

# Analysis of Inter-Event Times in Linear Systems under Event-Triggered or Self-Triggered Control

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**Abstract**—This paper deals with the analysis of evolution of inter-event times in linear systems under event-triggered control and region-based self-triggered control. First, we consider planar systems under control with a class of scale-invariant event triggering rules. In this setting, the inter-event time is a function of the “angle” of the state at an event. We analyze the properties of this inter-event time function such as periodicity and continuity. Then, we analyze the evolution of the inter-event times by studying the fixed points of the “angle” map which represents the evolution of the “angle” of the state between two successive events. For a specific triggering rule, we provide necessary conditions for the convergence of inter-event times to a steady state value. We also analyze stability and region of convergence of a fixed point of the “angle” map. Next, we consider n-dimensional systems under region-based self triggered control, in which the state space is partitioned into a finite number of conic regions and each region is associated with a fixed inter-event time. We analyze the evolution of inter-event times under the proposed self-triggered control method. In this framework, studying steady state behavior of the inter-event times is equivalent to studying the existence of a conic subregion, which is a positively invariant set under the map that gives the evolution of the state from one event to the next. We provide a sufficient condition for the existence of a special kind of positively invariant subregion called positively invariant ray. We also provide necessary and sufficient conditions for a positively invariant ray to be asymptotically stable. We also explore the existence of positively invariant subregions that do not include a positively invariant ray. We illustrate the proposed method of analysis and analytical results through numerical simulations.

**Index Terms**—Event-triggered control, Inter-event times, Self-triggered control, Networked Control Systems

## I. INTRODUCTION

Event-triggered and self-triggered control are popular methods for control under resource constraints. The main advantage of these control methods is the efficient state dependent sampling that is based on its necessity towards achieving the control objective. However, in these methods the inter-event times are opportunistic and implicitly determined by

the triggering rule. Thus, it is important to study about inter-event times for designing controllers that properly balance control objectives and constrained resources. For example, understanding the evolution of inter-event times helps to schedule multiple processes over a shared communication channel or to plan transmissions under constraints. Similarly, understanding inter-event times generated by an event-triggering rule can help in the analytical quantification of the improvement of average inter-event times for an event-triggered controller over that of a time-triggered controller. With these motivations, in this paper, we carry out a systematic analysis of the evolution of inter-event times for linear systems under a general class of event-triggering and self-triggering rules.

### A. Literature review

Event/self-triggered control has been an active area of research in the field of networked control systems. A quick introduction to these topics and a survey of the literature can be found in the references [1]–[4]. In the event-triggered control literature, typically the interest is only in showing the existence of a positive lower bound on the inter-event times to ensure the absence of Zeno behavior. Self-triggered control [5] and periodic event triggered control [6], which eliminate the need for continuous monitoring of the triggering condition, guarantee a positive minimum inter-event time by design. In all these settings, a detailed analysis of the inter-event times as a function of the state or time is typically missing.

Although it is not common, there are some works that analyze the average sampling rate [7], [8] or necessary and sufficient data rates for meeting the control goal with event-triggered control [9]–[12]. On the other hand, [13], [14] take a different approach and design event triggering rules that ensure better performance than periodic control for a given average sampling rate. Reference [15] proposes an event-triggering rule for exponential stabilization under a given interval constraints on event times.

In the literature, it has been observed that the inter-event times often settle to a steady state value. Reference [16] seeks to explain this phenomenon for planar linear systems with relative thresholding based event-triggering rule. In particular, the paper considers the “small” thresholding parameter scenario and talks about three different cases depending on the nature of the eigenvalues of the closed loop system matrix. It provides sufficient conditions under which the inter-event times either converge to some neighborhood of a given constant or lie in some neighborhood of a given constant

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for all positive times or oscillate in a near periodic manner. However, these results only show the existence of “small” enough thresholding parameters and neighborhoods for which the claims are true. Reference [17] designs self-triggering rules by studying isochronous manifolds - sets of points in the state space with a given inter-event time. As the aim of this work is to design self-triggering rules rather than to analyze inter-event times resulting from a given triggering rule, the triggering rule is suitably modified to aid the analysis. The recent paper [18] also proposes a self triggered control scheme that provides near-maximal average inter-sample time by using finite-state abstractions of a reference event-triggered control, and by using “early triggering”. The current paper is a continuation of our previous work [19], which analyzes the evolution of inter-event times for planar linear systems under scale invariant event-triggering rules.

## B. Overview of the Paper

This paper presents a framework for analyzing the evolution of inter-event times in linear systems under scale-invariant event-triggering or region-based self-triggering rules. We first consider a 2-dimensional event-triggered control system and analyze the inter-event time as a function of the state at a triggering instant. For planar systems under scale-invariant event-triggering rules, the inter-event time is solely a function of the “angle” of the state at the last event-triggering instant. We analyze this inter-event time function and provide sufficient conditions that guarantee its continuity. We then present a framework to analyze the evolution of the inter-event times indirectly by studying the evolution of the “angle” of the state from one event to the next. For example, steady state behavior of the inter-event times can be inferred from the fixed points, and their stability, of the nonlinear “angle” map that gives the evolution of the “angle” of the state from one event to the next. To demonstrate the usefulness of this framework, we present necessary conditions for the existence of a fixed point for the “angle” map for a specific event triggering rule. We also provide sufficient conditions for stability/instability of the fixed points and provide a method to determine their region of convergence. With such an analysis, we can precisely determine the value of the steady state inter-event time, if it exists.

Next, we consider  $n$ -dimensional systems under region-based self triggered control, in which the state space is partitioned into a finite number of conic regions and each region is associated with a fixed inter-event time. Then we analyze the evolution of inter-event times under the proposed self-triggered control method. In this framework, studying steady state behavior of the inter-event times is equivalent to studying the existence of a conic subregion that is a positively invariant set under the map that gives the evolution of the state from one event to the next. We provide a sufficient condition for the existence of a special kind of positively invariant subregion called positively invariant ray. We also provide necessary and sufficient conditions for a positively invariant ray to be asymptotically stable.

## C. Contributions

The major contribution of our work is that we provide a systematic way to analyze the inter-event times, as a function of state or time, for linear systems under event-triggered or region-based self-triggered control. While in the case of event-triggered control, we largely restrict ourselves to the planar systems setting, our results in the self-triggered control setting hold for arbitrary  $n$ -dimensional systems. Our results are applicable for a very broad class of event/self-triggering rules. We provide quantitative results regarding the steady state behavior of inter-event times and provide several necessary and sufficient conditions for the inter-event times to converge to a constant. In comparison, the results in [16] are largely qualitative in nature and restricted to planar linear systems under relative thresholding based event-triggering in the “small relative threshold parameter” setting. The references [7]–[15], [17], [18] and similar works are primarily concerned with designing event-triggering rules so that the average inter-event times or some other criterion on the inter-event times is satisfied. This is in contrast to our aim in this paper of analyzing the inter-event times generated by some popular event-triggering rules.

The main contribution of this paper with respect to our previous work [19] is that, here we also provide a framework to analyze the steady state behavior of the inter-event times for an  $n$ -dimensional region-based self-triggered control system whereas in the previous work we only consider planar event-triggered control systems. We provide several important new results, primarily in the self-triggered control setting. Other contributions are as follows. This paper includes the proofs of all the results proposed in the previous work. We also rectify some of the errors encountered in the previous work. For example, in the previous work we claim that if there does not exist a fixed point for the “angle” map then the inter-event times do not converge to a steady state value, which may not be true in general. We also improve the result on the algebraic necessary condition for the “angle” map to have a fixed point under a specific event-triggering rule. We also analyze the stability of the fixed points of the “angle” map under a specific event-triggering rule which was not done in the previous work. We also improve the numerical examples section by including more simulation results.

## D. Organization

Section II formally sets up the problem and states the objective of this paper. Section III and Section IV analyze the properties of the inter-event time as a function of the state at an event and the evolution of inter-event times under the event-triggered control method, respectively. In Section V we extend our analysis of inter-event times for a region-based self-triggered control systems. Section VI illustrates the results using numerical examples. Finally, we provide some concluding remarks in Section VII.

## E. Notation

Let  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ , and  $\mathbb{R}_{>0}$  denote the set of all real, non-negative real and positive real numbers, respectively.  $\mathbb{R} \setminus \{0\}$

and  $\mathbb{R}^n \setminus \{0\}$  denote the set of all non-zero real numbers and the set of all non-zero vectors in  $\mathbb{R}^n$ , respectively. Let  $\mathbb{N}$  and  $\mathbb{N}_0$  denote the set of all positive and non-negative integers, respectively. For an  $n \times n$  square matrix  $A$ , let  $\det(A)$ ,  $\text{tr}(A)$  and  $\lambda_{\min}(A)$  denote determinant, trace and smallest eigenvalue of  $A$ , respectively.  $B_\varepsilon(u)$  represents an  $n$ -dimensional ball of radius  $\varepsilon$  centered at  $u \in \mathbb{C}^n$ .

## II. PROBLEM SETUP

In this section, we present the dynamics of the system, the triggering rules and the objective of this paper.

### A. System Dynamics

Consider a continuous-time, linear time invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1a)$$

where  $x \in \mathbb{R}^n$  is the plant state and  $u \in \mathbb{R}^m$  is the control input, while  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are the system matrices. Consider a sampled data controller and let  $\{t_k\}_{k \in \mathbb{N}_0}$  be the sequence of event times at which the state is sampled and the control input is updated as follows,

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}). \quad (1b)$$

Let the control gain  $K$  be such that  $A_c := A + BK$  is Hurwitz.

### B. Triggering Rules

In this paper, we assume that the event times  $\{t_k\}_{k \in \mathbb{N}_0}$  are generated in an event-triggered or a self-triggered manner so as to implicitly guarantee asymptotic stability of the origin of the closed loop system. It is common to construct such event-triggering rules based on a candidate Lyapunov function. For example, consider a quadratic candidate Lyapunov function  $V(x) = x^T Px$ , where  $P \in \mathbb{R}^{n \times n}$  is a positive definite symmetric matrix that satisfies the Lyapunov equation

$$PA_c + A_c^T P = -Q, \quad (2)$$

for a given symmetric positive definite matrix  $Q$ . Following are three different event-triggering rules that are commonly used in the literature for stabilization tasks.

$$t_{k+1} = \min\{t > t_k : \dot{V}(x(t)) = 0\} \quad (3a)$$

$$t_{k+1} = \min\{t > t_k : \|x(t_k) - x(t)\| = \sigma \|x(t)\|\}, \quad (3b)$$

$$t_{k+1} = \min\{t > t_k : V(x(t)) = V(x(t_k))e^{-r(t-t_k)}\}. \quad (3c)$$

First two triggering rules render the origin of the closed loop system asymptotically stable, with  $\sigma$  sufficiently small in the latter rule. The third event-triggering rule ensures exponential stability for a sufficiently small  $r > 0$ , with a rate of convergence of the Lyapunov function  $V(x(t))$  more than  $r$ . From the event-triggering rules (3), one can also derive self-triggering rules for asymptotic stabilization.

