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Analysis of Inter-Event Times for Planar Linear Systems Under a General Class of Event Triggering Rules

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Abstract— **This paper analyzes the evolution of inter-event times for a planar linear system under a general class of scale-invariant event triggering rules. For scale-invariant event triggering rules, the inter-event time is a function of only the "angle" of the state at an event. We analyze the properties of this inter-event time function such as periodicity and continuity. In particular, the inter-event time function is continuous except for finitely many "angles" and we provide sufficient conditions under which the inter-event time function is continuous. Then, we analyze the evolution of the "angle" of the state from one event to the next and reduce the problem of studying the evolution of the inter-event times to that of studying the "angle" map and its fixed points. For a specific triggering rule, we provide necessary conditions for the convergence of inter-event time to a steady state value. We illustrate the proposed results through numerical simulations.**

I. Introduction

Event-triggered control has been an active area of research in the last decade. Its main advantage is efficient aperiodic state dependent sampling of the control while simultaneously achieving control objectives. One of the factors which plays an important role in balancing control objectives with efficient use of resources in networked control systems is the inter-event time. In this paper, we carry out a systematic analysis of the inter-event times for planar linear systems under a general class of event triggering rules.

Literature review: A lot of work has been done in the area of event triggered control so far and a comprehensive overview of it is not possible here. We refer the reader to references [1]–[4] for a quick introduction to the topic and survey of the literature. Most of the papers in event-triggered control literature ensure the absence of Zeno behavior by providing a positive lower bound on the inter-event times. Typically, the analysis of the inter-event times stops there and the efficiency of an event-triggered controller is demonstrated through simulations. Self-triggered control [5] and periodic event triggered control [6] guarantee a minimum positive inter-event times by design. Even in these settings, a detailed analysis of the inter-event times as a function of the state or time is typically missing.

However, there are papers that analyze the average sampling rate [7], [8] or necessary and sufficient data rates to meet the control objective with event-triggered control [9]–[12]. On the other hand, [13], [14] design event triggering rules that ensure better performance than periodic control for a given average sampling rate, while [15] designs event-triggering under interval constraints on event times. A recent work that analyzes the time evolution and steady state of inter-event times is [16]. This paper talks about three different cases depending on the nature of the eigenvalues of the closed loop system matrix. In particular, it says that the inter-event times either converge to some neighborhood of a given constant or lie in some neighborhood of a given constant for all positive times or oscillates in a near periodic manner. However, there is no quantification of the neighborhoods in any of the cases. Reference [17] seeks to study isochronous manifolds - set of points in the state space with a given inter-event time. However, the aim of [17] is to design self-triggering rules rather than to analyze inter-event times resulting from a given triggering rule. Hence the triggering rule is suitably modified to aid the analysis.

Contribution: The main contribution of this paper is a framework for analyzing the inter-event times of planar linear systems under a general class of event-triggering rules that are scale-invariant. We first analyze inter-event time as a function of the state at a triggering instant. In particular, for scale-invariant event-triggering rules, the inter-event time is determined completely by the "angle" of the state at the last event-triggering instant. We provide sufficient conditions that ensure the continuity of this inter-event time function. We then present a framework to analyze the evolution of the inter-event times by studying the evolution of the "angle" of the state. In this framework, studying steady state behavior of the inter-event times is equivalent to studying the fixed points and their stability of the nonlinear map that gives the evolution of the "angle" of the state from one event to the next. We present necessary conditions for the existence of a fixed point for the "angle" map for a specific event triggering rule. With this analysis, we can precisely determine the value of the steady state inter-event time if it exists. This is in contrast to [16], which shows only the existence of some bounds on the steady state inter-event times.

Organization: Section II formally states the objective of this paper. Section III and Section IV analyze the properties of the inter-event time as a function of the state and the evolution of inter-event time, respectively. Section V illustrates the results using numerical examples. Finally, we provide some concluding remarks in Section VI.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ denote the set of all real, non-negative real and positive real numbers, respectively. $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{0\}$ denote the set of all non-zero real numbers and the set of all non-zero vectors in $\mathbb{R}^2$, respectively. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive and non-

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negative integers, respectively. For an $n \times n$ square matrix A, let $\det(A)$, $\mathrm{tr}(A)$ and $\lambda_{min}(A)$ denote determinant, trace and smallest eigenvalue of A, respectively.

