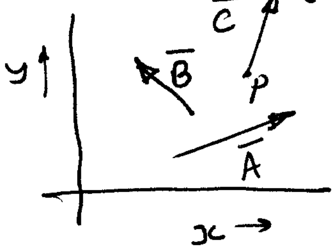
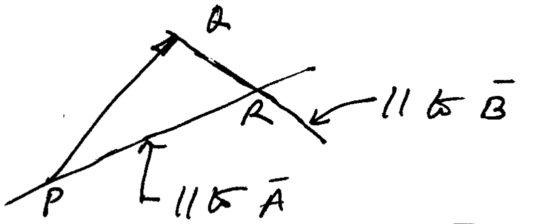
(ii) <u>The expansion theorem</u> :~ Suppose we have two vectors $\bar{A}$ & $\bar{B}$ such that (see fig.) $\bar{A} \neq \alpha \bar{B}$, α being a constant. Then we say that $\bar{A}$ & $\bar{B}$ are linearly independent. So,



any third vector $\bar{C}$ (we are considering 2-D vectors for the moment) can be expressed as a linear combination of $\bar{A}$ & $\bar{B}$, that is, we can find constants α & β such that $\bar{C} = \alpha\bar{A} + \beta\bar{B}$. This can be seen as follows:~ Let $\bar{C} = \overline{PQ}$.

Through P, draw a line parallel to $\bar{A}$ & through Q, draw a line parallel to $\bar{B}$. Let these intersect at R.



Then, $\bar{C} = \overline{PQ} = \overline{PR} + \overline{RQ}$
$= \alpha\bar{A} + \beta\bar{B}$

Now, we can represent $\bar{A}$ also as $\begin{Bmatrix} A_x \\ A_y \end{Bmatrix}$, $\bar{B}$ as $\begin{Bmatrix} B_x \\ B_y \end{Bmatrix}$ & $\bar{C}$ as $\begin{Bmatrix} C_x \\ C_y \end{Bmatrix}$ where A_x is the x-component of $\bar{A}$ etc. Then,

$$\begin{Bmatrix} C_x \\ C_y \end{Bmatrix} = \alpha \begin{Bmatrix} A_x \\ A_y \end{Bmatrix} + \beta \begin{Bmatrix} B_x \\ B_y \end{Bmatrix}.$$

→ So, if we have the modal vectors $\{X\}_1$ & $\{X\}_2$ linearly independent, then, any 2-D vector $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ can be expressed as:

$$\{u\} = c_1\{X\}_1 + c_2\{X\}_2$$

& this is called the <u>expansion theorem</u>. This theorem has far-reaching consequences in our studies. Actually if $\omega_1 \neq \omega_2$, then $\{x\}_1$ & $\{x\}_2$ will be linearly independent. What happens when $\omega_1 = \omega_2$ will be taken up later. So, in our example problems, $\omega_1 \neq \omega_2$ and $\{x\}_1 = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}$ & $\{x\}_2 = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$ are linearly independent. If this were not so, ~~$\{x\}_1$ & $\{x\}_2$~~ then $\{x\}_2 = \alpha\{x\}_1$ for some constant α & so, $\{x\}_2$ & $\{x\}_1$ would basically represent the <u>same</u> modal vector. This is so because if $\{x\}_1$ is a modal vector, corresponding to $\omega = \omega_1$, then $\alpha\{x\}_1$ is also a modal vector for $\omega = \omega_1$, since the elements of a modal vector can be multiplied by an arbitrary non-zero number & it still remains a modal vector. [Remember we said that in $\{X\}_1 = \begin{Bmatrix} x_{11} \\ \mu_1 x_{11} \end{Bmatrix}$, x_{11} is arbitrary? So, x_{11} could be replaced by αx_{11} without problem]

→ Now, we said any arbitrary $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ can be expanded as: $\{u\} = c_1\{X\}_1 + c_2\{x\}_2$ --(i)
How do we find c_1 & c_2? We shall apply a trick to bring into picture the orthogonality relation & this is what we do:
From (i), premultiplications will give:

$$\{x_1\}^T[m]\{u\} = c_1\{x\}_1^T[m]\{X\}_1 + c_2\{x\}_1^T[m]\{x\}_2$$

But by <u>mass orthogonality</u> of the modal vectors, we have

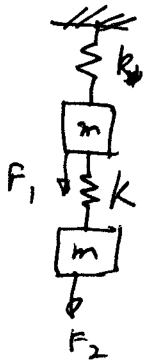
$\{x\}_1^T[m]\{x\}_2 = 0$. Hence, $c_1 = \dfrac{\{x\}_1^T[m]\{u\}}{\{x\}_1^T[m]\{x\}_1}$.