During the inter-event intervals, we can write the solution  $x(t)$  of system (1) as

$$x(t) = G(\tau)x(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

where  $\tau := t - t_k$  and

$$G(\tau) := e^{A\tau} + \int_0^\tau e^{A(\tau-s)} ds (A_c - A).$$

Using this structure of the solution, we can write the three triggering rules (3) as

$$t_{k+1} - t_k = \min\{\tau > 0 : f(x(t_k), \tau) := x^T(t_k)M(\tau)x(t_k) = 0\}, \quad (4)$$

where  $M(\tau)$  is a time varying symmetric matrix. In particular, for the triggering rules (3a)-(3c)  $M(\tau)$  is equal to  $M_1(\tau)$ ,  $M_2(\tau)$  and  $M_3(\tau)$ , respectively, where

$$M_1(\tau) := \frac{dG^T(\tau)}{d\tau}PG(\tau) + G^T(\tau)P\frac{dG(\tau)}{d\tau} \quad (5a)$$

$$M_2(\tau) := (1 - \sigma^2)G^T(\tau)G(\tau) - (G^T(\tau) + G(\tau)) + I \quad (5b)$$

$$M_3(\tau) := G^T(\tau)PG(\tau) - Pe^{-r\tau}. \quad (5c)$$

Note that if  $A$  is invertible, we can write

$$G(\tau) = I + A^{-1}(e^{A\tau} - I)A_c,$$

which simplifies the expression and computation of  $M(\tau)$  significantly. Thus, in order to communicate the main ideas easily, we make the following standing assumption.

**(A1)** The matrix  $A$  is invertible.

### C. Objective

The main objective of this paper is to analyze the evolution of inter-event times along the trajectories of system (1) for the general class of event triggering rules (4) and for conic region-based self triggering rules (which we specify formally in Section V). Moreover, we seek to provide analytical guarantees for the steady state behavior of inter-event times under these rules. The approach we take is to analyze inter-event time and the state at the next event as functions of the state at the time of the current event. First, we carry out the analysis for the event triggering rules and then we look at the self-triggering rules. In the next section, we begin with analysis of the inter-event time as a function of state for the case of event-triggered control in planar systems.

## III. INTER-EVENT TIME AS A FUNCTION OF THE STATE IN PLANAR SYSTEMS

In this section, we consider system (1) under the general class of event triggering rules (4), for which the inter-event time  $t_{k+1} - t_k$  can be represented as a function of  $x(t_k)$  for all  $k \in \mathbb{N}_0$ . We study this inter-event time function in the context of planar (two-dimensional) systems for tractability of the analysis.

Formally, we define the inter-event time function  $\tau_e : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_{>0}$  as

$$\tau_e(x) := \min\{\tau > 0 : f(x, \tau) = x^T M(\tau)x = 0\}. \quad (6)$$

We can write  $t_{k+1} - t_k = \tau_e(x(t_k))$  for all  $k \in \mathbb{N}_0$ . Next, we analyze the properties of this inter-event time function such as scale-invariance, periodicity and continuity.

### A. Properties of the inter-event time function

**Remark 1.** (The inter-event time function is scale-invariant). Note from (6) that  $f(\alpha x, \tau) = \alpha^2 f(x, \tau)$  for all  $\alpha \in \mathbb{R}$  and  $x \in \mathbb{R}^2$ . Hence,  $\tau_e(\alpha x) = \tau_e(x)$ , for any  $x \in \mathbb{R}^n \setminus \{0\}$  and for any  $\alpha \in \mathbb{R} \setminus \{0\}$ . •

The scale-invariance property implies that we can redefine the inter-event time function for planar systems ( $n = 2$ ) as a scalar function  $\tau_s : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ ,

$$\tau_s(\theta) := \min\{\tau > 0 : f_s(\theta, \tau) := x_\theta^T M(\tau) x_\theta = 0\}, \quad (7)$$

where  $x_\theta := [\cos(\theta) \ \sin(\theta)]^T$  so that  $\tau_e(x) = \tau_s(\theta)$  for  $x = \alpha x_\theta$  for all  $\alpha \in \mathbb{R} \setminus \{0\}$ . Hence for a planar system, the inter-event time  $t_{k+1} - t_k = \tau_s(\theta_k)$  for all  $k \in \mathbb{N}_0$ , where  $\theta_k$  is the angle between  $x(t_k)$  and the  $x_1$  axis. In Section III and section IV, we deal with planar systems and we use this definition of inter-event time function.

**Remark 2.** ( $\tau_s(\theta)$  is a periodic function with period  $\pi$ ). We know that for  $x_\theta = [\cos(\theta) \ \sin(\theta)]^T$ ,  $\tau_s(\theta) = \tau_e(x_\theta) = \tau_e(-x_\theta) = \tau_s(\theta + \pi)$  for all  $\theta \in \mathbb{R}$ . •

Periodicity of  $\tau_s(\theta)$  helps us to restrict our analysis to the domain  $[0, \pi)$ . Next, we present an important property of  $f_s(\theta, \tau)$  that plays a major role in the subsequent analysis.

**Lemma 3.** (For any fixed  $\tau$ ,  $f_s(\theta, \tau)$  is a sinusoidal function with a shift in phase and mean). Let  $m_{ij}(\tau)$  be the  $(ij)^{\text{th}}$  element of  $M(\tau) \in \mathbb{R}^{2 \times 2}$ . For any fixed  $\tau \in \mathbb{R}_{>0}$ ,

$$f_s(\theta, \tau) = \frac{\text{tr}(M(\tau))}{2} + a \sin(2\theta + \arctan(b)), \quad (8)$$

$$a := \frac{1}{2} \sqrt{(\text{tr}(M(\tau)))^2 - 4 \det(M(\tau))}, \quad b := \frac{m_{11}(\tau) - m_{22}(\tau)}{2m_{12}(\tau)}.$$

*Proof.* Here, we skip the time argument of  $m_{ij}(\tau)$  for brevity. Note that  $m_{12} = m_{21}$ . Then, for any fixed  $\tau \in \mathbb{R}_{>0}$ ,

$$\begin{aligned} f_s(\theta, \tau) &= [\cos(\theta) \ \sin(\theta)] M(\tau) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \\ &= m_{11} \cos^2(\theta) + m_{22} \sin^2(\theta) + 2m_{12} \cos(\theta) \sin(\theta), \\ &= m_{11} + (m_{22} - m_{11}) \sin^2(\theta) + m_{12} \sin(2\theta), \\ &= \frac{m_{11} + m_{22}}{2} + \frac{m_{11} - m_{22}}{2} \cos(2\theta) + m_{12} \sin(2\theta), \end{aligned}$$

from which we get the result by suitably expressing the coefficients. □

Using the structure of  $f_s(\theta, \tau)$  in (8), we can determine the number of solutions to  $f_s(\theta, \tau) = 0$  for any fixed  $\tau$ .

**Corollary 4.** (Number of solutions  $\theta$  to  $f_s(\theta, \tau) = 0$  for a fixed  $\tau$ ). For any fixed  $\tau \in \mathbb{R}_{>0}$ , if  $\det(M(\tau)) > 0$ , then  $f_s(\theta, \tau) = 0$  has no solutions; if  $\det(M(\tau)) = 0$  then  $f_s(\theta, \tau) = 0$  has a single solution  $\theta \in [0, \pi)$  or  $f_s(\theta, \tau) = 0$  for all  $\theta \in [0, \pi)$ ; if  $\det(M(\tau)) < 0$  then  $f_s(\theta, \tau) = 0$  has exactly two solutions  $\theta \in [0, \pi)$ . □

We can also obtain a necessary and sufficient condition for the event-triggering rule (4) to reduce to a periodic triggering rule, with inter-event times that are independent of the state.

**Corollary 5.** (Necessary and sufficient condition for the triggering rule (4) to reduce to periodic triggering).  $\tau_s(\theta) = \tau_1, \forall \theta \in [0, \pi)$  if and only if  $\det(M(\tau)) > 0$  for all  $\tau \in (0, \tau_1)$ ,  $\tau_1 = \min\{\tau > 0 : \det(M(\tau)) = 0\}$  and  $M(\tau_1) = 0$ , the zero matrix. □

### B. Continuity of the inter-event time function $\tau_s(\theta)$

In this subsection, we seek to obtain conditions under which the inter-event time function  $\tau_s(\theta)$  is continuous. Towards this aim, we make the following assumption about the matrix function  $M(\tau)$  since the general class of event-triggering rules (4) for an arbitrary  $M(\tau)$  is very broad.

**(A2)** Every element of the matrix  $M(\cdot)$  is a real analytic function of  $\tau$  and there exists a  $\tau_m$  such that  $M(\tau)$  is negative definite for  $(0, \tau_m)$ , where

$$\tau_m := \min\{\tau > 0 : \det(M(\tau)) = 0\}.$$

It is easy to verify that each  $M_i(\cdot)$  in (5), corresponding to the three triggering rules (3), satisfies Assumption (A2). This is because in  $M_1(\cdot)$  and  $M_2(\cdot)$  the only term dependent on  $\tau$  is the matrix exponential  $e^{A\tau}$ . In  $M_3(\cdot)$ , there is an additional exponential function  $e^{-r\tau}$  which is combined linearly with other terms dependent on  $\tau$ . Further, both  $M_1(0)$  and  $M_2(0)$  are negative definite. Though  $M_3(0) = 0$  the time derivative of  $M_3$  at  $\tau = 0$ ,  $\dot{M}_3(0)$ , is negative definite for suitable  $P$  and  $r$ .

Now, let  $\tau_{\min}$  and  $\tau_{\max}$  denote the global minimum and the global maximum of  $\tau_s(\theta)$ , respectively, that is,

$$\tau_{\min} := \min_{\theta \in [0, \pi)} \tau_s(\theta), \quad \tau_{\max} := \max_{\theta \in [0, \pi)} \tau_s(\theta).$$

For a matrix  $M(\cdot)$  that satisfies Assumption (A2), clearly  $\tau_{\min} = \tau_m$  as  $\det(M(\tau)) > 0$  in the interval  $(0, \tau_m)$  and according to Corollary 4,  $f_s(\theta, \tau) = 0$  has no solution for  $\tau \in (0, \tau_m)$  and has a solution for  $\tau = \tau_m$ . In general,  $\tau_{\max}$  may not exist, that is  $\tau_{\max} = \infty$ . In this case, it means that there exists a  $x_0 \in \mathbb{R}^2 \setminus \{0\}$  such that if  $x(t_k) = x_0$  then  $t_{k+1} = \infty$ . However, such an  $x_0$  cannot exist if  $A$  has positive real parts for both its eigenvalues and if the triggering rule (4) ensures  $x = 0$  is asymptotically stable. In such a case,  $\tau_{\max}$  is a finite quantity.

We approach the question of continuity of the inter-event time function  $\tau_s(\theta)$  by first analyzing the smoothness properties of the level set  $f_s(\theta, \tau) = 0$  in the  $(\theta, \tau)$  space. In particular, Assumption (A2) implies that the  $\det(M(\tau))$  is also real analytic and as a result it has finitely many zeros in the interval  $\tau \in [0, \tau_{\max}]$ . This observation, along with Corollary 4, can be used to say that  $f_s(\theta, \tau) = 0$  has finitely many connected branches in the set  $\{(\theta, \tau) \in [0, \pi) \times [0, \tau_{\max}]\}$ , which are arbitrarily smooth. We formally state this claim in the following result.

**Lemma 6.** (The level set  $f_s(\theta, \tau) = 0$  has finitely many connected branches, which are arbitrarily smooth). Suppose that  $M(\cdot)$  in (7) satisfies Assumption (A2) and  $\tau_{\max} < \infty$ . Then, the level set  $f_s(\theta, \tau) = 0$  has finitely many connected branches in the set  $\{(\theta, \tau) \in [0, \pi) \times [0, \tau_{\max}]\}$ . Each branch is an arbitrarily smooth curve in  $(\theta, \tau)$  space and can be parameterized by  $\tau$  in a closed interval.