II. Problem formulation

This section describes the system dynamics, the event triggering rules and the objective of this paper.

A. System description

Consider a continuous-time, linear time invariant planar system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{1a}$$

where $x \in \mathbb{R}^2$ denotes the state of the plant and $u \in \mathbb{R}^m$ denotes the control input, while A and B are system matrices of suitable dimensions. We consider a sampled data controller

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{1b}$$

where $\{t_k\}_{k\in\mathbb{N}_0}$ is the sequence of event or sampling times at which the state is sampled and the control input is updated. We assume that the control gain K is designed such that $A_c := A + BK$ is Hurwitz. We can analyze the stability of the closed loop system (1), using a quadratic candidate Lyapunov function $V(x) = x^T Px$, where $P \in \mathbb{R}^{2\times 2}$ is a positive definite matrix that satisfies the Lyapunov equation

$$PA_c + A_c^T P = -Q,$$

for some symmetric positive definite matrix Q. Such a candidate Lyapunov function can also be used to design the triggering rule for determining the event times $\{t_k\}_{k\in\mathbb{N}_0}$ that implicitly guarantee asymptotic stability of the origin of the closed loop system, possibly even with a desired convergence rate.

B. General class of event triggering rules

Depending on the control objective, different triggering rules can be used that determine the event times $\{t_k\}_{k\in\mathbb{N}_0}$. For example, as in [1], the recursive triggering rules

$$t_{k+1} = \min\{t > t_k : \dot{V}(x(t)) = 0\} \tag{2}$$
$$t_{k+1} = \min\{t > t_k : \|x(t_k) - x(t)\| = \sigma \|x(t)\|\}, \tag{3}$$

with σ sufficiently small, in the latter rule, each ensure asymptotic stability of the system (1). Another triggering rule is

$$t_{k+1} = \min\{t > t_k : V(x(t)) = V(x(t_k))e^{-r(t-t_k)}\}, \tag{4}$$

which ensures exponential stability of the system (1) with a rate of convergence of the Lyapunov function $V(x(t))$ more than r for a sufficiently small $r > 0$ (see [9] for example).

Note that for system (1), we can write the solution $x(t)$ as

$$x(t) = G(\tau)x(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

where $\tau := t - t_k$ and

$$G(\tau) := \mathrm{e}^{A\tau} + \int_0^\tau \mathrm{e}^{A(\tau-s)}\mathrm{d}s(A_c - A).$$

We can use this structure of the solution to write the three triggering rules (2)-(4) as

$$t_{k+1} - t_k = \min\{\tau > 0 : f(x(t_k), \tau) = 0\}, \tag{5}$$

where $f(x(t_k), \tau) := x^T(t_k)M(\tau)x(t_k)$ and $M(\tau)$ is a time varying symmetric matrix. In particular, for the triggering rules (2)-(4) $M(\tau)$ is equal to $M_1(\tau), M_2(\tau)$ and $M_3(\tau)$, respectively, where

$$M_1(\tau) := \frac{\mathrm{d}G^T(\tau)}{\mathrm{d}\tau} PG(\tau) + G^T(\tau)P\frac{\mathrm{d}G(\tau)}{\mathrm{d}\tau} \tag{6a}$$
$$M_2(\tau) := (1-\sigma^2)G^T(\tau)G(\tau) - (G^T(\tau) + G(\tau)) + I \tag{6b}$$
$$M_3(\tau) := G^T(\tau)PG(\tau) - P\mathrm{e}^{-r\tau}. \tag{6c}$$

In the special case where A is invertible, we can write

$$G(\tau) = I + A^{-1}(\mathrm{e}^{A\tau} - I)A_c,$$

which simplifies the expression and computation of $M(\tau)$ significantly. Thus, in order to communicate the main ideas easily, we make the following assumption.

(A1) The matrix A is invertible.

C. Objective

The main objective of this paper is to analyze the evolution of inter-event times along the trajectories of system (1) for a general class of event triggering rules (5). Moreover, we seek to provide analytical guarantees for the behavior of the inter-event times, such as the existence and value of steady state behavior of inter-event times.

III. Inter-event time as a function of the state

For the general class of triggering rules (5), we define the inter-event time function $\tau_e : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_{>0}$ as

$$\tau_e(x) := \min\{\tau > 0 : f(x, \tau) = x^T M(\tau)x = 0\}. \tag{7}$$

In this section, we analyze the properties of this inter-event time function. In Section III-A, we present some basic properties such as scale-invariance and periodicity, while in Section III-B we explore the issue of continuity of the inter-event time function.