Similarly, $c_2 = \dfrac{\{x\}_2^T[m]\{u\}}{\{x\}_2^T[m]\{x\}_2}$, check.

→ You could, of course, premultiply both sides of (i) by $\{x\}_1^T[K]$ instead of $\{x\}_1^T[m]$ & then obtain c_1 using stiffness orthogonality relation $\{x\}_1^T[K]\{x\}_2 = 0$.

But normally $[m]$ is simpler than $[K]$, so we used $[m]$ instead of $[K]$.

→ Now try to see this: Suppose ~~the~~ our example system is executing vibratory motion $\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$ under a set of complex forcing functions.



Then, at an instant $t = t_1$, the vector $\begin{Bmatrix} x_1(t_1) \\ x_2(t_1) \end{Bmatrix}$ can be expressed as:–

$$\begin{Bmatrix} x_1(t_1) \\ x_2(t_1) \end{Bmatrix} = c_1\{X\}_1 + c_2\{X\}_2$$

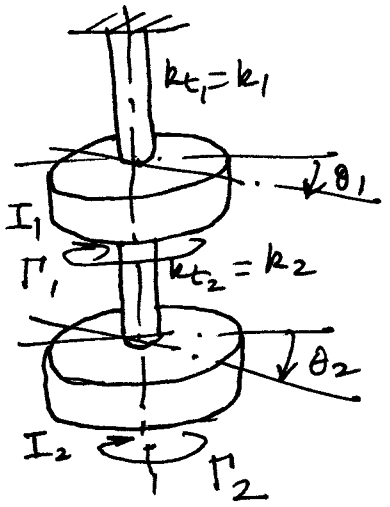
, by the expansion theorem. At $t = t_1 + dt$, $\begin{Bmatrix} x_1(t_1+dt) \\ x_2(t_1+dt) \end{Bmatrix} = c_1'\{X\}_1 + c_2'\{X\}_2$

Where c_1' is very little different from c_1 & c_2' is also very little different from c_2. So, may be we could write $c_1 = c_1(t)$ & $c_2 = c_2(t)$, so that

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = c_1(t)\{X\}_1 + c_2(t)\{X\}_2 \; ?$$

~~Actually~~ So, it seems we could obtain the forced response if we could find $c_1(t)$ & $c_2(t)$? Actually, when you will study vibration of continuous systems, you will see that several approximate methods for finding forced response depend on an expansion theorem like the above with the modal vectors replaced by the so called eigenfunctions.

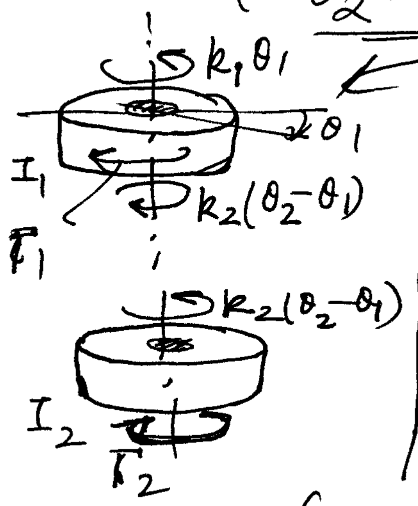
(§) The rotational 2-DOF system:~



The standard 2-DOF system for ~~torsional~~ vibration is shown here, damping neglected. To derive the DEOM, we might use the moment balance method as follows:~

