Partial Differential Equations (MA20103)

Assignment – 3

Second order PDE

Q1. Solve the following second order homogenous PDE with constant coefficients (symbols have usual meanings, say, $r = \frac{\partial z}{\partial x}; s = \frac{\partial^2 z}{\partial x \partial y}; t = \frac{\partial z}{\partial y}$)

(i) $25r - 40s + 16t = 0$

(ii) $r + (a+b)s + abt = xy$

(iii) $r - t = x - y$

(iv) $r + t + 2s = xy$

(v) $2r - 3s - 2t = 0$

(vi) $r - 4s + 4t = 0$

(vii) $r + 3s + 2t = 2x + 3y$

(viii) $r - s - 2t = (y-1)e^x$

(ix) $r - 5s + 4t = \sin(4x + y)$

(x) $r + t = \cos mx \cos ny$

Q2. Classify and reduce the following equations in to canonical form

(i) $\frac{\partial^2 z}{\partial x^2} - x^2 \frac{\partial^2 z}{\partial y^2} = 0$ (also find general solution)

(ii) $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ (also find general solution)

(iii) $16 \frac{\partial^2 z}{\partial x^2} - y^{10} \frac{\partial^2 z}{\partial y^2} = 5y^9 \frac{\partial z}{\partial y}$

(iv) $y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$

(v) $\left(y-1\right) \frac{\partial^2 z}{\partial x^2} - \left(y^2 - 1\right) \frac{\partial^2 z}{\partial x \partial y} + y\left(y-1\right) \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2ye^{2x}\left(1-y\right)^3$

Q3. Show that the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

satisfying the conditions

(i) $u \to 0$ as $t \to \infty, \forall\ x$

(ii) $u = 0$ for $x = 0$ and $x = a\ \forall\ t > 0$

(ii) $u = x$ when $t = 0$ and $0 < x < a$ is

$$u(x,\mathrm{t}) = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(x)\exp\left[-\left(\frac{n\pi}{a}\right)^2 t\right]$$

Q4. Solve $\nabla^2 u = 0$

subject to $u(x,0) = 0, u(x,a) = 0, u(x,y) \to 0$ as $x \to \infty$ where $x \geq 0$ and $0 \leq y \leq a$

Q5. Solve the 2 dimensional Laplace equation in polar co-ordinates r and θ in the region $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$ subject to

(i) u remains finite as $r \to 0$

(ii) $u = \sum_n c_n \cos(n\theta)$ on $r = a$

Q6. A tightly stretched string with fixed end point $x = 0$ and $x = l$ is initially in a position given by $u = u_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, show that the displacement is given by

$$u(x,t) = \frac{u_0}{4}\left(3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right)$$