A. Scale-invariance and periodicity of the inter-event time function

The function $f(x, \tau)$ possesses some basic properties, which are transferred to the inter-event time function $\tau_e(x)$. These properties follow directly from the form of $f(x, \tau)$.

Remark 1. *(Scale-invariance of the inter-event time function). Note from (7) that $f(\alpha x, \tau) = \alpha^2 f(x, \tau)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Hence, $\tau_e(\alpha x) = \tau_e(x)$, for any $x \in \mathbb{R}^2 \setminus \{0\}$ and for any $\alpha \in \mathbb{R} \setminus \{0\}$.* •

Using the scale-invariance property, we can redefine the *inter-event time function* for planar systems as a scalar function $\tau_s : \mathbb{R} \to \mathbb{R}_{>0}$,

$$\tau_s(\theta) := \min\{\tau > 0 : f_s(\theta, \tau) := x_\theta^T M(\tau)x_\theta = 0\}, \tag{8}$$

where $x_\theta := \begin{bmatrix}\cos(\theta) & \sin(\theta)\end{bmatrix}^T$. Thus, for $x = \alpha x_\theta$ for all $\alpha \in \mathbb{R} \setminus \{0\}$, $\tau_e(x) = \tau_s(\theta)$.

Remark 2. *($\tau_s(\theta)$ is a periodic function with period π). We know that for $x_\theta = \begin{bmatrix}\cos(\theta) & \sin(\theta)\end{bmatrix}^T$, $\tau_s(\theta) = \tau_e(x_\theta) = \tau_e(-x_\theta) = \tau_s(\theta+\pi)$ for all $\theta \in \mathbb{R}$.* •

As $\tau_s(\theta)$ is a periodic function with period π, we can restrict our analysis to the domain $[0,\pi)$. Next, by direct algebraic computations, we can show the following important property of $f_s(\theta,\tau)$.

Lemma 3. *(For any fixed τ, $f_s(\theta,\tau)$ is a sinusoidal function with a shift in phase and mean). Let $m_{ij}(\tau)$ be the element in row i and column j of $M(\tau) \in \mathbb{R}^{2\times 2}$. For any fixed $\tau \in \mathbb{R}_{>0}$,*

$$f_s(\theta,\tau) = \frac{\operatorname{tr}(M(\tau))}{2} + a\sin(2\theta + \arctan(b)), \tag{9}$$

$$a := \frac{1}{2}\sqrt{(\operatorname{tr}(M(\tau)))^2 - 4\det(M(\tau))},\ b := \frac{m_{11}(\tau) - m_{22}(\tau)}{2m_{12}(\tau)}.$$

□

From the quadratic structure of $f_s(\theta,\tau)$ in (8) as well as (9), we can directly make the following observation about the number of solutions to $f_s(\theta,\tau) = 0$ for any fixed τ. We can also use (9) to write a closed form expression for the solutions.

Corollary 4. *(Number of solutions θ to $f_s(\theta,\tau) = 0$ for a fixed τ). For any fixed $\tau \in \mathbb{R}_{>0}$, if $M(\tau)$ is positive or negative definite, then $f_s(\theta,\tau) = 0$ has no solutions; if $M(\tau)$ is singular then $f_s(\theta,\tau) = 0$ has a single solution $\theta \in [0,\pi)$ or $f_s(\theta,\tau) = 0$ for all $\theta \in [0,\pi)$; if $M(\tau)$ has one positive and one negative eigenvalue then $f_s(\theta,\tau) = 0$ has exactly two solutions $\theta \in [0,\pi)$.* □

We can also immediately provide a necessary and sufficient condition for an event-triggering rule to reduce to a periodic triggering rule that is independent of the state.

Corollary 5. *(Necessary and sufficient condition for the triggering rule (5) to reduce to periodic triggering). $\tau_s(\theta) = \tau_1, \forall\theta \in [0,\pi)$ if and only if $\det(M(\tau)) > 0$ for all $\tau \in (0,\tau_1)$, $\tau_1 = \min\{\tau > 0 : \det(M(\tau)) = 0\}$ and $M(\tau_1) = 0$, the zero matrix.* □

B. Continuity of the inter-event time function

Next, we explore if the inter-event time $\tau_s(\theta)$ function is continuous. The class of triggering rules (5) with an arbitrary function $M(\tau)$ is too broad. Thus, we make the following assumption about $M(\tau)$.

