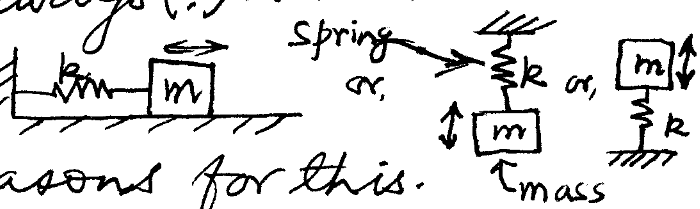
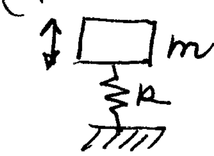
<u>Vibration Analysis</u> always(!) starts with the simple system:



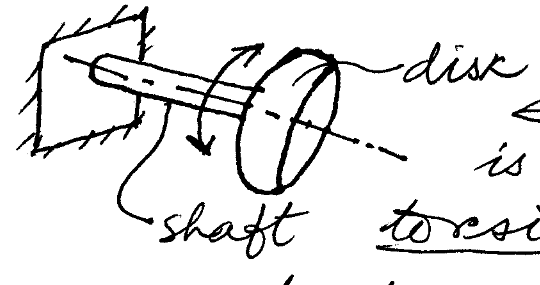
There are several reasons for this. Not only it is the simplest of vibratory mechanical systems but also because it represents many machines, to a first approximation!

For example, a milling m/c or an automobile (4- or 2-wheelers) can be roughly modeled as:



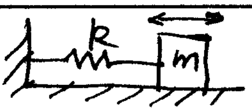
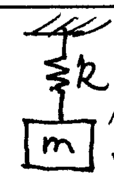
← This is a <u>translational</u> model.

→ There is a rotational counterpart of it:



← The disc (& shaft) is capable of executing <u>torsional</u> vibration.

This very simple torsional system is capable of representing many practical systems. For example, when a heavy electric motor is used to drive the impeller of a centrifugal pump which is quite lighter, the fundamental natural frequency of the system for torsional vibration can be approximately determined by assuming the motor-end of the shaft to be 'fixed' as shown in the above figure.

→ So, we propose to study the vibrational characteristics of the simple mass-spring system:  or  ← This indicates vibratory motion

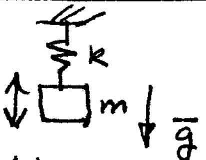
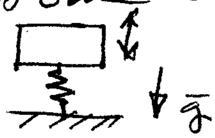
→ We first study the <u>free-vibrational characteristics</u>.

<u>'Free-vibration'</u> means the following: The block of mass is given an 'initial displacement', and/or, an 'initial velocity' at time $t=0$ and then left to itself. The motion that is now executed by the mass gives its free-vibration response.

In order to measure the displacement of the block at any time 't', we associate the variable x, which is actually $x(t)$, that is, a function of time. We usually take x in such a fashion that its value is zero at the static equilibrium position or <u>configuration</u> of the system. This is by no means necessary, but this results in a simpler <u>differential equation of motion</u> (DEOM) for our system.

Hence, for the horizontal system  the spring is in its 'free length', i.e., it is neither compressed nor extended in the static equilibrium configuration.

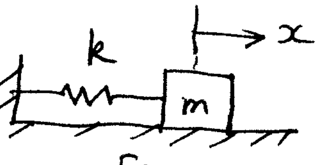
→ Note that we are <u>neglecting frictional effects</u> for this preliminary model! →

However, for the 'vertical' model  (the arrow indicating the direction of acceleration due to gravity), the spring is extended (in tension) in order to balance the weight of the block in static equilibrium. Similarly, the spring is 'compressed' by some amount in case of the system .

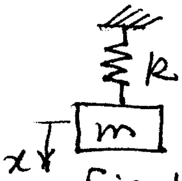
→ Why are we discussing these? One has to take care of these while drawing the free-body-diagrams while deriving the <u>DEOM</u> (<u>D</u>ifferential <u>E</u>quation <u>O</u>f <u>M</u>otion).

→ The first step to obtain vibrational characteristics of our system is to obtain the DEOM.

→ If you observe a little closely, you will find that quite a few <u>assumptions</u> are essential to obtain the well-known DEOM for free-vibrations, viz., $m\ddot{x}+kx=0$, which you must have come across in a Physics course somewhere! Here $\ddot{x}=\frac{d^2x}{dt^2}$ is the acceleration of the mass (block) at time t.

→ <u>The Model</u>:- 

<u>Fig. a</u>

or 

<u>Fig. b</u>

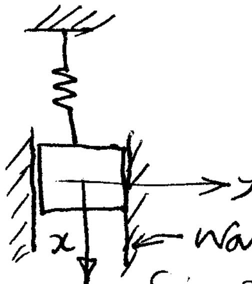
→

→ The assumptions:~

① The block is constrained to execute one-dimensional (1-D) translation only.

