Dt. 23/01/18

* Linearly independent set

$V \quad S \subseteq V$

$S = \{v_1, v_2, \ldots, v_K\}$

We say S is l.I. if for any l.c.

$c_1 v_1 + \ldots + c_K v_K = 0$

$\Rightarrow c_1 = 0 = c_2 \ldots = c_K \Rightarrow c_i = 0 \ \forall \ i = 1, \ldots k$

Eg: $S \subseteq V$.

$S = \{v_1, v_2, \ldots, v_k\}$ is l.d. iff atleast one element should be written as a linear combination of remaining elements.

S is l.d. $\exists$ set of scalars $c_1, c_2, \ldots, c_k$ not all zero such that (say $c_l \neq 0$)

$c_1 v_1 + \ldots + c_l v_l + \ldots + c_k v_k = 0$

$\Rightarrow v_l = \frac{1}{c_l}(-v_1 c_1 - c_2 v_2 + \ldots - c_{l-1} v_{l-1} - c_{l+1} v_{l+1} - \ldots - c_K v_K)$

Converse:

Suppose $v_i = d_1 v_1 + d_2 v_2 + \ldots + d_{i-1} v_{i-1} + d_{i+1} v_{i+1} + \ldots + d_K v_K$.

$d_1 v_1 + d_2 v_2 + \ldots + d_{i-1} v_{i-1} - v_i + d_{i+1} v_{i+1} + d_K v_K = 0$

$S = \{v_1, v_2, \ldots, v_K\}$ is l.d. $(d_i \neq 0)$

~~Que.~~ let V be a vector space over field F. let the dimension of V is finite and suppose $v_1, v_2, \ldots, v_m$ spans V then any set which contains more than 'm' elements (vectors) is linearly dependent.

Proof: (Soln):

let $S = \{w_1, w_2, \ldots, w_n\} \subseteq V$ be set s.t. $n > m$

We show S is l.d. Then $\exists$ scalars A_{ij} s.d.

eg: $w_j = \sum_{i=1}^{m} A_{ij} v_i$.

$\therefore w_1 = (A_{11} v_1 + A_{21} v_2 + \ldots + A_{m1} v_m)$

let $c_1 w_1 + c_2 w_2 + \ldots + c_n v_n = 0$.

$$\sum_{j=1}^{n} c_j w_j = 0.$$

$$\sum_{j=1}^{n} c_j \cdot \sum_{i=1}^{m} A_{ij} v_i = 0.$$

$$\sum_{i=1}^{m} \in \sum_{j=1}^{n} c_j A_{ij} v_i = 0.$$

$(A_{11} c_1 + A_{12} c_2 + \ldots + A_{1n} c_n) v_1 = 0$

$\vdots$

$(A_{m1} c_1 + A_{m2} c_2 + \ldots + A_{mn} c_n) v_n = 0$.

As $v_1 - v_n \neq 0$ $m < n$ the above system will have non-trivial soln. (Atleast one c_j's $\neq 0$).

Que. Any two bases of a finite dimensional vector space will have same no. of elements.

Suppose

$S = \{v_1, v_2, \ldots, v_n\}$

$S' = \{w_1, w_2 \ldots w_m\}$

is basis of V.

Let S be basis of V.

$\Rightarrow$ S is linearly independent.

but S' is also linearly independents.

$m \leq n$

Similarly $n \geq m$

$\Rightarrow m = n$

~~Que.~~

Remark: Let V be finite dimensional vector space of dimension 'n' Then no subsets of V which contains less than 'n' vectors can span V, any subset of V which contains n+1 vectors are more than that will be linearly dependent.

Que: ~~Show~~ Check ~~that~~ if the set ~~is~~ $\{(1,1,0),(1,3,2),(4,9,5)\}$ forms a basis of $\mathbb{R}^3$. $\dim(\mathbb{R}^3) = \underline{\underline{3}}$.

