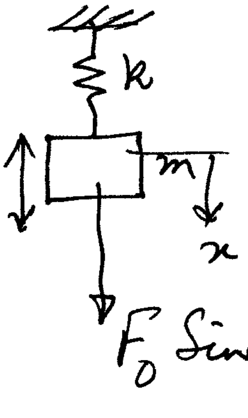
VA – Miscellaneous 1 ① @

§ The resonance response of the undamped single DOF system:–



The DEOM is : $m\ddot{x} + kx = F_0 \sin\omega_n t$ -- (1)

$\Rightarrow (mD^2 + k)x = F_0 \sin\omega_n t$; $D \equiv \frac{d}{dt}, D^2 \equiv \frac{d^2}{dt^2}$

$\Rightarrow (D^2 + \omega_n^2)x = \frac{F_0}{m} \sin\omega_n t$ (For particular integral)

$\Rightarrow x = \frac{F_0}{m} \frac{1}{(D^2+\omega_n^2)} (\sin\omega_n t)$ -- (2)

We can't substitute $D^2 = -\omega_n^2$ to get x. One way is to start with

$$(D^2 + \omega_n^2)x = \frac{F_0}{m} e^{i\omega_n t} \quad \text{-- (3)} \quad (i = \sqrt{-1})$$

$$= \frac{F_0}{m} [\cos\omega_n t + i \sin\omega_n t] \quad \text{-- (4)}$$

& solve for x. That way, we get the forced response for both excitations $F_0 \sin\omega_n t$ & $F_0 \cos\omega_n t$ by ~~equating~~ taking, respectively, the real & imaginary parts of the solution. $\boxed{\text{Check}}$

From (3), $x = \frac{F_0}{m} \cdot \frac{1}{(D+i\omega_n)(D-i\omega_n)} [e^{i\omega_n t}]$

$$= \frac{F_0}{m} \cdot \frac{1}{(2i\omega_n)} \cdot \left[\frac{1}{(D-i\omega_n)} (e^{i\omega_n t}) \right]$$

$$= \frac{-F_0 i}{2m\omega_n} \left[e^{i\omega_n t} \int e^{-i\omega_n t} \cdot e^{i\omega_n t} dt \right]$$

$$= -\frac{F_0 i}{2m\omega_n} \left[e^{i\omega_n t} \int dt \right]$$

$$= -\frac{F_0 i t}{2m\omega_n} (\cos\omega_n t + i \sin\omega_n t), \quad (5)$$

$$= + \frac{F_0 t}{2m\omega_n} [\sin\omega_n t] - i \left[\frac{F_0 t}{2m\omega_n} \cos\omega_n t \right]$$

Do you remember the solution of $f(D+P)x = Q$ or, $\frac{dx}{dt} + P(t)x = Q(t)$?
The integrating factor is $e^{+\int P dt}$ &

$d\{x e^{+\int P dt}\} = e^{\int P dt} Q(t) dt$

$\Rightarrow x = e^{-\int P dt} \int e^{\int P dt} Q(t) dt$

Here, $P = -i\omega_n$, $Q = e^{i\omega_n t}$

So, $\frac{1}{D - i\omega_n} e^{i\omega_n t}$ = etc

(2) (b)

From (4) & above expression (5), it should be clear that when $m\ddot{x}+kx=F_0\sin\omega_n t$,

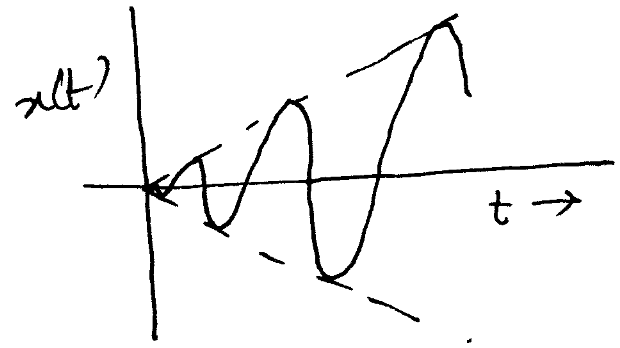
$$x_p = \boxed{x(t) = -\frac{F_0 t}{2m\omega_n}\cos\omega_n t} \quad \underline{\text{Remember}}$$