[This assumption is very important since, otherwise, a 2-D or 3-D motion ensues which may contain translational as well as rotational components & the resulting DEOM (Differential Equations of motion become quite involved]

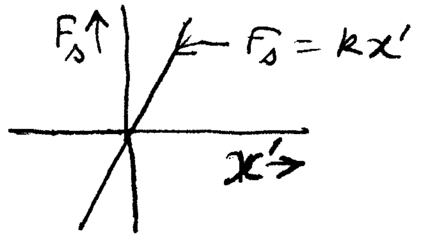
This can be achieved by placing the block (or imagining it is so placed) between frictionless walls so that motion in y- & z-directions are arrested, like the following:



Similarly, there are walls (or imaginary walls) to prevent motion in z-direction (Perpendicular to the plane of paper) also.

② The spring is linear and of negligible mass. This means that the spring force is given by $F_s = kx'$, where x' is the spring extension or shortening & k is the spring constant or spring stiffness or spring rate.

The SI unit for k is N/m, note.



($x' = 0$ at free length)

← Free-body-diagram of spring. The forces at its two ends must be equal & opposite at all times since, otherwise, there would occur an infinite acceleration of a spring element, the spring being massless!

This idealization of the spring, though appears to be vague (the spring has mass), works pretty well in many situations.

3. The mass executes 'small' oscillations, that is, the amplitude of vibration is small. This assumption is essential because the spring may not behave linearly if vibration amplitude is high.

4. All sorts of friction such as friction at solid surfaces of contact or air friction or internal friction (material damping or, hysteresis damping) are neglected for now. (These will be considered afterwards)

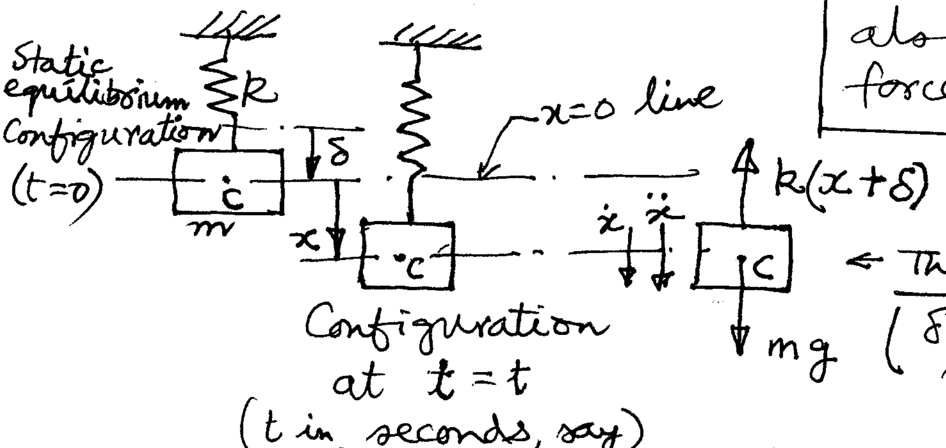
5. The 'mass' (or, 'block') is absolutely rigid, that is, it doesn't deform while in motion.

→

→ We now derive the required DEOM ~~reqd~~ by <u>Newton's Method</u>, that is, applying Newton's 2nd law of motion to the block.

For this, we first draw the 'Free ~~Body~~ Diagram' (FBD) of the block.

Newton's Method is also called the force-balance method



← <u>The FBD</u> at time t.
(δ = extension of the spring in static equilibrium)

x is the (vertical) displacement of block at time t.
x is taken positive downward here but it could be taken +ive upward for many problems, for convenience.

Here we also take $\dot{x}$ & $\ddot{x}$ +ive downward, this is the usual convention. ~~If x increases with time, $\frac{dx}{dt} = \dot{x}$ is positive~~

Since the block is in 1-dimensional (1-D) translation, the velocity of the centre of mass 'c' is $\dot{x}$ & its acceleration is $\ddot{x}$. ($\dot{x} = \frac{dx}{dt}$ & $\ddot{x} = \frac{d^2x}{dt^2}$)
(The terms 'displacement', 'velocity' & 'acceleration' are used a little <u>loosly</u> here, since x, $\dot{x}$ & $\ddot{x}$ are scalars but displacements velocity and acceleration are vectors. However, the context clarifies it all, it is hoped.)

Hence, applying Newton's 2nd law of motion to the centre of mass, we get:

$m\ddot{x}$ = Algebraic sum of external forces in the +ive $\ddot{x}$ direction $= mg - k(x+\delta)$

or, $m\ddot{x} + kx = mg - k\delta$ --- (i)

If we draw the FBD of the block in static equilibrium, we get the following:

