(5) The Concept of Complex Frequency Response:~

Let the DEOM be

$$m\ddot{x}+c\dot{x}+kx = F_0 e^{i\omega_f t} \quad \text{--- (1)}$$

$F_0 \cos\omega_f t + i(F_0 \sin\omega_f t)$

For the steady state solution, we assume $x(t) = \bar{X} e^{i\omega_f t}$ --- (2).

Now, $\bar{X} = X_0 e^{-i\psi}$, to take care of the phase lag of the response w.r.t. the excitation. So, $\bar{X}$ is complex.

Substitution of (2) in (1) leads to

$$(-m\omega_f^2 + ic\omega_f + k)\bar{X} = F_0$$

So, $$\bar{X} = \frac{F_0}{(k-m\omega_f^2)+i(c\omega_f)} = \frac{F_0/k}{(1-r^2)+i(2\zeta r)}$$

($r = \frac{\omega_f}{\omega_n}$, the frequency ratio)

→ The quantity $(k-m\omega_f^2)+i(c\omega_f)$ has a special name, it ~~is~~ called the '<u>Mechanical ~~Impedance~~ Impedance</u>' of our system. (See pg. 48, Mechanical vibrations, 2nd Edition; Tse, Morse, Hinkle

Also, $$H_1(\omega_f) = \frac{1}{(k-m\omega_f^2)+i(c\omega_f)}$$

is called the <u>complex frequency response</u>. Some authors modify this definition a little bit in order to make the complex frequency response

(2) (5)

non-dimensional. Hence, the following definition is also a complex frequency response:

$$H(r) = H(\omega_f) = \frac{1}{(1-r^2)+i(2\zeta r)}$$ [ω_n, $\zeta \to$ given. Hence, $H = H(\omega_f)$ with ω_f variable]

→ You should verify that $|H(\omega_f)|$ is nothing but the magnification factor MF and $\angle H(\omega_f) = (\psi)$ where

$$MF = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}, \quad \psi = \tan^{-1}\frac{(2\zeta r)}{(1-r^2)}$$

[In Automatic Control studies, you do come across the sinusoidal transfer function (STF). Check if our complex frequency response has anything to do with the STF]

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(5) The use of phasor (vector) diagrams in vibration studies:—

The ss response of

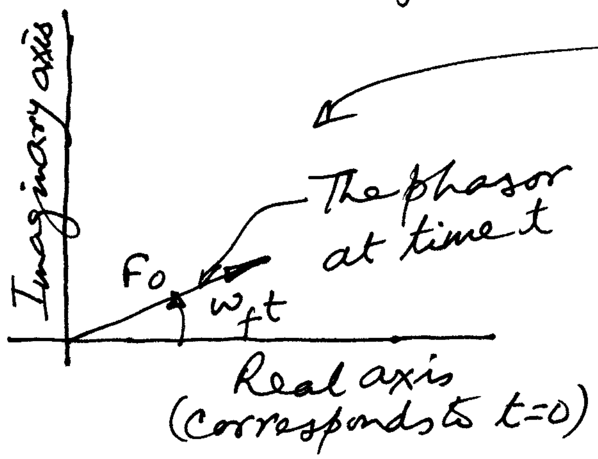
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_f t$$

or

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega_f t$$

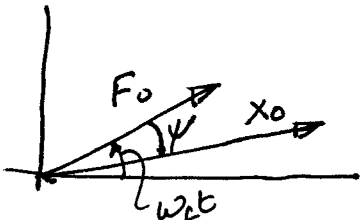
can also be obtained using a phasor diagram. This is done by many authors. You should be familiar with this method & so we introduce it. →

→ First of all, note that the forcing function $F_0 \sin\omega_f t$ can be represented by a phasor or a rotating vector as follows:–



So, the projection of the phasor on the imaginary axis gives $F_0 \sin\omega_f t$ and its projection on the real-axis gives $F_0 \cos\omega_f t$.

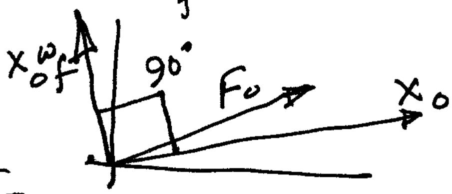
For $F(t) = F_0 \sin\omega_f t$ { We know that the steady-state response is $x = X_0 \sin(\omega_f t - \psi)$. Hence, the phasor corresponding to this $x(t)$ lags the phasor corresponding to $F_0 \sin\omega_f t$ by an angle ψ:



Also, $\dot{x} = X_0 \omega_f \cos(\omega_f t - \psi)$

$$= X_0 \omega_f \sin\left(\omega_f t - \psi + \frac{\pi}{2}\right)$$

Hence, $\dot{x}$ leads x by $\frac{\pi}{2}$ radians or, 90°.

