Problem Sheet
Mathematical Preliminaries

1. What is the order of the following tensors? Expand each of them and write the corresponding matrix form.

   (i) A_{ii} (ii) $A_{ij}b_j$ (iii) $A_{ji}b_j$ (iv) $\dfrac{\partial u_i}{\partial x_j}$ (v) $\dfrac{\partial u_p}{\partial x_p}$

   (vi) $\frac{\partial A_{ij}}{\partial x_j}$ (vii) $u_{p,p} + v_{q,q}$ (viii) $u_{p,q} + u_{q,p}$ (ix) $\dfrac{\partial u_i}{\partial t}$ (x) $\dfrac{\partial^2 u_i}{\partial x_i \partial x_j}$

2. Using the component form, rewrite each of the following in indical notation. Also write the corresponding matrix form.

   (i) $\mathbf{A}\boldsymbol{v}$ (ii) $\mathbf{A}^\mathsf{T}\boldsymbol{v}$ (iii) $\mathbf{A}\ \mathbf{B}$ (iv) $\mathbf{A}^\mathsf{T}\mathbf{B}$ (v) $\mathbf{A}\mathbf{B}^\mathsf{T}$ (vi) $\mathbf{A}^\mathsf{T}\mathbf{B}^\mathsf{T}$

3. Taking the trace of $\mathbf{AB}$, $\mathbf{A}^\mathsf{T}\mathbf{B}$, and so on result in scalar quantities. If we define $\mathbf{A} : \mathbf{B} := A_{ij}B_{ij}$ and $\mathbf{A} \cdot\cdot \mathbf{B} := A_{ij}B_{ji}$, then verify the following:

   (i) $\mathbf{A} : \mathbf{B} := A_{ij}B_{ij} = \mathrm{tr}(\mathbf{A}^\mathsf{T}\mathbf{B}) = \mathrm{tr}(\mathbf{B}^\mathsf{T}\mathbf{A}) = \mathrm{tr}(\mathbf{B}\mathbf{A}^\mathsf{T})$
   (i) $\mathbf{A} \cdot\cdot \mathbf{B} := A_{ij}B_{ji} = \mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}) = \mathrm{tr}(\mathbf{A}^\mathsf{T}\mathbf{B}^\mathsf{T})$

4. If A_{ij} is symmetric and B_{ij} is anti-symmetric, show that $A_{ij}B_{ij}$ is equal to 0.

5. The transformation of a second order tensor is brought about by the rule $A'_{ij} = Q_{ip}Q_{jq}A_{pq}$. Show using indical notation that the transformation rule in compact or equivalently in matrix form becomes $\mathbf{A}' = \mathbf{Q}\mathbf{A}\mathbf{Q}^\mathsf{T}$. Then carry out the transformation of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

   into a new coordinate system found through a rotation of 60° ($\pi/3$ radian) about the x_3-axis.

6. An isotropic properly is such that it is identical in all directions. Show by using transformation rules that $a\delta_{ij}$ is a second-order isotropic tensor.