$\uparrow k\delta$, C, $\downarrow mg$

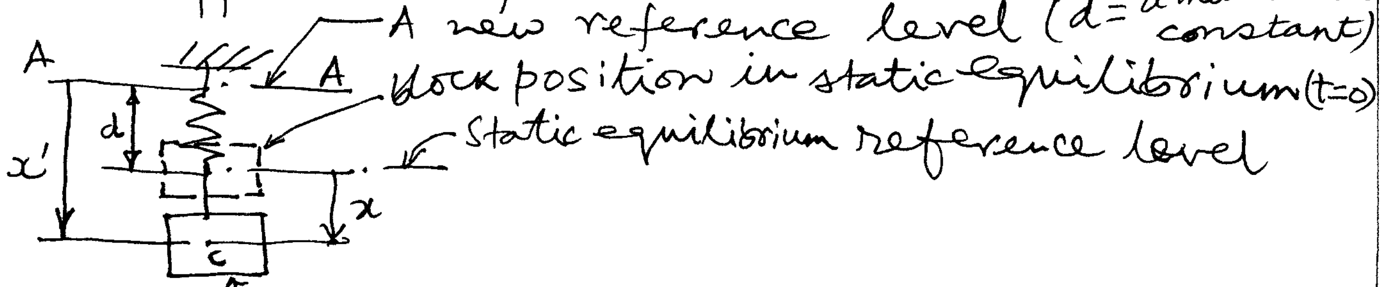
Hence, $\sum$ forces in x direction $= 0$ gives

$mg - k\delta = 0$ --- (ii)

Hence, from (i), using (ii), we get the final DEOM as:

$$\boxed{m\ddot{x} + kx = 0} \quad \text{--- (I)}$$

*A question:- What happens if the location of the mass is measured from a different reference level?



If $x' = x'(t)$ is the new coordinate measuring the location of the block, then, clearly, $x' = x + d$, or, $x = x' - d$

Hence, $\dot{x} = \frac{dx}{dt} = \frac{dx'}{dt} = \dot{x}'$ ($\because$ d = a constant)

& $\ddot{x} = \frac{d^2x'}{dt^2} = \ddot{x}'$

Then, ~~Hence~~, DEOM (I) becomes:

$$m\ddot{x}' + k(x' - d) = 0 \quad \text{or,} \quad \boxed{m\ddot{x}' + kx' = kd} \text{ --- (II)}$$

Now you can clearly see the difference.

DEOM (I) is homogeneous (It is a linear <u>homogeneous</u> 2nd order differential equation with constant coefficients) whereas DEOM (II) is inhomogeneous or, non-homogeneous. The former has a simpler solution compared with the latter. Though for such a simple system, this difference is hardly a matter of concern, for more complex systems, it is quite convenient to deal with homogeneous DEOM & hence, we shall usually measure coordinates representing the configuration of a dynamic system by taking reference levels at the static equilibrium configuration, that is, such geometric coordinates will have <u>zero values</u> at static equilibrium.

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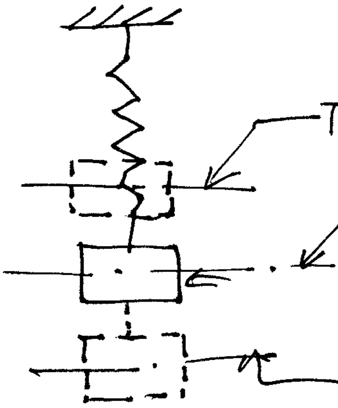
* <u>Other methods of obtaining the DEOM (I) (page 7)</u>

→ The **<u>energy method</u>** (or, the **<u>Power balance method</u>**) :- Our simple mass-spring system is conservative since frictional effects are neglected. Hence the total mechanical energy is conserved. Hence, Kinetic energy + Potential energy = a constant. (This constant depends upon the given initial conditions $x(0)$ and $\dot{x}(0)$, note)

Let T = Kinetic energy & 'V' represents the potential energy of the system at time t, while the system undergoes vibration. At the top & bottom positions, the velocity of mass will be zero and the velocity will be maximum, when the block passes through the equilibrium position. Do you visualize this?



(block in top position)

Top level, $\dot{x}=0$, $\ddot{x}$ downward, $T=0$, $V=V_{maxm}$

Equilibrium level, $\dot{x}$ maximum, $T=T_{max}$, $V=0$

Bottom level, $\dot{x}=0$, $\ddot{x}$ upward, $T=0$, $V=V_{maxm}$

(block in bottom position)

→ Note that we measure potential energies by assuming $V=0$ at the equilibrium configuration. Actually, the reference level ($V=0$) is quite arbitrary and what matters is the <u>change in potential energy over the equilibrium value</u>.

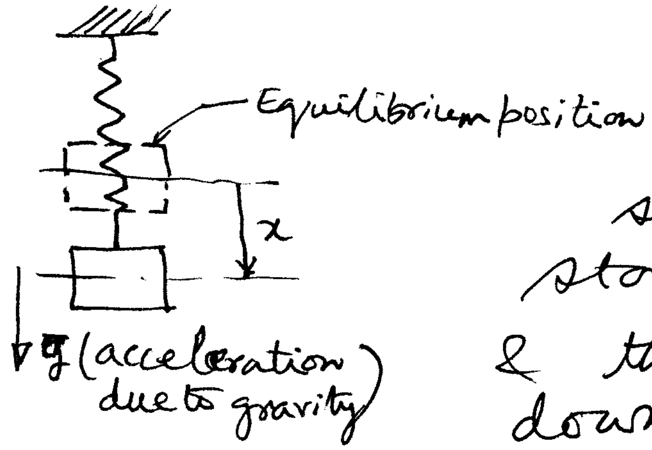
→ Here $T+V=$ constant

& so, $\frac{d}{dt}(T+V)=0$, i.e., $\frac{dT}{dt}+\frac{dV}{dt}=0$

& this relation can be used to obtain the DEOM.

