-at}f(t)\} = \bar{f}(s+a) \qquad \text{[B.5]}$$

Exa

Use Table B.1 and the first shifting theorem to calculate $\mathcal{L}\{e^{-\zeta\omega_n t}\cos\omega_d t\}$ where $\omega_d = \omega_n\sqrt{(1-\zeta^2)}$.

Solution:

Using the first shifting theorem and transform pair 5 from Table B.1,

$$\mathcal{L}\{e^{-\zeta\omega_n t}\cos\omega_d t\} = \left.\frac{s}{s^2+\omega_d^2}\right|_{s\to s+\zeta\omega_n} = \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} = \frac{s+\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

B.6 SECOND SHIFTING THEOREM

If $\bar{f}(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s) \qquad \text{[B.6]}$$

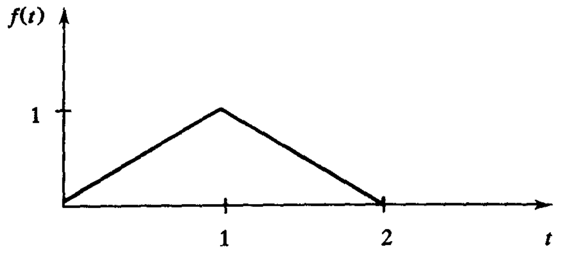


Figure B.1

.4 Use Table B.1 and the second shifting theorem to determine the Laplace transform of the function of Fig. B.1.

Solution:

The function of Fig. B.1 is written using unit step functions as

$$\begin{aligned} f(t) &= t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)] \\ &= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2) \end{aligned}$$

Use of transform pair 2 from Table B.1 with $n = 1$ and the second shifting theorem give

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - e^{-s}\frac{2}{s} + e^{-2s}\frac{1}{s} = \frac{1}{s}(1 - 2e^{-s} + e^{-2s})$$

B.7 INVERSION OF TRANSFORM

If $\bar{f}(s) = \mathcal{L}\{f(t)\}$, then $f(t) = \mathcal{L}^{-1}\{f(s)\}$ where

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)e^{st}\,ds \qquad \textbf{[B.7]}$$

is an integral carried out in the complex s plane. Inverse transforms are often obtained by using Table B.1 in conjunction with transform properties.

If

$$e^{-2s}\frac{s+5}{s^2+2s+5} = \bar{f}(s)$$

find $f(t)$, where $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$.

Completing the square of the denominator of $\bar{f}(s)$ gives

$$\bar{f}(s) = e^{-2s}\frac{s+5}{(s+1)^2+4} = e^{-2s}\left[\frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4}\right] = e^{-2s}\bar{g}(s)$$

Using linearity, the first shifting theorem, and transform pairs 4 and 5 from Table B.1 leads to

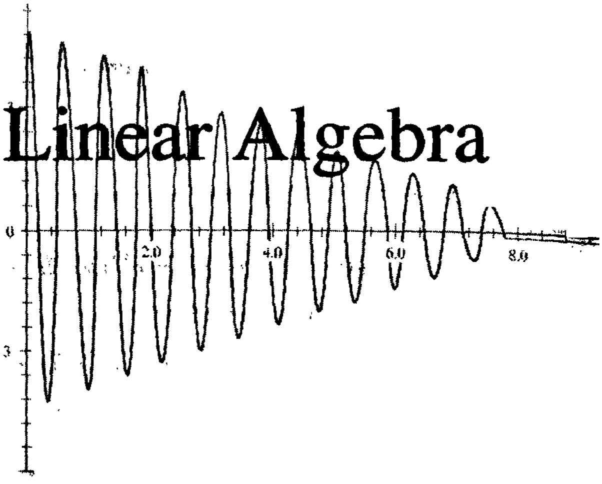
$$g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\} = e^{-t}(\cos 2t + 2\sin 2t)$$

Using the second shifting theorem leads to

$$f(t) = u(t-2)g(t-2) = e^{2-t}[\cos 2(t-2) + 2\sin 2(t-2)]u(t-2)$$

appendix

C Linear Algebra



C.1 Definitions

1. A *matrix* is a collection of numbers arranged in a specific order in rows and columns. If matrix **A** has n rows and m columns, then it is represented by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \tag{C.1}$$

Throughout this text a single capital letter in boldface is used to represent a matrix. The corresponding lowercase letter with two subscripts is used to refer to a specific element of the matrix. For example, the element a_{ij} resides in the ith row and jth column of **A**.

A matrix with n rows and m columns is called an $n \times m$ matrix. A square matrix has the same number of rows and columns.

2. A *column vector* is a matrix with only one column. A *row vector* is a matrix with only one row. Usually, a single lowercase letter in boldface is used to represent a column vector or a row vector. The letter with a single subscript refers to a specific element of the vector. A column vector with n rows or a row vector with n columns is said to be an n-dimensional vector. If **x** is an n-dimensional column vector then x_i, $i \leq n$, is the element in the ith row of the vector.

3. A *diagonal matrix* is a square matrix with all off-diagonal elements equal to zero. That is $a_{ij} = 0$ if $i \neq j$.

4. An *identity matrix* is a square diagonal matrix whose diagonal elements are all unity. That is $a_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

5. The *transpose* of the matrix **A**, denoted by $\mathbf{A}^T$, is the matrix obtained by interchanging the rows and columns of **A**. If $\mathbf{B} = \mathbf{A}^T$, then $b_{ij} = a_{ji}$. The transpose of a column vector is a row vector and vice versa.

