Problem Sheet
Kinematics

1. For arbitrary constants A, B, C, determine the strain tensor (ε_{ij}) and rotation tensor (ω_{ij}) for the following displacement fields:

   (a) $u = Axy$, $v = Bxz^2$, $w = C(x^2 + y^2)$

   (b) $u = Ax^2$, $v = Bxy$, $w = Cxyz$

   (c) $u = Ayz^3$, $v = Bxy^2$, $w = C(x^2 + y^2)$

   Find these using the symbolic facilities in MATLAB or Python (`sympy`).

$$(a) \quad \boldsymbol{\varepsilon} = \begin{bmatrix} Ay & \frac{Ax}{2} + \frac{Bz^2}{2} & Cx \\ \frac{Ax}{2} + \frac{Bz^2}{2} & 0 & Bxz + Cy \\ Cx & Bxz + Cy & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & \frac{Ax}{2} - \frac{Bz^2}{2} & -Cx \\ -\frac{Ax}{2} + \frac{Bz^2}{2} & 0 & Bxz - Cy \\ Cx & -Bxz + Cy & 0 \end{bmatrix}$$

$$(b) \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 2Ax & \frac{1}{2}By & \frac{1}{2}Cyz \\ \frac{1}{2}By & Bx & \frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & Cxy \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -\frac{1}{2}By & -\frac{1}{2}Cyz \\ \frac{1}{2}By & 0 & -\frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & 0 \end{bmatrix}$$

$$(c) \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \frac{Az^3}{2} + \frac{By^2}{2} & \frac{3A}{2}yz^2 + Cx \\ \frac{Az^3}{2} + \frac{By^2}{2} & 2Bxy & Cy \\ \frac{3A}{2}yz^2 + Cx & Cy & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & \frac{Az^3}{2} - \frac{By^2}{2} & \frac{3A}{2}yz^2 - Cx \\ -\frac{Az^3}{2} + \frac{By^2}{2} & 0 & -Cy \\ -\frac{3A}{2}yz^2 + Cx & Cy & 0 \end{bmatrix}$$

2. A two-dimensional problem of a rectangular bar stretched by uniform end loadings results in the following strain field:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & -C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where C_1 and C_2 are constants. Assuming the field depends only on x and y, integrate the strain-displacement relations to determine the displacement components and identify any rigid-body motion terms.

[$u_x = C_1 x + Ky + D$, $u_y = -C_2 y - Kx + E$; rigid body rotation about z-axis: $\omega_z = -K$; translation along x: D; translation along y: E]

3. Consider a cubical volume element with three of its edges at any one chosen vertex oriented along the principal directions. The original length of each side is $dX^{(k)}$ and the final length is $dx^{(k)}$, such that

$$dx^{(k)} = dX^{(k)}(1 + \varepsilon^{(1)}), \quad k = 1, 2, 3 \quad \text{(the superscripts are just labels; no summation implied)}$$

where $\varepsilon^{(k)}$ is the principal strain in the k-direction.

   (a) *Dilatation* is defined as the relative change in volume: (final vol. - initial vol.)/(initial vol.). Find an expression for *dilatation* assuming that $|\varepsilon^{(k)}| \ll 1$. Rewrite this expression in terms of the components of displacement gradient $u_{i,j}$.

   (b) If the initial and final volumes are V_0 and V_f respectively with corresponding densities ρ_0 and ρ_f (assumed uniform throughout the volumes), then mass conservation gives the relation $\rho_\mathrm{f} V_\mathrm{f} = \rho_0 V_0$. Using this relation and the result of part (a), relate ρ_0, ρ_f, and the divergence of the displacement, $u_{i,i}$. What happens when $\rho_0 = \rho_\mathrm{f}$, i.e. when the density is constant?

4. Since dilatation can be expressed solely in terms of the normal strain components (refer previous problem), these normal strain components are said to be responsible for *changes in volume* while the shearing strains are responsible for *changes in shape*. Often, the (infinitesimal) strain tensor is decomposed into two parts: the mean normal strain $\boldsymbol{\varepsilon}_\mathrm{M}$, which accounts for volumetric change, and the deviatoric strain ε_D, which accounts for shape change.

   (a) Define $\varepsilon_\mathrm{M} = \frac{1}{3}(\nabla \cdot \mathbf{u})\,\mathbb{I}$, where $\mathbb{I}$ is the identity tensor; or, equivalently, $(\varepsilon_M)_{ij} = \frac{1}{3} u_{i,i}\delta_{ij}$. Find the matrix representation of ε_D.

   (b) The definition of ε_M ensures that the mean normal strain represents a state of equal elongation in all directions. Under this state of strain the elemental volume deforms in such a way that the shape remains *similar* to the original shape. Since ε_M accounts for the volumetric strain, the volumetric change associated with ε_D should be zero. Check if this is so by finding the dilatation of ε_D.