Soln: $\overset{Ho}{(x,y,z)} = a(1,1,0) \overset{H_1}{+} b(1,3,2) + c(4,9,5)$

$$x = a+b+4c$$
$$y = a+3b+9c$$
$$z = 2b+5c.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & x \\ 1 & 3 & 9 & y \\ 0 & 2 & 5 & z \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & x \\ 0 & 2 & 5 & y-x \\ 0 & 2 & 5 & z \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & x \\ 0 & 2 & 5 & y-x \\ 0 & 0 & 0 & z-y+x \end{array}\right]$$

$z-y+x=0$.

$2b+5c=y-x$.

$a+b+4c=x$.

$a+b+4c=0$

$a+3b+9c=0$

$2b+5c=0$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Non Trivial soln

No, not ~~our~~ $\mathbb{R}^3$'s basis.

$W \subseteq \mathbb{R}^3$

$W = \text{span}\{(1,1,0),(0,-1,1)\}$

basis of W. $\boxed{\text{dimension}}$

W is subspace of V.

Symmetric Matrix: $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ $= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$a \quad b$

$b \quad d$

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$

Remark: if V is a finite dimensional vector space and W is a subspace of V then dimension of W will always be $<=$ dimension of V (if $W \subseteq V$ is a subspace of W.)

× show that $\{1, t, t^2, \ldots, t^n\}$ is basis of $P_n(t)$. (H/W)

Que Find dimension & bases of following subspace of $\mathbb{R}^3$.

$V = \mathbb{R}^3$.

$W = \{(a,b,c) : a+b+c=0\}$

$W = \{(a,b,c) \; a=b=c\}$

$\cancel{(1,1,0)}$ $\cancel{(0,0,0), (1,-1,1)}$

(a) $(1,2,3) \notin W \Rightarrow W$ is proper subspace ~~{a=b,~~

of V.

$S = \{\underline{(1,0,-1)}, \underline{(-1,1,0)}\}$ is l.I.

$(a,b,c) = x(1,0,-1) + y(-1,1,0)$

$= (x-y, y, -x)$ $\boxed{c=-x}$

$x - y = a$ $\boxed{b=y}$

$y = b$ $b+c=0$

$-x = c$ $\boxed{a=0}$

$= -\underline{\underline{c}}(1,0,-1) + \underline{\underline{b}}(-1,1,0)$

$-c-b = a$

$b = b$

$c = c$

$\dim = \underline{\underline{2}}$.

(b) $S = \{(1,1,1)\}$ is basis of W.

$\dim = \underline{\underline{1}}$.

Que. (H/W) Let S be a linearly independent subset of a vector space V. Suppose w is a vector in vector space V which doesn't belong to the subspace spanned by S then the new set obtained by adjoining w to S is linearly independent.

S L.I.

$w \notin \text{span } S$.

$S' = S \cup \{w\}$ is L.I.

DOUBT

$\exists c_1, c_2, \ldots, c_k, \tilde{c}$ not all zero

s.t. $\tilde{c}w + c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$

$\tilde{c} \neq 0$ ($\because$ if $\tilde{c} = 0 \Rightarrow c_1 = c_2 = \ldots = c_k = 0$ which is contradiction

else if $\tilde{c} = 0$ $c_i = k \neq 0$ $\Rightarrow$ contradicts S is L.I.)

$\Rightarrow \tilde{c} \neq 0$.

$w = \frac{-1}{\tilde{c}}(c_1 v_1 + \ldots + c_k v_k)$

w is l. combⁿ of $v_1, \ldots, v_k$.

$w \in \text{span } S$ contradⁿ.

Dt: 27/01/18

Que: W is a subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = (1, -2, 5, 3)$

$u_2 = (2, 3, 1, 4)$,

$u_3 = (3, 8, -3, +5)$.

Find basis & dim. of W. Extend the basis of W to form basis of $\mathbb{R}^4$.