6. A *symmetric matrix* is a square matrix whose transpose is equal to the matrix itself. If **A** is an $n \times n$ symmetric matrix, then $a_{ij} = a_{ji}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

C.2 DETERMINANTS

The *determinant* of a square $n \times n$ matrix is a number associated with the matrix that is often of great consequence. It is easiest to define the determinant of a 2×2 matrix and use this definition and properties of determinants to calculate the determinant of larger matrices.

The determinant of the 2×2 matrix **A** is

$$\det\{\mathbf{A}\} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{C.2}$$

The *minor* corresponding to the element in the ith row and jth column of an $n \times n$ matrix **A**, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column from **A**. The cofactor corresponding to the element in the ith row and jth column of **A**, denoted by C_{ij}, is

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{C.3}$$

For an i, $i = 1, \ldots, n$, the determinant of **A** is obtained by the following row expansion:

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{C.4}$$

The value of the determinant is the same regardless of the value of i. The determinant can also be calculated by a column expansion according to the formula

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ji} C_{ji} \tag{C.5}$$

Since the minors themselves are determinants, row or columns expansions can be used to express each of the minors in terms of the minors of their corresponding matrix. These expansions continue until the remaining minors are 2×2 determinants.

Calculate the determinant of the 4 × 4 matrix **A** where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

Solution:

The determinant is evaluated by a first-row expansion, using Eq. (C.4),

$$|\mathbf{A}| = (1)\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 2 & 0 & -2 \end{vmatrix}$$

Expansion by the first row is used to evaluate each of the 3 × 3 determinants, resulting in

$$|\mathbf{A}| = (2)\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 2\left[(1)\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}\right]$$

The 2 × 2 determinants are evaluated using Eq. (C.2), yielding

$$\begin{aligned} |\mathbf{A}| &= (2)[(3)(1) - (1)(-2)] + [(-1)(1) - (1)(0)] \\ &\quad - (2)\{[(-1)(-2) - (3)(0)] - (2)[(2)(-2) - (3)(2)] - [(2)(0) - (-1)(2)]\} \\ &= -31 \end{aligned}$$

The determinant of a matrix is zero if and only if the column vectors that form the matrix are linearly dependent. For example, the determinant of a matrix with a column of zeros is zero. A matrix whose determinant is zero is said to be *singular*. The row vectors of a singular matrix are also linearly dependent.

C.3 MATRIX OPERATIONS

If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then

$$c_{ij} = a_{ij} + b_{ij} \qquad \textbf{[C.6]}$$

If the number of columns of **A** equals the number of rows of **B**, then the matrix $\mathbf{C} = \mathbf{AB}$ is defined as a matrix with the number of rows of **A** and the number of columns of **B** and c_{ij} is the sum of the products of the corresponding elements in the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. That is,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \text{[C.7]}$$

Matrix multiplication is not commutative, but is associative and distributive. The transpose of the product has the following property. If $\mathbf{C} = \mathbf{AB}$, then

$$\mathbf{C}^T = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \qquad \text{[C.8]}$$

Example

Calculate $\mathbf{Ax}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & 6 & 2 \\ 0 & 2 & 3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}$$

Solution:

The product of a 4×4 matrix and a four-dimensional column vector is a four-dimensional column vector,

$$\mathbf{Ax} = \begin{bmatrix} (1)(1) + (2)(4) + (4)(-1) + (-1)(2) \\ (2)(1) + (3)(4) + (0)(-1) + (4)(2) \\ (1)(1) + (2)(4) + (6)(-1) + (2)(2) \\ (0)(1) + (2)(4) + (3)(-1) + (1)(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 7 \\ 7 \end{bmatrix}$$

C.4 SYSTEMS OF EQUATIONS

Consider the system of n simultaneous equations which are to be solved for the n unknowns $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n \end{aligned} \qquad \text{[C.9]}$$

Using the definitions of matrix addition and matrix multiplication, the system of Eq. (C.9) is written in matrix form as

$$\mathbf{Ax} = \mathbf{y}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{C.10}$$

Cramer's rule can be used to solve for the components of **x**,

$$x_i = \frac{|\mathbf{B}_i|}{|\mathbf{A}|} \tag{C.11}$$

where $\mathbf{B}_i$ is the matrix obtained by replacing the ith column of **A** with **y**. Thus if **A** is singular, a solution of Eq. (C.9) exists only for certain forms of **y**. Since its rows are linearly dependent when the matrix is singular, the solution corresponding to special forms of **y** is not unique.

An equation in a system of equations can be replaced, without affecting the solution of the system, by an equation obtained by multiplying the equation by a scalar and adding or subtracting it from another equation. The equations can be so manipulated until one of the equations only has one unknown. This is the basis of the Gauss elimination method.

Matrix formulation of the equations expedites the application of Gauss elimination. The $n \times n$ coefficient matrix is augmented with the right-hand side vector to form an $n \times (n+1)$ matrix. Each row of the augmented matrix represents one equation. The Gauss elimination procedure is applied by performing manipulations on the rows of the augmented matrix such that coefficients below the diagonal become zero. The elimination procedure results in a coefficient matrix with all zeros below its diagonal. Back substitution is used to determine the solution.