Here $T=\frac{1}{2}m\dot{x}^2$ & $V=\frac{1}{2}k(x+\delta)^2-\frac{1}{2}k\delta^2-mgx$

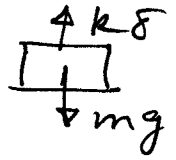
(Since, at location x at time t, the



strain (potential) energy stored in spring $= \frac{1}{2}k(x+\delta)^2$ & the mass has gone down by amount x in the earth's gravitational field & thus, it gravitation PE changes by $-mgx$. Also, the spring already had strain energy $= \frac{1}{2}k\delta^2$ in equilibrium. Hence, total change in PE over & above the PE in equilibrium position is our V which is equal to $\frac{1}{2}k(x+\delta)^2 - \frac{1}{2}k\delta^2 - mgx$

$$= \frac{1}{2}kx^2 + kx\delta + \cancel{\frac{1}{2}k\delta^2} - \cancel{\frac{1}{2}k\delta^2} - mgx$$

$$= \frac{1}{2}kx^2 + (k\delta - mg)x = \underline{\underline{\frac{1}{2}kx^2}}$$ (Since $k\delta = mg$ by equilibrium force balance for the mass :



)

Thus, $T+V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$$\therefore \frac{d}{dt}(T+V) = \frac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}k\frac{d}{dt}(x^2)$$

$$= \frac{1}{2}m.\left[\frac{d}{d\dot{x}}(\dot{x}^2)\right]\frac{d\dot{x}}{dt} + \frac{1}{2}k\left[\frac{d}{dx}(x^2)\right].\frac{dx}{dt}$$

$$= \frac{1}{2}m.2\dot{x}.\ddot{x} + \frac{1}{2}k.2x.\dot{x} = (m\ddot{x}+kx)\dot{x}$$

So, $\frac{d}{dt}(T+V)=0$ gives $(m\ddot{x}+kx)\dot{x}=0$
Since $\dot{x} \neq 0$ at all times, for above relation to be true at all times ($t>0$), we must have $m\ddot{x}+kx=0$ which is the required DEOM.

→ We now discuss a third method for deriving the DEOM. It is done by using the Lagrange Equation.

→ If you are not familiar with the Lagrange Equations (of the second kind, to be precise), do not bother. Just do it mechanically at this point. We shall take up this topic in detail at a later time.

→ Here x is called the 'generalized ~~coordin~~ coordinate' & ~~ẋ~~ $\dot{x}$ the generalized velocity. Also note that we are presently dealing with a single-degree-of-freedom system since only one generalized coordinate is required to define the configuration of the system. (A 'configuration' of our system is given by the location of the block at a particular time)

The Lagrange's Equation then can

be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \quad \text{--- (1)}$$

→ Just don't bother about the origin of this equation or the reason behind the partial derivative notation. All this & much more shall be taken up later. For the time being, just note that equ. (1) is the reqd. DEOM once the expressions for T & V are substituted & all differentiations are carried out.

Let us check this. Here $T = \frac{1}{2}m\dot{x}^2$ & $V = \frac{1}{2}kx^2$, as obtained already. Also, it is important to note that $\dot{x}$ & x <u>are independent</u> at this stage of DEOM derivation. Once the DEOM is solved and response $x(t)$ is obtained for a given set of initial conditions $x(0)$ & $\dot{x}(0)$, then $\dot{x}(t)$ will be given by $\frac{dx}{dt}$. (For now, just accept this, explanations will come later.)

A great advantage of $\dot{x}$ & x being independent is that for partial differentiations, you don't need to use any composite formula for differentiation, such as:

$$\frac{\partial(\dot{x}^2)}{\partial x} = \frac{\partial}{\partial \dot{x}}(\dot{x}^2) \cdot \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial t}\right) \quad \text{etc.}$$

[For our 1-DOF system, $\frac{\partial x}{\partial t} \equiv \frac{dx}{dt}$]

So, $\frac{\partial T}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2\right) = \frac{1}{2} m \times 2\dot{x} = m\dot{x}$

& $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$ — (2)

Now, $\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} m \dot{x}^2\right) = 0$, (3) since $\dot{x}$ & x are independent at this stage.