(The imaginary part of solution (5))

Also, if $m\ddot{x}+kx = F_0\cos\omega_n t$, then,

$$x(t) = \frac{F_0 t}{2m\omega_n}\sin\omega_n t.$$

Hence, at $\omega_f=\omega_n$, the forced response (at resonance) <u>doesn't</u> rise to a high value in an instant. The response grows as per above formulae and the system becomes nonlinear and behave differently and eventually may breakup. The response looks like:



You may also note the following interesting way to do it:-

$$x = \frac{F_0}{m}\left(\frac{1}{D^2+\omega_n^2}\right)\sin\omega_n t$$

$$= \frac{F_0}{m}\cdot\frac{\int 1.\,dt}{\frac{d}{dD}(D^2+\omega_n^2)}\sin\omega_n t$$

$$= \frac{F_0}{m}\cdot\frac{t}{2D}\sin\omega_n t$$

$$= \frac{F_0 t}{2m}\cdot\int\sin\omega_n t\,dt \quad \left[\because \frac{1}{D} \text{ means integration}\right.$$

$$= -\frac{F_0 t}{2m\omega_n}\cos\omega_n t, \text{ the required resonance response.}$$

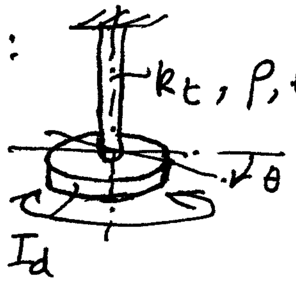
Also, there is a 3rd way: Write the complete solution $x=x_c+x_p$ for excitation $F_0\sin\omega_f t$ & then let $\omega_f\to\omega_n$ etc.

See S.S. Rao's book

Imp:- Actually, you need to remember the final formulae. The derivations are for sake of completeness

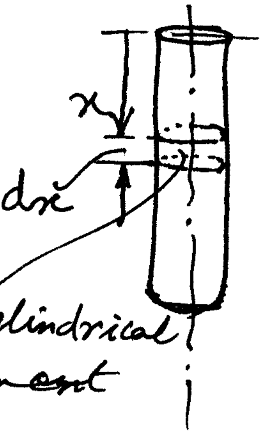
Q. What happens when the spring or shaft inertia is significant? How is the ω_n affected?

We shall takeup the torsional system:



ρ = mass density of shaft material
l = shaft length
$A = \pi R^2$ = Cross-sectional area of shaft
R = shaft radius.

→ What we want to do is compute the shaft kinetic energy and include it in the analysis.



Let us consider an element of the shaft at distance x from the top support with x varying from 0 to l. Clearly, the rotation of this element is $\frac{x}{l}\theta$ where θ is the rotation of the disc at $x=l$. Hence, angular velocity of the element is $\frac{x}{l}\dot{\theta}$.

So, Kinetic energy of this element

$$dT_s = \frac{1}{2}(dI_s)\left(\frac{x}{l}\dot{\theta}\right)^2$$

's' for Shaft

$$= \frac{1}{4}\frac{\pi\rho R^4\dot{\theta}^2}{l^2}x^2 dx$$

$\left[\right.$ dI_s = Moment of inertia of element about its axis $= \frac{1}{2}\times(\text{mass of element})\times R^2$

$= \frac{1}{2}\times\rho\times\pi R^2 dx \times R^2$

$= \frac{1}{2}\rho\pi R^4 dx$

$\therefore$ KE of shaft $= \int dT_s$

$$= \frac{1}{4}\frac{\pi\rho R^4\dot{\theta}^2}{l^2}\int_0^l x^2 dx = \frac{1}{4}\cdot\frac{\pi\rho R^4\dot{\theta}^2}{l^2}\times\frac{l^3}{3} = \cancel{\frac{1}{12}\times\pi\times}$$ →

or, $T_s = \frac{1}{12}(\rho\pi R^2 l)R^2\dot{\theta}^2 = \frac{1}{12} m_s R^2 \dot{\theta}^2$ [m_s = mass of shaft $= \rho\pi R^2 l$]

$= \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}m_s R^2\right)\dot{\theta}^2 = \frac{1}{2}\times\frac{1}{3}I_s\dot{\theta}^2$, where

$I_s = \frac{1}{2}m_s R^2$ is the moment of inertia of the whole shaft about its own axis.