C.5 INVERSE MATRIX

If **A** is a nonsingular $n \times n$ matrix, then a matrix $\mathbf{A}^{-1}$, called the *inverse* of **A**, exists such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{C.12}$$

If $\mathbf{A}^{-1}$ is known, Eq. (C.9) can be solved by premultiplying both sides by $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{C.13}$$

If **y** is a column vector with all zeros except $y_i = 1$, then $\mathbf{A}^{-1}\mathbf{y}$ is the ith column of $\mathbf{A}^{-1}$. This provides the basis of an extension of Gauss elimination which is used to determine $\mathbf{A}^{-1}$. The coefficient matrix is augmented by the $n \times n$ identity matrix. The procedure used in Gauss elimination is applied until the identity matrix appears in place of the original matrix. The matrix that augments the identity matrix is $\mathbf{A}^{-1}$.

Determine the inverse of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution:

Gauss elimination is applied to the following matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

Gauss elimination is used to develop zeros below the diagonal of the coefficient matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & -4 & 1 & 2 & 0 \\ 0 & 0 & \frac{7}{2} & 1 & 2 & \frac{5}{2} \end{bmatrix}$$

The procedure of Gauss elimination is used to eliminate the zeros above the diagonal of the coefficient matrix. Each row is divided by the value of the element along the diagonal of the matrix that has taken the place of the original coefficient matrix. The result is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{7} & \frac{3}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} & \frac{6}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{4}{7} & \frac{5}{7} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{6}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{5}{7} \end{bmatrix}$$

C.6 EIGENVALUE PROBLEMS

The *eigenvalues* of an $n \times n$ matrix, $\mathbf{A}$, are the values of λ such that the system of equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{[C.14]}$$

has a nontrivial solution. The nontrivial solution corresponding to an eigenvalue is called an *eigenvector*. Equation (C.14) can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \qquad \text{[C.15]}$$

From Cramer's rule, Eq. (C.11), the solution for x_i is

$$x_i = \frac{0}{|\mathbf{A} - \lambda\mathbf{I}|} \qquad i = 1, \ldots, n$$

Thus, for each $i = 1, \ldots, n$, $x_i = 0$, unless

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{C.16}$$

The determinant of Eq. (C.16) can be expanded by a row or column expansion. This yields an nth-order polynomial equation of the form

$$\lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + C_{n-1}\lambda + C_n = 0 \tag{C.17}$$

called the *characteristic equation.* Equation (C.17) has n roots, and $\mathbf{A}$ has n eigenvalues. Since the coefficients in Eq. (C.17) are all real, if complex eigenvalues occur, they occur as complex conjugate pairs.

If λ is an eigenvalue of $\mathbf{A}$, then Eq. (C.14) has a nontrivial solution, an eigenvector. From Eq. (C.16), the matrix $\mathbf{A} - \lambda I$ is singular. Thus the equations defining the components of the corresponding eigenvector are not all independent and the eigenvector is not unique. The eigenvector is unique only to an arbitrary multiplicative constant.

Determine the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution:

The eigenvalues of $\mathbf{A}$ are determined by finding the values of λ satisfying Eq. (C.16), which for this example become

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0$$

Expansion of the determinant by its first row gives

$$(2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

When the 2×2 determinants are expanded by using Eq. (C.2), the following cubic equation is obtained:

$$-\lambda^3 + 8\lambda^2 - 16\lambda + 7 = 0$$

The eigenvalues are the roots of the cubic equation which are 0.609, 2.227, and 5.164. The eigenvector corresponding to the smallest eigenvalue is obtained by solving

$$\begin{bmatrix} 1.391 & -1 & 0 \\ -1 & 2.391 & -2 \\ 0 & -2 & 2.391 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation gives $x_1 - 0.719x_2$. The third equation gives $x_3 = 0.836x_2$. When these relationships are substituted into the second equation, it is identically satisfied. Thus x_2 remains arbitrary and the eigenvector of **A** corresponding to $\lambda = 0.609$ is

$$C_1 \begin{bmatrix} 0.719 \\ 1 \\ 0.836 \end{bmatrix}$$

where C_1 is an arbitrary constant. The same procedure is followed yielding the eigenvectors corresponding to the second and third eigenvalues. These are

$$C_2 \begin{bmatrix} -4.41 \\ 1 \\ 2.59 \end{bmatrix} \qquad C_3 \begin{bmatrix} -0.316 \\ 1 \\ -0.924 \end{bmatrix}$$

respectively.

If **A** is an $n \times n$ singular matrix, then one of its eigenvalues is zero. If **A** is nonsingular, then the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of **A**. The eigenvectors of $\mathbf{A}^{-1}$ are the same as the eigenvectors of **A**.

C.7 SCALAR PRODUCTS

Let **u**, **v**, and **w** be arbitrary real n-dimensional column vectors. A *scalar product* is an operation among two of these vectors yielding a real value. The scalar product of **u** and **v** is denoted by $(\mathbf{u}, \mathbf{v})$. The scalar product must satisfy four requirements.

1. The scalar product is commutative. That is,

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u}) \qquad \text{[C.18]}$$

2. For any real α,

$$(\alpha\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u}, \mathbf{v}) \qquad \text{[C.19]}$$

3. The scalar product is distributive

$$(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}) \qquad \text{[C.20]}$$

4. $$(\mathbf{u}, \mathbf{u}) \geq 0 \tag{C.21}$$

and $$(\mathbf{u}, \mathbf{u}) = 0 \quad \text{if and only if } \mathbf{u} = \mathbf{0} \tag{C.22}$$

The definition of a scalar product is not unique. The standard scalar product is defined as

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{v} \tag{C.23}$$

Two vectors, **u** and **v**, are said to be *orthogonal* with respect to a scalar product if

$$(\mathbf{u}, \mathbf{v}) = 0 \tag{C.24}$$

A matrix **A** is said to be *positive definite* with respect to a scalar product if

$$(\mathbf{A}\mathbf{u}, \mathbf{u}) \geq 0 \tag{C.25}$$

and $$(\mathbf{A}\mathbf{u}, \mathbf{u}) = 0 \quad \text{if and only if } \mathbf{u} = \mathbf{0} \tag{C.26}$$

C.5 Show that if **A** is a positive-definite symmetric matrix, then

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = (\mathbf{A}\mathbf{u}, \mathbf{v}) \tag{C.27}$$

is a valid scalar product where $(\mathbf{u}, \mathbf{v})$ is the standard scalar product defined by Eq. (C.23).