(A2) Every element of the matrix $M(.)$ is a real analytic function of τ and there exists a τ_m such that $M(\tau)$ is negative definite for $(0,\tau_m)$, where

$$\tau_m := \min\{\tau > 0 : \det(M(\tau)) = 0\}.$$

Note that each $M_i(.)$ in (6), corresponding to the three triggering rules (2)-(4), satisfies Assumption **(A2)**. This is because in $M_1(.)$ and $M_2(.)$ the only dependence on τ comes from the matrix exponential $e^{A\tau}$. The matrix function $M_3(.)$ additionally combines linearly another exponential function. Thus, each element of $M_i(.)$ is a linear combination of products of exponential functions, polynomials (in case A is not diagonalizable) and sinusoidal functions (in case A has complex eigenvalues). Further, each of the matrices $M_1(0)$ and $M_2(0)$ is negative definite while $M_3(0) = 0$, though $\dot{M}_3(0)$ is negative definite for suitable P and r. Thus, we can say that each $M_i(.)$ in (6) satisfy Assumption **(A2)**.

Now, let $\tau_{\min}$ and $\tau_{\max}$ denote the global minimum and the global maximum of $\tau_s(\theta)$, respectively, that is,

$$\tau_{\min} := \min_{\theta\in[0,\pi)} \tau_s(\theta), \tag{10}$$

$$\tau_{\max} := \max_{\theta\in[0,\pi)} \tau_s(\theta). \tag{11}$$

For a matrix $M(.)$ that satisfies Assumption **(A2)**, clearly $\tau_{\min} > 0$ as $\det(M(\tau)) > 0$ in the interval $(0,\tau_m)$, and according to Corollary 4, $f_s(\theta,\tau) = 0$ has no solution in the interval $(0,\tau_m)$ and has a unique solution at $\tau = \tau_m$. Therefore we can say that $\tau_{\min} = \tau_m$. In general, $\tau_{\max}$ may not exist, that is $\tau_{\max} = \infty$. In this case, it means that there exists a $x_0 \in \mathbb{R}^2 \setminus \{0\}$ such that if $x(t_k) = x_0$ then $t_{k+1} = \infty$. In other words, this means that the solution to (1) with the initial condition x_0 and a constant control input converges to zero asymptotically. However, such an x_0 cannot exist if A has positive real parts for both its eigenvalues and if the triggering rule (5) ensures $x = 0$ is asymptotically stable. In such a case, $\tau_{\max}$ is a finite quantity.

Now, we are ready to discuss continuity of the inter-event time function $\tau_s(\theta)$. As is evident, $\det(M(\tau))$ plays a very significant role in the existence and the number of solutions θ to $f(\theta,\tau) = 0$ for any fixed $\tau \in \mathbb{R}_{>0}$. We can show that $\det(M(\tau))$ has finitely many zeros in any bounded interval and as a consequence of Corollary 4 we can also show that the level set $f_s(\theta,\tau) = 0$ has finitely many connected branches, each of which is a smooth curve in (θ,τ) space.

Lemma 6. *(The level set $f_s(\theta,\tau) = 0$ has finitely many connected branches, which are continuous). Suppose that $M(.)$ in (8) satisfies Assumption **(A2)** and $\tau_{\max} < \infty$. Then, the level set $f_s(\theta,\tau) = 0$ has finitely many connected branches in the set $\{(\theta,\tau) \in [0,\pi) \times [0,\tau_{\max}]\}$. Each branch is an arbitrarily smooth curve in (θ,τ) space and can be parameterized by τ in a closed interval.* □

As a result of Lemma 6, we can apply the implicit function theorem on $f_s(\theta,\tau) = 0$ at all $(\theta,\tau_s(\theta)) \in [0,\pi) \times [0,\tau_{\max}]$, except at finitely many points. This guarantees that $\tau_s(\theta)$ is continuously differentiable except at finitely many points in $[0,\pi)$. We state this claim formally in the following result.