$\Gamma_1=\Gamma_1(t)$ & $\Gamma_2=\Gamma_2(t)$ are the applied external torques

~~let~~ $\underline{\theta_2>\theta_1}$. The FBDs are:—



The DEOM are:–

$$I_1\ddot{\theta}_1=-k_1\theta_1+k_2(\theta_2-\theta_1)+\Gamma_1$$
$$I_2\ddot{\theta}_2=-k_2(\theta_2-\theta_1)+\Gamma_2$$

OR

$$I_1\ddot{\theta}_1+(k_1+k_2)\theta_1-k_2\theta_2=\Gamma_1$$
$$I_2\ddot{\theta}_2-k_2\theta_1+k_2\theta_2=\Gamma_2$$

Hence, in matrix form, the DEOM are:

$$[I]\{\ddot{\theta}\}+[k]\{\theta\}=\{\Gamma\} \quad \text{---(1) where}$$

$[I]=\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$ is the inertia matrix, $[k]=\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ is the stiffness matrix, $\{\theta\}=\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix}$ is the (angular) displacement vector, $\{\ddot{\theta}\}=\begin{Bmatrix} \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{Bmatrix}$ is the (angular) acceleration vector & $\{\Gamma\}=\begin{Bmatrix} \Gamma_1(t) \\ \Gamma_2(t) \end{Bmatrix}$ is the torque vector. $\longrightarrow$

For free vibration, the DEOM are:

$$I_1\ddot{\theta}_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0 \quad \text{--- (2)}$$

$$\& \quad I_2\ddot{\theta}_2 - k_2\theta_1 + k_2\theta_2 = 0 \quad \text{--- (3)}$$

Assume
$$\left.\begin{aligned}\theta_1 &= \Theta_1 \sin(\omega t+\phi)\\ \theta_2 &= \Theta_2 \sin(\omega t+\phi)\end{aligned}\right\} \Rightarrow \begin{aligned}\ddot{\theta}_1 &= -\omega^2\Theta_1\sin(\omega t+\phi)\\ \ddot{\theta}_2 &= -\omega^2\Theta_2\sin(\omega t+\phi)\end{aligned}$$

Substitution in (2) & (3) results in

$$\left.\begin{aligned}\left[(k_1+k_2) - I_1\omega^2\right]\Theta_1 - k_2\Theta_2 &= 0\\ \& \quad -k_2\Theta_1 + (k_2 - I_2\omega^2)\Theta_2 &= 0\end{aligned}\right\} \text{The amplitude equations}$$

For non-trivial Θ_1 & Θ_2,

$$\begin{vmatrix}(k_1+k_2)-I_1\omega^2 & -k_2\\ -k_2 & k_2-I_2\omega^2\end{vmatrix} = 0, \text{ giving the frequency equation.}$$

It's solution gives ω_1 & ω_2 as before.

The modal vectors are:-

$$\{\Theta\}_1 = \begin{Bmatrix}\Theta_{11}\\ \mu_1\Theta_{11}\end{Bmatrix} \ \& \ \{\Theta\}_2 = \begin{Bmatrix}\Theta_{12}\\ \mu_2\Theta_{12}\end{Bmatrix}$$

& the normalized modal vectors are:

$$\{\Theta\}_1 = \begin{Bmatrix}1\\ \mu_1\end{Bmatrix} \ \& \ \{\Theta\}_2 = \begin{Bmatrix}1\\ \mu_2\end{Bmatrix} \text{ where}$$