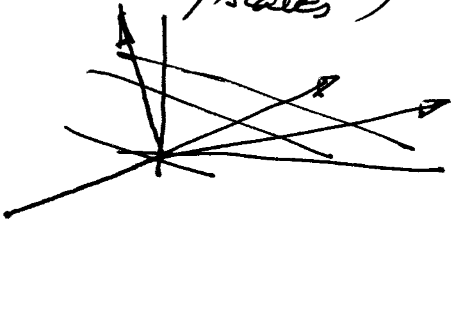


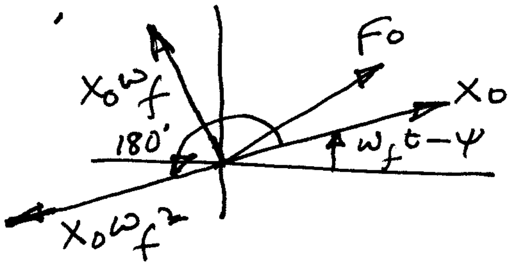
Finally, $\ddot{x} = -X_0 \omega_f^2 \sin(\omega_f t - \psi)$

$$= X_0 \omega_f^2 \sin(\omega_f t - \psi + \pi)$$

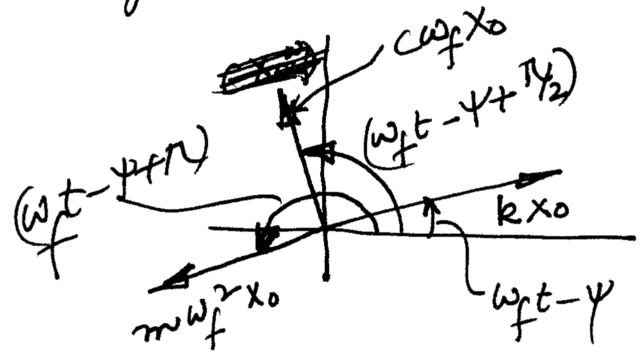
& so, $\ddot{x}$ leads x by π radians or, 180°;

(These figures have arbitrary scale)





Now, the forces in the spring & damper are kx & $c\dot{x}$, i.e., $kX_0 \sin(\omega_f t - \psi)$ and $cX_0\omega_f \sin(\omega_f t - \psi + \frac{\pi}{2})$. Also, the inertia force is $-mX_0 \omega_f^2 \sin(\omega_f t - \psi + \pi)$. Hence, these forces can be represented on a phasor diagram as:



Now, the DEOM is

$m\ddot{x}+c\dot{x}+kx=F_0\sin\omega_f t$

But $x_{ss}=x(t)=X_0\sin(\omega_f t-\psi)$

Hence,

$-m\omega_f^2 X_0 \sin(\omega_f t-\psi)$

$+c\omega_f X_0 \cos(\omega_f t-\psi)$

$+kX_0\sin(\omega_f t-\psi)$

$=F_0\sin\omega_f t$

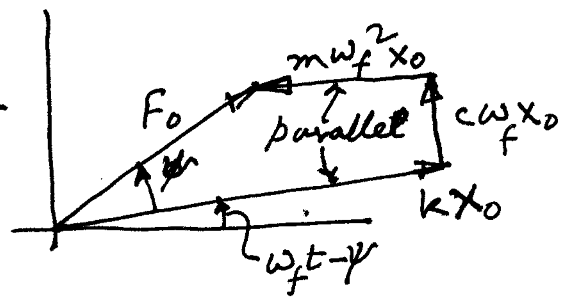
or, $m\omega_f^2 X_0\sin(\omega_f t-\psi+\pi)$

$+c\omega_f X_0\sin(\omega_f t-\psi+\frac{\pi}{2})$

$+kX_0\sin(\omega_f t-\psi)=F_0\sin\omega_f t.$

This relation can be graphically portrayed as follows:–

The Phasor Diagram →



← You need to remember this figure only for problem solving

This diagram is used by some authors to solve problems involving the ss response. This is how they do it:

Example:– See TU/HW sheet-3, problem 1

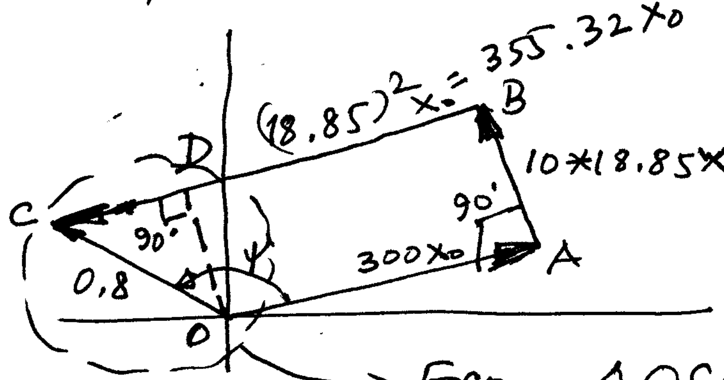
The aim is to obtain X_0 & ψ after the DEOM is obtained as: $\ddot{x}+10\dot{x}+300x=0.8\sin 18.85t$

(obtain this) (See solution sheet) Now, $x_{ss}=x=X_0\sin(18.85t-\psi)$

The phasor diagram is as follows:–

(If drawn to scale, it would look like the following. However, even without a figure to scale, you can compute X & ψ)

(fig. not to scale)



From ΔOCD, $OC^2 = OD^2 + CD^2$

or, $(0.8)^2 = (188.5 X_0)^2 + [(18.85)^2 - 300]^2 X_0^2$

check

$\Rightarrow X_0 = \cancel{4.07}\ 4.2 \times 10^{-3}$ m

Also, $\tan\left(\psi - \frac{\pi}{2}\right) = \frac{355.32 - 300}{188.5} = 0.2935$

$\Rightarrow \psi = 90° + 16.3° = \underline{106.3°}$

→ If you don't like this method, don't go for it. Use the analytical expressions for X_0 & ψ.

However, you should know about this method & that is why we presented it here!

→ ~~See for yourself how to use the phasor diagram~~

→ For some problems, the phasor method may prove very useful. See problem 4, Tu/Hw sheet 3.

(8) The concept of equivalent viscous damping:-

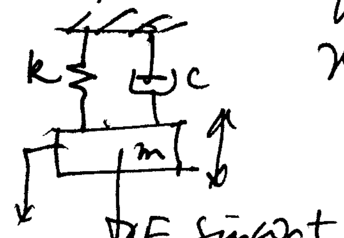
The damping present in a dynamic system is not linearly viscous in general. The damping force can be of the form: $F_d = c_1 \dot{x}^n$. For instance, a body moving in water or air with moderate velocity (in the range of 3 m/s to 20 m/s) is resisted by a damping force that is proportional to the square of the speed, i.e., $n=2$ in this case. c_1 is a constant. Also, there is solid damping or hysteresis damping or structural damping or internal damping in every material in vibration. We also have Coulomb friction as a special case of damping.

→ An accurate analysis of such dampings is very complex. Ordinarily, simplified analyses are made which are verified through experimentation. Such simplified analyses lead to the concept of an equivalent damping constant c_{eq} as described in the following.

→ The whole ~~theme~~ is based on the assumption that the forcing function is harmonic like $F_0 \sin\omega_f t$ or $F_0 \cos\omega_f t$ and the resulting vibration is also harmonic like we have seen for the linear viscous damping case.

Then what is done is this:— We set the Energy ~~dis~~ dissipated per cycle in the linear viscous case = energy dissipated per cycle in the other cases.

→ Let us first obtain W_d = energy dissipated per cycle of harmonic motion of the linear viscous damper.



We know that

$$x_{ss} = x(t) = X_0 \sin(\omega_f t - \psi)$$

$$\therefore \dot{x} = \omega_f X_0 \cos(\omega_f t - \psi)$$

Then, $W_d = \oint c\dot{x}\,dx$ (integral over a cycle of motion)

or, $$W_d = \int_0^{2\pi/\omega_f} c\dot{x}\frac{dx}{dt}dt = \int_0^{2\pi/\omega_f} c\,\dot{x}^2\,dt$$

$$= \int_0^{2\pi/\omega_f} c\,\omega_f^2 X_0^2 \cos^2(\omega_f t - \psi)\,dt = \pi c\,\omega_f X_0^2 \text{ (Show this)}$$