*Proof.* First note that under Assumption (A2), all elements of  $M(\cdot)$  are real analytic functions, which implies that  $\det(M(\tau))$  is also a real analytic function of  $\tau$ . This is true because the determinant of a matrix is a polynomial of its elements, and products and sums of real analytic functions are also real analytic. As a consequence, on the closed and bounded interval  $[0, \tau_{\max}]$ ,  $\det(M(\tau))$  has finitely many zeros  $\tau_i$ . This implies that there are finitely many sub-intervals  $[\tau_i, \tau_{i+1}]$  of  $[0, \tau_{\max}]$  such that  $\det(M(\tau_i)) = \det(M(\tau_{i+1})) = 0$  and  $\det(M(\tau)) < 0$  for all  $\tau \in (\tau_i, \tau_{i+1})$ . Then, Corollary 4 guarantees that in each of the intervals  $[\tau_i, \tau_{i+1}]$ ,  $f_s(\theta, \tau) = 0$  has exactly two solutions for each  $\tau \in [\tau_i, \tau_{i+1}]$ , with the two solutions coincident at  $\tau_i$  and  $\tau_{i+1}$  but nowhere else. Thus,  $f_s(\theta, \tau) = 0$  has finitely many branches on  $\{(\theta, \tau) \in [0, \pi) \times [0, \tau_{\max}]\}$ . Smoothness of the branches is a consequence of the fact that  $f_s(\theta, \tau)$  is an arbitrarily smooth function, which is also evident from (8).  $\square$

Lemma 6 allows us to apply the implicit function theorem on  $f_s(\theta, \tau) = 0$  at all  $(\theta, \tau_s(\theta)) \in [0, \pi) \times [0, \tau_{\max}]$ , except at finitely many points. From this, we guarantee that  $\tau_s(\theta)$  is continuously differentiable in  $[0, \pi)$ , except at finitely many points.

**Theorem 7.** (*Inter-event time function is continuously differentiable except for finitely many  $\theta$* ). Suppose that  $M(\cdot)$  in (7) satisfies Assumption (A2) and  $\tau_{\max} < \infty$ . Then, the inter-event time function  $\tau_s(\theta)$  defined as in (7) is continuously differentiable on  $[0, \pi)$  except at finitely many  $\theta$ .

*Proof.* Recalling Lemma 6, consider any one of the finitely many branches of the level set  $f_s(\theta, \tau) = 0$  in the set  $\{(\theta, \tau) \in [0, \pi) \times [0, \tau_{\max}]\}$ . We denote the smooth parameterization of the branch by  $\tau$  as  $\theta(\tau)$ . Then, by Theorem 1 in [20] (Morse-Sard Theorem for real analytic functions), we can say that the critical values  $\theta$  of the function  $\theta(\tau)$  form a finite set. We can infer two observations from this. First, in tracing out the  $\tau_s(\theta)$  function, there are finitely many jumps between the branches of the level set  $f_s(\theta, \tau) = 0$  in the set  $\{(\theta, \tau) \in [0, \pi) \times [0, \tau_{\max}]\}$ . Second, for all  $\theta \in [0, \pi)$  except at finitely many  $\theta$ ,  $\frac{\partial f_s(\theta, \tau)}{\partial \tau}|_{(\theta, \tau_s(\theta))} \neq 0$  and therefore the implicit function theorem guarantees continuous differentiability of  $\tau_s(\theta)$  on  $[0, \pi)$  except at finitely many  $\theta$ .  $\square$

Based on Theorem 7 and its proof, we provide a sufficient condition for  $\tau_s(\theta)$  to be continuously differentiable.

**Corollary 8.** (*Corollary to Theorem 7*). If  $\frac{\partial f_s(\bar{\theta}, \bar{\tau})}{\partial \tau} \neq 0$  for all  $(\bar{\theta}, \bar{\tau}) \in \mathbb{R} \times \mathbb{R}$  such that  $f_s(\bar{\theta}, \bar{\tau}) = 0$ , then the inter-event time function  $\tau_s : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  defined as in (7) is continuously differentiable.  $\square$

Since in simulations, we encounter  $\tau_s(\theta)$  functions that visually seem to be continuous quite often, we present the following result in the special case where  $\tau_s(\theta)$  is a continuous function.

**Proposition 9.** If the inter-event time function  $\tau_s(\theta)$  is a continuous function, then every local extremum of  $\tau_s(\theta)$  is a global extremum.

*Proof.* We prove this result by contradiction. Suppose there exists an extremum of  $\tau_s(\theta)$  at  $\theta_1$  with value  $\tau_1 \in (\tau_{\min}, \tau_{\max})$ .

That is, the extremum at  $\theta_1$  is not a global extremum. Then, the assumptions that  $\tau_s(\theta)$  is continuous and  $\theta_1$  is a local extremizer and the fact that  $\tau_s(\theta)$  is periodic with period  $\pi$  imply that there exist  $\theta_2, \theta_3 \in (\theta_1, \theta_1 + \pi)$  such that  $\tau_s(\theta_2) = \tau_s(\theta_3) = \tau_s(\theta_1) = \tau_1$ . However, this contradicts Corollary 4, which says that for any given  $\tau_1 > 0$ ,  $f_s(\theta, \tau_1) = 0$  and hence  $\tau_s(\theta) = \tau_1$  can at most have two solutions for  $\theta$ . Therefore, the claim in the result must be true.  $\square$

#### IV. EVOLUTION OF THE INTER-EVENT TIME IN PLANAR SYSTEMS

In this section, we provide a framework for analyzing the evolution of the inter-event time along the trajectories of the planar system (1), that is  $n = 2$ , under the general class of event-triggering rules (4). In the previous section we showed that, for scale-invariant event-triggering rules, the inter-event time is determined completely by the “angle” of the state at the current event-triggering instant. Thus, we now study the angle at the next event as a map of the angle at the current event. Then, we consider the inter-event time function and the angle map together to analyze the evolution of the inter-event times as a function of time, particularly their steady state behavior.

Note that we can represent the evolution of the angle of the state from one triggering instant to the next as

$$\theta_{k+1} = \phi(\theta_k) := \arg \left( G(\tau_s(\theta_k)) \begin{bmatrix} \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix} \right),$$

where

$$\arg(x) := \begin{cases} \arctan(\frac{x_2}{x_1}), & \text{if } x_1 \geq 0 \\ \pi + \arctan(\frac{x_2}{x_1}), & \text{otherwise,} \end{cases}$$

and  $\theta_k = \arg(x(t_k))$  denotes the angle between the state  $x(t_k)$  and the positive  $x_1$  axis. Thus, the analysis of the inter-event time function  $\tau_s(\theta)$  and the angle map  $\phi(\theta)$  helps us to understand the evolution of the inter-event time for an arbitrary initial condition  $x(t_0)$ . In particular, the analysis of fixed points of the angle map helps us to determine the steady state behavior of the inter-event times.

**Remark 10.** (*Procedure for analyzing the evolution of inter-event times for the general class of event-triggering rules (4)*). Consider the planar system (1) along with the event-triggering rule (4), for a general  $M(\cdot)$  that satisfies Assumption (A2). First, we compute the functions  $\tau_s(\theta)$  and  $\phi(\theta)$  for  $\theta \in [0, \pi)$ , possibly by simulating the dynamics given by (1) and (3), for a time interval up to the first event. Then, we identify the fixed points of the angle map  $\phi(\cdot)$ , that is points  $\theta$  such that  $\phi(\theta) = \theta$ . Note that for each fixed point  $\theta$  of the map  $\phi(\cdot)$ , the inter-event times  $t_{k+1} - t_k = \tau_s(\theta)$  for all  $k \in \mathbb{N}_0$  and for all initial conditions  $x(t_0) = \alpha [\cos(\theta) \ \sin(\theta)]^T$ , with  $\alpha \in \mathbb{R} \setminus \{0\}$ . We can then analyze stability of the fixed points of the angle map  $\phi(\cdot)$  and identify the region of attraction of the asymptotically stable fixed points to determine the steady state behavior of  $\tau_e(x(t_k))$  for  $x(t_0)$  with  $\arg(x(t_0))$  in the region of attraction of a stable fixed point. Even if the angle map  $\phi(\cdot)$  does not have any fixed points, the inter-event times may converge to a constant  $c$  if the level set  $\tau_s(\theta) = c$  is forward invariant under the angle map  $\phi(\cdot)$ . However, Corollary 4 implies that

the level set  $\tau_s(\theta) = c$  is either empty, or equal to  $[0, \pi]$  or consists of a single or two points only. Thus, in order to study the steady state behavior of the inter-event times it suffices to study the  $\phi^k(\cdot)$  maps for all  $k \in \{1, 2, \dots, 4\}$ . In particular, if  $\tau_s(\cdot)$  is not a constant function and the  $\phi^k(\cdot)$  maps, for all  $k \in \{1, 2, \dots, 4\}$ , do not have fixed points then we can guarantee that there is no steady state behavior of the inter-event times for any initial condition. The functions  $\tau_s(\cdot)$  and  $\phi(\cdot)$  together can also potentially tell us about the transient behavior of the inter-event times. Finally, the asymptotically stable fixed points of the  $\phi(\cdot)$  map determine lines in  $\mathbb{R}^2$  along which the state trajectories asymptotically converge to the origin. •

The procedure in Remark 10 can be used to analyze the evolution of inter-event times under the general class of event-triggering rules (3). But, it is difficult to say anything more specific that holds for all the triggering rules. Thus, in the following subsection, we consider the specific event-triggering rule (3b), or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5b). We analyze the inter-event times that are generated by this rule for the planar system (1), that is  $n = 2$ .

#### A. Analysis of fixed points of $\phi(\cdot)$ with $M(\cdot) = M_2(\cdot)$

Here, our goal is to provide necessary conditions for the existence of a fixed point for the angle map  $\phi(\cdot)$  under the specific event-triggering rule (3b) or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5b). First, in the following lemma, we present a necessary condition on a function of time that must be satisfied if the angle map is to have a fixed point. Building on this lemma, we then present an algebraic necessary condition.

**Lemma 11.** (Necessary condition for the angle map to have a fixed point under triggering rule (3b)). Consider the planar system (1), that is  $n = 2$ , under the event-triggering rule (3b) or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5b). Suppose that Assumption (A1) holds and that the parameter  $\sigma \in (0, 1)$  is such that the origin of the closed loop system is globally asymptotically stable. Then, there exists a fixed point for the angle map  $\phi(\cdot)$  only if  $\det(L(\tau)) = 0$  for some  $\tau \in \mathbb{R}_{>0}$ , where

$$L(\tau) := (1 - \alpha)I + A^{-1}(e^{A\tau} - I)A_c$$

and  $\alpha = (1 + \sigma)^{-1}$ .

