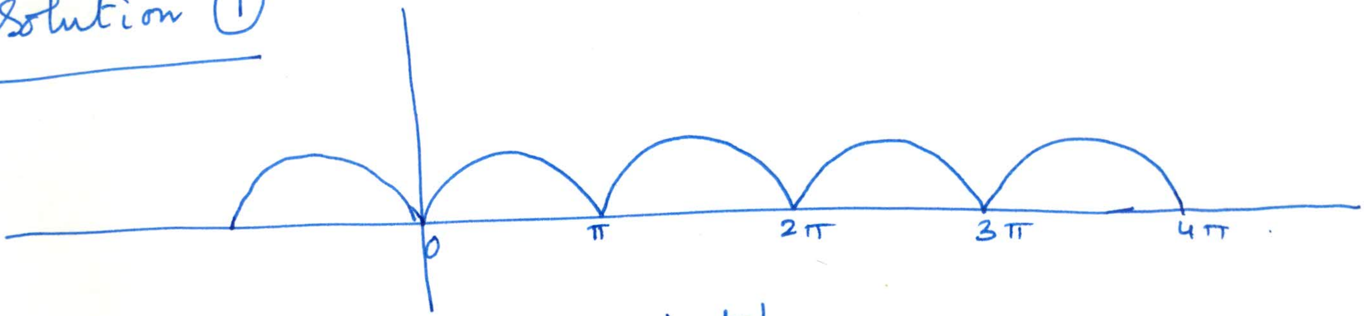
Surprise test - 1 - 31st August, 2017

① Find $L\{|\sin t|\}$

② Find $L\left\{\operatorname{erf}\left(\frac{3}{2\sqrt{t}}\right)\right\}$

Solution ①



$|\sin t|$

$\therefore$ $f(t) = |\sin t|$ is a periodic funct. of π.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)\,e^{-st}\,dt,$$

if $f(t)$ is a periodic funct. of period T.

$$\therefore L\{|\sin t|\} = \frac{1}{1-e^{-s\pi}}\int_0^{\pi} |\sin t|\,e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s\pi}}\int_0^{\pi} \sin t\,e^{-st}\,dt.$$

(Integrate by parts)

$$= \frac{1+e^{-\pi s}}{1-e^{-\pi s}}\cdot\frac{1}{1+s^2}.$$

Solution 2

$$L\left\{ \operatorname{erf}\left(\frac{3}{2\sqrt{t}}\right)\right\}$$

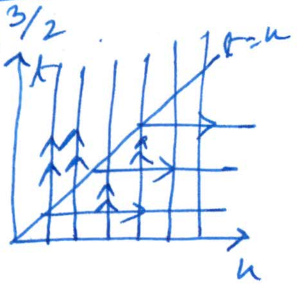
$$= \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{3/2\sqrt{t}} e^{-\alpha^2}\,d\alpha\right]dt$$

Put $\frac{4u}{9} = \frac{1}{\alpha^2}$; When $\alpha = \frac{3}{2\sqrt{t}}$, $u = t$; when $\alpha=0$, $u=\infty$

or, $\alpha^2 = \frac{9}{4u} \Rightarrow 2\alpha\, d\alpha = -\frac{9}{4u^2}du$.

$$\therefore\ d\alpha = -\frac{9}{8u^2\alpha}du = -\frac{\cancel{9}^{3}}{\cancel{8}_{4}}\cdot\frac{\cancel{2}}{\cancel{3}}\frac{\sqrt{u}}{u^2}du = -\frac{3}{4}\cdot\frac{du}{u^{3/2}}$$

$$\therefore\ L\{\cdots\} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-st}\int_t^\infty e^{-\frac{9}{4u}}\times\frac{3}{4}\frac{du}{u^{3/2}}$$



$$= \frac{3}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{9}{4u}}}{u^{3/2}}du\left(\int_0^u e^{-st}dt\right)$$

$$= \frac{3}{2\sqrt{\pi}\,s}\int_0^\infty \frac{e^{-\frac{9}{4u}}}{u^{3/2}}du\left(1-e^{-su}\right).$$

Put $\frac{9}{4u} = \beta^2 \Rightarrow -\frac{9}{4u^2}du = 2\beta\, d\beta$.

$$\Rightarrow du = -\frac{8\beta u^2 d\beta}{9}.$$

$$\therefore\ \frac{du}{u^{3/2}} = -\frac{8\beta\sqrt{u}}{9}d\beta = -\frac{\cancel{8}^{4}}{\cancel{9}_{3}}\times\frac{\cancel{3}}{\cancel{2}}d\beta = -\frac{4}{3}d\beta$$

$\therefore L\left\{ \operatorname{erf}\left(\frac{3}{2\sqrt{t}}\right)\right\}$

$$= \frac{\cancel{s}}{\cancel{2}\sqrt{\pi}\, s} \int_0^\infty e^{-p^2} \times \frac{\cancel{4}^{2}}{\cancel{s}}\, dp \left(1 - e^{-\frac{9s}{4p^2}}\right)$$

$$= \frac{2}{\sqrt{\pi}\, s} \int_0^\infty e^{-p^2}\left(1 - e^{-\frac{9s}{4p^2}}\right) dp.$$

$$= \frac{2}{\sqrt{\pi}\, s}\left[\int_0^\infty e^{-p^2} dp - \int_0^\infty e^{-\left(p^2 + \frac{9s}{4p^2}\right)} dp\right]$$

Check that,

$$\int_0^\infty e^{-\left(p^2+\frac{9s}{4p^2}\right)} dp = \frac{1}{2}\left[\int_0^\infty \left(1 - \frac{3\sqrt{s}}{2p^2}\right) \cdot e^{-\left\{\left(p + \frac{3\sqrt{s}}{2p}\right)^2 + 2\cdot\frac{3\sqrt{s}}{2}\right\}} dp + \int_0^\infty \left(1 + \frac{3\sqrt{s}}{2p^2}\right) e^{-\left\{\left(p - \frac{3\sqrt{s}}{2p}\right)^2 - 2\cdot\frac{3\sqrt{s}}{2}\right\}} dp\right].$$

Put $y = p \pm \frac{3\sqrt{s}}{2p}$ in the 1st and in the 2nd integral respectively. Then,

$$\int_0^\infty e^{-\left(p^2+\frac{9s}{4p^2}\right)} dp = \frac{1}{2}\left[e^{3\sqrt{s}}\underbrace{\int_\infty^\infty e^{-y^2} dy}_{=0} + e^{-3\sqrt{s}}\int_{-\infty}^\infty e^{-y^2} dy\right]$$

$$= e^{-3\sqrt{s}} \times \frac{1}{\cancel{2}} \times \cancel{2}\int_0^\infty e^{-y^2} dy = e^{-3\sqrt{s}} \times \frac{\sqrt{\pi}}{2}.$$

$$\therefore \text{Ans.} = \frac{1}{s}\left[\frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} \times e^{-3\sqrt{s}}\right] = \frac{1}{s}\left(1 - e^{-3\sqrt{s}}\right)$$

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