Solution:

In order for Eq. (C.27) to represent a valid scalar product, it is necessary to show that the four properties of Eqs. (C.18) through (C.22) are true, knowing that they are true for the standard scalar product.

1.
$$\begin{aligned}
(\mathbf{u}, \mathbf{v})_A &= (\mathbf{A}\mathbf{u})^T\mathbf{v} && \text{Eq. (C.27)}\\
&= \mathbf{u}^T\mathbf{A}^T\mathbf{v} && \text{Eq. (C.9)}\\
&= \mathbf{u}^T\mathbf{A}\mathbf{v} && \text{symmetry of } \mathbf{A}\\
&= (\mathbf{u}, \mathbf{A}\mathbf{v}) && \text{Eq. (C.23)}\\
&= (\mathbf{A}\mathbf{v}, \mathbf{u}) && \text{Eq. (C.18)}\\
&= (\mathbf{v}, \mathbf{u})_A && \text{Eq. (C.27)}
\end{aligned}$$

2. For any real α

$$\begin{aligned}
(\alpha\mathbf{u}, \mathbf{v})_A &= \alpha(\mathbf{A}\mathbf{u})^T\mathbf{v}\\
&= \alpha(\mathbf{u}, \mathbf{v})_A
\end{aligned}$$

3.
$$\begin{aligned}
(\mathbf{u}+\mathbf{v}, \mathbf{w})_A &= [\mathbf{A}(\mathbf{u}+\mathbf{v})]^T\mathbf{w}\\
&= \left[(\mathbf{A}\mathbf{u})^T + (\mathbf{A}\mathbf{v})^T\right]\mathbf{w}\\
&= (\mathbf{A}\mathbf{u})^T\mathbf{v} + (\mathbf{A}\mathbf{v})^T\mathbf{w}\\
&= (\mathbf{u}, \mathbf{w})_A + (\mathbf{v}, \mathbf{w})_A
\end{aligned}$$

4. The validity of the property 4 for this definition of the scalar product follows directly from the positive definiteness of **A**, Eqs. (C.25) and (C.26).

The concept of scalar products can be extended to continuous functions. Any operation between two continuous functions that results in a scalar and obeys Eqs. (C.18) through (C.22) is a valid scalar product. For example, for two functions $f(x)$ and $g(x)$ that are everywhere continuous between $x = 0$ and $x = 1$, a valid scalar product is

$$(f, g) = \int_0^1 f(x)g(x)\,dx \tag{C.28}$$

lix

Deflection of Beams Subject to Concentrated Loads

Consider a beam of total length L, subject to arbitrary end constraints. Let z be a coordinate along the neutral axis of the beam. The beam has n intermediate simple supports at $z = z_i$, $i = 1, 2, \ldots, n$. It is desired to calculate the deflection of the beam as a function of z due to a concentrated unit load applied at $z = a$. If $y(z)$ is the deflection of the neutral axis of the beam, measured positive downward from the horizontal, then use of the usual assumptions of linear elastic beam theory leads to

$$EI\frac{d^4y}{dz^4} = w(z) \qquad \text{[D.1]}$$

where $w(z)$ represents the loading, E is the elastic modulus of the beam, and I is the moment of inertia of the cross-sectional area about the neutral axis.

The intermediate supports are replaced by concentrated loads. Performance of the analysis requires the deflection to be zero at the intermediate supports.

The mathematical representation for a concentrated load of magnitude P applied at $z = a$ is $P\delta(z - a)$ where $\delta(z)$ is the unit impulse function. Thus the loading function $w(x)$ for the beam of Fig. D.1 is written as

$$w(z) = \delta(z-a) + \sum_{i=1}^{n} R_i\delta(z - z_i) \qquad \text{[D.2]}$$

where R_i, $i = 1, \ldots, n$, are the reactions at the intermediate supports. Equation (D.2) is substituted into Eq. (D.1) and the resulting equation is integrated three times, using Eq. (A.5), giving

$$EI\frac{d^3y}{dz^3} = u(z-a) + \sum_{i=1}^{n} R_i u(z - z_i) + C_1 \qquad \text{[D.3]}$$

$$EI\frac{d^2y}{dz^2} = (z-a)u(z-a) + \sum_{i=1}^{n} R_i(z - z_i)u(z - z_i) + C_1 z + C_2 \qquad \text{[D.4]}$$

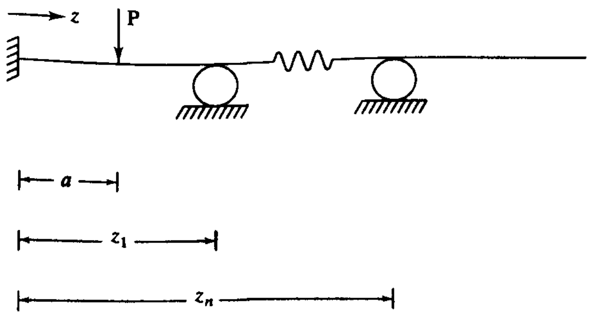


Figure D.1 Deflection equation for beam with intermediate supports due to a concentrated load is developed by representing the load and the support reactions using the unit impulse function.