*Proof.* Suppose there exists a fixed point for the  $\phi(\theta)$  map. This implies that there exists an  $x \in \mathbb{R}^2 \setminus \{0\}$  such that if  $x(t_k) = x$ , then  $x(t_{k+1}) = \alpha x$  for some  $\alpha > 0$ . Note that  $\alpha$  cannot be negative because then  $\|x(t_k) - x(t_{k+1})\| = (1 - \alpha^{-1})\|x(t_{k+1})\|$ . However, this is not possible for the event-triggering rule (3b) if  $\sigma \in (0, 1)$ . Further, if  $\alpha > 1$  then for the initial condition  $x(t_0) = x$ ,  $x(t_k) = \alpha^k x$  would grow unbounded, which violates the assumption that  $\sigma$  is such that the event-triggering rule (3b) guarantees global asymptotic stability. Thus, it must be that  $\alpha \in (0, 1)$ . Using this information, from the event-triggering rule (3b), we obtain  $\alpha = (1 + \sigma)^{-1}$ . Now, we can express

$$x(t_{k+1}) = G(\tau')x(t_k) = (I + A^{-1}(e^{A\tau'} - I)A_c)x(t_k) = \alpha x(t_k),$$

where  $\tau' = \tau_e(x(t_k))$ . This implies that  $L(\tau')x(t_k) = 0$ . Therefore  $\det(L(\tau')) = 0$  as  $x(t_k) \in \mathbb{R}^2 \setminus \{0\}$ . □

Lemma 11 is only a necessary condition and  $\det(L(\tau)) = 0$  for some  $\tau \in \mathbb{R}_{>0}$ , is not sufficient for the angle map  $\phi(\cdot)$  to have a fixed point. This is because even if  $\det(L(\tau_1)) = 0$  for some  $\tau_1 \in \mathbb{R}_{>0}$ , that  $\tau_1$  may not be in the interval  $[\tau_{\min}, \tau_{\max}]$ . Even if  $\tau_1 \in [\tau_{\min}, \tau_{\max}]$ , there may not exist  $x \neq 0$  in the null space of  $L(\tau_1)$  such that  $\tau_e(x) = \tau_1$ .

**Remark 12.** (Angle map for the triggering rule (3b) has a bounded number of fixed points). Note that  $\det(L(\tau))$  is an analytic function of  $\tau$ . Hence,  $\det(L(\tau))$  has a bounded number of zeros in the interval  $[\tau_{\min}, \tau_{\max}]$ . If there does exist a  $\tau \in [\tau_{\min}, \tau_{\max}]$  such that  $\det(L(\tau)) = 0$  then either  $\phi(\theta) = \theta$  for all  $\theta \in [0, \pi)$  or the angle map  $\phi(\cdot)$  has a bounded number of fixed points. •

While Lemma 11 provides a necessary condition for the angle map  $\phi(\cdot)$  to have a fixed point, it may not be easy to verify if  $\det(L(\tau)) = 0$  for some  $\tau \in [\tau_{\min}, \tau_{\max}]$ . Thus, we next present an algebraic necessary condition for the existence of fixed points for the angle map  $\phi(\cdot)$ .

**Proposition 13.** (Algebraic necessary condition for the angle map to have a fixed point under triggering rule (3b)). Consider the planar system (1), with  $n = 2$ , under the event-triggering rule (3b) or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5b). Suppose that Assumption (A1) holds and that the parameter  $\sigma \in (0, 1)$  is such that the origin of the closed loop system is globally asymptotically stable. Further, assume that both the eigenvalues of  $A$  have positive real parts. Let  $A =: SJS^{-1}$ , where  $J \in \mathbb{R}^{2 \times 2}$  is the real Jordan form of  $A$ . Then, there exists a fixed point for the angle map  $\phi(\cdot)$  only if

- $\|R\| > 1$ , when either  $A$  is diagonalizable or non-diagonalizable with eigenvalue  $\lambda \geq 0.5$
- $\|R\| \geq \sigma_m \left( \sqrt{\frac{1}{\lambda^2} - 4} \right)$ , when  $A$  is non-diagonalizable with eigenvalue  $0 < \lambda < 0.5$

where

$$R := S^{-1} [I - (1 - \alpha)AA_c^{-1}] S$$

and

$$\sigma_m(\tau) := e^{\lambda\tau} \sqrt{\frac{(\tau^2 + 2) - \tau\sqrt{\tau^2 + 4}}{2}}.$$

*Proof.* Suppose there exists a fixed point for the  $\phi(\theta)$  map. Then by Lemma 11, we know that there exists a  $\tau \in \mathbb{R}_{>0}$  such that  $L(\tau)x_0 = 0$  for some  $x_0 \in \mathbb{R}^2 \setminus \{0\}$ . However, this is equivalent to saying

$$(e^{A\tau} - I)z_0 = -(1 - \alpha)AA_c^{-1}z_0, \quad z_0 = A_c x_0.$$

Note that  $A_c$  is invertible because we have assumed it is Hurwitz. Thus, there exists a vector  $v := S^{-1}A_c x_0$  such that

$$e^{J\tau}v = Rv, \text{ for some } \tau > 0. \quad (9)$$

Note that  $\|e^{J\tau}v\| \geq \sigma_{\min}(e^{J\tau})\|v\|$ , where  $\sigma_{\min}(e^{J\tau})$  denotes the minimum singular value of  $e^{J\tau}$ . Thus,  $\|R\| \geq \sigma_{\min}(e^{J\tau})$  for some  $\tau > 0$ . Recall that we assumed that both the eigenvalues of  $A$  have positive real parts. We can show that if  $A$  is diagonalizable and has real positive eigenvalues, then

$\sigma_{\min}(e^{J\tau}) = e^{\lambda_1 \tau}$  where  $\lambda_1$  is the minimum eigenvalue of  $A$ . Similarly, if  $A$  has complex conjugate eigenvalues then  $\sigma_{\min}(e^{J\tau}) = e^{\mu \tau}$  where  $\mu > 0$  is the real part of the eigenvalues. On the other hand, if  $A$  is non-diagonalizable with eigenvalue  $\lambda$ , then  $\sigma_{\min}(e^{J\tau}) = e^{\lambda \tau} \sqrt{\frac{(\tau^2+2)-\tau\sqrt{\tau^2+4}}{2}} =: \sigma_m(\tau)$ . When  $A$  is diagonalizable or non-diagonalizable with eigenvalue  $\lambda \geq 0.5$ ,  $\sigma_{\min}(e^{J\tau})$  is a monotonically increasing function of  $\tau$ . Thus,  $\sigma_{\min}(e^{J\tau}) > 1$  for all  $\tau > 0$ . When  $A$  is non-diagonalizable with eigenvalue  $0 < \lambda < 0.5$ ,  $\sigma_{\min}(e^{J\tau})$  attains a minimum value when  $\tau = \sqrt{\frac{1}{\lambda^2} - 4}$ . Thus,  $\sigma_{\min}(e^{J\tau}) \geq \sigma_m(\sqrt{\frac{1}{\lambda^2} - 4})$  for all  $\tau > 0$ . This completes the proof of the theorem.  $\square$

Next we show that the algebraic necessary condition for the angle map to have a fixed point under event-triggering rule (3b) is always satisfied if  $A$  is diagonalizable with eigenvalues having positive real parts and  $A_c$  has real negative eigenvalues.

**Proposition 14.** (Algebraic necessary condition for the angle map to have a fixed point under triggering rule (3b) is always satisfied when  $A_c$  has real negative eigenvalues). Consider planar system (1) under the event-triggering rule (3b) or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5). Suppose that Assumption (A1) holds and that the parameter  $\sigma \in (0, 1)$  is such that the origin of the closed loop system is globally asymptotically stable. Further, assume that both the eigenvalues of  $A$  have positive real parts and  $A$  is diagonalizable. Let  $A =: SJS^{-1}$ , where  $J$  is the real Jordan form of  $A$  and let  $A_c$  have real negative eigenvalues. Then,  $\|R\| > 1$ , where

$$R := S^{-1} [I - (1 - \alpha)AA_c^{-1}] S.$$

*Proof.* Note that the induced 2-norm of matrix  $R$  can be expressed as  $\|R\| = \sup\{u^T R v : \|u\| = \|v\| = 1, u, v \in \mathbb{R}^2\}$ . Let  $(\lambda_c, x)$  be an eigen-pair of  $A_c$ , where  $\lambda_c < 0$ . Let  $y$  be a unit vector defined as  $y := \frac{S^{-1}x}{\|S^{-1}x\|}$ . Then,

$$\begin{aligned} \|R\| &\geq y^T R y = y^T y - (1 - \alpha)y^T J S^{-1} A_c^{-1} S y \\ &= 1 - \frac{1 - \alpha}{\lambda_c} y^T J y \geq 1 - \frac{1 - \alpha}{\lambda_c} \lambda > 1 \end{aligned}$$

where  $\lambda = \lambda_{\min}(A) > 0$  when  $A$  has real eigenvalues or  $\lambda = \text{Re}(\lambda(A)) > 0$  when  $A$  has complex conjugate eigenvalues. In the last inequality, we have used the facts that  $\alpha \in (0, 1)$  (see proof of Lemma 11) and that  $\lambda_c < 0$ .  $\square$

## B. Stability of the Fixed Points of the Angle Map

Next, we are interested in analyzing the stability of the fixed points of the angle map as this will help us understand the steady state behavior of the inter-event times. First we present the main result of this subsection that gives sufficient conditions for the stability and instability of the fixed points of the angle map. Then we make some observations that are used for further analysis.

In numerical examples, we have often observed that the angle map is monotonically increasing with a stable fixed point and an unstable fixed point in the interval  $[0, \pi)$ . We try to provide some analytical guarantees for this behavior in the following theorem.

**Theorem 15.** (Stability of fixed points of the angle map). Consider the planar system (1) under the event-triggering rule (3b). Assume that the angle map  $\phi(\cdot)$  is continuous and monotonically increasing.

Suppose that the angle map has two fixed points  $\theta_1$  and  $\theta_2$  in the interval  $[0, \pi)$  where  $\theta_1 < \theta_2$ . Assume also that there exists neighborhoods of  $\theta_1$  and  $\theta_2$  in which  $\phi(\theta) - \theta$  is monotonically increasing and decreasing, respectively. Then,

- $\theta_1$  is an unstable fixed point
- $\theta_2$  is a stable fixed point and the region of convergence of  $\theta_2$  is  $(\theta_1, \theta_1 + \pi)$

*Proof.* Let us prove the first statement of this theorem. Note that  $\phi(\theta) - \theta < 0$  for all  $\theta \in (0, \theta_1)$  and  $\phi(\theta) - \theta > 0$  for all  $\theta \in (\theta_1, \theta_2)$ . Hence there exists an  $\varepsilon > 0$  such that for all  $\theta_0 \in N_\varepsilon(\theta_1)$  the sequence generated by the angle map,  $\theta_{k+1} = \phi(\theta_k)$ , diverges away from  $\theta_1$ . Thus  $\theta_1$  is an unstable fixed point of  $\phi(\cdot)$ .

Next, we prove the second statement of this theorem. Note that  $\phi(\theta) - \theta > 0$  for all  $\theta \in (\theta_1, \theta_2)$  and  $\phi(\theta) - \theta < 0$  for all  $\theta \in (\theta_2, \theta_1 + \pi)$ . Then, for all  $\theta \in (\theta_1, \theta_2)$ ,  $\theta < \phi(\theta) < \phi(\theta_2) = \theta_2$  as the angle map  $\phi(\cdot)$  is monotonically increasing. Hence for any  $\theta_0 \in (\theta_1, \theta_2)$ , the sequence generated by the angle map,  $\theta_{k+1} = \phi(\theta_k)$ , is a monotonically increasing bounded sequence and it converges to  $\theta_2$ . Similarly,  $\theta > \phi(\theta) > \phi(\theta_2) = \theta_2$  for all  $\theta \in (\theta_2, \theta_1 + \pi)$ . Hence for any  $\theta_0 \in (\theta_2, \theta_1 + \pi)$ , the sequence generated by the angle map,  $\theta_{k+1} = \phi(\theta_k)$ , is a monotonically decreasing bounded sequence and it converges to  $\theta_2$ . This proves that the fixed point  $\theta_2$  is a stable fixed point and the region of convergence of  $\theta_2$  is  $(\theta_1, \theta_1 + \pi)$ .  $\square$

Theorem 15 is generalizable to the case where the angle map has more than one pair of stable and unstable fixed points in the interval  $[0, \pi)$ .