→ Hence, only a third of the moment of inertia of the shaft comes into play!

Now, T = KE of system $= \frac{1}{2}I_d\dot{\theta}^2 + \frac{1}{2}\times\frac{1}{3}I_s\dot{\theta}^2$

$= \frac{1}{2}\left(I_d + \frac{1}{3}I_s\right)\dot{\theta}^2$,

$V = PE = \frac{1}{2}k_t\theta^2$.

These give the DEOM as:

$\left(I_d + \frac{1}{3}I_s\right)\ddot{\theta} + k_t\theta = 0$ & hence,

$(\omega_n)_m$ = modified natural frequency $= \sqrt{\dfrac{k_t}{I_d + \frac{1}{3}I_s}}$.

So, apparently, neglecting the inertia of the shaft, we get a very inaccurate value of ω_n. In practice, this is not so in many-a-situation. ~~But the shaft~~

For instance, let us consider a situation when $I_s = \frac{1}{3}I_d$ (Remember that although the shaft is long, its elements are quite near the axis of rotation resulting in a low value of moment of inertia compared to that of the disc). In this situation,

→

$$(\omega_n)_m = \sqrt{\frac{k_t}{(1+\frac{1}{9})I_d}} = \sqrt{\frac{9k_t}{10 I_d}} = 0.9487\sqrt{\frac{k_t}{I_d}} \cancel{= 0.987}$$

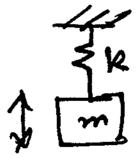
$$= 0.9487\,\omega_n$$

Hence, $(\omega_n)_m < \omega_n$ and taking $(\omega_n)_m$ to be more accurate than ω_n, the % error incurred in taking ω_n as the natural frequency is:

$$\%\ \text{error} = \left[\frac{\omega_n - (\omega_n)_m}{(\omega_n)_m}\right] \times 100 = \frac{(1-0.9487)}{0.9487}\times 100$$

$$= 5.41 \text{ only.}$$

In real situations, I_s is still much less & hence % error would be smaller. Thus, taking $\omega_n = \sqrt{\frac{k_t}{I_d}}$ is ok in many situations.

→ Note that in  , $(\omega_n)_m = \sqrt{\frac{k}{m+\frac{1}{3}m_s}}$, where m_s = mass of spring. Establish this.

x

→ In passing, note that when the above system is considered to be a continuous system having an infinitely many DOF, a more accurate method of evaluating the 'fundamental' natural frequency will evolve. You will study it later and compare the result obtained with the approximate values obtained above.

---

(8) A brief discussion on linearity (& nonlinearity) of differential equations (DEs)

This is an important topic since the methods (of solving) of linear DEs don't usually apply to the nonlinear DEs. So, it is very important that you are able to identify one from the other. In general, if we have a DE: $L(x)=0$ where L is a differential operator such as $m\frac{d^2}{dt^2}+c\frac{d}{dt}+k$, then if

$$L(c_1x_1+c_2x_2)=c_1L(x_1)+c_2L(x_2),$$

our DE would be linear. (c_1, c_2 constants, $x_1=x_1(t)$, $x_2=x_2(t)$)

Examples:- (i) $m\ddot{x}+c\dot{x}+kx=0$ --- (1)

Here $L=m\frac{d^2}{dt^2}+c\frac{d}{dt}+k$

$$\therefore L(c_1x_1+c_2x_2)=m\frac{d^2(c_1x_1+c_2x_2)}{dt^2}+c\frac{d(c_1x_1+c_2x_2)}{dt}+k(c_1x_1+c_2x_2)$$

$$=c_1\left(m\frac{d^2x_1}{dt^2}+c\frac{dx_1}{dt}+kx_1\right)+c_2\left(m\frac{d^2x_2}{dt^2}+c\frac{dx_2}{dt}+kx_2\right)$$

$$=c_1L(x_1)+c_2L(x_2)$$

& hence, (1) is linear.