Table D.1

End condition	Boundary condition	Boundary condition
Free	$EI\dfrac{d^2y}{dx^2} = 0$	$EI\dfrac{d^3y}{dx^3} = 0$
Fixed	$y = 0$	$\dfrac{dy}{dx} = 0$
Pinned	$y = 0$	$EI\dfrac{d^2y}{dx^2} = 0$

$$EI\frac{dy}{dz} = \frac{1}{2}(z-a)^2u(z-a) + \frac{1}{2}\sum_{i=1}^{n} R_i(z-z_i)^2u(z-z_i) + C_1\frac{z^2}{2} + C_2z + C_3 \quad \text{[D.5]}$$

$$EIy = \frac{1}{6}(z-a)^3u(z-a) + \frac{1}{6}\sum_{i=1}^{n} R_i(z-z_i)^3u(z-z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3z + C_4 \quad \text{[D.6]}$$

where C_1, C_2, C_3, and C_4 are constants of integration which are determined upon application of the appropriate boundary conditions.

The appropriate boundary conditions depend on the type of support at the boundaries. Table D.1 provides the boundary conditions for different types of support. Two boundary conditions are applied at each end of the beam. Thus $n + 4$ equations are applied to determine the $n+4$ unknowns, n intermediate support reactions, and four constants of integration.

D.1 Determine the deflection of a beam fixed at $x = 0$ and pinned at $z = L$ due to a concentrated load applied at $z = a$, $0 < a < L$.

Solution:

From Table D.1, the appropriate boundary conditions are

$$y(0) = 0 \quad (a) \qquad y(L) = 0 \quad (c)$$

$$\left.\frac{dy}{dz}\right|_{z=0} = 0 \quad (b) \qquad \left.\frac{d^2y}{dz^2}\right|_{z=L} = 0 \quad (d)$$

Application of (a) to Eq. (D.6) yields $C_4 = 0$. Application of (b) yields $C_3 = 0$. Application of (c) and (d) yields the following equations:

$$\frac{L^3}{6}C_1 + \frac{L^2}{2}C_2 = -\frac{1}{6}(L-a)^3$$

$$LC_1 + C_2 = -(L-a)$$

respectively. The preceding equations are solved simultaneously, yielding

$$C_1 = \frac{1}{2}\left(1 - \frac{a}{L}\right)\left[\left(\frac{a}{L}\right)^2 - 2\frac{a}{L} - 2\right]$$

$$C_2 = a\left(1 - \frac{a}{L}\right)\left(1 - \frac{a}{2L}\right)$$

Boundary conditions are applied to the beams of Table D.2, resulting in the evaluation of constants and, if applicable, intermediate reactions for each beam. Equation (D.6) is used to calculate the deflection of the beam at any point.

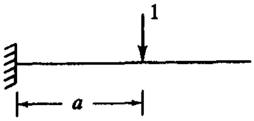
Table D.2

The deflection, $y(z)$, of a uniform beam of elastic modulus E and cross-sectional moment of inertia I due t concentrated load applied at $z = a$ is

$$y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3 u(z-a) + \frac{1}{6}\sum_{i=1}^{n} R_i (z-z_i)^3 u(z-z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4\right]$$

where R_i is the reaction at an intermediate support located at $z = z_i$. The forms of the constants and the inter reactions for common beams are given as follows.

1.

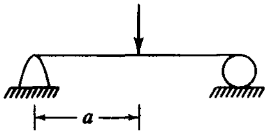


Fixed-free

$C_1 = -1 \qquad C_3 = 0$

$C_2 = a \qquad C_4 = 0$

2.

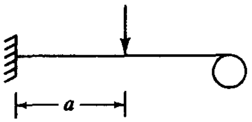


Pinned-pinned

$C_1 = \frac{a}{L} - 1 \qquad C_3 = \frac{aL}{6}\left(1 - \frac{a}{L}\right)\left(2 - \frac{a}{L}\right)$

$C_2 = 0 \qquad C_4 = 0$

3.

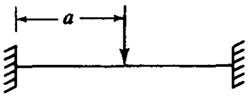


Fixed-pinned

$C_1 = \frac{1}{2}\left(1 - \frac{a}{L}\right)\left[\left(\frac{a}{L}\right)^2 - 2\frac{a}{L} - 2\right] \qquad C_3 = 0$

$C_2 = \frac{1}{2}a\left(1 - \frac{a}{L}\right)\left(2 - \frac{a}{L}\right) \qquad C_4 = 0$

4.

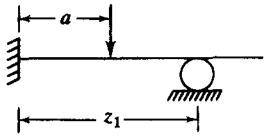


Fixed-fixed

$C_1 = -\left(1 - \frac{a}{L}\right)^2\left(1 + \frac{2a}{L}\right) \qquad C_3 = 0$

$C_2 = a\left(1 - \frac{a}{L}\right)^2 \qquad C_4 = 0$

5.