**Remark 16.** Let the angle map have a set of fixed points  $\{\theta_1, \theta_2, \dots, \theta_{2m}\}$  in the interval  $[0, \pi)$  where  $\theta_i < \theta_{i+1} \ \forall i \in \{1, 2, \dots, 2m-1\}$ . Let  $\theta_{2i-1}$  have a neighborhood in which  $\phi(\theta) - \theta$  increases monotonically and  $\theta_{2i}$  have a neighborhood in which  $\phi(\theta) - \theta$  decreases monotonically  $\forall i \in \{1, 2, \dots, m\}$ . Then  $\theta_{2i-1}$  is an unstable fixed point and  $\theta_{2i}$  is a stable fixed point  $\forall i \in \{1, 2, \dots, m\}$ . Region of convergence of  $\theta_{2i}$  is  $(\theta_{2i-1}, \theta_{2i+1}) \ \forall i \in \{1, 2, \dots, m-1\}$  and region of convergence of  $\theta_{2m}$  is  $(\theta_{2m-1}, \theta_1 + \pi)$ . •

Next, we present a geometric interpretation of the event-triggering rule (3b), which we then use to give bounds on the difference between an angle  $\theta$  and  $\phi(\theta)$ .

**Remark 17.** (Geometric interpretation of the event-triggering rule (3b)). The locus of points  $x$  which satisfy the equation  $\|x - \hat{x}\| = \sigma \|x\|$  for a fixed  $\hat{x}$  and  $\sigma$  is a circle with center at  $\frac{\hat{x}}{1-\sigma^2}$  and radius  $\frac{\sigma}{1-\sigma^2} \|\hat{x}\|$ . Also note that origin is always outside this circle. Hence, the event-triggering rule (3b) ensures that for all  $k \in \mathbb{N}_0$ ,  $x(t_k)$  and  $x(t_{k+1})$  satisfy  $\left\|x(t_{k+1}) - \frac{x(t_k)}{1-\sigma^2}\right\| = \frac{\sigma}{1-\sigma^2} \|x(t_k)\|$ . •

This observation leads us to an upper bound for  $|\phi(\theta_k) - \theta_k|$ ,  $\forall k \in \mathbb{N}_0$ , which we present in the following result.



**Lemma 18.** (Upper bound on  $|\phi(\theta_k) - \theta_k|$ ). Consider the planar system (1) under the event-triggering rule (3b) or equivalently (4) with  $M(\cdot) = M_2(\cdot)$  given in (5). Suppose that the parameter  $\sigma \in (0, 1)$  is such that the origin of the closed loop system is globally asymptotically stable. Then the evolution of the angle of the state from one sampling time to the next is uniformly bounded by  $\sin^{-1}(\sigma)$ . That is,  $|\phi(\theta_k) - \theta_k| \leq \sin^{-1}(\sigma)$ ,  $\forall k \in \mathbb{N}_0$ .

*Proof.* According to the geometric interpretation of the event-triggering rule (3b) provided in Remark 17,  $x(t_{k+1})$  is on the circle with center at  $\frac{x(t_k)}{1-\sigma^2}$  and radius  $\frac{\sigma}{1-\sigma^2} \|x(t_k)\|$ . Thus, the angle between  $x(t_k)$  and  $x(t_{k+1})$  is the maximum when  $x(t_{k+1})$  is on a tangent to this circle that passes through the origin. As a tangent to the circle is perpendicular to the radial line passing through the point of tangency, the maximum possible angle is exactly equal to  $\sin^{-1}(\sigma)$ . That is,  $|\phi(\theta_k) - \theta_k| \leq \sin^{-1}(\sigma)$ ,  $\forall k \in \mathbb{N}_0$ .  $\square$

We also make an observation regarding the periodicity of  $\phi(\theta) - \theta$  map which is presented as a remark here.

**Remark 19.** ( $\phi(\theta) - \theta$  is periodic with period  $\pi$ ). As the inter-event time function  $\tau_s(\theta)$  is periodic with period  $\pi$ ,  $\phi(\theta + \pi) = \arg(G(\tau_s(\theta + \pi))x_{\theta + \pi}) = \arg(-G(\tau_s(\theta))x_\theta) = \phi(\theta) + \pi$ . Thus  $\phi(\theta + \pi) - (\theta + \pi) = \phi(\theta) - \theta$  for all  $\theta \in \mathbb{R}$ . •

We use the results and observations in this section as a base to carry out our analysis in the self-triggered control case.

## V. EVOLUTION OF INTER-EVENT TIMES UNDER SELF-TRIGGERED CONTROL

In this section, we analyze the evolution of inter-event times along the trajectories of the  $n$ -dimensional system (1) under a region-based self-triggered control method. In the region-based self-triggered control method, we partition the state space into a finite number of regions  $R_i \subset \mathbb{R}^n$ , where  $i \in \{1, 2, \dots, r\}$ , each of which is a cone. We then associate each region with a fixed inter-event time  $\tau_i$ , which is chosen such that  $\tau_i \leq \tau_e(x)$  for all  $x \in R_i$ . Recall that  $\tau_e(x)$  is the event-triggered inter-event time function defined as in (6). An alternative way of partitioning the state-space is to first discretize the interval  $[\tau_{\min}, \tau_{\max}]$  as  $\{\tau_1, \tau_2, \dots, \tau_r\}$  where  $\tau_{\min}$  and  $\tau_{\max}$  denote the global minimum and the global maximum of  $\tau_e(x)$ , respectively, and  $\tau_1 \leq \tau_{\min} < \tau_2 < \dots < \tau_r \leq \tau_{\max} < \tau_{r+1}$ . Then we partition the state space into  $r$  regions as follows,

$$R_i := \{x \in \mathbb{R}^n : \tau_i \leq \tau_e(x) < \tau_{i+1}\} \quad \forall i \in \{1, \dots, r\}.$$

**(A3)** In either construction, we make the standing assumption that the regions  $R_i$ , for all  $i \in \{1, \dots, r\}$ , are cones.

Now, we define the region-based self-triggering rule by setting the inter-event time as

$$t_{k+1} - t_k = \tau_i, \quad \text{if } x(t_k) \in R_i, \quad i \in \{1, \dots, r\}. \quad (10)$$

Thus, the region to which  $x(t_k)$  belongs determines fully the inter-event time  $t_{k+1} - t_k$ , and as a result, the set of possible inter-event times is finite. This property of the inter-event times makes the analysis of their steady state behavior much easier than in event-triggered control setting.

We carry out the analysis of the inter-event times through a map that describes the evolution of the state from one event to the next. This is similar to our method in the event-triggered control setting for planar systems. We define the *inter-event state jump map* or simply the *gamma map* as

$$x(t_{k+1}) = \gamma(x(t_k)) := G(\tau_i)x(t_k), \quad \text{i.s.t. } x(t_k) \in R_i. \quad (11)$$

We can analyze the steady state behavior of the inter-event times along the trajectories of the system (1) under the region-based self-triggering rule (10), by studying about the existence of a subset of a region that is positively invariant under the  $\gamma$  map. We define such a set as a positively invariant subregion. A set  $\bar{R} \subseteq R_i$  for some  $i \in \{1, \dots, r\}$  is called a positively invariant subregion under the gamma map if  $G(\tau_i)x \in \bar{R}$  for all  $x \in \bar{R}$ . Given Assumption (A3), it suffices to look for positively invariant subregions that are cones. Note that if there does not exist a positively invariant subregion then we cannot observe steady state convergence of inter-event times.

**Remark 20.** (Positively invariant ray). Consider the system (1) under the self-triggering rule (10). If  $\exists x \in R_i$  such that  $G(\tau_i)x = \alpha x$  for some  $\alpha \in \mathbb{R}_{>0}$  and for some  $i \in \{1, \dots, r\}$ , then  $\{\beta x : \beta > 0\} \subseteq R_i$  is a positively invariant subregion under the gamma map (11), and we refer to it as a positively invariant ray. •

It is easy to find the existence of a positively invariant ray as we have finite, to be specific  $r$ , number of regions. We can determine the eigenvectors of  $G(\tau_i)$  for each  $i \in \{1, \dots, r\}$  and check if any of them belongs to the corresponding region. Next, we analyze the stability of a positively invariant ray. We recall the definition of stability of a set for convenience.

**Definition 21.** (Stability of a set). Under the discrete-time dynamics (11), a set  $\mathcal{M} \subset \mathbb{R}^n$  is said to be

- stable if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$d(\mathcal{M}, x(t_0)) < \delta \implies d(\mathcal{M}, x(t_k)) < \varepsilon \quad \forall k \in \mathbb{N},$$

$$\text{where } d(\mathcal{M}, x) := \inf_{y \in \mathcal{M}} \|y - x\|.$$

- unstable if it is not stable.
- asymptotically stable if it is stable and  $\delta$  can be chosen such that

$$d(\mathcal{M}, x(t_0)) < \delta \implies \lim_{k \rightarrow \infty} d(\mathcal{M}, x(t_k)) = 0. \quad \bullet$$

Next, we provide a necessary and sufficient condition for a positively invariant ray to be asymptotically stable under the assumption that the corresponding  $G$  matrix is diagonalizable. Note that while the essential idea of the following result comes from basic results in linear algebra, there are some important distinctions. First, we are showing asymptotic stability of a ray rather than a line or a subspace. Second, each of the matrices  $G(\tau_i)$  operate on only a conic region in  $\mathbb{R}^n$ .

**Theorem 22.** (Necessary and sufficient condition for a positively invariant ray to be asymptotically stable.). Consider system (1) under the self-triggering rule (10). Assume that the self-triggering rule (10) guarantees global asymptotic stability of the origin of the closed loop system. Let  $\exists x^* \in R_i$  such that  $G(\tau_i)x^* = \alpha x^*$  for some  $\alpha \in \mathbb{R}_{>0}$  and for some  $i \in \{1, \dots, r\}$ .



Assume that  $x^*$  is an interior point of  $R_i$  and that  $G(\tau_i)$  is diagonalizable. Then the positively invariant ray  $\mathcal{M} := \{\beta x^* : \beta > 0\}$ , under (11), is

- stable if and only if the magnitude of every eigenvalue of  $G(\tau_i)$  is less than or equal to  $\alpha$ .
- asymptotically stable if and only if  $\alpha$  is the dominant eigenvalue of  $G(\tau_i)$ , that is, algebraic multiplicity of  $\alpha$  is 1 and the magnitude of every eigenvalue of  $G(\tau_i)$ , except  $\alpha$ , is strictly less than  $\alpha$ .