Thus, for the linear viscous damper,

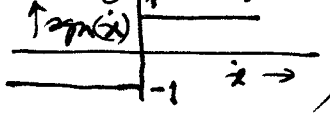
$$\boxed{W_d = \pi c\,\omega_f X_0^2} \quad \text{--- (1) (Remember)}$$

→ <u>Case (i)</u> <u>Quadratic (velocity squared) damping</u>:—

Here, $|F_{damper}| = c_1 \dot{x}^2$. The DEOM of this system can be written as:

$$m\ddot{x} + c_1 \dot{x}^2 \, sgn(\dot{x}) + kx = F_0 \sin\omega_f t$$

where $sgn(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}$ and it accounts for the sign change in the damping force as $\dot{x}$ changes from +ive to −ive value & vice-versa. (Do you remember sgn is the signum or sign-of function such that $sgn(\dot{x}) = +1$ for $\dot{x} > 0$ & $sgn(\dot{x}) = -1$ for $\dot{x} < 0$



) [The use of $sgn(\dot{x})$ is reqd. since $c_1 \dot{x}^2$ doesn't change sign as $\dot{x}$ changes sign]

The above DEOM is nonlinear, note.

Its forced response (PI) can be obtained by some perturbation method such as the MMS (Method of Multiple Scales), which you will study in a Course on Nonlinear Vibrations. (See 'Nonlinear Oscillations' by Nayfeh & Mook, for instance). However, a resonance still occurs when $\omega_f \cong \sqrt{\frac{k}{m}}$, the linear natural frequency of the system. The amplitude of vibration at resonance can be approximately obtained from the formula:

$$X_{res} \cong F_0/k \times \frac{1}{2\zeta} = \frac{F_0}{k \times \cancel{2} \times \frac{c}{\cancel{2}m\omega_n}} = \frac{F_0}{c\omega_n}$$

(for small damping)

if we substitute c by c_{eq} where c_{eq} is obtained as follows:—

$$W_d = 2\int_{-X_0}^{X_0} c_1 \dot{x}^2 \, dx = \frac{8}{3}\omega_f^2 c_1 X_0^3$$

(check) [Assuming $x = X_0 \sin\omega_f t$] →

Then, setting $\frac{8}{3} c_1 \omega_f^2 X_0^3 = \pi C_{eq} \omega_f X_0^2$, we get

$$\boxed{C_{eq} = \frac{8}{3\pi} c_1 \omega_f X_0}$$

→ At resonance, $\omega_f \approx \omega_n$ & $X_0 = X_{res}$, note.

∴ The amplitude at resonance (for small damping) $X_{res} = \frac{F_0}{C_{eq}\,\omega_n} = \frac{3\pi F_0}{8 c_1 \omega_n X_{res}\, \omega_n}$

$$\Rightarrow \quad X_{res} = \sqrt{\frac{3\pi F_0}{8 c_1 \omega_n^2}}$$

→ Case (ii) Velocity-nth Power Damping

We still assume there is harmonic motion since the forcing function is harmonic. This implicitly assumes that damping forces are not strong enough to make the forced vibration non-harmonic. This is seen to be true in most cases as experimentation indicates.

Here $W_d = 2\int_{-X_0}^{+X_0} (c_n \dot{x}^n)\, dx = 2\omega_f^n c_n X_0^{n+1} \int_{-\pi/2}^{+\pi/2} \cos^{(n+1)} \omega_f t \, d(\omega_f t)$ (check)

Damping force $= c_n \dot{x}^n$

or, $W_d = 2\omega_f^n c_n X_0^{n+1} * I \quad \left(I = \int_{-\pi/2}^{\pi/2} \cos^{(n+1)} \omega_f t \, d(\omega_f t)\right)$

$= \pi C_{eq} \omega_f X_0^2$

Then, $C_{eq} = \frac{2 c_n \omega_f^n X_0^{n+1} I}{\pi \omega_f X_0^2}$

or, $C_{eq} = \frac{2}{\pi} c_n \omega_f^{n-1} X_0^{n-1} I$ →

E 10

→ Case (iii) :- Coulomb damping

Here again, a sinusoidal forcing function is assumed to produce a response $x = X_0 \sin\omega_f t$. So, work done per cycle by the Coulomb friction force $= W_d = F_f * 4X_0$ where F_f = friction force during movement.