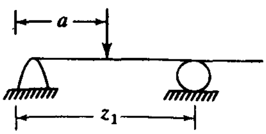
Fixed-free with overhang

$C_1 = -\frac{3}{2} + \frac{3a}{2z_1} + \frac{1}{2}\left(1 - \frac{a}{z_1}\right)^3 u(z_1 - a) \qquad C_3 = 0$

$C_2 = \frac{z_1}{2}\left(1 - \frac{a}{z_1}\right)\left[1 - \left(1 - \frac{a}{z_1}\right)^2 u(z_1 - a)\right] \qquad C_4 = 0$

$R_1 = \frac{1}{2} - \frac{3a}{2z_1} - \frac{1}{2}\left(1 - \frac{a}{z_1}\right)^3 u(z_1 - a)$

6.



Pinned-free with overhang

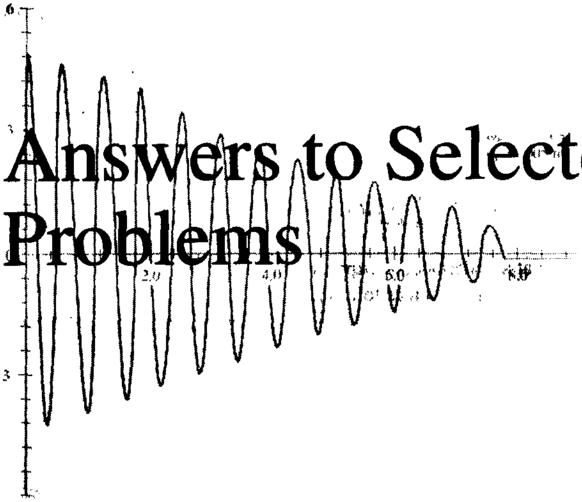
$C_1 = \frac{a}{z_1} - 1 \qquad C_3 = -\left(1 - \frac{a}{z_1}\right)\frac{z_1^2}{6}\left[\left(1 - \frac{a}{z_1}\right)^2 u(z_1 -\right.$

$C_2 = 0 \qquad C_4 = 0$

$R_1 = -\frac{a}{z_1}$

appendix

E Answers to Selected Problems



Chapter 1

1.1. (*a*) 0.565 m/s, (*b*) 106.6 m/s^2

1.3. $x(t) = 0.3\cos(30t + \pi/6)$ m

1.5. 100.8 m/s^2

1.7. $\mathbf{a}_P = (-66.5\mathbf{i} + 54.3\mathbf{j})$ m/s^2, $|\mathbf{a}_P| = 85.9$ m/s^2

1.9. 0.75 m/s^2

1.11. No

1.13. $|\mathbf{v}_A| = 91.4$ ft/s, $|\mathbf{a}_A| = 421.9$ ft/s^2

1.15. (*a*) $a(t) = -2.36 + 13.8e^{-t}$ ft/s^2, (*b*) $x = 22.98$ ft

1.17. (*a*) 2.68 rad/s^2, (*b*) 13.5 rad/s^2

1.19. $M = 2120$ N · m ↺, $O_x = 1110$ N ←, $O_y = 1400$ N ↑

1.21. $P < 6.36$ N

1.23. $\omega_{\max} = 0.837$ rad/s

1.25. $T = 297.4$ N · m

1.27. (*a*) $\delta = \dfrac{\mu mg}{K}$, (*b*) $\delta = \dfrac{2\mu mg}{K}$

1.29. $\theta_{\max} = 0.0721$ rad

1.31. (*a*) $\alpha = 15.29$ rad/s^2, (*b*) $\omega_{\max} = 0.322$ rad/s

1.33. $V = \dfrac{1}{2}k\left(2x^2 + \left(\dfrac{3L}{2}\right)x\theta + \dfrac{9}{16}L^2\theta^2\right)$

1.35. $v_2 = 2.86$ mph

1.37. $x_{\max} = 0.81$ m

1.39. $J = 340$ N · s · m

1.47. $v_{\max} = 5.60$ m/s

1.49. $\delta_{\max} = 1.62$ mm

1.57. $\Delta_{ST} = 0.22$ mm

1.61. $\Delta_{ST_1} = 4.41$ mm, $\Delta_{ST_2} = 1.78$ mm, $\Delta_{ST_3} = 3.57$ mm

1.63. $k_{eq} = 1.38 \times 10^8$ N/m

1.65. $k_{eq} = 92.3EI$

1.67. $k_t = 8.52 \times 10^6$ N · m/rad

1.69. $k_l = 9.67 \times 10^7$ N/m, $k_\theta = 1.93 \times 10^3$ N · m/rad, $k_y = 1.72 \times 10^4$ N/m

1.71. (*a*) 24.5 mm, (*b*) 132 g

1.73. $2m_s/3$

1.75. $m_{eq} = 0.486m_b$

1.77. $F = e^{-135t}[-2.54 - \sin 4t + 7.5\cos 4t]$ N

1.79. $C_t = \dfrac{T_1\mu R^4}{2h}$

1.83. m = 0.04 kg

Chapter 2

2.1 and 2.11. $\ddot{x} + \dfrac{3k}{m}x = 0$, $\omega_n = \sqrt{\dfrac{3k}{m}}$

2.3 and 2.13. $\ddot{\theta} + \frac{3c}{7m}\dot{\theta} + \left(\frac{27k}{7m} + \frac{12}{7}\frac{g}{L}\right)\theta = 0,$

$$\omega_n = \sqrt{\frac{27k}{7m} + \frac{12g}{7L}}$$

2.5 and 2.15. $\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{2k}{3m}x = 0, \ \omega_n = \sqrt{\frac{2k}{3m}}$