*Proof.* We prove the result in the transformed coordinates given by the diagonalization of  $G(\tau_i)$ . Let  $G(\tau_i) = V\Lambda V^{-1}$ , where  $\Lambda \in \mathbb{C}^{n \times n}$  is a diagonal matrix with  $(\Lambda)_{jj} = \lambda_j$  and let  $v_j$  denote the  $j^{\text{th}}$  column of  $V$  for all  $j \in \{1, 2, \dots, n\}$ . Let the eigen-pair  $(\lambda_1, v_1)$  be same as  $(\alpha, x^*)$ . Consider the change of variables  $z := V^{-1}x$  and the set  $\bar{R}_i := \{V^{-1}x : x \in R_i\}$ . Then,

$$z^* := V^{-1}x^* = V^{-1}v_1 = [1 \quad 0 \quad \dots \quad 0]^T$$

is an interior point of  $\bar{R}_i$  as  $x^*$  is an interior point of  $R_i$ . That is, we can find a  $\rho > 0$  such that  $B_\rho(z^*) \subset \bar{R}_i$ . As  $\bar{R}_i$  is a cone, the convex cone  $\mathcal{C} := \{cz : z \in B_\rho(z^*), c > 0\} \subset \bar{R}_i$ .

Now, we prove the first statement of the theorem. Suppose that the magnitude of every eigenvalue of  $G(\tau_i)$  is less than or equal to  $\alpha$ , that is  $|\lambda_j| \leq \alpha$  for all  $j \in \{1, 2, \dots, n\}$ . Then, for all  $z \in B_\delta(z^*)$ ,  $\frac{\Lambda z}{\lambda_1} \in B_\delta(z^*)$  as

$$\left\| \frac{\Lambda z}{\lambda_1} - z^* \right\| = \left\| \frac{\Lambda}{\lambda_1} (z - z^*) \right\| \leq \|z - z^*\| \leq \delta.$$

We can always choose a small enough  $\delta > 0$  such that the component of  $z$  along the  $z_1$  axis is nonzero for all  $z \in B_\delta(z^*)$ . Thus, if  $\delta \leq \rho$ , the convex cone generated by  $B_\delta(z^*)$  is positively invariant under the gamma map (11). As a result, given an  $\varepsilon > 0$ , we can guarantee that the ray spanned by  $z^*$  is stable. Equivalently, in the original  $x$  coordinates, the positively invariant ray  $\mathcal{M}$  is stable. For the converse, suppose that there exists at least one eigenvalue of  $G(\tau_i)$ , say  $\lambda_k$  for some  $k \in \{2, 3, \dots, n\}$ , with magnitude greater than  $\alpha$ . Then, using similar arguments as above, we can show that the iterates of  $G(\tau_i)^m x$  diverge away from the ray  $\mathcal{M}$  for  $x$  arbitrarily close to the ray  $\mathcal{M}$ . Thus, the positively invariant ray  $\mathcal{M}$  is unstable, which proves the first statement of the theorem.

Now, we prove the second statement of the theorem. Suppose that  $\alpha$  is the dominant eigenvalue of  $G(\tau_i)$ . Then, from the proof of the previous statement, we know that there exists a positively invariant convex cone  $\mathcal{S} \subseteq R_i$ , under the gamma map, such that  $x^* \in \mathcal{S}$  and the component of  $x$  along the dominant eigenspace of  $G(\tau_i)$  is nonzero for all  $x \in \mathcal{S}$ . Let  $x$  be an arbitrary vector in  $\mathcal{S}$ . We can represent  $x$  as a linear combination of eigenvectors of  $G(\tau_i)$ ,  $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ .

$$\begin{aligned} \lim_{m \rightarrow \infty} G(\tau_i)^m x &= \lim_{m \rightarrow \infty} (c_1 \lambda_1^m v_1 + c_2 \lambda_2^m v_2 + \dots + c_n \lambda_n^m v_n) \\ &= \lim_{m \rightarrow \infty} \lambda_1^m \sum_{j=1}^n c_j \left( \frac{\lambda_j}{\lambda_1} \right)^m v_j \\ &= \lim_{m \rightarrow \infty} \lambda_1^m c_1 v_1. \end{aligned}$$

The last equality follows from the fact that  $\lambda_1 \in (0, 1)$  as the self-triggering rule guarantees global asymptotic stability of

the origin of the closed loop system and  $\lim_{m \rightarrow \infty} \left( \frac{\lambda_i}{\lambda_1} \right)^m = 0$  for all  $i \in \{2, 3, \dots, n\}$  as  $\lambda_1$  is the dominant eigenvalue of  $G(\tau_i)$ . This implies that  $\mathcal{M}$  is asymptotically stable.

Now, suppose there exist at least two eigenvalues of  $G(\tau_i)$  with absolute value equal to  $\alpha$ , say  $|\lambda_1| = |\lambda_2| = \alpha$ . Then for any  $x \in \mathcal{S}$  with a nonzero component along  $v_2$ ,  $G(\tau_i)^m x$  does not converge to the subspace spanned by  $v_1$  as  $m$  tends to infinity. Hence  $\mathcal{M}$  is not asymptotically stable, which completes the proof of the second statement of this theorem.  $\square$

In the following proposition, we extend Theorem 22 and its proof, to provide a sufficient condition for the instability of a positively invariant ray even when the corresponding  $G$  matrix is not diagonalizable.

**Proposition 23.** (Sufficient condition for a positively invariant ray to be unstable). Consider system (1) under the self-triggering rule (10). Let  $\exists x^* \in R_i$  such that  $G(\tau_i)x^* = \alpha x^*$  for some  $\alpha \in \mathbb{R}_{>0}$  and for some  $i \in \{1, \dots, r\}$ . Assume that  $x^*$  is an interior point of  $R_i$ . If  $\alpha$  is a defective eigenvalue of  $G(\tau_i)$ , then the positively invariant ray  $\mathcal{M} := \{\beta x^* : \beta > 0\}$ , under (11), is unstable.

*Proof.* As in the proof of Theorem 22, we do a coordinate transformation and consider the change of variables  $z = V^{-1}x$  and  $z^* := V^{-1}x^*$ . Here the only difference is that now  $\Lambda$  is the Jordan normal form of  $G(\tau_i)$  with  $\Lambda_{11} = \lambda_1 = \alpha$ . Choose  $\delta > 0$  such that the component of  $z$  along the  $z_1$  axis is positive for all  $z \in B_\delta(z^*)$ . Then consider the set  $S := \{z \in B_\delta(z^*) : z_2 < 0, z_i = 0 \ \forall i \neq 1, 2\}$ . Let  $z$  be an arbitrary vector in  $S$ . Then we can write  $\Lambda z = \lambda_1 z + z_2 z^*$ . As  $z_2 < 0$ , angle between  $z$  and  $z^*$  is strictly less than the angle between  $\Lambda z$  and  $z^*$ . Using this fact, we can show that there exists  $x$  arbitrarily close to the ray  $\mathcal{M}$  such that the iterates of  $G(\tau_i)^m x$  diverge away from the ray  $\mathcal{M}$ . Thus, the positively invariant ray  $\mathcal{M}$  is unstable, which completes the proof of this proposition.  $\square$

Note that a positively invariant subregion need not always contain a positively invariant ray. We can have a positively invariant subregion, as a union of finite number of disjoint proper cones, which may or may not contain a positively invariant ray. First we present the formal definition of a proper cone and then we provide a result which presents a necessary condition for the existence of a positively invariant subregion as a union of finite number of disjoint proper cones.

**Definition 24.** (Proper cone). Let  $\mathcal{C}$  be a nonempty closed and convex subset of  $\mathbb{R}^n$ . We say that  $\mathcal{C}$  is a proper cone if it satisfies the following conditions:

- $\mathbb{R}_+ \mathcal{C} := \{\alpha x : \alpha \geq 0 \text{ and } x \in \mathcal{C}\} \subseteq \mathcal{C}$  (i.e.,  $\mathcal{C}$  is positively homogeneous);
- $\mathcal{C} \cap -\mathcal{C} = \{0\}$  (i.e.,  $\mathcal{C}$  is pointed or salient);
- $\text{span}(\mathcal{C}) = \mathbb{R}^n$  (i.e.,  $\mathcal{C}$  is full or solid).

**Proposition 25.** Consider system (1) under the self-triggering rule (10). Under the map (11), if there exists a positively invariant subregion,  $\bar{R} \subseteq R_i$  for some  $i \in \{1, 2, \dots, r\}$ , that is a union of a finite number of proper cones, then there exists  $m \in \mathbb{N}$  such that the spectral radius  $\rho(G(\tau_i)^m)$  is an eigenvalue

of  $G(\tau_i)^m$  and  $\bar{R}$  contains an eigenvector corresponding to  $\rho(G(\tau_i)^m)$ .

*Proof.* Let there exist a positively invariant subregion  $\bar{R} \subseteq R_i$  for some  $i \in \{1, 2, \dots, r\}$ , as a union of finite number of proper cones, that is,

$$\bar{R} := \bar{R}_1 \cup \bar{R}_2 \cup \dots \cup \bar{R}_p \subseteq R_i,$$

where each  $\bar{R}_l$  is a proper cone. By using continuity of the gamma map (11) on  $R_i$ , we can say that for each  $\bar{R}_j$

$$\{G(\tau_i)x : x \in \bar{R}_j\} \subseteq \bar{R}_l$$

for some  $l \in \{1, 2, \dots, p\}$ . As  $\bar{R}$  is a union of finite number of proper cones, according to pigeonhole principle, there should exist an  $m \in \mathbb{N}$  and some  $j \in \{1, 2, \dots, p\}$  such that

$$\{G(\tau_i)^m x : x \in \bar{R}_j\} \subseteq \bar{R}_j.$$

This implies that  $\bar{R}_j$  is positively invariant for  $G(\tau_i)^m$ . Then according to Theorem 4.7 in [21], the spectral radius  $\rho(G(\tau_i)^m)$  is an eigenvalue of  $G(\tau_i)^m$  and  $\bar{R}_j$  contains an eigenvector corresponding to  $\rho(G(\tau_i)^m)$ .  $\square$

Using Proposition 25, we provide some necessary conditions for the existence of a positively invariant subregion that is a union of finite number of proper cones and does not contain a positively invariant ray.

**Corollary 26.** (Necessary conditions for the existence of a positively invariant subregion that does not contain a positively invariant ray.). Consider system (1) under the self-triggering rule (10). There exists a positively invariant subregion,  $\bar{R} \subseteq R_i$  for some  $i \in \{1, 2, \dots, r\}$  that is a union of finite number of proper cones and does not contain a positively invariant ray only if one of the following conditions hold,

- $G(\tau_i)$  has complex conjugate eigenvalues  $a \pm jb$ , where  $j$  denotes the imaginary unit, with magnitude equal to the spectral radius  $\rho(G(\tau_i))$  and  $\exists m \in \mathbb{N}_{>1}$  such that  $m \tan^{-1}(\frac{b}{a}) = \pm 2\pi$ ;
- $G(\tau_i)$  has eigenvalues equal to  $\pm \rho(G(\tau_i))$ ;
- $G(\tau_i)$  has an eigenvalue equal to  $-\rho(G(\tau_i))$  but no eigenvalue equal to  $\rho(G(\tau_i))$ .