(ii) $\ddot{x}+(\alpha+\beta\sin\omega t)x=0$ --- (2) [α, β, ω constants]

This is the famous Mathieu equation which arises in numerous situations in Mechanical Engineering (& also Electrical Engg. etc.). It can't be solved by ordinary means! Is it linear or nonlinear? Let us test. Here $L \equiv \frac{d^2}{dt^2} + (\alpha + \beta \sin\omega t)$

So, $L(c_1 x_1 + c_2 x_2) = \frac{d^2}{dt^2}(c_1 x_1 + c_2 x_2)$
$+ (\alpha + \beta \sin\omega t)(c_1 x_1 + c_2 x_2)$

$$\cancel{= c_1 \frac{d^2}{dt^2}(} = c_1\left[\frac{d^2 x_1}{dt^2} + (\alpha + \beta \sin\omega t) x_1\right]$$
$$+ c_2\left[\frac{d^2 x_2}{dt^2} + (\alpha + \beta \sin\omega t) x_2\right]$$
$$= c_1 L(x_1) + c_2 L(x_2)$$

& hence, (2) is linear! (Although it has a variable coefficient $(\alpha + \beta \sin\omega t)$, but this is a function of the independent variable, note.

(iii) We next consider the famous Duffing equation: $m\ddot{x} + k_1 x + k_2 x^3 = 0$ --- (3)

So, if you visualize a spring-mass system with a nonlinear spring, then (3) could be its DEOM. Where else do we find such a DEOM? If, in the nonlinear simple pendulum DEOM $ml^2\ddot{\theta} + mgl\sin\theta = 0$, you put $\sin\theta \simeq \theta - \frac{\theta^3}{3!}$, you get a similar DEOM!

Let us check whether (3) is linear.

Here $L(x) \equiv m\frac{d^2 x}{dt^2} + k_1 x + k_2 x^3$

& so, $L(c_1 x_1 + c_2 x_2) = m\frac{d^2}{dt^2}(c_1 x_1 + c_2 x_2) + k_1(c_1 x_1 + c_2 x_2)$
$+ k_2(c_1 x_1 + c_2 x_2)^3 = c_1\left(m\frac{d^2 x_1}{dt^2} + k_1 x_1 + k_2 c_1 x^3\right) + \text{etc.}$

& so, (3) is nonlinear. $\longrightarrow$

(iv) Similarly, the following famous DEs are all linear: ($y=y(x)$ here)

→ $x^2y''+xy'+(x^2-p^2)y=0$ (Bessel's equation in normal form) (p = constant)

→ $(1-x^2)y''-2xy'+p(p+1)y=0$ (Legendre's equn)

→ $y''+xy=0$ (Airy's equation whose solutions are called Airy functions)

→ $(1-x^2)y''-xy'+p^2y=0$ (Chebyshev's equn.)

→ $y''-2xy'+2py=0$ (Hermite's equn).

$\left(y'=\frac{dy}{dx};\ y''=\frac{d^2y}{dx^2} \right)$

You don't need to rember these DEs but you should remember the names.

(v) The following partial DE is the DEOM for axial vibrations of a thin straight bar:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad \text{---(4)}$$

$u(x,t)$ is the axial deflection of the bar at location x & at time t. $c=\sqrt{E/\rho}$ is the speed of longitudinal elastic waves in the bar, $E\to$ Young's modulus, ρ = mass density of bar material. 4 can be put as:

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \quad \text{---(4')}$$

Replacing u by $c_1u_1+c_2u_2$ in (4'), LHS, we get,

$$L\equiv\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}$$

$$L(c_1u_1+c_2u_2)=\frac{\partial^2}{\partial t^2}(c_1u_1+c_2u_2)-c^2\frac{\partial^2}{\partial x^2}(c_1u_1+c_2u_2)$$

$$=c_1\left(\frac{\partial^2 u_1}{\partial t^2}-c^2\frac{\partial^2 u_1}{\partial x^2}\right)+c_2\left(\frac{\partial^2 u_2}{\partial t^2}-c^2\frac{\partial^2 u_2}{\partial t^2}\right)$$

$$=c_1L(u_1)+c_2L(u_2).$$ Hence, (4) is linear.

→

For your purpose, usually, a look at the DEOM should tell you, whether it is linear or nonlinear.

If there is a term which contains a product of a dependent variable and any of its derivatives, the DEOM is nonlinear. If it contains a term in which a dependent variable or any of its derivatives is raised to any power other than unity, then too it is nonlinear.