~~Soln~~:

Soln:

$$A = \begin{bmatrix} 1 & -2 & 5 & 3 \\ 2 & 3 & 1 & 4 \\ 3 & 8 & -3 & +5 \end{bmatrix}$$

A = A. rowReducedEchleon (Gauss Elimination);

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & -2 \\ 3 & 8 & -3 & +5 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & -2 \\ 0 & 14 & -18 & -24 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & -2 \\ 0 & 0 & 0 & 0 \ \cancel{+10} \end{bmatrix}$$

~~linear independent~~,

A. dim() = 2

$\cancel{=3}$ basis: $\{(1, -2, 5, 3), (0, 7, -9, -2)\}$ ~~(0, 0, 0, -10)~~.

~~basis~~ ~~u_1, u_2, u_3~~.

dd $e_3 = (0,0,1,0)$

$e_4 = (0,0,0,1)$

It'll span $\mathbb{R}^4$ and will be l.I.

Que. Show that the set

$S = \{1, t, t^2, \ldots, t^n\}$ $P_n(t)$.

dim = n+1

$c_1 + c_2 t + \ldots + c_{n+1} t^n = 0 \cdot t^0 + 0 t^1 + 0 \cdot t^2 + \ldots + 0 t^n$

$S = \{1, (t-c), \ldots, (t-c)^n\}$

basis of $P_n(t)$. dim = n+1

Theorem:

~~Que.~~ ~~If~~ Suppose V is a vector space over the field $\mathbb{F}$

$S = \{v_1, v_2, \ldots, v_n\}$ basis of V

then any $v \in V$ can be written uniquely ~~in~~ as $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$

Proof: Let

$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$

$v = d_1 v_1 + d_2 v_2 + \ldots + d_n v_n$

$c_1 v_1 + \ldots + c_n v_n = d_1 v_1 + \ldots + d_n v_n$

$\Rightarrow (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \ldots + (c_n - d_n) v_n = 0$

$\Rightarrow c_1 = d_1 \quad c_2 = d_2 \ldots c_n = d_n$

$[v]_S = [c_1\ c_2 \dots c_n]^T$

is called as co-ordinate vector of element v wrt the basis S.

Que. Find co-ordinate vector of $(V = \mathbb{R}^2)$

$v = (4, -3)$

$[v]_\beta$

$\beta = \{(1,1), (2,3)\}$

$c_1(1,1) + c_2(2,3) = (x, y)$

$c_1 + 2c_2 = x$

$c_1 + 3c_2 = y$. $\boxed{c_2 = y - x}$

$c_1 + 2y - 2x = x$

$\boxed{c_1 = 3x - 2y}$

$c_1 + 2c_2 = 0$

$c_1 + 3c_2 = 0$

$\boxed{c_2 = 0 \quad c_1 = 0}$ L.I.

$c_1 = 12 - 2(-3) = 18$

$c_2 = -3 - 4 = -7$.

$[v]_S = [18, -7]^T = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$

$[(a,b)]_\beta = [3a - 2b,\ b - a]^T$

Que. Find co-ordinate vector of

$v = 3t^3 - 4t^2 + 2t - 5$. $P_3(t)$.

$[v]_\beta$ $\quad \beta = \{1, t, t^2, t^3\}$

$[v]_{\tilde\beta}$ $\quad \tilde\beta = \{1, (t-1), (t-1)^2, (t-1)^3\}$

$[v]_\beta = [-5 \quad 2 \quad -4 \quad 3]^T$

$c_1 + c_2(t-1) + c_3(t-1)^2 + c_4(t-1)^3$

$= c_1 + c_2 t - c_2 + c_3(t^2 - 2t + 1)$
$\quad + c_4(t^3 - 3t^2 + 3t - 1)$

$= (c_1 - c_2 + c_3 - c_4)$
$\quad + (c_2 - 2c_3 + 3c_4)t$
$\quad + (c_3 - 3c_4)t^2 + c_4 t^3$