2.7 and 2.17. $\ddot{x} + \frac{25c}{3m} + \frac{38k}{3m}x = 0, \ \omega_n = \sqrt{\frac{38k}{3m}}$

2.9 and 2.19. $\dot{\phi} + \frac{5g}{7(R-r)}\phi = 0,$

$$\omega_n = \sqrt{\frac{5g}{7(R-r)}}$$

2.21. $\omega_n = 36.2$ rad/s
2.23. $\omega_n = 5.3 \times 10^3$ rad/s
2.25. $\omega_n = 5.48$ rad/s
2.27. S.G. = 12.8
2.29. $l = 0.248$ m
2.31. $\omega_n = 3.33$ rad/s
2.35. $h < 0.0176$ m and $h > 0.0230$ m
2.39. $x(t) = 16.83 \sin(2.32t)$ ft
2.41. $m_{max} = 1.25$ kg
2.43. $v_0 < \mu g \sqrt{\frac{6m}{k}}$
2.45. $\zeta = 0.013$
2.47. $\zeta = 1.26$
2.49. $t = 0.83$ s
2.51. $c = 1.50 \times 10^5$ kg/s
2.53. (a) $c_c = 154.9$ kg/s
2.57. $x_m = 0.812$ ft
2.61. $t = 0.132$ s
2.63. $\mu = 0.0233$
2.65. 128 cycles
2.67. $h = 0.00310, \ \zeta = 0.00155$

Chapter 3

3.7. $X = 0.187$ m
3.9. $M_0 < 248$ N · m
3.11. $M_0 < 130$ N · n
3.15. 1.13×10^{-5} m^4
3.17. $X = 0.112$ m
3.19. $X = 7.27$ mm,
3.21. $X = 1.52$ mm
3.25. $\zeta = 0.160, \ X$ =
3.27. $\zeta = 0.445, \ k$ =
3.29. $k = 9.31 \times 10^7$
3.31. $d < 21.1$ mm
3.33. $X = 0.36$ mm
3.37. $k > 1.83 \times 10^7$
3.39. $X = 9.4 \times 10^{-}$
3.41. $X_{max} = 1.75 \times$
3.45. $X = 17.2$ cm
3.47. $\Theta = 0.168$ rad
3.49. $A = 7.88$ m/s^2
3.53. $k < 130$ N/m
3.55. $X = 10.8$ mm
3.57. $F_0 = 165.5$ N
3.59. $X = 1.40$ mm
3.63. $X = 0.613$ mm
3.65. $X = 8.67$ mm

Chapter 4

4.7. $\theta(t) = \frac{9M_0}{mL^2\omega_n}$

$- \omega_n$

4.9. $t_0 < \frac{\pi}{\omega_n\sqrt{1-\zeta}}$
4.17. $x_{max} = 1.25$ m
4.19. $x_{max} = 1.16 \times$

Chapter 5

5.17. Statically coupl
5.19. Statically and d
5.21. Statically and d
5.23. Statically coupl

5.25. Statically coupled

5.27. Statically coupled

5.31. $T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$

5.33. $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}_2^2$

5.35. $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{6}m\dot{x}_2^2$

5.37. $V = \frac{1}{2}\left[3kx_1^2 - 4kx_1x_2 + 3kx_2^2 - 2kx_2x_3 + 2kx_3^2\right]$

5.39. $V = \frac{1}{2}\left[5kx_1^2 + 1.2kLx_1\theta - 4kx_1x_2 + 0.5kL^2\theta^2 - 0.4kL\theta x_2 + 2kx_2^2\right]$

5.41. $V = \frac{1}{2}\left[2kx_c^2 - 38kx_cx_D + 19kx_D^2\right]$

5.43. $V = \frac{1}{2}\left[3kx_1^2 + kLx_1\theta - kx_1x_2 + 3\frac{kL}{4}\theta^2 - \frac{1}{2}kx_2\theta + \frac{k}{4}x_2^2\right]$

5.45.
$$\mathbf{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 2k \end{bmatrix}$$

5.47.
$$\mathbf{K} = \begin{bmatrix} \frac{45}{16}kL^2 & -\frac{1}{2}kL & -2kL \\ -2kL & k & 0 \\ -2kL & 0 & 2k \end{bmatrix}$$

5.49.
$$\mathbf{K} = \begin{bmatrix} 5k & 0.6kL & -2k \\ 0.6kL & 0.5kL^2 & -0.2kL \\ -2k & -0.2kL & 2k \end{bmatrix}$$

5.51.
$$\mathbf{K} = \begin{bmatrix} 21k & -19k \\ -19k & 19k \end{bmatrix}$$

5.53.
$$\mathbf{K} = \begin{bmatrix} 2k & 0 & -kr \\ 0 & k & -2kr \\ -kr & -2kr & 5kr^2 \end{bmatrix}$$

5.55.
$$\mathbf{K} = \begin{bmatrix} m_1g\frac{l_1}{2} + m_2gl_1 & 0 \\ 0 & m_2g\frac{l_2}{2} \end{bmatrix}$$

5.57.
$$\mathbf{K} = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix}$$

5.59.
$$\mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

5.61.
$$\mathbf{A} = \begin{bmatrix} \frac{16}{kL^2} & \frac{8}{9kL} & \frac{16}{9kL} \\ \frac{8}{9kL} & \frac{13}{9k} & \frac{8}{9k} \\ \frac{16}{9kL} & \frac{8}{9k} & \frac{41}{18k} \end{bmatrix}$$