*Proof.* According to Proposition 25, there exists a positively invariant subregion,  $\bar{R} \subseteq R_i$  for some  $i \in \{1, 2, \dots, r\}$ , as a union of finite number of proper cones only if there exists  $m \in \mathbb{N}$  such that the spectral radius  $\rho(G(\tau_i)^m)$  is an eigenvalue of  $G(\tau_i)^m$  and  $\bar{R}$  contains an eigenvector  $v$  corresponding to  $\rho(G(\tau_i)^m)$ . As  $\bar{R}$  does not contain a positively invariant ray,  $m > 1$  and  $v$  should not be an eigenvector of  $G(\tau_i)$  corresponding to the eigenvalue  $\rho(G(\tau_i))$ , if it exists. This implies that the eigen subspace of  $G(\tau_i)^m$  corresponding to the eigenvalue  $\rho(G(\tau_i)^m)$  is not equal to the eigen subspace of  $G(\tau_i)$  corresponding to the eigenvalue  $\rho(G(\tau_i))$ , if it exists. This is possible only if either one of the three conditions in the result hold.  $\square$

Now consider the special case in which the region-based self-triggering rule (10) is designed in such a way that the inter-event time  $t_{k+1} - t_k \leq \tau_e(x(t_k))$ , where  $\tau_e(\cdot)$  is the inter-event time function corresponding to the specific event-triggering rule (3b). Using the geometric interpretation of

event-triggering rule (3b), we can say that the self-triggering rule (10) ensures that for all  $k \in \mathbb{N}_0$ ,  $x(t_k)$  and  $x(t_{k+1})$  satisfy  $\|x(t_{k+1}) - \frac{x(t_k)}{1-\sigma^2}\| \leq \frac{\sigma}{1-\sigma^2} \|x(t_k)\|$ . Thus, under this special case we can rule out the possibility of occurrence of the third condition in Corollary 26.

Also note that, if we consider a planar system (1), with  $n = 2$ , under this special case of region-based self triggered control method, then we can say that the evolution of the angle of the state from one sampling time to the next is uniformly bounded by  $\sin^{-1}(\sigma)$ . Thus if the state space is partitioned in such a way that the angle spanned by each conic region is greater than or equal to  $\sin^{-1}(\sigma)$ , then any vector  $x$  is mapped, under the gamma map (11), either into its own conic region or into one of the neighboring conic regions. Therefore under this special case, if there exists a positively invariant subregion then it always contains a positively invariant ray.

## VI. NUMERICAL EXAMPLES

In this section, we illustrate our results and highlight some interesting behavior of the inter-event times using numerical examples of several different systems of the form (1) with the event-triggering rule (3b). We also illustrate the self-triggered control scenario. In each case, we choose the control gain matrix  $K$  so that  $A_c = (A + BK)$  is Hurwitz. We choose a quadratic Lyapunov function  $V(x) = x^T P x$ , where  $P$  is the solution of the Lyapunov equation (2) with  $Q = I$ . Then, from an analysis as in [1], we set the thresholding parameter  $\sigma = \frac{0.99\lambda_{\min}(Q)}{2\|PBK\|}$  in the event-triggering rule (3b) for the sampled data controller (1b). Next, we describe specific cases in detail.

*Case 1:* Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: Ax + Bu.$$

The system matrix  $A$  has real eigenvalues at  $\{1, 2\}$ . We let the control gain  $K = [0 \quad -6]$  so that  $A_c$  has real eigenvalues at  $\{-1, -2\}$ . Figure 1 shows the simulation results for Case 1. For this case, the inter-event time function  $\tau_s(\theta)$  is continuous and periodic with period  $\pi$ . From Figure 1a and Figure 1b we can verify that  $\det(M(\tau)) = 0$  has exactly two solutions and they are precisely  $\tau_{\min}$  and  $\tau_{\max}$  respectively. Figure 1c shows that there are two points at which  $\det(L(\tau)) = 0$ . Figure 1d verifies that the angle map  $\phi(\cdot)$  has exactly two fixed points at  $\theta_1 = 1.9\text{rad}$  and  $\theta_2 = 2.52\text{rad}$ , where  $\theta_2$  is a stable fixed point. Figure 1e is the phase portrait of the closed loop system. Notice that the state trajectories converge to a radial line which makes an angle of 2.52 radian with the positive  $x_1$  axis, which is exactly the point at which the angle map  $\phi(\cdot)$  has the stable fixed point. From Figure 1f it is clear that the inter-event time converges to a steady state value of 0.281, which is exactly equal to  $\tau_s(\theta_2)$ .

*Case 2:* In this case, we use the same  $A$  and  $B$  matrices as in Case 1 but choose the control gain  $K = [0 \quad -5]$  so that  $A_c$  has complex conjugate eigenvalues at  $\{-1 + i, -1 - i\}$ . Figure 2 shows the simulation results for Case 2. For this case also the inter-event time function is continuous and periodic with period  $\pi$ . From Figure 2a and Figure 2b we can verify that  $\det(M(\tau)) = 0$  has exactly two solutions and these two

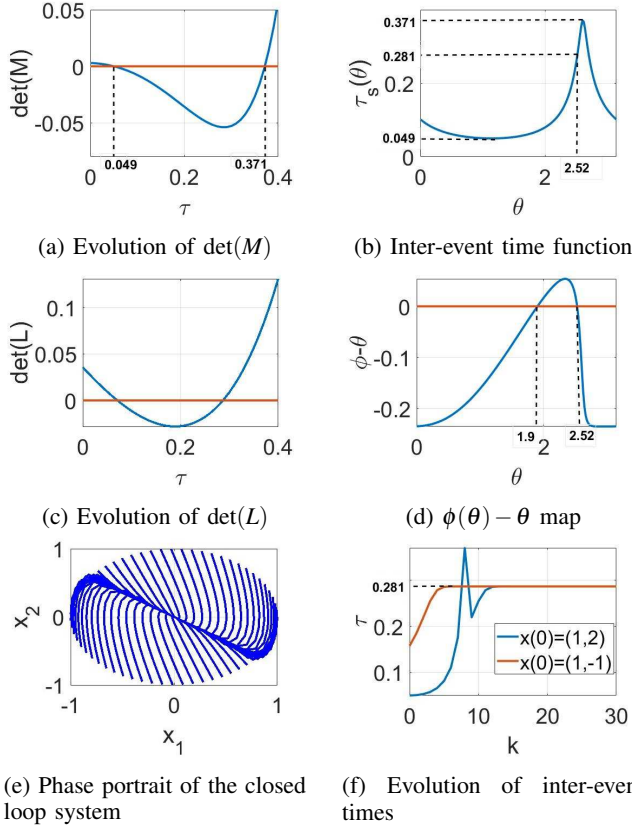


Fig. 1: Simulation results of case 1 when  $A_c$  has real eigenvalues at  $\{-1, -2\}$ .

points are  $\tau_{\min}$  and  $\tau_{\max}$  respectively. Figure 2c shows that  $\det(L(\tau))$  is always positive. Therefore the  $\phi$  map in Figure 2d has no fixed point. Figure 2e represents the phase portrait of the closed loop system. From Figure 2f it is clear that the inter-event time is not converging to a steady state value.

**Case 3:** Consider another system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u =: Ax + Bu.$$

The system matrix  $A$  has real eigenvalues at  $\{1, 2\}$ . The control gain  $K = \begin{bmatrix} -1 & -0.8 \\ 1.8 & -4 \end{bmatrix}$  so that  $A_c$  has complex conjugate eigenvalues at  $\{-1 + 0.2i, -1 - 0.2i\}$ . Figure 3 shows the simulation results of this system for the event triggering rule (3b). Figure 3a shows that the angle map  $\phi(\cdot)$  has two fixed points, where the larger one is a stable fixed point. In Figure 3b the inter-event time is converging to a steady state value for two different initial conditions. This example is important because it shows that even if  $A_c$  has only complex conjugate eigenvalues, the inter-event times may still converge to a steady state value.

**Case 4:** Now consider the system,

$$\dot{x} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: Ax + Bu.$$

$A$  has real and equal eigenvalues at  $\{1, 1\}$ . The control gain  $K = \begin{bmatrix} -2 & -4 \end{bmatrix}$  so that  $A_c$  has eigenvalues at  $\{-1 + 2i, -1 - 2i\}$ . Figure 4 shows the simulation results of this system for

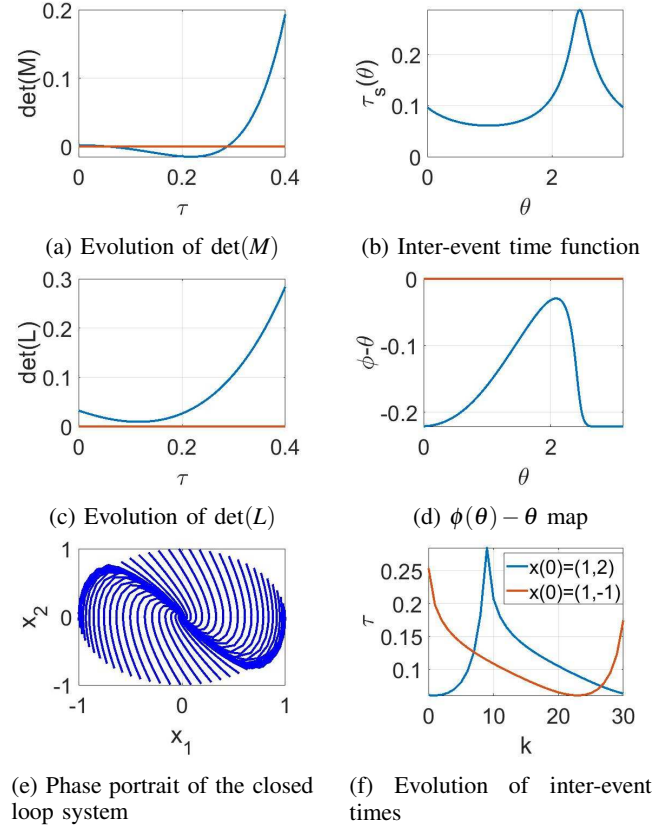


Fig. 2: Simulation results of case 2 when  $A_c$  has complex conjugate eigenvalues at  $\{-1 + i, -1 - i\}$ .

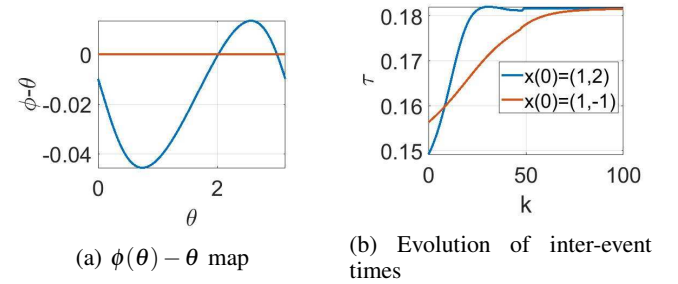
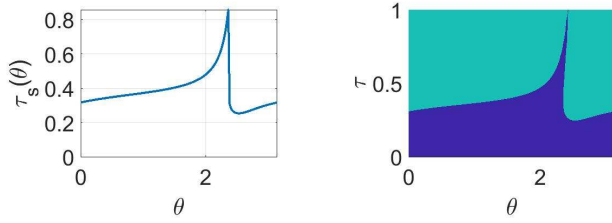


Fig. 3: Simulation results of case 3 when  $A_c$  has complex conjugate eigenvalues at  $[-1 + 0.2i, -1 - 0.2i]$ .

the triggering rule (3a). Figure 4a shows that the inter-event time function  $\tau_s(\theta)$  is discontinuous around  $\theta = 2.3$  radians. In Figure 4b, we can see that around  $\theta = 2.3$  radian there is a jump in the smallest  $\tau$  value at which  $f_s(\theta, \tau) = 0$ . This causes a point of discontinuity in the inter-event time function.