$\boxed{c_4 = 3}$

$c_3 - 9 = -4$

$\boxed{c_3 = 5}$

$c_2 - 10 + 9 = +2$

$\boxed{c_2 = 3}$

$c_1 - 3 + 5 - 3 = -5$

$c_1 - 1 = -5$

$\boxed{c_1 = 4}\boxed{(-1)}$

$[v]_\beta = [-4 \quad 3 \quad 5 \quad 3]^T$

<u>Theorem</u>: Let V be n-dimensional vector space & β and β' be two bases of V. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ $\tilde{\beta} = \{v_1', v_2', \ldots, v_n'\}$. Then there exists invertible matrix P such that

(1) $[v_\beta] = P[v]_{\beta'}$ (2) $[v]_{\beta'} = P^{-1}[v_\beta]$

where $P = [P_j]$ where ~~$P_j = \alpha_j$ j 1 to n~~

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P_j are column $P_j = [v_j']_\beta$ $j = 1, 2, \dots, n$ of P.

Que. $V = \mathbb{R}^2$

$\beta = \{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}\}$

$\beta' = \{\overset{v_1'}{(1,1)}, \overset{v_2'}{(2,3)}\}$

$v = (4, -3)$

$[v_1']_\beta = [1 \quad 1]$

$[v_2']_\beta = [2, 3]$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$[v]_\beta = [4, -3]$

$[v]_{\beta'} = [18, -7]$

$$P[v]_\beta = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 18 \\ -7 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = [4, -3]^T.$$

$$= \underline{\underline{[v]_{\beta'}}}.$$

H/W. Next Que to Prev Que's Prev. instance.

$\beta]$

Que. Show that if ~~V is~~ dimension of V is 'n' and V is a vector space over $\mathbb{F}$ then there exists one-to-one onto map between V and $\mathbb{F}^n$.

$\boxed{V \cong \mathbb{F}^n}$. ~~is~~ $\dim V = n$.

Soln: As $\dim V = n$

let $B = \{v_1, v_2, \ldots, v_n\}$ be basis of V

i.e. any $v \in V$ can be written UNIQUELY as

$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

Define $T: V \rightarrow \mathbb{F}^n$

$Tv = (c_1 \ c_2 \ \cdots \ c_n)$

let $Tv = Tw$

We have

$v = c_1 v_1 + \ldots + c_n v_n$

$w = d_1 v_1 + \ldots + d_n v_n$

$Tv = Tw$

$[c_1 \ c_2 \ldots \ c_n] = [d_1 \ d_2 \ldots \ d_n]$

$\boxed{c_i = d_i}$

$\Rightarrow$ $v = w$. One-to-one.

onto $\rightarrow$

let $(k_1 \ k_2 \ \ldots \ k_n) \in \mathbb{F}^n$,

we have

$v = k_1 v_1 + \ldots + k_n v_n \in V$.

$Tv = (k_1 \ldots k_n)$

Onto.

1/10

Que. Find dim. & basis of solⁿ space of homogeneous system

$$x+2y+2z-s+2t=0$$
$$x+2y+3z+s+t=0$$
$$3x+6y+8z+s+5t=0$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}_{3\times 5}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 4 & -1 \end{bmatrix}$$

$\boxed{z+2s=0.}$

$\boxed{t=0}$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{x+2y-5s=0.}$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -5 & 4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$r \neq n$ ∴ Trivial soln only

Q

$x+2y+2z-s+2t=0$

$x+2y+3z+s+t=0$

$3x+6y+8z+s+5t=0.$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}$$

Let $t=\alpha \quad s=\beta.$

$x+2y+2z=\beta-2\alpha.$

$x+2y+3z=-\alpha-\beta.$

$z=-\alpha-\beta+2\alpha-\beta.$

$\boxed{z=\alpha-2\beta}$

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Dt: 30/01/18

* Direct sum:

Let V be vector space over the field $\mathbb{F}$.

let U and W be subsets of V

$U+W = \{u+w \mid u \in U \ \& \ w \in W\}$

then $U+W$ is a subspace of V.