Theorem 7. *(Inter-event time function is continuously differentiable except for finitely many θ). Suppose that $M(.)$ in (8) satisfies Assumption **(A2)** and $\tau_{\max} < \infty$. Then, the inter-event time function $\tau_s(\theta)$ defined as in (8) is continuously differentiable on $[0,\pi)$ except at finitely many θ.* □

Based on Theorem 7, we can provide a sufficient condition for the function $\tau_s(\theta)$ to be continuously differentiable.

Corollary 8. *(Corollary to Theorem 7). If $\frac{\partial f_s(\bar{\theta},\bar{\tau})}{\partial \tau} \neq 0$ for all $(\bar{\theta},\bar{\tau}) \in \mathbb{R} \times \mathbb{R}$ such that $f_s(\bar{\theta},\bar{\tau}) = 0$, then the inter-event* *time function $\tau_s : \mathbb{R} \to \mathbb{R}_{>0}$ defined as in (8) is continuously differentiable.* □

Note that Theorem 7 and Corollary 8 hold for any $M(.)$ that satisfies Assumption **(A2)**. For specific matrix functions $M(.)$, such as $M_i(.)$ in (6), it may be possible to make far stronger claims. We leave such analysis on specific triggering rules to future work. It suffices to say here that we have observed that the triggering rules with $M_i(.)$ in (6) usually result in $\tau_s(\theta)$ that is continuous, or in certain cases, one that is discontinuous at only a couple of $\theta \in [0,\pi)$. We present the following important property in the special case where $\tau_s(\theta)$ is a continuous function.

Proposition 9. *If the inter-event time function $\tau_s(\theta)$ is a continuous function, then every local extremum of $\tau_s(\theta)$ is a global extremum.* □

IV. Evolution of the angle and inter-event time

In this section, we discuss how the analysis of inter-event time function helps us to determine the evolution (such as the steady state behavior) of the inter-event time along trajectories of the system (1) under the event-triggering rule (5). In particular, our analysis relies upon the inter-event time function $\tau_e(x)$ or $\tau_s(\theta)$ and on the analysis of the evolution of the angle θ_k, which denotes the angle between the system state $x(t_k)$ and the positive x_1 axis.

Note that the evolution of the state of the system (1) under the event-triggering rule (5) from one triggering instant to the next may be concisely expressed as

$$x(t_{k+1}) = G(\tau_s(\theta_k))x(t_k), \quad \theta_k = \arg(x(t_k)),$$
$$\arg(x) := \begin{cases} \arctan(\frac{x_2}{x_1}), & \text{if } x_1 \geq 0 \\ \pi + \arctan(\frac{x_2}{x_1}), & \text{otherwise.} \end{cases}$$

Thus, we can say that

$$\theta_{k+1} = \phi(\theta_k) := \arg\left(G(\tau_s(\theta_k)) \begin{bmatrix} \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix} \right).$$

As a result, it suffices to study the inter-event time function and the *angle map* $\phi(\theta)$ to understand the time evolution of the inter-event times for an arbitrary initial condition $x(t_0)$. For example, the fixed points of the angle map $\phi(.)$, that is, points θ such that $\phi(\theta) = \theta$ play a crucial role in the evolution of the inter-event times. Based on this idea, we present a procedure for analyzing the evolution of the inter-event time behavior for the system (1) under a general event-triggering rule (5).

Remark 10. *(Procedure for analyzing the evolution of inter-event times for a general triggering rule (5)). Consider the system (1) under the triggering rule (5) for a general $M(.)$ satisfying Assumption **(A2)**. We can compute the functions $\tau_s(\theta)$ and $\phi(\theta)$ for $\theta \in [0,\pi)$. For each fixed point θ of the map $\phi(.)$ the inter-event times $t_{k+1} - t_k = \tau_s(\theta)$ for all $k \in \mathbb{N}_0$ and for all initial conditions $x(t_0) = \alpha \begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix}^T$, for all $\alpha \in \mathbb{R} \setminus \{0\}$. One can identify the stable fixed points of the angle map $\phi(.)$ and their region of attraction to determine the steady state behavior of $\tau_e(x(t_k))$ for $x(t_0)$ with $\arg(x(t_0))$ in the region of attraction of a stable fixed point. One may also identify periodic points and their stability numerically or by analysis if possible. If the map $\phi(.)$ does not have any fixed points then we can guarantee that there is no steady state behavior of the inter-event times for any initial condition. The functions $\tau_s(.)$ and $\phi(.)$ together can also potentially tell us about the transient behavior of the inter-event times. Finally, the stable fixed points of the $\phi(.)$ map determine lines in $\mathbb{R}^2$ along which the state trajectories asymptotically converge to the origin.* •

While the ideas and procedure in Remark 10 are applicable to the triggering rule (5) for a general $M(.)$, it is difficult to say anything more specific in general. Thus, in the following, we analyze a specific event-triggering rule, namely (3) or equivalently (5) with $M(.) = M_2(.)$ given in (6).