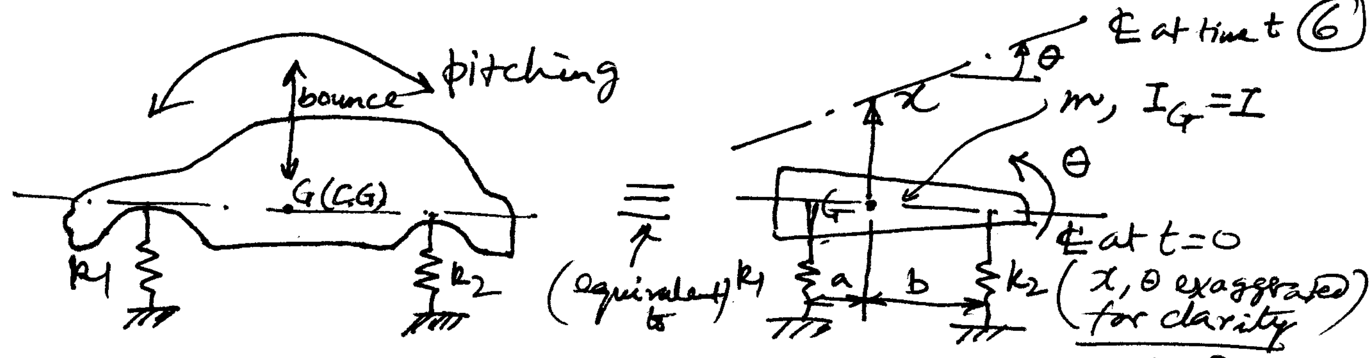
μ_1 & μ_2 are defined as in a translational system.

<u>An important example</u>:- In this example of a 2-DOF system, both translational <u>and rotational</u> vibrations are involved.

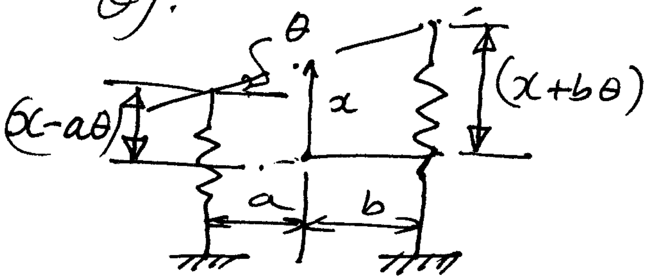
Here a 2-DOF model of a car is considered to study its bounce (up & down) & pitching motion (angular motion about a horizontal axis).

~~all~~ We consider free-vibration only.

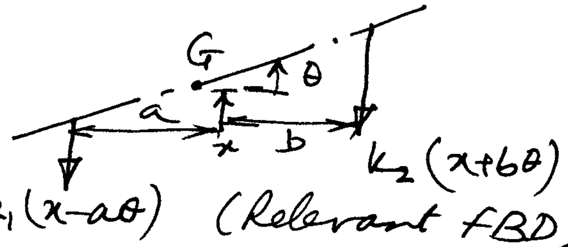
→



Here the assumptions are that the C.G. of the car moves up & down (denoted by generalized coordinate x) and the carbody rotates about a horizontal axis through G (rotation denoted by generalized coordinate θ). (x, θ are both actually small)



(Dimensions arbitrary for illustration)



Let m = mass of car
I = Moment of inertia about an appropriate axis through G.

So, $m\ddot{x} = -k_1(x-a\theta) - k_2(x+b\theta)$

& $I\ddot{\theta} = +k_1(x-a\theta)a - k_2(x+b\theta)b$

$\Rightarrow \quad m\ddot{x} + (k_1+k_2)x - (k_1 a - k_2 b)\theta = 0 \quad -(1)$

& $\quad I\ddot{\theta} - (k_1 a - k_2 b)x + (k_1 a^2 + k_2 b^2)\theta = 0 \quad -(2)$

(1) & (2) are the required DEOM.

In matrix form, these can be written as:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -(k_1 a - k_2 b) \\ -(k_1 a - k_2 b) & k_1 a^2 + k_2 b^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

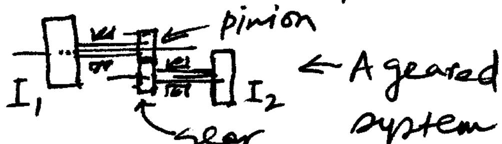
Check → For finding $x(t)$ & $\theta(t)$, we assume