We note $0 = 0+0$ where $0 \in U$ and $0 \in W$

$\Rightarrow 0 = 0+0 \in U+W$.

$z = u+w, z' = u'+w'$

$\alpha z + \beta z' = \alpha(u+w) + \beta(u'+w')$

$= (\alpha u + \beta u') + (\alpha w + \beta w')$

$\alpha u + \beta u' \in U \ \& \ \alpha w + \beta w' \in W$.

$\alpha z + \beta z' \in U+W$.

Question. $\left.\begin{array}{l} U \subseteq U+W \\ W \subseteq W+U \end{array}\right\}$ Observation.

* If $U \& W$ be subspace of V Then

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

Proof: let $\dim U = m$

$\dim W = n$.

Let $\dim(U \cap W) = r$

~~Let $S = \{w_1, \dots, w_r\}$ be basis of $U \cap W$~~

~~Let $S' = \{w_1 \dots w_r, u_1, u_2 \dots u_{m-r}\}$~~

Let $S = \{v_1, v_2, \dots v_r\}$ be basis of $U \cap W$.

~~Let $S' = \{w_1, \dots, w_r,$~~

~~Let $S' = \{v_1, v_2, \dots v_r, u_1, u_2 \dots u_{m-r}\}$~~

~~be basis of~~

Let $S' = \{v_1\ v_2 \dots v_r\ u_1\ u_2 \dots u_{m-r}\}$

$S'' = \{v_1\ v_2 \dots, v_r\ w_1\ w_2 \dots w_{n-r}\}$

be basis of U, W respectively.

claim $\beta = \{v_1\ v_2 \dots v_r, u_1, u_2 \dots u_{m-r}\ w_1\ w_2 \dots$

$r + m+n-2r = m+n-r$

Is basis of $U+W$ having $m+n-r$ elements

Let $z \in U+W$.

$z = u+w$ for some $u \in U$ and $w \in W$.

Then $u = c_1 v_1 + c_2 v_2 + \dots + c_r v_r + d_1 u_1 + \dots + d_{m-r} u_{m-r}$

$w = c_1' v_1 + c_2' v_2 + \dots + c_r' v_r + d_1' w_1 + \dots + d_{m-r}' w_{m-r}$

So $z = (c_1 + c_1')v_1 + (c_2 + c_2')v_2 + \dots + (c_r + c_r')v_r$

$\cancel{+ (d_1 + d_1')u_1 + \dots + (d_{m-r} + d_{m-r}) u_{m-r}}$

$+ d_1' w_1 + \dots + d_{n-r}' w_{n-r}$

$+ d_1 u_1 + \dots + d_{m-r} u_{m-r}$

$\in \text{span}\ \beta$.

Suppose

$a_1v_1 + \dots + a_r v_r + b_1 u_1 + \dots + b_{m-r} u_{m-r}$
$+ c_1 w_1 + \dots + c_{n-r} w_{n-r} = 0$

Denote

$v = a_1 v_1 + \dots + a_r v_r + b_1 u_1 + \dots + b_{m-r} u_{m-r}$.

$v = -(c_1 w_1 + \dots + c_{n-r} w_{n-r})$

$\Rightarrow v \in U \cap W$.

Then $v = d_1 v_1 + \dots + d_r v_r$

$\Rightarrow c_1 w_1 + \dots + c_{n-r} w_{n-r} + d_1 v_1 + \dots + d_r v_r = 0$

$\Rightarrow c_1 = 0 \dots c_{n-r} = 0$ ($\because$ S" is basis)

$d_1 = 0 \dots d_r = 0$

$v = c_1 w_1 + \dots + c_{n-r} w_{n-r} = 0 = a_1 v_1 + \dots + a_r v_r$
$+ b_1 u_1 + \dots + b_{m-r} u_{m-r}$

$a_1 = 0 \dots a_r = 0$

$b_1 = 0 \dots b_{m-r} = 0$.