A. Analysis of the fixed points of $\phi(.)$ with $M(.) = M_2(.)$

In this subsection, we analyze system (1) under the event-triggering rule (3) or equivalently (5) with $M(.) = M_2(.)$ given in (6). We present a necessary condition for the existence of a fixed point for the angle map $\phi(.)$ in the following Lemma.

Lemma 11. *(Necessary condition for the angle map to have a fixed point under triggering rule (3)). Consider system (1) under the event-triggering rule (3) or equivalently (5) with $M(.) = M_2(.)$ given in (6). Suppose that Assumption **(A1)** holds and that the parameter $\sigma \in (0,1)$ is such that the origin of the closed loop system is globally asymptotically stable. Then, there exists a fixed point for the angle map $\phi(.)$ only if $\det(L(\tau)) = 0$ for some $\tau \in \mathbb{R}_{>0}$, where*

$$L(\tau) := (1-\alpha)I + A^{-1}(e^{A\tau} - I)A_c$$

and $\alpha = (1+\sigma)^{-1}$. □

Note that $\det(L(\tau)) = 0$ for some $\tau \in \mathbb{R}_{>0}$ is not a sufficient condition for the existence of a fixed point for the angle map $\phi(.)$. This is because even if $\det(L(\tau_1)) = 0$ for some $\tau_1 \in \mathbb{R}_{>0}$, that τ_1 may not be in the interval $[\tau_{\min}, \tau_{\max}]$. Even if $\tau_1 \in [\tau_{\min}, \tau_{\max}]$, there may be no $x \neq 0$ in the null space of $L(\tau)$ such that $\tau_s(\arg(x)) = \tau_1$.

Remark 12. *(Angle map for the triggering rule (3) has a bounded number of fixed points). Note that $\det(L(\tau))$ is an analytic function of τ. Hence, $\det(L(\tau))$ has a bounded number of zeros in the interval $[\tau_{\min}, \tau_{\max}]$. If there does exist a $\tau \in [\tau_{\min}, \tau_{\max}]$ such that $\det(L(\tau)) = 0$ then either $\phi(\theta) = \theta$ for all $\theta \in [0,\pi)$ or the angle map $\phi(.)$ has a bounded number of fixed points.* •

While Lemma 11 has allowed us to conclude that, for the angle map $\phi(.)$, if every θ is not a fixed point then there are a bounded number of fixed points, verifying if $\det(L(\tau))$ has zeros in $[\tau_{\min}, \tau_{\max}]$ may not be easy. Thus, we next present an algebraic necessary condition for the existence of fixed points for the angle map $\phi(.)$.

Proposition 13. *(Algebraic necessary condition for the angle map to have a fixed point under triggering rule (3)).*

Consider system (1) *under the event-triggering rule* (3) *or equivalently* (5) *with* $M(.) = M_2(.)$ *given in* (6)*. Suppose that Assumption* ***(A1)*** *holds and that the parameter* $\sigma \in (0,1)$ *is such that the origin of the closed loop system is globally asymptotically stable. Further, assume that both the eigenvalues of A have positive real parts. Let* $A =: J\Lambda J^{-1}$*, where* Λ *is the Jordan form of A. Then, there exists a fixed point for the angle map* $\phi(.)$ *only if* $\|R\| > 1$*, where*

$$R := J^{-1}\left[I - (1-\alpha)AA_c^{-1}\right]J. \qquad \square$$

V. Numerical Examples

In this section, we illustrate our results using numerical examples. Consider the system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The system matrix A has real eigenvalues at $[1,2]$. The control gain K is chosen such that A_c has desired eigenvalues. The event triggering rule (3) is used to design a sampled data controller. With $Q = I$, the Lyapunov equation is solved to determine P. The thresholding parameter σ is chosen such that $\sigma = \frac{0.99\lambda_{\min}(Q)}{2\|PBK\|}$.