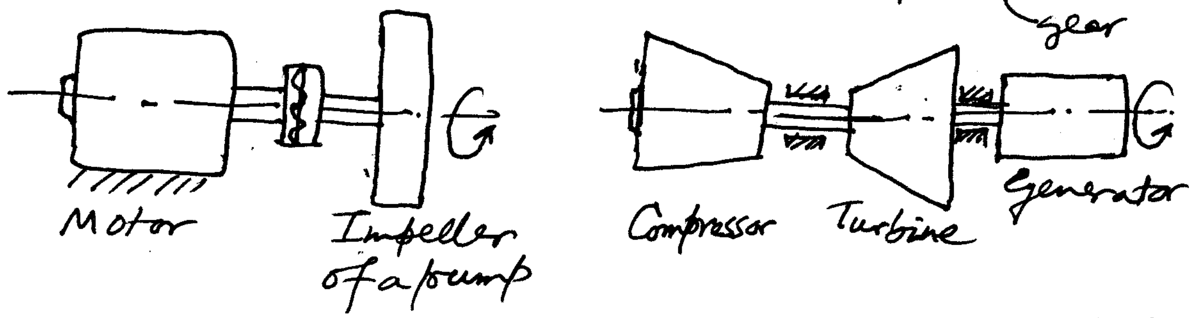
$$\left.\begin{aligned} x &= X \sin(\omega t+\phi) \\ \& \ \theta &= \Theta \sin(\omega t+\phi) \end{aligned}\right\}$$

& proceed as in the earlier cases.

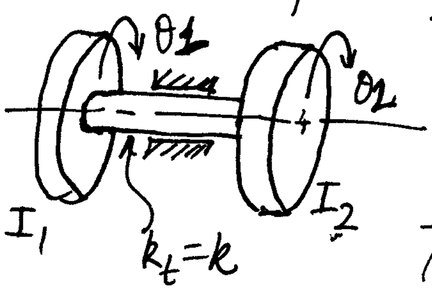
Example (from S.S. Rao, Mechanical Vibrations, 6th Ed, Pg 534)
Study & do example 5.7.

(8) **Semidefinite (free-free) systems:** — These are also known as ~~semidefinite~~ free-free systems and are very important in mechanical engineering. 3 ~~Two~~ examples are shown below & here →





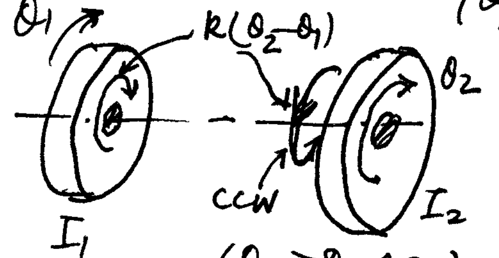
→ We consider the following simplified model for free-vibration studies:



$\theta_1(t)$ & $\theta_2(t)$ are the generalized coordinates which have zero values at static equilibrium. To visualize free vibration of this system, you may assume that I_1 is held fixed, I_2 is given an initial twist $\theta_2(0)$ & then both discs are released. The system would execute torsional free vibration. We, as usual, assume small amplitude oscillations. However, here there is an interesting feature of this problem: it has a zero natural frequency as you will see soon. →

→ <u>The DEOM</u> :– (Moment balance method)

(θ_1, θ_2 +ive CW as seen from the right)



($\theta_2 > \theta_1$, say)

$$I_1 \ddot{\theta}_1 = k(\theta_2 - \theta_1)$$

$$\& \quad I_2 \ddot{\theta}_2 = -k(\theta_2 - \theta_1)$$

So,

$$I_1 \ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0 \quad \text{—} (1)$$

$$I_2 \ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0 \quad \text{—} (2)$$

$[I] = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$, $[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

(1) & (2) are the reqd DEOM.

→ It is interesting to note that by adding (1) & (2), we get $I_1 \ddot{\theta}_1 + I_2 \ddot{\theta}_2 = 0$ & integrating this equation, we get $I_1 \dot{\theta}_1 + I_2 \dot{\theta}_2 = \text{Constant}$, which is the conservation of angular momentum relation for our free-free system.