5. The problem of finding the principal strains at a point reduces to the eigenvalue problem:

$$(\varepsilon_{ij} - \lambda_{ij})\, n_j = 0, \quad \text{or, equivalently, in matrix form} \quad ([\varepsilon] - \lambda[\mathbb{I}])\,[\hat{\mathbf{n}}] = 0.$$

Non-trivial solutions of this problem may be found by using the condition that $\det([\varepsilon] - \lambda[\mathbb{I}]) = 0$ which yields (note that $\det([A])$ means determinant of $[A]$)

$$\begin{vmatrix} \varepsilon_{11} - \lambda & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \lambda & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \lambda \end{vmatrix} = 0,$$

that reduces to

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0.$$

(a) Find expressions for J_1, J_2, and J_3 in terms of the components of ε_{ij}.

(b) J_1, J_2, and J_3 are referred to as the strain invariants. What do you think is the motivation behind calling them *invariants*? *Hint:* Principal strains pertain to the actual physical situation while the components of ε_{ij} are a consequence of the choice of our coordinate axes.

(c) The principal strain tensor is such that in its matrix representation the diagonal elements are the λ's while the off-diagonal elements are zero. Find expressions for J_1, J_2, and J_3 in terms of the principal strains λ_1, λ_2, and λ_3.

6. The strain field at a point $P(x, y, z)$ in an elastic body is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 20 & 3 & 2 \\ 3 & -10 & 5 \\ 2 & 5 & -8 \end{bmatrix} \times 10^{-6}.$$

Determine the following values:

(a) The strain invariants

(b) The principal strains

(c) The mean normal strain and the deviatoric strain

[(a) $J_1 = 2 \times 10^{-6}$, $J_2 = 318 \times 10^{-12}$, $J_3 = 1272 \times 10^{-18}$; (b) $\lambda_1 = 20.5 \times 10^{-6}$, $\lambda_2 = -14.1 \times 10^{-6}$, $\lambda_3 = -4.39 \times 10^{-6}$]

7. Consider a strain field such that

$$\varepsilon_{11} = Ax_2^2, \quad \varepsilon_{22} = Ax_1^2, \quad \varepsilon_{12} = Bx_1x_2, \quad \varepsilon_{33} = \varepsilon_{32} = \varepsilon_{31} = 0.$$

Find the relationship between A and B such that it is possible to obtain a single-valued continuous displacement field which corresponds to the given strain field. [$B = 2A$]

8. Consider the strain-displacement relations in a rectangular Cartesian coordinate system and verify that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{1}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{2}$$

Extend the ideas of these two equations to obtain

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \tag{3}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \tag{4}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{5}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) \tag{6}$$

These six equations are referred to as the compatibility equations.

9. The six compatibility equations in the previous question are not actually independent. To see this, first obtain from Eqs. (2), (5), and (6) the following:

$$\frac{\partial^4 \varepsilon_{xx}}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{7}$$

$$\frac{\partial^4 \varepsilon_{yy}}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{8}$$

$$\frac{\partial^4 \varepsilon_{zz}}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right). \tag{9}$$

Next, add Eqs. (7) and (8) and compare with what you obtain after differentiating Eq. (1) w.r.t z twice. *This comparison shows that Eqs.* (7), (8), *and* (9) *are really the three independent equations.*

10. Show that if the rotation is zero throughout a body (irrotational deformation), the displacement vector is the gradient of a scalar potential function. *Hint*: Use the idea from irrotational fluid flow.

11. This problem will involve two important results related to principal strains and principal directions

    (a) Using the property that the strain tensor is symmetric, show that eigenvectors corresponding to unequal eigenvalues are orthogonal to each other. *Hint:* Consider two different eigenvectors $n_j^{(1)}$ and $n_j^{(2)}$ corresponding to the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. Then write the eigenvalue problem equations for each case. Your aim is to show that $n_j^{(1)} n_j^{(2)} = 0$ when $\lambda^{(1)} \neq \lambda^{(2)}$; so multiply the equations appropriately to obtain this product, use the symmetry property, rearrange the indices, and proceed.

    (b) Show that the state of strain referred to a set of coordinate axes that are aligned along the principal directions is purely diagonal. In other words show that referred to these coordinate axes, the shear strains are zero.

12. In polar coordinates, the 2D-strains are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

Using these relations, determine the two-dimensional strains for the following displacement fields:

(a) $u_r = \frac{A}{r}$, $u_\theta = B\cos\theta$

(b) $u_r = Ar^2$, $u_\theta = Br\sin\theta$

(c) $u_r = A\sin\theta + B\cos\theta$, $u_\theta = A\cos\theta - B\sin\theta + Cr$

where A, B, C are arbitrary constants.

[(a) $\boldsymbol{\varepsilon} = \begin{bmatrix} -\frac{A}{r^2} & -\frac{B}{2r}\cos(\theta) \\ -\frac{B}{2r}\cos(\theta) & \frac{1}{r^2}(A - Br\sin(\theta)) \end{bmatrix}$ (b) $\boldsymbol{\varepsilon} = \begin{bmatrix} 2Ar & 0 \\ 0 & Ar + B\cos(\theta) \end{bmatrix}$ (c) $\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$]