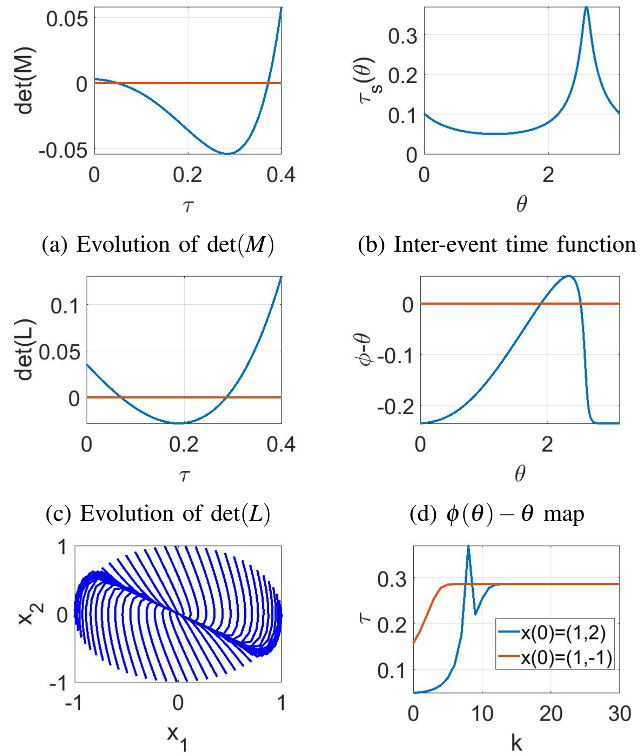
Case 1: The control gain $K = [0 \quad -6]$ so that A_c has real eigenvalues at $[-1,-2]$. Fig. 1 shows the simulation results for Case 1. For this case the inter-event time function is continuous and periodic with period π. From Fig. 1(a) and Fig. 1(b) we can verify that $\det(M(\tau)) = 0$ has exactly two solutions and these two points are $\tau_{\min}$ and $\tau_{\max}$ respectively. Fig. 1(c) shows that there are two points at which $\det(L(\tau)) = 0$. Fig. 1(d) verifies that the angle map $\phi(.)$ has exactly two fixed points, where the larger one is a stable fixed point. Fig. 1(e) represents the phase portrait of the closed loop system. The state trajectories are converging to a radial line which makes an angle of 2.5 radians (approximately) with the positive x_1 axis, which is exactly the point at which the angle map $\phi(.)$ has the stable fixed point. From Fig. 1(f) it is clear that the inter-event time is converging to a steady state value.

Case 2: The control gain $K = [0 \quad -5]$ so that A_c has complex conjugate eigenvalues at $[-1+i, -1-i]$. Fig. 2 shows the simulation results for Case 2. For this case also the inter-event time function is continuous and periodic with period π. From Fig. 2(a) and Fig. 2(b) we can verify that $\det(M(\tau)) = 0$ has exactly two solutions and these two points are $\tau_{\min}$ and $\tau_{\max}$ respectively. Fig. 2(c) shows that $\det(L(\tau))$ is always positive. Therefore the ϕ map in Fig. 2(d) has no fixed point. Fig. 2(e) represents the phase portrait of the closed loop system. From Fig. 2(f) it is clear that the inter-event time is not converging to a steady state value.

Case 3: Consider another system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u.$$

The system matrix A has real eigenvalues at $[1,2]$. The control gain $K = \begin{bmatrix} -1 & -0.8 \\ 1.8 & -4 \end{bmatrix}$ so that A_c has complex conjugate eigenvalues at $[-1+0.2i, -1-0.2i]$. Fig. 3 shows the simulation results of this system for the event triggering rule (3). Fig. 3(a) shows that the angle map $\phi(.)$ has two fixed points, where the larger one is a stable fixed point. In Fig. 3(b) the inter-event time is converging to a steady state value for two different initial conditions.



(a) Evolution of $\det(M)$ (b) Inter-event time function

(c) Evolution of $\det(L)$ (d) $\phi(\theta) - \theta$ map

(e) Phase portrait of the closed loop system (f) Evolution of inter-event times

Fig. 1: Simulation results of case 1 when A_c has real eigenvalues at $[-1,-2]$.