→ To get $\theta_1(t)$ & $\theta_2(t)$, we assume

$$\theta_1 = \Theta_1 \sin(\omega t + \phi), \quad \theta_2 = \Theta_2 \sin(\omega t + \phi).$$

Then substitution in (1) & (2) gives:

$$(k - I_1\omega^2)\Theta_1 - k\Theta_2 = 0 \quad \text{—} (3)$$

$$\& \quad -k\Theta_1 + (k - I_2\omega^2)\Theta_2 = 0 \quad \text{—} (4)$$

So, frequency eqn is

$$\begin{vmatrix} k - I_1\omega^2 & -k \\ -k & k - I_2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow k^2 - k(I_1 + I_2)\omega^2 + I_1 I_2 \omega^4 - k^2 = 0$$

$$\Rightarrow \omega^2 \left[I_1 I_2 \omega^2 - k(I_1 + I_2) \right] = 0$$

So, $\omega_1 = 0$, $\omega_2 = \sqrt{\dfrac{k(I_1 + I_2)}{I_1 I_2}}$.

This zero natural frequency actually means rigid body motion is possible, that is, our disc-shaft system can rotate as a single rigid body under proper initial conditions. This set up can also execute torsional oscillations at ω_2.

→ To get Θ_1 & Θ_2 for first principal mode, we substitute $\omega_1 = 0$ in (3) or (4) & get $\mu_1 = \dfrac{\Theta_{21}}{\Theta_{11}} = \dfrac{k}{k} = 1$

~~Hence~~ $\{\Theta\}_1$ = modal vector for first pr. mode

$$= \begin{Bmatrix} \Theta_{11} \\ \Theta_{11} \end{Bmatrix} \leftarrow \text{arbitrary}$$

The normalized

$$\{\Theta\}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \text{ (Remember).}$$

for $\omega = \omega_2$, (3) gives $\dfrac{\Theta_{22}}{\Theta_{12}} = \dfrac{k - I_1\omega_2^2}{k}$

or, $$\mu_2 = \frac{\Theta_{22}}{\Theta_{12}} = \frac{k - \dfrac{k(I_1+I_2)}{I_2}}{k} = -\frac{I_1}{I_2}$$

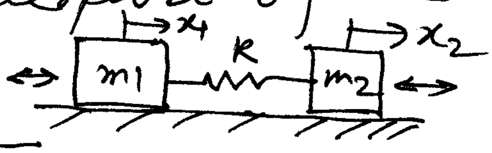
Hence, $$\{\Theta\}_2 = \begin{Bmatrix} \Theta_{12} \\ -\dfrac{I_1}{I_2}\Theta_{12} \end{Bmatrix} = \begin{Bmatrix} \Theta_{12} \\ \mu_2 \Theta_{12} \end{Bmatrix}, \ \Theta_{12} \text{ arbitrary.}$$

$$\left(\{\Theta\}_2\right)_{\text{normalized}} = \begin{Bmatrix} 1 \\ \mu_2 \end{Bmatrix}.$$

→ One interesting thing to note about the stiffness matrix ~~(k_1+k_2)~~ $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = [k]$ is that $\det[k] = k^2 - k^2 = \underline{\underline{0}}$.

This shows that $[k]$ is a semidefinite matrix & so our system is semidefinite.

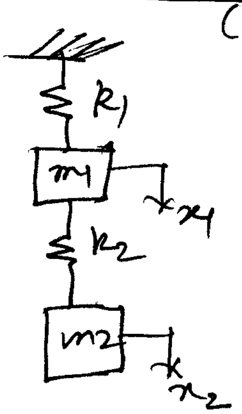
→ The translational counterpart of the semidefinite system is:



(8) To derive the DEOM for all the systems taken up so far by using Lagrange's Equns

(Free vibration only)

(1)



There are 2 Lagrange equns for a 2-DOF system. Here x_1 & x_2 are the generalized coordinates. The Lagrange equns are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0 \quad \text{--- (1)}$$

& $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = 0 \quad \text{--- (2)}$$

Now, $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

$V = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$

So, $\frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$, $\frac{\partial T}{\partial x_1} = 0$,

$\frac{\partial V}{\partial x_1} = k_1 x_1 - k_2(x_2 - x_1) = (k_1 + k_2)x_1 - k_2 x_2$

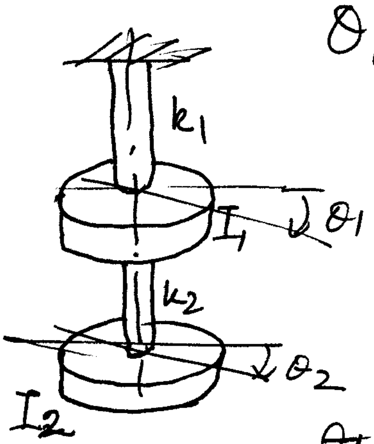
So, from (1), $m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$ (1st DEOM)

Also, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2$, $\frac{\partial T}{\partial x_2} = 0$, $\frac{\partial V}{\partial x_2} = k_2(x_2 - x_1)$

Hence, from (2), we get $m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$, which is the 2nd DEOM.