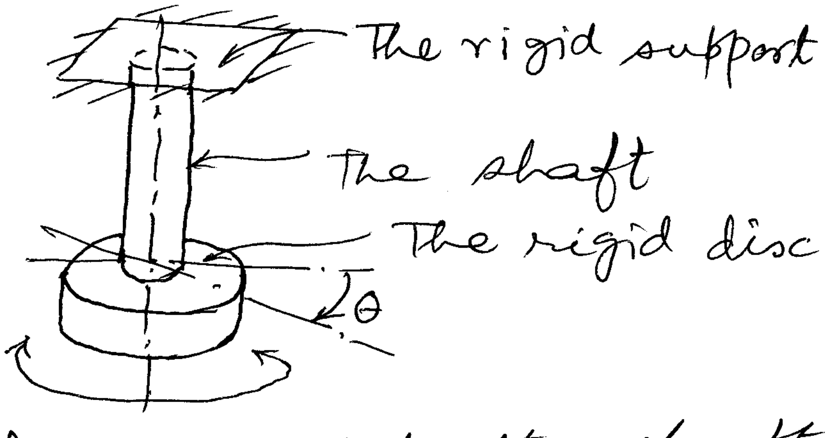
Finally, $\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} k x^2\right) = kx$ — (4)

→ Substituting (2), (3) & (4) in (1), we get

$\boxed{m\ddot{x} + kx = 0}$, the reqd. DEOM!

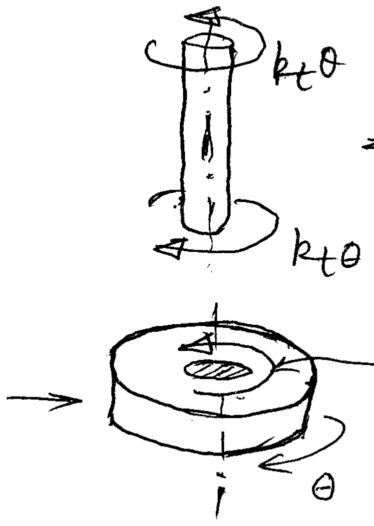
→ There are several other methods of obtaining the DEOM such as use of ~~the~~ D'Alembert's principle or even the ~~use of quantum mechanics~~ {~~the~~ Schrödinger Equation of quantum mechanics!

(8) The rotational counterpart of our translational model :~



→ In this model, the shaft, for now, is assumed to have negligible mass moment of inertia about its own axis. It acts like a rotational or torsional

→ The free-body diagrams:~



← FBD of shaft

This is the Moment of Momentum (MOM) method or, the moment balance method

$k_t\theta$ (Restoring torque on the disc, ccw as seen from above)

Hence, $I_d\ddot{\theta} = -k_t\theta$

or, $\boxed{I_d\ddot{\theta} + k_t\theta = 0}$, which is the required DEOM.

→ (II) The Energy or Power-balance method:~

T = Kinetic energy of the system = Kinetic energy of the rigid disc $= \frac{1}{2} I_d \dot{\theta}^2$ (Just like $\frac{1}{2}m\dot{x}^2$)

V = Potential energy = strain energy stored in the shaft $= \frac{1}{2} k_t \theta^2$ (Just like $\frac{1}{2}kx^2$)

Now, $T + V$ = constant.

Hence, $\frac{d}{dt}(T+V) = 0$ or, $\frac{dT}{dt} + \frac{dV}{dt} = 0$ — (a)

Now, $\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2} I_d \dot{\theta}^2\right) = \cancel{\frac{1}{2} I_d \frac{d}{d\theta}} = \frac{1}{2} I_d \frac{d(\dot{\theta}^2)}{d\dot{\theta}} \cdot \frac{d\dot{\theta}}{dt}$

$= I_d \dot{\theta}\ddot{\theta}$ ——— (b)

Also, $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2}k_t\theta^2\right) = \frac{1}{2}k_t \cdot \frac{d(\theta^2)}{d\theta} \cdot \frac{d\theta}{dt} = k_t \theta\dot{\theta}$ — (c)

Substitution of (b) & (c) in (a) gives: $(I_d\ddot{\theta} + k_t\theta)\dot{\theta} = 0$

Since $\dot{\theta} = 0$ is not true at all times we must have

~~of~~ spring with torsional stiffness k_t.

→ The disc is assumed to be rigid and has mass moment of inertia I_d about its own axis which is also the axis of rotation during torsional vibration.

→ We have chosen $\theta = \theta(t)$ as the generalized coordinate & assume $\theta = 0$ in the static equilibrium configuration. Note that this too is a single DOF system.

→ For free vibration to occur, initial conditions are applied to the disc & then the system is left to itself which then executes the so called free-vibration. These initial conditions are: an initial displacement $\theta(0)$, and/or, an initial angular velocity $\dot{\theta}(0)$.

→ (I) The DEOM by the <u>Moment of Momentum</u> method :~ Here the FBD of the disc is drawn at time t and $I_d\ddot{\theta}$ = Sum of ~~exter~~ torques on the disc, taken positive in the clockwise (CW) sense as seen from the above.

θ is also taken +ive in the same CW sense and so are $\dot{\theta} = \frac{d\theta}{dt}$ & $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.

$I_d\ddot{\theta} + k_t\theta = 0$, which is the required DEOM.

→ III **Using the Lagrange's Equation:-**

The Lagrange equation in this case is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{---} (1)$$

$$T = \frac{1}{2} I_d \dot{\theta}^2, \quad V = \frac{1}{2} k_t \theta^2$$

So, $\frac{\partial T}{\partial \dot{\theta}} = I_d \dot{\theta}$; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_d \ddot{\theta}$;

$\frac{\partial T}{\partial \theta} = 0$; $\frac{\partial V}{\partial \theta} = k_t \theta$.