Examples:– → $m\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 = 0$ is nonlinear since the dependent variable is raised to power 3 in one of the terms.

→ $\ddot{x} + (\alpha + \beta \sin\omega t)x = 0$ is linear since $(\alpha + \beta \sin\omega t)$ is a function of independent variable t.

→ $ml^2\ddot{\theta} + mgl \sin\theta = 0$ is nonlinear since $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$ & so, the dependent variable θ is raised to a power other than unity.

Why is it important to know if the DEOM you are handling linear or nonlinear? You may argue that you are not interested in knowing the nature of the DEOM because you have a powerful software (MATLAB, Mathcad, Maple, Mathematica) and will simulate your equation & give you the results you need.

Well, don't be too sure you will be able to interpret the result if you are not aware whether your system is linear or nonlinear. Nonlinear systems are peculiar. They exhibit phenomena which are not heard of in the linear systems behaviour, such as the limit cycle, the jump phenomenon, the combination resonance of sum type & difference type of various orders, subharmonic resonance, superharmonic resonance, internal resonance & so on.

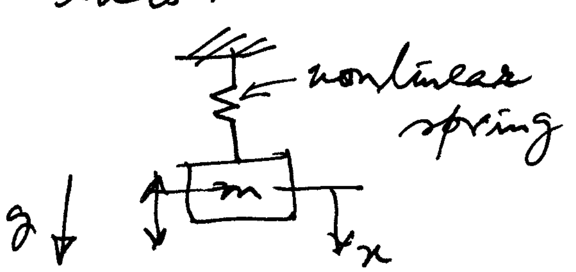
You can learn about these in a course on Nonlinear Vibrations.

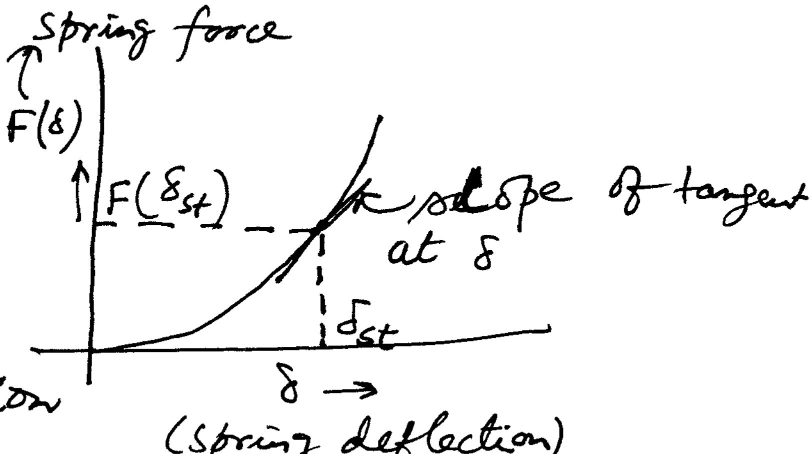
Anyway, I may make a fool of myself ~~yourself~~ to my boss if I try to solve a nonlinear DEOM by the means (methods) of solving linear DEOM (except for the first order equations, sometimes).

→ A few last words about linearizing

a DEOM. You already have seen that taking $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$ etc. helps. (Never linearize before differentiating, remember?)

→ Sometimes, you may linearize a nonlinear spring near the static equilibrium point. The idea is like this :-





The force vs deflection curve for the spring is as shown, say. Also $mg = F(\delta_{st})$.

We are interested in the free vibration response of the block about its static equilibrium position for small amplitude vibration. We can, as a first approximation, take the slope of the $F(\delta)$ vs δ curve at $\delta = \delta_{st}$ & use it as a linear spring stiffness & then solve the problem. So, we take $k = \left.\frac{dF}{d\delta}\right|_{\delta=\delta_{st}}$. Of course, this may introduce considerable error in some situations.

→ See & do Example 1.2, pg. 56 (Mechanical Vibrations, S.S. Rao, 6th Ed.)