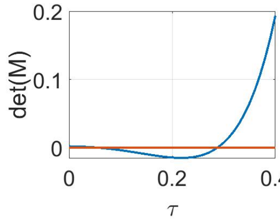
Case 4: Now consider the system,

$$\dot{x} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

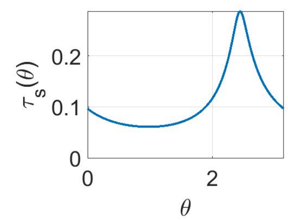
A has real and equal eigenvalues at $[1,1]$. The control gain $K = [-2 \quad -4]$ so that A_c has eigenvalues at $[-1+2i, -1-2i]$. Fig. 4 shows the simulation results of this system for the triggering rule (2). Fig. 4(a) shows that the inter-event time function $\tau_s(\theta)$ is discontinuous around $\theta = 2.3$ radians.

VI. Conclusion

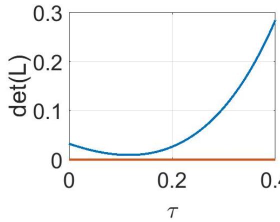
In this paper we analyzed the evolution of inter-event times along the trajectories of planar linear systems under a general class of event triggering rules that are scale-invariant. We analyzed the properties of the inter-event time as a function of the state at a event triggering instant, such as periodicity and continuity. Under some mild assumptions, we concluded that the inter-event time function is continuous except at finitely many angles and we found sufficient conditions which ensure continuity. We then analyzed the map that determines the



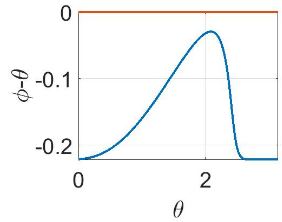
(a) Evolution of $\det(M)$



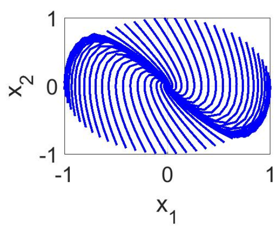
(b) Inter-event time function



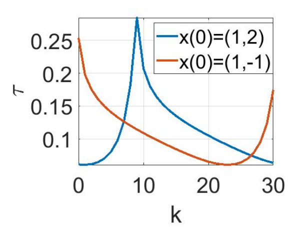
(c) Evolution of $\det(L)$



(d) $\phi(\theta)-\theta$ map

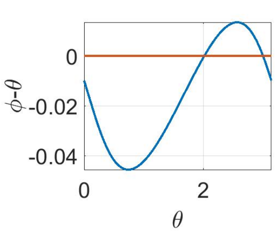


(e) Phase portrait of the closed loop system

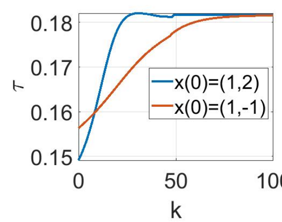


(f) Evolution of inter-event times

Fig. 2: Simulation results of case 2 when A_c has complex conjugate eigenvalues at $[-1+i,-1-i]$.



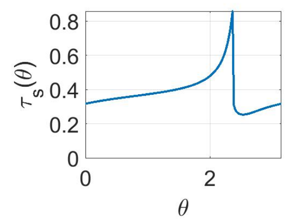
(a) $\phi(\theta)-\theta$ map



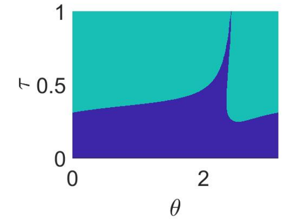
(b) Evolution of inter-event times

Fig. 3: Simulation results of case 3 when A_c has complex conjugate eigenvalues at $[-1+0.2i,-1-0.2i]$.

evolution of the angle of the state from one event to the next. Combining these two, we provided a framework for analyzing the evolution of the inter-event times for planar systems. For a specific event triggering rule, we determined a necessary condition for the convergence of inter-event time to a steady state value. We verified the proposed results through numerical simulations. Future work includes analysis of the angle map under specific triggering rules with regard to necessary and sufficient conditions for the existence of fixed points, their stability, region of convergence and rates of convergence. Extensions to periodic event-triggered control or self-triggered control are other avenues for future work.



(a) Inter-event time function



(b) Level set of $f_s(\theta,\tau)=0$

Fig. 4: Simulation results of case 4 with discontinuous inter-event time function.

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