5.63.
$$\mathbf{A} = \frac{1}{k}\begin{bmatrix} 1 & 2 & \frac{1}{r} \\ 2 & 9 & \frac{4}{r} \\ \frac{1}{r} & \frac{4}{r} & \frac{2}{r^2} \end{bmatrix}$$

5.79.
$$\mathbf{A} = 1 \times 10^{-8}\begin{bmatrix} 2.64 & 3.13 & 1.27 \\ 3.13 & 6.25 & 3.13 \\ 1.27 & 3.13 & 2.64 \end{bmatrix}\frac{\mathrm{m}}{\mathrm{N}}$$

5.81.
$$\mathbf{A} = 1 \times 10^{-8}\begin{bmatrix} 1.04 & 0.912 & -2.084 \\ 0.912 & 1.042 & -2.61 \\ -2.084 & -2.61 & 15.63 \end{bmatrix}\frac{\mathrm{m}}{\mathrm{N}}$$

5.83.
$$\frac{\rho AL^4}{3EI}(1 \times 10^{-3})\begin{bmatrix} 5.03 & 5.26 \\ 5.26 & 9.15 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \frac{F_0L^4}{6EI}(1 \times 10^{-3})\begin{bmatrix} 10.29 \\ 14.41 \end{bmatrix}$$

Chapter 6

6.1. $\omega_1 = 0.438\sqrt{\frac{k}{m}}$, $\omega_2 = 1.863\sqrt{\frac{k}{m}}$

6.3. $\omega_1 = 1.19 \times 10^3$ rad/s, $\omega_2 = 2.65 \times 10^3$ rad/s

6.5. $\omega_1 = 0.646\sqrt{\frac{k}{m}},\ \omega_2 = 1.892\sqrt{\frac{k}{m}}$

6.7. $\omega_1 = 0.796\sqrt{\frac{k}{m}},\ \omega_2 = 1.154\sqrt{\frac{k}{m}}$

$\omega_3 = 1.538\sqrt{\frac{k}{m}}$

6.9. $\omega_1 = 5.36 \times 10^3$ rad/s, $\omega_2 = 1.38 \times 10^4$ rad/s

6.11. $\omega_1 = 35.2$ rad/s, $\omega_2 = 42.3$ rad/s, $\omega_3 = 85.0$ rad/s

6.13. $\omega_1 = 18.7$ rad/s, $\omega_2 = 51.2$ rad/s, $\omega_3 = 98.9$ rad/s

6.15. $\omega_1 = 16.7$ rad/s, $\omega_2 = 255.4$ rad/s

6.19. $k_1 = 60.2$ N/m, $k_2 = 79.6$ N/m, $k_3 = 233.7$ N/m

6.21. $x_2(0) = 1.404x_1(0),\ \dot{x}_1(0) = \dot{x}_2(0) = 0$

6.25. $\omega_1 = 0,\ \omega_2 = 4.59$ rad/s, $\omega_3 = 13.0$ rad/s

6.27. $\omega_1 = 0,\ \omega_2 = 5.07 \times 10^2$ rad/s, $\omega_3 = 1.13 \times 10^3$ rad/s

6.31. $\omega = 1.26 \times 10^3$ rad/s

Chapter 7

7.1. $X_1 = 20.0$ mm, $X_2 = 1.72$ mm

7.3. $M_1 =$

$$\frac{1 - r_2^2}{\sqrt{4\zeta^2 r_1^2(1 - r_2^2) + (1 - r_1^2 - r_2^2 - \mu r_1^2 + r_1^2 r_2^2)^2}}$$

7.5. $X = 0.988$ mm

Chapter 8

8.1. $\Delta_{ST} = 17.9$ mm

8.3. $k = 8.83 \times 10^4$ N/m

8.5. 85.1 percent isolation

8.7. 200 kg

8.9. $k = 1.08 \times 10^5$ N/m

8.11. $k = 4.89 \times 10^6$ N/m

8.13. $k = 3.99 \times 10^3$ N/m, $Z = 6.96$ mm

8.17. $660.0 \text{ rad/s} < \omega < 733.2 \text{ rad/s}$

8.23. $\omega_n = 61.7$ rad/s

8.33. $F_{T_{max}} = 1.68 \times 10^5$ N

Chapter 9

9.1. $c = 5170$ m/s

9.5. $\omega_1 = 1.03 \times 10^4$ rad
$\omega_3 = 3.11 \times 10^3$ ra

9.7. $\omega_1 = 1.19 \times 10^3$ rad
$\omega_3 = 1.38 \times 10^4$ ra

9.9. $\Theta = 2.25 \times 10^{-5}$ r

9.11. $\omega^* = \beta \tan \omega^*,\ \beta$ =

Chapter 11

11.3. $T = \pi\sqrt{\frac{m}{k_2}} + \pi\sqrt{}$

11.9. (a) $\ddot{x} + x - \zeta x^2$ -

(b) $\zeta = \dfrac{1}{1 - a/n}$

(c) $t = w(1 + \frac{7}{24}$

11.13. $x = 0$ saddle poi

$x = \pm\sqrt{\frac{1}{\epsilon}}$ stabl

11.15. $x = 0$ stable foc

$x = \frac{1}{\epsilon}$ saddle p

11.17. $T = \dfrac{4\sqrt{2}}{x_0\xi}\int_0^1$ -

11.19. $T = 0.629$ s

11.25. (a) $\omega \approx 4.56$ r
(b) $\omega \approx 2.50$ r
(c) $\omega \approx 22.9$ r

11.27. $X = 0.0344$ m

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