(5) A few words(!) about Degrees of Freedom:-
(& constraints)

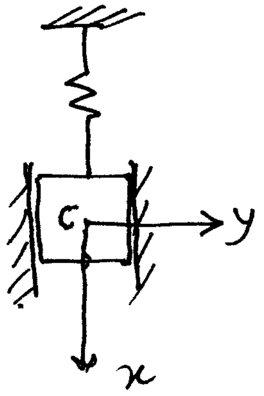
Definition:- The number of degrees of freedom (DOF) of a holonomic dynamic system is equal to the (minimum) ~~number~~ number of independent geometric coordinates required to specify the configuration of the system.

[A holonomic dynamic system is one that is subjected to constraints ~~each~~ which can be described by integrable differential equations. For example, the systems we consider in our course are all holonomic. However, the world is full of non-holonomic systems. Examples are: the man, the car, the bicycle, all other moving creatures of nature. Thus, basically, the world is non-holonomic. Only in man made machines & other systems, you find holonomicity. For example, the Lathe or milling m/c on a shop floor ~~are~~ are holonomic. But all these holonomic systems are very important to us. Hence, we study the dynamics of holonomic systems. To learn more about →

non-holonomic systems, see ~~the~~ 'Lectures in analytical mechanics' by F. Gantmacher]

We get back to our definition of DOF and consider a few examples to illustrate.

Ex. 1



Our simple spring-mass system requires only one geometric coordinate such as x, to know where the block is at any time t. This is so because the block is properly constrained. Look at the 'walls' we have drawn. There are similar 'walls' on the front & the back faces. Actually what we have drawn above is the mid plane of the block, half of it, imagine, is above the page (or, in front of the plane of the blackboard) & the rest half is ~~behind or~~ below the page (behind the blackboard).

~~If~~ The reference frame xyz is attached at the centre of mass location in static equilibrium & it doesn't move with the block. Then, if (x, y, z) are the coordinates of C at time t, then, clearly, $y=0$ & $z=0$ at all times.

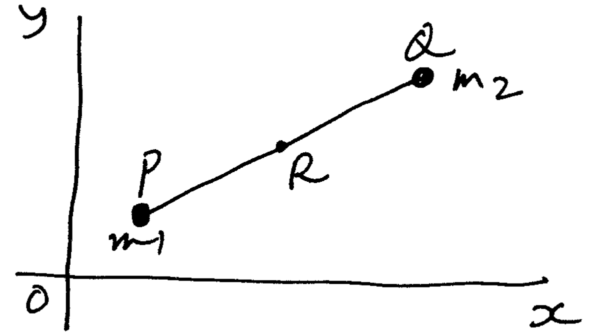
Now, $y=0$ & $z=0$ are two constraint equations our system is subjected to. →

These two constraints are in analytical geometrical form. These can also be represented as two differential equations, viz., $\dot{y}=0$ & $\dot{z}=0$ & these are the so called 'differential' constraints our system is subjected to, provided, after integration, i.e., after we get $y=c_1$ & $z=c_2$ after integrating $\dot{y}=0$ & $\dot{z}=0$, we take $c_1=0$ & $c_2=0$. Since these differential constraints are integrable, our system is holonomic.

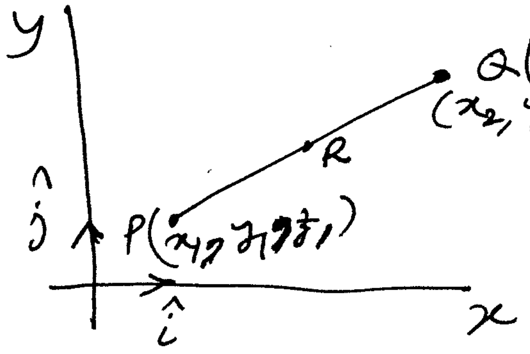
Important → You must not think all differential equations are integrable.

The following interesting example system is subjected to a differential constraint which is non-integrable.

Example:– PQ is a rigid bar with particle masses m_1 & m_2 at its two ends. It is allowed to move in xy-plane in such a way that the velocity of the mid point R is always in the direction of the bar.



(This system is supposed to model the motion of a skate on a plane, see Gantmacher's book) →



The question is: What are the constraint equations for this system?

Obviously, $z_1 = 0$ & $z_2 = 0$ are two such constraint equations. There is another: velocity of $R = \bar{V}_R = \alpha \overline{PQ}$ (α = Constant), by our requirement.

But R has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, 0\right)$.