Substitution of these in (1) leads to:

$I_d\ddot{\theta} + k_t\theta = 0$, which is the required DEOM.

(5) **Extracting information from the DEOM**

This is done by solving the DEOM $m\ddot{x} + kx = 0$ (for a translational system) or, the DEOM $I_d\ddot{\theta} + k_t\theta = 0$ (for a torsional system)

We assume $x = \hat{C}e^{st}$ (See any book on differential equations for getting the general solution of

$$m\ddot{x} + kx = 0 \quad \text{---} (1)$$

$$x = \hat{C}e^{st} \quad \text{---} (2)$$

$\hat{C}$ & s are unknowns at this stage.

Then, $\dot{x} = \hat{C}se^{st}$ & $\ddot{x} = \hat{C}s^2e^{st}$

Substitution in (1) now gives

$$\hat{C}ms^2e^{st} + \hat{C}ke^{st} = 0 \quad \text{or,} \quad (ms^2 + k)\hat{C}e^{st} = 0 \quad \text{--} (3)$$

But $e^{st} \neq 0$ for any t & c can't be zero (if so, the solution becomes trivial). Hence we must have $ms^2 + k = 0$ & so,

$s^2 = -k/m$ & $s = \pm\sqrt{-k/m} = \pm i\sqrt{\frac{k}{m}}$ where $i = \sqrt{-1}$.

Then, taking $s = s_1 = -i\sqrt{\frac{k}{m}}$ & $s = s_2 = +i\sqrt{\frac{k}{m}}$, the general solution of ① is:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$$= c_1 e^{-i\sqrt{\frac{k}{m}}} + c_2 e^{+i\sqrt{\frac{k}{m}}} \quad \text{--- ④}$$

Using the Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$, and noting that $x(t)$ must be real, we can simplify ④ & taking c_1 & c_2 to be complex conjugates, we arrive at the solution [Obtain it as home work]

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$= A\sin\omega_n t + B\cos\omega_n t \quad \text{--- ⑤}$$

with $\omega_n = \sqrt{\frac{k}{m}}$ & A & B arbitrary (constants of integration).

⑤ can also be written as:

$$x(t) = X_0 \sin(\omega_n t + \phi) \quad \text{--- ⑥}$$

where X_0 & ϕ are the two arbitrary constants of integration.

For given $x(0)$ & $\dot{x}(0)$, these two constants would take specific values & we get a particular motion of the system. $\sqrt{\frac{k}{m}} = \omega_n$ is called the undamped circular natural frequency of the system and has the unit radians per second (rad/s).

Then, $f_n = \frac{\omega_n}{2\pi}$ gives the natural frequency of the system in cycles/s or Hz.

However, you must remember that in the study of Mechanical Vibrations, ω_n is simply called the natural frequency and the term 'circular' is omitted. Thus, ω_n is the natural frequency in rad/s & f_n is the natural frequency in Hertz (Hz) or cycles/second.

Important:- In a numerical problem, express k in N/m (Newton per metre) and m in kg (kilogram). Then, $\sqrt{\frac{k}{m}}$ correctly gives you the value of ω_n in rad/s.

Example:- If $k = 2$ MN/m (Mega Newton per metre) & $m = 500$ g (grams), obtain ω_n & f_n

Solution:- $k = 2 \text{ MN/m} = 2 \times 10^6 \text{ N/m}$

$m = 500 \text{ g} = 500 \times 10^{-3} \text{ kg}$

Hence, $\omega_n = \sqrt{\frac{2 \times 10^6}{500 \times 10^{-3}}} \text{ rad/s} = 2000 \text{ rad/s}$

$f_n = \frac{\omega_n}{2\pi} \text{ Hz} = 318.31 \text{ Hz}$

Hence, the general solution of $m\ddot{x}+kx=0$ is:

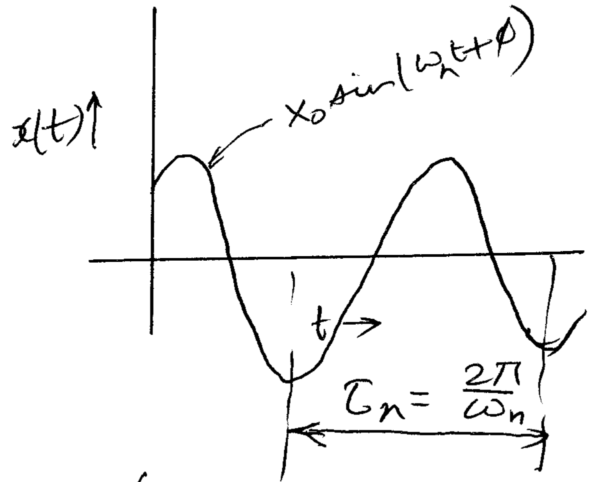
$$\left.\begin{aligned} x &= X_0 \sin(\omega_n t+\phi) \\ &\text{or,} \\ x &= A\sin\omega_n t + B\cos\omega_n t \end{aligned}\right\} \omega_n = \sqrt{\frac{k}{m}}\ (\text{rad/s})$$