→

(2)



$\theta_1, \theta_2 \rightarrow$ The generalized coordinates

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$V = \frac{1}{2} k_1 \theta_1^2 + \frac{1}{2} k_2 (\theta_2 - \theta_1)^2$$

The Lagrange equations in this case are (for free-vibration):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0 \quad \text{--- (1)}$$

& $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0 \quad \text{--- (2)}$$

So, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = I_1 \ddot{\theta}_1$, $\frac{\partial T}{\partial \theta_1} = 0$, $\frac{\partial V}{\partial \theta_1} = k_1\theta_1 - k_2(\theta_2 - \theta_1)$

From (1), $$I_1 \ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0 \quad \text{--- (3)}$$

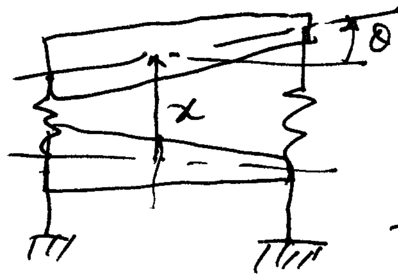
Also, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = I_2 \ddot{\theta}_2$, $\frac{\partial T}{\partial \theta_2} = 0$, $\frac{\partial V}{\partial \theta_2} = k_2(\theta_2 - \theta_1)$

From (2), $$I_2 \ddot{\theta}_2 - k_2\theta_1 + k_2\theta_2 = 0 \quad \text{--- (4)}$$

(3) & (4) are the required DEOM.

(3)

(See figs. for earlier method)



x, θ are the generalized coordinates.

The Lagrange eqns for free vibration are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \quad \text{--- (1)}$$

& $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad \text{--- (2)}$$

Here $$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 (x - a\theta)^2 + \frac{1}{2} k_2 (x + b\theta)^2$$

Hence, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$, $\frac{\partial T}{\partial x} = 0$, $\frac{\partial V}{\partial x} = k_1(x-a\theta)+k_2(x+b\theta)$

$$= (k_1+k_2)x-(k_1a-k_2b)\theta$$

Thus, from ①, we get

$$m\ddot{x} + (k_1+k_2)x - (k_1a-k_2b)\theta = 0 \quad \text{--- (3)}$$

Again, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I\ddot{\theta}$; $\frac{\partial T}{\partial \theta} = 0$, $\frac{\partial V}{\partial \theta} = -k_1a(x-a\theta) + k_2b(x+b\theta)$

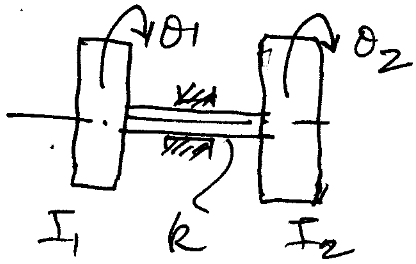
or, $\frac{\partial V}{\partial \theta} = -(k_1a-k_2b)x + (k_1a^2+k_2b^2)\theta.$

So, from ②, we get

$$I\ddot{\theta} - (k_1a-k_2b)x + (k_1a^2+k_2b^2)\theta = 0 \quad \text{--- (4)}$$

③ & ④ are the reqd DEOM.

(4)



θ_1, θ_2 are the generalized coordinates.

The Lagrange eqns are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0 \quad \text{--- (1)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0 \quad \text{--- (2)}$$

$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$, $V = \frac{1}{2}k(\theta_2-\theta_1)^2$.

Differentiations will lead to the same DEOM as before.

Complete the derivation.

End of VA-4, Part 3