Hence, $$\bar{V}_R = \frac{(\dot{x}_1+\dot{x}_2)}{2}\hat{i} + \frac{(\dot{y}_1+\dot{y}_2)}{2}\hat{j}$$

Also, $\overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$. Hence,

$$\frac{(\dot{x}_1+\dot{x}_2)}{2}\hat{i} + \frac{(\dot{y}_1+\dot{y}_2)}{2}\hat{j} = \alpha(x_2-x_1)\hat{i} + \alpha(y_2-y_1)\hat{j}$$

Thus, $$\frac{\dot{x}_1+\dot{x}_2}{2} = \alpha(x_2-x_1) \quad \&$$
$$\frac{\dot{y}_1+\dot{y}_2}{2} = \alpha(y_2-y_1)$$

So, $$\frac{\dot{x}_1+\dot{x}_2}{\dot{y}_1+\dot{y}_2} = \frac{x_2-x_1}{y_2-y_1}$$

or, $(\dot{x}_1+\dot{x}_2)(y_2-y_1) = (\dot{y}_1+\dot{y}_2)(x_2-x_1)$ & this is another (differential) constraint which is <u>non-integrable</u>!

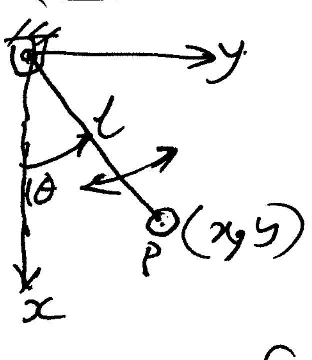
(How do we prove it?)

So, our system is subject to a non-holonomic constraint & our system is non-holonomic! →

→ We go back to holonomic systems.

→ By the way; we know the 'configuration' of a system if we know the location of each of the particles that form the system.

Ex. The simple pendulum :-

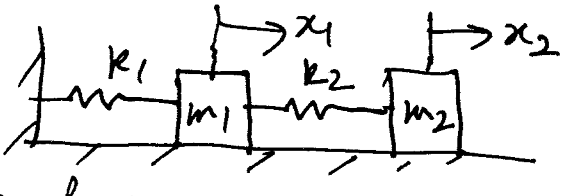


Apparently, two geometric coordinates (x, y) are required to specify the configuration of the system at any time t.

So, if you confidently say the system has 2-DOF, will you be right?

No. Because, x & y are not independent, i.e., they cannot vary independently, since, $x^2+y^2=l^2$ at all times. Hence, we can take either x or y as the generalized coordinate and the system has only one DOF. However, neither x nor y is a very convenient gen. coordinate for this problem. θ is much better.

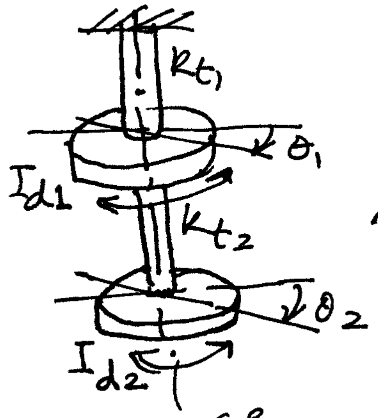
Ex.



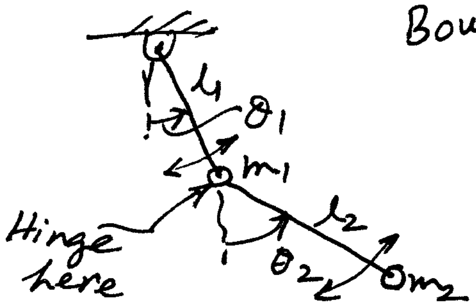
← This system has 2-DOF &

x_1 & x_2, measured from static equilibrium position, may serve as the generalized coordinates. x_1 & x_2 are independent of each other, for instance, we could hold x_1 at a particular value* & vary x_2 (quite) arbitrarily.