**Case 5:** Next, we illustrate the results from the self-triggered control section. Figure 5 represents the simulation results of the system as in Case 1, under the region based self-triggering rule (10). From Figure 1b we can verify that  $\tau_{\min} = 0.049$  and  $\tau_{\max} = 0.371$ . We choose the number of regions  $r = 7$  and discretize the interval  $[\tau_{\min}, \tau_{\max}]$  into 7 equal parts. We partition the state space into 7 regions and associate each region with the corresponding inter-event time. Figure 5a shows the state-space partitions and we can see that





(a) Inter-event time function (b) Level set of  $f_s(\theta, \tau) = 0$

Fig. 4: Simulation results of case 4 with discontinuous inter-event time function.

each partition is a union of finite number of convex cones. Figure 5b shows the plot of angle of the state vs the inter-event time. From Figure 5c it is clear that the angle map has two fixed points, at  $\theta_1 = 1.92$  radians and  $\theta_2 = 2.51$  radians, in the interval  $[0, \pi)$ . We can also numerically verify that the matrices  $G(\tau_1)$  and  $G(\tau_2)$ , where  $\tau_1$  and  $\tau_2$  are the respective inter-event times of  $\theta_1$  and  $\theta_2$ , have an eigenvector which makes an angle  $\theta_1$  and  $\theta_2$  respectively with the positive  $x_1$  axis. This implies the existence of two positively invariant rays for  $\theta \in [0, 2\pi)$ . We can also analyze the stability of these two positively invariant rays by analyzing the spectrum of the corresponding  $G$  matrices. Note that  $G(\tau_1)$  is diagonalizable and the magnitude of the eigenvalue of  $G(\tau_1)$  corresponding to the eigenvector which makes an angle  $\theta_1$  with the positive  $x_1$  axis is less than the magnitude of the other eigenvalue. Hence the positively invariant ray corresponding to the fixed point  $\theta_1$  is unstable, which also implies that  $\theta_1$  is an unstable fixed point of the angle map  $\phi_s(\cdot)$ . Similarly  $G(\tau_2)$  is also diagonalizable and the eigenvalue of  $G(\tau_2)$  corresponding to the eigenvector which makes an angle  $\theta_2$  with the positive  $x_1$  axis is the dominant eigenvalue. Hence the positively invariant ray corresponding to the fixed point  $\theta_2$  is asymptotically stable, which also implies that  $\theta_2$  is a stable fixed point of the angle map  $\phi_s(\cdot)$ . Figure 5d represents the region of convergence of the asymptotically stable positively invariant ray. Note that the whole  $\mathbb{R}^2$  plane except the unstable positively invariant radial line is the region of convergence of the asymptotically stable positively invariant radial line. We can numerically verify that none of the  $G$  matrices corresponding to the state-space regions satisfy the necessary conditions for the existence of a positively invariant subregion as a union of finite number of proper cones. Hence we can eliminate the possibility of existence of the same.

**Case 6:** Now consider the same system as in the previous case with a region based self-triggered controller designed in an alternate way. In this case, first the  $\mathbb{R}^2$  plane is partitioned uniformly into 24 convex cones with the angle spanned by each convex cone being 15 degrees. Then the inter-event time corresponding to each conic region is calculated by finding the minimum event-triggered inter-event time corresponding to the relative thresholding event triggering rule (3b) over all the states in that region. Figure 6 represents the simulation results of this case. Figure 6a represents the state-space partitions and Figure 6b plots the inter-event time as a function of the angle of the state. Figure 6c shows that the angle map

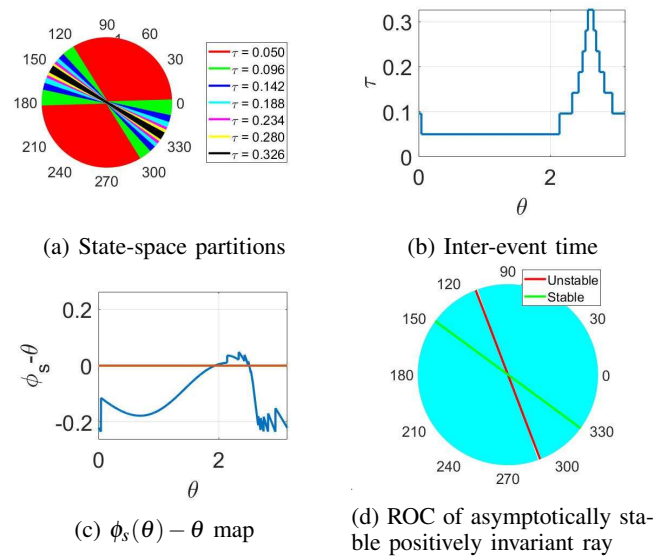


Fig. 5: Simulation results of case 5.

$\phi_s(\cdot)$  has two fixed points  $\theta_1$  and  $\theta_2$  in the interval  $[0, \pi)$ , which also implies the existence of two positively invariant rays. As in the previous case, we can analyze the stability of these two positively invariant rays. We can show that the positively invariant ray corresponding to the fixed point  $\theta_1$  is unstable and the positively invariant ray corresponding to the fixed point  $\theta_2$  is asymptotically stable, which also implies that  $\theta_1$  is an unstable fixed point and  $\theta_2$  is a stable fixed point of the angle map  $\phi_s(\cdot)$ . Figure 6d represents the region of convergence of the asymptotically stable positively invariant ray. As in the previous case the whole  $\mathbb{R}^2$  plane except the unstable positively invariant radial line is the region of convergence of the asymptotically stable positively invariant radial line. In this case, the state space is partitioned in such a way that the angle spanned by each conic region is greater than  $\sin^{-1}(\sigma\sqrt{1 - \frac{\sigma^2}{2}})$ , therefore we can eliminate the possibility of existence of a positively invariant subregion as a union of finite number of disjoint proper cones.

**Case 7:** Now let us consider a 3-dimensional system,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u =: Ax + Bu.$$

$A$  has real eigenvalues at  $\{1, 2, -3\}$ . The control gain  $K = [0 \ -18 \ -6]$  ensures that  $A_c$  has real eigenvalues at  $\{-1, -2, -3\}$ . Figure 7 represents the simulation results of the above case. Figure 7a shows the value of the inter-event time function  $\tau_e(\cdot)$  on the unit sphere under the event-triggering rule (3b). We determine the global minimum and global maximum of  $\tau_e(\cdot)$ , respectively, as  $\tau_{\min} = 0.0088$  and  $\tau_{\max} = 0.2655$ . We choose the number of regions  $r = 5$  and discretize the interval  $[\tau_{\min}, \tau_{\max}]$  into 5 equal parts. Then, we partition the state space into 5 regions and associate each region with the corresponding inter-event time. Figure 7b shows the state-space partitions and the corresponding inter-event time. Now, we can find numerically the existence of two positively invariant rays which are shown in Figure 7c.

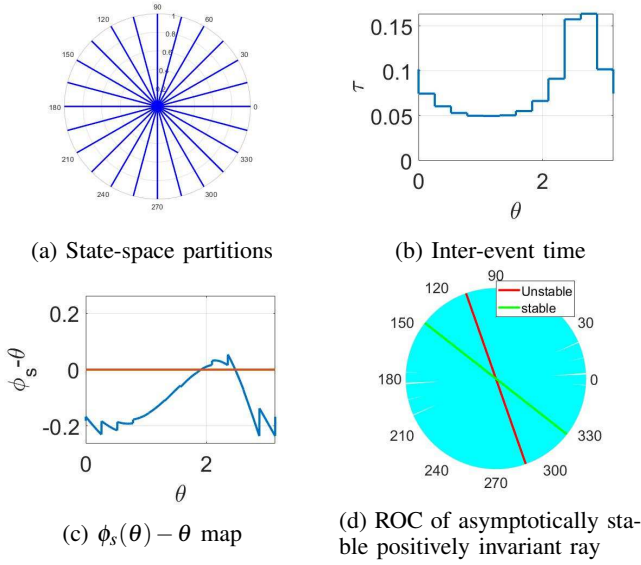


Fig. 6: Simulation results of case 6.

We can also analyze the stability of these two positively invariant rays by analyzing the spectrum of the corresponding  $G$  matrices. Note that, here one of the positively invariant rays is asymptotically stable and the other one is unstable. Next, we choose an arbitrary vector close to the asymptotically stable positively invariant ray as the initial state and Figure 7d represents the evolution of inter-event times under the region-based self-triggering rule (10). We can see that the inter-event times converges to a steady state value which is exactly equal to the inter-event time corresponding to the asymptotically stable positively invariant ray.

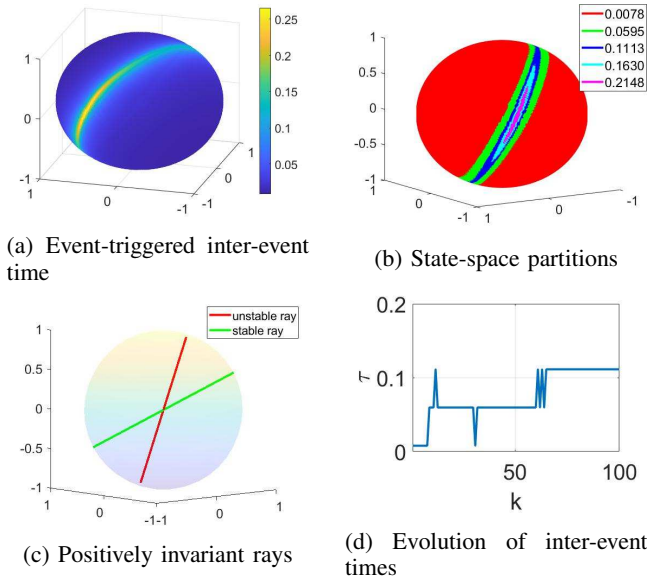


Fig. 7: Simulation results of case 7.

## VII. CONCLUSION

In this paper, we analyzed the evolution of inter-event times along the trajectories of linear systems under scale-invariant

event/self-triggering rules. In the case of event-triggered control, we mostly restricted our attention to planar systems. We analyzed the properties of the inter-event time as a function of the state at an event triggering instant, such as periodicity and continuity. Under some mild assumptions, we concluded that the inter-event time function is continuous except at finitely many angles and we found sufficient conditions that ensure continuity. We then analyzed the map that determines the evolution of the angle of the state from one event to the next. Combining these two, we provided a framework for analyzing the evolution of the inter-event times for planar systems. For a specific event triggering rule, we determined a necessary condition for the convergence of inter-event time to a steady state value. We also analyzed the stability and region of convergence of a fixed point of the angle map. Then, we considered a region-based self triggered control system in which the state space is partitioned into a finite number of conic regions and each region is associated with a fixed inter-event time. Then we analyzed the evolution of inter-event times under the proposed self-triggered control method. In this framework, studying steady state behavior of the inter-event times is equivalent to studying the existence of a conic subregion, which is a positively invariant set under the map that gives the evolution of the state from one event to the next. We provided a sufficient condition for the existence of a special kind of positively invariant subregion called positively invariant ray. We also provided necessary and sufficient conditions for a positively invariant ray to be asymptotically stable. We then analyzed region of convergence of a positively invariant subregion, which is asymptotically stable. We verified the proposed results through numerical simulations.

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