Similarly, the general solution of $I_d\ddot{\theta}+k_t\theta=0$ is:

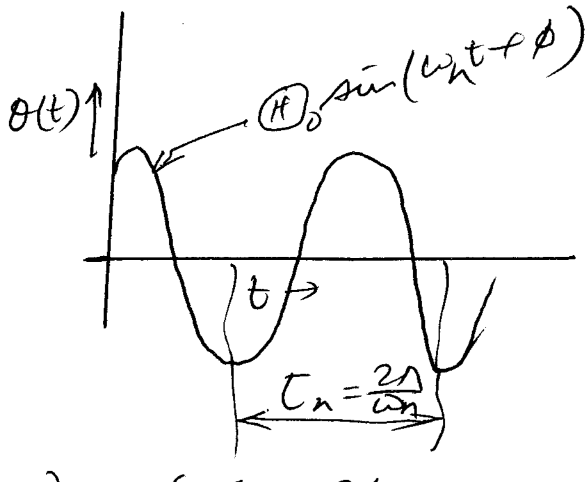
$$\left.\begin{aligned} \theta &= \Theta_0 \sin(\omega_n t+\phi) \\ &\text{or,} \\ \theta &= A\sin\omega_n t + B\cos\omega_n t \end{aligned}\right\} \omega_n = \sqrt{\frac{k_t}{I_d}}\ (\text{rad/s})$$

k_t is usually in N-m/rad (Torque per unit twist) & I_d is in $kg\,m^2$.

These free-vibration responses look like:



(when $x(0)>0$ & $\dot{x}(0)>0$)



(When $\theta(0)>0$ & $\dot{\theta}(0)>0$)

We could write the general responses in terms of the initial conditions $\{x(0)\ \&\ \dot{x}(0)\}$ or, $\{\theta(0), \dot{\theta}(0)\}$ as follows:-

Let $x = A\sin\omega_n t + B\cos\omega_n t$. So, $x(0)=B$ (at $t=0$)

& $\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$ & so, $\dot{x}(0)=A\omega_n \Rightarrow A = \frac{\dot{x}(0)}{\omega_n}$

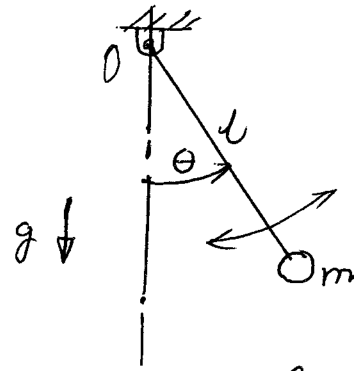
Thus, $x = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n(t)$

& Similarly, $\theta = \frac{\dot{\theta}(0)}{\omega_n}\sin\omega_n t + \theta(0)\cos\omega_n(t)$

Hence, for a particular motion (that is, for a given set of initial conditions), $x(t)$ or $\theta(t)$ can be obtained straightaway by using the above expressions.

→ Several important examples will now be taken up. You should practice these.

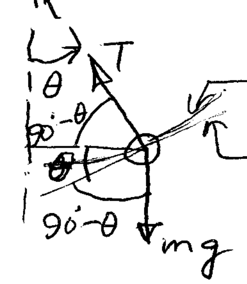
→ Example 1:- Obtain the DEOM of a simple pendulum and get the expression for its natural frequency for small amplitude oscillations.



The pendulum is vertical in static equilibrium and its configuration at time t is as shown in the figure. $\theta = \theta(t)$ is the generalized coordinate ($\theta = 0$ at static equilibrium)

(I) Newton's Method:- [θ +ive counter-clockwise]

FBD of the bob:-



T= tension in the string.

Note that the tangential acceleration of the bob is $l\ddot{\theta}$ in the direction of θ increasing. The net external force in the same direction is $-mg\cos(90°-\theta)$, i.e., $-mg\sin\theta$.

Hence, by Newton's 2nd law,

$$ml\ddot{\theta} = -mg\sin\theta \text{ or } ml\ddot{\theta} + mg\sin\theta = 0 \quad \text{--- (1)}$$

For small θ, $\sin\theta \cong \theta$ & hence (1) becomes $ml\ddot{\theta} + mg\theta = 0$, which is the reqd. DEOM for small oscillations. [Using $\sin\theta \cong \theta$, we linearize the DEOM, note.]