Que. let V be a vector space of 2×2 matrix.

$V = M_{2\times2}$. $U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| \, a \neq 0 \,||\, b \neq 0 \right\} \rightarrow$ basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

$W = \left\{ \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \middle| \, c \neq 0 \text{ OR } d \neq 0 \right\}$

Find $\dim(U+W)$

$\dim U = 2$

$\dim W = 2$.

$U+W = \left\{ \begin{bmatrix} a+c & b \\ d & 0 \end{bmatrix} \right\}$

$U \cap W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle| \, a \neq 0 \right\} \rightarrow \dim(U \cap W) = 1$

basis $\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\dim(U+W) = 2 + 2 - 1 = 4 - 1 = 3$.

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Direct sum: Vector space V is said to be direct sum of the subspaces U & W if any $v \in V$ can be written uniquely as $v = u + w$ for $u \in U$ and $w \in W$.

We denote it by $V = U \oplus W$.

eg: $V = U \oplus W$ iff

① $V = U + W$

② $U \cap W = \{0\}$.

Proof: Suppose $V = U \oplus W$

Then $v \in V$ can be written as

$v = u + w$ for $u \in U, w \in W$.

let $\cancel{v} \in (U \cap W)$ where $v \neq 0$.

$\Rightarrow v \in U \quad v \in W$.

$v = v + 0 \in U + W$.

$\Rightarrow v = 0 + v \in U + W$

$v = v + 0 = 0 + v$

$\Rightarrow \boxed{v = 0}$

CONVERSE: Let $V = U + W$

$U \cap W = \{0\}$

let $v \in U + W$.

Suppose $v = u_1 + w_1$ for $u_1, u_2 \in U$

$v = u_2 + w_2$ for $w_1, w_2 \in W$.

$u_1 + w_1 = u_2 + w_2$.

$u_1 - u_2 = w_2 - w_1 \Rightarrow u_1 - u_2 \in U \cap W = \{0\}$

$w_1 - w_2 \in U \cap W = \{0\}$

$\therefore u_1 = u_2, \; w_1 = w_2$.

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Que. $V = \mathbb{R}^3$

① $U = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$

$W = \{(0, b, c) \quad b, c \in \mathbb{R}\}$.

$\mathbb{R}^3 = U + W$ True.

$\mathbb{R}^3 = U \oplus W$

$U \cap W \neq \{0\}$

$(1,0,0) = (1,0,0) + (0,0,0)$

$(1,0,0) = (1,-1,0) + (0,1,0)$.

① $U = \{(a, b, c) \mid a = b = c\}$

$W = \{(0, b, c) \quad b, c \in \mathbb{R}\}$

Show that $\mathbb{R}^3 = U \oplus W$.

$\underset{U}{(a, b, c)} \quad \underset{W}{(0, b_1, c_1)}$

$u + w = (a, b + b_1, c + c_1) \in \mathbb{R}^3$ True.

$U \cap W = \{(0, 0, 0)\}$.

let $(a, b, c) \in U \cap W$.

$\left.\begin{array}{l}(a, b, c) \in U \Rightarrow a = b = c \\ (a, b, c) \in W \Rightarrow a = 0\end{array}\right\} \Rightarrow a = 0 = b = c$

$U \cap W = \{0\}$.

~~(x, y, z)~~

$(a, b, c) = (a, a, a) + (0, b - a, c - a)$.

H/W - 1)

Consider the following subspaces of $\mathbb{R}^5$

$V = \mathbb{R}^5$

$U = \text{span}\{(1,3,-2,2,3)(1,4,-3,4,2),$
$(2,3,-1,-2,9)\}$

$W = \text{span}\{(1,3,0,2,1),(1,5,-6,6,3),$
$(2,5,3,2,1)\}$

Find basis & dimension of $U+W$ and $U \cap W$.