Ex. of more 2-DOF systems:

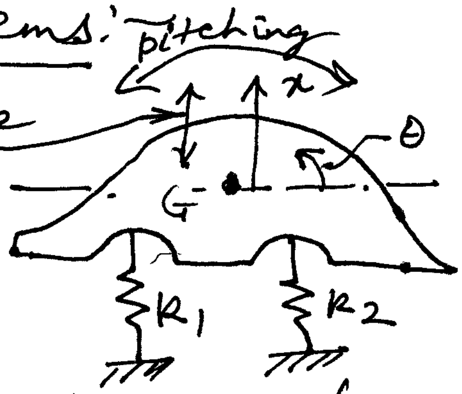


$\theta_1, \theta_2 \to$ gen. coordinates



The double-pendulum

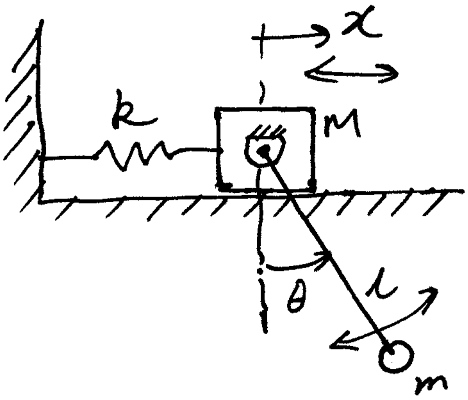
$\theta_1, \theta_2 \to$ gen. coordinates



The car for bounce & pitching motion.

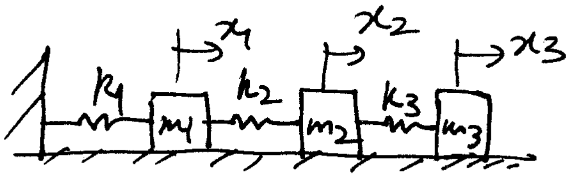
$(x, \theta) \to$ gen. coordinates

[The carbody looks more like a tortoise, sorry!]

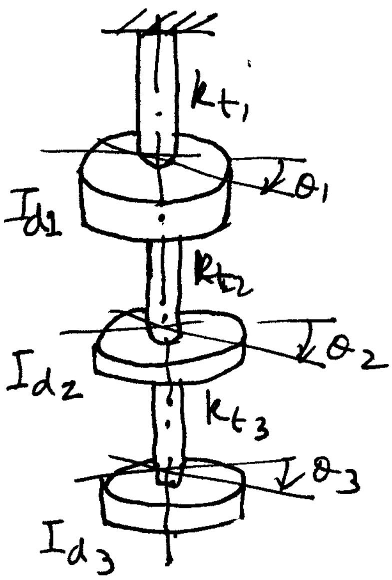


x & θ are the generalized coordinates

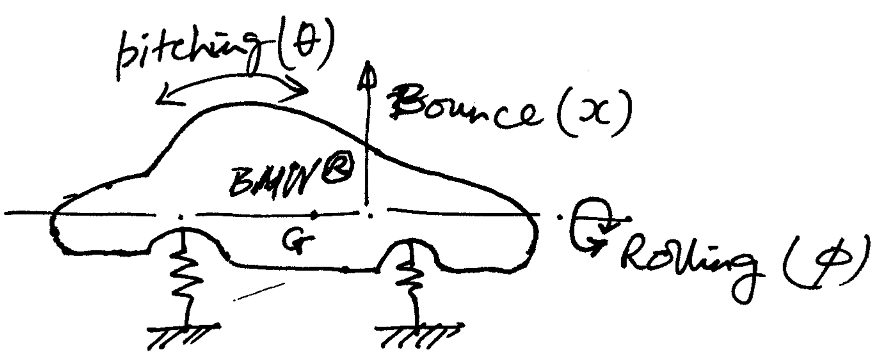
Ex. of 3-DOF systems:-



$x_1, x_2, x_3 \to$ gen. coords.



Gen. coords:- $\theta_1, \theta_2, \theta_3$



$x, \theta, \phi \to$ gen. coords.

[If you include fore-and-aft motion, lateral motion & yaw (rotation about a vertical axis), then DOF becomes 6]

Let us get back to the system

If we treat the block as a particle, it requires $\underline{3}$ geometric coordinates x, y, z to locate it in space. However, we have $\underline{2}$ constraint equations:

$$y = 0, \quad z = 0.$$

Hence, no. of DOF $= 3 - 2 = 1$.

This simple 'formula' may be extended to more complicated systems & if used carefully, could be a useful means for deciding upon the no. of DOF analytically.