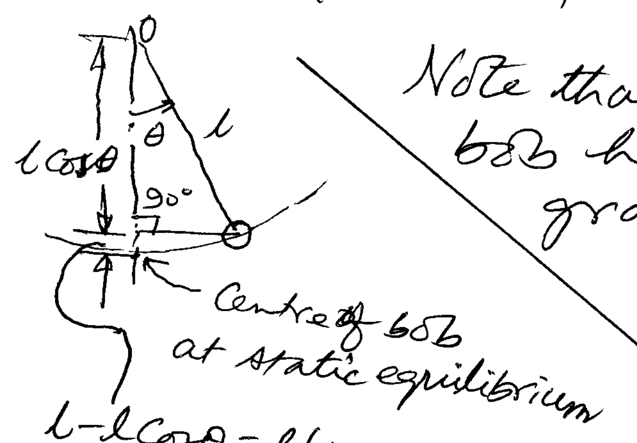
Comparing $ml\ddot{\theta} + mg\theta = 0$ with our standard DEOM $I_d\ddot{\theta} + K_t\theta = 0$ for rotational motion, we can see that $\omega_n = \sqrt{\frac{mg}{ml}} = \sqrt{\frac{g}{l}}$ (rad/s), the required natural frequency.

$$\therefore f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \text{ Hz}.$$

→ It is interesting to note that ω_n is independent of the mass of the bob.

(II) Energy Method :~ Note that the velocity of the bob is $l\dot{\theta}$. Hence its kinetic energy $= T = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$

[ml^2 is the moment of inertia of the bob, assumed to be a particle, about an axis through pivot 'O', perpendicular to the plane of oscillation (here perpendicular to the plane of the paper).]



Note that at time t, the bob has risen in the gravitational field by an amount $l(1-\cos\theta)$.

Hence, $V = mgl(1-\cos\theta)$

$\therefore \frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}^2\right)$

$= ml^2\dot{\theta}\ddot{\theta}$ & $\frac{dV}{dt} = \frac{dV}{d\theta}\cdot\frac{d\theta}{dt} = (mgl\sin\theta)\dot{\theta}$.

So, $T+V =$ Constant or, $\frac{dT}{dt} + \frac{dV}{dt} = 0$ gives:

$ml^2\dot{\theta}\ddot{\theta} + (mgl\sin\theta)\dot{\theta} = 0 \Rightarrow ml^2\ddot{\theta} + mgl\theta = 0$, which is the reqd DEOM, which is basically the same as the one obtained by Newton's method if we divide both sides by l. So, $\omega_n = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$, as before.

(II) Using the Lagrange Equation:

Here the Lagrange Equation is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{--- (1)}$$

as discussed beforehand.

$T = \frac{1}{2}ml^2\dot{\theta}^2 \Rightarrow \frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$; $\frac{\partial T}{\partial \theta} = 0$,

$V = mgl(1-\cos\theta) \Rightarrow \frac{\partial V}{\partial \theta} = mgl\sin\theta$.

Substituting these in (1), we get

$$ml^2\ddot{\theta} + mgl\sin\theta = 0.$$

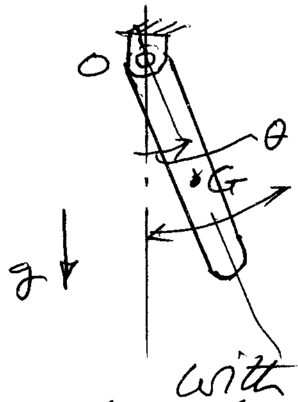
Linearizing ($\sin\theta \cong \theta$), we get

$ml^2\ddot{\theta} + mgl\theta = 0$ as the reqd DEOM, as before.

Note that in Newton's method, we considered forces in the tangential direction only thereby eliminating the unknown tension T in the string. In your dynamics course, you must have written a differential equation involving T by considering the centripetal acceleration of the bob. That equation may be used to obtain T as a function of time once $\theta(t)$ is obtained using the DEOM we derived.

The value of $T_{maximum}$ may be required in some design problem involving a simple pendulum!

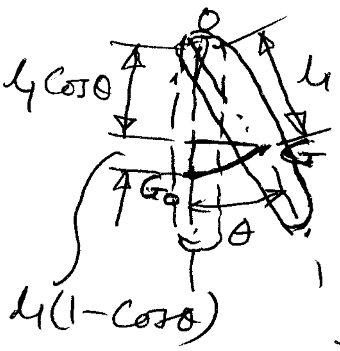
Example 2 :- The compound pendulum problem :- Obtain DEOM for small oscillations. Also obtain ω_n



In this, we consider the oscillation of a rigid body about pivotal axis at O. Let G be the centre of gravity (CG) of the body, which coincides with its centre of mass (CM) for this problem. Let I_G = moment of inertia of the body about centroidal axis at G.

If $OG = l_1$, then, $I_O = I_G + m l_1^2$ where m is the mass of the body.

Then, kinetic energy $T = \frac{1}{2} I_O \dot{\theta}^2$ where θ is the generalized coordinate with θ = 0 at static equilibrium position when G is vertically below O.



Also, $V = m g l_1 (1 - \cos\theta)$

(See figure. G_0 is the location of CG in static equilibrium)

The Lagrange equation here is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{--- (1)}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} I_O \dot{\theta}^2\right) = I_O \dot{\theta}; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_O \ddot{\theta},$$

$$\partial T/\partial \theta = 0; \quad \frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}\left[m g l_1 (1 - \cos\theta)\right] = m g l_1 \sin\theta .$$