Chance-constrained optimization

Chance-constrained optimization

- Anshuman Panda, 2019MT10463
- Ritvik Gupta, 2019MT10512

Suppose, $K \sim N(\mu, C)$,
If $X = (X_1, X_2, X_3, \dots, X_n)$ follows multivariate normal distribution, then $w^T\mu$ is also normally distributed.

$$K \sim N(\mu, C) \implies -w^T K \sim N(-w^T\mu, w^T C w)$$

Consider, $\mathbb{P}(-w^T K \leq d) \geq \alpha$,

$$\implies \mathbb{P}\left(\frac{-w^T K + w^T\mu}{\sqrt{w^T C w}} \leq \frac{d + w^T\mu}{\sqrt{w^T C w}}\right) \geq \alpha$$

Note that,

$$-w^T K \sim N(-w^T\mu, w^T C w) \implies \frac{-w^T K + w^t\mu}{\sqrt{w^T C w}} \sim N(0, 1)$$

$$\begin{aligned} \therefore \phi\left(\frac{d + w^t\mu}{\sqrt{w^T C w}}\right) &\geq \alpha \\ \implies \frac{d + w^t\mu}{\sqrt{w^T C w}} &\geq \phi^{-1}(\alpha) \\ \implies \phi^{-1}(\alpha)||C^{1/2}w||_2 &\leq w^T\mu + d \end{aligned}$$

Hence, we have the chance-constrained optimization problem as,

$$\begin{array}{ll} \max\limits_{w} & \mu^T w \\ \text{s.t.} & \phi^{-1}(\alpha)||C^{1/2}w||_2 \leq \mu^T w + d \\ & \sum\limits_{i=1}^{n} w_i = 1 \\ & w_i \geq 0, \quad i = 1, \dots, n \end{array}$$

This is a convex set for $\alpha \in [0.5, 1]$,

We have used CVXPY to solve the optimization problem. As the chance-constrained optmization problem is a Second Order Cone Problem (SOCP), we use the SOCP solver from CVXPY to find the optimum point of the problem. The SOCP solver solves the following problem:

$$
\begin{aligned}
\min \quad & f^T x \\
\text{s.t.} \quad & ||A_i x + b_i||_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, n \\
& Fx = g
\end{aligned}
$$

Comparing our formulation to this form, we get,

$$
x = w; \quad f = \mu; \quad F = [1, 1, 1, \ldots 1]; \quad g = 1
$$
$$
A_0 = C^{1/2} * (\phi^{-1}(\alpha)); \quad b_0 = 0; \quad c_0 = \mu; \quad d_0 = d;
$$