Then, $4F_f X_0 = \pi c_{eq} \omega_f X_0^2 \Rightarrow c_{eq} = \dfrac{4F_f}{\pi\omega_f X_0}$ ...(i)

The amplitude of forced vibration is given by: $X_0 = \dfrac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta_{eq} r)^2}}$ with $\zeta_{eq} = \dfrac{c_{eq}}{2m\omega_n}$

or, $2\zeta_{eq} r = \dfrac{4F_f}{\pi \cancel{\omega_f} X_0 * \cancel{2}m\omega_n} \times \cancel{2} \times \dfrac{\cancel{\omega_f}}{\omega_n}$ (using (i))

$= \dfrac{4F_f}{\pi k X_0}$

So, $X_0^2 = \dfrac{(F_0/k)^2}{(1-r^2)^2 + \left(\dfrac{4F_f}{\pi k X_0}\right)^2}$ check

Simplifying, we get $X_0 = \dfrac{F_0}{k} \cdot \dfrac{\sqrt{1 - \left(\dfrac{4F_f}{\pi F_0}\right)^2}}{1-r^2}$.

Thus, although damping is present, $X_0 \rightarrow$ grow indefinitely as $r \rightarrow 1$.

Also, $\dfrac{4F_f}{\pi F_0}$ must be < 1 for X_0 to be real, i.e. F_0 must be greater than $\dfrac{4F_f}{\pi}$ for the forced vibration to occur. →

→ Case (iv) :- Hysteretic/structural damping :-

Experiments have established that most of the structural materials like steel, aluminium etc. dissipate energy during vibration due to internal friction. (See 'Vibration Damping' by Nashif & Jones) When the vibration is sinusoidal, the energy dissipated per cycle is seen to be proportional to the square of vibration amplitude. However, an interesting observation is that this quantity is virtually independent of the frequency of vibration over a wide frequency range. Then, $W_d = \alpha X_0^2$ where α is a constant. So, $\alpha X_0^2 = \pi c_{eq} \omega_f X_0^2$

$\Rightarrow c_{eq} = \dfrac{\alpha}{\pi \omega_f}$. With this the DEOM becomes $m\ddot{x} + c_{eq}\dot{x} + kx = F_0 \sin\omega_f t$

or, $m\ddot{x} + \left(\dfrac{\alpha}{\pi\omega_f}\right)\dot{x} + kx = F_0 \sin\omega_f t$ — (a)

Now, while studying/calculating the flutter speeds (the speeds at which violent vibration occurs) of aeroplane wings etc., engineers found that the introduction of the concept of 'complex stiffness' was convenient. We can arrive at →

this complex stiffness by changing the excitation in DEOM ⓐ to $F_0 e^{i\omega_f t}$ $(i=\sqrt{-1})$

[Since $F_0 e^{i\omega_f t} = (F_0 \cos\omega_f t) + i(F_0 \sin\omega_f t)$, taking this new excitation ~~contains~~ is no loss of information, instead, it is more general & response due to both $F_0 \sin\omega_f t$ & $F_0 \cos\omega_f t$ are obtained simultaneously]

Then we have the following DEOM:

$$m\ddot{x} + \frac{\alpha}{\pi\omega_f}\dot{x} + kx = F_0 e^{i\omega_f t} \quad \text{--- (b)}$$

We now assume $x(t) = \tilde{X} e^{i\omega_f t}$ — (c)

($\tilde{X} = X e^{-i\psi}$, complex) Then, $\dot{x} = i\omega_f \tilde{X} e^{i\omega_f t} = i\omega_f x$

& (b) becomes:

$$m\ddot{x} + \frac{\alpha}{\pi\omega_f} \times i\omega_f x + kx = F_0 e^{i\omega_f t}$$

or, $$m\ddot{x} + \left(k + i\frac{\alpha}{\pi}\right)x = F_0 e^{i\omega_f t}$$

or, $$m\ddot{x} + k(1+i\gamma)x = F_0 e^{i\omega_f t} \quad \text{--- (d)}$$

In (d), $\gamma = \frac{\alpha}{\pi k}$ is called the <u>structural damping factor</u> and $k(1+i\gamma)$ is called the <u>complex stiffness</u>. The introduction of this may seem a little strange to you but the concept of complex stiffness, complex moduli of elasticity etc. are pretty common in structural vibration <u>studies</u>. (See 'Sandwich construction' by Plantema & the book by Nashif & Jones)